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# Integral foliated simplicial volume and ergodic decomposition



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# Introduction

The principle moving this thesis work is the analogy between the cost of a group and the integral foliated simplicial volume. A classical homotopy invariant for closed connected oriented manifolds is the integral simplicial volume $\|M\|_{\mathbb{Z}}$, computed by making use of the usual singular homology with integer coefficients. If $M$ is a closed connected oriented $n$-dimensional manifold with fundamental class $[M]_{\mathbb{Z}}$ and $c = \sum_{i=1}^{k} a_i\sigma_i \in C_n(M;\mathbb{Z})$ is a singular $n$-chain with integer coefficients, then the $\ell^1$-norm of $c$ is $\|c\|_1 = \sum_{i=1}^{k}|a_i| \in \mathbb{N}$ and the integral simplicial volume of $M$ is the infimum

$$\|M\|_{\mathbb{Z}} = \inf\{\|c\|_1 : c \in C_n(M;\mathbb{Z}) \text{ and } [c] = [M]_{\mathbb{Z}}\}$$

Gromov proved a relation of this invariant with $L^2$-Betti numbers through the inequality [8, p. 297]:

$$\sum_{i=0}^{n} b_i^{(2)}(M) \leq (n+1) \cdot \|M\|_{\mathbb{Z}}$$

Unfortunately a drawback of the integral simplicial volume is that it is a quite coarse invariant, because $\|M\|_{\mathbb{Z}} \geq 1$ for every manifold of positive dimension. This coarseness led first to the idea by Gromov and then to the formalization by Schmidt of integral foliated simplicial volume $|\!|\!| M |\!|\!|$ [28, Definition 5.22]. This invariant satisfies $\|M\|_{\mathbb{Z}} \geq |\!|\!| M |\!|\!|$ and still gives a finer upper bound

$$\sum_{i=0}^{n} b_i^{(2)}(M) \leq (n+1) \cdot |\!|\!| M |\!|\!|$$

for the $L^2$-Betti numbers [28, Corollary 5.28], [21, Theorem 6.4.5].

The philosophy behind the definition of integral foliated simplicial volume is to replace homology with integral coefficients with homology with twisted coefficients. Our new coefficient sets are now $L^\infty(X,\mu,\mathbb{Z})$ spaces, where the $(X,\mu)$'s are probability spaces on which the fundamental group of $M$ acts in a measure preserving way, say via an action $\alpha : \pi_1(M) \curvearrowright (X,\mu)$. We replace integral cycles $\sum_{i=1}^{k} a_i\sigma_i \in C_n(M;\mathbb{Z})$ with $(\alpha,\mu)$-parametrized cycles $\sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i \in C_n(M; L^\infty(X,\mu,\mathbb{Z}))$ and the integral $\ell^1$-norm $\sum_{i=1}^{k}|a_i|$ with the $(\alpha,\mu)$-parametrized $\ell^1$-norm $\sum_{i=1}^{k} \int_X |\varphi_i| d\mu$. We can define the $(\alpha,\mu)$-parametrized simplicial volume $|\!|\!| M |\!|\!|^{(\alpha,\mu)}$ of $M$ and then, by taking the infimum among all of the standard probability actions $\pi_1(M) \curvearrowright (X,\mu)$, the integral foliated simplicial volume $|\!|\!| M |\!|\!|$.

This particular definition via measure preserving group actions necessarily leads to the study of measurable group theory, i.e. the study of group actions on measure spaces or, in our case, standard probability actions. Levitt and Gaboriau introduced and developed the notions of measured equivalence relations and their cost, and then further the notion of cost of a group, which can be seen as the dynamical analogue of the group rank. Beginning from their definitions, the integral foliated simplicial volume and the cost of a group share many analogies. In particular the proofs for the results holding for the cost of measured equivalence relations can be used to model out proofs for results holding for the integral foliated simplicial volume. For instance:

1. the cost of a measured equivalence relation and the parametrized simplicial volume satisfy the same relation with respect to weak containment [13, Corollary 10.14], [4, Theorem 1.5];
2. amenable groups are cheap of fixed price and aspherical closed amenable manifolds are cheap of fixed price [22, Corollary 4.3.11], [4, Theorem 1.9];
3. the cost of a group (actually the cost minus one) and the integral foliated simplicial volume satisfy analogous proportionality relations with respect to finite index subgroups and finite index coverings, respectively [5, Théorème 3], [23, Theorem 4.22].

In the theory of dynamical systems, ergodic actions might be viewed as “indecomposable” systems, thus are of particular interest. It is already known that standard probability actions can be decomposed via particular collections of measures, called ergodic decompositions. A compatibility result for the cost of measured equivalence relations with respect to ergodic decompositions is already known [14, Proposition 18.4]: if $\alpha : \Gamma \curvearrowright (X, \mu)$ is a standard probability action and $\beta : X \to \mathrm{Erg}(\alpha)$ is an ergodic decomposition of it, then

$$\operatorname{cost}_\mu R_\alpha = \int_X \operatorname{cost}_{\beta_x}(R_\alpha) d\mu(x) \tag{1}$$

The main result proposed by Löh and then proved in this thesis as Theorem 5.2.26 is its counterpart for $(\alpha, \mu)$-parametrized simplicial volume:

**Theorem.** *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$, let $(X, \mu)$ be a standard Borel probability space with a measure preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$ and an ergodic decomposition $\beta : X \to \mathrm{Erg}(\alpha)$ of $\alpha$; then*

$$| M |^{(\alpha,\mu)} = \int_X | M |^{(\alpha,\beta_x)} d\mu(x)$$

The proof of this theorem uses the proof of the cost formula (1) as a guideline and relies on the existence of a countable subset

$$\left\{c(I, Q) \in C_n(M; L^\infty(X, \mu, \mathbb{Z})) : I \in P_{\mathrm{fin}}(\mathbb{N}), Q \in P_{\mathrm{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}\right\}$$

of the uncountable set of $(\alpha, \mu)$-parametrized fundamental cycles for $M$ that suffices to compute $| M |^{(\alpha,\mu)}$. More precisely in Lemma 5.2.25 we prove the following:

**Lemma.** *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$ and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard action. Let us fix $C > 0$. With the same notations as above, the following are equivalent:*

1. $\| M \|^{(\alpha,\mu)} < C$;
2. *there exist* $I \in P_{\text{fin}}(\mathbb{N})$ *and* $Q \in P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}$ *such that* $\| c(I, Q) \|^{(\alpha,\mu)} < C$.

A direct consequence of the ergodic decomposition formula for parametrized simplicial volume of Theorem 5.2.26 is Corollary 5.2.27:

**Corollary.** *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$; there exists an ergodic and essentially free standard probability action $\alpha : \Gamma \curvearrowright (X, \mu)$ such that*

$$\| M \| = \| M \|^{(\alpha,\mu)}$$

A still open question is the fixed price problem for a group (Question 3.2.15): we say that a group $\Gamma$ has fixed price if for all essentially free standard probability actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$, we have that $\operatorname{cost}_\mu R_\alpha = \operatorname{cost}_\nu R_\beta$. It is still unknown whether a countable group not of fixed price exists [5, Question 1.8]. The fixed price notion has a counterpart in algebraic topology: we say that an oriented closed connected manifold $M$ has fixed price if for all essentially free standard probability actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$, we have that $\| M \|^{(\alpha,\mu)} = \| M \|^{(\beta,\nu)}$ and again it is still an open problem, whether a manifold not of fixed price exists.

## Work organization

The contents of this thesis will be organized in the following way:

1. Chapter 1 contains a summary of basic results in measure theory we will use in the later chapters. Here we give the definition of standard Borel (probability) spaces, which are the measure spaces we will be dealing with;
2. Chapter 2 is an introduction to the theory of dynamical systems, with particular focus on ergodic actions and the ergodic decomposition;
3. in Chapter 3 we introduce the theory of measured equivalence relations and their cost and state and prove the cost formula (1);
4. Chapter 4 is about manifolds: we recall some definitions and introduce the classical invariant of integral simplicial volume;
5. Chapter 5 will be completely dedicated to the construction of integral foliated simplicial volume and the proof of the main theorem;
6. the Appendix contains the construction of two algebraic structures we used in the previous chapters and a short survey on $L^2$-Betti numbers.

# Contents

# Notations and conventions

| | |
|---|---|
| $\subseteq$ | Containment relation for sets: if $A$ and $B$ are sets, then we write $A \subseteq B$ if every element of $A$ belongs to $B$ |
| $\leq$ | Standard order relation in $\mathbb{R}$. Also used for the notion of subspace relation: if $M$ is an $R$-module and $N$ is an $R$-submodule of $M$, then we write $N \leq M$ |
| $\curvearrowright$ | If $\Gamma$ is a group and $X$ is a set, then we write $\Gamma \curvearrowright X$ to indicate that $\Gamma$ acts on $X$ |
| $\mathrm{Aut}_{\mathscr{C}}(X)$ | The group of automorphisms of the object $X$ in the category $\mathscr{C}$ |
| $\mathfrak{c}$ | the cardinality of the set $\mathbb{R}$ of real numbers |
| Countable | T set $S$ is countable if there exists an injective function $S \to \mathbb{N}$ |
| $\mathrm{Deck}(p)$ | The group of deck transformations of a covering $p : \tilde{X} \to X$ |
| $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ | The set of morphisms in the category $\mathscr{C}$ from the object $X$to the object $Y$ |
| IFSV | Integral foliated simplicial volume |
| $m$ | The Lebesgue measure on $\mathbb{R}^n$ |
| $\mathsf{Msrbl}$ | The category of measurable spaces with measurable functions |
| $R\text{-}\,\mathsf{Mod}$ | The category of left $R$-modules |
| $\mathsf{Mod}\text{-}R$ | The category of right $R$-modules |
| $\mathbb{N}$ | The set of natural numbers, containing 0 |
| $\mathrm{Ob}(\mathscr{C})$ | The class of objects of the category $\mathscr{C}$ |
| $P(A)$ | The power set of $A$: if $A$ is a set, then $P(A)$ is the set of all its subsets. There is a bijection between $P(A)$ and $\{0,1\}^A$ |
| $P_{\mathrm{fin}}(A)$ | The finite power sets: if $A$ is a set, then $P_{\mathrm{fin}}(A)$ is the set of all its subsets with finite cardinality |
| $P_n(A)$ | The finite power set of cardinality $n$: if $A$ is a set and $n \in \mathbb{N}$, then $P_n(A)$ is the set of all the subsets of $A$ with cardinality $n$ |
| PSV | Parametrized simplicial volume |
| $\mathbb{R}$ | The set of real numbers |
| $\mathsf{Set}$ | The category of sets |
| $\mathbb{Z}$ | The ring of integer numbers |
| $\pi_1(X, x)$ | The fundamental group of the topological space $X$ computed at the point $x$ |

# Dictionary

The notions in Ergodic group theory we will find in this work and their counterpart in Algebraic topology.

| ERGODIC GROUP THEORY | ALGEBRAIC TOPOLOGY |
|---|---|
| standard Borel probability space $(X, \mu)$ | oriented closed connected manifold $M$ |
| standard action $\alpha : \Gamma \curvearrowright (X, \mu)$ | standard action $\alpha : \pi_1(M) \curvearrowright (X, \mu)$ |
| measured equivalence relation $(R_\alpha, \mu)$ on $X$ | twisted chain complex $C_*(M; L^\infty(X, \mu, \mathbb{Z}))$ |
| graphing $\Phi \subseteq [\![R_\alpha]\!]$ | $(\alpha, \mu)$-fundamental cycle $c \in Z_n(M; L^\infty(X, \mu, \mathbb{Z}))$ |
| $\mathrm{cost}_\mu \Phi$ | $\boldsymbol{|}\, c \,\boldsymbol{|}^{(\alpha,\mu)}$ |
| $\mathrm{cost}_\mu(R_\alpha)$ | $\boldsymbol{|}\, M \,\boldsymbol{|}^{(\alpha,\mu)}$ |
| $\mathrm{cost}\, \Gamma$ | $\boldsymbol{|}\, M \,\boldsymbol{|}$ |
| for an ergodic decomposition $\beta : X \to \mathrm{Erg}(\alpha)$ | |
| $\mathrm{cost}_\mu(R_\alpha) = \int_X \mathrm{cost}_{\beta_x}(R_\alpha) d\mu(x)$ | $\boldsymbol{|}\, M \,\boldsymbol{|}^{(\alpha,\mu)} = \int_X \boldsymbol{|}\, M \,\boldsymbol{|}^{(\beta_x,\mu)} d\mu(x)$ |

# 1. Basic measure theory

We are refreshing some known definitions and facts in Measure theory with particular attention on a precise class of spaces, namely the standard Borel spaces, which are the ones we will use to develop our Ergodic group theory.

## 1.1. The category of measure spaces

Most of the basic definitions and propositions can be found in Folland's book [3, Chapter 1].

### 1.1.1. Measurable spaces

**Definition 1.1.1** (measurable spaces)**.** Let $X$ be any set;

1. a *$\sigma$-algebra* on $X$ is a subset $S \subseteq P(X)$ with the following properties:
   a) $\varnothing \in S$;
   b) closure under complement: if $A \in S$, then $X \smallsetminus A \in S$;
   c) closure under countable unions: if $A_n \in S$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in S$.
2. a *measurable space* is a pair $(X, S)$, where $X$ is a set and $S \subseteq P(X)$ is a $\sigma$-algebra on $X$;
3. let $(X, S)$ and $(Y, T)$ be measurable spaces; a *measurable function* $f : (X, S) \to (Y, T)$ is a set-theoretic function $f : X \to Y$ such that $f^{-1}(B) \in S$ for all $B \in T$.

We denote by Msrbl the category of measurable spaces with measurable maps. Usual examples of measurable spaces are the following.

**Example 1.1.2** (trivial $\sigma$-algebras)**.** Any set $X$ admits two trivial $\sigma$-algebras, namely the whole $P(X)$ and $\{\varnothing, X\}$.

If $X$ is a set and $S \subseteq P(X)$, then the $\sigma$-algebra generated by $S$ is the inclusion-wise smallest $\sigma$-algebra $\langle S\rangle$ on $X$ containing $S$.

**Example 1.1.3** (Borel $\sigma$-algebra)**.** When $X$ is a set with a topology $\tau \subseteq P(X)$, we can define the *$\sigma$-algebra of Borel subsets* of $X$ to be the $\sigma$-algebra $\mathscr{B}_X \subseteq P(X)$ generated by the topology $\tau$.

In the following chapters we will mainly deal with standard Borel spaces.

**Definition 1.1.4** (standard Borel spaces)**.**

1. a topological space $X$ is said to be *Polish* if it is completely metrizable (the topology is induce by a metric with respect to which $X$ is complete) and separable (it contains a countable dense subset);

2. a *standard Borel space* is any measurable space that is isomorphic in Msrbl, i.e. as a measurable space, to a Polish space with its Borel $\sigma$-algebra.

Standard Borel spaces have bounded cardinality.

**Lemma 1.1.5.** *If $(X, d)$ is a separable and metrizable topological space, then $|X| \leq \mathfrak{c}$.*

*Proof.* Let $A \subseteq X$ be a countable dense subset and take two distinct points $x, y \in X$. Suppose by absurd that $d(x, a) = d(y, a)$ for all $a \in A$. By the density of $A$ in $X$ we can pick $a \in A$ such that $d(x, a) = d(y, a) < d(x, y)/2$. But then the triangular inequality $d(x, y) \leq d(x, a) + d(y, a) < d(x, y)$ gives a contradiction. This means that every element of $X$ is uniquely determined by its distance from every point of $A$. In particular we get an injection $X \hookrightarrow \mathbb{R}^A$ and hence $|X| \leq |\mathbb{R}^A| \leq |\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}$. $\square$

### 1.1.2. Measures

**Definition 1.1.6** (measure spaces). Let $(X, S)$ be a measurable space;

1. a *measure* on $X$ is a function $\mu : S \to [0, +\infty]$ with the following properties:
   a) $\mu(\varnothing) = 0$;
   b) if $\{A_n\}_{n\in\mathbb{N}} \subseteq S$ is a family of pair-wise disjoint measurable subsets, then

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=0}^{\infty} A_n$$

2. a *measure space* is a triple $(X, S, \mu)$, where $(X, S)$ is a measurable space and $\mu : S \to [0, +\infty]$ is a measure on $X$;

3. let $(X, S, \mu)$ and $(Y, T, \nu)$ be measure spaces; a measurable function $f : (X, S) \to (Y, T)$ is *measure preserving* if $\mu(f^{-1}(B)) = \nu(B)$ for all $B \subseteq T$ measurable subsets.

When the $\sigma$-algebra $S$ on $X$ is clear, we write $(X, \mu)$ for $(X, S, \mu)$.

**Example 1.1.7** (Lebesgue measure). Let $\mathscr{B}_{\mathbb{R}}$ be the $\sigma$-algebra of Borel subsets of $\mathbb{R}$ with the usual topology. Then there exists a unique measure $m : \mathscr{B}_{\mathbb{R}} \to [0, +\infty]$ such that $m([a, b]) = b - a$ for all $a, b \in \mathbb{R}$ with $a < b$. Moreover there exists a $\sigma$-algebra $\mathscr{L}_{\mathbb{R}} \supseteq \mathscr{B}_{\mathbb{R}}$ and an extension of $m$ to $\mathscr{L}_{\mathbb{R}}$ which is also complete, i.e. if $N \in \mathscr{L}_{\mathbb{R}}$ and $m(N) = 0$, then every subset of $N$ belongs to $\mathscr{L}_{\mathbb{R}}$ as well. We call $m : \mathscr{L}_{\mathbb{R}} \to [0, +\infty]$ the *Lebesgue measure* of $\mathbb{R}$. A more precise construction can be found for instance in [3, p. 37].

**Example 1.1.8** (counting measure). Let $X$ be any set. The *counting measure* on $X$ is the function $\# : P(X) \to [0, +\infty]$ defined by

$$\#(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ +\infty & \text{if } A \text{ is infinite} \end{cases}$$

for all $A \subseteq X$.

A *probability measure* (or just a probability) on $X$ is a measure $\mu : S \to [0, +\infty]$ with the property that $\mu(X) = 1$. In such a case we call $(X, S, \mu)$ a *probability space* .

**Example 1.1.9** (Dirac's measure). Let $X$ be a set an let $z \in X$; the Dirac's probability measure centred at $z$ is the function $\delta_z : P(X) \to [0, 1]$ defined by

$$\delta_z(A) = \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \in X \setminus A \end{cases}$$

for all $A \subseteq X$.

More generally, a measure $\mu : S \to [0, +\infty]$ is *finite* if $\mu(X) < +\infty$. A criterion we will use to check whether two finite measures are the same is the following.

**Proposition 1.1.10** (uniqueness of finite measures, [9] Theorem 1.3.5). *Let $X$ be a set and let $S \subseteq P(X)$ be non-empty and closed under finite intersections ($S$ is said to be a $\pi$-system); if $\mu, \nu : \langle S \rangle \to [0, +\infty[$ are finite measures such that*

$$\mu|_S = \nu|_S \text{ and } \mu(X) = \nu(X)$$

*then $\mu = \nu$.*

Measure spaces with measure preserving functions define a category. We denote by $\mathsf{Meas_p}$ the category with:

- objects

$$\mathrm{Ob}(\mathsf{Meas_p}) = \{(X, S, \mu) : (X, S) \in \mathrm{Ob}(\mathsf{Msrbl}), \mu : S \to [0, +\infty] \text{ measure}\}$$

- morphisms

$$\mathrm{Hom}_{\mathsf{Meas_p}}((X, S, \mu), (Y, T, \nu)) = \\ = \{f \in \mathrm{Hom}_{\mathsf{Msrbl}}((X, S), (Y, T)) : f \text{ measure preserving}\}$$

In the same way we denote by $\mathsf{PMeas_p}$ the subcategory of $\mathsf{Meas_p}$ of probability spaces with probability preserving functions.

**Definition 1.1.11** (push-forward of measure)**.** Let $(X,S)$ and $(Y,T)$ be measurable spaces, let $f:(X,S)\to(Y,T)$ be a measurable function and let $\mu:S\to[0,+\infty]$ be a measure on $X$; the *push-forward* of $\mu$ via $f:X\to Y$ is the function

$$\begin{aligned} f_*\mu : T &\to [0,+\infty] \\ B &\mapsto \mu(f^{-1}(B)) \end{aligned}$$

which defines a measure on $Y$.

**Definition 1.1.12** ($\mu$-null sets)**.** Let $(X,\mu)$ be a measure space;

1. a set $A\subseteq X$ is said to be *null* with respect to $\mu$ (or $\mu$-null) if there exists a measurable subset $A\subseteq B\subseteq X$ such that $\mu(B)=0$;

2. let $\varphi$ be a property on $X$; we say that $\varphi$ holds $\mu$-almost everywhere on $X$ if the set

$$\{x\in X : \neg\varphi(x)\}$$

is $\mu$-null.

**Example 1.1.13** (equality almost everywhere)**.** Let $(X,\mu)$ be a measure space, let $Y$ be a set. We say that the maps $f,g:X\to Y$ are *equal almost everywhere* and write $f=_0 g$ if the set $\{x\in X : f(x)\neq g(x)\}$ is $\mu$-null. It is straightforward to see that $=_0$ is an equivalence relation on the set $\mathrm{Hom}_{\mathsf{Set}}(X,Y)$.

**Example 1.1.14** (almost everywhere defined maps)**.** Let $(X,S,\mu)$ and $(Y,T,\nu)$ be measure spaces; an *almost everywhere defined map* $f:(X,\mu)\dashrightarrow(Y,\nu)$ is a measurable map

$$f:(X',S',\mu')\to(Y,T,\nu)$$

such that:

1. $X'\in S$ and $\mu(X\smallsetminus X')=0$;

2. $S'$ is the $\sigma$-algebra $\{A\cap X' : A\in S\}$ on $X'$ induced by the $\sigma$-algebra $S$;

3. $\mu'$ is the restriction $\mu|_{S'}$.

Some basic properties of measures are summarized in the next theorem, a proof of which can be found for instance in [3].

**Theorem 1.1.15** (basic properties of measures, [3] Theorem 1.8)**.** *Let $(X,S,\mu)$ be a measure space; then the following properties hold:*

1. *monotonicity: if $A,B\in S$ and $A\subseteq B$, then $\mu(A)\leq\mu(B)$;*

2. *sub-additivity if $\{A_n\}_{n\in\mathbb{N}}\subseteq S$, then $\mu(\bigcup_{n\in\mathbb{N}}A_n)\leq\sum_{n=0}^{+\infty}\mu(A_n)$;*

3. *continuity from below: if $\{A_n\}_{n\in\mathbb{N}}\subseteq S$ and $A_n\subseteq A_{n+1}$ for all $n\in\mathbb{N}$, then $\lim_{n\to+\infty}\mu(A_n)=\mu(\bigcup_{n\in\mathbb{N}}A_n)$;*

4. *continuity from above: if* $\{A_n\}_{n\in\mathbb{N}} \subseteq S$, $A_n \supseteq A_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $\mu(A_0) < +\infty$ *then* $\lim_{n\to+\infty} \mu(A_n) = \mu(\bigcap_{n\in\mathbb{N}} A_n)$.

We call $x \in X$ an *atom* of the measure space $(X, \mu)$ if $\mu(\{x\}) > 0$. We say that $(X, \mu)$ is:

1. *atom-free* or *nonatomic* if for every $x \in X$, the singleton $\{x\} \subseteq X$ is measurable and $\mu(\{x\}) = 0$;

2. *purely atomic* if every measurable subset of $X$ with positive measure contains an atom.

Every measure space decomposes into a an atom-free part and a purely atomic part in the sense of the following:

**Theorem 1.1.16** (atomic decomposition, [11] Theorem 2.1). *Let* $(X, \mu)$ *be a measure space; then there exist two measures* $\mu_{pa}$ *and* $\mu_{af}$ *on* $X$ *such that*

1. $(X, \mu_{pa})$ *is purely atomic and* $(X, \mu_{af})$ *is atom-free;*

2. $\mu = \mu_{pa} + \mu_{af}$.

*Sketch of proof.* If $\mathfrak{M}$ is the family of all countable unions of atoms, then for all $A \subseteq X$ measurable subsets we define

$$\mu_{\text{pa}}(A) = \sup\{\mu(A \cap M) : M \in \mathfrak{M}\}$$

and

$$\mu_{\text{af}}(A) = \sup\{\mu(A \cap N) : \mu_{\text{pa}}(N) = 0\}$$ □

We will focus our interest mainly on standard Borel probability spaces. Indeed standard Borel spaces form a suitable category for measure theory in the sense of the following two theorems.

**Theorem 1.1.17** ([12, Corollary 15.2]). *Let* $(X, \mathscr{B}_X), (Y, \mathscr{B}_Y)$ *be standard Borel spaces and let* $f : X \to Y$ *be a measurable map; then*

1. *if* $f$ *is bijective, then* $f^{-1} : Y \to X$ *is measurable, which means that* $f$ *is an isomorphism in* **Msrbl**;

2. *if* $A \in \mathscr{B}_X$ *and* $f|_A : A \to Y$ *is injective, then* $f(A) \in \mathscr{B}_Y$ *and* $f|_A : A \to f(A)$ *is an isomorphism in* **Msrbl**.

**Theorem 1.1.18** ([15, Theorem A.20]). *Let* $(X, \mu)$ *be an atom-free standard Borel probability space; then* $(X, \mu)$ *is isomorphic in* $\mathsf{PMeas_p}$ *to the space* $([0, 1], m)$, *where* $m$ *is the usual Lebesgue measure.*

### 1.1.3. Products

The category $\mathsf{PMeas_p}$ admits arbitrary products and, to see this, we start by constructing products in the category $\mathsf{Msrbl}$. Let us take a set $I$ and a family $\{(X_i, S_i)\}_{i\in I} \subseteq \mathrm{Ob}(\mathsf{Msrbl})$ of measurable spaces. Every non-empty subset $J \subseteq I$ determines a canonical projection $\pi_J : \prod_{i\in I} X_i \to \prod_{i\in J} X_i$.

**Definition 1.1.19** (product $\sigma$-algebra)**.** Let $\{(X_i, S_i)\}_{i\in I} \subseteq \mathrm{Ob}(\mathsf{Msrbl})$; then

1. the *cylinder* associated with $(A_i)_{i\in J} \in \prod_{i\in J} S_i$ is the preimage $\pi_J^{-1}(\prod_{i\in J} A_i) \subseteq \prod_{i\in I} X_i$;
2. the *product $\sigma$-algebra* of the family $\{(X_i, S_i)\}_{i\in I}$ is the $\sigma$-algebra $\bigotimes_{i\in I} S_i \subseteq P(\prod_{i\in I} X_i)$ generated by all the cylinder sets.

Notice that the construction of $\bigotimes_{i\in I} S_i$ makes every projection $\pi_j : \prod_{i\in I} X_i \to X_j$ a measurable function. This construction indeed yields a product in the category $\mathsf{Msrbl}$.

**Proposition 1.1.20** (universal property of the product $\sigma$-algebra)**.** *Let $\{(X_i, S_i)\}_{i\in I} \subseteq \mathrm{Ob}(\mathsf{Msrbl})$ be a family of measurable spaces; the measurable space $(\prod_{i\in I} X_i, \bigotimes_{i\in I} S_i)$ (together with the projections on each factor) is a product of the family $\{(X_i, S_i)\}_{i\in I}$ in the category $\mathsf{Msrbl}$. More precisely, for every measurable space $(Y, T) \in \mathrm{Ob}(\mathsf{Msrbl})$ and for every family of measurable functions $\{f_i : (Y, T) \to (X_i, S_i)\}_{i\in I}$ there exists a unique measurable function $f : (Y, T) \to (\prod_{i\in I} X_i, \bigotimes_{i\in I} S_i))$ such that $\pi_j \circ f = f_j$.*

$$\begin{tikzcd}
(Y,T) \arrow[r, dashed, "f"] \arrow[dr, bend right, "f_j"'] & (\prod_{i\in I} X_i, \bigotimes_{i\in I} S_i) \arrow[d, "\pi_j"] \\
& (X_j, S_j)
\end{tikzcd}$$

When we move to the category $\mathsf{PMeas_p}$ of probability spaces, we also obtain product measures in the following way:

**Theorem 1.1.21** (existence and uniqueness of product measures, [2, Theorem 8.2.2])**.** *Let $I$ be a set and let $\{(X_i, S_i, \mu_i)\}_{i\in I} \subseteq \mathrm{Ob}(\mathsf{PMeas_p})$ be a family of probability spaces; then there exists a unique probability measure $\mu$ on the product space $(\prod_{i\in I} X_i, \bigotimes_{i\in I} S_i)$ extending the $\mu$'s. More precisely $\mu$ is the unique probability such that the equality*

$$\mu\left(\prod_{j\in J} A_j \times \prod_{i\in I\setminus J} X_i\right) = \prod_{j\in J} \mu_j(A_j)$$

*holds for every finite subset $J \subseteq I$ and for every $(A_j)_{j\in J} \in \prod_{j\in J} S_j$.*

Such a $\mu$ is called the *product measure* of the measures $\mu_i$'s and is denoted by $\bigotimes_{i\in I} \mu_i$.

## 1.2. Integrable functions

Integration of measurable functions allows us to build normed $\mathbb{Z}$-modules. We are particularly interested on the space $L^\infty(X, \mu, \mathbb{Z})$ of essentially bounded integer-valued functions, since they will be our new coefficients spaces for the twisted homology in Chapter 5.

### 1.2.1. Integrals

If $X$ is any set and $A \subseteq X$, then the *characteristic function* $\chi_A : X \to \{0, 1\}$ of $A$ is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \smallsetminus A \end{cases}$$

Also recall that a function $X \to \mathbb{Z}$ is *simple* if it is a $\mathbb{Z}$-linear combination $\sum_{i=1}^k m_i \chi_{A_i}$ of characteristic functions.

Since $\mathbb{Z}$ is discrete as a topological space, then its Borel $\sigma$-algebra is the trivial one, which means that every subset of $\mathbb{Z}$ is measurable. Hence if $(X, \mu)$ is a measure space, then a measurable function $f : X \to \mathbb{Z}$ admits a writing of the form

$$f = \sum_{i \in \mathbb{Z}} i \cdot \chi_{A_i}$$

where $A_i = f^{-1}(i) \subseteq X$. We call such a writing the standard representation of $f$.

**Definition 1.2.1** (integrals)**.** Let $(X, \mu)$ be a measure space and let $f = \sum_{i \in \mathbb{N}} i \cdot \chi_{A_i} : X \to \mathbb{N}$ be a positive measurable function in its standard representation; we define the *$\mu$-integral* of $f$ as the sum

$$\int_X f d\mu = \sum_{i=0}^{+\infty} i \cdot \mu(A_i) \in [0, +\infty]$$

For a measurable function $f : X \to \mathbb{Z}$ we can compute the integrals $\int_X f^+ d\mu$ and $\int_X f^- d\mu$. In the case at least one of the former is finite, we can define the integral

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

We say that a measurable function $f : X \to \mathbb{Z}$ is *$\mu$-integrable* (or just integrable) if $\int_X f d\mu < +\infty$. Since $|f| = f^+ + f^-$, $f$ is integrable if, and only if, $\int_X |f| d\mu < +\infty$. The set of integrable functions

$$\mathscr{L}^1(X, \mu, \mathbb{Z}) = \{f : X \to \mathbb{Z} : f \text{ is measurable and } \textstyle\int_X |f| d\mu < +\infty\}$$

inherits a natural $\mathbb{Z}$-module structure and the integral operator $\int_X (\cdot) d\mu : \mathscr{L}^1(X, \mu, \mathbb{Z}) \to \mathbb{R}$ is $\mathbb{Z}$-linear. Moreover the relation of equality almost-everywhere is compatible with the

$\mathbb{Z}$-module structure ($f =_0 g$ if, and only if, $f - g =_0 0$) and with the integration operator (if $f =_0 g$, then $\int_X f d\mu = \int_X g d\mu$). By quotienting out by this equivalence relation, we obtain the $\mathbb{Z}$-module

$$L^1(X, \mu, \mathbb{Z}) = \mathscr{L}^1(X, \mu, \mathbb{Z}) / =_0$$

and a well-defined induced $\mathbb{Z}$-linear function $\int_X (\cdot) d\mu : L^1(X, \mu, \mathbb{Z}) \to \mathbb{R}$.

### 1.2.2. $L^p$-spaces

The construction of the space $L^1(X, \mu, \mathbb{Z})$ can be generalized. Let us fix a real number $p \in ]0, +\infty[$ and a measure space $(X, \mu)$; the *p-norm* of a measurable function $f : X \to \mathbb{Z}$ is

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]$$

Again we can define the space

$$\mathscr{L}^p(X, \mu, \mathbb{Z}) = \{f \in \mathrm{Hom}_{\mathsf{Msrbl}}(X, \mathbb{Z}) : \|f\|_p < +\infty\}$$

of *$L^p$-integrable functions* and the quotient $L^p(X, \mu, \mathbb{Z}) = \mathscr{L}^p(X, \mu, \mathbb{Z}) / =_0$. In order to justify the therm “norm”, we first need to give a meaning to the concept of norm for $\mathbb{Z}$-modules.

**Definition 1.2.2** (normed $\mathbb{Z}$-modules)**.** Let $M$ be a $\mathbb{Z}$-module; a *$\mathbb{Z}$-module norm* (or just a norm) on $M$ is a function $\nu : M \to [0, +\infty[$ satisfying the following conditions:

1. $\nu(m) = 0$ if, and only if, $m = 0$;
2. homogeneity: $\nu(am) = |a| \cdot \nu(m)$ for every $m \in M$ and $a \in \mathbb{Z}$;
3. triangular inequality: $\nu(m + n) \leq \nu(m) + \nu(n)$ for every $m, n \in M$.

We call the pair $(M, \nu)$ a *normed $\mathbb{Z}$-module*. Some authors replace condition 2 with the weaker condition of $\nu(m) = \nu(-m)$ for all $m \in M$. By replacing the ring $\mathbb{Z}$ with a unitary normed ring $(R, |\cdot|)$, we can also define $R$-module norms in the same way.

Let $p \in ]0, +\infty[$ be a real number and let $(X, \mu)$ be a measure space. The function

$$\begin{aligned} \|\cdot\|_p : \mathscr{L}^p(X, \mu, \mathbb{Z}) &\to [0, +\infty[ \\ f &\mapsto \|f\|_p \end{aligned}$$

is a seminorm on $\mathscr{L}^p(X, \mu, \mathbb{Z})$ (every $\mu$-almost everywhere zero function has zero norm), which induces a $\mathbb{Z}$-module norm on the quotient $L^p(X, \mu, \mathbb{Z})$. The construction also extends for $p = +\infty$ and gives the case that is most interesting to us.

**Definition 1.2.3** ($\infty$-norm)**.** The *$\infty$-norm* (or essential supremum) of a measurable function $f : X \to \mathbb{Z}$ is

$$\|f\|_\infty = \inf\{c \geq 0 : \mu(\{x \in X : |f(x)| > c\}) = 0\}$$

and we say that $f$ is essentially bounded if $\|f\|_\infty < +\infty$.

The $\mathbb{Z}$-module of essentially bounded functions $X \to \mathbb{Z}$ is denoted by

$$\mathscr{L}^\infty(X,\mu,\mathbb{Z}) = \{f : X \to \mathbb{Z} : \|f\|_\infty < +\infty\}$$

and by quotienting out by the equivalence relation of equality almost-everywhere, we get $L^\infty(X,\mu,\mathbb{Z}) = \mathscr{L}^\infty(X,\mu,\mathbb{Z})/=_0$.

**Remark 1.2.4.** When $(X,\mu)$ is a space of finite measure (e.g. a probability space), it is straightforward to see that $L^\infty(X,\mu,\mathbb{Z}) \subseteq L^1(X,\mu,\mathbb{Z})$.

A better and explicit description of $L^\infty(X,\mu,\mathbb{Z})$ will help us in working with these classes of functions. We denote by $B(X,\mathbb{Z})$ the submodule of $\mathscr{L}^\infty(X,\mu,\mathbb{Z})$ of the globally bounded functions $X \to \mathbb{Z}$. An essentially bounded class of functions $\varphi \in L^\infty(X,\mu,\mathbb{Z})$ admits a representative $f \in B(X,\mathbb{Z})$ of the form a simple function

$$f = \sum_{i=M}^{N} i \cdot \chi_{A_i}$$

for some $M, N \in \mathbb{Z}$ and $A_i = f^{-1}(i) \subseteq X$ measurable subsets. This exhibits $\{\chi_B : B \in \mathscr{B}_X\}$ as a set of generators for $B(X,\mathbb{Z})$ as $\mathbb{Z}$-module. Let us denote by

$$N(X,\mu,\mathbb{Z}) = \{f \in B(X,\mathbb{Z}) : f =_0 0\} \leq B(X,\mathbb{Z})$$

the submodule of bounded functions that are equal to the zero function almost-everywhere.

**Proposition 1.2.5** (explicit description of $L^\infty(X,\mu,\mathbb{Z})$). *Let $(X,\mu)$ be a standard Borel space; there is an isomorphism of $\mathbb{Z}$-modules $L^\infty(X,\mu,\mathbb{Z}) \cong B(X,\mathbb{Z})/N(X,\mu,\mathbb{Z})$.*

*Proof.* The inclusion of universally bounded functions

$$B(X,\mathbb{Z}) \subseteq \mathscr{L}^\infty(X,\mu,\mathbb{Z})$$

induces a $\mathbb{Z}$-module homomorphism

$$\begin{aligned} \alpha : B(X,\mathbb{Z}) &\to L^\infty(X,\mu,\mathbb{Z}) \\ f &\mapsto [f]_{=_0} \end{aligned}$$

We claim that $\alpha$ is surjective: let us pick $\psi = [g]_{=_0} \in L^\infty(X,\mu,\mathbb{Z})$ for some $g \in \mathscr{L}^\infty(X,\mu,\mathbb{Z})$, which means that there is $A \in \mathscr{B}_X$ such that $\mu(A) = 0$ and $g|_{X \smallsetminus A}$ is bounded. Hence $g' = g \cdot \chi_{X \smallsetminus A} \in B(X,\mathbb{Z})$ and $\psi = \alpha(g')$, which shows the surjectivity. It is straightforward to see that $\operatorname{Ker}\alpha = N(X,\mu,\mathbb{Z})$. $\square$

# 2. Ergodic group theory

When we make a group act on a probability space in a way that is compatible with the measure space structure, we may ask which properties of the group translate into properties of the action and see algebraic/geometric problems under the dynamical point of view. Some of such actions, namely the ergodic actions, are particularly interesting in our survey because they form the indecomposable build blocks of every other dynamical system.

## 2.1. Dynamical systems

Dynamical systems are measure spaces together with a group (or monoid) action that is measure preserving. The usual classical theory of dynamical systems is constructed by considering actions of the group of integers $\mathbb{Z}$ or by the monoid of natural numbers $\mathbb{N}$ and leads to applications in the field of mathematical physics. Here we are giving the construction a more geometric flavour and are considering general groups: this different point of view is usually called Measurable group theory.

### 2.1.1. Measure preserving actions

If $\Gamma$ is a group, $\mathscr{C}$ is a category and $X \in \mathrm{Ob}(\mathscr{C})$ is an object, then we say that an *action* of $\Gamma$ on $X$ in the category $\mathscr{C}$ is a group homomorphism $\Gamma \to \mathrm{Aut}_{\mathscr{C}}(X)$. When $X$ is a set we have another equivalent description: a left action of $\Gamma$ on the set $X$ (in the category $\mathsf{Set}$) is a map

$$\begin{aligned} \alpha : \Gamma \times X &\to X \\ (\gamma, x) &\mapsto \gamma \cdot x \end{aligned}$$

(the notation $\gamma \cdot x$ is always preferred to $\alpha(\gamma, x)$) with the properties:

1. $\gamma \cdot (\delta \cdot x) = \gamma\delta \cdot x$;
2. $1_\Gamma \cdot x = x$;

for every $x \in X$ and for every $\gamma, \delta \in \Gamma$. In our measure-theoretic environment we must take the measure into account as well.

**Definition 2.1.1** (measure preserving group actions). Let $\Gamma$ be a countable group and let $(X, \mu)$ be a measure space; a left action $\alpha : \Gamma \times X \to X$ is measure preserving if the maps

$$\begin{aligned} \gamma \cdot\ : X &\to X \\ x &\mapsto \gamma \cdot x \end{aligned}$$

are (measurable and) measure preserving for all $\gamma \in \Gamma$.

We will usually write $\alpha : \Gamma \curvearrowright (X, \mu)$ to denote the action $\alpha$ of $\Gamma$ on $(X, \mu)$. In more generality we can define measure preserving monoid actions in the same way.

**Remark 2.1.2** (actions in $\mathsf{Meas_p}$)**.** Let $\Gamma$ be a countable group and let $(X, \mu)$ be a measure space. To give a measure preserving action $\Gamma \curvearrowright (X, \mu)$ is equivalent to give a group action in the category $\mathsf{Meas_p}$, i.e. a group homomorphism $\Gamma \to \mathrm{Aut}_{\mathsf{Meas_p}}(X, \mu)$.

From now on every group will be of countable cardinality, unless otherwise specified. The classical prototypical example are the actions generated by a measure preserving automorphism.

**Example 2.1.3** (action induced by automorphism)**.** Let $(X, \mu)$ be a measure space and let $f : X \to X$ be an automorphism in $\mathsf{Meas_p}$, which means that $f$ is a measure preserving bijection with measure preserving inverse. Then the group $\mathbb{Z}$ acts on $(X, \mu)$ via

$$\begin{aligned} \mathbb{Z} \times X &\to X \\ (n, x) &\mapsto f^n(x) \end{aligned}$$

where $f^0 = \mathrm{id}_X$ and $f^n = (f^{-1})^{-n}$ if $n < 0$. Since $f$ is measure preserving, then so are the $f^n$'s and hence the action $\mathbb{Z} \curvearrowright (X, \mu)$ is measure preserving. More generally every measure preserving $g \in \mathrm{End}_{\mathsf{Meas_p}}(X, \mu)$ induces a monoid action via

$$\begin{aligned} \mathbb{N} \times X &\to X \\ (n, g) &\mapsto g^n(x) \end{aligned}$$

**Terminology 2.1.4** (orbit, stabilizer and fixed space)**.** For $x \in X$ and $A \subseteq X$ we have the usual notions of:

1. *orbit* $\Gamma \cdot x = \{\gamma \cdot x \in X : \gamma \in \Gamma\} \subseteq X$ and $\Gamma \cdot A = \cup\{\Gamma \cdot a : a \in A\}$;
2. *stabilizer* $\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\} \leq \Gamma$;
3. *fixed space* $A^\Gamma = \{a \in A : \Gamma_a = \Gamma\}$.

Recall the an action $\Gamma \curvearrowright X$ is said to be *free* if $\Gamma_x = 1_\Gamma$ for all $\gamma \in \Gamma$. In our measure-theoretic environment the natural condition to ask is slightly different: we say that $\Gamma \curvearrowright (X, \mu)$ is *essentially free* if

$$\mu(\{x \in X : \Gamma_x \neq 1\}) = 0$$

**Remark 2.1.5.** If $\Gamma$ is an infinite group and $\Gamma \curvearrowright (X, \mu)$ is a measure preserving action that is essentially free, then the measure space $(X, \mu)$ cannot contain atoms.

**Definition 2.1.6** (standard actions)**.** Let $X$ be a topological space and let $\Gamma$ be a countable group;

1. a *standard Borel probability space* is a standard Borel space endowed with a probability measure;

2. a *standard Borel* $\Gamma$*-space* (or standard $\Gamma$-space) is a standard Borel probability space $(X, \mu)$ equipped with a measure preserving left action $\Gamma \curvearrowright (X, \mu)$. In such a case we say that $\Gamma \curvearrowright (X, \mu)$ is a *standard action* .

With an abuse of notation the terms "standard $\Gamma$-space" and "standard action" are used as synonyms. The category $\Gamma$-Standard of standard Borel $\Gamma$-spaces is the category whose:

1. objects are standard Borel $\Gamma$-spaces;

2. morphisms are Borel $\Gamma$-homomorphisms, i.e. measurable, $\Gamma$-equivariant, measure-preserving maps between standard Borel $\Gamma$-spaces.

Notice that by Theorem 1.1.17 measurable, $\Gamma$-invariant, measure-preserving bijections between standard Borel $\Gamma$-spaces are standard $\Gamma$-isomorphisms. In the following we are mainly dealing with standard actions.

### 2.1.2. Main examples

As usual we set $S^1 = \mathbb{R}/\mathbb{Z}$ to be the unit circle endowed with the quotient topology induced by the projection $\mathbb{R} \to S^1$. We view $S^1$ as a measure space with the Borel $\sigma$-algebra and the Lebesgue measure given by the push-forward of the Lebesgue measure on $[0, 1[$ via the map

$$\begin{aligned} [0,1[ &\to S^1 \\ x &\mapsto x + \mathbb{Z} = [x] \end{aligned}$$

A basis for the Borel $\sigma$-algebra on $S^1$ is given by the intervals $\{[a, b]_{S^1} : c, b \in [0, 1[\, , a < b\}$, where $[a, b]_{S^1} = \{x + \mathbb{Z} \in S^1 : x \in [a, b] \subseteq \mathbb{R}\}$.

**Example 2.1.7** (rotations on the circle). Let us fix $\vartheta \in \mathbb{R}$ and define the function

$$\begin{aligned} f : (S^1, m) &\to (S^1, m) \\ [x] &\mapsto [x + \vartheta] \end{aligned}$$

Then $f$ is well-defined, measurable and measure preserving. Indeed

$$m(f^{-1}([a, b]_{S^1})) = m([a - \vartheta, b - \vartheta]_{S^1}) = m([a, b]_{S^1})$$

for all $a, b \in [0, 1[$ with $a < b$. The *rotation action* on $S^1$ about $\vartheta$ is the action $r_\vartheta : \mathbb{Z} \curvearrowright (S^1, m)$ induced by the measurable isomorphism $f : S^1 \to S^1$.

**Example 2.1.8** (digit shifts on the cycle). Let $d \in \mathbb{N}$ and set

$$\begin{aligned} g : (S^1, m) &\to (S^1, m) \\ [x] &\mapsto [d \cdot x] \end{aligned}$$

which is well-defined and measurable. When $d \geq 1$, the function $g$ is also measure preserving. Indeed

$$m(g^{-1}([a,b]_{S^1})) = m\left(\bigcup_{j=0}^{d-1}\left[\frac{j}{d}+\frac{a}{d},\frac{j}{d}+\frac{b}{d}\right]\right) =$$

$$= \sum_{j=0}^{d-1} m\left(\left[\frac{j}{d}+\frac{a}{d},\frac{j}{d}+\frac{b}{d}\right]\right) = b-a = m([a,b]_{S^1})$$

Except for the case $d = 1$, $g$ is not injective, hence we only obtain a monoid action $D_d : \mathbb{N} \curvearrowright (S^1, m)$, called the *digit shift* action on $S^1$ of base $d$.

The explanation of the name "digit shift" is the following. When $d = 10$, the map $x \mapsto [10 \cdot x]$ shifts the decimal expansion of the fractional part of $x$ one digit to the left and forgets the first decimal digit after the dot. For $x = 2022.2022$ we obtain

$$2022.2022 \mapsto [0.2022] \mapsto [2.022] = [0.022]$$

The essentially freeness of rotations and digit shifts on the circle is strictly related to rationality/irrationality.

**Proposition 2.1.9** (essentially freeness of rotations and digit shifts, [22, Proposition 1.2.9]). *Let $z \in S^1$, $\vartheta \in \mathbb{R}$ and $d \in \mathbb{N}_{\geq 2}$; then*

1. *the stabilizer of $z$ with respect to the rotation $r_\vartheta : \mathbb{Z} \curvearrowright (S^1, m)$ is trivial if, and only if, $\vartheta$ is irrational. In particular the action $r_\vartheta : \mathbb{Z} \curvearrowright (S^1, m)$ is essentially free if, and only if, $\vartheta$ is irrational;*
2. *the monoidal orbit of $z$ in $S^1$ with respect to the digit shift $D_d : \mathbb{N} \curvearrowright (S^1, m)$ is finite if, and only if, one (whence every) representative of $z$ in $\mathbb{R}$ is irrational.*

One last important example is the Bernoulli shift.

**Example 2.1.10** (Bernoulli shift). Let $\Gamma$ be a countable group and let $(X, \mu)$ be a probability space. $\Gamma$ acts on the product probability spaces $(X, \mu)^\Gamma = \prod_{\gamma \in \Gamma}(X, \mu)$ via the action

$$\begin{aligned} \Gamma \times X^\Gamma &\to X^\Gamma \\ (\gamma, (x_\eta)_{\eta\in\Gamma}) &\mapsto (x_{\eta\cdot\gamma})_{\eta\in\Gamma} \end{aligned}$$

which is called the *Bernoulli shift* action and is measure preserving with respect to the product measure. When $X = \{0,1\}$ and $\mu : P(\{0,1\}) \to [0,1]$ is defined by $\mu(\{0\}) = \mu(\{1\}) = 1/2$, then the action $\Gamma \curvearrowright \{0,1\}^\Gamma$ is called the *standard Bernoulli shift.*

**Proposition 2.1.11** (essential freeness of the standard Bernoulli shift, [22, Proposition 1.2.43]). *Let $\Gamma$ be a countable group;*

1. *if $\Gamma \neq 1$, then the standard Bernoulli shift $\Gamma \curvearrowright \{0,1\}^\Gamma$ is not free;*

2. *if* $\Gamma$ *is finite, then the standard Bernoulli shift* $\Gamma \curvearrowright \{0,1\}^\Gamma$ *is not essentially free;*

3. *if* $\Gamma$ *is infinite, then the standard Bernoulli shift* $\Gamma \curvearrowright \{0,1\}^\Gamma$ *is essentially free.*

*Proof.* We only prove the third an more complicated claim, so let $\Gamma$ be a countable infinite group. For $\gamma \in \Gamma \smallsetminus \{1_\Gamma\}$ let $A_\gamma = \{x \in \{0,1\} < \Gamma : \gamma \cdot x = x\}$ be the set of the points fixed by $\gamma$, so that

$$\{x \in \{0,1\}^\Gamma : \Gamma_x \neq 1_\Gamma\} = \cup\{A_\gamma : \gamma \in \Gamma \smallsetminus \{1_\Gamma\}\}$$

Since $\Gamma$ is countable, it suffices to show that $A_\gamma$ is a $\mu^{\otimes\Gamma}$-null set for every $\gamma \in \Gamma \smallsetminus \{1_\Gamma\}$. For any $S \subseteq \Gamma$ we define

$$B(S) = \{x \in \{0,1\}^\Gamma : \text{ either } x_\gamma = 0 \text{ for all } \gamma \in S \text{ or } x_\gamma = 1 \text{ for all } \gamma \in S\}$$

and compute

$$\mu^{\otimes\Gamma}(B(S)) = \begin{cases} 2^{1-|S|} & \text{if } S \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$$

Let us now fix $\gamma \in \Gamma \smallsetminus \{1_\Gamma\}$ and let $S$ be the subgroup of $\Gamma$ generated by $\gamma$. We must distinguish two cases:

1. if $S$ is infinite, then $A_\gamma \subseteq B(S)$, hence $A_\gamma$ is $\mu^{\otimes\Gamma}$-null;

2. if $S$ is finite, then there are infinitely many cosets of $S$ in $\Gamma$. Let $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma^\mathbb{N}$ be a sequence in $\Gamma$ such that every $\gamma_n$ represents a different coset of $\Gamma/S$. We have that $A_\gamma \subseteq \cap\{B(\gamma_n \cdot S) : n \in \mathbb{N}\}$ and hence

$$\mu^{\otimes\Gamma}(A_\gamma) \leq \inf_{n\in\mathbb{N}} \bigcap_{i=0}^{n} B(\gamma_i \cdot S) = \inf_{n\in\mathbb{N}} \left(2^{1-|s|}\right)^n = 0$$

   showing that $A_\gamma$ is $\mu^{\otimes\Gamma}$-null. □

This construction yields the following useful result:

**Corollary 2.1.12** (existence of essentially free probability actions). *Every countable group admits an essentially free probability measure preserving action.*

Indeed, when $\Gamma$ is infinite, we can choose the standard Bernoulli shift $\Gamma \curvearrowright \{0,1\}^\Gamma$, whereas for $\Gamma$ finite we take the *left coset translation action* on the probability measure space $(\Gamma, \mu)$, where $\mu = \#/|\Gamma|$ is the normalized counting measure.

### 2.1.3. Comparing dynamical systems

After introducing our particular dynamical systems, we may ask, when possible, how to compare them and seek a proper notion of equivalence. Let us begin by comparing actions of the same group. Let $(X, \mu)$, $(Y, \nu)$ and $(Z, \rho)$ be measure spaces;

1. an almost everywhere defined map $f : X \dashrightarrow Y$ is measure preserving if

$$f_*(\mu|_{\mathrm{Dom}(f)}) = \nu$$

2. if $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are measure preserving almost everywhere defined maps, then their composition

$$g \circ f : \mathrm{Dom}(f) \cap f^{-1}(\mathrm{Dom}(g)) \to Z$$

is almost everywhere defined and measure preserving;

3. we denote by $\mathsf{Meas}_\mathsf{p}^\mathsf{ae}$ (resp. $\mathsf{PMeas}_\mathsf{p}^\mathsf{ae}$) the category whose objects are measure spaces (resp. probability spaces) and whose morphisms are almost measure preserving almost everywhere defined maps between measure spaces (resp. probability spaces).

**Definition 2.1.13** ($\Gamma$-equivariant maps)**.** Let $\Gamma$ be a group and let $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ be group actions on sets; a function $f : X \to Y$ is $\Gamma$-*equivariant* if

$$f(\gamma \cdot x) = \gamma \cdot f(x)$$

for every $x \in X$ and $\gamma \in \Gamma$.

In our measure theory setting we denote by ${}_\Gamma\,\mathsf{Meas}_\mathsf{p}^\mathsf{ae}$ the category whose:

1. objects are triplets $(X, \mu, \alpha)$, where $(X, \mu)$ is a measure space and $\alpha : \Gamma \curvearrowright (X, \mu)$ is a measure preserving $\Gamma$-action;

2. morphisms are $\Gamma$-equivariant measure preserving almost everywhere defined maps.

Again we say that two morphisms $f, g \in \mathrm{Hom}_{\Gamma\,\mathsf{Meas}_\mathsf{p}^\mathsf{ae}}((X, \mu), (Y, \nu))$ are equal almost everywhere if the set $\{x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) : f(x) \neq g(x)\}$ is $\mu$-null. In such a case we write $f =_0 g$ and the relation $=_0$ is an equivalence relation in the set $\mathrm{Hom}_{\Gamma\,\mathsf{Meas}_\mathsf{p}^\mathsf{ae}}((X, \mu), (Y, \nu))$, which is stable under composition of morphisms.

**Definition 2.1.14** (conjugacy)**.** Let $\Gamma$ be a group;

1. we denote by ${}_\Gamma\,\mathsf{Meas}_\mathsf{p}^0$ the homotopy category of ${}_\Gamma\,\mathsf{Meas}_\mathsf{p}^\mathsf{ae}$ with respect to the equivalence relation $=_0$. More explicitly:
   a) $\mathrm{Ob}({}_\Gamma\mathsf{Meas}_\mathsf{p}^0) = \mathrm{Ob}({}_\Gamma\mathsf{Meas}_\mathsf{p}^\mathsf{ae})$;
   b) if $(X, \mu, \alpha), (Y, \nu, \beta) \in \mathrm{Ob}({}_\Gamma\mathsf{Meas}_\mathsf{p}^0)$, then

$$\mathrm{Hom}_{\Gamma\,\mathsf{Meas}_\mathsf{p}^0}((X, \mu, \alpha), (Y, \nu, \beta)) = \mathrm{Hom}_{\Gamma\,\mathsf{Meas}_\mathsf{p}^\mathsf{ae}}((X, \mu, \alpha), (Y, \nu, \beta))/ =_0$$

2. two measure preserving actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$ are *conjugate* if the triplets $(X, \mu, \alpha)$ and $(Y, \nu, \beta)$ are isomorphic in ${}_\Gamma\,\mathsf{Meas}_\mathsf{p}^0$.

**Example 2.1.15** (digit shift and standard Bernoulli shift)**.** Let $\Gamma$ be the monoid $\mathbb{N}$ of natural numbers. The digit shift $D_2 : \mathbb{N} \curvearrowright (S^1, m)$ of base 2 and the standard Bernoulli shift $\mathbb{N} \curvearrowright \{0,1\}^\mathbb{N}$ are conjugate. To see this, we consider the two functions

1. $f : \{0,1\}^{\mathbb{N}} \to S^1$ defined by

$$f((x_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{+\infty} \frac{x_n}{2^{n+1}}$$

2. $g : \{[x] \in S^1 : x \in \mathbb{R} \smallsetminus \mathbb{Z}[1/2]\} \to \{0,1\}^{\mathbb{N}}$ sending $[x]$ to the 2-adic expansion of the fractional part of $x$.

Those maps are almost everywhere defined, measure preserving and $\mathbb{N}$-invariant. Moreover $g \circ f =_0 \mathrm{id}_{\{0,1\}^{\mathbb{N}}}$ and $f \circ g =_0 \mathrm{id}_{S^1}$.

**Definition 2.1.16** (weak containment). Let $\Gamma$ be a countable group and let $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$ be standard $\Gamma$-spaces; we say that $\alpha$ is *weakly contained* in $\beta$ and we write $\alpha \preccurlyeq \beta$ if the following condition holds: for every $\varepsilon > 0$, for every $F \subseteq \Gamma$ finite subset, for every $m \in \mathbb{N}$ and for every $A_1, \ldots, A_m \subseteq X$ measurable subsets there exist $B_1, \ldots, B_m \subseteq Y$ measurable subsets such that

$$|\mu(\gamma^\alpha(A_i) \cap A_j) - \nu(\gamma^\beta(B_i) \cap B_j)| < \varepsilon$$

for all $\gamma \in F$ and for all $i, j \in \{1, \ldots, m\}$.

A direct computation shows that weak containment is stable under the conjugacy relation. If we want to compare measure preserving actions by different groups as well, then we need to introduce the more general notion of orbit equivalence.

**Proposition 2.1.17** (orbit relation). *Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard action; the orbit relation*

$$R_\alpha = \{(x, \gamma \cdot x) : x \in X, \gamma \in \Gamma\} \subseteq X \times X$$

*is an equivalence relation and is measurable in $X \times X$.*

*Proof.* The fact that $R_\alpha$ is an equivalence relations descends from the definition of group action. To show that it is also measurable let us first notice that the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is measurable in $X^2$ with the product $\sigma$-algebra. Indeed if $X$ is a Polish space with its Borel $\sigma$-algebra, then $\Delta_X$ is closed in $X^2$, whence measurable. For every $\gamma \in \Gamma$ the function $\gamma\cdot : X \to X$ is measurable, hence the sets $\{(x, \gamma \cdot x) : x \in X\} = (\gamma \cdot \times \mathrm{id}_X)^{-1}(\Delta_X)$ are measurable for all $\gamma \in \Gamma$ and so is the countable union $R_\alpha = \bigcup_{\gamma\in\Gamma}(\gamma \cdot \times \mathrm{id}_X)^{-1}(\Delta_X)$. $\square$

**Definition 2.1.18** (orbit equivalence). Let $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Lambda \curvearrowright (Y, \nu)$ be standard actions. We say that $\alpha$ and $\beta$ are *orbit equivalent* and write $\alpha \sim_{\mathsf{OE}} \beta$ if there exist two mutually inverse isomorphisms $f : (X, \mu) \to (Y, \nu)$ and $g : (Y, \nu) \to (X, \mu)$ in $\mathsf{Meas}^0_{\mathsf{p}}$ such that

$$f(\Gamma \cdot x) \subseteq \Lambda \cdot f(x) \text{ and } g(\Lambda \cdot y) \subseteq \Gamma \cdot g(y)$$

for $\mu$-almost every $x \in X$ and for $\nu$-almost $y \in Y$.

It is straightforward to see that conjugate actions are also orbit equivalent, as one might expect.

## 2.2. Ergodicity

Ergodic actions can be seen as the building pieces of general dynamical systems. Indeed we will see in 2.2.14 that any standard action admits a decomposition into ergodic actions.

### 2.2.1. Ergodic actions

**Definition 2.2.1** (ergodic actions)**.** Let $\Gamma$ be a group and $(X, \mu)$ a measure space; a measure preserving action $\Gamma \curvearrowright (X, \mu)$ is said to be *ergodic* if either

$$\mu(A) = 0 \text{ or } \mu(X \smallsetminus A) = 0$$

for all measurable subsets $A \subseteq X$ such that $\Gamma \cdot A = A$.

For a measurable subset $A \subseteq X$ to check that $\Gamma \cdot A = A$ is equivalent to check that $\Gamma \cdot A \subseteq A$. Indeed, if we know that $\Gamma \cdot A \subseteq A$, then for every $a \in A$ and for every $\gamma \in \Gamma$ we have that $\gamma \cdot a = b$ for some $b \in A$. Thus $a = \gamma^{-1} \cdot b$ belongs to $\Gamma \cdot A$.

**Example 2.2.2** (coset translation)**.** Let $\Gamma$ be a countable group and $\Lambda \leq \Gamma$ a subgroup of finite index. The normalized counting measure $\#/|\Gamma/\Lambda|$ gives $\Gamma/\Lambda$ a probability space structure and the coset translation action

$$\begin{aligned} \Gamma \times \Gamma/\Lambda &\to \Gamma/\Lambda \\ (\gamma, \delta \cdot \Lambda) &\mapsto \gamma\delta \cdot \Lambda \end{aligned}$$

is transitive and hence ergodic. Indeed for every measurable subset $A \subseteq \Gamma/\Lambda$ we have that

$$\Gamma \cdot A = \begin{cases} \Gamma/\Lambda & \text{if } A \neq \varnothing \\ \varnothing & \text{if } A = \varnothing \end{cases}$$

Let us see why we call ergodic systems indecomposable: every $\Gamma$-invariant measurable subset $A \subseteq X$ induces a dynamical system $\alpha_A : A \curvearrowright (A, \mu|_A)$ by simply restricting the action to the measure subspace. Nevertheless this induced action tells us no new information when $\mu(A) \in \{0, 1\}$. The mixing condition is stronger than ergodicity but it sometimes easier to check.

**Definition 2.2.3** (mixing actions)**.** Let $\Gamma$ be an infinite group; a measure preserving action $\Gamma \curvearrowright (X, \mu)$ is *mixing* if

$$\lim_{g \in \Gamma} \mu(g \cdot (A \cap B)) = \mu(A) \cdot \mu(B)$$

for all measurable subsets $A, B \subseteq X$.

The limit in the definition explicitly means that for every $\varepsilon > 0$ there exists a finite subset $S \subseteq \Gamma$ such that

$$|\mu(g \cdot (A \cap B) - \mu(A) \cdot \mu(B))| < \varepsilon$$

for all $g \in \Gamma \smallsetminus S$.

**Proposition 2.2.4** (mixing implies ergodic). *Let $\Gamma$ be a countable infinite group and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a probability preserving action; if $\alpha : \Gamma \curvearrowright (X, \mu)$ is mixing, then it is ergodic.*

*Proof.* Let $A \subseteq X$ be a $\Gamma$-invariant measurable subset. Since $\Gamma \cdot A = A$, we have that $\gamma \cdot A \cap A = A$ for all $\gamma \in \Gamma$ and the mixing property yields

$$\mu(A) = \lim_{\gamma \in \Gamma} \mu(\gamma \cdot A \cap A) = \mu(A)^2$$

which gives that $\mu(A) \in \{0, 1\}$. □

**Proposition 2.2.5** (standard Bernoulli shifts are mixing, [22, Proposition 2.1.9]). *Let $\Gamma$ be an infinite group; the standard Bernoulli shift $\Gamma \curvearrowright \{0,1\}^\Gamma$ is a mixing probability preserving action.*

*Proof.* Let $A, B \subseteq \{0,1\}^\Gamma$ be measurable subsets. We first consider the case in which $A$ and $B$ are finite unions of cylinder sets, which means that there exist finite subsets $S, T \subseteq \Gamma$ and measurable subsets $A' \subseteq \{0,1\}^S$ and $B' \subseteq \{0,1\}^T$ such that $A = \pi_S^{-1}(A')$ and $B = \pi_T^{-1}(B')$. Then $T^{-1} \cdot S \subseteq \Gamma$ is finite and for all $\gamma \in \Gamma \setminus T^{-1} \cdot S$ we have

$$\begin{aligned}\mu(\gamma \cdot A \cap B) = \mu(\gamma \cdot \pi_S^{-1}(A') \cap \pi_T^{-1}(B')) = \mu(\pi_{S \cdot \gamma^{-1}}^{-1}(A') \cap \pi_T^{-1}(B')) \overset{(*)}{=} \\ = \mu(\pi_{S \cdot \gamma^{-1}}^{-1}(A')) \cdot \mu(\pi_T^{-1}(B')) = \mu(\gamma \cdot \pi_S^{-1}(A')) \cdot \mu(\pi_T^{-1}(B')) = \mu(A) \cdot \mu(B)\end{aligned}$$

where $(*)$ holds because $S \cdot \gamma^{-1} \cap T = \varnothing$. Sketch for the general case: every measurable subset of $\{0,1\}^\Gamma$ can be approximated by finite union of cylinders. More precisely for every $\varepsilon > 0$ and for every $A, B \subseteq \{0,1\}^\Gamma$ measurable subsets there exist $\tilde{A}, \tilde{B} \subseteq \{0,1\}^\Gamma$ which are finite union of cylinders and such that $\mu(A \triangle \tilde{A}), \mu(B \triangle \tilde{B}) < \varepsilon$. □

**Corollary 2.2.6** (existence of ergodic actions). *Let $\Gamma$ be a countable group; there exist a probability space $(X, \mu)$ and an essentially free and ergodic probability preserving action $\Gamma \curvearrowright (X, \mu)$.*

*Proof.* If $\Gamma$ is finite, then we use $(X, \mu) = (\Gamma, \#/|\Gamma|)$ with the coset translation action $\Gamma \curvearrowright \Gamma$, which is essentially free and transitive. If $\Gamma$ is infinite, then we use the standard Bernoulli shift $\Gamma \curvearrowright \{0,1\}^\Gamma$. □

### 2.2.2. Ergodicity and invariant bounded functions

Let $(X, \mu)$ be a measure space; the $\infty$-norm (or essential supremum) of a real-valued measurable function $f : X \to \mathbb{R}$ is

$$\|f\|_\infty = \inf\{c \geq 0 : \mu(\{x \in X : |f(x)| > a\}) = 0\}$$

and the $\mathbb{R}$-vector space of essentially bounded functions $X \to \mathbb{R}$ is

$$L^\infty(X, \mu) = \{f \in \mathrm{Hom}_{\mathsf{Msrbl}}(X, \mathbb{R}) : \|f\|_\infty < +\infty\} / =_0$$

A left measure preserving action $\Gamma \curvearrowright (X, \mu)$ induces a left action on the space $L^\infty(X, \mu)$ via

$$\begin{aligned} \Gamma \times L^\infty(X,\mu) &\to L^\infty(X,\mu) \\ (\gamma, f) &\mapsto \gamma \cdot f \end{aligned}$$

where $(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x)$ for all $x \in X$. A useful characterization of ergodicity holds in terms of $\Gamma$-invariant essentially bounded functions.

**Proposition 2.2.7** (ergodicity via invariant functions). *Let $(X, \mu)$ be a probability space and let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving action; then the following are equivalent:*

1. *the action $\Gamma \curvearrowright (X, \mu)$ is ergodic;*
2. *if $A \subseteq X$ is measurable and $\mu(\gamma \cdot A \triangle A) = 0$ for every $\gamma \in \Gamma$, then $\mu(A) \in \{0, 1\}$;*
3. *if $f \in L^\infty(X, \mu)$ satisfies $\gamma \cdot f = f$ for all $\gamma \in \Gamma$ (namely $f$ belongs to $L^\infty(X, \mu)^\Gamma$), then $f$ is essentially constant;*
4. *the inclusion of constant functions $\mathbb{R} \to L^\infty(X, \mu)^\Gamma$ is an isomorphism of $\mathbb{R}$-vector spaces.*

*Proof.* Let $\Gamma \curvearrowright (X, \mu)$ be a probability preserving action.

($1 \Rightarrow 2$) Let $A \subseteq X$ be measurable and such that $\mu(\gamma \cdot A \triangle A) = 0$ for all $\gamma \in \Gamma$. Define $A' = \Gamma \cdot A \supseteq A$, so that $\Gamma \cdot A' = A'$. We have

$$\mu(A) \leq \mu(A') = \mu\left(A \cup \bigcup_{\gamma \in \Gamma} \gamma \cdot A \triangle A\right) \leq \mu(A) + \sum_{\gamma \in \Gamma} \mu(\gamma \cdot A \triangle A) = \mu(A)$$

which gives $\mu(A) = \mu(A') \in \{0, 1\}$, because $\Gamma \curvearrowright (X, \mu)$ is ergodic.

($2 \Rightarrow 3$) Let $f \in L^\infty(X, \mu)$ such that $\gamma \cdot f = f$ for all $\gamma \in \Gamma$. Take a measurable subset $Y \subseteq \mathbb{R}$ and define $A = f^{-1}(Y)$. For $\gamma \in \Gamma$ we have that

$$\gamma \cdot A \triangle A \subseteq \bigcup_{n=1}^{+\infty} \{x \in X : |f(x) - (\gamma \cdot f)(x)| > 1/n\}$$

and $\mu(\{x \in X : |f(x) - (\gamma \cdot f)(x) > 1/n|\}) = 0$ for all $n \in \mathbb{N}_{\geq 1}$, since $f =_0 \gamma \cdot f$, thus $\mu(\gamma \cdot A \triangle A) = 0$. This means that $\mu(f^{-1}(Y)) = \mu(A) \in \{0, 1\}$ for all $Y \subseteq \mathbb{R}$ measurable subsets.

($3 \Rightarrow 1$) If $A \subseteq X$ is $\Gamma$-invariant, then so is $\chi_A \in L^\infty(X, \mu)$. Thus $\chi_A$ is essentially constant and hence $\mu(A) = 0$ if $\chi_A =_0 0$ and $\mu(A) = 1$ if $\chi_A =_0 1$.

($3 \Leftrightarrow 4$) By definition $L^\infty(X, \mu)^\Gamma = \{f \in L^\infty(X, \mu) : \gamma \cdot f = f \text{ for all } \gamma \in \Gamma\}$. □

### 2.2.3. Spaces of measures

The set of finite (resp. probability, ergodic, ...) measures on a measurable space carries a geometric structure. For a measurable space $X$ we denote by:

1. $\mathrm{SMeas}(X)$ the $\mathbb{R}$-vector space of all finite signed measures on $X$;

2. $\mathrm{Prob}(X) \subseteq \mathrm{SMeas}(X)$ the subset of probability measures on $X$.

Let $\Gamma$ be a group. An action $\alpha : \Gamma \curvearrowright X$ by measurable automorphisms induces an action $\Gamma \curvearrowright \mathrm{Prob}(X)$ via

$$\begin{aligned} \Gamma \times \mathrm{Prob}(X) &\to \mathrm{Prob}(X) \\ (\gamma, \mu) &\mapsto \gamma \cdot \mu \end{aligned}$$

where $(\gamma \cdot \mu)(A) = \mu(\gamma^{-1} \cdot A)$ for all measurable subsets $A \subseteq X$. It is straightforward to see that the fixed space $\mathrm{Prob}(X)^\Gamma$ is the set of the probabilities $\mu$ on $X$ such that the action $\alpha : \Gamma \curvearrowright (X, \mu)$ is measure preserving. We denote by:

1. $\mathrm{Prob}(\alpha) \subseteq \mathrm{Prob}(X)$ the set of all $\Gamma$-invariant probability measures on $X$, i.e. the set of all probabilities $\mu$ on $X$ such that $\alpha : X \curvearrowright (X, \mu)$ is measure preserving;

2. $\mathrm{Erg}(\alpha) \subseteq \mathrm{Prob}(\alpha)$ the set of all ergodic $\Gamma$-invariant probability measures on $X$.

These sets carry some geometric properties, which we are investigating. If $V$ is a real vector space, recall that:

1. a subset $C \subseteq V$ is *convex* if $tu + (1-t)v \in C$ for all $u, v \in C$ and for all $t \in [0, 1]$;

2. $w$ is an *extreme point* of the convex subset $C \subseteq V$ if the following holds: if $w = tu + (1-t)v$ for some $u, v \in C$ and $t \in [0, 1]$, then either $w = u$ or $w = v$;

3. the *convex hull* (or convex envelope) of $A \subseteq V$ is the inclusion-wise smallest convex subset of $V$ containing $A$.

**Proposition 2.2.8** (ergodicity via extreme points, [22, Proposition 2.3.3]). *Let $\Gamma \curvearrowright (X, \mu)$ be a standard probability action; then*

1. $\mathrm{Prob}(\Gamma \curvearrowright X)$ *is a convex subset of* $\mathrm{SMeas}(X)$*;*

2. *the action $\Gamma \curvearrowright (X, \mu)$ is ergodic if, and only if, the probability measure $\mu$ is an extreme point of* $\mathrm{Prob}(\Gamma \curvearrowright X)$*.*

If $X$ is a standard Borel space, we denote by $\mathscr{C}(X)$ the space of continuous functions $f : X \to \mathbb{R}$. We endow $\mathscr{C}(X)$ with the supremum norm $\|\cdot\|_\infty$ to get a topological vector space and we denote by $\mathscr{C}(X)^\#$ the functional analytic dual space of $\mathscr{C}(X)$, i.e. the $\mathbb{R}$-vector space of bounded linear functionals $F : \mathscr{C}(X) \to \mathbb{R}$.

**Theorem 2.2.9** (Riesz representation theorem). *Let $X$ be a standard Borel space; the Reisz map on $X$*

$$\begin{aligned} \mathrm{Riesz}_X : \mathrm{SMeas}(X) &\to C(X)^{\#} \\ \mu &\mapsto \int_X (\cdot) d\mu \end{aligned}$$

*is a bijection.*

*Proof.* The proof is based on the classical Riesz representation Theorem, see e.g. [3, Theorem 7.17]. □

Riesz's representation theorem allows us to bring a topology on the space $\mathrm{SMeas}(X)$. We equip $\mathscr{C}(X)^{\#}$ with the topology of point-wise convergence, which means that a sequence $\{F_n\}_n \subseteq \mathscr{C}(X)^{\#}$ converges to $F \in \mathscr{C}(X)^{\#}$ if

$$\lim_{n\to+\infty} F_n(f) = F(f)$$

for all $f \in \mathscr{C}(X)$, which in turn means that

$$\lim_{n\to+\infty} F_n(f)(x) = F(f)(x)$$

for all $f \in \mathscr{C}(X)$ and for all $x \in X$. Then we equip $\mathrm{SMeas}(X)$ with the weak* topology induced by $\mathrm{Riesz}_X$, namely the weakest (coarsest) topology on $\mathrm{SMeas}(X)$ making $\mathrm{Riesz}_X : \mathrm{SMeas}(X) \to \mathscr{C}(X)^{\#}$ continuous. More explicitly a sequence of signed finite measures $\{\mu_n\}_n \subseteq \mathrm{SMeas}(X)$ converges to $\mu \in \mathrm{SMeas}(X)$ if, and only if,

$$\lim_{n\to+\infty} \int_X f d\mu_n = \int_X f d\mu$$

for all $f \in C(X)$. If $\Gamma \curvearrowright X$ is an action of a countable group $\Gamma$ by measurable automorphisms, then we can also equip $\mathrm{Prob}(X), \mathrm{Prob}(\Gamma \curvearrowright X)$ and $\mathrm{Erg}(\Gamma \curvearrowright X)$ with the subspace topology.

**Corollary 2.2.10** (compactness of the space of probability measures, [22, Corollary 2.3.7]). *Let $X$ be a standard Borel space; then $\mathrm{Prob}(X)$ is compact. Moreover if $\Gamma$ is a countable group and $\Gamma \curvearrowright X$ is an action by measurable automorphisms, then $\mathrm{Prob}(\Gamma \curvearrowright X)$ is compact.*

**Corollary 2.2.11** (existence of ergodic measures). *Let $X$ be a standard Borel space, $\Gamma$ a countable group and $\Gamma \curvearrowright X$ an action by measurable isomorphisms, then $\mathrm{Prob}(\Gamma \curvearrowright X)$ is the convex hull of $\mathrm{Erg}(\Gamma \curvearrowright X)$. In particular:*

1. *if $\mathrm{Prob}(\Gamma \curvearrowright X)$ is non-empty, then so is $\mathrm{Erg}(\Gamma \curvearrowright X)$;*
2. *if $|\mathrm{Erg}(\Gamma \curvearrowright X)| = 1$, then $|\mathrm{Prob}(\Gamma \curvearrowright X)| = 1$.*

*Proof.* Let $\alpha : \Gamma \curvearrowright X$ be a standard action. The space $\mathrm{Prob}(\alpha)$ is compact and convex in $\mathrm{SMeas}(X)$, thus it is the convex hull of the set of its extreme points [17, p. 88]. Moreover Proposition 2.2.8 gives that the set of extreme points of $\mathrm{Prob}(\alpha)$ coincides with $\mathrm{Erg}(\alpha)$. □

### 2.2.4. Ergodic decomposition

**Definition 2.2.12** (ergodic decomposition). Let $\Gamma$ be a group, $(X, \mu)$ a probability space and $\alpha : \Gamma \curvearrowright (X, \mu)$ a probability preserving action; an *ergodic decomposition* of $\alpha : \Gamma \curvearrowright (X, \mu)$ is a a map $\beta : X \to \mathrm{Erg}(\alpha)$ (we are writing $\beta_x$ for $\beta(x)$) satisfying the following properties:

1. if $A \subseteq X$ is measurable, then the evaluation map
$$\begin{aligned} \mathrm{ev}_A : X &\to [0,1] \\ x &\mapsto \beta_x(A) \end{aligned}$$
is measurable;
2. if $A \subseteq X$ is measurable, then $\mu(A) = \int_X \beta_x(A) d\mu(x)$;
3. if $\gamma \in \Gamma$ and $x \in X$, then $\beta_{\gamma \cdot x} = \beta_x$;
4. for every $\nu \in \mathrm{Erg}(\alpha)$ the set $X_\nu = \{x \in X : \beta_x = \nu\}$ is measurable and $\nu(X_\nu) = 1$.

**Example 2.2.13** (ergodic decomposition of the Bernoulli shift). Let $\Gamma$ be a countable group and let $\mu : P(\{0,1\}) \to [0,1]$ be the probability measure defined by $\mu(\{0\}) = \mu(\{1\}) = 1/2$.

1. If $\Gamma$ is infinite, then by Proposition 2.2.5 the standard Bernoulli shift $\Gamma \curvearrowright (\{0,1\}^\Gamma, \mu^{\otimes\Gamma})$ is ergodic and thus an ergodic decomposition is given by $\beta_x = \mu^{\otimes\Gamma}$ for all $x \in \{0,1\}^\Gamma$.
2. Let us find an ergodic decomposition when $\Gamma = \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/2\mathbb{Z}$-orbits are $O_1 = \{(0,0)\}$, $O_2 = \{(1,1)\}$ and $O_3 = \{(1,0),(0,1)\}$. Let $\mu_i : P(\{0,1\}^{\mathbb{Z}/2\mathbb{Z}}) \to [0,1]$ be defined by
$$\mu_i = \begin{cases} \#/|O_i| & \text{on } O_i \\ 0 & \text{on } \{0,1\}^{\mathbb{Z}/2\mathbb{Z}} \setminus O_i \end{cases}$$
for $i = 1,2,3$. Then the actions $\mathbb{Z}/2\mathbb{Z} \curvearrowright (\{0,1\}^{\mathbb{Z}/2\mathbb{Z}}, \mu_i)$ are ergodic and $\beta : \{0,1\}^{\mathbb{Z}/2\mathbb{Z}} \to \mathrm{Erg}(\mathbb{Z}/2\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}/2\mathbb{Z}})$ defined by $\beta = \sum_{i=1}^3 \mu_i \cdot \chi_{O_i}$ is an ergodic decomposition.

Standard $\Gamma$-spaces admit an universal ergodic decomposition, i.e. a function $\beta : X \to \mathrm{Erg}(\alpha)$, which is an ergodic decomposition of every $\Gamma$-invariant probability measure on $X$.

**Theorem 2.2.14** (Ergodic decomposition theorem). *Let $\Gamma$ be a group, $X$ be a standard Borel space and $\alpha : \Gamma \curvearrowright X$ be an action by automorphisms in* Meas*. Suppose moreover that* $\mathrm{Prob}(\alpha) \neq \varnothing$ *(i.e. $X$ admits at least a $\Gamma$-invariant probability measure). Then there exists a map $\beta : X \to \mathrm{Erg}(\alpha)$, which is an ergodic decomposition for all of the $\Gamma$-invariant probability measures in* $\mathrm{Prob}(\alpha)$.

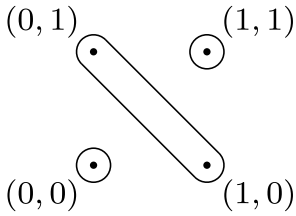


Figure 2.1.: Ergodic decomposition of $\mathbb{Z}/2\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}/2\mathbb{Z}}$

The proof, which we are not presenting, relies on results of functional analysis and probability theory. A sketch of it can be found in [22, Subsection 2.3.3].

**Remark 2.2.15.** The ergodic decomposition theorem often allows us to reduce problems for general dynamical systems to the ergodic ones.

# 3. Cost

When a group acts on a space, it identifies an equivalence relation, namely the one of belonging to the same orbit. In particular when a countable group acts on a probability space in a measure preserving way, we can talk about the cost of the equivalence relation, which is meant to encode the "amount of information" needed to build the relation. This datum was first introduced by Levitt ([19]) and further developed by Gaboriau. One can have an accurate overview of the topic in Gaboriau's paper [6].

## 3.1. Measured equivalence relations

In the measure-theoretic environment one has to consider only those equivalence relations that are compatible with the measurable space structure and that are (in a suitable sense) invariant with respect to the measure.

### 3.1.1. Standard equivalence relations

**Definition 3.1.1** (Standard equivalence relation)**.** Let $X$ be a standard Borel space; an equivalence relation $R \subseteq X \times X$ is said to be a *standard equivalence relation* if:

1. every equivalence class of $R$ is countable;

2. $R$ is measurable in $X \times X$.

**Terminology 3.1.2** (orbit, restriction and subrelation)**.** There is a particular terminology for standard equivalence relations miming the one of group actions. Let $R$ be a standard equivalence relation in $X$:

1. if $x \in X$, then the *$R$-orbit* of $x$ is the equivalence class

   $$R \cdot x = \{y \in X : (x, y) \in X\}$$

   of $x$. For a measurable subset $A \subseteq X$ we define $R \cdot A = \cup\{R \cdot a : a \in A\} \subseteq X$;

2. if $A \subseteq X$ is a measurable subset, then the *restriction* of $R$ to $A$ is the relation $R|_A = R \cap (A \times A)$, which is a standard equivalence relation in $A$;

3. a standard equivalence relation $R'$ on $X$ is a *subrelation* of $R$ if $R' \subseteq R$.

We have already seen an example of standard equivalence relation in Proposition 2.1.17. By the theorem of Feldman-Moore every standard equivalence relation is of this form.

**Example 3.1.3** (Orbit relation). Let $\Gamma$ be a countable group, $(X, \mu)$ a standard Borel space and $\Gamma \curvearrowright (X, \mu)$ an action by measurable isomorphisms. The of the subset

$$R_{\Gamma \curvearrowright X} = \{(x, \gamma \cdot x) : x \in X, \gamma \in \Gamma\}$$

is a standard equivalence relation. Notice that $R_{\Gamma \curvearrowright X} \cdot x = \Gamma \cdot x$ for all $x \in X$, justifying the notation for the $R$-orbits.

A standard equivalence relation $R$ on $X$ is said to be *aperiodic* if all of the $R$-orbits are infinite.

### 3.1.2. The Feldman-Moore theorem

Every standard equivalence relation is essentially an orbit equivalence relation.

**Theorem 3.1.4** (Feldman-Moore, [22, Theorem 3.1.6]). *Let $X$ be a standard Borel space and let $R \subseteq X^2$ be a standard equivalence relation; then there exists a countable group $\Gamma$ and an action $\alpha : \Gamma \curvearrowright X$ by measurable automorphisms such that*

$$R = R_\alpha$$

*Sketch of proof.* Let $\pi : R \to X$ be the projection on the first component.

1. $\pi$ is measurable and $\pi^{-1}(x) = R \cdot x$ for all $x \in X$. There are $\{X_n\}_{n \in \mathbb{N}} \subseteq P(X)$ measurable subsets and $\{f_n : X_n \to X\}_{n \in \mathbb{N}}$ measurable functions such that $R = \cup\{F_n : n \in \mathbb{N}\}$ for $F_n = \mathrm{Graph}(f_n)$ and such that the restrictions $\pi|_{F_n} : F_n \to X$ are injective [27, Theorem 1.3]. By inductive elimination one can always assume that $F_m \cap F_n = \varnothing$ for all $m \neq n$.

2. Let $F_n^{-1} = \{(y, x) : (x, y) \in F_n\}$ and $F_{m,n} = F_m \cap F_n^{-1}$. Note that if $F_m = \mathrm{Graph}(f_m : A_m \to X)$ and $F_n = \mathrm{Graph}(f_n : A_n \to X)$, then $F_{m,n}$ is the graph of the restriction $f_m|_{X_{m,n}} : X_{m,n} \to X_{n,m}$ (here $X_{m,n} = \{x \in X_m : x = (f_n \circ f_m)(x)\}$) which is a measurable isomorphism.

3. Since $X$ is a standard Borel space, there exist families $\{A_k\}_{k \in \mathbb{N}}$ and $\{B_k\}_{k \in \mathbb{N}}$ of measurable subset such that $X^2 \setminus \Delta_X = \cup\{A_k \times B_k : k \in \mathbb{N}\}$. In particular $A_k \cap B_k = \varnothing$ for all $k \in \mathbb{N}$. Define $F_{m,n,k} = F_{m,n} \cap (A_k \times B_k)$ for $m, n, k \in \mathbb{N}$ so that $F_{m,n,k}$ is the graph of a measurable isomorphism $f_{m,n,k} : D_{m,n,k} \to R_{m,n,k}$ with $D_{m,n,k} \cap R_{m,n,k} = \varnothing$. We can construct global measurable automorphisms $h_{m,n,k} : X \to X$ via

$$h_{m,n,k}(x) = \begin{cases} f_{m,n,k}(x) & \text{if } x \in D_{m,n,k} \\ f_{m,n,k}^{-1}(x) & \text{if } x \in R_{m,n,k} \\ x & \text{otherwise} \end{cases}$$

and by construction $R = \Delta_X \cup \bigcup_{m,n,k \in \mathbb{N}} \mathrm{Graph}(h_{m,n,k})$. If $\Gamma \leq \mathrm{Aut}_{\mathsf{Meas}}(X)$ is the subgroup generated by $\{h_{m,n,k} : m, n, k \in \mathbb{N}\}$, then $\Gamma$ is countable and $R = R_{\Gamma \curvearrowright X}$. $\square$

**Warning 3.1.5.** In most situations standard equivalence relations do not arise canonically from an orbit relation via the Feldman-Moore construction. For instance the notions of stabilizers and essentially free actions have no natural counterpart for standard equivalence relations.

### 3.1.3. Measured equivalence relations

We want to translate the notion of measure preserving group action in the setting of standard equivalence relations.

**Definition 3.1.6** (full group). Let $X$ be a standard Borel space and let $R$ be a standard equivalence relation on $X$.

1. The *full group* of $R$ is
$$[R] = \{f \in \mathrm{Aut}_{\mathsf{Meas}}(X) : (x, f(x)) \in R \text{ for all } x \in X\}$$
Composition of morphisms gives $[R]$ a group structure.
2. A *partial $R$-automorphism* of $X$ is a measurable isomorphism $f : A \to B$ for some measurable subsets $A, B \subseteq X$ such that $(a, f(a)) \in R$ for all $a \in A$.
3. The *full groupoid* of $R$ is the set $[\![R]\!]$ of all partial $R$-automorphisms of $X$. Composition of morphisms gives $[\![R]\!]$ a groupoid structure (i.e. a set with a partial binary operation satisfying the group axioms).

**Definition 3.1.7** (measured equivalence relation). Let $X$ be a standard Borel space; a *measured equivalence relation* on $X$ is a pair $(R, \mu)$, where:

1. $R \subseteq X^2$ is a standard equivalence relation on $X$;
2. $\mu$ is an $R$-invariant measure on $X$, which means that every $f \in [R]$ is $\mu$-preserving.

**Proposition 3.1.8** (characterisation of measured equivalence relations). *Let $X$ be a standard Borel space, let $\Gamma \curvearrowright X$ be an action by measurable isomorphisms and let $\mu$ be a measure on $X$; the following are equivalent:*

1. *the measure $\mu$ is $R_{\Gamma \curvearrowright X}$-invariant;*
2. *the measure $\mu$ is invariant with respect to the action $\Gamma \curvearrowright X$;*
3. *if $f : A \to B$ is a partial $R_{\Gamma \curvearrowright X}$-automorphism of $X$, then $\mu(A) = \mu(B)$.*

*Proof.*

$(1 \Rightarrow 2)$ For every measurable subset $A \subseteq X$ and every $\gamma \in \Gamma$, the function $\gamma\cdot : A \to A$ is an $R_{\Gamma \curvearrowright X}$-partial automorphism of $X$, hence measure preserving.

(2 ⇒ 3) Let $f : A \to B$ be a partial $R_{\Gamma \curvearrowright X}$-automorphism of $X$. $f$ acts by $\gamma \in \Gamma$ on the measurable subset $A_\gamma = \{a \in A : f(a) = \gamma \cdot a\}$ and we have that $A = \cup\{A_\gamma : \gamma \in \Gamma\}$ and $B = \cup\{f(A_\gamma) : \gamma \in \Gamma\}$. With inductive elimination we can extract measurable subsets $A'_\gamma \subseteq A_\gamma$ for every $\gamma \in \Gamma$ such that $A = \sqcup\{A'_\gamma : \gamma \in \Gamma\}$ and $B = \sqcup\{f(A'_\gamma) : \gamma \in \Gamma\}$. Since $\Gamma$ is countable and $\Gamma \curvearrowright X$ is $\mu$-preserving, we get

$$\mu(A) = \sum_{\gamma \in \Gamma} \mu(A'_\gamma) = \sum_{\gamma \in \Gamma} \mu(\gamma \cdot A'_\gamma) = \mu(B)$$

(3 ⇒ 1) If $f \in [R_{\Gamma \curvearrowright X}]$ and $A \subseteq X$ is measurable, then the restriction $f|_{f^{-1}(A)} : f^{-1}(A) \to A$ is a partial $R$-automorphism of $X$ and hence $f_*\mu(A) = \mu(f^{-1}(A)) = \mu(A)$. Thus $\mu$ is $R_{\Gamma \curvearrowright X}$-invariant. □

**Example 3.1.9** (orbit relations as measured equivalence relations)**.** Let $\Gamma \curvearrowright (X, \mu)$ be a standard action. The pair $(R_{X \curvearrowright X}, \mu)$ is a measured equivalence relation on $X$.

**Definition 3.1.10** (morphisms of measured equivalence relations)**.** Let $X$ and $Y$ be standard Borel spaces and let $(R, \mu)$ and $(S, \nu)$ be measured equivalence relations on $X$ and $Y$ respectively; a *morphism of measured equivalence relations* $f : (R, \mu) \to (S, \nu)$ is a measure preserving map $f : X' \to Y$ such that:

1. $X' \subseteq X$ is measurable and $\mu(X \setminus X') = 0$, i.e. $f$ is defined almost everywhere on $X$;
2. if $(a, b) \in R \cap (X' \times X')$, then $(f(a), f(b)) \in S$.

A morphism $f : (R, \mu) \to (S, \nu)$ of measured equivalence relations is an isomorphism if there exists a morphism of measured equivalence relations $g : (S, \nu) \to (R, \mu)$ such that

$$g \circ f =_0 \mathrm{id}_X \text{ and } f \circ g =_0 \mathrm{id}_Y$$

The notion of ergodicity has a counterpart for measured equivalence relations.

**Definition 3.1.11** (ergodic measured equivalence relation)**.** Let $X$ be a standard Borel space; a measured equivalence relation $(R, \mu)$ on $X$ is *ergodic* if

$$\mu(A) = 0 \text{ or } \mu(X \setminus A) = 0$$

for all $A \subseteq X$ measurable subset such that $R \cdot A = A$.

**Proposition 3.1.12** (ergodicity of orbit relations)**.** *The following are equivalent for a standard action $\Gamma \curvearrowright (X, \mu)$:*

1. *the action $\Gamma \curvearrowright (X, \mu)$ is ergodic;*
2. *the measured equivalence orbit relation $R_{\Gamma \curvearrowright (X,\mu)}$ is ergodic.*

*Proof.* $R_{\Gamma \curvearrowright X} \cdot A = \Gamma \cdot A$ holds for every measurable subset $A \subseteq X$, so the notions of ergodicity for dynamical systems and measured equivalence relations coincide. □

## 3.2. Cost of measured equivalence relations

The cost of a measured equivalence relation is the measure-theoretic counterpart of the size of a minimal generating subset. When the relation is the orbit relation of a measure-preserving group action, then its cost is also connected with the group rank.

### 3.2.1. Generating equivalence relations

Graphings are the generating sets of standard equivalence relations.

**Definition 3.2.1** (graphings). Let $X$ be a standard Borel space and let $R \subseteq X \times X$ be a standard equivalence relation.

1. Let $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I} \subseteq [\![R]\!]$ a family of partial $R$-automorphisms of $X$, which means that $\mathrm{Graph}(\varphi_i) \subseteq R$ for all $i \in I$; the *equivalence relation generated by* $\Phi$ is the inclusion-wise minimal equivalence relation $\langle \Phi \rangle \subseteq X \times X$ such that $\mathrm{Graph}(\varphi_i) \subseteq \langle \Phi \rangle$ for all $i \in I$.

2. A *graphing* of $R$ is a countable family $\Phi \subseteq [\![R]\!]$ of partial $R$-automorphisms of $X$ generating $R$, i.e. such that $\langle \Phi \rangle = R$.

3. A family $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I} \subseteq [\![R]\!]$ is *reduced* if for all $i \in I$ we have that:
   a) $\varphi_i(x) \neq x$ for all $x \in A_i$;
   b) if $j \in I \smallsetminus \{i\}$ and $x \in A_i \cap A_j$, then $\varphi_i(x) \neq \varphi_j(x)$;
   c) if $j \in I \smallsetminus \{i\}$ and $x \in A_i \cap B_j$, then $\varphi_i(x) \neq \varphi_j^{-1}(x)$.

**Example 3.2.2** (orbit equivalence relation). Let $X$ be a standard Borel space and let $\alpha : \Gamma \curvearrowright X$ be an action by measurable automorphisms. If $S \subseteq \Gamma$ is a generating set, then

$$\Phi_S = \{s \cdot\ : X \to X\}_{s \in S}$$

is a graphing for $R_\alpha$.

This example combined with Feldman-Moore theorem shows that every standard equivalence relation admits a graphing.

The notion of generating a standard equivalence relation can be seen under a more geometric point of view. Let $X$ be a standard Borel space and $R$ a standard equivalence relation on $X$; the (simple undirected) graph $G(\Phi) = (V, E)$ associated to the family $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I} \subseteq [\![R]\!]$ is is the graph with:

1. set of vertices $V = X$;

2. set of edges $E = \{\{x, \varphi_i(x)\} : x \neq \varphi_i(x), x \in A_i, i \in I\}$.

By the construction of $G(\Phi)$ the following conditions are equivalent for a family $\Phi \subseteq [\![R]\!]$:

1. $\Phi$ is a graphing of $R$;

2. the connected components of the graph $G(\Phi)$ are exactly the equivalence classes of $R$.

A simple undirected graph $G = (V, E)$ is a tree if every two vertices of $G$ are connected by a unique path. We say that a graphing $\Phi \subseteq [\![R]\!]$ of $R$ is a *treeing* of $R$ if all of the connected components of $G(\Phi)$ are trees.

### 3.2.2. Cost of measured equivalence relations

The cost encodes measure-wise the amount of information needed to generate a measured equivalence relation.

**Definition 3.2.3** (cost of a measured equivalence relation). Let $X$ be a standard Borel space and let $(R, \mu)$ be a measured equivalence relation on $X$;

1. if $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I} \subseteq [\![R]\!]$ is a graphing of $R$, then we define the $\mu$-*cost* (or just the cost, if the measure is clear) of $\Phi$ as
$$\mathrm{cost}_\mu(\Phi) = \sum_{i \in I} \mu(A_i) \in [0, +\infty]$$
2. the *cost* of the measured equivalence relation $R$ is the infimum
$$\mathrm{cost}_\mu(R) = \inf\{\mathrm{cost}_\mu(\Phi) : \Phi \text{ is a graphing of } R\}$$

**Example 3.2.4** (identity relation). The identity relation $\Delta_X = \{(x, x) : x \in X\}$ on a standard measure space $(X, \mu)$ is generated by the graphing $\varnothing$, hence $\mathrm{cost}_\mu \Delta_X = 0$.

**Example 3.2.5** (orbit relation). Let $\Gamma$ be a countable group generated by $S$, let $X$ be a standard Borel space and let $\Gamma \curvearrowright X$ be an action by measurable automorphisms. The family
$$\{s \cdot : X \to X\}_{s \in S}$$
is a graphing for the orbit relation $R_{\Gamma \curvearrowright X}$. In particular we obtain that
$$\mathrm{cost}_\mu(R_{\Gamma \curvearrowright X}) \leq |S|$$
and by taking the infimum among all of the generating subsets of $\Gamma$ we get that
$$\mathrm{cost}_\mu(R_{\Gamma \curvearrowright X}) \leq d(\Gamma)$$

**Proposition 3.2.6** (cost is attained by treeings, [22, Proposition 3.2.11]). *Let $(R, \mu)$ be a measured equivalence relation on a standard Borel space $X$ with finite cost and let $\Phi \subseteq [\![R]\!]$ be a graphing of $R$ that is not a treeing, modulo $\mu$-null sets; then there exists a treeing $\Psi \subseteq [\![R]\!]$ such that*
$$\mathrm{cost}_\mu(\Psi) < \mathrm{cost}_\mu(\Phi)$$
*In particular, if $\Phi$ is a graphing of $R$ such that $\mathrm{cost}_\mu(\Phi) = \mathrm{cost}_\mu(R)$, then $\Phi$ is a treeing.*

**Definition 3.2.7.** Let $X$ be a standard Borel space and let $R$ be a standard equivalence relation on $X$;

1. a measurable subset $A \subseteq X$ is a *section* of $R$ if every orbit of $R$ intersects $A$, namely $A \cap (R \cdot x) \neq \varnothing$ for all $x \in X$;
2. a measurable subset $A \subseteq X$ is a *fundamental domain* of $R$ if every orbit of $R$ intersects $A$ in exactly one point, namely $|A \cap (R \cdot x)| = 1$ for all $x \in X$;
3. the standard equivalence relation $R$ is said to be *smooth* if it admits a fundamental domain.

**Lemma 3.2.8** ([19, Proof of Proposition 1]). *Let $\Gamma \curvearrowright (X, \mu)$ be a standard probability action and suppose that every orbit of $\Gamma$ contains exactly $q \in \mathbb{N}$ points; then there exists a family $\{A_n\}_{n \in \mathbb{N}}$ of measurable subsets of $X$ such that:*

1. *for all $x \in X$ there is exactly one $n \in \mathbb{N}$ such that $\Gamma \cdot x \cap A_n \neq \varnothing$;*
2. *for every $i \in \mathbb{N}$ there exists $q-1$ elements $g_1^i, \dots, g_{q-1}^i \in \Gamma$ such that the sets $A_i, g_1^i \cdot A_i, \dots, g_{q-1}^i \cdot A_i$ are pairwise disjoint.*

In the case $\Gamma$ is a finite group, the set $A = \bigcup_{n \in \mathbb{N}} A_n$ is a fundamental domain for the orbit equivalence relation $R_{\Gamma \curvearrowright X}$ and $\mu(A) = 1/q$. When the cardinality of the orbits of $\Gamma$ in $X$ is not constant, a similar argument shows that $R_{\Gamma \curvearrowright X}$ admits a fundamental domain as well.

**Proposition 3.2.9** (cost of sections, [22, Proposition 3.2.13]). *Let $X$ be a standard Borel space and $(R, \mu)$ be a measured equivalence relation on $X$; if $A \subseteq X$ is a section of $R$, then*

$$\operatorname{cost}_\mu R = \operatorname{cost}_{\mu|_A}(R|_A) + \mu(X \setminus A)$$

**Corollary 3.2.10** (lower bound for aperiodic relations). *Let $X$ be a standard Borel space and let $(R, \mu)$ be a measured equivalence relation on $X$ such that $\mu(X) < +\infty$; then*

$$\operatorname{cost}_\mu R \geq \mu(X)$$

*Proof.* Let $\Phi \subseteq [\![R]\!]$ be a graphing. Since $R$ is an aperiodic standard equivalence relation, then $R$ admits a sequence $(A_n)_{n \in \mathbb{N}}$ of measurable subsets of $X$ such that

1. the sequence is decreasing: $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$;
2. the sequence vanishes asymptotically: $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$;
3. $A_n \cap R \cdot x \neq \varnothing$ for every $n \in \mathbb{N}$ and for every $x \in X$.

Such a sequence is called a *vanishing sequence of markers* for $R$ and its existence is granted by the Marker lemma [14, Lemma 6.7]. Each marker $A_n$ is a section for $R$ and hence Proposition 3.2.9 yields

$$\begin{aligned}
\operatorname{cost}_\mu \Phi &\geq \sup_{n\in\mathbb{N}} \Big\{ \operatorname{cost}_{\mu|_{A_n}}(R|_{A_n}) + \mu(X \smallsetminus A_n) \Big\} \\
&= \sup_{n\in\mathbb{N}} \{ \operatorname{cost}_{\mu|_{A_n}}(R|_{A_n}) + \mu(X) - \mu(A_n) \} && (\text{because } \mu(X) < +\infty) \\
&= \sup_{n\in\mathbb{N}} \{ \mu(X) - \mu(A_n) \} && (\text{because } \textstyle\bigcap_{n\in\mathbb{N}} A_n = \varnothing) \\
&= \mu(X) && (\text{continuity form above})
\end{aligned}$$

Then taking the infimum among all graphings gives $\operatorname{cost}_\mu R \geq \mu(X)$. □

**Proposition 3.2.11** (cost of smooth relations). *Let $X$ be a standard Borel space and let $(R, \mu)$ be a smooth measured equivalence relation on $X$ with fundamental domain $A \subseteq X$ and finite measure $\mu(X) < +\infty$; then*

$$\operatorname{cost}_\mu R = \mu(X \smallsetminus A)$$

*Proof.* Since $A$ is a fundamental domain for $R$, the restricted equivalence relation $R|_A = R \cap (A \times A)$ is the identity relation $\Delta_A$ and hence Proposition 3.2.9 gives

$$\operatorname{cost}_\mu R = \operatorname{cost}_{\mu|_A}(\Delta_A) + \mu(X \smallsetminus A) = \mu(X \smallsetminus A)$$ □

Putting together Lemma 3.2.8 and Proposition 3.2.11 we have that if $\alpha : \Gamma \curvearrowright (X, \mu)$ is a standard action, then the orbit relation $R_\alpha$ admits a fundamental domain $A \subseteq X$ and

$$\operatorname{cost}_\mu R_\alpha = \mu(X \smallsetminus A) = 1 - \mu(A)$$

### 3.2.3. Cost of groups

We use the notion of cost for a measured equivalence relation to introduce a measure-theoretic version of the group rank.

**Definition 3.2.12** (cost of groups). Let $\Gamma$ be a countable group; the *cost* of the group $\Gamma$ is

$$\begin{aligned}
\operatorname{cost}(\Gamma) = \inf\{ \operatorname{cost}_\mu(R_\alpha) : \alpha : \Gamma \curvearrowright (X, \mu) \\
\text{is an essentially free standard probability action}\} \in [0, +\infty]
\end{aligned}$$

Notice that the definition of cost is well posed, since, if $\Gamma$ is a countable group, then there always exist a measure space $(X, \mu)$ and a free measure preserving action $\Gamma \curvearrowright (X, \mu)$.

**Remark 3.2.13.** For a fixed group $\Gamma$ the family of all isomorphism classes of standard probability actions $\Gamma \curvearrowright (X, \mu)$ is a proper set. A quite rough estimate would work as follows: from Lemma 1.1.5 we get that every Polish space has at most the cardinality of the continuum, hence is in a bijection with $\mathbb{R}$. The family of all possible $\sigma$-algebras over

$\mathbb{R}$ is a subset of $PP(\mathbb{R})$, and hence has cardinality bounded by $|2^{2^{\mathbb{R}}}|$. If $S$ is a $\sigma$-algebra on $\mathbb{R}$, then there are at most $|[0,1]^S|$ many probability measures on the measurable space $(\mathbb{R}, S)$ and there are at most $|\mathbb{R}^{\Gamma\times\mathbb{R}}|$ many actions $\Gamma \curvearrowright \mathbb{R}$. Of course this estimate is very coarse and could be refined, for instance by recalling that atom-free standard Borel probability spaces are measurably isomorphic with $([0,1], m)$ (Theorem 1.1.18).

**Terminology 3.2.14** (cheap groups and fixed price)**.** We say that the group $\Gamma$:

1. is *cheap* if $\Gamma$ is infinite and $\operatorname{cost}\Gamma = 1$;
2. has *fixed price* if $\operatorname{cost}\Gamma = \operatorname{cost}(R_{\Gamma\curvearrowright(X,\mu)})$ holds for all $\Gamma \curvearrowright (X,\mu)$ essentially free standard probability actions.

**Question 3.2.15** (fixed price problem)**.** It is still an open question, whether there exists or not a countable group not of fixed price [5, Question 1.8].

**Example 3.2.16** (cost of finite groups)**.** Let $\Gamma \curvearrowright (X,\mu)$ be a standard action by a finite group. When the action $\Gamma \curvearrowright (X,\mu)$ is free, one can find a fundamental domain $A \subseteq X$ for $R_{\Gamma\curvearrowright X}$ constructed as in Lemma 3.2.8 of measure $1/|\Gamma|$ and hence $\operatorname{cost}\Gamma = 1 - 1/|\Gamma|$.

**Example 3.2.17** (cost of $\mathbb{Z}$)**.** The group $\mathbb{Z}$ is cheap of fixed price. Let $\alpha : \mathbb{Z} \curvearrowright (X,\mu)$ be an essentially free standard probability action. On one hand $\operatorname{cost}_\mu R_\alpha \leq 1 = d(\mathbb{Z})$. On the other hand $\alpha : \mathbb{Z} \curvearrowright (X,\mu)$ is aperiodic because essentially free, hence $\operatorname{cost}_\mu R_\alpha \geq \mu(X) = 1$ by Corollary 3.2.10.

**Example 3.2.18** (cost of free groups, [5, Corollary 1])**.** The free group of rank $k$ has fixed price and cost $k$.

## 3.3. Cost and ergodic decomposition

A decomposition formula holds for the cost of measured equivalence relations with respect to ergodic decompositions. In order to prove this formula, we first need to show that countably many graphings suffice to compute the cost.

We need to introduce some notation. Let $\Gamma$ be a finitely generated group and let us choose a generating subset $S \subseteq \Gamma$ of finite cardinality. Let $\Gamma \curvearrowright (X, \mathscr{B}_X, \mu)$ be a standard probability action. Let us fix:

1. an enumeration $(\gamma_n)_{n\in\mathbb{N}}$ of $\Gamma$;
2. a countable algebra $\{A_n : n \in \mathbb{N}\} \subseteq \mathscr{B}_X$ of measurable subsets that is dense in the $\sigma$-algebra $\mathscr{B}_X$.

Let us clarify condition 2: $\{A_n : n \in \mathbb{N}\}$ is dense in $\mathscr{B}_X$ if for every $B \in \mathscr{B}_X$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\mu(B \triangle A_n) < \varepsilon$. For instance one could take the algebra generated by a countable basis for the topology on $X$.

For $I \subseteq \mathbb{N}\times\mathbb{N}$, $M \in \mathbb{N}$ and $\gamma \in \Gamma$ we introduce the notation:

1. $\Theta_I = \{\gamma_n \cdot \ : A_m \to X : (m,n) \in I\}$;

2. $\Theta_{I,M} = \{\vartheta_1^{\varepsilon_1} \circ \cdots \circ \vartheta_k^{\varepsilon_k} : 0 \leq k \leq M, \vartheta_1, \cdots, \vartheta_k \in \Theta_I, \varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}\}$;

3. $D(\gamma, I, M) = \{x \in X : \gamma \cdot x \neq \vartheta(x) \text{ for all } \vartheta \in \Theta_{I,M}\}$;

4. $\Phi_{I,M} = \Theta_{I,M} \sqcup \{s \cdot |_{D(s,I,M)} : D(s,I,M) \to X : s \in S\}$.

**Remark 3.3.1** (cost estimate for $\Phi_{I,M}$). The construction of $\Phi_{I,M}$ gives a nice explicit estimate for $\operatorname{cost}_\mu(\Phi_{I,M})$. Indeed

$$\begin{aligned}\operatorname{cost}_\mu \Phi_{I,M} &= \operatorname{cost}_\mu(\Theta_I \sqcup \{(s \cdot |_{D(s,I,M)}) : s \in S\}) = \\ &= \operatorname{cost}_\mu \Theta_I + \sum_{s \in S} \operatorname{cost}_\mu(s \cdot |_{D(s,I,M)}) = \\ &= \sum_{(m,n) \in I} \mu(A_m) + \sum_{s \in S} \mu(D(s,I,M)) \leq |I| + |S|\end{aligned}$$

**Proposition 3.3.2** ([14, Proposition 18.3]). *Let $\Gamma$ be a countable group and let $\alpha : \Gamma \curvearrowright X$ be a measurable action on a standard Borel space; then the set*

$$\{\mu \in \operatorname{Prob}(\alpha) : \operatorname{cost}_\mu R_\alpha < +\infty\}$$

*is analytic (continuous image of a Polish space) and the map*

$$\begin{aligned}\{\mu \in \operatorname{Prob}(\alpha) : \operatorname{cost}_\mu R_\alpha < +\infty\} &\to [0, +\infty[ \\ \mu &\mapsto \operatorname{cost}_\mu R_\alpha\end{aligned}$$

*is measurable.*

**Lemma 3.3.3** (finitary characterisation of cost). *With the above notation the following are equivalent for $c > 0$:*

1. $\operatorname{cost}_\mu R_\alpha < c$;

2. *there exist a finite subset $I \subseteq \mathbb{N} \times \mathbb{N}$ and $M \in \mathbb{N}$ such that $\operatorname{cost}_\mu(\Phi_{I,M}) < c$.*

*Proof.* First of all we have to show that the $\Phi_{I,M}$'s are indeed graphings for $R_\alpha$. Let us take $(x, \gamma \cdot x) \in R_\alpha$ with $x \in X$ and $\gamma \in \Gamma$. If $\gamma \cdot x = \vartheta(x)$ for some $\vartheta \in \Theta_{I,M}$, then $(x, \gamma \cdot x) \in \operatorname{Graph}(\vartheta)$, whereas if $\gamma \cdot x \neq \vartheta(x)$ for every $\vartheta \in \Theta_{I,M}$, then $(x, \gamma \cdot x)$ belongs to the equivalence relation generated by $\{s \cdot \ : D(s,I,M) \to X : s \in S\}$, because $S$ a generating set of $\Gamma$.

The implication ($2 \implies 1$) is trivial, so we are only proving ($1 \implies 2$). Let us fix $0 < \varepsilon < (c - \operatorname{cost}_\mu R_\alpha)/(1 + |S|)$. Then we can find $I \subseteq \mathbb{N} \times \mathbb{N}$ such that [14, Poposition 18.1]:

1. there exists $Y \subseteq X$ such that $Y^c$ is $\mu$-null and $\langle \Theta_I|_Y \rangle = R_\alpha|_Y$;

2. $\text{cost}_\mu \Theta_I \leq \text{cost}_\mu R_\alpha + \varepsilon$.

Since in general $I \subseteq \mathbb{N} \times \mathbb{N}$ can be an infinite set, let us set $I_n = I \cap \{0, \ldots, n\}^2$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ and every $s \in S$ let us set

$$E(s, I, n) = \{x \in X : (x, s \cdot x) \notin \langle \Theta_{I_n} \rangle\}$$

Since $E(s, I, n) \supseteq E(s, I, n+1)$ for every $n \in \mathbb{N}$ and $\langle \Theta_I \rangle|_Y = R_\alpha|_Y$, we have that

$$\lim_{n \to +\infty} \mu(E(s, I, n)) = \mu(\cap\{E(s, I, n) : n \in \mathbb{N}\}) = 0$$

and hence we can find $n \in \mathbb{N}$ such that $\mu(E(s, I, n)) < \varepsilon$ for every $s \in S$. Moreover, since $E(s, I, n) = \cap\{D(s, I_n, M) : M \in \mathbb{N}\}$, we can also find an $M \in \mathbb{N}$ such that $\mu(D(s, I_n, M)) < \varepsilon$ for every $s \in S$. Then

$$\begin{aligned}
\text{cost}_\mu(\Phi_{I_n, M}) &= \text{cost}_\mu(\Theta_{I_n}) + \text{cost}_\mu(\{s \cdot\ : D(s, I_n, M) \to X : s \in S\}) \\
&\leq \text{cost}_\mu(\Theta_{I_n}) + \varepsilon \cdot |S| \\
&\leq \text{cost}_\mu R_\alpha + \varepsilon \cdot (1 + |S|) \\
&< c
\end{aligned}$$

proves the claim. $\square$

**Theorem 3.3.4** (cost and ergodic decomposition). *Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard probability action and let $\beta : X \to \text{Erg}(\alpha)$ be an ergodic decomposition of the action; then*

$$\text{cost}_\mu(R_\alpha) = \int_X \text{cost}_{\beta_x}(R_\alpha) d\mu(x)$$

*Proof.* Let us only consider the case of $\Gamma$ a countable group with finite set of generators $S \subseteq \Gamma$. Moreover we assume that the action $\alpha : \Gamma \curvearrowright X$ is essentially free. From Proposition 3.3.2 we get that the function

$$\begin{aligned}
X &\to [0, +\infty] \\
x &\mapsto \text{cost}_{\beta_x}(R_\alpha)
\end{aligned}$$

is measurable and bounded by $|S|$ (Example 3.2.5), hence integrable. Let us prove the claimed equality via two inequalities.

($\geq$) For every graphing $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I}$ of $R_\alpha$ we have

$$\begin{aligned}
\text{cost}_\mu(\Phi) = \sum_{i \in I} \mu(A_i) = \sum_{i \in I} \int_X \beta_x(A_i) d\mu(x) = \\
= \int_X \sum_{i \in I} \beta_x(A_i) d\mu(x) = \int_X \text{cost}_{\beta_x}(R_\alpha) d\mu(x)
\end{aligned}$$

By taking the infimum among all graphings $\Phi \subseteq [\![R]\!]$ we obtain

$$\text{cost}_\mu(R_\alpha) = \inf_{\Phi \subseteq [\![R]\!]} \int_X \text{cost}_{\beta_x}(\Phi) d\mu(x) \geq \int_X \text{cost}_{\beta_x}(R_\alpha) d\mu(x)$$

($\leq$) Let us fix $\varepsilon > 0$ and define the measurable set

$$A_{I,M} = \{x \in X : \operatorname{cost}_{\beta_x}(\Phi_{I,M}) < \operatorname{cost}_{\beta_x}(R_\alpha) + \varepsilon\}$$

for $M \in \mathbb{N}$ and $I \subseteq \mathbb{N} \times \mathbb{N}$. The finitary characterization of cost of Lemma 3.3.3 gives that

$$X = \cup\{A_{I,M} : M \in \mathbb{N} \text{ and } I \in P_{\text{fin}}(\mathbb{N} \times \mathbb{N})\}$$

By an enumeration $h : P_{\text{fin}}(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$ we can get a disjoint union

$$X = \sqcup\{B_{I,M} : M \in \mathbb{N} \text{ and } I \in P_{\text{fin}}(\mathbb{N} \times \mathbb{N})\}$$

into measurable subsets with the following properties:

a) $B_{I,M} \subseteq A_{I,M}$;

b) $\Gamma \cdot B_{I,M} = B_{I,M}$;

c) if $x \in B_{I,M}$, then $X_{\beta_x} \subseteq B_{I,M}$;

for all $M \in \mathbb{N}$ and $I \in P_{\text{fin}}(\mathbb{N} \times \mathbb{N})$. For instance one could take for $B_{I,M}$ the set of all $x \in X$ such that $h(I, M) = \min\{n \in \mathbb{N} : x \in A_{h^{-1}(n)}\}$. Let us define

$$\Phi = \cup\{\Phi_{I,M}|_{B_{I,M}} : M \in \mathbb{N} \text{ and } I \in P_{\text{fin}}(\mathbb{N} \times \mathbb{N})\}$$

Then $\Phi$ is a graphing for $R_\alpha$ such that, if $x \in B_{I,M}$, then

$$\begin{aligned}
\operatorname{cost}_{\beta_x} \Phi = \operatorname{cost}_{\beta_x}(\Phi|_{X_{\beta_x}}) &= & \beta_x(X_{\beta_x}) = 1 \\
\operatorname{cost}_{\beta_x}(\Phi_{I,M}|_{X_{\beta_x}}) &= & \text{disjoint union} \\
\operatorname{cost}_{\beta_x} \Phi_{I,M} &\leq & \beta_x(X_{\beta_x}) = 1 \\
\operatorname{cost}_{\beta_x} R_\alpha + \varepsilon & & B_{I,M} \subseteq A_{I,M}
\end{aligned}$$

Hence we get that

$$\operatorname{cost}_\mu \Phi = \int_X \operatorname{cost}_{\beta_x}(\Phi) d\mu(x) \leq \int_X \operatorname{cost}_{\beta_x}(R_\alpha) d\mu(x) + \varepsilon$$

and we can take the limit for $\varepsilon \to 0^+$ to end the proof. $\square$

**Corollary 3.3.5.** *Let $\Gamma$ be a countable group; there exists an essentially free and ergodic standard probability action $\alpha : \Gamma \curvearrowright (X, \mu)$ such that*

$$\operatorname{cost} \Gamma = \operatorname{cost}_\mu R_\alpha$$

*Proof.* If $\operatorname{cost} \Gamma = +\infty$, then there is nothing to prove, so let us assume that $\Gamma$ has finite cost. By definition of cost as an infimum, for every $n \in \mathbb{N} \setminus \{0\}$ there exists an essentially free standard probability action $\alpha_n : \Gamma \curvearrowright (X_n, \mu_n)$ such that

$$\operatorname{cost}_{\mu_n} R_{\alpha_n} \leq \operatorname{cost} \Gamma + \frac{1}{n}$$

Up to adjustments of the measure spaces $(X_n, \mu_n)$'s we may assume that the actions $\alpha_n : \Gamma \curvearrowright (X_n, \mu_n)$ are also free. Let us consider the product $(X, \mu) = (\prod_{n=1}^{+\infty} X_n, \bigotimes_{n=1}^{+\infty} \mu_n)$ endowed with the free diagonal action $\alpha : \Gamma \curvearrowright (X, \mu)$ constructed from the $\alpha_n$'s (which means that $\gamma \cdot (x_n)_{n=1}^{+\infty} = (\gamma \cdot x_n)_{n=1}^{+\infty}$). Then

$$\operatorname{cost} \Gamma \leq \operatorname{cost}_\mu R_\alpha \leq \inf\{\operatorname{cost}_{\mu_n} R_{\alpha_n} : n \in \mathbb{N} \setminus \{0\}\} = \operatorname{cost} \Gamma$$

For an ergodic decomposition $\beta : X \to \operatorname{Erg}(\alpha)$ by Theorem 3.3.4 we can find $x \in X$ such that $\operatorname{cost}_{\beta_x} R_\alpha = \operatorname{cost}_\mu R_\alpha$. $\square$

# 4. Manifolds and simplicial volume

Manifolds are topological spaces that locally "look like" euclidean spaces and are the objects of study of algebraic topology. After some basic definitions we are proving that, under certain reasonable conditions, countably many singular simplices suffice to generate the homology of the manifold (Lemma 4.3.4). In the last part of the chapter we will define two classical homotopy invariants for manifolds: the simplicial volume and the integral simplicial volume.

## 4.1. Singular homology

Let us recall some basic definitions and facts about homology we will need in our survey.

### 4.1.1. The singular homology modules

**Definition 4.1.1** (standard simplices)**.** Let $n \in \mathbb{N}$;

1. the *standard simplex* of dimension $n$ (or standard $n$-simplex) is the subset
$$\Delta^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0, \dots, x_n \geq 0 \text{ and } \textstyle\sum_{i=0}^n x_i = 1\right\} \subseteq \mathbb{R}^{n+1}$$
2. for all $0 \leq i \leq n$, the *i-th face* of the standard $n$-simplex $\Delta^n$ is the image of the map $\varepsilon_n^i : \Delta^{n-1} \to \Delta^n$ defined by
$$\varepsilon_n^i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$
for all $(x_0, \dots, x_{n-1}) \in \Delta^{n-1}$.

**Definition 4.1.2** (singular simplices)**.** Let us fix a topological space $X$, a commutative ring $R$ with unit and $n \in \mathbb{N}$;

1. a *singular n-simplex* of $X$ is any continuous function $\Delta^n \to X$. We denote by $S_n(X)$ the set of all $n$-simplices of $X$. If $\sigma : \Delta^n \to X$ is a singular $n$-simplex of $X$ and $0 \leq i \leq n$, then the $i$-th face of $\sigma$ is the singular $(n-1)$-simplex $\sigma \circ \varepsilon_n^i : \Delta^{n-1} \to X$;
2. the $R$-module of *singular n-chains* of $X$ with coefficients in the ring $R$ is the free $R$-module $C_n(X; R)$ generated by $S_n(X)$;
3. the *n-th boundary* is the $R$-module homomorphism $\partial_n : C_n(X; R) \to C_{n-1}(X; R)$ defined by
$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \cdot \sigma \circ \varepsilon_n^i$$
on the generators $\sigma \in S_n(X)$ and then extended by $R$-linearity.

The sequence of $R$-modules and $R$-module homomorphisms $\{(C_n(X;R),\partial_n) : n \in \mathbb{N}\}$ defines a chain complex, which means that $\partial_{n-1} \circ \partial_n = 0$ for all $n \geq 1$. With abuse of notation we shorten this condition with $\partial^2 = \partial \circ \partial = 0$.

**Terminology 4.1.3** (cycles and boundaries)**.** There is a traditional nomenclature:

1. $\operatorname{Ker}\partial_n$ is called the submodule of singular $n$-*cycles* of $X$ and is denoted by $Z_n(X;R)$;

2. $\operatorname{Im}\partial_{n+1}$ is called the submodule of singular $n$-*boundaries* of $X$ and is denoted by $B_n(X;R)$

**Definition 4.1.4** (singular homology)**.** Let $X$ be a topological space; the *singular homology* of $X$ is the homology $H_*(X;R)$ of the chain complex $C_*(X;R)$. More explicitly, for all $n \in \mathbb{N}$ the singular homology $n$-th $R$-module of $X$ is defined by

$$H_n(X;R) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \frac{Z_n(X;R)}{B_n(X;R)}$$

We can generalize the former construction by replacing the ring $R$ with an arbitrary $R$-module $G$. We simply construct the chain complex

$$C_*(X;G) = C_*(X;R) \otimes_R G$$

and then define the singular homology of $X$ with coefficients in $G$ by

$$H_n(X;G) = H_n(C_*(X;G))$$

In the case $R = \mathbb{Z}$ it is common to write $C_*(X)$ for $C_*(X;\mathbb{Z})$ and $H_n(X)$ for $H_n(X;\mathbb{Z})$.

### 4.1.2. Computing homology

If $X$ is a topological space and $A \subseteq X$ is a topological subspace (i.e. the inclusion $A \hookrightarrow X$ is a continuous map), then we have a short exact sequence

$$0 \to C_*(A;R) \to C_*(X;R) \to C_*(X;R)/C_*(A;R) \to 0$$

of $R$-modules chain complexes. The homology $n$-th $R$-module of $X$ relative to $A$ is

$$H_n(X,A;R) = H_n(C_*(X;R)/C_*(A;R))$$

The excision lemmas and the Mayer-Vietoris long exact sequence are useful tools for computing the homology modules by exploiting the short exact sequence above.

**Lemma 4.1.5** (first excision lemma, [26, p. 106])**.** *Let $X$ be a topological space and $U, A \subseteq X$ topological subspaces such that $\overline{U} \subseteq A^\circ$; then*

$$H_n(X \setminus U, A \setminus U) \cong H_n(X,A)$$

*for every $n \in \mathbb{N}$.*

**Lemma 4.1.6** (second excision lemma, [26, p. 106]). *Let $X$ be a topological space and $A, B \subseteq X$ topological subspaces such that $X = A^\circ \cup B^\circ$; then*

$$H_n(A, A \cap B) \cong H_n(X, B)$$

*for every $n \in \mathbb{N}$.*

**Theorem 4.1.7** (Mayer-Vietoris long exact sequence, [26, Theorem 6.3]). *Let $X$ be a topological space and $A, B \subseteq X$ topological subspaces such that $X = A^\circ \cup B^\circ$; then there exists a long exact sequence of $R$-modules*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

## 4.2. Triangulations

Let $v_0, \ldots, v_q \in \mathbb{R}^N$ be affinely independent points (or points in general position), which means that the vectors $v_1 - v_0, \ldots, v_q - v_0$ are $\mathbb{R}$-linearly independent. We denote by $[v_0, \ldots, v_q] \subseteq \mathbb{R}^N$ the *$q$-simplex* spanned by $v_0, \ldots, v_q$, i.e. the convex envelope

$$\left\{\sum_{i=0}^q t_i v_i \in \mathbb{R}^N : t_i \geq 0 \text{ for all } i = 0, \ldots, q \text{ and } \sum_{i=0}^q t_i = 1\right\}$$

of the points $v_0, \ldots, v_q$. If $s = [v_0, \ldots, v_q]$ is a $q$-simplex, then:

1. its *vertex* set is $\mathrm{Vert}(s) = \{v_0, \ldots, v_q\}$;
2. a simplex $s'$ is a *face* of $s$ if $\mathrm{Vert}(s') \subseteq \mathrm{Vert}(s)$. A face $s'$ is said to be proper if $\mathrm{Vert}(s') \subset \mathrm{Vert}(s)$;
3. $q = \dim s$ is the *dimension* of $s$.

If $v_0, \ldots, v_q \in \mathbb{R}^N$ are not necessarily in general position, then of course one can define a simplex $s = [v_0, \ldots, v_q]$ spanned by these points, but its dimension might be smaller or equal than $q$. In such a case we say that $s$ is an $r$-simplex if there exist $u_0, \ldots, u_r \in \mathbb{R}^N$ in general position spanning $s$.

**Example 4.2.1** (standard simplices). The standard $q$-simplex of Definition 4.1.1 is the simplex spanned by the $q+1$ points $0$, $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots$ in $\mathbb{R}^q$.

**Definition 4.2.2** (simplicial complex). A *simplicial complex* in $\mathbb{R}^N$ is a finite family $K$ of simplices of $\mathbb{R}^N$ such that:

1. if $s \in K$, then every face of $s$ belongs to $K$;
2. if $s, t \in K$ then $s \cap t$ is either empty or a face of both $s$ and $t$.

For $i \in \mathbb{N}$ we denote $K^{(i)} = \{s \in K : \dim s = i\}$ the family of $i$-simplices of $K$. The 0-simplices of $K$ are also called the vertices of $K$. If $K$ is a simplicial complex in $\mathbb{R}^N$, then its underlying space is the union

$$|K| = \bigcup_{s \in K} s \subseteq \mathbb{R}^N$$

of its simplices.

**Definition 4.2.3** (triangulations). A topological space $X$ is said to be *triangulable* (or a polyhedron) if there are a simplicial complex $K$ and a homeomorphism $h : |K| \to X$. The pair $(K, h)$ is called a *triangulation* of $X$.

**Example 4.2.4** (triangulation of the sphere). If $s$ is an $n$-simplex and $\dot{s}$ is the simplicial complex formed by all of the proper faces of $s$, then there exists a triangulation $(\dot{s}, h)$ of the sphere $S^{n-1}$.

Let $s$ be a $q$-simplex; the open $q$-simplex associated to $s$ is the interior of $s$, hence defined by

$$s^\circ = \begin{cases} s & \text{if } q = 0 \\ s \smallsetminus \dot{s} & \text{if } q \geq 1 \end{cases}$$

If $K$ is a simplicial complex, then its geometric realizations admits a disjoint decomposition as $|K| = \sqcup \{s^\circ : s \in K\}$.

**Definition 4.2.5** (star of a point). If $K$ is a simplicial complex and $p \in \mathrm{Vert}(K)$, then we define the *star* of $p$ as

$$\mathrm{Star}(p) = \cup \{s^\circ : s \in K \text{ such that } p \in \mathrm{Vert}(s)\} \subseteq |K|$$

In other words $\mathrm{Star}(p)$ is the union of all of the simplices of $K$ that are neighbourhoods of $p$.

**Definition 4.2.6** (simplicial maps). Let $K$ and $L$ be simplicial complexes; a *simplicial map* $f : K \to L$ is a function $f : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ on the vertices such that $[f(v_0), \ldots, f(v_q)] \in L$ for all $[v_0, \ldots, v_q] \in K$.

Notice that the points $f(v_0), \ldots, f(v_q)$ might not be affinely independent, even when the points $v_0, \ldots, v_q$ are.

Simplicial complexes together with simplicial maps form a category $\mathsf{Simp}$, where composition of simplicial maps is the usual composition of functions. The geometric realization $|\cdot| : \mathsf{Simp} \to \mathsf{Top}$ gives a functor in the following way. If $f : K \to L$ is a simplicial map, then for all $s \in K$ let $f_s : s \to |L|$ be the affine (hence continuous) map determined by $f|_{\mathrm{Vert}(s)}$, i.e. $f_s(\sum_{i=0}^q t_i v_i) = \sum_{i=0}^q t_i f(v_i)$, where $s = [v_0, \ldots, v_q]$. Condition 2 of Definition 4.2.2 says that $f_s = f_t$ in the overlapping $s \cap t$ and hence the $f_s$'s can be glued to a unique continuous function $|f| : |K| \to |L|$. If $f : K \to L$ is a simplicial map, then the geometric realization $|f| : |K| \to |L|$ is called a *piecewise linear* function.

**Definition 4.2.7** (simplicial approximation). Let $K$ and $L$ be simplicial complexes, $\varphi : K \to L$ be a simplicial map and let $f : |K| \to |L|$ be continuous; then we say that $\varphi$ is a *simplicial approximation* of $f$ if $f(\mathrm{Star}(p)) \subseteq \mathrm{Star}(\varphi(p))$ for every vertex $p$ of $K$.

Notice that if $\varphi : K \to L$ is a simplicial map, then the geometric realization $|\varphi| : |K| \to |L|$ is a simplicial approximation of $\varphi$.

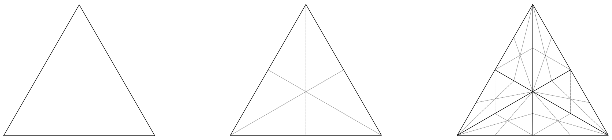

Figure 4.1.: The zeroth, first and second barycentric subdivisions of the 2-simplex

**Definition 4.2.8** (barycentric subdivision). Let $s = [v_0, \dots, v_q]$ be a $q$-simplex and let $K$ be a simplicial complex; then:

1. the *barycentre* of $s$ is the point
$$b^s = \sum_{i=0}^{q} \frac{1}{1+q} v_i \in s$$
2. the *barycentric subdivision* of $K$ is the simplicial complex $\mathrm{Sd}(K)$ with $\mathrm{Vert}(\mathrm{Sd}(K)) = \{b^s : s \in K\}$ and simplices $[b^{s_1}, \dots, b^{s_q}]$ for $q \in \mathbb{N}$, $s_1, \dots, s_q \in K$ and $s_1 < \dots < s_q$.

**Remark 4.2.9.** Some remarks about the above construction:

1. $s = b^s$ holds for every 0-simplex and hence $\mathrm{Vert}(K) \subseteq \mathrm{Vert}(\mathrm{Sd}(K))$;
2. $|\mathrm{Sd}\, K| = |K|$ holds for every simplicial complex $K$;
3. the identity $\mathrm{id}_{|K|} : |\mathrm{Sd}\, K| \to |K|$ admits a simplicial approximation $\varphi : \mathrm{Sd}\, K \to K$. One defines $\varphi : \mathrm{Vert}(\mathrm{Sd}\, K) \to \mathrm{Vert}(K)$ to be the identity on $\mathrm{Vert}(K)$ and with the property that $\varphi(b^s) \in s$ for all $s \in K$.

One can define iterated barycentric subdivisions by setting $\mathrm{Sd}^0(K) = K$, $\mathrm{Sd}^1(K) = \mathrm{Sd}(K)$ and $\mathrm{Sd}^i(K) = \mathrm{Sd}(\mathrm{Sd}^{i-1}(K))$ for $i \geq 2$.

**Theorem 4.2.10** (simplicial approximation theorem, [26, Theorem 7.3]). *Let $K$ and $L$ be simplicial complexes and let $f : |K| \to |L|$ be a continuous function; then there exists a simplicial approximation $\varphi : \mathrm{Sd}^q\, K \to L$ of $f$ for some $q \in \mathbb{N}$.*

## 4.3. Manifolds

Manifolds are topological spaces that are locally similar to a euclidean space and the main object of algebraic topology.

### 4.3.1. Topological manifolds

Let $n \in \mathbb{N} \smallsetminus \{0\}$; we say that a non-empty topological space $M$ is an $n$-dimensional *manifold* (or manifold of dimension $n$ or simply $n$-manifold) if:

1. $M$ is an Hausdorff and second countable;

2. every $x \in M$ admits an open neighbourhood homeomorphic with the $n$-dimensional open ball $B(0,1) = \{v \in \mathbb{R}^n : |v| < 1\}$.

The dimension $n$ of a manifold $M$ is intrinsically characterized by the local homology $\mathbb{Z}$-modules: for all $x \in M$ and $i \geq 1$ we have that

$$\begin{aligned} H_i(M, M \smallsetminus \{x\}) &\cong H_i(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}) && \text{(excision)} \\ &\cong H_{i-1}(\mathbb{R}^n \smallsetminus \{0\}) && (\mathbb{R}^n \text{ is contractible}) \\ &\cong H_{i-1}(S^{n-1}) && (\mathbb{R}^n \smallsetminus \{0\} \simeq S^{n-1}) \end{aligned}$$

and hence

$$H_i(M, M \smallsetminus \{x\}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Compact manifolds are usually called *closed manifolds*. Though a little confusing, this distinguishes from the concept of compact manifold with boundary, which we will not talk about.

**Remark 4.3.1** (connected manifolds). Connectedness for manifolds is equivalent to path-wise connectedness. Indeed every manifold is locally path-connected, being locally homeomorphic to an open ball in $\mathbb{R}^N$, and spaces that are connected and locally path-wise connected are also (globally) path-wise connected.

### 4.3.2. Orientation

If we want to give a definition of orientation for manifolds, then we may start by defining an orientation on $\mathbb{R}^n$, which should be compatible with the following principle: an orientation must be preserved by rotations and must be reversed by reflections. Let us take $x \in \mathbb{R}^n$ and observe that

$$H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x\}) \cong H_{n-1}(\mathbb{R}^{n-1}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

where we may see $S^{n-1}$ as a sphere centred at $x$. A rotation in $S^{n-1}$ is homotopic to $\mathrm{id}_{S^{n-1}}$, whereas a reflection has degree $-1$. Let us say that a local orientation of $\mathbb{R}^n$ at the point $x$ is one of the two generators of $H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x\})$. Notice that if $y \in \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ is a ball containing $x$ and $y$, then

$$H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{y\})$$

which means that a local orientation at $x$ determines a local orientation at $y$. We can generalize this definition to the manifold case.

**Definition 4.3.2** (Orientation). Let $M$ be an $n$-dimensional manifold and let $x \in M$;

1. a *local orientation* of $M$ at the point $x$ is a generator $\omega_x$ of $H_n(M, M \smallsetminus \{x\})$;

2. an *orientation* (or global orientation) of $M$ is a function
$$\omega : M \to \coprod_{x\in M} H_n(M, M \setminus \{x\})$$
satisfying the following conditions:
   a) $\omega(x) = \omega_x$ is a local orientation of $M$ at $x$ for all $x \in M$;
   b) for all $x \in M$ there exists a radius $r > 0$ for an open $n$-ball $B = B(x, r) \subseteq M$ such that the natural map
   $$H_n(M, M \setminus B) \to H_n(M, M \setminus \{y\})$$
   sends $\omega_B$ to $\omega_y$ for all $y \in B$, being $\omega_B$ the generator of $H_n(M, M \setminus B)$ corresponding to $\omega_x$ under the map
   $$H_n(M, M \setminus B) \to H_n(M, M \setminus \{x\})$$

A manifold $M$ is said to be *orientable* if it admits an orientation and, if $\omega : M \to \coprod_{x\in M} H_n(M, M \setminus \{x\})$ is an orientation on $M$, then we call the pair $(M, \omega)$ an *oriented manifold* .

### 4.3.3. Countable characterization of manifold homology

Under certain reasonable hypotheses countably many singular complexes in a manifold suffices to compute the homology.

**Remark 4.3.3** (fundamental group of closed manifolds)**.** As a side result of the proof of Lemma 4.3.4 we will see that every closed connected orientable manifold has countable fundamental group. Indeed we will see (thou not prove) that every closed manifold is homotopy equivalent to a simplicial complex. Simplicial complexes have a finitely presented fundamental group, hence countable in particular.

**Lemma 4.3.4.** *Let $M$ be a closed connected orientable manifold with fundamental group $\Gamma$ and universal covering $p : \tilde{M} \to M$; for every $k \in \mathbb{N}$ there exists a subset of singular $k$-simplices $S'_k(\tilde{M}) \subseteq \mathrm{Hom}_{\mathsf{Top}}(\Delta^k, \tilde{M}) = S_k(\tilde{M})$ with the following properties:*

1. *$S'_k(\tilde{M})$ is countable for every $k \in \mathbb{N}$;*
2. *the family $\{S'_k(\tilde{M})\}_{k\in\mathbb{N}}$ is closed under taking faces, i.e. if $\tilde{\sigma} \in S'_k(\tilde{M})$, then $\tilde{\sigma} \circ \varepsilon^i_k \in S'_{k-1}(\tilde{M})$ for every $k \in \mathbb{N}$ and for every $0 \le i \le k$;*
3. *$S'_k(\tilde{M})$ is closed under the action of $\Gamma$ for every $k \in \mathbb{N}$, i.e. if $\tilde{\sigma} \in S'_k(\tilde{M})$ and $\gamma \in \Gamma$, then $\gamma \circ \tilde{\sigma} \in S'_k(\tilde{M})$;*
4. *let $C'_*(\tilde{M}; \mathbb{Z}) = \{(C'_k(\tilde{M}; \mathbb{Z}), \partial_k)\}_{k\in\mathbb{N}}$ be the chain complex of the free $\mathbb{Z}$-modules generated by $\{S'_k(\tilde{M})\}_{k\in\mathbb{N}}$; the inclusion*
$$C'_*(\tilde{M}; \mathbb{Z}) \hookrightarrow C_*(\tilde{M}, \mathbb{Z})$$
*is a homotopy equivalence of $\mathbb{Z}[\Gamma]$-modules chain complexes.*

*Proof.* One can show that every closed manifold is homotopy equivalent to a simplicial complex. This result is highly non-trivial and it is based on the following facts:

1. every closed manifold is homotopy equivalent to a countable CW-complex [25, Corollary 1];
2. a refinement of the former point gives that every closed manifold is homotopy equivalent to a finite CW-complex [16];
3. every finite CW-complex is homotopy equivalent to a simplicial complex with the same dimension [10, Theorem 2C.5].

This shows that we may restrict to the case $M$ is a triangulable manifold. As a side result we get that $M$ has a countable fundamental group $\Gamma$ in particular. If $(K, h)$ is a triangulation of $M$, then for every $i \in \mathbb{N}$ there also exists a triangulation $(K_i, h_i)$ of $M$, where $K_i = \operatorname{Sd}^i K$. The universal covering map $p : \tilde{M} \to M$ lifts every triangulation $(K_i, h_i)$ to a triangulation $(\tilde{K}_i, \tilde{h}_i)$ of $\tilde{M}$. We set

$$A_k = \bigcup_{i \in \mathbb{N}} (\tilde{K}_i)^{(k)}$$

If $s_k \in A_k$, then $s_k \in (\tilde{K}_i)^{(k)}$ for some $i \in \mathbb{N}$ and is homeomorphic to the standard $k$-simplex $\Delta^k$, say via the homeomorphism $\alpha_{s_k} : \Delta^k \to s_k$. Then we associate to $s_k$ the singular $k$-simplex $\tilde{\sigma}_k = h_i \circ \alpha_{s_k} : \Delta^k \to s_k \to \tilde{M}$. We define $S'_k(\tilde{M})$ to be the family of all simplicial $k$-simplices associated to the elements of $A_k$. Clearly $S'_k(\tilde{M})$ is countable and $S'_*(\tilde{M}) = \{S'_k(\tilde{M})\}_{k \in \mathbb{N}}$ is closed under taking faces. Moreover $S'_k(\tilde{M})$ is closed under the action of $\Gamma \cong \operatorname{Deck}(p)$, since $p : \tilde{M} \to M$ lifts $(K_i, h_i)$ to $(\tilde{K}_i, \tilde{h}_i)$. The last point of the proof is much more delicate and is treated in Theorem 4.3.5. $\square$

The statement and the proof of Theorem 4.3.5 are modelled out the ones of a lemma that Lee uses to prove that smooth homology and singular homology coincide for smooth manifolds [18, Lemma 18.8]. Since it is very technical, we are only sketching this proof. Here $I = [0, 1]$ denotes the unit interval.

**Theorem 4.3.5.** *With the same notations and hypotheses of Lemma 4.3.4: for every $k \in \mathbb{N}$ and every singular $k$-simplex $\tilde{\sigma} : \Delta^k \to \tilde{M}$ there exists a continuous map $h : \Delta^k \times I \to \tilde{M}$ such that:*

1. *there exists $\tilde{\sigma}' \in S'_k(\tilde{M})$ such that $h$ is a homotopy between $\tilde{\sigma} = h(\,\cdot\,, 0)$ and $\tilde{\sigma}' = h(\,\cdot\,, 1)$;*
2. *for every face map $\varepsilon^i_k : \Delta^{k-1} \to \Delta^k$ we have*

$$h_{\tilde{\sigma} \circ \varepsilon^i_k} = h_{\tilde{\sigma}} \circ (\varepsilon^i_k \times \mathrm{id}_I)$$

   *which explicitly means $h_{\tilde{\sigma} \circ \varepsilon^i_k}(x, t) = h_{\tilde{\sigma}}(\varepsilon^i_k(x), t)$ for all $(x, t) \in \Delta^{k-1} \times I$;*

3. *if $\tilde{\sigma} \in S'_k(\tilde{M})$, then $h_{\tilde{\sigma}}(\,\cdot\,, t) = \tilde{\sigma}$ for all $t \in I$.*

*Sketch of proof.* Let us fix a singular $k$-simplex $\tilde{\sigma} : \Delta^k \to \tilde{M}$. We are proving the result by induction on $k \in \mathbb{N}$. If $k = 0$, then $\tilde{\sigma} : \{x\} \to \tilde{M}$ is a point. Let us fix another point $q \in \tilde{M} \cap A_0$ and let $\varphi : I \to \tilde{M}$ be an arc connecting $\tilde{\sigma}(x)$ and $q$ ($\tilde{M}$ is connected). Then we set $h_{\tilde{\sigma}}(x,t) = \varphi(t)$ for every $t \in I$. Note that condition 2 is trivially satisfied, since there are no face maps form $\Delta^0$. Let now be $k \geq 1$ and suppose that the theorem holds for every singular $k'$-simplex for every $0 \leq k' \leq k-1$. If $\tilde{\sigma} \in S'_k(\tilde{M})$, then we just set $h_{\tilde{\sigma}}(\cdot, t) = \tilde{\sigma}$ for every $t \in I$, so we may assume that $\tilde{\sigma} \notin S'_k(\tilde{M})$.

1. Let
$$S = (\Delta^k \times \{0\}) \cup (\partial\Delta^k \times I) \subseteq \Delta^k \times I$$
and define a function $h_0 : S \to \tilde{M}$ via
$$h_0(x,t) = \begin{cases} \tilde{\sigma}(x) & \text{if } x \in \Delta^k \text{ and } t = 0 \\ h_{\tilde{\sigma}\circ\varepsilon_k^i}((\varepsilon_k^i)^{-1}(x), t) & \text{if } (x,t) \in \varepsilon_k^i(\Delta^{k-1}) \times I \end{cases}$$
The inductive hypothesis for $\{\tilde{\sigma} \circ \varepsilon_k^i : \Delta^{k-1} \to \tilde{M}\}_{i=0}^k$ shows that $h_0$ is well-defined on the overlapping $(\Delta^k \times \{0\}) \cap (\partial\Delta^k \times I)$, hence continuous in particular by the Gluing lemma.

2. The space $\Delta^k \times I$ admits a retraction $r : \Delta^k \times I \to S$ and hence we can extend $h_0$ to some continuous map $h : \Delta^k \times I \to \tilde{M}$ by setting $h = h_0 \circ r$. Then $h$ is a homotopy between $\tilde{\sigma}$ and another singular $k$-simplex $\tilde{\sigma}'$, which does not necessarily lie in $S'_k(\tilde{M})$ but satisfies the condition 2: the restriction of $h$ to each boundary face $\varepsilon_k^i(\Delta^{k-1}) \times \{1\}$ belongs to $S'_{k-1}(\tilde{M})$ for all $0 \leq i \leq k$ by the inductive hypothesis applied to $\tilde{\sigma} \circ \varepsilon_k^i$.

3. Our final task is now to modify $h$ to get a homotopy between $\tilde{\sigma}$ and a simplex lying in $S'_k(\tilde{M})$. By the same argument of Whitney's approximation theorem [18, Theorem 6.26], there exists a homotopy relative to $\partial\Delta^k = \bigcup_{i=0}^k \varepsilon_k^i(\Delta^{k-1})$ between $\tilde{\sigma}'$ and some $\tilde{\sigma}'' \in S'_k(\tilde{M})$. A convenient gluing of this latter homotopy with $h$ gives a homotopy $h_{\tilde{\sigma}}$ between $\tilde{\sigma}$ and $\tilde{\sigma}''$ satisfying the conditions 1-2-3. □

### 4.3.4. Simplicial volume

The simplicial volume is a homotopy invariant for manifolds, whose definition is due to Gromov [7, p. 8].

**Definition 4.3.6** ($\ell^1$-norm). Let $X$ be a topological space and let $(R, |\cdot|)$ be a normed ring; the $\ell^1$-norm of the singular $m$-th chain $\sum_{i=1}^k r_i\sigma_i \in C_m(X;R)$ (where all of the $\sigma_i$'s are distinct) is defined by
$$\left\| \sum_{i=1}^k r_i\sigma_i \right\|_1^R = \sum_{i=1}^n |r_i|$$

**Remark 4.3.7.** It is straightforward to see that the function $\|\cdot\|_1^R : C_m(X;R) \to [0, +\infty[$ is an $R$-module norm.

The norm $\|\cdot\|_1^R$ on $C_m(X;R)$ induces a seminorm-like function (again denoted by $\|\cdot\|_1^R$) on the homology $H_m(X;R)$ by

$$\|\alpha\|_1^R = \inf\{\|a\|_1^R : a \in Z_m(X;R) \text{ such that } a + B(X;R) = \alpha\}$$

for all $\alpha \in H_m(X;R)$. The function $\|\cdot\|_1^R : H_m(X;R) \to [0,+\infty[$ generally fails to satisfy the homogeneity of the norm (Example 4.3.12).

**Definition 4.3.8** (simplicial volume)**.** Let $M$ be an oriented closed connected $n$-dimensional manifold;

1. let $[M]_{\mathbb{Z}}$ be the *integral fundamental class* of $M$ , i.e. the positive generator of $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$; the *integral simplicial volume* of $M$ is
$$\|M\|_{\mathbb{Z}} = \|[M]_{\mathbb{Z}}\|_1^{\mathbb{Z}} \in \mathbb{N}$$
2. let $[M]_{\mathbb{R}}$ be the *real fundamental class* of $M$, i.e. the image of $[M]_{\mathbb{Z}}$ via the change of coefficients homomorphism $H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{R})$; the (real) *simplicial volume* of $M$ is
$$\|M\|_{\mathbb{R}} = \|[M]_{\mathbb{R}}\|_1^{\mathbb{R}} \in [0,+\infty[$$

**Terminology 4.3.9** (fundamental cycles)**.** Let $M$ be an oriented closed connected $n$-dimensional manifold; we say that:

1. $c \in C_n(M;\mathbb{Z})$ is an *integral fundamental cycle* (or just a fundamental cycle) if $c + B(M;\mathbb{Z}) = [M]_{\mathbb{Z}}$ in $H_n(M;\mathbb{Z})$;
2. $c \in C_n(M;\mathbb{R})$ is a *real fundamental cycle* if $c + B(M;\mathbb{R}) = [M]_{\mathbb{R}}$ in $H_n(M;\mathbb{R})$.

**Example 4.3.10** (simplicial volume of $S^1$)**.** For every $d \in \mathbb{N} \smallsetminus \{0\}$ let

$$\begin{aligned} \sigma_d : \Delta^1 &\to S^1 \\ (1-t,t) &\mapsto [d\cdot t] \end{aligned}$$

be the singular 1-simplex that wraps $d$ times around the circle $S^1$. Then $1/d \cdot \sigma_d \in C_1(S^1;\mathbb{R})$ is a fundamental cycle for $S^1$ and thus

$$\|S^1\| \le \inf\{\|1/d\cdot\sigma_d\|_1^{\mathbb{R}} : d \in \mathbb{N}\smallsetminus\{0\}\} = \inf\{1/d : d \in \mathbb{N}\smallsetminus\{0\}\} = 0$$

**Remark 4.3.11.** By construction $\|M\| \le \|M\|_{\mathbb{Z}}$ holds for every manifold. Indeed every integral fundamental cycle is also a real fundamental cycle.

**Example 4.3.12** (lack of homogeneity)**.** The singular 1-simplex $\sigma_1 : \Delta^1 \to S^1$ is an integral fundamental cycle for $S^1$, as well as a real fundamental cycle, thus $\|S^1\|_{\mathbb{Z}} = 1$. The cycle $2\cdot\sigma_1$ represents $2\cdot[S^1]_{\mathbb{Z}}$, hence $\|2\cdot[S^1]_{\mathbb{Z}}\|_1^{\mathbb{Z}} \le 2\cdot\|S^1\|_{\mathbb{Z}}$. Nevertheless an equality does not hold: let us define

$$\begin{aligned} f_2 : S^1 &\to S^1 \\ [t] &\mapsto [2t] \end{aligned}$$

and consider the singular 2-chain

$$\begin{aligned}\tau : \Delta^2 &\to S^1 \\ (t_0, t_1, t_2) &\mapsto [t_2 - t_0]\end{aligned}$$

Let us compute the boundary of $\tau$:

$$\begin{aligned}\partial_2 \tau &= \tau \circ \varepsilon_2^0 - \tau \circ \varepsilon_2^1 + \tau \circ \varepsilon_2^2 \\ &= ((u,v) \mapsto [v]) - ((u,v) \mapsto [v-1]) + ((u,v) \mapsto [-u]) \\ &= \sigma_1 - ((u,v) \mapsto [v - (1-v)]) + ((u,v) \mapsto [-(1-v)]) \\ &= s \cdot \sigma_1 - f_2 \circ \sigma_1\end{aligned}$$

and hence $2 \cdot \sigma_1 + B_1(S^1) = f_2 \circ \sigma_1 + B(S^1) = H_1(f_2)(\sigma_1 + B_1(S^1))$ in the homology $H_1(S^1)$. In particular $\|2 \cdot [S^1]_{\mathbb{Z}}\|_1^{\mathbb{Z}} = \|[f_2 \circ \sigma_1]\|_1^{\mathbb{Z}} = 1$.

# 5. Integral foliated simplicial volume

The integral foliated simplicial volume is a homotopy invariant for manifolds introduced by Gromov and could be viewed as a "dynamical version" of the simplicial volume. The integral coefficients for the singular complexes are replaced by spaces $L^\infty(X, \mu, \mathbb{Z})$ of essentially bounded functions, where the $(X, \mu)$'s are standard probability spaces on which the fundamental group of the manifold acts. In the last part of the chapter we will state and prove the decomposition formula for the parametrized simplicial volume with respect to ergodic decompositions, which is the aim of this thesis work.

## 5.1. Twisted coefficients

We are following Hatcher's construction for the homology with local coefficients [10, Section 3.H]. The definition and properties of the group ring are given in A.1.2 in the Appendix.

In the following let $X$ be a path-connected topological space with universal covering $p : \tilde{X} \to X$ and fundamental group $\Gamma$. From the isomorphism $\mathrm{Deck}(p) \cong \Gamma$ it is already known that $\Gamma$ acts on $\tilde{X}$ on the left: if $\tilde{x} \in \tilde{X}$, $x = p(\tilde{x}) \in X$ and $g \in \pi_1(X, x)$, then one sets $g \cdot \tilde{x} = f_g(\tilde{x})$, where $f_g : \tilde{X} \to \tilde{X}$ is the unique deck transformation identified by $\tilde{x} \mapsto \tilde{g}(1)$, being $\tilde{g} : [0, 1] \to \tilde{X}$ the unique $p$-lift of $g$ such that $\tilde{g}(0) = \tilde{x}$.

**Proposition 5.1.1.** *The left action $\Gamma \curvearrowright \tilde{X}$ induces a left action $\Gamma \curvearrowright C_k(\tilde{X}; \mathbb{Z})$ for every $k \in \mathbb{N}$, which makes $C_*(\tilde{X}; \mathbb{Z})$ into a chain complex of left modules over the ring $\mathbb{Z}[\Gamma]$.*

*Proof.* The action $\Gamma \curvearrowright \tilde{X}$ induces an action $\Gamma \curvearrowright C_k(\tilde{X}; \mathbb{Z})$ by sending a singular $k$-complex $\tilde{\sigma} : \Delta^n \to \tilde{X}$ to the composition $g \circ \tilde{\sigma} : \Delta^n \to \tilde{X}$ for $g \in \Gamma$. This action makes $C_k(\tilde{X}; \mathbb{Z})$ into a left $\mathbb{Z}[\Gamma]$-module in a natural way:

$$\sum_{i=1}^{n} a_i g_i \cdot \sum_{j=1}^{m} b_j \tilde{\sigma}_j = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j (g_i \circ \tilde{\sigma}_j)$$

for all $\sum_{i=1}^{n} a_i g_i \in \mathbb{Z}[\Gamma]$ and for all $\sum_{j=1}^{m} b_j \tilde{\sigma}_j \in C_k(\tilde{X}; \mathbb{Z})$. To see that $C_*(\tilde{X}; \mathbb{Z})$ is a chain complex in $\mathbb{Z}[\Gamma]$-$\mathsf{Mod}$ it suffices to prove that

$$\partial_k : C_k(\tilde{X}; \mathbb{Z}) \to C_{k-1}(\tilde{X}; \mathbb{Z})$$

is a $\mathbb{Z}[\Gamma]$-module homomorphism for every $n \in \mathbb{N} \setminus \{0\}$. Let us take $\sum_{i=1}^{M} a_i g_i$, $\sum_{k=1}^{p} c_k h_k \in$

$\mathbb{Z}[\Gamma]$ and $\sum_{j=1}^{N} b_j\sigma_j$, $\sum_{l=1}^{q} d_l\tau_l \in C_n(\tilde{X};\mathbb{Z})$ and let us compute

$$\begin{aligned}\partial_n(\textstyle\sum_{i=1}^{M} a_i g_i \cdot \sum_{j=1}^{N} b_j\sigma_j + \sum_{k=1}^{p} c_k h_k \cdot \sum_{l=1}^{q} d_l\tau_l) = \\ = \partial_n(\textstyle\sum_{i,j} a_i b_j \cdot g_i \circ \sigma_j + \sum_{k,l} c_k d_l \cdot h_k \circ \tau_l) = \\ = \textstyle\sum_{i,j} a_i b_j \partial_n(g_i \circ \sigma_j) + \sum_{k,l} c_k d_l \partial_n(h_k \circ \tau_l)\end{aligned}$$

and recall the commutativity of the diagram

$$\begin{tikzcd} {} \arrow[r] & C_n(\tilde{X};\mathbb{Z}) \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1}(\tilde{X};\mathbb{Z}) \arrow[r] \arrow[d, "f_{n-1}"] & {} \\ {} \arrow[r] & C_n(\tilde{X};\mathbb{Z}) \arrow[r, "\partial_n"'] & C_{n-1}(\tilde{X};\mathbb{Z}) \arrow[r] & {} \end{tikzcd}$$

for all continuous maps $f : \tilde{X} \to \tilde{X}$ to derive that $\partial_n(g_i \circ \sigma_j) = g_i \circ \partial_n(\sigma_j)$ and $\partial_n(h_k \circ \tau_l) = h_k \circ \partial_n(\tau_l)$. The relation $\partial^2 = 0$ trivially still holds. □

**Definition 5.1.2** (homology with local coefficients). Let $X$ be a path-connected topological space with universal covering $p : \tilde{X} \to X$ and fundamental group $\Gamma$; if $A$ is a right $\mathbb{Z}[\Gamma]$-module with structure given by the representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(A)$, then we define

$$C_n(X; A_\rho) = A \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{X};\mathbb{Z})$$

The homology $H_*(X; A_\rho)$ of the complex $C_*(X; A_\rho)$ is called the homology of $X$ with *local coefficients* in $A$ or, more precisely, the homology of $X$ with *coefficients twisted* by the representation $\rho$.

**Example 5.1.3** (trivial module structure). Let us consider the case in which $A$ is a $\mathbb{Z}$-module with the trivial right $\mathbb{Z}[\Gamma]$-module structure, which means that $a\gamma = a$ for all $\gamma \in \mathbb{Z}[\Gamma]$ and for all $a \in A$, or equivalently the representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(A)$ is given by $\rho(g) = \mathrm{id}_A$ for all $g \in \Gamma$. In such a case we want to show that

$$H_n(X; A_\rho) \cong H_n(X; A)$$

is just the ordinary homology with coefficients in the $\mathbb{Z}$-module $A$.

Indeed if $\sigma : \Delta^n \to X$ is a singular $n$-simplex, then the set of all of its $p$-lifts $\tilde{\sigma} : \Delta^n \to \tilde{X}$ forms an orbit of the action $\Gamma \curvearrowright C_n(\tilde{X};\mathbb{Z})$. The tensor product relation of $A \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{X};\mathbb{Z})$ gives

$$a \otimes \tilde{\sigma} = a\gamma \otimes \tilde{\sigma} = a \otimes \gamma\tilde{\sigma}$$

which means that all of the lifts $\tilde{\sigma} : \Delta^n \to \tilde{X}$ get identified in $C_n(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} A$. Hence we can identify $A \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{X};\mathbb{Z})$ with the usual chain complex $A \otimes_{\mathbb{Z}} C_n(X;\mathbb{Z}) = C_n(X; A)$, namely the singular chain complex of $X$ with coefficients in the $\mathbb{Z}$-module $A$.

## 5.2. Integral foliated simplicial volume

We begin this section by constructing the integral foliated simplicial volume as the infimum of parametrized simplicial volumes: the main source for the definitions and the first basic properties is Schmidt's work [28, Section 5.2]. In the second part we are looking at the relation between the parametrized volume and the ergodic decomposition of the actions. In particular in Theorem 5.2.26 we get a decomposition formula for the parametrized simplicial volume.

### 5.2.1. Definition of integral foliated simplicial volume

**Definition 5.2.1.** Let $M$ be a closed connected oriented $n$-dimensional manifold with fundamental group $\Gamma$ and universal covering $\tilde{M} \to M$:

1. let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard $\Gamma$-space; then the $\Gamma$-space structure on $X$ induces a right action of $\Gamma$ on the space $L^\infty(X, \mu, \mathbb{Z})$ by

$$\begin{aligned} L^\infty(X, \mu, \mathbb{Z}) \times \Gamma &\to L^\infty(X, \mu, \mathbb{Z}) \\ (f, \gamma) &\mapsto f \cdot \gamma \end{aligned}$$

   where $(f \cdot \gamma)(x) = f(\gamma \cdot x)$ for all $x \in X$. The inclusion of constant functions $\mathbb{Z} \hookrightarrow L^\infty(X, \mu, \mathbb{Z})$ induces the change of coefficients homomorphism

$$\begin{aligned} i^\alpha_M : C_*(M; \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C_*(\tilde{M}; \mathbb{Z}) &\to L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(\tilde{M}; \mathbb{Z}) \\ a \otimes \tilde{\sigma} &\mapsto a \cdot \chi_X \otimes \tilde{\sigma} \end{aligned}$$

2. the $(\alpha, \mu)$-*parametrized fundamental class* of $M$ is the image

$$[M]^{(\alpha,\mu)} = H_n(i^\alpha_M)([M]_\mathbb{Z}) \in H_n(M; L^\infty(X, \mu, \mathbb{Z}))$$

   and all of the cycles in the chain complex

$$C_*(M; L^\infty(X, \mu, \mathbb{Z})) = C_*(\tilde{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} L^\infty(X, \mu, \mathbb{Z})$$

   representing $[M]^{(\alpha,\mu)}$ are called the $(\alpha, \mu)$-*parametrized fundamental cycles* of $M$.

**Remark 5.2.2** (canonical form for the twisted chains)**.** By the $\mathbb{Z}[\Gamma]$-balancedness of the map

$$\otimes : L^\infty(X, \mu, \mathbb{Z}) \times C_m(\tilde{M}; \mathbb{Z}) \to C_m(\tilde{M}; L^\infty(X, \mu, \mathbb{Z}))$$

we may always write the elements of $C_m(M; L^\infty(X, \mu, \mathbb{Z}))$ in the form $\sum_{i=1}^k \varphi_i \otimes \tilde{\sigma}_i$ for some $\varphi_i \in L^\infty(X, \mu, \mathbb{Z})$ and $\tilde{\sigma}_i \in S_m(\tilde{M})$ with the property that $\tilde{\sigma}_i \neq \tilde{\sigma}_j$ for every $i, j \in \{1, \dots, k\}$ with $i \neq j$. The explicit algorithm is the following: for every $a, a' \in \mathbb{Z}$, $\varphi, \varphi' \in L^\infty(X, \mu, \mathbb{Z})$ and $\tilde{\sigma}, \tilde{\sigma}' \in S_m(\tilde{M})$ we have that

1. writings of the form $\varphi \otimes (a \cdot \tilde{\sigma} + a' \cdot \tilde{\sigma}')$ are always replaced with $\varphi \cdot a \otimes \tilde{\sigma} + \varphi \cdot a' \otimes \tilde{\sigma}'$;

2. writing of the form $\varphi \otimes \tilde{\sigma} + \varphi' \otimes \tilde{\sigma}$ are always replaced with $(\varphi + \varphi') \otimes \tilde{\sigma}$.

Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$ and universal covering $p : \widetilde{M} \to M$ and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard $\Gamma$-space;

**Definition 5.2.3** (reduced form). A chain $\sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i \in C_*(M; L^\infty(X, \mu, \mathbb{Z}))$ is in *reduced form* if $p \circ \tilde{\sigma}_i \neq p \circ \tilde{\sigma}_j$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

The condition of being in reduced form means that the singular simplices $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ do not lift from the same singular simplex in $M$.

**Lemma 5.2.4** (existence of reduced form). *Every $(\alpha, \mu)$-parametrized chain admits a representative in reduced form;*

*Proof.* Let $c \in C_m(M; L^\infty(X, \mu, \mathbb{Z}))$ be represented by $\sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i$. If $\sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i$ is not in reduced form, then $k \geq 2$ (because chains in the form $\varphi \otimes \tilde{\sigma}$ are trivially reduced) and there must exist two different indices $i, j \in \{1, \ldots, k\}$ such that $p \circ \tilde{\sigma}_i = p \circ \tilde{\sigma}_j$. After a permutation of the indices we may assume that $i = k - 1$ and $j = k$. In such a case $\tilde{\sigma}_k = \gamma \cdot \tilde{\sigma}_{k-1}$ for some $\gamma \in \Gamma$. Then the expression

$$\varphi_1 \otimes \tilde{\sigma}_1 + \cdots + \varphi_{k-2} \otimes \tilde{\sigma}_{k-2} + (\varphi_{k-1} + \varphi_k \cdot \gamma) \otimes \tilde{\sigma}_{k-1}$$

represents $c$ and involves $k - 1$ summands. By iterating the argument we obtain a representative in reduced form. $\square$

The twisted counterpart of the integral $\ell^1$-norm is the $(\alpha, \mu)$-parametrized $\ell^1$-norm.

**Definition 5.2.5** (parametrized $\ell^1$-norm). The $(\alpha, \mu)$-*parametrized $\ell^1$-norm* of the $m$-chain $c \in C_m(M; L^\infty(X, \mu, \mathbb{Z}))$ is the infimum

$$| c |^{(\alpha,\mu)} = \inf \left\{ \sum_{i=1}^{k} \int_X |\varphi_i| d\mu : \sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i \text{ reprsents } c \text{ in } C_m(M; L^\infty(X, \mu, \mathbb{Z})) \right\}$$

**Lemma 5.2.6** (properties of parametrized $\ell^1$-norm). *Let $c \in C_m(M; L^\infty(X, \mu, \mathbb{Z}))$ be an $(\alpha, \mu)$-parametrized chain of $M$; then*

1. *the infimum defining the $(\alpha, \mu)$-parametrized $\ell^1$-norm is attained by the representatives in reduced form, namely*

$$| c |^{(\alpha,\mu)} = \inf \left\{ \sum_{i=1}^{k} \int_X |\varphi_i| d\mu : \sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i \textit{ is in reduced form and represents } c \right\}$$

2. *if $\sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i$ and $\sum_{j=1}^{k'} \varphi'_j \otimes \tilde{\sigma}'_j$ are both in reduced form and represent $c$, then*

$$\sum_{i=1}^{k} \int_X |\varphi_i| d\mu = \sum_{j=1}^{k'} \int_X |\varphi'_j| d\mu$$

*3. the function* $|\!\cdot\!|^{(\alpha,\mu)} : C_m(M; L^\infty(X,\mu,\mathbb{Z})) \to [0,+\infty[$ *is a* $\mathbb{Z}$*-module norm.*

*Proof.*

1. let $\sum_{i=1}^k \varphi_i \otimes \tilde{\sigma}_i$ be a representative of $c$. If it is not in reduced form, then $k \geq 2$ and (up to a permutation of the indices) $\tilde{\sigma}_k = \gamma \cdot \tilde{\sigma}_{k-1}$ for some $\gamma \in \Gamma$. Then

$$\varphi_1 \otimes \tilde{\sigma}_1 + \cdots + \varphi_{k-2} \otimes \tilde{\sigma}_{k-2} + (\varphi_{k-1} + \varphi_k \cdot \gamma) \otimes \tilde{\sigma}_{k-1}$$

represents $c$, involves $k-1$ summands and

$$\int_X |\varphi_{k-1} + \varphi_k \cdot \gamma| d\mu \leq \int_X |\varphi_{k-1}| d\mu + \int_X |\varphi_k \cdot \gamma| d\mu \stackrel{(*)}{=} \int_X |\varphi_{k-1}| d\mu + \int_X |\varphi_k| d\mu$$

where $(*)$ holds because the action is measure preserving. We can repeat this argument a finite number of times until we obtain a representative of $c$ in reduced form.

2. let $\sum_{i=1}^k \varphi_i \otimes \tilde{\sigma}_i$ be a representative of $c$ in reduced form. By an explicit construction of the tensor product every other representative of $c$ in $L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_m(\tilde{M};\mathbb{Z})$ is obtained from $\sum_{i=1}^k \varphi_i \otimes \tilde{\sigma}_i$ by adding $\mathbb{Z}$-linear combinations of the expressions

$$\varphi \otimes (\tilde{\sigma} + \tilde{\sigma}') - \varphi \otimes \tilde{\sigma} - \varphi \otimes \tilde{\sigma}'$$
$$(\varphi + \varphi') \otimes \tilde{\sigma} - \varphi \otimes \tilde{\sigma} - \varphi' \otimes \tilde{\sigma}$$
$$\varphi \cdot \gamma \otimes \tilde{\sigma} - \varphi \otimes \gamma \cdot \tilde{\sigma}$$

for $\varphi, \varphi' \in L^\infty(X,\mu,\mathbb{Z})$, $\tilde{\sigma}, \tilde{\sigma}' \in S_m(\tilde{M})$ and $\gamma \in \Gamma$. According to Remark 5.2.2, expressions of the first and second type in the form

$$\varphi \otimes (\tilde{\sigma} + \tilde{\sigma}') \text{ and } \varphi \otimes \tilde{\sigma} + \varphi' \otimes \tilde{\sigma}$$

are not taken into account in our description of the elements of $C_m(\tilde{M}; L^\infty(X,\mu,\mathbb{Z}))$, whereas changing $\varphi \cdot \gamma \otimes \tilde{\sigma}$ with $\varphi \otimes \gamma \cdot \tilde{\sigma}$ does not change the quantity $\int_X |\varphi| d\mu$, because the action of $\Gamma$ is measure preserving.

3. we need to check the $\mathbb{Z}$-module norm axioms:
   a) the zero-chain is represented by $0 \otimes \tilde{\sigma}$ for any $\tilde{\sigma} \in S_m(\tilde{M})$ and thus has zero norm. If $c \neq 0$, then a reduced representative of $c$ contains at least a summand of the form $\varphi \otimes \tilde{\sigma}$ for some $\varphi \in L^\infty(X,\mu,\mathbb{Z}) \smallsetminus \{0\}$ and some $\tilde{\sigma} \in S_m(\tilde{M})$, hence has strictly positive norm.
   b) let $\sum_{i=1}^k \varphi_i \otimes \tilde{\sigma}_i$ be a representative of $c$ in reduced form and let $a \in \mathbb{Z}$. Then $\sum_{i=1}^k a \cdot \varphi_i \otimes \tilde{\sigma}_i$ represents $a \cdot c$ and

$$|a \cdot c|^{(\alpha,\mu)} = \sum_{i=1}^k \int_X |a \cdot \varphi_i| d\mu = |a| \cdot \sum_{i=1}^k \int_X |\varphi_i| d\mu = |a| \cdot |c|^{(\alpha,\mu)}$$

c) let $c, c' \in C_m(M; L^\infty(X,\mu,\mathbb{Z}))$ and let $\sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i$ and $\sum_{j=k+1}^{k+k'} \varphi_j \otimes \tilde{\sigma}_j$ be representatives of $c$ and $c'$ respectively in reduced form. Then $\sum_{i=1}^{k+k'} \varphi_i \otimes \tilde{\sigma}_i$ represents $c + c'$ but is generally not in reduced form and hence

$$\left| c + c' \right|^{(\alpha,\mu)} \leq \sum_{i=1}^{k+k'} |\varphi_i| d\mu = \left| c \right|^{(\alpha,\mu)} + \left| c' \right|^{(\alpha,\mu)}$$

proving the triangular inequality. □

**Definition 5.2.7** (integral foliated simplicial volume)**.** Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$ and universal covering $p : \tilde{M} \to M$;

1. let $\alpha : \Gamma \curvearrowright (X,\mu)$ be a standard $\Gamma$-space and let $[M]^{(\alpha,\mu)}$ be the $(\alpha,\mu)$-parametrized fundamental class of $M$; the $(\alpha,\mu)$-*parametrized simplicial volume* of $M$ is the infimum
$$\left| M \right|^{(\alpha,\mu)} = \inf\{ \left| c \right|^{(\alpha,\mu)} : c \in C_n(M; L^\infty(X,\mu,\mathbb{Z})), [c] = [M]^{(\alpha,\mu)}\}$$
of all of the $(\alpha,\mu)$-parametrized $\ell^1$-norms of the $(\alpha,\mu)$-parametrized fundamental cycles of $M$;

2. the *integral foliated simplicial volume* of $M$ is the infimum
$$\left| M \right| = \inf_{\alpha:\Gamma\curvearrowright(X,\mu)} \left| M \right|^{(\alpha,\mu)}$$
where $\alpha : \Gamma \curvearrowright (X,\mu)$ ranges in the set of Borel $\Gamma$-isomorphism classes of standard $\Gamma$-spaces.

**Remark 5.2.8.** Let $\Gamma$ be a countable group. The class of isomorphism classes of standard $\Gamma$-spaces forms a set [28, Remark 5.26]. This claim relies on the fact that there exists a universal standard $\Gamma$-space $U_\Gamma$, i.e. a standard $\Gamma$-space in which every standard $\Gamma$-space embeds by a morphism in $\Gamma$-Standard [1, Theorem 2.6.1].

**Terminology 5.2.9** (cheap manifolds and fixed price)**.** Similarly to the cost of groups we say that an oriented closed connected manifold $M$:

1. is *cheap* if $\left| M \right| = 0$;

2. has *fixed price* if $\left| M \right|^{(\alpha,\mu)} = \left| M \right|^{(\beta,\nu)}$ holds for all essentially free standard actions $\alpha : \Gamma \curvearrowright (X,\mu)$ and $\beta : \Gamma \curvearrowright (Y,\nu)$.

**Question 5.2.10** (fixed price problem for IFSV)**.** We can state an analogous fixed price problem for integral foliated simplicial volume. Let $M$ be an oriented closed connected manifold with fundamental group $\Gamma$; if $\alpha : \Gamma \curvearrowright (X,\mu)$ and $\beta : \Gamma \curvearrowright (Y,\nu)$ are free standard probability actions, then is it always true that $\left| M \right|^{(\alpha,\mu)} = \left| M \right|^{(\beta,\nu)}$? As in the case of the group cost, the fixed price problem for manifolds is still open [20, Definition 1.3].

### 5.2.2. First properties of the integral foliated simplicial volume

**Proposition 5.2.11** (comparison with (integral) simplicial volume). *Let $M$ be a closed connected oriented $n$-dimensional manifold with fundamental group $\Gamma$ and universal covering $p : \tilde{M} \to M$; then*

$$\|M\|_{\mathbb{R}} \leq \boldsymbol{|}\, M \,\boldsymbol{|} \leq \|M\|_{\mathbb{Z}}$$

*Proof.*

1. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard $\Gamma$-space and consider the function

$$\begin{aligned} \Phi : C_n(M; L^\infty(X, \mu, \mathbb{Z})) &\to C_n(M; \mathbb{R}) \\ \varphi \otimes \tilde{\sigma} &\mapsto \left(\textstyle\int_X \varphi d\mu\right) \cdot (p \circ \tilde{\sigma}) \end{aligned}$$

   Then $\Phi$ is well-defined: if $\gamma \in \Gamma$, then $\varphi \cdot \gamma \otimes \tilde{\sigma}$ and $\varphi \otimes \gamma \cdot \tilde{\sigma}$ are both mapped to $\int_X \varphi d\mu$. Moreover it is a $\mathbb{Z}$-modules homomorphism by the $\mathbb{Z}$-linearity of the integral operator $\int_X (\cdot) d\mu : L^\infty(X, \mu, \mathbb{Z}) \to \mathbb{R}$ and makes the diagram

$$\begin{tikzcd} C_n(M;\mathbb{Z}) \arrow[r, "i_M^\alpha"] \arrow[dr, bend right, "i"'] & C_n(M; L^\infty(X,\mu,\mathbb{Z})) \arrow[d, "\Phi"] \\ & C_n(M;\mathbb{R}) \end{tikzcd}$$

   commutative, where $i : C_n(M; \mathbb{Z}) \to C_n(M; \mathbb{R})$ is induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. In particular

$$H_n(\Phi)([M]^{(\alpha,\mu)}) = H_n(\Phi \circ i_M^\alpha)([M]_{\mathbb{Z}}) = H_n(i)([M]_{\mathbb{Z}}) = [M]_{\mathbb{R}}$$

   which says that $\Phi$ maps $(\alpha, \mu)$-parametrized fundamental cycles to real fundamental cycles. Let $c = \sum_{i=1}^k \varphi_i \otimes \tilde{\sigma}_i \in C_n(M; L^\infty(X, \mu, \mathbb{Z}))$ be an $(\alpha, \mu)$-parametrized fundamental cycle of $M$ in reduced form. Then

$$\|M\|_{\mathbb{R}} \leq \|\Phi(c)\|_1^{\mathbb{R}} = \sum_{i=1}^k \left| \int_X \varphi_i d\mu \right| \leq \sum_{i=1}^k \int_X |\varphi_i| d\mu = \boldsymbol{|}\, c \,\boldsymbol{|}^{(\alpha,\mu)}$$

   and by taking the infimum among all of the $(\alpha, \mu)$-parametrized fundamental cycles of $M$ we obtain $\|M\|_{\mathbb{R}} \leq \boldsymbol{|}\, M \,\boldsymbol{|}^{(\alpha,\mu)}$ for every standard probability action $\alpha : \Gamma \curvearrowright (X, \mu)$.

2. The change of coefficients $C_n(M; \mathbb{Z}) \to C_n(M; L^\infty(X, \mu, \mathbb{Z}))$ induced by the inclusion of constant functions $\mathbb{Z} \hookrightarrow L^\infty(X, \mu, \mathbb{Z})$ is an isometry with respect to the $(\alpha, \mu)$-parametrized $\ell^1$-norm and maps integral fundamental cycles to $(\alpha, \mu)$-parametrized fundamental cycles. Hence $\boldsymbol{|}\, M \,\boldsymbol{|}^{(\alpha,\mu)} \leq \|M\|_{\mathbb{Z}}$. □

**Proposition 5.2.12** (integral foliated simplicial volume of simply connected manifolds, [28, Proposition 5.29]). *Let $M$ be an oriented closed connected $n$-dimensional manifold; if $M$ is simply connected, then*

$$\boldsymbol{|}\, M \,\boldsymbol{|} = \|M\|_{\mathbb{Z}}$$

*Proof.* Let $(X,\mu)$ be any standard Borel probability space. Then $\pi_1(M)=1$ acts on $(X,\mu)$ via the only possible action, namely trivial action, and hence

$$L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} C_n(\tilde{M};\mathbb{Z}) \cong L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z})$$

A singular chain $c=\sum_{i=1}^k \varphi_i \otimes \sigma_i \in L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z})$ identifies an integral singular chain $c_x=\sum_{i=1}^k \varphi_i(x)\cdot\sigma_i \in C_n(M;\mathbb{Z})$ for almost every $x\in X$ and, if $c$ and $c'$ are homologous in $L^\infty(X,\mu,\mathbb{Z})\otimes_{\mathbb{Z}} C_n(M;\mathbb{Z})$, then so are $c_x$ and $c'_x$ in $C_n(M;\mathbb{Z})$ for almost every $x\in X$. Moreover parametrized fundamental cycles induce integral fundamental cycles for almost every $x\in X$, hence for every $c=\sum_{i=1}^k \varphi_i\otimes\sigma$ we have that

$$\|c_x\|_{\mathbb{Z}} = \sum_{i=1}^k |\varphi_i(x)| \geq \|M\|_{\mathbb{Z}}$$

for almost every $x\in X$. We obtain that

$$\begin{aligned}
\mid c \mid^{(\alpha,\mu)} &= \sum_{i=1}^k \int_X |\varphi_i(x)| d\mu(x)\\
&= \int_X \sum_{i=1}^k |\varphi_i(x)| d\mu(x)\\
&\geq \int_X \|M\|_{\mathbb{Z}} d\mu(x)\\
&= \|M\|_{\mathbb{Z}}
\end{aligned}$$

for every $(\alpha,\mu)$-parametrized fundamental cycle $c\in C_n(M;L^\infty(X,\mu,\mathbb{Z}))$. The inequality $\mid M \mid \leq \|M\|_{\mathbb{Z}}$ is always true. ☐

**Proposition 5.2.13** (comparing parameter spaces). *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$, let $\alpha:\Gamma \curvearrowright (X,\mu)$ and $\beta:\Gamma\curvearrowright (Y,\nu)$ be standard $\Gamma$-spaces and let $f:X\to Y$ be a measurable $\Gamma$-equivariant map with the property that*

$$\mu(f^{-1}(B)) \leq \nu(B)$$

*for all measurable subsets $B\subseteq Y$. Then*

$$\mid M \mid^{(\alpha,\mu)} \leq \mid M \mid^{(\beta,\nu)}$$

*Proof.* The $\mathbb{Z}[\Gamma]$-chain homomorphism

$$\begin{aligned}
F = f^*\otimes \mathrm{id}_{C_*(M;\mathbb{Z})} : C_*(M;L^\infty(Y,\nu,\mathbb{Z})) &\to C_*(M;L^\infty(Y,\nu,\mathbb{Z}))\\
\varphi\otimes\tilde{\sigma} &\mapsto \varphi\circ f\otimes\tilde{\sigma}
\end{aligned}$$

together with the property $\mu(f^{-1}(B))\leq\nu(B)$ for all measurable subsets $B\subseteq Y$ gives that $\mid F(c) \mid^{(\alpha,\mu)} \leq \mid c \mid^{(Y,\nu)}$ for every chain $c\in C_*(M;L^\infty(Y,\nu,\mathbb{Z}))$. By using $B=f(X)$ we obtain that

$$1=\mu(X)=\mu(f^{-1}(f(X)))\leq\nu(f(X))$$

showing that $f$ is $\nu$-almost surjective. In particular $f^*$ maps $\nu$-almost constant functions to $\mu$-almost constant functions with the same value. Hence $F$ maps integral fundamental cycles to integral fundamental cycles and hence $(\beta, \nu)$-parametrized fundamental cycles to $(\alpha, \mu)$-parametrized fundamental cycles, since every $(\beta, \nu)$-parametrized fundamental cycle can be written in the form $c_{\mathbb{Z}} + \partial d$ for some integral fundamental cycle $c_{\mathbb{Z}}$ and for some $(n+1)$-cycle $d$. Taking the infimum over all $(\beta, \nu)$-parametrized fundamental cycles of $M$ we get $| M |^{(\beta,\nu)} \geq | M |^{(\alpha,\mu)}$. $\square$

**Proposition 5.2.14** (products of parameter spaces). *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$;*

1. *if $\{\alpha_i : \Gamma \curvearrowright (X_i, \mu_i)\}_{i\in\mathbb{N}}$ is a family of standard $\Gamma$-spaces, then also the product*
$$(Z, \zeta) = \left( \prod_{i\in\mathbb{N}} X_i, \bigotimes_{i\in\mathbb{N}} \mu_i \right)$$
*with diagonal action $\alpha : \Gamma \curvearrowright (Z, \zeta)$ (where $\gamma \cdot (x_i)_{i\in\mathbb{N}} = (\gamma \cdot x_i)_{i\in\mathbb{N}}$) is a standard $\Gamma$-space and*
$$| M |^{(\alpha,\zeta)} \leq \inf_{i\in\mathbb{N}} | M |^{(\alpha_i,\mu_i)}$$

2. *let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard $\Gamma$-space and let $(Y, \nu)$ be a standard Borel probability space; then*
$$| M |^{(\alpha',\mu\otimes\nu)} = | M |^{(\alpha,\mu)}$$
*where the action $\alpha' : \Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ is induced by $\alpha : \Gamma \curvearrowright (X, \mu)$.*

*Proof.*

1. One just needs to apply Proposition 5.2.13 to the projection $p_i : Z \to X_i$ for every $i \in I$.

2. The trivial action $\Gamma \curvearrowright (Y, \nu)$ gives $(Y, \nu)$ a standard $\Gamma$-spaces structure, so that $| M |^{(\alpha',\mu\otimes\nu)} \leq | M |^{(\alpha,\mu)}$. On the other hand let
$$c = \sum_{i=1}^{k} \varphi_i \otimes \tilde{\sigma}_i \in C_n(M; L^\infty(X \times Y, \mu \otimes \nu, \mathbb{Z}))$$
be an $(\alpha', \mu \otimes \nu)$-parametrized fundamental cycle for $M$ in reduced form. If we fix an integral fundamental cycle $c_{\mathbb{Z}} \in C_n(M; \mathbb{Z})$, then there exists an $(n + 1)$-cycle $d \in C_{n+1}(M; L^\infty(X \times Y, \mu \times \nu, \mathbb{Z}))$ such that $c = c_{\mathbb{Z}} + \partial d$. For $\nu$-almost every $y \in Y$ the chain
$$c_y = \sum_{i=1}^{k} \varphi_i(\cdot, y) \otimes \tilde{\sigma}_i \in C_n(M; L^\infty(X, \mu, \mathbb{Z}))$$

is well-defined and an $(\alpha,\mu)$-parametrized fundamental cycle of $M$, being equal to $c_{\mathbb{Z}} + \partial(d_y)$. Fubini's theorem gives

$$\begin{aligned} \| c \|^{(\alpha',\mu\otimes\nu)} &= \sum_{i=1}^{k} \int_{X\times Y} |\varphi_i| d(\mu\otimes\nu) \\ &= \int_Y \int_X \sum_{i=1}^{k} |\varphi_i(x,y)| d\mu(x) d\mu(y) \\ &= \int_Y \| c_y \|^{(\alpha,\mu)} d\nu(y) \end{aligned}$$

and in particular there exists $y \in Y$ such that $\left\| c_y \right\|^{(\alpha,\mu)} \leq \| c \|^{(\alpha',\mu\otimes\nu)}$. Taking the infimum among all of the $(\alpha',\mu\otimes\nu)$-parametrized fundamental cycles gives $\| M \|^{(\alpha',\mu\otimes\nu)} = \| M \|^{(\alpha,\mu)}$. □

**Corollary 5.2.15.** *Let $M$ be a closed connected oriented $n$-dimensional manifold with fundamental group $\Gamma$; there exists a standard $\Gamma$-space $\alpha : \Gamma \curvearrowright (X,\mu)$ with essentially free action such that*

$$\| M \| = \| M \|^{(\alpha,\mu)}$$

*Proof.* Let $\alpha_0 : \Gamma \curvearrowright (X_0,\mu_0)$ be a standard $\Gamma$-space with essentially free action, whose existence is granted by Corollary 2.1.12. For every $k \in \mathbb{N} \smallsetminus \{0\}$ let $\alpha_k : \Gamma \curvearrowright (X_k,\mu_k)$ be a standard $\Gamma$-space satisfying

$$\| M \|^{(\alpha_k,\mu_k)} \leq \| M \| + \frac{1}{k}$$

Then the diagonal action $\alpha : \Gamma \curvearrowright (\prod_{k\in\mathbb{N}} X_k, \bigotimes_{k\in\mathbb{N}} \mu_k)$ given by $\gamma\cdot(x_k)_{k\in\mathbb{N}} = (\gamma\cdot x_k)_{k\in\mathbb{N}}$ is essentially free and

$$\| M \|^{(\alpha,\otimes_k\mu_k)} \leq \inf_{k\in\mathbb{N}} \| M \|^{(\alpha_k,\mu_k)} \leq \inf_{k\in\mathbb{N}} \| M \| + \frac{1}{k} = \| M \|$$

by the first point of Proposition 5.2.14. □

As we may guess, ergodic actions are of particular interest in this context as well.

**Proposition 5.2.16.** *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$ and let $\alpha : \Gamma \curvearrowright (X,\mu)$ be an ergodic standard probability action; then the change of coefficients $i_M^\alpha : C_n(M;\mathbb{Z}) \to C_n(M; L^\infty(X,\mu,\mathbb{Z}))$ induces an isomorphism*

$$H_n(i_M^\alpha) : H_n(M;\mathbb{Z}) \to H_n(M; L^\infty(X,\mu,\mathbb{Z}))$$

*in the $n$-th homology $\mathbb{Z}$-module.*

*Proof.* Recall that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$, because $M$ has dimension $n$. The generalized Poincaré duality for local coefficients gives $H_n(M; L^\infty(X,\mu,\mathbb{Z})) \cong H^0(M; L^\infty(X,\mu,\mathbb{Z}))$, being the right hand side isomorphic with $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, L^\infty(X,\mu,\mathbb{Z}))$ by the universal coefficients theorem for cohomology, having $\mathbb{Z}$ the trivial $\mathbb{Z}[\Gamma]$-module structure [26, Theorem 12.11]. Let us study the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, L^\infty(X,\mu,\mathbb{Z}))$: a $\mathbb{Z}[\Gamma]$-module homomorphism $F : \mathbb{Z} \to L^\infty(X,\mu,\mathbb{Z})$ (which is uniquely determined by the image of the generator $1 \in \mathbb{Z}$) must be $\mathbb{Z}$-linear and satisfy

$$F(1 \cdot \gamma) = F(1) = F(1) \cdot \gamma$$

for every $\gamma \in \Gamma$. In particular $F(1) \in L^\infty(X,\mu,\mathbb{Z})^\Gamma$. On the other hand let $\varphi \in L^\infty(X,\mu,\mathbb{Z})^\Gamma$ be a $\Gamma$-invariant essentially bounded class of functions. Then the function

$$\begin{aligned} F_\varphi : \mathbb{Z} &\to L^\infty(x,\mu,\mathbb{Z}) \\ i &\mapsto i \cdot \varphi \end{aligned}$$

is a homomorphism of $\mathbb{Z}[\Gamma]$-modules. Thus the associations

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, L^\infty(X,\mu,\mathbb{Z})) &\to L^\infty(X,\mu,\mathbb{Z})^\Gamma \\ F &\mapsto F(1) \\ F_\varphi &\leftarrow\!\shortmid \varphi \end{aligned}$$

are mutually inverse $\mathbb{Z}$-modules homomorphism. Since the action $\alpha : \Gamma \curvearrowright (X,\mu)$ is ergodic, we also have that the inclusion $\mathbb{Z} \hookrightarrow L^\infty(X,\mu,\mathbb{Z})$ is an isomorphism by Proposition 2.2.7, showing the claim. □

### 5.2.3. Integral foliated simplicial volume and ergodic decomposition

We wish to investigate the relation between the $(\alpha,\mu)$-parametrized simplicial volume and an ergodic decomposition $\beta : X \to \mathrm{Erg}(\alpha)$ of the action $\alpha : \Gamma \curvearrowright (X,\mu)$. What we already know is that the infimum defining the integral foliated simplicial volume can be computed taking into account only standard $\Gamma$-spaces with an ergodic action.

**Proposition 5.2.17** (ergodic parameters, [4, Proposition 4.14]). *Let $M$ be an oriented closed connected manifold with fundamental group $\Gamma$ and let $(X,\mu)$ be a standard Borel probability space; for every measure preserving action $\alpha : \Gamma \curvearrowright (X,\mu)$ and for every $\varepsilon > 0$ there exists a probability measure $\mu'$ on $X$ such that the action $\alpha : \Gamma \curvearrowright (X,\mu')$ is (measure preserving and) ergodic and*

$$| M |^{(\alpha,\mu')} \leq | M |^{(\alpha,\mu)} + \varepsilon$$

*In particular for every $\varepsilon > 0$ there exists a standard $\Gamma$-space $(Y,\nu)$ with ergodic action $\beta : \Gamma \curvearrowright (Y,\nu)$ such that*

$$| M |^{(\beta,\nu)} \leq | M | + \varepsilon$$

**Remark 5.2.18** (explicit construction of $C_*(M; L^\infty(X,\mu,\mathbb{Z}))$). Let us denote by $B(X,\mathbb{Z})$ the $\mathbb{Z}[\Gamma]$-module of bounded functions $X \to \mathbb{Z}$ and by

$$N(X,\mu,\mathbb{Z}) = \{f \in B(X,\mathbb{Z}) : f = 0 \ \mu\text{-almost everywhere}\}$$

the submodule of globally bounded functions that are zero $\mu$-almost everywhere, so that $L^\infty(X,\mu,\mathbb{Z}) = B(X,\mathbb{Z})/N(X,\mu,\mathbb{Z})$ as $\mathbb{Z}[\Gamma]$-modules. Since the $C_i(\tilde{M};\mathbb{Z})$'s are free $\mathbb{Z}[\Gamma]$-module, we can tensor

$$0 \to N(X,\mu,\mathbb{Z}) \to B(X,\mathbb{Z}) \to L^\infty(X,\mu,\mathbb{Z}) \to 0$$

to get an isomorphism of chain complexes

$$\frac{B(X,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(\tilde{M};\mathbb{Z})}{N(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(\tilde{M};\mathbb{Z})} \cong L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(\tilde{M};\mathbb{Z})$$

explicitly given on the generators by $(f + N(X,\mu,\mathbb{Z})) \otimes \tilde{\sigma} \mapsto [f]_{=0} \otimes \tilde{\sigma}$ for $f \in B(X,\mathbb{Z})$ and $\tilde{\sigma} \in S_i(\tilde{M})$.

**Remark 5.2.19** (integrals and ergodic decomposition)**.** Let $\alpha : \Gamma \curvearrowright (X,\mu)$ be a standard $\Gamma$-space and let $\beta : X \to \operatorname{Erg}(\alpha)$ be an ergodic decomposition of $\alpha$. Let us take $f \in B(X,\mathbb{Z})$ and write it in the form $f = \sum_{i=M}^{N} i \cdot \chi_{A_i}$ for some $M, N \in \mathbb{Z}$, where $A_i = f^{-1}(i)$. By the definition of ergodic decomposition we obtain

$$\begin{aligned}
\int_X f d\mu &= \int_X \sum_{i=M}^{N} i \cdot \chi_{A_i} d\mu \\
&= \sum_{i=M}^{N} i \cdot \mu(A_i) \\
&= \sum_{i=M}^{N} i \cdot \int_X \beta_x(A_i) d\mu(x) \\
&= \int_X \sum_{i=M}^{N} i \cdot \beta_x(A_i) d\mu(x) \\
&= \int_X \left( \int_X f d\beta_x \right) d\mu(x)
\end{aligned}$$

*Proof. (of Proposition 5.2.17).* Let us set $n = \dim M$ and take $\varepsilon > 0$. By the definition of $| M |^{(\alpha,\mu)}$ as an infimum there exists an $(\alpha,\mu)$-parametrized fundamental cycle $c = \sum_{i=1}^{k} (f_i + N(X,\mu,\mathbb{Z})) \otimes \tilde{\sigma}_i \in C_n(M; L^\infty(X,\mu,\mathbb{Z}))$ such that

$$\sum_{j=i}^{k} \int_X |f_i| d\mu \leq | M |^{(\alpha,\mu)} + \varepsilon$$

Let us fix an integral fundamental cycle $c_\mathbb{Z} \in C_n(M;\mathbb{Z})$. Since both $c$ and $c_\mathbb{Z}$ represent $[M]^{(\alpha,\mu)}$, we can find an $(n+1)$-chain $d \in C_{n+1}(M; L^\infty(X,\mu,\mathbb{Z}))$ such that $c - c_\mathbb{Z} = \partial d$. By the above explicit description of $C_n(M, L^\infty(X,\mu,\mathbb{Z}))$ we may suppose that the coefficients $f_1, \ldots, f_k$ of $c$ and the coefficients of $d$ are in $B(X,\mathbb{Z})$ and that there exist a $\Gamma$-invariant $\mu$-null subset $A \subseteq X$ and $c' \in B(X,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{M};\mathbb{Z})$ such that

$$c - c_\mathbb{Z} = \partial d + \chi_A \cdot c' \in B(X,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{M};\mathbb{Z})$$

By Theorem 2.2.14, there exists an ergodic decomposition $\beta : X \to \mathrm{Erg}(\alpha)$ for $\alpha : \Gamma \curvearrowright (X, \mu)$. In particular

$$0 = \mu(A) = \int_X \beta_x(A) d\mu(x) \text{ and } \int_X \sum_{i=1}^{k} |f_i| d\mu = \int_X \int_X \sum_{i=1}^{k} |f_i| d\beta_x d\mu(x)$$

and hence we can find $z \in X$ such that

$$\beta_z(A) = 0 \text{ and } \int_X \sum_{i=1}^{k} |f_i| d\beta_z \leq \int_X \sum_{i=1}^{k} |f_i| d\mu$$

Let us define $c_z$ to be the class of (the representative $c_{\mathbb{Z}} + \partial d + \chi_A \cdot c'$ of) $c$ in

$$\frac{B(X,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{M};\mathbb{Z})}{N(X,\beta_z,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{M};\mathbb{Z})} \cong C_n(M; L^\infty(X, \beta_z, \mathbb{Z}))$$

From the equality $c - c_{\mathbb{Z}} = \partial d + \chi_A \cdot c'$ in $B(X,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{M};\mathbb{Z})$ and the fact that $\beta_z(A) = 0$ we obtain that

$$c_z - c_{\mathbb{Z}} = \partial d_z \in C_n(M; L^\infty(X, \beta_z, \mathbb{Z}))$$

which says that $c_z$ is a $(X, \beta_z)$-parametrized fundamental cycle of $M$. By the inequality

$$| c_z |^{(\alpha,\beta_z)} \leq \sum_{i=1}^{k} \int_X |f_i| d\beta_z \leq \int_X \sum_{i=1}^{k} |f_i| d\mu \leq | M |^{(\alpha,\mu)} + \varepsilon$$

we conclude. $\square$

**Warning 5.2.20.** In the former proof to a cycle $c \in C_n(M; L^\infty(X, \mu, \mathbb{Z}))$ we associated a cycle $c_z \in C_n(M; L^\infty(X, \beta_z, \mathbb{Z}))$. This association is not unique, since $c_z$ does depend on the bounded functions representing the coefficients of $c$. The next basic example will clarify the issue.

**Example 5.2.21.** If $(f + N(X, \mu, \mathbb{Z})) \otimes \tilde{\sigma}$ is an $(\alpha, \mu)$-chain, then $(f + \chi_A + N(X, \mu, \mathbb{Z})) \otimes \tilde{\sigma}$ is the same chain in $C_n(M; L^\infty(X, \mu, \mathbb{Z}))$ for every measurable subset $A \subseteq X$ with $\mu(A) = 0$, but it is generally not true that $(f + N(X, \beta_z, \mathbb{Z})) \otimes \tilde{\sigma}$ and $(f + \chi_A + N(X, \beta_z, \mathbb{Z})) \otimes \tilde{\sigma}$ are the same chain in $C_n(M; L^\infty(X, \beta_z, \mathbb{Z}))$. Indeed the equality $0 = \mu(A) = \int_X \beta_x(A) d\mu(x)$ only implies that $\beta_x(A) = 0$ for $\mu$-almost every $x \in X$.

The following construction will overcome this problem. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard $\Gamma$-space and let $\beta : X \to \mathrm{Erg}(\alpha)$ be an ergodic decomposition of $\alpha$.

1. Let us fix an integral fundamental cycle $c_{\mathbb{Z}}$ of $M$, so that every $(\alpha, \mu)$-parametrized fundamental cycle $c$ of $M$ is of the form $c = c_{\mathbb{Z}} + \partial d$ for some $(n+1)$-chain $d \in C_{n+1}(M; L^\infty(X, \mu, \mathbb{Z}))$.

2. Explicitly let $\sum_{i=1}^{k}(f_i + N(X,\mu,\mathbb{Z})) \otimes \tilde{\tau}_i$ be a representative of $d$, so that

$$c_{\mathbb{Z}} + \partial d = c_{\mathbb{Z}} + \sum_{i=1}^{k}(f_i + N(X,\mu,\mathbb{Z})) \otimes \partial\tilde{\tau}_i$$

represents $c$ (we will not explicit a representative for $c_{\mathbb{Z}}$ to avoid a heavy notation).

3. For every $h_1,\dots,h_k \in N(X,\mu,\mathbb{Z})$ and every $x \in X$,

$$\sum_{i=1}^{k}(f_i + h_i + N(X,\beta_x,\mathbb{Z})) \otimes \tilde{\tau}_i$$

represents an $(\alpha,\beta_x)$-parametrized $(n+1)$-chain for $M$ and hence

$$\begin{aligned} c_{\mathbb{Z}} + \textstyle\sum_{i=1}^{k}(f_i + h_i + N(X,\beta_x,\mathbb{Z})) \otimes \partial\tilde{\tau}_i &= \\ &= c_{\mathbb{Z}} + \partial\left(\textstyle\sum_{i=1}^{k}(f_i + h_i + N(X,\beta_x,\mathbb{Z})) \otimes \tilde{\tau}_i\right)\end{aligned}$$

represents an $(\alpha,\beta_x)$-parametrized fundamental cycle for $M$.

4. Let us define

$$\begin{aligned} n_{\mu}^{\beta_x}(c) = \inf\Big\{ \Big| c_{\mathbb{Z}} + \partial\left(\textstyle\sum_{i=1}^{k}(f_i + h_i + N(X,\beta_x,\mathbb{Z})) \otimes \tilde{\tau}_i\right)\Big|^{(\alpha,\beta_x)} : \\ h_i \in N(X,\mu,\mathbb{Z}) \text{ and } \textstyle\sum_{i=1}^{k}(f_i + N(X,\mu,\mathbb{Z})) \otimes \tilde{\tau}_i \text{ represents } d\Big\}\end{aligned}$$

as the infimum of the $(\alpha,\beta_x)$-parametrized $\ell^1$-norms of all of the $(\alpha,\beta_x)$-parametrized fundamental cycles constructed from $c = c_{\mathbb{Z}} + \partial d$.

**Remark 5.2.22** (inf of the $n_{\mu}^{\beta_x}(c)$'s). Since every $(\alpha,\mu)$-parametrized fundamental cycle $c = c_{\mathbb{Z}} + \partial d \in Z_n(M; L^{\infty}(X,\mu,\mathbb{Z}))$ of $M$ defines a subset

$$\begin{aligned}\Big\{ c_{\mathbb{Z}} + \partial\left(\textstyle\sum_{i=1}^{k}(f_i + h_i + N(X,\beta_x,\mathbb{Z})) \otimes \tilde{\tau}_i\right) : \\ h_i \in N(X,\mu,\mathbb{Z}) \text{ and } \textstyle\sum_{i=1}^{k}(f_i + N(X,\mu,\mathbb{Z})) \otimes \tilde{\tau}_i \text{ represents } d\Big\}\end{aligned}$$

of $(\alpha,\beta_x)$-parametrizes fundamental cycles for $M$, we obtain an upper bound

$$\inf\{n_{\mu}^{\beta_x}(c) : c \text{ is an } (\alpha,\mu)\text{-parametrized fundamental cycle of } M\} \geq |M|^{(\alpha,\beta_x)}$$

for the $(\alpha,\beta_x)$-parametrized simplicial volume.

**Remark 5.2.23.** While the quantity $\int_X |f| d\beta_x$ is not an invariant for the equivalence class $f + N(X,\mu,\mathbb{Z}) \in L^{\infty}(X,\mu,\mathbb{Z})$, its integral $\int_X(\int_X |f| d\beta_x) d\mu(x)$ is. Indeed in view of Remark 5.2.19 we have

$$\int_X \left(\int_X |f| d\beta_x\right) d\mu(x) = \int_X |f| d\mu = \int_X |f+h| d\mu = \int_X \left(\int_X |f+h| d\beta_x\right) d\mu(x)$$

for every $h \in N(X,\mu,\mathbb{Z})$.

**Proposition 5.2.24.** *Let $\alpha : \Gamma \curvearrowright (X,\mu)$ be a standard $\Gamma$-space with ergodic decomposition $\beta : X \to \mathrm{Erg}(\alpha)$ and let $c \in Z_n(M; L^\infty(X,\mu,\mathbb{Z}))$; then*

$$| c |^{(\alpha,\mu)} = \int_X n_\mu^{\beta_x}(c) d\mu(x)$$

*Proof.* The proof is a matter of elementary but tedious computation. Let $c = c_{\mathbb{Z}} + \partial d$ as usual with $c_{\mathbb{Z}} = \sum_{i=1}^q a_i \otimes \tilde{\sigma}_i \in \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C_n(\tilde{M};\mathbb{Z})$ in reduced form and $d = \sum_{j=1}^r (f_j + N(X,\mu,\mathbb{Z})) \otimes \tilde{\tau}_j$. Computing the differential of $d$ we get

$$\begin{aligned}\partial d &= \sum_{j=1}^r \left( (f_j + N(X,\mu,\mathbb{Z})) \otimes \sum_{k=0}^{n+1} (-1)^k \cdot \tilde{\tau}_j \circ \varepsilon_{n+1}^k \right) \\ &= \sum_{j=1}^r \sum_{k=0}^{n+1} ((-1)^k \cdot f_j + N(X,\mu,\mathbb{Z})) \otimes \tilde{\tau}_j \circ \varepsilon_{n+1}^k\end{aligned}$$

Up to a permutation of $\{1,\dots,q\}$ we can find $q' \in \{0,\dots,q\}$ such that:

1. $p\circ\tilde{\sigma}_i \neq p\circ(\tilde{\tau}_j\circ\varepsilon_{n+1}^k)$ for every $1 \le i \le q'$ and every $(j,k) \in \{1,\dots,r\}\times\{0,\dots,n+1\}$;
2. for every $q'+1 \le i \le q$ there exists $(j,k) \in \{1,\dots,r\} \times \{0,\dots,n+1\}$ such that $p\circ\tilde{\sigma}_i = p\circ(\tilde{\tau}_j\circ\varepsilon_{n+1}^k)$. In such a case there exists $\gamma_i \in \Gamma$ such that $\tilde{\sigma}_i = \gamma_i\cdot(\tilde{\tau}_j\circ\varepsilon_{n+1}^k)$ and
$$\begin{aligned}a_i \cdot \chi_X \otimes \tilde{\sigma}_i &= a_i \cdot \chi_X \otimes \gamma_i \cdot (\tilde{\tau}_j \circ \varepsilon_{n+1}^k) \\ &= a_i \cdot \chi_X \cdot \gamma_i \otimes (\tilde{\tau}_j \circ \varepsilon_{n+1}^k) \\ &= a_i \cdot \chi_X \otimes (\tilde{\tau}_j \circ \varepsilon_{n+1}^k)\end{aligned}$$
because $X$ is trivially $\Gamma$-invariant.

By an inductive reducing process we can find $\{(j_i,k_i)\}_{i=q'+1}^q \subseteq \{1,\dots,r\}\times\{0,\dots,n+1\}$ and $J \subseteq (\{1,\dots,r\} \times \{0,\dots,n+1\}) \setminus \{(j_i,k_i)\}_{i=q'+1}^q$ such that

$$\begin{aligned}&\sum_{i=1}^{q'} (a_i + N(X,\mu,\mathbb{Z})) \otimes \tilde{\sigma}_i + \sum_{i=q'+1}^{q} (u_{j_i} + N(X,\mu,\mathbb{Z})) \otimes (\tilde{\tau}_{j_i} \circ \varepsilon_{n+1}^{k_i}) + \\ &\qquad + \sum_{(j,k)\in J} (v_j + N(X,\mu,\mathbb{Z})) \otimes (\tilde{\tau}_j \circ \varepsilon_{n+1}^k)\end{aligned}$$

is a representative for $c$ in reduced form, where the $u_{j_i}$'s are of the form $a_i + f_1 \cdot \gamma_1 + \cdots + f_m \cdot \gamma_m$ and the $v_j$'s are of the form $f_1 \cdot \gamma_1' + \cdots + f_l \cdot \gamma_l'$ for some $\gamma_i, \gamma_j' \in \Gamma$. For every $\{h_{q'+1},\dots,h_q\} \cup \{h_j : (j,k) \in J\} \subseteq N(X,\mu,\mathbb{Z})$ and for every $x \in X$, the $(\alpha,\beta_x)$-parametrized fundamental cycle

$$\begin{aligned}&\sum_{i=1}^{q'} (a_i + N(X,\beta_x,\mathbb{Z})) \otimes \tilde{\sigma}_i + \sum_{i=q'+1}^{q} (u_{j_i} + h_i + N(X,\beta_x,\mathbb{Z})) \otimes (\tilde{\tau}_{j_i} \circ \varepsilon_{n+1}^{k_i}) + \\ &\qquad + \sum_{(j,k)\in J} (v_j + h_j + N(X,\beta_x,\mathbb{Z})) \otimes (\tilde{\tau}_j \circ \varepsilon_{n+1}^k)\end{aligned}$$

is in reduced form. The invariance for the integrals granted by Remark 5.2.23 gives

$$\begin{aligned}
|c|^{(\alpha,\mu)} &= \sum_{i=1}^{q'}|a_i| + \sum_{i=q'+1}^{q}\int_X |u_{j_i}|d\mu + \sum_{(j,k)\in J}\int_X |v_j|d\mu && \text{(reduced form)}\\
&= \int_X\Big(\sum_{i=1}^{q'}|a_i| + \sum_{i=q'+1}^{q}\int_X |u_{j_i}|d\beta_x + \sum_{(j,k)\in J}\int_X |v_j|d\beta_x\Big)d\mu(x) && \text{(ergodic decomposition)}\\
&= \int_X n_\mu^{\beta_x}(c)d\mu(x) && \text{(Remark 5.2.23)}
\end{aligned}$$

and allows us to conclude. □

### 5.2.4. An ergodic decomposition formula for $|M|^{(\alpha,\mu)}$

We aim to achieve an ergodic decomposition formula for the parametrized simplicial volume that is analogue to the ergodic decomposition formula of Theorem 3.3.4 we have for cost. Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$ and universal covering $\tilde{M} \to M$.

Let $(X, \mathscr{B}_X, \mu)$ be a standard Borel probability space and let $\alpha : \Gamma \curvearrowright X$ be a standard action. Let us fix:

1. a family $S'_*(\tilde{M}) = (S'_k(\tilde{M}))_{k\in\mathbb{N}}$ of countably many singular simplices of $\tilde{M}$ with the properties of Lemma 4.3.4. Let us enumerate the set of $(n+1)$-simplices $S'_{n+1}(\tilde{M}) = (\tilde{\tau}'_i : \Delta^{n+1} \to \tilde{M})_{i\in\mathbb{N}}$;
2. an integral fundamental $n$-cycle $c_\mathbb{Z} \in C_n(\tilde{M};\mathbb{Z})$ for $M$;
3. a countable algebra $\mathscr{A} = \{A_j : j \in \mathbb{N}\} \subseteq P(\mathscr{B}_X)$ that is dense in the $\sigma$-algebra $\mathscr{B}_X$ in the following sense: for all measurable subsets $B \subseteq X$ and for all $\varepsilon > 0$ there is a $j \in \mathbb{N}$ such that $\mu(B \triangle A_j) < \varepsilon$. Replacing $\mathscr{A}$ with $\{g \cdot A_j \subseteq X : g \in \Gamma, j \in \mathbb{N}\}$, we may always assume that $\mathscr{A}$ is closed under the action of $\Gamma$.

With these new data we want to give an analogue of $B(X,\mathbb{Z})$, $N(X,\mu,\mathbb{Z})$ and $C_*(M; L^\infty(X,\mu,\mathbb{Z}))$, hence we denote by:

1. $Z'_k(\tilde{M};\mathbb{Z})$ and $B'_k(\tilde{M};\mathbb{Z})$ the submodules of singular $k$-cycles and $k$-boundaries of $\tilde{M}$ with respect to the chain complex $C'_*(\tilde{M};\mathbb{Z})$ for all $k \in \mathbb{N}$;
2. $B'(X,\mathbb{Z})$ the submodule of $B(X,\mathbb{Z})$ generated by $\{\chi_{A_j} : j \in \mathbb{N}\}$;
3. $N'(X,\mu,\mathbb{Z})$ the submodule $B'(X,\mathbb{Z}) \cap N(X,\mu,\mathbb{Z})$ of $B'(X,\mathbb{Z})$ of bounded functions that are zero $\mu$-almost everywhere. The inclusion $u : B'(X,\mathbb{Z}) \hookrightarrow B(X,\mathbb{Z})$ yields a well-defined $\mathbb{Z}$-linear map on the quotients

$$\begin{aligned}
\bar{u} : B'(X,\mathbb{Z})/N'(X,\mu,\mathbb{Z}) &\to B(X,\mathbb{Z})/N(X,\mu,\mathbb{Z})\\
f + N'(X,\mu,\mathbb{Z}) &\mapsto f + N(X,\mu,\mathbb{Z})
\end{aligned}$$

making the diagram

$$\begin{array}{ccc} B'(X,\mathbb{Z}) & \xrightarrow{u} & B(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \frac{B'(X,\mathbb{Z})}{N'(X,\mu,\mathbb{Z})} & \xrightarrow[\bar{u}]{} & \frac{B(X,\mathbb{Z})}{N(X,\mu,\mathbb{Z})} \end{array}$$

commutative. Moreover $\bar{u}$ is a monomorphism: if $\bar{u}(f + N'(X, \mu, \mathbb{Z})) = f + N(X, \mu, \mathbb{Z}) = N(X, \mu, \mathbb{Z})$, then $f \in N(X, \mu, \mathbb{Z}) \cap B'(X, \mathbb{Z}) = N'(X, \mu, \mathbb{Z})$. Hence we are allowed to view classes in $B'(X, \mathbb{Z})/N'(X, \mu, \mathbb{Z})$ as classes in $L^\infty(X, \mu, \mathbb{Z})$.

4. $C'_k(M; B'(X, \mathbb{Z})/N'(X, \mu, \mathbb{Z}))$ the tensor product of $\mathbb{Z}[\Gamma]$-modules

$$\frac{B'(X,\mathbb{Z})}{N'(X,\mu,\mathbb{Z})} \otimes_{\mathbb{Z}[\Gamma]} C'_k(\tilde{M}; \mathbb{Z})$$

for all $k \in \mathbb{N}$. Note that $B'(X, \mathbb{Z})$ is indeed a right $\mathbb{Z}[\Gamma]$-module, since $\mathscr{A}$ is closed under the action of $\Gamma$.

Our aim is to extract and parametrize countably many $(\alpha, \mu)$-parametrized fundamental cycles of $M$ that suffice to compute $| M |^{(\alpha,\mu)}$. Let us proceed in the following way:

1. take $I \in P_{\text{fin}}(\mathbb{N})$;
2. take $Q = (Q_i)_{i\in I} \in P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}$;
3. denote by $d(I, Q)$ the element

$$\sum_{i\in I} \left( \sum_{(q,r)\in Q_i} q \cdot \chi_{A_r} + N'(X, \mu, \mathbb{Z}) \right) \otimes \tilde{\tau}_i \in \frac{B'(X,\mathbb{Z})}{N'(X,\mu,\mathbb{Z})} \otimes_{\mathbb{Z}[\Gamma]} C'_{n+1}(\tilde{M}; \mathbb{Z})$$

4. every cycle of the form $c(I, Q) = c_{\mathbb{Z}} + \partial d(I, Q)$ is an $(\alpha, \mu)$-parametrized fundamental cycle of $M$.

In such a way we get

$$| d(I,Q) |^{(\alpha,\mu)} \leq \sum_{i\in I} \sum_{(q,r)\in Q_i} |q| \cdot \mu(A_r)$$

with equality when $d(I, Q)$ is in reduced form.

**Lemma 5.2.25** (finitary characterization of PSV). *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$ and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a standard action. Let us fix $C > 0$. With the same notations as above, the following are equivalent:*

1. *$| M |^{(\alpha,\mu)} < C$;*
2. *there exist $I \in P_{\text{fin}}(\mathbb{N})$ and $Q \in P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}$ such that $| c(I,Q) |^{(\alpha,\mu)} < C$.*

*Proof.* The implication ($2 \implies 1$) is trivial, thus we are only proving ($1 \implies 2$). From Lemma 4.3.4 the inclusion $\iota_* : C'_*(\tilde{M};\mathbb{Z}) \to C_*(\tilde{M};\mathbb{Z})$ is a $\mathbb{Z}[\Gamma]$-chain homotopy equivalence. Let us denote by $\varphi_* : C_*(\tilde{M};\mathbb{Z}) \to C'_*(\tilde{M};\mathbb{Z})$ a $\mathbb{Z}[\Gamma]$-chain homotopy inverse of $\iota_*$. By tensoring $C_*(\tilde{M};\mathbb{Z})$ and $C'_*(\tilde{M};\mathbb{Z})$ with $L^\infty(X,\mu,\mathbb{Z})$ we obtain that also $C_*(\tilde{M};L^\infty(X,\mu,\mathbb{Z}))$ and $C'_*(\tilde{M};L^\infty(X,\mu,\mathbb{Z})) = L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C'_*(\tilde{M};\mathbb{Z})$ are chain homotopy equivalent via $\mathrm{id}_{L^\infty(X,\mu,\mathbb{Z})} \otimes \varphi_*$ (see Proposition A.1.4) and the commutativity of the diagram

$$\begin{tikzcd} Z_n(M;L^\infty(X,\mu,\mathbb{Z})) \arrow[r, "\mathrm{id}_{L^\infty(X,\mu,\mathbb{Z})} \otimes\varphi_n"] \arrow[d] & Z'_n(M;L^\infty(X,\mu,\mathbb{Z})) \arrow[d] \\ H_n(M;L^\infty(X,\mu,\mathbb{Z})) \arrow[r, "\cong"'] & H'_n(M;L^\infty(X,\mu,\mathbb{Z})) \end{tikzcd}$$

shows that, if $c \in Z_n(M;L^\infty(X,\mu,\mathbb{Z}))$ is an $(\alpha,\mu)$-parametrized fundamental cycle for $M$, then so is $(\mathrm{id}_{L^\infty(X,\mu,\mathbb{Z})} \otimes\varphi_n)(c) \in Z'_n(M;L^\infty(X,\mu,\mathbb{Z}))$.

Let us fix an $(\alpha,\mu)$-parametrized fundamental cycle $c \in Z_n(M;L^\infty(X,\mu,\mathbb{Z}))$ with $| c |^{(\alpha,\mu)} < C$. We claim that we can always assume that $c \in Z'_n(M;L^\infty(X,\mu,\mathbb{Z}))$: indeed if $c = \sum_{j=1}^k \psi_i \otimes \tilde{\sigma}_i$ is in reduced form, then the image $\sum_{j=1}^k \psi_i \otimes \tilde{\sigma}'_i$ of $c$ under $\mathrm{id}_{L^\infty(X,\mu,\mathbb{Z})} \otimes\varphi_n$ might not necessarily be in reduced form and hence

$$\begin{aligned} | (\mathrm{id}_{L^\infty(X,\mu,\mathbb{Z})} \otimes\varphi_n)(c) |^{(\alpha,\mu)} &= \left| \sum_{j=1}^k \psi_i \otimes \tilde{\sigma}'_i \right|^{(\alpha,\mu)} \leq \sum_{i=1}^k \int_X |\psi_i| d\mu = \\ &= \left| \sum_{j=1}^k \psi_i \otimes \tilde{\sigma}_i \right|^{(\alpha,\mu)} = | c |^{(\alpha,\mu)} < C \end{aligned}$$

Since $c$ is an fundamental $n$-cycle, it is of the form $c = c_\mathbb{Z} + \partial d$ for some $d = \sum_{i\in I} h_i \otimes \tilde{\tau}'_i \in C'_{n+1}(M;L^\infty(X,\mu,\mathbb{Z}))$. By writing all of the $h_i$'s in the explicit form

$$h_i = \sum_{(q_i,r_i)\in Q_i} q_i \cdot \chi_{B_{r_i}} + N(X,\mu,\mathbb{Z}) \in L^\infty(X,\mu,\mathbb{Z}) \cong \frac{B(X,\mathbb{Z})}{N(X,\mu,\mathbb{Z})}$$

for some $Q_i \in P_{\mathrm{fin}}(\mathbb{Z}\times\mathbb{N})$ with measurable subsets $B_{r_i} \subseteq X$ for all $(q_i,r_i) \in Q_i$ and for all $i \in I$, we obtain a new description of $c$ in the form

$$c = c_\mathbb{Z} + \sum_{i\in I} \left( \sum_{(q_i,r_i)\in Q_i} q_i \cdot \chi_{B_{r_i}} + N(X,\mu,\mathbb{Z}) \right) \otimes \partial\tilde{\tau}'_i$$

Let us take $0 < \varepsilon < C - | c |^{(\alpha,\mu)}$, let $q = \max\{|Q_i| \in \mathbb{N} : i \in I\}$ and let $m = \max\{|q_i| \in \mathbb{N} : (q_i,r_i) \in Q_i, i \in I\}$. The density property of the algebra $\mathscr{A}$ gives that for every measurable subset $B_{r_i}$ there exists an $A \in \mathscr{A}$ such that

$$\mu(B_{r_i} \triangle A) < \frac{\varepsilon}{|I| \cdot q \cdot m}$$

By changing the enumeration $(A_j)_{j\in\mathbb{N}}$ of $\mathscr{A}$, we may always assume that $A_{r_i} \in \mathscr{A}$ approximates $B_{r_i}$ for all $(q_i, r_i) \in Q_i$ and for all $i \in I$. Let us consider

$$c(I,Q) = c_{\mathbb{Z}} + \partial d(I,Q) = c_{\mathbb{Z}} + \sum_{i\in I}\left(\sum_{(q_i,r_i)\in Q_i} q_i \cdot \chi_{A_{r_i}} + N(X,\mu,\mathbb{Z})\right) \otimes \partial\tilde{\tau}_i'$$

Then we have that

$$\begin{aligned}
\left| c - c(I,Q) \right|^{(\alpha,\mu)} &= \left| \sum_{i\in I}\sum_{(q_i,r_i)\in Q_i} q_i \cdot (\chi_{B_{r_i}} - \chi_{A_{r_i}}) \otimes \partial\tilde{\tau}_i' \right|^{(\alpha,\mu)} \\
&\leq \sum_{i\in I}\sum_{(q_i,r_i)\in Q_i} \int_X \left| q_i \cdot (\chi_{B_{r_i}} - \chi_{A_{r_i}}) \right| d\mu \\
&= \sum_{i\in I}\sum_{(q_i,r_i)\in Q_i} |q_i| \cdot \mu(B_{r_i} \triangle A_{r_i}) < \varepsilon
\end{aligned}$$

and finally the reverse triangle inequality

$$\left| \left| c \right|^{(\alpha,\mu)} - \left| c(I,Q) \right|^{(\alpha,\mu)} \right| \leq \left| c - c(I,Q) \right|^{(\alpha,\mu)} < \varepsilon$$

gives that $\left| c(I,Q) \right|^{(\alpha,\mu)} < \varepsilon + \left| c \right|^{(\alpha,\mu)} < C$. $\square$

**Theorem 5.2.26** (ergodic decomposition formula for PSV). *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$, let $\alpha : \Gamma \curvearrowright (X,\mu)$ be a standard action with an ergodic decomposition $\beta : X \to \mathrm{Erg}(\alpha)$; then*

$$\left| M \right|^{(\alpha,\mu)} = \int_X \left| M \right|^{(\alpha,\beta_x)} d\mu(x)$$

As a strategy of proof we will follow the proof of Theorem 3.3.4 as a guideline.

*Proof.* Let $c = \sum_{i=1}^{k}(f_i + N(X,\mu,\mathbb{Z})) \otimes \tilde{\sigma}_i \in C_n(M; L^\infty(X,\mu,\mathbb{Z}))$ be an $(\alpha,\mu)$-parametrized fundamental cycle in reduced form. In view of Proposition 5.2.24 the map

$$x \mapsto n_\mu^{\beta_x}(c)$$

is measurable and the infimum of measurable functions is still measurable [3, Proposition 2.7]. Hence the function

$$\begin{aligned}
X &\to [0,+\infty[ \\
x &\mapsto \left| M \right|^{(\alpha,\beta_x)}
\end{aligned}$$

is measurable and it is bounded by $\|M\|_{\mathbb{Z}}$ by Proposition 5.2.11. Hence it is integrable in particular. Let us prove the equality via two inequalities.

($\geq$) By Proposition 5.2.24 for an $(\alpha, \mu)$-parametrized fundamental cycle $c \in C_n(M; L^\infty(X, \mu, \mathbb{Z}))$ we have

$$|c|^{(\alpha,\mu)} = \int_X n_\mu^{\beta_x}(c) d\mu(x)$$

Since $n_\mu^{\beta_x}(c) \geq |M|^{(\alpha,\beta_x)}$ for all $x \in X$ and for every $(\alpha, \mu)$-parametrized fundamental cycle $c$ (as pointed out in Remark 5.2.22), we also have

$$|c|^{(\alpha,\mu)} = \int_X n_\mu^{\beta_x}(c) d\mu(x) \geq \int_X |M|^{(\alpha,\beta_x)} d\mu(x)$$

by monotonicity of the integral operator. Taking the infimum among all of the $(\alpha, \mu)$-parametrized fundamental cycles gives

$$|M|^{(\alpha,\mu)} = \inf_c |c|^{(\alpha,\mu)} = \inf_c \int_X n_\mu^{\beta_x}(c) d\mu(x) \geq \int_X |M|^{(\alpha,\beta_x)} d\mu(x)$$

($\leq$) Fix $\varepsilon > 0$ and define the subsets

$$A_{I,Q} = \left\{x \in X : n_\mu^{\beta_x}(c(I,Q)) < |M|^{(\alpha,\beta_x)} + \varepsilon\right\}$$

which are measurable, since the functions $x \mapsto |M|^{(\alpha,\beta_x)}$ and $x \mapsto n_\mu^{\beta_x}(c(I,Q))$ are measurable. By Lemma 5.2.25 (for every $x \in X$, use $\mu = \beta_x$ and $C = |M|^{(\alpha,\beta_x)} + \varepsilon$) we obtain

$$X = \cup\left\{A_{I,Q} : I \in P_{\text{fin}}(\mathbb{N}) \text{ and } Q \in P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}\right\}$$

With an enumeration $h : \bigcup_{n\in\mathbb{N}} P_n(\mathbb{N}) \times P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^n \to \mathbb{N}$ we can obtain $X$ as a disjoint union of measurable subsets

$$X = \sqcup\left\{B_{I,Q} : I \in P_{\text{fin}}(\mathbb{N}) \text{ and } Q \in P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}\right\}$$

with the properties that:

a) $B_{I,Q} \subseteq A_{I,Q}$;

b) $\Gamma \cdot B_{I,Q} = B_{I,Q}$;

c) if $x \in B_{I,Q}$, then $X_{\beta_x} \subseteq B_{I,Q}$

for all $I \in P_{\text{fin}}(\mathbb{N})$ and $Q \in P_{\text{fin}}(\mathbb{Z} \times \mathbb{N})^{|I|}$. For instance one could take

$$B_{I,Q} = \left\{x \in X : h(I,Q) = \min\{n \in \mathbb{N} : x \in A_{h^{-1}(n)}\}\right\}$$

Hence one gets

$$1 = \mu(X) = \mu\left(\bigsqcup_{n\in\mathbb{N}} B_{h^{-1}(n)}\right) = \sum_{n=0}^{+\infty} \mu\left(B_{h^{-1}(n)}\right)$$

Since the series is convergent, there must exist $N \in \mathbb{N}$ such that

$$\mu(B_{h^{-1}(0)}) + \cdots + \mu(B_{h^{-1}(N)}) \geq 1 - \varepsilon$$

Define the $(\alpha, \mu)$-parametrized fundamental cycle $c_N$ as

$$c_{\mathbb{Z}} + \sum_{n=0}^{N} \partial \left( d(h^{-1}(n))|_{B_{h^{-1}(n)}} \right) = c_{\mathbb{Z}} + \sum_{n=0}^{N} \sum_{i \in I} \left( \sum_{(q,r) \in Q_i} q \cdot \chi_{A_r \cap B_{I,Q}} + N(X, \mu, \mathbb{Z}) \right) \otimes \partial \tilde{\tau}_i'$$

where $(I, Q) = h^{-1}(n)$. Define $B = B_{h^{-1}(0)} \sqcup \cdots \sqcup B_{h^{-1}(N)}$, so that $\mu(B) \geq 1 - \varepsilon$ and hence $\mu(B^c) \leq \varepsilon$. Observe that for $x \in B_{I,Q}$ we have

$$\begin{aligned}
n_\mu^{\beta_x}(c_N) &= n_\mu^{\beta_x}\left(c_N|_{X_{\beta_x}}\right) && (\beta_x(X_{\beta_x}) = 1) \\
&= n_\mu^{\beta_x}(c(I, Q)|_{X_{\beta_x}}) && \text{(disjoint union)} \\
&= n_\mu^{\beta_x}(c(I, Q)) && (\beta_x(X_{\beta_x}) = 1) \\
&< | M |^{(\alpha, \beta_x)} + \varepsilon && (B_{I,Q} \subseteq A_{I,Q})
\end{aligned}$$

whereas

$$n_\mu^{\beta_y}(c_N) = n_\mu^{\beta_y}(c_N|_{X_{\beta_y}}) = n_\mu^{\beta_y}(c_{\mathbb{Z}}) \leq | c_{\mathbb{Z}} |^{(\alpha, \beta_y)} = \|c_{\mathbb{Z}}\|_{\mathbb{Z}}$$

for all $y \in B^c$. Finally we obtain

$$\begin{aligned}
| c_N |^{(\alpha, \mu)} &= \int_X n_\mu^{\beta_x}(c_N) d\mu(x) \\
&= \int_B n_\mu^{\beta_x}(c_N) d\mu(x) + \int_{B^c} n_\mu^{\beta_x}(c_N) d\mu(x) \\
&\leq \int_B ( | M |^{(\alpha, \beta_x)} + \varepsilon) d\mu(x) + \|c_{\mathbb{Z}}\|_{\mathbb{Z}} \cdot \mu(B^c) \\
&\leq \int_B | M |^{(\alpha, \beta_x)} d\mu(x) + \varepsilon \cdot \mu(B) + \|c_{\mathbb{Z}}\|_{\mathbb{Z}} \cdot \varepsilon \\
&\leq \int_X | M |^{(\alpha, \beta_x)} d\mu(x) + (1 + \|c_{\mathbb{Z}}\|_{\mathbb{Z}}) \cdot \varepsilon
\end{aligned}$$

and letting $\varepsilon \to 0^+$ we conclude that $| M |^{(\alpha, \mu)} \leq \int_X | M |^{(\alpha, \beta_x)} d\mu(x)$. $\square$

**Corollary 5.2.27.** *Let $M$ be an oriented closed connected $n$-dimensional manifold with fundamental group $\Gamma$; there exists an ergodic and essentially free standard probability action $\alpha : \Gamma \curvearrowright (X, \mu)$ such that*

$$| M | = | M |^{(\alpha, \mu)}$$

*Proof.* Corollary 5.2.15 gives the existence of an essentially free standard probability action $\alpha : \Gamma \curvearrowright (X, \nu)$ such that $| M | = | M |^{(\alpha, \nu)}$. Let $\beta : X \to \mathrm{Erg}(\alpha)$ be an ergodic decomposition of this action. Then:

1. since the action is essentially free, there exists a measurable subset $A \subseteq X$ such that $\nu(A) = 1$ and $\beta_a(\{x \in X : \Gamma_x \neq 1\}) = 0$ for every $a \in A$;

2. there exists a measurable subset $B \subseteq X$ such that $\nu(B) \in ]0,1]$ and $| M |^{(\alpha,\beta_x)} \leq | M |$ for every $x \in B$. Indeed, if we assume by contradiction that $| M |^{(\alpha,\beta_x)} > | M |$ for $\nu$-almost every $x \in X$, then
$$| M | = | M |^{(\alpha,\nu)} = \int_X | M |^{(\alpha,\beta_x)} d\nu(x) > | M |$$
gives an absurd.

Since $\nu(A \cap B) = 1 - \nu(A^c \cup B^c) \geq 1 - (\nu(A^c) + \nu(B^c)) > 1 - (0+1) = 0$, in particular $A \cap B \neq \varnothing$ and thus we can find $x \in X$ such that the action $\alpha : \Gamma \curvearrowright (X, \beta_x)$ is ergodic and essentially free. $\square$

# A. Appendix

This appendix is meant to collect some useful notion, which will not be developed in depth.

## A.1. Some algebraic structures

### A.1.1. Tensor products

Here we want to recall the notion of tensor product for modules over a fixed ring $R$, which may not be commutative.

**Definition A.1.1** ($R$-balanced maps). Let $M \in \mathsf{Mod}\text{-}R$, let $N \in R\text{-}\mathsf{Mod}$ and let $G \in \mathbb{Z}\text{-}\mathsf{Mod}$; we say that a function $\tau : M \times N \to G$ is an *$R$-balanced map* if

$$\begin{aligned} \tau(m_1 + m_2, n) &= \tau(m_1, n) + \tau(m_2, n) \\ \tau(m, n_1 + n_2) &= \tau(m, n_1) + \tau(m, n_2) \\ \tau(mr, n) &= \tau(m, rn) \end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.

**Definition A.1.2** (tensor product). Let $M \in \mathsf{Mod}\text{-}R$ and let $N \in R\text{-}\mathsf{Mod}$; a *tensor product* of $M$ and $N$ is a pair $(T, \tau)$, where $T \in \mathbb{Z}\text{-}\mathsf{Mod}$ and $\tau : M \times N \to T$ is an $R$-balance map, satisfying the following universal property: for every $G \in \mathbb{Z}\text{-}\mathsf{Mod}$ and for every $R$-balance map $f : M \times N \to G$ there exists a unique $\mathbb{Z}$-modules homomorphism $\tilde{f} : T \to G$ such that $\tilde{f} \circ \tau = f$.

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\tau"'] & G \\ T \arrow[ur, dashed, bend right, "\tilde{f}"'] & \end{tikzcd}$$

A tensor product of modules exists and is essentially unique.

**Proposition A.1.3** (existence and uniqueness of the tensor product). *Let $M \in \mathsf{Mod}\text{-}R$, let $N \in R\text{-}\mathsf{Mod}$; then*

1. *a tensor product $(T, \tau)$ of $M$ and $N$ exists;*

2. *if $(T', \tau')$ is another tensor product of $M$ and $N$, then there is a unique isomorphism $\varphi : T \to T'$ in $\mathbb{Z}\text{-}\mathsf{Mod}$ such that $\varphi \circ \tau = \tau'$.*

By this existence-uniqueness result it is common to denote by $(M \otimes_R N, \otimes)$ "the" tensor product of $M$ and $N$. We state and proof a result we used in the proof of Lemma 5.2.25.

**Proposition A.1.4.** *Let $C_* = \{(C_n, c_n)\}_{n\in\mathbb{Z}}$ and $D_* = \{(D_n, d_n)\}_{n\in\mathbb{Z}}$ be chain complexes in the category* Mod-$R$ *and let $A \in R$-*Mod *with associated chain complex $A_* = \{(A, \mathrm{id}_A)\}_{n\in\mathbb{Z}}$; if $C_*$ and $D_*$ are homotopy equivalent, then $C_* \otimes_R A_*$ and $D_* \otimes_R A_*$ are homotopy equivalent.*

Here $C_* \otimes_R A_*$ denotes the chain complex $\{(C_n \otimes_R A, c_n \otimes \mathrm{id}_A)\}_{n\in\mathbb{Z}}$.

*Proof.* As $C_*$ and $D_*$ are homotopy equivalent, there exist chain maps $f : C_* \to D_*$ and $g : D_* \to C_*$ such that $g \circ f \sim \mathrm{id}_{C_*}$ and $f \circ g \sim \mathrm{id}_{D_*}$. This means that there exists a family of $R$-linear homomorphism $\{h_n : C_n \to C_{n+1}\}_{n\in\mathbb{Z}}$ such that $g_n \circ f_n - \mathrm{id}_{C_n} = h_{n-1} \circ c_n + c_{n+1} \circ h_n$ for all $n \in \mathbb{Z}$. We get that

$$
\begin{aligned}
&(g_n \otimes \mathrm{id}_A) \circ (f_n \otimes \mathrm{id}_A) - \mathrm{id}_{C_n} \otimes \mathrm{id}_A = \\
&\qquad = (g_n \circ f_n - \mathrm{id}_{C_n}) \otimes \mathrm{id}_A = (h_{n-1} \circ c_n + c_{n+1} \circ h_n) \otimes \mathrm{id}_A = \\
&\qquad\qquad = (h_{n-1} \otimes \mathrm{id}_A) \circ (c_n \otimes \mathrm{id}_A) + (c_{n+1} \otimes \mathrm{id}_A) \circ (h_n \otimes \mathrm{id}_A)
\end{aligned}
$$

which shows that $(g \otimes \mathrm{id}_{A_*}) \circ (f \otimes \mathrm{id}_{A_*}) \sim \mathrm{id}_{C_*} \otimes \mathrm{id}_{A_*}$. The other relation is analogue. □

### A.1.2. The group ring

The notions of group ring is the essential algebraic structure for the definition of homology with twisted coefficients in Section 5.1.

**Definition A.1.5** (group ring)**.** Let $R$ be a ring with unity and $(\Gamma, \cdot)$ be a group; the *group ring* of $\Gamma$ over $R$ is the free $R$-module $R[\Gamma]$ generated by $\Gamma$, endowed with the multiplication that is the $R$-linear extension of the multiplication of $(\Gamma, \cdot)$, i.e.

$$
\sum_{i=1}^{m} a_i g_i \cdot \sum_{j=1}^{n} b_j h_j = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i b_j)(g_i \cdot h_j)
$$

for all $a_i, b_j \in R$ and for all $g_i, h_j \in \Gamma$.

The triple $(R[\Gamma], +, \cdot)$ is indeed a ring. There are two remarkable functions associated to this construction:

1. the ring homomorphism
$$
\begin{aligned}
R &\to R[\Gamma] \\
r &\mapsto r \cdot 1_\Gamma
\end{aligned}
$$
which exhibits an $R$-algebra structure for $R[\Gamma]$;

2. the set-theoretic inclusion of $\Gamma$ as standard basis
$$
\begin{aligned}
\iota : \Gamma &\to R[\Gamma] \\
g &\mapsto 1_R \cdot g
\end{aligned}
$$

**Example A.1.6.** When $\Gamma = 1$ is the trivial group, we have that $R[1] \cong R$.

**Example A.1.7.** When $(\Gamma, \cdot) = (\mathbb{Z}, +)$ is the ring of integers, we obtain that the group ring $R[\mathbb{Z}]$ is isomorphic with the $R$-algebra $R[t, t^{-1}]$ of Laurent polynomials.

**Warning A.1.8.** Let us assume $R$ to be commutative ring. Then $R[\Gamma]$ is a commutative ring if, and only if, $\Gamma$ is an abelian group.

**Lemma A.1.9** (universal property of group rings, [24, Proposition 3.2.7]). *Let $R$ be a ring and let $\Gamma$ be a group; the group ring $R[\Gamma]$ satisfies the following universal property: for every $R$-algebra $S$ and for every group homomorphism $f : \Gamma \to S^\times$ there exists a unique $R$-algebra homomorphism $\tilde{f} : R[\Gamma] \to S$ such that $f = \tilde{f} \circ \iota$.*

$$\begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, "\iota"'] & S^\times \arrow[d, hook] \\ R[\Gamma] \arrow[r, dashed, "\tilde{f}"'] & S \end{tikzcd}$$

## A.2. $L^2$-Betti numbers

This part is meant to be a very short summary on the concept of $L^2$-Betti numbers, which was the base point for Gromov to define the integral foliated simplicial volume.

### A.2.1. The von Neumann algebra

We consider now group rings over the field $\mathbb{C}$. Every such a group ring $\mathbb{C}[\Gamma]$ inherits an scalar product $\langle \cdot, \cdot \rangle : \mathbb{C}[\Gamma] \times \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ defined by

$$\left\langle \sum_{\gamma \in \Gamma} z_\gamma \gamma, \sum_{\gamma \in \Gamma} w_\gamma \gamma \right\rangle = \sum_{\gamma \in \Gamma} \bar{z}_\gamma w_\gamma$$

where the coefficients $z_\gamma$ and $w_\gamma$'s are almost all vanishing.

**Definition A.2.1** (space $\ell^2\Gamma$). We denote by $\ell^2\Gamma$ the completion of $\mathbb{C}[\Gamma]$ with respect to (the metric induced by) the scalar product $\langle \cdot, \cdot \rangle : \mathbb{C}[\Gamma] \times \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$.

More explicitly $\ell^2\Gamma$ can be realized as the complex Hilbert space $L^2(\Gamma, \#, \mathbb{C})$ of square-summable functions $\Gamma \to \mathbb{C}$ with respect to the counting measure $\#$ endowed with the scalar product $\langle \cdot, \cdot \rangle : \ell^2\Gamma \times \ell^2\Gamma \to \ell^2\Gamma$. More often the space $L^2(\Gamma, \#, \mathbb{C})$ is denoted by $\ell^2(\Gamma, \mathbb{C})$.

**Example A.2.2** (finite groups). $\ell^2\Gamma \cong \mathbb{C}[\Gamma]$ whenever $\Gamma$ is a finite group.

**Example A.2.3** (the group $\mathbb{Z}$). Fourier analysis shows that $\ell^2\,\mathbb{Z} \cong L^2([-\pi, \pi], m, \mathbb{C})$ as Hilbert spaces.

The group $\Gamma$ acts on $\ell^2\Gamma$ from the left via

$$\begin{aligned} \Gamma \times \ell^2\Gamma &\to \ell^2\Gamma \\ (\gamma, f) &\mapsto f_\gamma \end{aligned}$$

where $f_\gamma(x) = f(x\gamma)$ for all $x \in \Gamma$. By extending this action by $\mathbb{C}$-linearity we can see $\ell^2\Gamma$ as a $\mathbb{C}[\Gamma]$-module.

**Definition A.2.4** (Hilbert modules). Let $\Gamma$ be a countable group;

1. an *Hilbert $\Gamma$-module* is a pair $(V, \Gamma \curvearrowright V)$, where:
   a) $V$ is a complex Hilbert space;
   b) $\Gamma \curvearrowright V$ is a $\mathbb{C}$-linear isometric left group action, i.e. $\gamma\cdot : V \to V$ is a $\mathbb{C}$-linear isometry for all $\gamma \in \Gamma$;
   c) there is an isometric embedding of $\Gamma$-spaces $V \to (\ell^2\Gamma)^n$ for some $n \in \mathbb{N}$.
2. let $V, W$ be Hilbert $\Gamma$-modules; a morphism $V \to W$ of Hilbert $\Gamma$-modules is a $\Gamma$-equivariant bounded $\mathbb{C}$-linear map $V \to W$.

We denote by $B(\ell^2\Gamma)$ the $\mathbb{C}$-algebra of the bounded $\mathbb{C}$-linear operators $\ell^2\Gamma \to \ell^2\Gamma$. We endow $\ell^2\Gamma$ with the weak topology, which means the weakest topology on $\ell^2\Gamma$ making the functions $\langle f, \cdot\rangle : \ell^2\Gamma \to \mathbb{C}$ continuous for all $f \in \ell^2\Gamma$.

**Definition A.2.5** (group von Neumann algebra). Let $\Gamma$ be a countable group; the *group von Neumann algebra* of $\Gamma$ is the weak closure $N\Gamma$ of $\mathbb{C}[\Gamma]$ in $B(\ell^2\Gamma)$.

Here $\mathbb{C}[\Gamma] \hookrightarrow B(\ell^2\Gamma)$ by associating to every $a \in \mathbb{C}[\Gamma]$ the multiplication $M_a : \ell^2\Gamma \to \ell^2\Gamma$, which is a left $\Gamma$-equivariant isometric $\mathbb{C}$-linear map.

**Definition A.2.6** (von Neumann trace). Let $\Gamma$ be a countable group; the *von Neumann trace* associated to $\Gamma$ is the function

$$\begin{aligned} \operatorname{tr}_\Gamma : N\Gamma &\to \mathbb{C} \\ a &\mapsto \langle \chi_{\{1_\Gamma\}}, a(\chi_{\{1_\Gamma\}})\rangle \end{aligned}$$

The von Neumann trace defined above is indeed a trace:

**Lemma A.2.7** (properties of $\operatorname{tr}_\Gamma$, [21, Theorem 1.1.12]). *Let $\Gamma$ be a countable group; then $\operatorname{tr}_\Gamma : N\Gamma \to \mathbb{C}$ satisfies the following properties:*

1. *trace property:* $\operatorname{tr}_\Gamma(a \circ b) = \operatorname{tr}_\Gamma(b \circ a)$ *for all $a, b \in N\Gamma$;*
2. *faithfulness:* $\operatorname{tr}_\Gamma(a^* \circ a) = 0$ *if, and only if, $a = 0$;*
3. *positivity:* $\operatorname{tr}_\Gamma(a) \geq 0$ *for all $a \in N\Gamma$ such that $\langle f, a(f)\rangle \geq 0$ for all $f \in \ell^2\Gamma$.*

**Definition A.2.8** (von Neumann dimension). Let $\Gamma$ be a countable group and let $V$ be a Hilbert $\Gamma$-space with isometric embedding $i : V \to (\ell^2\Gamma)^n$; the *von Neumann dimension* of $V$ is defined as $\dim_{N\Gamma} V = \operatorname{tr}_\Gamma p$, where $p : (\ell^2\Gamma)^n \to (\ell^2\Gamma)^n$ is the orthogonal projection onto $i(V)$.

**Lemma A.2.9** (well-definition of $\dim_{N\Gamma} V$, [21, Proposition 1.2.1]). *Let $\Gamma$ be a countable group and let $V$ be a Hilbert $\Gamma$-space with isometric $\Gamma$-embedding $i : V \to (\ell^2\Gamma)^n$; then*

1. $\dim_{N\Gamma} V$ *is well-defined: if $j : V \to (\ell^2\Gamma)^m$ is another isometric $\Gamma$-embedding and $q : (\ell^2\Gamma)^m \to (\ell^2\Gamma)^m$ is the orthogonal projection on $j(V)$, then* $\operatorname{tr}_\Gamma p = \operatorname{tr}_\Gamma q$*;*
2. $\dim_{N\Gamma} V \in [0, +\infty[$.

### A.2.2. $L^2$-Betti numbers

**Definition A.2.10** (equivariant CW-complexes). Let $\Gamma$ be a group;

1. a *free $\Gamma$-CW-complex* is a pair $(X, \alpha)$, where $X$ is a CW-complex and $\alpha : \Gamma \curvearrowright X$ is a free action such that:
   a) $\alpha$ permutes the open cells of $X$;
   b) if $e$ is an open cell of $X$ and $\gamma \in \Gamma \smallsetminus \{1\}$, then $\gamma \cdot e \cap e \neq \varnothing$.
2. let $(X, \alpha)$ and $(Y, \beta)$ be free $\Gamma$-CW-complexes; a morphism $(X, \alpha) \to (Y, \beta)$ of $\Gamma$-CW-complexes is a $\Gamma$-equivariant cellular map $X \to Y$.

For $\Gamma$ a group recall the following notions of finite type:

1. a CW-complex is *of finite type* if for every $n \in \mathbb{N}$ it has finitely many open $n$-cells;
2. a free $\Gamma$-CW-complex is *of finite type* if for each $n \in \mathbb{N}$ it has finitely many $\Gamma$-orbits of the open $n$-cells.

**Definition A.2.11** ($L^2$-Betti numbers of spaces). Let $\Gamma$ be a countable group and let $(X, \alpha)$ be a free $\Gamma$-CW-complex of finite type;

1. the action $\alpha : \Gamma \curvearrowright X$ induces an action on the cellular chain complex $C_*^{\mathrm{cell}}(X; \mathbb{Z})$. The cellular $L^2$-chain complex of $X$ is defined as the twisted chain complex
$$C_*^{(2)}(\Gamma \curvearrowright X) = \ell^2\Gamma \otimes_{\mathbb{C}[\Gamma]} C_*^{\mathrm{cell}}(X; \mathbb{Z})$$
2. for $n \in \mathbb{N}$, the reduced $L^2$-homology in degree $n$ is
$$H_n^{(2)}(\Gamma \curvearrowright X) = \operatorname{Ker} \partial_n^{(2)} / \overline{\operatorname{Im} \partial_{n+1}^{(2)}}$$
where $\partial_*^{(2)} = \mathrm{id}_{\ell^2\Gamma} \otimes \partial_n$ is the bounday operator of $C_*^{(2)}(\Gamma \curvearrowright X)$;
3. the $n$-th *$L^2$-Betti number* of $X$ is the von Neumann dimension
$$b_n^{(2)}(\Gamma \curvearrowright X) = \dim_{N\Gamma} H_n^{(2)}(\Gamma \curvearrowright X)$$

Notice that in the definition of reduced $L^2$-homology we are quotienting by the closure of $\operatorname{Im}\partial_{n+1}^{(2)}$ so that $H_n^{(2)}(\Gamma \curvearrowright X)$ is a complete space, and thus an Hilbert $\Gamma$-module.

When $X$ is a CW-complex of finite type with fundamental group $\Gamma$ and universal covering $\tilde{X} \to X$, $\Gamma$ acts on $\tilde{X}$ by deck transformations and we usually write $b_n^{(2)}(X)$ for $b_n^{(2)}(\Gamma \curvearrowright \tilde{X})$. The basic estimate moving the definition and study of integral foliated simplicial volume is the following.

**Proposition A.2.12** ($L^2$-Betti numbers estimates, [21, Theorem 6.4.5]). *Let $M$ be an oriented closed connected manifold; then*

$$b_k^{(2)}(M) \leq \left| M \right|$$

*for every $k \in \mathbb{N}$.*

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# Index

Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in § 26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Regensburg, July 7th 2022 Giovanni Sartori