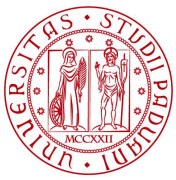

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CBI and Hawkes processes: theory and application to power markets

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Introduction

Energy markets and, in particular, electricity markets, exhibit very peculiar features. Electricity can be considered a commodity, just as oil and grain are; however, electricity markets are particularly different from other commodity markets.
Firstly, electricity cannot be stored economically (the problem is essentially due to the physical difficulty in storing large amount of energy and also to the fact that this procedure is extremely expensive); this implies that it is purchased primarily for consumption and then a relatively small change in demand or capacity can cause an immediate jump in price. Although producers can adjust their supply, there is always a time delay and therefore it is quite common to observe spikes (a comparatively large upward or downward movement of a price in a short period of time) in electricity prices.
Secondly, electricity consumption exhibits strong seasonality and sometimes also the capacity is affected by seasonality like hydro power generation which varies with seasons. Due to the non-storability and seasonality electricity prices are highly volatile and exhibit jumps and spikes from time to time.
Due to this properties and also to the fact that empirical evidence suggests that in many asset prices often jumps appear in cluster, to model the dynamics describing the spot price (or the future price) we need to introduce jump processes exhibiting a clustering or self-exciting behavior. There is a wide class of works in the literature with several different approaches to describe power prices evolution and a comprehensive literature review until 2008 is presented in [2].
In this work we aim to investigate if self exciting features arise in power forward prices evolution and we will focus our attention on two classes of stochastic processes: continuous branching processes with immigration (CBI) and Hawkes processes. CBI processes are commonly used in modeling population dynamics and their self exciting features describe the growth of the population due to the reproduction of the previous generations. Hawkes processes, instead, are particular extension of Poisson processes with self exciting properties, where points shows clustering effects; they have been introduced by Hawkes (1979) and the fact that they are extremely versatile makes them interesting both from a theoretical and from a practical point of view.
This work is organized in the following way: in the first chapter we set up the theoretical framework and we define and give the main properties of CBI and Hawkes processes, in the second chapter we recall some stochastic analysis notions (Girsanov Theorem) and we set up the modeling framework for forward prices based on Hawkes and CBI processes and, finally, in the third chapter we describe the algorithm to determine forward curves and then we give the parameter estimation for the model proposed. We also implement the algorithm for jump detection in forward curves and we apply it to detect jumps in historical series of data of future prices of other commodities (gold and crude oil).

Chapter 1

Properties of CBI and Hawkes processes

The first section of this chapter is devoted to some important definitions about random measures that will be used in the construction of the model given in Chapter 2. The references for this part are [5], [6] and [16]. The second section is devoted to the construction and the presentation of some important properties of Continuous state Branching processes (CB processes) and of Continuous state Branching processes with Immigration (CBI processes); here the main references are [20], [21] and [22]. The last section is dedicated to a review of Point processes and Poisson processes that are used to define Hawkes processes. The references for this part are [7], [19], and [26].

1.1 Introductory definitions

We by start describing the model which is based on stochastic differential equations (SDEs) driven by Levy random fields. First of all we set some important definitions:

Definition 1.1 (Random Measure with transition kernels)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ a measurable space. A Random Measure X is a function:*

$$X : \Omega \times \mathcal{E} \to [0, \infty)$$

such that:

- *$\forall \omega \in \Omega$ the map:*

  $$X(\omega, \cdot) : \mathcal{E} \to [0, \infty)$$

  is a measure;

- *For every $B \in \mathcal{E}$ the map:*

  $$X(\cdot, B) : \Omega \to [0, \infty)$$

  is $\mathcal{F}-$measurable random variable.

An alternative definition is the following:

Definition 1.2 (Random Measure as random element)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ a measurable space. A Random Measure X is a measure valued element from $(\Omega, \mathcal{F}, P)$ to $(M, \mathcal{M})$ where M is the space of all measures on $(E, \mathcal{E})$ and $\mathcal{M}$ is its σ-algebra.*

Definition 1.3. *A stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is said to be a Lévy process if:*

- *$X_0 = 0$ almost surely.*
- *for all $t_0 < t_1 < ... < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are mutually independent.*
- *for any $s < t$ the distribution of $(X_t - X_s)$ is the same of X_{t-s}.*
- *for any $\varepsilon > 0$ and $h \geq 0$ it holds that:*

$$\lim_{h\to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$$

Finally a Lévy random field is a multi-dimensional generalization of a Lévy process

In particular we have that:

Definition 1.4 (Gaussian random measure)**.** *A Gaussian random measure is a random measure such that $\forall\omega \in \Omega$ the function:*

$$X(\omega, \cdot) : \mathcal{E} \to [0, \infty)$$

is a Gaussian measure, i.e., it is either the Dirac delta at a point $a \in \mathbb{R}$ or it has density:

$$p(\cdot, a, \sigma^2) : x \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(x-a)^2}{2\sigma^2} \right)$$

with respect to the Lebesgue measure where the parameters a and σ are called mean and variance of the measure.

We consider now the case of a Gaussian random measure and we introduce the concept of white noise:

Definition 1.5. *A Gaussian random measure $W : \Omega \times \mathcal{B}(\mathbb{R}^2_+) \to [0, \infty)$ is said to be a white Noise on $\mathbb{R}^2_+$ if:*

- *$\forall A \in \mathcal{B}(\mathbb{R}^2_+)$ for which $\lambda^2(A) < \infty$ (λ^2 denotes the Lebesgue measure on $\mathbb{R}^2$) we have that $W(\cdot, A)$ is a Gaussian random variable with zero mean and variance equal to $\lambda^2(A)$.*
- *Given $A_1, ..., A_n$ disjoint Borel set with finite Lebesgue measure, then the random variables*

$$W(\cdot, A_1), \ldots, W(\cdot, A_n)$$

are mutually independent.

We also need the following:

Definition 1.6. *We say that $N : \Omega \times \mathcal{B}(\mathbb{R}^3_+) \to [0,\infty)$ is a Poisson random measure with intensity λ (with λ Borel measure on $\mathbb{R}^3_+$) if it is the product measure of the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with a Borel measure μ on $\mathbb{R}_+$ such that:*

$$\int_0^\infty (z \wedge z^2)\mu(dz) < \infty$$

In particular the Borel measure μ in Definition (1.6) is a Levy measure since:

$$1 \wedge z^2 \leq (z \wedge z^2)\mathbf{1}_{(0,1]}(z) + (z \wedge z^2)\mathbf{1}_{(1,\infty)}(z) = z \wedge z^2$$

This implies that:

$$\int_0^\infty (1 \wedge z^2)\mu(dz) \leq \int_0^\infty (z \wedge z^2)\mu(dz) < \infty$$

We also introduce:

Definition 1.7. *Let N be a Poisson random measure, we define the compensated Poisson random measure of intensity λ as $\widetilde{N} = N - \lambda$.*

At this point we introduce the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ as the natural filtration generated by Lévy's random field, namely for any Borel subset $A \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\mathbb{R}^2)$ of finite Lebesgue measure the processes: $(W([0,t] \times A), t \geq 0)$ and $(\tilde{N}([0,t] \times B), t \geq 0)$ are P-Martingales.
At this point we are ready to set up the stochastic differential equation that we are going to use: we set $a, b, \sigma, \gamma \in \mathbb{R}_+$ positive constants and:

- $W(ds, du)$ a white noise on $\mathbb{R}^2$ with unit covariance matrix.
- $\widetilde{N}(ds, du, dz)$ an independent compensated Poisson random measure on $\mathbb{R}^3$ with intensity $dsdu\mu(dz)$, with μ Lévy measure on $\mathbb{R}^+$ satisfying $\int_0^\infty (z \wedge z^2)\mu(dz) < \infty$.

1.2 Laplace transform of a measure

We start recalling some basic properties of the Laplace transform of a measure and then we continue defining the continuous state branching process (CB) and continuous state branching process with immigration (CBI). Let $\mathcal{B}([0,\infty))$ be the Borel σ-algebra on the positive half line and let $B_b([0,\infty))$ the set of bounded measurable functions on $[0,\infty)$. Given a finite measure μ on $[0,\infty)$ we define the Laplace transform L_μ of μ by:

$$L_\mu(\lambda) = \int_{[0,\infty)} e^{-\lambda x}\mu(dx) \quad \lambda \geq 0 \tag{1.1}$$

Theorem 1.1. *A finite measure on $[0,\infty)$ is uniquely determined by its Laplace transform.*

Proof. Suppose that μ_1 and μ_2 are finite measures on $[0,\infty)$ and that $L_{\mu_1}(\lambda) = L_{\mu_2}(\lambda)$ for all $\lambda \geq 0$. Then set $\mathcal{H} = \{x \mapsto e^{-\lambda x} : \lambda \geq 0\}$ and $\mathcal{L}$ the set of functions in $B_b([0,\infty))$ such that:

$$\int_{[0,\infty)} F(x)\mu_1(dx) = \int_{[0,\infty)} F(x)\mu_2(dx)$$

Then $\mathcal{H}$ is closed under multiplication and $\mathcal{L}$ is a monotone vector space containing $\mathcal{H}$. Moreover we have that $\sigma(\mathcal{H}) = \mathcal{B}([0,\infty))$ and hence we have that by the monotone class theorem:

$$\mathcal{L} \supset \sigma_b(\mathcal{H}) = B_b([0,\infty))$$

which proves the result. □

We state now a theorem about converging result of Laplace transform of a measure.

Theorem 1.2. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of finite measures on $[0,\infty)$ and $\lambda \mapsto L(\lambda)$ be a continuous function on $[0,\infty)$. If $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L(\lambda)$ for every $\lambda \geq 0$ then there is a finite measure μ such that $L_\mu = L$ and $\lim_{n\to\infty} \mu_n = \mu$ by weak convergence.*

Corollary 1.1. *Let $\mu_1, ..., \mu_n$ be finite measures on $[0,\infty)$. Then $\mu_n \to \mu$ weakly if and only if $L_{\mu_n}(\lambda) \to L(\lambda)$.*

Proof. If $\mu_n \to \mu$ weakly we have that $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L_\mu(\lambda)$ for every $\lambda \geq 0$, indeed being $x \mapsto e^{-\lambda x}$ continuous and bounded for every $\lambda \geq 0$ we have that:

$$\lim_{n\to\infty} L_{\mu_n}(\lambda) = \lim_{n\to\infty} \int_0^\infty e^{-\lambda x}\mu_n(dx) = \int_0^\infty e^{-\lambda x}\mu(dx) = L_\mu(\lambda)$$

The converse is a consequence of the Theorem 1.2. □

We conclude this section adding some additional results:

- given two probability measures μ_1, μ_2 we define $\mu_1 \times \mu_2$ their product measure on $[0,\infty)^2$.
- The image of $\mu_1 \times \mu_2$ under the mapping $(x_1, x_2) \mapsto x_1 + x_2$ is called convolution of μ_1 and μ_2 and it is denoted by $\mu_1 * \mu_2$ and it is a probability measure on $[0,\infty)$

For any $F \in_b B([0,\infty))$ we have that:

$$\int_{[0,\infty)} F(x)(\mu_1 * \mu_2)(dx) = \int_{[0,\infty)} \mu_1(dx) \int_{[0,\infty)} F(x_1 + x_2)\mu_2(dx) \tag{1.2}$$

If X_1 and X_2 are random variables with distributions μ_1 and μ_2 then $X_1 + X_2$ is a random variable with distribution $\mu_1 * \mu_2$. Finally we have that:

$$L_{\mu_1*\mu_2}(\lambda) = L_{\mu_1}(\lambda)L_{\mu_2}(\lambda) \quad \lambda \geq 0 \tag{1.3}$$

Let now:

$$\mu^{*0} = \delta_0$$
$$\mu^{*n} = \mu^{*(n-1)} * \mu \quad \text{for } n \geq 1$$

We say that a probability distribution μ on $[0,\infty)$ is infinitely divisible if for each integer $n \geq 1$ there is a probability μ_n such that $\mu = \mu_n^{*n}$; in this case μ_n is called the n-th root of μ.
A positive random variable X is said to be infinitely divisible if it admits infinitely divisible distribution on $[0,\infty)$.

1.3 Construction of a CB process

CB and CBI processes are a particular class of stochastic processes commonly used in modelling population dynamics. The self-exciting features describe the growth of the population due to the reproduction of the previous generations. Our aim is to use them to describe jumps generated by previous jumps. In this section we present a possible construction of a CB-Process.

1.3.1 Galton Watson Branching process

Let $\{p(j) : j \in \mathbb{N}\}$ be a probability distribution of a r.v. X on the space of positive integers $\mathbb{N}$. It is well known that $\{p(j) : j \in \mathbb{N}\}$ is uniquely determined by its generating function $g : \mathbb{N} \to \mathbb{R}$ defined by:

$$g(z) = E(z^X) = \sum_{j=0}^{\infty} p(j) z^j \quad |z| \leq 1$$

At this point we suppose to have $\{\xi_{n,i} : n, i = 1, 2, ...\}$ a family of $\mathbb{N}$-valued random variables with distribution $\{p(j) : j \in \mathbb{N}\}$. Given an $\mathbb{N}$-valued random variable Z_0 independent on $\{\xi_{i,n}\}$ we define inductively for $n = 1, 2, ...$:

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n,i} \quad Z_0 = 0 \tag{1.4}$$

Of course we set $\sum_{i=1}^{0} = 0$. For each $i \in \mathbb{N}$ we denote with $\{Q(i, j) : j \in \mathbb{N}\}$ the i-fold convolution of $\{p(j) : j \in \mathbb{N}\}$, that is, $Q(i, j) = p^{*i}(j)$ for $i, j \in \mathbb{N}$. For any $n \geq 1$ and $\{i_0, i_1, i_2, ..., i_{n-1} = i, j\} \subset \mathbb{N}$ we have that:

$$\begin{aligned} P(Z_n = j | Z_0 = i_0, ..., Z_{n-1} = i) &= P\left(\sum_{i=1}^{Z_{n-1}} \xi_{n,i} = j \,\middle|\, \sum_{i=0}^{Z_{n-2}} \xi_{n-1,i} = i_{n-1}, ..., Z_0 = i_0 \right) \\ &= P\left(\sum_{i=1}^{Z_{n-1}} \xi_{n,i} = j \,\middle|\, \sum_{i=0}^{Z_{n-2}} \xi_{n-1,i} = i_{n-1} \right) \\ &= P\left(\sum_{k=1}^{i} \xi_{n,k} = j \right) = p^{*i}(j) = Q(i, j). \end{aligned}$$

Then $Z_n : n \in \mathbb{N}$ is an $\mathbb{N}$-valued Markov chain with one step transition matrix $Q = Q(i, j) : i, j \in \mathbb{N}$; in particular the random variable Z_n can be tought as the number of individuals in the generation n of an evolving population system. After one time unit each individual of the population splits independently on the others into a random number of offspring according to the distribution $\{p(j) : j \in \mathbb{N}\}$. Clearly we have for $i \in \mathbb{N}$ and for $|z| \leq 1$:

$$\sum_{j=0}^{\infty} Q(i, j) z^j = \sum_{j=0}^{\infty} p^{*i}(j) z^j = g(z)^i. \tag{1.5}$$

Moreover the transition matrix Q satisfies the branching property:

$$Q(i_1 + i_2, \cdot) = Q(i_1, \cdot) * Q(i_2, \cdot). \tag{1.6}$$

Namely, that different individuals of the population propagate independently on each other. A Markov chain with state space $\mathbb{N}$ with one step transition matrix defined by (1.5) is called a *Galton Watson branching process* (GW-process) or a *Bienaymé-Galton-Watson branching process* with branching distribution given by g.
By a general result on the theory of Markov chains we have that for $n \geq 1$ the n-step transition matrix of the GW-process is just the n-fold product matrix $Q^n = \{Q^n(i,j) : i,j \in \mathbb{N}\}$.

Proposition 1.1. *For $n \geq 1$ and $i \in \mathbb{N}$ we have:*

$$\sum_{j=0}^{\infty} Q^n(i,j) z^j = g^{\circ n}(z)^i \tag{1.7}$$

where $g^{\circ n}(z)$ is defined iteratively by: $g^{\circ 0}(z) = z$ and:

$$g^{\circ n} = g \circ g^{\circ(n-1)}(z) = g(g^{\circ(n-1)}(z)).$$

Proof. We know that the previous property holds for $n = 1$. Suppose now that it holds for some $n \geq 1$, then we have that:

$$\begin{aligned}\sum_{j=0}^{\infty} Q^{n+1}(i,j) z^j &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} Q(i,k) Q^n(k,j) z^j \\ &= \sum_{k=0}^{\infty} Q(i,k) g^{\circ n}(z)^i\end{aligned}$$

And this proves inductively the thesis. □

It is easy to see that 0 is a trap for the GW-process; indeed if at some $n \geq 1$ we have that $Z_n = 0$, then:

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i} = 0$$

And therefore by induction on n we conclude that $Z_k = 0 \ \forall k \geq n$.
If $g'(-1) < \infty$ by differentiating both sides of (1.7) we see that the first moment of the distribution $\{Q^n(i,j) : j \in \mathbb{N}\}$ is given by:

$$\sum_{j=1}^{\infty} j Q^n(i,j) = i g'(1-)^n \tag{1.8}$$

1.3.2 Continuous extension of a Galton-Watson process

Suppose now to have a sequence of GW-processes $\{X_k(n) : n \geq 0\}_{k \geq 1}$ with branching distribution given by the probability generating function g_k, for $k = 1, 2, \ldots$. Consider now:

$$Z_k(n) = \frac{1}{k} X_k(n) \quad n \geq 0 \tag{1.9}$$

the process defined by(1.9) is a Markov chain with state space $E_k = \{0, k^{-1}, 2k^{-1}, ...\}$ and n-step transition probability given by $Q_k^n(x, dy)$ determined by:

$$\int_{E_k} e^{\lambda y} Q_k^n(x, dy) = g_k^{\circ n}(e^{-\lambda/k})^{kx}. \tag{1.10}$$

Suppose now that $\{\gamma_k\}_k$ is a positive sequence $\gamma_k \to \infty$ as $k \to \infty$ and let $\lfloor \gamma_k t \rfloor$ denote the integer part of $\gamma_k t$. Given $Z_k(0) = x$ we know that the random variable $Z_k(\lfloor \gamma_k t \rfloor) = k^{-1} X_k(\lfloor \gamma_k t \rfloor)$ has distribution $Q_k^{\lfloor \gamma_k t \rfloor}(x, \cdot)$ on E_k determined by:

$$\int_{E_k} e^{-\lambda y} Q_k^{\lfloor \gamma_k t \rfloor}(x, dy) = \exp\{-x v_k(t, \lambda)\} \tag{1.11}$$

where:

$$v_k(t, \lambda) = -k \log \left(g_k^{\circ \lfloor \gamma_k t \rfloor}(e^{-\lambda/k}) \right) \tag{1.12}$$

We are interested in the asymptotic behavior of the sequence of continuous time processes: $\{Z_k(\lfloor \gamma_k t \rfloor) : t \geq 0\}$ as $k \to \infty$. By (1.12) for $\gamma_k^{-1}(i-1) \leq t \leq \gamma_k^{-1} i$ we have:

$$v_k(t, \lambda) = v_k(\gamma_k^{-1} \lfloor \gamma_k t \rfloor, \lambda) = v_k(\gamma_k^{-1}(i-1), \lambda)$$

It follows that (the following chain of calculations is taken from [20]):

$$\begin{aligned}
v_k(t, \lambda) &= v_k(0, \lambda) + \sum_{j=1}^{\lfloor \gamma_k t \rfloor} [v_k(\gamma_k^{-1} j, \lambda) - v_k(\gamma_k^{-1}(j-1), \lambda)] \\
&= \lambda - k \sum_{j=1}^{\lfloor \gamma_k t \rfloor} [\log(g_k^{\circ j}(e^{-\lambda/k})) - \log(g_k^{\circ (j-1)}(e^{-\lambda/k}))] \\
&= \lambda - k \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \log[(g_k(g_k^{\circ (j-1)}(e^{-\lambda/k}))(g_k^{\circ (j-1)}(e^{-\lambda/k}))^{-1}] \\
&= \lambda - \gamma_k^{-1} \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \varphi_k(-k \log g_k^{\circ (j-1)}(e^{-\lambda/k})) \\
&= \lambda - \gamma_k^{-1} \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \varphi_k(v_k(\gamma_k^{-1}(j-1))) \\
&= \lambda - \int_0^{\gamma_k^{-1} \lfloor \gamma_k t \rfloor} \varphi_k(v_k(s, \lambda)) ds
\end{aligned}$$

where:

$$\varphi_k(z) = k \gamma_k \log[g_k(e^{-z/k}) e^{z/k}] \tag{1.13}$$

We can rewrite the previous function as:

$$\varphi_k(z) = k \gamma_k \log[1 + (k \gamma_k)^{-1} \tilde{\varphi}_k(z) e^{z/k}] \tag{1.14}$$

where:

$$\tilde{\varphi}_k(z) = k \gamma_k [g_k(e^{-z/k}) - e^{-z/k}] \tag{1.15}$$

We state now a lemma which shows that the two sequences (1.13) and (1.15) are not very different:

Lemma 1.1. *Suppose that the sequence* $\{\varphi_k\}_{k\in\mathbb{N}}$ *is given by* (1.13) *and that* $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ *is given by* (1.15)*. Assume moreover that* $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ *is bounded on each bounded interval. an Then we have:*

- $\lim_{k\to\infty} |\tilde{\varphi}_k(z) - \varphi_k(z)| = 0$ *uniformly on each bounded interval.*
- $\{\varphi_k\}_k$ *is uniformly Lipschitz on each bounded interval if and only if so is* $\{\tilde{\varphi}_k\}_k$

This lemma allow us to deduce that if either $\{\varphi_k\}_k$ or $\{\tilde{\varphi}_k\}_k$ is uniformly Lipschitz on each bounded interval then they converge or diverge simultaneously and in the convergent case they have the same limit. For convenience we formulate the following condition:

Assumption 1.1. *The sequence* $\{\tilde{\varphi}_k\}_k$ *is uniformly Lipschitz on* $[0, a]$ *for every* $a \geq 0$ *and there is a function* φ *on* $[0, \infty)$ *so that* $\tilde{\varphi}_k(z) \to \varphi(z)$ *uniformly on* $[0, a]$ *as* $k \to \infty$*.*

Proposition 1.2. *Suppose that Assumption 1.1 is satisfied, then the limit function of the* $\lim_{k\to\infty} \varphi_k = \varphi$ *has representation:*

$$\varphi(z) = bz + cz^2 + \int_0^\infty (e^{-zu} - 1 + zu)m(du), \quad z \geq 0 \tag{1.16}$$

where $c \geq 0$*,* b *is a real constant and* $m(du)$ *is a* σ*-finite measure such that:*

$$\int_{[0,\infty)} (u \wedge u^2)m(du) < \infty.$$

Proof. For each $k \geq 1$ we define the function φ_k on $[0, k]$ by:

$$\varphi_k(z) = k\gamma_k[g_k(1 - z/k) - (1 - z/k)]. \tag{1.17}$$

From (1.15) and (1.17) we have:

$$\tilde{\varphi}'_k(z) = \gamma_k e^{-z/k}[1 - g'_k(e^{-z/k})] \quad z \geq 0$$

and:

$$\varphi'_k(z) = \gamma_k[1 - g'_k(1 - z/k)] \quad 0 \leq z \leq k.$$

Since $\{\tilde{\varphi}_k\}_k$ is uniformly Lipschitz on each bounded interval the sequence $\{\tilde{\varphi}_k\}_k$ is uniformly bounded on each bounded interval too. Then also $\{\varphi'_k\}_k$ is uniformly bounded on each bounded interval and so the sequence $\{\varphi_k\}_k$ is uniformly Lipschitz on each bounded interval. Let now $a \geq 0$, for $k \geq a$ and $0 \leq z \leq a$, by the mean value theorem we have:

$$\begin{aligned}\tilde{\varphi}_k(z) - \varphi_k(z) &= k\gamma_k\left[g_k(e^{-z/k}) - g_k\left(1 - \frac{z}{k}\right) - e^{-z/k} + \left(1 - \frac{z}{k}\right)\right] \\ &= k\gamma_k[g'_k(\eta_k) - 1]\left(e^{-z/k} - 1 + \frac{z}{k}\right)\end{aligned}$$

where:

$$1 - \frac{a}{k} \leq 1 - \frac{z}{k} \leq \eta_k \leq e^{-z/k} \leq 1$$

Choose now $k_0 \geq a$ so that $e^{-2a/k_0} \leq 1 - \frac{a}{k_0}$. Then $e^{-2a/k} \leq 1 - \frac{a}{k}$ for all $k \geq k_0$ and hence:

$$\gamma_k |g_k'(\eta_k) - 1| \leq \sup_{0 \leq z \leq 2a} \gamma_k |g_k'(e^{-z/k}) - 1| = \sup_{0 \leq z \leq 2a} e^{-z/k} |\tilde{\varphi}_k'(z)|$$

Since $\{\tilde{\varphi}_k'\}$ is uniformly bounded on $[0, 2a]$ the sequence $\{\gamma_k |g_k'(\eta_k) - 1| : k \geq k_0\}$ is bounded. Then $\lim_{k\to\infty} |\tilde{\varphi}_k(z) - \varphi_k(z)| = 0$ uniformly on each bounded interval and this implies that:

$$\lim_{k\to\infty} \varphi_k(z) = \varphi(z)$$

uniformly on each bounded interval. The result follows from Corollary 1.46 in [21] □

Proposition 1.3. *For any function φ with representation* (1.16) *there is a sequence $\{\tilde{\varphi}_{k\in\mathbb{N}}\}_k$ satisfying Assumption 1.1.*

Proof. By the proof of proposition 1.2 it suffices to construct a sequence $\{\varphi_k\}_k$ via expression (1.17) that is uniformly Lipschitz on $[0, a]$ and $\varphi_k(z) \to \varphi(z)$ uniformly on $[0, a]$ for every $a \geq 0$. To simplify the formulations we decompose the function φ into two parts. Let $\varphi_0(z) = \varphi(z) - bz$. We first define:

$$\gamma_{0,k} = (1 + 2c)k + \int_{(0,\infty)} u(1 - e^{-ku}) m(du)$$

and:

$$g_{0,k}(z) = z + k^{-1}\gamma_{0,k}^{-1}\varphi_0(k(1 - z)) \quad |z| \leq 1.$$

The function $z \mapsto g_{0,k}(z)$ is analytic, satisfies $g_{0,k}(1) = 1$ and:

$$\frac{d^n}{dz^n} g_{0,k}(0) \geq 0 \quad n \geq 0.$$

Therefore $g_{0,k}(\cdot)$ is a probability generating function. Let $\varphi_{0,k}$ be defined by (1.17) with the pair (γ_k, g_k) replaced by $(\gamma_{0,k}, g_{0,k})$. Then $\varphi_{0,k}(z) = \varphi_0(z)$ if $0 \leq z \leq k$ and this completes the proof if $b = 0$.
If $b \neq 0$ we set:

$$g_{1,k}(z) = \frac{1}{2}\left(1 + \frac{b}{|b|}\right) + \frac{1}{2}\left(1 - \frac{b}{|b|}\right) z^2.$$

Let $\gamma_{1,k} = |b|$ and let $\varphi_{1,k}(z)$ defined by (1.17) with the pair (γ_k, g_k) replaced by $(\gamma_{1,k}, g_{1,k})$. Then:

$$\varphi_{1,k}(z) = bz + \frac{1}{2k}(|b| - b)z^2.$$

Finally let $\gamma_k = \gamma_{0,k} + \gamma_{1,k}$ and:

$$g_k = \gamma_k^{-1}(\gamma_{0,k} g_{0,k} + \gamma_{1,k} g_{1,k})$$

Then the sequence $\varphi_k(z)$ defined by (1.17) is equal to $\varphi_{0,k}(z) + \varphi_{1,k}(z)$ which satisfies the required condition. □

Lemma 1.2. *Suppose that the sequence $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ defined by* (1.15) *is uniformly Lipschitz on $[0, 1]$. Then there exist constants $B, N \geq 0$ such that*

$$v_k(t, \lambda) \leq \lambda e^{Bt}$$

for every $t, \lambda \geq 0$ and $k \geq N$.

Proof. Let $b_k := \varphi_k'(0+)$ for $k \geq 1$. Since $\{\tilde{\varphi}_k\}_k$ is uniformely Lipschitz on $[0,1]$ the sequence $\{b_k\}_k$ is bounded. From (1.15) we have that $b_k = \gamma_k[1 - g_k'(1-)]$. Then:

$$\int_{E_k} yQ_k^{\lfloor \gamma_k t \rfloor}(x, dy) = xg_k'(1-)^{\lfloor \gamma_k t \rfloor} = x\left(1 - \frac{b_k}{\gamma_k}\right)^{\lfloor \gamma_k t \rfloor}$$

Let B a constant such that $2|b_k| \leq B$ for all $k \geq 1$. Since $\gamma_k \to \infty$ as $k \to \infty$ there is $N \geq 1$ such that:

$$0 \leq \left(1 - \frac{b_k}{\gamma_k}\right)^{\gamma_k/B} \leq \left(1 + \frac{B}{2\gamma_k}\right)^{\gamma_k/B} \leq e \quad k \geq N.$$

It follows that for $t \geq 0$ and $k \geq N$:

$$\int_{E_k} yQ_k^{\lfloor \gamma_k t \rfloor}(x, dy) \leq x \exp\{B\lfloor \gamma_k t \rfloor / \gamma_k\} \leq xe^{Bt}.$$

The estimate required can be obtained from (1.8) and Jensen's inequality. □

Theorem 1.3. *Suppose that Assumption 1.1 holds. Then for every $a \geq 0$ we have that $v_k(t,\lambda) \to v_t(\lambda)$ uniformly on $[0,a]^2$ and the limit function solves the integral equation:*

$$v_t(\lambda) = \lambda - \int_0^t \varphi(v_s(\lambda))ds \quad \lambda, t \geq 0 \tag{1.18}$$

Proof. Recall that:

$$v_k(t,\lambda) = \lambda - \int_0^{\gamma_k^{-1}\lfloor \gamma_k t \rfloor} \varphi_k(v_k(s,\lambda))ds,$$

so that we can write:

$$v_k(t,\lambda) = \lambda + \varepsilon_k(t,\lambda) - \int_0^t \varphi_k(v_k(s,\lambda))ds, \tag{1.19}$$

where:

$$\varepsilon_k(t,\lambda) = (t - \gamma_k^{-1}\lfloor \gamma_k t \rfloor)\varphi_k\big(v_k(\gamma_k^{-1}\lfloor \gamma_k . t \rfloor, \lambda)\big)$$

By lemma 1.1 and Assumption 1.1 for any $0 \leq \varepsilon \leq 1$ we can choose $N \geq 1$ so that $|\varphi_k(z) - \varphi(z)| \leq \varepsilon$ for $k \geq N$ and $0 \leq z \leq ae^{Ba}$. It follows that for $0 \leq t \leq a$ and $0 \leq \lambda \leq a$:

$$|\varepsilon_k(t,\lambda)| \leq \gamma_k^{-1}|\varphi_k\big(v_k(\gamma_k^{-1}\lfloor \gamma_k t \rfloor, \lambda)\big)| \leq \gamma_k^{-1}(1 + \sup_{0 \leq z \leq ae^{Ba}} |\varphi(z)|) = \gamma_k^{-1}M. \tag{1.20}$$

For $n \geq K \geq N$ let:

$$K_{k,n}(t,\lambda) = \sup_{0 \leq s \leq t} |v_n(s,\lambda) - v_k(s,\lambda)|.$$

By (1.19) and (1.20) we obtain for $(t,\lambda) \in [0,a]^2$:

$$\begin{aligned} K_n(t,\lambda) &\leq 2\gamma_k^{-1}M + \int_0^t |\varphi_k\big(v_k(s,\lambda) - \varphi_n\big(v_n(s,\lambda)\big)|ds \\ &\leq 2(\gamma_k^{-1}M + \varepsilon a) + \int_0^t |\varphi_k\big(v_k(s,\lambda) - \varphi_n\big(v_n(s,\lambda)\big)|ds \\ &\leq 2(\gamma_k^{-1}M + \varepsilon a) + L\int_0^t K_{k,n}(s,\lambda)ds \end{aligned}$$

where $L = \sup_{0\leq z\leq ae^{Ba}} |\varphi'(z)|$. By Gronwall's [1] inequality:

$$K_{k,n}(t,\lambda) \leq 2(\gamma_k^{-1}M + \varepsilon a)\exp Lt \quad 0 \leq t, \lambda \leq a$$

Then $v_k(t,\lambda) \to v_t(\lambda)$ uniformly on $[0,a]^2$ as $k \to \infty$ and from (1.19) we get the thesis □

The following theorem (whose proof can be founded on [20]) shows some important properties of the function $v_t(\lambda)$:

Theorem 1.4. *Suppose that φ is a function given by (1.16). Then for any $\lambda \geq 0$ there is a unique positive solution $t \mapsto v_t(\lambda)$ to (1.18); moreover the solution satisfies the semigroup property:*

$$v_{r+t}(\lambda) = v_r \circ v_t(\lambda) \tag{1.21}$$

Theorem 1.5. *Suppose that φ is a function given by (1.2). For any $\lambda \geq 0$ let $t \mapsto v_t(\lambda)$ be the unique positive solution of (1.18). Then we define a transition semigroup $(Q_t)_{t\geq 0}$ on $[0,\infty)$ by:*

$$\int_{[0,\infty)} e^{-\lambda y} Q_t(x,dy) = e^{-xv_t(\lambda)} \quad \lambda \geq 0, x \geq 0 \tag{1.22}$$

Proof. There is a sequence $\{\tilde{\varphi}_k\}_k$ in form (1.15) satisfying Assumption 1.1. By Theorem 1.3 we have that $v_k(t,\lambda) \to v_t(\lambda)$ uniformly on $[0,a)^2$ as $k \to \infty$ for every $a \geq 0$. Taking $x_k \in E_k$ such that $x_k \to x$ as $k \to \infty$ we see by theorem 1.2 that (1.22) defines a probability measure $Q_t(x,dy)$ on $[0,\infty)$ and that:

$$\lim_{k\to\infty} Q_k^{\lfloor \gamma_k t \rfloor}(x_k,\cdot) = Q_t(x,\cdot)$$

by weak convergence. By a monotone class argument we can see that $Q_t(x,dy)$ is a kernel on $[0,\infty)$. The semigroup property of the family of kernels follows from (1.21) and (1.22). □

Proposition 1.4. *For every $t \geq 0$ the function $\lambda \mapsto v_t(\lambda)$ is strictly increasing on $[0,\infty)$*

Proof. By the continuity of $t \mapsto v_t(\lambda)$, for any $\lambda_0 > 0$ there is $t_0 > 0$ so that $v_t(\lambda_0) > 0$ for $0 \leq t \leq t_0$. Then (1.22) implies $Q_t(x,\{0\}) < 1$ for $x > 0$ and $0 \leq t \leq t_0$, and so $\lambda \mapsto v_t(\lambda)$ is strictly increasing for $0 \leq t \leq t_0$. By the semigroup property (1.21) we infer $\lambda \mapsto v_t(\lambda)$ is strictly increasing for all $t \geq 0$. □

Assume now that E is a metrizable locally compact topological space. We also assume that E is countable at infinity, meaning that E is a countable union of compact sets. Let $C_0(E)$ the space of real valued continuous functions that vanishes at infinity (or uniformly smaller than ε) outside of a compact set); this space is a Banach space with the standard supremum norm.

Definition 1.8. *Let $\{Q_t\}_{t\geq 0}$ a transition semigroup on E, we say that $\{Q_t\}_{t\geq 0}$ is a Feller semigroup if:*

[1] Let I denote an interval on the real line of the type $[a,b]$ and let β, u two real functions continuous in the interior of I. If u is differential in the interior of I and satisfies $u'(t) \leq \beta(t)u(t)$ then u is bounded by the solution of the differential equation $\nu'(t) = \beta(t)\nu(t)$, specifically $u(t) \leq u(a)\int_0^t \beta(s)ds$

- $Q_t f \in C_0(E)$ *for all* $f \in C_0(E)$;
- $\lim_{t\to 0} \|Q_t f - f\| = 0$ *for all* $f \in C_0(E)$.

A Markov process on E is a Feller process if its semigroup is a Feller semigroup.

Theorem 1.6. *The transition semigroup $(Q_t)_{t\geq 0}$ defined by Equation*(1.22) *is a Feller semigroup.*

Proof. For $\lambda \geq 0$ and $x \geq 0$ set $e_\lambda(x) = e^{-\lambda x}$. Denote with $\mathcal{D}_0$ the linear span of $e_\lambda : \lambda \geq 0$. By Proposition 1.4 the operator Q_t preserves $\mathcal{D}_0$ for every $t \geq 0$. By the continuity of $t \mapsto v_t(\lambda)$ it is easy to show that $t \mapsto Q_t e_\lambda(x)$ is continuous for $\lambda \geq 0$ and $x \geq 0$. This implies that $t \mapsto Q_t f(x)$ is continuous for every $f \in \mathcal{D}_0$ and $x \geq 0$.
Let $C_0([0,\infty))$ the space of continuous functions on $[0,\infty)$ vanishing at ∞. By the Stone-Weierstrass theorem the set $\mathcal{D}_0$ is uniformly dense in $C_0([0,\infty))$. Then each operator Q_t preserves $C_0([0,\infty))$ and $t \mapsto Q_t f(x)$ is continuous for all $x \geq 0$ and $f \in C_0([0,\infty))$. This gives the Feller property of the semigroup $(Q_t)_{t\geq 0}$. □

At this point we have all the ingredients to define a CB-process:

Definition 1.9. *A Markov process on $[0,\infty)$ is called a continuous state Branching process (CB-Process) with branching mechanism φ if it has transition semigroup $(Q_t)_{t\geq 0}$ defined by* (1.22). *Moreover the family of functions $(v_t)_{t\geq 0}$ defined by Equation* (1.18) *is called the cumulant semigroup of the CB-process.*

It is easy to see that $(Q_t)_{t\geq 0}$ satisfies the branching property:

$$Q_t(x_1 + x_2, \cdot) = Q_t(x_1, \cdot) * Q_t(x_2, \cdot)$$

To end this section we state a proposition which states an important property of CB-processes:

Proposition 1.5. *Suppose that $\{(X_1(t), \mathcal{F}_t^1) : t \geq 0\}$ and $\{(X_2(t), \mathcal{F}_t^2) : t \geq 0\}$ are two independent CB-processes with branching mechanism φ and $\mathcal{F}_t^1, \mathcal{F}_t^2$ their associated σ-algebras. Let $X(t) = X_1(t) + X_2(t)$ and $\mathcal{F}_t = \sigma(\mathcal{F}_t^1 \cup \mathcal{F}_t^2)$. Then the process $\{(X(t), \mathcal{F}_t) : t \geq 0\}$ is also a CB-process with branching mechanism φ.*

Proof. Let $t \geq r \geq 0$ and F_r^i be $\mathcal{F}_r^i$-measurable random variables for $i = 1, 2$. For any $\lambda \geq 0$ we have:

$$\begin{aligned}
\mathbb{P}(F_1 F_2 e^{-\lambda X(t)}) &= \mathbb{P}(F_1 e^{-\lambda X_1(t)}) \mathbb{P}(F_2 e^{-\lambda X_2(t)}) \\
&= \mathbb{P}(F_1 e^{-X_1(r) v_{t-r}(\lambda)}) \mathbb{P}(F_2 e^{-X_2(r) v_{t-r}(\lambda)}) \\
&= \mathbb{P}(F_1 F_2 e^{-X(r) v_{t-r}(\lambda)})
\end{aligned}$$

A monotone class argument shows that:

$$\mathbb{P}(F e^{-\lambda X(t)}) = \mathbb{P}(F e^{-X(r) v_{t-r}(\lambda)})$$

For any bounded $\mathcal{F}_r$-measurable random variable F. Then $\{(X(t), \mathcal{F}_t) : t \geq 0\}$ is also a CB-process with transition semigroup $(Q_t)_{t\geq 0}$. □

1.4 Construction of a CBI processe

In this section we present the construction of a CBI process starting from a Galton-Watson process with immigration; we follow [20].

1.4.1 Galton-Watson process with immigration

Let $\{p(j) : j \in \mathbb{N}\}$ and $\{q(j) : j \in \mathbb{N}\}$ be probability distribution on the space of positive integers $\mathbb{N}$ with generating functions g and h respectively. Suppose that $\{\xi_{n,i} : n, i = 1, 2, ...\}$ is a family of $\mathbb{N}$-valued random variables with distribution $\{p(j) : j \in \mathbb{N}\}$ and $\{\eta_n : n = 1, 2, ...\}$ is a family on $\mathbb{N}$-valued random variables with distribution $\{q(j) : j \in \mathbb{N}\}$. Assume moreover that the two families are mutually independent. Given another random variable $Y(0)$ independent of $\{\xi_{n,i}\}$ and $\{\eta_n\}$ we define inductively:

$$Y(n) = \sum_{i=1}^{Y(n-1)} \xi_{n,i} + \eta_n \tag{1.23}$$

This is clearly a generalization of Equation (1.4). For $i \in \mathbb{N}$ let $\{Q(i, j) : j \in \mathbb{N}\}$ denote the i-fold convolution of $\{p(j) : j \in \mathbb{N}\}$ by:

$$P(i, j) = (Q(i, j) * q)(j) = (p^{*i} * q)(j)$$

Now for $n \geq 1$ and $\{i_0, ..., i_{n-1} = i, j\}$ with $i_j \in \mathbb{N}$ we have:

$$\begin{aligned} P(Y(n) = j | Y(0) = i_0, ..., Y(n-1) = i_{n-1}) &= P\left(\sum_{k=1}^{Y(n-1)} \xi_{n,k} + \eta_n = j \middle| Y(n-1) = i \right) \\ &= P\left(\sum_{k=1}^{Y(n-1)} \xi_{n,k} + \eta_n = j \right). \end{aligned}$$

By the above computation $\{Y(n) : n \in \mathbb{N}\}$ is a Markov chain with one step transition matrix given by: $P = \{P(i, j) : i, j \in \mathbb{N}\}$.

Remark 1.1. *The random variable $Y(n)$ can be thougth as the number of individuals in generation n of a population system with immigration; after one unit of time each of the $Y(n)$ individuals splits independently of others into a random number of offspring according to the distribution $\{p(j) : j \in \mathbb{N}\}$ and a random number of immigrants are added to the system according to the distribution $\{q(j) : j \in \mathbb{N}\}$.*

It is easy to see that:

$$\sum_{j=0}^{\infty} P(i, j) z^j = g(z)^i h(z) \quad |z| \leq 1. \tag{1.24}$$

A Markov chain in $\mathbb{N}$ with one step transition matrix defined by (1.24) is called a *Galton-Watson* branching process with immigration (GWI-process) or a BienaymÃ¨-Galton-Watson branching process with immigration (BGWI-process) with branching distribution given by g and immigration distribution h. When $h \equiv 1$ this reduces to the previous GW process defined before. For any $n \geq 1$ the n-step transition matrix of the GWI-process is just the n-fold product $P^n = \{P^n(i, j) : i, j \in \mathbb{N}\}$.

Proposition 1.6. *For any $n \geq 1$ and $i \in \mathbb{N}$ we have:*

$$\sum_{j=0}^{\infty} P^n(i,j)z^j = g^{\circ n}(z)^i \prod_{j=1}^{n} h(g^{\circ(j-1)}(z)) \quad |z| \leq 1. \tag{1.25}$$

Proof. From (1.24) we have that the statement is true for $n = 1$; suppose now that it holds for some $n \geq 1$, then:

$$\begin{aligned}
\sum_{j=0}^{\infty} P^{n+1}(i,j)z^j &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P(i,k)P^n(k,j)z^j \\
&= \sum_{k=0}^{\infty} P(i,k)g^{\circ n}(z)^k \prod_{j=1}^{n} h(g^{\circ(j-1)}(z)) \\
&= g(g^{\circ n}(z))^i h(g^{\circ n}(z)) \prod_{j=1}^{n} h(g^{\circ(j-1)}(z)) \\
&= g^{\circ(n+1)}(z)^i \prod_{j=1}^{n+1} h(g^{\circ(j-1)}(z))
\end{aligned}$$

□

Suppose now to have a sequence of GWI-processes $\{Y_k(n) : n \geq 0\}$ with branching distribution given by the probability generating function g_k and immigration distribution given by the probability generating function h_k. Let:

$$Z_k(n) := \frac{Y_k(n)}{k} \quad k \in \mathbb{N},\ n \in \mathbb{N}$$

Then $\{Z_k(n) : n \geq 0\}$ is a Markov chain with state space $E_k = \{0, k^{-1}, 2k^{-1}, ...\}$ and n-step transition probability $P_k^n(x, dy)$ determined by:

$$\int_{E_k} e^{-\lambda y} P_k^n(x, dy) = g_k^{\circ n}(e^{-\lambda x})^k x \prod_{j=1}^{n} h_k\big(g_k^{\circ(j-1)}(e^{-\lambda/k})\big) \tag{1.26}$$

Suppose now that $\{\gamma_k\}$ is a positive real sequence so that $\gamma_k \to \infty$ as $k \to \infty$. Let $\lfloor \gamma_k t \rfloor$ denote the integer part of $\gamma_k t$. In view of (1.26) given $Z_k(0) = x \in \mathbb{E}_\neg$ the random variable:

$$Z_k(\lfloor \gamma_k t \rfloor) = \frac{Y_k(\lfloor \gamma_k t \rfloor)}{k}$$

has distribution $P_k^{\lfloor \gamma_k t \rfloor}(x, \cdot)$ determined by:

$$\begin{aligned}
\int_{E_k} e^{\lambda y} P_k^{\lfloor \gamma_k t \rfloor}(x, dy) &= g_k^{\lfloor \gamma_k t \rfloor}(e^{-\lambda/k})^{kx} \prod_{j=1}^{\lfloor \gamma_k t \rfloor} h_k(g_k^{\circ(j-1)}(e^{-\lambda/k})) \\
&= \exp\{xk \log\big[g_k^{\lfloor \gamma_k t \rfloor}(e^{-\lambda/k})\big]\} \exp\left\{ \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \log\big[h_k(g_k^{\circ(j-1)})(e^{-\lambda/k})\big] \right\} \\
&= \exp\left\{ - x v_k(t,\lambda) - \int_0^{\gamma_k^{-1}\lfloor \gamma_k t \rfloor} \psi_k(v_k(s,k))ds \right\},
\end{aligned}$$

where:

$$v_k = -k \log \left(g_k^{\circ \lfloor \gamma_k t \rfloor}(e^{-\lambda/k}) \right)$$

and

$$\psi_k(z) = -\gamma_k \log h_k(e^{-z/k}) \tag{1.27}$$

As in Section 1.3 we can rewrite Equation (1.27) in term of a function $\tilde{\psi}(z)$:

$$\psi_k(z) = -\gamma_k \log[1 - \gamma_k^{-1} \tilde{\psi}_k(z)] \tag{1.28}$$

where:

$$\tilde{\psi}_k(z) = \gamma_k[1 - h_k(e^{-z/k})]. \tag{1.29}$$

Lemma 1.3. *Suppose that the sequence* $\{\tilde{\psi}_k\}_k$ *is uniformely bounded on each bounded interval. Then we have that* $\lim_{k\to\infty} |\tilde{\psi}_k(z) - \psi_k(z)| = 0$ *uniformely on each bounded interval.*

We set up now a condition that will be used in the following theorems:

Assumption 1.2. *There is a function* ψ *on* $[0, \infty)$ *such that* $\tilde{\psi}_k(z) \to \psi(z)$ *uniformly on* $[0, a]$ *for every* $a \geq 0$ *as* $k \to \infty$.

Proposition 1.7. *Suppose that Assumption 1.2 is satisfied. Then the limit function* ψ *has representation:*

$$\psi(z) = \beta z + \int_{(0,\infty)} (1 - e^{-zu}) \nu(du) \tag{1.30}$$

where $\beta \geq 0$ *is a positive constant and* ν *is a* σ*-finite measure on* $(0, \infty)$ *such that:*

$$\int_{(0,\infty)} (1 \wedge u) \nu(du) < \infty$$

Proof. It is well known that φ has representation (1.30) if and only if $e^{-\psi} = L_\mu$ is the Laplace transform of an infinitely divisible distribution μ on $[0, \infty)$ (see [21] Theorem 1.39). In view of (1.29) the function can be represented by a special form (1.30), so $e^{-\tilde{\psi}_k} = L_{\mu_k}$ is the Laplace transform of an infinitely divisible distribution μ_k on $[0, \infty)$. By the previous lemma and Assumption 1.2 we have that: $\tilde{\psi}_k(z) \to \psi(z)$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$. By Theorem 1.2 there is a probability distribution μ on $[0, \infty)$ such that $\mu = \lim_{k\to\infty} \mu_k$ weakly and $e^{-\psi} = L_\mu$ clearly μ is also infinitely divisible and hence ψ has representation (1.30). □

The following two theorems have a proof which is the complete analoguous to the one presented in the previous section so it is omitted.

Proposition 1.8. *For any function* ψ *with representation* (1.30) *there is a sequence* $\{\tilde{\psi}_k\}_k$ *in the form* (1.29) *satisfying Assumption 1.2.*

Theorem 1.7. *Suppose that* φ *and* ψ *are given by* (1.16) *and* (1.30) *respectively. For any* $\lambda \geq 0$ *let* $t \mapsto v_t(\lambda)$ *be the unique solution of* (1.18). *Then there is a Feller semigroup* $(P_t)_{t\geq 0}$ *on* $[0, \infty)$ *defined by:*

$$\int_{[0,\infty)} e^{-\lambda y} P_t(x, dy) = exp\Big\{ - x v_t(\lambda) - \int_0^t \psi(v_s(\lambda)) ds \Big\} \tag{1.31}$$

We have all the ingredients to define a CBI process:

Definition 1.10. *A Markov process on* $[0,\infty)$ *with transition semigroup defined by* (1.31) *is called a Continuous-state Branching process with Immigration (CBI-Process) with branching mechanism* φ *given by Equation* (1.16) *and defined on* $[0,\infty)$ *and immigration mechanism* ψ *given by* (1.30) *defined again on* $[0,\infty)$.

The following proposition states an additivity property of CBI processes:

Proposition 1.9. *Suppose that* $\{(Y_1(t),\mathcal{G}_t^1) : t \geq 0\}$ *and* $\{(Y_2(t),\mathcal{G}_t^2) : t \geq 0\}$ *are two independent CBI processes with branching mechanism* φ *and immigration rate* ψ_1 *and* ψ_2. *Let:*

$$\begin{aligned} Y(t) &= Y_1(t) + Y_2(t) \\ \mathcal{G}_t &= \sigma(\mathcal{G}_t^1 \cup \mathcal{G}_t^2) \end{aligned}$$

Then the process $\{(Y(t),\mathcal{G}_t) : t \geq 0\}$ *is a CBI-process with branching mechanism* φ *and immigration rate* $\psi = \psi_1 + \psi_2$.

Proof. Let $t \geq r \geq 0$ and for $i = 1,2$ let F_i a bounded positive $\mathcal{G}_t^i$-measurable random variable. For any $\lambda \geq 0$ we have:

$$\begin{aligned} \mathbb{P}[F_1F_2e^{-\lambda Y(t)}] &= \mathbb{P}[F_1e^{-\lambda Y_1(t)}]\mathbb{P}[F_2e^{-\lambda Y_2(t)}] \\ &= \mathbb{P}\left[F_1 \exp\left\{ - Y_1(t)v_{t-r}(\lambda) - \int_0^{t-r} \psi_1(v_s(\lambda))ds \right\}\right] \\ &\cdot \mathbb{P}\left[F_2 \exp\left\{ - Y_2(t)v_{t-r}(\lambda) - \int_0^{t-r} \psi_2(v_s(\lambda))ds \right\}\right] \\ &= \mathbb{P}\left[F_1F_2 \exp\left\{ - Y(t)v_{t-r}(\lambda) - \int_0^{t-r} \psi(v_s(\lambda))ds \right\}\right] \end{aligned}$$

As in the proof of proposition 1.5 one can see that $\{(Y(t),\mathcal{G}_t) : t \geq 0\}$ is a CBI process with branching mechanism φ and immigration rate ψ. □

To conclude this Section we give an additional definition of CBI processes which will be the base of the models that we are going to build in the following chapter.

Definition 1.11. *A Markov process* Y *with state space* $\mathbb{R}^+$ *is called a CBI process characterized by branching mechanism* φ *and immigration rate* ψ *if its characteristic representation, for* $p \geq 0$, *is given by:*

$$E_y[e^{-pY(t)}] = \exp\left(- y\nu(t,p) - \int_0^t \Phi\big(\nu(s,p)\big)ds \right) \tag{1.32}$$

where E_y *denotes the conditional expectation of with respect to the initial value* $Y(0) = y$. *The function* $\nu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_+$ *satisfies the following differential equation:*

$$\begin{cases} \dfrac{\partial \nu(t,p)}{\partial t} = -\psi(\nu(t,p)) \\ \nu(0,p) = p \end{cases} \tag{1.33}$$

where ψ and Φ are functions of the variable $q \geq 0$ given by:

$$\varphi(q) = aq + \frac{1}{2}\sigma^2 q^2 + \gamma \int_0^\infty (e^{-qu} - 1 + qu)\pi(du)$$
$$\psi(q) = abq + \int_0^\infty (1 - e^{-qu})\nu(du)$$

with $a, b \in \mathbb{R}$, $\sigma, \gamma \geq 0$ and π, ν being two Lévy measures such that:

$$\int_0^\infty (u \wedge u^2)\pi(du) < \infty.$$
$$\int_0^\infty (1 \wedge u)\nu(du) < \infty.$$

An important theorem was proved by Li (see [12] Theorem 3.1) and it guarantees the uniqueness od a solution to the SDE given by (**??**):

Theorem 1.8. *There is a unique non-negative strong solution of the SDE:*

$$Y(t) = Y(0) + \int_0^t a(b - Y(s))ds + \sigma \int_0^t \int_0^{Y(s)} W(ds, du) + \gamma \int_0^t \int_0^{Y(s^-)} \int_{\mathbb{R}^+} z\tilde{N}(ds, du, dz)$$

Moreover the solution $\{Y_t : t \geq 0\}$ is a CBI process with branching mechanism:

$$\psi(q) = aq + \frac{1}{2}\sigma^2 q^2 + \gamma \int_0^\infty (e^{-q\gamma z} - 1 + q\gamma z)\mu(dz)$$

And immigration rate given by:

$$\Phi(q) = abq \quad q \geq 0$$

1.5 Hawkes Processes

The self exciting point process, which is commonly known as the Hawkes process, it is a point process on the real line introduced by Hawkes (1971). The distinguishing feature of such processes is that they allow all past events to affect the intensity function at the current time. In this section we present the definition and some important features of Hawkes processes and then we provide the definition of Hawkes processes with exponential kernel that will be used in the following chapter.

1.5.1 Unmarked point processess

An unmarked random point process on the nonnegative real line $[0, \infty)$, where the non-negative line is taken to represent time, is a random process whose realisations consists on a sequence of times $T_1, T_2, ...$ where T_n is the n-th occurrence of an event. We make moreover the assumption of non-explosion:

$$T_\infty = \lim_{n \to \infty} T_n = \infty \tag{1.34}$$

this means that almost surely we do not have an accumulation of events in finite time. Given a random point process $\{T_n\}_{n\in\mathbb{N}}$ the sequence of inter-events is defined as $\{S_n\}_{n\in\mathbb{N}}$:

$$S_n = T_n - T_{n-1}. \tag{1.35}$$

To represent a point process we can introduce the following:

Definition 1.12. *Let $N : \big([0,\infty), \mathcal{B}[0,\infty)\big) \to \mathbb{N}$ we say that N is the counting measure associated to an unmarked point process if for any $A \in \mathcal{B}([0,\infty)$ we have that:*

$$N(A) = \#\{i : T_i \in A\}.$$

Clearly if $A = [a, b)$ then we have that:

$$N(A) = \sum_{i=1}^{\infty} \mathbf{1}_{\{a\le T_i<b\}}$$

In particular we set:

$$N_t = N([0,t)) = \sum_{i=1}^{\infty} \mathbf{1}_{\{0\le T_i<t\}}.$$

The point process defined above may be represented equivalently via its associated counting process $\{N_t\}_{t\ge0}$ where:

$$N_t = n \text{ if } t \in [T_n, T_n + 1) \tag{1.36}$$

In particular the random variable N_t counts the number of events up to time t and the non-explosion condition becomes $N_t < \infty$. Both the processes $\{T_n\}_{n\in\mathbb{N}}$ and $\{N_t\}_{t\ge0}$ are the defined on some probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}\}_{t\ge0}$ for which N is $\{\mathcal{F}_t\}_{t\ge0}$ adapted. One important example of a random point process is:

Definition 1.13. *A point process $\{N_t\}_{t\ge0}$ is called a Poisson point process if:*

1. *$N_0 = 0$*
2. *N_t is a process with independent increments i.e. $(N_t - N_s) \perp \mathcal{F}_s \ \forall t > s$.*
3. *$(N_t - N_s)$ is a Poisson random variable with parameter $\Lambda_{s,t} = \lambda(t-s) = \int_s^t \lambda(u)du$*

Usually one assumes that $\Lambda_{s,t} = \int_s^t \lambda_u du$ for some λ_t deterministic function called *intensity* of the Poisson Point Process. In particular if $\{\mathcal{F}_t\}_{t\ge0}$ is the filtration generated by N_t i.e. $\mathcal{F}_t = \sigma\{N_s : s \le t\}$ and $\lambda_t \equiv 1$ then N_t is called *standard Poisson process*. One can also prove that if $\lambda_t = \lambda$ then the sequence of inter-events $\{S_n\}_{n\in\mathbb{N}}$ is made of i.i.d. exponential random variables with parameter λ. We now focus our attention to the case on which the intensity is λ constant. By definition we have that:

$$\begin{aligned}
P((N_t - N_s) = 0) &= e^{-\lambda(t-s)}\frac{\lambda^0(t-s)^0}{0!} = e^{\lambda(t-s)} = 1 - \lambda\cdot\Delta + o(\Delta)\\
P((N_t - N_s) = 1) &= e^{-\lambda(t-s)}\frac{\lambda(t-s)}{1!} = (t-s)e^{\lambda(t-s)} = \lambda\cdot\Delta + o(\Delta)\\
P((N_t - N_s) \ge 2) &= \sum_{k\ge2} e^{-\lambda(t-s)}\frac{\lambda^k(t-s)^k}{k!} = o(\Delta)
\end{aligned}$$

where $\Delta = t - s$.

Remark 1.2. *The above characterization for a Poisson process is parallel to the one of a Wiener process:*

- *both are processes with independent increment; the increments of a Wiener process are normally distributed while those of a Poisson process are Poisson distributed.*
- *The Wiener process is a basic building block for processes with continuous trajectories, the Poisson process is a basic building block for processes with jumping trajectories.*
- *the Wiener process is itself a martingale while the Poisson process is not, nevertheless it becomes a martingale if one subtracts from N_t its Compensator.*

Lemma 1.4. *Given $(\Omega, \mathcal{F}, P)$ a probability space with filtration $\mathcal{F}_t$, $\{N_t\}_{t\geq 0}$ Poisson process with intensity $\lambda_s \in L^1(0,s)$ which is $\mathcal{F}_t$ adapted we have that the process:*

$$M_t = N_t - \int_0^t \lambda_s ds \tag{1.37}$$

is an $\mathbb{P}$ martingale

Proof. We have to check that $E(M_t|\mathcal{F}_s) = M_s \ \forall t > s$, in particular we have:

$$\begin{aligned} E(M_t - M_s|\mathcal{F}_s) &= E\left(N_t - N_s - \int_s^t \lambda_u du \middle| \mathcal{F}_s\right) = E(N_t - N_s|\mathcal{F}_s) - \int_s^t \lambda_u du \\ &= E(N_t - N_s) - \int_s^t \lambda_u du = 0 \end{aligned}$$

Where we used that $(N_t - N_s) \perp \mathcal{F}_s$ and $(N_t - N_s) \sim \text{Pois}(\Lambda_{s,t})$ and in particular $\Lambda_{s,t} = \int_s^t \lambda_u du$ □

1.5.2 Marked point process

A Marked point process is a point process with a random variable or a random vector attached to each point; each of the times T_i has a mark M_i associated with it and a possible realization of a marked point process is given by a sequence:

$$(T_1, M_1), ..., (T_i, M_i), ...$$

with $T_1 < T_2, ...$ and $M_i \in \mathcal{M}$ the space of marks (we usually work with non negative marks). The counting process associated is the process $\{N_t\}_{t\geq 0}$ which, as before, counts the number of occurrence of the sequence of the T_i in the interval $[0, t)$.

Remark 1.3. *The generalization of unmarked point processes to marked point processes subsumes several important point processes. For example, a marked point process can be used to define a point process with multiple occurrences; the marks in this case would give the number of occurrences at each point event. A marked point process may also be used to define a multi-type point process with the marks identifying the type of a point event, i.e. $M = \{1, 2, ..., k\}$ for a multi-type point process with k types of point event.*

1.5.3 Hawkes processes

We start defining the univariate Hawkes process recalling that $\{\mathcal{F}_t\}_{t\geq 0}$ is σ-algebra generated containing the complete history of the process:

Definition 1.14. *The univariate Hawkes process N with intensity $\lambda(\cdot|\mathcal{F}_t)$ is defined for all $t \geq 0$ and $h \to 0^+$ by:*

$$\begin{aligned} P(N[t,t+h) = 1|\mathcal{F}_t) &= \lambda(t|\mathcal{F}_t)h + o(h) \\ P(N[t,t+h) > 1|\mathcal{F}_t) &= o(h) \end{aligned} \tag{1.38}$$

where the complete intensity is defined by:

$$\begin{aligned} \lambda(t|\mathcal{F}_t) &= \tau + \int_{[0,t)} \omega(t-u)N(du) \\ &= \tau + \sum_{j:t_j<t} \omega(t-t_j) \end{aligned} \tag{1.39}$$

with $\tau in \mathbb{R}$, $\tau > 0$ and $\omega(s) \geq 0$

The complete intensity is a stochastic process and can be tougth as a Shot Noise process (see [10] p.74) where all the past point events can contribute to the current value of the complete intensity.
The self-exciting nature of the Hawkes process arises via the integral in Equation (1.39). The contribution from a point event at time $t_i < t$ to the complete intensity at time t is $\omega(t-t_i)$, and all points before time t contribute in such a way to the complete intensity at time t. The function $\omega(\cdot)$ controls the effect that past point events have on the intensity and it is often assumed to be a monotonically decreasing function so that the latest point events have the greatest influence on the current value of the intensity.
For a monotonically decreasing $\omega(\cdot)$, the intensity will increase immediately after a point event and as time passes the effect from the point event dies off. As a result, the risk of further point events occurring increases immediately following a point event and this increased risk dies off as time passes.
A Markovian decay function is the exponential decay function which has the form:

$$\omega_e(s) = \psi \exp(-\gamma s) \tag{1.40}$$

Where $\psi \geq 0$, $\gamma > 0$ and $\psi < \gamma$. An other popular example of of a decay function used in several application is the power-law decay function which has the form:

$$\omega_d(s) = \frac{\psi}{(s+\gamma)^{n+1}} \tag{1.41}$$

Where $\psi \geq 0$, $\eta, \gamma > 0$ and $\psi < \eta\gamma^{\eta}$.
To conclude this Section we add a slightly different definition of Hawkes processes that will be used to in the following chapter:

Definition 1.15. *We have:*

- *A point process with associated counting process $\{N_t\}_{t\geq 0}$,*

- *$J(ds, dz)$ a Poisson random measure with intensity $\lambda(t)$ satisfying the SDE:*

$$\begin{aligned}\lambda(t) &= \lambda(0) - \beta \int_0^t \lambda(s)ds + \alpha \int_0^t \int_0^\infty zJ(ds, dz) \\ &= \exp(-\beta t)\lambda(0) + \alpha \sum_{i=1}^{N_t} \exp[-\beta(t - t_i)]Z_i\end{aligned} \tag{1.42}$$

  with $\beta > 0$ is the rate of exponential decay of the influence of previous jumps on the intensity level, α is the amplitude of the of the memory kernel, T_i the jump times and Z_i the jump sizes.

Then we define an Hawkes process with exponential kernel a process that can be written as:

$$Y(t) = Y(0) + \sum_{i=0}^{N_t} Z_i = Y(0) + \int_0^t \int_0^\infty zJ(ds, dz) \tag{1.43}$$

Moreover we assume that the jump sizes are distributed according to an exponential density with parameter δ (so that only positive jumps appear in Equations (1.42) and (1.43)) and we can write:

$$\tilde{J}(ds, dz) = J(ds, dz) - \lambda(s)\mu(dz)ds$$

and $\mu(dz) = \delta \exp(-\delta z)dz$. The last Equation denotes the compensated version of the Poisson measure $J(ds, dz)$. Finally, in order to guarantee the non explosiveness of of the Hawkes process we assume that $\beta - \frac{\alpha}{\delta} > 0$.

Chapter 2

Hawkes and CBI to model forward prices

In this chapter the first part is devoted to a review of some notions martingale approach and arbitrage theory in the Black & Scholes market; the main references for this part are the books [4] and [14]. In the second part I will present two models for forward prices, where the dynamic of the underlying factor is driven by a CBI process and by an Hawkes process, applied to the power marke; the main references for this part will be the articles [8], [15] and [3].

2.1 Martingale approach to arbitrage theory

2.1.1 The general case

We will start by considering the special case when one of the assets on the market is a risk free asset with zero rate of return. As the basic setup we thus consider a financial market consisting of N given risky traded assets, and the asset price vector is as usual denoted by:

$$S(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{pmatrix} \tag{2.1}$$

We also assume that there exists a risk free asset with price process $S_0(t)$ which satisfies the following:

$$S_0(t) > 0 \quad \text{P-a.s. } \forall t \geq 0 \tag{2.2}$$

The main problem is to give condition for absence of arbitrage in this model and this are easily obtained moving to the "normalized" economy where we use S_0 as a numeraire. Thus instead of looking at the price vector process $S = [S_0, S_1, \ldots, S_n]$, we look to the relative price vector process $S(t)/S_0(t)$ where we have used S_0 as a numeraire price.

Definition 2.1. *The normalized economy (also referred as "Z-economy") is defined by the price vector process Z where:*

$$Z(t) = \frac{S(t)}{S_0(t)}$$

i.e.

$$Z(t) = [Z_0(t), \ldots, Z_n(t)] = \left[1, \frac{S_1(t)}{S_0(t)}, \ldots, \frac{S_n(t)}{S_0(t)}\right] \tag{2.3}$$

Note that at this point we have two price systems to keep track of: the S-system and the Z-system; the following definition clarifies the the relations between the two systems:

Definition 2.2. *We define:*

- *A portfolio strategy is any adapted $N+1$-dimensional process:*

$$h(t) = [h_0(t), h_1(t), \ldots, h_n(t)]$$

- *The S-value process $V^S(t,h)$ corresponding to the portfolio h is given by:*

$$V^S(t,h) = \sum_{i=0}^{N} h_i(t) S_i(t) \tag{2.4}$$

- *The Z-value process $V^Z(t,h)$ corresponding to the portfolio h is given by:*

$$V^Z(t,h) = \sum_{i=0}^{N} h_i(t) Z_i(t) \tag{2.5}$$

- *A portfolio is said to be admissible (as a Z-portfolio) if there exist a non negative real number α such that:*

$$\int_0^t h_S(u) dZ(u) \geq -\alpha \quad \textit{for all } t \in [0,T] \tag{2.6}$$

- *An admissible portfolio is said to be S-self financing if:*

$$dV^S(t,h) = \sum_{i=0}^{N} h_i(t) dS_i(t) \tag{2.7}$$

- *An admissible portfolio is said to be Z-self financing if:*

$$dV^Z(t,h) = \sum_{i=0}^{N} h_i(t) dZ_i(t) \tag{2.8}$$

We have the following important lemma:

Lemma 2.1. *With assumption and notations of the previous definition we have that:*

- *A portfolio h is S-self financing if and only if it is Z-self financing.*
- *The value process V^Z and V^S are connected by:*

$$V^Z(t,h) = \frac{1}{S_0(t))} \cdot V^S(t,h)$$

- *A claim $\mathcal{Y}$ is S-reachable if and only if the claim:*

$$\frac{\mathcal{Y}}{S_0(t)}$$

is Z-reachable.

- *The model is S arbitrage free if and only if it is Z arbitrage free.*

Proof. We just prove the third item since the other two are obvious. We assume moreover that all processes have stochastic differentials driven by a finite number of Wiener processes and that the portfolio h is S-self financing. We set $\beta = S_0$, then we have:

$$\begin{aligned} Z &= \beta^{-1} S \\ V^S &= h \cdot S \\ V^Z &= \beta^{-1} V^S \\ dV^S &= h \cdot dS \end{aligned}$$

And we want to prove that:

$$dV^Z = h \cdot dZ$$

Using the Ito formula on $Z = \beta^{-1} S$ we thus want to prove that:

$$dV^Z = \beta^{-1} h \cdot dS + h \cdot S dB^{-1} + h \cdot dS d\beta^{-1}$$

From $V^Z = \beta^{-1} V^S$ we have:

$$\begin{aligned} dV^Z &= \beta^{-1} dV^S + V^S d\beta^{-1} + d\beta^{-1} dV^S \\ &= \beta^{-1} h \cdot dS + h \cdot S d\beta^{-1} + d\beta^{-1} h \cdot dS \end{aligned}$$

where we used $dV^S = h \cdot dS$ and $V^S = h \cdot S$. □

We formulate the first fundamental theorem concerning absence of arbitrage:

Theorem 2.1. *Consider the market model $S_0, S_1, \ldots, S_N$ where we assume that $S_0(t) > 0$ P-a.s. for all $t \geq 0$. Assume furthermore that $S_0, S_1, \ldots, S_N$ are locally bounded. The model is arbitrage free if and only if there exist a martingale measure $Q \sim P$ such that the processes:*

$$Z_0, Z_1, \ldots, Z_n$$

are local martingales under Q.

2.1.2 Completeness

In this section we assume absence of arbitrage i.e. that there exist a local martingale measure. We now turn to the possibility of replicating a given contingent claim in terms of a portfolio based on the underlying assets. We introduce an important lemma:

Lemma 2.2. *Consider a given T-claim X. Fix a martingale measure Q and assume that the normalized claim $X/S_0(t)$ is integrable. If the Q-martingale M defined by*

$$M(t) = E^Q\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right] \tag{2.9}$$

admits an integral representation of the form:

$$M(t) = x + \sum_{i=1}^{N}\int_0^t h_i(s)dZ_i(s) \tag{2.10}$$

then X can be hedged in the S-economy. Furthermore the replicating portfolio $(h_0, h_1, \ldots, h_N)$ is given by equation (2.10) for the part $(h_1, \ldots, h_N)$ whereas:

$$h_0(t) = M(t) - \sum_{i=1}^{N} h_i(t)Z_i(t) \tag{2.11}$$

Proof. We want to hedge X in the S-economy i.e. we want to hedge $X/S_0(t)$ in the Z-economy. In terms of normalized prices we are looking for a process $(h_0, h_1, \ldots, h_N)$ such that:

$$V^Z(T,h) = \frac{X}{S_0(T)} \quad \text{P-a.s.} \tag{2.12}$$

and

$$dV^Z = \sum_{i=1}^{N} h_i dZ_i \tag{2.13}$$

where the normalized value process is given by:

$$V^Z(t,h) = h_0(t)\cdot 1 + \sum_{i=1}^{N} h_i(t)Z_i(t) \tag{2.14}$$

A reasonable guess is that $M = V^Z$ so let M be defined by (2.9) and set $(h_1, \ldots, h_N)$ by (2.10) and h_0 by:

$$h_0(t) = M(t) - \sum_{i=1}^{N} h_i(t)Z_i(t)$$

Now from (2.14) we obviously have $M = V^Z$ and from (2.10) we get:

$$dV^Z = dM = \sum_{i=1}^{N} h_i Z_i(t)$$

which shows that the portfolio is self financing. Furthermore we have:

$$V^Z(T,h) = M(T) = E^Q\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_T\right] = \frac{X}{S_0(T)}$$

which shows that X is replicated by h. □

We thus see that, modulo some integrability considerations, completeness is equivalent to the existence of a martingale representation theorem for the discounted price process. Now we are ready to set an important theorem which is the base of the proof of the second fundamental theorem of asset pricing:

Theorem 2.2. *Let $\mathcal{M}$ denote the convex set of equivalent martingale measures. Then for any fixed $Q \in \mathcal{M}$ the following are equivalent:*

- *Every Q local martingale M has dynamics of the form:*

$$dM(t) = \sum_{i=1}^{N} h_i(t) dZ_i(t)$$

- *Q is an extremal point of $\mathcal{M}$*

Then we have:

Theorem 2.3 (Second fundamental theorem of asset pricing)**.** *Assume that the market is arbitrage free and consider a fixed numeraire asset S_0. Then the market is complete if and only if the martingale measure corresponding to the numeraire S_0 is unique.*

Proof. If the martingale measure is unique then $\mathcal{M}$ is a singleton $\mathcal{M} = Q$ so Q is trivially an extremal point of $\mathcal{M}$ and thus for the previous theorem (Jacod) we have a stochastic integral representation of every Q martingale and then it follows from lemma 2.2 that the model is complete. The other implication is trivial. □

2.1.3 Martingale pricing

We turn to the pricing problem for contingent claims. We thus consider the primary market $S_0, S_1, \ldots, S_N$ and we fix a T-claim X. Our task is to determine a reasonable price process $\Pi(t, X)$ assuming that the market is arbitrage free. There are two possible approaches:

- The derivative should be priced in a way that it is consistent with the price of the underlying assets. More precisely we should demand that the extended market $\Pi(t, X), S_0, S_1, \ldots, S_N$ if free of arbitrage opportunities.
- If the claim is attainable with hedging portfolio h then the only reasonable price is given by $\Pi(t, X) = V(t, h)$.

In the first approach the aim is to find a martingale measure Q for the extended market $\Pi(\cdot, X), S_0, S_1, \ldots, S_N$. Assuming that such a measure Q does exist, and assuming enough integrability conditions, by definition of martingale measure we have that:

$$\frac{\Pi(t, X)}{S_0(t)} = E^Q\left[\frac{\Pi(T; X)}{S_0(T)}\middle|\mathcal{F}_t\right] = E^Q\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right] \tag{2.15}$$

We thus have the following:

Theorem 2.4 (General pricing formula)**.** *The arbitrge free price process for the T-claim X is given by:*

$$\Pi(t,X) = S_0(t)E^Q\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right] \tag{2.16}$$

Where Q is the (not necessarily unique) martingale measure for the a-priori given market $S_0, S_1, \ldots, S_N$ with numeraire S_0.

If we assume that S_0 is the money account:

$$S_0(t) = S_0(0) \cdot e^{\int_0^t r(s)ds} \tag{2.17}$$

where r denotes the short rate, then equation (2.16) reduces to the "risk-neutral valutation formula":

Theorem 2.5. *Assuming the existence of a short rate, the pricing formula takes the form:*

$$\Pi(t,X) = E^Q\left[Xe^{\int_t^T r(s)ds}\middle|\mathcal{F}_t\right] \tag{2.18}$$

Where Q is the (not necessarily unique) martingale measure with money account as numeraire.

For the second approach to pricing let us assume that X can be replicated by h. Since the holding of the derivative contract and the holding of the replicating portfolio are equivalent from a financial point of view, we see that the price of the derivative must be given by the formula:

$$\Pi(t,X) = V(t,h) \tag{2.19}$$

One problem is what happen in case when X can be replicated by two different portfolios and one would also know how this formula is connected to (2.16).
Defining $\Pi(t,X)$ by (2.19) we see that $\Pi(t,X)/S_0(t) = V^Z(t)$ and since, assuming enough integrability, V^Z is a Q martingale we see that also $\Pi(t,X)/S_0(t,X)$ is a Q-martingale. Thus we obtain the formula:

$$V(t,h) = S_0(t)E^Q\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right] \tag{2.20}$$

which will hold for any replicating portfolio and any martingale measure Q.

2.2 The mathematics of Martingale approach

2.2.1 Stochastic integral representation

Let us consider a fixed time interval $[0,T]$, a probability space $(\Omega,\mathcal{F},P)$ with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and an adapted vector Wiener process $W = (W_1,\ldots,W_d)^T$. Fix now a vector process $h = (h_1,\ldots,h_d)$ which is "integrable enough" and a real number x_0.If we define the process:

$$M(t) = x_0 + \sum_{i=1}^{d}\int_0^t h_i(s)dW_i(s), \quad t\in[0,T] \tag{2.21}$$

then we know that M is a martingale. (Recall that under mild integrability condition every stochastic integral w.r.t a Wiener process is an $\mathcal{F}_t$ martingale). A natural and important question is whether the converse holds, i.e. if every $\mathcal{F}_t$-adapted martingale M can be written in the form (2.21); if this is the case we say that M admits a stochastic integral representation.
It is not hard to see that in the completely general case, there is no hope for a stochastic integral representation w.r.t. to W for a general martingale. As a counterexample consider the case in which $d = 1$ so W is scalar, we consider a Poisson process N and we assume that $\mathcal{F}_t$ contains all the information generated by W and N in the interval $[0, t]$. We know that:

$$M(t) = N(t) - \lambda t$$

is an $\mathcal{F}_t$-martingale. If we look at the trajectories they consist into straight lines with downward slope λ interrupted at exponentially distributed points in time by positive jumps of unit size. From this it is obvious that M can posses no stochastic integral representation of the form (2.21) since any such representation has continuous trajectories.
It is clear from the previous example that we can only hope for a stochastic integral representation in the case when $\{\mathcal{F}_t\}_{t\geq 0}$ is the internal filtration generated by the Wiener process W itself.
We state now:

Theorem 2.6. *Let W a d-dimensional Wiener process and let X a stochastic variable such that:*

- $X \in \mathcal{F}_T^W$
- $E[|X|] < \infty$

There exist a uniquely determined $\mathcal{F}_t^W$-adapted process $h_1, \ldots, h_d$ such that X has representation:

$$X = E[X] + \sum_{i=1}^{d} \int_0^T h_i(s) dW_i(s) \tag{2.22}$$

Proof. For simplicity of notations we consider the scalar case $d = 1$. Recall the Geometric Brownian motion equation:

$$\begin{aligned} dX_t &= \sigma X_t dW_t \\ X_0 &= 1 \end{aligned}$$

and its solution:

$$X_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t} \tag{2.23}$$

The previous SDE in integral form becomes:

$$X_t = 1 + \int_0^t \sigma X_s dW_s \tag{2.24}$$

Plugging now (2.23) into (2.24) we obtain that:

$$e^{\sigma W_t} = e^{\frac{1}{2}\sigma^2 t} + \sigma \int_0^t e^{-\frac{1}{2}\sigma^2 (u-t) + \sigma W_u} dW_u$$

Using the same argument we obtain for $s \geq t$ that:

$$e^{\sigma(W_t - W_s)} = e^{\frac{1}{2}\sigma^2(t-s)} + \sigma \int_s^t e^{-\frac{1}{2}\sigma^2(u-t+s)+\sigma W_u} dW_u \tag{2.25}$$

Thus any stochastic variable of the type:

$$Z = \exp\{\sigma(W_t - W_s)\}$$

will have representation of the form:

$$Z = E[Z] + \int_0^T h_u dW_u$$

with $h \equiv 0$ outside $[s,t]$. From this we have easily that any random variable Z of the form:

$$Z = \prod_{k=1}^{n} \exp\{\sigma_k(W_{t_k} - W_{t_{k-1}})\} \tag{2.26}$$

where $0 \leq t_0 \leq \cdots \leq t_n \leq T$ has representation of the form:

$$Z = E[Z] + \int_0^T h_u dW_u \tag{2.27}$$

It is now fairly straightforward to see that any variable of the form:

$$Z = \prod_{k=1}^{n} \exp\{i\sigma_k(W_{t_k} - W_{t_{k-1}})\} \tag{2.28}$$

where i is the imaginary unit, has representation of the form (2.27). Using now some Fourier techniques we can prove that the set of variables of the form (2.28) is dense in $L^2(\mathcal{F}_T)$ and from this we can conclude that any variable in $L^2(\mathcal{F}_T)$ has a representation of the type (2.27). □

From this theorem we easily obtain the:

Theorem 2.7 (Martingale representation theorem)**.** *Let W a d-dimensional Wiener process and assume that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is defined as:*

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\{W_s : s \leq t\} \quad t \in [0,T]$$

Let M any $\mathcal{F}_t$ adapted martingale, then there exist a uniquely determined $\mathcal{F}_t$ adapted process $h_1, \ldots, h_d$ such that M has the representation:

$$M(t) = M(0) + \sum_{i=1}^{d} \int_0^t h_i(s) dW_i(s) \tag{2.29}$$

The proof follows immediately from theorem 2.6.

2.2.2 The Girsanov Theorem

In this section we discuss the effect that an absolutely continuous measure transformation will have upon a Wiener process.
Assume that our space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ carries a P-Wiener process W^P and that for some fixed T we have changed to a new measure Q on $\mathcal{F}_T$ by choosing a non-negative random variable $L_T \in \mathcal{F}_T$ and defining Q by:

$$dQ = l_T dP \quad \text{on } \mathcal{F}_\mathcal{T}$$

This measure transformation will generate a likelihood process $\{L_t : t \geq 0\}$ defined by:

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_\mathcal{T}$$

and we know that the previous process is a P-martingale.
Since L is a P-martingale and since any stochastic integral with respect to W is a martingale it is natural to define L as the solution of the stochastic differential equation:

$$\begin{cases} dL_t = \varphi_t L_t dW_t^P \\ L_0 = 1 \end{cases} \tag{2.30}$$

With this procedure it seems that we can generate a large class of natural measure transformations from P to Q by following:

1. Choose an arbitrary process W.
2. Define a likelihood process L by:
$$\begin{cases} dL_t = \varphi_t L_t dW_t^P \\ L_0 = 1 \end{cases}$$
3. Define a new measure Q setting:
$$dQ = L_t dP$$
on $\mathcal{F}_t$ for all $t \in [0, T]$.

Applying Ito formula we easily find that

$$L_t = e^{\int_0^t \varphi_s dW_s^P - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

Thus L is non-negative and under some integrability conditions on φ (Novikov condition) we easily derive that L is a martingale with $E^P[L_t] = 1$.
We recall now an important theorem known as "Abstract Bayes formula", (for a complete proof we remind to [4], Appendix B, proposition B.41)

Theorem 2.8. *Assume that X is a random variable on $(\Omega, \mathcal{F}, P)$ and let Q a probability measure with radon Nikodym derivative:*

$$L = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \tag{2.31}$$

Assume that $X \in L^1(\Omega)$ and that $\mathcal{G}$ is a sigma algebra s.t. $\mathcal{F} \subseteq \mathcal{G}$ then

$$E^Q[X|\mathcal{G}] = \frac{E^P[L \cdot X|\mathcal{G}]}{E^P[L|\mathcal{G}]} \quad Q - a.s. \tag{2.32}$$

Our aim is now to compute what are the dynamics of W^P under Q. Recall that if X has dynamics:

$$dX_t = \mu_t dt + \sigma_t dW_t^P$$

and we have that:

$$\begin{aligned} E^P[dX_t|\mathcal{F}_t] &= \mu_t dt \\ E^P[(dX_t)^2|\mathcal{F}_t] &= \sigma_t^2 dt \end{aligned} \tag{2.33}$$

with the informal interpretation $dX_t = X_{t+dt} - X_t$.
Let now $X = W^P$ (i.e. $\mu = 0$ and $\sigma = 1$), using theorem 2.8 and the fact that L is a P-martingale we obtain

$$\begin{aligned} E^Q[dX_t\mathcal{F}_t] &= \frac{E^P[L_{t+dt}dX_t\mathcal{F}_t]}{E^P[L_{t+dt}\mathcal{F}_t]} \\ &= \frac{E^P[L_{t+dt}dX_t\mathcal{F}_t]}{L_t} \\ &= \frac{E^P[L_t dX_t\mathcal{F}_t]}{L_t} + \frac{E^P[dL_t dX_t\mathcal{F}_t]}{L_t} \end{aligned}$$

Since L is adapted and X has zero drift under P we have that:

$$\frac{E^P[L_t dX_t\mathcal{F}_t]}{L_t} = L_t \cdot \frac{E^P[dX_t\mathcal{F}_t]}{L_t} = 0 \cdot dt$$

Furthermore we have that:

$$dL_t dX_t = L_t\varphi_t dW_t^P(0 \cdot dt + 1 \cdot dW_t^P) = L_t\varphi_t(dW_t^P)^2$$

and hence since $L_t\varphi_t \in \mathcal{F}_t$:

$$\frac{E^P[dL_t dX_t|\mathcal{F}_t]}{L_t} = \varphi_t dt$$

finally using that $dX_t^2 = dt$ we can compute the quadratic variation of X under Q:

$$\begin{aligned} E^Q[(dX_t)^2|\mathcal{F}_t] &= \frac{E^P[L_{t+dt}(dX_t)^2|\mathcal{F}_t]}{L_t} \\ &= \frac{E^P[L_{t+dt}dt|\mathcal{F}_t]}{L_t} \\ &= \frac{E^P[L_{t+dt}|\mathcal{F}_t]}{L_t}dt \\ &= dt \end{aligned}$$

We found that the process X that was, under P, a standard Wiener process with unit diffusion term and zero drift, under Q defined above the drift process has changed from zero to φ and the diffusion remains the same. In other words we have that:

$$dW_t^P = \varphi_t dt + dW_t^Q$$

Theorem 2.9 (Girsanov Theorem). *Let W^P a d-dimensional P-Wiener process defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and let φ be any d-dimensional adapted column vector process. Choose a fixed T and define the process L on $[0,T]$ by:*

$$\begin{cases} dL_t = \varphi_t^\star L_t dW_t^P \\ L_0 = 1 \end{cases} \tag{2.34}$$

i.e.

$$L_t = e^{\int_0^t \varphi_s^\star dW_s^P - \frac{1}{2}\int_0^t \|\varphi_s\|^2 ds} \tag{2.35}$$

Assume that:

$$E^P[L_T] = 1 \tag{2.36}$$

and define a new probability measure Q on $\mathcal{F}_T$ by:

$$\frac{dQ}{dP} \quad on\ \mathcal{F}_T \tag{2.37}$$

then

$$dW^P = \varphi_t dt + dW_t^Q \tag{2.38}$$

where W^Q is a Q-Wiener process.

Observation 2.1. *In the previous theorem we adopted the following notation:*

- *we denote with $\varphi^\star$ the transpose of the vector φ and we can rewrite equation (2.34) and (2.35) by components as:*

$$dL(t) = L(t)\sum_{i=1}^d \varphi_i(t) dW_i^P(t)$$

$$L(t) = \exp\left\{\sum_{i=1}^d \int_0^t \varphi_i(s) dW_i^P(s) - \frac{1}{2}\int_0^t \sum_{i=1}^d \varphi_i^2(s) ds\right\}$$

- *We often refer to the process φ as the Girsanov Kernel of the measure transformation*

Definition 2.3. *For any Wiener process W and any kernel process φ the Doleans-Dade exponential process $\mathcal{E}$ is defined by:*

$$\mathcal{E}(\varphi^\star W)(t) = \exp\left\{\int_0^t \varphi^\star(s) dW^P(s) - \frac{1}{2}\int_0^t \|\varphi\|^2(s) ds\right\} \tag{2.39}$$

Observe that in Girsanov theorem we have to assume an ad-hoc condition thet φ is such that $E^P(L_T) = 1$ or, in other words, that L is a martingale. We can remove this hypothesis in case in which it is satisfied the following condition:

Lemma 2.3 (Novikov Condition). *Assume that the Girsanov kernel φ is such that:*

$$E^P\left[e^{\frac{1}{2}\int_0^T \|\varphi_t\|^2 dt}\right] < \infty \tag{2.40}$$

Then L is a martingale and in particular $E^P[L_T] = 1$

2.3 Forward price modeling

In this section we are going to introduce two alternative models for the forward prices and we are going to test them against electricity market data. We assume that the spot price process evolves according to the following dynamics:

$$S(t) = \alpha(t) + X(t) \tag{2.41}$$

where $\alpha(t)$ is a function representing the seasonality and the process $\{X(t) : t \geq 0\}$ is a superposition of the factors Y_i; we can rewrite the previous equation as:

$$S(t) = \alpha(t) + \sum_{i=1}^{n} Y_i(t) \tag{2.42}$$

2.4 The model based on CBI

In this subsection we will focus to the case in which the underlying factors follows a dynamic which is given by the stochastic differential equation:

$$\begin{aligned} Y_i(t) = Y_i(0) - \int_0^t \alpha_i(b_i - Y_i(s))ds + \sigma_i \int_0^t \int_0^{Y_i(s)} W_i(ds, du) \\ + \gamma_i \int_0^t \int_0^{Y_i(s-)} \int_{\mathbb{R}^+} x\tilde{N}(ds, du, dz) \end{aligned} \tag{2.43}$$

where $\alpha_i, \sigma_i, \gamma_i \in \mathbb{R}^+$ are constant parameters for $i = 1, \ldots, n$. Assuming that the factors Y_i follows the dynamic given by equation (2.43) we obtain that:

Lemma 2.4. *For any $T \geq 0$ and $\tau \leq T$ we have that:*

$$E[Y_i(T)|\mathcal{F}_\tau] = b_i + (Y_i(\tau) - b_i)e^{-a_i(T-\tau)} \tag{2.44}$$

Proof. First of all we observe that:

$$\begin{aligned} E(Y_i(T)) &= Y_i(0) + E\left[\int_0^T a_i(b_i - Y_i(s))ds\right] \\ &= Y_i(0) + \int_0^T a_i(b_i - E(Y_i(s)))ds \end{aligned}$$

set now $\varphi(t) = E(Y_i(t))$, we can differentiate both sides to get:

$$\frac{\partial}{\partial t}\varphi(t) = a_i(b_i - \varphi(t))$$

whose solution is:

$$E(Y_i(t)) = b_i + (Y_i(0) - b_i)e^{-a_i t} \tag{2.45}$$

A direct application of Markov property allow us to conclude. □

We discuss now the procedure of choosing a suitable risk neutral probability measure, in particular in the electricity market, the risk-neutral probability $\mathbb{Q}$ is often chosen by introducing a drift adjustment in the dynamics of an underlying asset. The following proposition tells us that after this change of measure the spot process remains in the same class (i.e. CBi processes with modified coefficients).

Proposition 2.1. *Let $Y_1, \dots, Y_n$ be independent CBI processes where for each $i = \{1, \dots, n\}$ Y_i is a CBI process under the historical probability measure $\mathbb{P}$. Assume that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the random fields $W_1, \dots, W_n$ and $\tilde{N}_i, \dots, \tilde{N}_n$. For each i fix $\eta_i \in \mathbb{R}$ and $\xi_i \in \mathbb{R}^+$ and define:*

$$U_t = \sum_{i=1}^{n} \eta_i \int_0^t \int_0^{Y_i(s)} W_i(ds, du) + \sum_{i=1}^{n} \int_0^t \int_0^{Y_i(s-)} \int_0^{\infty} (e^{-\xi_i z} - 1)\tilde{N}(ds, du, dz) \tag{2.46}$$

Then the Doleans-Dade exponential $\mathcal{E}(U)$ is a martingale under P and the probability measure defined by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(U) \tag{2.47}$$

Moreover under $\mathbb{Q}$ the processes Y_i are independent on each other and for each i they are CBI processes with parameters $(a_i', b_i', \sigma_i', \gamma_i', \mu_i')$ given by:

$$\begin{aligned}
a_i' &= a_i - \sigma_i\eta_i - \gamma_i \int_0^{\infty} z(e^{-\theta_i z} - 1)\mu_i(dz) \\
b_i' &= \frac{a_i b_i}{a_i'} \\
\sigma_i' &= \sigma_i \\
\gamma_i' &= \gamma_i \\
\mu_i'(dz) &= e^{-\theta_i z}\mu_i(dz)
\end{aligned}$$

Proof. The process $(Y_1, \dots, Y_n, U)$ is a time homogeneous affine process ([13] Theorem 2.12). The Doleans-Dade exponential $\mathcal{E}(U)$ is a true martingale by checking that the conditions in ([17], Corollary 3.2) are satisfied, so it defines an equivalent probability measure $\mathbb{Q}$. Note that $Z = \mathcal{E}(U)$ is the unique strong solution of

$$dZ_t = Z_t dU_t.$$

Then for any function $F \in C^2(\mathbb{R}^n_+)$ the process:

$$\begin{aligned}
&Z_t F(Y_1(t), \dots, Y_n(t)) - \sum_{i=1}^{n} \frac{\sigma_i^2}{2} \int_0^t Z_s F_{ii}''(Y_1(s), \dots, Y_n(s))Y_i(s)ds \\
&- \sum_{i=1}^{n} \int_0^t Z_s F_i'(Y_1(s), \dots, Y_n(s))\left\{a_i b_i - \left[a_i - \sigma_i\eta_i - \gamma_i \int_0^{\infty} z(e^{-\theta_i z} - 1)\mu_i(dz)\right]Y_i(s)\right\} \\
&- \sum_{i=1}^{n} \int_0^t Z_s Y_i(s)ds \int_0^{\infty} \big[F(Y_1(s-), \dots, Y_i(s-) + \gamma_i z, \dots, Y_n(s-) - \\
&- F(Y_1(s-), \dots, Y_n(s-) + F_i'(Y_1(s-), \dots, Y_n(s-)Y_i(s-)\gamma_i z\big]e^{-\theta_i z}\mu_i(dz)
\end{aligned}$$

is a local martingale which implies that under $\mathbb{Q}$ the process $Y_1, \dots, Y_n$ are independent of each other and for each $i = 1, \dots, n$ Y_i is a CBI process with parameters $(a_i', b_i', \sigma_i', \gamma_i', \mu_i')$. □

Applying now the result proved in Lemma 2.4 we can provide an explicit expression for the instantaneous forward contract in the present modelling framework. By definition of a forward contract we have that:

$$F(\tau, T) = E^{\mathbb{Q}}[S(T)|\mathcal{F}_\tau] \tag{2.48}$$

The model parameters that we used so far were those defined by the hystorical dynamics in such a way that a_i, b_i and μ_i denote respectively the mean-reversion speed, the long term value and the jump measure with respect to $\mathbb{P}$. Applying the equivalent change of probability measure described in Proposition 2.1 we move to parameters a_i', b_i' and μ_i' with respect to $\mathbb{Q}$.
Since the factors Y_i remains CBI processes also under $\mathbb{Q}$ we can apply the result given by Lemma 2.4:

Proposition 2.2. *The price of the forward contract written on S is given by:*

$$F(\tau, T) = S(\tau) + (\alpha(T) - \alpha(\tau)) + \sum_{i=1}^{n} ()e^{-a_i'(T-\tau)} - 1)(Y_i(\tau) - b_i') \tag{2.49}$$

Proof. Applying Lemma 2.4 we have that under the probability measure $\mathbb{Q}$:

$$\begin{aligned} F(\tau, T) &= \alpha(T) + \sum_{i=1}^{n} E^{\mathbb{Q}}[Y_i(T)|\mathcal{F}_t] \\ &= \alpha(T) + \sum_{i=1}^{n} b_i' + (Y_i(\tau) - b_i')e^{-a_i'(T-\tau)} \end{aligned}$$

and in particular:

$$S(\tau) = F(\tau, \tau) = \alpha(\tau) + \sum_{i=1}^{n} b_i' + (Y_i(\tau) - b_i')$$

Hence:

$$\begin{aligned} F(\tau, T) &= S(\tau) + F(\tau, T) - F(\tau, \tau) \\ &= \alpha(T) - \alpha(t) + S(\tau) + \sum_{i=1}^{n} (e^{-a_i'(T-\tau)} - 1)(Y_i(\tau) - b_i') \end{aligned}$$

□

2.5 The forward model based on Hawkes processes

In this section we will provide an alternative model where our factor are assumed to follow an Hawkes-type dynamic. Let n the number of factors in our model and let $c_i, \sigma_i \in \mathbb{R}^+$

with $i = 1, \ldots, n$ constant parameters. We assume that under the historical probability measure $\mathbb{P}$ the dynamics of the X_i is given by the following SDE:

$$X_i(t,T) = X_i(0,T) - \int_0^t c_i X_i(s,T)ds + \sigma_i \int_0^t \sqrt{X_i(s,T)}dW_i(s) + \int_0^t \int_0^\infty z\tilde{J}_i(dz,ds) \tag{2.50}$$

where $\tilde{J}_i(dz,ds)$ are compensated marked point process with intensity $\lambda_i(t)$ satisfying the SDE:

$$\lambda_i(t) = \lambda_i(0) - \beta_i \int_0^t \lambda_i(s)ds + \alpha_i \int_0^t \int_0^\infty z J_i(ds,dz) \tag{2.51}$$

where in the previous equation β_i is the rate of exponential decay of the influence of the previous jumps on the intensity level and α_i is the amplitude of the memory kernel of each factor X_i. We also assume that the jump size is distributed according to an exponential density with parameter δ_i for each (λ_i, X_i) so we can write:

$$\begin{aligned} \tilde{J}_i(ds,dz) &= J_i(ds,dz) - \lambda_i(s)\mu(dz)ds \\ &= J_i(ds,dz) - \lambda_i(s)\delta_i \exp(-\delta_i z)(dz)ds \end{aligned} \tag{2.52}$$

We observe, as a remark, that the choice of a square root process for the diffusion part of the forward curves dynamics is motivated by the positivity requirement as well as the choice of the exponential distribution for the sizes of the jumps.
In view of this preliminary work we obtain that formulation for the price of the forward contract written on n underlying factors X_i:

$$\begin{aligned} F(t,T) = \Lambda(t) - \Lambda(0) + f(0,T) - \sum_{i=1}^n \int_0^t c_i X_i(s,T)ds + \sum_{i=1}^n \int_0^t \sigma_i \sqrt{X_i(s,T)}dW_i(s) \\ + \sum_{i=1}^n \int_0^t \int_0^\infty z\tilde{J}_i(ds,dz) \end{aligned} \tag{2.53}$$

An alternative representation for the for the Hawkes type dynamic given by Dawson and Li, under the Historical probability measure $\mathbb{P}$ is given by the following SDE:

$$\begin{aligned} X_i(t,T) = X_i(0,T) - \int_0^t c_i X_i(s,T)ds + \int_0^t \int_0^{X_i(s,T)} \sigma_i W_i(du,ds) \\ + \int_0^t \int_0^{X_i(s_-,T)} \int_{\mathbb{R}^+} z\tilde{N}_i(dz,du,ds) \end{aligned} \tag{2.54}$$

Also in this case the intensity evolves according to equation (2.51). It appears immediate that the dynamics described by the two models look almost identical when written in the Dawson-Li representation and the unique difference is given by the equation governing the evolution of the intensity process.
At this point in order to have a description with respect to a risk neutral probability measure $\mathbb{Q}$ we need to introduce a measure change. The following proposition provides a measure change which preserves the Hawkes type dynamic.

Proposition 2.3. *Let (λ_i, X_i) be described by equations (2.51) and (2.50) under the historical probability $\mathbb{P}$. Fix $(\eta, \xi) \in \mathbb{R} \times (-\delta_i, \infty)$ and define:*

$$U_t := \sum_{i=1}^{n} \eta_i \sigma_i \int_0^t \int_0^{X_i(s,T)} \sigma_i W_i(du, ds) + \sum_{i=1}^{n} \int_0^t \int_0^{X_i(s_-,T)} \int_{\mathbb{R}^+} (e^{-\xi_i z} - 1) \tilde{N}_i(dz, du, ds) \tag{2.55}$$

Then the Doleans-Dade exponential $\mathcal{E}(U)$ is a martingale under $\mathbb{P}$ and the probability measure defined by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \mathcal{E}(U)_t \tag{2.56}$$

is equivalent to $\mathbb{P}$. Moreover the dynamic with respect to $\mathbb{Q}$ takes the following form:

$$X_i(t,T) = X_i(0,T) + \int_0^t \int_0^{X_i(s,T)} \sigma_i' W_i(du, ds) + \int_0^t \int_0^{X_i(s_-,T)} z \tilde{N}_i'(dz, du, ds)$$
$$\lambda_i(t) = \lambda_i(0) - \int_0^t \beta_i' \lambda(s) ds + \alpha_i' \int_0^t \int_0^{\infty} \exp(-\beta_i'(t-s)) J_i'(ds, dz)$$

(the symbol $'$ denotes the parameters under the probability measure $\mathbb{Q}$) where:

$$c_i' = c_i - \sigma_i \eta_i - \int_0^{\infty} z(e^{-\theta_i z} - 1)\mu_i(dz)$$
$$\sigma_i' = \sigma_i$$
$$\alpha_i' = \alpha_i$$
$$\beta_i' = \beta_i$$
$$\mu_i'(dz) = e^{-\theta_i z}\mu_i(dz)$$

We end this section with some remarks:

- parameters η_i and ξ_i can be interpreted as the market price of risk associated with the diffusion/jump part of the i-th factor X_i.
- We assume that the de-seasonalized dynamics of X_i is a local martingale under $\mathbb{Q}$ which makes automatically the mean reversion speed $c_i = 0$ under $\mathbb{Q}$.
- From the formulas in the previous lines, specifying the relations between the model parameters under the risk-neutral measure $\mathbb{Q}$ and the historical measure $\mathbb{P}$, it is clear that in the Hawkes modeling framework a mean reversion speed coefficient c_i can be nonzero under $\mathbb{P}$ and zero under $\mathbb{Q}$.

Chapter 3

Financial model

In this chapter we discuss an application of the previous two types of processes to power markets; in particular we want to give an estimate of forward curves starting from future prices and then use this forward curves to test the two models based on CBI and Hawkes processes. For this chapter we mainly refer to [8] and also to [2] for the part related on forward curves estimation.

3.1 From futures prices to forward curves

In this section we focus our attention on how to determine the historical forward prices starting from future prices observed in the market. First of all we need to set a definition:

Definition 3.1. *The price at time $t \geq 0$ of a futures contract with delivery period $[T_1, T_2]$ with $t \leq T_1 \leq T_2$ is given by:*

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, x) dx$$

where $f(t, x)$ is the price of the forward contract to be paid upon delivery.

In our modeling framework the value at time t of a future contract with delivery period $[T_1, T_2]$ is given by:

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, x) dx = \Lambda(t) - \Lambda(0) + \frac{1}{T_2 - T_1} \left(\sum_{i=1}^{n} \int_{T_1}^{T_2} X_i(t, x) dx \right) \quad (3.1)$$

Introducing now the dynamics of factors X_i in Equation (3.1) under the risk neutral probability $\mathbb{Q}$, we obtain that in the CBI framework:

$$\begin{aligned} F(t, T_1, T_2) = \Lambda(t) - \Lambda(0) &+ \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(0, x) dx \\ &+ \frac{1}{T_2 - T_1} \sum_{i=1}^{n} \sigma_i \int_{T_1}^{T_2} \int_0^t \int_0^{X_i(s,x)} W_i(ds, dy) dx \\ &+ \frac{1}{T_2 - T_1} \sum_{i=1}^{n} \gamma_i \int_{T_1}^{T_2} \int_0^t \int_0^{X_i(s-,x)} \int_{\mathbb{R}^+} z \tilde{N}_i(ds, dy, dz) dx. \end{aligned} \quad (3.2)$$

and in the Hawkes setting:

$$\begin{aligned} F(t,T_1,T_2) = \Lambda(t) - \Lambda(0) + \frac{1}{T_2-T_1}\int_{T_1}^{T_2} f(0,x)dx \\ + \frac{1}{T_2-T_1}\sum_{i=1}^{n}\sigma_i \int_{T_1}^{T_2}\int_0^t \sqrt{X_i(s,x)}dW_i(ds)dx \\ + \frac{1}{T_2-T_1}\sum_{i=1}^{n}\gamma_i \int_{T_1}^{T_2}\int_0^t \int_{\mathbb{R}^+} z\tilde{J}_i(dz,ds)dx. \end{aligned} \tag{3.3}$$

In order to avoid arbitrage opportunities the prices of futures with different delivery period must satisfy some specific time consistency relations. In particular if we have a contract with delivery period $[T_1, T_n]$ it is linked to the values of the contract in intervals $[T_i, T_i+1]$ with $i = 1, \ldots, n-1$ by the following:

$$F(t,T_1,T_n) = \frac{1}{T_n - T_1}\sum_{i=1}^{n-1}(T_{i+1}-T_i)F(t,T_i,T_{i+1}). \tag{3.4}$$

From a theoretical point of view the contracts are settled continuously over the delivery period but in practice they are settled at discrete times; assuming settlement at N points in time $u_1 < u_2 < \cdots < u_n$ with $u_1 = T_1$ and $u_N = T_2$, thus the discrete version of Equation (3.1) becomes:

$$F(t,T_1,T_2) = \frac{1}{T_2-T_1}\sum_{i=1}^{N} f(t,u_i)\Delta_i \tag{3.5}$$

with $\Delta_i = u_{i+1} - u_i$. The main goal is to provide a forward dynamics formulation starting from the futures prices that we observe in the market.

Notation 3.1. *We use T_i^s and T_i^e to denote respectively the first and the last day of the delivery period for the i-th contract*

From the market we observe the quantity $F(0, T^s, T^e)$ for every contract with different possible choices of T^s and T^e with $T^e - T^s = 7, 30, 90$ and 365 days according to the type of the contract and where 0 denotes the current date; in other words for any day in an historical horizon we observe the value $F(0, T^s, T^e)$ which corresponds to the price of a future contract with delivery period $[T^s, T^e]$.
The initial condition that we need to use the Heat-Jarrow-Morton approach when modeling forward is a smooth curve describing today forward prices which must be extracted from futures prices observed in the market. Following the approach given in [2] we set:

Assumption 3.1. *The forward curve can be rapresented as the sum of two continuous functions $\Lambda(u)$ and $\varepsilon(u)$:*

$$f(u) = \Lambda(u) - \Lambda(0) + \varepsilon(u), \quad u \in [T^s, T^e] \tag{3.6}$$

where T^s is the starting day of the settlement period for the contract with the closest delivery period and T^e is the first day of the settlement period for the contract with the farthest delivery period.

We interpret $\Lambda(u)$ as a seasonality function and $\varepsilon(u)$ as an adjustment function that captures the forward curve's deviation from the seasonality.
For the seasonality function, following [2], we assume that:

$$\Lambda(u) = a\cos\left((u-b)\frac{2\pi}{365}\right) \tag{3.7}$$

where $a \in \mathbb{R}^+$ is the minimum of the prices all over the contracts and $b \in \mathbb{R}$ is the normalized distance between the end of the last day of the year from the day when the minimum occurs. Following again the approach from [2] we require for the adjustment function that it is twice continuously differentiable and horizontal at time T^e, that is:

$$\varepsilon'(T^e) = 0$$

This flatness condition is due to the fact that the long end of the curve may be several years ahead and obviously the market's view on risk becomes less and less sensitive as time goes by.

Definition 3.2. *Let:*

$$C_0^2([T^s,T^e]) = \{\varepsilon : [T^s,T^e] \to \mathbb{R} \text{ } s.t. \text{ } \varepsilon'' \in C_0(T^s,T^e) \text{ } and \text{ } \varepsilon'(T^e) = 0\}. \tag{3.8}$$

Furthermore we consider $\mathcal{C}$ the set of polynomial spline functions of order four which belong to $C_0^2([T^s,T^e])$. Then we define the adjustment function on an interval $[T^s,T^e]$ the function that minimizes over $\mathcal{C}$ the integral:

$$\int_{T^s}^{T^e} [\varepsilon''(u)]^2 du. \tag{3.9}$$

To determine the adjustment function, and therefore the smoothest forward curve a detailed algorithm is presented in [8], other approaches can be found in [2].
At this point, applying choosing different initial day (the 0 in the formulation of $f(0,u)$) in a fixed (sufficiently long) temporal horizon, we find a certain amount of different forward curves $f(0,u)$ s.t. $u \in [T^s,T^e]$ as in Assumption 3.1.
Given this collection of curves we want to detect if there are times such that positive jumps in the price occur. For a fixed maturity T we define:

$$V_t = f(t,T).$$

If we plot the forward curves previously determined as functions of time to maturity (x-axes) we can visualize the quantity V_t as a vertical section of such graph for a fixed time to maturity T. Clearly t ranges over the number of curves that we dispose. The simplest way to detect if a jump occurs is to fix a certain $\Theta \in \mathbb{R}^+$, then we say that a jump occurs at time $\hat{t}$ if:

$$|V_{\hat{t}+1} - V_{\hat{t}}| \geq \Theta.$$

An other possible approach is given by an iterative least square algorithm. Set n the number of forward curves available and $\mathcal{N} = \{0,1,\ldots,n-1\}$ then the algorithm reads as follows:

1. Set σ_1^2 by:
$$\sigma_1^2 = \frac{1}{n-2}\sum_{t\in\mathcal{N}}\frac{(V_{t+1}-V_t)^2}{V_t}$$

2. Define $\mathcal{M}_1 \subseteq \mathcal{N}$ the set of elements t such that $\frac{V_{t+1}-V_t}{\sqrt{V_t}} \geq 3\sigma_1$

3. Set $m_1 = |\mathcal{M}_1|$ and $i = 1$

4. While $m_i \neq 0$ repeat the computation:

   - set $i = i + 1$ and then:
   $$k = \sum_{j=1}^{i-1} m_j \ , \quad K = \bigcup_{j=1}^{i-1} \mathcal{M}_j$$
   and
   $$\sigma_i^2 = \frac{1}{n-k-1}\sum_{t\in\mathcal{N}\setminus K}\frac{(V_{t+1}-V_t)^2}{V_t} \tag{3.10}$$
   - Define $\mathcal{M}_i \subseteq \mathcal{N} \setminus K$ the set of elements t such that $\frac{V_{t+1}-V_t}{\sqrt{V_t}} \geq 3\sigma_i$
   - set $m_i = |\mathcal{M}_i|$.

Looking at the algorithm we can notice that, as i increases, the number σ_i^2 decreases and so the number of jumps detected; it is reasonable, looking at our data set, to stop the algorithm after a certain amount of iteration in order to focus our attention on larger jumps; if one waits until when $m_k = 0$ a lot of very small sized jumps occur.

3.2 Testing the model

3.2.1 Parameter estimation

Before proceeding with the Kolmogorov-Smirnov test (KS) on the two models we need to estimate the sizes of jumps (whose occurrence have been computed in Section 3.1)and the parameters characterizing the drift and the volatility coefficients.
Since we assumed for both CBI and Hawkes model that the size of jumps is distributed as an exponential random variable we can estimate the size of jumps δ via its maximum likelihood estimator; let z_i the size of the i-th jump, and L the number of jump estimated with the algorithm at the end of Section 3.1, then the estimator is:

$$\hat{\delta} = \frac{L}{\sum_{i=1}^{n} z_i}. \tag{3.11}$$

At this point to estimate the parameters appearing in the drift coefficient of our forward dynamics a_1 in Equation (2.43) and c_1 in Equation (2.50) (recall that we are assuming to work with a unique factor X) we first identify them by $\tilde{a}$ and then we recall that:

Remark 3.1. *Under the historical probability measure $\mathbb{P}$ both the CBI and Hawkes process in discrete form reads as (assuming no seasonality and no jumps):*

$$X(t+1,T) = X(t,T) - \int_t^{t+1} \tilde{a}X(s,T)ds$$

where X is the unique factor in the forward dynamic.

Thus to estimate $\tilde{a}$ we use, for a fixed maturity T:

$$\hat{\tilde{a}} = 1 - \frac{X(t+1,T)}{X(t,T)} \tag{3.12}$$

Finally to estimate the volatility parameter σ we take:

$$\hat{\sigma} = \sigma_2, \tag{3.13}$$

where σ_2 is given by Equation (3.10) of the algorithm to detect jumps at the end of Section 3.1.

3.2.2 KS test for the models

We want now to perform statistical tests on the models described in Chapter 2 and we want to establish which one of the two model proposed (the one based on CBI processes and the other based on Hawkes processes) is better. To perform such tests we will use Kolmogorov-Smirnov (KS) test, it is a non-parametric goodness-of-fit test and is used to determine whether two distributions differ, or whether an underlying probability distribution differs from a hypothesized distribution. In our case we will test the null hypothesis:

$H_0 =$ "the distribution of the data is the same of the one coming from the model"

against the alternative H_1.
Before using this test we need to estimate the intensity from our data set. We have as input the occurrences of jumps $\tau_1 < \dots \tau_n = T$ then in case of the CBI model we know the stochastic intensity $\lambda(t)$ is proportional to $X(t,T)$ (see [15]) and the constant of proportional is estimated using the number of jumps and the cumulative de-seasonalized forward prices. In case of the Hawkes dynamic to estimate the stochastic intensity we have to determine the values of parameters α, β and $\lambda(0)$ in Equation (2.51). The log-likelihood of an Hawkes process is given by:

$$\log L(\tau_1, \dots, \tau_N) = -\lambda_0 \tau_N + \sum_{i=1}^{N} \frac{\alpha}{\beta}\left(e^{-\beta(\tau_n - \tau_i)} - 1\right) + \sum_{i=1}^{N} \log(\lambda(0) + \alpha A(i)) \tag{3.14}$$

with $A(i) = \sum_{\tau_j < \tau_i} e^{-\beta(\tau_i - \tau_j)}$ and $A(1) = 0$. To determine the values of parameters α, β and $\lambda(0)$ we have to compute the maximum of the previous function as a real function of three variables.
At this point everything is set to perform the KS test to see which of the distribution best models the jumps observed in our set of data. This procedure is entirely described in [8]; in particular starting from an historical set of data on French power futures prices and after the implementation of the algorithms to compute forward prices and to estimates jumps on such prices and their sizes, the KS statistical test is performed (with a significance level of 0.05) to test the to models based on CBI and Hawkes processes and an additional toy model based on Poisson processes.

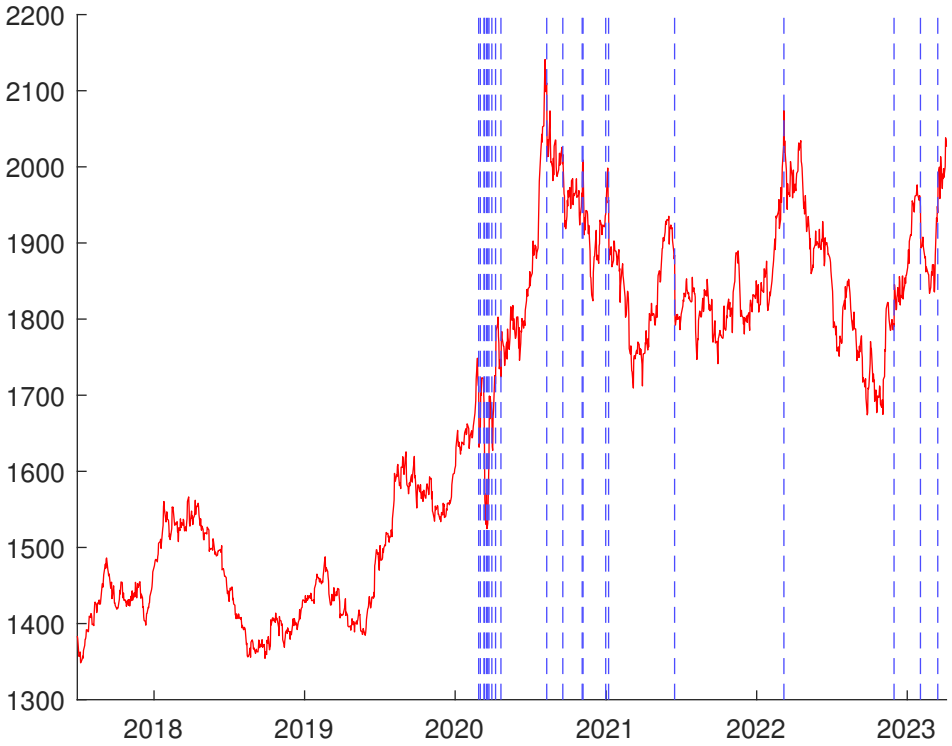

Figure 3.1: This figure shows the evolution of gold future prices (red line) and estimated jump times (blue vertical lines)

3.3 Focus on jump detection

In this subsection we focus our attention on the algorithm to estimate the number of jumps presented ad the end of Section 3.1 and we apply it to three different set of data:

- A simulated geometric Brownian motion trajectory with jumps given by an Hawkes process
- an historical set of data on gold futures[1]
- an historical set of data on crude oil futures[2]

In particular starting from this set of data (the first one is a toy model) we want first to apply the algorithm (whose code is available in Appendix C) to detect jumps and then we want to see if we can say something about the correlation of this jumps or not.

Figure 3.1 shows the evolution of the future price of gold from the starting date 2017/6/29 to 2023/4/6 in red line and the jumps in the future prices detected using the algorithm of jump detection (blue line). In particular looking at the position of jumps we see an high concentration of jumps around the first months of 2020, this is due to the beginning of corona-virus pandemic and in particular by the global collapse of the economies as a result of policies of home confinements and the clear contagion effect between markets of very different natures and geographies. After this gold experiment an

[1] GCM23.CMX on "https://finance.yahoo.com/", data are choose from 2017/6/29 to 2023/4/6
[2] CLN23.NYM on "https://finance.yahoo.com/"data are choose from 2017/11/21 to 2023/4/6

Figure 3.2: This figure shows the evolution of crude oil (red line) and estimated jump times (blue vertical lines)

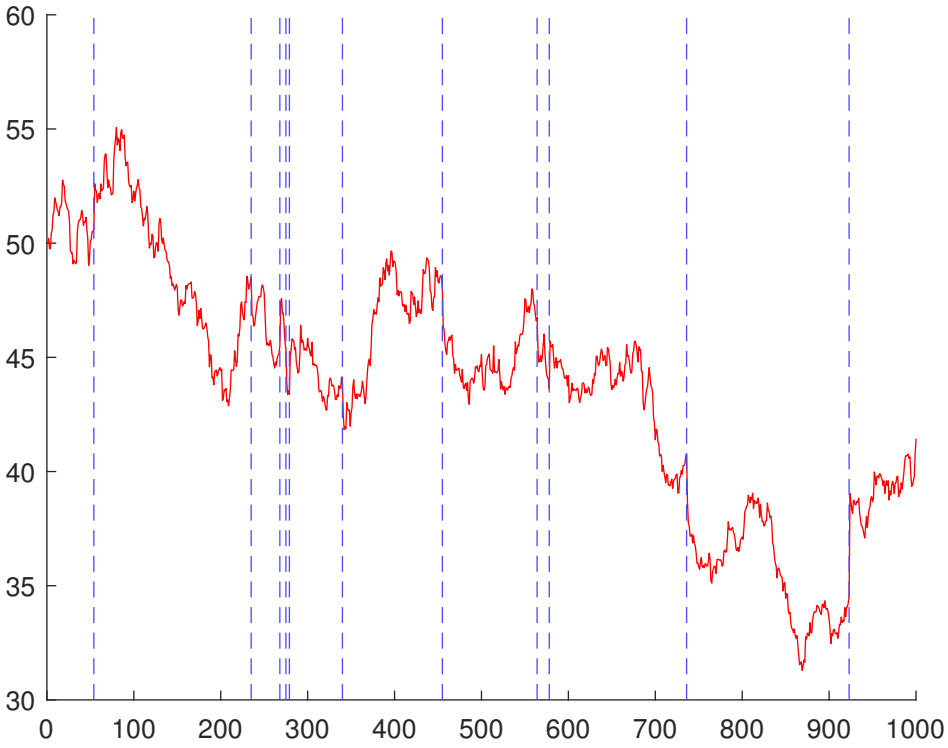

Figure 3.3: This figure shows the evolution of a simulated geometric Brownian motion with Hawkes jumps and the jumps detected with the algorithm

exponential recovery and the major driver behind the strong purchasing of Gold during the year 2020 lies in its store of value nature. As time goes by just few jumps are experimented but without an apparently fixed pattern.
Figure 3.2 shows the evolution of the future prices of crude oil from the starting date 2017/11/21 to 2023/4/6 (red line) and the jumps in the price detected using the algorithm of jump detection (blue line). We can see that around February/March 2020 there is a big downward jump due to corona-virus pandemic and the consequent global collapse of the economies while an high concentration of jumps, due to the explosion of the war between Ukraine and Russia, is present starting from the first months of 2022.

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Appendix A

Useful results

A.1 Doubly Stochastic Or Conditional Poisson Processes

A nonmathematical definition of a doubly stochastic Poisson process consists in describing a two step randomization procedure: in the first step one draws at random the trajectory of a driving process say Y_t and once the trajectory is selected one generates a Poisson process of intensity $f(t, Y_t)$. Of course such an intensity is random, since Y_t is a random process and thereforethe point process is Poisson only conditionally with respect to Y_t.

Definition A.1. *Let $\{N_t\}_{t\geq 0}$ a point process adapted to a σ-algebra $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\{\lambda_t\}_{t\geq 0}$ a nonnegative measurable process. Suppose that:*

$$\lambda_t \text{ is } \mathcal{F}_0 - \text{measurable } \forall t \geq 0$$

and that:

$$\int_0^t \lambda_s ds < \infty \ P - a.s. \quad \forall t \geq 0$$

If for all $0 \leq s \leq t$ and $u \in \mathbb{R}$:

$$E[e^{iu(N_t - N_s)}|\mathcal{F}_s] = exp\left\{(e^{iu} - 1)\int_s^t \lambda_v dv\right\} \tag{A.1}$$

then N_t is called a $(P, \mathcal{F}_t)$-doubly stochastic Poisson process.

In particular if λ_t is deterministic then N_t is a $(P, \mathcal{F}_t)$-Poisson process.
As an example we consider the case in which $\lambda_t = f(t, Y_t)$ for some properly measurable and non-negative function f and for some measurable process Y_t and if $\mathcal{F}_0$ contains $\mathcal{F}_\infty^Y$ then N_t is a doubly stochastic Poisson process driven by Y_t. We mention now an important theorem which gives a characterization for doubly stochastic Poisson processes:

Theorem A.1. *Let $\{N_t\}_{t\geq 0}$ a point process adapted to some history $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{\lambda_t\}_{t\geq 0}$ a non negative process such that $\forall t \geq 0$:*

1. *λ_t is $\mathcal{F}_t$-measurable.*
2. *$\int_0^t \lambda_s ds < \infty$ P-a.s.*

Then if the equality:

$$E\left[\int_0^\infty C_s dN_s\right] = E\left[\int_0^\infty C_s \lambda_s ds\right] \tag{A.2}$$

Is verified for all nonnegative $\mathcal{F}_t$ predictable processes C_t then N_t is a doubly stochastic Poisson process with the $\mathcal{F}_t$-intensity λ_t.

A.2 Multivariate and Marked Point Processes

Let $\{T_n\}_{n\in\mathbb{N}}$ a univariate point process and $\{Y_n\}_{n\in\mathbb{N}}$ a sequence of random variables with values in $\{1, 2, ..., K\}$ all defined in $(\Omega, \mathcal{F}, P)$. For each k we can consider:

$$N_t(k) = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{T_n\leq t\}}\mathbf{1}_{\{Y_n=k\}}$$

That is a random variable which counts the number of jumps of length k in the interval $(0, t]$. Moreover each $N_t(k)$ is a Poisson process and $\forall k \neq h$ the processes $N_t(k)$ and $N_t(h)$ have no jumps in common. We have two possible representation to describe this process called multivariate Poisson process; either with $(T_n, Y_n)_{n\in\mathbb{N}}$ or with a K-dimensional vector $(N_t(1), ..., N_t(K))_{t\geq 0}$. Considering the first representation $(T_n, Y_n)_{n\in\mathbb{N}}$ we may interpret T_n as the time of occurrence of some event and Y_n as an attribute or a *mark* of that event.

Definition A.2. *An E-marked point process is a double sequence $(T_n, Y_n)_{n\in\mathbb{N}}$ where:*

1. *T_n is a univariate point process.*
2. *Y_n is a sequence of random variables with values on $(E, \mathcal{E})$.*

Generalizing the previous representation of a multivariate point process we associate for any measurable set $A \in \mathcal{E}$ the counting process:

$$N_t(A) = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{T_n\leq t\}}\mathbf{1}_{\{Y_n\in A\}}$$

And we let simply $N_t := N_t(E)$. Then we consider the filtration

$$\mathcal{F}_t^N = \sigma\{N_s(A) : s \leq t, A \in \mathcal{E}\}$$

And we define the counting measure:

$$p((0, t], A) = N_t(A)\ t \geq 0,\ A \in \mathcal{E}$$

this measure is σ-finite under the assumption of non explosivity of T_n. Moreover it allows to obtain more coincise expressions for integrals of the type:

$$\int_0^t \int_E H(s, y)p(ds, dy) = \sum_{n\in\mathbb{N}} H(T_n, Y_n)\mathbf{1}_{\{T_n\leq t\}} \tag{A.3}$$

Assume now that for each $A \in \mathcal{E}$ we have that the point process $N_t(A)$ admits the intensity $\lambda_t(A)$; this leads to a measure valued intensity $\lambda_t(dy)$, we can then generalize equation (A.2):

$$E\left[\int_0^\infty \int_E H(s, y)p(ds, dy)\right] = E\left[\int_0^\infty \int_E H(s, y)\lambda_s(dy)ds\right] \tag{A.4}$$

That has to be valid for all nonnegative $\mathcal{F}_t$ predictable E-marked process where in this case $\mathcal{F}_t$-predictable means measurable w.r.t. $\mathcal{P}(\mathcal{F}_t) \otimes \mathcal{E}$ where $\mathcal{P}(\mathcal{F}_t)$ is the predictable σ-algebra on $(0,\infty) \times \Omega$. We have also the generalization of (1.37) in the form:

$$q(ds, dy) = p(ds, dy) - \lambda_s(dy)ds \tag{A.5}$$

where $q(ds, dy)$ is a signed measure valued martingale in the sense that:

$$\int_0^t \int_E H(s, y)q(ds, dy)$$

is a $(P, \mathcal{F}_t)$ martingale for each $\mathcal{F}_t$ predictable E-marked process satisfying appropriate integrability conditions. The most common form of intensity is:

$$\lambda_t(dy) = \lambda_t m_t(dy)$$

where λ_t is non negative and $\mathcal{F}_t$-predictable and represents the intensity of the process $N_t(E)$ and $m_t(dy)$ is a probability measure on E. The pair $(\lambda_t, m_t(dy))$ is called the $(P, \mathcal{F}_t)$-*characteristic*.

A.3 Martingale representation and generalized Ito formula

Martingale representation results are widely used in finance especially when it comes to solving hedging problems. For pure "Wiener-martingales" we have in fact the well known result that every square integrable martingale with respect to a filtration generated by a Wiwener process is, up to an additive constant, a stochastic integral of the Ito type. We recall now an important theorem:

Theorem A.2. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space where $\mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{F}_t^p$ with $\mathcal{F}_t^p$ filtration generated by a marked point process represented by the counting measure $p(dt, dy)$. Then any $(P, \mathcal{F}_t)$-martingale M_t admits the representation:*

$$M_t = M_0 + \int_0^t \int_E H(s, y)q(ds, dy) \tag{A.6}$$

with $q(\cdot, \cdot)$ as in (A.5) *and H an integrable (with respecto to $\lambda_t(dy)$) $\mathcal{F}_t$-predictable E-marked process. This representation is essentially unique.*

Now using the definition of a marked point process and of integrals in the form of (A.3) we may now consider processes of the general type:

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dw_s + \int_0^t \int_E \gamma(s, y)p(ds, dy) \tag{A.7}$$

that are called jump diffusion processes and where the coefficients satisfy the implicit integrability condition, β_t is $\mathcal{F}_t$ adapted and γ_t is predictable in the sense of the previous definition. We may rewrite the previous equation in differential form as:

$$dX_t = X_{t-}\left(\alpha_t dt + \beta_t dw_t + \int_E \gamma(t, y)p(dt, dy)\right) \tag{A.8}$$

In the previous equation $t-$ specifies the predictability requirement on the last term. As we saw before the last term can be rewritten as:

$$\int_E \gamma(t,y)p(dt,dy) = \gamma(t,Y_t)dN_t \tag{A.9}$$

where $N_t = N_t(E) = p((0,t],E)$ is the total number of jumps and Y_t denotes the piecewise constant, left continuous time interpolation of the sequence Y_n. A particular solution to (A.7) is given by the following exponential formula:

$$\begin{aligned} X_t &= X_0 \exp\left\{ \int_0^t \left(\alpha_s - \frac{1}{2}\beta_s^2\right)ds + \int_0^t \beta_s dw_s + \int_0^t \log(1+\gamma(s,Y_s))dN_s \right\} \\ &= X_0 \exp\left\{ \int_0^t \left(\alpha_s - \frac{1}{2}\beta_s^2\right)ds + \int_0^t \beta_s dw_s \right\} \prod_{n=1}^{N_t}(1+\gamma(T_n,Y_n)) \end{aligned} \tag{A.10}$$

While the diffusion part in the expression follows from the usual Ito formula the jump part follows from the so called exponential formula of Lebesgue-Stieltjes Calculus but can be also obtained from the generalized Ito formula.
Let X_t a process satisfying (A.7) and let $F \in C^{1,2}$ function $F(t,X_t)$; the generalized Ito formula:

$$\begin{aligned} dF(t,X_t) =& F_t(\cdot)dt + F_X(\cdot)\alpha_t dt + \frac{1}{2}F_{XX}(\cdot)\beta_t^2 dt + F_X(\cdot)\beta_t dw_t \\ &+ [F(t,X_{t-}+\gamma(t,Y_t)) - F(t,X_{t-})]dN_t \end{aligned} \tag{A.11}$$

that in the specific case of (A.8) becomes:

$$\begin{aligned} dF(t,X_t) =& F_t(\cdot)dt + F_X(\cdot)X_t\alpha_t dt + \frac{1}{2}F_{XX}(\cdot)X_t^2\beta_t^2 dt + F_X(\cdot)X_t\beta_t dw_t \\ &+ [F(t,X_{t-}(1+\gamma(t,Y_t))) - F(t,X_{t-})]dN_t \end{aligned} \tag{A.12}$$

and where again $N_t = N_t(E) = p((0,t],E)$ and $(\cdot)$ stands for (t,X_t) the pedices denotes partial derivatives. If we write the last equation in the integral form we have two equivalent representations for the last term:

$$\int_0^t [F(s,X_{s-}(1+\gamma(s,Y_s))) - F(s,X_{s-})]dN_s = \sum_{n=1}^{N_t}[F(T_n,X_{T_n}) - F(T_n,X_{T_n^-})] \tag{A.13}$$

Now choosing as $F(t,X) = \log X$ we have:

$$dF = \alpha_t dt - \frac{1}{2}\beta_t^2 dt + \beta_t dw_t + \log(1+\gamma(t,Y_t))dN_t \tag{A.14}$$

from which:

$$\log X_t = \log X_0 + \int_0^t \left(\alpha_s - \frac{1}{2}\beta_s^2\right)ds + \int_0^t \beta_s dw_s + \int_0^t \log(1+\gamma(s,Y_{s-}))dN_s \tag{A.15}$$

And taking exponential on both sides we reobtain (A.10).

Appendix B

Hawkes process simulation

In this section we discuss the algorithm to simulate an Hawkes process and we provide also the matlab code of such simulation. The main references are [13] and [24].
This algorithm is able to exactly generate the point process and the intensity process by sampling inter-arrival times directly via the underlying analytic distribution functions without numerical inverse and hence avoids simulating intensity paths and introducing discretization bias.

B.1 The algorithm

Assume to have a one dimensional Hawkes process $\{(N(t), \lambda(t)), t \geq 0\}$ where $\lambda(t) = \lambda(t|\mathcal{H}_t)$ conditional on λ_0 and $N(0) = 0$ with jump size distribution $Y \sim G$ and K jump times $\{T_1, \ldots, T_K\}$, the algorithm is based on the following steps:

1. Set the initial conditions $T_0 = 0$, $\lambda(T_0) = \lambda_0 > a$, $N(0) = 0$ and $k \in \{0, 1, 2, \ldots, K-1\}$.

2. Simulate the $(k+1)^{th}$ inter-arrival time S_{k+1} by:

$$S_{k+1} = \begin{cases} S^1_{k+1} \wedge S^2_{k+1} & D_{k+1} > 0 \\ S^2_{k+1} & D_{k+1} < 0 \end{cases}$$

   where:

$$D_{k+1} = 1 + \frac{\delta \ln U_1}{\lambda(T_k^+) - a} \quad U_1 \sim \text{Unif}[0,1]$$

   and:

$$S^1_{k+1} = -\frac{1}{\delta} \ln D_{k+1} \qquad S^2_{k+1} = -\frac{1}{a} \ln U_2 \quad U_2 \sim \text{Unif}[0,1]$$

3. record the $(k+1)^{th}$ jump-time T_{k+1} by:

$$T_{k+1} = T_k + S_{k+1}$$

4. record the change at the jump time T_{k+1} in the process $\lambda(t)$ by

$$\lambda(T_{k+1}) = \lambda(T^-_{k+1}) + Y_{k+1} \quad Y_{k+1} \sim G \tag{B.1}$$

   where:

$$\lambda(T^-_{k+1}) = (\lambda(T_k) - a)e^{-\delta(T_{k+1} - T_k)} + a$$

5. Record the change at the jump-time T_{k+1} in the point process $N(t)$ by

$$N(T_{k+1}) = N(T_k^-) + 1 \tag{B.2}$$

We prove now that this algorithm produces an Hawkes process with exponential decaying intensity

Proof. Given the k-th jump time T_k the point process has the intensity $\{\lambda(t) : T_k \leq t < T_{k+1}\}$ following the ordinary differential equation (between the times of the k-th and the $k+1$-th jumps):

$$\begin{cases} \dfrac{d\lambda(t)}{dt} = -\delta(\lambda(t) - a) \\ \lambda(t)|_{t=T_k} = \lambda(T_k) \end{cases} \tag{B.3}$$

The solution of the above ODE is given by:

$$\lambda(t) = a + C_2 e^{-\delta t} \tag{B.4}$$

and imposing theboundary conditions we obtain:

$$C_2 = (\lambda(T_k) - a)e^{\delta T_k} \tag{B.5}$$

Finally we obtain that for $T_k \leq t \leq T_k + S_{k+1}$

$$\lambda(t) = a + (\lambda(T_k) - a)e^{-\delta(t - T_k)} \tag{B.6}$$

We want now to evaluate the cumulative distribution function of the $k+1$-th inter arrival time S_{k+1}

$$\begin{aligned} F_{S_{k+1}}(s) &= P(S_{k+1} \leq s) \\ &= 1 - P(S_{k+1} > s) \\ &= 1 - P(N(T_k + s) - N(T_k) = 0 \end{aligned}$$

Since we know that $\{N(T_k + s) - N(T_k), s \geq 0\}$ is a Poisson random variable with mean:

$$E(N(T_k + s) - N(T_k)) = \int_{T_k}^{T_k + s} \lambda(u)du$$

Therefore we obtain that:

$$F_{S_{k+1}}(s) = 1 - e^{-\int_{T_k}^{T_k+s} \lambda(u)du}$$

we make now the change of variable $v = u - T_k$ to find:

$$F_{S_{k+1}}(s) = 1 - e^{-\int_0^s \lambda(T_k+v)dv}$$

and finally replacing the expression obtained for λ in equation (B.6) we derive:

$$\begin{aligned} F_{S_{k+1}}(s) &= 1 - e^{-\int_0^s [a+(\lambda(T_k)-a)e^{-\delta(T_k+v-T_k)}]dv} \\ &= 1 - e^{-\int_0^s [a+(\lambda(T_k)-a)e^{-\delta v}]dv} \\ &= 1 - e^{-(\lambda(T_k)-a)\frac{1-e^{-\delta s}}{\delta} - as} \end{aligned}$$

At this point we decompose S_{k+1} into two simpler random variables S^1_{k+1} and S^2_{k+1} via:

$$P(S^1_{k+1} > s) = e^{-(\lambda(T_k)-a)\frac{1-e^{-\delta s}}{\delta}}$$
$$P(S^2_{k+1} > s) = e^{-as}$$

We need now to simulate S^1_{k+1} and S^2_{k+1} since we have that:

$$P(S_{k+1} > s) = P(S^1_{k+1} \wedge S^2_{k+1} > s) \tag{B.7}$$

Recall that:

$$F_{S^1_{k+1}}(s) = P(S^1_{k+1} < s) = 1 - e^{-(\lambda(T_k)-a)\frac{1-e^{-\delta s}}{\delta}}$$

then we set:

$$e^{-(\lambda(T_k)-a)\frac{1-e^{-\delta S^1_{k+1}}}{\delta}} = U_1$$

and we invert explicitely the function:

$$\begin{aligned}
& e^{-(\lambda(T_k)-a)\frac{1-e^{-\delta S^1_{k+1}}}{\delta}} = U_1 \\
& \Rightarrow -(\lambda(T_k)-a)\frac{1-e^{-\delta S^1_{k+1}}}{\delta} = \ln U_1 \\
& \Rightarrow -\frac{1-e^{-\delta S^1_{k+1}}}{\delta} = \frac{\ln U_1}{\lambda(T_k)-a} \\
& \Rightarrow e^{-\delta S^1_{k+1}} = \frac{\ln U_1}{\lambda(T_k)-a} \\
& \Rightarrow -\delta S^1_{k+1} = 1 + \frac{\delta \ln U_1}{\lambda(T_k)-a} \\
& S^1_{k+1} = -\frac{1}{\delta}\ln\left(1 + \frac{\delta \ln U_1}{\lambda(T_k)-a}\right)
\end{aligned}$$

Note that the random variable S^1_{k+1} may take with positive probability the value ∞. Moreover the condition for simulate a valid S^1_{k+1} is that:

$$1 + \frac{\delta \ln U_1}{\lambda(T_k)-a} > 0$$

thus we introduce the random variable D_{k+1} setting:

$$D_{k+1} = 1 + \frac{\delta \ln U_1}{\lambda(T_k)-a}$$

Now for the simulation of S^2_{k+1}, since we have that $S^2_{k+1} \sim \text{Exp}(a)$ we use the standard simulation method:

$$S^2_{k+1} = \frac{1}{a}\ln U_2$$

Hence for the simulation of S_{k+1} we have:

$$S_{k+1} = \begin{cases} S^1_{k+1} \wedge S^2_{k+1} & \text{if } D_{k+1} > 0 \\ S^2_{k+1} & \text{if } D_{k+1} < 0 \end{cases}$$

Therefore the $(k+1)$-th jump in the Hawkes process is given by:

$$T_{k+1} = T_k + S_{k+1}$$

and the change in $\lambda(t)$ and $N(t)$ at time T_{k+1} can be easily derived as given by (B.1) and (B.2). □

B.2 Code of the simulation

In the sequel the code of the simulation of the algorithm previously described

```
clear all; clc;

% setting initial conditions

tsimul=0;
tmax=100;
npts=500;

a=0.5;
alpha=1;
b=1.3;
k=1;

t=linspace(tsimul,tmax,npts);
Rt=zeros(1,length(t));
Rt(1)=a;
jump=Rt(1);

[Rt,Nt]=hawkesSimulation(tsimul,tmax,a,alpha,b,t);

figure(1)
hold on
title('Intensity Hawkes process')
plot(t,Rt,'r-')
stairs(t,0.05*Nt,'b-')
hold off
figure(2)
hold on
title('Counting process')
stairs(t,Nt,'b-')
hold off
```

This function implements the algorithm and produces two vectors, the first one containing the evolution of the intensity of the process and the second one containing the evolution of the counting process.

```
function [Rt,Nt]=hawkesSimulation(tsimul,tmax,a,alpha,b,t)

Rt=zeros(1,length(t));
```

```
Nt=zeros(1,length(t));
Rt(1)=a;
jump=Rt(1);
k=1;

while tsimul(k,:)<tmax

tjump=tsimul(k,:);
D=1+b*log(rand)/(jump(k,:)-a);
interev=[-log(D)/b,-log(rand)/a];
if D>0

interevNextJump=min(interev);

else

interevNextJump=interev(2);

end

tnextJump=tjump+interevNextJump;
tsimul=[tsimul;tnextJump];
ind=find((tsimul(k,:)<t)&(t<tsimul(k+1,:)));
tind=t(ind);
indbelow=find(t<=tsimul(k,:));
tbelow=t(indbelow);
indabove=find((tsimul(k+1,:)<=t));
tabove=t(indabove);
RtInd=a+(jump(k,:)-a)*exp(-b*(tind-tjump));
RtBelow=zeros(1,length(tbelow));
RtAbove=zeros(1,length(tabove));
Rt=Rt+[RtBelow, RtInd, RtAbove];
Rtmin=a+(jump(k,:)-a)*exp(-b*(tnextJump-tjump));
jump=[jump;Rtmin+alpha];
k=k+1;

end

for i=1:length(tsimul)

iabove=find(t>=tsimul(i,:));
nabove=t(iabove);
ibelow=find(t<tsimul(i,:));
nbelow=t(ibelow);
zer=zeros(1,length(tbelow));
Nt=Nt+[zeros(1,length(nbelow)) ones(1,length(nabove))];

end
```

Appendix C

Matlab simulations

In this section we include some Matlab codes where we implement the algorithm to detect jumps described in Section 3.1; in particular we use it to detect jumps in case of a simulated geometric Brownian motion with jumps and in case in which we dispose of historical series of Gold crude oil future prices (data are chosen in the range from 2017/6/29 until 2023/4/6 for gold and from 2017/11/21 until 2023/4/6 for crude oil).

C.1 Matlab codes

The following are the main code of the simulation of a geometric Brownian motion with jumps (chosen randomly in the interval $[-5, 5]$) and the function that performs such computation:

```
% simulation geometric brownian motion with jumps

Tin=0;
Tf=10;
npts=1000;
mu=0.03;
sigma=0.1;
inV=50;
nsimul=20;
lambda=1.5;

t=linspace(Tin,Tf,npts);
a=0.5;
alpha=1;
b=1.3;
k=1;

[Rt,Nt,T]=hawkesSimulation(Tin,Tf,a,alpha,b,t);

[GBM,GBM1,time]=simulateGBMHawkes(inV,Tin,Tf,mu,sigma,npts,T);

hold on
title('Comparison GBM and GBM with Hawkes type jumps')
```

```
plot(time,GBM,'-b')
plot(time,GBM1,'-r')
xline(T(1:end-1))
legend('geometric brownian motion with jumps', ...
'geometric brownian motion', 'time of occurrence of a jump')
hold off
```

```
function [S,S1,T]=simulateGBMHawkes(inV,Tin,Tf,mu,sigma,npts,times)

t=sort(Tin+rand(1,npts)*(Tf-Tin));
T=t;
%t=linspace(Tin,Tf,npts);
r=normrnd(0,1,[1,length(t)]);
S=zeros(1,length(t));
S1=zeros(1,length(t));
S(1)=inV;
S1(1)=inV;
j=1;

for i=2:length(t)

S(i)=S(i-1)*exp((mu-sigma^2/2)*(t(i)-t(i-1))+ ...
sigma*sqrt(t(i)-t(i-1))*r(i));
S1(i)=S1(i-1)*exp((mu-sigma^2/2)*(t(i)-t(i-1))+ ...
sigma*sqrt(t(i)-t(i-1))*r(i));

if (times(j)<t(i))&& (times(j)>=t(i-1))

jump=-5+rand*10;
S(i)=S(i)+jump;
j=j+1;

end

end
```

The following codes are the main code and the function to detect jumps on the set of data that we dispose (simulated GBM with jumps, gold future prices and crude oil future prices).

```
clear all; clc;

%JUMP DETECTION ALGORITHM

%Construction of the futures price vector (random entries)
% n=4000;
% xmin=90;
% xmax=95;
% x=xmin+rand(1,n)*(xmax-xmin);
```

```
% POSSIBLE CHOOSING AS COMMODITY THE SPOT PRICE
% OF GOLD AND CRUDE OIL
% gold=futureprices("futures_gold1.xlsx", "Foglio4", ...
% [2, Inf]);
% comm=fliplr(gold')';
% date=readtable("date_gold.xlsx");

% oil=futureprices("futures_gold1.xlsx", "Foglio5", ...
% [2, Inf]);
% comm=fliplr(oil')';
% date=readtable("date_imp.xlsx");

dat=table2array(date);

% SPOT PRICE GENERATED SIMULATING A GBM WITH
% POISSONIAN JUMPS
% setting data to simulate a GBM with poissonian jumps
% Tin=0;
% Tf=10;
% npts=1000;
% mu=0.03;
% sigma=0.1;
% inV=50;
% lambda=1.5;
% [interev,times]=homoPoisson(Tf,lambda);
% [comm,t]= ...
% simulateGBMPoisson(inV,Tin,Tf,mu,sigma,npts,times);

% SPOT PRICE GENERATED SIMULATING A GBM WITH HAWKES JUMPS
% Tin=0;
% Tf=10;
% npts=1000;
% mu=0.03;
% sigma=0.1;
% inV=50;
% nsimul=20;
% lambda=1.5;
% t=linspace(Tin,Tf,npts);
% a=0.5;
% alpha=1;
% b=1.5;
% k=1;
% [Rt,Nt,T]=hawkesSimulation(Tin,Tf,a,alpha,b,t);
% [comm,comm1,time]=...
% simulateGBMHawkes(inV,Tin,Tf,mu,sigma,npts,T);

[jumptimes,numberjumpstep,sigmavec]=jumpDetect(comm);
```

```
hold on
plot(dat,comm,'-r')
xline(dat(jumptimes(1:numberjumpstep(1))),'--b')
hold off

% print -depsc CrudeOil.eps
% print -depsc Gold.eps
% print -depsc GBM.eps
```

```
function [jumptimes,numberjumpstep,sigmavec]=jumpDetect(comm)

par=3;

x=comm;
y=1:1:length(comm);
MatComm=[x';y];
n=length(x);
sigma2=0;
M1=[];
jumptimes=[];
numberjumpstep=[];
sigmavec=[];
Vt=zeros(2,length(comm)-1)';

for i=1:n-1

sigma2=sigma2+1/(n-2)* ...
(MatComm(1,i+1)-MatComm(1,i))^2/MatComm(1,i);

end

for j=1:n-1

test=abs((MatComm(1,j+1)-MatComm(1,j))/sqrt(MatComm(1,j)));
Vt(j,1)=j;
Vt(j,2)=test;
if test>=par*sqrt(sigma2)

M1=[M1,MatComm(2,j)];

end

end

jumptimes=[jumptimes,M1];
m1=length(M1);
numberjumpstep=[numberjumpstep,m1];
sigmavec=[sigmavec,sqrt(sigma2)];
```

```
% updating x vector
MatComm(:,M1)=[];

k=1;
% counts the number of jumps at each step
counter=0;

while m1 ~= 0

sigma2=0;
M1=[];
M2=[];
k=k+1;
counter=counter+m1;

for i=1:length(MatComm)-1

sigma2=sigma2+1/(n-counter-1)*(MatComm(1,i+1) ...
-MatComm(1,i))^2/MatComm(1,i);

end

for j=1:length(MatComm)-1

test=abs((MatComm(1,j+1)-MatComm(1,j))/sqrt(MatComm(1,j)));

if test>=par*sqrt(sigma2)

M1=[M1,MatComm(2,j)];
M2=[M2,j];

end
end

jumptimes=[jumptimes,M1];
m1=length(M1);
numberjumpstep=[numberjumpstep,m1];
sigmavec=[sigmavec,sqrt(sigma2)];

if k==3

m1=0;

end

MatComm(:,M2)=[];

end
```