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Semi-analytical estimates for the speed of diffusion in the second fundamental model of resonance: a Jeans-Landau-Teller approach

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Preface

The problem of the slow chaotic diffusion in Hamiltonian systems is very well known since the pioneering work of Arnol'd [Arn64]. In systems with d=2 degrees of freedom, the presence of 2-dimensional KAM tori constitutes a topological barrier preventing long excursions of the orbits in the phase space. On the contrary, when $d > 2$, the structure of the phase space in principle allows the dynamics to explore the whole energy shell in a sufficiently long time. In this work we face the problem of the diffusion near the separatrices of a resonance model with 3 degrees of freedom, inspired by the model of mean motion resonance in celestial mechanics. As pointed out by Chirikov [Chi79] long-term instabilities in near-integrable systems have in fact a diffusive character. The diffusion coefficient can be estimated using, for example, Melnikov estimates. Our approach in the present work is based on a combination of two methods proposed by Benettin et al in [BCF97] and by Guzzo et al in [GEP19]. In particular, we remark the fact that the diffusion along a guiding resonance does not proceed uniformly, but rather by a sequence of impulsive "kicks" or "jumps" at each homoclinic loop. Computing the size of the jumps allows then to quantify the rate of diffusion.

Thesis structure

In this work, we propose to study the speed of the slow chaotic diffusion of the adiabatic actions near the separatrices of a non-linear Hamiltonian dynamical system, corresponding to the so-called "second fundamental model of resonance". In particular, we develop an analytical procedure for computing the rate of diffusion, exploiting rigorous estimates of Melnikov-type integrals via a stationary phase approach. We finally compare the estimates obtained by this procedure with numerical estimates of the size of the jump in the evolution of the adiabatic action variables in an example of the 3:2 mean motion resonance in the spatial elliptic restricted three-body problem.

The structure of the thesis is as follows:

Chapter 1 contains mathematical definitions related to the notion of fundamental model of resonance. In particular, in this work we deal with the second fundamental model, described by the well-known Andoyer Hamiltonian [HL83]. We analyse the general features of this Hamiltonian and we present an analytical solution for the motion equation along the separatrices. Such an analytical solution is an essential ingredients in the subsequent estimates.

Chapter 2 performs an analysis of weakly perturbed Hamiltonians, following the Jeans-Landau-Teller approach(see [BCF97] for a review or the original work of Landau et al. [Lan36]), originally implemented for the study of energetic exchanges between fast internal degree of freedom (say, vibrational) and slow translational motions. The Hamiltonians we consider have the general form

$$H(S, J_1, J_2, \sigma, \alpha_1, \alpha_2) = H_0(S, \sigma) + K_1(S, J_1, J_2) + K_2(S, J_1, J_2, \sigma, \alpha_1, \alpha_2) \quad (1)$$

where

$$H_0 = aS^2 + bS + c\sqrt{2S}\cos s \quad (2)$$

is the Andoyer Hamiltonian,

$$K_1 = \eta_1 S J_1 + \eta_2 S J_2 + \eta_3 J_1 J_2 + \ldots$$

are coupling terms between the adiabatic action variables J_1, J_2 and

$$K_2 = \sum_{k,j,l} C_{k,j,l}(S, J_1, J_2)\cos(k\sigma + j\alpha_1 + l\alpha_2)$$

is the Fourier expansion of the perturbing function.
Following the idea of Chirikov [Chi79], the main step consists of substituting the

motion *near* the separatrix of the resonance by the Fourier transform of the associated explicit solution. The method leads to the evaluation of Melnikov-type integrals. Taking profit of Poincaré variables and substituing the Fourier representation of the separatrix, we arrive at an explicit rigorous expression that measures the size of the jump

$$\Delta J_i = \sum_{k,j,l} \int_0^T C_{k,j,l} \sin(f_{k,j,l}(t) \pm \omega t) dt$$

Each integral is estimated via a stationary-phase method as in [GEP19]. The method allows to identify those terms in the remainder the Hamiltonian which give the largest contributions, namely those which satisfy a stationary phase condition.

Chapter 3 provided all needed estimates on Melnikov integrals used in the above analysis. As in [GEP19], such estimates are derived on the basis of the well-known stationary phase approximations.

Chapter 4 is devoted to an application of our analysis to a model taken from celestial mechanics. Thus, we construct a hamiltonian model for the problem of first order mean motion resonances in a restricted three-body system. Moreover, we recall some basic notions from celestial dynamics used in the process. A key remark regards the need to perform a normalisation process via an algebraic manipulator, in order to eliminate the so-called "deformation" effect [Nek77]. This procedure allows to identify clearly the diffusive chain of the adiabatic variables. In particular, we are interested in the evolution of the adiabatic action variables in an example of the 3:2 mean motion resonance. We compare the analytical estimates developed in the previous chapters with the results of numerical integrations. In this chapter we also discuss briefly the phenomenology of the slow chaotic diffusion in the phase space, by means of numerical simulations of ensembles of trajectories.

1 Second fundamental model of resonance

In this chapter we discuss the general notion of fundamental model of resonance[HL83]. We study in details the so-called Andoyer Hamiltonian, which provides a basic model of the so-called *second* fundamental model of resonance.

1.1 Fundamental models

In a variety of problems arising in celestial mechanics one has to deal with resonances, i.e. commensurabilities between two or more orbital frequencies of the interacting bodies. Generally speaking, resonances lead to particular features for the phase space and one wonders if is it possible to model it in a proper but simple way. The notion of *fundamental model* answers this type of question. A fundamental model is a one dimensional Hamiltonian of the form $H(I, \phi)$, in which I is an action variable (see [AKN07]), while the canonically congiugate angle ϕ represents the resonant combination of the original angles.
A widely applicable model of resonance is the pendulum Hamiltonian, $H = \frac{a}{2} I^2 - b \cos \phi$. It is quite simple and it has been extensively used in many contexts, but it presents some drawbacks we will enlight in a while. To overcome such issues, Henrard and Lemaitre introduced a second class of models [HL83], i.e the *second fundamental model*, that deal with the d'Alemebertian properties of the Hamiltonian. One can consider also some extensions of the second fundamental models, able to describe some important behaviour, for example separatrix bifurcation. These are reffered to as them as *extended*, or *third*, fundamental models [Bre03].

1.1.1 The first fundamental model: the pendulum

The procedure reducing a problem to a fundamental model is quite general. Suppose we are given a N-degree of freedom Hamiltonian of the form

$$H(A,\alpha) = H_0(A) + \epsilon H_1(A,\alpha) \tag{1.1}$$

with $A \in D \subset \mathbb{R}^N$, $\alpha \in \mathbb{T}^N$ The model (1.1) is called simply-resonant when there is a canonical transformation

$$\big(A,\alpha\big) \to \big(S, J_1, \ldots, J_{N-1}, s, \phi_1, \ldots, \phi_{N-1}\big) \tag{1.2}$$

with $s = \bar{m}\cdot\bar{\alpha}$, $\bar{m} \in \mathbb{Z}^N$, $|\bar{m}| \neq 0$, such that, in the new variables $\big(S, \bar{J}, s, \bar{\phi}\big)$ the Hamiltonian assumes the form

$$K = K_0(S,\bar{J}) + \epsilon K_1(S,s,\bar{J}) \tag{1.3}$$

Suppose now that for $S = 0$ we have $\frac{\partial K_0}{\partial S} \approx 0$. Rescaling the variables (S,s) through the canonical transformation

$$S = m\Phi \tag{1.4}$$

$$s = \frac{\phi}{m} \tag{1.5}$$

and performing a Taylor expansion, we get the Hamiltonian

$$K = K_0(0) + m\,K_0'(0)\Phi + \frac{m^2}{2}K_0''(0)\Phi^2 + \epsilon K_1(0)\cos\phi + \ldots \tag{1.6}$$

that is, for suitable constants β,γ,δ and neglecting higher order terms

$$K = \beta\Phi + \gamma\,\Phi^2 + \delta\,cos\phi \tag{1.7}$$

A straightforward translation, namely $\Phi - \frac{\beta}{2\gamma}$, leads to the usual form of pendulum Hamiltonian. With a bit of calculation, it is possibile, via suitable rescaling, to show that in fact the model is free of parameters,

$$K = \Phi^2 - \cos\,\phi \tag{1.8}$$

This is the form of the first fundamental model of resonance.

1.1.2 The second fundamental model

A Hamiltonian of the form (1.3) is said to be D'Alambertian (or to possess the D'Alambertian characteristic) if its expansion in Poincaré variables

$$x = \sqrt{2\,S}\,\cos s \quad y = \sqrt{2\,S}\,\sin s \tag{1.9}$$

is analytic at the origin. Formally, its amplitude function must be of the form

$$K_1(S, s) = a_1 S^{\frac{m}{2}} + a_2 S^{\frac{m+1}{2}} + \dots \tag{1.10}$$

with $m > 0$ and odd, in order to guarantee the analyticity in the variables x, y. A difference emerges with respect to the previous case: translating momentum and rescaling the angle would destroy the form of the expansion, so the D'Alambertian property would fail. In order to preserve such a property, we can expand directly around $S* = 0$, getting the Hamiltonian

$$K = b\,S^2 + a\,S + \epsilon\,c\,(2S)^{\frac{m}{2}}\cos m\,s \tag{1.11}$$

This defines a family of fundamental models, according to the integer m, and we can refer to it as the family of second fundamental models, SFMm. A detailed description of the cases $m > 1$ can be found in [Fer07]. In the sequel, we will focus instead in the case $m = 1$, which naturally arises in problems of celestial mechanics, pertinent to the so-called "first order mean motion resonances" (see [Fer07]). As in the case of pendulum, it is always possible to proceed with time and length rescaling in order to reduce the number of parameters, up to one[HL83]. Also, varying the parameter from negative to positive produce a so-called saddle-node bifurcation.

1.2 Andoyer Hamiltonian

The Hamiltonian describing the second fundamental model of resonances is the so-called Andoyer Hamiltonian. As in the case of pendulum Hamiltonian, an analytic solution of the equations of motion is possible, along the separatrices of the model (1.11). We briefly discuss its features in the following paragraphs.

1.2.1 Critical points and bifurcations

Consider the Andoyer Hamiltonian

$$K = b\,S^2 + a\,S + c\,\sqrt{2S}\cos s \tag{1.12}$$

Using Poincaré variables

$$x = \sqrt{2S}\ \cos\ s \qquad y = \sqrt{2S}\ \sin\ s \tag{1.13}$$

allows to investigate the position of critical points and detect the presence of bifurcations, to varying the value of the parameters. Without loss of generality we can always assume $b > 0$, $c > 0$. In Poincaré variables the Hamiltonian reads

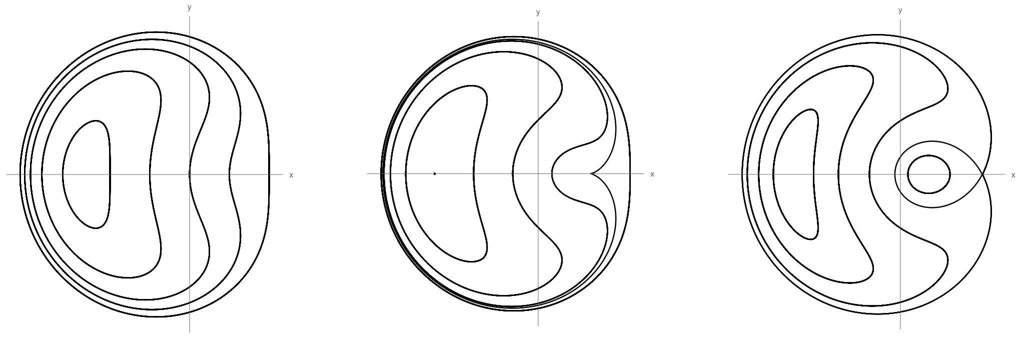
$$H = \frac{a}{2}(x^2 + y^2) + \frac{1}{4}b\,(x^2 + y^2)^2 + cx \tag{1.14}$$

so critical points lay on the x-axis, with abscissas given by the solutions of the cubic equation

$$a\,x + b\,x^3 + c = 0 \tag{1.15}$$

This equation has one, two, or three real solutions, according to the value of a, b and c, that in principle may be expressed via classical algebraic formulas. However, it is possible to find a boundary value a^* imposing the critical point of the polynomial, $x = \sqrt{-\frac{a}{3b}}$, to be a root of the polynomial. Namely

$$a\Big(-\frac{a}{3b}\Big)^{\frac{1}{2}} + b\Big(-\frac{a}{3b}\Big)^{\frac{3}{2}} + c = 0 \tag{1.16}$$



(a) $a = -1$ (b) $a = a^* = -\frac{3}{2}\sqrt[3]{2}$ (c) $a = -2.5$

Figure 1.1: An illustration of the sequence of bifurcations in the phase space, for different values of a and fixed b, c.

implies

$$a = a^* = -\frac{3}{2}\sqrt[3]{2b\,c^2} \tag{1.17}$$

In summary:

- if $a > a^*$ there is one equilibrium. It is stable and the motion is a rotation with a noise.
- if $a = a^* = -\frac{3}{2}\sqrt[3]{2b\,c^2}$ there are two equlibria: one is stable and one is a degenerate saddle-point.
- if $a < a^*$ there are three equilibria. The cusp splits in a stable point and a saddle. Two homoclinic curves stem from the saddle point and inviduate a *resonance* zone.

With simple calculations one finds that the saddle lays in the interval $\left(\sqrt{-\frac{b}{3a}}, \sqrt{-\frac{b}{a}}\right)$.

1.2.2 Integration along the separatrix

It is possible to integrate explicitly the equations of motion, exploiting the properties of Weierstrass elliptic function. Following [Fer07], the preliminary step consists of a reparametrization of the system, reducing the equations of motion to a one-parameter problem. Fixing the constants

$$K^* = \frac{1}{4}\, b\left(\frac{4c}{b}\right)^{\frac{4}{3}}, \quad \alpha = \frac{a}{a^*} \tag{1.18}$$

and after the rescaling

$$H = \frac{K}{K^*}, \quad J = S\left(\frac{b}{4c}\right)^{\frac{2}{3}}, \quad t' = -\frac{1}{3}\, a^*\, t \tag{1.19}$$

the equations read

$$H = -3\alpha\, J + 4J^2 + \sqrt{2J}\cos\, s \tag{1.20}$$

and

$$\dot{J} = \sqrt{2J}\sin\, s \tag{1.21}$$

$$\dot{s} = -3\alpha + 8J + \frac{1}{\sqrt{2J}}\, \cos\, s \tag{1.22}$$

Combining these equations with the conservation of energy, $\big(H = H_1 = \text{constant}\big)$, the problem reduces to a straightforward integration and an inversion

$$t' - t'_0 = \int_{J_0}^{J} \frac{d\,J}{P(J)} \tag{1.23}$$

where $P(I)$ is a quartic polynomial

$$P(I) = \left(\frac{d\,J}{d\,t'}\right)^2 = 2\,J - (H_1 + 3\alpha\,J - 4\,J^2)^2 \tag{1.24}$$

The inversion of this integral is given by an appropriate Weierstrass elliptic function (see [WW96] for a complete review)

$$J - J_0 = \frac{1}{4}\, P'(J_0)\Big(\wp(t' - t'_0,\, g_2,\, g_3) - \frac{1}{24}\, P''(J_0)\Big)^{-1} \tag{1.25}$$

where

$$g_2 = \frac{1}{3}\left(8H_1 + \frac{9}{2}\alpha^2\right)^2 - 12\alpha \tag{1.26}$$

$$g_3 = \frac{1}{27}\left(8H_1 + \frac{9}{2}\alpha^2\right)^3 - 2\alpha\,(8H_1 + \frac{9}{2}\,\alpha^2) + 4 \tag{1.27}$$

In order to study the solution of $P(J) = 0$ one introduces the discriminant $\Delta = g_2^3 - 27\,g_3^2$. The sign of Δ separates the zone of *librations* from the zone of *circulations*, namely, the inner and the outer part with respect to the separatrix homoclinic curves. The value $\Delta = 0$ corresponds to the solution of the separatrix. In this case the integral turns out to be pseudo-elliptical: the solutions can be written via trascendental functions.
In fact, for $\Delta = 0$ we get

$$\wp(t' - t'_0) = c + \frac{3c}{\sinh^2\left(\sqrt{3c}(t' - t'_0)\right)} \tag{1.28}$$

with

$$c = -\frac{3\,g_3}{2\,g_2} \tag{1.29}$$

Thus, in view of (1.25), the solution asymptotic to the saddle is

$$J - J_0 = \frac{A\,\sinh^2\left(\sqrt{3c}(t' - t'_0)\right)}{1 - B\,\sinh^2\left(\sqrt{3c}(t' - t'_0)\right)} \tag{1.30}$$

where

$$A = \frac{P'(J_0)}{12c} \tag{1.31}$$

$$B = \frac{P''(J_0)}{72c} - \frac{1}{3} \tag{1.32}$$

For $\Delta = 0$, $P(J) = 0$ has three solutions: the doubly-degenerate one gives the position of the saddle, while the lowest and the greatest individuate the intersection between, respectively, the inner and the outer separatrix with the x-axis. Since all the parameterizations are explicit, eventually one can always proceed backward getting an expression in terms of original parameters a, b and c. It remains to calculate the explicit expression for the motion of the angle $s(t)$. It can be recovered by straightforward derivation from the equation of motion for J:

$$\sin\, s = \frac{1}{\sqrt{2J}} \cdot \frac{\mathrm{d}J}{\mathrm{dt}'} = \frac{\mathrm{d}}{\mathrm{dt}'}\sqrt{2J} \tag{1.33}$$

2 Jeans-Landau-Teller approximation

As pointed out in [GEP19], in certain systems the evolution of the adiabatic action variables does not proceed uniformly. Rather, the slow-time diffusion motion is produced by subsequent "jumps" in correspondence with the resonant action-angle pair completing a sequence of succesive homoclinic loops. We will discuss numerical examples of this behavior in chapter 4. In the present chapter, instead, we discuss the so-called "Jeans-Landau-Teller approximation", which can be exploited in order to arrive at analytical estimates of the size of the typical jump. Exploiting the explicit expression of the motion along the separatrices in the second fundamental model, it is possible to express the one-period dynamics of adiabatic variables by means of some Melnikov-type integrals. In particular, profiting of the simmetries of the systems (namely, the D'Alambertian character of the Hamiltonian), we develop a Fourier decomposition of each Melnikov integral appearing. We deal with such integrals in the chapter 3, via a classical stationary-phase approach.

2.1 Landau-Teller approximation

In their celebrated paper [Lan36] about the theory of sound dispersion, Landau and Teller implemented a rigorous method to derive an exponential law, $\Delta E \approx Ke^{-a\omega}$, for energy exchanges between vibrational and rotational molecular degrees of freedom. Later on, this method was revised by Rapp [Rap60] and by Benettin, Carati et al. [BCS93], for studying the evolution of adiabatic invariants, for example, in a system of coupled rotators [BCF97]. Following this pathway we propose a similar approach, leading to analytical estimates for the evolution of action-variables as forced by the terms in the remainder of the Hamiltonian. The main idea of our approach consists of substituing the motion *near* the separatrices produced by the non-linear resonance with the motion along the separatrices. A similar idea, in the case of pendulum resonance model, was proposed by Chirikov [Chi79]. Our work can be interpreted as an extention of this idea, in the case of the second fundamental model of resonance.

2.1.1 Impulsive homoclinic dynamics

We consider a general Hamiltonian model with three degrees of freedom of the form

$$H(S, J_1, J_2, \sigma, \alpha_1, \alpha_2) = H_0(S, \sigma) + K_1(S, J, F_1, J_2) + \\ + R(S, J_1, J_2, \sigma, \alpha_1, \alpha_2) \tag{2.1}$$

where

$$H_0 = aS + bS^2 + c\sqrt{2S}\cos\sigma \tag{2.2}$$

$$K_1 = \eta_1 S J_1 + \eta_2 S J_2 + \eta_3 J_1 J_2 + \gamma_1 J_1 + \gamma_2 J_1^2 + \dots \tag{2.3}$$

and

$$R = \sum_{\bar{n}=(l,m,n)} C_{l,m,n}(\sqrt{S}, J_1, J_2) \mathrm{e}^{i\bar{n}\cdot\Phi}, \qquad \Phi = (\sigma, \alpha_1, \alpha_2) \tag{2.4}$$

The remainder function R contains the coupling terms producing the dynamics of the variables J_1, J_2.
The dynamics produced by these Hamiltonians can be qualitatevely predicted from the one-dimensional dynamics of the second fundamental model. For suitable values of the initial data, the six-dimensional phase space is characterized by the presence of a hyperbolic point. The position of this unstable point is indeed predicted by the same condition undergoing for the one dimensional Andoyer Hamiltonian H_0.
For an initial datum near the unstable point, that is, inside the stochastic layer produced by the splitting of the separatrices, the dynamics has an impulsive behavior. It remains very close to the unstable equilibirum for a long time, then, following

the unstable manifold stemming from the hyperbolic point, the dynamics perform a relatively fast pulse. Eventually, the orbit comes back towards the saddle, driven by the stable manifold. Thus, the motion is quasi-periodic, since the process repeats with a stochasticity due to the varying of the starting point. The clue of this analysis is to substitute such a quasi-periodic motion with the exact solution along the separatrices of the Andoyer Hamiltonian.
This approximation is meaningful provided that the perturbation due to coupling terms is small. In the example we present in the results of chapter 4, this is assured by the theory of solar sistem dynamics.
The presence of coupling terms depending on α_1 and α_2 forces the evolution of J_1 and J_2. If, as we are assuming, such perturbation is small, the evolution can be considered *slow*. This fact descents directly from the hamiltonian motion equations

$$\dot{J}_k = \frac{\partial}{\partial \alpha_k} \sum_{\bar{n}=(l,m,n)} C_{l,m,n}(\sqrt{S}, J_1, J_2) \mathrm{e}^{i\bar{n}\cdot\Phi} \tag{2.5}$$

So, we refer to the variables J_1 and J_2 as the "adiabatic" variables. Such a terminology generally indicates an observable whose evolution is slower with respect to the dynamics involed in the system. The aim of this work is to provide an analytical estimate for their rate of dffusion.
We first deal with the impulsive evolution of the variable S, profiting of the explicit expression discussed in the chapter 1. Then, defining a mean period for the quasi-periodic near the separatrices, we discuss its Fourier representation, in the spirit of the Landau-Teller approach. This procedure leads to a Melnikov-type expression for the evolution of the adiabatic variables.

2.1.2 Mean period of circulation

The small perturbation, due to the presence of the coupling terms, produces a chaotic layer in the vicinity of the separatrices. A quite natural question is to characterize the finite period of a single circulation within the stochastic layer. The period circulation depends very sensitively on initial data, namely on the position within the layer; however, we can define a circulation mean period, parametrized by the initial energy. So, since by continuity we can assume that, for $\eta << 1$, the unstable equilibrium

(J^*_η, u_η) is close to the saddle point of the unperturbed fundamental model, the equation

$$H_0 + \eta K_1 = E(1+\kappa) \tag{2.6}$$

is solved for $\eta = 0$, to obtain the function

$$\sin\,\sigma = \sqrt{1 - \left(\frac{E(1+\kappa) - aS^2 - bS}{c\sqrt{2S}}\right)^2} \tag{2.7}$$

Using the equation of motion $\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\partial H}{\partial \sigma}$, and fixing E, the value of the energy at the saddle point, the half-period of a circulation in the stochastic layer can be estimated, for any $\kappa > 0$ by

$$\frac{T_\kappa}{2} = \int_{S_0}^{S^*} \frac{\mathrm{d}S}{c\sqrt{2S}\sin\,\sigma} = \tag{2.8}$$

$$\int_{S_0}^{S^*} \frac{dS}{\sqrt{2c^2S - \big(M(1+\kappa) - aS^2 + bS\big)^2}} \tag{2.9}$$

where S^* and S_0 are, respectively, the unstable equilibrium and the middle-point of the pulse. Considering the norm of the coupling terms as the effective energy perturbation, we can define the mean period of an homoclinic circulation.

Definition 2.1. Fixing $\kappa = \|R\|$, "mean period of circulation in the stochastic layer" is called the quantity

$$T_\kappa = \int_{S_0}^{S^*} \frac{2\,\mathrm{d}S}{\sqrt{2c^2S - \big(M(1+\kappa) - aS^2 + bS\big)^2}} \tag{2.10}$$

The latter integral can be evaluated via cumbersome elliptic functions or by numerical integration. However, even a straightforward approximation, suggested by Chirikov for the case of pendulum,

$$T \approx 2\frac{\log\left(\frac{32}{|w|}\right)}{\omega_0} \tag{2.11}$$

where w is the relative energy, turns out to be sufficiently precise, at least at an euristical level, for the estimate of the typical circulation time in the second fundamental model. In particular, the latter approximation is quite in agreement with numerical experiments, as shown in chapter 4.

Remark 2.2. The mean period $T_\kappa \to \infty$ as the size of the remainder terms goes to zero. That is, we recover the asymptotical evolution along the separatrices.

2.1.3 Fourier analysis of the pulse

In (2.1) we defined a mean period for a circulation in the stochastic layer, while in earlier paragraphs we discussed the impulsive dynamics within the homoclinic layer. We describe this dynamics with the pulse along the separatrix of the second fundamental model. An explicit expression for this function is given in (1.30). Fixing T_κ, it is possible to represent the pulse along the separatrices via its Fourier decomposition. Namely

$$S_j = \frac{1}{2T}\int_0^T S(t)\, e^{-ij\Omega t}\,\mathrm{dt}, \qquad \Omega = \frac{2\pi}{T_\kappa} \tag{2.12}$$

However, as shown in the following paragraphs, we are more interested in the cartesian representation of the pulse. So, introducing Poincaré variables, namely

$$x(t) = \sqrt{2S}\cos\sigma \tag{2.13}$$

we model the pulse with a Gaussian function, following the idea in [BCF97],

$$\tilde{x}(t) = x_0 + \mathcal{C}\mathrm{e}^{-\delta^2 t^2} \tag{2.14}$$

In particular, the parameters x_0, δ and τ are fixed such that:

- the asymptotical limit is $\tilde{x}(t) \to x^*$, the coordinate of the saddle point.
- the second derivatives at the top of the pulse coincide.
- the height of the peaks coincide.

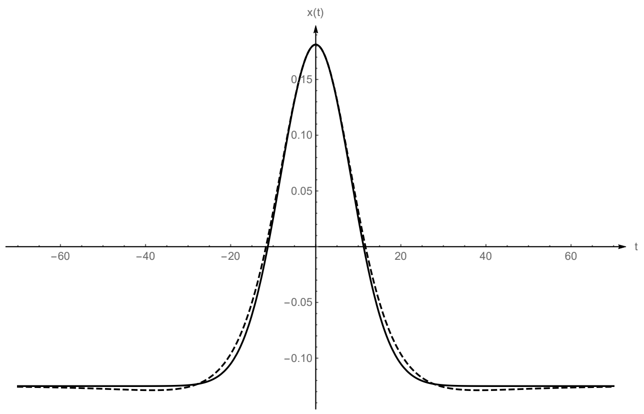


Figure 2.1: The homoclinic pulse, dashed, modellized by a Gaussian function, in black.

Hence, the Fourier coefficient of $x(t)$ can be estimated, analytically computing standard Gaussian integrals of the form

$$X_j = \frac{1}{2T_\kappa} \int_{-\frac{T_\kappa}{2}}^{\frac{T_\kappa}{2}} \tilde{x}(t)\, \cos(j\Omega t)\mathrm{d}t \tag{2.15}$$

That is, after a straightforward calculation

$$X_j = \mathcal{C}\frac{\sqrt{\pi}}{\delta}\mathrm{e}^{-\frac{(\Omega j)^2}{4\delta^2}}\,\mathrm{erf}\left(\frac{\delta\, T}{2}\right) \qquad \text{for } j > 1 \tag{2.16}$$

Remark 2.3. The latter calculation assures the fast decay of Fourier coefficient as the wave number j grows up. Such a behavior of Fourier coefficient of smooth function may be predicted by Paley-Wiener theorem, see [Tre96] for details. This fact allows us to introduce a cut-off in the analysis of the Fourier decomposition.

2.2 One-period adiabatic evolution

We now exploit the Landau-Teller approach for describing, via Melnikov integrals, the one-period evolution of J_1 and J_2.

2.2.1 Melnikov integrals

In the context of the Landau-Teller approach, the one-period evolution of the adiabatic actions can be deduced by a straightforward integration of the Hamiltonian equations. Thus, the evolution of adiabatic variables F_j forced by the coupling terms, rigorously estimated by Melnikov-type integrals.

Remark 2.4. The real evolution $S(t)$ is substitued by the a finite time cut-off of the asymptotical pulse along the separatrices of the fundamental model,

$$S(t) = S(0) + \frac{A\sinh^2(t)}{1 - B\sinh^2(t)} \tag{2.17}$$

The cut-off time is defined by the mean circulation period T_κ.

Remark 2.5. Assuming $J_1,\ J_2$ to be constant along an homoclinic loop and dropping out higher order terms of coupling, the normal form of the Hamiltonian produces the following motion equations for the angle variables,

$$\dot{\alpha_1} = \eta_1 S(t) + \eta_3 J_2 + \gamma_1 + 2\,\gamma_2 J_1 \tag{2.18}$$

and

$$\dot{\alpha_2} = \eta_2 S(t) + \eta_3 J_1 + \gamma_3 + 2\,\gamma_4 J_2 \tag{2.19}$$

Thus, the approximate evolution of the angle variables associated to adiabatic actions is given by two straightforward integrations, namely

$$\alpha_1(t) = \alpha_1(0) + \omega_2 t + \eta_1 W(t) \tag{2.20}$$

$$\alpha_2(t) = \alpha_2(0) + \omega_3 t + \eta_2 W(t) \tag{2.21}$$

where

$$W(t) = \int S(t)\,\mathrm{dt} = \frac{A}{B}\left(\frac{\arctan\left(\sqrt{1+B}\tanh(t)\right)}{\sqrt{1+B}}\right) \tag{2.22}$$

and

$$\omega_2 = \eta_3 J_2 + \gamma_1 + 2\,\gamma_2 J_1 + \eta_1 S(0) \tag{2.23}$$

$$\omega_3 = \eta_3 J_1 + \gamma_3 + 2\,\gamma_4 J_2 + \eta_2 S(0) \tag{2.24}$$

Thus, the one-period evolution of the adiabatic action variable is provided by several contributions described by Melnikov integrals, of the form

$$\Delta J_k = J_k(T) - J_k(0) =$$
$$= \sum_{l,m,n} \int_0^T \bar{n}_k, C_{l,m,n}\big(I(t)\big)\ \sin(l\sigma(t)\ +\ m\alpha_1(t)\ +\ n\alpha_2(t)) \tag{2.25}$$

2.2.2 Fourier decomposition of Melnikov integrals

In a wide class of systems, one can assume the perturbing function must respect the so-called D'Alambertian rules. In particular, it means that the coupling terms may be written in a polynomial form, profiting of the Poincaré variables

$$x = \sqrt{2S}\cos\,\sigma \tag{2.26}$$

$$y = \sqrt{2S}\sin\,\sigma \tag{2.27}$$

That is, the coefficient $C_{l,m,n}\big(\sqrt{2S}\big)$ of each Melnikov integrals in (2.25) can be expressed as a polynomial in $x(t)$, $y(t)$.

Example 2.6. Fixing l=3, we have

$$(2\,S)^{\frac{3}{2}}\ \cos(3\sigma + m\alpha_1 + n\alpha_2) =$$
$$(2S)^{\frac{3}{2}}\left[\big(\cos^3(\sigma) - 3\,\cos(\sigma)\,\sin^2(\sigma)\big)\ \cos(m\alpha_1 + n\alpha_2) + \right.$$
$$\left. + \big(\sin^3(\sigma) - 3\,\cos^2(\sigma)\,\sin(\sigma)\big)\ \sin(m\alpha_1 + n\alpha_2)\right] =$$
$$= \big(x^3 - 3x\,y^2\big)\ \cos(m\alpha_1 + n\alpha_2) + \big(y^3 - 3x^2y\big)\ \sin(m\alpha_1 + n\alpha_2)$$

So, exploiting this observation, in fact a direct consequence of the symmetries of the problem we are dealing with, we pass to a cartesian representation of equations.

Then, through trigonometric relations as those in the last example, we re-write the Melnikov integrals in the form

$$\Delta J_k = \sum_{l,m,n} \int_0^T \Big\{ \bar{n}_k \, C_{l,m,n}(J_1, J_2) \times \tag{2.28}$$

$$\times \big[P\big(x(t), y(t)\big) \sin\big(m\,\alpha_1(t) + n\,\alpha_2(t)\big) + D\big(x(t), y(t)\big) \cos\big(m\,\alpha_1(t) + n\,\alpha_2(t)\big) \big] \Big\} \, \mathrm{dt}$$

where P are suitable even polynomials and D are odd polynomials. Profiting of the Fourier decomposition (2.15), we proceed with a decomposition of each Melnikov integral, namely

$$\Delta J_k = \sum_{l,m,n} \int_0^T \Big\{ \bar{n}_k \, C_{l,m,n}(J_1, J_2) \times \tag{2.29}$$

$$\times \sum_j \big[P_j \sin\big(m\,\alpha_1(t) + n\,\alpha_2(t)\big) + D_j \cos\big(m\,\alpha_1(t) + n\,\alpha_2(t)\big) \big] \cdot \mathrm{e}^{\imath j \Omega t} \Big\} \mathrm{dt} \tag{2.30}$$

Further expanding trigonometric functions, eventually each Melnikov integral decomposes along the Fourier expansion of the homoclinic pulse

$$\Delta J_k = \sum_{m,n} \int_0^T \bar{n}_k \, C_{m,n}(J_1, J_2) \times$$

$$\times \sum_j \frac{1}{2} P_j \big[\sin\big(j\Omega t + m\alpha_1(t) + n\alpha_2(t)\big) - \sin\big(j\Omega t - m\alpha_1(t) - n\alpha_2(t)\big) \big] +$$

$$+ \frac{1}{2} D_j \big[\sin\big(m\alpha_1(t) + n\alpha_2(t) + j\Omega t\big) - \sin\big(m\alpha_1(t) + n\alpha_2(t) - j\Omega t\big) \big] \mathrm{dt} \tag{2.31}$$

Inspired by the analysis in [GEP19], in the following chapters we develop and implement a procedure capable to identify those terms responsible of the greatest contributions among all terms in 2.31. This methodology allows also to arrive at analytical estimates for the size of each contribution.

3 Analytical estimates

The aim of this chapter is to develop a methodology to identify the harmonics in the Fourier decomposition of the remainder that give the major contributions to the evolution of the adiabatic variables. This procedure leads quite naturally to an algorithm, inspired by [GEP19], one can implement in practical applications. The main analytical tool is based on the well-known principle of the stationary phase. The first part of the chapter is devoted to a review of classical asymptotical analysis through the stationary phase method. For a complete treatment of stationary phase asymptotical developments, one can refer for example to [BH86] or [Won01]. Moreover, such an approach produce naturally analytically estimates of all the contributions of Melnikov integrals. A similar approach to Melnikov integrals via stationary phase method, in the more usual context of separatrices splitting, is presented in [ELP19].

3.1 Principle of stationary phase

The principle of stationary phase applies to integrals of the form

$$\mathcal{I}(\lambda) = \int_a^b f(t)\,\mathrm{e}^{i\lambda g(t)}\mathrm{dt} \tag{3.1}$$

Since for $g'(t) \neq 0$, in the limit $\lambda >> 1$, rapid oscillations compensate each other, one can infer that major contributions to the integral 3.1 come from stationary points of the phase, i.e. points such that $g'(t_0) = 0$.

$$\mathcal{I}(\lambda) \approx \int_{t_0-\epsilon}^{t_0+\epsilon} f(t)\,\mathrm{e}^{i\lambda g(t)} \tag{3.2}$$

In the non-stationary case, the value of 3.1 simply depends on end points a, b. A straightforward integration by parts produces an asympotical expansion. The first step of the development comes from

$$\begin{aligned}\mathcal{I}(\lambda) &= \int_a^b f(t)\,\mathrm{e}^{i\lambda g(t)} \cdot \frac{g'(t)}{g(t')} = \\ &= \frac{\mathrm{e}^{i\lambda g(b)}}{i\lambda g'(b)} f(b) - \frac{\mathrm{e}^{i\lambda g(a)}}{i\lambda g'(a)} f(a) - \frac{1}{i\lambda}\int_a^b \mathrm{e}^{i\lambda g(t)} \cdot \frac{\mathrm{d}}{\mathrm{dt}} \frac{f(t)}{g'(t)} \mathrm{dt} \end{aligned} \tag{3.3}$$

But now the second part of 3.3 is again of the form 3.1. So iterating this procedure, higher order terms in $\frac{1}{\lambda}$ can be recovered. At order one, one gains an explicit estimate for the integral, namely

$$\mathcal{I}(\lambda) = \frac{\mathrm{e}^{i\lambda g(t)}}{i\lambda g'(t)} f(t)\Big|_a^b + \mathcal{O}(\frac{1}{\lambda}) \tag{3.4}$$

Suppose now the phase has a non-degenerate stationary point t_0 . For the sake of simplicity, let assume $g(t)$ has one stationary point, $t_0 \neq a$ and $t_0 \not{b}$. At an heuristic level, we can assume that the stationary point gievs the major contribution. A straightforward calculation then shows that

$$\begin{aligned}\mathcal{I}(\lambda) &\approx f(t_0) \int_{t_0-\epsilon}^{t_0+\epsilon} \mathrm{e}^{i\lambda\left[g(t_0)+\frac{g''(t_0)}{2}(t-t_0)^2\right]} \mathrm{dt} = \\ &= f(t_0)\sqrt{\frac{2\pi}{\lambda|g''(t_0)|}}\mathrm{e}^{i\left[\lambda g(t_0)\pm\frac{\pi}{4}\right]}\end{aligned} \tag{3.5}$$

according to sign of $g''(t_0)$. A rigorous deduction of a such estimates is presented in following paragraphs.

3.2 Estimates for Melnikov integrals

Definition 3.1. For any triplet of wave numbers of the remainder's Fourier decomposition (2.31)

$$\mathcal{I}_{j,m,n} := \int_0^T \sin\big(g_{j,m,n}(t)\big)\mathrm{dt} \tag{3.6}$$

with

$$\begin{aligned} g_{j,m,n}(t) &= j\Omega t + mu_2(t) + nu_3(t) \\ &= g_0 + \omega t + \mathcal{M}\, W(t) \end{aligned} \tag{3.7}$$

and

$$W(t) = \frac{A}{B}\left(\frac{\arctan\left(\sqrt{1+B}\tanh(t)\right)}{\sqrt{1+B}}\right) \tag{3.8}$$

Remark 3.2. It is worthwhile to notice that the constants ω and $\mathcal{M}$ can be derived directly from the coefficient of the hamiltonian model, as follows by equations (2.21). The coefficient A, B come from the integration of the fundamental model of resonance performed in (1.30).

3.2.1 Detection of stationary phases

The first lemma allows to characterize the stationary phases in terms of the mean period of circulation and the fixed parameters of the Hamiltonian.

Lemma 3.3. *Suppose we are given* $\Omega = \frac{2\pi}{T_\kappa}$, ω_2 *and* ω_3 *fixed and*

$$u_k(t) = u_k(0) + \omega_k t + \eta_k W(t) \tag{3.9}$$

Then the phases $g_{j,m,n}(t)$*, whose wave-numbers satisfy*

$$\frac{\omega}{-\omega B - \mathcal{M}\, A} > 0 \tag{3.10}$$

have two stationary points, symmetric with respect to $\frac{T_\kappa}{2}$*.*
Here

$$S(t) = \frac{A\sinh^2(t)}{1 + B\sinh^2(t)} \tag{3.11}$$

is the analytic expression of the homoclinic pulse and $\frac{dW}{dt} = S(t)$*.*

Proof. The stationary phase points are solutions of

$$\frac{\mathrm{d}}{\mathrm{dt}} g_{j,m,n}(t) = \omega + \mathcal{M}\, S(t) = 0 \tag{3.12}$$

Now, $S(t)$ is an even-simmetric impulsive function, so it is clear that the solutions of equation 3.12 must be simmetric, as pointed out by figure 4.1. Thus, the number of solutions depend just on the constant value of ω. A straightforward computation allows to identify the critical values.

$$\omega + \mathcal{M}\,\frac{A\sinh^2(t)}{1+B\sinh^2(t)} = 0 \iff \sinh^2(t) = \frac{\omega}{-\omega B - \mathcal{M}A} \tag{3.13}$$

That is, the equation has two solution if and only if

$$\frac{\omega}{-\omega B - \mathcal{M}A} > 0 \tag{3.14}$$

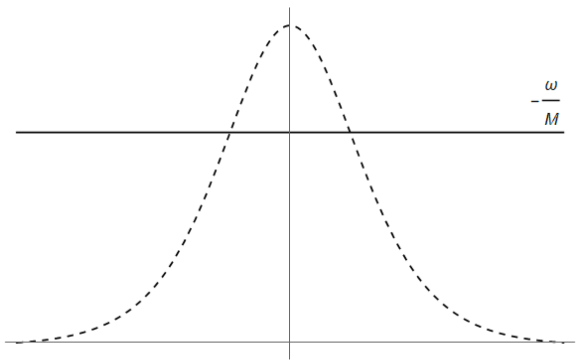


Figure 3.1: A graphical solution of the equation 3.12

□

3.2.2 Analytical estimates

We consider, at the outset, integrals whose phases are non-stationary. According to the stationary phase principle, these provide negligible contributions.

Proposition 3.4. *Consider an integral of the form $\mathcal{I}_{j,m,n}$, such that its phase function does not satisfy the condition in (3.3), namely such that*

$$\frac{\omega}{\omega B - \mathcal{M}A} < 0 \tag{3.15}$$

Then,

$$\mathcal{I}_{j,m,n} = -P_j \frac{\cos\left(g_{j,m,n}(t)\right)}{\omega + \mathcal{M}W(t)}\Bigg|_0^T + \mathcal{O}\left(\frac{1}{\omega^2}\right) \tag{3.16}$$

Proof. Since condition (3.15) assures g'(t) to be non-zero in the whole integration path, the thesis follows from a straightforward integration by parts

$$\begin{aligned} \mathcal{I}_{j,m,n} = \int_0^T \sin(g_{j,m,n}(t))\mathrm{dt} = \\ = -\frac{\cos(g_{j,m,n}(t))}{g'_{j,m,n}(t)}\Bigg|_0^T + \int_0^T \frac{\cos(g_{j,m,n}(t))\, S'(t)}{\big(\omega + S(t)\big)^2}\,\mathrm{d}t \end{aligned} \tag{3.17}$$

But since $S > 0$,

$$\left|\int_0^T \frac{\cos(g_{j,m,n}(t))\, S'(t)}{\big(\omega + S(t)\big)^2}\right| < T \cdot \frac{K}{\omega^2} \tag{3.18}$$

where

$$K = \max|S'(t)| \tag{3.19}$$

□

For the estimate of stationary-phase integrals, we need two lemmas we come to state. The proofs are straightforward calculations of complex-variable integrals. We refer to [Olv97] for the details.

Lemma 3.5. *If* $0 < \alpha < 1$ *and* $x > 0$*, then*

$$\int_0^\infty e^{ix\nu}\nu^{\alpha-1}\,d\nu = \frac{e^{\frac{\alpha\pi i}{2}}\Gamma(\alpha)}{x^\alpha} \tag{3.20}$$

Proof. It follows by the clockwise integration along the path in figure 4.2, letting $r \to 0$ and $R \to \infty$.

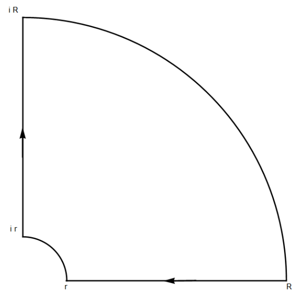


Figure 3.2: The path for the integration in 3.20

□

Lemma 3.6. *If $\alpha < 1$ and $\beta > 0$, then*

$$\int_{\beta}^{\infty} e^{ix\nu}\nu^{\alpha-1}\, d\nu = \mathcal{O}(\frac{1}{x}) \tag{3.21}$$

Proof. It suffices to integrate by parts

$$\left|\int_{\beta}^{\infty} \mathrm{e}^{ix\nu}\nu^{\alpha-1}\, d\nu\right| = \left|\left[\frac{\mathrm{e}^{ix\nu}}{ix}\nu^{\alpha-1}\right]_{\beta}^{\infty} - \frac{\alpha-1}{ix}\int_{\beta}^{\infty} \mathrm{e}^{ix\nu}\nu^{\alpha-1}\, d\nu\right| \leq$$
$$\leq \frac{2\beta^{\alpha-1}}{x} \tag{3.22}$$

□

Exploiting the last computations, we provide a estimate for the integrals with a stationary phase. The proof in the general case, is based on the so-called neutralization, see [Olv97].

Proposition 3.7. *Fix $\epsilon > 0$ and suppose*

$$0 < \omega < -\frac{\mathcal{M}A}{B} \tag{3.23}$$

then $g_{j,m,n}(t)$ has two non-degenere stationary points $t_0^{1,2}$.
Moreover, if ω is bounded away from zero, $|\omega| > \epsilon$, then

$$\mathcal{I}_{j,m,n} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{O}\left(\frac{1}{\omega}\right) \tag{3.24}$$

where

$$\mathcal{I}_k = \sin\left(g_{j,m,n}(t_0^k) \pm \frac{\pi}{4}\right) \cdot \sqrt{\frac{2\pi}{|g''_{j,m,n}(t_0^k)|}} \tag{3.25}$$

Proof. From lemma 3.3, it is clear that if (3.23) holds, then $g_{j,m,n}$ has two stationary points. In particular, a straightforward calculation assures they are non-degenere, or of order one. So, for $t \to t_0$, one has

$$g(t) - g(t_0) \sim G(t - t_0)^2 \tag{3.26}$$

The idea is to split the integral, namely

$$\mathcal{I}_{j,m,n} = \left[\int_0^{t_0^1} + \int_{t_0^1}^{\frac{T}{2}} + \int_{\frac{T}{2}}^{t_0^2} + \int_{t_0^2}^{T}\right] \sin\big(g_{j,m,n}(t)\big)\mathrm{d}t \tag{3.27}$$

such that in each interval we can assume the derivative goes to zero just at one extreme of integration.
So, we are dealing with an integral of the form

$$\int_a^b \mathrm{e}^{ix\,p(t)}\mathrm{dt} \tag{3.28}$$

such that, without loss of generality,

- for $t \to a^+$, $p(t) - p(a) \sim P(t-a)^\mu$
- $p'(t) > 0$ in the interior of the range of integration

So, introducing the change of variable

$$\nu = p(t) - p(a) \tag{3.29}$$

we find

$$\mathcal{I}(x) = \int_a^b \mathrm{e}^{ix\,p(t)}\mathrm{dt} = \mathrm{e}^{ixp(a)} \int_0^\beta \mathrm{e}^{i\,x\nu}\, f(\nu)\mathrm{d}\nu \tag{3.30}$$

where

$$\beta = p(b) - p(a) \tag{3.31}$$

and

$$f(\nu) \sim \frac{\nu^{\frac{1-\mu}{\mu}}}{\mu P^{\frac{1}{\mu}}} \tag{3.32}$$

as $t \to a^+$ (that is, $\nu \to 0^*$).

It is worthwhile to notice that since the derivative of $p(t)$ is not zero in the whole range, the change of variable is in fact a one-to-one correspondence. Now we write

$$\mathcal{I}(x) = \frac{1}{\mu P^{\frac{1}{\mu}}} \left(\int_0^\infty \mathrm{e}^{ix\nu} \nu^{\frac{1}{\mu}-1} \mathrm{d}\nu - \int_\beta^\infty \mathrm{e}^{ix\nu} \nu^{\frac{1}{\mu}-1} \mathrm{d}\nu \right) = \frac{1}{\mu P^{\frac{1}{\mu}}} (\mathcal{A} + \mathcal{B}) \tag{3.33}$$

The first term can be computed through the lemma 3.5, namely

$$\mathcal{A} = \frac{\mathrm{e}^{\frac{\pi i}{2\mu}} \Gamma(\frac{1}{\mu})}{x^{\frac{1}{\mu}}} \tag{3.34}$$

The second term can be estimated by means of the lemma 3.6, giving an error of order $\mathcal{O}\left(\frac{1}{x}\right)$. Eventually, coming back to our function,

$$\begin{aligned} P &= \frac{1}{2} g''_{j,m,n}(t_0^{1,2}), \\ \mu &= 2 \\ x &= \omega \end{aligned} \tag{3.35}$$

and summing up all the contribution in 3.27, we deduce

$$\mathcal{I}_k = \sin\left(g_{j,m,n}(t_0^k) \pm \frac{\pi}{4} \right) \cdot \sqrt{\frac{2\pi}{|g''_{j,m,n}(t_0^k)|}} \tag{3.36}$$

where the sign of $\pm$ depends on the sign of the second derivative at the stationary points. □

Remark 3.8. By means of lemma 3.3, each Melnikov integral $\mathcal{I}_{j,m,n}$ lies in one of the two possible cases and thus can be evaluated by means of the last propositions. However, in the transitions between stationary and non-stationary phases, the reliability of such estimates must be discussed and it is maybe necessary to introduce an intermediate class of *quasi*-stationary terms. We will come back to this problem in the next chapter.

Remark 3.9. In any case, such a procedure is indeed useful, since it allows to reduce drastically the amount, in principle very large, of terms in 2.31 we have to compute. In fact, many of them, namely those with non-stationary phase, give negligible contributions that can be estimate via lemma 4.2, or simply ignored. This allows, for example, to cut significately the time required for numerical checks.

3.3 Analytical estimate for the speed of diffusion

Profiting of analytical tools we developed in the previous paragraphs, we come back to discuss Melnikov integrals, responsible of the evolution of adiabatic action variables. Recalling (2.31), we write, for example for k=2,

$$\Delta F_2 = \sum_{m,n,j} P_j C_{m,n}(F_2, F_3)\, \mathcal{I}_{j,m,n} =$$
$$\sum_{m,n} C_{m,n} \sum_{j} P_j \int_0^{T_\kappa} \sin\big(g_{j,m,n}(t)\big)\, \mathrm{d}t \tag{3.37}$$

Thus, for any fixed harmonic labelled by the couple m, n, we check condition 3.15 at the varying of the value of j.

Remark 3.10. The coefficients in the remainder $C_{m,n}$ drop down for the harmonic order $|m+n|$ getting larger. So, in the application, it is possible to fix a proper cut-off and consider harmonics with $|m+n| < L$.

It remains to deal with the constant coefficient P_J. As shown in (2.30), they result from the Fourier decomposition of polynomial in Poincaré variables $x(t)$, $y(t)$. Thus, they can be estimated by suitable combinations of the coefficient of the Fourier representation of the homoclinic pulse. Following (2.16), they decay exponentially fast,

$$P_j \sim b_j \mathrm{e}^{-c_j j^2} \tag{3.38}$$

Remark 3.11. Such an exponentially fast decay of coefficients P_j suggests to introduce another cut-off, in practical applications, and consider just the first terms in the devolpement. Namely, we fix $\hat{J}$ sufficiently large and we consider

$$\sum_{j=0}^{\hat{J}} \mathcal{I}_{j,m,n} \tag{3.39}$$

Eventually, it is possible to sum up the results of this discussion in following proposition.

Proposition 3.12. *The value of the jump of the the adiabatic action variable* F_2

$$\Delta F_2 = \sum_{m,n,j} P_j C_{m,n}(F_2, F_3)\, \mathcal{I}_{j,m,n} \tag{3.40}$$

is well-approximated, in the Landau-Teller approach, by

$$\Delta F_2 \approx \sum_{j,m,n} C_{m,n} e^{-c_j j^2} \big(\mathcal{E}_1 + \mathcal{E}_2\big) \tag{3.41}$$

where $\mathcal{E}_1$ *contains all the stationary-phase terms, estimated by*

$$\mathcal{I}_{j,m,n} = \sin\left(g_{j,m,n}(t_0^k) \pm \frac{\pi}{4}\right) \cdot \sqrt{\frac{2\pi}{|g''_{j,m,n}(t_0^k)|}} \tag{3.42}$$

and $\mathcal{E}_2$ *contains all the non-stationary-phase terms.*

4 Numerical results

In celestial mechanics, mean motions resonances (MMR) of celestial bodies play a central role and a variety of observations can be explained by the presence of resonances. A typical example is the presence of Kirkwood gap in the main asteroidal belt in our solar system (see [Moo96] or [FNM98] for a review), observed for the first time in 1866. These gaps in the distribution of the semi-major axis of main-belt asteroids arise exactly in correspondence of some resonances with Jupiter mean motion. However, such a mechanism of expulsion by resonance seems to be not generic. Indeed, groups of bodies can be observed in resonant zones: for example the so called Hilda's group is located at 3:2 resonance with Jupiter, that corresponds to a semi-major axis $a \approx 3.97$ AU. In this chapter we present numerical results on the slow chaotic evolution of the orbital elements in the case of the 3:2 asteroidal MMR, implementing the estimates computed in previous chapters.

4.1 Restricted three-body problem

The three-body problem is a cornerstone of celestial dynamics (see [MD99] for a complete review). Consider three bodies, say for example the Sun, Jupiter and a sufficiently small body, such that its gravitational potential does not affect the dynamics of the others, while itself being subject to their gravitational force. Newton's equations, with respect to a fixed reference system, are

$$m\ddot{\vec{R}} = -\frac{GMm}{||\vec{R} - \vec{R_\odot}||^3}(\vec{R} - \vec{R_\odot}) - \frac{Gm'm}{||\vec{R} - \vec{R}'||^3}(\vec{R} - \vec{R}') \tag{4.1}$$

$$m'\ddot{\vec{R}}' = -\frac{GMm'}{||\vec{R}' - \vec{R_\odot}||^3}(\vec{R}' - \vec{R_\odot}) - \frac{Gm'm}{||\vec{R} - \vec{R}'||^3}(\vec{R} - \vec{R}') \tag{4.2}$$

$$M\ddot{\vec{R}}_\odot = -\frac{GMm}{||\vec{R} - \vec{R_\odot}||^3}(\vec{R} - \vec{R_\odot}) - \frac{GMm'}{||\vec{R_\odot} - \vec{R}'||^3}(\vec{R_\odot} - \vec{R}') \tag{4.3}$$

where the symbol $\odot$ stands for the primary body (Sun) and the prime for the secondary body (say Jupiter). Passing to a heliocentric system, the equation for the negligible mass reads:

$$\ddot{\vec{r}} = -\frac{GM}{r^3}\vec{r} - \frac{Gm'}{\Delta^3}\vec{\Delta} - \frac{Gm'}{r'^3}\vec{r'} \tag{4.4}$$

where $\vec{r} = R - R_\odot$ and $\vec{R}' + \vec{\Delta} = \vec{R}$. Before moving to an hamiltonian description we fix unity of time and length in terms of the keplerian orbit of the primaries:

$$a_J = 1; \text{ the semi-major axis of Jupiter}$$

$$\eta_J = \sqrt{\frac{G(M+m')}{a_J^3}} = 1; \text{ the mean motion frequency}$$

$$\frac{Gm'}{G(M+m')} = \mu; \text{ the reduced mass parameter of Jupiter}$$

$$Gm' = \mu \qquad GM = 1 - \mu$$

The equations of motion follow from the Hamiltonian [Mor02])

$$H = \frac{\vec{p}^2}{2} - \frac{GM}{r} - \frac{Gm'}{\Delta} + \frac{Gm'}{r'^3}\vec{r'} \cdot \vec{r} = \frac{\vec{p}^2}{2} - \frac{1}{r} - \mu\left(\frac{1}{\Delta} - \frac{\vec{r'} \cdot \vec{r}}{r'^3} - \frac{1}{r}\right) \tag{4.5}$$

where, in the Hamiltonian, we distinguish the Keplerian part and the disturbing function R:

$$H_0 = \frac{\vec{p}^2}{2} - \frac{1}{r} \tag{4.6}$$

$$R = \mu\left(\frac{1}{\Delta} - \frac{\vec{r'} \cdot \vec{r}}{r'^3} - \frac{1}{r}\right) \tag{4.7}$$

At this point, it is useful to introduce an appropiate set of coordinates used to settle the position and the motion of a celestial body. Relative motion of bodies are commonly described by orbital parameters and the position of the orbital plane with respect to a fixed orthonormal frame of references. These spatial coordinates are called *orbital elements*. A detailed description can be found in [Gio] or in [Mor02]. The usual orbital elements (see figure 4.1 and 4.2) are

- a, the semi-major axis of the ellipse.
- e, the eccentricity of ellipse.
- ω, the argument of pericenter. It defines the angular position of the pericenter with respect to the ascending node.
- f, the true anomaly, or E, the eccentric anomaly. They define the position of the body along the orbit, or its projection on the circle tangent to the ellipse.
- i, the inclination of the orbital plane with respect to the fixed plane.
- Ω, the longitude of nodes. It orients the ascending node with respect to the reference fixed plane.

The well-known Kepler equation

$$\sqrt{\frac{G(M_\odot + m')}{a^3}} \cdot (t - t_0) = E - e \sin E \tag{4.8}$$

allows to determine all coordinates along the planar ellipse as a function of time. This can further parametrized in terms of the *mean anomaly*

$$M = \sqrt{\frac{G(M_\odot + m')}{a^3}} \cdot (t - t_0) \tag{4.9}$$

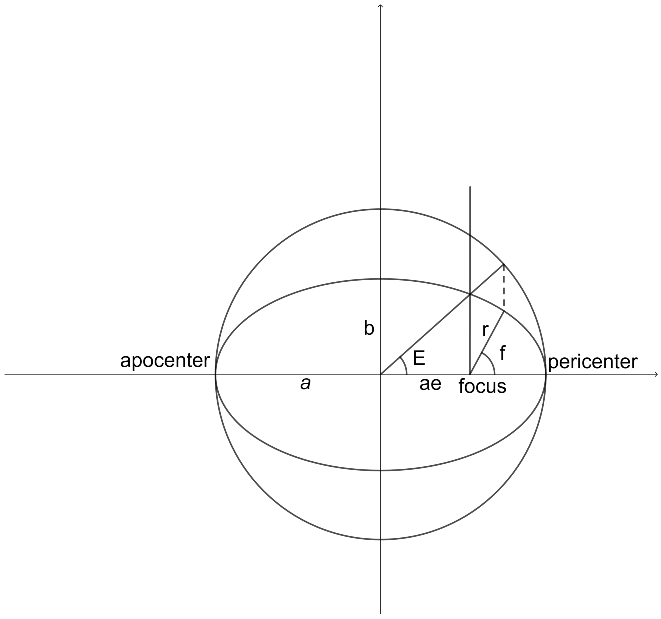


Figure 4.1: Definition of true anomaly and eccentric anomaly.

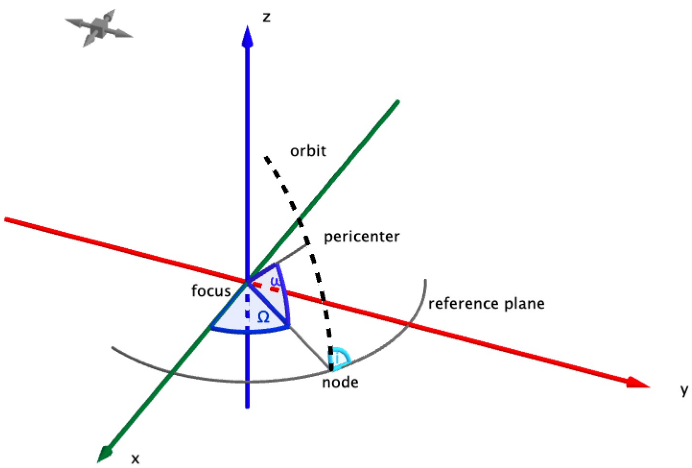


Figure 4.2: Spatial disposition of orbital plane described by inclination and longitude of nodes.

4.1.1 The disturbing function in Delaunay variables

A straightforward application of Arnold-Liouville-Jost theorem [Arn13] to the integrable two-body problem permits to define a set of action-angle variables depending on orbital elements. Since they present singularities, often they are replaced by the so-called *modified* Delaunay variables. The entire construction of these variables via Arnold-Jost theorem can be found, for example, in [Mor02]. The modified Delaunay variables are

$$\lambda = M + \omega + \Omega \qquad \Lambda = \sqrt{a} \tag{4.10}$$

$$-\bar{\omega} = \gamma = -\omega - \Omega \qquad \Gamma = \sqrt{a}(1 - \sqrt{1 - e^2}) \tag{4.11}$$

$$\theta = -\Omega \qquad \Theta = \sqrt{a}(1 - \sqrt{1 - e^2})(1 - \cos\, i) \tag{4.12}$$

These variables represent the most natural way to deal with N-body gravitational problems.

In order to arrive at an expression for the disturbing function in terms of the Delaunay action-angle variables, one usually expands the disturbing function in terms of the orbital elements i and e, considered small. See for example [MD99, chapter 6].

After the expansion in series, eventually one gets an expression of the very general form

$$R = \sum_{m,s} c(\Lambda, \Lambda') e^{s_1} (\sin\, i)^{s_2} e'^{s_3} (\sin\, i')^{s_4} \cos(m_1\lambda + m_2\lambda' + m_3\bar{\omega} + m_4\bar{\omega}' + m_5\Omega + m_6\Omega') 4.1.1 \tag{4.13}$$

The expression () obey the following rules, known as *D'Alambert rules.*

- R is invariant under rotation around z-axis. Since any rotation affects each angle, it must be $\sum_{i=1}^{6} m_i = 0$.
- a simultaneous change of sign of the inclinations does not affect the system, so $s_2 + s_4$ must be even.
- the Hamiltonian must be analtyic in Poincaré (cartesian) variables, say e.g. $x = e\,\cos\,\gamma$. In other words, $s_1 \geq |m_3|$ and they must have the same parity. Same restrictions go also for any s_i.

As we will see, this observations are very useful in practical computations involving the disturbing function. The problem naturally involves two timescale, namely a fast dynamics, represented by the mean motion, M, and a secular dynamics, much slower. By an averaging argument it is possible to remove, with a small error, all terms with fast evolution. Practically, one can pick up any term that has λ or λ' in the harmonics , taking care about non-resonance condition, since $\lambda \approx M$, namely the mean motion frequency. We discuss this fact in the next paragraph.

4.1.2 Mean motion resonances

A number of results and techniques in the theory of quasi-integrable systems, starting with the work of Lagrange and Laplace for orbital dynamics [Lap99], go under the name of *averaging methods* (see for example [Arn13] or [AKN07]). If a system contains an angular variable evolving much faster than the others, in principle it is possible to decompose the dynamics in fast oscillations (say, mean motions) and a slow evolution (say, secular evolutions). Performing a suitable time average, fast oscillations can be eliminated with a small enough error.
To fix ideas consider the following system with single periodic motion

$$\dot{\phi} = \omega(I) + \epsilon f(I,\phi), \qquad \phi \in S^1 \tag{4.14}$$

$$\dot{I} = \epsilon g(I,\phi), \qquad I \in U \subset \mathbb{R}^n \tag{4.15}$$

and the average

$$\bar{g}(J) = \frac{1}{2\pi}\int_0^{2\pi} g(J,\phi)d\phi \tag{4.16}$$

Following the averaging principle, the original evolution is substituted by the averaged evolution, namely

$$\dot{J} = \epsilon \bar{g}(J) \tag{4.17}$$

Under suitable hypothesis, see for example [Arn13], one can prove that, for small enough ϵ, in the time interval $0 \leq t \leq \frac{1}{\epsilon}$

$$|I(t) - J(t)| < C\epsilon \tag{4.18}$$

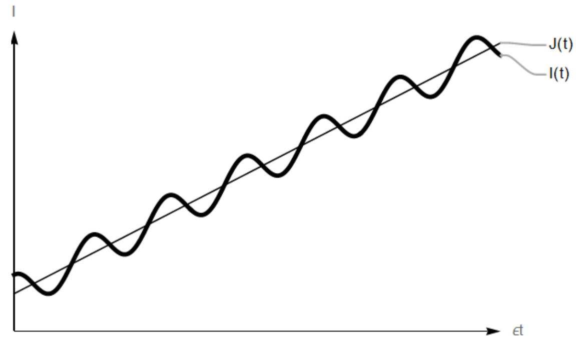


Figure 4.3: A toy illustration of the averaging principle.

Averaging methods can be applied in 2n-dimensional Hamiltonian systems, leading to rigorous statements. See for example the Anosov's Averaging Theorem [Ano60] or a possible extension in [DGL94]. Analogous results can be obtained introducing a normal form for the Hamiltonian via canonical perturbation theory, leading to so-called *secular normal form* (see [Mor02] for a review).
Recalling the general form of terms in the perturbing function

$$c(\Lambda, \Lambda')e^{s_1}(\sin\, i)^{s_2}e'^{s_3}(\sin\, i')^{s_4}\cos(m_1\lambda + m_2\lambda' + m_3\bar{\omega} + m_4\bar{\omega}' + m_5\omega + m_6\omega') \tag{4.19}$$

the frequencies of the mean motions, $\lambda \approx M$ and $\lambda' \approx M'$, correspond to the fast ones. So, in the study of the dynamics, the terms that depend on these angles can be formally ignoreted (or normalized). However, the resonant combinations of mean motions can not be normalized.
If mean motion frequencies are commensurable, namely

$$n_1\lambda + n_2\lambda' \approx 0 \tag{4.20}$$

for fixed integers n_1, n_2, their evolution must be considered as slow, and so it can not be eliminated by averaging arguments or by normalization processes. Since $\lambda \approx M$ represents the orbital mean motion, fixing a particular resonance corresponds to fixing the distance of the small body with respect to the orbital center (third Kepler's law). For example, considering the three-body restricted system Sun-Jupiter-asteroid, the 3:2 resonance corresponds to a semi-major axis of $a \approx 3.97\,\text{AU}$.
Summing up, two class of terms can appear:

- purely secular, terms that do not depend on λ and λ'
- resonant terms that depend on $2\lambda - 3\lambda'$ or multiple combinations.

while the other terms are ignoreted by averaging.

4.2 The Hamiltonian model

Before presenting the general form of the model, a proper change of variables is very useful to enlight some features of the problem. Via the second-type generating function

$$\mathcal{S} = (3\lambda' - 2\lambda + \gamma) \cdot J_R + (3\lambda' - 2\lambda + \theta) \cdot F_2 + (-3\lambda' + 2\lambda) \cdot F_3 + \lambda \cdot F_p \tag{4.21}$$

we introduce the following variables

$$\begin{aligned} \lambda &\to \frac{u_3}{2} + \frac{3u_p}{2} & d\Lambda &\to -2F_2 + 2F_3 - 2J_R \\ \gamma &\to u_3 + u_R & \Gamma &\to J_R \\ \theta &\to u_2 + u_3 & \Theta &\to F_2 \\ \lambda' &\to u_p & \Lambda' &\to 3F_2 - 3F_3 + F_p + 3J_R \end{aligned}$$

Remark 4.1. New variables have a straightforward interpretation in term of orbital elements. For example:

$$F_2 = \Theta = \sqrt{a}(\sqrt{1-e^2})(1 - \cos\, i) = \sqrt{a}\,\frac{i^2}{2} + \mathcal{O}_4 \tag{4.22}$$

at where $\mathcal{O}_4$ denotes terms of degree four in the eccentricity and inclination.

Thus, summing up, we have to study Hamiltonians of the form

$$H(J_R, F_2, F_3, u_R, u_2, u_3) = H_0(J_R, u_R) + K_1(J_R, F_1, F_2) + \tag{4.23}$$
$$+ K_2(J_R, F_2, F_3, u_R, u_2, u_3) \tag{4.24}$$

where

$$H_0 = aJ_R + bJ_R^2 + c\sqrt{2J}\cos\, u_R \tag{4.25}$$

is the second fundamental model of resonance,

$$K_1 = \eta_1 J_R F_2 + \eta_2 J_R F_3 + \eta_3 F_2 F_3 + \dots \tag{4.26}$$

are coupling terms and

$$K_2 = \sum_{k,j,l} C_{k,j,l}(\sqrt{J_R}, F_2, F_3) \cos(k u_R + j u_2 + l u_3) \tag{4.27}$$

are the terms coming from the expansion of perturbing function. Thus, we recover an Hamiltonian dealing with the general model (2.1.1). We propose to apply our methodology to this concrete example. Before doing that, a further consideration involving the so-called normal form of the Hamiltonian is to be discussed.

4.3 Canonical perturbation theory

For many inital conditions, the one-period evolution of the action variables is dominated by the so-called "deformation" effect, namely rapid oscillations around the drift motion, as shown for example in figure 4.4. In order to reduce deformation effects, we implement a standard Lie series normalisation algorithm. Regardless the optimal order in the perturbative analysis, actually a single step in normalising process resulted to be sufficient to identify clearly the feature of the dynamics, for a collection of initial data nearby the unstable point. We give here a short review about canonical perturbation theory via the Lie series method. A complete description can be found for example in [Fer07].

4.3.1 Lie series canonical transformation

The general aim of the canonical perturbation theory is the construction of suitable canonical transformations, such that the dynamics of the system in the new variables is somehow *simpler*. The idea, undergoing the so-called Lie series method, consists of producing a canonical transformation solving, by means of a Taylor series, the motion equations given by a suitable function χ. Indeed, as it is well known in the theory of hamiltonian systems, for every time t the flow Φ_t generated by a certain

Hamiltonian χ is a canonical transformation. Recalling the definition of the Poisson bracket operator

$$\mathcal{L}_\chi f = \frac{\mathrm{d}f}{\mathrm{d}t} = \{f, \chi\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial \chi}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \chi}{\partial q_i} \right) \tag{4.28}$$

the canonical flow generated by χ is formally expressed by the so-called Lie series operator

$$p(t) = \exp(t\mathcal{L}_\chi)p = p_0 + \left(\mathcal{L}_\chi p_0\right)t + \frac{1}{2}\left(\mathcal{L}_\chi^2 p_0\right)t^2 + \ldots \tag{4.29}$$

$$q(t) = \exp(t\mathcal{L}_\chi)q = q_0 + \left(\mathcal{L}_\chi q_0\right)t + \frac{1}{2}\left(\mathcal{L}_\chi^2 q_0\right)t^2 + \ldots \tag{4.30}$$

where the dots stand for higher order terms in the definition of the exponential operator

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{4.31}$$

Thus, the last expression defines an explicit canonical transformation, for every fixed t,

$$p = \exp(t\mathcal{L})p' \qquad q = \exp(\mathcal{L})q' \tag{4.32}$$

It is worthwhile to recall that, by Grobner exchange theorem, it is possible to express a function $f(q, p)$ in terms of the new variables (q', p') without performing substitutions. Rather, one can apply the exponential operator directly on the function, namely,

$$f(q, p)\big|_{p=\exp(t\mathcal{L})p',\, q=\exp(\mathcal{L})q'} = \exp(t\mathcal{L})f\big|_{p=p',\, q=q'} \tag{4.33}$$

It remains to explain how the choose of a suitable function χ is performed, by means of the so-called homological equation. Let consider, for the sake of simplicity, an Hamiltonian of the form

$$H = Z_0 + \lambda H_1 \tag{4.34}$$

and suppose we want to normalize it up to order one in the parameter λ. By the exchange theorem, we have

$$H' = \exp(\mathcal{L})H = Z_0 + \lambda H_1 + \{Z_0, \chi\} + \lambda\{H_1, \chi\} + \frac{1}{2}\{\{H, \chi\}, \chi\} + \dots \tag{4.35}$$

That is, just the second and the third term are of order one in λ. Thus, the following equation must be solved, in order to eliminate them. This is the so-called homological equation

$$\lambda H_1 + \{Z_0, \chi\} = 0 \tag{4.36}$$

It is possibile to iterate the procedure to get an higher order normal form for the Hamiltonian, taking care of the appeareance of small denominators produced by resonances. However, in the study of the model we are interested in, a first order normalisation turns out to be sufficient.

4.3.2 First order normalisation of the model

We apply a single step of the normalisation process to our model describing mean motion resonances in the restricted three-body system. So, considering the Hamiltonian discussed in the chapter 1

$$H = Z_0(I) + K_1(I, u_R) + K_2(I, \Phi) \tag{4.37}$$

where

$$Z_0 = \omega \cdot \vec{I}, \qquad \vec{I} = (J_R, F_2, F_3) \tag{4.38}$$

and

$$K_2 = \sum_{\bar{n}\,=\,(l,m,n)} C_{\bar{n}}(I)\, \cos \bar{n} \cdot \Phi, \qquad \Phi = (u_R, u_2, u_3) \tag{4.39}$$

and solving the homological equation

$$\{Z_0, \chi_1\} + K_2 = 0 \tag{4.40}$$

one produces an explicit canonical transformation generated by the flow of

$$\chi_1 = \sum_{\bar{n}} \frac{C_{\bar{n}}(I)\, \sin \bar{n} \cdot \Phi}{\bar{n} \cdot \omega} \tag{4.41}$$

That is, the new Hamiltonian, given by

$$H' = \exp(\mathcal{L}_{\chi_1})\, H = H + \{H, \chi_1\} + \frac{1}{2}\{\{H, \chi_1\}, \chi_1\} + \dots \tag{4.42}$$

takes the form

$$H' = Z_0(I') + K_1(I', u'_R) + R(I', \Phi') \tag{4.43}$$

where the terms of the remainder R are of the form

$$R = \sum_{\bar{n}} C'_{\bar{n}}(I') r_{l,m,n} \cos(\bar{n} \cdot \Phi') \tag{4.44}$$

It is worthwhile to observe that the coefficient $r_{l,m,n}$ are explicitly known as a result of the normalisation process. This fact assures that the size of the remainder is controlled. The Lie series method provides also straightforward formulas for computing the evolution of new variables in terms of the evolution of the old ones

$$I'_i = \exp(-\mathcal{L}_{\chi_1}) I_i, \qquad i = 1, 2, 3 \tag{4.45}$$

$$\Phi'_i = \exp(-\mathcal{L}_{\chi_1}) \Phi_i, \qquad i = 1, 2, 3 \tag{4.46}$$

In principle, after the normalisation process, the remainder terms is composed by a finite, but large, amount of terms. In the following, we will develope a methodology for picking up most relevant contributions that force the slow evolution of the adiabatic action variables.

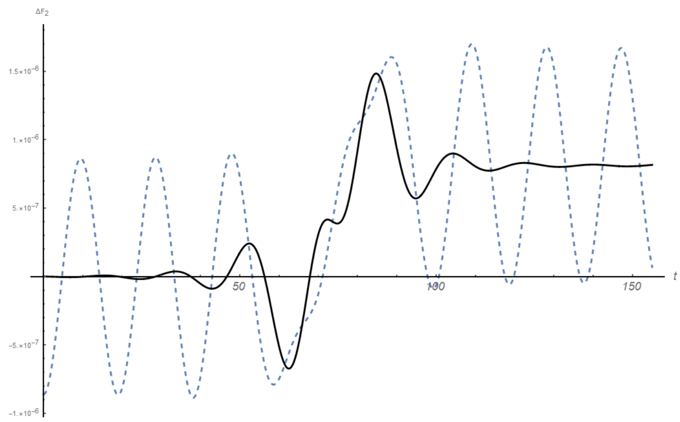


Figure 4.4: An example of the evolution of F_2 along an homoclinic loop, before and after one step of the normalisation process. The "jump" is clearly visible after one normalisation step.

4.4 Results

We now apply the methodology developed in the previous chapters to a realistic model arising from the study of mean motion resonances in the context of the well-known three-body restricted problem in celestial mechanics. For the description of the construction of such a model, we refer to chapter 1. In particular, we are interested in the 3 : 2 mean motion resonance, that coincides, in the Main Belt of asteroids, with the so-called Hilda's group.

We focus on the diffusive behavior of the action-variable F_2. Recalling the meaning of the coordinates we introduce in chapter 1 (see, 4.1), this variable is proportional to the inclination of the orbital plane. So, a spread in the value F_2 produces a spread also in the positions of the celestial body. In particular, large excursions of the dynamics in the phase space may modify significately the eccentricity of the orbits, so that the small body can approach the gravity center and be expulsed from the orbit. Thus, the knowledge of the speed of diffusion allows to predict, for example, the mean life of asteroidal groups within the Main Belt. Results of this type are obtained for example in [CCF10]. A detailed analysis about a possible theoretical explanation of Kirkwood gaps in the Main Belt by the diffusion along mean motion resonance in the restricted three-body problem can be found in [Fej+16].

In the first part of the paragraph we discuss, profiting of some numerical simulations,

the effective diffusive character for the long-time dynamics of F_2. In particular, we point out clearly the non-uniformity of the evolution. The evolution, rather, proceeds by kicks, as already highlighted in [GEP19]. This impulsive behavior is far to be a peculiarity of the model we are considering. On the contrary, the generality of such a character of the diffusion was predicted by Chirikov in [Chi79], suggesting the definition of a diffusion rate. We also discuss the essential difference elapsing between the circular and the elliptical problem, essentially due to some topological consideration involving KAM tori.
Then, we implement the methodology for computing the rate of diffusion we developed in chapters 3 and 4 and we compare the analytical estimates with numerical predictions.

4.4.1 Evidences of a slow chaotic diffusion

Already Poincaré showed that the separatrices associated to integrable systems near simple resonances split under the effect of small perturbation, that we can identify with the remainder in the normalised Hamiltonian. Within the so-called stochastic layer, the chaotic motion ensures an appreciable slow diffusion in the evolution of the adiabatic action variables. Apart from that, the presence of KAM tori renders the diffusion very slow, and then hard to detect numerically. A crucial observation regards the character of the diffusive evolution. In fact, the spread turns out to be non-uniform in time. Rather, the diffusion is forced by a periodic sequence of impulsive "kicks", in correspondence to each homoclinic loop. Assuming the diffusion to have a normal character, it is possible to deduce the typical excursion from the knowledge of the typical size of the one-period impulse (see [EH13], [KK89] for a discussion of the assumption of normal diffusion).
The motion in original variables is affected by the so-called *deformation* effect, that completely dominates over the feature of the jump. Thus, it is necessary to introduce new variables through a standard normalisation method, discussed in the chapter 4.3. Putting aside questions about order optimality, a single step in the normalisation process turns out to be quite sufficient to clearly identify the one-period jump in the one period evolution of F_2, as shown by figure 4.6 for a couple of homoclinic loops.

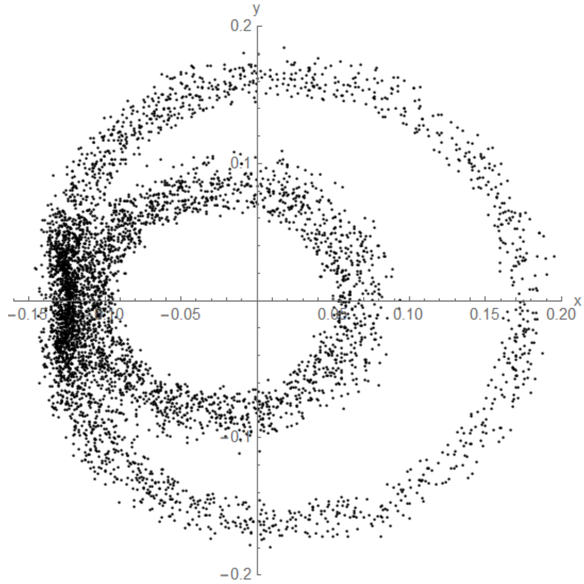


Figure 4.5: Poincarè section, at plane $u_2 = 0$, of the stochastic layer for the spatial elliptical problem.

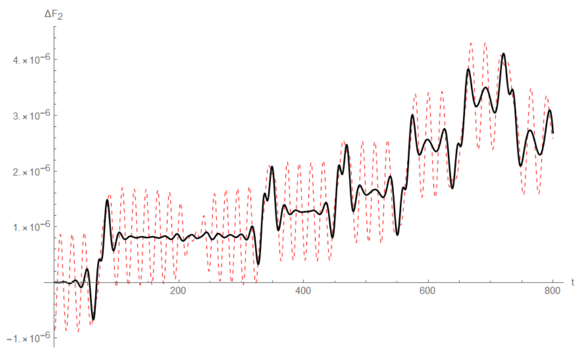


Figure 4.6: Numerical evolution of F_2 after eight homoclinic loops in the spatial elliptical RTBP. The dynamics proceed by subsequent impulses. The equation of motion are integrated in the original non-normalised variables. Then, through explicit formulas given by the Lie series normalising process, it is possible to achieve the normalised dynamics. The black line represents the dynamics after the normalisation.

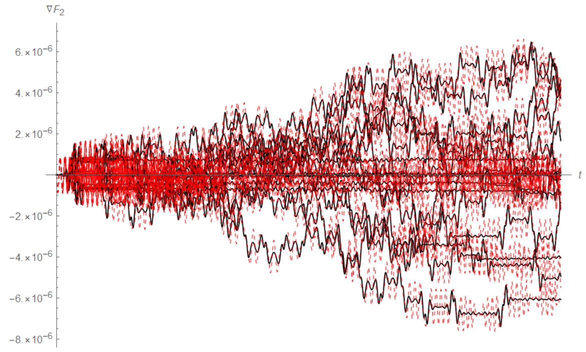


Figure 4.7: Diffusive behavior for a swarn (18) of initial data in the spatial elliptic model. Again, black lines stand for the normalised dynamics.

It is worthwhile to notice that, due to the ergodic nature of the dynamics inside the stochastic layer, the long-time evolution strongly depends on initial data. In particular, the dynamics is strongly affected by a change, even small, in initial phases, as shown in figure 4.8.

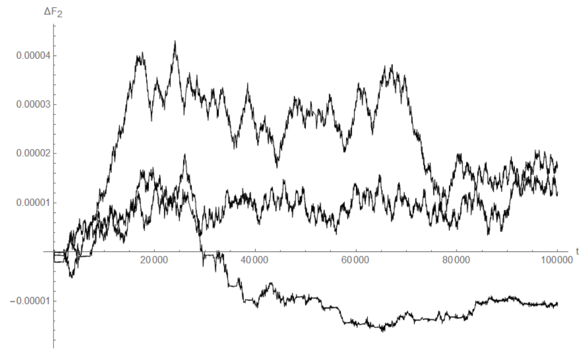


Figure 4.8: Long-time evolution of F_2 for three different initial phase $u_2(0)$.

4.4.2 Comparison between the circular and the elliptic RTBP

As known since the original work of Arnol'd [Arn64], in systems with two degrees of freedom, the motion in the phase space can not transit from one unstable zone to another one. In fact, the presence of two dimensional KAM tori acts as a topological barrier on diffusion. These topological barriers prevent significant excursions in a long time evolution. On the contrary, if a system has three or more degrees of freedom, the connection between unstable zones, surrounded by the KAM tori, in

principle allows large excursion for the evolution of adiabatic variables.
We illustrate such essential difference by means of figure 4.9. The circular model, obtained assuming the eccentricity of the primar body's orbit to be zero, is a two dimensional system (that is, F_3 is a constant of motion). On the contrary, the elliptic model has three degrees of freedom. As clearly shown by the figure, the evolution of F_2 in the circular model is clearly restricted and no large excursion seems to be possible.

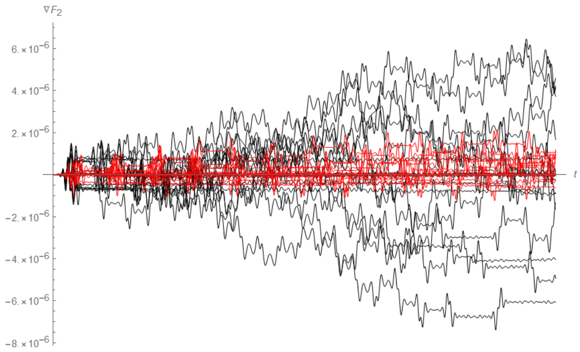


Figure 4.9: Diffusion of F_2 for a swarn of initial data. Red lines stand for the circular problem, black lines for the elliptic.

4.4.3 Jeans-Landau-Teller approach

In chapter 3 we deduced an analytical estimate for the value of ΔF_2 through suitable Melnikov integrals. The deduction, referred to as a Jeans-Landau-Teller approach, is based on the following assumption: it is possible, with a meaningful error, to substitute the motion in the stochastic layer with a proper cut-off of the asymptotic solution of the second fundamental model of resonance. The proper period, identified with the "mean circulation period" in 2.1 and depending on the size of the remainder, is in also good agreements with numerical observation. In this paragraph, we discuss the reliability of this approach by comparison with numerical experiments.

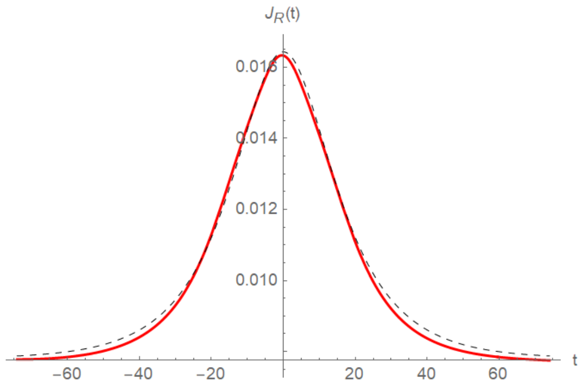


Figure 4.10: A comparison between the homoclinic pulse computed numerically, in red, and the analytical solution of the second fundamental model, dashed in black.

Figure 2.5 shows the level of error in dropping out the effect of the remainder on the one-loop evolution of the angle variables congiugate to the adiabatic actions F_2, F_3. Figure 4.11 shows the analytic prediction for the one-loop evolution of the action F_2 by the Jeans-Landau-Teller approach, compared with the numerical evolution.

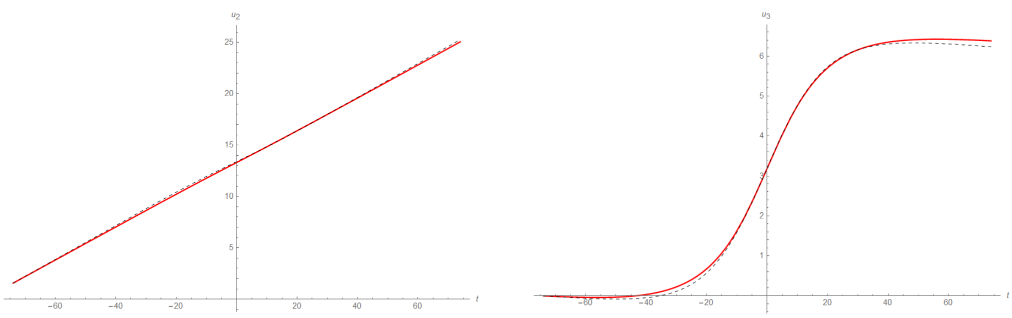


Figure 4.11: One-period evolution angle variables u_2, u_3 computed numerically, in red, and approximately deduced in the JLT approach, dashed.

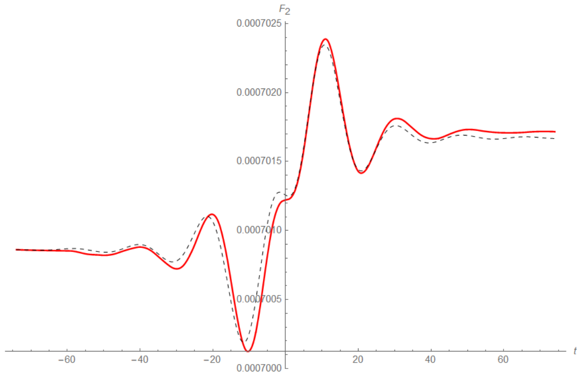


Figure 4.12: A typical jump of F_2 computed numerically, in red, and the same jump deduced by the JLT approach, dashed.

4.5 Semi-analytical estimates for the speed of diffusion

The following paragraphs are devoted to implementing the procedure, described in chapter 3 and 4, that leads to analytical estimates for the variation of action variables during a one-period evolution. That is, we implement an algorithm via an algebraic manipulator, such that:

1. for each fixed m, n, the correspondent Melnikov integral is decomposed on the Fourier series of the Poincarè representaiton of the pulse, producing the integral

$$\mathcal{I}_{j,m,n} = C_{j,m,n} \int_0^T \sin\left(j\Omega t + m u_2(t) + n u_3(t)\right) \quad \text{for j} = 0,\, 1 \ldots,\, \hat{J} \tag{4.47}$$

2. via the condition in lemma 3.3, the algorithm recognizes the stationary-phase terms,

3. depending on this condition, each integral is substitued with the suitable estimated given in propositions 3.4 and 3.7,

4. after a "term-by-term" numerical comparison, the procedure collects all the contributions to reconstruct the effective size of the jump.

It should be stressed that, since the phase depends linearly on j, stationary-phase terms appear for j sufficiently small. Thus, combining this observation with the exponentially fast decay of the coefficient of the Fourier representation, only a couple of integrals give in fact the real effective contribution. Thus, in principle, the analysis can be restricted to a small portion of the whole amount of integrals.

4.5.1 Spatial circular RTBP

In the case of non-planar circular problem, the system has two degree of freedom, since F_3 is a constant of motion. Even if, as discussed in (4.4.2), broad excursions for the long time dynamics of F_2 are topologically prevented, it is still possible to study the evolution along a single homoclinic loop applying the methodology we developed. For gaining the circular problem one fixes the eccentricity of the primar body's orbit to be zero. Thus, it amounts to exclude in the remainder all the harmonics depending on the angle u_3.
So, each term in the Fourier decomposition of Melnikov integrals, defined in 2.31, assumes the form

$$\mathcal{I}_{j,m} = C_{j,m} \int_0^T \sin\big(j\Omega t \pm m u_2(t)\big)\mathrm{d}t \tag{4.48}$$

and the size of the jump is estimated, in this setting, by the sum

$$\Delta F_2 = \sum_{j,m} \mathcal{I}_{j,m} \tag{4.49}$$

Thus, for every term in the remainder (that is, for every fixed value of n), the algorithm verifies the condition stated by lemma 3.3 at the varying of j, to detecet those terms whose phase is stationary. The coefficients $C_{j,m}$ depends on suitable products of Fourier coefficient of the homoclinic pulse and coefficients $r_{j,m,n}$ coming from the remainder of the normal form. Since they decay exponentially fast (see 2.16) as the wave-number j grows up, it is possible to fix a proper finite cut-off in the development. In this example, we choose $\hat{J} = 20$. A larger value for the cut-off does not affect the results appreciably.
We recognize two groups of integrals, depending on conditions 3.3:

i) non-stationary-phase terms, estimated via the proposition 3.4

ii) stationary-phase terms, estimated via the proposition 3.7. They produce, as theoretically predicted by the principle of stationary phase, the major contributions.

It is wortwhile to notice that the considerably fast evolution of the angle u_2 ensures the transition from the case i) to the case ii) not to produce *quasi*-stationary, or degenere, phases. Thus, the analytical estimates result to be quite in agreement

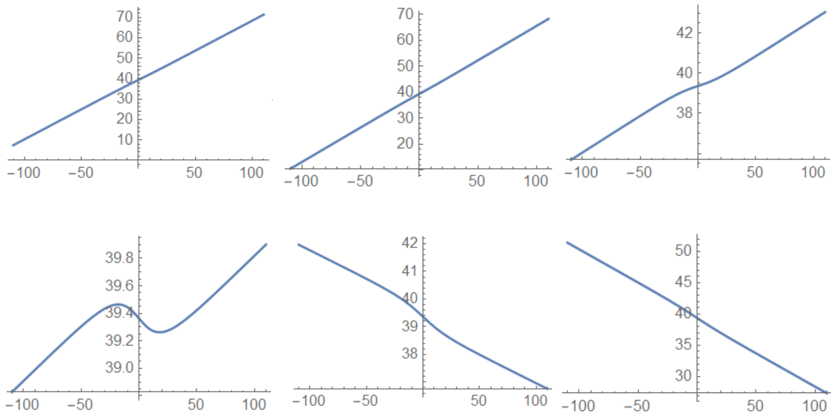


Figure 4.13: An example of the transition of the phase from stationary to non-stationary, for j=1,2,10,11,12,15 and n=2. For j=11 the phase has two stationary points.

with numerical computations. In particular, we report an explicative result of such an analysis in the case of maximal jump with respect to initial data. The size of the effective jump computed via analytical estimates results to be

$$\Delta F_2 = 3.80938 \cdot 10^{-7} \tag{4.50}$$

to be compared with the numerical prediction

$$\Delta F_2 = 4.16845 \cdot 10^{-7} \tag{4.51}$$

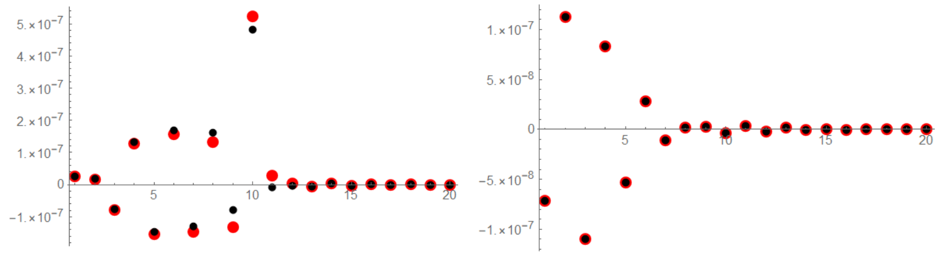


Figure 4.14: A comparison, for n=2, between numerical, in black, and analytical value of the Fourier decomposition of a Melnikov integral for the circular problem. One stationary-phase term is identified at j=10 and it produces the major contribution. Similar results are found in the analysis of Melnikov integrals labelled by different value of the wave-number n.

4.5.2 Spatial elliptic RTBP

The same procedure can be applied to the most general case of the problem, namely the spatial, elliptic, restricted three-body problem, so for decomposition of Melnikov integrals of the form

$$\Delta F_2 = \sum_{j,m,n} \mathcal{I}_{j,m,n} = C_{j,m,n} \int_0^T \sin\big(j\Omega t \pm m u_2(t) \pm n u_3(t)\big) \mathrm{d}t \qquad (4.52)$$

We first notice that, beacuse of the form of the canonical transformation involved in the normalisation (see 4.3), the coefficient $\mathcal{I}_{j,m,n}$ in the remainder converge to zero for increasing values $|m+n|$. Thus, in addiction to the cut-off $\hat{J}$ in the Fourier development, one is allowed to consider harmonics such that $|m+n| < \hat{N}$, for a priori fixed $\hat{N}$. Moreover, it turns out that very few harmonics produce in fact relevant contributions.

A further consideration regarding the stationarity condition should be pointed out. Since the angle u_3, introduced in the three degree of freedom problem, evolves much slowly with respect to u_2, the transition from stationary to non-stationary terms involves terms for which a stationary-phase approach is arguable. The presence of those terms, in fact, affects the reliability of the analytical estimates for a few terms. As for the circular problem, we test our methodology for computing the size of the maximal spread of F_2 with respect to the initial phase $u_2(t)$. The analytical estimate for the size of the jump result to be

$$\Delta F_2 = 5.48584 \cdot 10^{-7} \tag{4.53}$$

quite in agreement with the numerical prediction

$$\Delta F_2 = 6.7796 \cdot 10^{-7} \tag{4.54}$$

Figure 4.15 shows a comparison of the analtyical versus numerical estimates for the most relevant terms in the above analysis.

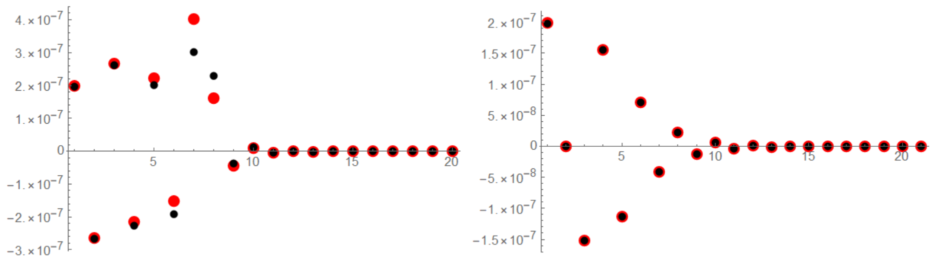


Figure 4.15: An example of some out of the most relevant integrals, analytically estimated, in black, with a numerical comparison, in red.

5 Conclusions

In the present thesis, we combined estimates obtained via the Jeans-Landau-Teller approximation with those of the method of stationary phase in order to provide upper bounds for the rate of slow chaotic diffusion in models of so-called "second fundamental model of resonance". Furthermore we applied these results in a numerical example corresponding to the asteroidal 3:2 mean motion resonance in the spatial elliptic restricted three body problem. Our main conclusions are the following.

In chapter 1 we perform a complete analysis of the second fundamental model. In particular, following [Fer07], we express the solution of the equation of motions in terms of explicit trascendental functions.
In chapter 2 we implement the Jeans-Landau-Teller approximation. Profiting of the available analytical expression for the separatrix solution, we describe the dynamics within the stochastic layer produced by a small perturbation. By means of the Fourier representation of the separatrices, we are able to express the size of the one-period homoclinic jump of the adiabatic action variables in terms of Melnikov-type integrals.
In chapter 3 we state the propositions providing analytical estimates for the Melnikov integrals. Those estimates are based on the well-known method of stationary phase. We also provide a condition able to identify those integrals which give larger contributions, namely those which present a stationary phase.
In chapter 4 we apply our results to the slow chaotic diffusion of the orbital elements in the case of the 3:2 asteroidal mean motion resonance. First, we construct a hamiltonian model describing mean motion resonances in the restricted three-body problem. We find that the second fundamental model naturally arises in this context. The impulsive character of the diffusion is explicitly revealed by a canonical

transformation produced via the Lie series method, which allows to remove all deformation effects in the time series yielding the evolution of the adiabatic action variables. The slow chaotic diffusion proceeds by subsequent "jumps" of stochastic nature. Thus, the rate of diffusion is provided by the size of the typical one-period jump. Evidence of KAM tori's bounding effect in the two dimensional systems are also pointed out via numerical integrations. Eventually, we implement the analytical methodology developed in the chapter 2 and 3 for the study of an example of a 3:2 resonant orbits. Thus, we are able to measure the size of the one-period jump of the adiabatic action variable F_2, for a given inital datum. The results are quite in agreement with numerical predictions, in both cases of the circular and of the elliptic three-body problem.

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