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EFFECTS OF ENVIRONMENTAL COLORED NOISE
IN THE CONTACT PROCESS AND ITS
IMPLICATIONS ON BET-HEDGING

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CONTENTS

1	THE CONTACT PROCESS	1
1.1	An introduction to the contact process	1
1.1.1	The CP model	1
1.1.2	The CP equation of motion and its mean-field approximation	1
1.2	CP properties: criticality, absorbing states and extinction times	3
1.2.1	Criticality in the CP	3
1.2.2	Absorbing states	5
1.2.3	Extinction times in the CP	6
2	NOISE IN THE CONTACT PROCESS	8
2.1	Bringing noise in the scheme	8
2.2	Spatial noise and spatial Griffiths phases	8
2.3	Temporal noise	10
2.3.1	White noise, Langevin, master and Fokker-Planck equations	10
2.3.2	Demographic noise	13
2.3.3	Environmental white noise and temporal Griffiths phases	15
2.3.4	Colored noise	17
3	REVIEW OF THE SIMULATION TECHNIQUES	20
3.1	Basic simulation method for the CP	20
3.2	Gillespie methods	20
3.3	Anderson's algorithm	21
4	STATIONARY STATE DISTRIBUTIONS	25
4.1	Introduction	25
4.2	The UCNA	26
4.2.1	Review of the method	26
4.2.2	How to interpret the UCNA method?	27
4.3	Implementing the UCNA	29
4.3.1	The stationary p.d.f.	29
4.3.2	Comparison with simulations	31
5	EXTINCTION TIME ANALYSIS	34
5.1	The backward Fokker-Planck method	34
5.2	The path-integral method	35
5.2.1	The method	35
5.2.2	Applying the path-integral method	36

5.2.3	Comparison with simulations and comment on the result	42
6	BET-HEDGING	45
6.1	What is bet-hedging about?	45
6.2	Bet-hedging with white noise	46
6.2.1	A note on the simulations	47
6.3	Bet-hedging with colored noise	47
7	CONCLUSIONS	51
A	ITÔ-STRATONOVICH DILEMMA	53
A.1	Introduction to the problem	53
A.2	The Interpretations	54
A.2.1	Stratonovich interpretation	54
A.2.2	Itô interpretation	54
A.3	Which interpretation should be chosen? Internal and external noise	55
	BIBLIOGRAPHY	57

LIST OF FIGURES

Figure 1	A schematic representation of the contact process in dimension 1.	2
Figure 2	An example of CP phase diagram	3
Figure 3	Stability of the fixed points for the CP with fixed rates	3
Figure 4	The three different behaviors for the MET of the CP with fixed rates	7
Figure 5	The potential of equation (10) depicted for different values of λ and $\mu = 1$	7
Figure 6	A schematic phase diagram of a randomly diluted Ising model	9
Figure 7	Phase diagrams for one-dimensional CP on finite-size lattices	14
Figure 8	A schematic representation of the phase diagram obtained with the UCNA approach	30
Figure 9	Stationary p.d.f. obtained with the UCNA method	32
Figure 10	Comparison of the analytical p.d.f. and the simulated CP	33
Figure 11	A representation of $V(u)$	39
Figure 12	A representation of $p_u(u)$ trajectories in the phase space	39
Figure 13	The comparison between the simulations and the analytical results of MET for different τ and λ	44
Figure 14	Effects of colored-noise bet-hedging for different values of τ and $\bar{\lambda}$	50

LIST OF TABLES

Table 1	Some known values of the critical exponents of the CP with $\mu = 1$	5
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ABSTRACT

The contact process is one of the simplest models for studying population dynamics on a lattice, whose properties have been well studied when birth and death rates are constant in time. However, non-trivial effects arise when such rates fluctuate, as for instance due to environmental variability. In this work we develop a general framework to analyze the case of colored (i.e. temporally correlated) environmental noise; by means of the well-established "unified colored noise approximation" and a path-integral approach, we calculated the stationary density distributions as well as the asymptotic extinction time behavior. At the same time, we implemented exact simulation techniques which proved to be a robust tool to check our analytical results. Finally we apply our results to a topic of interest in the context of living systems. We focus on the problem of bet-hedging in population dynamics, in which a community of individuals reproduces through to two different spreading strategies: a constant one with a low benefit and a more risky one which depends on the external environmental conditions. In particular we study the role that temporal correlations of such environmental conditions may have on the overall dynamics, since they are an intrinsic characteristic of every real system.

INTRODUCTION

In this Master Thesis I present the work done by me together with Professor Miguel Ángel Muñoz and Dr. Jorge Hidalgo at the Department of Electromagnetism and Condensed Matter Physics of University of Granada, under the supervision of Professor Amos Maritan of University of Padua. In an effort to introduce temporally correlated environmental noise in the previous work of Hidalgo and Muñoz [17] we ended up putting together a series of techniques, both analytical and numerical, developing an optimal framework to tackle the problem of correlated noise in a large number of stochastic models. Most importantly we understood the importance of sound simulation techniques when dealing with time-varying rates which are often underestimated, leading to the possibility of contradictory results. In this work we first explain such techniques in general and we apply them in the case of the classic contact-process dynamics with the addition of the environmental correlated noise. Then, we study the case of a bet-hedging dynamics to understand if a temporal correlation of the noise leads to different effects respect to the white noise case.

THE FIRST CHAPTER consists of a brief introduction to the contact process as well as to some main concepts in statistical physics such as critical behaviour, absorbing states, phase diagrams and extinction times.

THE SECOND CHAPTER is meant to be a small review of the known effects of noise in the contact-process dynamics.

THE THIRD CHAPTER is a review of the different types of simulation techniques.

THE FOURTH CHAPTER is where we explain the main analytical approximation we used to obtain the stationary density distribution for the contact process with environmental colored-noise: the Unified Colored Noise Approximation (UCNA)

THE FIFTH CHAPTER is devoted to the problem of the extinction times, with the calculation of their scaling as a function of the population size.

THE SIXTH CHAPTER illustrates the bet-hedging theory and the application of the results of the previous chapters.

THE SEVENTH CHAPTER concludes the work with a summary of the results.

APPENDIX A is a basic summary of the Itô-Stratonovich dilemma, whose understanding is mandatory when dealing with absorbing states.

1

THE CONTACT PROCESS

1.1 AN INTRODUCTION TO THE CONTACT PROCESS

1.1.1 The CP model

The *contact process* (CP) is a *toy-model* for activity propagation on a lattice [26]. It was first introduced as a simple model for epidemics by Harris in 1974 [15] and it can be used to model the spreading of infections [27], the survival or extinction of populations [21], forest fires [1], transport in random media [16] etc. *Spreading processes* share a common dynamics emerging from the competition between two main effects. Taking the case of an infection as example, we have individuals infecting neighbors and individuals recovering from the infection; depending on the respective infection and recovering rates we can have two outcomes: one in which the infection spreads through the population, eventually reaching a stable percentage of it whose value depends on the balance between the rates and one in which the recovering is much faster decreasing the infected percentage until it vanishes. The CP is important because its relative simplicity makes it easy to study, while it still presents many physical properties; the most important of these are the *absorbing states*. The CP represents a paradigm for systems with such states and related non-equilibrium problems. Moreover, the CP belongs to the same *universality class* (i.e. it shares the same critical properties) of other significant models in theoretical physics such as the directed percolation [26] or the Reggeon field theory [26]. Finally the versatility of the CP makes it easy to study how the introduction of small modifications, both micro- or macroscopic, can lead to different dynamics respect to the basic case, as done in this work with the introduction of environmental noise.

1.1.2 The CP equation of motion and its mean-field approximation

In a lattice, at time t a site x is either *occupied* ($\sigma_x(t) = 1$) or *empty* ($\sigma_x(t) = 0$): an occupied site can be for instance a member of a population or an infected cell. Calling q the coordination number of the lattice, in the CP an occupied site reproduces at rate λ/q and dies at rate μ , as represented in figure 1. We can write an equation of motion

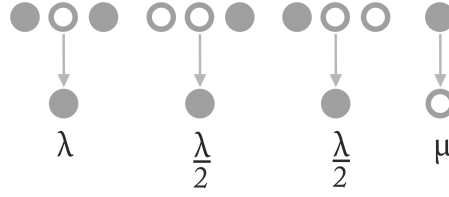


Figure 1: A schematic representation of the contact process in dimension 1.

for the probability of a site to be occupied $\rho(\mathbf{x}, t) \equiv \text{Prob}[\sigma_{\mathbf{x}}(t) = 1]$ which we will use as an order parameter:

$$\dot{\rho}(\mathbf{x}, t) = \frac{\lambda}{q} \sum_{\mathbf{y}} \text{Prob}[\sigma_{\mathbf{x}}(t) = 0, \sigma_{\mathbf{y}}(t) = 1] - \mu\rho(\mathbf{x}, t) \quad (1)$$

The sum is taken upon the nearest neighbors of the site \mathbf{x} .

We can already see that although being very simple the CP is not exactly solvable. The argument of the sum in the right side of equation (1) is an infinite hierarchy of probabilities. The two-site probability leads to the three-site one and so on: solving the system requires the knowledge of these probabilities. However if we eliminate the site-dependence with a *mean-field* approximation, meaning that every site can see every other and not just the nearest neighbors, the model is solvable. In the mean-field approximation we set $\rho(\mathbf{x}) \rightarrow \rho$ and $\text{Prob}[\sigma_{\mathbf{x}}(t) = 0, \sigma_{\mathbf{y}}(t) = 1] \rightarrow (1 - \rho)$ and we can see that the CP's equation of motion becomes nothing more than the Malthus-Verhulst equation:

$$\dot{\rho} = (\lambda - \mu)\rho - \lambda\rho^2 \quad (2)$$

This approximation is a strong one but keeps the main properties of the CP. We can then use it to illustrate these properties as well as some important concepts of statistical mechanics. Equation (2) tells us that ρ will gradually reach a stationary value $\bar{\rho}(\lambda, \mu)$. If $\lambda < \mu$, $\bar{\rho}$ vanishes. If ρ represents a population's density this means the population goes extinct since there are more deaths than births per unit time. We call this phase *inactive* and this type of state of the system *absorbing*. Since there are no beings left to reproduce there is no possibility for the dynamics to start again and the system gets trapped in the state. When the birth rate λ is bigger than the death rate μ there are now two possible stationary states: $\bar{\rho} = 0$ and $\bar{\rho} = (\lambda - \mu)/\lambda$. Moreover the non-vanishing stationary state is an attractor for the system, meaning that starting from any density bigger than zero the system will eventually reach this stationary density (figure 2). We then call this phase of the system *active*. We can picture this behavior in a *phase diagram* in which we plot our order parameter ρ against our control parameter λ as represented in figure 3. The phase transition between the two phases is called an *absorbing phase transition* because of the presence of the absorbing state. CP's phase

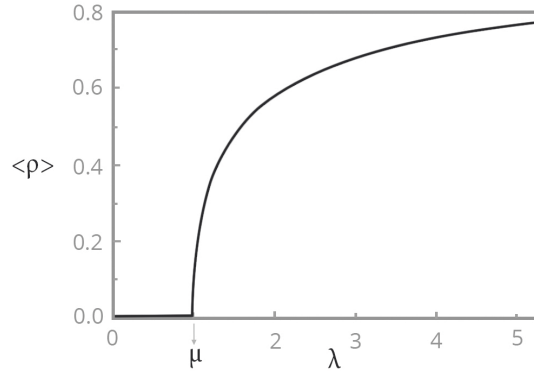


Figure 2: An example of phase diagram for the Cp described by equation (2). The active phase starts when $\lambda > \mu$.

transitions belong to a class of transitions which is continuous but singular. The value of the control parameter at which the transition happens is called the *critical point* λ_c . The critical point together with the phase diagram provide a rough description of how a system behaves and which phase transitions it is subjected to. This overall description of the CP keeps its validity even when moving away from the mean-field approximation, although just qualitatively.

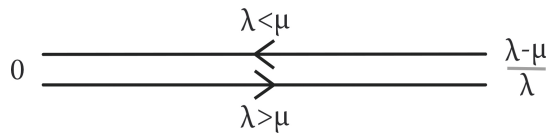


Figure 3: This image illustrates the stability of the fixed points for the two regions of the control parameter space $\lambda > \mu$ and $\lambda < \mu$.

1.2 CP PROPERTIES: CRITICALITY, ABSORBING STATES AND EXTINCTION TIMES

1.2.1 Criticality in the CP

Criticality is a fundamental concept in statistical mechanics and refers to the set of laws that many models share at the critical point. At the critical point the system's properties are usually governed by power-laws whose exponents, called *critical exponents*, are the same independently of the small differences between the models. Moreover at the critical point different quantities of the system obey *scaling* relations and correlation lengths or susceptibilities diverge. A famous and studied example is the 2D Ising model [4] where at the critical point the stationary magnetization obeys the power-law $M \sim |t|^{1/8}$ and the magnetic susceptibility diverges as $\chi \sim |t|^{-7/4}$ where $t = (T - T_c)/T_c$ is the scaled control parameter (temperature in this case). The same

considerations apply to the CP, for which we can use equation (2) to deduce a simple critical exponent in the mean-field approximation. Defining $\Delta \equiv \lambda - \lambda_c$, our stationary active density becomes

$$\begin{aligned}\rho &= \frac{\lambda - \lambda_c}{\lambda} \\ &= \frac{\Delta}{\lambda_c + \Delta} \\ &= \frac{\Delta}{\lambda_c} + O(\Delta^2)\end{aligned}\quad (3)$$

from which we can see that

$$\rho \sim |\Delta|^\beta \quad \text{with } \beta = 1 \quad (4)$$

There are many more critical exponents related to the CP, for instance:

- the relaxation time (or survival probability time) exponent δ : near the critical point there is a characteristic slowing-down of the time dependence of the density

$$\rho(t) \sim t^{-\delta} \quad (5)$$

We can see this by the fact that at the critical point equation (2) becomes simply

$$\dot{\rho}|_{\lambda=\mu} = -\mu\rho^2 \quad (6)$$

so that $\rho \sim t^{-1}$.

- the correlation time critical exponent ν_{\parallel} : it regulates the temporal behavior of the stationary time correlation function $C_s(t) \equiv \text{Prob}[\sigma_0(t_0 + t) = 1; \sigma_0(t) = 1] - \bar{\rho}^2$ where we have

$$C_s(t) \propto e^{-t/\tau} \quad \text{with } \tau \sim |\Delta|^{-\nu_{\parallel}} \quad (7)$$

- similarly to ν_{\parallel} , ν_{\perp} is related to the correlation length ξ of the two-point equal time correlation function $C(\mathbf{x}) \equiv \langle \sigma_{\mathbf{x}} \sigma_0 \rangle - \bar{\rho}^2$

$$C(\mathbf{x}) \propto e^{-|\mathbf{x}|/\xi} \quad \text{with } \xi \sim |\Delta|^{-\nu_{\perp}} \quad (8)$$

Clearly this spatial correlation function has a meaning for systems on a lattice and not in the mean-field approximation.

One important thing that influences the numerical values of the critical exponents is the dimension of the system. As we can see from table 1, one, two and three dimensional systems have their own values for the critical point and critical exponents and for dimension bigger

exponent	d = 1	d = 2	d = 3	d = 4
λ_c	3.29785(2)	1.6488(1)	1.3169(1)	1
β	0.27649(4)	0.583(4)	0.805(10) ^f	$\frac{1}{2}$
δ	0.15947(3) ^f	0.4505(10)	0.730(4)	1

Table 1: In this table are collected some known values of the critical exponents of the CP with $\mu = 1$.

than three we retrieve the mean field values. We then say that the *upper critical dimension* of the CP is four. Models who share these properties and the numerical values of the critical exponents and scaling relations are said to belong to the same *universality class*. Universality is a very important concept, linking different models that apparently have nothing in common in a unique scheme. A conjecture by Janssen and Grassberger [12] states that all the different systems and models with a unique absorbing state, a single order parameter and no extra symmetry or conservation law belong to the same universality class called "direct percolation class". The basic CP belongs then to this class, but there are others [29] that arise when introducing certain modifications. One of these is multiplicative noise such the one we inserted in our CP model. This type of systems seem to belong to the *multiplicative noise* universality class.

1.2.2 Absorbing states

We've seen that the presence of an absorbing state defines by itself a universality class. But what are absorbing states exactly and what is the difference between the other states? Absorbing states are characteristic of *non-equilibrium processes*. In fact, in normal equilibrium processes a strong condition known as *detailed balance* holds. Detailed balance is a concept related to microscopic reversibility and we can reassume it by saying that in a system near equilibrium, around any close cycles of states there is no net flow of probability. When a system such ours reaches an absorbing states (which in CP is usually an empty lattice), the state acts as a probability sink thus creating a net flow. The CP is then a non-equilibrium process. In chapter 2 we will visualize this concept more in detail showing how the effective potential relative to a stationary Fokker-Planck distribution has a singularity at the absorbing state. In the noisy case, where fluctuations can move the system from one state to another, we will see how an absorbing state can be seen as a fluctuation-independent one.

1.2.3 Extinction times in the CP

Apart from phase diagrams and critical properties, a significant way to characterize a spreading process such the CP is to study the *mean extinction times* (MET). MET are the mean time necessary to reach the absorbing state and provide another way to look at the different phases of the model. All the considerations made so far are based on the assumption that the population is infinite in size. Equation (2) describes a dynamical system and its solutions are unique and separate: the absorbing state represents a singular trajectory itself and we then know the time required to reach it is infinite. But real systems are finite. Finite-size effects then modify the dynamics, as we will see more in detail in chapter 2. What is important for the present argument is that a finite population *always* goes extinct at some point. What differentiate the phases is *how* it does so as a function of the lattice size N . When a system is in the inactive phase, the MET scale *logarithmically* with N . When it is in the critical state, MET scale linearly and when the phase is active MET scale exponentially. All of these behaviors are depicted in figure 4. To explain this we will refer again to equation (2). The right hand side of this equation of motion can be rewritten as the spatial derivative of a potential $V(\rho, t)$:

$$\dot{\rho}(t) = -\frac{\partial V}{\partial \rho} \quad (9)$$

The potential $V(\rho)$ is then:

$$V(\rho) = \frac{1}{6}\rho^2(3\mu + \lambda(2\rho - 3)) \quad (10)$$

In figure 5 it is represented for values of the birth rate in the inactive, critical and active phase. In the inactive phase, whatever the initial condition a particle will fall towards the absorbing state. Near extinction the density is small so that equation (2) can be approximated with

$$\dot{\rho} = -|\lambda - \mu|\rho \quad (11)$$

whose solution is an exponential decay:

$$\rho(t) = \rho_0 \exp(-|\lambda - \mu|t) \quad (12)$$

To obtain the scaling function we consider the absorbing state reached when there is just one particle left, that is, $\rho = 1/N$. We can then invert the previous relation to obtain the time required to reach it that is

$$t|_{\rho=N^{-1}} = \frac{\log(N)}{|\lambda - \mu|} \quad (13)$$

The critical scaling behavior was already given in section 1.2.1: we know that at the critical point $\rho \sim t^{-1}$ meaning that the MET scales

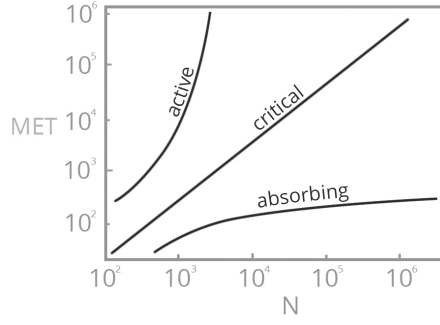


Figure 4: The three different behaviors for the MET of the CP with fixed rates depicted schematically in a log-log plot.

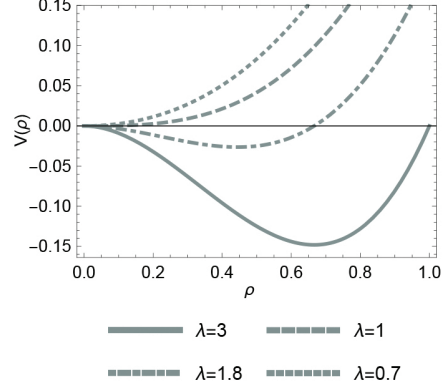


Figure 5: The potential (10) depicted for different values of λ and $\mu = 1$. The barrier appears for values of $\lambda > 1$

linearly with N . Finally, the exponential scaling of the active phase is due to the Arrhenius law. As shown in figure 5 when $\lambda > \mu$ the second derivative of the potential has an inverted sign respect to the inactive phase. It forms a potential wall between the stationary state and the absorbing one. We know from Arrhenius law [9] that the rate at which the potential wall is hit by casual fluctuations of the variable is inversely proportional to the height of the wall. In the case of our potential, such height is the difference between the value of the potential at the stationary state and at the state $\rho = 1/N$, all in absolute value.

$$\begin{aligned} \Delta V &\equiv \left| V\left(\frac{\lambda - \mu}{\mu}\right) \right| - \left| V(1/N) \right| \\ &= \frac{(\lambda(N-1) - \mu N)^2 (\lambda(N+2) - \mu N)}{6\lambda^2 N^3} \end{aligned} \quad (14)$$

which goes as N^{-1} when $N \rightarrow \infty$. Since the rate is inversely proportional to the MET we obtain the exponential scaling. This mandatory introduction was propedeutic to the understanding of how noise modify the classic MET behavior as we will see in chapter 5.

2

NOISE IN THE CONTACT PROCESS

2.1 BRINGING NOISE IN THE SCHEME

Noise is an unavoidable aspect of every real system and this is especially true in population dynamics. Birth rates are not the same for every individual but vary in time and space due to multiple causes: environmental conditions, intrinsic genetic modifications, finite-size of the population and so on. Introducing noise makes the models *non-deterministic* but it also produces non-trivial effects that make them more accurate in their description of reality. Noise can be divided in two big categories: *spatial* noise and *temporal* noise. Spatial noise refers to a spatial disorder of the control parameters of the model, which may or may not depend also on time. When it is time-independent and fixed it is called *quenched disorder*, whereas when it also changes in time we speak of *annealed disorder*. Temporal noise, on the other hand, refers to a type of disorder that affects every site equally. Among the possible temporal noises that can affect a system, two of them play a fundamental role in this work and will be discussed separately: *environmental* noise and *demographic* noise.

2.2 SPATIAL NOISE AND SPATIAL GRIFFITHS PHASES

Among the two types of disorder, quenched disorder is the most interesting one. Dealing with annealed disorder in fact usually means to reabsorb its contributions in the thermodynamic averaging. Quenched disorder, on the other hand, plays the role of an externally given background and studying it mathematically is not an easy task. This type of disorder is fixed in time and varies spatially: as an example we can think of a model for epidemics in which the infection rate depends on the lattice site x . Such a system may represent an infection over a large and geographically different environment, where local environmental differences or genetic variability influence the effectiveness of the infection's virus. Historically though the study of the effects of quenched disorder was born in the field of statistical mechanics with an article from Griffiths [13]. In his 1969 paper he studied how randomly placed spin vacancies could modify the dynamics of the classic Ising model. He concluded that quenched disorder strongly modified the Ising phase diagram introducing a new phase, now called a *Griffiths phase*. This phase, located between the inactive and active ones but *before* the critical point (schematically depicted in figure 6)

turned out to have very peculiar properties. [33]. These region display critical-like behavior such as power-laws and correlation divergence, even if the system is not at the critical state (which we recall is *a point* in the parameters space). Griffiths phases's critical exponents though are not numerically fixed but slowly vary with the control parameter. Quenched disorder then smears the phase diagram of the system. This is a remarkable property because of two main reasons: first of all, as already said noise is unavoidable so Griffiths phases must be present in many real systems, such as brain networks (see [28] and references). A second reason is that critical properties are sometimes *required* to explain certain processes [28] and Griffiths phases represent a natural way to provide them without requiring a fine-tuning necessary for maintaining the system in a critical state. During the years, the study of Griffiths phases gave more insight on the reason for this peculiar behavior. In particular it was discovered how the formation of rare regions with a unique dynamics influenced the whole system. These *rare-regions effects* are considered to be the main contributors to the Griffiths phases critical properties [33]. To better explain this let us refer again to the disordered systems studied by Griffiths. In such a system, the vacancies reduce the tendency towards a magnetic long-range order and thus the critical temperature of the system. When the system is very large though there is the possibility for large regions without vacancies to form. These regions will behave ferromagnetically even if the overall system is in the paramagnetic state. The probability of the formation of a rare region is exponentially small in its volume, but the activity *lingers* in this regions for exponentially long times. As Griffiths demonstrated these effects combined are important as they produce a singularity in the free-energy. This singularity is the phase transition from the inactive phase to the Griffiths phase. The relevance of the Griffiths regions is stronger the stronger the correlation length of the disorder.

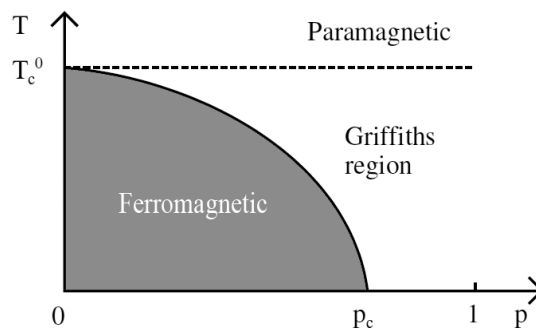


Figure 6: This image, taken from [33] represents a schematic phase diagram of a randomly diluted Ising model as function of temperature T and impurity concentration p . p_c is the geometric percolation threshold of the lattice and T_c^0 is the clean critical temperature.

2.3 TEMPORAL NOISE

Temporal noise is different from spatial noise and we can say that they represent two sides of the same coin. Contrary to quenched spatial noise, temporal noise affects the system as a whole but vary in time: it is then ideal to represent global influences on a system such as environmental effects. They can be periodical, such as seasons cycling or alternation of drought and rainy periods or aperiodical such as random weather modifications or resources casual fluctuations.

2.3.1 White noise, Langevin, master and Fokker-Planck equations

THE LANGEVIN EQUATION To model ecosystems with noise the usual approach in the physical sciences is to write an effective equation of motion for the order parameter [22]. To simplify the argumentation we will use the density ρ as in the previous chapter referring to population dynamics but the method is general. The equation for the density, called a *Langevin* equation, contains a function of time representing the noise

$$\dot{\rho}(t) = f(\rho(t)) + g(\rho(t))\eta(t) \quad (15)$$

$\eta(t)$ is an unknown function but whose averaged properties over an ensemble of systems are simple. This systems are in our case the populations on which the noise acts on: this means $\eta(t)$ can be treated as a *stochastic process*. Moreover, to be treated mathematically the function $\eta(t)$ must have a defined mean value and *autocorrelation function* independent of $\rho(t)$. The autocorrelation function defines how the value of $\eta(t)$ at a time t is correlated to its value at a time t' . The simplest type of noise one can imagine is a totally uncorrelated function with zero mean

$$\langle \eta(t) \rangle = 0 \quad (16)$$

$$\langle \eta(t)\eta(t') \rangle = \Gamma\delta(t - t') \quad (17)$$

Such a function is called *white noise* and it is at the base of Brownian motion, representing a random force acting on a particle. White noise has no time scale and exists independently of any physical system. The adjective “white” derives from the fact that it can informally be seen as a Gaussian process whose spectral function is constant, although a sound mathematical definition is complex and is beyond the scope of this introduction. Temporal noise can be divided in two categories depending on $g(\rho)$. If $g(\rho)$ is constant we speak of *additive* noise. It is the simplest form for the Langevin equation and the easier to work with. When $g(\rho)$ is not a constant function then we speak of *multiplicative* noise and a series of problems arise. Multiplicative

noise is strongly coupled with the dynamics of the system and the problem is generally more difficult to handle. The first and most important consequence of this coupling is that one has to decide *when* the noise influences the density $\rho(t)$ and which value insert in the function $g(\rho)$. Without this choice the Langevin description is incomplete. This mathematical problem has deep physical consequences and each interpretation represents a different physical process; this is particularly important when absorbing states are present. Historically two different interpretations arose: the Itô interpretation and the Stratonovich interpretation. Appendix A is an introduction to their mathematical representation, meaning and comparison. However, although being used frequently for its convenience, the Langevin equation is in any case a continuous approximation of a *discrete step-process* which obeys his own rate equation.

THE MASTER EQUATION The real system is discrete: a population is made of n individuals and the birth or death of one of them produces a jump of $\pm 1/n$ in the density. Since at every time-step it is possible to change the density just by one such jump, the contact process belongs to the category of so called *one-step processes*. So, instead of the density ρ it is appropriate to start with a discrete formulation using as variable the number of occupied sites n on a lattice made of N total sites. The evolution over time of $n(t)$ depends on the concurrent happening of births and deaths. $n(t)$ can then be described as a stochastic process with its *probability density* distribution function $P(n, t)$. It is possible to write an *exact* rate equation for the probability density function $P(n, t)$, called *master equation*, starting from the death and birth rates:

$$\lambda_n = \lambda n \left(1 - \frac{n}{N}\right) \quad (18)$$

$$\mu_n = \mu n = n \quad (19)$$

In this equation we set a unitary death rate $\mu = 1$ as in the classic CP dynamics. We will keep it so throughout the entire work, although the results are valid for any constant μ . The master equation then reads

$$\begin{aligned} \dot{P}(n, t) = & \mu_{n+1} P(n+1, t) - \mu_n P(n, t) \\ & + \lambda_{n-1} P(n-1, t) - \lambda_n P(n, t) \quad (20) \end{aligned}$$

The right hand side of this equation is nothing more than the net flux of probability through the state of the system with n occupied sites. Although being exact, this equation cannot be solved directly. It is then necessary to make an approximation to make it more tractable, as we will see in the next paragraph.

THE FOKKER-PLANCK EQUATION the Fokker-Planck equation is a special type of master equation used as approximation for the exact one. It is a second order partial differential equation for the probability density function $P(y, t)$ in the *continuous* variable y . It has the following general form:

$$\dot{P}(y, t) = -\frac{\partial}{\partial y} A(y)P + \frac{1}{2} \frac{\partial^2}{\partial y^2} B(y)P \quad (21)$$

The coefficients $A(y)$ and $B(y)$ can be any kind of differentiable functions, with the only constraint that $B(y) > 0$. For one-step processes the Fokker-Planck equation is an approximation in the continuous limit of the order parameter y as the Langevin approach. The two are in fact equivalent since from the Langevin equation one can obtain the Fokker-Planck equation and vice-versa. For a Stratonovich Langevin equation of the form (15) the corresponding Fokker-Planck equation reads

$$\dot{P}(\rho, t) = -\frac{\partial}{\partial \rho} [f(\rho) + \frac{\Gamma}{2} g(\rho)g'(\rho)]P + \frac{\Gamma}{2} \frac{\partial^2}{\partial \rho^2} g(\rho)^2 P \quad (22)$$

Again, different interpretations of the Langevin equation lead to different Fokker-Planck equations as explained in appendix A. The Fokker-Planck equation can be derived from the exact master equation in several manners. We will now illustrate a general derivation method particularly suited for one-step processes [22]. First we change variable in equation (20) from n to the discrete density $\rho = \frac{n}{N}$. Such density is not continuous but proceeds by jumps $\pm \frac{1}{N}$. Then we notice that it can be rewritten in terms of an operator \mathbb{E} whose action on a *smooth* function $f(\rho)$ is

$$\mathbb{E}f(\rho) = f\left(\rho + \frac{1}{N}\right) \quad (23)$$

so that equation (20) becomes

$$\dot{P}(\rho, t) = (\mathbb{E} - 1)\mu_{N\rho}P(\rho, t) + (\mathbb{E}^{-1} - 1)\lambda_{N\rho}P(\rho, t) \quad (24)$$

Now we make the *approximation* of considering ρ *continuous*; this is justified by noting that the rates $\lambda_{N\rho}$ and $\mu_{N\rho}$ considered as functions of ρ , $\lambda(N\rho)$ and $\mu(N\rho)$, are continuous and differentiable and that in the limit of large population size the jumps in density are small enough. We can then expand the operator \mathbb{E} in a Taylor expansion

$$\mathbb{E} = 1 + \frac{1}{N} \frac{\partial}{\partial \rho} + \frac{1}{2N^2} \frac{\partial^2}{\partial \rho^2} + \dots \quad (25)$$

obtaining our Fokker-Planck equation

$$\begin{aligned} \dot{P}(\rho, t) = & -\frac{1}{N} \frac{\partial}{\partial \rho} (\lambda(N\rho) - \mu(N\rho)) P(\rho, t) \\ & + \frac{1}{2N^2} \frac{\partial^2}{\partial \rho^2} (\lambda(N\rho) + \mu(N\rho)) P(\rho, t) \end{aligned} \quad (26)$$

From this equation it is possible to derive a stationary solution for the long-time limit $P^{\text{stat}}(\rho)$ through the following procedure: we first put the time derivative to zero and then rewrite equation (26) as

$$0 = \frac{\partial}{\partial \rho} [(\mu(N\rho) - \lambda(N\rho))P] + \frac{\partial}{\partial \rho} (\lambda(N\rho) + \mu(N\rho))P \quad (27)$$

This means that the quantity in square brackets is constant. Setting the value of this constant to zero thanks to the detailed balance property we now have a first order ordinary differential equation whose solution is

$$P^{\text{stat}}(\rho) = \frac{Z}{\lambda(N\rho) + \mu(N\rho)} \exp \left[2 \int_{\rho_0}^{\rho} \frac{\lambda(N\rho') - \mu(N\rho')}{\lambda(N\rho') + \mu(N\rho')} d\rho' \right] \quad (28)$$

where Z is a constant. We underline that for this function to be a probability density it must be integrable so that it can be normalized.

2.3.2 Demographic noise

A fundamental type of temporal noise, characteristic of discrete systems such as in population dynamics models is *demographic* noise. This noise is an intrinsic *internal* noise for the systems and arises directly from an expansion of the *fixed-rates* CP master equation. It represents those casual fluctuations of the density ρ due to the concurring birth and death processes. Demographic noise can be introduced in a Langevin equation by means of a multiplicative term proportional to the square root of ρ

$$\dot{\rho} = f(\rho) + \alpha \sqrt{\rho} \eta(t) \quad (29)$$

Because of this intrinsic fluctuation a finite-size system will eventually get trapped in the absorbing state. Since in population dynamics $f(\rho)$ usually contains terms in ρ , ρ^2 or higher powers, demographic noise leads the dynamics near the absorbing state. However, the multiplicative term α turns out to be inversely proportional to square root of the size N of the system so that demographic noise effects vanish when $N \rightarrow \infty$. However, we underline that most simulations are carried out on finite systems and so this effect is always present. Now we can also give a better interpretation of the absorbing state: any state different from the absorbing one cannot be stable since the presence

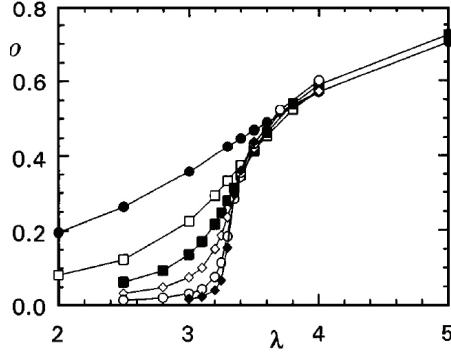


Figure 7: This image can be found in [26]. From left to right it shows phase diagrams for the one-dimensional CP with lattice size of 20, 50, 100, 20, 50, 1000 to appreciate the sharpening at the transition.

of demographic noise constantly causes jumps in the stochastic variable. The absorbing state (in this case $\rho = 0$) is the one at which such fluctuations vanish. The absorbing state is a *fluctuation-free state*. Using the method explained in section 2.3 we can show how the square root appears. Inserting the noiseless rates (18) we obtain the following master equation

$$\dot{P}(\rho, t) = N(\mathbb{E} - 1)\rho P(\rho, t) + \lambda N(\mathbb{E}^{-1} - 1)\rho(1 - \rho)P(\rho, t) \quad (30)$$

whose Fokker-Plank approximation is

$$\dot{P}(\rho, t) = -\frac{\partial}{\partial \rho}(\rho(\lambda - 1)(1 - \rho)P) + \frac{1}{2N} \frac{\partial^2}{\partial \rho^2}(\rho(\lambda + 1)(1 - \rho))P \quad (31)$$

from equation (22) we know that it must be

$$\begin{aligned} g(\rho) &= \sqrt{\frac{\rho(\lambda + 1)(1 - \rho)}{N}} \\ &\approx \sqrt{\frac{\lambda + 1}{N}} \sqrt{\rho} \end{aligned} \quad (32)$$

We stress again that this noise is unavoidable since it is an intrinsic property of the system. It is also a finite-size effect since it vanishes in the limit of large populations. Because simulations are carried out on finite-size systems, the phase diagram obtained presents a smooth phase transition that gets sharper at the critical point as the size of the system grows, as depicted in figure 7. Another way to see that a finite-size system always ends up in the absorbing state consists in using equation (28) to show that the only stable state is the absorbing one. Inserting the rates and changing variable to ρ we obtain

$$p^{\text{stat}}(\rho) = \frac{ZN}{\rho(\lambda + 1)(1 - \rho)} \exp\left[-2N \frac{\lambda - 1}{\lambda + 1} \rho\right]. \quad (33)$$

We can see that this distribution is peaked at $\rho = 0$ for any N .

2.3.3 Environmental white noise and temporal Griffiths phases

With the term *environmental* noise we refer to a type of noise which does not arise naturally from the master equation as the demographic noise but it is instead imposed on the system. Such noise is *external* and is usually used to model the effects of changing environmental conditions in population dynamics, thus the name environmental noise. Environmental noise can be any type of function of time. Like spatial noise, temporal noise can greatly influence the dynamics and produce changes in the system's phase diagram, depending on the type of time-dependence of the noise itself. Noises with correlation times and complex time dependencies are very hard to treat mathematically and the usual approach is to use white noise coupled with some function of the order parameter, since Gaussian white noise allows to use the techniques illustrated in the previous sections which do not work for non-Markovian systems. The study on the effects of environmental noise on stochastic population models started with Leigh [25] who showed how it can change the MET of a system from exponential to power-law. A fair amount of literature flourished after Leigh's work showing how time-dependent fluctuations of the parameters of a model can shift critical points [8] and change universal features in and out of equilibrium [2, 19]. An extensive analysis of the effects of environmental white noise on the CP of [31] showed how it gives origin to a phase of the system with a behavior similar to the spatial Griffiths phase: for extended regions in the parameter space the system presents critical-like properties. These new phase has been called *temporal Griffiths phase* (TGP). The main difference between usual Griffiths phases is that they take place *before* the critical point while TGPs are found *after*. These findings allow to insert TGPs and normal GPs into a coherent framework of the effects of noises depending on both space and time.

VELASQUEZ ET AL. WORK We will now go briefly through Velasquez et al. analytical work since their conclusions are important to understand ours. In [31] they explored the properties of a classic CP system with the addition of multiplicative white noise for dimensions 1, 2 and for the mean-field approximation, both analytically and through simulations. We note here that they didn't actually use a Gaussian white noise, whose variance is technically infinite, but a noise $b(t)$ randomly sampled from an uniform distribution instead. The distribution is centered at a value b_0 and its half-width is σ . b_0 represents the control parameter for this system. A birth occurs with rate $b(t)\rho(1 - \rho)$ and a death with rate $\rho(1 - b(t))$. These rates are normalized so they can also be interpreted as probabilities for the birth and death processes. From the master equation an Itô Langevin equation is derived.

$$\dot{\rho} = a\rho - b\rho^2 + \alpha\sqrt{\rho}\eta(t) + \sigma\rho\xi(t) \quad (34)$$

where $a = 2b_0 - 1 + \sigma^2/2$, $b = b_0$, $\alpha = N^{-1/2}$ (N is the size of the system) and the noise $\xi(t) = 2[b(t) - b_0]/\sigma$. We can see in this equation the presence of the demographic noise, proportional to the square root of the density. From this equation a Fokker-Planck equation can be derived, whose stationary solution in the limit of large N is:

$$P(\rho) = \rho^{2(\frac{a}{\sigma^2}-1)} \exp\left(-\frac{2b_0\rho}{\sigma^2}\right) \quad (35)$$

This probability density function exhibits a singularity at $\rho = 0$ depending on the value of a/σ^2 . When $a < a_c = \sigma^2/2$ (or equivalently $b < b_{0,c} = 1/2$ for every σ) such singularity is non-integrable and the only acceptable solution for the probability density is a delta distribution peaked at $\rho = 0$. This means that for values of b_0 below $1/2$ the system is in the absorbing state. With a change of variable to $z = \log(\rho)$ equation(34) becomes

$$\dot{z} = -\tilde{a}z + b \exp(z) + \sigma\xi(t) \quad (36)$$

This equation describes a random walker trapped in a potential as already showed in section 1.2.3. We can then identify three different regimes:

- I When $\tilde{a} > 0$ the phase is active and the same argument of section 1.2.3 holds, so we get

$$\begin{aligned} \text{MET}(N) &\sim \exp[(V(z(\rho = 1/N)))/(\sigma^2/2)] \\ &\sim \exp\left(\frac{2\tilde{a} \log(N)}{\sigma^2}\right) \\ &\sim N^{\frac{2\tilde{a}}{\sigma^2}} \end{aligned} \quad (37)$$

This means that due to the noise the MET exhibits *generic algebraic scaling* with continuously varying exponents.

- II When $\tilde{a} = 0$ the system is at the critical point and the barrier vanishes for small values of z , meaning that near the absorbing state the variable represents a free random walker. A general free random walk covers a distance \sqrt{t} in a time t . Equating this distance to the one required to reach the absorbing state which is $z = \log(N)$ we obtain

$$\text{MET}(N) \sim [\log(N)]^2 \quad (38)$$

- III When $\tilde{a} < 0$ the system is in the absorbing phase. Since in this phase the density ρ decays exponentially in time, z decays linearly and thus the time required to reach the state $z = \log(N)$ is

$$\text{MET}(N) \sim \log(N) \quad (39)$$

The analogy between TGPs and normal GPs is strong. In GPs rare active regions exist even when the overall state is absorbing, whereas in TGPs there exist rare time intervals when the system behaves like in the absorbing state, even if the overall phase is active. The roles of active and absorbing phases are exchanged by the different noises. It can be stated that in GPs noise is quenched in space and in TGPs it is “quenched” in time. In GPs the probability for an active region of size s to form is exponential, $P(s) \sim \exp(-\alpha s)$; the same holds for the probability in TGP for an absorbing interval of length T to happen, $P(T) \sim \exp(-\alpha T)$. Moreover, in GPs the leading contribution of the decay at time t comes from a rare region of size $s^* \sim \log(t\beta/\alpha)/\beta$ which combined with the exponential probability of finding one gives the generic power-law decay in time of the order parameter $t^{-\alpha/\beta}$. In TGPs the time required to reach the absorbing state in a rare absorbing interval is given by $t^* \sim \log(N)/\beta$ which again combined with the exponential probability of finding one gives rise to a generic algebraic decay in the system’s size $\text{MET}(N) \sim N^{\alpha/\beta}$. The conclusion is that TGPs are the exact analogues of GPs, just with switched roles of space and time.

2.3.4 Colored noise

Completely uncorrelated noise is never an exact representation of *real* noise. In the early studies of Brownian motion, when the first stochastic methods were developed, inertial forces could be neglected since the damping of suspended particles was much larger compared to the motion of the fluid’s molecules. White noise it is a very good approximation in these cases and it simplifies the problem making it *Markovian*: nowadays a large amount of literature exists on how to deal with Markovian processes. However, in many physical models making such approximation is a mistake since the dynamics of the noise and of the order parameter are not easily separable. In these cases the time-correlation of the noise is comparable with the relaxation time of the system and can play an important role that cannot be neglected. This argument is valid above all for biological systems in which changes usually evolve at the same time scale of the environment: temperature rises and drops, humidity modifications or DNA genetic alterations are smooth functions of time and do not usually vary randomly and rapidly in a short time interval (except for catastrophic events like volcanoes eruptions which are a separate matter). The term *colored* was introduced to distinguish the correlated case from the white noise case. Depending on the strength of the colored noise the system’s behavior can deviate significantly from the Markovian one and this poses serious limitations on the techniques

available to study it. In such problems the markovianity can be retrieved by extending the number of variables considered but this is not a solution by itself since this procedure can greatly complicate the problem. If the noise one is interested in is not a specific type of time function but rather a general effect of several coupled environmental degrees of freedom, thanks to the *central limit theorem* they can be treated as Gaussian and if the noise itself is Markovian then Doob's theorem [7] states that the only possible time-correlated noise of this kind is a *Ornstein-Uhlenbeck* process (OU). A OU process $\xi(t)$ is a process with an exponentially decaying autocorrelation function and zero mean[22]

$$\langle \xi(t) \rangle = 0 \quad (40)$$

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{\tau} \exp\left(-\frac{|t-t'|}{\tau}\right) \quad (41)$$

where D is the dimensionless noise intensity and τ is the correlation time. The first to study intensively this type of noise were Stratonovich and coworkers who established a *small τ approximation* [24]. Several methods have been developed since: [14] presents a fair review of some of these. Referring to [22] we will now outline the main differences between a proper white noise Langevin equation and a colored noise one as well as introducing the Fokker-Planck equation in the extended variable space. We start with the following general equation:

$$\dot{y} = f(y) + g(y)\xi(t) \quad (42)$$

in which ξ is a stationary process with

$$\langle \xi(t) \rangle = 0 \quad (43)$$

$$\langle \xi(t)\xi(t') \rangle = \kappa(t-t') \quad (44)$$

where $\kappa(t-t')$ is not a delta function. An equation of this type is a *stochastic differential equation*, meaning a differential equation whose coefficients are random numbers or random functions of an independent variable (time t in this case). The variable y itself is not a Markovian process since its value at a time t bigger than the initial time t_0 depends on the quantity $\xi(t)$ whose value, being time-correlated, depends on the values *before* t_0 . Analytic solutions for $y(t)$ are not easy to obtain and they usually require making use of some kind of approximation meant to retrieve markovianity, as we will do in this work. If there is the need to retrieve the Markovian problem without any approximation the only way is to increase the number of stochastic variables and consider the complete system

$$\dot{y} = f(y) + g(y)\xi(t) \quad (45)$$

$$\dot{\xi} = h(\xi, t) \quad (46)$$

This *bivariate* process (y, ξ) is Markovian and its joint probability $P(y, \xi, t)$ obeys the master equation

$$\dot{P}(y, \xi, t) = -\frac{\partial}{\partial y} \{f(y) + g(y)\xi\}P + \mathbb{W}P \quad (47)$$

where \mathbb{W} is the matrix regulating the master equation of the ξ process alone [22]. In the case of ξ being a OU process the system is of this type:

$$\dot{y} = f(y) + g(y)\xi(t) \quad (48)$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \eta(t) \quad (49)$$

with

$$\langle \eta(t) \rangle = 0 \quad (50)$$

$$\langle \eta(t)\eta(t') \rangle = \Gamma\delta(t-t') \quad (51)$$

We stress here the fact that in this system the noise is additive since $\xi(t)$ in the first equation is not a delta-correlated function of time. In this case the Itô-Stratonovich interpretation is not an issue and the problem is well defined as it is. The related master equation reads

$$\dot{P}(y, \xi, t) = -\frac{\partial}{\partial y} \{f(y) + g(y)\xi\}P + \frac{1}{\tau} \frac{\partial}{\partial \xi} \xi P + \frac{\Gamma}{2} \frac{\partial^2}{\partial \xi^2} \quad (52)$$

In chapter 4 we will introduce the Markovian approximation scheme we used, while in chapter 5 we will make use of equation (52) in order to apply a path-integral approach to the computation of the MET for our system.

3

REVIEW OF THE SIMULATION TECHNIQUES

Being able to correctly simulate a dynamical system is of paramount importance: little details that affect the dynamics may or may not appear depending on the method used to simulate the equations, in particular when dealing with noise; bad simulations may introduce correlations difficult to spot that can predict non-physical results.

3.1 BASIC SIMULATION METHOD FOR THE CP

The simplest way one can imagine to simulate a birth-death process on a lattice is to actually simulate it directly in the classical sequential way using probabilities instead of rates [26]. One constructs a lattice and fills it with particles. Then for each time step a cell is chosen randomly. If there is a particle there it dies with probability $p_{\text{die}} = \frac{\mu}{(\lambda+\mu)}$ or reproduces with probability $p_{\text{breed}} = \frac{\lambda}{(\lambda+\mu)}$, where λ and μ are the birth and death rates respectively. If a reproduction event is selected a neighbor is chosen randomly and, if empty, a particle is placed. The time is then updated with time-step $dt = \frac{1}{N}$ where N is the number of particles in the system before the update. This assures that on average each particle will be processed at least once. The process in itself is a continuous Markov process and this discrete formulation is not exact in the short time, but the two share the same stationary properties and long-time dynamics. If the rates are continuous functions of time though ($(\lambda = \lambda(t), \mu = \mu(t))$) this simulation method is not correct. A rate indicates the number of events per second, but in translating it into probabilities this meaning is lost. A process with doubled rates proceeds at doubled speed, but such doubled rates lead to the same probabilities. Moreover for p_{die} and p_{breed} to be probabilities they must always sum to one. When rates change in time as in our case this is not always assured. We then need simulation techniques whose solutions are exact solutions of the master equation.

3.2 GILLESPIE METHODS

With the term *Gillespie* methods we generally refer to a way of simulating a continuous deterministic process as a discrete stochastic one. This way of approaching the problem was first invented by Doob and others around 1945 and popularized by Gillespie years after when

he used it to simulate chemical systems of reactions. In his 1977 paper he describes the algorithm whose pivotal points are separately extracting *when* the next reaction will occur and *which* reaction occurs [11]. To show this, consider a system of N chemical species X_1, \dots, X_N undergoing M chemical reactions with rates $a_1(\mathbf{X}) \dots a_M(\mathbf{X})$. Now the entire algorithm is based on the fundamental premise of chemical kinetics that states that the probability that reaction k takes place in a small time interval $[t, t + \Delta t]$ is given by $a_k(\mathbf{X}(t))\Delta t + o(\Delta t)$. Then the algorithm reads:

1. Initialize the system, setting the number of molecules and $t = 0$.
2. Calculate the rates for every reaction and the total rate $a_0(\mathbf{X}) = \sum_{k=1}^M a_k(\mathbf{X})$
3. Sample a uniform random number r_1 in the interval $(0,1)$. Extract the time-step to the next reaction Δ sampling it from an exponential distribution weighted by the total rate: $\Delta = a_0^{-1} \log(\frac{1}{r_0})$
4. Sample another uniform random number in $(0,1)$ r_2 to chose the reaction μ that occurs. The choice is made such that

$$\sum_{k=1}^{\mu-1} a_k < r_2 a_0 \leq \sum_{k=1}^{\mu} a_k$$

5. set $t = t + \Delta$ and update the number of each molecular species depending on the reaction μ , then return to step 2 or quit.

This methods overcomes the problem stated in the previous section by computing the next reaction time extracting it from an exponential distribution based on the assumption that the rate can be considered constant at least during the time-step. Then it chooses the next reaction essentially as the previous method. Although being the correct approach for simulating models with fixed rates, it cannot work for continuously varying rates as it is. In fact, at each step it computes the time-step Δ using the *current* value of the rates which can change a lot before the next reaction occurs. Nevertheless we tried it and it actually works quite well for rates with medium to large correlation time, since on average they remain almost constant for large chunks of time. As expected it fails for short correlation times and then it cannot be used to confirm our theoretical predictions.

3.3 ANDERSON'S ALGORITHM: A MODIFIED NEXT REACTION METHOD

The method we finally used it's a modified version of the Gillespie algorithm developed by Anderson [3]. To illustrate the algorithm the

first step is to show how the Gillespie algorithm can be represented using Poisson processes. Calling ν_k and ν'_k the vectors representing the number of each species created and consumed in the k th reaction we can write an equation for the evolution over time of the chemicals

$$\mathbf{X}(t) = \mathbf{X}(0) + \sum_{k=1}^M R_k(t)(\nu_k - \nu'_k) \quad (53)$$

where $R_k(t)$ represents the number of times that the k th reaction has taken place up to time t . It can be shown [23] that, based on the fundamental premise, $R_k(t)$ is a counting process with intensity $a_k(\mathbf{X}(t))$ so that it can be restated as an independent unit rate Poisson process $Y_k(t)$ with *internal time* $T_k(t)$. The latter is given by the integrated rate

$$T_k(t) = \int_0^t a_k(\mathbf{X}(s)) ds \quad (54)$$

$$R_k(t) = Y_k(T_k(t)) \quad (55)$$

Then equation 53 can be rewritten

$$\mathbf{X}(t) = \mathbf{X}(0) + \sum_{k=1}^M Y_k(T_k(t))(\nu_k - \nu'_k) \quad (56)$$

This representation of the problem shows that the Poisson processes account for the whole randomness in the system and that there are $M + 1$ significant time frames: the absolute one in which the system evolves and one for each of the Poisson processes.

Now, denoting by P_k the first firing time of the Poisson process Y_k in its time frame

$$P_k = \min\{s > T_k : Y_k(s) > Y(T_k)\} \quad (57)$$

we can use equation 55 to obtain the absolute time interval required for the Poisson process to fire, *assuming the rate a_k remains constant*:

$$\Delta t_k = \frac{P_k - T_k}{a_k} \quad (58)$$

In this way we know that the next reaction to take place and the absolute time steps required are given by the minimum Δt_k over all values of $k = 1, \dots, M$. The algorithm goes as follows:

1. Initialize time and number of molecules.
2. Calculate the rates a_k .
3. Generate M independent and uniform random numbers in the interval $(0,1)$ r_k .

4. Calculate P_k from an exponential distribution $P_k = -\log(1/r_k)$ for each k .
5. Compute $\Delta t_k = \frac{P_k - T_k}{a_k}$ for each k .
6. Find the minimum time-step $\Delta = \min_k \{\Delta t_k\}$. We denote μ the corresponding reaction.
7. Update the number of molecules corresponding to reaction μ and time t to $t + \Delta$.
8. For each k , set $T_k = T_k + a_k \Delta$.
9. Update the internal firing time for reaction μ generating a uniform random number in $(0,1)$ r and setting $P_\mu = P_\mu + \log(1/r)$
10. Recalculate the rates and start again from step 5.

So far we assumed that the rates don't depend explicitly on time but just on the number of molecules. This algorithm is completely equivalent in terms of computational expense and results to the next reaction method by Gibson and Bruck [10]. The main and most important difference though is that Anderson's algorithm can be easily adapted to *any* type of time dependence of the rates whereas the next reaction method can't apart from special cases. Anderson's method remains the same but it may not be possible to solve equation 55 directly. Δt_k is then found by solving

$$\int_t^{t+\Delta t_k} a_k(\mathbf{X}(s)) ds = P_k - T_k \quad (59)$$

Now that we have a proper way to simulate a contact-process like system with rates depending explicitly on time we need to apply it to our system. In our case we have two rates, one of which is constant. We obtain the following equations

$$\Delta t_{\text{death}} = P_{\text{death}} - T_{\text{death}} \quad (60)$$

$$\bar{\lambda} \Delta t_{\text{birth}} + \int_t^{t+\Delta t_{\text{birth}}} \xi(s) ds = P_{\text{birth}} - T_{\text{birth}} \quad (61)$$

Here a small complication arises: our birth rate depends on time stochastically. Moreover, the method has an anticipating nature. For example imagine we compute the two time steps and Δt_{death} results to be the shortest. Then a death will occur and time will start again from $t + \Delta t_{\text{death}}$. But then Δt_{birth} has been calculated with a O.U. path that goes longer than $t + \Delta t_{\text{death}}$ and must not be forgotten. We then need to compute the O.U. before and keep track of it's values on a temporal grid with fixed time-step. The grid goes from the present t to a t' large enough and it is continuously updated as the time moves. The grid's time-step has to be small enough so that the integral can be evaluated properly and must be quite smaller than the

O.U. correlation time τ . To compute the OU at any time step, we use a linear interpolation from the grid's stored values. Since the grid's time-step is usually very small this is a legit procedure.

4

STATIONARY STATE DISTRIBUTIONS

4.1 INTRODUCTION

We are now ready to present the main subject of this work: a contact process with unitary death rate and a birth rate subjected to environmental OU noise. The decision to study this type of system originated from the need to introduce temporal correlations in the study of bet-hedging dynamics. Chapter 6 contains a full introduction to the subject; in the present and next chapters we will focus on the statistical properties of this model without any reference to its applications. The stochastic equation for this models is the following.

$$\dot{\rho} = (\lambda + \xi(t))\rho(1 - \rho) - \rho \quad (62)$$

The control parameter is represented by the mean value of the birth rate λ , since the OU noise has zero mean as defined by equation (43). This equation does not contain the term accounting for the demographic noise, introduced in section 2.3.2. This does not mean our equation is incorrect: it means it is an approximation valid for large-size populations. For the OU noise $\xi(t)$ we are using a particular form which guarantees we can control the correlation time and the standard deviation of this Gaussian process separately. Written as in equation (49) the OU process has stationary variance $\frac{\Gamma\sigma^2}{2}$, where Γ comes from the autocorrelation function as in equation (51). By taking $\Gamma = 2$ and rewriting the equation in the following form

$$\dot{\xi} = -\frac{\xi}{\tau} + \frac{\sigma}{\sqrt{\tau}}\eta(t) \quad (63)$$

The variance is now σ^2 and it is not influenced by the correlation time τ . The autocorrelation function now reads

$$\langle \xi(t)\xi(t') \rangle = \sigma^2 \exp\left(-\frac{|t-t'|}{\tau}\right) \quad (64)$$

The main goal of this section is to derive a stationary probability density distribution for the density ρ alone. The Markovian system of the joint process (ρ, ξ) made by equations (62) and (63) can be translated into equation (52). We could ideally use it to find an equation for the marginal probability $P(\rho, t)$ alone, but this approach turned out too complex. Instead, we used a Markovian approximation for the system to retrieve a proper Langevin equation.

4.2 THE UCNA

4.2.1 Review of the method

The U.C.N.A acronym stands for *Unified Colored-Noise Approximation* and was developed by Jung and Hänggi [20]. It will allow us to use general Markovian techniques to tackle the problem and most importantly it will do this for *any* correlation time, in contrast with the usual small τ approximations. The method basically consists in an adiabatic elimination scheme combined with a time rescaling. To explain it we will start from a general system of this form

$$\dot{y} = f(y) + \xi(t) \quad (65)$$

$$\dot{\xi} = -\frac{\xi(t)}{\tau} + \frac{\sigma}{\sqrt{\tau}}\eta(t) \quad (66)$$

where $\langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$. The method requires to start with a stochastic equation (65) with additive noise. The first step consists in deriving equation (65) with respect to time and then substitute $\dot{\xi}$ from equation (66). Then, we introduce a new time scale $\hat{t} = t\tau^{-1/2}$ obtaining the following equation

$$\ddot{y} + \gamma(y, \tau)\dot{y} - f(x) = \sigma\sqrt{\tau}\xi(\tau^{1/2}\hat{t}) \quad (67)$$

where

$$\gamma(y, \tau) = \tau^{-1/2} - \tau^{1/2} \frac{df}{dy} \quad (68)$$

It is important to notice that $\gamma(y, \tau)$ is positive whenever $f'(y) < 0$ and that it approaches infinity both in the limit of vanishing τ and infinite τ . The adiabatic approximation is carried out by setting $\ddot{y} = 0$. Thanks to the behavior of $\gamma(y, \tau)$ this approximation is valid when $\tau \rightarrow 0$ and $\tau \rightarrow \infty$, that is, for a white noise and for a fixed-rates system. Jung and Hänggi expect to verify its validity for any τ through the confrontation with simulations, but as we will see on the next section another interpretation sheds some light on the matter. From this procedure we obtain another Langevin equation, but with multiplicative noise $\chi(\hat{t})$ instead:

$$\dot{y} = \frac{f(y)}{\gamma(y, \tau)} + \frac{\sigma\sqrt{\tau}}{\gamma(y, \tau)}\chi(\hat{t}) \quad (69)$$

Such equation must be interpreted as Stratonovich since we used the usual calculus rules to obtain it. The corresponding Fokker-Planck equation reads:

$$\dot{P}(x, \tau)_{\hat{t}} = -\frac{\partial}{\partial x} \left\{ \frac{f(x)}{\gamma(x, \tau)} - \frac{\sigma\sqrt{\tau} \frac{d\gamma(x, \tau)}{dx}}{\gamma^3(x, \tau)} \right\} P(x, \tau)_{\hat{t}} + \sigma\sqrt{\tau} \frac{\partial^2}{\partial x^2} \left\{ \frac{P(x, \tau)_{\hat{t}}}{\gamma^2(x, \tau)} \right\} \quad (70)$$

whose stationary solution is:

$$P^{\text{stat}}(y, \tau) = Z |1 - \tau f'(y)| \exp \left[-\frac{\sqrt{\tau}}{2\sigma} f^2(x) \right] \exp \left\{ \frac{\sqrt{\tau}}{\sigma} \int^y f(y') dy' \right\} \quad (71)$$

where Z is as always the normalizing constant.

4.2.2 How to interpret the UCNA method?

Although the validity of this probability density can be ensured each time by simulating the system, a more solid interpretation is desirable. The above brief explanation on the validity of the UCNA method does not clarify how exactly it interpolates the limit regimes of white noise and constant noise. In [5] the authors interpret the UCNA method from a path integral point of view. We provide a more detailed discussion of this approach in the next chapter, although the path integral formulation of stochastic processes is nowadays a standard technique. We now show of this interpretation is derived. First we combine equations (62) and (63) in the limits of $\tau \rightarrow 0, \infty$ and obtain the following Langevin equations:

I [$\tau \rightarrow 0$]

$$\dot{y} = f(y) + \sigma\sqrt{\tau}\eta(t) \quad (72)$$

whose corresponding Fokker-Planck equation is

$$\dot{P}(y, t) = -\frac{\partial}{\partial y} f(y)P + \sigma^2\tau \frac{\partial^2}{\partial y^2} P \quad (73)$$

II [$\tau \rightarrow \infty$]

$$\dot{y} = -\frac{f(y)}{\tau f'(y)} - \frac{\sigma}{\sqrt{\tau f'(y)}} \eta(t) \quad (74)$$

whose corresponding Fokker-Planck equation is

$$\dot{P}(y, t) = -\frac{\partial}{\partial y} \left\{ -\frac{f(y)}{\tau f'(y)} - \frac{\sigma f''(y)}{\tau f'^3(y)} \right\} P + \frac{\sigma^2}{\tau} \frac{\partial^2}{\partial y^2} f'^{-2}(y) P \quad (75)$$

In these limits we can see that the markovianity is naturally retrieved. Associated with this Fokker-Planck equations are two Langrangians \mathcal{L}_0 for the white noise and \mathcal{L}_∞ for the constant noise that read

$$\mathcal{L}_0(x, \dot{x}) = \frac{[\dot{x} - f(x)]^2}{4\sigma^2\tau} + \frac{f'(x)}{2} \quad (76)$$

$$\mathcal{L}_\infty(x, \dot{x}) = \frac{[\tau f'(x)\dot{x} + f(x)]^2}{4\sigma^2\tau} - \frac{1}{2\tau} \quad (77)$$

In the same way, a Langrangian related to the UCNA Fokker-Planck equation (70) can be found:

$$\begin{aligned} \mathcal{L}_{\text{UCNA}}(x, \dot{x}) = \frac{1}{4\sigma^2\tau} \{ [1 - \tau f'(x)]\dot{x} - f(x) \}^2 \\ + \frac{1}{2} [1 - \tau f'(x)]^{-1} f'(x) \end{aligned} \quad (78)$$

We now observe that if we can find an appropriate function $\theta[\tau f'(x)]$ that satisfies the limit conditions

$$\lim_{\tau \rightarrow 0} \theta[\tau f'(x)] = 1 \quad (79)$$

$$\lim_{\tau \rightarrow \infty} \theta[\tau f'(x)] = -[\tau f'(x)]^{-1} \quad (80)$$

then an *interpolating Langrangian* can be defined

$$\mathcal{L}_{\text{Int}}(x, \dot{x}) = \frac{1}{4\sigma^2\tau} \left\{ \frac{\dot{x}}{\theta[\tau f'(x)]} - f(x) \right\}^2 - \frac{1}{2} \theta[\tau f'(x)] f'(x) \quad (81)$$

Comparing this expression with the UCNA's Langrangian it follows that

$$\theta[\tau f'(x)] = [1 - \tau f'(x)]^{-1} \quad (82)$$

From this point of view, the UCNA method can be interpreted as the choice of an appropriate interpolating function between the Langrangians associated with the extreme regimes $\tau \rightarrow 0, +\infty$. This interpretation shows also how a general Markovian approximation can be seen as a choice of θ and this can be used to compare different approximations through the comparison of different interpolating functions.

4.3 IMPLEMENTING THE UCNA

4.3.1 The stationary p.d.f.

We now apply the UCNA procedure to our system. We rewrite it in this form

$$\dot{\rho} = \lambda\rho(1-\rho) - \rho + \rho(1-\rho)\xi(t) \quad (83)$$

$$\dot{\xi} = -\frac{\xi}{\tau} + \frac{\sigma}{\sqrt{\tau}}\eta(t) \quad (84)$$

Since in equation (83) the noise is multiplicative, we first have to change it into additive through a change of variable. The new variable we will be working with is

$$\begin{aligned} u &= \int^{\rho} \frac{d\rho'}{\rho'(1-\rho')} \\ &= \log\left(\frac{\rho}{1-\rho}\right) \end{aligned} \quad (85)$$

This change of variable modifies our working domain from $\rho \in [0, 1]$ to $u \in [-\infty, +\infty]$. Eq. 83 then becomes

$$\dot{u} = \lambda - 1 - e^u + \xi(t) \quad (86)$$

In the notation of section 4.2 we then have

$$f(u) = \lambda - 1 - e^u \quad (87)$$

Substituting this function into equation (71) and changing back to ρ we obtain the stationary p.d.f.

$$\begin{aligned} P(\rho) &= Z^{-1} \rho^{\frac{\lambda-\sigma^2\tau-1}{\sigma^2\tau}} (\rho(\tau-1)+1)(1-\rho)^{\frac{1-\lambda}{\sigma^2\tau}-2} \times \\ &\quad \times \exp\left(-\frac{\frac{(\lambda(\rho-1)+1)^2}{(\rho-1)^2} + \frac{2\rho}{\tau(1-\rho)}}{2\sigma^2}\right) \end{aligned} \quad (88)$$

By means of this p.d.f. we can build a phase diagram and characterize our system. The term in this function that determines its behavior is $\rho^{\frac{\lambda-\sigma^2\tau-1}{\sigma^2\tau}}$: when $\lambda < \lambda_c = 1$, a non-integrable singularity in $\rho = 0$ appears. This value of λ delimits then the absorbing phase since as already said in the previous chapters the only solution for equation (88) is a delta function peaked on $\rho = 0$ for $\lambda < 1$. The first result of this analytical study is that the influence of correlated environmental noise does *not* shift the critical point, at least in the contact process. For values of $\lambda > 1$ the function $P(\rho)$ can be integrated successfully and

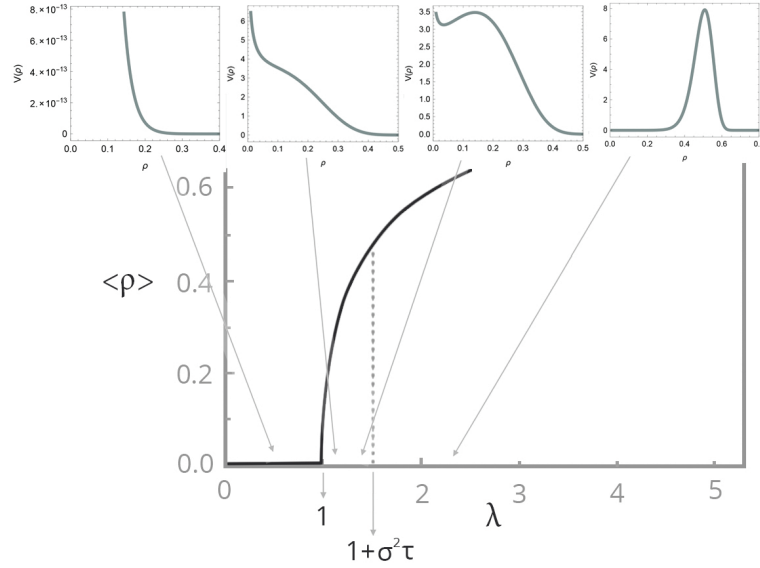


Figure 8: A schematic representation of the phase diagram obtained with the UCNA approach. From left to right are depicted the different shapes of the p.d.f. for $\lambda < 1$, $1 < \lambda < \lambda_{\text{bim}}$, $\lambda_{\text{bim}} < \lambda < 1 + \sigma^2\tau$, $\lambda > 1 + \sigma^2\tau$,

so it represents a proper p.d.f. In this case though emerges a difference from the system with fixed rates. In the fixed-rates case, as soon as the active phase is reached the probability $P(\rho = 0)$ of reaching the absorbing state is zero in the thermodynamic limit $N \rightarrow \infty$. With the presence of environmental noise the active phase gets split in two regions. The first spanning $1 < \lambda < 1 + \sigma^2\tau$ has $P(\rho = 0) \neq 0$ and the second, for values $\lambda > 1 + \sigma^2\tau$, has $P(\rho = 0) = 0$. We remind that we excluded the effects of demographic noise which may play a fundamental role and change the present picture. The shape of this p.d.f. rises a new question: is the condition of integrability sufficient to define the active phase? Can the first region be properly called “active”? Moreover, this region is itself divided into two subregions; after a certain value λ_{bim} the distribution becomes bimodal, only to recover its monomodality when $\lambda > 1$. The value of λ_{bim} does not have a simple analytical form and must be found numerically. The description done so far is summarized in figure 8 and in figure 9 on page 32. We can see now how noise *smeared* the usual phase diagram into a more variegated one in which the absorbing behavior slowly blends into the active one. We also notice that by just looking at the phase diagram it is not possible to understand if $P(\rho)$ is bimodal or not, and if a $\langle \rho \rangle > 0$ has still a physical significance in such case, since we know that once the absorbing state is reached the system cannot escape from it. We know also that the description provided by the Fokker-Planck equation is not too precise near the absorbing and reflecting states so all of this could be a problem of approximation. To

better understand the situation we computed the extinction times, as illustrated in the next chapter.

4.3.2 Comparison with simulations

In figure 10 is represented a comparison between the analytical p.d.f. (solid lines) and the simulated ones (colored histograms). To produce the histograms we used Anderson's simulation algorithm illustrated in section 3.3 *removing* the absorbing state. Without doing so the simulated systems would rapidly die as explained in the previous chapters because of intrinsic demographic noise. To remove the absorbing point we simply forced the birth of a particle every time the system reached zero density: in the jargon of statistical physics these are called *quasistationary* p.d.f.. The results are in very good agreement with the analytical ones except for minor differences which do not modify our conclusions. The graph in which analytical functions and simulations seem to disagree is the one for $\tau = 0.1$ but this is clearly due to numerical errors since the region $1 < \lambda < 1 + \sigma^2\tau$ spans a very small interval.

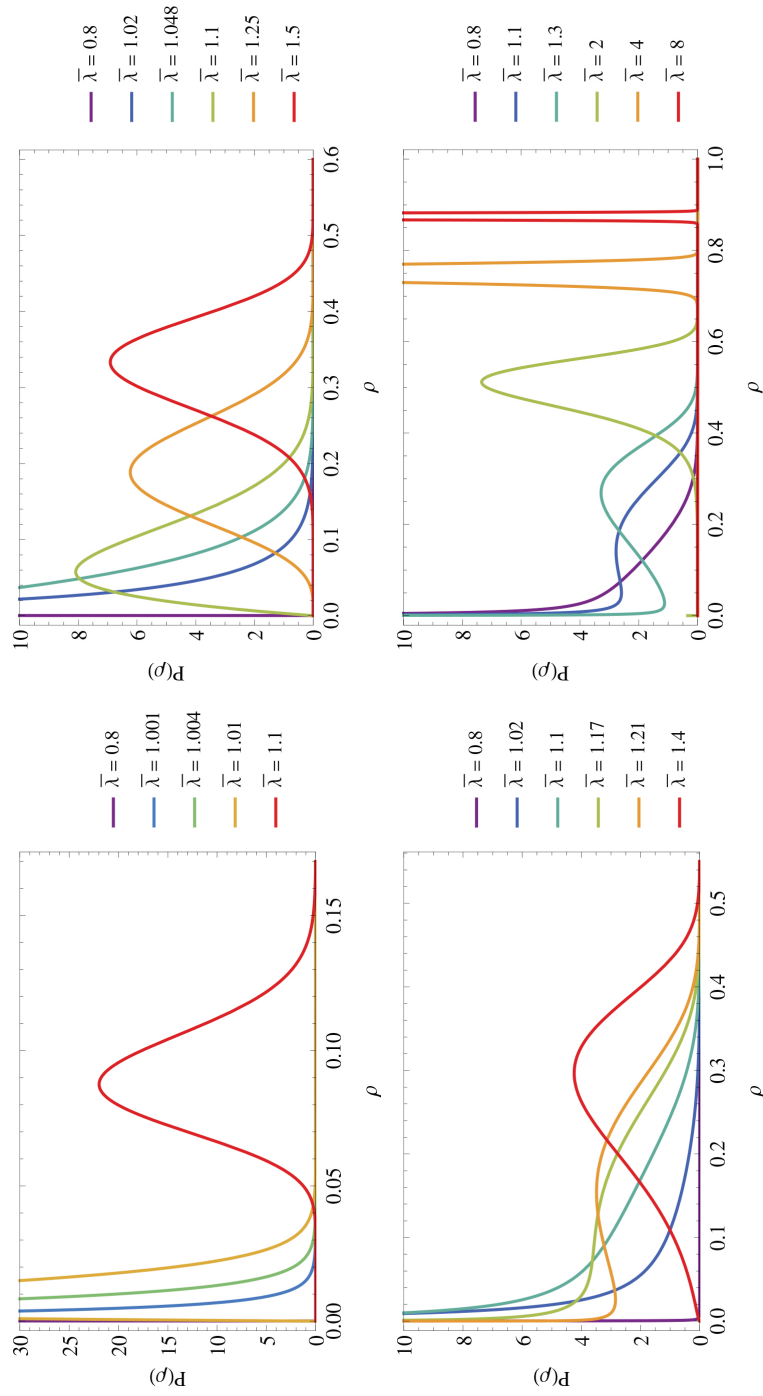


Figure 9: In these images are collected some stationary p.d.f. for different values of λ and τ . In clockwise order from the upper left graph: $\tau = 0.1$, $\tau = 1$, $\tau = 5$, $\tau = 100$.

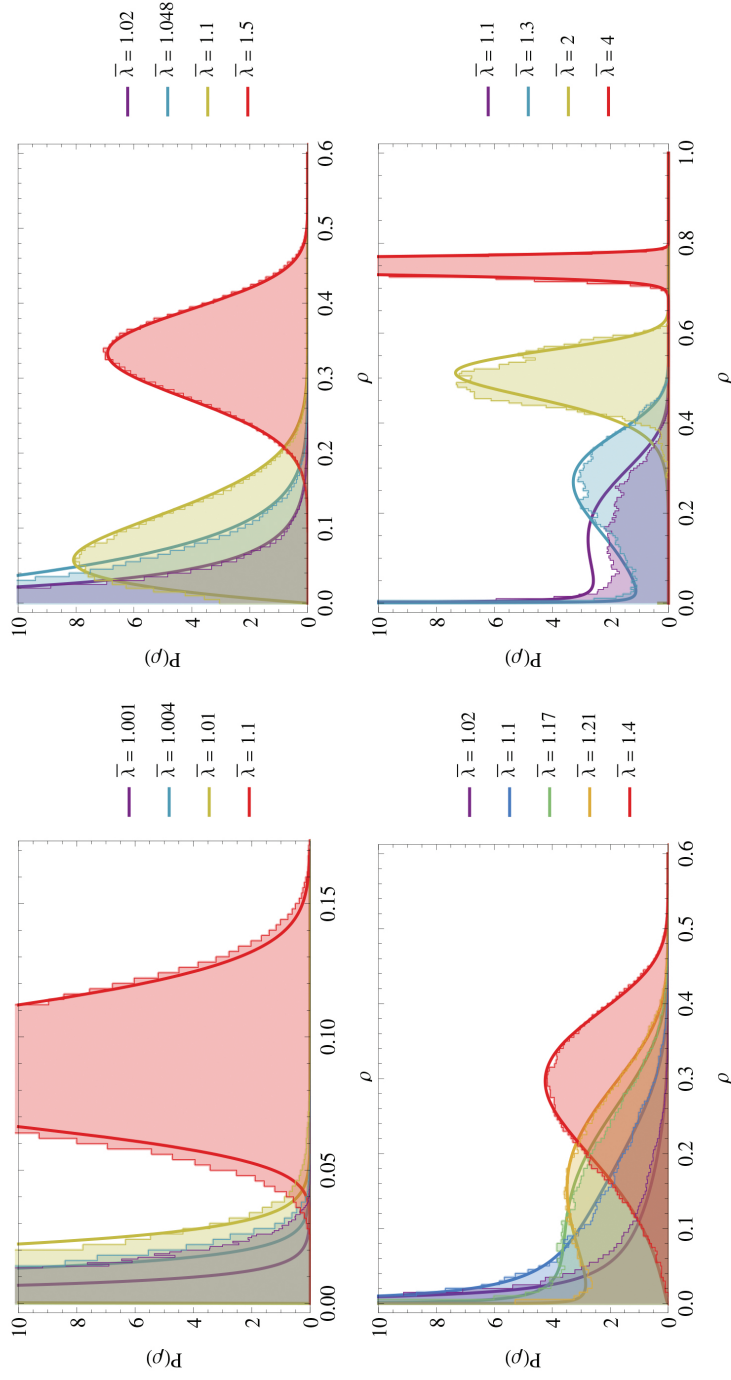


Figure 10: Comparison of the analytical p.d.f. and the simulated CP. In clockwise order from the upper left graph: $\tau = 0.1, \tau = 1, \tau = 5, \tau = 100$. All histogram columns have negligible error bars.

5

EXTINCTION TIME ANALYSIS

In this chapter we analyze the problem of the extinction times of the contact process with environmental noise. MET have already been introduced in section 1.2.3. In this chapter we will present some approaches that can be adopted to solve the problem, show the results in our case and compare them with simulations.

5.1 THE BACKWARD FOKKER-PLANCK METHOD

As a first try to find the MET for our model, we adopted a classic approach based on the backward Fokker-Planck equation [9]. We know that our density ρ lives in the interval $[0, 1]$. $\rho = 0$ is an *absorbing* extreme since it is a sink for the probability flux and $\rho = 1$ is a *reflecting* extreme since we cannot have densities bigger than one. We then ask ourselves what is the values of the time T during which ρ lives in the interval before “exiting” through zero. We consider then the probability

$$\text{Prob}(T > t) = G(\rho, t) = \int_0^1 P(\rho', t|\rho, 0) d\rho' \quad (89)$$

The probability $P(\rho', t|\rho, 0)$ obeys the *backward* Fokker-Planck equation which is an equivalent form of equation (21). In the case of homogeneous processes it is

$$\partial_t P(\rho', t|\rho, 0) = [A(\rho)\partial_\rho + \frac{1}{2}B(\rho)\partial_\rho^2]P(\rho', t|\rho, 0) \quad (90)$$

The same equation holds then for $G(\rho, t)$. We then define the MET, or mean first passage time to the absorbing state starting from an initial state ρ as

$$\begin{aligned} T(\rho) &= - \int_0^\infty t dG(x, t) \\ &= - \int_0^\infty t \partial_t G(\rho, t) dt \\ &= - \int_0^\infty G(\rho, t) dt \quad (\text{after integrating by parts}) \end{aligned} \quad (91)$$

Equation (90) can be translated into a differential equation for $T(\rho)$:

$$A(\rho)\partial_\rho T(\rho) + \frac{1}{2}B(\rho)\partial_\rho^2 T(\rho) = -1 \quad (92)$$

This equation is useless without the appropriate boundary conditions. It can be showed that the proper boundary conditions in this case are [9]

$$\partial_\rho G(\rho = 1, t) = 0 \quad (93)$$

$$G(\rho = 0, t) = 1 \quad (94)$$

A closed form for $T(\rho)$ is then

$$T(\rho) = 2 \int_0^\rho \frac{dy}{\psi(y)} \int_y^1 \frac{\psi(z)}{B(z)} dz \quad (95)$$

with

$$\psi(\rho) = \exp \left[2 \int_0^1 \frac{A(\rho')}{B(\rho')} d\rho' \right] \quad (96)$$

Although this is the classic way of approaching extinction times problems, since in our case we were looking for an analytical result equation (95) proved to be too difficult to solve.

5.2 THE PATH-INTEGRAL METHOD

5.2.1 The method

Many stochastic processes can be seen from the alternative point of view of the path-integrals. Every stochastic equation defines an ensemble of paths that the dynamical variable can take depending on the values of the random function. Without the random function (the noise in our case), the equation would be deterministic and the path the variable would take would be the one of minimum energy. The basic idea behind the path-integral approach is that in the phase space defined by the set $(\mathbf{X}, \dot{\mathbf{X}})$ of stochastic variables and their velocities the paths still obey the constraints defined by the Langrangian function $\mathcal{L}(\mathbf{X}, \dot{\mathbf{X}})$. The probability density function defined in such phase space and associated with each particular path will be exponential in the action of the path itself, in WKB-like approximation :

$$P((\mathbf{X}(t), \dot{\mathbf{X}}(t))) \propto \exp(-S[\mathbf{X}(t), \dot{\mathbf{X}}(t)]) \quad (97)$$

where the action $S[\mathbf{X}(t), \dot{\mathbf{X}}(t)]$ is

$$S[\mathbf{X}(t), \dot{\mathbf{X}}(t)] = \int dt \mathcal{L}(\mathbf{X}(t), \dot{\mathbf{X}}(t)) \quad (98)$$

In this framework, we can then interpret extinction as a special set of paths which pass through the absorbing state. Between all of these, the most probable ones will be the ones *minimizing the action*. We restate the interpretation as follows: the extinction time is governed

by *rare* paths that go rapidly to the extinction since they follow the line of minimum action (the deterministic one), instead of bumping back and forth because of random fluctuations. The extinction time is then given by the inverse of the probability of this paths, that is, the exponential of the action calculated upon them up to a prefactor

$$\text{MET} \approx \exp (S[\mathbf{X}(t)_{\min}, \dot{\mathbf{X}}(t)_{\min}]) \quad (99)$$

5.2.2 Applying the path-integral method

As we will see to make use of the path integral method it is not necessary to derive a Langrangian function: one can use directly the an Hamiltonian function associated with the Fokker-Planck equation of the system. However, the Langrangian function highlights the important variables that really weight the paths in the phases space. To gain a deeper understanding of the topic in the next two paragraph we will write the Langrangian for a OU noise and for the whole system of equations (83) and (84) and show that they are the same function, because of the non-markovianity of the $\rho(t)$ process.

THE LANGRANGIAN OF O.U. NOISE Lets consider now the simple OU process of equation (63). Which is the associated Langrangian? To obtain it, we first write (63) in the corresponding Fokker-Planck form:

$$\dot{P}(\xi, t) = \frac{1}{\tau} \frac{\partial}{\partial \xi} \xi P(\xi, t) + \frac{\sigma^2}{\tau} \frac{\partial^2}{\partial \xi^2} P(\xi, t) \quad (100)$$

Similarly to the Schrödinger equation, by setting $p_\xi = -\partial_\xi$ we can rewrite this Fokker-Planck equation in the form $\dot{P} = HP$ where the Hamiltonian $H = H(\xi, p_\xi)$ reads:

$$H(\xi, p_\xi) = -\frac{p_\xi \xi}{\tau} + \frac{\sigma^2 p_\xi^2}{\tau} \quad (101)$$

Interpreting p_ξ as a *variable* instead of an *operator* we treat $H(\xi, p_\xi)$ as a proper Hamiltonian. This allows us to recover the Langrangian equation associated with equation (63) through a Legendre transform:

$$\mathcal{L}(\xi, \dot{\xi}) = p_\xi(\xi, \dot{\xi}) \dot{\xi} - H(\xi, p_\xi(\xi, \dot{\xi})) \quad (102)$$

To find $p_\xi(\xi, \dot{\xi})$ we invert the Hamilton equation for $\dot{\xi}$:

$$\begin{aligned} \dot{\xi} &= \frac{\partial H}{\partial p_\xi} \\ &= -\frac{\xi}{\tau} + \frac{2\sigma^2 p_\xi}{\tau} \end{aligned} \quad (103)$$

obtaining

$$p_\xi = \frac{\tau}{2\sigma^2} \left(\dot{\xi} + \frac{\xi}{\tau} \right) \quad (104)$$

now we can write

$$\mathcal{L}(\xi, \dot{\xi}) = \frac{\tau}{4\sigma^2} \left(\dot{\xi} + \frac{\xi}{\tau} \right)^2. \quad (105)$$

Finally we can find the most probable path: it is the one minimizing the action and the solution of the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \xi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\xi}} = 0 \Leftrightarrow \ddot{\xi} + \frac{\xi}{\tau^2} = 0 \quad (106)$$

THE LANGRANGIAN FOR THE WHOLE SYSTEM The very same procedure applied in the previous paragraph can be used to derive the Langrangian function of the whole system made by equations (83) and (84). However, it is possible to know the Langrangian just by looking at the system: the stochasticity of the equations is all inside equation (84). Equation (83) is "deterministic" once $\xi(t)$ is known. Thus, we expect the Langrangian to be the OU one. The Fokker-Planck equation corresponding to the system

$$\begin{aligned} \dot{P}(\rho, \xi, t) = & -\frac{\partial}{\partial \rho} (\lambda\rho(1-\rho) - \rho + \xi\rho(1-\rho)) P(\rho, \xi, t) \\ & + \frac{1}{\tau} \frac{\partial}{\partial \xi} \xi P(\rho, \xi, t) + \frac{\sigma^2}{\tau} \frac{\partial^2}{\partial \xi^2} P(\rho, \xi, t) \end{aligned} \quad (107)$$

Again, setting ($p_\rho = -\partial_\rho$, $p_\xi = -\partial_\xi$) and interpreting them as variables instead of operators, the Hamiltonian associated with the Fokker-Planck equation is

$$\begin{aligned} H(\rho, p_\rho, \xi, p_\xi) = & p_\rho (\lambda\rho(1-\rho) - \rho + \xi\rho(1-\rho)) \\ & - \frac{\xi p_\xi}{\tau} + \frac{\sigma^2 p_\xi^2}{\tau} \end{aligned} \quad (108)$$

The Legendre transform immediately shows that the Langrangian function is independent of ρ and $\dot{\rho}$:

$$\begin{aligned} \mathcal{L}(\rho, \dot{\rho}, \xi, \dot{\xi}) = & p_\rho(\rho, \dot{\rho})\dot{\rho} + p_\xi(\xi, \dot{\xi})\dot{\xi} - H(\rho, p_\rho(\rho, \dot{\rho}), \xi, p_\xi(\xi, \dot{\xi})) \\ = & p_\rho(\rho, \dot{\rho})\dot{\rho} + p_\xi(\xi, \dot{\xi})\dot{\xi} \\ & - p_\rho(\rho, \dot{\rho})(\rho(1-\rho)(\lambda-1) + \xi\rho(1-\rho)) \\ & + \frac{\xi p_\xi(\xi, \dot{\xi})}{\tau} - \frac{\sigma^2 p_\xi(\xi, \dot{\xi})^2}{\tau} \\ = & \frac{\tau}{4\sigma^2} \left(\dot{\xi} + \frac{\xi}{\tau} \right)^2 = \mathcal{L}(\xi, \dot{\xi}) \end{aligned} \quad (109)$$

This result is a natural consequence of the non-Markovianity of the process $\rho(t)$. We can use equation (83) to translate $\mathcal{L}(\xi, \dot{\xi})$ into $\mathcal{L}(\rho, \dot{\rho})$, obtaining

$$\mathcal{L}(\rho, \dot{\rho}) = \frac{1}{4\sigma^2\tau(\rho-1)^4\rho^4} ((\tau\dot{\rho}^2 + \rho^2(\lambda + \tau\ddot{\rho} + \dot{\rho} - 1) - \rho(\tau\ddot{\rho} + 2\tau\dot{\rho}^2 + \dot{\rho}) + (\lambda-1)\rho^4 - 2(\lambda-1)\rho^3)^2) \quad (110)$$

which is not a proper Lagrangian since it contains second derivatives of ρ . Setting these derivatives to zero to obtain a true Lagrangian function in the variables $(\rho, \dot{\rho})$ corresponds to a pretty strong Markovian approximation.

THE UCNA To make use of the path-integral approach we then need to use a Markovian approximation for our system. We already developed such approximation in the previous chapter and we can already derive a complete Hamiltonian for the variables (u, p_u) from the UCNA Fokker-Planck (70) with $f(u)$ given by equation (87):

$$H(u, p_u) = \frac{\sqrt{\tau}p_u}{(\tau e^u + 1)^3} (\sigma^2\tau p_u(\tau e^u + 1) + (\lambda - e^u - 1)(\tau e^u + 1)^2 + \sigma^2\tau^2(-e^u)). \quad (111)$$

This equation contains just u and p_u so it is not necessary to compute the Lagrangian to understand which variables will effectively contribute. We now show how we can directly use the Hamiltonian function without first deriving the Lagrangian: since the path minimizing the action is also the one of minimum energy, by setting $H(u, p_u(u)) = 0$ we can obtain $p_u(u)$ and write

$$\begin{aligned} S(u(t)) &= \int \mathcal{L}(u) dt \\ &= \int \{p_u(u(t))\dot{u}(t) - H(u(t), p_u(u(t)))\} dt \quad (112) \\ &= \int p_u(u(t))\dot{u}(t) dt. \end{aligned}$$

Solving $H(u, p_u) = 0$ leads to

$$\begin{aligned} p_u(u) &= \frac{1}{\sigma^2\tau(\tau e^u + 1)} (-\lambda - \lambda\tau^2 e^{2u} - 2\lambda\tau e^u \\ &\quad + \sigma^2\tau^2 e^u + \tau^2 e^{2u} \\ &\quad + \tau^2 e^{3u} + 2\tau e^u + 2\tau e^{2u} + e^u + 1). \quad (113) \end{aligned}$$

In figure 12 we can see $p_u(u)$ represented with others paths in the (u, p_u) space as well with the direction of the trajectories. Now, to

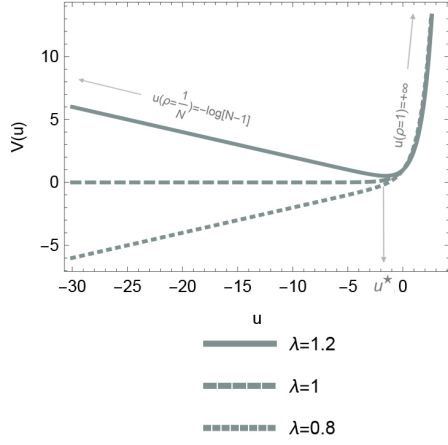


Figure 11: A representation of $V(u) = \int \{(\lambda - 1) + e^u\} du$ in the active, critical and absorbing phases.

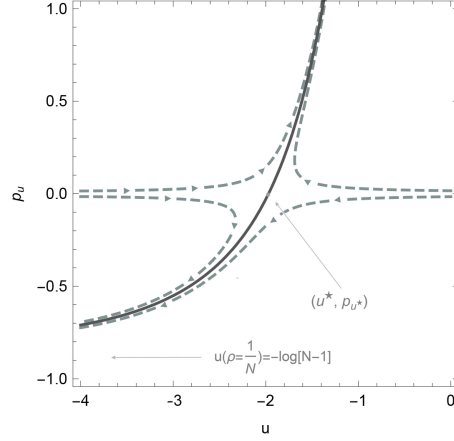


Figure 12: This image shows the path of zero energy $p_u(u)$ as well with the trajectories of different energies and their directions. For this example we used $\lambda = 1.2$, $\tau = 5$ and $\sigma = \sqrt{0.05}$.

compute the action we do not need to integrate over the whole domain of u $[-\infty, +\infty]$: as we already stated in section 1.2.3 the dynamical variable is confined in a well. This well has a finite wall before the absorbing point $\rho = 0$ in the variable ρ (see figure 5). When $\lambda > 1$ the potential in the variable u , obtained integrating the function $f(u)$ in equation (87), has an infinite wall instead as depicted in figure 11. Since we are interested in the scaling behavior of the action and not in the proper values of the integral, we can restrict the interval of integration to $[u(\rho = 1/N) = -\log(N-1), u^*]$ where u^* can be any value as long as it is kept fixed and it is not a singular one (in this case, $u(\rho = 1) = +\infty$). This allows us to write

$$S(N) = \int_{t_0}^{t_{\text{fin}}} p_u(u(t)) \dot{u}(t) dt \quad (114)$$

$$= \int_{u(t_0)=u^*}^{u(t_{\text{fin}})=\log(N-1)} p_u(u) du \quad (115)$$

$$= - \int_{\log(N-1)}^{u^*} p_u(u) du \quad (116)$$

$$= \frac{1}{\sigma^2 \tau} \left(\frac{-\lambda \tau + \tau + 1}{N-1} + (\lambda - 1) \log(N-1) \right) \quad (117)$$

$$+ \sigma^2 \tau \log\left(\frac{\tau}{N-1} + 1\right) + \frac{\tau}{2(N-1)^2} \quad (118)$$

$$+ e^{u^*} ((\lambda - 1)\tau - 1) + \lambda u^* - \sigma^2 \tau \log(\tau e^{u^*} + 1) \quad (119)$$

$$- \frac{1}{2} \tau e^{2u^*} - u^*). \quad (120)$$

where the minus sign before the integration arises because of the change of variable $t \rightarrow u$: the actual path starts at u^* and arrives at $\log(N-1)$. Since u^* does not depend on the system's size N , when $N \gg 1$ we get the following asymptotic behavior:

$$S(N) \sim \frac{\lambda-1}{\sigma^2\tau} \log N. \quad (121)$$

Thus by taking the exponential of the action we obtain our second main result: the asymptotic behavior for large populations of the extinction times for the contact process with environmental OU noise:

$$\text{MET}(N) \sim N^{\frac{(\lambda-1)}{\sigma^2\tau}}. \quad (122)$$

KAMENEV ET AL. WORK The analysis we carried out thanks to the path-integral approach was inspired by an article by Kamenev et al. [21]. They were some of the early authors pointing out how colored environmental noise could strongly reduce the MET of a model for population dynamics. We will now apply their mathematical procedure to obtain again result (122) *in the limit of small* τ . In their analytical work they consider the noise $\xi(t)$ and the density ρ separately: first they derive the Hamiltonian for a “fixed-rates” system without extending ξ to the role of variable and then they *weight* the paths with the Lagrangian of the OU noise (105). We start by writing the Fokker-Planck equation expanding the proper fixed-rates master equation as explained in chapter 2, with the modification $\lambda \rightarrow (\lambda + \xi(t))$:

$$\begin{aligned} \dot{P}(\rho, t) = & -\frac{\partial}{\partial \rho} ((\lambda + \xi(t))\rho(1-\rho) - \rho)P(\rho, t) \\ & + \frac{1}{2N} \frac{\partial^2}{\partial \rho^2} ((\lambda + \xi(t))\rho(1-\rho) + \rho)P(\rho, t) \end{aligned} \quad (123)$$

The associated Hamiltonian is then:

$$\begin{aligned} H_{\xi(t)}(\rho, p_\rho) = & p_\rho((\lambda + \xi(t))\rho(1-\rho) - \rho) \\ & + \frac{p_\rho^2}{2N}((\lambda + \xi(t))\rho(1-\rho) + \rho) \end{aligned} \quad (124)$$

whose term proportional to p_ρ^2 vanishes in the large population size limit, in which we can rewrite it

$$\begin{aligned} H_{\xi(t)}(\rho, p_\rho) = & p_\rho((\lambda + \xi(t))\rho(1-\rho) - \rho) \\ = & H(\rho, p_\rho) + \xi(t)p_\rho\rho(1-\rho) \end{aligned} \quad (125)$$

where $H(\rho, p_\rho) = H_{\xi(t)=0}(\rho, p_\rho)$. Taking now the Legendre transform of the Hamiltonian but keeping the variables $\rho(t)$ and $p_\rho(t)$ we can write the total action in this way:

$$S_{\xi(t)}[\rho(t), p_\rho(t)] = \int dt \frac{\tau}{4\sigma^2} \left(\dot{\xi}(t) + \frac{\xi(t)}{\tau} \right)^2 + p_\rho \dot{\rho} - p_\rho ((\lambda + \xi(t))\rho(1 - \rho) - \rho) \quad (127)$$

and now *derive* with respect to $\rho(t)$, $p_\rho(t)$ and $\xi(t)$ obtaining the following three equations

$$\dot{\rho} = \frac{\partial H(\rho, p_\rho)}{\partial p_\rho} + \xi(t)\rho(1 - \rho) \quad (128)$$

$$\dot{p}_\rho = -\frac{\partial H(\rho, p_\rho)}{\partial \rho} - \xi(t)\rho(1 - 2\rho) \quad (129)$$

$$\ddot{\xi}\tau^2 + \xi = 2\sigma^2\tau p_\rho\rho(1 - \rho) \quad (130)$$

where equations (128) and (129) are the Hamilton equations for the paths $\rho(t)$ and $p_\rho(t)$. In the limit of vanishing τ equation (130) becomes

$$\xi = 2\sigma^2\tau p_\rho\rho(1 - \rho). \quad (131)$$

Substituting ξ into the Hamiltonian equations we obtain

$$\dot{\rho} = \frac{\partial H(\rho, p_\rho)}{\partial p_\rho} + 2\sigma^2\tau p_\rho[\rho(1 - \rho)]^2 \quad (132)$$

$$\dot{p}_\rho = -\frac{\partial H(\rho, p_\rho)}{\partial \rho} - 2\sigma^2\tau p_\rho^2(1 - 2\rho)\rho(1 - \rho) \quad (133)$$

which can be seen as the Hamiltonian equations of an *effective* Hamiltonian

$$H(\rho, p_\rho)_{\text{eff}} = H(\rho, p_\rho) + \sigma^2\tau p_\rho^2[\rho(1 - \rho)]^2. \quad (134)$$

Now we can repeat the procedure illustrated previously. The path of zero energy reads:

$$p_\rho(\rho) = \frac{\lambda\rho - \lambda + 1}{(\rho - 1)^2\rho\sigma^2\tau} \quad (135)$$

We can then write

$$S(N) = \int_{t_0}^{t_{\text{fin}}} p_{\rho(t)} \dot{\rho}(t) dt \quad (136)$$

$$= \int_{\rho(t_0)=\rho^*}^{\rho(t_{\text{fin}})=\frac{1}{N}} p(\rho) d\rho \quad (137)$$

$$= - \int_{\frac{1}{N}}^{\rho^*} \left\{ \frac{\lambda\rho - \lambda + 1}{(\rho - 1)^2\rho\sigma^2\tau} \right\} d\rho. \quad (138)$$

where again we integrate from a starting point ρ^* whose value is not important as long as it is kept fixed and it is not singular for the path (such as $\rho = 1$). We obtain

$$S(N) = \frac{N\rho^* - 1}{\sigma^2\tau(N-1)(\rho^*)} + \frac{(\lambda - 1)}{(\sigma^2\tau)} \log \left[\frac{(N-1)\rho^*}{1 - \rho^*} \right] \quad (139)$$

which in the limit of $N \gg 1$ corresponds to (122).

5.2.3 Comparison with simulations and comment on the result

The path-integral approach gave us a critical exponent for the MET

$$\varkappa = \frac{\lambda - 1}{\sigma^2\tau} \quad (140)$$

Before commenting on the result, we stress again the fact that in our analysis we excluded the term responsible of the demographic noise because it would have introduced many complications in the Markovian approximation. Our system is valid for population of large size in which the environmental noise is the sole responsible for extinction. As we can see from figure 13 this assumption is correct since the analytical scaling trend and the simulations are in very good agreement. What conclusions can we take from exponent \varkappa ?

- I The first important thing is that environmental noise induces algebraic scaling on the *whole* active phase. This is in contrast not only with the fixed-rates case but also with the TGPs originating from white noise.
- II The second consequence is that the active region gets divided into two sub-regions. For $1 < \lambda < 1 + \sigma^2\tau$ the MET scale *sublinearly* while for $\lambda > 1 + \sigma^2\tau$ they scale *superlinearly*.
- III The third conclusion is that the effects of σ and τ are combined in a unique *effective variance* $\text{var} = \sigma^2\tau$.

Again, we spoke of “active” phase, but this results confirm what we already knew from the analysis of the stationary distribution: we must be cautious when speaking of a proper active phase since as we can easily see a value of λ bigger than one doesn’t mean the phase is active. The exponent grows linearly with λ so that there is no actual threshold after which we can say that the phase is active or still absorbing. This behavior makes sense because there is always the possibility that our birth rate samples from the absorbing phase values for a certain amount of time. If we take a look at the exponent, is nothing more than the ratio between the distance from the absorbing phase and the fluctuations of the birth rate. When σ grows this has

the obvious meaning that there are more possibilities for the birth rate to sample from the absorbing phase, but when τ grows it means that despite being not so probable, if the birth rate samples from the absorbing phase it keeps doing that for a greater amount of time.

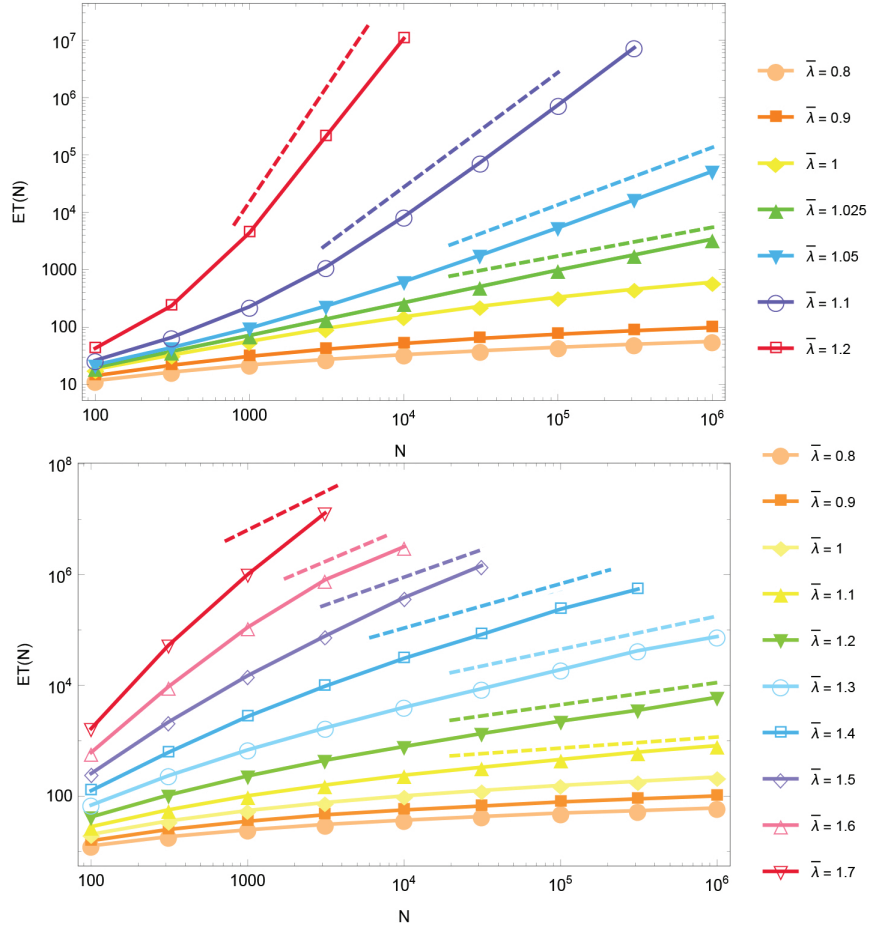


Figure 13: In these images we can appreciate the comparison between the simulations and the analytical results of MET for different τ and λ . The upper graph refers to OU noise with $\tau = 1$ while the lower to a OU noise with $\tau = 10$. As we can see they are *almost all* in very good agreement with the analytical trend, depicted as a dashed line. Although the predicted behavior is clearly wrong for $\lambda = 1.6$ and is not in good agreement for $\lambda = 1.5, 1.4$ the simulations seem to suggest that the MET curves gets less steep as the size grows. We then make the reasonable hypothesis that the analytical results and simulated systems still agree but for larger N . To certify it requires a computational effort beyond our current means. All error bars are smaller than point markers.

6 | BET-HEDGING

6.1 WHAT IS BET-HEDGING ABOUT?

It is a common practice in biology and evolutionary ecology to classify at species and populations from their *evolutionary strategies*. A species in a particular environment will develop a series of characteristics that together form their strategy to survive. In a steady environment this strategy is either a winning one or one that leads to extinction. In such a case, once a successful strategy has been found the species should just stick to it. In terms of classical contact process we could encode this in a fixed birth rate above the critical threshold. When environmental conditions are rapidly changing though this picture does not hold anymore. It's a long time conjecture that in these type of situations, either when the environment conditions vary with time or there is insufficient knowledge to find the optimal strategy, implementing a *bet-hedging* strategy may save the species from extinction. A bet-hedging strategy consists in the parallel application of two different strategies: the first one is a relatively safe one that guarantees a really slow but steady growth rate while the second one is a rather risky one; its goodness depends on the environmental condition. On their own these two strategies lead to extinction on the long time. Recalling the example of bacteria introduced at the beginning of this report, if the population chooses to continue to process the same sugar that's running out it can only slow down the extinction process. On the other hand, just randomly starting to process other sugars without a real knowledge of their global concentration in the environment is rather risky: what if it will not be enough? A move such this one can be a deadly stab for the survival of a population. Although it may seem paradoxical, bet-hedging strategies in nature show that a combination of two potentially losing strategies makes a winning one. Nowadays there's been a large empirical confirmation that bet-hedging-like strategies are actually implemented in nature such as by bacterial and viral communities, insects or plants [6, 18, 32]. In game theory this effect has been formalized under the name of Parrondo's paradox [30]. The paradox states that a winning strategy can be obtained by *alternating* two losing ones.

6.2 BET-HEDGING WITH WHITE NOISE

In the work by Hidalgo et al [17] the benefits of bet-hedging have been studied computationally for dimension 1, 2, 3 and mean-field underlying the positive effects of stochastic demographics fluctuations; a mean-field analysis supports and explains the point: bet-hedging benefits are strongly enhanced in low-dimensional environments. Their work is important since it shows that an effect of modest magnitude in a fully connected network such as bet-hedging can make the difference when dimension decreases. They simulate their systems using the basic technique explained in section 3.1. Here, they encoded the low-gain strategy as a p_0 fixed birth *probability* placed slightly below the active phase threshold. The risky one is then represented by an uncorrelated Gaussian white noise $p(t) = \bar{p} + \sigma\xi(t)$ centered at \bar{p} with variance σ^2 . Since it is also a probability σ cannot be too large to not lose physical significance. In the simulation a selected cell chooses with probability α the risky strategy and with probability $1 - \alpha$ the low-gain one, with $0 \leq \alpha \leq 1$. As we can see in figure ??, they found that there exist values of α for which the system passes from the absorbing to the active phase. An explanation of how this can happen can be given in the mean-field approximation: the corresponding Langevin equation for this system reads

$$\begin{aligned} \dot{\rho} = & \alpha[p(t)\rho(1 - \rho) - (1 - p(t))\rho] \\ & + (1 - \alpha)[p_0\rho(1 - \rho) - (1 - p_0)\rho] \end{aligned} \quad (141)$$

This equation is interpreted in the Itô sense. It can be rewritten keeping just the linear leading terms in the form

$$\dot{\rho} \approx (2p_{av}(\alpha) - 1)\rho + 2\alpha\sigma\rho\xi(t) \quad (142)$$

where $p_{av}(\alpha) = \alpha\bar{p} + (1 - \alpha)p_0$ is the average spreading probability. This form is quite valid near the critical point when ρ is still very small. By a change of variable and using Itô's calculus this equation can be put in term of the *exponential growth rate* $G(\alpha)$ [17]

$$\frac{d}{dt} \langle \log \rho \rangle = G(\alpha) \quad (143)$$

$$= -2\sigma^2\alpha^2 + 2p_{av}(\alpha) - 1 \quad (144)$$

The region in which $G(\alpha)$ is positive characterize the active phase, where it's negative the absorbing phase and where it's zero the critical point. What this tells us is that the critical point $p_c(\alpha)$ interpolates quadratically between the critical point of the pure strategies

$$p_c(\alpha) = \frac{1}{2} + \sigma^2\alpha^2 \quad (145)$$

where $p_c(0) = \frac{1}{2}$ and $p_c(0) = \frac{1}{2} + \sigma^2$. Since $p_{av}(\alpha)$ interpolates linearly with α this opens up the possibility for a window of values in which the pure strategies are in the absorbing phase but the combined strategy is in the active one, as depicted in figure ???. The important thing to notice here is that these strong effect is due to the fact that the critical point *moves*. This goes against the analysis done by Vezquez and our results and we will discuss this in the concluding chapter of this work. By fixing all parameters and maximizing $G(\alpha)$ with respect to α it's possible to obtain the optimum value α^* . When $\bar{p} < p_0$ we have $\alpha^* = 0$, when $\bar{p} > p_0 + 2\sigma^2$ we have $\alpha^* = 1$. For values of \bar{p} in between those the optimum values is $\alpha^* = (\bar{p} - p_0)/2\sigma^2$. The conclusion we can take is that a larger variability in the environment, given by σ , widens the window of values in which is possible to have an active phase by means of a bet-hedging strategy.

6.2.1 A note on the simulations

When simulating this processes there are two possible ways to update the system: *in parallel* or *sequentially*. Sequential update is the one already described in section 3.1. Parallel update, on the contrary, means that *every* cell in the lattice is processed at each time step. In the two cases the time step is different. The time step in the sequential update is changed every time depending on the number of active sites in the system. The main difference between the two methods is that in parallel updating every cell is subjected to the same environmental conditions (when they depend *only* on time) while in sequential updating the conditions change since different cells are evaluated at different times. The second difference is that in parallel updating *every cell is processed* at each time step while the same holds for sequential updating just *on average*: it is possible that some cells may not be processed at all. We could then say that parallel and sequential updating correspond to different physical dynamics. This important facts can lead to very different outcomes and makes it hard to compare systems simulated with one method or the other. Although in the fixed-rate CP they produce very similar results, further investigation is required when temporal noise is added.

6.3 BET-HEDGING WITH COLORED NOISE

When we introduce bet-hedging in the colored noise scheme it still provides some advantages but not as strong ones as in Hidalgo's work. The Lagevin equation (141) is modified in this way

$$\dot{\rho} = \alpha[(\bar{\lambda} + \xi(t))\rho(1 - \rho) - \rho] + (1 - \alpha)[\lambda_0\rho(1 - \rho) - \rho] \quad (146)$$

which can be rewritten into

$$\dot{\rho} = \lambda^*(\alpha)\rho(1 - \rho) - \rho + \alpha\xi(t)\rho(1 - \rho) \quad (147)$$

where $\lambda^*(\alpha) = \alpha\lambda + (1 - \alpha)\lambda_0$. Once the equation is written in this way we can just take the results of the previous chapters making the substitutions $\lambda \rightarrow \lambda^*(\alpha)$ and $\sigma \rightarrow \alpha\sigma$. While the first substitution is obvious, the second derives from a basic property of Gaussian functions. In our analysis we are forced to exclude zero as a possible values for α since this would mean a zero variance noise and the change of variable needed by the UCNA method would not be possible. what we find applying the method showed in the previous chapters is that the critical point remains fixed to $\lambda^* = 1$ and that the “second” critical point is placed in $1 + \alpha^2\sigma^2\tau$. The coefficient of the MET scaling N^\varkappa turns into

$$\varkappa = \frac{\lambda^*(\alpha) - 1}{\alpha^2\sigma^2\tau} \quad (148)$$

Then in this case we can say that bet-hedging can have the positive effect that taken two strategies, a fixed strategy λ_0 sampled from the absorbing regime $\lambda^*(\alpha) \leq 1$ and a time-varying one with mean $\bar{\lambda}$ sampled in the sub-linear scaling regime $1 < \lambda^*(\alpha) \leq 1 + \alpha^2\sigma^2\tau$, a proper choice of α can get the bet-hedged one $\lambda^*(\alpha)$ to be in the super-linear scaling regime. Also, even if $\bar{\lambda}$ is already in the super-linear phase a bet-hedging strategy strongly increases the scaling exponent. To prove this we can maximize the exponent with respect to α to find the α^* that gives the best results. It turns out it's independent of both σ and τ

$$\alpha^* = 2 \frac{(\lambda_0 - 1)}{(\lambda_0 - \bar{\lambda})} \quad (149)$$

Now we first notice that for α^* to belong to $(0, 1]$ it must be $\lambda_0 < 1$. Otherwise α^* would be negative, meaning that the best bet is $\alpha = 0$. This obviously makes sense, meaning that when the fixed strategy λ_0 (which we underline represents a *fixed strategy in a evolving environment*) is already a winning one there is no need to adopt bet-hedging. Then, in the case $\lambda_0 < 1$ for α^* to be less then one, which means the best strategy is the pure noisy one, we must choose $\bar{\lambda} > 2 - \lambda_0$. Evaluating the exponent \varkappa at α^* we obtain

$$\varkappa_{\alpha^*} = \frac{(\bar{\lambda} - \lambda_0)^2}{4\sigma^2\tau(1 - \lambda_0)} \quad (150)$$

This equation tells us is that the greater the distance between $\bar{\lambda}$ and λ_0 and the nearer λ_0 to the critical point, the grater the exponent is. Now how to choose $\bar{\lambda}$ and λ_0 properly depends on several parameters since the super-linear threshold $1 + \alpha^2\sigma^2\tau$ moves also with τ and σ . When $\sigma^2\tau \ll 1$ the bet-hedging strategy does not really have any advantage since this represents the limit of zero variance noise and thus we recover the noiseless dynamics. The sub-linear phase shrinks so much

that any $\bar{\lambda}$ bigger just a bit more than the critical point would have a very large exponent and thus the pure noisy strategy would be already enough. The bet-hedging strategy would increase the exponent if λ_0 approaches one but this wouldn't make a difference in practice. As $\sigma^2\tau$ grows the sub-linear region widens and bet-hedging effects starts to be of practical value. Rising the fluctuations permits to lower λ_0 (allowing more and more losing strategies) although this must be compensated by a larger $\bar{\lambda}$. Parallel to this α^* decreases because it is inversely proportional to the distance between λ_0 and $\bar{\lambda}$, as we can see in figures 14. This leads to the paradox that for $\sigma^2\tau \gg 1$ some extreme behaviors are allowed, in which λ_0 gets smaller and smaller and also α^* , while $\bar{\lambda}$ grows larger and larger. These values represent the unrealistic situation in which a population with a really losing strategy is constantly saved by rare but very effective events in which the environmental conditions are so good that the growth in this periods compensates the losses. These cases are obviously extreme and do not represent a bet-hedging strategy in the sense we described.

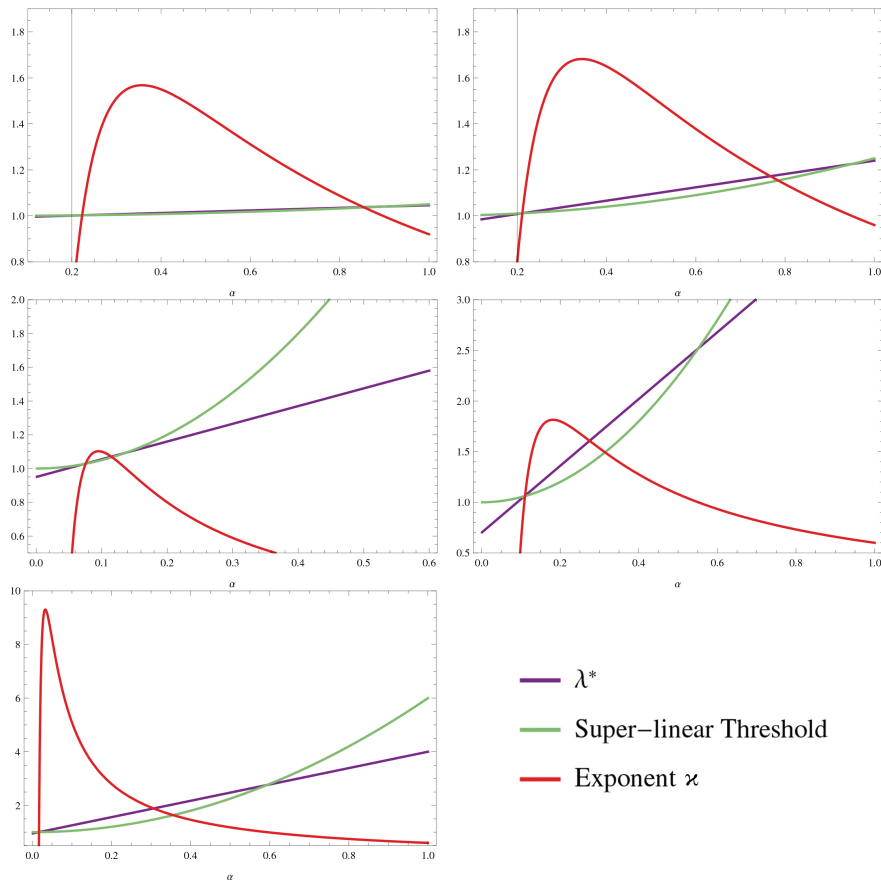


Figure 14: This images show the effects of colored-noise bet-hedging for different values of τ , $\bar{\lambda}$ and λ_0 with fixed $\sigma^2 = 0.05$. In the graph are depicted the values of $\lambda^*(\alpha)$, the super-linear threshold and the variation of the exponent ζ . In clockwise order we have $[\tau=1, \bar{\lambda}= 1.04, \lambda_0=0.99]$, $[\tau=5, \bar{\lambda}= 1.24, \lambda_0=0.95]$, $[\tau=100, \bar{\lambda}= 4, \lambda_0=0.95]$, $[\tau=100, \bar{\lambda}= 4, \lambda_0=0.7]$, $[\tau=100, \bar{\lambda}= 2, \lambda_0=0.95]$.

7 | CONCLUSIONS

This work was born with the idea that time-correlation in the noise would have added small or negligible effects to the already sound conclusions of Hidalgo and Muñoz work “*Stochasticity enhances the gaining of bet-hedging strategies in contact-process-like dynamics*” [17]. Under suggestion of Prof. Muñoz we decided to introduce them since they are an intrinsic property of every system in nature, being it a population of plants or animals, the spreading of an infection or a fire etc. We then studied a contact-process system with a birth rate subjected to a Ornstein-Uhlenbeck noise and a fixed unitary death rate. The choice of an OU noise was due to the fact that it represents the most general time-correlated noise and can account for many different environmental effects.

Carrying out this work not only we were able to obtain a series of results on how time-correlations modify the classic CP dynamics and to know how wrong we were in our assumption, but most importantly we gained a deeper understanding of temporal noise in general: on how to treat it analytically, on the details and problems of simulating it, on the nature of temporal Griffiths phases. We were able to gather in a unique and coherent framework a series of techniques and approaches which can be readily applied to any other model. On the analytical side we first studied the known effects of noise to confront our analysis with the previous literature. From this point of view, this work belongs to the general field of study of the effects of temporal noise in dynamical systems began with Leigh [25] and to which belong also the works of Velasquez et al. [31] and Kamenev et al. [21] that we directly confronted with ours in this Thesis.

To tackle the non-markovianity of our system we used a markovian approximation developed by Jung and Hänggi (UCNA) [20] which allowed us to use a large amount of traditional markovian techniques to characterize our system. We concentrated our efforts into two different but complementary directions of work: obtaining the long time steady probability distributions for the density and finding the scaling function of the mean extinction times in the limit of large populations. The results gave us a coherent picture in which the fixed-rate CP behavior is strongly modified: in the presence of environmental time-correlated noise the whole active phase behaves as the temporal Griffiths phases described in Velasquez’s work. Because of the exponent function the scaling is so slow in the subcritical region that this

lead us to question ourselves if we can still speak of a proper active phase when noise is present. Finally, we applied the results to the problem of bet-hedging with time-correlated noise which was at the origin of the work. Our conclusion differ slightly from the one for the white noise case. While a white-noise bet-hedged strategy can move a population from the absorbing to the active phase, in the case of colored-noise bet-hedging the effect is weaker: the colored-noise bet-hedged strategy can move the population from the sub-linear scaling region to the super-linear scaling region.

On the computational side we adopted exact simulation techniques, adapting to our goal the method developed by Anderson [3]. This gave us access to rapid confirmation of the analytical results having the full control of the simulation process. This knowledge proved to be of fundamental importance when needing to understand how the little details of the model concurred to the final outcome and confronting this results with the literature. The simulations corresponded with very good agreement to our analytical results both for the stationary probability distributions and the mean extinction times trend for large populations. This represents an additional confirmation of the goodness of the UCNA markovian approximation.

We conclude by underlining how works such as ours and several others addressing the problem of time-correlations in stochastic models are of great importance since they allow to overcome the white-noise limit and allow a better description of reality, in which such time-correlations are unavoidable and may play fundamental and unexpected roles.

A

ITÔ-STRATONOVICH DILEMMA

A.1 INTRODUCTION TO THE PROBLEM

In this appendix we outline the problem of interpretation of a Langevin equation, generally referred to as *Itô-Stratonovich dilemma*, following the great in-depth chapter from [22]. Let us consider the following general Langevin equation:

$$\dot{y} = A(y) + C(y)L(t) \quad (151)$$

with $L(t)$ a zero-mean, delta-correlated function of time. When $C(y)$ is a constant function of time $C(y) = \bar{C}$, each sample function $L(t)$ uniquely determines the process $y(t)$ for a given $y(0)$. $y(t)$ is a Markovian process, obeys a proper master equation and its associated Fokker-Planck equation reads

$$\dot{P}(y, t) = -\frac{\partial}{\partial y} A(y)P(y, t) + \frac{\Gamma \bar{C}^2}{2} \frac{\partial^2}{\partial y^2} P(y, t) \quad (152)$$

In this situation equation (151) can be readily rewritten in an integral form

$$\Delta y = y(t + \Delta t) - y(t) = \int_t^{t+\Delta t} A(y(s)) ds + \bar{C} \int_t^{t+\Delta t} L(s) ds \quad (153)$$

It is this form which clearly provides all the information related to the *physical process*. For a general non-constant function $C(y)$ in fact equation (151) as it stands has no well-defined meaning. In order to put it into an integral form we must additionally choose which value of $y(t)$ use to evaluate the function $C(t)$. The problem is due to the particular mathematical complications given by the nature of white noise: $L(t)$ is a *singular* stochastic function. It can be informally visualized as a sequence of consecutive delta peaks arriving at random times that modify the process $y(t)$. Which value of $y(t)$ should be inserted in function $C(y)$? We have an infinite set of interpretations depending on the value of the variable $\alpha \in [0, 1]$ parameterizing the interval $[t, t + \Delta t]$ such that

$$\begin{aligned} \Delta y(\alpha) = y(t + \Delta t) - y(t) = & \int_t^{t+\Delta t} A(y(s)) ds \\ & + C(\alpha y(t) + (1 - \alpha)y(t + \Delta t)) \int_t^{t+\Delta t} L(s) ds \quad (154) \end{aligned}$$

Although there are countless available choices, the problem is generally reduced to two main interpretations: the Itô interpretation $\alpha_{\text{Itô}} = 0$ and the Stratonovich interpretation $\alpha_{\text{Strat}} = 1/2$.

A.2 THE INTERPRETATIONS

A.2.1 Stratonovich interpretation

The integral form of the Stratonovich interpretation is

$$\begin{aligned} \Delta y(\alpha) = y(t + \Delta t) - y(t) = & \int_t^{t+\Delta t} A(y(s)) ds \\ & + C(\alpha y(t) + (1 - \alpha)y(t + \Delta t)) \int_t^{t+\Delta t} L(s) ds \quad (155) \end{aligned}$$

The Stratonovich interpretation can be related to the *physical* procedure of considering a stochastic differential equation in the form (151) with finite time-correlated noise $L(t)$ associated with a real system (which has *always* some kind of time correlation) and then *physically* reduce the correlation time τ : in the limit of $\tau \rightarrow 0$ the *Wong-Zakai theorem* states that the stochastic solution of such an equation becomes the solution of a Langevin equation of the same form interpreted as Stratonovich. The associated Fokker-Planck equation reads

$$\begin{aligned} \dot{P}(y, t) = -\frac{\partial}{\partial y} [A(y) + \frac{\Gamma}{2} C(y) C'(y)] P(y, t) \\ + \frac{\Gamma}{2} \frac{\partial^2}{\partial y^2} [C(y)]^2 P(y, t) \quad (156) \end{aligned}$$

A.2.2 Itô interpretation

The integral form of the Itô interpretation is

$$\begin{aligned} \Delta y(\alpha) = y(t + \Delta t) - y(t) = & \int_t^{t+\Delta t} A(y(s)) ds \\ & + C(\alpha y(t)) \int_t^{t+\Delta t} L(s) ds \quad (157) \end{aligned}$$

and the associated Fokker-Planck equation reads

$$\dot{P}(y, t) = -\frac{\partial}{\partial y} A(y) P(y, t) + \frac{\Gamma}{2} \frac{\partial^2}{\partial y^2} [C(y)]^2 P(y, t) \quad (158)$$

This interpretation brings in an unexpected complication: it makes the Langevin equation incompatible with the usual calculus rules.

A new calculus, called Itô's calculus, must be developed. We stress the fact that because of Wong-Zakai theorem the Itô interpretation cannot be obtained as a limit of the correlated case: it cannot be formulated unless τ is strictly zero.

A.3 WHICH INTERPRETATION SHOULD BE CHOSEN? INTERNAL AND EXTERNAL NOISE

There is no correct or incorrect interpretation of the Langevin equation, but each of them is more suited in certain cases than others. It is not just a matter of mathematical rigor: every interpretation is associated with a different *physical* system. One must consider first of all the nature of the noise studied and then decide which interpretation is the most appropriate one. Then it is possible to switch between different interpretations [9] in order to ease the solution of problem because the nature of the coefficients $A(y)$ and $C(y)$ is well-defined. We can divide noise in two classes: *external* noise and *internal* noise. A noise belongs to the external class when it represents a random force applied on an otherwise deterministic system, such as the influence of the environment in the growth of populations or a noise generator inserted in an electric circuit. In such a case, the coefficient $A(y)$ comes directly from the deterministic equation of the isolated system and the noise term can be added with or without the coupling $C(y)$. In this situations the Stratonovich interpretation is a better choice since the correlation time of this type of noise can be physically amplified or reduced externally. Also, it is legit to model this kind of systems setting $A(y)$ from macroscopic properties and then add the noise, as it is usually done by physicists. However as we have seen for the fixed-rate CP master equation there exist cases in which the noise is an intrinsic property of the system and cannot be "macroscopically tuned". The demographic noise is an exemple of such internal noise. The origin of internal noise is due to the fact that real systems are composed of *discrete individuals* (or particles, sites etc.) and it proceeds through jumps given by the concurring effects influencing the order parameter such as births and deaths in population dynamics. These random jumps are then translated into a random noise when switching to the continuous formulation given by the Langevin equation as we have seen in section 2.3.2. Internal noise *cannot be turned off* as long as the dynamics proceeds and this has the important consequence that the system cannot be "isolated" from internal noise. Therefore, it does not exist a proper deterministic $A(y)$ that can be derived from macroscopic considerations. Choosing a proper interpretation for internal noise is a delicate matter and this is particularly true if a system has an absorbing state. As we saw in the first chapters an absorbing state is a fluctuation-free one. Depending on the

type of coupling the system has with the noise, the interpretations that may allow the system to recover from the absorbing state must be excluded. The most evident case for which this happens is for the Langevin equation with demographic noise (29) in which we put $f(\rho) = \gamma\rho$ for sake of simplicity. In this equation the internal noise is coupled with the square root or the density and should then disappear at the absorbing state $\rho = 0$. If we interpret the equation as Stratonovich though, there is the possibility for the dynamic to escape the absorbing state due to the *anticipating nature* of this interpretation: if at time t the density reaches zero, in the Itô interpretation we have

$$\rho(t + \Delta t) = \gamma\rho(t) + \alpha\sqrt{\rho(t)} + \rho(t) = 0 \quad (159)$$

while in the Stratonovich we have

$$\rho(t + \Delta t) = \gamma\rho(t) + \alpha\sqrt{\frac{\rho(t) + \rho(t + \Delta t)}{2}} + \rho(t) \quad (160)$$

which can be different from zero. This can also be seen with the corresponding Fokker-Planck equations: the Itô equation gives the following Fokker-Planck

$$\dot{P}(\rho, t) = -\gamma\frac{\partial}{\partial\rho}\rho P(\rho, t) + \frac{\alpha^2\Gamma}{2}\frac{\partial}{\partial\rho^2}P(\rho, t) \quad (161)$$

and the Stratonovich interpretation gives

$$\dot{P}(\rho, t) = -\frac{\partial}{\partial\rho}(\gamma\rho - \frac{\alpha^2\Gamma}{2})P(\rho, t) + \frac{\alpha^2\Gamma}{2}\frac{\partial}{\partial\rho^2}\rho P(\rho, t) \quad (162)$$

Since the coefficient of the first derivative corresponds to $\langle\rho(t)\rangle$ [22] we can see that with the Stratonovich equation we obtain a wrong value meaning the process actually escapes from the absorbing state. These considerations are valid in general for every coupling that should *vanish* at the absorbing state.

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