Deligne-Lusztig reduction in the case of an affine Weyl group

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Introduction

The *affine Deligne-Lusztig varieties* were introduced for the first time in 2005 by M. Rapoport in "A guide to the reduction modulo p of Shimura varieties" (see [1]). Understanding the emptiness/nonemptiness and the dimension of these objects is fundamental to examine certain aspects of the reduction of Shimura varieties. In general, the affine Deligne-Lusztig varieties (abbreviated ADLV) are difficult to handle. However, Xuhua He in [6] managed to bring back the question of the dimension for arbitrary ADLV to some well-studied ADLV. His solution is based on the *affine Deligne-Lusztig reduction* and gives rise to a concrete algorithm for the calculation of the dimension. Our first goal was then to create a computer program for the dimensions of ADLV following his strategy. Indeed, such a program was not yet implemented. We decided to use the mathematics software SageMath (see [2]). Our program can be applied in several cases and many examples of computations are reported in the thesis. In particular, calculating the dimensions of a specific subset of ADLV it is possible to find the dimension of the *supersingular locus* of the moduli space of principally polarized abelian varieties of dimension g with Iwahori level structure at p over $\overline{\mathbb{F}}_p$. Using this strategy we managed to find the dimension of the supersingular locus in the unknown case '$g = 5$'. In the next pages we are going to present both theoretical and practical tools used to write the programs and to get the results. Since the ADLV are strictly connected to Weyl groups, the first chapter is dedicated to these groups. In the first section we recall the most important results and objects related to a root systems Φ in a euclidean space V. In particular, we introduce the *classical Weyl group* W, which is a finite subgroup of $GL(V)$. In the second section, we enlarge the classical Weyl group considering also some affine reflections. The resulting infinite group $W_a \subset \text{Aff}(V)$ is called *affine Weyl group*. Similarly, we can define the *extended affine Weyl group* $\widetilde{W} \subset \text{Aff}(V)$ which contains W_a as normal subgroup. In the first four subsections we examine the relations between these groups. In particular, we find a finite set of generators for W_a and we define a length function on $\widetilde{W}$. In the last two subsections we present some properties satified by W_a as Coxeter group. The second chapter begins with a brief introduction to the ADLV. Each of these varieties is constructed starting from some algebraic groups over $\mathbb{F}_q((\epsilon))$ or $\mathbb{Q}_p$ and choosing two elements from an associated extended Weyl group $\widetilde{W}$. In order to find the dimension we have to consider also an induced automorphism δ on $\widetilde{W}$. In the following subsection, the three main results used in the solution of He are presented: the Deligne-Lusztig reduction, the dimension in the case of minimal length elements and the theorem of He about minimal elements in $\delta-$conjugacy classes. At the beginning of the second section we describe the idea of the algorithm and we explain how the Weyl groups are implemented in Sage. Then we speak about the programs for δ=id and for arbitrary δ. In the third chapter some possible applications of the program are presented. In the first section we consider a conjecture regarding the dimension of ADLV calculating some examples. At the beginning of the last paragraph we explain how the dimension of the supersingular locus S_I is related to the dimensions of some ADLV. Then we report the calculations we have done. In particular in the case $g = 5$ we prove that $\dim S_I = 10$. The Python scripts used for the dimension of the ADLV and the supersingular locus can be found in the appendix.

Contents

Chapter 1

Weyl groups

In this chapter we are going to introduce the classical Weyl groups, the affine Weyl groups and the extended affine Weyl groups.

1.1 Root systems

In the first section we recall the main results and properties of root systems, following [3, Chapter III].

1.1.1 Root systems and classical Weyl groups

In order to define a root system we need the following definitions:

Definition 1.1.1. A *Euclidean space* V is a finite dimensional vector space over $\mathbb{R}$ endowed with a positive define symmetric bilinear form.
A *reflection* in V is an invertible linear transformation leaving pointwise fixed some hyperplane and sending any vector orthogonal to that hyperplane into its negative.

Let V be a euclidean space with associated bilinear form $(\cdot,\cdot)$. Given a non-zero vector $\alpha \in V$ we denote by s_α the reflection which fixes the hyperplane $H_\alpha := \{\beta \in V \mid (\beta,\alpha) = 0\}$, i.e. $s_\alpha(\beta) := \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ for $\beta \in V$. From now on we abbreviate the expression $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ by $\langle \beta, \alpha \rangle$.

Definition 1.1.2. A subset Φ of the Euclidean space V is called a *root system* in V if the following axioms are satisfied:

(R1) Φ is finite, spans V, and doesn't contain 0.

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection s_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

The *rank* of Φ is defined as $n = \dim(V)$ and the vectors in Φ are called *roots*.

Let Φ a root system in V and α, $\beta \in \Phi$. Then the axioms $(R3)$ and $(R4)$ state that the root $s_\alpha(\beta)$ and β differ by an integer multiple of α. It is useful to calculate the possible values for $\langle \beta, \alpha \rangle$ and which angles and ratios of moduli between two pairs of roots can occur. Indeed, we have just a limited number of possibilities. Now suppose that $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$. Recall that the cosine of the angle θ between two vectors α, $\beta \in V$ is given by the formula $\|\alpha\|\|\beta\|cos\theta = (\alpha, \beta)$. Hence, $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}cos\theta$ and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4cos^2\theta$. From axiom (R4) it follows that $4cos^2\theta$ should be a non-negative integer. Since $0 \leq cos^2\theta \leq 1$ we get that $cos^2\theta \in \{0, 1/4, 1/2, 3/4, 1\}$. Considering that $\langle \alpha, \beta, \rangle \langle \beta, \alpha \rangle$ have the same sign, the following are the only possibilities:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	undeterminated
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Table 1.1: Pair of roots

Using this table, we can prove the following rule.

Proposition 1.1.3. *Let Φ a root system and $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$. If $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.*

Proof. If $(\alpha, \beta) > 0$ both the integers $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ are postive and looking at Table 1.1 we see that one of them is equal to 1. Suppose $\langle \alpha, \beta \rangle = 1$ then $s_\beta(\alpha) = \alpha - \beta \in \Phi$. If $\langle \alpha, \beta \rangle = 1$ then $s_\alpha(\beta) = \beta - \alpha \in \Phi$. Thus also $-(\beta - \alpha) = \alpha - \beta \in \Phi$. Similarly, if $(\alpha, \beta) < 0$ we get that $\langle \alpha, \beta \rangle = -1$ or $\langle \beta, \alpha \rangle = -1$ (or both). In the first case we have $s_\beta(\alpha) = \alpha + \beta \in \Phi$. In the second we get $s_\alpha(\beta) = \alpha + \beta \in \Phi$. □

Every root system gives rise to an important subgroup of $GL(V)$:

Definition 1.1.4. Let Φ be a root system in V. The *classical Weyl group* of Φ is the subgroup of $GL(V)$ generated by the reflections s_α with $\alpha \in \Phi$. We denote it by W.

Notice that we can identify W with a subgroup of the symmetric group on Φ, indeed by (R3) W permutes the set Φ, which by $(R1)$ is finite and spans V. In particular, the group W is finite. We have a natural notion of isomorphism between root systems.

Definition 1.1.5. Two root systems Φ, Φ', in respective euclidean spaces V and V', are said to be *isomorphic* if there exists a vector space isomorphism $\phi : V \to V'$ such that ϕ induces a bijection $\Phi \to \Phi'$ and $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for every $\alpha, \beta \in \Phi$.

Let ϕ, α, β as above. From the definition of the reflections associated with the roots it follows that $s_{\phi(\alpha)}(\phi(\beta)) = \phi(s_\alpha(\beta))$, in particular $s_{\phi(\alpha)}(\beta) = \phi(s_\alpha(\phi^{-1}(\beta)))$. Thus we obtain a natural isomorphism of the Weyl groups associated with Φ and Φ' considering the map: $w \mapsto \phi \circ w \circ \phi^{-1}$. However, the 'vice versa' is not true: in Remark 2 we will present two root systems which are not isomorphic, but whose Weyl groups are isomorphic. It is easily verified that an automorphism of Φ is the same thing as an automorphism of V leaving Φ invariant. Hence we can regard W as a subgroup of $Aut(\Phi)$. It is practical to consider also the 'dual' root system of Φ.

Definition 1.1.6. Let Φ be a root system. The vectors $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ with $\alpha \in \Phi$ are called *coroots*. We call $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ the *dual* of Φ. It is in fact a root system in V, whose Weyl group is canonically isomorphic to the Weyl group of Φ (since $s_\alpha = s_{\alpha^\vee}$).

Notice that $(\alpha^\vee)^\vee = \alpha$. Thus Φ is the dual root system of $\Phi^\vee$. We see also that $\langle \alpha, \beta \rangle = (\alpha, \beta^\vee)$ for α, β in Φ.

1.1.2 Bases

Throughout this subsection we fix a root system Φ of rank n in a euclidean space V, with Weyl group W.

Definition 1.1.7. A subset Δ of Φ is called a *base* if:

(B1) Δ is a basis of V.

(B2) Each root β can be written as $\beta = \sum k_\alpha \alpha$, $\alpha \in \Delta$ with integral coefficients k_α all non-negative or all non-positive.

The roots in Δ are called *simple.*

By Proposition 1.1.3 we have that $(\alpha, \alpha') \leq 0$ for α, α' in Δ. Clearly, the cardinality of Δ is n and the expression for β in $(B2)$ is unique. If all k_α are non-negative (resp. non-positive) we call β *positive* (resp. *negative*) and we write $\beta > 0$ (resp. $\beta < 0$). The collection of positive (resp. negative) roots is denoted by Φ^+ (resp. Φ^-). We can define a partial order on V, compatible with the notation $\alpha > 0$, setting $\beta < \alpha$ iff $\alpha - \beta$ is a sum of positive roots or $\beta = \alpha$. The definition of a base doesn't guarantee the existence, however it can be proved easily that we can always find a base. To be more precise, for every $\gamma \in V$ *regular*, i.e $\gamma \in V \setminus \bigcup_{\alpha \in \Phi} H_\alpha$, we define $\Phi^+(\gamma) := \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$ and call $\alpha \in \Phi^+(\gamma)$ *indecomposable* if $\alpha \neq \beta_1 + \beta_2$ for every $\beta_i \in \Phi^+(\gamma)$. Then we have the following theorem:

Theorem 1.1.8. *Let γ in V regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ, and every base is obtainable in this manner.*

The connected components of $V \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ are called the *Weyl chambers* of V. Since every regular $\gamma \in V$ belongs to a chamber and if γ and γ' belong to the same chamber $\Delta(\gamma) = \Delta(\gamma')$, the Weyl chambers are in bijection with the bases. The chamber corresponding to a base Δ is called *the fundamental Weyl chamber* relative to the base Δ, and it is denoted by $C(\Delta)$. It consists of all $\lambda \in V$ such that $(\lambda, \alpha) > 0$ for every $\alpha \in \Delta$. Regarding the dual system of Φ we can state:

Proposition 1.1.9. *Let Δ a base for Φ. Then $\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$ is a base for $\Phi^\vee$.*

Proof. Suppose that $\Delta = \Delta(\gamma)$ for some γ regular. Since $H_\alpha = H_{\alpha^\vee}$ for every α in Φ, the vector γ is regular also for $\Phi^\vee$. Since $(\gamma, \alpha) > 0$ iff $(\gamma, \alpha^\vee) > 0$, we have $(\Phi^\vee)^+(\gamma) = \{\alpha^\vee \mid \alpha \in \Phi^+\}$ and $\Delta^\vee \subseteq (\Phi^\vee)^+$. Let $\Delta'(\gamma)$ be the set of all the indecomposable roots in $(\Phi^\vee)^+(\gamma)$. By the previous theorem $\Delta'(\gamma)$ is base for $\Phi^\vee$. We want to prove that $\Delta^\vee = \Delta'(\gamma)$. It is enough to show that $\Delta^\vee \subseteq \Delta'(\gamma)$ since the two sets have the same cardinality. Suppose by contradiction that $\alpha^\vee \notin \Delta'(\gamma)$ with $\alpha \in \Delta$, i.e. α is not indecomposable. Then we can write $\alpha^\vee = \beta_1^\vee + \beta_2^\vee$ for some $\beta_i \in \Phi^+$ distinct. Thus $\alpha = a_1\beta_1 + a_2\beta_2$ with $a_i \in \mathbb{R}_{>0}$. Since Δ is a basis for V and β_i are positive roots, we get $\alpha = \sum_{\delta \in \Delta} k_\delta \delta + \sum_{\delta' \in \Delta} k_{\delta'} \delta'$ with all k_δ and $k_{\delta'}$ in $\mathbb{R}_{\geq 0}$. Since Δ is a basis for the vector space V, both β_i should be multiple of α against the hypothesis. $\square$

Studying the behavior of the simple roots, the following result can be proved.

Theorem 1.1.10. *Let Δ be a base of Φ. Then*

1. *if Δ' is another base of Φ, then $w(\Delta') = \Delta$ for some w in W.*
2. *W is generated by s_α with $\alpha \in \Delta$.*
3. *If $w(\Delta) = \Delta$, $w \in W$ then $w = 1$.*

The theorem tells us that W acts simply transitively on the bases. Thus the cardinality of W is equal to the number of bases (or equivalently to the number of Weyl chambers). The reflections s_α with α in Δ are called *simple reflections*. Thanks to point 2 we can give the following definition

Definition 1.1.11. The *length* of $w \in W$ with respect to Δ is the minimal natural number t such that $w = s_1 \cdots s_t$ with s_i in $S := \{s_\alpha \mid \alpha \in \Delta\}$ for every i. We write $l'(w) = t$ and the expression $s_1 \cdots s_t$ is called a *reduced* expression for w.

Another consequence is the following remark which is used in Section 1.2.4.

Remark 1. Let $\Delta = \{\alpha_1, ..\alpha_n\}$ a base for Φ. By the previous theorem there exists a unique element w_Δ in W such that $w_\Delta(\Delta) = -\Delta$ since $-\Delta$ is again a base for Φ. Furthermore, since $w_\Delta(-\alpha_i) = -w_\Delta(\alpha_i)$, we have $w_\Delta w_\Delta(\Delta) = \Delta$. Then by point 3 of Theorem 1.1.10 it follows that $w_\Delta^2 = 1$. Now we fix $\alpha_i \in \Delta$. Let $\Delta_i = \Delta \setminus \{\alpha_i\}$ and $W_i = \langle s_{\alpha_j} \mid j \neq i\rangle$. We want to show that there exists a unique element $w_{\Delta_i} \in W_i$ such that $w_{\Delta_i}(\Delta_i) = -\Delta_i$. Let V' be the subspace $\langle \alpha_j \in \Delta \mid j \neq i\rangle$ and $\Phi_i := \Phi \cap V'$. Then Φ_i is a root system for V' (notice that all the axioms are satisfied) with base Δ_i. We denote by $\tilde{s}_{\alpha_j}$ with $j \neq i$ the restriction of s_{α_j} to the subspace V'. We easily see that $W_i' = \langle \tilde{s}_{\alpha_j} \mid j \neq i\rangle$ is the Weyl group of Φ_i. Furthermore, the surjective projection $p : W_i \to W_i'$ is injective, since all the elements in W_i fix pointwise the subvector space orthogonal to V' (it follows from the definition of s_{α_j} $j \neq i$). By the previous reasoning there exists a unique w in the Weyl group W_i', such that $w(\Delta_i) = -\Delta_i$. Let $w_{\Delta_i} = p(w)^{-1}$. Then $w_{\Delta_i}(\Delta_i) = -\Delta_i$ and the order of w_{Δ_i} is two, since it is equal to the order of w. Now let $w' \in W_i$ such that $w'(\Delta_i) = -(\Delta_i)$. We get that $p(w')(\Delta_i) = -\Delta_i$. Hence $p(w') = w$, so $w' = p^{-1}(w) = w_{\Delta_i}$.

Finally, it can be verified that:

Proposition 1.1.12. *Let Δ be a base for Φ. Then the subset $\overline{C(\Delta)}$ of V is a fundamental domain for the action of W on V.*

1.1.3 Classification Theorem

In order to classify the root systems, we need to work with irreducible root systems, that is:

Definition 1.1.13. A root system Φ is called *irreducible* if it cannot be partioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

To check if Δ is irreducible is enough to check if Δ can be partioned in the same way. In the following lemma we present some fundamental properties of irreducible systems. They are used in the proof of the classification theorem and we will need them also in the following chapters.

Lemma 1.1.14. *Let Φ be irreducible. Then:*

1. *Relative to the partial order $<$, there exists a unique maximal root $\tilde{\alpha}$.*
2. *At most two root lengths occur in Φ. If Φ has two distinct lengths, we speak of long and short roots. The maximal root $\tilde{\alpha}$ is always long.*

Let Φ be a root system of rank n and Δ a base. We fix an order on the simple roots, $\Delta = \{\alpha_1, ..., \alpha_n\}$. The matrix $(\langle \alpha_i, \alpha_j \rangle)_{i,j}$ is then called the *Cartan matrix* of Φ. Its entries are called *Cartan integers*. Of course the matrix depends on the choice of the order of the roots. However, thanks to the fact that W acts transitively on the collection of bases, the Cartan matrix is independent of the choice Δ. Indeed, let Δ' another base for Φ and let $w \in W$ such that $w(\Delta) = \Delta'$. Let us consider the order $\Delta' = \{\alpha'_1, \cdots, \alpha'_n\}$ with $\alpha'_i = w(\alpha_i)$. Then we get the same Cartan Matrix as before because $(\langle \alpha_i, \alpha_j \rangle)_{i,j} = (\langle w(\alpha_i), w(\alpha_j) \rangle)_{i,j} = (\langle \alpha'_i, \alpha'_j \rangle)_{i,j}$ since w preserves the bilinear product. The Cartan matrix determines Φ up to isomorphism:

Proposition 1.1.15. *Let $\Phi' \subset V'$ be another root system with base $\Delta' = \{\alpha'_1, ..., \alpha'_n\}$. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for all $1 \le i, j \le n$ then the bijection $\alpha_i \mapsto \alpha'_i$ extends uniquely to an isomorphism $\phi : V \to V'$ mapping Φ onto Φ' and satisfying $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$. Therefore the Cartan matrix of Φ determines Φ up to isomorphism.*

It is practical to represent the Cartan matrix through a graph. Recall that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3$ if α and β are roots with $\alpha \neq \pm\beta$.

Definition 1.1.16. The *Coxeter graph* of Φ is the graph having n vertices and where the i-th is joined to the j-th ($i \neq j$) by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.

The Coxeter graph determines the numbers $\langle \alpha_i, \alpha_j \rangle$ in case all roots have equal length, since then $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$ and $\langle \alpha_i, \alpha_i \rangle = 2$. In case two lengths occur the graph fails to tell us which vertex correspond to the short simple root, which to the long. However, when a double or triple edge occurs in the Coxeter graph of Φ, we can add an arrow pointing to the shorter of the two roots. With this additional information we can recover the Cartan integers looking at the graph. The resulting figure is called the *Dynkin diagram* of Φ. Of course, Φ is irreducible if and only if its Coxeter graph is connected. In general, there will be a number of connected components of the Coxeter graph; let $\Delta = \Delta_1 \cup ... \cup \Delta_t$ be the corresponding partition of Δ into mutually orthogonally sets. If V_i is the span of Δ_i we can write $V = V_1 \oplus ... \oplus V_t$. Furthermore, the $\mathbb{Z}$-linear combinations of Δ_i which are in Φ (we call this set Φ_i) form a root system in V_i, whose Weyl group is the restriction to V_i of the subgroup of W generated by all s_α, with $\alpha \in \Delta_i$. We have that each V_i is W-invariant: if $\alpha \notin \Delta_i$ then s_α acts trivially on V_i, if $\alpha \in \Delta_i$ we get an automorphism of V_i. Notice that if $V' \le V$ is stable under s_α with $\alpha \in \Phi$ then either $\alpha \in V'$ or $V' \subset H_\alpha$. From this it follows that for every $\alpha \in \Phi$ we have $\alpha \in V_i$ for some i (since not all V_i can be contained in H_α). Thus we obtain $\Phi = \Phi_i \cup ... \cup \Phi_t$.

Proposition 1.1.17. *A root system Φ decomposes uniquely as the union of irreducible root systems Φ_i of subspaces V_i of V such that $V = V_1 \oplus ... \oplus V_t$.*

Thanks to the above proposition it is enough to classify the irreducible root systems or equivalently the connected Dynkin diagrams. It is possible to obtain the classification using first some elementary euclidean geometry in order to find the admissible Dynkin diagrams, then constructing a root system for every admissible graph. The result is the following theorem.

Theorem 1.1.18. *If Φ is an irreducible root system of rank n, its Dynkin diagram is one of the following:*

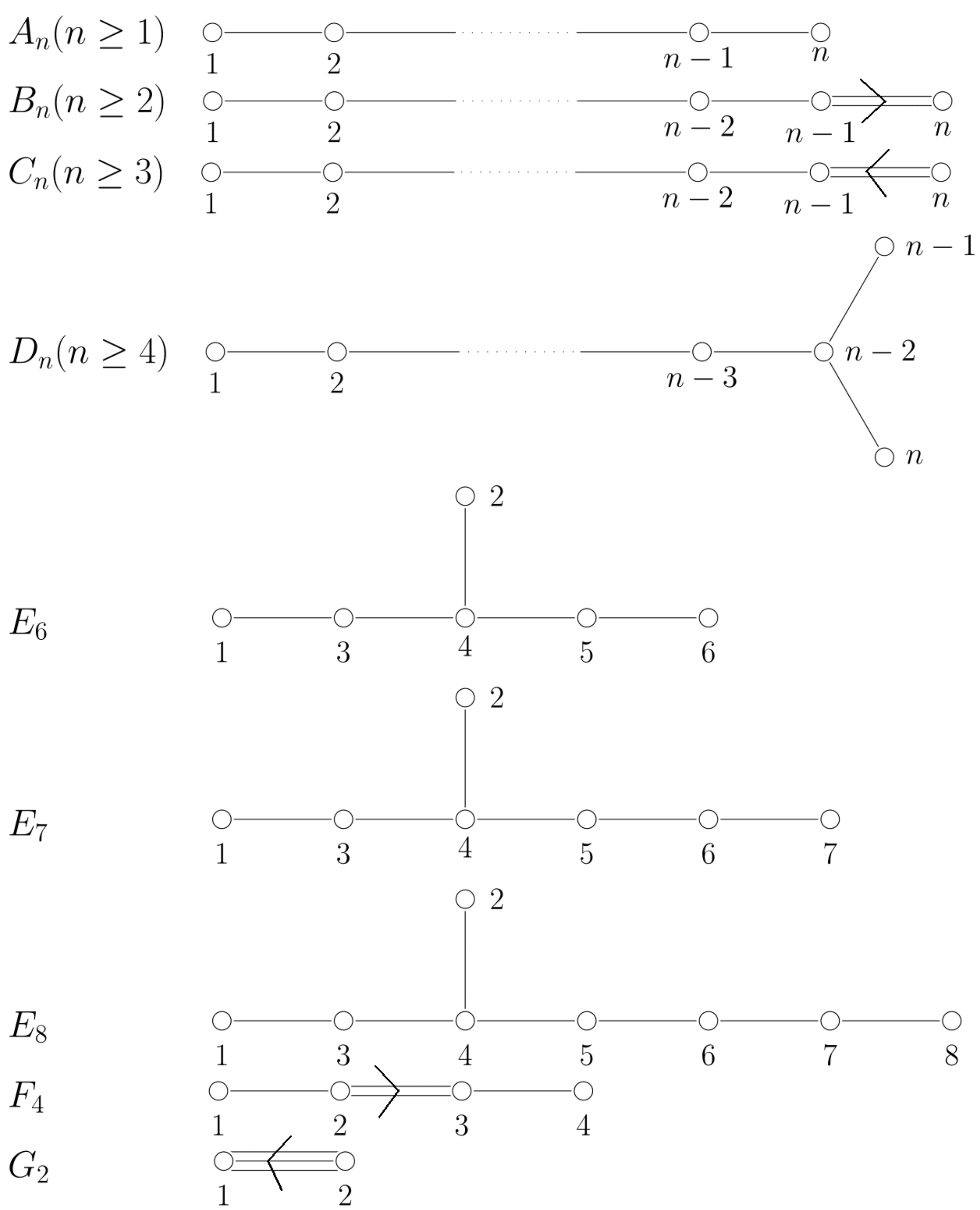

Figure 1.1: Dynkin diagrams

In the above figure, the condition $n \geq i$, $i \in \mathbb{N}$ is added in order to avoid repetitions. The name used in the picture to identify the graphs (or equivalently the associated irreducible root systems), are called *Cartan types*. In order to have some examples in mind, we present the construction of the root systems of Cartan type A_n, B_n, C_n.

Example 1.1.19. **($\boldsymbol{A_n, n \geq 1}$)**. Consider the vector space $\mathbb{R}^{n+1}$ with the usual inner product and canonical basis $\{\epsilon_1, ..., \epsilon_{n+1}\}$. Let V be the subspace of $\mathbb{R}^{n+1}$ orthogonal to

the vector $\epsilon_1 + ... + \epsilon_{n+1}$ with the induced inner product. We take $\Phi = \{\epsilon_i - \epsilon_j$ with $i \neq j\}$. The axioms (R1)-(R4) are immediately verified, thus Φ is a root system in V. The vectors $\alpha_i = \epsilon_i - \epsilon_{i+1}$ with $1 \leq i \leq n$ form a base for Φ. Indeed, they are a basis for V and $\epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$ if $i < j$ and $\epsilon_i - \epsilon_j = -\alpha_i - \alpha_{i+1} - ... - \alpha_{j-1}$ if $i > j$. Looking at the Cartan integers we obtain the desired Dynkin diagram. Furthermore, we have $\tilde{\alpha} = \epsilon_1 - \epsilon_{n+1}$. Now we want to give a description of the associated Weyl group. Notice that every s_{α_i} exchanges ϵ_i with ϵ_{i+1} and fixes the other ϵ_k. So we can identify s_{α_i} with the transposition $(i, i+1)$ of the symmetric group S_{n+1} associated to the canonical basis. Since these transpositions generate S_{n+1} we get a natural isomorphism between W and S_{n+1}.
$(\boldsymbol{B_l, n \geq 2})$. Let $V = \mathbb{R}^n$ with the usual inner product and canonical basis $\{\epsilon_1, ..., \epsilon_n\}$. Consider the set $\Phi = \{\pm\epsilon_i$ with $1 \leq i \leq n\} \bigcup \{\pm(\epsilon_i \pm \epsilon_j)$ with $i \neq j\}$ which satisfies the conditions (R1)-(R4). As Δ we can take $\alpha_1 = \epsilon_1 - \epsilon_2, ..., \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$ and $\alpha_n = \epsilon_n$. Then we have $\tilde{\alpha} = \epsilon_1 + \epsilon_2$. Notice that the reflection s_{ϵ_i} sends ϵ_i to $-\epsilon_i$ and fixes ϵ_k with $k \neq i$. Then W acts as the group of permutations and signed changes on the set $\{\epsilon_1, .., \epsilon_n\}$, so it is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$.
$(\boldsymbol{C_n, n \geq 3})$ We can define C_n as the dual system of B_n. We obtain $\Phi = \{\pm 2\epsilon_i$ with $1 \leq i \leq n\} \bigcup \{\pm(\epsilon_i \pm \epsilon_j)$ with $i \neq j\}$. A possible base is $\alpha_1 = \epsilon_1 - \epsilon_2, ..., \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$ and $\alpha_n = 2\epsilon_n$. Since it is the dual system of B_n, we obtain the same Weyl group as before. In particular we can represent $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ as subgroup of S_{2n} in the following way. If we consider the permutation group over the set $\{\epsilon_1, ..., \epsilon_n, -\epsilon_l, ..., -\epsilon_n\}$, we get this identification: $\rho(s_{\alpha_1}) = (1, 2)(2n-1, 2n)$, $\rho(s_{\alpha_2}) = (2, 3)(2n-2, 2n-1)$, ..., $\rho(s_{\alpha_{n-1}}) = (n-1, n)(n+1, n+2)$ and $\rho(s_{\alpha_n}) = (n, n+1)$. Then the map ρ extends to a group morphism $W \hookrightarrow S_{2n}$.

Remark 2. The root systems B_n and C_n with $n > 2$ are the only irreducible root systems which are not isomorphic to their dual.

1.1.4 Weights

Let Φ be a root system of rank n. We consider now the set

$$P := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z},\ \forall\ \alpha^\vee \in \Phi^\vee\}.$$

Since (λ, α) depends linearly on λ, the set P is a subgroup of V whose elements are called *weights*. We want to show that P is a lattice and find a basis. Let $\Delta = \{\alpha_1, .., \alpha_n\}$ a base for Φ. Since $\Delta^\vee$ is a base for $\Phi^\vee$ we can write $P = \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z},\ \mid \alpha^\vee \in \Delta^\vee\}$. Let $\omega_1, ..., \omega_l$ be the dual basis of $\Delta^\vee$ relative to the inner product on V, i.e. $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$. The ω_i are called *fundamental weights.* Let $\lambda \in V$ with $\lambda = \sum_{i=1}^n k_i\omega_i$. Since $(\lambda, \alpha_i^\vee) = k_i$ we get that $\lambda \in P$ iff all the k_i are integers. Thus P is the lattice generated by the fundamental coweights which form a basis. It is called the *weight lattice.* Now we define Q to be the lattice generated by Φ and call it *root lattice.* For our purpose it is useful to know the index of Q in P when Φ is an irreducible system. It is possible to find it looking at its Cartan matrix. If we write $\alpha_i = \sum_j m_{ij}\omega_j$ with $m_{ij} \in \mathbb{Z}$, then $\langle \alpha_i, \alpha_k \rangle = (\alpha_i, \alpha_k^\vee) = \sum_j m_{ij}(\omega_j, \alpha_k^\vee) = m_{ij}$. In other words the Cartan matrix expresses the change of basis. From this it follows that the index of Q in P is the determinant of the Cartan matrix. Looking at the Cartan matrices is then possible to find the following table where for every irreducible root system we have written the order of P/Q.

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$n+1$	2	2	4	3	2	1	1	2

Table 1.2: order of P/Q

We define $\lambda \in P$ to be *dominant* if all the integers $(\lambda, \alpha^\vee)$ with $\alpha \in \Delta$, are non negative, i.e. $\lambda \in \overline{C(\Delta)}$. Of course the fundamental weights are dominant. We denote by P_+ the set of all dominant weights.

Remark 3. Notice that the Weyl group preserves the inner product on V, hence it leaves P invariant. In particular from Proposition 1.1.12 it follows that every weight is conjugate under the action of W to one and only one dominant weight.

For the next section it is important to keep in mind the following definition:

Definition 1.1.20. Let Φ a root system of rank n. The *coweight lattice* is the lattice $P^\vee := \{\lambda \in V \mid (\lambda, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. If we fix a base Δ, the elements in the dual basis of Δ with respect to the bilinear product, are called *fundamental coweights* and denoted by $\omega_1^\vee, ..., \omega_n^\vee$.

In other words, the coweight and coroot lattices are respectively the root and weight lattices of the dual root system of Φ. If we fix a base Δ, the fundamental coweights are the fundamental weights relative to the base $\Delta^\vee$ of $\Phi^\vee$. We conclude this section with an image from SageMath of the irreducible root systems in the plane. In the figure you can see a base $\Delta = \{\alpha_1, \alpha_2\}$, the fundamental weights $\omega_1 = \Lambda_1$, $\omega_2 = \Lambda_2$, the hyperplanes H_{α_1}, H_{α_2} and in gray the fundamental Weyl chamber.

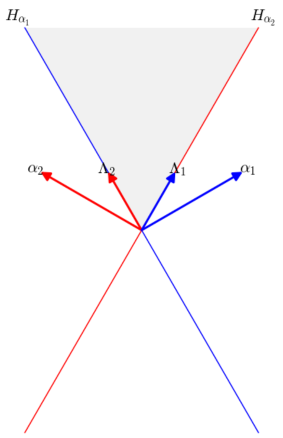

Figure 1.2: A2

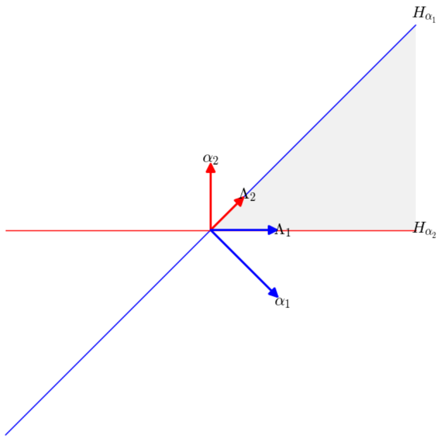

Figure 1.3: B2

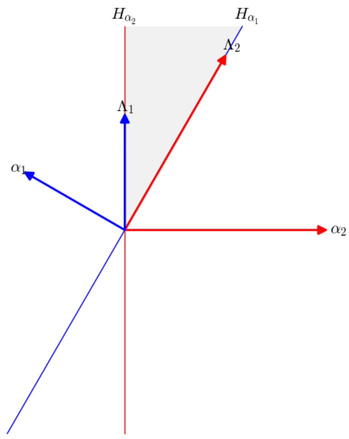

Figure 1.4: G2

1.2 Affine Weyl groups

In this chapter we are going to introduce the affine Weyl groups following mainly [4, Chapters IV and V].

1.2.1 Affine reflections

Let V be a euclidean space. Now we consider the group of affine transformations $\mathrm{Aff}(V)$, i.e. the semidirect product of $GL(V)$ and the group of translations by elements of V. For every λ in V let $t(\lambda)$ be the translation which sends $\mu \in V$ to $\mu + \lambda$. Let Φ be a root system in V. For every α in Φ and every k in $\mathbb{Z}$ we define the affine hyperplane $H_{\alpha,k} := \{\lambda \in V \mid (\lambda, \alpha) = k\}$ and the corresponding affine reflection $s_{\alpha,k}(\lambda) := \lambda - ((\lambda, \alpha) - k)\alpha^\vee$. Note that $s_{\alpha,k} = t(k\alpha^\vee)s_{\alpha,0}$, $H_{\alpha,0} = H_{\alpha,}$ and $s_{\alpha,0} := s_\alpha$ where H_α and s_α are the Hyperplane and the reflection defined in the previous chapter. We consider the following infinite subgroup of $\mathrm{Aff}(V)$.

Definition 1.2.1. The *affine Weyl group* W_a is the subgroup of $\mathrm{Aff}(V)$ generated by the affine reflections $s_{\alpha,k}$, with $\alpha \in \Phi$, $k \in \mathbb{Z}$.

From the definition it follows that the classical Weyl group W is a subgroup of W_a. We can say more about the relations of this two groups.

Proposition 1.2.2. *W_a is the inner semidirect product of W and the translation group corresponding to the coroot lattice $Q^\vee$.*

Proof. Let T be the translation group corresponding to $Q^\vee$. Then W normalizes T because for every $\lambda \in Q^\vee$, $wt(\lambda)w^{-1} = t(w\lambda)$ and $Q^\vee$ is stabilized by W. Since W and T have trivial intersection, we can consider their semidirect product W'. Writing $s_{\alpha,k} = t(k\alpha^\vee)s_\alpha$, we see immediately that all the generators of W_a lie in W'. On the other hand, since $t(k\alpha^\vee) = s_{\alpha,k}s_\alpha$, we have that both $Q^\vee$ and W are contained in W_a. □

We can also say that $W_a = Q^\vee \rtimes W$ (external semidirect product) where the action of w on λ is $w(\lambda)$. Following the convention, we will then denote an element $w \in Q^\vee \rtimes W$ as $t^\lambda w$ with $\lambda \in Q^\vee$ and $w \in W$. Since the translation group corresponding to $P^\vee$ is also normalized by W, we can give the following definition.

Definition 1.2.3. The *extended affine Weyl group* $\widetilde{W}$ is the semidirect product of W and the translation group corresponding to the coweight lattice $P^\vee$.

Also in this case we can directly write $\widetilde{W} = P^\vee \rtimes W$. From the definition it follows that W_a is a subgroup of $\widetilde{W}$. Now we want to understand how W and $P^\vee$ act on $\mathcal{H} := \{H_{\alpha,k} \mid \alpha \in \Phi \text{ and } k \in \mathbb{Z}\}$ and on the affine reflections $s_{\alpha,k}$. Easy calculations show that:

Proposition 1.2.4. *1. If $w \in W$ then $wH_{\alpha,k} = H_{w\alpha,k}$ and $ws_{\alpha,k}w^{-1} = s_{w\alpha,k}$.*

2. If $\lambda \in P^\vee$ then $t(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha)}$ and $t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k+(\lambda,\alpha)}$.

Thus we get

Corollary 1.2.5. *Let $\widetilde{w} \in \widetilde{W}$, with $\widetilde{w} = t(\lambda)w$ for $w \in W$ and $\lambda \in P^\vee$. Let $H_{\alpha,k} \in \mathcal{H}$, then $\widetilde{w}H_{\alpha,k} = H_{w(\alpha),k+(\lambda,w(\alpha))}$ and $ws_{\alpha,k}w^{-1} = s_{w(\alpha),k+(\lambda,w(\alpha))}$.*

In particular, it follows that W_a is a normal subgroup of $\widetilde{W}$. The index of W_a in $\widetilde{W}$ is the equal to $|P^\vee/Q^\vee| = |P'/Q'|$ where P' and Q' are the root lattice and weight lattice of $\Phi^\vee$. We can find this number looking at Table 1.2. In the following example we will consider the case where Φ is of type A_{n-1}.

Example 1.2.6. Let $\mathbb{R}^n$ be the usual euclidean space with canonical basis $\{\epsilon_1, ..., \epsilon_n\}$. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, ..., n-1$. In Example 1.1.19 we saw that $\Phi = \langle \epsilon_i - \epsilon_j \mid j \neq i \rangle$ is a root system of type A_{n-1} for $V = \langle \epsilon_1 + ... + \epsilon_n \rangle^\perp$. We checked also that $\Delta = \{\alpha_i \mid i = 1, ..., n-1\}$ is a base for Φ. Recall that $\Phi^\vee = \Phi$, so $Q^\vee = Q = \oplus_{i=1}^{n-1}\mathbb{Z}\alpha_i$ and $P^\vee = P$. Let

$M = \{\sum_{i=1}^n a_i\epsilon_i \in \mathbb{R}^n \mid a_i \in \mathbb{Z}\ \forall i \text{ and } \sum_i^n a_i = 0\}$. We have that $M \subseteq Q^\vee$. Indeed, let $v = \sum_i^n a_i\epsilon_i \in M$ then $v = \sum_{i=1}^{n-1}\sum_{j=1}^i a_j\alpha_i$. Clearly, all the $\mathbb{Z}$-linear combinations of simple roots lie in M. Then we have $Q^\vee = M$. In Example 1.1.19 we have identified the Weyl group W with S_n, thus we get an action of S_n on $Q^\vee$. Explicitly, let $v = \sum_{i=1}^n a_i\epsilon_i \in Q^\vee$, then we have $(i, i+1)(v) = s_{\alpha_i}(v) = a_{i+1}\epsilon_i + a_i\epsilon_{i+1} + \sum_{j\neq i,i+1} a_j\epsilon_j$. Hence we obtain

$$W_a \cong \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n \mid \sum_i^n \lambda_i = 0\} \rtimes S_n$$

where $\sigma(\lambda) = (\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n)})$ for σ in S_n and $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$. Now we consider the lattice $P^\vee$. Let $N = \{\sum_{i=1}^n a_i\epsilon_i \in \mathbb{R}^n \mid a_i \in \mathbb{Z}\ \forall\ i\}$. Let p the orthogonal projection from $\mathbb{R}^n$ to V. Note that $p(N) \subseteq P^\vee$. Indeed, $p(\epsilon_1) = \epsilon_1 - \frac{\epsilon_1+...+\epsilon_n}{n}$ thus $(p(\epsilon_1), \alpha_i) = 1$ and $(p(\epsilon_1), \alpha_j) = 0$ when $j \neq 1$. If $1 < i \leq n$ we get $(p(\epsilon_i), \alpha_{i-1}) = -1$, $(p(\epsilon_i), \alpha_i) = 1$ and $(p(\epsilon_i), \alpha_j) = 0$ for $j \neq i, i+1$. Then the inclusion follows from the linearity of p and of the scalar product. From the previous calculations, we see that $\omega_i^\vee = p(\sum_{j=1}^i \epsilon_j)$. Thus the map $p : N \to P^\vee$ is a surjective group morphism (the coweights are a basis for the lattice $P^\vee$). Clearly the kernel is the subgroup $\langle(1, ..., 1)\rangle$ of N. Hence, the usual isomorphism $S_n \cong W$, induces an action of S_n on $P^\vee \cong \frac{\mathbb{Z}^n}{\langle(1,...,1)\rangle}$. Morovere, we have that $p(s_{\alpha_i}(v)) = s_{\alpha_i}(p(v))$ for v in $\mathbb{R}^n$ and $i = 1, ..., n-1$. Thus we obtain

$$\widetilde{W} \cong \frac{\mathbb{Z}^n}{\langle(1, ..., 1)\rangle} \rtimes S_n$$

where $\sigma(\lambda+\mathbb{Z}(1, ..., 1)) = (\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n)})+\mathbb{Z}(1, ..., 1)$ for σ in S_n and $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$. Finally, by construction, we can identify $W_a \subset \widetilde{W}$ with the image of the following map

$$\{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n \mid \sum_i^n \lambda_i = 0\} \rtimes S_n \to \frac{\mathbb{Z}^n}{\langle(1, ..., 1)\rangle} \rtimes S_n$$

$$(z, \sigma) \mapsto (\overline{z}, \sigma),$$

where $\overline{z}$ is the image of z under the canonical projection $\mathbb{Z}^n \to \frac{\mathbb{Z}^n}{\langle(1,...,1)\rangle}$.

1.2.2 Generators

Now we want to find a finite set of generators for the affine Weyl groups. Let Φ a root system in V and $\Phi = \Phi_1 \cup ... \cup \Phi_t$ its decomposition in irreducible root systems (see 1.1.17). Let V_i the subvector space generated by Φ_i. Let $\alpha \in \Phi_i$ and $\beta \in \Phi_j$ with $i \neq j$. From the definition of the affine reflection, we get easily that $s_{\alpha,k}s_{\beta,j} = s_{\beta,j}s_{\alpha,k}$ for every j, k in $\mathbb{Z}$. Thus $W_a = W_1 \times ... \times W_n$ where $W_i = \langle s_{\alpha,k} \mid \alpha \in \Phi_i \text{ and } k \in \mathbb{Z}\rangle$. Furthermore, the restriction of the subgroup W_i to V_i is clearly injective and it coincides with the affine Weyl group of Φ_i. To sum up we get

Proposition 1.2.7. *Let Φ a root system and W_a its Weyl group. Let $\Phi = \Phi_1 \cup ... \cup \Phi_t$ its decomposition in irreducible root systems. Then*

$$W_a \cong W_a^1 \times ... \times W_a^t$$

where W_a^i is the affine Weyl group of Φ_i.

Thanks to the previous proposition it is enough to study only affine Weyl groups of irreducible root systems. From now on we suppose that Φ is an irreducible root system in V of rank n and we fix a base $\Delta = \{\alpha_1, .., \alpha_n\}$. In order to find a finite number of generators for the associated affine Weyl group W_a we need to study how W_a permutes the connected components of $V^\circ = V \setminus \bigcup_{H \in \mathcal{H}} H$. These components are called *alcoves* and their collection is denoted by $\mathcal{A}$. It is convenient to fix a particular alcove:

$$A_\circ := \{\lambda \in V \mid 0 < (\lambda, \alpha) < 1 \text{ for all } \alpha \in \Phi^+\}.$$

Note that $A_\circ$ is an alcove. Indeed it is connected (since it is convex) and contained in V°. Furthermore, every element outside $A_\circ$ is separated from it by H_α or $H_{\alpha,1}$. Let $\tilde{\alpha}$ be the maximal root of Φ, then we can write:

$$A_\circ = \{\lambda \in V \mid 0 < (\lambda, \alpha) \text{ for all } \alpha \in \Delta \text{ and } (\lambda, \tilde{\alpha}) < 1\}. \tag{1.1}$$

We see immediately that $A_\circ$ is contained in the above RHS. On the other hand, if λ satisfies the above inequality, for evey α in Φ^+ we have $(\lambda, \alpha) > 0$. Since $\tilde{\alpha} - \alpha$ is a sum of simple roots we get $(\lambda, \tilde{\alpha} - \alpha) > 0$. Then $(\lambda, \alpha) < (\lambda, \tilde{\alpha}) < 1$ and this implies that $\lambda \in A_\circ$.

Example 1.2.8. When $|\Delta| = 2$, we obtain three affine Weyl groups coming from the classical Weyl groups associated with the root systems A_2, B_2 and G_2. We denote them by $\widetilde{A_2}$, $\widetilde{B_2}$ and $\widetilde{G_2}$. The alcoves are represented in the following figures. The alcove $A_\circ$ is traced in black and $\alpha_0 = \tilde{\alpha}$.

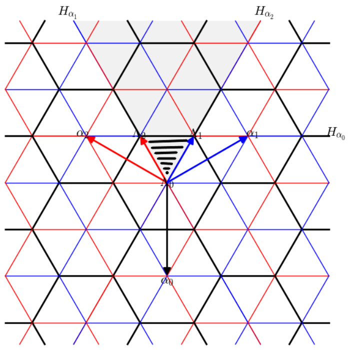

Figure 1.5: $\widetilde{A_2}$

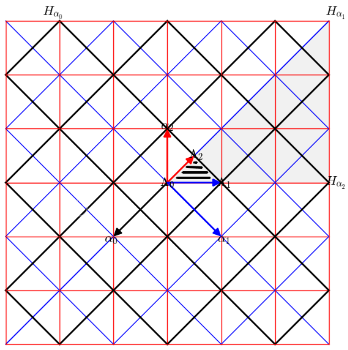

Figure 1.6: $\widetilde{B_2}$

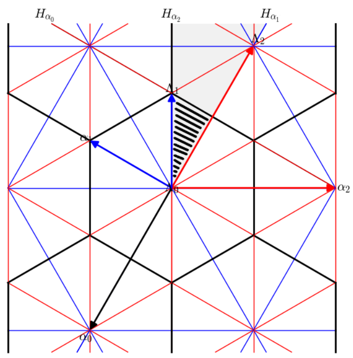

Figure 1.7: $\widetilde{G_2}$

We define the *walls* of $A_\circ$ to be the hyperplanes H_α, $\alpha \in \Delta$ and $H_{\tilde{\alpha},1}$. Let S_a be the corresponding set of reflections, called *simple affine reflections*:

$$S_a := \{s_\alpha, \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}.$$

The walls of the alcove $wA_\circ$ with $w \in W_a$ are defined to be the images of the walls of $A_\circ$ under w. They are defined for every alcove thanks to the following important result.

Proposition 1.2.9. *The group W_a permutes the collection of alcoves $\mathcal{A}$ transitively and it is generated by the set S_a.*

Proof. Let W' be the subgroup of W_a generated by S_a. We want to show that W' acts transitively on $\mathcal{A}$. It is enough to show that for every A in $\mathcal{A}$ there exists w' in W' such that $w'A = A_\circ$. Let λ and μ two elements in $A_\circ$ and A. Notice that the orbit of μ under the translations associated to $P^\vee$ is a discrete subset of V because $P^\vee$ is a lattice in V. Then also the orbit of μ under the group W_a is discrete since it is an extension of the previous orbit by the action of a finite group of reflections (the Weyl group W) which stabilizes $P^\vee$. Since W' is a subgroup of W_a, also the W'-orbit of μ is discrete. Let $\delta = w'(\mu)$ with $w' \in W'$ be an element in this orbit with the minimal distance from λ. If we show that $\delta \in A_\circ$, then $w'A \cap A_\circ \neq \emptyset$ and thus $w'A = A_\circ$. Suppose that $\delta \notin A_\circ$ then λ and μ must lie in different half-spaces relative to some wall H of $A_\circ$. Let s be the corresponding reflection, so $s \in W'$. We can consider the isosceles trapezoid with vertices $\delta, s(\delta), s(\lambda), \lambda$. Since the length of a diagonal is greater than the length of the two nonparallel sides we get that $\|s(\delta) - \lambda\| < \|\delta - \lambda\|$. But this contradicts the choice of δ, since also $s(\delta)$ lies in the W'-orbit of μ. We have then that W' acts transitively on $\mathcal{A}$ and thus every alcove has well defined walls. To show that $W' = W_a$ we have to prove that $s_{\alpha,k} \in W'$ for every α and k. Let $H_{\alpha,k}$ be the corresponding hyperplane and A be an alcove having $H_{\alpha,k}$ as a wall. Then there is $w' \in W'$ such that $w'A = A_\circ$. Then $w'H_{\alpha,k} = H$ with H a wall of $A_\circ$ with associated reflection $s \in W'$. By Corollary 1.2.5 we have $w's_{\alpha,k}w'^{-1} = s$, forcing $s_{\alpha,k} \in W'$. □

Since S_a generates W_a we can define the *length* $l(w)$ of an element w in W_a to be the smallest r for which w is a product of r elements of S_a. Such expression is called *reduced.*

1.2.3 Geometric interpretation of the length function

Using the alcoves we are going to define an integer value function on $\widetilde{W}$. We will then see that its restriction on W_a coincides with the length function $l(w)$. This geometric point of view will be useful to study deeply the relations between the Weyl groups. Clearly, if we consider a hyperplane $H \in \mathcal{H}$, then each alcove $A \in \mathcal{A}$ lies in just one of the half-spaces identified by H. We say that H separates two alcoves if they lie in different half-spaces relative to H. Note that there is only a finite number of hyperplanes that separates two alcoves A and A'. By Corollary 1.2.5, we know that also $\widetilde{W}$ permutes the hyperplanes in $\mathcal{H}$, thus it induces a bijections in the set $\mathcal{A}$. We can then define the following function on $\widetilde{W}$:

$$n(w) := \#\mathcal{L}(w) \text{ with } \mathcal{L}(w) := \{H \in \mathcal{H} \mid H \text{ separates } A_\circ \text{ and } wA_\circ\}.$$

From the definition it follows that $n(1) = 0$, $n(s) = 1$ with $s \in S_a$ since $\mathcal{L}(s) = \{H_s\}$. Furthermore $n(w^{-1}) = n(w)$ because H separates $A_\circ$ from $w\mathcal{A}_\circ$ if and only if $w^{-1}H$ separates $w^{-1}A_\circ$ from $A_\circ$. Then $\mathcal{L}(w^{-1}) = w^{-1}\mathcal{L}(w)$. First we need to study how the function $n()$ changes when we multiply by elements of S_a.

Proposition 1.2.10. *Let w in $\widetilde{W}$ and s in S_a.*

1. *H_s belongs to exactly one of the sets $\mathcal{L}(w), \mathcal{L}(sw)$.*
2. *$s(\mathcal{L}(w) \setminus \{H_s\}) = \mathcal{L}(sw) \setminus \{H_s\}$.*
3. *$n(ws) = n(w) - 1$ if $wH_s \in \mathcal{L}(w)$ and $n(ws) = n(w) + 1$ otherwise.*

Proof. 1)First suppose that H_s belongs to both sets. Then H_s separates $wA_\circ$ from $A_\circ$. Thus applying w^{-1}, we get that $w^{-1}H_s$ separates $w^{-1}A_\circ$ from $A_\circ$. Since H_s lies in $\mathcal{L}(sw)$, we have that $swA_\circ$ and $A_\circ$ are separated by H_s. Hence the hyperplane $w^{-1}sH_s = w^{-1}H_s$ sepates $w^{-1}sA_\circ$ from $A_\circ$. It follows that $w^{-1}A_\circ$ and $w^{-1}sA_\circ$ are not separated by $w^{-1}H_s$. Appling w we get that $sA_\circ$ and $A_\circ$ lie on the same side with respect to H_s and this is

absurd. Similarly, if we suppose that H_s belongs to none of the two set, we get that $w^{-1}A_\circ$ and $w^{-1}sA_\circ$ are not separated by $w^{-1}H_s$. Thus we obtain again a contradiction.
2)Suppose that $H \in \mathcal{L}(w)$ with $H \neq H_s$. Then sH separates $sA_\circ$ from $swA_\circ$. Clearly, $sH \neq H_s$. Suppose by contradiction that sH does not separate $swA_\circ$ from $A_\circ$. It follows that sH separates $sA_\circ$ from $A_\circ$. Thus H separates $A_\circ$ from $sA_\circ$. It is absurd because $H \neq H_s$. Then we get that sH separates $swA_\circ$ from $A_\circ$, i.e. $sH \in \mathcal{L}(sw) \setminus \{H_s\}$. Now we want to prove that $\mathcal{L}(sw) \setminus \{H_s\} \subseteq s(\mathcal{L}(w) \setminus \{H_s)\}$ or equivalently $s(\mathcal{L}(sw) \setminus \{H_s\}) \subseteq \mathcal{L}(w) \setminus \{H_s\}$. Note that for the last inclusion it is enough to repeat the same reasoning of before replacing w with ws. Then we have proved the statement.
3) We have already noticed that $n(ws) = n(sw^{-1})$. If we apply point 1) and point 2) to w^{-1} and s we get that $n(sw^{-1}) = n(w^{-1}) - 1 = n(w) - 1$ if $H_s \in \mathcal{L}(w^{-1})$, otherwise $n(sw^{-1}) = n(w^{-1}) - 1 = n(w) - 1$. Since $L(w) = w\mathcal{L}(w^{-1})$, we get that $H_s \in \mathcal{L}(w^{-1})$ iff $wH_s \in \mathcal{L}(w)$. Then we get the desidered statement. □

Corollary 1.2.11. *For any w in W_a we have $n(w) \leq l(w)$.*

Proof. Clearly, it is true for $w = 1$. Otherwise let $w = s_1 ... s_r$ be a reduced expression for w. We use induction on r. Part 3) of the previous proposition implies that the value of $n()$ can increase at most 1 every time we multiply by s with $s \in S_a$. So $n(w) \leq r = l(w)$. □

1.2.4 Simple transitivity

In order to prove that $n(w) = l(w)$ we will describe explicitly the set $\mathcal{L}(w)$.

Theorem 1.2.12. *1. Let $w \neq 1$ in W_a with a reduced expression $w = s_1 ... s_r$. Then we have (setting $H_i := H_{s_i}$)*

$$\mathcal{L}(w) = \{H_1, s_1H_2, s_1s_2H_3, ..., s_1...s_{r-1}H_r\},$$

where the r hyperplanes are all distinct.

2. The function $n()$ on W_a coincides with the length function $l()$.

Proof. 1) We have already noticed that $\mathcal{L}(s) = \{H_s\}$ when $s \in S_\alpha$. Now we use induction on $r = l(w)$. When $r > 1$, the induction hypothesis says that

$$\mathcal{L}(s_2...s_r) = \{H_2, s_2H_3, ..., s_2...s_{r-1}H_r\}.$$

where the $r - 1$ hyperplanes are distinct. Suppose that H_1 is in the list. Then we can write

$$H_1 = s_1H_1 \in \{s_1H_2, s_1s_2H_3, ..., s_1...s_{r-1}H_r\}.$$

Thus $H_1 = s_1 \cdots s_{j-1}H_j$ for some $2 \leq j \leq r$. Applying Proposition 1.2.4, we obtain $s_1 = s_1 \cdots s_{j-1}s_js_{j-1} \cdots s_1$. Then we get $s_2 \cdots s_{j-1} = s_1 \cdots s_j$. This allows us to decrease the length of the reduced expression for w but this is not possible. It follows that $H_1 \notin \mathcal{L}(s_1w)$. Now part 1) of Proposition 1.2.10, forces H_1 to lie in $\mathcal{L}(w)$, while part 2) forces $\mathcal{L}(w)$ to be the desired set of r distinct hyperplanes.
2)From the previous point it follows that $\#\mathcal{L}(w) = r$. Thus we get $n(w) = r = l(w)$ □

Let w be an element of the classical Weyl group $W \subset W_a$. Notice that every H in $\mathcal{H}$ which divides $wA_\circ$ and $A_\circ$ goes through the origin, since $0 \in \overline{wA_\circ} \cap \overline{A_\circ}$. Let $w = s_1 \cdots s_t$ with $s_i \in S_a$ be a reduced expression for w. Then $s_i \neq s_{\tilde{\alpha},1}$ for every i. Suppose that it is not true. Let j the smallest index such that $s_j = s_{\tilde{\alpha},1}$. Then by Theorem 1.2.12 we get that $s_1 \cdots s_{j-1} H_j$ divides $wA_\circ$ from $A_\circ$. This is absurd since it is not a hyperlane through the origin. In particular, we get that the function $l'(w)$ on W of Definition 1.1.11 coincides with the restriction of $n()$ to W. Let $w \in \widetilde{W}$. From now on we set $l(w) := n(w)$ and we call it the length of w, without ambiguity with the previous definitions of length in the subgroups W_a and W. A fundamental consequence of the previuos theorem is the following corollary.

Corollary 1.2.13. *The group W_a acts simply transitively on $\mathcal{A}$.*

Proof. We have already seen that W_a acts transitively on $\mathcal{A}$, so it is enough to show that the only element which fixes an alcove, say $A_\circ$, is 1. Suppose that $wA_\circ = A_\circ$ with $w \in W_a$. Then we have $\mathcal{L}(w) = \emptyset$ then $l(w) = 0$. It implies $w = 1$. □

Now we can say something more about the structure of $\widetilde{W}$. First we introduce a subgroup of $\widetilde{W}$ which plays an important role in the next chapters.

Definition 1.2.14. The *fundamental group* Ω is the subgroup of $\widetilde{W}$:

$$\Omega := \{w \in \widetilde{W} \mid wA_\circ = A_\circ\},$$

or equivalently $\Omega := \{w \in \widetilde{W} \mid l(w) = 0\}$.

Let w be an element in $\widetilde{W}$. The transitivity of the normal subgroup $W_a \subset \widetilde{W}$ implies that $wA_\circ = w'A_\circ$ for some w' in W_a. Thus $(w')^{-1}wA_\circ = A_\circ$, so we can write $w = w'\tau = \tau\tau^{-1}w'\tau = \tau w''$ with τ in Ω and $w'' \in W_a$. We get that $\widetilde{W} = \Omega W_a$ and from the simple transitivity of W_a it follows that $\Omega \cap W_a = 1$. So $\widetilde{W} = \Omega \ltimes W_a$ and $\Omega \cong \widetilde{W}/W_a \cong P^\vee/Q^\vee$. Now we want to examine the fundamental group using some results of [5]. First we show that the elements of Ω are in bijection with a subset of the vertices of $\overline{A_0}$. Suppose that $\tilde{\alpha} = \sum_i^n k_i\alpha_i$ with k_i positive integers. Then by the definition of $A_\circ$ we get that the vertices of the simplex $\overline{A_\circ}$ are 0 and $\omega_i^\vee/k_i$ with $1 \leq i \leq n$.

Definition 1.2.15. A vertex of $\overline{A_\circ}$ is said to be *special* if it belongs to n distinct hyperplanes of the set $\mathcal{H}$.

Notice that 0 is always a special vertex and $v = \omega_i^\vee/k_i$ is special if $k_i = 1$, i.e. v is a fundamental coweight. Equivalently, we can say that the special vertices are the 0 and the fundamental coweights $\omega_i^\vee$ such that $(\omega_i^\vee, \tilde{\alpha}) = 1$.

Proposition 1.2.16. *The elements of the fundamental group are in bijection with the special vertices.*

Proof. Let S be the set of special vertices and $\tau \in \Omega$. From the definition of the fundamental group it follows that τ induces a permutation of the vertices of $\overline{A_0}$. Notice that $\tau(0)$ is again a special vertex because the elements of $\widetilde{W}$ induce a bijection on the set $\mathcal{H}$, i.e. the number of hyperplanes going through a vertex is preserved. We can then define a map: $\pi : \Omega \to S$, $\tau \mapsto \tau(0)$. First we want to show that π is injective. Let τ, τ' in Ω with $\tau(0) = \tau'(0)$. Then the affine reflection $\tau\tau'^{-1}$ fixes the origin. Hence it is an element in the classic Weyl group W. Since $\Omega \cap W = 1$, we get that $\tau = \tau'$. On the other hand let $v \neq 0$ a special vertex, so $v = \omega_i^\vee$ for some $1 \leq i \leq n$. Let w_Δ be the unique element of W such that $w_\Delta(\Delta) = -\Delta$ and let $w_{\Delta_i} \in W_i = \langle s_{\alpha_j} \mid j \neq i\rangle$ such that $w_{\Delta_i}(\Delta_i) = -\Delta_i$ where

$\Delta_i = \Delta \setminus \alpha_i$. By Remark 1 we know that these two elements exists and they have order two. Now we want to show that $\tau := t^v w_{\Delta_i} w_\Delta$ is an element of the fundamental group, i.e. $\tau(A_\circ) = A_\circ$. Let λ in $-A_\circ$ it is enough to check that $t^v w_{\Delta_i}(\lambda) = w_{\Delta_i}(\lambda) + v \in A_\circ$. First notice that since w_{Δ_i} is a product of s_{α_j} with $j \neq i$, we get that $w_{\Delta_i}(\alpha_i) = \alpha_i + \sum_{j\neq i} a_j\alpha_j$ with $a_j \in \mathbb{Z}$. Hence $w_{\Delta_i}(\alpha_i) > 0$. This implies that also $w_{\Delta_i}(\tilde{\alpha}) > 0$. Indeed, $w_{\Delta_i}(\tilde{\alpha}) = w_{\Delta_i}(\alpha_i + \sum_{j\neq i} k_j\alpha_j) = w_{\Delta_i}(\alpha_i) + w_{\Delta_i}(\sum_{j\neq i} k_j\alpha_j)) = \alpha_i + \sum_{j\neq i} b_j\alpha_j$ with $k_j, b_j \in \mathbb{Z}$, since $w_{\Delta_i}(\Delta_i) = -\Delta_i$. Now if $j \neq i$ we have $(\alpha_j, w_{\Delta_i}(\lambda) + v) = (\alpha_j, w_{\Delta_i}(\lambda)) = (w_{\Delta_i}(\alpha_j), \lambda) > 0$ since $w_{\Delta_i}(\alpha_j) \in -\Delta$ and $\lambda \in -A_\circ$. If we consider α_i, we have $(\alpha_i, w_{\Delta_i}(\lambda) + v) = 1 + (w_{\Delta_i}(\alpha_i), \lambda) > 0$ since $w_{\Delta_i}(\alpha_i) \in \Phi^+$ and $\lambda \in -A_\circ$ implies that $(w_{\Delta_i}(\alpha_i), \lambda) > -1$. Finally $(\tilde{\alpha}, w_{\Delta_i}(\lambda)) = 1 + (w_{\Delta_i}(\tilde{\alpha}), \lambda) < 1$ because $w_{\Delta_i}(\tilde{\alpha}) \in \Phi^+$ and $\lambda \in -A_\circ$ implies that $(w_{\Delta_i}(\tilde{\alpha}), \lambda) < 0$. These inequalities imply that $\tau(A_\circ) = A_\circ$. Since $\tau(0) = v$, we get that the map π is a bijection. □

From the proof of the previous proposition, we get the following description of Ω:

$$\Omega = \{t^{\omega_i^\vee} w_{\Delta_i} w_\Delta \mid (\omega_i^\vee, \tilde{\alpha}) = 1\} \cup \{1\}.$$

For every irreducible root system, it is useful to construct the so called *extended Dynkin diagram*, that is the usual Dynkin diagram plus the root $-\tilde{\alpha}$ with relative arrows and number of edges $\langle -\tilde{\alpha}, \alpha\rangle\langle \alpha, -\tilde{\alpha}\rangle$ with $\alpha \in \Delta$. Before giving a picture of the graphs we want to prove that the fundamental group can be identified with a subgroup of the automorphisms of the extended Dynkin diagram of Φ or with a subset of the nodes of the extended Dynkin diagram. Let $\tau \in \Omega$. We can consider the automorphism of W_a given by $w \mapsto \tau w \tau^{-1}$. Since $l(\tau w \tau^{-1}) = l(w)$, it induces a permutation on the set of the simple affine reflections. Thus we get a group homomorphism $\Omega \to S_{n+1}$. This homomorphism is injective. Indeed, suppose that the image of $\tau = t^{\omega_i^\vee} w_{\Delta_i} w_\Delta$ with $(\omega_i^\vee, \tilde{\alpha}) = 1$ is the identity, i.e. $\tau s \tau^{-1} = s$ for every s in S_a. In particular we have:

$$s_{\alpha_i} t^{\omega_i^\vee} w_{\Delta_i} w_\Delta s_{\alpha_i} = s_{\alpha_i} t^{\omega_i^\vee} s_{\alpha_i} s_{\alpha_i} w_{\Delta_i} w_\Delta s_{\alpha_i} = t^{\omega_i^\vee} w_{\Delta_i} w_\Delta, \ \textit{with}\ 1 \leq i \leq l,$$

From the last equality it follows that $s_{\alpha_i} t^{\omega_i^\vee} s_{\alpha_i} = t^{\omega_i^\vee}$ for every i. This implies that $s_{\alpha_i}(\omega_i^\vee) = \omega_i^\vee$, i.e. $(\alpha_i, \omega_i^\vee) = 0$. Thus $\omega_i^\vee = 0$, which is a contradiction.

Proposition 1.2.17. *1. Let $\tau = t^{\omega_i^\vee} w_{\Delta_i} w_\Delta$ with $(w_i^\vee, \tilde{\alpha}) = 1$, then $\tau s_{\tilde{\alpha},1} \tau^{-1} = s_{\alpha_i}$.*

2. Let ρ the projection $P^\vee \rtimes W \to W$, then ρ is injective on Ω and the set $\{\alpha_1, ..., \alpha_n, -\tilde{\alpha}\}$ is stable under the action of $\rho(\Omega)$.

Proof. 1. First notice that $\tau s_{\tilde{\alpha},1} \tau^{-1} \in W$, i.e. $\tau s_{\tilde{\alpha},1} \tau^{-1}(0) = 0$. It is enough to show that $\tau^{-1}(0) \in H_{\tilde{\alpha},1}$. This is easily verified because $\tau^{-1}(0)$ is a special vertex different from 0. Since $\Omega \cap T = 1$, where T is the translation group associated to $P^\vee$, the projection $P^\vee \rtimes W \to W$ is injective. Then to determine the element $\tau s_{\tilde{\alpha},1} \tau^{-1}$ it is enough to check the image under this homomorphism. We have $\tau s_{\tilde{\alpha},1} \tau^{-1} = w_{\Delta_i} w_\Delta s_{\tilde{\alpha}} w_\Delta w_{\Delta_i} = w_{\Delta_i} s_{\tilde{\alpha}} w_{\Delta_i}$ since $w_\Delta(\tilde{\alpha}) = -\tilde{\alpha}$ and $w_\Delta s_{\tilde{\alpha}} w_\Delta = s_{-\tilde{\alpha}} = s_{\tilde{\alpha}}$. Thus the image is equal to w_β where $\beta = w_{\Delta_i}(\tilde{\alpha})$. On the other hand $\beta \in \pm\Delta$ since $\tau s_{\tilde{\alpha},1} \tau^{-1} \in \{s_{\alpha_1}, ..., s_{\alpha_n}\}$. But we have already noticed that $w_{\Delta_i}(\tilde{\alpha}) > 0$, so $\beta \in \Delta$. Since α is of the form $\alpha_i + \sum_{j\neq i} m_j\alpha_j$ and w_{Δ_i} is a product of s_{α_j} with $j \neq i$, β is also of the form $\alpha_i + \sum_{j\neq i} a_j\alpha_j$. Then we get $\beta = \alpha_i$ and $\tau s_{\tilde{\alpha},1} \tau^{-1} = s_{\alpha_i}$.

2. Let $\tau = t^{\omega_i^\vee} w_{\Delta_i} w_\Delta$ be a non-trivial element in Ω. We have already seen in point 1) that $\rho(\tau)(-\tilde{\alpha}) = w_{\Delta_i} w_\Delta(-\tilde{\alpha}) = \alpha_i$. Let $\tau^{-1} = t^{\omega_j^\vee} w_{\Delta_j} w_\Delta$. Then $w_{\Delta_j} w_\Delta =$

$(w_{\Delta_i} w_\Delta)^{-1} = (w_\Delta w_{\Delta_i})$. Hence $\rho(\tau)(\alpha_j) = w_{\Delta_i} w_\Delta(\alpha_j) = w_\Delta w_{\Delta_j}(\alpha_j) = -\tilde{\alpha}$, since $w_{\Delta_j} w_\Delta(-\tilde{\alpha}) = \alpha_j$. For every α_k with $k \neq j$ we get: $\rho(\tau)(\alpha_k) = w_\Delta w_{\Delta_j}(\alpha_k) \in w_\Delta(-\Delta_j) \subset \Delta$. Thus $\rho(\tau)$ permutes the roots $\{\alpha_1, ..., \alpha_n, -\tilde{\alpha}\}$.

□

Remark 4. Note that the permutation on $\{s_{\tilde{\alpha},1}, ..., s_{\alpha_n}\}$ given by $s \mapsto \tau s \tau^{-1}$ coincides with the permutation on the set $\{-\tilde{\alpha}, \alpha_1, ..., \alpha_n\}$ given by $\rho(\tau)$. Furthermore, since $\rho(\tau)$ preserves the angles, it induces an automorphism of the extended Dynkin diagram of $\{-\tilde{\alpha}, ..., \alpha_n\}$. For this reason we can identify the fundamental group with a subgroup of the automorphism of the extended Dynkin diagram. A node corresponding to α is said to be *special* if $\rho(\tau(\alpha)) = -\tilde{\alpha}$ for some τ in Ω. Or equivalently, if $\alpha = -\tilde{\alpha}$ or $\alpha = \alpha_i$ with $(\omega_i^\vee, \tilde{\alpha}) = 1$. We can then say that the special nodes are in bijection with the elements of Ω, considering the map: $-\tilde{\alpha} \mapsto 1$ and $\alpha_i \mapsto \tau_i := t^{\omega_i^\vee} w_{\Delta_i} w_\Delta$.

To find the special nodes we look at the nodes that are sent to 0 by an automorphism of the extended Dynkin diagram. Using Table 1.2 we check in each case that they are as many as the elements of the fundamental group. Hence we have found all the special nodes. Equivalently one could look at the k_i=1 in the expression $\tilde{\alpha} = \sum_i^n k_i \alpha_i$. The extended Dynkin diagrams are presented in the figure of the next page. The node associated to $-\tilde{\alpha}$ is in red and the other special nodes are colored in blue. For type A_n, B_n, C_n we can find the graphs looking at Example 1.1.19. The data about the other root systems can be found for example in [4, Section 2.10]. For the root system of type A_1 ($\Phi = \{\alpha, -\alpha\}$) we have that $\tilde{\alpha} = \alpha$. In this case the edge connecting the node 0 to 1 is label with ∞. We will understand why in the next subsection.

Remark 5. If we look at the Dynkin diagrams $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{E}_7$ we see immediately which is the automorphism of the diagram associated to the element $\tau_i \neq id$ where i is a special node. Indeed, we have just one possibility. For the type $\widetilde{E}_6$ since Ω has order three, $\Omega = \langle \tau_1 \rangle$, we get that $p(\tau_1)(\alpha_1) = \alpha_6$ and $p(\tau_1)(\alpha_6) = -\tilde{\alpha}$. However, for the type $\widetilde{D}_n$ we can't determine, just looking at the image, the automorphisms induced by Ω.

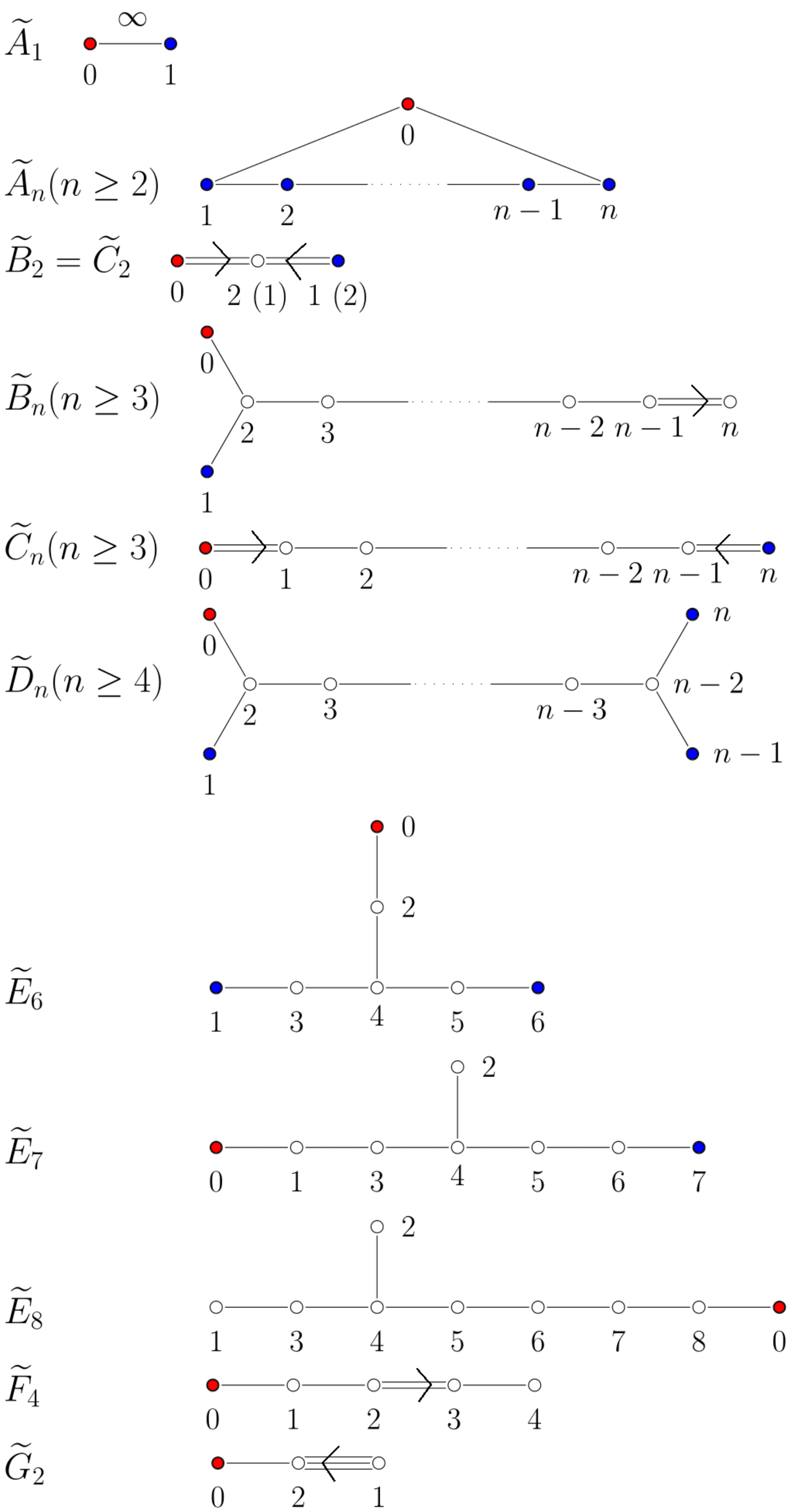

Figure 1.8: Extended Dynkin diagrams

This the usual numbering of roots. In the third dynkin diagram we have writen the numbering for respectively $\widetilde{B}_2$ and $\widetilde{C}_2$.

1.2.5 Coxeter groups

In this subsection we want to point out that the affine Weyl groups belong to a larger class of groups called *Coxeter groups*. First we need to introduce an important property of the affine Weyl group. From now on the hat over an element of S_a means that it is omitted.

Proposition 1.2.18 (Exchange condition). *Let w in W_a have reduced expression $w = s_1...s_r$ with $s_i \in S_a$. If $l(ws) < l(w)$ with $s \in S_a$, then there exists an index $1 \leq i \leq r$ for which $ws = s_1...\widehat{s_i}...s_r$.*

Proof. By Theorem 1.2.12 we know that

$$\mathcal{L}(w) = H_1, s_1H_2, s_1s_2H_3, ..., s_1...s_{r-1}H_r.$$

Using part 3) of Proposition 1.2.10 together with $l(ws) < l(s)$, we get that $wH_s \in \mathcal{L}(w)$. So $s_1 \cdots s_r H_s = s_1...s_{i-1}H_i$ for $1 \leq i \leq r$. Thus $H_s = s_r \cdots s_{i+1}H_i$ and Corollary 1.2.5 implies $(s_r...s_{i+1})s_i(s_{i+1}...s_r) = s$. Then we get $s_is_{i+1}...s_r = s_{i+1}...s_rs$. From the last equality it follows that $ws = s_1...\widehat{s_i}...s_r$. □

The exchange condition is equivalent to the so called 'Deletion Condition'.

Proposition 1.2.19 (Deletion Condition). *Let $w = s_1...s_r$ be an expression for w which is not reduced, then there exist indices $1 \leq i < j \leq r$ such that $w = s_1...\widehat{s_i}...\widehat{s_j}...s_r$*

Proof. Since the expression $s_1...s_r$ is not reduced and s_1 is reduced, there exists an index j such that $s_1...s_{j-1}$ is reduced but $s_1...s_j$ is not. By the Exchange Condition applied to $s_1...s_{j-1}$ and s_j, there exists an index $i < j$ such that $s_1...s_j = s_1..\widehat{s_i}..s_{j-1}$. Substituting we get $w = s_1..\widehat{s_i}...\widehat{s_j}...s_r$. □

It follows that given an expression for $w \in W_a$ we can obtain a reduced expression by successive omissions of pairs of factors. Furthermore, it is easily checked that the Deletion Condition implies the Exchange Condition.

Definition 1.2.20. A *Coxeter system* is a pair (W, S) consisting of a group W and a set $S \subset W$ of generators, subject only to relations of the form $(ss')^{m(s,s')} = 1$ where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for s, s' in S. In case no relation occurs between a pair of roots we write $m(s, s') = \infty$. Formally, W is the quotient of F/N where F is the free group on the set S and N is the normal subgroup generated by all the elements $(ss')^{m(s,s')}$, with $m(s, s')$ finite. Every group G which admits a presentation (F, N) of this type is called *Coxeter group.*

Just using the Deletion Condition one can prove that W_a is a Coxeter group (using the same argument of [4, Section 1.9]).

Theorem 1.2.21. *The pair (W_a, S_a) is a Coxeter system.*

Remark 6. It is possible to prove that also the classical Weyl group satifies the Deletion Condition. Then the pair (W, S) where $S = \{s_\alpha \mid \alpha \in \Delta\}$ is a Coxeter system as well.

It is common to represent all the information of a Coxeter system (W, S) in a graph. The vertices are in one-to-one correspondence with the elements of S. Each pair of vertices corresponding to the elements $s \neq s'$ has an edge with label $m(s, s')$ whenever $m(s, s') \geq 3$ is finite, or one edge with the label ∞ whenever $m(s, s') = \infty$. Then if there are no edges between two vertices s, s' this means that $m(s, s') = 2$. We want to show how the Coxeter graphs for the irreducible affine Weyl groups are connected with the extended Dynkin diagrams of Figure 1.8. Suppose that we want to find the order of $s_\alpha s_\beta$ with $\alpha, \beta \in \Phi$ and $\alpha \neq \pm\beta$. Notice that both s_α and s_β stabilize the subvector space $\langle\alpha, \beta\rangle$ and fix pointwise the space orthogonal to this. It is then enough to study the order of the induced morphism in $\langle\alpha, \beta\rangle$. In this space we obtain a rotation of 2θ where θ is the acute angle between α and the line $\mathbb{R}\beta$. The order of $s_\alpha s_\beta$ is then π/θ. Notice that the order of $s_{\tilde{\alpha},1} s_\beta$ with $\tilde{\alpha} \neq \pm\beta$ is equal to the order to $s_{\tilde{\alpha}} s_\beta$ with β in Φ. Looking at Table 1.1, we obtain the following correspondence between the number of edges connecting the vertices associated to α and β in the extended Dynkin diagram and the number $m(s_\alpha, s_\beta)$.

$\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle$	$m(s_\alpha, s_\beta)$
0	2
1	3
2	4
3	6

In the root system A_1 we have that $s_{\tilde{\alpha},1} s_\alpha = s_{\alpha,1} s_\alpha = t^{\alpha^\vee}$. The order of the product is then ∞ and this explains the label in Figure 1.8. Hence, we can obtain the Coxeter graph from the extended Dynkin diagram forgetting the arrows and relabelling the edges as shown in the previous table. The same reasoning can be applied to the Dynkin diagrams associated to the classical Weyl groups. This explains the name 'Coxeter graph' used in Chapter 1.

1.2.6 Bruhat ordering

For every Coxeter group it is possible to define a partial order (see [4, Chapter V] for the general case). We give now the definition in the case of the Coxeter system (W_a, S_a) where W_a is the affine Weyl group associated to an irreducible system Φ.

Definition 1.2.22. Let T the set of reflections $s_{\alpha,k}$ with $\alpha \in \Phi$ and k in $\mathbb{Z}$. We write $w' \to w$ if $w = w't$ for some $t \in T$ and $l(w) > l(w')$. Set $w' < w$ if there exists a sequence $w' = w_0 \to w_1 \to ... \to w_m = w$. We can then define a partial order on W_a writing $w \leq w'$ if $w < w'$ or $w = w'$. It is called *Bruhat Ordering*.

To give an interesting characterisation of the Bruhat ordering we need a stronger version of the Exchange Condition presented in the previous subsection. Indeed, it can be proved that every Coxeter group satisfies the following property (see [4, Section 5.8]).

Theorem 1.2.23 (Strong Exchange Condition)**.** *Let $w = s_1 ... s_r$ not necessarily a reduced expression. Suppose a reflection $t \in T$ satisfies $l(wt) < l(w)$. Then there exists an index i for which $wt = s_1 ... \widehat{s_i} ... s_r$. If the expression for w is reduced then i is unique.*

Now we can prove the following result.

Proposition 1.2.24. *Let $w' \leq w$ and $s \in S$. Then either $w's \leq w$ or else $w's \leq ws$ (or both).*

Proof. First suppose that $w' \to w$, where $w = w't$ $(t \in T)$ and $l(w) > l(w')$. If $s = t$ there is nothing to prove so assume $s \neq t$. Two cases have to be analyzed:

(a) if $l(w's) = l(w' - 1)$, then $w's \to w' \to w$ forcing $w's \leq w$.

(b) If $l(w's) = l(w') + 1$, we want to prove that $w's < ws$. Since $(w's)t' = ws$ for the reflection $t' = sts$, it is enough to show that $l(w's) < l(ws)$. Suppose the contrary, i.e. $l(ws) < l(w's)$. Then for any reduced expression $w' = s_1...s_r$, the expression $w's = s_1, ..., s_r s$ is also reduced, since $l(w's) > l(w')$. If we apply the Strong Exchange Condition to the pair $t', w's$ we get that $ws = (w's)t'$ is obtained from $w's$ by omitting one factor in the above reduced expression. This factor cannot be s since $s \neq t$. Hence $ws = s_1...\widehat{s_i}...s_r s$ for some i and $w = s_1...\widehat{s_i}...s_r$, contradicting $l(w) > l(w')$.

Now suppose that $w' = w_0 \to w_1 \to ... \to w_m = w$. One possibility is that $w_i s \leq w_{i+1} s$ for every $0 \leq i \leq m - 1$ which implies $w's \leq ws$. Otherwise there exists j such that $w's \leq w_j s \leq w_{j+1} \leq w$ which implies $w's \leq w$. □

Let $w \in W_a$. The *subexpressions* of a reduced expression $w = s_1...s_r$ are the products (not necessarily reduced and possibly empty) of the form $s_{i_1}...s_{i_q}$ with $1 \leq i_1 < i_2 < ... < i_q \leq r$.

Theorem 1.2.25. *Let $w = s_1...s_r$ be a reduced expression for w. Then $w' \leq w$ if and only if w' can be obtained as a subexpression of this reduced expression.*

Proof. First suppose that $w' < w$ and $w' \to w$, say $w = w't$. Since $l(w') < l(w)$, the Strong Exchange Condition can be applied to the pair t, w to yield $w' = wt = s_1...\widehat{s_i}...s_r$ for some i. This argument can be iterated. Indeed, suppose that $w'' \to w'$, with $w' = w''t'$, applying again the Strong Exchange Condition we obtain $w'' = w't' = s_1...\widehat{s_i}...\widehat{s_j}...s_r$ or else $w'' = s_1...\widehat{s_j}...\widehat{s_i}...s_r$ for some j. Thus, if $w' < w$ with $w \to w_1 \to ... \to w_{m-1} \to w$ using this argument m times we get that w' is a subexpression of w. On the other hand, let $s_{i_1}...s_{i_q}$ be a subexpression. We want to show that it is $\leq w$. We use induction on $r = l(w)$. If $r = 0$ it is trivially verified. Suppose first that $i_q < r$, then the induction hypothesis can be applied to the reduced expression $s_1...s_{r-1}$. So we get: $s_{i_1}...s_{i_q} \leq s_1...s_{r-1} = ws_r < w$. If $i_q = r$ we can first use induction to get $s_{i_1}..s_{i_{q-1}} \leq s_1...s_{r-1}$ then Proposition 1.2.24 to get either $s_{i_1}...s_{i_q} \leq s_1...s_{r-1} < w$ or $s_{i_1}...s_{i_q} \leq s_1...s_r = w$. □

For every w in W_a we can define the *support* of w,

$$\mathrm{supp}(w) := \{s \in S_a \mid s \leq w\}.$$

Thanks to the previous theorem these are all the simple affine reflections that occur in every reduced expression for w.

Chapter 2

Calculation of the dimension of ADLV

The aim of this chapter is to explain in detail the tools used to design and implement our program for the calculation of the dimension of the affine Deligne-Lusztig varieties. From the theoretical point of view, we have followed the idea of the reduction method in [6, Chapter VI]. For the implementation of the algorithm, we have used the functionalities for Weyl groups of the software Sage. For the Weyl groups we use the notation introduced in chapter 1. When we say that $\widetilde{W}$ is an extended affine Weyl group we implicitly require that it is associated to an irreducible root system.

2.1 Affine Deligne-Lusztig varieties

Since the target of this thesis is neither to give an introduction to affine Deligne-Lusztig varieties nor to study in detail the results about the dimension, we will just briefly introduct the ADLV and state the main results without proofs. If the reader wants to know more about the definition of ADLV, he can refer to the note [11]. What we want to point out in this section is that the problem of finding the dimension of an affine Deligne-Lustig variety can be reformulated just in terms of Weyl groups. For this reason, the knowledge of the first chapter will be enough to understand how the program works.

2.1.1 Definition of ADLV

Before talking about ADLV it is worth to mention the classical Deligne-Lusztig varieties. These varieties were introduced in 1976 by Deligne and Lusztig in [7] and have an important application in the study of linear representations of finite groups of Lie-type. Given an element g in a quotient group, say G/N, we will denote by $\dot{g} \in G$ a representative of the class g.

Definition 2.1.1. Let G be a 'connected reductive' algebraic group over a finite field $\mathbb{F}_q$. Let B be a Borel subgroup containing a maximal torus T. Let σ be the field automorphism $\sigma : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q,\ x \mapsto x^q$. We denote the automorphism $G(\overline{\mathbb{F}}_q) \to G(\overline{\mathbb{F}}_q)$ induced by σ by the same symbol. Let W be the group $N(\overline{\mathbb{F}}_q)/T(\overline{\mathbb{F}}_q)$ where N is the normalizer of T in G. W is the classical Weyl group of some root system in the sense of Chapter 1. Then for every w in W we can define the set

$$X_w := \{gB(\overline{\mathbb{F}}_q) \in G(\overline{\mathbb{F}}_q)/B(\overline{\mathbb{F}}_q)\} \mid g^{-1}\sigma(g) \in B(\overline{\mathbb{F}}_q) \cdot \dot{w} \cdot B(\overline{\mathbb{F}}_q)\}.$$

It can be proven (see [7, Section 1.3]) that:

Proposition 2.1.2. *The set X_w is a locally closed subvariety of the flag variety $G(\overline{\mathbb{F}}_q)/B(\overline{\mathbb{F}}_q)$, called (classical) Deligne-Lusztig variety. Furthermore, X_w is always nonempty and it is a smooth variety of dimension $l(w)$.*

Furthermore, we have the *Bruhat decomposition* which says $G = \bigsqcup_{w\in\tilde{W}} B(\overline{\mathbb{F}}_q)\dot{w}B(\overline{\mathbb{F}}_q)$.. Now we want to give an example where there is a nice interpretation of the set X_w.

Example 2.1.3. We consider $F = \mathbb{F}_p$ where p is a prime number, $G = GL_n$, T= subgroup of the diagonal matrices, B = subgroup of the upper triangular matrices. In this case W is the Weyl group associated with A_{n-1}. Indeed, N is the subgroup of the matrices which have for every column and every row exactly one entry different from zero. Hence, when we consider the quotient $N(\overline{\mathbb{F}}_p)/T(\overline{\mathbb{F}}_p)$ we remember only the 'shape' of the matrices, i.e. where are the entries different from zero. We can then identify W with S_n considering the map $\mu : W \to S_n, aT(\overline{\mathbb{F}}_p) \mapsto \sigma_a$ where $a = (a_{ij})_{i,j} \in N(\overline{\mathbb{F}}_p)$ and for every $1 \leq i \leq n$ $\sigma_a(i) = j$ where j is the only index such that $a_{j,i} \neq 0$. In Example 1.1.19 we saw that S_n is isomorphic to the classical Weyl group associated with A_{n-1}. Let $\mathrm{Flag}(\overline{\mathbb{F}}_p)$ be the set of all complete flags in $\overline{\mathbb{F}}_p^n$, i.e.

$$\mathrm{Flag}(\overline{\mathbb{F}}_p) := \{0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_{n-i} \subsetneq U_n = \overline{\mathbb{F}}_p^n \mid U_i \text{ is a subvector space}, \dim(U_i) = i\}.$$

It is possible to give to this set a structure of algebraic variety thanks to the closed embedding $\mathrm{Flag}(\overline{\mathbb{F}}_p) \to Gr(1, \overline{\mathbb{F}}_p^n) \times Gr(2, \overline{\mathbb{F}}_p^n) \times \cdots \times Gr(n-1, \overline{\mathbb{F}}_p^n)$ where $Gr(i, \overline{\mathbb{F}}_p^n)$ denote the Grassmannian variety of i-dimensional subvector spaces of $\overline{\mathbb{F}}_p^n$. Now we can consider the natural action of $G(\overline{\mathbb{F}}_p)$ on $\mathrm{Flag}(\overline{\mathbb{F}}_p)$. The action is transitive and the stabilizer of the flag $E = 0 \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle, \ldots, \subsetneq \langle e_1, \ldots, e_{n-1} \rangle \subsetneq \overline{\mathbb{F}}_p^n$ where $e_1, .., e_n$ is the standard basis, is the subgroup of the upper triangular matrices. Thus we get a bijection between $G(\overline{\mathbb{F}}_p)/B(\overline{\mathbb{F}}_p)$ and $\mathrm{Flag}(\overline{\mathbb{F}}_p)$. Furthermore, one can prove that it is an isomorphism of varieties. Let $\sigma : \mathrm{Flag}(\overline{\mathbb{F}}_p) \to \mathrm{Flag}(\overline{\mathbb{F}}_p)$ be the isomorphism obtained by applying σ to all the subspaces U_i of the flags. Then we can prove the following

Proposition 2.1.4. *Under the previous identification, we get*

$$X_w = \{U \in \mathit{Flag}(\overline{\mathbb{F}}_p) \mid \dim(U_i \cap \sigma(U_j)) = d(w, i, j)\ \forall i, j\},$$

where $d(w, i, j) = \#\{1 \leq k \leq j \mid \mu(w)(k) \leq i\}$.

Proof. Let $\dot{w}$ be the representative for w with the entries equal to 0 or 1. With abuse of notation, we will identify w with $\mu(w) \in S_n$. Let U in $\mathrm{Flag}(\overline{\mathbb{F}}_p)$ with $U = g(E)$ for some $g \in GL(\overline{\mathbb{F}}_p)$. Suppose that $gB(\overline{\mathbb{F}}_p) \in X_w$. Then we have $g^{-1}\sigma(g) = b_1\dot{w}b_2$ with b_1 and b_2 in $B(\overline{\mathbb{F}}_p)$. Let $h = gb_1$ and $h' = \sigma(g)b_2^{-1}$. Let h_i be the $i-$th column of H. Then we have $h^{-1}h' = \dot{w}$, $U = h(E)$ and $\sigma(U) = h'(E)$. It implies $\sigma(U_j) = h'(E_j) = h\dot{w}(E_j) = \langle h_{w(1)}, ..., h_{w(j)} \rangle$ while $U_i = \langle h_1, .., h_i \rangle$. Thus we get $\dim(U_i \cap \sigma(U_j)) = d(w, i, j)$. On the other hand, suppose that $U = g(E)$ satisfies the RHS of the equality in the statement. By the Bruhat decomposition there exists a unique v in $\widetilde{W}$ such that $g^{-1}\sigma(g) \in B(\overline{\mathbb{F}}_p)\dot{v}B(\overline{\mathbb{F}}_p)$. Then using the same reasoning as before, we get $\dim(U_i \cap \sigma(U_j)) = d(v, i, j)\ \forall i, j$. Thus we have $d(w, i, j) = d(v, i, j)$ for every i, j and this implies $v = w$. Then we get that $U \in X_w$. $\square$

The first mathematician to use the term "affine Deligne-Lusztig varieties" was Rapport in 2005 in his article "A guide to the reduction modulo p of Shimura varieties" ([1]). The word affine highlights that this time the definition depends on affine Weyl groups. To introduce these varieties we need first to fix some setting.
Let G be a 'split semisimple connected ' algebraic group over $F := \mathbb{F}_q((\epsilon))$ and let L an algebraic closure of F, $L := \overline{\mathbb{F}}_q((\epsilon))$. If we consider a 'split' maximal torus $T \subset G$ we can get an extended affine Weyl group setting $\widetilde{W} := N(L)/T(O_L)$ where N is the normalizer of T in G and O_L is the ring of integers of L. Let B be a Borel subgroup containing T and π be the projection $G(O_L) \to G(k_L)$ where k_L is the residue field of L. We define the *Iwahori subgroup* to be $I := \pi^{-1}(B(k_L))$. Let J be a proper subset of the set of simple affine reflections (determined by choice of B), and let P be the subgroup of $G(L)$ generated by $I \cup J$. Let σ be the automorphism

$$\sigma : L \to L, \sum_{i \geq N} a_i \epsilon^i \mapsto \sum_{i \geq N} a_i^q \epsilon^i.$$

We denote by σ also the endomorphism induced in $G(L)$.

Definition 2.1.5. For every $b \in G(L)$ and for every $w \in \widetilde{W}$, we can define the set

$$X_w(b) := \{gP \in G(L)/P \mid g^{-1}b\sigma(g) \in P\dot{w}P\}.$$

There is the *Iwahori-Bruhat decomposition* which says

$$G(L) = \bigsqcup_{w \in \tilde{W}} I\dot{w}I.$$

When P is larger then I, in order to write $G(L)$ as a disjoint union of P-double cosets we need to consider a subset of $\widetilde{W}$. For example, we can write:

$$G(L) = \bigsqcup_{w \in W_J \backslash \widetilde{W}/W_J} P\dot{w}P,$$

where W_J is the subgroup of $\widetilde{W}$ generated by the set J. It is possible to give to the set $X_w(b)$ a structure of variety. We will give a rough idea in the following examples. First we have to point out that the ADLV over F can be defined in a more general setting than the one of Definition 2.1.5.

Remark 7. Let G a 'connected reductive' group over F. For a suitable choice of maximal torus $T \subset G$ we can define a group $\widetilde{W}$ which is usually called 'extended affine Weyl group'. If G is semisimple $\widetilde{W}$ is an extended affine Weyl group in the sense of Chapter 1, otherwise it is a slightly more general notion. However, we still have that $\widetilde{W} = \Omega \rtimes W_a$ where W_a is an affine Weyl group and Ω is the subgroup of the length 0 elements with respect to an extension of the usual length function of W_a. For every b in $G(L)$ and $w \in \widetilde{W}$ one can define the ADLV $X_w(b)$. However, we are not going to give more details about this definition.

Example 2.1.6. We consider $J = \emptyset$ thus we get $P = I$ in Definition 2.1.5. For every w and b in $\widetilde{W}$ the set $X_w(b)$ carries a natural structure of variety over k_L. In particular, it is possible to prove that IwI/I is in bijection in a natural way with $k_L^{l(w)}$. Hence we can give a structure of variety to IwI/I and get $IwI/I \cong \mathbb{A}^{l(w)}(k_L)$. Then one defines

$$\dim X_w(b) = \max_{v\in\widetilde{W}}\{\dim(X_w(b) \cap IvI/I)\}.$$

Indeed, one can show that $X_w(b) \cap IvI/I$ is a subvariety of IvI/I for every v in $\widetilde{W}$ and the maximum of the dimensions is a finite number. As in the example of the classical DLV, we can take $G = GL_n$, as maximal torus T we consider the subgroup of diagonal matrices and as Borel subgroup B the subgroup of upper triangular matrices. Even if G is not semisimple we can follow the construction of Definition 2.1.5 to define the associated ADLV. We get that $\widetilde{W} = N(L)/T(O_L) \cong \mathbb{Z}^n \rtimes S_n$. Indeed, we have that $T(O_L)$ are the diagonal matrices with coefficients in $O_L^\times$ and $L/O_L^\times$ is isomorphic to $\mathbb{Z}$ (since $O_L^\times$ is the kernel of the map $l \mapsto v(l)$ with $l \in L$ and $v(l)$ the valution of l). Then we can identify $\widetilde{W}$ with the matrices which have coefficients in $\mathbb{Z}$ and exactly one entry different from 0 for every row and every column. It is easily seen that this group is isomorphic to $\mathbb{Z}^n \rtimes S_n$ which is closely related to the extended affine Weyl group of A_{n-1}. For pratical purpose one can consider the 'adjoint group' of G, which is denoted by PGL_n. This algebraic group is the quotient of GL_n by its center which is the subgroup of the scalar diagonal matrices. This time we get that $\widetilde{W} = \mathbb{Z}^n/(1, ..., 1)\mathbb{Z} \rtimes S_n$ which is the extended affine Weyl group of A_{n-1} (see Example 1.2.6). The canonical projection $N(L)/T(O_L) \to N(L)/T(L)$ is the usual projection $\widetilde{W} \to W$. Similarly, if we consider $G = Sp_{2n}$, $T=$ set of diagonal matrices and $B=$ set of upper triangular matrices, we get that $\widetilde{W}$ is the affine Weyl group associated to C_n.

Example 2.1.7. If J is the set of all simple reflections, we get $P = G(O_L)$ and $P\dot{\tilde{w}}P = P\dot{\tilde{w}}'P$ if and only if $\tilde{w} = t^\lambda w$, $\tilde{w}' = t^{\lambda'} w'$ where w, $w' \in W$ and λ and λ' are conjugate under the action of W. For this reason a set of representatives for $W_J\backslash\tilde{W}/W_J$ is $\{t^\lambda \mid \lambda \text{ is a dominant coweight}\}$. In this case we have that the algebraic structure of $G(L)/G(O_L)$ is given by the 'affine Grassmannian'. In order to give a rough idea, we set $G = GL_n$. Then we have a bijection:

$$G(L)/G(O_L) \to \{\mathcal{L} \mid \mathcal{L} \text{ is a lattice in } L^n \text{ over } O_L\},\ gG(O_L) \mapsto gO_L^n.$$

Recall that a $\mathcal{L}$ is lattice over O_L if $\mathcal{L} = \langle v_1, ..., v_n\rangle_{O_L}$ where $v_1, ..., v_n$ is an L-basis for L^n. We can write $\{\mathcal{L} \subset L^n \mid \mathcal{L} \text{ lattice}\} = \bigcup_{N\geq 0} g_N$ where $g_N = \{\mathcal{L} \subset L^n \mid \mathcal{L} \text{ lattice }, t^N O_L^n \subseteq \mathcal{L} \subseteq t^{-N}O_L^n\}$. Now we have the following injection

$$g_N \hookrightarrow \bigsqcup_{d=0}^{2nN} Gr(d, \overline{\mathbb{F}}_q^{2nN}),\ \mathcal{L} \mapsto \mathcal{L}/t^N O_L^n,$$

where we have that $\mathcal{L}/t^N O_L^n$ is a subvector space of $t^{-N}O_L^n/t^N O_L$ which is a $\overline{\mathbb{F}_q}$-vector space of dimension $2Nn$ (basis $\{t^i e_j \mid 1 \leq j \leq n,\ -N \leq i \leq N-1\}$). Furthermore, the image of g_N is a closed subvariety of $\bigsqcup_{d=0}^{2nN} Gr(d, \overline{\mathbb{F}}_q^{2nN})$. Thus we can give to g_N the structure of a variety and write $GL_n(L)/GL_n(O_L)$ as a union of varieties. The dimension of $X_w(b)$ is defined as $\dim X_w(b) := \max_{N\geq 0}\{\dim(X \cap g_N)\}$ which can be proved to be different from infinity.

An important "variant" of ADLV is with $F := \mathbb{Q}_p$, $L := \widehat{\mathbb{Q}_p^{nr}}$ and $\sigma : L \to L$ the Frobenius morphism. The advantage of working with $\mathbb{F}_q((\epsilon))$ is that it is easier to understand in which way $X_w(b)$ is a variety. However, the ADLV defined over $\mathbb{Q}_p$ are more directly related to Shimura varieties and thanks to the introduction of the p-adic affine Grassmannian by Zhu in [15] it is now possible to treat them as algebro-geometric objects, namely as so-called perfect schemes. We decided to give the definition over $\mathbb{F}_q((\epsilon))$ because it is the one used in [6]. However, the results in the next sections can be similarly proved for $\mathbb{Q}_p$.

2.1.2 Dimension of ADLV

From now on, we consider only ADLV over $F = \mathbb{F}_q((\epsilon))$ which satisfy the hypothesis of [6, Section 1.1]. In particular the algebraic group G is 'quasi-split' over F. If G is split, we have that $P = I$ in Definition 2.1.5. To be coherent with our definition of extended affine Weyl group, we assume also that G is semisimple. As an example, the reader can consider $G = PGL_n$ and T, B as in Example 2.1.6. From now on, we will denote by δ the automorphism of $\widetilde{W}$ induced by $\sigma : G(L) \to G(L)$. With an appropriate choice of T and B, we have that δ is an automorphism of $\widetilde{W}$ such that

1. it restricts to an automorphism of Ω and it induces an automorphism $P^\vee \to P^\vee$ which stabilizes $Q^\vee$.

2. It induces a bijection on the simple affine reflection associated with the base Δ identified by the Borel group.

3. It fixes the affine reflection s_0 (for this last property we need that G is quasi-siplit over F).

It follows that δ induces an automorphism of W_a and W. If G is split we obatin $\delta = id$. From now on we fix the setting used to define an ADLV (G, $\widetilde{W}$, σ, the torus T, the borel subgroup B, the normalizer of T denoted by N). Furthermore, we suppose that δ satifies the previous properties. We denote by Δ the base identified by the Borel subgroup and by S_a the set of corresponding simple affine reflections. We denote again by δ the automorphism induced on $P^\vee$ and $Q^\vee$. Let b and b' in $G(L)$, we write $b \sim_\sigma b'$ if $b = xb'\sigma(x)^{-1}$ for some $x \in G(L)$. Let τ be the projection $N(L) \to \widetilde{W}$. To simplify the presentation of the theorems about the dimension, we have considered the following important result:

Proposition 2.1.8. *1. For every b in $G(L)$, there exists b' in $N(L)$ such that $b \sim_\sigma b'$ (see [8, Corollary 7.2.2]). If $b = xb'\sigma(x)^{-1}$ for $x \in G(L)$ the map $g \mapsto x^{-1}g$ gives $X_w(b) \cong X_w(b')$ for every w in $\widetilde{W}$.*

2. Let b and b' in $N(L)$. If $\tau(b) = \tau(b')$ then $b \sim_\sigma b'$ (see [9, Proposition 3]).

Thanks to the first part of the proposition we will restrict to the case $X_w(b)$ with b in $N(G)$. Furthermore, the second part allows us, when we consider the variety $X_w(b)$, to 'identify' b with its image on $\widetilde{W}$. Sometimes with abuse of notation we will then write $X_w(b)$ with $b \in \widetilde{W}$. Before stating the main theorems, we have to give some definitions.

Definition 2.1.9. Two elements w, w' in $\widetilde{W}$ are said to be δ-conjugate if $w' = xw\delta(x)^{-1}$ for some $x \in \widetilde{W}$.

The relation of δ-conjugacy is an equivalence relation, and the equivalence classes are called δ-conjugacy classes. It is possible that $\dot{w}$ and $\dot{w}'$ are $\sigma-$conjugate without w and w' being $\delta-$conjugate. Let Ω be as usual the fundamental group of $\widetilde{W}$. We denote by $(\Omega)_\delta$ the coinvariants of δ on Ω, i.e. the quotient of Ω by the image of the group morphism $id \cdot (\delta)^{-1} : \Omega \to \Omega$. We consider the natural projection

$$k : \widetilde{W} \to \widetilde{W}/W_a = \Omega \to (\Omega)_\delta.$$

The map k is called *Kottwitz map* (from [6, Section 1.7]). Note that k is constant in each $\delta-$coniugacy class. Indeed, let w', w and x in $\widetilde{W}$ with $w' = xw\delta(x)^{-1}$ and let τ the projection of x in Ω. Then $k(x) = [\tau] = [\delta(\tau)] = k(\delta(x))$ since $\tau\delta(\tau)^{-1}$ belongs to the image of the map $id \cdot (\delta)^{-1} : \Omega \to \Omega$. Thus we get $k(w') = k(xw\delta(x)^{-1}) = k(x)k(x)^{-1}k(w) = k(w)$ since the group Ω is commutative (because $\Omega \cong P^\vee/Q^\vee$). Recall that there is an action of W on $P^\vee$ and every element of $P^\vee$ is conjugate with one and just one dominant coweight. Let $P^\vee_{\mathbb{Q}} := P^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ and $P^\vee_{\mathbb{Q}}/W$ the quotient of $P^\vee_{\mathbb{Q}}$ by the extended action of W. Let $P^\vee_+$ be the sets of dominant coweights and $P^\vee_{\mathbb{Q},+} := \{\lambda \otimes q \mid \lambda \in P^\vee_+,\ q \in \mathbb{Q}, q \geq 0\}$. Then one can identify $P^\vee_{\mathbb{Q}}/W$ with the set $P^\vee_{\mathbb{Q},+}$. We consider the following action of δ on $P^\vee_{\mathbb{Q},+}$: $\delta(\lambda \otimes q) = \overline{\delta(\lambda)} \otimes q$ where $\overline{\delta(\lambda)}$ is the only element in $P^\vee_+$ coniugate to $\delta(\lambda)$. We denote by $P^{\vee,\delta}_{\mathbb{Q},+}$ the set of δ-invariants points in $P^\vee_{\mathbb{Q},+}$. Now we define a map from $\widetilde{W}$ to $P^{\vee,\delta}_{\mathbb{Q},+}$ as follows. For every element $\tilde{w} = t^\chi w$ in $\widetilde{W}$ with $w \in W$ and $\chi \in P^\vee$, there is a positive natural number n such that $\delta^n = 1$ and $w\delta(w)\delta^2(w)\cdots\delta^{n-1}(w) = 1$. Indeed, let s be the order of the automorphism δ and k the order of the element $(w\delta(w)\cdots\delta^{s-1}(w))$. For every choice of $l \in \mathbb{N}_{>0}$ we have that $n = skl$ satisfies the required properties since $\delta^n = 1$ and $w\delta(w)\cdots\delta^{skl-1}(w) = (w\delta\cdots\delta^{s-1}(w))^{kl} = 1$. Furthermore, these are all the possible choices for n. Indeed, let n with the required properties. From the fact that $\delta^n = 1$ we get $n = ps$ for some $p \in \mathbb{N}_{>0}$. Since $(w\cdots\delta^{s-1}(w))^p = 1$ it follows that $p = lk$ for some $l \in \mathbb{N}_{>0}$. Now we consider the element $\tilde{w}\delta(\tilde{w})...\delta^{n-1}(\tilde{w}) = t^\lambda$ with λ in $P^\vee$. Let $\nu_{\tilde{w}} = \lambda/n \in P^\vee_{\mathbb{Q}}$ and let $\overline{\nu}_{\tilde{w}}$ the corresponding element in $P^\vee_{\mathbb{Q},+}$. We see easily that $\nu_{\tilde{w}}$ is independent of the choice of n: if $\tilde{w}\delta(\tilde{w})...\delta^{sk-1}(\tilde{w}) = t^{\lambda_0}$ then $\tilde{w}\delta(\tilde{w})...\delta^{skl-1}(\tilde{w}) = t^{l\lambda_0}$ for the chosen $l \in \mathbb{N}_{>0}$. So we get that $\nu_{\tilde{w}} = \lambda_0/sk = l\lambda_0/skl = \nu'_{\tilde{w}}$. Now we want to show that $\overline{\nu}_{\tilde{w}}$ belongs to $P^{\vee,\delta}_{\mathbb{Q},+}$. From the definition of t^λ, applying δ to both sides of the equation, we get that $t^\lambda = \tilde{w}t^{\delta(\lambda)}\tilde{w}^{-1}$. This implies that $t^\lambda = t^{w\delta(\lambda)}$. Thus λ and $\delta(\lambda)$ correspond to the same element in $P^\vee_+$. From this it follows that $\delta(\overline{\nu}_{\tilde{w}}) = \overline{\nu}_{\tilde{w}}$. We call the map

$$\eta : \widetilde{W} \to P^{\vee,\delta}_{\mathbb{Q},+},\ \tilde{w} \mapsto \overline{\nu}_{\tilde{w}}$$

the *Newton map* (from [6, Section 1.7]). Also this map is constant in each $\delta-$coniugacy class. Indeed, let $\tilde{w}', \tilde{w}, \tilde{x}$ in $\widetilde{W}$ with $\tilde{w}' = \tilde{x}\tilde{w}\delta(\tilde{x})^{-1}$ and w, w', x the projections of respectively $\widetilde{w}, \widetilde{w}', \widetilde{x}$ in W. Let n such that $\delta^n = 1$ and $w\delta(w)\cdots\delta^{n-1}(w) = 1$. Let $t^\lambda = \tilde{w}\delta(\tilde{w})\cdots\delta^{n-1}(\tilde{w})$. Then $w'\cdots\delta^{n-1}(w') = xw\delta(x)^{-1}\delta(x)\delta(w)\delta^2(x)^{-1}\cdots\delta^{n-1}(x)\delta^{n-1}(w)\delta^n(x)^{-1}$ $= x1x^{-1} = 1$ and $\tilde{x}\tilde{w}\delta(\tilde{x})^{-1}\delta(\tilde{x})\delta(\tilde{w})\cdots\delta^{n-1}(\tilde{x})\delta^{n-1}(\tilde{w})\delta^n(\tilde{x})^{-1} = \tilde{x}t^\lambda\tilde{x}^{-1} = t^{x(\lambda)}$. Thus we get $\nu_{\tilde{w}} = \lambda/n$ and $\nu_{\tilde{w}'} = x(\lambda)/n$, so that $\overline{\nu}_{\tilde{w}} = \overline{\nu}_{\tilde{w}'}$. For convenience, we define also the map

$$f : \widetilde{W} \to P^{\vee,\delta}_{\mathbb{Q},+} \times (\Omega)_\delta,\ \tilde{w} \mapsto (\overline{\nu}_{\tilde{w}}, k(\tilde{w})).$$

Now we present a theorem which plays a key role in the idea of the algorithm. It is a special property of the extended affine Weyl groups proved in [12, Theorem 2.10]. Let us first give some definitions.

Definition 2.1.10. For w, w' in $\widetilde{W}$ and s_i in S_a, we write $w \xrightarrow{s_i}_\delta w'$ if $w' = s_i w \delta(s_i)$ and $l(w') \leq l(w)$. We write $w \to_\delta w'$ if there exists a sequence $w = w_0, ..., w_n = w'$ of elements in $\widetilde{W}$ such that for any k, $w_{k-1} \xrightarrow{s_i}_\delta w_k$ for some s_i in S_a. We write $w \approx_\delta w'$ if $w \to_\delta w'$ and $w' \to_\delta w$.

Notice that $w \approx_\delta w'$ if and only if $w \to_\delta w'$ and $l(w)= l(w')$. Then we have:

Theorem 2.1.11. *Let O be a δ-conjugacy class in $\widetilde{W}$ and O_{min} be the set of minimal length elements in O. Then for each element $w \in O$, there exists $w' \in O_{min}$ such that $w \to_\delta w'$.*

This is a non-trivial result which generalizes a previous theorem for classical Weyl groups. The other fundamental ingredient is the Deligne-Lusztig reduction for affine Weyl groups, see [13, Corollary 2.5.3] for the proof in the case $\delta = 1$. The general case can be proved similarly.

Theorem 2.1.12 (Deligne-Lusztig reduction)**.** *Let $w,\ b \in \widetilde{W}$, and let $s \in S_a$.*

1. *If $l(sw\delta(s)) = l(w)$, then there exists a universal homeomorphism $X_w(b) \to X_{sw\delta(s)}(b)$.*
2. *If $l(sw\delta(s)) = l(w) - 2$, then $X_w(b)$ can be written as a disjoint union $X_w(b) = X_1 \sqcup X_2$, where X_1 is closed and X_2 is open, and such that there exist morphisms $X_1 \to X_{sw\delta(s)}(b)$ and $X_2 \to X_{sw}(b)$ that are compositions of Zariski-locally trivial fiber boundle with one-dimensional fibers and a universal homeomorphism.*

Remark 8. What is important for us is that in case 1 we have:

$$\dim X_w(b) = \dim X_{sw\delta(s)}(b).$$

Instead in case 2 we get

$$\dim X_w(b) = \max\{\dim X_{sw\delta(s)}(b), \dim X_{sw}(b)\} + 1.$$

In general, understanding if the variety $X_w(b)$ is nonempty and then determining the dimension, is a difficult task. However, when w is a minimal length element in its δ-conjugacy class it can be easily computed. We denote by ρ half of the sum of the positive roots of the root system Φ associated to $\widetilde{W}$ with respect to the base Δ chosen at the beginning. As usual we denote by $(\cdot, \cdot)$ the bilinear form on V. We can now state the theorem by Xuhua He, see [6, Theorem 4.8].

Theorem 2.1.13. *Let $w \in \widetilde{W}$ be a minimal length element in its δ-conjugacy class. Let $b \in N(L)$ then*

1. *If $f(w) \neq f(\tau(b))$, $X_w(b) = \emptyset$.*
2. *If $f(w) = f(\tau(b))$, $\dim X_w(b) = l(w) - (\overline{\nu}_{\tau(b)}, 2\rho)$.*

Thanks to this last result, it is possible to define a recursion function for $\dim X_w(b)$ with $w \in \widetilde{W}$ and $b \in N(G)$.

2.2 Program

In this section we explain how the program works and we provide some examples of computation.

2.2.1 Algorithm

This is the idea of the algorithm for the dimension. Suppose that we want to find $\dim X_w(b)$ for the setting chosen at the beginning of the chapter. The output -inf means that the variety is empty.
Input: (Φ, δ, b, w)
where Φ is the irreducible root system associated to $\widetilde{W}$, δ is the usual automorphism of $\widetilde{W}$, b and w elements in $\widetilde{W}$.
Output: $\dim X_{b'}(w)$
where $\tau(b') = b$.

- dimension(b,w):
  1. if $k(w) \neq k(b)$: **return** -inf
  2. else: **return** dl_reduction(b, w)
- dl_reduction(b,w):
  1. C=[w], S=[w]
  2. while S $\neq \emptyset$:
     (a) N=[ ].
     (b) for every $w' \in S$ and every $s_i \in S_a$:
        i. if $l(s_i w'\delta(s_i))< l(w)$:
           return max{dl_reduction(b, $s_i w'\delta(s_i)$),dl_reduction(b, $s_i w'$)}+1.
        ii. if $l(s_i w'\delta(s_i)) = l(w)$: save $s_i w'\delta(s_i)$ in N.
     (c) $S = N \setminus C$, $C = C \cup S$
  3. If $\overline{\nu}_w = \overline{\nu}_b$: **return** $l(w) - (\overline{\nu}_b, 2\rho)$.
  4. else: **return** -inf

Exactness of the algorithm

First suppose that w is a minimal length element in its conjugacy class. If $k(w) \neq k(b)$ the function dimension(b,w) return the correct result for Theorem 2.1.13. If $k(w) = k(b)$ it calls the function dl_reduction(b,w). Since it is a minimal length element the step (i) in the while loop is never true. During the iterations of the while loop we manage to find all the elements $w' \in \widetilde{W}$ such that $w \approx_\delta w'$ and we memorize them in the list C. Indeed, we test all the possible combinations of simple affine reflections thanks to the for loop in (b). When the list C is complete, in step (c) we obtain S=[ ] and the while loop terminates. Then by Theorem 2.1.13 step 3 and 4 return the right dimension of $X_w(b)$. Now suppose that w is not minimal and $k(w) = k(b)$. Then by Theorem 2.1.11, in step (i) we find (after a finite number of iterations) an element w' such that $w \approx_\delta w'$ and a simple affine reflection s_i such that $l(s_i w'\delta(s_i)) < l(w')$. By Theorem 2.1.12 we know that $\dim X_w(b) = \dim X_{w'}(b)$ and $\dim X_{w'}(b) = \max\{\dim X_{s_i w'\delta(s_i)}(b), \dim X_{s_i w'}(b)\} + 1$. So in step (i) we return the correct dimension of $X_w(b)$. Since also $l(s_i w') < l(w')$, we have decreased the length of the elements we are going to examine in the function dl_reduction(b,$\cdot$). Thus after a finite number of recursions we find minimal length elements and the algorithm terminates. Finally, from the definition of the Kottwitz map it follows that $k(w) = k(s_i w'\delta(s_i)) = k(s_i w')$. This explains why we don't have to check again the value of the

Kottwitz map when we reach a minimal length element. It is also the reason why if at the beginning we have $k(w) \neq k(b)$ we can say immediately that the variety is empty (step 1 in dimension(b, w)).

2.2.2 Extended affine Weyl groups in Sage

To implement the algorithm we have used the software system "SageMath". It is often called simply Sage and it is a free open-source mathematics software with features covering many aspects of mathematics, including algebra, number theory, combinatorics, graph theory and calculus. It integrates many already existing mathematics packages (NumPy, Scip, matplotlib, etc.) written in different languages (C, C++, Fortran, etc.). This integration is possible through a common Python-based language or directly via interfaces or wrappers.

Now we want to explain how extended affine Weyl groups are implemented in Sage-Math and the most important methods needed for the program. When we write the commands, we will use the notations of the scripts. In particular, we have used the letter E for the extended affine Weyl group. To define this group, we have to choose a Cartan Type of Figure 1.8. In Sage a Cartan type for an affine Weyl group is represented by a list with three elements: a letter for the type, a positive integer for the rank and the number one to indicate that the type is untwisted (our case). For example, if we have chosen the Cartan type ['A',5,1], then we can create the associated extended Weyl group through the command: E = ExtendedAffineWeylGroup(['A',5,1], affine="s", fundamental="F"). In the command we have also specified the prefixes used for the simple affine reflections ("si" with $0 \leq i \leq 5$) and for the elements of the fundamental group ("$F[i]$" with i special node). Indeed, to present the elements of the fundamental group, Sage uses the special nodes following the identification of Remark 4. The numbers for the special nodes are the same as in Figure 1.8. In our example we get Ω = F[0], F[1], F[2], $\cdots$, F[5]. We can create the fundamental group through the command f = E.fundamental_group(). Then one has two options to call the elements of Ω. If we use the round brackets, we get f(i)=F[i] for every special node i. If we use the square bracket we get $\Omega = \{f[i] \mid 0 \leq i \leq |\Omega| - 1\}$, where the elements are arranged in ascending order with respect to the associated special nodes. Thus if for example we are working with ['C',n,1], we get f(0)=f[0]= F[0] and f(n)=f[1]=F[n]. With the command Wa=E.affine_weyl(), we introduce the affine Weyl group associated with E. Writing s = Wa.simple_reflections() we obtain a dictionary where the number i is the key for the reflection s_i. The simple affine reflections are ordered as in Figure 1.8. In Sage an element of the extended affine Weyl group E is presented following one of the six possible representations for E:

1. as semidirect product of the coweight lattice and the classical Weyl group (acting on the left or on the right);

2. as semidirect product of the fundamental group and the Affine Weyl group (acting on the left or on the right);

3. as semidirect product of the dual lattice and the classical Weyl group (acting on the left or on the right).

The commands that point out which representation we are working with are respectively: E.PW0(), E.W0P(), E.WF(), E.FW(), E.PvW0(), E.W0Pv(). Following [6] we will use only E.FW() and E.PW0().

E.FW() $\cong \Omega \rtimes W_a$ If u is an element in the fundamental group or in the affine Weyl group we have to write E.FW(u) if we want to consider its embedding in E. We can then define an element w in E writing for example w = E.FW(f[4])*E.FW(s[2]*s[1]*s[3]). The other way to define an element is to consider the numbers associated to the simple affine reflections and to the special nodes. For example we can write: w = E.FW.from_fundamental(4)*E.FW.from_reduced_word([2, 1, 3]). The projections from E.FW to Ω and W_a are then given by w.to_affine_weyl_right() and w.to_fundamental_group(). It is important to know that the elements in this representation are not hashable.

E.PW() $\cong P^\vee \ltimes W$ Given an element w in E.FW() we can change representation writing E.PW(w). Then it is possible to find the projection to $P^\vee$ and to the classical Weyl group using the commands: w.to_translation_left() and w.to_classical_weyl(). We can introduce the groups W and $P^\vee$ writing W0=E.classical_weyl() and L = E.lattice(). An element in L is presented through the basis of fundamental coweights $\omega_i^\vee$ which in Sage are denoted by "Lambdacheck[i]". It is important to know that the elements in E.FW() are hashable.

2.2.3 Program for $\delta = \mathrm{id}$

Now we show the first script we wrote. It is for the case $\delta = id$ and it was possible to test it calculating some already known dimensions (we used [8], [13] and [6, Corollary 12.2]). In the following code we have omitted the comments but they can be found in the extended version on the appendix (Script 1). Notice that when $\delta = id$ the Kottwitz map is just the usual projection to the fundamental group.

```
# Give a Cartan type, for example Ca = ['A',5,1].
Ca = ['A', 5, 1]
n = Ca[1]
Ra = Ca[0:2]
Y = RootSystem(Ra).coweight_space()
R = RootSystem(Ra).root_lattice()
rho2 = sum(R.positive_roots())
E = ExtendedAffineWeylGroup(Ca,affine="s",fundamental="F")
FW = E.FW()
PW0 = E.PW0()
Wa = E.affine_weyl()
s = Wa.simple_reflections()
f = E.fundamental_group()

def newton_vector(w):
    global Y

    p = PW0(w).to_classical_weyl().order()
    w3 = PW0(w**p)
    return (1/p) * Y(w3.to_translation_left().to_dominant_chamber())
```

```
def dimension(b, w):
    """
    Calculation of the dimension of the variety $X_{b}(w)$ """

    if w.to_fundamental_group() != b.to_fundamental_group():
        return(float("-inf"))

    return dl_reduction(PW0(b), PW0(w))

def dl_reduction(b, w):
    """
    The Deligne-Lusztig reduction. """

    global n, s

    l = w.length()
    S=[w]
    C = [w]

    while not S == []:
        N = []
        for x in S:
            for i in range (n+1):
                wi = s[i] * x * s[i]
                li = wi.length()
                if li < l:
                    d1 = dl_reduction(b, wi)
                    vi = s[i] * x
                    d2 = dl_reduction(b, vi)
                    return max(d1,d2) + 1
                if (li == l) and (wi not in N):
                    N.append(wi)

        S = [i for i in N if i not in C]
        C = C + S

    nu_b = newton_vector(b)

    if (newton_vector(w) == nu_b):
        return w.length() - nu_b.scalar(rho2)
    else:
        return(float("-inf"))

# Give as input b and w in the form FWa to use dimension(b,w).
```

Given w it is possible to visualize the recursions of the algorithm through a binary tree, which is denoted by $T(w)$. We take as root the node associated with w. Thus the leaves are the nodes associated with minimal length elements. We present an example in the following figure. In every level of the trees there are elements with the same length. Every time we go down by one level the length decreases by 1. We have considered: Cartan type $= ['A', 5, 1]$, $b = 1$, $w = s_3s_4s_3s_2s_1s_0s_5s_1s_2s_3s_4$. For every call of the function dl_reduction(b,w') we have written the simple affine reflections used to get w' from the previous element, the reduced expression used by Sage to represent w' and the dimension of $X_{w'}(b)$.

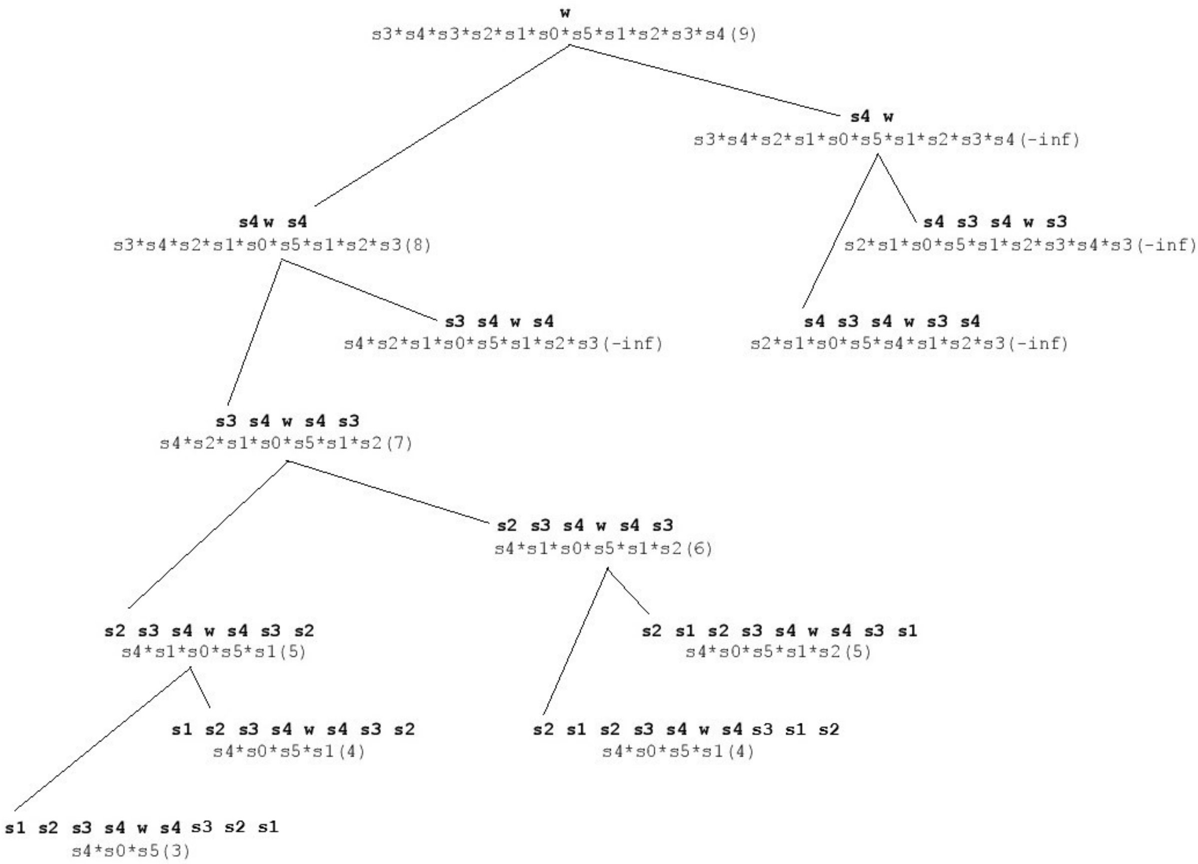

Figure 2.1: Tree of recursions

Notice that for the elements $s_3s_4s_2s_1s_0s_5s_1s_2s_3s_4$ and $s_4s_1s_0s_5s_1s_2$ it was not possible to decrease the length just considering one simple affine reflection. We had to multiply them by respectively s_4s_3 and s_2s_1. Suppose that $w = 1$ then $(\overline{\nu}_w, 2\rho) = 0$ for every $\widetilde{W}$. Thus $X_w(b) = \emptyset$ if $f(b) \neq f(1) = (1,0)$ otherwise we get $\dim X_w(b) = 0$. Furthermore, we can give the following bounds for the dimension of an ADLV with $\delta = 1$:

Proposition 2.2.1. *Let b and w in $\widetilde{W}$ with $w \neq 1$. Suppose that $X_w(b) \neq \emptyset$. Then*

$$\lfloor l(w)/2 \rfloor + 1 - (\overline{\nu}_b, 2\rho) \leq \dim X_w(b) \leq l(w) - (\overline{\nu}_b, 2\rho). \tag{2.1}$$

Proof. We consider the binary tree $T(w)$. We denote by M the set of the leaves of $T(w)$, i.e. the nodes associated with minimal length elements. With abuse of notation we will identify the nodes with the corresponding elements in $\widetilde{W}$. Recall that for every node $v \notin M$, we have $\dim X_v(b) = \max\{\dim X_{v'}(b), \dim X_{v''}(b)\} + 1$ where v' and v'' are the children of v. If $v \in M$ we know that $\dim X_v(b) = l(v) - (\overline{\nu}_b, 2\rho)$ or $\dim X_v(b) =$ -inf. Thus we get the following equality

$$\dim X_w(b) = \max_{v \in M}\{\dim X_v(b) + d^v\}, \tag{2.2}$$

where d^v is the length of the unique path from w to v. We can write $d^v = d_1^v + d_2^v$ where d_1^v (resp. d_2^v) is the number of nodes $u \neq w$ in the path from w to v such that $l(u) = l(u') - 1$ (resp. $l(u') - 2$) where u' is the parent of u. Thus for every $v \in M$ we get: $\dim X_v(b) + d^v \leq l(v) + d^v - (\overline{\nu}_b, 2\rho) = l(w) - d_1^v - 2d_2^v + d_1^v + d_2^v - (\overline{\nu}_b, 2\rho) = l(w) - d_2^v - (\overline{\nu}_b, 2\rho) \leq l(w) - (\overline{\nu}_b, 2\rho)$. Thus by Equation 2.2 we get $\dim X_w(b) \leq l(w) - (\overline{\nu}_b, 2\rho)$. Since we have supposed $X_w(b) \neq$ -inf there exists a leaf v such that $\dim X_v(b) \neq$ -inf. Thus we have

$$\dim X_w(b) \geq \dim X_v(b) + d^v = l(v) - (\overline{\mu}_b, 2\rho) + d^v = l(w) - d_2^v - (\overline{\nu}_b, 2\rho).$$

Since $0 \leq l(v) \leq l(w) - 2d_2^v$ we have $d_2^v \leq \lfloor l(w)/2 \rfloor$. Thus if $l(w)$ is odd we get $\dim X_w(b) \geq l(w) - \lfloor l(w)/2 \rfloor - (\overline{\nu}_b, 2\rho) = \lfloor l(w)/2 \rfloor + 1 - (\overline{\nu}_b, 2\rho)$. If $l(w)$ is even and $d_2^v = l(w)/2$ we obtain $l(v) = 0$, i.e. $v = 1$. Since $w = x^{-1}ux$ for some $x \in W_a$, we get $w = 1$ against the hypothesis. It implies that $d_2^u \leq l(w)/2 - 1$. So $\dim X_w(b) \geq l(w)/2 + 1$. □

Remark 9. Similarly, when $\delta \neq id$ we get $\lceil l(w)/2 \rceil - (\overline{\nu}_b, 2\rho) \leq \dim X_w(b) \leq l(w) - (\overline{\nu}_b, 2\rho)$. Indeed the only difference in the proof of 2.2.3 is that for $l(w)$ even, we can have $d_2^v = l(w)/2$ for some leaf v. Indeed, we obtain $w = x^{-1}\delta(x)$ for some x in W_a.

Remark 10. If $f(w) = f(b)$ then $X_w(b) \neq \emptyset$ for arbitrary δ. Indeed, there is always a leaf v in $T(w)$ such that $d_1^v = 0$. Thus we get $v = xw\delta(x)^{-1}$ for some x in W_a. Since f is constant in each δ-coniugacy class we get $f(v) = f(w) = f(b)$ thus $X_v(b) \neq \emptyset$. For Equation 2.2 we obtain that $X_w(b) \neq \emptyset$.

Let us consider the program of the previous script. Notice that the dimension of $X_w(b)$ depends only on the Cartan Type associated to $\widetilde{W}$, $f(b)$ and w. Even if it is a deterministic algorithm, the computational cost of the program limits its applications. Of course as the rank of the irreducible root system increases, the time used for the calculation increases as well, since we have more simple affine reflections to consider. The length of w is another important parameter that influences the computational cost of the program. Indeed, with increasing length, the number of recursions and the amount of elements to examinate during the while loop may grow considerably. We present now some examples were we have taken always $b = 1$ and we have changed the other parameters. In addition to the dimension we have written down the number of times that the function dl_reduction() has been called and the seconds used for the calculation (Using Asus F555L).

Type = ['A', 5, 1], $w = s_3s_4s_3s_2s_1s_0s_5s_1s_2s_3s_4$,
$l(w) = 11$, dim = 9, rec = 13, time = 10.315313
Type = ['A', 5, 1], $w = s_2s_1s_0s_5s_0s_1s_2s_3s_1s_2s_1$,
$l(w) = 11$, dim = 11, rec = 55, time = 9.804502
Type = ['A', 5, 1], $w = s_3s_4s_1s_0s_5s_2s_3s_4s_1s_0s_5s_2s_3s_2s_0s_1s_0$,
$l(w) = 17$, dim = 12, rec = 73, time = 142.531811
Type = ['A', 5, 1], $w = s_3s_4s_0s_5s_2s_3s_4s_2s_1s_0s_5s_3s_4s_1s_2s_3s_2$,
$l(w) = 17$, dim = 13, rec = 91, time = 179.876699
Type = ['A', 5, 1], $w = s_0s_1s_2s_3s_4s_2s_1s_0s_5s_0s_1s_2s_3s_4s_1s_2s_3s_1s_2$,
$l(w) = 19$, dim = 14, rec = 221, time = 512.561199
Type = ['A', 5, 1], $w = s_0s_5s_1s_2s_3s_4s_2s_1s_0s_5s_0s_1s_2s_3s_4s_2s_3s_1s_2s_1$,

$l(w) = 20$, dim = -inf, rec = 9, time = 87.307117
Type = ['A', 5, 1], $w = s_0s_5s_2s_3s_4s_2s_1s_0s_5s_2s_3s_4s_2s_1s_0s_5s_4s_2s_3s_0$,
$l(w) = 20$, dim = 14, rec = 189, time = 449.063352
Type = ['A', 5, 1], $w = s_4s_5s_3s_4s_0s_5s_0s_1s_2s_3s_4s_0s_5s_3s_4s_1s_2s_3s_2s_1s_0$,
$l(w) = 21$, dim = 14, rec = 89, time = 303.597700
Type = ['A', 5, 1], w = $s_1s_2s_3s_4s_5s_0s_1s_2s_3s_4s_5s_4s_0s_1s_2s_3s_1s_2s_0s_1s_0$,
$l(w) = 21$, dim = 13, rec = 61, time = 203.818952
Type = ['A', 5, 1], w = $s_2s_3s_4s_0s_5s_1s_2s_3s_4s_0s_5s_1s_2s_3s_4s_2s_1s_0s_5s_3s_4s_2$ $s_0s_1s_0$, $l(w) = 25$, dim = -inf, rec = 37, time = 251.142763
Type = ['A', 5, 1], $w = s_4s_0s_5s_0s_1s_2s_3s_4s_0s_5s_0s_1s_2s_3s_4s_0s_5s_1s_2s_3s_1s_2$ $s_0s_1s_0$, $l(w) = 25$, dim = 16, rec = 153, time = 569.078516
Type = ['A', 7, 1], $w = s_0s_7s_1s_2s_3s_4s_5s_4s_0s_1s_2s_3s_2s_1s_0$,
$l(w) = 15$, dim = 15, rec = 135, time = 255.121628
Type = ['A', 7, 1], $w = s_6s_4s_3s_2s_1s_0s_7s_5s_6s_0s_1s_2s_3s_4s_5s_4$,
$l(w) = 16$, dim = -inf, rec = 5, time = 149.403329
Type = ['B', 5, 1], $w = s_1s_2s_3s_4s_5s_1s_0s_2s_3s_4s_5s_3s_2s_1s_0s_2s_3s_4s_5s_4s_3$ $s_1s_0s_2s_1$, $l(w) = 25$, dim = 17, rec = 339, time = 893.253320
Type = ['B', 5, 1], $w = s_4s_5s_2s_3s_4s_5s_0s_2s_3s_4s_5s_1s_2s_3s_4s_5s_1s_2s_3s_4s_2s_1$ $s_0s_2s_0$, $l(w) = 25$, dim = 23, rec = 1771, time = 2162.928026
Type = ['B', 5, 1], $w = s_5s_2s_3s_4s_5s_0s_2s_3s_4s_5s_2s_3$,
$l(w) = 12$, dim = 12, rec = 47, time = 29.447427
Type = ['B', 5, 1], $w = s_5s_3s_4s_5s_0s_2s_3s_4s_5s_4s_3s_1s_2$,
$l(w) = 13$, dim = 9, rec = 11, time = 72.522725
Type = ['B', 5, 1], $w = s_5s_0s_2s_3s_4s_5s_1s_0s_2$,
$l(w) = 9$, dim = 8, rec = 11, time = 11.062305
Type = ['B', 5, 1], $w = s_5s_3s_4s_0s_2s_3s_0s_2s_1$,
$l(w) = 9$, dim = 7, rec = 5, time = 11.318997
Type = ['C', 5, 1], $w = s_5s_4s_5s_3s_4s_5s_2s_1s_0s_1s_2s_3s_4s_5s_3s_4s_0s_1s_2s_3s_2$,
$l(w) = 21$, dim = 14, rec = 51, time = 182.839021
Type = ['C', 5, 1], $w = s_3s_4s_5s_1s_0s_1s_2s_3s_4s_5s_1s_0s_1s_2s_3s_4s_5s_3s_1s_0s_1$,
$l(w) = 21$, dim = 15, rec = 177, time = 268.621457
Type = ['C', 5, 1], $w = s_4s_5s_2s_3s_4s_5s_4s_3s_2s_1s_0s_1s_2s_3s_4s_5s_3s_0s_1s_2s_1$ $s_0s_1s_0$, $l(w) = 24$, dim = 17, rec = 141, time = 276.830408
Type = ['C', 5, 1], $w = s_5s_2s_3s_4s_5s_3s_2s_1s_0s_1s_2s_3s_4s_5s_4s_2s_1s_0s_1s_2s_3$ $s_1s_2s_0$, $l(w) = 24$, dim = 20, rec = 369, time = 691.505575
Type = ['D', 5, 1], $w = s_4s_0s_2s_3s_5s_1s_2s_3s_4s_0s_2s_3s_5s_2s_1s_0s_2s_3s_4s_1s_2$ $s_3s_1s_2s_1$, $l(w) = 25$, dim = -inf, rec = 17, time = 318.962278
Type = ['D', 5, 1], $w = s_3s_4s_1s_2s_3s_5s_0s_2s_3s_4s_2s_1s_0s_2s_3s_5s_2s_1s_0s_2s_3$ $s_4s_3s_2s_1$, $l(w) = 25$, dim = 18, rec = 329, time = 1406.090060
Type = ['E', 6, 1], $w = s_4s_1s_3s_2s_4s_5s_6s_4s_3s_0s_2s_4s_3s_0$,
$l(w) = 14$, dim = 12, rec = 13, time = 211.465417
Type = ['E', 6, 1], $w = s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6s_4s_3s_2s_4s_1$,
$l(w) = 15$, dim = 15, rec = 113, time = 169.132179
Type = ['E', 6, 1], $w = s_4s_5s_6s_0s_2s_4s_5s_3s_2s_4s_1s_3s_0s_2s_4s_5s_6s_3s_0$,
$l(w) = 19$, dim = -inf, rec = 13, time = 1078.834167
Type = ['E', 6, 1], $w = s_6s_5s_4s_1s_3s_0s_2s_4s_5s_6s_5s_4s_3s_0s_2s_4s_5s_4s_2s_1$,
$l(w) = 20$, dim = 19, rec = 453, time = 1991.075369
Type = ['E', 7, 1], $w = s_6s_5s_4s_3s_2s_4s_5s_6s_7s_3s_2s_0s_1s_0$,

$l(w) = 14$, dim $= 13$, rec $= 47$, time $= 234.241124$
Type = ['E', 7, 1], $w = s_2s_4s_5s_6s_7s_4s_5s_6s_3s_4s_5s_0s_1s_3$,
$l(w) = 14$, dim $= 12$, rec $= 13$, time $= 535.717511$
Type = ['E', 7, 1], $w = s_7s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7s_6s_2s_4s_2s_0s_1$,
$l(w) = 18$, dim $= 17$, rec $= 65$, time $= 920.746553$
Type = ['E', 7, 1], $w = s_6s_7s_5s_4s_1s_3s_2s_4s_5s_6s_3s_4s_5s_4s_1s_3s_2s_1$,
$l(w) = 18$, dim $= 18$, rec $= 215$, time $= 1109.261556$
Type = ['E', 8, 1], $w = s_0s_8s_3s_2s_4s_5s_6s_4s_3s_2s_4s_5s_3s_0$,
$l(w) = 14$, dim $= 14$, rec $= 63$, time $= 225.399089$
Type = ['E', 8, 1], $w = s_3s_4s_5s_6s_7s_0s_8s_4s_5s_6s_2s_4s_5s_4$,
$l(w) = 14$, dim $= 14$, rec $= 27$, time $= 775.574401$

Notice that in the previous calculations there are some examples where $\dim X_w(1) \neq \dim X_{w'}(1)$ for $w, w' \in \widetilde{W}$ with $l(w) = l(w')$.

2.2.4 Program for arbitrary δ

Now we want to consider also the cases when $\delta \neq id$. Thus we have to find the possible automorphisms with the properties listed at the beginning of subsection 2.1.2. Let $\widetilde{W}$ be an extended affine Weyl group and $\Delta = \{\alpha_1, ..., \alpha_n\}$ be a base for the associated root system Φ. Let ϕ' be an automorphism of the extended Dynkin diagram associated with $\widetilde{W}$, which fixes the node 0 (the node associated with the root $\alpha_0 = -\tilde{\alpha}$). Let ϕ be the induced permutation on the simple roots and on the associated nodes. Since the simple roots are a basis for the underlying euclidean space V, ϕ defines an automorphism of the vector space V. We denote it again by ϕ. Then we have:

Proposition 2.2.2. *The map*

$$\delta_\phi : \widetilde{W} \to \mathit{Aff}(E),\ \tilde{w} \mapsto \phi^{-1}\tilde{w}\phi$$

is an automorphism of $\widetilde{W}$ such that

1. *it restricts to an automorphism of Ω and it induces an automorphism on $P^\vee$ which stabilizes $Q^\vee$.*

2. *It induces a permutation on the simple reflections and it fixes the reflection s_0.*

Furthermore, every automorphism of $\widetilde{W}$ which satisfies the previous listed properties is of this form (is equal to δ_ϕ for a suitable ϕ).

Proof. First note that ϕ preserves the inner product of V, since it preserves the integers $\langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle$ with $i \neq j$ and sends short (long) roots to short (long) roots. Hence it induces also a bijection on the coroots and fundamental coweights ($\alpha_i^\vee \mapsto \alpha_{\phi(i)}^\vee$, $\omega_i^\vee \mapsto \omega_{\phi(i)}^\vee$). Note that $\phi s_{\alpha_i}\phi^{-1} = s_{\phi(\alpha_i)}$ for every α_i with $1 \leq i \leq n$ and $\phi t^\lambda \phi^{-1} = t^{\phi(\lambda)}$ for every λ in $P^\vee$. Thus the map

$$\delta_\phi :\ \widetilde{W} \to \widetilde{W},\ \tilde{w} \mapsto \phi\tilde{w}\phi^{-1},$$

is an automorphism of $\widetilde{W}$ which fixes the classical Weyl group and the translations associated to the coweights. Since it induces a bijection on the coroots also the group of translations associated to $Q^\vee$ is fixed. Hence, δ induces a group morphism on $P^\vee$ and on $Q^\vee$. Since ϕ preserves the Cartan integers ($\langle\alpha_i,\alpha_j\rangle = \langle\phi(\alpha_i),\phi(\alpha_j)\rangle$), we can apply Proposition 1.1.15 which implies that $\phi(\Phi) = \Phi$. Since $\phi(\Delta) = \Delta$, we get $\phi(\Phi^+) = \Phi^+$. To prove that $\phi(\tilde{\alpha}) = \tilde{\alpha}$ it is enough to show that $\phi(\tilde{\alpha}) - \beta = \phi(\tilde{\alpha} - \phi^{-1}(\beta))$ is a sum of positive roots for every β in Φ. But this is clearly true since $\tilde{\alpha} - \phi^{-1}(\beta)$ is a sum of positive roots and $\phi(\Phi^+) = \Phi^+$. If we write $\tilde{\alpha} = \sum_{i=1}^n k_i\alpha_i$, we obtain $k_i = k_{\phi(i)}$ for every i. This implies that ϕ induces a permutation on the vertices of $A_\circ$. Recall that they are equal to $\omega_i^\vee/k_i$. Let τ in Ω. Then $\delta_\phi(\tau) = \phi^{-1}\tau\phi$ is an element of $\widetilde{W}$ which permutes the vertices of $\overline{A_\circ}$. So we have that $\delta_\phi(\tau) \in \Omega$, i.e. δ_ϕ fixes the fundamental group. Notice that if τ_i is the element of Ω associated to the special node i then $\delta_\phi(\tau_i) = \tau_{\phi(i)}$ (since $\delta_\phi(\tau)(0) = \phi\tau(0)$). Finally, we have to show that $\delta_\phi(s_0) = s_0$. But this follows easily since $\delta_\phi(s_0) = \delta_\phi(t^{\tilde{\alpha}^\vee}s_{\tilde{\alpha}}) = \delta_\phi(t^{\tilde{\alpha}^\vee})\delta_\phi(s_{\tilde{\alpha}}) = t^{\phi(\tilde{\alpha}^\vee)}s_{\phi(\tilde{\alpha})} = s_0$. Now Let δ be an automorphism of $\widetilde{W}$ which fixes s_0, induces a bijection on the simple reflections and an automorphism on the fundamental group. Naturally, the induced automorphism of W_a corresponds to an automorphism of the Coxeter graph of W_a which fixes the node corresponding to s_0. However, we have seen at the end of subsection 1.8 that such an automorphism is also as an automorphism of the non-oriented graph associated to the corresponding extended Dynkin diagram. Now if we look at Figure 1.8 we see that it is also an automorphism of the extended Dynkin diagram. Let ϕ be the induced bijection on the simple roots. Then δ acts on W_a as δ_ϕ. Let τ in Ω, $\tau = t^\lambda w$ with $\lambda \in P^\vee$ and $w \in W$. We have $\delta(\tau) = \delta(t^\lambda)\delta(w) \in \Omega$ and $\delta_\phi(\tau) = \delta_\phi(t^\lambda)\delta_\phi(w) \in \Omega$. Since $\delta(w) = \delta_\phi(w)$ and the projection $\Omega \to W$ is injective, we get $\delta(\tau) = \delta_\phi(\tau)$. It follows that δ=δ_ϕ. □

It can be shown that these automorphisms really occur for some linear algebraic groups over local fields (see [14]) and hence for some ADLV. Now, looking at the extended Dynkin diagrams, we want to write down all the possibilities for ϕ with $\phi \neq id$. As usual the numbers of the nodes are the ones in Figure 1.8 which are the same used in Sage.

1. ($A_n, n \geq 2$) We can consider the permutation that sends the node i to $n - i + 1$ with $1 \leq i \leq n$.

2. (D_4) We have five possible permutations: $\phi_1 = (1,3)$, $\phi_2 = (1,4)$, $\phi_3 = (3,4)$, $\phi_4 = (1,3,4)$, $\phi_5 = (1,4,3)$.

3. ($D_n, n \geq 5$) There is one admissible permutation $\phi = (n-1,n)$.

4. (E_6) In this situation, we have $\phi = (1,6)$.

Now that we have identified and studied all the possible automorphisms for $\widetilde{W}$, we can include them in our program for the dimension. After having chosen a Cartan type, it is now possible to choose one of the bijections ϕ of the previous list or $\phi = \mathrm{id}$. (see Script 2 in the appendix for the code). It was quite easy to define δ_ϕ on Sage because we know exactly how it acts on the classic Weyl group, affine Weyl group and fundamental group. These are some examples of computations with $b = 1$:
Type = ['A', 5, 1], phi = (1,5)(2,4), $w = s_2s_1s_0s_5s_0s_1s_2s_3s_1s_2s_1$,
$l(w) = 11$, dim = 8, rec = 13, time = 50.672914
Type = ['A', 7, 1], phi = (1,7)(2,6)(3,5), $w = s_3s_4s_5s_4s_0s_1s_2s_0$,
$l(w) = 8$, dim = 7, rec = 11, time = 38.065987
Type = ['D', 4, 1], phi = (1,3), $w = s_2s_3s_0s_2s_4s_3s_2s_1$,
$l(w) = 8$, dim = 7, rec = 11, time = 6.389851

Type = ['D', 4, 1], phi = (1,4), $w = s_2s_3s_0s_2s_4s_3s_2s_1$,
$l(w)$ = 8, dim = -inf, rec = 1, time = 3.571901
Type = ['D', 4, 1], phi = (3,4), $w = s_2s_3s_0s_2s_4s_3s_2s_1$,
$l(w)$ = 8, dim = -inf, rec = 1, time = 2.752002
Type = ['D', 4, 1], phi = (1,3,4), $w = s_2s_3s_0s_2s_4s_3s_2s_1$,
$l(w)$ = 8, dim = 5, rec = 7, time = 3.571096
Type = ['D', 4, 1], phi = (1,4,3), $w = s_2s_3s_0s_2s_4s_3s_2s_1$,
$l(w)$ = 8, dim = 6, rec = 7, time = 3.269639
Type = ['D', 6, 1], phi = (5,6), $w = s_5s_1s_2s_3s_4s_6s_5s_1$,
$l(w)$ = 8, dim = 8, rec = 11, time = 13.414901
Type = ['E', 6, 1], phi = (1,6), $w = s_4s_1s_3s_2s_4s_5s_6s_4s_3s_0s_2s_4s_3s_0$,
$l(w)$ = 14, dim = 13, rec = 87, time = 207.043472

Chapter 3

Some applications

The aim of this chapter is to use our program to test a conjecture regarding the ADLV and to calculate the dimension of the 'supersingular locus' S_I in some unknown cases.

3.1 Generalization of the Reumann conjecture

We search evidence for a particular case of [8, Conjecture 9.5.1]. We have analysed the following case:

Conjecture 3.1.1. *Let $b = \tau w'$ with $\tau \in \Omega$ and $w' \in W_a$. Then there exist $N_b \in \mathbb{N}$ and $C_b \in \mathbb{Z}$ such that for all w in $\widetilde{W}$ with $l(w) \geq N_b$ we have*

$$X_w(b) \neq \emptyset \iff X_w(\tau) \neq \emptyset,$$

and in this case we have

$$\dim X_w(b) - \dim X_w(\tau) = C_b.$$

Roughly speaking the conjecture states that if the length of w increases we can 'forget' about the 'affine part' of b. We have then decided to calculate $\dim X_w(b)$ and $\dim X_w(\tau)$ for some b and w. We have taken $\delta = id$ and root systems of type A_n with $n = 6, 7, 8$. After having chosen $b \in \widetilde{W}$, we have considered elements in $\widetilde{W}$ with different lengths and with the same projection of b in Ω. We have used a slight variation of the program of Script 1 in order to save time. Indeed, for every chosen w we have found $\dim X_w(b)$ and $\dim X_w(\tau)$ calculating only once the while loop in the function dl_reduction(). We have used a computer of the University (4 Intel Xeon 2.4 GHz processors with 4 cores each, 128 GB of Ram).

Example 1

We have considered: Cartan type = ['A',6,1], $b = F[1]s_0s_1s_2$. We have that $\eta(b) = 1/5\omega_5^\vee$ and $(\eta(b), 2\rho)$= 2. For every i between 3 and 23 we have picked 30 random elements in $\widetilde{W}$ of length i. For every such element, say w, we have found $\dim X_w(b)$ and $\dim X_w(F[1])$. In our sample we got that either $\dim X_w(b) = \dim X_w(F[1])$ or $\dim X_w(F[1])$ = -inf and $\dim X_w(F[1]) \neq$ -inf. Up to $i = 18$ we found elements with $\dim X_w(b) \neq \dim X_w(F[1])$. Then we have considered other 40 elements of length 19 and 40 elements of length 20. In every case we got $\dim X_w(b) = \dim X_w(F[1])$. Here are some results where the first dimension is $\dim X_w(b)$ and the second is $\dim X_w(F[1])$.

$w = F[1]s_5s_6s_4s_5s_3s_2s_1s_0s_6s_1s_2$, $l(w) = 11$,
dim = [5, -inf], time = 52.210507

$w = F[1]s_5s_6s_3s_4s_5s_0s_6s_4s_5s_3s_4s_2$, $l(w) = 12$,
dim = [7, -inf], time = 12.062589
$w = F[1]s_1s_0s_6s_5s_3s_4s_0s_1s_2s_3s_1s_2s_1$, $l(w) = 13$,
dim = [7, -inf], time = 101.594934
$w = F[1]s_6s_5s_0s_6s_4s_5s_1s_0s_6s_4s_5s_3s_1s_2s_1$, $l(w) = 15$,
dim = [9, -inf], time = 249.989452
$w = F[1]s_1s_0s_6s_5s_2s_1s_0s_6s_5s_4s_1s_2s_3s_2s_0s_1s_0$, $l(w) = 17$,
dim = [11, -inf], time = 427.096439
$w = F[1]s_3s_4s_5s_4s_3s_2s_1s_0s_6s_2s_3s_4s_5s_3s_4s_2s_3s_2$, $l(w) = 18$,
dim = [12, -inf], time = 529.062200
$w = F[1]s_4s_5s_6s_3s_4s_5s_0s_6s_3s_4s_5s_3s_4s_1s_2s_3s_2s_0s_1$, $l(w) = 19$,
dim = [10, 10], time = 701.600925
$w = F[1]s_0s_6s_5s_4s_3s_2s_1s_0s_6s_4s_5s_4s_3s_2s_1s_0s_6s_4s_2$, $l(w) = 19$,
dim = [-inf, -inf], time = 224.058512
$w = F[1]s_2s_3s_4s_5s_6s_0s_1s_2s_3s_4s_5s_6s_3s_4s_1s_2s_3s_1s_2s_0$, $l(w) = 20$,
dim = [-inf, -inf], time = 904.944162
$w = F[1]s_5s_2s_1s_0s_6s_3s_4s_5s_2s_1s_0s_6s_5s_3s_4s_2s_3s_1s_2s_1$, $l(w) = 20$,
dim = [13, 13], time = 1315.524028

Example 2

We have considered: Cartan type = ['A', 7, 1], $b = F[1]s_0s_1s_2$. We have $\eta(b) = 1/6\ \omega_6^\vee$ and $(\eta(b), 2\rho) = 2$. We took 20 random elements with length i for $i = 10, ..., 25$. Again, either the two dimensions were equal or $\dim X_w(b) = -\text{inf}$. Up to length 20, we found w such that $X_w(b) \neq X_w(F[1])$. Some results:
$w = F[1]s_7s_5s_6s_0s_7s_4s_5s_6s_4s_5s_4s_2s_3s_2s_0s_1s_0$, $l(w) = 17$,
dim = [10, -inf], time = 310.897546
$w = F[1]s_5s_6s_3s_2s_1s_0s_7s_2s_3s_4s_5s_6s_4s_5s_2s_3s_4s_2$, $l(w) = 18$,
dim = [11, -inf], time = 2459.584924
$w = F[1]s_1s_0s_7s_6s_5s_4s_3s_2s_1s_0s_7s_5s_0s_1s_2s_3s_0s_1s_2s_0$, $l(w) = 20$,
dim = [13, -inf], time = 1838.931162
$s_6s_2s_1s_0s_7s_5s_6s_3s_2s_1s_0s_7s_1s_2s_3s_4s_5s_2s_3s_0s_1s_2s_0$, $l(w) = 23$,
dim = [13, 13], time = 7818.037942
$w = F[1]s_3s_4s_5s_6s_2s_1s_0s_7s_2s_3s_4s_5s_6s_2s_3s_4s_5s_0s_1s_2s_3s_4s_3s_2s_1$,
$l(w) = 25$, dim = [15, 15], time = 13426.784137

Example 3

We have considered: Cartan type = ['A', 8, 1], $b = F[1]s_0s_1s_2$. We have $\eta(b)$ =$1/7\ \omega_7^\vee$ and $(\eta(b), 2\rho) = 2$. We took 20 random elements of length i with $i = 13, ..., 21$. We got, as before, that either the two dimensions were equal or $\dim X_w(b) = -\text{inf}$. For every i there are examples where the two dimensions are different. Some results:
$w = F[1]s_8s_7s_0s_8s_4s_5s_6s_7s_4s_5s_6s_4s_5s_4s_3s_2s_1$, $l(w) = 17$,
dim = [9, -inf], time = 1520.073336
$w = F[1]s_6s_7s_3s_2s_1s_0s_8s_5s_0s_1s_2s_3s_4s_1s_2s_3s_1s_0$, $l(w) = 18$,
dim = [11, -inf], time = 9092.632414
$w = F[1]s_0s_8s_6s_7s_6s_5s_4s_3s_2s_1s_0s_8s_3s_4s_3s_0s_1s_2s_1$, $l(w) = 19$,
dim = [11, -inf], rec = 90, time = 6393.354397
$w = F[1]s_7s_0s_8s_5s_6s_7s_6s_5s_4s_3s_2s_1s_0s_8s_5s_6s_4s_5s_1s_0$, $l(w) = 20$,
dim = [12, -inf], time = 8759.840022
$w = F[1]s_8s_7s_0s_8s_6s_7s_5s_4s_3s_2s_1s_0s_8s_7s_6s_1s_2s_3s_0s_1s_2$, $l(w) = 21$,
dim = [13, -inf], time = 15075.369537

Example 4
We have considered: Cartan type = ['A', 6, 1], $b = F[1]s_5s_4s_0$. We have $\eta(b) = 1/4\ \omega_4^\vee$ and ($\eta(b)$, 2ρ)=3. Recalling example 1, we have tested only elements with length greater then 19. Precisely we have considered 30 elements with length i with $i = 19, 20, 21$. The two associated dimensions were always equal. Some results:
$w = F[1]s_6s_5s_4s_3s_2s_1s_0s_6s_4s_5s_1s_2s_3s_4s_2s_3s_2s_0s_1s_0$, $l(w) = 20$,
dim = [12, 12], time = 225.491332
$w = F[1]s_6s_5s_1s_0s_6s_3s_4s_5s_4s_3s_2s_1s_0s_6s_4s_5s_1s_2s_3s_4s_2$, $l(w) = 21$,
dim = [12, 12], time = 1019.948641
Example 5 We have considered: Cartan type = ['A', 6, 1], $b = F[3]s_5s_4s_3$. We have $\eta(b) = 1/6\ \omega_4^\vee$, $(\eta(b), 2\rho) = 2$. We took 30 elements of length i for every $i = 3, ..., 23$. We found an element of length 3 where $\dim X_w(b)$ =-inf and $\dim X_w(F[3]) = 2$ and few cases where $\dim X_w(b) \neq \dim X_w(F[3])$ with $l(w) \leq 5$ and $\dim X_w(F[3])$ =-inf. However, in all the other samples the two dimensions were the same. Some results:
$w = F[3]s_6s_3s_2$, $l(w) = 3$, dim = [-inf, 2], time = 0.700693
$w = F[3]s_4s_5s_6s_4s_0$, $l(w) = 5$, dim = [2, -inf], time = 2.323644
$w = F[3]s_0s_1s_2s_3s_1$, $l(w) = 5$, dim = [2, -inf], time = 2.346560
Example 6
We have considered: Cartan type = ['A', 7, 1], $b = F[1]s_5s_4s_0$. We have $\eta(b) = 1/5\ \omega_5^\vee$ and $(\eta(b), 2\rho) = 3$. We took 30 elements of length i with $i = 16, ..., 23$. Up to length 19 we found cases where $X_w(b) \neq \emptyset$ but $X_w(F[1]) = \emptyset$. In the other examples the two dimensions were equal. Some results:
$w = F[1]s_3s_2s_1s_0s_7s_6s_2s_3s_4s_5s_2s_3s_4s_2s_3s_1s_2$, $l(w) = 17$,
dim = [10, -inf], time = 972.855964
$w = F[1]s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1s_0s_7s_6s_3s_4s_2s_3s_2$, $l(w) = 18$,
dim = [11, -inf], time = 1504.888439
$w = F[1]s_7s_6s_2s_1s_0s_7s_5s_6s_4s_3s_2s_1s_0s_7s_6s_0s_1s_2s_0$, $l(w) = 19$,
dim = [12, -inf], time = 1860.914333
$w = F[1]s_0s_7s_6s_2s_1s_0s_7s_6s_5s_0s_1s_2s_3s_4s_1s_2s_3s_2s_1$, $l(w) = 19$,
dim = [12, -inf], time = 2353.338240
$w = F[1]s_1s_0s_7s_3s_4s_5s_6s_0s_1s_2s_3s_4s_5s_0s_1s_2s_3s_4s_0s_1s_2s_0s_1$, $l(w) = 23$,
dim = [17, 17], time = 11809.361937
Even if these computations don't prove or contradict the conjecture, they highlight the possibility of an interesting relation between the two dimensions.

3.2 Dimension of supersingular locus

We fix g in $\mathbb{N}_{\geq 1}$ and p a prime number. Let A_I be the moduli space of abelian varieties of dimension g 'with Iwahori level structure at p', over $\overline{\mathbb{F}}_p$. Let S_I be the 'supersingular locus' inside A_I. What is known about the dimension of this algebraic object is the following result from [10, Theorem 1.1].

Theorem 3.2.1. *If g is even then* $\dim S_I = g^2/2$.
If g is odd, then
$$\frac{g(g-1)}{2} \leq \dim S_I \leq \frac{(g+1)(g-1)}{2}.$$

When $g = 3$ it was proved that $\dim S_I = 3$, however the dimension is unknown in all the remaining cases (g odd, $g > 3$). Since there is a 'link' between the ADLV and $\dim S_I$, we can use our program to find the dimension in the case $g = 5$. In order to explain the connection between these two algebraic objects, we need to introduce a partial order on the extended affine Weyl groups.

Definition 3.2.2. Let $\widetilde{W}$ be an extended affine Weyl group. Let $\tilde{w} = \tau w$ and $\tilde{w}' = \tau' w'$ be two elements in $\widetilde{W}$, with $\tau, \tau' \in \Omega$ and $w, w' \in W_a$. Then $\tilde{w} \leq \tilde{w}'$ iff $\tau = \tau'$ and $w \leq w'$ with respect to the Bruhat ordering on W_a.

Thanks to the work of Zhu in [15] and Rapoport-Zink in [16], it is possible to relate the supersingular locus S_I to a 'Rapoport-Zink space' and then to connect this space with the ADLV in the p-adic case. Let Φ be the root system of type C_g and let $\Delta = \{\alpha_1, ..., \alpha_g\}$ be a base for Φ ordered as in Figure 1.8. Let $\widetilde{W}$ be the associated extended affine Weyl group. The following description of $\dim S_I$ was proved:

Proposition 3.2.3. *We define the "admissible set" (in [10] $\mu-$admissible set) to be:*

$$Adm := \{w \in \widetilde{W} \mid \exists\, v \in W : w \leq t^{v(\omega_g^\vee)}\}.$$

We write $\omega_g^\vee = \tau w$ with $\tau \in \Omega$ and $w \in W_a$. Then we have

$$\dim S_I = \max_{x \in Adm} \{\dim X_x(\tau)\}.$$

Hence it is possible to find $\dim S_I$ just computing the dimension of a finite number of ADLV.

Remark 11. Notice that for every g we have two possibilities for τ, or $\tau = 1$ or $\tau = \tau_g$ (using the notation of Remark 4). Notice that since $\omega_g^\vee \in P^\vee$, τ=1 if and only if $\omega_g^\vee \in Q^\vee$. By Example 1.1.19 we know that the coroot lattice of C_g (equivalently the root lattice of B_g) is generated by the vectors $\epsilon_1, ..., \epsilon_g$ and we easily see that $\omega_g^\vee = \frac{1}{2}\epsilon_1 + \cdots + \frac{1}{2}\epsilon_g$. It follows that $\omega_g^\vee \notin Q^\vee$, so $\tau = \tau_g$.

It was not a problem to find the set Adm in Sage. We first have considered the orbit O of $\omega_g^\vee$ under the action of W. Then for every t^o with o in O, we have taken the projection o' in W_a and we have computed all $w \in W_a$ such that $w \leq o'$ using the command Wa.bruhat_interval$(1, o')$. Hence the elements $\tau w \in \widetilde{W}$, where τ is the projection of $\omega_g^\vee$ in Ω, form the desired set. Since the cardinality of Adm increases rapidly with g, we have used some other theoretical results from [10] to decrease the size of the set Adm. From now on we fix the setting of Proposition 3.2.3 ($\widetilde{W}$, $\omega_g^\vee$, τ).

Definition 3.2.4. Let $\tilde{w} = t^\lambda w$ with $\lambda \in P^\vee$ and $w \in W$, be an element of $\widetilde{W}$. We define the *p-rank* of $\tilde{w}$ to be $\#\{i \in \{1, ..., g\} | \rho(w)(i) = i\}$, where ρ is the usual group morphism $\rho : W \to S_{2g}$ of Example 1.1.19.

Then we have:

Proposition 3.2.5. *Let $\tilde{w} \in Adm$ with p-rank different from zero, then $X_{\tilde{w}}(\tau) = \emptyset$.*

The previous result follows from the interpretation of the statement in terms of the moduli space of abelian varieties ("The Newton stratification is a refinement of the p-rank stratification"). Thanks to this proposition, it is enough to calculate the dimension for the elements in $\mathrm{Adm}^{(0)} := \{w \in \widetilde{W} \mid w \text{ has } p\text{-rank} = 0\}$. Furthermore, there is a subset of $\mathrm{Adm}^{(0)}$ that can be omitted from the computation, the so called 'superspecial elements'.

Definition 3.2.6. Let $\tilde{w}$ in $\widetilde{W}$ and let w its projection in the affine Weyl group W_a. Then $\tilde{w}$ is called *superspecial* if $\exists\ 0 \leq i \leq g$ such that neither s_i nor s_{g-i} are contained in $\text{supp}(w)$.

To explain why these elements can be omitted we need the following remark.

Remark 12. Let $J \subsetneq S_a$ where S_a are the simple affine reflections associated with Δ. Let $W_J := \langle s \mid s \in J \rangle$. We want to show that W_J is a finite group. It is enough to prove that $|W_J| < \infty$ for $J = S_a \setminus \{s\}$ for s in S_a. Notice that the induced subgraph obtained removing one node from the extended Dynkin diagram of $\widetilde{C}_g$ has one or two connected componets which are Dynkin diagrams of some irreducible root systems. Recall that, forgetting the arrows, these diagrams are the Coxeter graphs of the corresponding classical Weyl groups. Then we get that W_J is the direct product of classical Weyl groups, thus is finite. In general, this result holds also for the other root systems (one can use the same reasoning).

Let $\tilde{w} \in \widetilde{W}$ with $\tilde{w} = \tau w$ with w in W_a. We define $\text{supp}_\sigma(\tilde{w})$ the smallest subset J of S_a such that $\text{supp}(w) \subset J$ and $\tau(J) = J$ where $\tau(s_i) = \tau s_i \tau^{-1}$. By Remark 11 we know that $\tau = \tau_g$. Looking at Figure 1.8 we see that there is only one automorphism ϕ of the Dynkin diagram of $\widetilde{C}_g$ sending the node 0 to the node g. We have that $\phi(i) = g - i$ for every $i = 0, ..., g$. By Remark 4, it follows that $\tau s_i \tau^{-1} = s_{g-i}$ for every i. Then $\text{supp}_\sigma(\tilde{w}) = \{s_i \in S_a \mid s_i \textit{ or } s_{g-i} \in \text{supp}(w)\}$. Thus if $\tilde{w}$ in Adm is superspecial, we have that $\text{supp}_\sigma(\tilde{w}) \subsetneq S_a$ and by Remark 12 $W_{\text{supp}_\sigma(\tilde{w})} := \langle s \mid s \in \text{supp}_\sigma(\tilde{w}) \rangle$ is finite. Then by the proof of [6, Theorem 4.8] we get that $\dim X_{\tilde{w}}(\tau) = l(\tilde{w})$. Then after finding the maximal length between the superspecial elements in Adm (see [18, Proposition 4.6]), we can then state:

Proposition 3.2.7. *:*

1. *If g is even:*

$$\max\{\dim X_w(\tau) \mid w \in Adm, w \textit{ is superspecial }\} = g^2/2.$$

2. *If g is odd:*

$$\max\{\dim X_w(\tau) \mid w \in Adm, w \textit{ is superspecial }\} = g(g-1)/2.$$

We set $\text{Adm}' := \{w \in \text{Adm} \mid w \text{ has p-rank 0 and is not superspecial}\}$. Then, when g is odd, we get:

$$\dim S_I = \max\{g(g-1)/2, \max_{w \in \text{Adm}'}\{\dim X_w(\tau)\}\}.$$

In Sage we can easily check if an element has p-rank 0 and if it is superspecial. In the first case we just have to define the group morphism ρ from W to S_{2g}, in the second case to get the set $\text{supp}(w)$ with w in W_a we can use the command w.reduced_word()(see Script 1 in appendix for details). Now we would like to give an upper bound for the cardinality of Adm' finding the cardinality of $\text{Adm}^{(0)}$. First we need to study better the map $\rho : W \to S_{2g}$. Recall that $W \cong S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$.

Remark 13. We want to show that for every σ in $\rho(W)$, $\sigma(i) = i$ if and only if $\sigma(2g - i + 1) = (2g - i + 1)$. If we consider the group $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, we get that $\rho(W)$ is the image of the morphism ϕ: $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g \to S_{2g}$, $(i, i+1) \mapsto (i, i+1)(2g - i, 2g - i + 1)$ and

$(0, ..., i, ...0) \mapsto (i, 2g-i+1)$ with $1 \leq i \leq g$. From the definition of ϕ follows that an element σ in $\rho(W)$ can be written as $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_1 is a permutation of the elements $\{1, ..., g\}$, σ_2 is a permutation of $\{g+1, ..., 2g\}$ and σ_3 is a product of transpositions of type $(i, 2g-i+1)$. Furthermore, $\sigma_2 = \gamma^{-1}\sigma_1\gamma$ where γ is the permutation sending $i \mapsto 2g-i+1$ for $1 \leq i \leq 2g$. From this description follows that if $(i, 2g-i+1)$ appears in σ_3 neither i nor $2g-i+1$ can be fixed. If $(i, 2g-i+1) \notin \sigma_3$, we get $\sigma(2g-i+1) = \gamma(\sigma(i))$. This implies that $(2g-i+1)$ is fixed if and only if i is fixed.

Let $W^{(0)}$ be the elements in $\rho(W) \subset S_{2g}$ with no fixed points (derangements). In [10, Lemma 8.1] it is proved the following:

Proposition 3.2.8. *Let π be the projection $\widetilde{W} \to W$. Then the following map is a bijection:*

$$Adm^{(0)} \to W^{(0)},\ w \mapsto \rho(\pi(w))$$

Notice that the image of the function is contained in $W^{(0)}$ by Remark 13. Now we want to find $\#W^{(0)}$. In order to express the cardinality in a 'fancy' way we will use the principle of exclusion-inclusion explained in [17, Chapter II]. This is usually used in the following combinatoric context.

Proposition 3.2.9. *Let A be a finite set. Let S be a set of properties that an element in A may or may not have. For any subset T in S let $f_=(T)$ be the number of objects in A which have the properties in T and don't have the properties in $S \setminus T$. Let $f_\geq(T)$ be the set of elements which have the properties in T, i.e. $f_\geq(T) = \sum_{Y \supseteq T} f_=(Y)$. Then we can write*

$$f_=(T) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} f_\geq(Y).$$

Proof. Notice that

$$\sum_{Y \supseteq T} (-1)^{\#(Y-T)} f_\geq(Y) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} \sum_{Z \supseteq Y} f_=(Z) = \sum_{Z \supseteq T} \Big(\sum_{Z \supseteq Y \supseteq T} (-1)^{\#(Y-T)} \Big) f_=(Z). \tag{3.1}$$

Let Z and T be fixed and $m = \#(Z-T)$. We have

$$\sum_{Z \supseteq Y \supseteq T} (-1)^{\#(Y-T)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} = \delta_{0m},$$

where the last equality comes from the expression $(1+x)^m = \sum_{i=0}^m \binom{m}{i} x^i$ setting $x = -1$. If we substitute δ_{0m} in Equation 3.1 we get the statement. □

Now observe that we can find the number of elements in A which don't satify any of the properties in S, setting $T = \emptyset$. We obtain

$$f_=(\emptyset) = \sum_{Y \subseteq S} (-1)^{\#(Y)} f_\geq(Y) \tag{3.2}$$

where Y ranges over all the subsets in S. Using this last equality, we can prove the following proposition

Proposition 3.2.10.

$$\#W^{(0)} = 2^g g! \sum_{i=0}^{g} \frac{(-1)^i}{2^i i!}.$$

Proof. We want to apply Formula 3.2 with $A = \rho(W)$ and S the set of properties $\{\sigma(i) = i \mid i \in \{1, ..., g\}\}$. Indeed, from Remark 13 follows that $\#W^{(0)} = f_{=}(\emptyset)$. Now let $Y \subset S$ with $|Y| = i$. We want to find $f_{\geq}(Y)$, i.e. the elements in $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$ whose image fixes the numbers determined by Y, say $J = \{j_1, ..., j_i\}$ with $1 \leq j_k \leq g$. We see that we have $(g-i)!$ choices for the part in S_g (we can permute freely the numbers not in J) and 2^{g-i} choices for the part in $(\mathbb{Z}/2\mathbb{Z})^g$ (the permutations $(j_k, 2g - j_k + 1)$ can't occur, see proof of Remark 13). Since there are $\binom{g}{i}$ subsets of S of cardinality i we get:

$$\#W^{(0)} = \sum_{i=0}^{g} \binom{g}{i} (-1)^i (g-i)! 2^{g-i} = \sum_{i=0}^{g} (-1)^i \frac{g!}{i!} 2^{g-i} = 2^g g! \sum_{i=0}^{g} \frac{(-1)^i}{2^i i!}.$$

□

Therefore, we can estimate $|W^{(0)}|$ with $|W|(\sqrt{e})^{-1} = \frac{2^g g!}{\sqrt{e}}$. It is a good estimate, indeed:

$$|\,|W|(\sqrt{e})^{-1} - |W^{(0)}|| = |2^g g! \sum_{i=g+1}^{\infty} \frac{(-1)^i}{2^i i!}| < \frac{2^g g!}{2^{g+1}(g+1)!} (\sqrt{e})^{-1} = \frac{1}{2(g+1)\sqrt{e}}.$$

After having reduced the cardinality of Adm we wanted also to increase the speed of the computations. Hence we used the decorator **@lru_cache(maxsize=None)** for the function dl_reduction(·,·). This decorator creates a dictionary and every time we call the function it saves in the dictionary a new element: the parameters (b,w) are the key and the dimension is the value. In this way, if we call again the function with the same parameters, it returns the result avoiding computations. Note that we have to give b and w as input in the form PW (not FW), since the keys of a dictionary should be hashable. Furthermore, we wanted to use in parallel the 16 CPU of our computer. It is possible to do it through the decorator **@parallel()**. For applying it properly, we have divided the elements of Adm′ in 16 lists and we have put these lists in a unique list, say l. Then we used the code:

```
@parallel(ncpus=16)
def dim_for_list(l):
    return [dimension(b, w) for b, w in l]
```

The resulting program can be found in the appendix (see Script 1). Now we present the cases that we have examined.
Case g = 2
We have :

$$\#\mathrm{Adm} = 13,\ \#\mathrm{Adm}' = 0,\ \#\mathrm{Adm}^{(0)} = 5,\ \#Sp = 5.$$

The program gave $\max_{w \in Adm}\{\dim X_w(\tau)\} = 2$, as stated in Theorem 3.2.1. The only element with the maximum dimension is the superspecial element $F[2]s_2s_0$.
Case g = 3
The first interesting case we have examined was $g = 3$. We have

$$\#\mathrm{Adm} = 79,\ \#\mathrm{Adm}' = 20,\ \#\mathrm{Adm}^{(0)} = 29,\ \#Sp = 9.$$

We found $\max_{w \in Adm}\{\dim X_w(\tau)\} = 3$ which agrees with the expectations. The elements in Adm with the maximum dimension are:

$F[3]s_1s_2s_1$ (superspecial), $F[3]s_2s_3s_0s_1$, $F[3]s_3s_2s_3s_0$, $F[3]s_3s_1s_2s_0$, $F[3]s_3s_0s_1s_0$.

Case g = 4

In this case we have that

$$\#\mathrm{Adm} = 633, \ \#\mathrm{Adm}' = 146, \ \#\mathrm{Adm}^{(0)} = 233, \ \#Sp = 87.$$

We found that $\max_{w \in Adm}\{\dim X_w(\tau)\} = 8$ as expected. The only element in Adm with the maximum dimension is the superspecial element $F[4]s_4s_3s_4s_3s_1s_0s_1s_0$.

Case g = 5

In the unknow case g = 5, we get that

$$\#\mathrm{Adm} = 6331, \ \#\mathrm{Adm}' = 2134, \ \#\mathrm{Adm}^{(0)} = 2329, \ \#Sp = 195.$$

After 30 minutes of computation we obtained that $\max_{w \in \mathrm{Adm}'}\{\dim X_w(\tau)\} = 10$. The elements in Adm with the maximum dimension are:

$F[5]s_1s_2s_3s_4s_1s_2s_3s_1s_2s_1$ (superspecial), $F[5]s_2s_3s_4s_5s_0s_1s_2s_3s_1s_2s_1$,
$F[5]s_4s_5s_3s_4s_5s_4s_1s_0s_1s_2s_0s_1$, $F[5]s_5s_4s_5s_3s_4s_5s_4s_1s_0s_1s_2s_0$,
$F[5]s_5s_4s_5s_3s_4s_5s_3s_4s_1s_0s_1s_0$, $F[5]s_5s_1s_2s_3s_4s_1s_2s_3s_1s_2s_0$,
$F[5]s_5s_4s_5s_2s_3s_4s_5s_3s_1s_0s_1s_0$, $F[5]s_5s_4s_5s_1s_2s_3s_4s_3s_1s_0s_1s_0$,
$F[5]s_5s_2s_3s_4s_5s_0s_1s_2s_3s_1s_2s_0$, $F[5]s_5s_4s_5s_4s_0s_1s_2s_3s_1s_0s_1s_0$,
$F[5]s_5s_4s_5s_4s_1s_0s_1s_2s_1s_0s_1s_0$, $F[5]s_5s_3s_4s_5s_4s_1s_0s_1s_2s_0s_1s_0$,

We can then state the following result:

RESULT

For $g = 5$, $\dim S_I = 10$.

This gives a hope that for g odd the dimension is always equal to the lower bound. Clearly the program can be used to check some other cases, but one should be patient due to the amount of operations involved.

Appendix A

Programs

These are the Python scripts used for the calculations.

A.1 Script 1

The first code can be used to find $\dim X_w(b)$ when $\delta = id$ and the dimension of the supersingular locus.

```
from functools32 import lru_cache
import time


NUM_PROCESSORS = 16

# Give a Cartan type, for example Ca = ['A',5,1].
Ca = ['C', 7, 1]

# Fix names for some objects related to the root system.
n = Ca[1]
Ra = Ca[0:2]
Y = RootSystem(Ra).coweight_space()
R = RootSystem(Ra).root_lattice()
rho2 = sum(R.positive_roots())

# Creation of the Extended Weyl group specifying the prefix for
# s.a. reflections and special nodes. Creation of the two
# presentations FW and PW0, the coweight lattice and the
# list of fundamental coweights.
E = ExtendedAffineWeylGroup(Ca, affine="s",fundamental="F")
FW = E.FW()
PW0 = E.PW0()
L = E.lattice()
Fu = sorted(L.fundamental_weights())
Wa = E.affine_weyl()

# Let s be the dictionary with the s.a.reflections.
s = Wa.simple_reflections()
```

```

# Let f be the fundamental group where the elements are in
# ascending order with respect to the associated nodes.
f = E.fundamental_group()


def newton_vector(w):
    global Y

    p = PW0(w).to_classical_weyl().order()
    w3 = PW0(w**p)
    return (1/p) * Y(w3.to_translation_left().to_dominant_chamber())


@parallel(ncpus=NUM_PROCESSORS)
def dim_for_list(l):
    return [dimension(b, w) for b, w in l]


def dimension(b, w):
    """
    Calculation of the dimension of the variety $X_{b}(w)$ """

    if w.to_fundamental_group() != b.to_fundamental_group():
        return(float("-inf"))

    return dl_reduction(PW0(b), PW0(w))


@lru_cache(maxsize=None)
def dl_reduction(b, w):
    """
    The Deligne-Lusztig reduction.
    We assume here that b and w map to the same element
    in the fundamental group.
    """

    global n, s

    l = w.length()

    # Use S to store the elements w' such that w' $\approx$ w and that
    # we haven't calculated yet the conjugation by s. a. reflections.
    S=[w]

    # Use C to store the elements w' such that w' $\approx$ w and we have
    # already computed (or we are going to compute in the next while loop)
    # the conjugation by s. a. reflections.
    C = [w]

    while not S == []:
```

```
        N = [] # Use N to save the elements with the same length of w.
        for x in S:
            for i in range (n+1):
                wi = s[i] * x * s[i]
                li = wi.length()
                if li < l:
                    d1 = dl_reduction(b, wi)
                    vi = s[i] * x
                    d2 = dl_reduction(b, vi)
                    return max(d1,d2) + 1 # D.L. reduction, [8, Cor. 2.53].
                if (li == l) and (wi not in N):
                    N.append(wi)

        S = [i for i in N if i not in C]
        C = C + S

    # The while loop ended with S == [], hence w is minimal in its conjugacy
    # class. Then we can use the dimension formula from [3, Theorem 4.8].

    nu_b = newton_vector(b)

    if (newton_vector(w) == nu_b):
        return w.length() - nu_b.scalar(rho2)
    else:
        return(float("-inf"))


# Give as input b and w in the form FWa. We show two possibilities.
# For example to define b we can use the lists f,s and the mode FW.
# b = FW(f[4])*FW(s[2]*s[1]*s[2]*s[1]*s[2]*s[1]*s[0]*s[2]*s[1]*s[2])
# b = E.one()  # b = id

# For example to define w we can use the numbers of the special nodes
# and the numbers for simple reflections (0,1,---.n).
# w = FW.from_fundamental(4) * FW.from_reduced_word([0, 2, 1, 3, 0])


# The following functions are for Cartan type Cn.


def proj_to_symmetricgroup(w):
    """
    The projection of an elements w in the classical Weyl group
    into the Symmetric group Sn.
    """
    global n

    Sy = SymmetricGroup(2*n)
    G = [] # Use G to save the imagine of the s. reflections.
    for j in range (1,n):
        r = Sy([(j,j+1),(2*n-j,2*n-j+1)]) # Map defined in example 1.1
```

```
            G.append(r)
    r = Sy([(n,n+1)])
    G.append(r)
    R = w.reduced_word()
    si = Sy([])
    for j in range(len(R)):
        e = G[R[j]-1]
        si =  si * e
    return si


def is_p_rank_0(w):
    global n

    w1 = PW0(w).to_classical_weyl()
    si = proj_to_symmetricgroup(w1)
    if any (si(k) == k for k in range (1,n+1)):
        return false
    return true


def is_superspecial(w):
    global n

    w1 = w.to_affine_weyl_right()
    R1 = w1.reduced_word()
    if any ((k not in R1) and (n-k not in R1) for k in range((n//2)+1)):
            return true
    return false


def list_of_adm():
    global n, Fu

    c0 = Fu[n-1]
    O = c0.orbit()
    b1 = FW(PW0.from_translation(c0))
    b = FW(b1.to_fundamental_group())
    Adm1 = [] # Use to save the projection of Adm's elements in Wa.
    for w in O:
        w1 = FW(PW0.from_translation(w))
        w2 = w1.to_affine_weyl_right()
        I = Wa.bruhat_interval(1,w2)
        I1 = [r for r in I if r not in Adm1]
        Adm1 = Adm1 + I1
    Adm = [b*FW(r) for r in Adm1]
    return Adm


def list_of_adm_reduced():
    Adm = list_of_adm()
```

```
    Adm1 = []
    for w in Adm:
        if not is_p_rank_0(w):
            continue
        if not is_superspecial(w):
            Adm1.append(w)
    return Adm1


def max_dim_adm_reduced():
    Adm = list_of_adm_reduced()
    b = FW(Adm[0].to_fundamental_group())
    Adm1 = [(b,w) for w in Adm]
    lists = [Adm1[i::NUM_PROCESSORS] for i in range(NUM_PROCESSORS)]
    d1 = (max([ max(x[1]) for x in dim_for_list(lists)]))
    return d1


def dim_supersingular_locus():
    """
    The dimension of the supersingular locus Sn.
    We suppose n odd and the Cartan type C.
    Return also the time used and the maximum dimension in Adm'.
    """
    global n

    start_time = time.time()
    d1 = max_dim_adm_reduced()
    print('Maximum dimension between the elements in Adm\': %f.' % (d1))
    d2 = (n * (n-1)) // 2
    dim = max(d1,d2)
    print ('Dimension of supersingular locus: %f' % (dim))
    print('Seconds: %f' % (time.time() - start_time))
```

A.2 Script 2

The following code can be used to find $\dim X_w(b)$ when δ is one of the automorphisms described in subsection 2.2.4.

```
# Give a Cartan type
Ca = ['D', 4, 1]

# Creation of the simmetric group
n = Ca[1]
Sym = SymmetricGroup(n)

# Give delta, i.e. give the permutation of the nodes 1,2,....,n.
phi = Sym([]) # Case delta = identity
# The other possibilities are the following:
# phi = Sym([(i,n-i+1) for i in range (1,n//2+1)])
# for Cartan type ['A', n, 1].
```

```python
# phi = Sym([(1,3)]), phi = Sym([(1,4)]), phi = Sym([(3,4)]),
# phi = Sym([(1,3,4)]), phi = Sym([(1,4,3)]) for Cartan type ['D', 4, 1].
# phi = Sym([(n-1, n)]) for Cartan type ['D', n, 1].
# phi = Sym([(1,6)]) for Cartan type ['E', 6, 1].

# Fix names for some objects related to the root system.

Ra = Ca[0:2]
Y = RootSystem(Ra).coweight_space()
R = RootSystem(Ra).root_lattice()
rho2 = sum(R.positive_roots())

# Creation of the Extended Weyl group specifying the prefix for s.a.
# reflections and special nodes. Creation of the two presentations
# FW and PW0, the coweight lattice and the list of fundamental coweights.

E = ExtendedAffineWeylGroup(Ca, affine="s", fundamental="F")
FW = E.FW()
PW0 = E.PW0()
L = E.lattice()
Fu = sorted(L.fundamental_weights())
Wa = E.affine_weyl()
W0 = E.classical_weyl()

# Let s be the dictionary with the s.a.reflections.
s = Wa.simple_reflections()

# Let f be the fundamental group where the elements are in
# ascending order with respect to the associated nodes.
f = E.fundamental_group()


# Permutation of the nodes.
def delta_nodes(i):
    global phi

    if i == 0:
        return 0
    else:
        return phi(i)


def delta_fundamental(z):
    """
    Compute the action of delta on the elements
    of the fundamental group.
    """
    return f(delta_nodes(z.value()))


def delta_classic(w):
```

```
    """
    Compute the action of delta on the elements
    of the classical Weyl group.
    """
    global W0

    R = w.reduced_word()
    R1 = [delta_nodes(i) for i in R]
    w1 = W0.from_reduced_word(R1)
    return w1


def delta_affine(w):
    """
    Compute the action of delta on the elements
    of the affine Weyl group.
    """
    global Wa

    R = w.reduced_word()
    R1 = [delta_nodes(i) for i in R]
    w2 = Wa.from_reduced_word(R1)
    return w2


def delta_extended(w):
    """
    Compute the action of delta on the elements
    of the extended affine Weyl group.
    """
    global f

    w1 = w.to_affine_weyl_right()
    w2 = FW(delta_affine(w1))
    w3 = delta_fundamental(w.to_fundamental_group())
    return w3 * w2


# Calculation of the normal subgroup
# N = < delta(f)/f | for f in the fundamental group >.
if phi == Sym([]):
    N = [f[0]]
else:
    N = []
    for a in f:
        b = a * delta_fundamental(a)**(-1)
        if b not in N:
            N.append(b)


def newton_vector(w):
```

```
    global Y, phi

    w1 = w.to_classical_weyl()
    w2 = w1
    for i in range(5):
        w1 = delta_classic(w1)
        w2 = w2 * w1
    p = w2.order()
    w3 = w
    w4 = w
    for i in range(5):
        w3 = delta_extended(w)
        w4 = w4 * w3
    w5 = PW0(w4 **(p))
    return (1/(6*p) * Y(w5.to_translation_left().to_dominant_chamber()))


def dimension(b, w):
    """
    Calculation of the dimension of the variety $X_{b}(w)$ """
    global N

    if (b.to_fundamental_group()*(w.to_fundamental_group())**(-1)) not in N:
        return(float("-inf"))

    return dl_reduction(PW0(b), PW0(w))


def dl_reduction(b, w):
    """
    The Deligne-Lusztig reduction.
    We assume here that b and w map to the same element
    in the fundamental group.
    """

    global n, s

    l = w.length()

    # Use S to store the elements w' such that w' $\approx$ w and that
    # we haven't calculated yet the conjugation by s. a. reflections.
    S=[w]

    # Use C to store the elements w' such that w' $\approx$ w and we have
    # already computed (or we are going to compute in the next while loop)
    # the conjugation by s. a. reflections.
    C = [w]

    while not S == []:
        N = [] # Use N to save the elements with the same length of w.
        for x in S:
```

```
            for i in range (n+1):
                wi = s[i] * x * s[delta_nodes(i)]
                li = wi.length()
                if li < l:
                    d1 = dl_reduction(b, wi)
                    vi = s[i] * x
                    d2 = dl_reduction(b, vi)
                    return max(d1,d2) + 1 # D.L.reduction, [8, Corollary 2.53].
                if (li == l) and (wi not in N):
                    N.append(wi)

        S = [i for i in N if i not in C]
        C = C + S

    # The while loop ended with S == [], hence w is minimal in its conjugacy
    # class. We can then use the dimension formula from [3, Theorem 4.8].

    nu_b = newton_vector(b)

    if (newton_vector(w) == nu_b):
        return w.length() - nu_b.scalar(rho2)
    else:
        return(float("-inf"))
```

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