UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Probing black hole microstates with holographic correlators

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Academic Year 2018/2019

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Introduction

Often referred to as the most beautiful physical theory ever invented, General Relativity is universally accepted as the most comprehensive description of classical gravitation. The amount of evidence supporting the theory collected in this last century is ever-growing, from the anomalous precession of Mercury's perihelion (1915) to the direct detection of gravitational waves (2017) and of the first image of a black hole (2019). As of today, the story of General Relativity is a success one, and it seems very determined at remaining so in the foreseeable future.

But, no matter how many experimental verifications we will collect, we know that General Relativity cannot be the last word, as the theory itself predicts its own doom. Spacetime singularities, that is points where divergences of gravitational fields occur, arise in solutions of General Relativity which are of fundamental importance for Astrophysics and Cosmology, such as the Big Bang singularity and the gravitational collapse of massive stars. This is nothing but a reflection of the classical nature of the theory. General Relativity is not expected to predict physics at scales below $\ell_P \sim 10^{-33}$ cm, where a more fundamental quantum-mechanical description should be necessary. The smallness of such scale is related to the weakness of gravitational interaction compared to the other fundamental forces, which is conjectured to be a cardinal principle of any candidate theory of gravity. On the other hand, it is still possible that quantum gravity effects can be detected indirectly at higher scales, for instance in the details of primordial gravitational waves predicted by modern inflationary models.

There is yet another sector of General Relativity where one might find hints about the quantum nature of gravity at higher scales, and that is Black Hole physics. Even though in this case direct observation of quantum effects seems to be very unlikely, there are a lot of strong theoretical reasons for their presence, with a unique source: the key fact that black holes admit a thermodynamic description and its consequent clash with classical results. In particular, the fact that black holes can be assigned an entropy, the Bekenstein-Hawking entropy S_{BH}, seems hard to reconcile with the no-hair theorem, and Hawking's claim that they emit thermal black-body radiation at temperature T_H due to pair creation at the horizon implies that black holes slowly lose mass and eventually disappear.
These unexpected features make a number of puzzling situations arise. For instance, it is not clear how one should interpret S_{BH}: does it come from degeneracy of microscopic configurations that are compatible with the macroscopic thermodynamic properties, like for any other statistical system, or is it perhaps of completely different origin? If the former is true, what are such microscopic configurations, and how could their presence be in agreement with the no-hair theorem? Additionally, S_{BH} is not a function of the volume of the black hole but

rather of its outer (horizon) area, which is untypical.
Another delicate question comes from the thermal nature of Hawking's radiation. Thermality implies that the emitted particles are in a maximally entangled state with the corresponding infalling antiparticles, hence the radiation cannot carry information. When the black hole fully evaporates, the details about the initial configuration that lead to the formation of the black hole via gravitational collapse seem then to be lost, and we are left with quanta entangled with nothing. Information loss and pure-to-mixed state evolution are irreconcilable with our knowledge of Quantum Mechanics. At this point we are faced with a deeper dilemma: should we admit that the semiclassical Quantum Field Theory approximation breaks down for some reason near the black hole horizon to preserve unitarity, or should we discard the statistical description despite the compelling evidence for Black Hole Thermodynamics?
Any candidate theory of Quantum Gravity must ultimately answer these questions; unfortunately, we do not know which theory is the correct one. Quantization of gravity is a difficult task, and radically different approaches are still pursued. The most promising option seems to be String Theory, which contains General Relativity - or better, its supersymmetric extension, Supergravity - in a natural way.

Within String Theory, one can form supersymmetric bound states of strings and branes, which are solitonic objects extended in extra dimensions, which in the effective lower-dimensional theory reduce to black holes. The Strominger-Vafa black hole, a supersymmetric five dimensional black hole on which we will focus in this thesis, is an example of such bound states. It is a solution of type IIB Supergravity in ten spacetime dimensions compactified on $S^1 \times T^4$, and it can be obtained from a brane configuration where n_1 D1-branes with n_p units of momentum are wrapped around S^1 and n_5 D5-branes are wrapped around the whole compact space $S^1 \times T^4$. The degeneracy of excitation modes of the D-branes bound state is then a natural explanation for black hole entropy of the corresponding Supergravity solution, and one can check explicitly that the two ways of counting states are equivalent. Still, it is unclear how the degeneracy should be reproduced at the gravitational level, and if it could help solving the information paradox.

In a number of simple cases the geometries related to different microscopic configurations of D-branes (or "microstates") have been explicitly computed. Such geometries show notable features, namely they form a class of smooth, horizonless geometries, that asymptote the corresponding black hole metric at infinity but display a far richer structure at the horizon scale and below. If this is to be true for any kind of black hole, as conjectured by the fuzzball proposal, then the question of the meaning of S_{BH} as well as the dispute about information loss are solved. For the former, the horizonlessness of the microstate geometries allows to picture a black hole as a coarse-grained sum over fuzzball configurations; whereas concerning the latter, since the microstate geometries are different from the "naïve" black hole and from each other already at the horizon scale, it might be possible that some information about microscopic configurations is carried away by emitted radiation quanta, so that unitarity is maintained.

Support for the fuzzball proposal and unitarity comes also from AdS/CFT duality. One of the most important novel ideas in Physics, AdS/CFT correspondence states that a string theory in $d+1$-dimensional Anti-de-Sitter space is dual to a d-dimensional Conformal Field Theory. Since the Strominger-Vafa black hole has a near horizon geometry which asymptotes to $AdS_3 \times S^3$, it allows for an equivalent description as a two-dimensional Conformal Field Theory, the so-called D1D5 CFT. Within this framework, supergravity fields are dual to a subset of supersymmetric CFT operators. We can distinguish two types of such operators in the large central charge limit. Heavy operators, that is operators whose conformal dimension is of order of the central charge, source a strong gravitational backreaction and are dual to a given microstate geometry in the bulk theory; whereas, operators whose conformal dimension is of order one are called light operators, and are dual to linear perturbations around the background geometry. Even though in the Supergravity limit the bulk description is linked to a strongly coupled CFT, in this thesis we will work at a specific point in moduli space, the orbifold point, where the D1D5 CFT becomes a free theory. We will consider a special class of operators, chiral primary operators, whose n-point functions on a subset of heavy states are protected by non-renormalization theorems and thus allow us to establish contact with the gravitational side of the correspondence. In particular, we focus on four-point correlation functions in the D1D5 CFT, involving two heavy operators acting as asymptotic states and two light chiral primary operators acting as probes: this class of Heavy-Heavy-Light-Light (HHLL) correlators is relevant for the analysis of a version the information loss problem for non-evaporating black holes.

A general technique to compute holographically the above kind of correlators has been developed in a recent series of papers [1–6]. In this thesis we will extend this work to the case where light operators are taken to be the global $SO(4)$ symmetry currents J^I that arise from the S^3 factor of the geometry. On the gravity side, the correlator can be obtained by studying the equations of motion of the dual Supergravity field: we thus identify these fields and work out the corresponding equations. For convenience, we select heavy operators that admit a particularly simple dual geometry, which can be locally reduced to $AdS_3 \times S^3$ via diffeomorphisms. This simplicity allows us to solve the Supergravity equations analytically. The main corollary of our analysis is that chiral current operators in the boundary theory are dual to topological gauge degrees of freedom in the AdS bulk. This is not a novel result in literature, but to our knowledge the explicit realization of this correspondence for the D1D5 CFT has not been carried out before. The check is concretely performed by computing the Heavy-Heavy-Current-Current (HHJJ) correlation functions in the CFT, as well as in the dual Supergravity description by making use of the dictionary dictated by AdS/CFT, and carefully comparing the two results.

Outline of the thesis. The work is organized as follows.
In Chapter 1 we will review in some detail the state of black holes in General Relativity and of the puzzles that arise from their statistical description. In order to address the issue within Quantum Gravity, in Chapter 2 we shall provide the salient features of String Theory and Supergravity. With these tools, in Chapter 3 we will move towards a string-theoretical description of black holes as bound states of branes and show how this picture is naturally completed by the fuzzball proposal. In Chapter 4 we will discuss the other important tool for our analysis, namely Ads/CFT correspondence, and we will apply it to construct in detail the D1D5 CFT dual to the Strominger-Vafa black hole. In Chapter 5 we will build carefully the holographic mapping between CFT states and operators and the bulk Supergravity fields that arise from fuzzball configurations; we will also provide a general method to compute HHLL correlation functions from the gravitational side and discuss their importance for addressing the information paradox.

In Chapter 6 we finally use all the previously gathered machinery to analyze the special case where light operators are taken to be chiral currents: this is the most original part of the thesis. Here we will identify the Supergravity fields dual to the $SO(4)$ R-currents of the D1D5 CFT and derive the corresponding equations of motion over a set of simple geometric backgrounds. By solving the equations and thus extracting the holographically computed HHJJ correlators, we will be able to compare them with the same correlators computed at the free orbifold point of the CFT.
We conclude with a summary of our work and a discussion of possible future directions.
A short introduction to two-dimensional Conformal Field Theories is given in Appendix A.

Chapter 1

Black Holes

Ever since Black Holes were discovered as solutions of General Relativity, they have been a fertile soil for further innovative ideas in Theoretical Physics. Apart from their intrinsic appeal and their phenomenological exploitation in Astrophysics, Black Holes are widely studied for their puzzling features. The fact that Black Holes admit a thermodynamic description, in particular the discovery of their entropy [7] and of Hawking radiation [8], challenges our very fundamental concepts of Quantum Physics and spacetime itself. Black Holes, with their strong gravitational field, are also the most natural places where to look for hints at a theory of Quantum Gravity.
In this Chapter we shall give a review of the theory of Black Holes in General Relativity and of their thermodynamic description, highlighting their most controversial aspects. We will follow for the most part [9].

1.1 Black holes in General Relativity

In General Relativity spacetime is treated as a legitimate dynamical entity. The field content of the theory is the spacetime metric $g_{\mu\nu}$, as well as matter and the other gauge fields residing in the spacetime itself. Gravity couples with every field that carries energy, including the metric itself. Together with the assumption of the strong equivalence principle, this selects a unique field theory, which action is given by

$$S = S_{EH} + S_M = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}R + \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_M, \tag{1.1}$$

where S_{EH} is the so-called Einstein-Hilbert action, and $\mathcal{L}_M$ is the Lagrangian of every other field except gravity; $R = R_{\mu\nu}{}^{\rho}{}_{\sigma}\delta^{\mu}_{\rho}g^{\nu\sigma}$ is the Ricci scalar curvature and G_N is Newton's constant. Varying the action with respect to the metric leads to the Einstein's equations of motion,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor curvature and the stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}\mathcal{L}_M\right). \tag{1.3}$$

Einstein's equation may be thought of as a set of second-order nonlinear differential equations for the metric $g_{\mu\nu}$. They are extremely complicated: the Ricci scalar and tensor involve derivatives and products of Christoffel symbols, which are given in terms of the inverse metric and derivatives of the metric; furthermore, the stress-energy tensor typically contains the metric as well. Consequently, it is not possible to solve Einstein's equations in full generality. It is instead possible to solve them if some simplifying assumptions are made, typically by requiring that the metric has a sufficient amount of symmetries.

1.1.1 The Schwarzschild solution

We turn our attention to the simple - yet important - case of spherical symmetry in the vacuum. A theorem by Birkhoff states that there exists a unique vacuum solution with spherical symmetry and that such solution is also static: this is nothing but the well-known Schwarzschild solution, where the line element is given by

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.4}$$

with $d\Omega^2$ denoting the metric on a unit two-sphere. M is usually interpreted as the mass of the gravitating object; this identification is not quite accurate, since we are solving Einstein's equation in the absence of matter: M is rather the energy contained in the metric itself.
One notices that the metric coefficients become infinite at $r = 0$ and $r = 2G_N M \equiv r_S$. Since the metric ultimately depends to the choice of coordinates, it might as well be that such singular behaviour is just an artifact of our choice of coordinates. The general rule of thumb is that one has to worry about singularities only when they appear as divergences of curvature invariants. For instance, the square of the Riemann tensor for the Schwarzschild metric is given by

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G_N^2 M^2}{r^6}, \tag{1.5}$$

which is finite for $r = r_S$, but clearly diverges for $r \to 0$. Hence $r = r_S$ does not seem to be a real singularity, and actually one can remove it with an appropriate change of coordinates. Our original coordinates are in fact adapted to an observer standing still at infinite radius, whereas to stand still at any finite r coordinate one needs to accelerate; such acceleration turns out to diverge for $r = r_S$.
This implies that to remove the coordinate singularity at $r = r_S$ we must give up our stubbornness in resisting the gravitational pull and let ourselves fall towards the $r = 0$ singularity. It is better to use a coordinate adapted to ingoing null geodesics in place of coordinate time,

$$v = t + r + r_S \log\left|\frac{r - r_S}{r_S}\right|, \tag{1.6}$$

and the line element now reads

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{1.7}$$

which gives an invertible metric, the Schwarzschild metric in ingoing Eddington-Finkelstein coordinates, for $r = r_S$.
Let us elaborate further on the physical properties of the $r = r_S$ locus. We notice that in the metric (1.4) for $0 < r < r_S$ the radial coordinate becomes timelike and the time coordinate becomes spacelike. This means that the $r = 0$ singularity is not a point in space, but rather some instant in time, making for an infalling observer impossible to avoid their miserable fate:

once they are set on an impact trajectory, they will ultimately crash against the singularity. Their only chance of salvation would have been to never have crossed the $r = r_S$ surface in the first place. Such a surface is said to be an event horizon. This also means that nothing from inside the event horizon could possibily ever come out of it, not even if it travelled at the speed of light: that is why this geometry is referred to as a black hole.
On the other hand, the infalling observer has no means of understanding that they are crossing the event horizon by local experiments. Therefore it would be useful to have a rigorous coordinate-independent way of defining the event horizon. This is achieved with the more general definition of Killing horizon.

1.1.2 Null hypersurfaces and Killing horizons

Let us consider the metric (1.4) once again: it is clearly independent of time, hence it is invariant under time translations. Another way of stating this is that the vector $\xi = \partial_t$ is a Killing vector. A Killing vector ξ is defined as a vector whose Lie derivative of the metric with respect to it is zero, or equivalently

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{1.8}$$

The notion of Killing vector allows us to give a more precise definition of the parameter M appearing in the Schwarzschild metric. To a Killing vector ξ^μ we can associate a Killing one-form by lowering the index with the metric. If we pick $\xi^\mu\,\partial_\mu = \partial_t$, then $\xi_\mu\,\mathrm{d}x^\mu = -(1 - r_s/r)\,\mathrm{d}t$, and it easy to show that M is the charge associated with the symmetry generated by the time translational Killing form,

$$M = \frac{1}{8\pi G_N} \int_{S^2} \star\,\mathrm{d}\xi, \tag{1.9}$$

where S^2 is the two-sphere at spatial infinity and $\star$ is the Hodge dual.

If we compute the square of $\xi = \partial_t$ in Eddington-Finkelstein coordinates, we see that ξ switches from being timelike for $r > r_S$ to being spacelike for $r < r_S$. At $r = r_S$ one has $\xi^2 = 0$, and hence we say that $r = r_S$ is a null hypersurface.
More generally, if ξ^μ is a Killing vector and it is normal to a null hypersurface $\mathcal{N}$, then $\mathcal{N}$ is said to be a Killing horizon. In the Schwarzschild case, the event horizon is the Killing horizon associated to the Killing vector that generates times translational invariance.

A null hypersurface has the interesting property that its normal vector, which is by construction everywhere lightlike, is also a tangent vector and hence it generates geodesic flow on the surface. This allows us to further define another quantity on a Killing horizon, the surface gravity,

$$\kappa = -\frac{1}{2}\,\nabla_\mu \xi_\nu\,\nabla^\mu \xi^\nu \Big|_{\mathcal{N}} \tag{1.10}$$

which is actually constant on the Killing horizon $\mathcal{N}$. The exact value of κ is fixed by the normalization of ξ. For instance, if we choose $\xi = \partial_t$ and require that $\xi^2 = -1$ at spatial infinity, we would get for the Schwarzschild solution

$$\kappa = \frac{1}{4G_N M}, \tag{1.11}$$

which is nothing but the Newtonian acceleration at $r = r_S$, i.e. the acceleration of an observer standing at the horizon as seen from infinity.
We will see later how surface gravity plays an important role in the thermodynamics of black holes.

1.1.3 The Reissner-Nordström solution

Let us now turn on the following matter action

$$S_M = S_{U(1)} = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.12}$$

that is we are allowing for the presence of some $U(1)$ gauge field A_μ.
We want to find the solution of Einstein's equation (1.2) with spherical symmetry in space. This can be expressed in terms of the charges of the theory,

$$Q = \frac{1}{4\pi}\int_{S^2} \star F, \qquad P = \frac{1}{4\pi}\int_{S^2} F, \tag{1.13}$$

where Q, P are the electric and magnetic charges respectively.
The Reissner-Nordström solution for a charged, spherically symmetric black hole is given by

$$ds^2 = -\Delta\,\mathrm{d}t^2 + \Delta^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \qquad \Delta = 1 - \frac{2G_N M}{r} + \frac{G_N(Q^2+P^2)}{r^2}. \tag{1.14}$$

For $r = 0$ the metric still has a true singularity, but the horizon structure is more complicated than in the Schwarzschild case. Horizons are located at zeroes of the function Δ, which are given by the r coordinates

$$r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - G_N(Q^2+P^2)}. \tag{1.15}$$

There are three possibilities, depending on the relative values of $G_N M^2$ and $Q^2 + P^2$.

- $G_N M^2 < Q^2 + P^2$. Then Δ never vanishes, and the metric displays no horizons. The $r = 0$ singularity is thus exposed, and it is called a naked singularity. This solution is generally considered to be unphysical, as it could produce a spacetime with closed time-like curves. The cosmic censorship conjecture states that it is impossible to dynamically generate such configuration.

- $G_N M^2 > Q^2 + P^2$. In this case Δ is positive both at small and large r, and negative for $r_- < r < r_+$. Those are both coordinate singularities; actually, they are also both null hypersurfaces and event horizons. The $r = 0$ singularity is a spacelike singularity, unlike the Schwarzschild case: the radial coordinate is timelike only for $r_- < r < r_+$.

- $G_N M^2 = Q^2 + P^2$. This case is referred to as the extremal Reissner-Nordström solution. There is only one event horizon, located at $r = G_N M$, but the radial coordinate is never timelike. The $r = 0$ singularity is once again spacelike.

1.1.4 The Kerr solution

The Kerr solution to Einstein's equations represents the metric generated by a rotating black hole. This is of particular phenomenological interest because rotating black holes are expected to be typical in nature. To find the exact solution is now much more difficult, because we have to give away spherical symmetry as well as staticity. The solution is instead only axially symmetric and stationary: it still has an asymptotically timelike Killing vector K_t, and it has only one spacelike Killing vector K_ϕ that generates rotational invariance. The resulting metric is the following:

$$\begin{aligned} ds^2 = &-\left(1-\frac{2G_N M r}{\rho^2}\right)\mathrm{d}t^2 - \frac{2G_N M a r\sin^2\theta}{\rho^2}(\mathrm{d}t\,\mathrm{d}\phi+\mathrm{d}\phi\,\mathrm{d}t)+ \\ &+\frac{\rho^2}{\Delta}\,\mathrm{d}r^2+\rho^2\,\mathrm{d}\theta^2+\frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2-a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2, \end{aligned} \tag{1.16}$$

where

$$\Delta = r^2 - 2G_N M r + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \qquad a = \frac{J}{M}. \tag{1.17}$$

J is the charge associated to the rotational Killing vector,

$$J = \frac{1}{4\pi}\int_{S^2} \star\,\mathrm{d}K_\phi, \tag{1.18}$$

and a is then interpreted as the angular momentum per unit mass.
This choice of coordinates makes it so that the event horizons occur at those values of r for which $\Delta = 0$. As in the Reissner-Nordström solution, there are three possibilities: $G_N M > a$, $G_N M = a$, $G_N M < a$. The last case features a naked singularity, and the extremal case turns out to be unstable.
In the $G_N M > a$ case there are once again two horizons, given by

$$r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - a^2}. \tag{1.19}$$

Notice, however, that none of these is the Killing horizon defined by the vanishing of the norm of the Killing vector K_t. In fact its norm vanishes at

$$\Delta = a^2\sin^2\theta \implies r_{1,2} = G_N M \pm \sqrt{G_N^2 M^2 - a^2\cos^2\theta}. \tag{1.20}$$

The solution with smaller radius $r = r_2$, however, is inside the inner event horizon, and we do not consider it. The other one, $r = r_1$, lies outside the outer event horizon, and it is called ergosurface. In the region between the outer horizon and the ergosurface, called the ergosphere, there cannot be any static observer, because K_t has become spacelike: they are forced to move, dragged by the rotation of the black hole.

The event horizons $r = r_\pm$ are still Killing horizons, but they are generated by the Killing vectors

$$\xi = K_t + \Omega^H_\pm K_\phi, \qquad \Omega^H_\pm = \frac{a}{r_\pm^2 + a^2}. \tag{1.21}$$

$\Omega^H_\pm$ is the angular velocity of the horizons.

1.1.5 Uniqueness theorem

As mentioned earlier, Birkhoff's theorem selects the Schwarzschild metric as the only static, spherically symmetric vacuum solution to Einstein's equations. This is similar to the situation in electromagnetism, where the only static spherically symmetric field configuration in empty

space is a Coulomb field. If we were to give up spherical symmetry we would expect, much like in electromagnetism, that the metric could be decomposed in multipole moments and that an infinite number of coefficients would have to be specified to describe the gravitational field exactly.
However, it turns out that this is not true for objects like black holes. Stationary black hole solutions are always described by a small amount of parameters. This is more precisely stated by the following

No-hair theorem. Stationary, asymptotically flat black hole soulutions to General Relativity coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters M (mass), Q, P (electric and magnetic charge), and J (angular momentum).

Stationary solutions are of special interest because they are the end states of gravitational collapse. However this no-hair theorem states that, irrespective of the original initial conditions before the collapse, we could end up with the same gravity solution. Information appears then to be lost in the process. In the classical theory this might not be much of a problem, because we could still think that the information is stored behind the event horizon. In the quantum theory this gets even worse, because black holes evaporate and eventually disappear. This leads to the so-called black hole information paradox. We will discuss further the paradox in Section 1.3, together with the entropy puzzle.

1.2 Black Hole Thermodynamics

From the no-hair theorem we know that black holes are described by a small number of parameters. This is reminiscent of what happens for thermodynamic systems: the behaviour of the system is described by macroscopic variables, irrespective from the microscopic details of the theory. As we will see in this Section, this is more than just a loose analogy.

1.2.1 The Penrose process

By definition, a black hole is a "region of no escape": no matter or light can ever be extracted from a black hole. Nevertheless, it is possible to extract energy from a black hole with an ergosphere such as the Kerr black hole. This procedure is known as the Penrose process.

As mentioned earlier, the Killing vector K_t is spacelike inside the ergosphere. Thus, for a test particle of momentum $p^\mu = m\dot{x}^\mu$, the energy

$$E = -K_t^\mu g_{\mu\nu} p^\nu \tag{1.22}$$

need not be positive in the ergosphere. Therefore, by making a black hole absorb a particle with negative total energy, it is possible to extract energy from a black hole.
Let us see this in more detail. Suppose that a particle of momentum p_0^μ and energy E_0 moves along geodesic motion (hence E_0 stays fixed) towards a Kerr black hole of mass M. After it crosses the ergosurface, we could make it so that the particle breaks up in two fragments: by local conservation of energy and momentum, we will have

$$p_0^\mu = p_1^\mu + p_2^\mu, \qquad E_0 = E_1 + E_2, \tag{1.23}$$

and since inside the ergosphere energies can be negative we could also arrange the process such that

$$E_1 < 0. \tag{1.24}$$

If the other fragment escapes from the ergosphere following geodesic motion, then it will have an energy $E_2 > E_0$. One can verify that the negative energy fragment always falls into the black hole. Thus, the energy $|E_1|$ has been extracted from the black hole, which mass is now given by $M - |E_1|$.
Of course, this process cannot go on indefinitely: the obvious limitation is that we need to have $M - |E_1| > 0$ to still have a black hole. But we also need it to keep a nonzero angular momentum J in order to have an ergosurface in the first place. The negative energy fragment also carries negative angular momentum L_1,

$$-\xi^\mu p_\mu = -(K_t^\mu p_\mu + \Omega_+^H K_\phi^\mu p_\mu) = E_1 - \Omega_+^H L_1 \geq 0 \implies L_1 \leq \frac{E_1}{\Omega_+^H} < 0. \tag{1.25}$$

Thus the angular momentum of the black hole changes by $\delta J = L_1$, and the mass changes by $\delta M = E_1$. The above restriction also reads

$$\delta J \leq \frac{\delta M}{\Omega_+^H}, \tag{1.26}$$

which can be rewritten in the form

$$\delta\left(M^2 + \sqrt{M^4 - \frac{J^2}{G_N^2}}\right) = \delta\left(\frac{r_+^2 + a^2}{2G_N^2}\right) \geq 0. \tag{1.27}$$

The physical meaning of this quantity can be understood computing the area of the outer even horizon:

$$A_H = \int \mathrm{d}\theta\, \mathrm{d}\phi\, \sqrt{|\gamma|}\Big|_{r=r_+} = 8\pi G_N^2 \left(M^2 + \sqrt{M^4 - \frac{J^2}{G_N^2}}\right), \tag{1.28}$$

where γ is the induced metric on the horizon. Comparing with (1.27), one reads

$$\delta A_H \geq 0. \tag{1.29}$$

1.2.2 Black hole temperature and Hawking radiation

In the following we will show how it is possible to define a concept of temperature for black holes. In order to do so it is natural to switch to Euclidean formalism. Define imaginary time by Wick rotation,

$$t = i\tau, \tag{1.30}$$

and by performing analytic continuation to real values of τ we can write the (Euclidean) Schwarzschild metric as [10]

$$ds_E^2 = \left(1 - \frac{2G_N M}{r}\right) d\tau^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.31}$$

Let us now expand the metric near the horizon $r = r_S$: defining the parameter x in such a way that

$$r - r_S = \frac{x^2}{8G_N M} = \frac{x^2}{2\kappa}, \tag{1.32}$$

the metric reads, at leading order in x,

$$ds_E^2 \approx (\kappa x)^2\,\mathrm{d}\tau^2 + \mathrm{d}x^2 + \frac{1}{4\kappa^2}\,\mathrm{d}\Omega^2, \tag{1.33}$$

where κ is the surface gravity. We recognize that it is of the form of a 2-dimensional Euclidean Rindler space times S^2. If we allow the coordinate τ to be periodic, namely

$$\tau \sim \tau + \frac{2\pi}{\kappa}, \tag{1.34}$$

then the Rindler factor is just the Euclidean plane in polar coordinates, and $x = 0$ is a coordinate singularity, as we want it to be.

On the other hand, the Euclidean partition function of a thermal quantum field theory is defined as

$$Z_E = \mathrm{Tr}\left(e^{-\beta H}\right) = \int D\phi\, e^{-S_E[\phi]}, \tag{1.35}$$

where the path integral is taken on field configurations that are periodic in Euclidean time, with period $\beta\hbar$. The temperature is defined by $\beta = (k_B T)^{-1}$ as usual.

Putting everything together, this suggest that we should associate to the Schwarzschild black hole of mass M the temperature

$$T_H = \frac{\hbar\kappa}{2\pi k_B} = \frac{\hbar}{8\pi k_B G_N M}, \tag{1.36}$$

which goes by the name of Hawking temperature.

There is yet another simple way of seeing that it makes sense to define a temperature for a black hole. In the following, let us work with natural $c = \hbar = k_B = 1$ units. Let us compute the horizon area of the Schwarzschild black hole (1.4): explicitly,

$$A_H = r_S \int_{S^2} \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi = 4\pi r_S^2 = 16\pi G_N^2 M^2. \tag{1.37}$$

Exploiting this, we can express the mass M of the black hole in terms of its area,

$$M^2 = \frac{A_H}{16\pi G_N^2}. \tag{1.38}$$

Now let us assume that somehow one manages to extract energy from a Schwarzschild black hole. Then the horizon area will also decrease, and the relation between the variations will be

$$\delta M = \frac{1}{32\pi G_N^2 M}\delta A_H = \frac{\kappa}{2\pi}\frac{\delta A_H}{4G_N}. \tag{1.39}$$

By staring at this identity one can recognize an analogy with the first law of thermodynamics, that is

$$\delta U = T\delta S. \tag{1.40}$$

Comparing with the definition of T_H given in (1.36), this means that it should make sense also to define an entropy for the black hole [7],

$$S_{BH} = \frac{A_H}{4G_N}. \tag{1.41}$$

S_{BH} goes by the name of Bekenstein-Hawking entropy.

This thermodynamic picture is made even more suggestive by the presence of another phenomenon. A black hole as we have described it so far behaves as a black body. If we have to assign a temperature to a black body, this means that it should emit radiation. This is indeed the case, as has been discovered by Hawking [8]. Roughly speaking, when applying quantum field theory on a black hole background, due to the absence of a global timelike Killing vector one obtains outgoing particle states starting from the ingoing vacuum, and the spectral distribution of outgoing particles is perfectly thermal, with temperature given by (1.36). The emitted particles are created in pairs with negative energy quanta, which fall into the black hole decreasing its mass, and effectively make the black hole evaporate.

1.2.3 The laws of black hole mechanics

During this Section we have been deriving in an intuitive way all the elements which allow us to write down the laws of black hole mechanics, in complete analogy with the laws of thermodynamics. A classical reference is [11]. A more modern derivation exploiting covariant phase space formalism of the first and second laws can be found in [12].

Zeroth law. The surface gravity κ of a stationary black hole is constant over the event horizon.

First law. For a stationary, axisymmetric black hole of mass M, angular momentum J and $U(1)$ charge Q,

$$G_N \,\mathrm{d}M = \frac{\kappa}{2\pi}\,\mathrm{d}\frac{A_H}{4} + \phi_H \,\mathrm{d}Q + \Omega_H \,\mathrm{d}J, \tag{1.42}$$

where ϕ_H is the value of the electrostatic potential at the horizon and Ω_H is the angular velocity of an observer standing at the horizon.

Second law. The horizon area of an asymptotically flat black hole is non-decreasing, i.e.

$$\delta A_H \geq 0. \tag{1.43}$$

Third law. It is not possible to bring the surface gravity of a black hole κ to zero with a finite sequence of transformations.

Let us comment briefly on these laws. The zeroth law can be proved to be valid in full generality under the assumption of the dominant energy condition, otherwise it holds only for static and axisymmetric black holes. Similarly, the first law holds for stationary black holes and the second law holds only under the assumption of the weak energy condition and the cosmic censorship conjecture.
The version that we have given here of the second law (1.43) is purely classical, because it is in contrast with the existence of Hawking radiation: since black holes evaporate, their area must be allowed to decrease. To solve this contradiction, Bekenstein [13] conjectured the

Generalized second law. The sum of the black hole entropy and the entropy in the black hole exterior never decreases, i.e.

$$\delta(S_{BH} + S_{out}) \geq 0. \tag{1.44}$$

1.3 Two puzzles about black holes

Even though on the one hand the arguments for a thermodynamic description of black holes are compelling, on the other hand if one takes it seriously then a number of puzzling points arises after further investigation. In this Section we shall address the two main problems within black hole physics, which give us the motivation to go beyond General Relativity: the entropy puzzle and the information paradox.

1.3.1 The entropy puzzle

The most remarkable feature that stems out from the analogy between black holes and thermodynamic system is for sure the existence of Bekenstein-Hawking entropy (1.41). If one takes the analogy further, they might ask themselves what is the meaning of such a quantity, i.e. where does it come from?
Let us recall what entropy is for statistical systems. In a microcanonical ensemble, and assuming equilibrium, the (microscopic) entropy is approximately given by

$$S_{micro} \sim \log \mathcal{N}, \tag{1.45}$$

where $\mathcal{N}$ is the number of microscopic configurations compatible with the macroscopic thermodynamic properties. On the other hand, in classical General Relativity we know the value of $\mathcal{N}$: the no-hair theorem asks for $\mathcal{N} = 1$, and the microscopic entropy must vanish. It is clear then that if one wishes to reproduce the Bekenstein-Hawking formula from microscopic arguments they must go quantum.

It is important to stress that the Bekenstein-Hawking formula prescribes the entropy to be proportional to the black hole area, rather than to its volume. This is in contrast with the usual intuition that the entropy is an extensive quantity with respect to the volume of the system. This striking feature lead to a formulation of the holographic principle for black hole physics, and for gravitational theories more in general: the relevant degrees of freedom for the thermodynamic description of the black hole should live on a lower dimensional theory. We will return to this point in Section 4.1.

1.3.2 The information paradox

A deeper problem with the semiclassical description of black holes is the so-called information paradox. First described by Hawking [14], in its most common formulation it goes as follows.

Suppose that one starts with some initial pure state $|\phi\rangle_{in}$ that collapses and forms a black hole. Black holes emit Hawking radiation, and therefore lose mass. Hawking radiation is completely thermal: its spectrum depends only on the temperature of the black hole, as well as possibly to its charges. The radiation quanta are emitted in pairs, with the emitted quantum being in a (maximally) entangled state with its partner which falls into the hole. From the point of view of an outside observer, that has no access to physics inside the horizon, the outgoing quanta are effectively in a mixed state, but this is just an artifact due to having integrated over the phase space of the infalling quanta, and the full system is still in a pure state. However, when the process comes to an end the black hole has completely evaporated, together with the infallen quanta, and we are left with radiation which is entangled with nothing. We have thus created a mixed state starting from a pure state: we have broken unitarity of quantum evolution, and this comes with dire consequences.

So far there seems to be no reason to call this a paradox rather than just an argument for information loss, and it might as well be circumvented by stating that evaporation is halted before reaching completion, or that small quantum corrections to Hawking computations could solve the problem. Actually this is not the case: there is a sharper formulation of the problem that makes the paradox evident, and that does not need the black hole to fully evaporate. A nice exposition has been given by Page [15], and further reviewed in [16–18].
Let us still suppose that the original state before collapse $|\phi\rangle_{in}$ is pure. Let us also require the dynamics to be unitary, thus the system composed of the emitted radiation and the black hole is pure at any moment during the evolution. As mentioned, by thermality of Hawking radiation the emitted photons will be in a mixed state, and so for the total state to be pure each outgoing mode of radiation has to be entangled with a radiation mode inside the event horizon.
More concretely, if the system is in a pure state then the von Neumann entropy of the total outgoing radiation is always equal to the von Neumann entropy of the black hole. If we want a statistical description of black hole thermodynamics, we can define black hole states of given energy within a microcanonical ensemble on some Hilbert space $\mathcal{H}$. A property of the von Neumann entropy of a system is that it is bounded from above by $\log \dim \mathcal{H}$, that is the microcanonical entropy. As the system cools and the black holes evaporates losing energy, the Hilbert space shrinks. Thus the microcanonical entropy decreases roughly with the same rate as the emission rate of Hawking quanta: the same rate at which the von Neumann entropy of the outgoing radiation increases. The bound on the von Neumann entropy of the black hole is thus saturated at about halfway through the evaporation process, and after that moment the radiation cannot be exactly thermal: as a consequence late-time radiation will have to be entangled with early-time radiation, contradicting either Hawking computations or unitarity. This version of the information paradox is truly paradoxical; moreover, it does not need full evaporation, and small corrections to the von Neumann entropy cannot help [16].

Even more astonishingly, one can reformulate the information paradox also for black holes that do not evaporate at all. This is more easily done using holography. We will follow the discussion in [18–20].

In an AdS spacetime, we can build black holes that exist forever and thus never fully evaporate. It has been pointed out that the thermality of Hawking radiation implies that all the two point functions probing a large AdS black hole must decay exponentially at late times [19]. That is,

$$C(t) \equiv \langle \widehat{O}(t)\widehat{O}(0)\rangle_\rho \equiv \mathrm{Tr}(\rho \widehat{O}(t)\widehat{O}(0)) \sim e^{-c\beta t}, \tag{1.46}$$

where ρ is the thermal state of the radiation, $\widehat{O}$ is some operator on the boundary, and c is a dimensionless constant.

But if the thermodynamic interpretation of black holes is correct, then the full system must have a discrete energy spectrum, because the entropy of the black hole is finite. We would write the correlator as

$$C(t) \equiv \frac{1}{Z(\beta)} \sum_{i,j} e^{-\beta E_i} |\langle i|\widehat{O}|j\rangle|^2 e^{-it(E_i - E_j)}, \tag{1.47}$$

where $\widehat{H}|i\rangle = E_i|i\rangle$ is a complete set of eigenfunctions of the Hamiltonian. One has $Z(\beta) = \sum_i e^{-\beta E_i}\langle i|\widehat{O}|i\rangle \sim e^{S-\beta\langle E\rangle}$, where S is the entropy and $\langle E\rangle$ is the ensemble average of the energy. The long time average of the correlator is

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |C(t)|^2 dt \approx \frac{1}{Z^2(\beta)}\sum_{i,j} e^{-2\beta E_i}|\langle i|\widehat{O}|j\rangle|^2 \approx \frac{Z(2\beta)}{Z^2(\beta)} \sim e^{-S}. \tag{1.48}$$

This is nonvanishing. Of course occasionally the correlators will become large again at some time when the phases in the sum interfere constructively, or arbitrarily small when they interfere destructively. Nevertheless, the correlators should not drop exponentially, but they should stabilize at a value around e^{-S} after a long time.

In this formulation the paradox is even more evident. Both the arguments validating the statistical description of black holes and the ones validating the gravitational one are compelling. On the one hand, as we will see, the statistical description provides a reproduction of the entropy formula in low-energy Supergravity, as well as the recovery of Black Hole Thermodynamics from AdS/CFT duality. On the other hand, there is no a priori reason why the semiclassical Quantum Field Theory description should break down in the description of an evaporating black hole.

Arguments from AdS/CFT tell us that the thermodynamic description, as well as unitarity, should be preserved and our understanding of Quantum Field Theory has to be modified. How this should happen, however, is still not clear, and must be addressed by any candidate theory of quantum gravity.
So far, the most promising option seems to be String Theory, whose low-energy effective theory reproduces General Relativity. In the following Chapter we shall review its prominent features.

Chapter 2

Strings and Supergravity

2.1 String Theory in a nutshell

2.1.1 Bosonic strings

String theory, as the name suggests, is a theory of strings, i.e. extended one-dimensional objects. We are used to theories of particles, which are pointlike. The action for a point particle moving in D-dimensional Minkowski spacetime is

$$S_{\text{particle}} = -m \int \mathrm{d}s = -m \int \mathrm{d}\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}, \quad \mu, \nu = 0, \ldots, D-1, \tag{2.1}$$

where s is proper time, and τ is any timelike parameter parametrizing the worldline X^μ. Notice that the action is proportional to the proper length of the worldline. However this action is not well defined in the massless limit, and the presence of a square root makes it difficult to define a saddle point expansion of the action upon quantization. To overcome these difficulties we can introduce an auxiliary field, the einbein $e(\tau) = \sqrt{-g_{\tau\tau}(\tau)}$, where $g_{\tau\tau}(\tau)$ is the induced metric on the worldline. An action equivalent to (2.1) is given by

$$S'_{\text{particle}} = \frac{1}{2} \int \mathrm{d}\tau \left(e^{-1}\dot{X}^2 - em^2\right), \tag{2.2}$$

where we have introduced the shorthand notation $\dot{X}^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}$. This action is still reparametrization invariant, is quadratic in X^μ (which is convenient for path integral quantization), has a well defined massless limit, and has the same equations of motion as (2.1) for the fields X^μ, whereas the equation of motion for e is purely algebraic: integrating e away, one obtains S_{particle}.

For strings we can write similar expressions. Now $X^\mu = X^\mu(\tau, \sigma)$ is not representing a worldline, but rather a world*sheet*. τ is still taken to be a timelike parameter, whereas σ is a spacelike parameter. We can have either open strings or closed strings: conventionally, we let $\sigma \in [0, \pi]$ for open strings and $\sigma \in [0, 2\pi[$ for closed strings. Moreover, for closed strings we require periodicity in σ: $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$.
The straightforward generalization of (2.1) is to take an action which is proportional to the proper area of the worldsheet, i.e.

$$S_{NG} = -T \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\det \gamma_{\alpha\beta}}. \tag{2.3}$$

This goes by the name of Nambu-Goto action. Here,

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \tag{2.4}$$

is the induced metric on the worldsheet, with the convention $\sigma^0 = \tau, \sigma^1 = \sigma$ (in the following, we will write just $\sigma \equiv (\tau, \sigma)$, and its meaning will be clear from the context). T is the tension of the string, that is its potential energy per unit length. It is customary to define a new parameter α', the Regge slope, as

$$T = \frac{1}{2\pi\alpha'}. \tag{2.5}$$

The action (2.3) is manifestly reparametrization invariant on the worldsheet as well as spacetime Poincarè invariant.
Still, the action (2.3) is not always a convenient choice, and one can try to obtain something similar to (2.2). Introducing as auxiliary field the worldsheet metric $g_{\alpha\beta}$, we can write the equivalent action

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{2.6}$$

which is called Polyakov action. The Polyakov action still features worldsheet reparametrization invariance and spacetime Poincarè invariance, but it has an additional symmetry: Weyl invariance. A Weyl transformation is defined as a rescaling of the (worldsheet) metric,

$$g_{\alpha\beta}(\sigma) \mapsto \Omega^2(\sigma)g_{\alpha\beta}(\sigma), \tag{2.7}$$

and it is a consequence of the fact that the Polyakov action is an action for a theory on two dimensions.
The equations of motion for the X^μ fields and for the worldsheet metric read

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\,\partial_\beta X^\mu) = 0, \tag{2.8}$$

$$T_{\alpha\beta} \equiv -\frac{1}{\alpha'}\left(\partial_\alpha X^\mu\,\partial_\beta X_\mu - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\,\partial_\gamma X^\mu\,\partial_\delta X_\mu\right) = 0. \tag{2.9}$$

Notice that $T_{\alpha\beta}$ is automatically symmetric, $T_{\alpha\beta} = T_{\beta\alpha}$, and traceless, $T^\alpha{}_\alpha = 0$.

Closed strings. Typically, one exploits both Weyl invariance and reparametrization invariance to fix the metric to be the flat two-dimensional Minkowski metric: $g_{\alpha\beta} = \eta_{\alpha\beta}$. With this gauge choice and with the coordinates $\sigma^\pm = \tau \pm \sigma$, the equations of motion for X^μ read

$$\partial_+\,\partial_- X^\mu = 0, \tag{2.10}$$

and the general solution is

$$X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-), \tag{2.11}$$

with

$$\begin{aligned} X^\mu_L\left(\sigma^+\right) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \\ X^\mu_R\left(\sigma^-\right) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}. \end{aligned} \tag{2.12}$$

It is usual to define also the zero modes

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{2.13}$$

However, one still has to impose $T_{\alpha\beta} = 0$, which in these coordinates read $T_{\pm\pm} = 0$. Using the mode expansion, we can also express them as

$$L_n \equiv \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{n-m}^\mu \alpha_m^\nu \eta_{\mu\nu} = 0, \quad \tilde{L}_n \equiv \frac{1}{2} \sum_{m \in \mathbb{Z}} \tilde{\alpha}_{n-m}^\mu \tilde{\alpha}_m^\nu \eta_{\mu\nu} = 0. \tag{2.14}$$

$L_n, \tilde{L}_n$ are called Virasoro generators. The constraints arising from $L_0 = 0$ and $\tilde{L}_0 = 0$ are not independent, because $\alpha_0^\mu = \tilde{\alpha}_0^\mu \sim p^\mu$. We can thus read off the mass squared of the string solution,

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_{-n}^\mu \alpha_n^\nu \eta_{\mu\nu} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu \eta_{\mu\nu}. \tag{2.15}$$

This goes by the name of level matching condition.
So far our discussion has been purely classical. To quantize the theory, one has to promote the $\alpha, \tilde{\alpha}$ modes to operators and impose the correct commutation relations. This can be done in different ways, and its detailed exposition is marginal to the scope of the thesis. For an exhaustive treatment see [21]. The main result is that we can build a Fock space, starting from vacuum states $|0; p\rangle$, which satisfy

$$\alpha_n^\mu |0; p\rangle = \tilde{\alpha}_n^\mu |0; p\rangle = 0 \; \forall n > 0, \quad \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu |0; p\rangle = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0^\mu |0; p\rangle = p^\mu |0; p\rangle. \tag{2.16}$$

The level matching condition gets shifted, and becomes

$$M^2 = \frac{4}{\alpha'} \left(-1 + \sum_{n>0} \alpha_{-n}^\mu \alpha_n^\nu \eta_{\mu\nu} \right) = \frac{4}{\alpha'} \left(-1 + \sum_{n>0} \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu \eta_{\mu\nu} \right). \tag{2.17}$$

Moreover, and most surprisingly, consistency of the theory will require the number of spacetime dimensions to be fixed to

$$D = 26. \tag{2.18}$$

We will be interested in the low-energy spectrum of String Theory. Thus we look for the lowest possible values of M^2. If no mode excitations are present, i.e. we consider a vacuum state $|0, p\rangle$, we have a negative mass squared state. This is a tachyon, and it signals an instability of the vacuum. We can however discard the tachyon in Superstring theories, as we will see later.
The next allowed value of M^2 is zero: we obtain thus massless spacetime fields. The states are of the form

$$\xi_{ij} \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0; p\rangle, \quad i, j = 1, \ldots, 24. \tag{2.19}$$

In $D = 26$ massless fields should live in an irreducible representation of $SO(24)$, their little group. The fields ξ_{ij} live in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$, which is however not irreducible. Decomposing into irreducibles we obtain, schematically,

$$\mathbf{24} \otimes \mathbf{24} = \begin{pmatrix} \text{traceless} \\ \text{symmetric} \end{pmatrix} \oplus \begin{pmatrix} \text{anti-} \\ \text{symmetric} \end{pmatrix} \oplus (\text{trace}). \tag{2.20}$$

The traceless symmetric part is interpreted as the spacetime metric $G_{\mu\nu}(X)$; the antisymmetric part is a 2-form field, called Kalb-Ramond field $B_{\mu\nu}(X)$; the trace part is a scalar field $\Phi(X)$ called dilaton.
States with positive mass squared have masses of the order of the Planck mass: they are thus too heavy to be observed at the energy scales we are interested into.

Open strings. For open strings, $\sigma \in [0, \pi]$, the discussion is slightly different. Their dynamics is still described by the Polyakov action (2.6), but to the equations of motion one has to add appropriate boundary conditions. In flat metric gauge, the variation of the action under δX^μ reads

$$\delta S_P = -T \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\, \partial^\alpha X^\mu\, \partial_\alpha \delta X_\mu = T \int \mathrm{d}^2\sigma\, \partial^\alpha\, \partial_\alpha X^\mu \delta X_\mu + T \int_{\tau_i}^{\tau_f} \mathrm{d}\tau\, \left[X'^\mu \delta X_\mu\right]_{\sigma=0}^{\sigma=\pi}, \tag{2.21}$$

where we assumed that $\delta X^\mu(\tau = \tau_i) = \delta X^\mu(\tau = \tau_f) = 0$, and we have defined $X'^\mu \equiv \partial_\sigma X^\mu$. For (2.21) to vanish we need thus $X'^\mu \delta X_\mu = 0$ at the endpoints $\sigma = 0, \pi$. This leads to

- Neumann boundary condition: $\partial_\sigma X^\mu = 0$ at the endpoint,
- Dirichlet boundary condition: $\delta X^\mu = 0$ (equivalently, $\partial_\tau X^\mu = 0$) at the endpoint.

To grasp the physical meaning of these boundary conditions, let us consider an open string such that

$$\begin{cases} \partial_\sigma X^I(\sigma = 0, \pi) = 0, & I = 0, \dots, p \\ X^a(\sigma = 0, \pi) = c^a, & a = p+1, \dots, D-1. \end{cases} \tag{2.22}$$

The string endpoints lie in a $(p+1)$-dimensional hypersurface, where they can move freely. These hypersurfaces are called Dp-branes. It turns out that Dp-branes are dynamical objects (they are solitonic solutions of the equations of motion), and they obey an effective action which is an higher dimensional generalization of Nambu-Goto action, the Dirac action,

$$S_{Dp} = -T_p \int \mathrm{d}^{p+1}\xi \sqrt{-\det \gamma}, \tag{2.23}$$

where ξ^I, $I = 0, \dots, p$ are worldvolume coordinates and γ_{IJ} is the induced metric on the worldvolume. However, in string perturbation theory, the branes are very heavy objects and thus can be treated as static.

The equations of motion for open strings are solved in a similar fashion to the closed string case, except from the fact that rather than periodicity in σ we need to impose the correct boundary conditions. Irrespective of the choice of boundary conditions, it turns out that only one set of modes α_n^μ survives. We can still read off the mass squared of states from the level matching condition, but now we must add a term coming from the tension of the open string if it stretches between two different branes.

Upon quantization, for open strings attached to a single Dp-brane we have, at the massless level, the states

$$\xi_I \alpha_{-1}^I |0; p\rangle, \quad \xi_a \alpha_{-1}^a |0; p\rangle, \quad I = 1, \dots p-1,\, a = p+1, \dots D-1. \tag{2.24}$$

They represent, respectively, a massless brane vector A_I and $D-p-1$ Goldstone bosons ϕ_a, related to the spontaneous breaking of spacetime translational invariance by the Dp-brane.

Adding more branes makes the spectrum more interesting, and one is also able to describe gauge theories on branes.
Let us first consider two parallel branes and open strings with endpoints on each brane. The physical states at level one are still of the form

$$\xi_I \alpha^I_{-1}|0;p;1,2\rangle, \quad \xi_a \alpha^a_{-1}|0;p;1,2\rangle, \quad I=1,\dots p-1,\ a=p+1,\dots D-1, \tag{2.25}$$

where $(1,2)$ denotes that the oriented string stretches from the first brane to the second, but now their masses are given by

$$M^2 = \frac{(X_2-X_1)^2}{(2\pi\alpha')^2}, \tag{2.26}$$

where $X_{1,2}$ denote the coordinates of the two branes. Notice that M^2 vanishes in the limit where the two branes collide.
The states appearing in (2.25) represent, respectively, $p-1$ components of a massive bulk vector $V_{I(1,2)}$ and $D-p-1$ massive brane scalars $s_{a(1,2)}$. The missing p-th component of the bulk vector is given by the combination of brane scalars

$$\sum_a (X_2^a - X_1^a)\alpha^a_{-1}|0;p;1,2\rangle. \tag{2.27}$$

Putting everything together, this gives a massive vector and $D-p-2$ massive scalars.

Looking at the full system of two parallel branes, we can have all possible combinations of open strings ending on different branes, even with different orientations. Concerning vectors, in total we have two massless vectors (one from each brane) and two massive vectors (one for each orientation of the open string connecting the branes). If we take the limit of colliding branes, we end up with four massless vectors and $4(D-p-1)$ massless scalars.
It turns out that the vectors interact with each other, and behave like gauge fields. When the branes coincide, the system describes a $U(2)$ Yang-Mills theory; when the branes are apart, the gauge symmetry breaks down to $U(1)\times U(1)$, because we are giving an expectation value to some of the scalars.
More generally, for N coincident D-branes, we can describe $U(N)$ gauge theories, and also gauge theories related to proper subgroups such as $SO(N)$, $Sp(N)$. For more details, see for instance [22].

2.1.2 String interactions and effective actions

Just as it is usually done for point particles, one would like to calculate amplitudes for processes like

$$\{\text{initial strings state}\} \longrightarrow \{\text{final strings state}\}. \tag{2.28}$$

This is more easily thought of in the path integral formalism. The amplitude will be given by the sum over all worldsheets connecting the initial state and the final state, weighted by the exponential of the action.

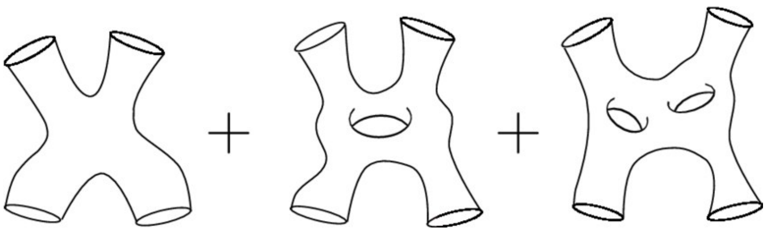

Figure 2.1: First terms contributing to the scattering amplitude of two closed strings.

Notice that there is no definite interaction location, in contrast to the point particle case. Locally, interaction worldsheets look like free ones, and only the global properties of the worldsheets capture interactions.
The general formula for the S-matrix of any n closed strings in Euclidean time is, schematically,

$$S_{j_1,\ldots,j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\\text{topologies}}} \int \frac{DXDg}{V_{diff}V_{Weyl}} e^{-S} \prod_{i=1}^{n} \int \mathrm{d}^2\sigma_i \sqrt{g(\sigma_i)} V_{j_i}(k_i,\sigma_i). \tag{2.29}$$

This is a highly nontrivial statement, thus some explanation is mandatory. k_i^μ is the D-dimensional momentum of the ith string, the integration is over the possible field configurations for the X^μ fields and the worldsheet metric $g_{\alpha\beta}$. The division by $V_{diff}V_{Weyl}$, the volume of the local symmetry group, is a schematical way to avoid overcounting gauge equivalent configurations. $V_{j_i}(k_i,\sigma)$ are called vertex operators, and they represent the insertion of a free string worldsheet at the point σ_i. The sum is over compact topologies, because the interaction takes place at finite coordinates in spacetime. The action S appearing in the path integral is not just the Polyakov action: rather, it is given by

$$S = S_P + \lambda\chi, \tag{2.30}$$

where $\lambda = \langle\Phi\rangle$ is the dilaton expectation value and χ is the Euler characteristics of the worldsheet. For an oriented, closed surface of genus g and with b boundaries,

$$\chi = 2 - 2g - b. \tag{2.31}$$

The sum over compact topologies can thus be seen as a perturbative expansion in powers of $g_s \equiv e^{\langle\Phi\rangle}$. If $g = b = 0$, like for the sphere S^2, we say that the interaction is at tree level.

For our purposes, however, we are mainly interested in the effective dynamics for massless fields. It is possible to obtain an ansatz for the spacetime action by looking at scattering amplitudes of strings. The right effective action for the metric $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton Φ turns out to be, at tree level,

$$S_{26} = \frac{1}{2k_0^2} \int \mathrm{d}^{26}X\sqrt{-G}e^{-2\Phi}\left[R^{(26)} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\,\partial_\mu\Phi\,\partial^\mu\Phi + O(\alpha')\right], \tag{2.32}$$

where $H = \mathrm{d}B$.

The action (2.32) might look somewhat unusual. The coefficient of the 26-dimensional Einstein-Hilbert term is field dependent, and the kinetic term of the dilaton appears with the "wrong" sign. This is just an arifact of our choice of fields: we are describing fields and the action in the string frame. We could define a new metric, $\tilde{G}_{\mu\nu} = e^{-\frac{4}{D-2}(\Phi-\langle\Phi\rangle)}G_{\mu\nu} \equiv e^{-\tilde{\Phi}/6}G_{\mu\nu}$, such that the action will read

$$S_{26} = \frac{1}{2k^2}\int \mathrm{d}^{26}X \left[\tilde{R} - \frac{1}{12}e^{-\tilde{\Phi}/3}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\,\partial_\mu\tilde{\Phi}\,\partial^\mu\tilde{\Phi} + O(\alpha')\right], \tag{2.33}$$

where $k = k_0 e^{\langle\Phi\rangle}$. This defines the so-called Einstein frame, which is the correct frame where to formulate questions about the spacetime geometry.

2.1.3 Supersymmetry

Bosonic string theory has two main limitations. First of all, it contains a tachyon, which makes the theory unstable if left untreated. Even if one were able to somehow remove the tachyon, still the theory would not contain spacetime fermions, which we know to be the building blocks of matter. A solution to both these problems is to include Supersymmetry in String Theory.

Broadly speaking, supersymmetry is a symmetry (or a set of symmetries) that relates bosonic and fermionic states in a quantum theory. Our exposition will be very sketchy, and we refer to [23] for a detailed exposition.

Let us work with $\mathcal{N}$ supersymmetries: they are defined in terms of fermionic generators Q^I_A, the supercharges, where $I = 1, \ldots, \mathcal{N}$ and $A, B, \ldots$ denote spinor indices. Schematically, they act on bosonic and fermionic states as

$$\delta_\epsilon B = \bar{\epsilon}F, \quad \delta_\epsilon F = \epsilon\gamma^\mu\partial_\mu B, \tag{2.34}$$

respectively, where ϵ_I is the parameter of the infinitesimal transformation (we have suppressed the copy index I in the above), and γ^μ are gamma-matrices obeying the usual Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Notice that ϵ must be a Grassmann number by consistency. The supersymmetry transformations have nontrivial commutation relations, e.g.

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,B \sim (\bar{\epsilon}_1\gamma^\mu\epsilon_2)\,\partial_\mu B. \tag{2.35}$$

Since $P_\mu = -i\,\partial_\mu$, we see that composing two supersymmetry transformations we obtain a spacetime translation. We can thus expand the Poincarè algebra to include the supercharges and their (anti-)commutation relations. The detailed structure of the extension depends on the number of spacetime dimensions and of the supersymmetries, but we can write down, schematically

$$[P, Q^I] = 0, \quad [M, Q^I] \sim Q^I, \quad \{Q^I, Q^J\} \sim Z^{IJ}, \quad \{Q^I, \bar{Q}^J\} \sim \delta^{IJ}P. \tag{2.36}$$

Here P, M denote the generators of translations and Lorentz transformations respectively, whereas Z^{IJ} are operators that commute with all the generators, called central charges. Notice that the central charges can be present only if the number of supersymmetries $\mathcal{N}$ is greater than one.

A naïve, but suggesting way to picture supersymmetry transformations is to add to the spacetime coordinates x^μ the "fermionic directions" θ_A, where A is still the spinor index cited above. This is called superspace formalism, and we can think of supersymmetry and Poincarè transformations as being spacetime "Poincarè" transformations in this extended space. Consider now a function of the superspace coordinates $\Phi(x,\theta)$: the fact that the θ coordinates are fermionic allows us to perform a Taylor expansion around $\theta = 0$ that terminates:

$$\Phi(x,\theta) \sim \phi(x) + \theta\psi(x) + \ldots + \theta\cdots\theta f(x). \tag{2.37}$$

The coefficients in front of the various combinations of θ are fields with different spin: for instance, $\phi(x)$ is a bosonic field and $\psi(x)$ is a fermionic field. Upon a supersymmetry transformation, $\phi(x)$ will transform into $\psi(x)$. This is very schematic, but shows how bosonic fields get partnered to fermionic fields.

2.1.4 Superstrings

There are various ways of putting String Theory and Supersymmetry together: we will follow the traditional approach of implementing supersymmetry on the worldsheet. We refer to [24] for further details.

With Supersymmetry, there comes the supersymmetric partners of the fields we have encountered in String Theory: the spacetime fields X^μ will be partnered to the spacetime spinors ψ^μ_A, whereas the worldsheet metric $g_{\alpha\beta}$ will be partnered to the gravitino $\chi_{\alpha A}$. Here $A, B, \ldots$ are two-dimensional spinor indices.
The Polyakov action (2.6) can be generalized to Superstring theory including the fermions.

$$S = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-g}\left[g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + \frac{i}{2}\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu + \frac{i}{2}\left(\chi_\alpha\rho^\beta\rho^\alpha\psi^\mu\right)\left(\partial_\beta X_\mu - \frac{i}{4}\chi_\beta\psi_\mu\right)\right], \tag{2.38}$$

where ρ^α are the two-dimensional Dirac matrices and we have suppressed spinor indices. As additional local symmetries, it features supersymmetry and "super-Weyl" invariance, the analogous of Weyl invariance for the gravitino. It is convenient to work in the so-called superconformal gauge,

$$g_{\alpha\beta} = \eta_{\alpha\beta}, \qquad \chi_\alpha = 0. \tag{2.39}$$

With this gauge choice, the generalized Polyakov action reads

$$S = -\frac{1}{4\pi\alpha}\int \mathrm{d}^2\sigma\,\partial_\alpha X^\mu\,\partial^\alpha X_\mu - \frac{i}{2\pi\alpha'}\int \mathrm{d}^2\sigma\psi^\mu\rho^\alpha\,\partial_\alpha\psi_\mu, \tag{2.40}$$

Using light-cone coordinates, the equations of motion will read

$$\partial_+\partial_- X^\mu = 0, \qquad \partial_+\psi^\mu_- = 0 = \partial_-\psi^\mu_+, \tag{2.41}$$

where we have split the two dimensional spinors ψ^μ into their chiral components $\psi^\mu_\pm$. For the bosonic fields the same discussion goes throught as in bosonic string theory, whereas for spinors

$$\psi^\mu_\pm = \psi^\mu_\pm(\sigma^\pm). \tag{2.42}$$

One still has to impose the correct boundary conditions and constraints. Upon variation of the action, we obtain now a piece involving fermions that does not vanish on-shell,

$$\delta S \sim \int \mathrm{d}\tau\left[\psi^\mu_+\delta\psi_{+\mu} - \psi^\mu_-\delta\psi_{-\mu}\right]_{\sigma=0}^{\sigma=\pi \text{ or } 2\pi}. \tag{2.43}$$

Let us examine the closed string case. Then the term at $\sigma = 0$ must cancel the term at $\sigma = 2\pi$. Thus, in full generality, $\psi^\mu_\pm$ can be either periodic or antiperiodic, i.e.

$$\begin{aligned} \psi^\mu_+(\tau, \sigma = 2\pi) &= \pm\psi^\mu_+(\tau, \sigma = 0), \\ \psi^\mu_-(\tau, \sigma = 2\pi) &= \pm\psi^\mu_-(\tau, \sigma = 0). \end{aligned} \tag{2.44}$$

In the periodic case we say that the fermions live in the Ramond (R) sector, whereas antiperiodic fermions live in the Neveu-Schwarz (NS) sector. Notice that we can pick different sectors for the two different chiralities.

The constraints arising from the action are

$$T_{\alpha\beta} = 0, \qquad G_\alpha = 0, \tag{2.45}$$

where G_α is the supersymmetric counterpart of the stress-energy tensor, the supercurrent. In light cone coordinates, they read

$$\begin{aligned} 0 = T_{\pm\pm} &= \frac{1}{2}\left(\partial_\pm X^\mu \partial_\pm X_\mu + i\psi^\mu_\pm \partial_\pm \psi_{\pm\mu}\right), \\ 0 = G_\pm &= \frac{1}{2}\psi^\mu_\pm \partial_\pm X_\mu. \end{aligned} \tag{2.46}$$

At this point, one expands the solutions to the equations of motion in modes, quantizes the theory by imposing canonical commutation (and anticommutation) relations for the modes, and implements the constraints. Also in this case consistency of the theory fixes the number of spacetime dimensions, this time to

$$D = 10. \tag{2.47}$$

When analyzing the spectrum, a further requirement forces us to discard half of the Fock space, via a procedure called GSO projection. This is because not all combinations of the choice of R/NS in the left and right chirality sector are consistent. It turns out that some of the allowed combinations are also free of tachyons: they are called type IIA and type IIB Superstring theories.
At the massless level, type II Superstring theories contain in their spectrum:

- NS-NS sector: the bosonic fields of bosonic string theory, i.e. the metric $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton Φ;
- R-NS and NS-R sector: the spinors ψ^μ and the dilatino λ, the fermionic superpartner of the dilaton;
- R-R sector: (bosonic) gauge p-forms C_p. Here we have to make a distinction between type IIA and type IIB theories: in type IIA there are a one-form C_1 and a three-form C_3, whereas in type IIB there are a zero-form C_0, a two-form C_2 and a self-dual four-form C_4.

The low-energy effective action for the bosonic part of type IIA and type IIB, in the string frame, are [25]

$$S_{IIA/B} = S_{NSNS} + S_{RR} + S_{CS}, \tag{2.48}$$

where S_{NSNS} is the same for both type IIA and IIB and is formally identical to the effective action arising from bosonic string theory,

$$S_{NS} = \frac{1}{2k_{10}^2}\int \mathrm{d}^{10}X\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{2.49}$$

with $H_3 = \mathrm{d}B_2$. S_{RR} contains the kinetic terms for the C_p gauge forms and S_{CS} contains certain topological terms. For type IIA,

$$\begin{aligned} S_R &= -\frac{1}{4k_{10}^2}\int \left(F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star \tilde{F}_4\right), \\ S_{CS} &= -\frac{1}{4k_{10}^2}\int B_2 \wedge F_4 \wedge F_4, \end{aligned} \tag{2.50}$$

with $F_{p+1} = \mathrm{d}C_p$ and $\tilde{F}_4 = F_4 - C_1 \wedge F_3$; whereas for type IIB

$$\begin{aligned} S_R &= -\frac{1}{4k_{10}^2}\int \left(F_1 \wedge \star F_1 + \tilde{F}_3 \wedge \star \tilde{F}_3 + \tilde{F}_5 \wedge \star \tilde{F}_5\right), \\ S_{CS} &= -\frac{1}{4k_{10}^2}\int C_4 \wedge H_3 \wedge F_3, \end{aligned} \tag{2.51}$$

where again $F_{p+1} = \mathrm{d}C_p$, and $\tilde{F}_3 = F_3 - C_0 \wedge H_3$, $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 = \star\tilde{F}_5$.

For completeness we mention rapidly the issue of open strings. In Superstring theories it is not as easy to build a consistent theory containing open strings as it was in the bosonic string case. It is nevertheless possible, and the resulting theory is called type I theory. We will not make use of this theory.

2.1.5 Kaluza-Klein mechanism

We have seen how consistency of superstring theories requires the number of spacetime dimensions to be fixed to $D = 10$. Thus, to make contact with our seemingly $D = 4$ Universe we need to get rid of unwanted extra dimensions. The strategy is to ask them to be very small, such that they cannot be noticed with experiments: this goes by the name of compactification, and follows an old idea in field theory, the Kaluza-Klein mechanism. Here we follow the presentation in [25].

Consider a toy field theory in $D = 5$ dimensional spacetime. Let the fifth dimension, described by the coordinate $x^4 \equiv y$, be a circle of radius r, i.e. $y \sim y + 2\pi R$. This has three consequences.

- There appears a Kaluza-Klein tower of massive states in $(D-1)$ dimensions. Let $M, N = 0, \ldots, 4$ and $\mu, \nu = 0, \ldots, 3$. A massless scalar field in $D = 5$ dimensions satisfies

$$\partial_M \partial^M \Phi(x^M) = 0. \tag{2.52}$$

  Periodicity in $x^4 = y$ prescribes the Fourier decomposition

$$\Phi(x^\mu, y) = \sum_n \phi_n(x^\mu) e^{iny/R}, \tag{2.53}$$

  and plugging this inside the five-dimensional equation of motion gives

$$\left(\partial_\mu \partial^\mu - \frac{n^2}{R^2}\right)\phi_n(x^\mu) = 0. \tag{2.54}$$

  The $n-$th Fourier mode $\phi_n(x^\mu)$ is a lower dimensional scalar field of mass $m_n^2 = n^2/R^2$. Notice that the zero mode ϕ_0 is massless, and it is the only state that survives in the $R \to 0$ limit.

- There appears an extra $U(1)$ symmetry in $(D-1)$ dimensions. The gauge potential arises from the components $G_{\mu y}$ of the 5-dimensional metric. The most general ansatz for the metric reads

$$ds^2 = G_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dy + A_\mu dx^\mu)^2. \tag{2.55}$$

  We can write down an expansion similar to (2.53) for the fields appearing in the metric. Let us consider only the zero modes. Diffeomorphism invariance along the compact direction, $y \longmapsto y' = y + \lambda(x^\mu)$, translates into

$$A_\mu \longmapsto A'_\mu = A_\mu - \partial_\mu \lambda, \tag{2.56}$$

  therefore A_μ is a gauge potential in 4 dimensions.

- There appear massless scalar fields, called moduli, in $(D-1)$ dimensions. In this setup the field σ is the modulus, and it sets the volume of the internal space,

$$\mathrm{Vol}(S^1) = \int_0^{2\pi R} dy e^\sigma = 2\pi R e^\sigma. \tag{2.57}$$

  In general, if there is a potential for a modulus then its expectation value will determine the geometric properties of the compact space.

If the five-dimensional theory contained only gravity, with dimensional reduction it becomes

$$S_5 = \frac{1}{2k_5^2}\int \mathrm{d}^5x\sqrt{-G}R^{(5)} = \frac{2\pi r}{2k_5^2}\int \mathrm{d}^4x\sqrt{-g}e^\sigma\left(R^{(4)} - \frac{e^{2\sigma}}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma\,\partial^\mu\sigma\right). \tag{2.58}$$

Let us now discuss compactification from the point of view of bosonic string theory. Now $D = 26$, and let us assume that X^{25} describes a compact direction with period R. We have now to think about string excitations. The first consequence is that p^{25} is quantized: in fact,

$$e^{2\pi i R p^{25}}|X^{25}\rangle = |X^{25} + 2\pi R\rangle = |X^{25}\rangle \iff p^{25} = \frac{n}{R},\ n \in \mathbb{Z}. \tag{2.59}$$

Another consequence is that closed strings can wind around the compact direction many times,

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi m R, \quad m \in \mathbb{Z}. \tag{2.60}$$

Therefore, X^{25} allows the mode expansion

$$X^{25}(\tau,\sigma) = x_0^{25} + \alpha'\frac{n}{R}\tau + mR\sigma + (\text{oscillators}), \tag{2.61}$$

and the string spectrum from the 25-dimensional point of view can be read from the level matching condition,

$$M^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} + (\text{oscillators})\,. \tag{2.62}$$

Notice that this formula is symmetric in the simultaneous exchange of $n \longleftrightarrow m$, $R \longleftrightarrow \alpha'/R$. This is the first hint of the presence of dualities in string theory, which we will address in the more specific framework of Supergravity.

2.2 Supergravity theories

When one takes the effective low-energy limit of Superstring theories one obtains Supergravity (SUGRA), the supersymmetric extension of General Relativity. However one does not need to start from String Theory to define Supergravity, and of course historically they have been developed independently. The motivation to develop Supergravity was that supersymmetry makes the divergences that appear when trying to quantize gravity more tractable.

The basic idea is to promote supersymmetry from a global symmetry to a local one, i.e. $\delta_\epsilon = \epsilon_I Q^I \longmapsto \epsilon_I(x) Q^I$. In this way, (2.34, 2.35) become

$$\delta_\epsilon B = \bar{\epsilon}(x) F, \quad \delta_\epsilon F = \epsilon(x)\gamma^\mu \, \partial_\mu B, \quad [\delta_{\epsilon_1}, \delta_{\epsilon_2}] B \sim (\bar{\epsilon}_1 \gamma^\mu \, \epsilon_2)(x) \, \partial_\mu B. \tag{2.63}$$

By the last of these relations, we see that a theory that is invariant under local supersymmetry is invariant also under local diffeomorphisms, as happens in General Relativity.

The features of a Supergravity theory depend mainly on the number of spacetime dimensions D and on the quantity of supersymmetries $\mathcal{N}$ present in the theory. The field content of the theory is divided into multiplets: each multiplet is closed under supersymmetry transformation. We are interested in the multiplets that contain a spin-2 massless particle, the graviton, which we will call supergravity multiplet.

The number of spacetime dimensions is relevant due to the distinct structure of spinorial representations in different D. Let us recall some properties of spinor representations [26]. Let us parametrize the number of dimensions by $D = 2k + 2$ if D is even, and $D = 2k + 3$ if D is odd.
In any dimension we can define the gamma matrices γ_μ obeying the usual Clifford algebra; the corresponding spinors are called Dirac spinors and transform under a representation of complex dimension 2^{k+1}. For D even, however, we can further define another matrix $\tilde{\gamma}$, that obeys

$$\tilde{\gamma} = i^k \gamma_0 \gamma_1 \dots \gamma_{D-1}, \quad \{\gamma_\mu, \tilde{\gamma}\} = 0, \quad \tilde{\gamma}^2 = -1. \tag{2.64}$$

This means that the Dirac representation is not irreducible, and we can split one Dirac spinor into two Weyl spinors. The corresponding representation has complex dimension 2^k.
There is also another way of splitting spinors: by looking at real representations rather than complex representations. If $D \equiv 0, 1, 2, 3, 4 \mod 8$ then one can define Majorana (or pseudo-Majorana) spinors. Moreover, if D is even and the decompositions into Majorana and Weyl spinors are compatible, then one has Majorana-Weyl spinors: this happens only if $D \equiv 2 \mod 8$. The corresponding irreducible representation for Majorana-Weyl spinors has real dimension 2^k. The results are summarized in Table 2.1 below, where $\dim_{\mathbb{R}} R$ denotes the real dimension of the corresponding irreducible spinor representation.

The total number of supercharges Q^I_A in a supergravity theory is $\mathcal{N}$ times $\dim_{\mathbb{R}} R$. The quantity of supersymmetries $\mathcal{N}$ is however not arbitrary, because the final theory should be free of particles with spin greater than 2, which would lead to instability. One finds [23] that the maximum allowed amount of supercharges in a supergravity theory is 32: if there are exactly 32 supercharges then the theory is said to be maximal. For instance, if $D = 4$ then $\mathcal{N} \leq 8$, whereas the highest number of dimensions achievable with $\mathcal{N} = 1$ is $D = 11$.

D	2	3	4	5	6	7	8	9	10	11
k	0	0	1	1	2	2	3	3	4	4
Weyl?	✓		✓		✓		✓		✓	
(pseudo-) Majorana?	✓	✓	✓				✓	✓	✓	✓
Majorana-Weyl?	✓								✓	
$\dim_{\mathbb{R}} R$	1	2	4	8	8	16	16	16	16	32

Table 2.1: Admissable spinorial structures in D dimensions.

2.2.1 Eleven- and ten-dimensional supergravity

Since it turns out [23] that maximal supergravites in different D are related by dimensional reduction, we can think of $D = 11$ maximal supergravity as being the most fundamental theory of supergravity. It has 256 degrees of freedom, which are evenly divided into 128 bosonic and 128 fermionic degrees of freedom thanks to supersymmetry. Massless particles in $D = 11$ fall into irreducible representations of $SO(9)$: we can fit all the degrees of freedom of the theory into

- the metric G_{MN}, a symmetric traceless field (44 bosonic d.o.f.s),
- a gauge 3-form A_{MNP} (84 bosonic d.o.f.s),
- the gravitino ψ^A_M (128 fermionic d.o.f.s).

The bosonic part of the action is given by

$$S_{11} = \frac{1}{2k_{11}^2} \int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12k_{11}^2}\int A_3 \wedge F_4 \wedge F_4, \tag{2.65}$$

where $F_4 = \mathrm{d}A_3$, whereas the fermionic part of the action is fixed by supersymmetry. The last term in the bosonic action is also required by supersymmetry: it is a topological term, called Chern-Simons term.[1]

If we want to make contact with Superstring theory, however, we should require the number of spacetime dimensions to be ten, not eleven. We want to obtain maximal supergravity in $D = 10$. Maximality will require $\mathcal{N} = 2$, since the dimension of irreducible spinor representation is halved in ten dimensions with respect to the eleven-dimensional case.

An option is to start from $D = 11$, $\mathcal{N} = 1$ supergravity and allow the space direction $y \equiv x^{10}$ to be a compact direction. We will interpret then eleven-dimensional fields from a ten-dimensional perspective: explicitly, we let

$$\begin{aligned} ds_{11}^2 &= e^{2\sigma}\left(dy + C_{1\mu}dx^\mu\right)^2 + ds_{10}^2, \\ A_3 &= B_2 \wedge dy + C_3. \end{aligned} \tag{2.66}$$

Here greek indices run from 0 to 9. C_1 and C_3 are RR gauge forms, B_2 is the NSNS gauge form, and the field σ is related to the dilaton Φ by

$$\Phi = \frac{3}{2}\sigma. \tag{2.67}$$

[1]Despite its appearence, the Chern-Simons term in (2.65) is gauge invariant.

We thus recover the massless spectrum of type IIA superstring theory: the resulting supergravity theory is also referred to as type IIA. The dimensionally reduced action reads

$$\begin{aligned} S_{IIA} = & \frac{1}{2k_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^{\sigma} R^{(10)} + e^{\sigma} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} e^{3\sigma} \left| F_2 \right|^2 \right) + \\ & - \frac{1}{4k_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^{-\sigma} \left| H_3 \right|^2 + e^{\sigma} \left| \tilde{F}_4 \right|^2 \right) - \frac{1}{4k_{10}^2} \int B_2 \wedge F_4 \wedge F_4, \end{aligned} \tag{2.68}$$

with $F_{p+1} = \mathrm{d}C_p$, $H_3 = \mathrm{d}B_2$, $\tilde{F}_4 = \mathrm{d}C_3 - C_1 \wedge F_3$. However, this is neither in the string frame nor in the Einstein frame. To pass between frames one can use

$$(g_E)_{\mu\nu} = e^{\sigma/4} (g_{10})_{\mu\nu} = e^{\Phi/6} (g_{10})_{\mu\nu}, \qquad (g_s)_{\mu\nu} = e^{\sigma} (g_{10})_{\mu\nu} = e^{2\Phi/3} (g_{10})_{\mu\nu}. \tag{2.69}$$

In the string frame, one recovers the action (2.48) that comes from superstring theory.

One can also describe type IIB Supergravity, i.e. the low-energy effective theory arising from type IIB Superstring theory. This theory cannot be obtained from eleven-dimensional maximal Supergravity via dimensional reduction, but it can be obtained from type IIA theory via T-duality. As we have seen in Section 2.1.5, if one wraps a closed bosonic string around a circle of radius R, the string receives contributions to the mass coming from winding modes (proportional to R/α') and momentum modes (proportional to $1/R$). If the string were wrapped around a circle of radius $\tilde{R} = \alpha'/R$ instead, one can swap winding and momentum modes, but the spectrum stays identical. It turns out [25] that the two scenarios are equivalent also at the interacting level. This is T-duality.
Something similar happens for superstrings. Let y denote now a compact direction in ten dimensions. Let us parametrize the fields appearing in type IIA supergravity, in the string frame, as

$$\begin{aligned} ds^2 &= g_{yy} \left(dy + A_\mu dx^\mu \right)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu, \\ B_2 &= B_{\mu y} dx^\mu \wedge \left(dy + A_\mu dx^\mu \right) + \hat{B}_2, \\ C_p &= (C_{p-1})_y \wedge \left(dy + A_\mu dx^\mu \right) + \hat{C}_p. \end{aligned} \tag{2.70}$$

T-duality maps these fields to their type IIB analogs,

$$\begin{aligned} ds^2 &= g_{yy}^{-1} \left(dy + B_{\mu y} dx^\mu \right)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu, \\ e^{2\Phi'} &= g_{yy}^{-1} e^{2\Phi}, \\ B_2' &= A_\mu dx^\mu \wedge dy + \hat{B}_2, \\ C_p' &= \hat{C}_{p-1} \wedge \left(dy + B_{\mu y} dx^\mu \right) + (C_p)_y. \end{aligned} \tag{2.71}$$

Notice that the C_p' forms have now odd p. With these redefinitions, one recovers the low-energy action (2.48) of type IIB superstring theory.

There is yet another duality concerning type II theories, and that is S-duality. In particular, S-duality relates two different type IIB theories. It is relevant to study this duality beacause it gives us information about the strong coupling limit of type II theories. We know that the dilaton Φ is related to the string coupling by

$$g_s = e^{\langle \Phi \rangle}. \tag{2.72}$$

S-duality acts on the fields of type IIB supergravity as

$$\begin{aligned} \Phi' &= -\Phi, \\ g'_{\mu\nu} &= e^{-\Phi} g_{\mu\nu}, \\ B'_2 &= C_2, \\ C'_2 &= -B_2, \end{aligned} \tag{2.73}$$

thus effectively changing the coupling to $g'_s = 1/g_s$. We see then that a strongly coupled type IIB theory is S-dual to a weakly coupled one, which can be analyzed at the perturbative level.
S-duality is useful also because it will allow us to generate new solutions in type IIB supergravity starting from known ones.

2.2.2 Branes and charges

As we have mentioned when addressing open strings, string theory is more than just a theory of strings: it contains also other extended objects, branes. By studying branes one finds that they couple to gauge forms, as we will explain in the following.
Before addressing branes, let us recall how a gauge 1-form A_1 can give charges to a particle in $D = 4$. The interaction between the particle and the gauge field is given by the term

$$\mathcal{L}_{int} = q \int A_1, \tag{2.74}$$

where the integral is over the particle's worldline, and q is a coupling constant. If we let $F_2 = \mathrm{d}A_1$ be the field strength of the gauge field generated by the particle, the electric and magnetic charges of the particle are defined in a way similar to (1.13),

$$Q_e = \int_{S^2} \star F_2, \qquad Q_m = \int_{S^2} F_2. \tag{2.75}$$

The generalization to branes living in D dimensions is straightforward. The interaction with a p-form A_p is still of the form

$$\mathcal{L}_{int} = \mu_p \int A_p, \tag{2.76}$$

but now the integral runs over the p-dimensional worldvolume of a $(p-1)$-brane. Letting $F_{p+1} = \mathrm{d}A_p$, we see that A_p couples electrically to a $(p-1)$-brane (with electric charge Q_e), and magnetically to a $(D-p-3)$-brane (with magnetic charge Q_m). The charges of the branes are, respectively,

$$Q_e = \int_{S^{D-p-1}} \star F_{p+1}, \qquad Q_m = \int_{S^{p+1}} F_{p+1}. \tag{2.77}$$

For instance, the field A_3 in $D = 11$ maximal supergravity couples electrically to a 2-brane and magnetically to a 5-brane.

When applying this to type II theories we obtain, respectively,

- Type IIA. The RR gauge form C_1 couples electrically to a D0-brane and magnetically to a D6-brane, whereas C_3 couples electrically to a D2-brane and magnetically to a D4-brane. The NSNS gauge form B_2 couples electrically to a 1-brane and magnetically to a 5-brane, which are called F1- and NS5-brane respectively. F1 stands for fundamental string. A NS5-brane is another different fundamental object: it can be thought of as the analogue of a magnetic monopole for B_2.

- Type IIB. The RR gauge forms C_2 couples electrically to a D1-brane and magnetically to a D5-brane, whereas C_4 couples both electrically and magnetically to a D3-brane. C_0 is a scalar, and thus it does not couple to branes. The NSNS gauge form B_2 behaves as in type IIA.

We can see how dualities swap the role of branes. An S-duality for type IIB theories acts on the branes by exchanging D1,D5-branes with F1,NS5-branes and viceversa. A T-duality acts in a more complicated way. Winding modes of a fundamental string become momentum modes along the string in the duality direction, whereas for branes sourcing RR gauge forms the action of the duality depends wheter it is performed along the brane or not. Schematically,

$$F1 \longleftrightarrow P, \quad Dp \overset{\parallel}{\longleftrightarrow} D(p-1), \quad Dp \overset{\perp}{\longleftrightarrow} D(p+1). \tag{2.78}$$

The presence of branes in the theory breaks some of the symmetries of the theory: translation invariance perpendicular to the branes, as well as supersymmetry to a certain amount. If the brane is not excited it breaks only half of the supersymmetries, because left- and right-movers are no more independent due to boundary conditions on the branes. If more than one type of branes are present, each kind of brane halves the number of supersymmetries preserved in the theory.

Chapter 3

Black Holes from String Theory

Now that we have String Theory, and in particular its low-energy limit, Supergravity, in our toolbox, it is natural to ask how one is supposed to describe black holes within this new framework.
The first step is to form bound state out of objects in String Theory that are at least reasonably stable. A good way of doing so is to pick those objects to be branes. Branes are solitonic solutions of String Theory which are naturally long-lived at the energy scale of Supergravity. String theory then allows to give a fairly simple explanation of how the black hole entropy arises, by giving a description of the microscopic states (microstates) that account for a given macroscopic configuration.

3.1 Brane solutions in Supergravity

In this Section we will derive solutions of the supergravity equations for fixed configurations of p-branes in the background.
We will focus on a particular class of solutions, BPS solutions, for which all fermionic fields are set to zero (hence, they are bosonic solutions), and that are invariant under a fraction of the supersymmetries characterizing the theory. The general rule for BPS bound states of p-branes is that each type of parallel brane present in the background configuration halves the number of supersymmetries in the theory; this is easily understood by looking, for instance, at the bosonic Poincaré generators. We are interested in BPS solutions mainly because they have two nice properties [26]. First, BPS states have charge equal to their mass in appropriate units, and hence they represent natural candidates for generating extremal black holes. Secondly, their microscopic degeneracy is a quantity protected by supersymmetry, i.e. it does not vary when one changes the couplings in the theory, thus the entropy arising from the supergravity computation (free theory) should match the microscopic entropy at strong coupling.
There are two possible ways of deriving the solutions:

Direct method. Solve the equations of motion of the supegravity theory. This is in general impractical to perform analytically, but just like what happens for Einstein's equations the presence of symmetries makes it more manageable. BPS solutions are then obtained imposing supersymmetry of the fields configuration: since the fermionic fields are set to zero in BPS configurations, then the bosonic fields are invariant under supersymmetry.

Indirect method. Start from a trivial solution of the equations of motion, and derive less trivial solutions applying symmetries and S,T-dualities. To add charges to the solution it is possible to perform boosts along compact directions. In order to obtain a BPS solution, however, one must take an appropriate decoupling limit.

In the following we will provide an explicit construction of BPS brane solutions with different number of charges using the indirect method. In particular, when the third charge is added we will find a description of the Strominger-Vafa black hole [27].

3.1.1 1-charge solution

The starting point is 10D Supergravity on flat Minkowski space compactified on $S^1 \times T^4$. Let us denote by (x^i, t) the coordinates along the noncompact directions $\mathbb{R}^{4,1}$, let y be the coordinate along the S^1 and (z^a) be the coordinates of T^4.
In this framework, the trivial solution of type II Supergravity equations from which we start is the 5D Schwarzschild solution in the noncompact directions,

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + dy^2 + dz_a dz^a, \tag{3.1}$$

where we have set $G_N = 1$ for convenience. Let us pick Hopf coordinates for Ω_3,

$$x^1 = r\sin\theta\cos\phi, \quad x^2 = r\sin\theta\cos\phi, \quad x^3 = r\cos\theta\cos\psi, \quad x^4 = r\cos\theta\sin\psi, \tag{3.2}$$

where $\theta \in \left[0, \frac{\pi}{2}\right]$, $\phi, \psi \in [0, 2\pi[$. All other supergravity fields are switched off, namely

$$\Phi = B_2 = C_p = 0. \tag{3.3}$$

Let us add the first charge by performing a boost of parameter α along S^1:

$$y \to y\cosh\alpha + t\sinh\alpha \equiv y, \qquad t \to t\cosh\alpha + y\sinh\alpha \equiv t. \tag{3.4}$$

The metric (3.1) becomes

$$\begin{aligned} ds^2 = &\left(1 + \frac{2M\sinh^2\alpha}{r^2}\right) dy^2 + \left(-1 + \frac{2M\cosh^2\alpha}{r^2}\right) dt^2 + \\ &+ \sinh 2\alpha \frac{2M}{r^2} dy dt + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + dy^2 + dz_a dz^a. \end{aligned} \tag{3.5}$$

This is a solution of type IIA Supergravity generated by a wave carrying momentum along S^1: let us call this charge P_y.
To obtain a charge that is related to a bound state of branes, we need to perform T-duality along the y direction. This brings us to a type IIB supergravity solution describing a fundamental string, which we call $\mathrm{F1}_y$. The result is

$$\begin{cases} ds^2 &= S_\alpha^{-1}\left[dy^2 + \left(-1 + \frac{2M}{r^2}\right) dt^2\right] + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + dz_a dz^a, \\ e^{2\Phi} &= S_\alpha^{-1}, \\ B_2 &= S_\alpha^{-1} \frac{M}{r^2} \sinh 2\alpha \, \mathrm{d}t \wedge \mathrm{d}y, \\ C_p &= 0, \end{cases} \tag{3.6}$$

where

$$S_\alpha \equiv \left(1 + \frac{2M\sinh^2\alpha}{r^2}\right). \tag{3.7}$$

To obtain a BPS solution, we must perform the so-called BPS limit,

$$M \to 0, \qquad \alpha \to \infty, \qquad Me^{2\alpha} \equiv 2Q. \tag{3.8}$$

Here Q has the meaning of the winding charge of the fundamental string. S_α becomes

$$S_\alpha \to 1 + \frac{Q}{r^2} \equiv Z(r), \tag{3.9}$$

and the BPS solution obtained from (3.6) reads

$$\begin{cases} ds^2 &= Z^{-1}(r)\left(dy^2 - dt^2\right) + dr^2 + r^2 d\Omega_3^2 + dz_a dz^a, \\ e^{2\Phi} &= Z^{-1}(r), \\ B_2 &= -Z^{-1}(r)\, \mathrm{d}t \wedge \mathrm{d}y, \\ C_p &= 0. \end{cases} \tag{3.10}$$

3.1.2 2-charge solution

We wish to add another charge to the solution. To do this, let us start again from (3.6)[1], and perform another boost along S^1, of parameter β. This describes a $\mathrm{F1}_y$ string carrying momentum P_y,

$$\begin{cases} ds^2 &= S_\beta S_\alpha^{-1}\left(dy + \frac{M\sinh 2\beta}{r^2+2M\sinh^2\beta}dt\right)^2 + S_\beta^{-1}S_\alpha^{-1}\left(-1+\frac{2M}{r^2}\right)dt^2 + \\ &\quad + \left(1-\frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2 + dz_a dz^a, \\ e^{2\Phi} &= S_\alpha^{-1}, \\ B_2 &= S_\alpha^{-1}\frac{M}{r^2}\sinh 2\alpha\, \mathrm{d}t \wedge \mathrm{d}y, \\ C_p &= 0. \end{cases} \tag{3.11}$$

Before taking the BPS limit, let us move to a more convenient duality frame: the D1D5 frame. It can be generated from the F1P frame via the following chain of dualities,

$$\begin{pmatrix}\mathrm{F1}_y \\ \mathrm{P}_y\end{pmatrix} \xrightarrow{S_y} \begin{pmatrix}\mathrm{D1}_y \\ \mathrm{P}_y\end{pmatrix} \xrightarrow{T_{T^4}} \begin{pmatrix}\mathrm{D5}_{yT^4} \\ \mathrm{P}_y\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\mathrm{NS5}_{yT^4} \\ \mathrm{P}_y\end{pmatrix} \xrightarrow{T_y} \begin{pmatrix}\mathrm{NS5}_{yT^4} \\ \mathrm{F1}_y\end{pmatrix} \xrightarrow{T_{z^1}+S} \begin{pmatrix}\mathrm{D5}_{yT^4} \\ \mathrm{D1}_y\end{pmatrix}. \tag{3.12}$$

After all these dualities, (3.11) reads

$$\begin{cases} ds^2 &= S_\beta^{-1/2}S_\alpha^{-1/2}\left[dy^2 - \left(1-\frac{2M}{r^2}\right)dt^2\right] + S_\beta^{1/2}S_\alpha^{1/2}\left[\left(1-\frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2\right] + \\ &\quad + S_\beta^{1/2}S_\alpha^{-1/2}dz_a dz^a, \\ e^{2\Phi} &= S_\beta S_\alpha^{-1}, \\ B_2 &= 0, \\ C_2 &= -S_\beta^{-1}\frac{M}{r^2}\sinh 2\beta\, \mathrm{d}t \wedge \mathrm{d}y - f(\theta,\alpha,\beta)\, \mathrm{d}\phi \wedge \mathrm{d}\psi, \end{cases} \tag{3.13}$$

[1] Recall that the BPS limit must always be taken at the end.

where $f(\theta, \alpha, \beta)$ is a function whose exact form before the BPS limit we are not interested into. It can be shown, however, that in the BPS limit one has

$$S_\alpha \to 1 + \frac{Q_1}{r^2} \equiv Z_1(r), \quad S_\beta \to 1 + \frac{Q_5}{r^2} \equiv Z_5(r), \quad f(\theta, \alpha, \beta) \to -Q_5 \sin^2\theta, \tag{3.14}$$

and the 2-charge BPS solution is thus given by

$$\begin{cases} ds^2 &= Z_1^{-1/2}(r) Z_5^{-1/2}(r) \left(dy^2 - dt^2\right) + Z_1^{1/2}(r) Z_5^{1/2}(r) \left(dr^2 + r^2 d\Omega_3^2\right) + \\ & \quad + Z_1^{1/2}(r) Z_5^{-1/2}(r) dz_a dz^a, \\ e^{2\Phi} &= Z_1(r) Z_5^{-1}(r), \\ B_2 &= 0, \\ C_2 &= -(1 - Z_5^{-1}(r))\, \mathrm{d}t \wedge \mathrm{d}y + Q_5 \sin^2\theta\, \mathrm{d}\phi \wedge \mathrm{d}\psi. \end{cases} \tag{3.15}$$

Q_1 and Q_5 have the meaning of winding charges of the D1, D5 branes respectively along the compact directions. Notice that Q_5 comes from the winding of the fundamental string around S^1 and that Q_1 comes from its momentum instead.

3.1.3 3-charge solution: the Strominger-Vafa black hole

The three-charge solution is obtained by performing another boost along the S^1 direction in the D1D5 system before taking the BPS limit. Let γ be the boost parameter; the metric in (3.13) becomes

$$\begin{aligned} ds^2 = S_\gamma S_\beta^{-1/2} S_\alpha^{-1/2} \left(dy + S_\gamma^{-1} \frac{M}{r^2} \sinh 2\gamma dt\right)^2 - S_\gamma^{-1} S_\beta^{-1/2} S_\alpha^{-1/2} \left(1 - \frac{2M}{r^2}\right) dt^2 + \\ + S_\beta^{1/2} S_\alpha^{1/2} \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right] + S_\beta^{1/2} S_\alpha^{-1/2} dz_a dz^a, \end{aligned} \tag{3.16}$$

and the other fields do not change. Taking the BPS limit (3.14), as well as

$$S_\gamma \to 1 + \frac{Q_P}{r^2} \equiv Z_P(r) \equiv 1 + K(r), \tag{3.17}$$

the full 3-charge BPS solution reads

$$\begin{cases} ds^2 &= Z_1^{-1/2}(r) Z_5^{-1/2}(r) \left(dy^2 - dt^2 + K(r)(dt + dy)^2\right) + Z_1^{1/2}(r) Z_5^{1/2}(r) \left(dr^2 + r^2 d\Omega_3^2\right) + \\ & \quad + Z_1^{1/2}(r) Z_5^{-1/2}(r) dz_a dz^a, \\ e^{2\Phi} &= Z_1(r) Z_5^{-1}(r), \\ B_2 &= 0, \\ C_2 &= -(1 - Z_5^{-1}(r))\, \mathrm{d}t \wedge \mathrm{d}y + Q_5 \sin^2\theta\, \mathrm{d}\phi \wedge \mathrm{d}\psi. \end{cases} \tag{3.18}$$

As we will see in the following, this solution describes a black hole, the Strominger-Vafa black hole, with a finite horizon area.

3.1.4 Computing S_{BH}

We now want to compute the entropy of the 1,2,3-charge BPS black holes via the Bekenstein-Hawking relation (1.41).

The metrics appearing in (3.10, 3.15, 3.18) are given in the string frame. In order to compute the correct horizon area, one must switch to the Einstein frame first,

$$(g_E)_{\mu\nu} = e^{-\Phi/2}(g_s)_{\mu\nu}. \tag{3.19}$$

Let us start from the 1-charge case (3.10). Since $e^{2\Phi} = Z^{-1}$, then the metric in the Einstein frame reads

$$ds_E^2 = Z^{1/4} ds_s^2 = Z^{-3/4}(dy^2 - dt^2) + Z^{1/4}(dr^2 + r^2 d\Omega_3^2 + dz_a dz^a). \tag{3.20}$$

The metric is regular everywhere but at $r = 0$, so the horizon must be located there. The area of the horizon in 10D is given by

$$A_{10} = V_{S^1} V_{T^4} \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \sqrt{-g_E}\big|_{r=0} \,, \tag{3.21}$$

but

$$\sqrt{-g_E} = Z^{1/4} r^3 \sin\theta \cos\theta \xrightarrow{r\to 0} 0. \tag{3.22}$$

A similar computation shows that $A_{10} = 0$ also for the 2-charge solution[2]. Thus, we cannot interpret such solutions as black holes with thermodynamic properties.

For the 3-charge solution, however, the number of charges as well as the number of compact dimensions makes it so that an exact cancellation happens. Also in this case the horizon is located at $r = 0$, but this time

$$A_{10} = V_{S^1} V_{T^4} \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, r^3 \sin\theta \cos\theta \sqrt{Z_1 Z_5 Z_P}\Big|_{r=0} = 2\pi^2 V_{S^1} V_{T^4} \sqrt{Q_1 Q_5 Q_P}, \tag{3.23}$$

and the entropy reads

$$S_{BH} = \frac{A_{10}}{4G_N^{(10)}} = \frac{\pi^2 V_{S^1} V_{T^4} \sqrt{Q_1 Q_5 Q_P}}{2G_N^{(10)}}, \tag{3.24}$$

where $G_N^{(10)}$ is Newton's constant in ten dimensions.
One could argue that an observer which cannot resolve the compact directions might observe a different entropy. This would be a problem, because the entropy of a system is a physical quantity and should not depend on the properties of the observer. In five dimensions, one would write

$$S_{BH} = \frac{A_5}{4G_N^{(5)}}. \tag{3.25}$$

Recall however the relation between G_N before and after compactification:

$$G_N^{(5)} = \frac{G_N^{(10)}}{V_{S^1} V_{T^4}}, \tag{3.26}$$

and of course $A_5 = A_{10}/(V_{S^1} V_{T^4})$: the two computations do match indeed.

Another issue at hand is whether the 3-charge solution is really a black hole: we need its mass to be sufficiently large to form an horizon with finite area in the first place. In order

[2]Actually, for the 2-charge solution the curvature blows up at $r = 0$, so the classical Supergravity limit is no more reliable. We shall return on this point later.

to answer this question we have to compute the five-dimensional metric coming from the full ten-dimensional solution (3.18), because the definition of the mass involves spatial infinity. We can use the standard Kaluza-Klein recipe to obtain [28]

$$ds_5^2 = -(Z_1 Z_5 Z_P)^{-2/3} dt^2 + (Z_1 Z_5 Z_P)^{1/3}(dr^2 + r^2 d\Omega_3^2). \tag{3.27}$$

In order to estract the mass of the solution from the metric, we can use an expression of the mass for asymptotically flat d-dimensional spacetimes [28],

$$g_{tt} \xrightarrow{r\to\infty} -1 + \frac{16\pi G_N^{(d)}}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}}, \tag{3.28}$$

where in our case $d = 5$. A comparison yields

$$M = \frac{Q_1 + Q_5 + Q_P}{4G_N^{(5)}}. \tag{3.29}$$

Thus, in order to have a large mass one needs to have large value of the charges. The resulting black hole is also indeed extremal.
The relation (3.29) tells us also another thing. The 3-charge solution (3.18) has been obtained by considering bound states of branes. Since for BPS states charge is equal to mass, each charge Q_i is proportional to the degeneracy of the associated kind of branes present in the background configuration. The exact relations for Q_1, Q_5, Q_P are (see for instance [29])

$$Q_1 = (2\pi)^4 \frac{g_s \alpha'^3}{V_{T^4}} n_1, \quad Q_5 = g_s \alpha' n_5, \quad Q_P = (2\pi)^4 \frac{g_s \alpha'^4}{V_{T^4} R^2} n_p. \tag{3.30}$$

Inserting these relations in (3.24), and using also the relation between Newton's constant and the couplings,

$$G_N^{(10)} = 8\pi^6 g_s^2 \alpha'^4, \tag{3.31}$$

one obtains

$$S_{BH} = 2\pi\sqrt{n_1 n_5 n_p}. \tag{3.32}$$

Thus, the Bekenstein-Hawking entropy does not depend on the moduli of the theory. This is a remarkable and important feature of this solution, because in this case we can compare the black hole entropy with the microscopic entropy coming from the degeneracy of states. This is going to be the task of the following Section.

3.2 The microstate geometries

In the previous Section we have obtained solutions in Supergravity that admit an interpretation as Black Holes by using dualities. We therefore inquire in this Section whether it is possible to construct them also by starting directly from String Theory and its symmetries. The answer will turn out to be positive, as first suggested by Susskind [30]. Our goal is to provide a recipe for the microscopic computation of Bekenstein-Hawking entropy from string-theoretical arguments.

3.2.1 1-charge states

Let us start by looking at BPS states corresponding to a fundamental string F1. For concreteness, let us consider type II Superstring theory, where the y direction has been compactified to a circle S^1 of radius R, and let us wrap the fundamental string along this circle n times. This is a BPS state, because no oscillators are excited on the string. If we further compactify 4 dimensions to T^4, the Supergravity metric produced by this string is given by (3.10),

$$ds^2 = Z^{-1}(dy^2 - dt^2) + dr^2 + r^2 d\Omega_3^2 + dz_a dz^a, \qquad Z(r) = 1 + \frac{Q}{r^2}. \tag{3.33}$$

Since $Q \propto n$, if the winding is large then the mass is also large, and then this is indeed a macroscopic black hole. As it is manifest from the metric in the Einstein frame (3.20), the only singularity is located at $r = 0$. For $r \to 0$, the dilaton field Φ goes to $-\infty$ and the radius of S^1 goes to zero. The horizon is also located at $r = 0$, its area vanishes, and $S_{BH} = 0$.
This can also be seen with the microscopic counting. The fundamental string is an oscillator ground state, so its degeneracy matches the total number of zero modes of the oscillators. There are 128 bosonic and 128 fermionic possible states. Hence $S_{micro} = \ln(256)$, which does not grow with n: in the thermodynamic limit, $n \to \infty$, one would write $S_{micro} = 0$, in agreement with S_{BH}.

3.2.2 2-charge states

As mentioned in the previous Section, the supergravity solution for the 2-charge states has vanishing S_{BH}. On the other hand, we have also noticed that in the limit $r \to 0$ the curvature of the solution diverges. Since the supergravity limit is reliable only as long as the curvature R is much smaller than the string length ℓ_s, it is not guaranteed that the supergravity solution that we have found is the low-energy limit of some stringy solution. Let us then try to build an explicit solution from String Theory.
For simplicity, let us consider again a fundamental string F1 wrapped around S^1, that now carries momentum P as well. But now, remember that in String Theory no longitudinal modes are allowed: thus all the momentum must be carried by transverse oscillators. This means then that the string must bend away from its axis at some point, and it is not confined to the $r = 0$ region anymore.
From this we conclude that the corresponding Supergravity solution cannot be generated from String Theory. Let us call this the "naïve" metric for the F1-P system.
We wish to overcome our naïveté and to understand which solution is produced by a fundamental string wounded n_1 times carrying n_p units of momentum. For a single strand the solution is known, and reads [31]

$$ds^2 = H(-du dv + K dv^2 + 2A_i dx^i dv) + dx_i dx^i + dz_a dz^a, \qquad e^{2\Phi} = H, \tag{3.34}$$

where $u = t + y$, $v = t - y$, and the functions H, K, A_i are defined by

$$H^{-1}(\vec{x}, v) = 1 + \frac{Q_1}{|\vec{x} - \vec{F}(v)|^2}, \quad K(\vec{x}, v) = \frac{Q_1 |\dot{\vec{F}}(v)|^2}{|\vec{x} - \vec{F}(v)|^2}, \quad A_i(\vec{x}, v) = -\frac{Q_1 \dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2}. \tag{3.35}$$

The function $\vec{F}$ represents the profile of the string. Due to supersymmetry, $\vec{F}$ can be a function of v or of u, but not both. The metric is singular along the curve $\vec{x} = \vec{F}(v)$ that represents the position of the string.

Since the string is multiwound, one can obtain the full solution by summing the contribution from each strand. The black hole must also be macroscopic, thus one must have $n_1, n_p \to \infty$. In this limit the sum over strands can be replaced with an integral, and the functions become

$$H^{-1} = 1 + \frac{Q_1}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^2}, \quad K = \frac{Q_1}{L}\int_0^L \frac{dv|\dot{\vec{F}}(v)|^2}{|\vec{x} - \vec{F}(v)|^2}, \quad A_i = -\frac{Q_1}{L}\int_0^L \frac{dv\dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2}, \tag{3.36}$$

where $L = 2\pi R n_1$.
To make a comparison with the naïve geometry obtained in Supergravity, it is convenient to follow again the chain of dualities (3.12) and to rewrite the solution in the D1D5 frame. In this frame, the function $\vec{F}$ does not have a clear interpretation as the profile of the branes. The calculation is not particularly illuminating, and can be found in [32, 33]. The resulting metric is

$$ds^2 = (Z_1Z_5)^{-1/2}\left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2\right] + (Z_1Z_5)^{1/2}dx_i dx^i + \left(\frac{Z_1}{Z_5}\right)^{1/2} dz_a dz^a, \tag{3.37}$$

with

$$\begin{gathered} Z_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{dv|\dot{\vec{F}}(v)|^2}{|\vec{x} - \vec{F}(v)|^2}, \quad Z_5 = 1 + \frac{Q_5}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^2}, \\ A_i = -\frac{Q_5}{L}\int_0^L \frac{dv\dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2}, \quad dB = -\star_4 dA. \end{gathered} \tag{3.38}$$

Here $Q_5 = RL/(2\pi)$ is the D5 charge, $\star_4$ is the Hodge dual on spatial noncompact directions, and the D1 charge is given by $Q_1 = \frac{Q_5}{L}\int_0^L |\dot{\vec{F}}|^2 dv$.

In this duality frame, the geometry is smooth everywhere: the singularities at $\vec{x} = \vec{F}(v)$ are just coordinate singularities [34]. The geometry is flat at infinity, and instead of having a singularity at $r = 0$ the geometry ends in a smooth cap. Different profile functions $\vec{F}(v)$ provide different caps. We thus claim that there is an ensemble of solutions rather than just the naïve F1-P solution, and then we see explicitly the distinction between different microstates.

Let us investigate our claim further. Let us return to the F1-P duality frame, where $\vec{F}$ has a clear interpretation. The different profile functions arise from the excitation of different momentum-carrying harmonics. We can partition the momentum in many different ways, and this gives a large number of states for a given choice of n_1, n_p.
One way to count the degeneracy is the following. Since every state must be BPS, we can consider without loss of generality that the excited oscillators are all left-movers with level N_L, whereas $N_R = 0$. The mass of the string state is

$$m^2 = \left(\frac{1}{\alpha'}Rn_1 - \frac{n_p}{R}\right)^2 + \frac{4}{\alpha'}N_L = \left(\frac{1}{\alpha'}Rn_1 + \frac{n_p}{R}\right)^2 + \frac{4}{\alpha'}N_R, \tag{3.39}$$

and if $N_R = 0$ then $N_L = n_1 n_p$, $m = \frac{1}{\alpha'}Rn_1 + \frac{n_p}{R}$. This oscillator level is partitioned among 8 bosonic oscillators and 8 fermionic oscillators, thus the total central charge is $c = 8 + 4 = 12$. We can compute the number of states using Cardy's formula [35], which for our case reads

$$\mathcal{N} \sim e^{2\pi\sqrt{\frac{c}{6}N_L}} = e^{2\sqrt{2}\pi\sqrt{n_1 n_p}}. \tag{3.40}$$

From this, we read off the microscopic entropy

$$S_{micro} = \ln \mathcal{N} = 2\sqrt{2}\pi\sqrt{n_1 n_p}, \tag{3.41}$$

which accounts for the correct entropy of the 2-charge solution.
This is of course different from the Bekenstein-Hawking entropy of the naïve two-charge configuration, which is vanishing. However, we might hope that higher order Supergravity corrections to S_{BH} could account for the full microscopic entropy. It turns out that this is not the case, as the first corrections to A_H are of order $O(R^4)$ in the curvature. On the other hand, it has been argued in [36] that compactifying over $K3$ rather than on T^4 can solve the problem, as the corrections to the entropy are of order $O(R^2)$; moreover, trading T^4 for K^3 does not seem to affect the properties of the solution in the D1D5 frame [37], thus our construction stays qualitatively valid also in this framework. The question whether this is enough to reproduce S_{micro} is still debated.

3.2.3 3-charge states

Adding the momentum charge to the D1D5 system, one obtains the D1D5P bound state, whose Supergravity solution we have found in the previous Section, and whose metric reads

$$ds^2 = (Z_1 Z_5)^{-1/2}\left(dy^2 - dt^2 + K(dt+dy)^2\right) + (Z_1 Z_5)^{1/2}\left(dr^2 + r^2 d\Omega_3^2\right) + \left(\frac{Z_1}{Z_5}\right)^{1/2} dz_a dz^a, \tag{3.42}$$

where

$$Z_{1,5} = 1 + \frac{Q_{1,5}}{r^2}, \qquad K = \frac{Q_p}{r^2}. \tag{3.43}$$

This solution represent a physical black hole because it has nonvanishing horizon area and Bekenstein-Hawking entropy, which in terms of the integer charges reads

$$S_{BH} = 2\pi\sqrt{n_1 n_5 n_p}. \tag{3.44}$$

We can compute the total number of microstates once again using Cardy's formula as in the 2-charge case. The interpretation of c, N_L is not transparent in the D1D5 duality frame: we can however reason as follows. The bound state of D1D5 branes will generate an effective string with total winding number $n_1 n_5$. This effective string can give rise to many states where m_i of the component strings of the state have windinds k_i, with

$$\sum_i m_i k_i = n_1 n_5. \tag{3.45}$$

The extremal cases are $m = n_1 n_5$ singly ($k = 1$) wound component strings and $m = 1$ maximally wound ($k = n_1 n_5$) component string. Our claim is that the latter configuration gives the leading contribution to the entropy in the large $n_1 n_5$ limit. In fact, in the former configuration there is only one state: all the component strings are identical, thus when adding one unit of momentum we must consider the superposition of all states where only one component string is excited. On the other hand, when the winding is maximal one unit of momentum becomes an excitation at level $n_1 n_5$. If we put n_p units of momentum, the total level is $n_1 n_5 n_p$. A less trivial argument coming from holography says that in each component string there are 4 bosonic and 4 fermionic oscillators, whence

$$c = 6. \tag{3.46}$$

Using again Cardy's formula, one has

$$S_{micro} = \ln \mathcal{N} = 2\pi\sqrt{n_1 n_5 n_p}, \tag{3.47}$$

in perfect agreement with (3.44).

This result is for sure remarkable, but still it does not imply that the three-charge solution is also an allowed solution in Superstring theory. Thus we can think of the above three-charge metric again as a naïve metric, and to see how it gets modified in the string-theoretical regime we can ideally repeat the same reasoning that has been done for the two charge case, but adding momentum to the D1 branes.
This has proved to be not trivial to perform in full generality, but a subset of these states has been constructed, see for instance [38]. In these cases one starts from a special, simple class of two charge solution, called "seed solutions", and then adds momentum in the form of a perturbation. We shall not give here a detailed exposition of the derivation, but let us just briefly summarize the salient results and features.

The general BPS solution of type IIB supergravity on $\mathbb{R}^{4,1} \times S^1 \times T^4$ (possibly with $K3$ instead of T^4), assuming invariance under T^4 rotations, that preserves the same charge as the D1D5P system is,

$$\begin{aligned}
ds_{10}^2 &= \frac{1}{\sqrt{\alpha}} ds_6^2 + \sqrt{\frac{Z_1}{Z_2}} dz_a dz^a, \\
ds_6^2 &= -2\frac{1}{\sqrt{\mathcal{P}}}(dv+\beta)\left[du + \omega + \frac{\mathcal{F}}{2}(dv+\beta)\right] + \sqrt{\mathcal{P}} d\tilde{s}_4^2, \\
e^{2\Phi} &= \frac{Z_1^2}{\mathcal{P}}, \\
B &= -\frac{Z_1}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_4 \wedge (dv+\beta) + \delta_2, \\
C_0 &= \frac{Z_4}{Z_1}, \\
C_2 &= -\frac{Z_2}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_1 \wedge (dv+\beta) + \gamma_2, \\
C_4 &= \frac{Z_4}{Z_2}\mathrm{vol}_{T^4} - \frac{Z_4}{\mathcal{P}}\gamma_2 \wedge (du+\omega)\wedge(dv+\beta) + x_3 \wedge (dv+\beta) + \mathcal{C}, \\
C_6 &= \mathrm{vol}_{T^4} \wedge \left[-\frac{Z_1}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_2 \wedge (dv+\beta) + \gamma_1\right] + \\
&\qquad - \frac{Z_4}{\mathcal{P}}\mathcal{C} \wedge (du+\omega)\wedge(dv+\beta),
\end{aligned} \tag{3.48}$$

where ds_{10}^2 is the ten-dimensional metric in the string frame, ds_6^2 is the six-dimensional metric on $\mathbb{R}^{4,1} \times S^1$ in the Einstein frame, $d\tilde{s}_4^2$ corresponds to some asymptotically flat Euclidean metric on the spatial $\mathbb{R}^4$. We have let

$$\alpha \equiv \frac{Z_1 Z_2}{\mathcal{P}}, \qquad \mathcal{P} = Z_1 Z_2 - Z_4^2. \tag{3.49}$$

Φ is the dilaton, B, C_p are the NSNS and RR gauge forms. $Z_1, Z_2, Z_4, \mathcal{F}$ are four scalar functions; $\beta, \omega, a_1, a_2, a_4$ are one-forms, $\gamma_1, \gamma_2, \delta_2$ two-forms, x_3 a three-form, all defined on spatial $\mathbb{R}^4$; $\mathcal{C}$ is also a four-form on $\mathbb{R}^4$ that can be set to zero via field redefinition. We have introduced also light-cone coordinates from time t and the compact S^1 direction y,

$$u = \frac{t-y}{\sqrt{2}}, \qquad v = \frac{t+y}{\sqrt{2}}. \tag{3.50}$$

The naïve three-charge geometry (3.18) has the form of (3.49), with

$$Z_1 = 1 + \frac{Q_1}{r^2}, \quad Z_2 = 1 + \frac{Q_5}{r^2}, \quad \mathcal{F} = -2Q_P, \quad Z_4 = 0 = \beta = \omega, \quad d\tilde{s}_4^2 = dx_i dx^i. \tag{3.51}$$

Nevertheless, the class of solution that is obtained is made of smooth, horizonless, asymptotically flat geometries with the same charges as the supersymmetric D1D5P black hole that we have analyzed above. Instead of having a singularity at $r=0$ there is a smooth cap, and different profiles provide different caps, just as we had for the two charge case.

3.3 The Fuzzball proposal

Let us compare the microstates that we have found, e.g. for the simple two charge case (3.37), with the corresponding naïve geometry (3.15).

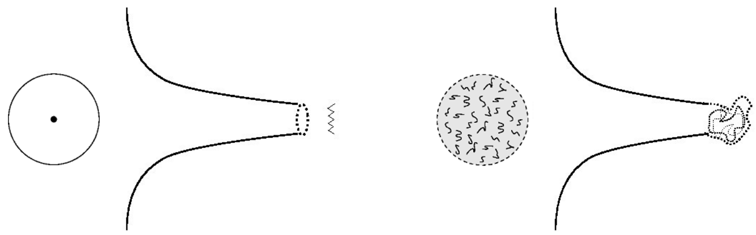

Figure 3.1: Comparison between naïve geometry (left) and fuzzball geometry (right).

We can distinguish four regions in both geometries:

- asymptotic region: $r^2 \gg Q_1 Q_5$. Both the microstates and the naïve geometry are asymptotically flat, and the geometry reduces to $\mathbb{R}^{4,1} \times S^1 \times T^4$;
- throat region: $r^2 \sim Q_1 Q_5 \gg |\vec{F}|^2$, where $|\vec{F}|^2$ is the characteristic amplitude of the excitation profile. In this region, for the microstates we have $Z_1 \sim 1 + \frac{Q_1}{r^2}$, $Z_2 \sim 1 + \frac{Q_5}{r^2}$ and $A \sim 0 \sim B$, thus they still look like the naïve geometry;
- "near-horizon" region: $|\vec{F}|^2 \ll r^2 \ll Q_1 Q_5$. Here the naïve geometry and the microstates start to differ. The functions Z_1, Z_2 for the microstates receive contributions from higher powers of $1/r$. In the asymptotic limit, the microstates are no more flat: the geometry becomes $AdS_3 \times S^3 \times T^4$,

$$\begin{aligned} ds^2 &\approx_{r\to\infty} \frac{r^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \frac{\sqrt{Q_1 Q_5}}{r^2} dr^2 + \sqrt{Q_1 Q_5} d\Omega_3^2 + \sqrt{\frac{Q_1}{Q_5}} dz_a dz^a \\ &= \sqrt{Q_1 Q_5}\left[\frac{d\tilde{r}^2}{\tilde{r}^2} + \tilde{r}^2(-dt^2 + dy^2)\right] + \sqrt{Q_1 Q_5} d\Omega_3^2 + \sqrt{\frac{Q_1}{Q_5}} dz_a dz^a, \end{aligned} \tag{3.52}$$

  where we have let $\tilde{r} = r/\sqrt{Q_1 Q_5}$.

- "cap" region: $r^2 \lesssim |\vec{F}|^2$. The microstates are completely smooth and horizonless all the way to $r = 0$. The naïve geometry does not have this region: it ends with a singularity at $r = 0$.

In the previous sections we have seen how the degeneracy of brane solutions in String Theory reproduces the macroscopic black hole entropy. We have also see what is the correct geometric interpretation that we should give, in this framework, to the microstates: they describe smooth solutions with different excitation profiles and no horizon. In this way, the naïve black hole geometry arises as a coarse-grained description of the ensemble of microscopic states.

This is the scenario depicted by the fuzzball proposal [29]. The proposal states that, for a macroscopic black hole of entropy S, there are $\sim e^S$ such microstates, thus solving the entropy puzzle.

The fuzzball proposal conveniently prescribes horizonless microstates. If it were not the case each microstate should be given an entropy, but this would go against the thermodynamic interpretation of black holes as a statistical ensemble. In the macroscopic description entropy arises with a coarse-graining procedure, whereas each microstate gives a full description of the state and should not be given an entropy.
This implies that the microstate geometry must be smooth in the black hole interior, all the way up to the "horizon". The consequences are radical: while it was commonly expected that quantum gravity effects should become relevant only when the curvature exceeds some fixed microscopical scale, e.g. the Planck length ℓ_P or the string length $\ell_s \sim \sqrt{\alpha'}$, the fuzzball proposal asks the classical solution to be modified already at the macroscopic horizon scale.

The fuzzball proposal could also give an answer to the information paradox as well. In the original computation by Hawking, it is assumed that General Relativity holds at the horizon scale, and the fuzzball proposal claims that this is not true. The microstates differ from each other and from the classical solution at the horizon scale , thus the creation of particle pairs near the horizon is in principle sensible to the properties of each microstate, and the emitted quanta can carry information about the microscopic configuration.

We should mention, however, that the fuzzball proposal is not entirely accepted as a description of black holes. Completion of the fuzzball program still requires to solve some crucial technical issues. While the description of two-charge extremal black holes has been largely achieved, this is not the case for three-charge configurations: it is even still unclear whether Supergravity is rich enough to admit their full characterization. Other subtle points involve the behaviour of non-extremal black holes as well as the typicality of microstates that allow for a Supergravity description within the thermodynamic ensemble.

In the thesis we will work in the fuzzball framework in order to study the behaviour of correlation functions on some black hole microstates. In order to formulate our query we need to introduce the appropriate language: this is done by means of AdS/CFT correspondence.

Chapter 4

Holography and the D1D5 CFT

Gravitational holography has been one of the most prolific ideas in Theoretical Physics for the last 25 years. The idea behind this theory is that a gravitational theory might as well be formulated as a nongravitational theory living in one dimension less [39]. Up to date it still remains a conjecture, but the empirical evidence for it is overwhelming. This is true in particular for its sharpest formulation, AdS/CFT correspondence, which provides many successful ways of testing itself.
In this Chapter we will present AdS/CFT correspondence, together with the arguments that lead to its statement. This framework will allow us to write a new, equivalent formulation of the D1D5 geometries: the D1D5 CFT.

4.1 AdS/CFT

AdS/CFT is the most powerful instance of Gauge/Gravity duality. It relates gravitational theories in AdS_{d+1} spaces with lower dimensional conformal field theories, CFT_d.

Before AdS/CFT was proposed, evidence for the existence of gravitational holography had already been collected. The simplest argument was made by 't Hooft and Susskind [39, 40]. Consider a gravitational theory, in which we somehow have managed to condense in a region of space an amount of entropy S greater than the Bekenstein-Hawking entropy of a black hole occupying the same region of space. Suppose now that we start throwing matter at our region until it forms a black hole that fills the region. In typical physical systems the entropy grows with the energy, thus we would expect the entropy to diminish outside the region and to grow inside the region: we would form a black hole with more entropy than its Bekenstein-Hawking entropy. So either the Bekenstein-Hawking formula is not true, or we are violating the second law of thermodynamics. To keep both we must ask our hypothesis to be false, i.e. that any given region of space can contain at most an amount of entropy given by

$$S_{max} = \frac{A}{4G_N}. \tag{4.1}$$

Since entropy is related to the number of degrees of freedom, we might hope that a $d+1$-dimensional gravitational system could be equivalently described by a d-dimensional non-gravitational theory.

Another hint at the relationship between AdS spaces and conformal field theories comes from the work of Brown and Henneaux [41], which focuses on the case of asymptotically AdS_3 spaces. When analyzing the asymptotic symmetries of such spaces, i.e. the diffeomorphisms that preserve the asymptotic AdS structure of the metric, one realizes that those symmetries from a group isomorphic to the conformal group in $d = 2$, and that the algebra of the diffeomorphisms is nothing but the Virasoro algebra with central charge

$$c = \frac{3R_{AdS}}{2G_N^{(3)}}, \tag{4.2}$$

where R_{AdS} is the radius of AdS_3 and $G_N^{(3)}$ is Newton's constant.

Further evidence for the correspondence is given by the study of large N gauge theories and their relationship with String Theory [42, 43]. Let us consider $SU(N)$ Yang-Mills gauge theories in four spacetime dimensions, where N is the number of colors. In such theories the only dimensionless parameter is N, since the coupling constant g_{YM} gets dimensionally transmuted to Λ_{QCD}. Thus one hopes that it is possible to study physics at the Λ_{QCD} scale in the large N limit, by means of a $1/N$ series expansion. The beta function equation reads

$$\mu\frac{dg_{YM}}{d\mu} = -\frac{11}{3}N\frac{g_{YM}^3}{16\pi^2} + \mathcal{O}\left(g_{YM}^5\right), \tag{4.3}$$

hence the leading terms are of the same order if we take the limit $N \to \infty$ while keeping the 't Hooft coupling $\lambda \equiv Ng_{YM}^2$ fixed. It turns out that the generating functional for connected correlation functions can be written as a power series in $1/N$,

$$\log Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda). \tag{4.4}$$

Here $f_g(\lambda)$ are functions of the 't Hooft coupling only. The index g has the meaning of the genus of the corresponding Feynman diagram, which is a closed, connected, oriented surface. For N large, the leading contribution will come from diagrams with minimal genus. This is exactly what happens in String Theory. Comparing with the discussion in Subsection 2.1.2, we see the identification

$$N \longleftrightarrow g_s = e^{\langle\Phi\rangle}. \tag{4.5}$$

But the main precursor of AdS/CFT correspondence arises from the physics of D-branes. As we have mentioned, D-branes are nonperturbative objects in String Theory, nevertheless we can consider perturbative expansions around brane backgrounds.
Let us consider then N parallel D3-branes in type IIB superstring theory. In the low-energy limit, the effective spacetime action is given by, schematically,

$$S = S_{bulk} + S_{branes} + S_{interaction}. \tag{4.6}$$

As we have seen in Subsection 2.1.1, we can describe gauge theories by looking at the low-energy spectrum of open strings ending on the stack of branes. As the presence of a given kind of D-brane halves the number of supersymmetries, in the limit of coinciding branes, S_{branes} describes a $\mathcal{N} = 4$ super-Yang-Mills gauge theory with gauge group $U(N)$ in $3 + 1$ dimensions, with Yang-Mills coupling given by [44]

$$g_{YM} = 4\pi g_s. \tag{4.7}$$

Thanks to supersymmetry, the theory is scale invariant also at the quantum level: its beta function vanishes exactly.
In the action (4.6), the term S_{bulk} describes closed string physics far from the branes and $S_{interaction}$ contains interaction terms between closed strings and branes. Since these interaction terms are proportional to powers of $g_s\alpha'^2 \sim \sqrt{G_N}$, in the classical limit $\alpha' \to 0$ we can neglect them. S_{bulk} is then just free gravity in the bulk.
On the other hand, D3-branes can act as sources for the gauge form field C_4. The ten-dimensional supergravity metric generated by a single D3-brane is given by

$$\begin{aligned} &ds^2 = f^{-\frac{1}{2}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_5^2\right), \\ &F_5 = \mathrm{d}C_4 = (1+\star)\,\mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}f^{-1}, \\ &f = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s \alpha^2 N, \end{aligned} \tag{4.8}$$

where we have denoted with x_i the spacelike coordinates along the brane, and with r, Ω_5 the polar coordinates in the remaining six-dimensional space.
The metric (4.8) is asymptotically flat, and in the "near-horizon" limit $r \sim R$ one has $f \sim (R/r)^4$. Thus in this limit the geometry becomes asymptotically $AdS_5 \times S^5$, with AdS radius given by R.
Consider now an object living in this geometry. Its energy E_p as measured by an observer at constant position r is related to its energy E as measured by an observer at infinity by a factor $E = f^{-\frac{1}{4}}E_p$. Thus if the object is located near $r \sim 0$ it will have very low E. This is important because we want to consider the low-energy limit of the theory as seen from the observer at infinity. Therefore we will have two kind of excitations in this limit: the ones living near $r \sim 0$ and the low-energy bulk excitations. It turns out that the two types of excitation decouple [42], and the low-energy theory consist then of two sectors: free bulk supergravity and near-horizon physics.

We see then that from the point of view of a field theory of open strings living on the branes, as well as from the point of view of the supergravity description, we have two decoupled theories in the low-energy limit. In both cases one of the systems is supergravity in flat space. It is natural to identify the other systems: we conjecture that $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in $3+1$ dimensions is equivalent to type IIB superstring theory on $AdS_5 \times S^5$. The relations between the parameters appearing on the two sides of the conjecture are

$$\lambda = N g_{YM}^2 \longleftrightarrow \left(\frac{R}{\ell_s}\right)^4, \qquad N \longleftrightarrow \frac{\pi^2}{\sqrt{2}}\left(\frac{R}{\ell_P}\right)^4. \tag{4.9}$$

The two theories are tractable in different parameter regimes. When $\lambda \ll 1$ one has $R \ll \ell_s$, so that the supergravity approximation of superstring theory is not reliable, whereas the Yang-Mills theory is weakly coupled. When $\lambda \gg 1$, conversely, one has $R \gg \ell_s$: the perturbative description of Yang-Mills breaks down, but the supergravity approximation holds. This makes the correspondence a powerful tool, but at the same time makes it hard to prove.

4.1.1 Statement of the correspondence

In its most famous formulation, AdS/CFT duality determines an equivalence between a gravity theory on AdS_{d+1}, the "bulk theory", and a local CFT living in d dimensions, the "boundary theory". In this Section it is our aim to make the equivalence manifest. We will follow the presentation of [42, 45].

Let $S_{AdS}[\Phi]$ be the action of the bulk theory, where we have denoted by Φ the fields living in the bulk theory. Typically, we will be interested in supergravity theories, so $S_{AdS}[\Phi]$ will be the effective supergravity action in $d+1$-dimensional AdS space, and Φ will denote the massless fields of supergravity: the dilaton, the graviton, R-R and NS-NS gauge fields, gauge fields sourced by branes.
Let us focus on the behaviour of the bulk theory near the AdS boundary, which is located at $z=0$ in its Poincaré patch. Let us suppose that the asymptotic expansion of bulk fields near the boundary satisfies

$$\Phi(x,z) \sim_{z\to 0} f(z)\phi_0(x), \tag{4.10}$$

with $f(0)=1$. The partition function of the bulk theory is then

$$Z_{AdS}\left[\Phi(z=0,x)=\phi_0(x)\right] = \int D\Phi e^{-S_{AdS}[\Phi]}, \tag{4.11}$$

where the path integral is taken over field configurations satisfying the boundary conditions (4.10).
We are now ready to state the correspondence. Let $\mathcal{O}$ be an operator on the conformal boundary theory. We say that $\mathcal{O}$ is dual to the bulk field Φ if it couples to ϕ_0 and the following equality holds,

$$\left\langle e^{\int \phi_0(x)\mathcal{O}(x)}\right\rangle_{CFT} = Z_{AdS}\left[\Phi(z=0,x)=\phi_0(x)\right]. \tag{4.12}$$

The correspondence thus depicts an equivalence between the generating function of correlation functions for $\mathcal{O}$ in the boundary field theory and the bulk generating functional.

Let us stress the fact that the quantities appearing on the left hand side of the duality, namely $\int \phi_0\mathcal{O}$, must be invariant under conformal transformations. This tells us something about the map between bulk fields and boundary operators. For instance, if Φ is a scalar (respectively vector, tensor) field in the bulk, then the dual operator $\mathcal{O}$ is also a scalar (respectively vector, tensor) operator on the boundary. Moreover, if $\Phi = A^a$ is a gauge field in the bulk theory, then it is sensible to expect that its dual operator in the boundary theory is J_a, the conserved current related to the corresponding global symmetry.

4.2 The D1D5 CFT

In Section 3.1 we have derived the Supergravity solution of the D1D5(P) system, and later in Section 3.2 we have described its string-theoretical formulation.
In light of gravitational holography, we wish to find its field-theoretical dual description as well. Our exposition will be based upon [46–48].

Once again, the starting point is type IIB theory compactified on $S^1\times T^4$. We assume to work in a region of moduli space where the radius R of S^1 is much larger than the radii of T^4, which we assume to be of order of the string length ℓ_s.
Let us include n_1 D1-branes wrapping around S^1, as well as n_5 D5-branes wrapping around the whole compact space. In the low-energy limit, we can safely neglect the contributions to the mass coming from winding modes and momentum modes along the torus, whereas the

momentum modes running around S^1 have to be considered, because $R \gg \ell_s$. The winding modes around S^1 can be discarded as well.
On the other hand, we have seen that the supergravity description of the D1D5 black hole has a near horizon geometry which is asymptotically $AdS_3 \times S^3 \times T^4$, and by AdS/CFT correspondence we expect that an equivalent description in terms of a two-dimensional superconformal field theory with 8 supersymmetries should exist. The central charge of the dual conformal field theory will be given by [47]

$$c = \frac{3R_{AdS}}{2G_N^{(3)}} = 6n_1n_5. \tag{4.13}$$

Following [46], we can give two field theoretical descriptions of the D1D5 system.

One possibility is to study the field theory that arises from open strings that have endpoints on either D1 or D5 branes. Explicitly, there are three possibilities: 5-5 strings with both endpoints on D5 branes, that give rise to a $U(n_5)$ gauge theory on 5+1 dimensions, with 16 supercharges; 1-1 strings with both endpoints on D1 branes, that give rise to a $U(n_1)$ gauge theory on 1+1 dimensions, still with 16 supercharges; and 1-5, 5-1 strings ending on different kind of branes at each endpoint, that transform in the fundamental representation of $U(n_1)$ (respectively $U(n_5)$) and in the antifundamental of $U(n_5)$ (respectively $U(n_1)$), with only 8 supercharges.
The Lagrangian for the bosonic part of the theory can be obtained via dimensional reduction: we can discard the T^4 and the flat transverse directions and work with a two-dimensional theory, parametrized by time and the S^1 coordinate. Being the theory supersymmetric, the vacua are obtained by imposing vanishing of the resulting potential for the fields, see for instance [47] for an explicit expression. There are two classes of vacua, that select two different sectors of the theory. In the so-called Coulomb branch, the 1-1 and 5-5 transverse string states acquire a nonvanishing expectation value, causing the stack of branes to separate and breaking the $U(n_1)$, $U(n_5)$ gauge groups into smaller ones. In the other sector, the Higgs branch, the transverse states of 1-5 and 5-1 strings get nonvanishing expectation values, the branes do not separate and form bound states: this is what we are looking for.

Alternatively, we can choose to describe D1-branes as solitonic[1] configurations inside the six-dimensional $U(n_5)$ gauge theory on D5-branes, that is n_1 strings wrapping around S^1 and localized on T^4. These solutions break one half of the sixteen supersymmetries of the D5-brane theory and have zero modes which form a moduli space. From our discussion we see then that the low-energy effective theory for the D1D5 system in the Higgs branch is a two-dimensional $\mathcal{N} = (4, 4)$ sigma-model with target space given by the instanton moduli space. The structure of moduli space is in general complicated, but there exist a choice of the parameters of the supergravity theory such that it takes the simple form

$$\frac{(T^4)^{n_1n_5}}{S_{n_1n_5}}, \tag{4.14}$$

where S_n is the symmetric group of degree n. This point in parameter space is called orbifold point.

[1]In literature they are usually referred to as "instantonic", even though they are not localized in time.

We can visualize the CFT at the free orbifold point as a collection of $N \equiv n_1 n_5$ strings, or "strands", each one with four bosons and four doublets of fermions,

$$\left(X^{A\dot{A}}_{(r)}(\tau,\sigma), \psi^{\alpha\dot{A}}_{(r)}(\tau+\sigma), \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\tau-\sigma)\right), \tag{4.15}$$

where $r = 1, \ldots, N = n_1 n_5$ is the strand index and (τ, σ) are the coordinates in the CFT, which will correspond to the coordinates (t, y) on the bulk side. $\alpha, \dot{\alpha} = 1, 2$ are spinorial indices for the R-symmetry group $SU(2)_L \times SU(2)_R$, which can be identified with the rotations in the S^3 factor of the bulk metric, whereas $A, \dot{A} = 1, 2$ are indices for the $SU(2)_1 \times SU(2)_2 \approx SO(4)_I$ rotations acting on the tangent space of T^4. As said, the CFT at the free orbifold point the theory has a further S_N discrete symmetry, that corresponds to permutations of the strands. Each strand of the CFT consists of four bosons and four fermions, and hence gives a contribution to the central charge $c_{(r)} = 6$.
For our purposes it will be useful to switch to Euclidean time in the CFT,

$$\tau \to -i\tau_E, \tag{4.16}$$

so that left- and right-moving fermions can be written as

$$\psi^{\alpha\dot{A}}_{(r)}(\tau_E + i\sigma), \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\tau_E - i\sigma). \tag{4.17}$$

The boundary of AdS_3 is a cylinder, thus σ will be periodic, with periodicity

$$\sigma \sim \sigma + 2\pi. \tag{4.18}$$

As usual, one maps the (τ_E, σ) cylinder onto the complex plane by letting

$$z = e^{\tau_E + i\sigma}, \qquad \bar{z} = e^{\tau_E - i\sigma}. \tag{4.19}$$

The spinor fields can then be rewritten as holomorphic and antiholomorphic,

$$\psi^{\alpha\dot{A}}_{(r)}(z), \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z}), \tag{4.20}$$

and the derivatives of bosons can also be split into holomorphic and antiholomorphic components,

$$\partial X^{A\dot{A}}_{(r)}(z), \qquad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}). \tag{4.21}$$

As mentioned the theory is an orbifold theory under the permutation group S_N. This means that we can consider the different twist sectors of the theory, corresponding to length k cycles in S_N.

4.2.1 The untwisted ($k = 1$) sector

The easiest case is the untwisted sector. In this sector each strand is completely independent of the others, thus we can think of the theory as a collection of N "singly wound" strands. With singly wound we mean that the boundary conditions on fields have to be imposed upon taking $\sigma \to \sigma + 2\pi$ on the cylinder or, equivalently, $z \to e^{2\pi i} z$ on the plane. The boundary conditions for bosons are periodic,

$$X^{A\dot{A}}_{(r)}(\tau_E, \sigma + 2\pi) = X^{A\dot{A}}_{(r)}(\tau_E, \sigma), \tag{4.22}$$

or, equivalently,

$$\partial X^{A\dot{A}}_{(r)}\left(e^{2\pi i}z\right)=\partial X^{A\dot{A}}_{(r)}(z),\quad \bar{\partial} X^{A\dot{A}}_{(r)}\left(e^{-2\pi i}\bar{z}\right)=\bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}). \tag{4.23}$$

Fermions instead can either be in the Ramond (R) sector or in the Neveau-Schwarz (NS) sector. Fermions in the R sector are periodic on the cylinder, but antiperiodic on the plane; viceversa, fermions in the NS sector are antiperiodic on the cylinder but become periodic on the plane. The difference arises because of the Jacobian factor coming from the change of coordinates. Explicitly, for holomorphic fermions in the R sector

$$\psi^{\alpha\dot{A}}_{(r)}(\tau,\sigma+2\pi)=\psi^{\alpha\dot{A}}_{(r)}(\tau,\sigma),\quad \psi^{\alpha\dot{A}}_{(r)}\left(e^{2\pi i}z\right)=-\psi^{\alpha\dot{A}}_{(r)}(z), \tag{4.24}$$

whereas in the NS sector

$$\psi^{\alpha\dot{A}}_{(r)}(\tau,\sigma+2\pi)=-\psi^{\alpha\dot{A}}_{(r)}(\tau,\sigma),\qquad \psi^{\alpha\dot{A}}_{(r)}\left(e^{2\pi i}z\right)=\psi^{\alpha\dot{A}}_{(r)}(z). \tag{4.25}$$

Analogous relations hold in the antiholomorphic sector replacing $(z\to e^{2\pi i}z)$ with $(\bar{z}\to e^{-2\pi i}\bar{z})$.

The mode expansion of the fields respect the boundary conditions. For bosons we have

$$\partial X^{A\dot{A}}_{(r)}(z)=\sum_{n\in\mathbb{Z}}\alpha^{A\dot{A}}_{(r)n}z^{-n-1},\qquad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z})=\sum_{n\in\mathbb{Z}}\tilde{\alpha}^{A\dot{A}}_{(r)n}\bar{z}^{-n-1}, \tag{4.26}$$

for the fermions in the R sector we have

$$\psi^{\alpha\dot{A}}_{(r)}(z)=\sum_{n\in\mathbb{Z}}\psi^{\alpha\dot{A}}_{(r)n}z^{-n-\frac{1}{2}},\qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})=\sum_{n\in\mathbb{Z}}\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}\bar{z}^{-n-\frac{1}{2}}, \tag{4.27}$$

whereas for fermions in the NS sector

$$\psi^{\alpha\dot{A}}_{(r)}(z)=\sum_{n\in\mathbb{Z}+\frac{1}{2}}\psi^{\alpha\dot{A}}_{(r)n}z^{-n-\frac{1}{2}},\qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})=\sum_{n\in\mathbb{Z}+\frac{1}{2}}\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}\bar{z}^{-n-\frac{1}{2}}. \tag{4.28}$$

The OPE of fermions and bosons are

$$\begin{aligned}
\psi^{1\dot{A}}_{(r)}(z)\psi^{2\dot{B}}_{(s)}(w)&=\frac{\epsilon^{\dot{A}\dot{B}}\delta_{rs}}{z-w}+[\text{reg}\cdot],\\
\psi^{1\dot{A}}_{(r)}(\bar{z})\psi^{2\dot{B}}_{(s)}(\bar{w})&=\frac{\epsilon^{\dot{A}\dot{B}}\delta_{rs}}{\bar{z}-\bar{w}}+[\text{reg}\cdot],\\
\partial X^{A\dot{A}}_{(r)}(z)\partial X^{B\dot{B}}_{(s)}(w)&=\frac{\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\delta_{rs}}{(z-w)^2}+[\text{reg.}],\\
\bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z})\bar{\partial} X^{B\dot{B}}_{(s)}(\bar{w})&=\frac{\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\delta_{rs}}{(\bar{z}-\bar{w})^2}+[\text{reg.}],
\end{aligned} \tag{4.29}$$

where $\epsilon_{AB},\epsilon_{\dot{A}\dot{B}}$ are totally antisymmetric with $\epsilon_{12}=\epsilon_{\dot{1}\dot{2}}=-\epsilon^{12}=-\epsilon^{\dot{1}\dot{2}}=1$, $\epsilon_{AB}\epsilon^{BC}=\delta^{C}_{A}$. They imply the mode algebras

$$\begin{aligned}
\left[\alpha^{A\dot{A}}_{(r)n},\alpha^{B\dot{B}}_{(s)m}\right]&=\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}n\delta_{n+m,0}\delta_{rs}, &\qquad \left[\tilde{\alpha}^{A\dot{A}}_{(r)n},\tilde{\alpha}^{B\dot{B}}_{(s)m}\right]&=\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}n\delta_{n+m,0}\delta_{rs},\\
\left\{\psi^{1\dot{A}}_{(r)n},\psi^{2\dot{B}}_{(s)m}\right\}&=\epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs}, &\qquad \left\{\tilde{\psi}^{\dot{1}\dot{A}}_{(r)n},\psi^{\dot{2}\dot{B}}_{(s)m}\right\}&=\epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs},
\end{aligned} \tag{4.30}$$

which hold for fermions both in the R sector and in the NS sector.

On each one of the N copies we can define a vacuum state $|0\rangle_{(r)}$. Strictly speaking, it will be the product of a vacuum state for bosons and one for fermions, and each in turn will be a product of a vacuum state in the holomorphic sector and one in the antiholomorphic sector. However, as we will see shortly, we have to pay attention whether we are considering the vacuum state for fermions in the R sector or in the NS sector. Nevertheless, we assume the vacuum state to be normalized, i.e.

$$_{(r)}\langle 0|0\rangle_{(s)} = \delta_{r,s}. \tag{4.31}$$

By definition, the vacuum is annihilated by positive modes of fermionic and bosonic operators. For the bosons we will assume

$$\alpha^{A\dot{A}}_{(r)n}|0\rangle_{(r)} = 0, \quad \tilde{\alpha}^{A\dot{A}}_{(r)n}|0\rangle_{(r)} = 0, \quad \forall n \geq 0, \quad \forall A, \dot{A}, \tag{4.32}$$

that is, that also the zero modes annihilate the vacuum. This is because zero modes of bosons are related to momentum excitations along some direction on T^4, which we do not want to have. Fermions in the NS sector do not have zero modes. Letting $|0\rangle_{(r),\mathrm{NS}}$ be the fermionic (normalized) NS vacuum, we can safely write

$$\psi^{\alpha\dot{A}}_{(r)n}|0\rangle_{(r),\mathrm{NS}} = 0, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}|0\rangle_{(r),\mathrm{NS}} = 0, \quad \forall n > 0, \quad \forall \alpha, \dot{\alpha}, \dot{A}. \tag{4.33}$$

Fermions in the R sector do have zero modes, and only half of them will annihilate the vacuum. Let us denote by $|++\rangle_{(r),\mathrm{R}}$ the (normalized) R vacuum. Explicitly,

$$\begin{gathered}\psi^{\alpha\dot{A}}_{(r)n}|++\rangle_{(r),\mathrm{R}} = 0, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}|++\rangle_{(r),\mathrm{R}} = 0, \quad \forall n > 0, \quad \forall \alpha, \dot{\alpha}, \dot{A}, \\ \psi^{1\dot{A}}_{(r)0}|++\rangle_{(r),\mathrm{R}} = 0, \qquad \tilde{\psi}^{\dot{1}\dot{A}}_{(r)0}|++\rangle_{(r),\mathrm{R}} = 0,\end{gathered} \tag{4.34}$$

whereas when acting on the R vacuum with the zero modes $\psi^{2\dot{A}}_{(r)0}, \tilde{\psi}^{\dot{2}\dot{A}}_{(r)0}$ we get other degenerate vacua with the same energy. We shall return on the matter of degenerate vacua later.

With bosons and fermions we can build composite operators.
For us, operators of particular importance will be the current operators related to the R-symmetry group $SU(2)_L \times SU(2)_R$. In the holomorphic part of the R sector they are

$$\begin{aligned} J^{+}_{(r)} &= \frac{1}{2} : \psi^{1\dot{A}}_{(r)}\psi^{1\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} := J^{1}_{(r)} + iJ^{2}_{(r)}, \\ J^{-}_{(r)} &= -\frac{1}{2} : \psi^{2\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} := J^{1}_{(r)} - iJ^{2}_{(r)}, \\ J^{3}_{(r)} &= -\frac{1}{2}\left(: \psi^{1\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} : -1\right), \end{aligned} \tag{4.35}$$

where colons denote normal ordering with respect to the $|{+}{+}\rangle_{(r)}$ Ramond vacuum. Analogous definitions apply for antiholomorphic currents. Their mode expansion is

$$J^{a}_{(r)}(z) = \sum_{n\in\mathbb{Z}} J^{a}_{(r)n} z^{-n-1}, \tag{4.36}$$

and the modes satisfy the affine algebra

$$\left[J^{a}_{(r)n}, J^{b}_{(s)m}\right] = i\epsilon^{abc}J^{c}_{(r)n+m}\delta_{r,s} + \frac{c_{1\text{ copy}}}{12} n d^{ab}\delta_{r,s}\delta_{m+n,0}. \tag{4.37}$$

where d^{ab} is the $SU(2)$ flat metric and ϵ^{abc} is totally antisymmetric, with $\epsilon^{123}=1$. The mode algebra implies the OPE

$$J^a_{(r)}(z)J^b_{(s)}(w)=\frac{\delta_{r,s}}{(z-w)^2}d^{ab}+\frac{\delta_{r,s}}{z-w}i\epsilon^{abc}J^c_{(r)}(w)+[\text{reg}\cdot]. \tag{4.38}$$

The currents are built out of R fermions, and have zero modes which do not annihilate the R vacuum. The constant term in $J^3_{(r)}$ is such that $|++\rangle_{(r)}$ has $(1/2,1/2)$ eigenvalues under $(J^3_{(r),0},\tilde{J}^3_{(r),0})$. The R vacuum is a highest weight state under the (zero modes of the) raising operators $J^+_{(r)}, J^-_{(r)}$, whereas by acting on $|++\rangle_{(r)}$ with the zero modes of the lowering operators we get

$$|-+\rangle_{(r)}\equiv J^-_{(r)0}|++\rangle_{(r)},\quad |+-\rangle_{(r)}\equiv \tilde{J}^-_{(r)}|++\rangle_{(r)},\quad |--\rangle_{(r)}\equiv J^-_{(r)}\tilde{J}^-_{(r)0}|++\rangle_{(r)}, \tag{4.39}$$

where a minus now signals that the eigenvalue of to the zero mode of the corresponding diagonal current operator is $-1/2$.

We can define a further R vacuum state by introducing the operators

$$O^{\alpha\dot{\alpha}}_{(r)}(z,\bar{z})\equiv\frac{-i}{\sqrt{2}}:\psi^{\alpha\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{\alpha}\dot{B}}_{(r)}:(z,\bar{z})=\sum_{n,m\in\mathbb{Z}}O^{\alpha\dot{\alpha}}_{(r)mn}z^{-n-\frac{1}{2}}\bar{z}^{-m-\frac{1}{2}}, \tag{4.40}$$

which are still operators of total conformal dimension 1, so that

$$|00\rangle_{(r)}\equiv\lim_{z\to 0}O^{2\dot{2}}_{(r)}(z,\bar{z})|++\rangle_{(r)}=O^{2\dot{2}}_{(r)}|++\rangle_{(r)}=\frac{-i}{\sqrt{2}}\psi^{2\dot{A}}_{(r)0}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)0}|++\rangle_{(r)}. \tag{4.41}$$

Another important operator that is present in every CFT is of course the stress-energy operator,

$$\begin{aligned}T_{(r)}(z)&=T^B_{(r)}(z)+T^F_{(r)}(z)=\textstyle\sum_{n\in\mathbb{Z}}L_{(r)n}z^{-n-2}\\ T^B_{(r)}(z)&=\tfrac{1}{2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\cdot\partial X^{A\dot{A}}_{(r)}(z)\partial X^{B\dot{B}}_{(r)}(z):=\textstyle\sum_{n\in\mathbb{Z}}L^B_{(r)n}z^{-n-2}\\ T^F_{(r)}(z)&=\tfrac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}:\psi^{\alpha\dot{A}}_{(r)}(z)\partial\psi^{\beta\dot{B}}_{(r)}(z):=\textstyle\sum_{n\in\mathbb{Z}}L^F_{(r)n}z^{-n-2}\end{aligned} \tag{4.42}$$

where B,F refer to the bosonic and fermionic part, respectively. The modes $L_{(r),n}$ satisfy the usual Virasoro algebra

$$\left[L_{(r)n},L_{(s)m}\right]=(n-m)L_{(r)n+m}\delta_{r,s}-\frac{c_{1\text{ copy}}}{12}n\left(n^2-1\right)\delta_{n+m,0}\delta_{r,s}, \tag{4.43}$$

which is reproduced by the OPE

$$T_{(r)}(z)T_{(s)}(w)=\delta_{r,s}\frac{c_{1\text{ copy}}/2}{(z-w)^4}+\delta_{r,s}\frac{2T_{(r)}(w)}{(z-w)^2}+\delta_{r,s}\frac{\partial T_{(r)}(w)}{z-w}+[\text{reg.}]. \tag{4.44}$$

The currents and the stress-energy operator satisfy

$$\left[J^a_{(r)n},L_{(s)m}\right]=nJ^a_{n+m}\delta_{r,s}, \tag{4.45}$$

which, OPE-wise, is written as

$$J^a_{(r)}(z)T_{(s)}(w)=\delta_{r,s}\frac{J^a_{(r)}(w)}{z-w}+[\text{reg.}] \tag{4.46}$$

The supersymmetry of the theory implies the presence of another class of operators, the supercurrents,

$$G^{\alpha A}_{(r)} = \psi^{\alpha\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\partial X^{A\dot{B}}_{(r)} = \sum_n G^{\alpha A}_{(r)n} z^{-n-\frac{3}{2}}, \tag{4.47}$$

where $n \in \mathbb{Z}$ in the R sector, and $n \in \mathbb{Z}+\frac{1}{2}$ in the NS sector. They have conformal dimension $(h, \bar{h}) = (\frac{3}{2}, 0)$, and satisfy the mode algebra

$$\begin{aligned}\left\{G^{\alpha A}_{(r)m}, G^{\beta B}_{(s)n}\right\} = &-\frac{c_{1\text{copy}}}{6}\left(m^2 - \frac{1}{2}\right)\epsilon^{AB}\epsilon^{\alpha\beta}\delta_{m+n,0}\delta_{r,s}+\\ &+(m-n)\epsilon^{AB}\epsilon^{\beta\gamma}\left(\sigma^{*a}\right)^{\alpha}_{\gamma} J^a_{m+n}\delta_{r,s} - \epsilon^{AB}\epsilon^{\alpha\beta}L_{m+n}\delta_{r,s},\end{aligned} \tag{4.48}$$

where σ^a are the usual Pauli matrices.

It is possible to obtain the operators in the full CFT from operators defined on single strands by summing over copies,

$$\mathcal{O} = \sum_{r=1}^{N}\mathcal{O}_{(r)}, \tag{4.49}$$

where it is intended that the operator $\mathcal{O}_{(r)}$ acts trivially on every copy of the CFT except from the rth. The expressions that we have given so far will still be valid, provided that we replace $c_{1\text{ copy}} = 6$ by $c = 6N$.

4.2.2 The twisted $(k > 1)$ sector

In general, we can have m_i strands of length k_i such that $\sum_i m_i k_i = N$. Thus we must discuss also strands with length $k > 1$, which are multiwound. By multiwound we mean that by sending $\sigma \to \sigma + 2\pi$, or $z \to e^{2\pi i}z$, we end up on a differend copy of the CFT, hence we are changing boundary condition for fields. For bosons living on a length k strand, this means

$$\partial X^{A\dot{A}}_{(r)}\left(e^{2\pi i}z\right) = \partial X^{A\dot{A}}_{(r+1)}(z), \quad \bar{\partial} X^{A\dot{A}}_{(r)}\left(e^{-2\pi i}\bar{z}\right) = \bar{\partial} X^{A\dot{A}}_{(r+1)}(\bar{z}), \tag{4.50}$$

where the copy index $r = 1, \dots k$ is defined modulo k.
The new boundary conditions are thus non-diagonal in copy indices, and this is inconvenient because it does not allow us to write a mode expansion for the fields. A natural choice is then to diagonalize the boundary conditions on a length k strand. This can be done by switching to a new basis, which we label by $\rho = 0, \dots, k-1$. For bosons in the holomorphic sector, this reads

$$\begin{aligned}\partial X^{1\dot{1}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\partial X^{1\dot{1}}_{(r)}(z), \quad \partial X^{2\dot{2}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\partial X^{2\dot{2}}_{(r)}(z),\\ \partial X^{1\dot{2}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\partial X^{1\dot{2}}_{(r)}(z), \quad \partial X^{2\dot{1}}_{\rho}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\partial X^{2\dot{1}}_{(r)}(z),\end{aligned} \tag{4.51}$$

with monodromy conditions

$$\begin{aligned}\partial X^{1\dot{1}}_{\rho}\left(e^{2\pi i}z\right) &= e^{2\pi i\frac{\rho}{k}}\partial X^{1\dot{1}}_{\rho}(z), \quad \partial X^{2\dot{2}}_{\rho}\left(e^{2\pi i}z\right) = e^{-2\pi i\frac{\rho}{k}}\partial X^{2\dot{2}}_{\rho}(z),\\ \partial X^{12}_{\rho}\left(e^{2\pi i}z\right) &= e^{-2\pi i\frac{\rho}{k}}\partial X^{12}_{\rho}(z), \quad \partial X^{2\dot{1}}_{\rho}\left(e^{2\pi i}z\right) = e^{2\pi i\frac{\rho}{k}}\partial X^{2\dot{1}}_{\rho}(z),\end{aligned} \tag{4.52}$$

and mode expansions

$$\begin{aligned}
\partial X_{\rho}^{1\dot{1}}(z) &= \sum_{n\in\mathbb{Z}} \alpha_{\rho,n-\frac{\rho}{k}}^{1\dot{1}} z^{-n-1+\frac{\rho}{k}}, & \partial X_{\rho}^{2\dot{2}}(z) &= \sum_{n\in\mathbb{Z}} \alpha_{\rho,n+\frac{\rho}{k}}^{2\dot{2}} z^{-n-1-\frac{\rho}{k}}, \\
\partial X_{\rho}^{1\dot{2}}(z) &= \sum_{n\in\mathbb{Z}} \alpha_{\rho,n+\frac{\rho}{k}}^{1\dot{2}} z^{-n-1-\frac{\rho}{k}}, & \partial X_{\rho}^{2\dot{1}}(z) &= \sum_{n\in\mathbb{Z}} \alpha_{\rho,n-\frac{\rho}{k}}^{2\dot{1}} z^{-n-1+\frac{\rho}{k}}.
\end{aligned} \tag{4.53}$$

Analogous relations hold in the antiholomorphic sector by letting $(z, \partial, \alpha, i) \to (\bar{z}, \bar{\partial}, \tilde{\alpha}, -i)$. The mode algebra for bosons is now realized in the ρ basis,

$$\left[\alpha_{\rho_1,n}^{A\dot{A}}, \alpha_{\rho_2,m}^{B\dot{B}}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}} n\delta_{n+m,0}\delta_{\rho_1,\rho_2}, \quad \left[\tilde{\alpha}_{\rho_1,n}^{A\dot{A}}, \tilde{\alpha}_{\rho_2,m}^{B\dot{B}}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}} n\delta_{n+m,0}\delta_{\rho_1,\rho_2}. \tag{4.54}$$

Something similar happens for fermions, for which we still have to distinguish between the R sector and the NS sector. The R sector is analogous to the bosonic sector. The boundary conditions for fermions in the r basis are

$$\psi_{(r)}^{\alpha\dot{A}}\left(e^{2\pi i}z\right) = -\psi_{(r+1)}^{\alpha\dot{A}}(z), \quad \tilde{\psi}_{(r)}^{\dot{\alpha}\dot{A}}\left(e^{-2\pi i}\bar{z}\right) = -\tilde{\psi}_{(r+1)}^{\dot{\alpha}\dot{A}}(\bar{z}). \tag{4.55}$$

We can diagonalize them by switching to the ρ basis,

$$\begin{aligned}
\psi_{\rho}^{1\dot{A}}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\psi_{(r)}^{1\dot{A}}(z), & \psi_{\rho}^{2\dot{A}}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\psi_{(r)}^{2\dot{A}}(z), \\
\tilde{\psi}_{\rho}^{\dot{1}\dot{A}}(\bar{z}) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\tilde{\psi}_{(r)}^{\dot{1}\dot{A}}(\bar{z}), & \tilde{\psi}_{\rho}^{\dot{2}\dot{A}}(\bar{z}) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\tilde{\psi}_{(r)}^{\dot{2}\dot{A}}(\bar{z}),
\end{aligned} \tag{4.56}$$

where they read

$$\begin{aligned}
\psi_{\rho}^{1\dot{A}}\left(e^{2\pi i}z\right) &= -e^{-2\pi i\frac{\rho}{k}}\psi_{\rho}^{1\dot{A}}(z), & \psi_{\rho}^{2\dot{A}}\left(e^{2\pi i}z\right) &= -e^{2\pi i\frac{\rho}{k}}\psi_{\rho}^{2\dot{A}}(z), \\
\tilde{\psi}_{\rho}^{\dot{1}\dot{A}}\left(e^{-2\pi i}\bar{z}\right) &= -e^{2\pi i\frac{\rho}{k}}\tilde{\psi}_{\rho}^{\dot{1}\dot{A}}(\bar{z}), & \tilde{\psi}_{\rho}^{\dot{2}\dot{A}}\left(e^{-2\pi i}\bar{z}\right) &= -e^{-2\pi i\frac{\rho}{k}}\tilde{\psi}_{\rho}^{\dot{2}\dot{A}}(\bar{z}).
\end{aligned} \tag{4.57}$$

Subsequently, the mode expansions of fermions in the R sector will be

$$\begin{aligned}
\psi_{\rho}^{1\dot{A}}(z) &= \sum_{n\in\mathbb{Z}} \psi_{\rho,n+\frac{\rho}{k}}^{1\dot{A}} z^{-n-\frac{1}{2}-\frac{\rho}{k}}, & \psi_{\rho}^{2\dot{A}}(z) &= \sum_{n\in\mathbb{Z}} \psi_{\rho,n-\frac{\rho}{k}}^{2\dot{A}} z^{-n-\frac{1}{2}+\frac{\rho}{k}}, \\
\tilde{\psi}_{\rho}^{\dot{1}\dot{A}}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \tilde{\psi}_{\rho,n+\frac{\rho}{k}}^{\dot{1}\dot{A}} \bar{z}^{-n-\frac{1}{2}-\frac{\rho}{k}}, & \tilde{\psi}_{\rho}^{\dot{2}\dot{A}}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \tilde{\psi}_{\rho,n-\frac{\rho}{k}}^{\dot{2}\dot{A}} \bar{z}^{-n-\frac{1}{2}+\frac{\rho}{k}}.
\end{aligned} \tag{4.58}$$

The monodromy condition on the entire strand in the ρ basis is

$$\psi_{\rho}^{\alpha\dot{A}}\left(e^{2\pi ik}z\right) = (-1)^k\psi_{\rho}^{\alpha\dot{A}}(z), \quad \tilde{\psi}_{\rho}^{\dot{\alpha}\dot{A}}\left(e^{-2\pi ik}\bar{z}\right) = (-1)^k\tilde{\psi}_{\rho}^{\dot{\alpha}\dot{A}}(\bar{z}). \tag{4.59}$$

The fermion modes satisfy the algebra

$$\left\{\psi_{\rho_1,n}^{1\dot{A}}, \psi_{\rho_2,m}^{2\dot{B}}\right\} = \epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{\rho_1,\rho_2}, \quad \left\{\tilde{\psi}_{\rho_1,n}^{\dot{1}\dot{B}}, \tilde{\psi}_{\rho_2,m}^{\dot{2}\dot{B}}\right\} = \epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{\rho_1,\rho_2}. \tag{4.60}$$

For fermions in the NS sector, the boundary conditions in the r basis read

$$\psi_{(r)}^{\alpha\dot{A}}\left(e^{2\pi i}z\right) = \psi_{(r+1)}^{\alpha\dot{A}}(z), \quad \tilde{\psi}_{(r)}^{\dot{\alpha}\dot{A}}\left(e^{-2\pi i}\bar{z}\right) = \tilde{\psi}_{(r+1)}^{\dot{\alpha}\dot{A}}(\bar{z}), \tag{4.61}$$

but since we are in the NS sector we have to enforce "antiperiodicity" by defining $\psi_{(k+1)}^{\alpha\dot{A}} \equiv (-1)^{k+1}\psi_{(1)}^{\alpha\dot{A}}$ and $\tilde{\psi}_{(k+1)}^{\dot{\alpha}\dot{A}} \equiv (-1)^{k+1}\tilde{\psi}_{(1)}^{\dot{\alpha}\dot{A}}$. The ρ basis is thus not convenient for fermions in the NS sector. Let us introduce a new basis, labelled by $l = -\frac{k-1}{2}, -\frac{k-1}{2}+1, \dots \frac{k-1}{2}$,

$$\begin{aligned}
\psi_{l}^{1\dot{A}}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{rl}{k}}\psi_{(r)}^{1\dot{A}}(z), & \psi_{l}^{2\dot{A}}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{rl}{k}}\psi_{(r)}^{2\dot{A}}(z), \\
\tilde{\psi}_{l}^{\dot{1}\dot{A}}(\bar{z}) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{rl}{k}}\tilde{\psi}_{(r)}^{\dot{1}\dot{A}}(\bar{z}), & \tilde{\psi}_{l}^{\dot{2}\dot{A}}(\bar{z}) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{rl}{k}}\tilde{\psi}_{(r)}^{\dot{2}\dot{A}}(\bar{z}),
\end{aligned} \tag{4.62}$$

where the monodromy conditions are diagonal,

$$\begin{aligned} \psi_l^{1\dot{A}}\left(e^{2\pi i}z\right) &= e^{-2\pi i\frac{l}{k}}\psi_l^{1\dot{A}}(z), & \psi_l^{2\dot{A}}\left(e^{2\pi i}z\right) &= e^{2\pi i\frac{l}{k}}\psi_l^{2\dot{A}}(z), \\ \tilde{\psi}_l^{\dot{1}\dot{A}}\left(e^{-2\pi i}\bar{z}\right) &= e^{2\pi i\frac{l}{k}}\tilde{\psi}_l^{\dot{1}\dot{A}}(\bar{z}), & \tilde{\psi}_l^{\dot{2}\dot{A}}\left(e^{-2\pi i}\bar{z}\right) &= e^{-2\pi i\frac{l}{k}}\tilde{\psi}_l^{\dot{2}\dot{A}}(\bar{z}), \end{aligned} \tag{4.63}$$

and the mode expansions read

$$\begin{aligned} \psi_l^{1\dot{A}}(z) &= \textstyle\sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{1\dot{A}}_{l,n+\frac{l}{k}} z^{-n-\frac{1}{2}-\frac{l}{k}}, & \psi_l^{2\dot{A}}(z) &= \textstyle\sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{2\dot{A}}_{l,n-\frac{l}{k}} z^{-n-\frac{1}{2}+\frac{l}{k}}, \\ \tilde{\psi}_l^{\dot{1}\dot{A}}(\bar{z}) &= \textstyle\sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^{\dot{1}\dot{A}}_{l,n+\frac{l}{k}} \bar{z}^{-n-\frac{1}{2}-\frac{l}{k}}, & \tilde{\psi}_l^{\dot{2}\dot{A}}(\bar{z}) &= \textstyle\sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^{\dot{2}\dot{A}}_{l,n-\frac{l}{k}} \bar{z}^{-n-\frac{1}{2}+\frac{l}{k}}. \end{aligned} \tag{4.64}$$

The monodromy condition on the entire strand in the l basis is

$$\psi_l^{\alpha\dot{A}}\left(e^{2\pi ik}z\right) = (-1)^{k+1}\psi_l^{\alpha\dot{A}}(z), \quad \tilde{\psi}_l^{\dot{\alpha}\dot{A}}\left(e^{-2\pi ik}\bar{z}\right) = (-1)^{k+1}\tilde{\psi}_l^{\dot{\alpha}\dot{A}}(\bar{z}). \tag{4.65}$$

Vacuum states in the twisted sector are analogous to the ones in the untwisted sector, except from the difference in monodromy conditions. Let us denote by $|0\rangle_k$ the bosonic vacuum on a strand of length k; we ask that

$$\alpha^{A\dot{A}}_{\rho,n}|0\rangle_k = 0, \quad \tilde{\alpha}^{A\dot{A}}_{\rho,n}|0\rangle_k = 0, \quad \forall n \geq 0, \quad \forall A, \dot{A}, \tag{4.66}$$

as we did in the untwisted sector.
Similarly, for fermions in the R sector we have the set of vacua

$$|\pm\pm\rangle_k, \quad |00\rangle_k, \tag{4.67}$$

with the highest weight state $|++\rangle_k$ that is annihilated by the positive modes and half of the zero modes in the ρ basis,

$$\begin{aligned} &\psi^{\alpha\dot{A}}_{\rho,n}|++\rangle_k = 0, \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho,n}|++\rangle_k = 0, \quad \forall n > 0, \quad \forall \alpha, \dot{\alpha}, \dot{A}, \\ &\psi^{1\dot{A}}_{\rho,0}|++\rangle_k = 0, \quad \tilde{\psi}^{\dot{1}\dot{A}}_{\rho,0}|++\rangle_k = 0 \end{aligned} \tag{4.68}$$

and the other vacua are obtained from $|++\rangle_k$ by acting with $J^-, \tilde{J}^-, O^{2\dot{2}}$ in an analogous way as in the untwisted sector.
For fermions in the NS sector there exist only one vacuum,

$$|0\rangle_{k,\mathrm{NS}}, \tag{4.69}$$

which is annihilated by positive modes of the fermions. Notice that the NS vacuum behaves as a scalar under $SU(2)_L \times SU(2)_R$, like in the untwisted sector.

Let us generalize also the current operators, the $O^{\alpha\dot{\alpha}}$ operators and the stress-energy operator to the case of a strand of length k. It is again natural to work in the ρ basis. For the holomorphic part of current operators in the R sector we have

$$\begin{aligned} J^+ &= \sum_{r=1}^k J^+_{(r)} = \frac{1}{2}\sum_{r=1}^k :\psi^{1\dot{A}}_{(r)}\psi^{1\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{1}{2}\left(:\psi^{1\dot{A}}_{\rho=0}\psi^{1\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{1\dot{A}}_{\rho}\psi^{1\dot{B}}_{k-\rho}\epsilon_{\dot{A}\dot{B}}: \right) \\ J^- &= \sum_{r=1}^k J^-_{(r)} = -\frac{1}{2}\sum_{r=1}^k :\psi^{2\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = -\frac{1}{2}\left(:\psi^{2\dot{A}}_{\rho=0}\psi^{2\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{2\dot{A}}_{\rho}\psi^{2\dot{B}}_{k-\rho}\epsilon_{\dot{A}\dot{B}}: \right) \\ J^3 &= \sum_{r=1}^k J^3_{(r)} = -\frac{1}{2}\sum_{r=1}^k \left(:\psi^{1\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: -1 \right) = -\frac{1}{2}\sum_{\rho=1}^{k-1} \left(:\psi^{1\dot{A}}_{\rho}\psi^{2\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: -1 \right). \end{aligned} \tag{4.70}$$

In passing from fermions in the r basis to fermions in the ρ basis we have used the inverse transformations of the ones in (4.58) and

$$\sum_{r=1}^{k} e^{2\pi i \frac{r}{k}(\rho_1+\rho_2)} = k\delta_{\rho_1+\rho_2,0}. \tag{4.71}$$

Their antiholomorphic version are analogous. Similarly, for the $O^{\alpha\dot{\alpha}}$ operators,

$$\begin{aligned}
O^{1\dot{1}} &= \sum_{r=1}^{k} O^{1\dot{1}}_{(r)} = \frac{-i}{\sqrt{2}} \sum_{r=1}^{k} :\psi^{1\dot{A}}_{(r)}\tilde{\psi}^{\dot{1}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-i}{\sqrt{2}} \sum_{\rho=0}^{k-1} :\psi^{1\dot{A}}_{\rho}\tilde{\psi}^{\dot{1}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}:,\\
O^{2\dot{2}} &= \sum_{r=1}^{k} O^{2\dot{2}}_{(r)} = \frac{-i}{\sqrt{2}} \sum_{r=1}^{k} :\psi^{2\dot{A}}_{(r)}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-i}{\sqrt{2}} \sum_{\rho=0}^{k-1} :\psi^{2\dot{A}}_{\rho}\tilde{\psi}^{\dot{2}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}:,\\
O^{1\dot{2}} &= \sum_{r=1}^{k} O^{1\dot{2}}_{(r)} = \frac{-i}{\sqrt{2}} \sum_{r=1}^{k} :\psi^{1\dot{A}}_{(r)}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-i}{\sqrt{2}} \left(:\psi^{1\dot{A}}_{\rho=0}\tilde{\psi}^{\dot{2}\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{1\dot{A}}_{\rho}\tilde{\psi}^{\dot{2}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: \right),\\
O^{2\dot{1}} &= \sum_{r=1}^{k} O^{2\dot{1}}_{(r)} = \frac{-i}{\sqrt{2}} \sum_{r=1}^{k} :\psi^{2\dot{A}}_{(r)}\tilde{\psi}^{\dot{1}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-i}{\sqrt{2}} \left(:\psi^{2\dot{A}}_{\rho=0}\tilde{\psi}^{\dot{1}\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{2\dot{A}}_{\rho}\tilde{\psi}^{\dot{1}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: \right).
\end{aligned} \tag{4.72}$$

The stress-energy operator on the length k strand can also be expressed in the ρ basis,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = T_B(z) + T_F(z) = \sum_{\rho=0}^{k-1} \left(T^B_\rho(z) + T^F_\rho(z)\right), \tag{4.73}$$

with

$$\begin{aligned}
T^B_\rho(z) &= \frac{1}{2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}} : \partial X^{A\dot{A}}_\rho \partial X^{B\dot{B}}_\rho : (z),\\
T^F_\rho(z) &= \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}} : \psi^{\alpha\dot{B}}_\rho \partial \psi^{\beta\dot{A}}_\rho : (z).
\end{aligned} \tag{4.74}$$

The modes L_n still satisfy the Virasoro algebra (4.43) on the strand of length k, and the OPE of $T(z)$ with itself is still the usual one, with central charge $c = 6k$.

4.2.3 Twist operators

In the D1D5 CFT there is also a special family of operators that relate different twist sectors, that is they relate strands with different k. They are called twist operators. With the strand picture in mind, the twist operators are operators that perform the operation of sewing k untwisted copies of the CFT into a single strand of length k, on which fields acquire the monodromy conditions that we have discussed in the previous Section. More precisely, they act on a k untwisted vacua of one kind (bosonic, R-fermionic, NS-fermionic) to obtain a single twisted vacua of the same kind.

In the bosonic sector we can define the twist operators σ^X_k, $\tilde{\sigma}^X_k$. They create a bosonic ground state of length k out of k length 1 bosonic ground states,

$$\lim_{z\to 0} \sigma^X_k(z)\tilde{\sigma}^X_k(\bar{z}) \left[\otimes^k_{r=1}|0\rangle_{(r)}\right] = |0\rangle_k, \tag{4.75}$$

and have conformal dimension $h = \frac{1}{6}\left(k - \frac{1}{k}\right)$, $\bar{h} = \frac{1}{6}\left(k - \frac{1}{k}\right)$ respectively. Since we end up in a twist k sector, we might as well decompose the operators σ_k^X, $\tilde{\sigma}_k^X$ as a product of operators in the ρ basis,

$$\sigma_k^X = \otimes_{\rho=0}^{k-1}\sigma_\rho^X, \quad \tilde{\sigma}_k^X = \otimes_{\rho=0}^{k-1}\tilde{\sigma}_\rho^X, \tag{4.76}$$

with $h_{\sigma_\rho^X} = \bar{h}_{\tilde{\sigma}_\rho^X} = \frac{\rho}{k}\left(1 - \frac{\rho}{k}\right)$. Moreover, using $\otimes_{r=1}^{k}|0\rangle_{(r)} = \otimes_{\rho=0}^{k-1}|0\rangle_\rho$, we can also write (4.75) as

$$\lim_{z\to 0}\otimes_{\rho=0}^{k-1}\sigma_\rho^X(z)\tilde{\sigma}_\rho^X(\bar{z})|0\rangle_\rho = |0\rangle_k. \tag{4.77}$$

In the fermionic NS sector things are similar to the bosonic case. We can define the twist fields $\Sigma_k(z,\bar{z})$, which are scalars under $SU(2)_L \times SU(2)_R$, and have conformal dimension $h = \frac{1}{12}\left(k - \frac{1}{k}\right)$, $\bar{h} = \frac{1}{12}\left(k - \frac{1}{k}\right)$. Decomposing them in the ρ basis,

$$\Sigma_k(z,\bar{z}) = \otimes_{\rho=0}^{k-1}\Sigma_\rho(z,\bar{z}), \tag{4.78}$$

and using the fact that $\otimes_{r=1}^{k}|0\rangle_{(r),\mathrm{NS}} = \otimes_{\rho=0}^{k-1}|0\rangle_{\rho,\mathrm{NS}}$, we can obtain the fermionic vacuum of length k in the NS sector as

$$\lim_{z\to 0}\Sigma_k(z,\bar{z})\otimes_{r=1}^{k}|0\rangle_{(r),\mathrm{NS}} = \lim_{z\to 0}\otimes_{\rho=0}^{k-1}\Sigma_\rho(z,\bar{z})|0\rangle_{\rho,\mathrm{NS}} = |0\rangle_{k,\mathrm{NS}}. \tag{4.79}$$

In the fermionic R sector we can define the twist fields $\Sigma_k^{s_1\dot{s}_2}$. The indices s_1, $\dot{s}_2$ will transform under a representation of spin $\left(\frac{k-1}{2}, \frac{k-1}{2}\right)$ of $SU(2)_L \times SU(2)_R$. The conformal dimension of $\Sigma_k^{s_1\dot{s}_2}$ is

$$(h,\bar{h}) = \left(\frac{(k-1)(2k-1)}{6k}, \frac{(k-1)(2k-1)}{6k}\right). \tag{4.80}$$

As in the other cases, $\Sigma_k^{s_1\dot{s}_2}$ can be written as

$$\Sigma_k^{s_1\dot{s}_2}(z,\bar{z}) = \bigotimes_{\rho=0}^{k-1}\Sigma_\rho^{s_1\dot{s}_2}(z,\bar{z}), \tag{4.81}$$

and the conformal dimension of $\Sigma_\rho^{s_1\dot{s}_2}$ is $h = \bar{h} = \rho^2/k^2$.
In the R sector we have to be more careful because there are degenerate vacua, and the fact that these can or cannot be generated with the operators $\Sigma_k^{s_1\dot{s}_2}$ depends on the spins.

The vacuum state on the length k strand is then defined via a product of bosonic and fermionic vacua, thus we need to perform the two twists simultaneously,

$$\sigma_k^X\tilde{\sigma}_k^X\Sigma_k^{s_1\dot{s}_2} = \bigotimes_{\rho=0}^{k-1}\sigma_\rho^X\tilde{\sigma}_\rho^X\Sigma_\rho^{s_1\dot{s}_2}. \tag{4.82}$$

The total conformal dimension of the full twist operator is $h = \bar{h} = \frac{k-1}{2}$.

4.2.4 Spectral flow

The building blocks of the D1D5 superconformal algebra on the complex plane are the $SU(2)$ chiral currents, the supercurrents and the stress-energy operator. It turns out that the algebra is not uniquely defined: there are some transformations that seem nontrivial, but nevertheless

lead to an isomorphic algebra.
One can act on holomorphic generators with an $SU(2)_L$ transformation, which has the effect of adding a phase

$$\eta(z) = i\nu \log z. \tag{4.83}$$

This is called spectral flow by ν units. Notice that $\log z$ has a branch cut, and thus the spectral flow operation can lead to nontrivial monodromies.
To better grasp the meaning of the transformation, let us consider fermions. Under spectral flow, they transform as

$$\begin{aligned} \psi^{1\dot{A}}(z) &\mapsto \psi^{1\dot{A}\prime}(z) = e^{+\frac{i}{2}\eta(z)}\psi^{1\dot{A}}(z) = z^{-\frac{\nu}{2}}\psi^{1\dot{A}}(z), \\ \psi^{2\dot{A}}(z) &\mapsto \psi^{2\dot{A}\prime}(z) = e^{-\frac{i}{2}\eta(z)}\psi^{2\dot{A}}(z) = z^{+\frac{\nu}{2}}\psi^{2\dot{A}}(z). \end{aligned} \tag{4.84}$$

If ν is odd, then $\psi'(z)$ have opposite periodicity with respect to $\psi(z)$: spectral flow by odd units swaps the NS sector and the R sector. In general, if an operator $\mathcal{O}(z)$ has $SU(2)_L$ charge m, then spectral flow acts on it as

$$\mathcal{O}(z) \mapsto z^{-\nu m}\mathcal{O}(z). \tag{4.85}$$

The only exception applies to the currents mentioned above, which turn out to be anomalous under spectral flow. Explicitly, [47]

$$\begin{aligned} J^3(z) &\mapsto J^3(z) - \frac{c\nu}{12z}, \\ J^{\pm}(z) &\mapsto z^{\mp\nu}J^{\pm}(z), \\ G^{\pm A}(z) &\mapsto z^{\mp\frac{\nu}{2}}G^{\pm A}(z), \\ T(z) &\mapsto T(z) - \frac{\nu}{z}J^3(z) + \frac{c\nu^2}{24z^2}, \end{aligned} \tag{4.86}$$

or equivalently, when acting on modes,

$$\begin{aligned} J^3_m &\mapsto J^3_m - \frac{c\nu}{12}\delta_{m,0}, \\ J^{\pm}_m &\mapsto J^{\pm}_{m\mp\nu}, \\ G^{\pm A}_m &\mapsto G^{\pm A}_{m\mp\frac{\nu}{2}}, \\ L_m &\mapsto L_m - \nu J^3_m + \frac{c\nu^2}{24}\delta_{m,0}. \end{aligned} \tag{4.87}$$

Spectral flow also acts on states. Starting with a state in the NS sector, with spectral flow we end up with states in the R sector with different conformal dimension and charge,

$$\begin{aligned} NS &\mapsto R, \\ h &\mapsto h' = h + \nu m + \frac{c\nu^2}{24}, \\ m &\mapsto m' = m + \frac{c\nu}{12}. \end{aligned} \tag{4.88}$$

In particular, under spectral flow the NS vacuum state is mapped to a R vacuum state,

$$|0\rangle_{NS} \mapsto |++\rangle_R. \tag{4.89}$$

4.2.5 Bosonization

On a (1+1)-dimensional field theory it is possible to realize the properties of interacting fermions by writing them in terms of a set of free bosons. Let us introduce, on a strand of length $k = 1$, the holomorphic bosons $H_{(r)}(z), K_{(r)}(z)$ and their antiholomorphic counterparts $\tilde{H}_{(r)}(\bar{z}), \tilde{K}_{(r)}(\bar{z})$. We ask that they satisfy the OPE

$$\begin{aligned}
H_{(r)}(z)H_{(s)}(w) &= -\delta_{r,s}\log(z-w) + [\text{reg.}],\\
K_{(r)}(z)K_{(s)}(w) &= -\delta_{r,s}\log(z-w) + [\text{reg.}],\\
\tilde{H}_{(r)}(\bar{z})\tilde{H}_{(s)}(\bar{w}) &= -\delta_{r,s}\log(\bar{z}-\bar{w}) + [\text{reg.}],\\
\tilde{K}_{(r)}(\bar{z})\tilde{K}_{(s)}(\bar{w}) &= -\delta_{r,s}\log(\bar{z}-\bar{w}) + [\text{reg.}].
\end{aligned} \tag{4.90}$$

The fermions can be written in terms of these bosons as

$$\begin{aligned}
&\psi^{1\dot{1}}_{(r)} = i : e^{iH_{(r)}} :, &&\psi^{2\dot{2}}_{(r)} = i : e^{-iH_{(r)}} :,\\
&\psi^{1\dot{2}}_{(r)} =: e^{iK_{(r)}} :, &&\psi^{2\dot{1}}_{(r)} =: e^{-iK_{(r)}} :,\\
&\tilde{\psi}^{\dot{1}\dot{1}}_{(r)} = i : e^{i\tilde{H}_{(r)}} :, &&\tilde{\psi}^{\dot{2}\dot{2}}_{(r)} = i : e^{-i\tilde{H}_{(r)}} :,\\
&\tilde{\psi}^{\dot{1}\dot{2}}_{(r)} =: e^{i\tilde{K}_{(r)}} :, &&\tilde{\psi}^{\dot{2}\dot{1}}_{(r)} =: e^{-i\tilde{K}_{(r)}} : .
\end{aligned} \tag{4.91}$$

Notice however that the bosonized language does not automatically implement the full algebra of fermionic modes. This means that we should take into account the anticommutation rules of fermions before switching to the bosonized language. It is also possible to define bosonized fermions with the correct anticommutation relations rigorously through the use of cocycles [49].

Nevertheless, we can now see the reason why bosonization is helpful: it transforms fermions into "vertex operators", whose correlation functions are particularly simple to compute. In general, if $X(z)$ is an operator like H or K, the vertex operator

$$: e^{i\alpha X(z)} : \tag{4.92}$$

has conformal dimension $(h, \bar{h}) = (\alpha^2/2, 0)$ and spin $(j, \bar{j}) = (\alpha/2, 0)$. The OPE of two such operators is

$$\begin{aligned}
: e^{i\alpha X(z)} :: e^{i\beta X(w)} : &=: \exp\Big((i\alpha)(i\beta)\overbrace{X(z)X}(w) + i\alpha X(z) + i\beta X(w)\Big) :\\
&= (z-w)^{\alpha\beta} : \exp\left(i(\alpha+\beta)X(w) + \sum_{n=1}^{+\infty}\frac{(z-w)^n}{n!}\partial^n X(w)\right) :,
\end{aligned} \tag{4.93}$$

where we have performed the contraction of the X with itself, and expanded $X(z)$ around w.

The H, K bosons are useful also for defining the spectral flow operator. As we have seen in Subsection 4.2.4, spectral flow is an operation that maps the fermionic NS vacuum $|0\rangle_{\text{NS}}$ to the R vacuum $|++\rangle$. For a single copy we have

$$|++\rangle_{(r)} = \lim_{z\to 0} e^{\frac{i}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar{z})+\tilde{K}_{(r)}(\bar{z})\right)}|0\rangle_{(r),\text{NS}}, \tag{4.94}$$

and for the R vacuum in the full CFT it suffices to take the tensor product over $r = 1, \ldots, N$ of the above expression,

$$\bigotimes_{r=1}^{N} |++\rangle_{(r)} = \bigotimes_{r=1}^{N} \left(\lim_{z\to 0} e^{\frac{i}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar{z})+\tilde{K}_{(r)}(\bar{z})\right)} |0\rangle_{(r),\mathrm{NS}} \right). \tag{4.95}$$

The spectral flow operator thus has conformal dimension $h = \bar{h} = \frac{N}{4} = \frac{c}{24}$. Since the NS vacuum (that is, the $SL(2,\mathbb{C})$ invariant vacuum) has zero conformal dimension, this is also the conformal dimension of the R vacuum $\otimes_{r=1}^{N} |++\rangle_{(r)}$.

In the twisted sector, most of the above properties concerning bosonization hold in the ρ basis. We let

$$\begin{aligned}
\psi_\rho^{1\dot{1}} &= i : e^{iH_\rho} :, & \psi_\rho^{2\dot{2}} &= i : e^{-iH_\rho} : \\
\psi_\rho^{1\dot{2}} &=: e^{iK_\rho} :, & \psi_\rho^{2\dot{1}} &=: e^{-iK_\rho} : \\
\tilde{\psi}_\rho^{\dot{1}\dot{1}} &= i : e^{i\tilde{H}_\rho} :, & \tilde{\psi}_\rho^{\dot{2}\dot{2}} &= i : e^{-i\tilde{H}_\rho} : \\
\tilde{\psi}_\rho^{\dot{1}\dot{2}} &=: e^{i\tilde{K}_\rho} :, & \tilde{\psi}_\rho^{\dot{2}\dot{1}} &=: e^{-i\tilde{K}_\rho} :,
\end{aligned} \tag{4.96}$$

with OPE rules

$$\begin{aligned}
H_{\rho_1}(z)H_{\rho_2}(w) &= -\delta_{\rho_1,\rho_2}\log(z-w) + [\text{reg.}], \\
K_{\rho_1}(z)K_{\rho_2}(w) &= -\delta_{\rho_1,\rho_2}\log(z-w) + [\text{reg.}], \\
\tilde{H}_{\rho_1}(\bar{z})\tilde{H}_{\rho_2}(\bar{w}) &= -\delta_{\rho_1,\rho_2}\log(\bar{z}-\bar{w}) + [\text{reg.}], \\
\tilde{K}_{\rho_1}(\bar{z})\tilde{K}_{\rho_2}(\bar{w}) &= -\delta_{\rho_1,\rho_2}\log(\bar{z}-\bar{w}) + [\text{reg.}].
\end{aligned} \tag{4.97}$$

We can also use the bosonized language to define the twist operators in the R sector; we will need only the lowest weight state, $\Sigma_k^{-\frac{k-1}{2},-\frac{k-1}{2}}$. First of all, notice that

$$\begin{aligned}
\bigotimes_{r=1}^{k} |++\rangle_{(r)} &= \bigotimes_{r=1}^{k} \left[\lim_{z\to 0} e^{\frac{i}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar{z})+\tilde{K}_{(r)}(\bar{z})\right)} |0\rangle_{(r),\mathrm{NS}} \right] \\
&= \bigotimes_{\rho=0}^{k-1} |++\rangle_\rho \\
&= \bigotimes_{\rho=0}^{k-1} \left[\lim_{z\to 0} e^{\frac{i}{2}\left(H_\rho(z)+K_\rho(z)+\tilde{H}_\rho(\bar{z})+\tilde{K}_\rho(\bar{z}\right)} |0\rangle_{\rho,\mathrm{NS}} \right].
\end{aligned} \tag{4.98}$$

Thus, we can define the lowest weight R-twist operator as

$$\Sigma_k^{-\frac{k-1}{2},-\frac{k-1}{2}} = \bigotimes_{\rho=0}^{k-1} \Sigma_\rho^{-\frac{k-1}{2},-\frac{k-1}{2}} = \bigotimes_{\rho=0}^{k-1} e^{-i\frac{\rho}{k}\left(H_\rho+K_\rho+\tilde{H}_\rho+\tilde{K}_\rho\right)}, \tag{4.99}$$

from which the length k R vacuum will read

$$
\begin{aligned}
|++\rangle_k &= \lim_{z\to 0} \Sigma^{-\frac{k-1}{2},-\frac{k-1}{2}}(z,\bar{z}) \bigotimes_{\rho=0}^{k-1} |++\rangle_\rho \\
&= \lim_{\substack{z\to 0\\ w\to 0}} \bigotimes_{\rho=0}^{k-1} \left[e^{-i\frac{\rho}{k}\left(H_\rho(z)+K_\rho(z)+\tilde{H}_\rho(\bar{z})+\tilde{K}_\rho(\bar{z})\right)} e^{\frac{i}{2}\left(H_\rho(w)+K_\rho(w)+\tilde{H}_\rho(\bar{w})+\tilde{K}_\rho(\bar{w})\right)} |0\rangle_{\rho,\mathrm{NS}} \right] \\
&= \lim_{\substack{z\to 0\\ w\to 0}} |z-w|^{-(k-1)} \bigotimes_{\rho=0}^{k-1} \left[e^{i\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(H_\rho(w)+K_\rho(w)+\tilde{H}_\rho(\bar{w})+\tilde{K}_\rho(\bar{w})\right)} |0\rangle_{\rho,\mathrm{NS}} \right],
\end{aligned}
\tag{4.100}
$$

where we have used the OPE of vertex operators (4.93).

4.2.6 Chiral primaries

In this Section we have been describing the behaviour of the D1D5 CFT at the free orbifold point. On the other hand, thanks to AdS/CFT duality, we know that the moduli spaces of the gravity theory and of the field theory are related.

On the gravitational side we are interested in the supergravity limit, where the AdS radius is much larger than the string length ℓ_s. This is a particular region in moduli space. The CFT description dual to this scenario however does not lie at the free orbifold point, as we have argued in Section 4.1: rather, it lies at a point in CFT moduli space where the field theory is strongly interacting.

It might seem that our program is then doomed to fail, but fortunately it is not the case. It is true that, in general, the quantities computed in the CFT depend on moduli, but there is also a class of observables which are protected by supersymmetry. For such observables there is no moduli dependence, and thus they can be computed at any point in moduli space, even at the free orbifold point. AdS/CFT then will require the matching of the CFT results with the ones obtained in Supergravity.

Let us then describe this class of observables. To do so we will need some basic representation theory of the D1D5 CFT in the NS sector. Recall that a Virasoro primary state is a state $|\phi\rangle$ such that

$$
L_0|\phi\rangle = h|\phi\rangle, \qquad L_n|\phi\rangle = 0 \;\forall n>0. \tag{4.101}
$$

By definition $|\phi\rangle = \phi(0)|0\rangle_{NS}$, where ϕ is a primary field of conformal dimension h. Now, the D1D5 superconformal algebra includes the relations

$$
\begin{aligned}
\left\{G^{-A}_{+\frac{1}{2}}, G^{+B}_{-\frac{1}{2}}\right\} &= \epsilon^{AB}\left(J^3_0 - L_0\right), \\
\left\{G^{+A}_{+\frac{1}{2}}, G^{-B}_{-\frac{1}{2}}\right\} &= \epsilon^{AB}\left(J^3_0 + L_0\right).
\end{aligned}
\tag{4.102}
$$

If $|\phi\rangle$ is a state with quantum numbers $(h, j(j+1), m)$ under $(L_0, (J^a_0)^2, J^3_0)$, then by the above relations we obtain

$$\begin{aligned}\sum_B \left| G^{+B}_{-\frac{1}{2}} |\phi\rangle \right|^2 + \sum_B \left| G^{-B}_{\frac{1}{2}} |\phi\rangle \right|^2 &= 2(h-m),\\ \sum_B \left| G^{-B}_{-\frac{1}{2}} |\phi\rangle \right|^2 + \sum_B \left| G^{+B}_{\frac{1}{2}} |\phi\rangle \right|^2 &= 2(h+m).\end{aligned} \tag{4.103}$$

Unitarity requires the left hande sides to be nonegative, thus

$$h \geq |m| \implies h \geq j. \tag{4.104}$$

For a state $|\phi\rangle$ that saturates the bound, say $h = m$, the equalities 4.103 require

$$\begin{gathered} G^{\pm A}_n |\phi\rangle = L_n |\phi\rangle = 0 \quad \forall n > 0,\\ G^{+A}_{-\frac{1}{2}} |\phi\rangle = 0. \end{gathered} \tag{4.105}$$

Such a state is said to be a chiral primary state, and the corresponding operator is called chiral primary operator. A similar result holds choosing the bound $h = -m$: the resulting state/operator will then be an anti-chiral state/operator. Chiral primary operators are naturally highest weight states of both the Virasoro algebra and of the $SU(2)_L$ chiral algebra.

Chiral primary operators are operators whose expectation values on R ground states turn out to be protected by supersymmetry. More generally, all three-point functions of (anti-)chiral primary operators and their descendants do not depend on moduli thanks to a non-renormalization theorem [50, 51]. By (4.88), anti-chiral primary operators are in one-to-one correspondence with R vacua, which are in turn dual to microstate geometries. Thus we can compute expectation values of chiral primary operators on CFT states dual to two- and three-charge microstates and, once we have identified the Supergravity fields which are dual to such operators, compare them with the results obtained in the gravity picture.

In the following chapter we will make the correspondences between CFT operators and Supergravity fields more precise, so that we will be able to compute correlation functions of a special class of chiral primary operators, the chiral currents, in Chapter 6.

Chapter 5

Applying the holographic principle

It is widely known that on the CFT side black hole microstates correspond to "heavy" states, that is states which have conformal dimension of order c, belonging to the Ramond-Ramond sector. For instance, the simple $(|++\rangle_{k=1})^N$ state is the product of N Ramond vacua obtained after spectral flow from N Neveau-Schwarz ones, its conformal dimension is equal to the conformal dimension of the spectral flow operator itself and it is therefore of order c. The same holds for all Ramond vacua.
A natural way to probe these states is to compute the holographic n-point functions of different chiral primary operators. In particular, we are interested in correlators of two "light" operators in a heavy state. An operator is said to be light if its conformal dimension is finite in the $c \to \infty$ limit. In the $n = 1$ case, one obtains the expectation value of the chiral primary operator in the given background geometry. For the case $n = 2$, it would describe the emission and absorption of "light" quanta from the D1D5 brane. On the CFT side one needs to compute a 3- or 4-point function with two heavy and one or two light operators, whereas on the bulk side one needs to study the equation of motion of a light field in the nontrivial geometry dual to the heavy state at hand.

In this chapter we are presenting a systematic method for obtaining the expectation values of chiral primary operators. It is not trivial to understand which bulk fields are going to be dual to the chosen chiral primary. However, as soon as one has a mapping between operators and fields it becomes easier to compute higher n-point functions.
Our aim is to study two-point correlators of light chiral primary operators taken in a heavy state (thus, effectively, a four-point function). These Heavy-Heavy-Light-Light (HHLL) correlation functions are of particular interest because, as we will argue, they allow for a direct probing of information loss.

5.1 Holographic dictionary for light operators

In Section 4.1 we have seen how a Supergravity theory on the interior of AdS space is equivalent to a Conformal Field Theory living on the boundary of AdS. In this Section we will make the correspondence more precise for the D1D5 black hole. It has been shown [52, 53] that microstate geometries encode the expectation values of chiral primary operators in the dual CFT state. As their one point functions are protected, the expectation values obtained with this procedure should match the results computed at the free orbifold point of the CFT.

The first step is to determine how the solutions dual to the D1D5P system can be embedded into six-dimensional Supergravity in their decoupling limit. Then, after expanding these solutions near the $AdS_3 \times S^3$ boundary, we will be able to explain how the expectation values of the CFT operators can be extracted from these expansions.

For simplicity let us focus on two charge fuzzball solution. The fields appearing in the solution solve the equations of motion of type IIB supergravity whose action reads, in the string frame,

$$S = \frac{1}{2k_{10}} \int \mathrm{d}^{10}x\sqrt{-g_{10}} \left(e^{-2\Phi} \left(R_{10} + 4(\partial\Phi)^2 \right) - \frac{1}{12} F_3^2 + \dots \right), \tag{5.1}$$

where the dots represent higher order corrections, $F_3 = \mathrm{d}C_2$ and $2k_{10} = (2\pi)^7 g_s^2 \alpha'^4$. The functions Z_1, Z_5, A, B appearing in the solution can be expressed in terms of the profile function $\vec{F}(v)$, as in 3.38.

We wish to perform compactification over the compact space T^4 (or $K3$). The effective six-dimensional metric that one obtains, in the Einstein frame, reads

$$ds_6^2 = (Z_1 Z_5)^{-1/2} \left(-(dt - A)^2 + (dy + B)^2 \right) + (Z_1 Z_5)^{1/2} dx_i dx^i, \tag{5.2}$$

At leading order in the large r expansion, this class of metrics reduces always to $AdS_3 \times S^3$, that in its Poincaré patch reads

$$ds^2 \approx_{r\to\infty} \sqrt{Q_1 Q_5} \left(\frac{dr^2}{r^2} + d\Omega_3^2 \right) + \frac{r^2}{\sqrt{Q_1 Q_5}} (-dt^2 + dy^2), \tag{5.3}$$

and the three-form and the dilaton are asymptotic to

$$F_{rty} = \frac{2r}{Q_1}, \qquad F_{\Omega_3} = 2Q_5, \qquad e^{2\Phi_0} = \frac{Q_1}{Q_5}. \tag{5.4}$$

The effective six-dimensional action is given by

$$S = \frac{1}{2k_6} \int \mathrm{d}^6x\sqrt{-g} \left(R - (\partial\Phi)^2 - \frac{1}{12} e^{2\Phi} F_3^2 \right), \tag{5.5}$$

where R is now the six dimensional curvature. The equations of motion following from the dimensionally reduced action are

$$\begin{aligned} R_{MN} &= \frac{1}{4} e^{2\Phi} \left(F_{MPQ} F_N{}^{PQ} - \frac{1}{6} F^2 g_{MN} \right) + \partial_M\Phi\, \partial_N\Phi, \\ D_M \left(e^{2\Phi} F^{MNP} \right) &= 0, \\ \Box\Phi &= \frac{1}{12} e^{2\Phi} F^2. \end{aligned} \tag{5.6}$$

These equations of motion can be embedded into those of $d = 6$, $\mathcal{N} = 4b$ supergravity coupled to n_t tensor multiplets on $AdS_3 \times S^3$ background [54–56]. The bosonic field content of theories consist of a supergravity multiplet containing a graviton g_{MN} and five self-dual tensor fields B^m_{MN}, as well as n_t tensor multiplets containing an anti-self-dual tensor field B^r_{MN} and five scalars ϕ^{mr}. Let us denote with $M, N = 1, \dots, 6$ the spacetime indices, with $m, n = 1, \dots, 5$ the $SO(5)$ vector indices, and with $r, s = 1, \dots n_t$ the $SO(n_t)$ vector indices. Let us also

introduce $I, J = 1, \ldots, 5 + n_t$ $SO(5, n_t)$ vector indices.
The scalar sector of the theory is a sigma model over

$$\frac{SO(5, n_t)}{SO(5) \times SO(n_t)}. \tag{5.7}$$

The scalars parametrize a vielbein (V_I^m, V_J^r), satisfying

$$V_I^m V_J^m - V_I^r V_J^r = \eta_{IJ}, \tag{5.8}$$

where $\eta_{IJ} = (+++++-\cdots-)$. The associated Maurer-Cartan form is

$$dVV^{-1} = \begin{pmatrix} Q^{mn} & \sqrt{2}P^{ms} \\ \sqrt{2}P^{nr} & Q^{rs} \end{pmatrix}. \tag{5.9}$$

In the bosonic sector, the field equations are

$$\begin{aligned} R_{MN} &= H^m_{MPQ} H_N^{mPQ} + H^r_{MPQ} H_N^{r\,PQ} + 2P_M^{mr} P_N^{mr}, \\ D^M P_M^{mr} &= \frac{\sqrt{2}}{3} H^{mMNP} H^r_{MNP}, \\ H^m_{MNP} &= \frac{1}{3!}\epsilon_{MNPQRS} H^{mQRS}, \\ H^r_{MNP} &= -\frac{1}{3!}\epsilon_{MNPQRS} H^{rQRS}. \end{aligned} \tag{5.10}$$

The three-form field strenghts H are given by

$$H^m = G^I V_I^m, \qquad H^r = G^I V_I^r, \tag{5.11}$$

where $G^I = \mathrm{d}B^I$ are the elementary field strengths.
The embedding of the original equations of motion (5.6) is done as follows. Let

$$V_5^{m=5} = \cosh\Phi, \quad V_6^{m=5} = \sinh\Phi, \quad V_5^{r=1} = \sinh\Phi, \quad V_6^{r=1} = \cosh\Phi, \tag{5.12}$$

so that $\sqrt{2}P^{56} = \mathrm{d}\Phi$. Now, let

$$G^5 = \frac{1}{4}(F_3 + e^{2\Phi} \star_6 F_3), \qquad G^6 = \frac{1}{4}(F_3 - e^{2\Phi} \star_6 F_3), \tag{5.13}$$

which implies that

$$H^{m=5} = \frac{1}{4}e^{\Phi}(F_3 + \star_6 F_3), \qquad H^{r=1} = \frac{1}{4}e^{\Phi}(F_3 - \star_6 F_3). \tag{5.14}$$

Substituting in (5.10) one obtains again the original equations (5.6).

Given the asymptotic behaviour of fuzzball solutions, it is natural to pick as a background solution the supersymmetric vacuum soluton with the geometry of $AdS_3 \times S^3$. To do so, it is convenient to shift $\Phi \to \Phi - \Phi_0$, $G^{5,6} \to e^{\Phi_0} G^{5,6}$. Then

$$G^{0,m=5} = H^{0,m=5} = \frac{r}{Q_1 Q_5}\,\mathrm{d}r \wedge \mathrm{d}t \wedge \mathrm{d}y + \sqrt{Q_1 Q_5}\,\mathrm{vol}_{S^3}, \quad V_5^{0,m=5} = V_6^{0,r=1} = 1, \tag{5.15}$$

whereas the other fields are set to zero.

We wish to analyze the expansion around the AdS boundary of the fields in order to obtain the expectation values of chiral primaries in the dual CFT. From the asymptotic solutions one gets six-dimensional gauge-invariant fields that must be reduced to three-dimensional fields via Kaluza-Klein reduction on S^3. The expectation values can then be extracted using holographic renormalization [57].

We parametrize the linearized fluctuation around the $AdS_3 \times S^3$ background as

$$g_{MN} = g^0_{MN} + h_{MN}, \quad G^I = G^{0I} + g^I, \quad \phi^{mr}, \tag{5.16}$$

Since S^3 admits a complete set of spherical harmonics, we can further expand

$$\begin{aligned}
h_{\mu\nu} &= \sum h^{(\ell,i)}_{\mu\nu}(x) Y^i_\ell(y), \\
h_{\mu a} &= \sum \left(h^{(\ell,i_v)}_{\mu}(x) Y^{i_v}_{\ell a}(y) + h^{(\ell,i)}_{(s)\mu}(x) D_a Y^i_\ell(y) \right), \\
h_{(ab)} &= \sum \left(\rho^{(\ell,i_t)}(x) Y^{i_t}_{\ell(ab)}(y) + \rho^{(\ell,i_v)}_{(v)}(x) D_a Y^{i_v}_{\ell b}(y) + \rho^{(\ell,i)}_{(s)}(x) D_{(a} D_{b)} Y^i_\ell(y) \right), \\
h^a_a &= \sum \pi^{(\ell,i)}(x) Y^i_\ell(y), \\
g^I_{\mu\nu\rho} &= \sum 3 D_{[\mu} b^{I(\ell,i)}_{\nu\rho]}(x) Y^i_\ell(y), \\
g^I_{\mu ab} &= \sum \left(D_\mu U^{I(\ell,i)}(x) \epsilon_{abc} D^c Y^i_\ell(y) + 2 Z^{I(\ell,i_v)}_\mu D_{[b} Y^{i_v}_{\ell a]} \right), \\
g^I_{abc} &= \sum \left(-\epsilon_{abc} \Lambda^{(\ell,i)} U^{I(\ell,i)}(x) Y^i_\ell(y) \right), \\
\phi^{mr} &= \sum \phi^{mr(\ell,i)}(x) Y^i_\ell(y).
\end{aligned} \tag{5.17}$$

We have labeled with late greek indices and with x the AdS components and with early latin indices and with y the S^3 components; here (ab) stands for traceless symmetrization. The labeling of harmonic functions is such that ℓ labels the degree of the harmonic and i, i_v, i_t label generically the degeneracy of scalar, vector, tensor harmonics respectively, for fixed ℓ. Notice that the vectors $Y^{i_v}_{\ell=1}$ are the six $SO(4)$ Killing vectors given in Appendix B.

Now consider the expansion at large radius r of the fuzzball solutions. We can still expand the leading terms into spherical harmonics,

$$\begin{aligned}
Z_1 &= \frac{Q_1}{r^2} \left(1 + \frac{f^1_{1i}}{r} Y^i_1 + \dots \right), \\
Z_5 &= \frac{Q_5}{r^2} \left(1 + \frac{f^5_{1i}}{r} Y^i_1 + \dots \right), \\
A &= \frac{Q_5}{r^2} + \frac{\sqrt{Q_1 Q_5}}{r^2} \left(a_{\alpha -} Y^{\alpha -}_1 + a_{\alpha +} Y^{\alpha +}_1 \right) + \cdots .
\end{aligned} \tag{5.18}$$

for some coefficients $f^1_{1A}, f^5_{1i}, a_{\alpha\pm}$; the ellipses denote higher order terms. The index $i = \pm\pm, \pm\mp$ labels the degeneracy of the degree one scalar harmonics, whereas we have let $i_v = (\alpha, \pm)$ with $\alpha = \pm, 0$, in accordance with the notation in Appendix B.
One can show [53] that one can always choose appropriate coordinates such that

$$f^1_{1i} + f^5_{1i} = 0. \tag{5.19}$$

Plugging the expansions (5.18) in the fuzzball solution, one obtains that the leading perturbations are, at first order in the coefficients $f^5_{1i}, a_{\alpha\pm}$,

$$
\begin{aligned}
h_{ta} &= \left(a_{\alpha-}Y_1^{\alpha-} + a_{\alpha+}Y_1^{\alpha+}\right),\\
h_{ya} &= \left(a_{\alpha-}Y_1^{\alpha-} - a_{\alpha+}Y_1^{\alpha+}\right),\\
g_{tab}^{m=5} &= -\left(a_{\alpha-}D_{[a}\left(Y_1^{\alpha-}\right)_{b]} - a_{\alpha+}D_{[a}\left(Y_1^{\alpha+}\right)_{b]}\right),\\
g_{yab}^{m=5} &= -\left(a_{\alpha-}D_{[a}\left(Y_1^{\alpha-}\right)_{b]} + a_{\alpha+}D_{[a}\left(Y_1^{\alpha+}\right)_{b]}\right),\\
g_{tyr}^{r=1} &= \frac{1}{2}f^5_{1i}Y_1^i,\\
g_{tya}^{r=1} &= \frac{r}{2}D_a f^5_{1i}Y_1^i,\\
g_{rab}^{r=1} &= \frac{1}{2r^2}\epsilon_{ab}{}^c f^5_{1i}D_cY_1^i,\\
g_{abc}^{r=1} &= \frac{3}{2r}\epsilon_{abc}f^5_{1i}Y_1^i,\\
\phi^{51} \equiv \Phi &= -\frac{f^5_{1i}}{r}Y_1^i.
\end{aligned}
\tag{5.20}
$$

All other fluctuations are vanishing at linear order.

With this highly nontrivial procedure we have been able to define a dictionary between the asymptotic properties of fuzzball solutions and the asymptotic properties of the effective low-energy supergravity solution. Since the asymptotic structure is preserved also in the classical limit, we have found the so-called hair of the D1D5 black hole.

Let us now move to a general three-charge state for completeness. In this case there are more fields characterizing our solution, but we can still perform the expansion in spherical harmonics. Explicitly, let

$$
\begin{aligned}
&Z_1 = \frac{Q_1}{r^2}\left(1+\frac{f^1_{1i}}{r}Y_1^i + O\left(r^{-2}\right)\right), \quad Z_2 = \frac{Q_5}{r^2}\left(1+\frac{f^5_{1i}}{r}Y_1^i + O\left(r^{-2}\right)\right),\\
&Z_4 = \frac{\sqrt{Q_1Q_5}}{r^3}\mathcal{A}_{1i}Y_1^i + O\left(r^{-4}\right), \quad \mathcal{F} = -\frac{2Q_p}{r^2} + O\left(r^{-3}\right), \quad ds_4^2 = dx^idx^i + O\left(r^{-4}\right),\\
&\beta = -\frac{\sqrt{2Q_1Q_5}}{r^2}a_{\alpha-}Y_1^{\alpha-} + O\left(r^{-3}\right), \quad \omega = -\frac{\sqrt{2Q_1Q_5}}{r^2}a_{\alpha+}Y_1^{\alpha+} + O\left(r^{-3}\right).
\end{aligned}
\tag{5.21}
$$

In principle, one can repeat the procedure that has been done for the two charge states and obtain the relevant supergravity fields for each perturbation. We shall not repeat the construction here.

The coefficients $f^{1,5}_{1i}, a^{\alpha\pm}, \mathcal{A}_{1i}$ encode the expectation values of chiral primary operators of total conformal dimension one. They are the $SU(2)_L \times SU(2)_R$ chiral currents $J^\alpha, \tilde{J}^\alpha$, with $(h,\bar{h}) = (j,\bar{j}) = (1,0)$ and $(0,1)$ respectively, the twist fields $\Sigma_2^{\alpha\dot{\alpha}}$ and the operators $O^{\alpha\dot{\alpha}}$, with $(h,\bar{h}) = (j,\bar{j}) = (\frac{1}{2},\frac{1}{2})$.
The relation between one point functions of these operators in a state $|s\rangle$ dual to the fuzzball

geometry and the coefficients appearing in (5.21) has been shown to be [53, 58]

$$\begin{aligned}
\langle s|J^{\pm}|s\rangle &= c_J a_{\mp+}, & \langle s|\tilde{J}^{\pm}|s\rangle &= c_{\tilde{J}} a_{\mp-},\\
\langle s|J^{3}|s\rangle &= c_J a_{0+}, & \langle s|\tilde{J}^{3}|s\rangle &= c_{\tilde{J}} a_{0-},\\
\langle s|O^{++}|s\rangle &= -\sqrt{2}c_O \mathcal{A}_{1--}, & \langle s|O^{+-}|s\rangle &= -\sqrt{2}c_O \mathcal{A}_{1-+},\\
\langle s|O^{-+}|s\rangle &= \sqrt{2}c_O \mathcal{A}_{1+-}, & \langle s|O^{--}|s\rangle &= \sqrt{2}c_O \mathcal{A}_{1++},\\
\langle s|\Sigma_2^{++}|s\rangle &= -\sqrt{2}c_\Sigma f^1_{1--}, & \langle s|\Sigma_2^{+-}|s\rangle &= -\sqrt{2}c_\Sigma f^1_{1-+},\\
\langle s|\Sigma_2^{-+}|s\rangle &= \sqrt{2}c_\Sigma f^1_{1+-}, & \langle s|\Sigma_2^{--}|s\rangle &= \sqrt{2}c_\Sigma f^1_{1++},
\end{aligned} \tag{5.22}$$

The further coefficients $c_J, c_{\tilde{J}}, c_O, c_\Sigma$ appearing in these relations do not depend on the state $|s\rangle$ but only on the global parameters of the theory: N, R, Q_1, Q_5, and are difficult to determine a priori. They will not play a role in our analysis.

The above relations let us lay down a map between operators and bulk fields in type IIB Supergravity. If we want to have a nonvanishing expectation value for some chiral primary operator $\mathcal{O}$ of dimension one, we should allow the corresponding coefficient on the right hand side of (5.22) to be nonzero, that is we should consider fuzzball solutions whose fields possess the corresponding term in the asymptotic expansion.
Using the map derived previously between asymptotics of fuzzballs and of Supergravity fields, we can read off which fields must be switched on in the effective low-energy description if we want to have nonzero expectation value of the operator $\mathcal{O}$ in the state dual to the chosen background geometry.

5.2 Holographic dictionary for heavy operators

We wish now to describe the relation between microstate geometries and heavy CFT operators (or, equivalently, states). In this Section we will only report the results which will be necessary for our discussion. For further details we refer to [38, 59, 60].

As we have argued in Section 3.3, the D1D5 CFT is dual to a gravitational theory on asymptotically $AdS_3 \times S^3$ spacetimes. The S^3 factor is crucial to implement the $SO(4) \approx SU(2)_L \times SU(2)_R$ R-symmetry in the CFT. The geometries generated by generic heavy operators are typically complicated, as only asymptotically they factorize into the trivial S^3 fibration of AdS_3.

In this thesis we focus on a particularly simple set of states, whose dual geometries are locally isometric to $AdS_3 \times S^3$ by a diffeomorphism that is however nontrivial at the boundary. Let us denote with $|s,k\rangle$ this set of states. They are generated by the action of some set of heavy operators on the conformal invariant vacuum,

$$|s,k\rangle \equiv \lim_{z,\bar{z}\to 0} O_H(s,k;z,\bar{z})|0\rangle. \tag{5.23}$$

We postpone the explicit expression of the operators O_H appearing in this definition to the next Chapter. For the present discussion, what is important is that operators of conformal dimension of the order of the central charge backreact strongly on the geometry, thus the states $|s,k\rangle$ admit a dual gravity description.

The six-dimensional Einstein metric corresponding to those states can be written as

$$\begin{aligned} ds_6^2 &= \sqrt{Q_1 Q_5}\left(ds^2_{AdS_3} + ds^2_{S^3}\right) \\ ds^2_{AdS_3} &= \frac{dr^2}{a^2k^{-2}+r^2} - \frac{a^2k^{-2}+r^2}{Q_1Q_5}dt^2 + \frac{r^2}{Q_1Q_5}dy^2 \\ ds^2_{S^3} &= d\theta^2 + \sin^2\theta d\hat{\phi}^2 + \cos^2\theta d\hat{\psi}^2 \end{aligned} \tag{5.24}$$

The coordinates t, y are identified with the time and space coordinates of the conformal field theory. In particular, we ask the coordinate y to be periodic, with period $2\pi R$. The angles $\hat{\phi}, \hat{\psi}$ are some state-dependent linear combination of the S^3 Hopf angles ϕ, ψ and the coordinates t, y. Q_1 and Q_5 are just the usual D1 and D5 charge, which are related to the corresponding integer charges by (3.30). The parameter a is linked to the charges and to the period of the y coordinate by

$$a = \frac{\sqrt{Q_1Q_5}}{R}, \tag{5.25}$$

whereas k is a positive integer. If $k > 1$, then the AdS_3 geometry includes a conical defect, i.e. the space is effectively the quotient $AdS_3/\mathbb{Z}_k$.

The dual supergravity solution also includes the RR two-form C_2, whose field strength $F_3 = \mathrm{d}C_2$ is given by

$$\begin{aligned} F_3 &= 2Q_5\left(-\mathrm{vol}_{AdS_3} + \mathrm{vol}_{S^3}\right) \\ \mathrm{vol}_{AdS_3} &= \frac{r}{Q_1Q_5} dr\wedge dt \wedge dy, \quad \mathrm{vol}_{S^3} = \sin\theta\cos\theta d\theta\wedge d\hat{\phi}\wedge d\hat{\psi} \end{aligned} \tag{5.26}$$

F_3 is anti-self-dual in the six dimensional Einstein metric (5.24),

$$\star_6 F_3 = -F_3. \tag{5.27}$$

Two charge states. The states $|s=0,k\rangle$ carry D1 and D5 charges only. The geometries dual to these states are the same as (5.24), with

$$\hat{\phi} = \phi - \frac{t}{Rk}, \quad \hat{\psi} = \psi - \frac{y}{Rk}. \tag{5.28}$$

Recall that the coordinates (t, y, ϕ, ψ) are defined up to the equivalence relation given by

$$(t, y, \phi, \psi) \sim (t, y + 2\pi l R, \phi + 2\pi m, \psi + 2\pi n), \tag{5.29}$$

with l, m, n integers. When $k = 1$, we can replace (ϕ, ψ) with $(\hat{\phi}, \hat{\psi})$ in the above identification. In that case, the coordinate transformation $(t, y, \phi, \psi) \to (t, y, \hat{\phi}, \hat{\psi})$ is the transformation that gives spectral flow from $|s=0, k=1\rangle$ to the $SL(2,\mathbb{C})$-invariant vacuum $|0\rangle$, that is just plain global $AdS_3 \times S^3$.
For $k > 1$, the identifications become less trivial,

$$(t, y, \hat{\phi}, \hat{\psi}) \sim \left(t, y + 2\pi l R, \hat{\phi} + 2\pi m, \hat{\psi} - 2\pi\frac{l}{k} + 2\pi n\right), \tag{5.30}$$

and the geometry dual to the state $|s=0,k\rangle$ is the $\mathbb{Z}_k$ orbifold of AdS_3 times S^3.

Three charge states. The states with $|s,k\rangle$, with s positive integer, carry also momentum charge $n_p = N\frac{s(s+1)}{k}$. The geometries dual to these states are still the same as (5.24), but this time with

$$\hat{\phi} = \phi - \frac{t}{Rk} - s\frac{t+y}{Rk}, \quad \hat{\psi} = \psi - \frac{y}{Rk} - s\frac{t+y}{Rk}. \tag{5.31}$$

As for the two charge case, the identifications (5.29) are still preserved for $k = 1$, but they are different for $k > 1$. For $k > 1$ the geometry is still locally $(AdS_3/\mathbb{Z}_k) \times S^3$, but the orbifold action is realized differently than in the previous case,

$$(t, y, \hat{\phi}, \hat{\psi}) \sim \Big(t, y + 2\pi l R, \hat{\phi} - 2\pi s\frac{l}{k} + 2\pi m, \hat{\psi} - 2\pi(s+1)\frac{l}{k} + 2\pi n\Big). \tag{5.32}$$

5.3 Computing HHLL correlators holographically

We are interested in computing two point functions of light operators in a heavy state. Let us forget for a moment about the nature of the particular state and be completely general.

In a generic quantum field theory, we know how to compute correlation functions with the functional formalism. For instance, suppose that one wishes to compute the two point correlation function $\langle\bar{O}(x)O(y)\rangle$. What one usually does is to couple the QFT to an external source $\int J\bar{O}$ and then they compute the response function,

$$\langle\bar{O}(x)O(y)\rangle = \frac{\delta}{\delta J(x)}\langle O(y)\rangle_J\bigg|_{J=0}. \tag{5.33}$$

The holographic computation is performed in an identical manner [1, 61], where one uses the holographic dictionary built in Section 5.1 to read off the expectation value of the operator O in the presence of a source.

Let us make this more concrete. Consider a bulk scalar field Φ with mass m, dual to the boundary operator O with conformal dimension Δ. In the AdS_{d+1} bulk, Φ satisfies the free equation of motion

$$(\Box + m^2)\Phi(r, t, \vec{y}) = 0. \tag{5.34}$$

Consider now the solution to this equation. Near the AdS boundary $r \to \infty$, the solution admits the expansion

$$\Phi(r, t, \vec{y}) \sim \beta(t, \vec{y})r^{\Delta-d}(1 + \dots) + \alpha(t, \vec{y})r^{-\Delta}(1 + \dots), \tag{5.35}$$

where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}, \tag{5.36}$$

and the ellipses denote subleading terms. The first term $\sim r^{\Delta-d}$ is the non-normalizable term, and its coefficient β is identified with the source; the second term $\sim r^{-\Delta}$ is the normalizable term, and its coefficient is identified with the expectation value of O.
The functions α, β are not independent of each other: they are related through boundary conditions imposed on the behaviour in the bulk of the full solution. On smooth geometries we must impose regularity in the bulk, that is for $r \sim 0$. The holographic two point function

is then given by $\frac{\delta\alpha}{\delta\beta}$. Since we are inserting the operator $\bar{O}$ at the specific point x in the boundary, if we pick β to be a delta function at the point x the correlator (5.33) is simply given by α.

Some considerations are in order. First of all, we are interested in correlation functions involving four operators, two of which are heavy. The heavy operators however only play the role of modifying the background geometry, as we have exemplified in the previous Section. Thus we need to solve the equations of motion in more general backgrounds rather than just plain AdS.
Second, here the setup which we are considering is particularly simple, since Φ is a minimally coupled scalar, and we could be able to perform full computations analytically if the geometry is not too complicated. Typically this is not the case: not all light operators are necessarily dual to minimally coupled scalars, making the equations much more intricated, and the background geometry can be very complex too. This is usually bypassed by performing an expansion of the geometry around an easier background, and considering only the corresponding linearized equations.
Third, the expansion (5.35) might take slightly different forms whenever $2\Delta \in \mathbb{Z}$ (in particular this happens if $\Delta = d - \Delta$): in this case, logarithms appear in the expansion and it becomes more difficult (yet still possible, see [57]) to obtain the physical correlators.
Last, if one considers not a scalar field but a field with different Poincarè quantum numbers, then the dimensions are shifted. For instance, a massive p-form C_p on AdS_{d+1} couples to an operator O on the boundary that has conformal dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} - p, \tag{5.37}$$

whereas a massless p-form couples to a $d-p$-form operator on the boundary, of conformal dimension $\Delta = d - p$. This last one is the case for bulk gauge potentials and boundary chiral currents, that we will study in the following Chapter: $d = 2$, $p = 1$, $m = 0$ leads to $\Delta = 1$.

5.4 HHLL correlators and information loss

In Subsection 1.3.2 we have presented a version of the information paradox involving correlators of operators in a CFT dual to an AdS black hole. We have claimed that on a black hole background all correlators with infalling matter must decay exponentially. This behaviour is exactly what is present in a thermal conformal field theory on a non-compact space [20]. For instance, the thermal two-point function of scalar primary operators in the two-dimensional CFT on the line is

$$\langle \mathcal{O}_\Delta(t)\mathcal{O}_\Delta(0)\rangle = \left(\frac{\pi T}{\sinh(\pi T t)}\right)^{2\Delta} \sim e^{-2\pi\Delta T t}, \tag{5.38}$$

where t is (real) time. However, this is not what happens on compact spaces, like a CFT on the cylinder. In a sense, thus, our AdS black hole is "more thermal" than a thermal state in an unitary theory.

The same exponential behaviour is present in a field theory with an infinite number of local degrees of freedom, i.e. in the $c \to \infty$ limit. We can thus hope to further analyze the problem by studying HHLL correlators,

$$\langle \bar{O}_H(\infty)\bar{O}_L(z,\bar{z})O_L(1)O_H(0)\rangle \approx \langle \bar{O}_L(z,\bar{z})O_L(1)\rangle_{T_H}, \tag{5.39}$$

where $\langle\cdot\rangle_{T_H}$ means that we are computing correlators in the black hole microstate created by O_H and belonging to the ensemble with approximate Hawking temperature T_H, $(z, \bar{z})$ are coordinates on the plane, and let $\tau = -\log(1-z)$ be an Euclidean time coordinate on the cylinder. If the correlator (5.39) is thermal, then it is periodic in τ. This periodicity is connected with the exponential decay in real Lorentzian time t discussed above.
However, periodicity is not allowed in correlation functions of local operators, because from operator-state correspondence CFT correlators can be singular only in the OPE limit, i.e. when operators are inserted at the same point. If the correlator were to be periodic, it would display additional singularities at periodic images of the OPE singularities.

Chapter 6

Holography for HHJJ correlators

In this Chapter we devote our attention to the study of a class of correlators in the D1D5 CFT and in its dual $AdS_3 \times S^3 \times T^4$ description, following the strategy outlined in the previous section and applied in [1] to a simpler class of operators.
The CFT correlators involve two heavy operators O_H and two conjugate chiral current operators J^I. The CFT computation is performed at the free orbifold point.
On the gravity side, the heavy states are described by regular, asymptotically $AdS_3 \times S^3 \times T^4$ solutions of Supergravity and the correlators are obtained by solving the equations of motions for the dual fields.

6.1 Correlators from CFT

In this Section we compute four-point correlators in the D1D5 CFT involving two heavy operators O_H, which have conformal dimension of order c, and two $SU(2)$ chiral current operators, which are also light operators, i.e. they have conformal dimension of order one. The structure of the correlators is

$$\langle \bar{O}_H(z_1) O_H(z_2) \bar{J}^I(z_3) J^I(z_4) \rangle = \frac{1}{z_{12}^{2h_H} z_{34}^{2h_I}} \frac{1}{\bar{z}_{12}^{2\bar{h}_H} \bar{z}_{34}^{2\bar{h}_I}} \mathcal{G}_I(z, \bar{z}) \tag{6.1}$$

where $z_{ij} = z_i - z_j$ and

$$z = \frac{z_{14} z_{23}}{z_{13} z_{24}}. \tag{6.2}$$

Here and throughout this Section I is an $SO(4)$ index which is held fixed, i.e. no summation is intended. $(h_H, \bar{h}_H)$ and $(h_I, \bar{h}_I)$ are the holomorphic and antiholomorphic conformal dimensions of the heavy operators O_H and of the current operators J^I respectively.
For simplicity, we will use heavy operators in the Ramond-Ramond sector of the CFT that are related to chiral primaries by a chiral algebra transformation that acts only on the left sector. The current operators are by definition chiral primaries. Moreover, we will work at the free orbifold point of the CFT moduli space, where all the machinery of Section 4.2 applies.

6.1.1 Untwisted ($k = 1$) sector

The computations in the untwisted sector of the symmetric orbifold are easily carried out. We will leave s generic since it does not add any substantial complication in the calculation.

We will use the standard bosonization approach. The background in the untwisted sector is represented by heavy operators of the form

$$O_H = \bigotimes_{r=1}^{N} O^H_{(r)}, \qquad O^H_{(r)} = S^{\dot 1}_{s,(r)} S^{\dot 2}_{s,(r)} \tilde S^{\dot 1}_{s=0,(r)} \tilde S^{\dot 2}_{s=0,(r)}, \tag{6.3}$$

where

$$S^{\dot 1}_{s,(r)} = e^{i\left(s+\frac{1}{2}\right)H_{(r)}}, \qquad S^{\dot 2}_{s,(r)} = e^{i\left(s+\frac{1}{2}\right)K_{(r)}}, \tag{6.4}$$
$$\tilde S^{\dot 1}_{s,(r)} = e^{i\left(s+\frac{1}{2}\right)\tilde H_{(r)}}, \qquad \tilde S^{\dot 2}_{s,(r)} = e^{i\left(s+\frac{1}{2}\right)\tilde K_{(r)}}. \tag{6.5}$$

The conformal dimension and angular momentum of O_H is

$$h_H = N\left(s+\frac{1}{2}\right)^2, \qquad j_H = N\left(s+\frac{1}{2}\right) \tag{6.6}$$

The perturbations are represented by light operators. As said, here we choose as light operators the chiral current operators,

$$O_L = \sum_{r=1}^{N} O^L_{(r)}, \qquad O^L_{(r)} = J^I_{(r)}, \tag{6.7}$$

where $J^I_{(r)}$ can be written as

$$\begin{aligned}
J^+_{(r)} &= \frac{1}{2} : \psi^{1\dot A}_{(r)}\psi^{1\dot B}_{(r)}\epsilon_{\dot A\dot B} := i : e^{i\left(H_{(r)}+K_{(r)}\right)} :, \\
J^-_{(r)} &= -\frac{1}{2} : \psi^{2\dot A}_{(r)}\psi^{2\dot B}_{(r)}\epsilon_{\dot A\dot B} := i : e^{-i\left(H_{(r)}+K_{(r)}\right)} :, \\
J^3_{(r)} &= -\frac{1}{2}\left(: \psi^{1\dot A}_{(r)}\psi^{2\dot B}_{(r)}\epsilon_{\dot A\dot B} : -1\right) = \frac{1}{2}\left(: e^{iH_{(r)}} :: e^{-iH_{(r)}} : + : e^{iK_{(r)}} :: e^{-iK_{(r)}} : -1\right).
\end{aligned} \tag{6.8}$$

Analogous definition hold for the antiholomorphic generators.
The four-point functions $\langle O_H(z_1)\bar O_H(z_2)J^I(z_3)\bar J^I(z_4)\rangle$ are easily computed to be

$$\begin{aligned}
\langle O_H(z_1)\bar O_H(z_2)J^3(z_3)J^3(z_4)\rangle &= \frac{N}{4}\frac{1}{z_{12}^{2N(s+1/2)^2}\,\bar z_{12}^{N/2}}\frac{1}{z_{34}^2}, \\
\langle O_H(z_1)\bar O_H(z_2)J^+(z_3)J^-(z_4)\rangle &= N\frac{1}{z_{12}^{2N(s+1/2)^2}\,\bar z_{12}^{N/2}}\frac{1}{z_{34}^2}z^{-(1+2s)}, \\
\langle O_H(z_1)\bar O_H(z_2)J^-(z_3)J^+(z_4)\rangle &= N\frac{1}{z_{12}^{2N(s+1/2)^2}\,\bar z_{12}^{N/2}}\frac{1}{z_{34}^2}z^{+(1+2s)}, \\
\langle O_H(z_1)\bar O_H(z_2)\tilde J^3(z_3)\tilde J^3(z_4)\rangle &= \frac{N}{4}\frac{1}{z_{12}^{2N(s+1/2)^2}\,\bar z_{12}^{N/2}}\frac{1}{\bar z_{34}^2}, \\
\langle O_H(z_1)\bar O_H(z_2)\tilde J^+(z_3)\tilde J^-(z_4)\rangle &= N\frac{1}{z_{12}^{2N(s+1/2)^2}\,\bar z_{12}^{N/2}}\frac{1}{\bar z_{34}^2}\bar z^{-1}, \\
\langle O_H(z_1)\bar O_H(z_2)\tilde J^-(z_3)\tilde J^+(z_4)\rangle &= N\frac{1}{z_{12}^{2N(s+1/2)^2}\,\bar z_{12}^{N/2}}\frac{1}{\bar z_{34}^2}\bar z,
\end{aligned} \tag{6.9}$$

where

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}}. \tag{6.10}$$

Let us compute explicitly, for instance,

$$\langle O_H(z_1)\bar{O}_H(z_2)J^+(z_3)J^-(z_4)\rangle. \tag{6.11}$$

Exploiting the factorization of the operators into holomorphic and antiholomorphic part, we can write it as

$$\langle O_H(z_1)\bar{O}_H(z_2)J^+(z_3)J^-(z_4)\rangle = \frac{1}{4}\sum_{r=1}^{N} F^{\dot{A}\dot{B}\dot{C}\dot{D}}_{s,(r)}(z_i)\tilde{F}_0(\bar{z}_i)\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}, \tag{6.12}$$

where

$$\begin{aligned} F^{\dot{A}\dot{B}\dot{C}\dot{D}}_{s,(r)}(z_i) = \Big\langle & e^{i(s+\frac{1}{2})(H+K)_{(r)}(z_1)}e^{-i(s+\frac{1}{2})(H+K)_{(r)}(z_2)} \times \\ & \times :\psi^{1\dot{A}}_{(r)}(z_3)\psi^{1\dot{B}}_{(r)}(z_3) :: \psi^{2\dot{C}}_{(r)}(z_4)\psi^{2\dot{D}}_{(r)}(z_4): \Big\rangle \times \\ & \times \prod_{r'\neq r}\langle e^{i(s+\frac{1}{2})(H+K)_{(r')}(z_1)}e^{-i(s+\frac{1}{2})(H+K)_{(r')}(z_2)}\rangle, \end{aligned} \tag{6.13}$$

$$\tilde{F}_0(\bar{z}_i) = \prod_r \langle e^{i(s+\frac{1}{2})(\tilde{H}+\tilde{K})_{(r)}(\bar{z}_1)}e^{-i(s+\frac{1}{2})(\tilde{H}+\tilde{K})_{(r)}(\bar{z}_2)}\rangle. \tag{6.14}$$

Notice that in principle there should be two sum over strands, but by spin conservation the only nonzero contributions must come from the cases in which both currents act on the same strand. Moreover,

$$F^{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} = 4F^{\dot{1}\dot{2}\dot{1}\dot{2}}, \tag{6.15}$$

thus

$$\langle O_H(z_1)\bar{O}_H(z_2)J^+(z_3)J^-(z_4)\rangle = \sum_{r=1}^{N} F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s,(r)}(z_i)\tilde{F}_0(\bar{z}_i) = NF^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s,(r)}(z_i)\tilde{F}_0(\bar{z}_i). \tag{6.16}$$

Contracting every possible pair of fields, we find

$$F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s,(r)} = \frac{1}{z_{12}^{2Nh}}\frac{z_{13}^{2s+1}z_{24}^{2s+1}}{z_{14}^{2s+1}z_{23}^{2s+1}}\frac{1}{z_{34}^2} = \frac{1}{z_{12}^{2Nh}z_{34}^2}z^{-(2s+1)}, \tag{6.17}$$

$$\tilde{F}_0 = \frac{1}{\bar{z}_{12}^{2N\bar{h}}}, \tag{6.18}$$

where $h = (s+1/2)^2$, $\bar{h} = 1/4$. Since $Nh = h_H$, $N\bar{h} = \bar{h}_H$, we find the result given in (6.9).

For the correlator involving $J^3(z_3)J^3(z_4)$ we have to be careful when taking the contraction between the heavy operators and the currents. We have to include contractions of the kind

$$e^{i(s+\frac{1}{2})(H+K)_{(r)}(z_1)} \times \left(: e^{iH_{(r)}(z_3)} :: e^{-iH_{(r)}(z_3)} :\right). \tag{6.19}$$

Using (4.93), we see that this is regular in the OPE limit. Thus, the contraction

$$e^{i(s+\frac{1}{2})(H+K)_{(r)}(z_1)}\left(: e^{iH_{(r)}(z_3)} :: e^{-iH_{(r)}(z_3)} : + : e^{iK_{(r)}(z_3)} :: e^{-iK_{(r)}(z_3)} : -1\right) \tag{6.20}$$

gives a constant, and $\mathcal{G}(z,\bar{z})$ is a constant too.

6.1.2 Twisted $(k > 1)$ sector

We consider now correlators in the twisted sector of the CFT. As we have done in Subsection 4.2.2, it is more natural to express the operators in the ρ basis. After the change of basis, the left-moving current operators read

$$J^+ = \frac{1}{2}\left(:\psi^{1\dot{A}}_{\rho=0}\psi^{1\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{1\dot{A}}_{\rho}\psi^{1\dot{B}}_{k-\rho}\epsilon_{\dot{A}\dot{B}}: \right) = \left(:\psi^{1\dot{1}}_{\rho=0}\psi^{1\dot{2}}_{\rho=0}: + \sum_{\rho=1}^{k-1} :\psi^{1\dot{1}}_{\rho}\psi^{1\dot{2}}_{k-\rho}: \right), \tag{6.21}$$

$$J^- = -\frac{1}{2}\left(:\psi^{2\dot{A}}_{\rho=0}\psi^{2\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{2\dot{A}}_{\rho}\psi^{2\dot{B}}_{k-\rho}\epsilon_{\dot{A}\dot{B}}: \right) = -\left(:\psi^{2\dot{1}}_{\rho=0}\psi^{2\dot{2}}_{\rho=0}: + \sum_{\rho=1}^{k-1} :\psi^{2\dot{1}}_{\rho}\psi^{2\dot{2}}_{k-\rho}: \right), \tag{6.22}$$

$$J^3 = -\frac{1}{2}\sum_{\rho=0}^{k-1}\Big(:\psi^{1\dot{A}}_{\rho}\psi^{2\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: -1 \Big) = -\frac{1}{2}\sum_{\rho=0}^{k-1}\Big(:\psi^{1\dot{1}}_{\rho}\psi^{2\dot{2}}_{\rho}: - :\psi^{1\dot{2}}_{\rho}\psi^{2\dot{1}}_{\rho}: -1 \Big). \tag{6.23}$$

The definition for right-movers are analogous. In the following, we will write schematically

$$J^a \equiv \sum_{\rho=0}^{k-1} J^a_\rho, \tag{6.24}$$

where $J^{\pm}_\rho$ for $\rho > 0$ are intended to be defined as written in the last equalities of (6.21, 6.22).

As before, we consider s momentum-carrying excitations in the holomorphic sector. We characterize the heavy operators O_H by the integers s, k. Their conformal dimension and spin read

$$h_H = \frac{N}{k}\left(\frac{k}{4} + \frac{s(s+1)}{k}\right), \quad j_H = \frac{N}{k}\left(s + \frac{1}{2}\right). \tag{6.25}$$

These states have $s(s+1)/k$ units of momentum on each strand, which must be integer. Thus, assuming that k is prime for simplicity, we have that either $s = pk$ or $s = pk - 1$, with $p \in \mathbb{N}$.

Let us focus on the $s = pk$ case. On each strand we have k operators $S^{\dot{A}}_{k,s,\rho}$ in the left sector and k operators $\tilde{S}^{\dot{A}}_{k,\rho}$ in the right sector. The total heavy operator is

$$O_H(s = pk, k) = \left[\Sigma_k\tilde{\Sigma}_k \otimes_{\rho=0}^{k-1} S^{\dot{1}}_{k,pk,\rho}S^{\dot{2}}_{k,pk,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}. \tag{6.26}$$

Notice that $h_H \neq \bar{h}_H$ for $s > 0$ and thus heavy states can carry momentum. In the bosonized language,

$$S^{\dot{1}}_{k,pk,\rho} = e^{i\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)H_\rho}, \quad S^{\dot{2}}_{k,pk,\rho} = e^{i\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)K_\rho}, \tag{6.27}$$

with the right part given by analogous definitions with $s = 0$. The states generated by these operators are

$$|s = pk, k\rangle \equiv \left[\left(J^+_{-2p}\dots J^+_{-2/k}\right)\lim_{z,\bar{z}\to 0}\Sigma_k\bigotimes_{\rho=0}^{k-1} S^{\dot{1}}_{k,\rho}S^{\dot{2}}_{k,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}|0\rangle. \tag{6.28}$$

We can compute the correlators following the same logic as in the untwisted sector. For instance, let us consider the correlator

$$\langle O_H(z_1)\bar{O}_H(z_2)J^+(z_3)J^-(z_4)\rangle = -\frac{N}{k}\sum_{\rho=0}^{k-1} F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s=pk,k,\rho}(z_i)\tilde{F}_{s=0,k}(\bar{z}_i), \tag{6.29}$$

where now we have defined

$$\begin{aligned} F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s=pk,k,\rho=0}\left(z_i\right) \equiv \Big\langle & e^{i\left(\frac{1}{2}+p\right)\left(H_{\rho=0}(z_1)+K_{\rho=0}(z_1)\right)} e^{-i\left(\frac{1}{2}+p\right)\left(H_{\rho=0}(z_2)+K(z_2)_{\rho=0}\right)} \times \\ & \times :\psi^{+\dot{1}}_{\rho=0}\psi^{+\dot{2}}_{\rho=0}\left(z_3\right)::\psi^{-\dot{1}}_{\rho=0}\psi^{-\dot{2}}_{\rho=0}\left(z_4\right): \Big\rangle \times \\ & \times \prod_{\rho'\neq 0}\left\langle e^{i\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_1)+K_{\rho'}(z_1)\right)} e^{-i\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_2)+K_{\rho'}(z_2)\right)}\right\rangle \times \\ & \times \left\langle \Sigma_k\left(z_1\right)\Sigma_k\left(z_2\right)\right\rangle, \end{aligned} \tag{6.30}$$

$$\begin{aligned} F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s=pk,k,\rho>0}\left(z_i\right) \equiv \Big\langle & e^{i\left(-\frac{\rho}{k}+\frac{1}{2}+p\right)\left(H_{\rho}(z_1)+K_{\rho}(z_1)\right)} e^{-i\left(-\frac{\rho}{k}+\frac{1}{2}+p\right)\left(H_{\rho}(z_2)+K(z_2)_{\rho}\right)} \times \\ & \times e^{i\left(\frac{\rho}{k}-\frac{1}{2}+p\right)\left(H_{k-\rho}(z_1)+K_{k-\rho}(z_1)\right)} e^{-i\left(\frac{\rho}{k}-\frac{1}{2}+p\right)\left(H_{k-\rho}(z_2)+K(z_2)_{k-\rho}\right)} \times \\ & \times :\psi^{+\dot{1}}_{\rho}\psi^{+\dot{2}}_{k-\rho}\left(z_3\right)::\psi^{-\dot{1}}_{k-\rho}\psi^{-\dot{2}}_{\rho}\left(z_4\right): \Big\rangle \times \\ & \times \prod_{\rho'\neq \rho,k-\rho}\left\langle e^{i\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_1)+K_{\rho'}(z_1)\right)} e^{-i\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_2)+K_{\rho'}(z_2)\right)}\right\rangle \times \\ & \times \left\langle \Sigma_k\left(z_1\right)\Sigma_k\left(z_2\right)\right\rangle, \end{aligned} \tag{6.31}$$

$$\tilde{F}_{s=0,k} = \prod_{\rho}\left\langle e^{i\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(H_{\rho}(z_1)+K_{\rho}(z_1)\right)} e^{-i\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(H_{\rho}(z_2)+K_{\rho}(z_2)\right)}\right\rangle \left\langle \Sigma_k\left(z_1\right)\Sigma_k\left(z_2\right)\right\rangle. \tag{6.32}$$

The factor N/k comes from the summation over strands. Notice that also here we should have had a double sum over ρ, but the only nonzero contributions come from the cases in which all the fermions can be contracted. We have to be careful in the $\rho > 0$ terms because we will have more contractions appearing between heavy operators and the fermions. Contracting every possible pair of fields, we find

$$F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s=pk,k,\rho=0}\left(z_i\right) = -\frac{1}{z_{12}^{2h_H}}\frac{1}{z_{34}^2} z^{-(2p+1)}, \tag{6.33}$$

$$F^{\dot{1}\dot{2}\dot{1}\dot{2}}_{s=pk,k,\rho>0}\left(z_i\right) = -\frac{1}{z_{12}^{2h_H}}\frac{1}{z_{34}^2} z^{-2p}, \tag{6.34}$$

$$\tilde{F}_{s=0,k} = \frac{1}{\bar{z}_{12}^{2h_H}}. \tag{6.35}$$

Putting everything together in (6.29), then, for $s = pk$, the full set of correlators reads

$$\begin{aligned}
\langle O_H(z_1)\bar{O}_H(z_2)J^3(z_3)J^3(z_4)\rangle &= \frac{N}{4k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^2}, \\
\langle O_H(z_1)\bar{O}_H(z_2)J^+(z_3)J^-(z_4)\rangle &= \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^2}z^{-(2p+1)}[1+(k-1)z], \\
\langle O_H(z_1)\bar{O}_H(z_2)J^-(z_3)J^+(z_4)\rangle &= \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^2}z^{2p}[z+(k-1)], \\
\langle O_H(z_1)\bar{O}_H(z_2)\tilde{J}^3(z_3)\tilde{J}^3(z_4)\rangle &= \frac{N}{4k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{\bar{z}_{34}^2}, \\
\langle O_H(z_1)\bar{O}_H(z_2)\tilde{J}^+(z_3)\tilde{J}^-(z_4)\rangle &= \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{\bar{z}_{34}^2}\bar{z}^{-1}[1+(k-1)\bar{z}], \\
\langle O_H(z_1)\bar{O}_H(z_2)\tilde{J}^-(z_3)\tilde{J}^+(z_4)\rangle &= \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{\bar{z}_{34}^2}[\bar{z}+k-1].
\end{aligned} \tag{6.36}$$

For correlators involving J^3 the computation is the same as in the untwisted sector, because J^3 is diagonal also in the ρ basis. Notice also that by letting $k=1$ the $\rho>0$ terms disappear and we recover the results in the untwisted sector.

Let us move now to the $s=pk-1$ case. The heavy state $|s=pk-1,k\rangle$ is obtained from the one in the $s=pk$ case (6.28) by acting on it with J^-_{2p},

$$|s=pk-1,k\rangle \equiv \left[\left(J^-_{2p}J^+_{-2p}\dots J^+_{-2/k}\right)\lim_{z,\bar{z}\to 0}\Sigma_k\bigotimes_{\rho=0}^{k-1}S^{\dot{1}}_{k,\rho}S^{\dot{2}}_{k,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}|0\rangle. \tag{6.37}$$

This only changes the operator (6.26) in the holomorphic $\rho=0$ sector, where now we have

$$\hat{S}^{\dot{1}}_{k,0} = e^{i\left(-\frac{1}{2}+p\right)H_{\rho=0}}, \quad \hat{S}^{\dot{2}}_{k,0} = e^{i\left(-\frac{1}{2}+p\right)K_{\rho=0}}, \tag{6.38}$$

and thus O_H reads

$$O_H(s=pk-1,k) = \left[\Sigma_k\tilde{\Sigma}_k\hat{S}^{\dot{1}}_{k,0}\hat{S}^{\dot{2}}_{k,0}\otimes_{\rho=1}^{k-1}S^{\dot{1}}_{k,pk,\rho}S^{\dot{2}}_{k,pk,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}. \tag{6.39}$$

The full set of correlators will read now

$$\langle O_H(z_1)\bar{O}_H(z_2)J^3(z_3)J^3(z_4)\rangle = \frac{N}{4k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^2}, \tag{6.40}$$

$$\langle O_H(z_1)\bar{O}_H(z_2)J^+(z_3)J^-(z_4)\rangle = \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^2}z^{-2p}[z+(k-1)], \tag{6.41}$$

$$\langle O_H(z_1)\bar{O}_H(z_2)J^-(z_3)J^+(z_4)\rangle = \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^2}z^{2p-1}[1+(k-1)z], \tag{6.42}$$

$$\langle O_H(z_1)\bar{O}_H(z_2)\tilde{J}^3(z_3)\tilde{J}^3(z_4)\rangle = \frac{N}{4k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{\bar{z}_{34}^2}, \tag{6.43}$$

$$\langle O_H(z_1)\bar{O}_H(z_2)\tilde{J}^+(z_3)\tilde{J}^-(z_4)\rangle = \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{\bar{z}_{34}^2}\bar{z}^{-1}[1+(k-1)\bar{z}], \tag{6.44}$$

$$\langle O_H(z_1)\bar{O}_H(z_2)\tilde{J}^-(z_3)\tilde{J}^+(z_4)\rangle = \frac{N}{k}\frac{1}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{\bar{z}_{34}^2}[\bar{z}+k-1]. \tag{6.45}$$

Notice that the correlators involving antiholomorphic currents are formally identical to the $s = pk$ case: this is because in the antiholomorphic sector there are no momentum-carrying excitations.

6.2 Kaluza-Klein reduction with $SO(4)$ Yang-Mills fields

We now turn our attention to the gravity side of the duality, to which we will devote this and the following sections of this chapter.

As anticipated in Section 5.3, we can think of the four-point correlation functions computed in the previous Section as two-point functions of chiral current operators,

$$\langle s,k|J^I(1)\bar{J}^I(z)|s,k\rangle = \frac{1}{(1-z)^{2h_I}(1-\bar{z})^{2\bar{h}_I}}\mathcal{G}_I(z,\bar{z}). \tag{6.46}$$

In the $c \to \infty$ limit, the state $|s,k\rangle$ is dual to a background geometry. Here we will compute these two-point functions for values of the CFT moduli for which the geometry is well approximated by Supergravity. These values for the moduli, however, lie far from the free orbifold point of the CFT, where the CFT correlators have been previously computed. On the other hand, the current operators are chiral primary operators, and of a very special kind. They obey the affine Kač-Moody algebra (4.37), and their OPEs contain only the identity, the chiral current operators themselves, and their descendants. This implies that CFT and gravity results must agree.

From equations (5.20) and (5.22), we see that in the six-dimensional theory the fields which are dual to the chiral current operators J^I are vector perturbations of the metric ds_6^2 (5.24), which we recall to be generated by the backreaction of the heavy operator on the background, and of the RR gauge two-form C_2, or better, of its field strength $F_3 \equiv \mathrm{d}C_2$. All other Supergravity fields are switched off.
We want to find and solve the perturbation equations for those fields. The procedure outlined in Section 5.3 will then give us an alternative strategy for obtaining the value of the correlators (6.1).

6.2.1 The metric

We want to consider vector perturbations of the background metric (5.24), which we write in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{mn}dy^m dy^n, \tag{6.47}$$

referring to the Anti-de-Sitter part and to the 3-sphere part respectively.
The Kaluza-Klein ansatz for the perturbed metric is schematically given by [62]

$$d\hat{s}^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)Dy^m Dy^n, \qquad Dy^m = dy^m - Y^{Im}(y)A^I_\mu(x)dx^\mu. \tag{6.48}$$

The Kaluza-Klein gauge fields A^I_μ are associated with the six Killing vectors Y^{Im} of S^3. Such Killing vectors arise from the $SO(4)$ isometry group of the 3-sphere, and the metric is manifestly invariant under $SO(4)$ gauge transformations,

$$\delta x^m = \epsilon^I Y^{Im}, \qquad \delta x^\mu = 0, \qquad \delta A^I_\mu = \partial_\mu\epsilon^I + f^I{}_{JK}A^J_\mu\epsilon^K, \tag{6.49}$$

where ${f^I}_{JK}$ are the SO(4) structure constants,

$$[Y^I, Y^J] = {f^{IJ}}_K Y^K. \tag{6.50}$$

Since $SO(4) \approx SU(2)_L \times SU(2)_R$, the Killing vectors of $SO(4)$ are nothing but the vector spherical harmonics on S^3, $Y_1^{\alpha\pm}$, which are labeled by $(\alpha, \pm)$, where $\pm$ labels the left- and right-handed $SU(2)$ respectively and $\alpha = 0, \pm$. An explicit form is given in Appendix B.

The $SO(4)$ gauge invariance of the metric follows from the transformations of g_{mn} and Dx^m,

$$\delta Dx^m = \epsilon^I\, \partial_n Y^{Im} Dx^n, \qquad \delta g_{mn} = \epsilon^I Y^{Ir}\, \partial_r g_{mn} = -g_{rn}\epsilon^I\, \partial_m Y^{Ir} - g_{mr}\epsilon^I\, \partial_n Y^{Ir}. \tag{6.51}$$

Notice that Dx^m transforms under a local gauge transformation in the same way as dx^m under a global gauge transformation: D is like a covariant exterior derivative.
In order to write down the equations of motion of our theory we need to compute the curvature data related to the perturbed metric $d\hat{s}^2$. It is more convenient to rewrite the metric using the vielbein formalism [63],

$$d\hat{s}^2 = e^\alpha \otimes e^\beta \eta_{\alpha\beta} + (e^a - Y^{Ia}A^I) \otimes (e^b - Y^{Jb}A^J)\delta_{ab} \equiv \hat{e}^\alpha \otimes \hat{e}^\beta \eta_{\alpha\beta} + \hat{e}^a \otimes \hat{e}^b \delta_{ab}, \tag{6.52}$$

where $e^\alpha = \hat{e}^\alpha$ is a dreibein in the AdS_3 factor, e^a is a dreibein in the internal, unperturbed S^3 and $\hat{e}^a = e^a - Y^{Ia}A^I$. Here we adopt the shorthand notation $A^I = e^\alpha A^I_\alpha$ for the Yang-Mills potential. Let us also define the field strength of such potential,

$$F^I = \frac{1}{2} F^I_{\alpha\beta} e^\alpha \wedge e^\beta = \mathrm{d}A^I + \frac{1}{2} {f^I}_{JK} A^J \wedge A^K. \tag{6.53}$$

The spin connection 1-form for the metric $d\hat{s}^2$ is given by

$$\hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} + \frac{1}{2} F^I_{\alpha\beta} Y^I_a \hat{e}^a, \tag{6.54}$$

$$\hat{\omega}_{ab} = \omega_{ab} + A^I\, \nabla_a Y^I_b, \tag{6.55}$$

$$\hat{\omega}_{\alpha b} = \frac{1}{2} Y^I_b F^I_{\alpha\beta} \hat{e}^\beta, \tag{6.56}$$

where ω is the spin connection 1-form for the unperturbed metric ds^2. From the curvature 2-form we get that the only nonzero independent components of the Riemann tensor are

$$\hat{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{4} Y^I_a Y^J_b \delta^{ab} (F^I_{\alpha\gamma} F^J_{\beta\delta} - F^I_{\alpha\delta} F^J_{\beta\gamma} + 2F^I_{\alpha\beta} F^J_{\gamma\delta}), \tag{6.57}$$

$$\hat{R}_{abcd} = R_{abcd}, \tag{6.58}$$

$$\hat{R}_{\alpha\beta\gamma d} = \frac{1}{2} D_\gamma F^I_{\alpha\beta} Y^J_d \delta_{IJ}, \tag{6.59}$$

$$\hat{R}_{\alpha\beta cd} = F^I_{\alpha\beta}(\nabla_c Y^J_d) - \frac{1}{2} F^I_{\alpha\gamma} {F^J_\beta}^\gamma Y^I_{[c} Y^J_{d]}. \tag{6.60}$$

where D_γ is the gauge covariant derivative,

$$D_\gamma F^I_{\alpha\beta} = \nabla_\gamma F^I_{\alpha\beta} + {f^I}_{JK} A^J_\gamma F^K_{\alpha\beta}. \tag{6.61}$$

The Ricci tensor is given by

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} Y^I_a Y^J_b \delta^{ab} F^I_{\alpha\gamma} F^{J\gamma}_{\beta}, \tag{6.62}$$

$$\hat{R}_{ab} = R_{ab} + \frac{1}{4} Y^I_a Y^J_b F^I_{\alpha\beta} F^{J\alpha\beta}, \tag{6.63}$$

$$\hat{R}_{\alpha b} = \hat{R}_{b\alpha} = -\frac{1}{2} Y^I_b D_\beta F^{I\beta}_{\alpha}, \tag{6.64}$$

and finally the Ricci scalar is

$$\hat{R} = R_{\alpha\beta}\eta^{\alpha\beta} + R_{ab}\delta^{ab} - \frac{1}{4} Y^I_a Y^J_b \delta^{ab} F^I_{\gamma\delta} F^{J\gamma\delta}. \tag{6.65}$$

6.2.2 The 3-form

We also have to write down an ansatz for the field strength $F_3 = \mathrm{d}C_2$.[1] Since we want to keep invariance under $SO(4)$ transformations, our ansatz must be gauge invariant [62]. Let

$$V(y)\epsilon_{mnr}\,\mathrm{d}y^m \wedge \mathrm{d}y^n \wedge \mathrm{d}y^r = \frac{1}{3!}\hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c \epsilon_{abc}, \quad W(x)\epsilon_{\mu\nu\rho}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\mathrm{d}x^\rho = \frac{1}{3!}\hat{e}^\alpha \wedge \hat{e}^\beta \wedge \hat{e}^\gamma \epsilon_{\alpha\beta\gamma} \tag{6.66}$$

be the volume forms on the tilted S^3 and on AdS_3 repsectively. A gauge invariant ansatz is

$$F_3 = \frac{1}{3!}\hat{e}^\alpha \wedge \hat{e}^\beta \wedge \hat{e}^\gamma \epsilon_{\alpha\beta\gamma} + \frac{1}{3!}\hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c \epsilon_{abc}. \tag{6.67}$$

However, we should find a proposal for the 2-form field C_2 rather than for the field strength $F_3 = \mathrm{d}C_2$, which is possible only if $\mathrm{d}F_3 = 0$. Using

$$\mathrm{d}\hat{e}^a = -\hat{\omega}^a{}_B \wedge \hat{e}^B = -F^I Y^{Ia} - A^I_\alpha\,\partial_b Y^{Ia} \hat{e}^b \wedge \hat{e}^\alpha, \tag{6.68}$$

we obtain

$$\mathrm{d}F_3 = -\frac{1}{2}\epsilon_{abc} Y^{Ia} F^I \wedge \hat{e}^b \wedge \hat{e}^c. \tag{6.69}$$

Our ansatz for F_3 is not closed for a generic field strength F^I. To make it so we must add a gauge-invariant contribution to F_3 that cancels out $\mathrm{d}F_3$. Consider the 2-form $\omega_I = \frac{1}{3!}\epsilon_{abc} Y^{Ia} e^b \wedge e^c$: it is an inner derivative, as $\omega_I = \iota_{Y^I}\mathrm{vol}_{S^3}$. We now make use of Cartan's magic formula,

$$\mathcal{L}_{Y^I}\mathrm{vol}_{S^3} = \iota_{Y^I}\,\mathrm{d}\mathrm{vol}_{S^3} + \mathrm{d}\iota_{Y^I}\mathrm{vol}_{S^3}. \tag{6.70}$$

Since the volume form on S^3 is $SO(4)$ invariant, and $\mathrm{d}\mathrm{vol}_{S^3} = 0$, then ω_I is closed and hence exact, S^3 being simply connected. Therefore, $\omega_I = dN_I$ for a globally defined one form $N_I = N_{Ia}\,\mathrm{d}y^a$. It is straightforward to compute such a form: notice that $\omega_I = 2 \star Y^I$, where $\star$ is the Hodge dual on S^3. Using the properties of the vector spherical harmonics (see Appendix B), and the fact that $\star^2 = +1$ on 1- and 2- forms on S^3, then one has simply

$$N_I = \pm Y^I, \tag{6.71}$$

where the sign $\pm$ hold in the left- and right-chirality sector respectively.
Hence a candidate Kaluza-Klein ansatz for a closed 3-form field strength is

$$\hat{F}_3 = F_3 + 3F^I \wedge N_{Ia}\hat{e}^a. \tag{6.72}$$

This is manifestly gauge invariant and it is closed by construction.

[1] Given the unfortunate state of conventions, to avoid mistaking $F_3 = \mathrm{d}C_2$ for the $SO(4)$ field strength F^I we always keep the subscript explicit for the former.

6.2.3 Equations of motion

So far we have obtained a consistent $SO(4)$ invariant ansatz for Kaluza-Klein reduction. We can now compute the six-dimensional equations of motion for our ansatz. We are interested in obtaining the AdS_3 equations of motion for the gauge potential A^I.
In our construction we are interested only in first order perturbations around the $AdS_3 \times S^3$ background. Hence we can discard every quantity which is quadratic in the gauge potential A^I. This has also the consequence that at leading order the field strength becomes $F^I = \mathrm{d}A^I$, an abelian field strength. Similarly, we can replace the gauge covariant derivative of the field strength $D_\gamma F^I_{\alpha\beta}$ with the simple covariant derivative $\nabla_\gamma F^I_{\alpha\beta}$.
The equations of motion are given by the Einstein equations,

$$\hat{R}_{AB} - \frac{1}{2}\eta_{AB}\hat{R} = \hat{T}_{AB} = -2\frac{\delta}{\delta g^{AB}}\left(-\frac{1}{12}\hat{F}_3^2\right)\Big|_{g=\eta} - \frac{1}{12}\eta_{AB}\hat{F}_3^2, \tag{6.73}$$

together with the conditions $\mathrm{d}\hat{F}_3 = 0$, which we imposed already in the above discussion, and $\mathrm{d}\star\hat{F}_3 = 0$.
The relevant equation of motion that contains the gauge field strength at linear order is the Einstein equation with mixed indices,

$$\hat{R}_{\alpha b} = -2\frac{\delta}{\delta g^{\alpha b}}\left(-\frac{1}{12}\hat{F}_3^2\right) = \frac{1}{2}(\hat{F}_3)_{\alpha AB}(\hat{F}_3)_b{}^{AB}, \tag{6.74}$$

where we recall that, at linear order in the gauge potential,

$$\hat{R}_{\alpha b} = -\frac{1}{2}Y^I_b D_\beta F^{I\beta}_\alpha = -\frac{1}{2}Y^I_b\,\nabla_\beta F^{I\beta}_\alpha. \tag{6.75}$$

One computes the right hand side to be

$$(\hat{F}_3)_{\alpha AB}(\hat{F}_3)_b{}^{AB} = -\frac{1}{4}\epsilon_{\alpha\beta\gamma}F^{I\beta\gamma}N_{Ib}. \tag{6.76}$$

We make use now of the relation between N_I's and Y^I's (6.71): let us denote by A^{IJ} the map from $SO(4)$ onto itself, that acts as $+1$ on $SU(2)_L$ and as -1 on $SU(2)_R$. With this notation,

$$\delta_{ab}Y^{Ib} = A^{IJ}N_{Ja}. \tag{6.77}$$

This allows us to rewrite in the left hand side $Y^I_b = A^I_J N^J_b$, and then Einstein equation becomes

$$A^{IJ}\,\nabla_\beta F^{I\beta}_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}F^{J\beta\gamma}, \tag{6.78}$$

or equivalently, by taking its AdS_3 Hodge dual,

$$A^{IJ}\,\nabla_\beta F^{I\beta}_\alpha \epsilon^{\alpha\gamma\delta} = -\frac{1}{2}F^{J\gamma\delta}, \tag{6.79}$$

which is more conveniently rewritten in the language of differential forms as

$$\mathrm{d}\star F^I + A^I_J F^J = 0. \tag{6.80}$$

In order to keep the full $SO(4)$ gauge structure, it suffices to send $\mathrm{d} \to \mathrm{D}$ and let F^I be the non-abelian field strength. This is nothing but the equations of motion of a Yang-Mills Chern-Simons theory.

6.3 A closer look at Yang-Mills Chern-Simons theories

In this Section we shall analyze in more detail some peculiarities of Chern-Simons theories that will play a crucial role in obtaining a consistent solution of the equations of motion (6.80).

6.3.1 The space of solutions

Let us consider again the equations of motion,

$$\mathrm{D} \star F^I \pm F^I = 0, \tag{6.81}$$

where the upper and lower sign apply to the left and right chirality sector respectively.

We are interested in describing the phase space of the theory, that is the space of gauge inequivalent solutions of the equations of motion.
In the present theory, thanks to linearity of the equations the space of (not necessarily gauge inequivalent) solutions of the equations of motion is a product [64],

$$\mathcal{H} = \mathcal{H}_f \times \mathcal{H}_{nf}, \tag{6.82}$$

where $\mathcal{H}_f$ is the space of "flat" solutions $F^I = 0$. These are solutions of the topological sector of the theory, that come from a Chern-Simons term in the low-energy effective action. $\mathcal{H}_{nf}$ is instead the space of connections with nontrivial field stregth.

As we have already pointed out, at linear order in the fields the theory is abelian. Let us work in the linearized framework in the following, and for sake of notation let us also suppress the $SO(4)$ index. The fact that the space of solutions is a product means that, given a solution A of the equations of motion, it is always possible to split it into

$$A = A^0 + B, \qquad F = \mathrm{d}A = \mathrm{d}B, \qquad \mathrm{d}A^0 = 0. \tag{6.83}$$

The splitting is clearly not unique, due to gauge redundancy. We can however fix the gauge for B in a clever way [65]. The dimensionality of our spacetime makes it so that

$$B = \mp \star F \tag{6.84}$$

is a legitimate expression, as well as a good gauge fixing condition. Notice also that by taking $\mathrm{d}\star$ of both sides it is manifest that it implies the Lorenz gauge condition,

$$\mathrm{d} \star B = 0. \tag{6.85}$$

Furthermore, we can use (6.84) and the fact that $\star^2 = -1$ on 1- and 2- forms on Lorentzian AdS_3 to rewrite the equations of motion as

$$\star\, \mathrm{d}B \pm B = 0 \implies \epsilon^{\alpha\beta\gamma}\, \nabla_\beta B_\gamma \pm B^\alpha = 0. \tag{6.86}$$

This is a Maxwell-Chern-Simons equation of motion with a topological mass term. For this reason, the $\mathcal{H}_{nf}$ sector of the space of solution is also referred to as the "massive" sector. To convince ourselves that the excitations are indeed massive, applying $(\star\mathrm{d} \mp 1)$ to both members of the equation for B one gets

$$(\Box + 1) B_\alpha = 0, \tag{6.87}$$

where $\Box$ is the scalar AdS_3 Laplacian.

It is interesting to compare this intermediate result with the ones appearing in [1]. In [1] the authors were interested in obtaining and solving the wave equation for a scalar field B in the bulk, which is dual to the operator $O^{++} \equiv O^{1\dot{1}}$ defined in (4.72). Comparing (6.87) with the equation (4.16) for B (with $\ell = 1$) of [1], we see that they are the same equation, with the only exception that here B_α is a spin-1 field. Nevertheless, B_α still has only one single degree of freedom also in this case. Our claim is then that the massive sector of our equations of motion is "equivalent"[2] to a theory of scalars in AdS_3.

For our purposes, this means that the solution of (6.87) has the same form of the solution of (4.16) of [1]. Without giving the details of the computation, we just quote the result for B obtained in [1]: the coefficient of the non renormalizable term appearing in the solution for the wave equation for the scalar field B is

$$b_1(z,\bar{z}) = -i\left(\frac{z}{\bar{z}}\right)^{\frac{\hat{s}}{2k}} \frac{1}{|z|^{\frac{1}{k}} - |z|^{-\frac{1}{k}}}\left[\frac{\bar{z}}{\bar{z}-1}|z|^{-\frac{\hat{s}}{k}} + \frac{1}{z-1}|z|^{\frac{\hat{s}}{k}}\right], \tag{6.88}$$

where $\hat{s} \equiv s \mod k$. The relevant degree of freedom in B_α will have a solution with exactly the same form. Applying the holographic dictionary, this would translate into an expectation for some vector operator in the boundary CFT that is neither holomorphic nor antiholomorphic, and hence such operator cannot be a chiral $SU(2)_{L,R}$ current.

If the massive vectors B_α are not the bulk fields dual to chiral currents, then flat gauge connections A^0_α must be. This might get us worried for two reasons.
First, naïvely we would be lead to think that pure gauge configurations do not yield any interesting physics, but this cannot be more wrong. This is because $\mathcal{H}_f$ is not a connected set, rather it is the union of countably many disjoint pieces. Two field configurations in the same connected component of $\mathcal{H}_f$ are related by a "small" gauge transformation ($A^0 \to A^0 + \mathrm{d}\Lambda$ for some regular function Λ), whereas one can move to another component of $\mathcal{H}_f$ by means of a "large" gauge transformation (a gauge transformation with singular gauge function Λ). What distinguishes between the various components of $\mathcal{H}_f$ is the topological charge of the solutions, namely the winding around S^1_y,

$$\frac{1}{2\pi R_y}\int_{S^1_y} A^0 \in \pi_1\left(S^1_y\right) \simeq \mathbb{Z}. \tag{6.89}$$

Second, and this is a legitimate worry, if we accept that nontrivial pure gauge configurations are dual to chiral currents, then it seems that we have lost the possibility to distinguish between the left- and right- chirality sector. In fact, the equations

$$F^I = 0 \tag{6.90}$$

do not depend on the choice of the $SO(4)$ index I. But this would be a problem, because in the boundary CFT holomorphic and antiholomorphic currents do not mix.
Actually, there is a very natural solution to this issue, and that is that we still have to impose appropriate boundary conditions on solutions of the equations of motion. Recall that AdS_3 has a boundary, and that boundary terms might spoil gauge invariance. We shall review the matter in detail in the next Subsection.

[2]While we do not need a rigorous statement for our purposes, this equivalence can be indeed made more precise, see for instance [66, 67].

6.3.2 Boundary conditions for Chern-Simons on AdS_3

Consistent boundary conditions have to be imposed on the gauge potential in order to restore gauge invariance at the boundary. Our exposition will be based upon [68–70].

Let us consider a gauge theory on a 3-dimensional AdS space. For simplicity we let the gauge group be $U(1)$, but the following arguments will go through almost untouched also for non abelian gauge theoeries.
Since spacetime is odd-dimensional, besides the usual Maxwell term the action can have Chern-Simons terms. The bulk action reads

$$S = S_M + S_{CS} = -\frac{1}{4\pi}\int_{AdS_3} F \wedge \star F + \frac{\kappa}{4\pi}\int_{AdS_3} A \wedge F, \tag{6.91}$$

where κ is the Chern-Simons level. Taking the variation of the action with respect to the gauge field A, and assuming that variations vanish at the boundary, we easily get the equations of motion in the bulk,

$$\mathrm{d} \star F + \kappa F = 0, \tag{6.92}$$

which is nothing but the linearized form of (6.81) if $\kappa = \pm 1$.

Let us stress that the Chern-Simons term in the action is not gauge invariant: under a gauge transformation $\delta_\Lambda A = \mathrm{d}\Lambda$, S_{CS} varies by a boundary term,

$$\delta_\Lambda S_{CS} = \frac{\kappa}{4\pi}\int_{\partial AdS_3} \Lambda \, \mathrm{d}A. \tag{6.93}$$

This implies that gauge orbit degrees of freedom live in the 2-dimensional boundary. With the holographic principle and our previous discussion in mind, we are encouraged to study the space of flat gauge connections.

In the flat sector, we expect a boundary current (a "source") to be obtained from the on-shell variation of the action with respect to the (flat) gauge potential A^0,

$$\delta S = \frac{1}{2\pi}\int_{\partial AdS_3} \mathrm{d}^2 x \sqrt{-g}\Big|_{\partial AdS_3} J^\alpha \delta A^0_\alpha. \tag{6.94}$$

To avoid cluttering let us suppress the superscript 0 that denotes flatness of the gauge connection.

We need to define the appropriate boundary conditions that make the action fully gauge invariant.
Let (t, y) be a coordinatization of ∂AdS_3 such that they are timelike and spacelike coordinates respectively. The structure of the boundary of AdS_3 is the cylinder with the identification $y \sim y + 2\pi R_y$. Naively, one might guess that in the variational principle one could hold fixed both the components A_t and A_y at the boundary. But this is too strong, and will typically result in the absence of smooth solutions for the fields. The obstruction here is the holonomy around the y circle,

$$\int_{S^1_y} \mathrm{d}y A_y. \tag{6.95}$$

When the y circle is contracted we need this to either vanish or to match onto an appropriate source to avoid singularities.

So it is only A_t that really can take generic values. If we define the usual null coordinates (u, v) on the boundary, then an appropriate variational principle is to hold fixed either A_u or A_v, but not both. The choice of which component must be held fixed is however not up to our desire, and it will depend on the sign of κ, as we will see shortly.

In order to restore gauge symmetry of the theory, we might try adding a boundary term to the action, which we will call S_{CT} (as for "counterterm"). Our claim is that a good choice for the counterterm is

$$S_{CT} = -\frac{|\kappa|}{4\pi}\int_{\partial AdS_3} \mathrm{d}u\,\mathrm{d}v A_u A_v. \tag{6.96}$$

Of course, S_{CT} is not gauge invariant, but this is legitimate because the original action is neither.

For simplicity let us assume that $\kappa > 0$. Then

$$\delta_\Lambda(S_{CS} + S_{CT}) = -\frac{\kappa}{2\pi}\int_{\partial AdS_3} \mathrm{d}u\,\mathrm{d}v A_v \delta_\Lambda A_u \tag{6.97}$$

implies that the correct boundary condition that makes this vanish is $A_v = 0$, whereas A_u is left unconstrained at the boundary. This is equivalent to saying that

$$J_v = 0, \tag{6.98}$$

i.e. the only source that is present is the purely left-moving J_u. If we were to assume $\kappa < 0$, then the same argument goes through switching u with v.

We want now to show that the choice of the sign of S_{CT} is the only consistent one. Note that the integrand in S_{CT} depends on the metric, and so it will contribute to the stress-energy tensor. The stress-energy tensor is readily computed by taking variation with respect to the metric, and it amounts to

$$T_{\alpha\beta} = \frac{|\kappa|}{8\pi}\left(A_\alpha A_\beta - \frac{1}{2}A_\gamma A^\gamma g_{\alpha\beta}\right), \tag{6.99}$$

or, in (u, v) coordinates,

$$T_{uu} = \frac{|\kappa|}{8\pi}A_u A_u, \qquad T_{vv} = \frac{|\kappa|}{8\pi}A_v A_v, \qquad T_{uv} = T_{vu} = 0. \tag{6.100}$$

We can now see why the sign of S_{CT} is important: if we took a different sign then the energy would have been unbounded from below. So S_{CT} is really the only consistent improvement of the action that makes it gauge invariant.

To make contact with the problem at hand, we want to find solutions of (6.92) in the flat sector for $\kappa = \pm 1$. Since the sign of κ prescribes the correct boundary condition, in each chirality sector only the corresponding gauge field can be nontrivial at the conformal boundary, and AdS/CFT can still be exploited.

6.4 Correlators from gravity

As argued above, we are interested in solutions of the AdS_3 equations of motion

$$\mathrm{d}\star F^I \pm F^I = 0, \tag{6.101}$$

that belong to the "trivial", gauge-flat sector

$$F^I = \mathrm{d}A^I = 0. \tag{6.102}$$

This equation for the gauge potential A^I, however, cannot be inverted as is. The strategy then is to choose a gauge fixing in the bulk that preserves the boundary conditions and that admits nontrivial solutions to the equations of motion. Lorenz gauge turns out to be a good choice,

$$\mathrm{d}\star A^I = 0. \tag{6.103}$$

Here $\star$ is the Hodge dual taken in the $AdS_3/\mathbb{Z}_k$ metric given in (5.24),

$$ds^2_{AdS_3} = \frac{dr^2}{a^2k^{-2}+r^2} - \frac{a^2k^{-2}+r^2}{Q_1Q_5}dt^2 + \frac{r^2}{Q_1Q_5}dy^2, \tag{6.104}$$

but for our purposes we will rather use the coordinates

$$\rho = \frac{rk}{a}, \qquad \tau = \frac{at}{k\sqrt{Q_1Q_5}} = \frac{t}{Rk}, \quad \sigma = \frac{ay}{k\sqrt{Q_1Q_5}} = \frac{y}{Rk}, \tag{6.105}$$

where R is the S^1_y radius. Hence the metric reads

$$ds^2 = \frac{\mathrm{d}\rho^2}{(1+\rho^2)} - (1+\rho^2)\,\mathrm{d}\tau^2 + \rho^2\,\mathrm{d}\sigma^2. \tag{6.106}$$

Notice that the coordinate σ is periodic with period $2\pi/k$.

The most general solution can be decomposed into a sum over Fourier modes,

$$A^I_\mu = \frac{1}{(2\pi)^2}e^{i\tilde{s}\sigma}\sum_{l\in\mathbb{Z}}\int \mathrm{d}\omega e^{-i\omega\tau}e^{ilk\sigma}g(l,\omega)a^I_\mu(l,\omega;\rho), \tag{6.107}$$

where the choice of $g(l,\omega)$ encodes a particular boundary data, and $\tilde{s}$ is defined in order to make sure that A^IY^I is invariant under $y \to y + 2\pi R$. In fact, the expectation value of the currents is encoded in the component of the full metric deformation proportional to the vector spherical harmonic $Y^I(\theta,\hat{\phi},\hat{\psi})$ [53]. Explicitly, since

$$\hat{\psi}+\hat{\phi} = \psi+\phi-(2s+1)\frac{t+y}{Rk}, \qquad \hat{\psi}-\hat{\phi} = \psi-\phi+\frac{t-y}{Rk}, \tag{6.108}$$

then (see Appendix B for full expressions)

$$\begin{aligned}
&Y^{\pm+}_1 \sim e^{\pm(2s+1)i\frac{t+y}{Rk}} && \Longrightarrow \tilde{s} \equiv \mp(2s+1) \mod k,\\
&Y^{\pm-}_1 \sim e^{\mp i\frac{t-y}{Rk}} && \Longrightarrow \tilde{s} \equiv \mp 1 \mod k,\\
&Y^{0,\pm}_1 \sim 1 && \Longrightarrow \tilde{s} \equiv 0 \mod k,
\end{aligned} \tag{6.109}$$

whereas $\tilde{s} \equiv 0$ always for $k = 1$. We will choose $0 \leq \tilde{s} < k$. To avoid unnecessary cluttering, let us suppress the group index in the following.

Substituting the expansion for the fields in the equations of motion and in the gauge fixing conditions one obtains, respectively,

$$\omega a_\sigma + (lk + \tilde{s}) a_\tau = 0, \qquad a'_\sigma - i(lk + \tilde{s}) a_\rho = 0, \qquad a'_\tau + i\omega a_\rho = 0, \tag{6.110}$$

and

$$(1 + 3\rho^2) a_\rho + \rho(1 + \rho^2) a'_\rho - \frac{\rho}{1 + \rho^2} i\omega a_\tau + \frac{1}{\rho} i(lk + \tilde{s}) a_\sigma = 0. \tag{6.111}$$

Let us now switch to null coordinates,

$$u = \tau + \sigma, \qquad v = \tau - \sigma, \tag{6.112}$$

so that $a_u = a_\tau + a_\sigma$, $a_v = a_\tau - a_\sigma$. We can thus recast the equations of motion as

$$0 = (\omega + (lk + \tilde{s}))\, a_u - (\omega - (lk + \tilde{s}))\, a_v, \tag{6.113}$$
$$0 = a'_u + i\,(\omega - (lk + \tilde{s}))\, a_\rho, \tag{6.114}$$
$$0 = a'_v + i\,(\omega + (lk + \tilde{s}))\, a_\rho. \tag{6.115}$$

Going to second order in derivatives and exploiting the gauge fixing condition we can decouple the last two equations, obtaining

$$\rho(1 + \rho^2) a''_u + \rho(1 + 3\rho^2) a'_u + \left(\frac{\rho^2}{1 + \rho^2} \omega^2 - (lk + \tilde{s})^2 \right) a_u = 0, \tag{6.116}$$
$$\rho(1 + \rho^2) a''_u + \rho(1 + 3\rho^2) a'_u + \left(\frac{\rho^2}{1 + \rho^2} \omega^2 - (lk + \tilde{s})^2 \right) a_u = 0. \tag{6.117}$$

Setting $x = \sin^2 \arctan \rho$, we can rewrite the equations as

$$f'' + \frac{f'}{x} + \frac{1}{4} \left(\frac{\omega^2}{x(1 - x)} - \frac{(lk + \tilde{s})^2}{x^2(1 - x)} \right) f = 0, \tag{6.118}$$

where $f = a_u$ or a_v and prime now means derivation with respect to x. The solution is

$$f = x^{|lk+\tilde{s}|/2} {}_2F_1 \left(\frac{1}{2}(|lk + \tilde{s}| - \omega), \frac{1}{2}(|lk + \tilde{s}| + \omega), 1 + |lk + \tilde{s}|; x \right). \tag{6.119}$$

f is regular everywhere in the bulk and its expansion near the $x \sim 1$ boundary reads

$$f \approx \frac{\Gamma(1 + |lk + \tilde{s}|)}{\Gamma\left(\frac{1}{2}(2 + |lk + \tilde{s}| - \omega)\right) \Gamma\left(\frac{1}{2}(2 + |lk + \tilde{s}| + \omega)\right)} \left\{ 1 + (1 - x) \left[\frac{|lk + \tilde{s}|}{2} - \frac{(\omega^2 - (lk + \tilde{s})^2)}{4} \times \right.\right.$$
$$\left.\left. \times \left(2\gamma + \psi\left(\frac{|lk + \tilde{s}| + \omega + 2}{2} \right) + \psi\left(\frac{|lk + \tilde{s}| - \omega + 2}{2} \right) + \log(1 - x) - 1 \right) \right] \right\} + O((1-x)^2). \tag{6.120}$$

Recall however that boundary conditions require either that $A_u = 0$ or $A_v = 0$ at the boundary. Since f approaches a constant at the boundary instead, the only way to have a solution that is consistent with the boundary conditions is to have

$$a_u = f\delta(\omega + (lk + \tilde{s})), \qquad a_v = 0, \qquad a_\rho = i\frac{d}{d\rho}f, \tag{6.121}$$

or

$$a_u = 0, \qquad a_v = f\delta(\omega - (lk + \tilde{s})), \qquad a_\rho = i\frac{d}{d\rho}f, \tag{6.122}$$

respectively.

The boundary conditions also imply that the term proportional to $\left(\omega^2 - (lk + \tilde{s})^2\right)$ in f vanishes, thanks to the delta functions. We recognize a constant non-normalizable term (the source) and a normalizable term (the expectation value) proportional to $(1 - x) \sim \rho^{-2}$.

We are interested in seeing what happens when the source is a delta function at the boundary. This can be obtained by tuning the function $g(l, \omega)$ in the Fourier expansion in such a way that the non-normalizable term has constant Fourier transform: explicitly, let us choose

$$g(l, \omega) = \frac{\Gamma\left(\frac{1}{2}(2 + |lk + \tilde{s}| - \omega)\right)\Gamma\left(\frac{1}{2}(2 + |lk + \tilde{s}| + \omega)\right)}{\Gamma(1 + |lk + \tilde{s}|)}. \tag{6.123}$$

Let us choose the boundary condition $A_v = 0$, which selects the left-moving currents. Then the expansion for A_u is just

$$A_u = \frac{1}{(2\pi)^2}\sum_l e^{i(lk+\tilde{s})u}\left[1 + \frac{|lk + \tilde{s}|}{2}(1 - x)\right] + O((1 - x)^2), \tag{6.124}$$

where we have removed the ω integration with the delta function. The coefficient of the normalizable term can easily be computed to be

$$a_1^+(u) = ke^{i\tilde{s}u}\frac{e^{iku}}{(1 - e^{iku})^2} + \tilde{s}e^{i\tilde{s}u}\frac{1}{1 - e^{iku}}. \tag{6.125}$$

where the $+$ superscript denotes that we are considering the left-moving sector and we have dropped the 2π factors. Going back to the original (t, y) coordinates, this reads

$$a_1^+(t + y) = ke^{i\tilde{s}\frac{t+y}{Rk}}\frac{e^{i\frac{t+y}{R}}}{\left(1 - e^{i\frac{t+y}{R}}\right)^2} + \tilde{s}e^{i\tilde{s}\frac{t+y}{Rk}}\frac{1}{1 - e^{i\frac{t+y}{R}}}. \tag{6.126}$$

A similar computation for the right-moving sector yields

$$a_1^-(t - y) = ke^{-i\tilde{s}\frac{t-y}{Rk}}\frac{e^{i\frac{t-y}{R}}}{\left(1 - e^{i\frac{t-y}{R}}\right)^2} + \tilde{s}e^{i\tilde{s}\frac{t-y}{Rk}}\frac{1}{1 - e^{i\frac{t-y}{R}}}. \tag{6.127}$$

The two-point correlator of the current operators in the state $|s, k\rangle$ is given by the full metric perturbation

$$\begin{aligned}\langle s,k|J^3(0,0)J^3(t,y)|s,k\rangle &\simeq a_1^{0+}(t,y)Y_1^{0+}(\theta,\hat\phi,\hat\psi),\\ \langle s,k|J^+(0,0)J^-(t,y)|s,k\rangle &\simeq a_1^{++}(t,y)Y_1^{++}(\theta,\hat\phi,\hat\psi),\\ \langle s,k|J^-(0,0)J^+(t,y)|s,k\rangle &\simeq a_1^{-+}(t,y)Y_1^{-+}(\theta,\hat\phi,\hat\psi),\\ \langle s,k|\tilde J^3(0,0)\tilde J^3(t,y)|s,k\rangle &\simeq a_1^{0-}(t,y)Y_1^{0-}(\theta,\hat\phi,\hat\psi),\\ \langle s,k|\tilde J^+(0,0)\tilde J^-(t,y)|s,k\rangle &\simeq a_1^{+-}(t,y)Y_1^{+-}(\theta,\hat\phi,\hat\psi),\\ \langle s,k|\tilde J^-(0,0)\tilde J^+(t,y)|s,k\rangle &\simeq a_1^{--}(t,y)Y_1^{--}(\theta,\hat\phi,\hat\psi),\end{aligned} \tag{6.128}$$

where we have reinstated the $SU(2)$ index $\alpha=0,\pm$ on the functions $a_1^{\alpha\pm}$, because they differ for the choice of $\tilde s$, and $\simeq$ means that we should take only the t,y dependence appearing in front of the vector spherical harmonics on the right hand side.

To compare the bulk results with the CFT results, one should transform from the cylinder coordinates (t,y) to the Euclidean plane coordinates $z,\bar z$,

$$z=e^{i\frac{t+y}{R}},\qquad \bar z=e^{i\frac{t-y}{R}}, \tag{6.129}$$

and remember that, for an operator O of conformal weights $(h,\bar h)$,

$$O(z,\bar z)=z^{-h}\bar z^{-\bar h}O(t,y). \tag{6.130}$$

6.4.1 Comparing with the CFT results

On the CFT side, for the twisted sector we have studied the cases $s=pk$ and $s=pk-1$, with k prime and $p\in\mathbb{N}$. In general, we will also need to distinguish those cases on the gravity side, because they can affect the monodromy of the solutions.

Let us start however from the simplest correlator, involving only the diagonal chiral current J^3, which is already monodromous by itself. In this case $\tilde s=0$ and the spherical harmonic Y^{0+} has no t,y dependence, thus

$$\langle s,k|J^3(1)J^3(z)|s,k\rangle = z^{-1}a_1^{0+}(z,\tilde s=0)=k\frac{1}{(1-z)^2}, \tag{6.131}$$

and for its antiholomorphic counterpart, similarly,

$$\langle s,k|\tilde J^3(1)\tilde J^3(\bar z)|s,k\rangle = \bar z^{-1}a_1^{0-}(\bar z,\tilde s=0)=k\frac{1}{(1-\bar z)^2}. \tag{6.132}$$

Let us consider now the correlators involving J^+ and J^-. Here we will need to distinguish between $s=pk$ and $s=pk-1$, and the order is going to be important as well. Let us start from

$$\langle s,k|J^+(1)J^-(z)|s,k\rangle. \tag{6.133}$$

The relevant spherical harmonic is $Y_1^{++}\propto z^{-(2s+1)/k}$. For $s=pk$, we will have $\tilde s\equiv 2pk+1 \mod k\equiv 1 \mod k$. Then,

$$\begin{aligned}\langle s=pk,k|J^+(1)J^-(z)|s=pk,k\rangle &= z^{-1}z^{-2p-1/k}\left[kz^{1/k}\frac{z}{(1-z)^2}+\frac{z^{1/k}}{1-z}\right]\\ &=\frac{z^{-(2p+1)}}{(1-z)^2}[1+(k-1)z].\end{aligned} \tag{6.134}$$

If $s = pk - 1$ instead, then $\tilde{s} \equiv 2pk - 1 \mod k \equiv k - 1 \mod k$, and

$$\begin{aligned} \langle s = pk-1, k | J^+(1) J^-(z) | s = pk-1, k \rangle &= z^{-1} z^{-2p+1/k} \left[k z^{1-1/k} \frac{z}{(1-z)^2} + (k-1) \frac{z^{1-1/k}}{1-z} \right] \\ &= \frac{z^{-2p}}{(1-z)^2} [z + (k-1)]. \end{aligned} \tag{6.135}$$

If we switch the two operators, namely

$$\langle s, k | J^-(1) J^+(z) | s, k \rangle. \tag{6.136}$$

then the relevant spherical harmonic is now $Y_1^{-+} \propto z^{(2s+1)/k}$.
For $s = pk$, we will have now $\tilde{s} \equiv -(2pk+1) \mod k \equiv k - 1 \mod k$, thus

$$\begin{aligned} \langle s = pk, k | J^-(1) J^+(z) | s = pk, k \rangle &= z^{-1} z^{2p+1/k} \left[k z^{1-1/k} \frac{z}{(1-z)^2} + \frac{z^{1-1/k}}{1-z} \right] \\ &= \frac{z^{2p}}{(1-z)^2} [z + (k-1)], \end{aligned} \tag{6.137}$$

whereas for $s = pk - 1$ one has $\tilde{s} \equiv 1 \mod k$, and

$$\begin{aligned} \langle s = pk-1, k | J^-(1) J^+(z) | s = pk-1, k \rangle &= z^{-1} z^{2p-1/k} \left[k z^{1/k} \frac{z}{(1-z)^2} + \frac{z^{1/k}}{1-z} \right] \\ &= \frac{z^{2p-1}}{(1-z)^2} [1 + (k-1)z]. \end{aligned} \tag{6.138}$$

For the correlators involving the antiholomorphic currents $\tilde{J}^+, \tilde{J}^-$ there is no distinction between $s = pk$ and $s = pk - 1$, because $\tilde{s}$ does not depend on s. We can use the results obtained in the $s = pk$ case and formally let $p \to 0$, $z \to \bar{z}$ while keeping everything else fixed. We thus obtain

$$\langle s, k | \tilde{J}^+(1) \tilde{J}^-(z) | s, k \rangle = \frac{\bar{z}^{-1}}{(1-\bar{z})^2} [1 + (k-1)\bar{z}], \tag{6.139}$$

$$\langle s, k | \tilde{J}^-(1) \tilde{J}^+(z) | s, k \rangle = \frac{1}{(1-\bar{z})^2} [\bar{z} + (k-1)]. \tag{6.140}$$

In each case, a direct comparison with the CFT results (6.9, 6.36) shows a match for the values of the correlators, up to some overall numerical factors. These numerical factors are not relevant for us: they arise from choosing different normalizations for the operators and the fields, which we did not keep track of.

Conclusions

In this thesis we have computed a special class of four-point correlators among chiral primary operators in the D1D5 CFT at the free orbifold point, where two of such operators were chosen to be heavy (i.e. their conformal dimension is of the order of the central charge), and the other two were chosen to be light (i.e. they have conformal dimension of order one).
In general, n-point correlation function of chiral primary operators with $n \geq 4$ are not protected, as in the OPEs between the various operators non-supersymmetric operators can be present as well. However, if the OPE of heavy operators contains only affine descendants of the identity, then it is possible that some HHLL correlators are protected. In our case at hand, the light operators were taken to be the chiral currents related to the $SO(4) \approx SU(2)_L \times SU(2)_R$ affine symmetry of the CFT. This suggests that the correlation functions are protected by Supersymmetry and thus motivates a Supergravity analysis via AdS/CFT duality, which we have presented in this work.
Thanks to the simplicity of the geometries dual to the chosen heavy states, also the gravity calculation was expected to be easy; however, obtaining the correct set of equations and boundary conditions required some additional care, as the bulk duals of chiral currents were found to be flat $SO(4)$ gauge field configurations with nontrivial topology.
By solving the equations of motion for such fields in $AdS_3/\mathbb{Z}_k$ and comparing with the CFT results, we have found agreement between the two computations. This is remarkable because the Supergravity description is related to a different point in the CFT moduli space, and confirms our previous prediction based on Supersymmetry.

Our work hints at some possible future developments.

Concerning the CFT computation of correlators involving chiral currents, it is reasonable to expect that they can be equivalently determined by means of a Ward identity. However this seems not entirely trivial to implement for HHJJ correlators, especially when heavy states belong to the twisted sector of the CFT: further analysis might clarify this aspect.

To generalize our results, it would be interesting to address the case of heavy states whose dual geometry is not related to global $AdS_3/\mathbb{Z}_k$ by a change of coordinates, i.e. that display a richer microscopic structure than a conical defect. Also, choosing new light operators to probe the heavy background could provide a more complete description of the D1D5 holographic dictionary.
So far, in this thesis as well as in known literature, only the lowest level of the Kaluza-Klein field towers has been studied, and completion of the program requires to address perturbations proportional to $\ell > 1$ spherical harmonics. For instance, for the tower laying on top of $SO(4)$ gauge fields it is already knwon that the dual operators are of the kind $J\Sigma_\ell$, where

Σ_ℓ is the ℓ-twist operator defined in Subsection 4.2.3. On the other hand, for what concerns the gravity computation, it seems hard to generalize the procedure followed in this thesis, as the guiding principle of gauge invariance is lost.

Finally, one could inquire about the nature of the massive solutions of the Yang-Mills Chern-Simons equations of motion (6.81), in particular what is their proper holographic interpretation in terms of operators on the D1D5 CFT.

Appendix A

Basics of Conformal Field Theories in $d = 2$

In this Appendix we review rapidly the salient features of CFTs, focusing on the $d = 2$ case. For further details we refer to [71–73].

A.1 The conformal group

By definition, conformal diffeomorphisms are coordinate transformations $x \longmapsto x'$ which leave the metric invariant up to some scale,

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{A.1}$$

They obviously form a group, called the conformal group.
Let us suppose $d \geq 3$ for the moment. One finds that the the transformations that make up the conformal group are

$$\begin{aligned}
&\text{translations} && x'^{\mu} = x^{\mu} + a^{\mu}, && \text{(A.2)}\\
&\text{dilations} && x'^{\mu} = \lambda x^{\mu}, && \text{(A.3)}\\
&\text{rotations} && x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \quad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}, && \text{(A.4)}\\
&\text{SCTs} && x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}, && \text{(A.5)}
\end{aligned}$$

where SCTs are the so-called special conformal transformaitons, that can also be written as a translation preceded and followed by an inversion,

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu}. \tag{A.6}$$

The generators of such transformations are given by

$$\begin{aligned}
&\text{translations} && P_{\mu} = -i\partial_{\mu}, && \text{(A.7)}\\
&\text{dilations} && D = -ix^{\mu}\partial_{\mu}, && \text{(A.8)}\\
&\text{rotations} && M_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right), && \text{(A.9)}\\
&\text{SCTs} && K_{\mu} = -i\left(2x_{\mu}x^{v}\partial_{v} - x^2\partial_{\mu}\right), && \text{(A.10)}
\end{aligned}$$

which satisfy the algebra

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu}D - M_{\mu\nu}\right), \\
[K_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right), \\
[P_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right).
\end{aligned}
\tag{A.11}
$$

This algebra can be shown to be isomorphic to $SO(d, 2)$.
In $d = 2$, it turns out that the conformal group is much bigger. Suppose to start with flat Euclidean space, with coordinates x^1, x^2. Define the complex coordinates

$$
z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2. \tag{A.12}
$$

Then any coordinate transformation of the form

$$
z \longmapsto f(z), \qquad \bar{z} \longmapsto \bar{f}(\bar{z}), \tag{A.13}
$$

keeps the metric flat up to a factor,

$$
ds^2 = dzd\bar{z} \longmapsto \left|\frac{df}{dz}\right|^2 dzd\bar{z}. \tag{A.14}
$$

The conformal group in $d = 2$ is thus the group of holomorphic/antiholomorphic functions, which is infinite dimensional and is generated by

$$
\ell_n = -z^{n+1}\,\partial, \qquad \tilde{\ell}_n = -\bar{z}^{n+1}\bar{\partial}, \qquad n \in \mathbb{Z}. \tag{A.15}
$$

The generators $\ell_n, \tilde{\ell}_n$ form two independent copies of the Witt algebra

$$
[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \qquad [\tilde{\ell}_m, \tilde{\ell}_n] = (m-n)\tilde{\ell}_{m+n}, \qquad [\ell_m, \tilde{\ell}_n] = 0. \tag{A.16}
$$

However, not all the generators $\ell_n, \tilde{\ell}_n$ give rise to globally defined transformations. Regularity at $z = 0$ and $z = \infty$ tells us that the only globally defined subalgebra of the conformal algebra is generated by $\{\ell_{0,\pm1}, \tilde{\ell}_{0,\pm1}\}$. Upon exponentiation, they give rise to the finite transformations

$$
z \longmapsto \frac{az - b}{cz - d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}, \tag{A.17}
$$

which form a group isomorphic to[1] $PSL(2, \mathbb{C})$.

The Witt algebra (A.16) admits a central extension: the Virasoro algebra. The Virasoro algebra is the algebra that the generators of the conformal group satisfy upon quantization. Let us denote by $L_n, \tilde{L}_n$ the generators of the Virasoro algebra; their commutation relations read

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}, \\
[L_m, \tilde{L}_n] &= 0,
\end{aligned}
\tag{A.18}
$$

where $c, \bar{c}$ are called central charges of the central extensions.

[1]Notice that the transformations are defined up to a sign for a, b, c, d.

A.2 Primary fields

The concept of "field" in CFTs is broader than the one we are used to: with the term field we refer to any local expression of observables. Primary fields are thus fields (in the above sense) that transform in a representation of the conformal group.

In $d = 2$ primary fields $\Phi(z, \bar{z})$ are classified by two numbers $h, \bar{h} \in \mathbb{R}$ such that, under a conformal transformation $z \longmapsto z' = f(z)$,

$$\Phi(z, \bar{z}) \longmapsto \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \Phi(z', \bar{z}'). \tag{A.19}$$

It turns out that we can be less restrictive, and it is enough to ask that (A.19) holds under a locally defined conformal transformation. In this case, Φ is said to be a quasi-primary field.

Moreover, we say that a field $\Phi(z, \bar{z})$ is chiral if it is purely holomorphic, i.e. $\Phi = \phi(z)$, respectively anti-chiral if it is purely antiholomorphic $\Phi = \phi(\bar{z})$. Notice that if Φ is also primary[2] this means that $\bar{h} = 0$ or $h = 0$, respectively. A chiral quasi-primary field $\phi(z)$ admits the expansion

$$\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n-h}, \tag{A.20}$$

where the shift by the conformal dimension in the exponent of z is conventional. A similar expression holds for anti-chiral quasi-primary fields.

Among quasi-primary fields, a special role is played by the stress-energy operators. In theories with a conformal symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$ in any dimension, we can define a conserved current $j_\mu = T_{\mu\nu}\epsilon^\nu$ for a symmetric tensor $T_{\mu\nu}$, the stress-energy tensor, which is conserved: $\partial^\mu T_{\mu\nu} = 0$. One can check that conformal invariance requires also tracelessness of $T_{\mu\nu}$, i.e. $T_\mu{}^\mu = 0$. For the case $d = 2$ and using complex coordinates $z, \bar{z}$ this means that $T_{z\bar{z}} = T_{\bar{z}z} = 0$, and that the other two components of the stress-energy tensor are chiral and anti-chiral, respectively:

$$T(z) \equiv T_{zz}(z, \bar{z}), \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(z, \bar{z}). \tag{A.21}$$

They are the holomorphic and anti-holomorphic stres-energy operators, and they have $(h, \bar{h}) = (2, 0)$ and $(0, 2)$ respectively. They generate the Virasoro algebra via their Laurent series, i.e.

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \qquad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \tilde{L}_n \bar{z}^{-n-2}. \tag{A.22}$$

A.3 Radial quantization

In our two-dimensional Euclidean theory, let x^0 be the Euclidean time direction and x^1 the Euclidean space direction. Compactify the latter on a circle of unit radius. The CFT obtained in this way is defined on an infinite cylinder, described by the complex coordinate

$$w = x^0 + ix^1, \qquad w \sim w + 2\pi i. \tag{A.23}$$

[2] A field that is both chiral and primary must not be confused with a "chiral primary" field, see Subsection 4.2.6.

This is the most natural framework, for instance, in String Theory, where the worldsheet of a closed string is a cylinder in Euclidean coordinates, and in AdS_3/CFT_2, because the boundary of AdS_3 is also a cylinder.
We now proceed to map the CFT from the cylinder to the complex plane. To do so, we define the coordinate

$$z = e^w = e^{x^0 + ix^1}. \tag{A.24}$$

In these coordinates, time translations are mapped to dilations. Thus it is natural to define time evolution on the plane via the dilation operator D. The infinite past on the cylinder is mapped to the origin $z = 0$, whereas the infinite future is mapped to the point $z = \infty$. Similarly, time ordering on the cylinder becomes radial ordering on the plane.
We can thus define the equivalent of asymptotic states in CFT. Let $\phi(z, \bar{z})$ be a field of conformal dimensions $(h, \bar{h})$. For an in-state, we define

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle, \tag{A.25}$$

whereas for an out-state we need to take the hermitian conjugate of this,

$$\langle\phi| = \lim_{z,\bar{z}\to 0} \langle 0|\phi^\dagger(z, \bar{z}) = \lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}}\langle 0|\phi(w, \bar{w}). \tag{A.26}$$

A.4 Correlation functions

The global conformal $PSL(2, \mathbb{C})$ symmetry fixes uniquely the structure of two- and three-point functions for chiral quasi-primary fields. We will always assume radial ordering inside correlation functions.

Let us start with the two-point function of two chiral quasi-primary fields

$$\langle\phi_1(z)\phi_2(w)\rangle = g(z, w). \tag{A.27}$$

Translation invariance requires $g(z, w) = g(z - w)$. Scale invariance requires that, under a dilation $z \longmapsto \lambda z$,

$$\langle\phi_1(z)\phi_2(w)\rangle \longmapsto \langle\lambda^{h_1}\phi_1(\lambda z)\lambda^{h_2}\phi_2(\lambda w)\rangle = \lambda^{h_1+h_2}g(\lambda(z - w)) = g(z - w), \tag{A.28}$$

from which we obtain

$$g(z - w) = \frac{d_{12}}{(z - w)^{h_1+h_2}} \tag{A.29}$$

for some constant d_{12}. Similarly, invariance under inversions $z \longmapsto -1/z$ implies

$$\langle\phi_1(z)\phi_2(w)\rangle \longmapsto \left\langle \frac{1}{z^{2h_1}w^{2h_2}}\phi_1\left(-\frac{1}{z}\right)\phi_2\left(-\frac{1}{w}\right)\right\rangle = \frac{1}{z^{2h_1}w^{2h_2}}g\left(-\frac{1}{z} + \frac{1}{w}\right) = g(z - w), \tag{A.30}$$

that can be satisfied only if $h_1 = h_2$. Therefore,

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}}{(z - w)^{2h_i}}\delta_{h_i,h_j}. \tag{A.31}$$

In a similar way, one can show that conformal invariance constrains the three-point function chiral quasi-primary fields to be of the form

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_1+h_3-h_2}}. \tag{A.32}$$

Higher order correlation functions are not completely constrained by conformal invariance. However, they are at least partially fixed. Consider, for instance,

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle, \tag{A.33}$$

and focus on the case $h_1 = h_4$, $h_2 = h_3$. Then conformal invariance states that

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \frac{1}{z_{14}^{2h_1} z_{23}^{2h_2}} \mathcal{G}(z), \tag{A.34}$$

where $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ is a conformally invariant quantity called conformal cross ratio, and $\mathcal{G}(z)$ is a function of z only.

A.5 Operator Product Expansion

A remarkable feature of CFTs (in any dimension) is that their spectrum can be described by a complete set of states. As we have seen, states correspond to fields, or operators. Thus also the product of two operators, which would usually be an ill-defined quantity, can be expressed in terms of linear combinations of states. Using once again the correspondence between operators and states, we can write

$$\mathcal{O}_1(z)\mathcal{O}_2(w) = \sum_{\mathcal{O}} C_{12}^{\mathcal{O}}(z-w)\mathcal{O}(w)\Big|_{x=w}. \tag{A.35}$$

This goes by the name of operator product expansion (OPE). The sum is over all possible operators $\mathcal{O}$ appearing in the theory, but it is a convergent sum in a finite neighborhood of $z \sim w$. The coefficients $C_{12}^{\mathcal{O}}$ are called OPE coefficients, and are functions of $(z-w)$ only.

We can use OPE to reduce four-point functions to three-point functions and subsequently to two point functions. However, inside radial ordering (which is intended in correlation functions) we can change the order of operators. This allows us to compute OPEs between different operators. We require crossing symmetry, that is,

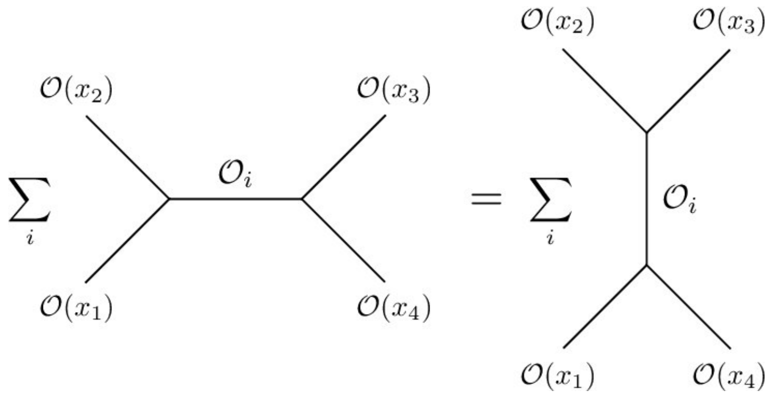

Figure A.1: Crossing symmetry equation for the correlator $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$.

The resulting set of constraints on OPE coefficients goes by the name of conformal bootstrap equations.

A.6 Kač-Moody algebras

We can have more operators in the CFT whose Laurent series coefficient satisfy some symmetry algebra. A Kač-Moody (or affine) algebra is defined via the commutation relations

$$[j^a_m, j^b_n] = i \sum_c f^{abc} j^c_{m+n} + km\delta^{ab}\delta_{m+n,0}, \tag{A.36}$$

where f^{abc} are structure constants of some Lie algebra $\mathfrak{g}$ and k is a constant, called the level of the algebra. Moreover, we see that the zero modes j^a_0 of the currents form a finite subalgebra of the Kač-Moody algebra that is isomorphic to the underlying Lie algebra $\mathfrak{g}$. The commutation relations (A.36) can be equivalently expressed in terms of the OPE

$$j^a(z)j^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \sum_c \frac{if^{abc}}{z-w} j^c(w) + \cdots . \tag{A.37}$$

A.7 Supersymmetric Conformal Field Theories

Two-dimensional CFTs can be generalized to respect Supersymmetry. This means that for any operator in the CFT we must include also its supersymmetric counterpart.
Just like any other field theory, a SCFT is characterized by the number of its supersymmetries $\mathcal{N}$.

Let us focus on the easiest $\mathcal{N} = 1$ case first. The Virasoro algebra (A.18) gets extended to its supersymmetric version,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},
\end{aligned} \tag{A.38}$$

where

$$G(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}} \tag{A.39}$$

is the superpartner of the stress-energy operator $T(z)$, the supercurrent.

The next to easiest case has $\mathcal{N} = 2$. Already here we find the same ingredients that are present in the $\mathcal{N} = 4$ D1D5 CFT, presented in Section 4.2. We express the corresponding algebra in terms of the Laurent modes L_m of the stress-energy tensor, its superpartners $G^\pm_r$ and in terms of the modes j_n of a $U(1)$ current. For half-integer moding of $G^\pm_r$ this algebra is also known as the Neveu–Schwarz algebra, whereas for integer moded $G^\pm_r$ it is called the

Ramond algebra.

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[L_m, j_n] &= -n j_{m+n}, \\
\left[L_m, G_r^{\pm}\right] &= \left(\frac{m}{2} - r\right) G_{m+r}^{\pm}, \\
[j_m, j_n] &= \frac{c}{3} m \delta_{m+n,0}, \\
\left[j_m, G_r^{\pm}\right] &= \pm G_{m+r}^{\pm}, \\
\left\{G_r^{+}, G_s^{-}\right\} &= 2L_{r+s} + (r-s)j_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\
\left\{G_r^{+}, G_s^{+}\right\} &= \left\{G_r^{-}, G_s^{-}\right\} = 0.
\end{aligned}
\tag{A.40}
$$

The superscript on $G^{\pm}(z)$ denotes its $U(1)$ charge. The generalization to affine algebras is straightforward.

Appendix B

Spherical harmonics on S^3

The spherical harmonics on S^3 are a representation of the isometry group of the 3-sphere $SO(4) \approx SU(2)_L \times SU(2)_R$. We use spherical coordinates in the $\mathbb{R}^4$ base space that are related to the Cartesian coordinates by

$$\begin{aligned} x^1 &= r \sin\theta \cos\phi, \qquad & x^2 &= r \sin\theta \sin\phi, \\ x^3 &= r \cos\theta \cos\psi, \qquad & x^4 &= r \cos\theta \sin\psi, \end{aligned} \tag{B.1}$$

with $\theta \in [0, \pi/2]$ and $\phi, \psi \in [0, 2\pi[$. With this coordinatization, the metric on S^3 reads

$$ds^2_{S^3} = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 + \cos^2\theta\, \mathrm{d}\psi^2, \tag{B.2}$$

and we choose the orientation $\epsilon_{\theta\phi\psi} = 1$.

The generators of the isometry group of S^3, written in terms of the standard $SU(2)$ generators, are

$$\begin{aligned} J^{\pm} &= \frac{1}{2} e^{\pm i(\phi+\psi)} \left(\pm\, \partial_\theta + i \cot\theta\, \partial_\phi - i \tan\theta\, \partial_\psi \right), \qquad & J^3 &= -\frac{i}{2} \left(\partial_\phi + \partial_\psi \right), \\ \tilde{J}^{\pm} &= \frac{1}{2} e^{\pm i(\phi-\psi)} \left(\mp\, \partial_\theta - i \cot\theta\, \partial_\phi - i \tan\theta\, \partial_\psi \right), \qquad & \tilde{J}^3 &= -\frac{i}{2} \left(\partial_\phi - \partial_\psi \right). \end{aligned} \tag{B.3}$$

A degree ℓ scalar spherical harmonic is denoted by $Y_\ell^{m,\tilde{m}}$, where $(m, \tilde{m})$ are the spin charges under $(J^3, \tilde{J}^3)$. They are usually taken to be normalized in such a way that

$$\int_{S^3} Y_{\ell_1}^{*m_1,\tilde{m}_1} Y_{\ell_2}^{m_2,\tilde{m}_2} = 2\pi^2 \delta_{\ell_1,\ell_2} \delta^{m_1,m_2} \delta^{\tilde{m}_1,\tilde{m}_2}. \tag{B.4}$$

In the thesis we have made use of degree $\ell = 1$ spherical harmonics, given by

$$\begin{aligned} Y_1^{++} &\equiv Y_1^{+1/2,+1/2} = \sqrt{2} \sin\theta e^{i\phi}, \\ Y_1^{+-} &\equiv Y_1^{+1/2,-1/2} = \sqrt{2} \cos\theta e^{i\psi}, \\ Y_1^{-+} &\equiv Y_1^{-1/2,+1/2} = -\sqrt{2} \sin\theta e^{-i\psi}, \\ Y_1^{--} &\equiv Y_1^{-1/2,-1/2} = \sqrt{2} \sin\theta e^{-i\phi}. \end{aligned} \tag{B.5}$$

We have also introduced degree $\ell = 1$ vector spherical harmonics $Y_1^{\alpha\pm}$, $(\alpha = \pm, 0)$,

$$
\begin{aligned}
Y_1^{++} &= \frac{1}{\sqrt{2}} e^{+i(\phi+\psi)} \left[-i\,\mathrm{d}\theta + \sin\theta\cos\theta\,\mathrm{d}(\phi-\psi)\right], \\
Y_1^{-+} &= \frac{1}{\sqrt{2}} e^{-i(\phi+\psi)} \left[i\,\mathrm{d}\theta + \sin\theta\cos\theta\,\mathrm{d}(\phi-\psi)\right], \\
Y_1^{0+} &= -\cos^2\theta\,\mathrm{d}\psi - \sin^2\theta\,\mathrm{d}\phi, \\
Y_1^{+-} &= \frac{1}{\sqrt{2}} e^{+i(\phi-\psi)} \left[i\,\mathrm{d}\theta - \sin\theta\cos\theta\,\mathrm{d}(\phi+\psi)\right], \\
Y_1^{--} &= -\frac{1}{\sqrt{2}} e^{-i(\phi-\psi)} \left[i\,\mathrm{d}\theta + \sin\theta\cos\theta\,\mathrm{d}(\phi+\psi)\right], \\
Y_1^{0-} &= \cos^2\theta\,\mathrm{d}\psi - \sin^2\theta\,\mathrm{d}\phi,
\end{aligned}
\tag{B.6}
$$

normalized according to

$$
\int_{S^3} \left(Y_1^{\alpha A}\right)_a \left(Y_1^{\beta B}\right)^b = 2\pi^2 \delta^{\alpha,\beta}\delta^{A,B}. \tag{B.7}
$$

Vector spherical harmonics also have nice properties under differential operators, namely

$$
\star\,\mathrm{d}Y_1^{\alpha\pm} = \pm 2 Y_1^{\alpha\pm}, \tag{B.8}
$$

where $\star$ is the Hodge star operator on S^3 with the metric (B.2).

Bibliography

[1] Andrea Galliani et al. “Correlators at large c without information loss”. In: *Journal of High Energy Physics* 2016.9 (2016), p. 65. arXiv: `1606.01119 [hep-th]`.

[2] Andrea Galliani, Stefano Giusto, and Rodolfo Russo. “Holographic 4-point correlators with heavy states”. In: *JHEP* 10 (2017), p. 040. arXiv: `1705.09250 [hep-th]`.

[3] Alessandro Bombini et al. “Unitary 4-point correlators from classical geometries”. In: *Eur. Phys. J.* C78.1 (2018), p. 8. arXiv: `1710.06820 [hep-th]`.

[4] Alessandro Bombini and Andrea Galliani. “AdS_3 four-point functions from $\frac{1}{8}$ -BPS states”. In: *JHEP* 06 (2019), p. 044. arXiv: `1904.02656 [hep-th]`.

[5] Jia Tian, Jue Hou, and Bin Chen. *Holographic Correlators on Integrable Superstrata.* 2019. arXiv: `1904.04532 [hep-th]`.

[6] Stefano Giusto et al. “Holographic correlators in AdS_3 without Witten diagrams”. In: (2019). arXiv: `1905.12314 [hep-th]`.

[7] J. D. Bekenstein. “Black holes and the second law”. In: *Lett. Nuovo Cim.* 4 (1972), pp. 737–740.

[8] S. W. Hawking. “Particle Creation by Black Holes”. In: *Commun. Math. Phys.* 43 (1975). [,167(1975)], pp. 199–220.

[9] Sean M. Carroll. *Spacetime and geometry: An introduction to general relativity.* Cambridge University Press, 2019.

[10] P. K. Townsend. *Black holes: Lecture notes.* 1997. arXiv: `gr-qc/9707012 [gr-qc]`.

[11] James M Bardeen, Brandon Carter, and Stephen W Hawking. “The four laws of black hole mechanics”. In: *Communications in mathematical physics* 31.2 (1973), pp. 161–170.

[12] Robert M Wald. “Black hole entropy is the Noether charge”. In: *Physical Review D* 48.8 (1993), R3427. arXiv: `gr-qc/9307038 [gr-qc]`.

[13] Jacob D. Bekenstein. “Generalized second law of thermodynamics in black hole physics”. In: *Phys. Rev.* D9 (1974), pp. 3292–3300.

[14] Stephen W Hawking. “Black holes and thermodynamics”. In: *Physical Review D* 13.2 (1976), p. 191.

[15] Don N Page. “Information in black hole radiation”. In: *Physical review letters* 71.23 (1993), p. 3743.

[16] Samir D. Mathur. “The Information paradox: A Pedagogical introduction”. In: *Class. Quant. Grav.* 26 (2009), p. 224001. arXiv: `0909.1038 [hep-th]`.

[17] Ahmed Almheiri et al. “Black Holes: Complementarity or Firewalls?” In: *JHEP* 02 (2013), p. 062. arXiv: `1207.3123 [hep-th]`.

[18] David Wallace. *Why black hole information loss is paradoxical.* 2017. arXiv: `1710.03783 [hep-th]`.

[19] Juan Maldacena. "Eternal black holes in anti-de Sitter". In: *Journal of High Energy Physics* 2003.04 (2003), p. 021. arXiv: `hep-th/0106112 [hep-th]`.

[20] A Liam Fitzpatrick et al. "On information loss in AdS 3/CFT 2". In: *Journal of High Energy Physics* 2016.5 (2016), p. 109. arXiv: `1603.08925 [hep-th]`.

[21] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[22] David Tong. *String Theory.* 2009. arXiv: `0908.0333 [hep-th]`.

[23] Henning Samtleben. "Introduction to supergravity". In: *Lectures presented at the 13th Saalburg School on Fundamentals and New Methods in Theoretical Physics* (2007), pp. 3–14.

[24] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[25] Timo Weigand. *Introduction to String Theory.*

[26] Luca Mattiello. *Black Hole Microstates in String Theory.*

[27] A. Strominger and C. Vafa. "Microscopic origin of the Bekenstein-Hawking entropy". In: *Physics Letters B* 379.1-4 (1996), pp. 99–104. arXiv: `hep-th/9601029 [hep-th]`.

[28] A.W. Peet. "TASI lectures on black holes in string theory". In: *Strings, branes and gravity.* World Scientific, 2001, pp. 353–433. arXiv: `hep-th/0008241 [hep-th]`.

[29] Samir D Mathur. "The fuzzball proposal for black holes: An elementary review". In: *Fortschritte der Physik: Progress of Physics* 53.7-8 (2005), pp. 793–827. arXiv: `hep-th/0502050 [hep-th]`.

[30] Leonard Susskind. "Some speculations about black hole entropy in string theory". In: *Teitelboim, C. (ed.): The black hole* (1993), pp. 118–131. arXiv: `hep-th/9309145 [hep-th]`.

[31] C.G. Callan Jr, J.M. Maldacena, and A.W. Peet. "Extremal black holes as fundamental strings". In: *Nuclear Physics B* 475.3 (1996), pp. 645–676. arXiv: `hep-th/9510134 [hep-th]`.

[32] Oleg Lunin and Samir D Mathur. "Metric of the multiply wound rotating string". In: *Nuclear Physics B* 610.1-2 (2001), pp. 49–76. arXiv: `hep-th/0105136 [hep-th]`.

[33] O. Lunin and S. Mathur. "AdS/CFT duality and the black hole information paradox". In: *Nuclear Physics B* 623.1-2 (2002), pp. 342–394. arXiv: `hep-th/0109154 [hep-th]`.

[34] Oleg Lunin, Juan Maldacena, and Liat Maoz. *Gravity solutions for the D1-D5 system with angular momentum.* 2002. arXiv: `hep-th/0212210 [hep-th]`.

[35] John L. Cardy. "Operator Content of Two-Dimensional Conformally Invariant Theories". In: *Nucl. Phys.* B270 (1986), pp. 186–204.

[36] Atish Dabholkar. "Exact counting of supersymmetric black hole microstates". In: *Physical review letters* 94.24 (2005), p. 241301. arXiv: `hep-th/0409148 [hep-th]`.

[37] Stefano Giusto and Samir D Mathur. "Fuzzball geometries and higher derivative corrections for extremal holes". In: *Nuclear Physics B* 738.1-2 (2006), pp. 48–75. arXiv: `hep-th/0412133 [hep-th]`.

[38] Iosif Bena et al. "Habemus Superstratum! A constructive proof of the existence of superstrata". In: *Journal of High Energy Physics* 2015.5 (2015), p. 110. arXiv: `1503.01463 [hep-th]`.

[39] Leonard Susskind. "The World as a hologram". In: *J. Math. Phys.* 36 (1995), pp. 6377–6396. arXiv: `hep-th/9409089 [hep-th]`.

[40] Gerard 't Hooft. "Dimensional reduction in quantum gravity". In: *Conf. Proc.* C930308 (1993), pp. 284–296. arXiv: `gr-qc/9310026 [gr-qc]`.

[41] J. David Brown and M. Henneaux. "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity". In: *Commun. Math. Phys.* 104 (1986), pp. 207–226.

[42] Ofer Aharony et al. "Large N field theories, string theory and gravity". In: *Phys. Rept.* 323 (2000), pp. 183–386. arXiv: `hep-th/9905111 [hep-th]`.

[43] Gerard 't Hooft. "A Planar Diagram Theory for Strong Interactions". In: *Nucl. Phys.* B72 (1974). [,337(1973)], p. 461.

[44] Eric D'Hoker and Daniel Z. Freedman. "Supersymmetric gauge theories and the AdS / CFT correspondence". In: *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings.* 2002, pp. 3–158. arXiv: `hep-th/0201253 [hep-th]`.

[45] Edward Witten. "Anti-de Sitter space and holography". In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 253–291. arXiv: `hep-th/9802150 [hep-th]`.

[46] Justin R. David, Gautam Mandal, and Spenta R. Wadia. "Microscopic formulation of black holes in string theory". In: *Phys. Rept.* 369 (2002), pp. 549–686. arXiv: `hep-th/0203048 [hep-th]`.

[47] Steven G. Avery. "Using the D1D5 CFT to Understand Black Holes". PhD thesis. Ohio State U., 2010. arXiv: `1012.0072 [hep-th]`.

[48] Emanuele Moscato. "Black Hole microstates and Holography in the D1D5 CFT". PhD thesis. Queen Mary, U. of London, 2017-09.

[49] V. Alan Kostelecky et al. "Conformal Techniques, Bosonization and Tree Level String Amplitudes". In: *Nucl. Phys.* B288 (1987), pp. 173–232.

[50] Jan de Boer et al. "The Chiral ring of AdS(3)/CFT(2) and the attractor mechanism". In: *JHEP* 03 (2009), p. 030. arXiv: `0809.0507 [hep-th]`.

[51] Marco Baggio, Jan de Boer, and Kyriakos Papadodimas. "A non-renormalization theorem for chiral primary 3-point functions". In: *JHEP* 07 (2012), p. 137. arXiv: `1203.1036 [hep-th]`.

[52] Stefano Giusto, Emanuele Moscato, and Rodolfo Russo. "AdS 3 holography for 1/4 and 1/8 BPS geometries". In: *Journal of High Energy Physics* 2015.11 (2015), p. 4. arXiv: `1507.00945 [hep-th]`.

[53] Ingmar Kanitscheider, Kostas Skenderis, and Marika Taylor. "Holographic anatomy of fuzzballs". In: *Journal of High Energy Physics* 2007.04 (2007), p. 023. arXiv: `hep-th/0611171 [hep-th]`.

[54] LJ Romans. "Self-duality for interacting fields: Covariant field equations for six-dimensional chiral supergravities". In: *Nuclear Physics B* 276.1 (1986), pp. 71–92.

[55] S Deger et al. “Spectrum of D= 6, N= 4b supergravity on AdS3× S3”. In: *Nuclear Physics B* 536.1-2 (1998), pp. 110–140. arXiv: hep-th/9804166 [hep-th].

[56] Samir D Mathur, Ashish Saxena, and Yogesh Srivastava. “Constructing ‘hair’for the three charge hole”. In: *Nuclear Physics B* 680.1-3 (2004), pp. 415–449. arXiv: hep-th/0311092 [hep-th].

[57] Kostas Skenderis. “Lecture notes on holographic renormalization”. In: *Classical and Quantum Gravity* 19.22 (2002), p. 5849. arXiv: hep-th/0209067 [hep-th].

[58] Ingmar Kanitscheider, Kostas Skenderis, and Marika Taylor. “Fuzzballs with internal excitations”. In: *JHEP* 06 (2007), p. 056. arXiv: 0704.0690 [hep-th].

[59] Stefano Giusto and Rodolfo Russo. “Superdescendants of the D1D5 CFT and their dual 3-charge geometries”. In: *JHEP* 03 (2014), p. 007. arXiv: 1311.5536 [hep-th].

[60] Sami Rawash. *A Holographic Description of Black Hole Microstates.*

[61] Iosif Bena et al. *Thermal Decay without Information Loss in Horizonless Microstate Geometries.* 2019. arXiv: 1905.05194 [hep-th].

[62] Vijay Balasubramanian et al. “Supersymmetric conical defects: Towards a string theoretic description of black hole formation”. In: *Physical Review D* 64.6 (2001), p. 064011. arXiv: hep-th/0011217 [hep-th].

[63] M. J. Duff, B. E. W. Nilsson, and C. N. Pope. “Kaluza-Klein Supergravity”. In: *Phys. Rept.* 130 (1986), pp. 1–142.

[64] Sergei Gukov et al. “Chern-Simons gauge theory and the AdS_3/CFT_2 correspondence”. In: *From Fields to Strings: Circumnavigating Theoretical Physics: Ian Kogan Memorial Collection (In 3 Volumes).* World Scientific, 2005, pp. 1606–1647. arXiv: hep-th/0403225 [hep-th].

[65] Tomás Andrade, Juan I Jottar, and Robert G Leigh. “Boundary conditions and unitarity: the Maxwell-Chern-Simons system in AdS 3/CFT 2”. In: *Journal of High Energy Physics* 2012.5 (2012), p. 71. arXiv: 1111.5054 [hep-th].

[66] Stanley Deser, Roman Jackiw, and Stephen Templeton. “Topologically massive gauge theories”. In: *Annals of Physics* 140.2 (1982), pp. 372–411.

[67] Sheer El-Showk, Yu Nakayama, and Slava Rychkov. “What Maxwell theory in D$\neq$ 4 teaches us about scale and conformal invariance”. In: *Nuclear Physics B* 848.3 (2011), pp. 578–593. arXiv: 1101.5385 [hep-th].

[68] Per Kraus. “Lectures on black holes and the AdS3/CFT2 correspondence”. In: *Supersymmetric Mechanics-Vol. 3.* Springer, 2008, pp. 1–55. arXiv: hep-th/0609074 [hep-th].

[69] Rashmish K Mishra and Raman Sundrum. “Asymptotic symmetries, holography and topological hair”. In: *Journal of High Energy Physics* 2018.1 (2018), p. 14. arXiv: 1706.09080 [hep-th].

[70] Kristan Jensen. “Chiral anomalies and AdS/CMT in two dimensions”. In: *Journal of High Energy Physics* 2011.1 (2011), p. 109. arXiv: 1012.4831 [hep-th].

[71] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory.* Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997.

[72] Slava Rychkov. *EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions*. SpringerBriefs in Physics. 2016. arXiv: `1601.05000 [hep-th]`.

[73] Ralph Blumenhagen and Erik Plauschinn. "Introduction to conformal field theory". In: *Lect. Notes Phys.* 779 (2009), pp. 1–256.