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Non-abelian R-symmetries defects from probe branes configurations in holography

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Abstract

In quantum field theory, symmetries are traditionally associated with conserved currents and charges. A more modern and more general characterization—valid also for discrete and higher-form symmetries—identifies symmetry generators with topological defects: extended operators whose correlation functions are invariant under smooth deformations of their worldvolumes. In this framework, conservation laws are encoded in the topological nature of these defects rather than in local currents.

The aim of this thesis is to investigate how such symmetry defects, and in particular those associated with non-abelian R-symmetries, are realized in the context of holography. Understanding their holographic realization is especially important for clarifying the ultraviolet fate of symmetry defects in a theory of quantum gravity, where purely field-theoretic notions of topological operators may receive nontrivial gravitational corrections. In this respect, top-down holographic models provide a controlled and predictive playground in which these questions can be addressed.

Recent proposals suggest that these defects admit a gravitational description in terms of specific probe brane configurations that are anchored at the AdS boundary but extend a finite distance into the bulk. These objects can be viewed as a regularized version of a brane–antibrane system and are conjectured to generate the full non-abelian R-symmetry group through their topological properties.

In this work we test this proposal in an explicit top-down holographic background that has not previously been explored from this perspective, namely type IIA string theory on $AdS_7 \times M_3$. This geometry is dual to families of six-dimensional superconformal field theories with non-abelian R-symmetry. By constructing and analyzing the corresponding probe brane defects, we provide new evidence for their role as holographic realizations of non-abelian symmetry generators and clarify how R-symmetry is encoded in the bulk through topological brane configurations.

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1 Introduction

Symmetries have always played a central role in theoretical physics, from the classification of particle spectra to the identification of conserved quantities and the construction of low-energy effective descriptions. In ordinary quantum field theory, global symmetries are described by group actions on local operators, and their associated conserved charges are supported on codimension-one surfaces. Over the past decade, however, it has become clear that this picture is only a special case of a much richer structure. The framework of generalized global symmetries [34] extends the notion of symmetry well beyond the group-theoretic setting traditionally associated with it: symmetry operators need not act on local operators alone, but can act on extended objects of arbitrary dimension (higher-form symmetries); several such symmetries can combine into structures that are not simply direct products (higher-group symmetries); and, in certain circumstances, symmetry operators need not even be invertible (non-invertible symmetries). This generalized perspective has proven remarkably powerful, providing new tools to constrain renormalization group flows, classify gapped and gapless phases of matter, and organize the possible global structures of a given quantum field theory.

A natural arena in which to realize and study generalized symmetries geometrically is string theory and, more specifically, the holographic correspondence. In the context of AdS/CFT, symmetry operators of the boundary field theory can often be constructed explicitly as branes wrapping suitable cycles of the bulk geometry, or extending along its radial direction, [5]. This holographic dictionary has been used to systematically derive the higher-form and higher-group symmetries of theories engineered from D-brane and M-brane setups, and, more recently, has been extended to non-invertible symmetries [6]. While this technology is well established for discrete symmetries, the holographic realization of continuous symmetries - abelian and, in particular, non-abelian - has only begun to be explored. Non-BPS branes have been proposed as candidates for continuous abelian symmetry operators [16], later reinterpreted as regulated brane/anti-brane pairs [25], while the first steps towards a holographic construction of non-abelian symmetry operators - realized as branes hanging from the conformal boundary of AdS_{d+1} - have been taken in [11, 10].

The purpose of this thesis is to apply and test this emerging holographic construction of non-abelian symmetry operators in a setup that has not been previously considered: type IIA string theory on $AdS_7 \times M_3$, where M_3 is an S^2 -fibration over an interval. This family of backgrounds is holographically dual to six-dimensional $(1,0)$ superconformal field theories, whose R-symmetry contains an $SU(2)$ factor arising geometrically from the isometry of M_3 , [4], [55]. We construct the charged operators associated with this non-abelian R-symmetry, together with the corresponding symmetry operators, following the general procedure of [11, 10], and we show how the process of charge measurement is realized microscopically in terms of a Hanany-Witten transition between the probe branes and the brane sourcing the charge.

The thesis is organized as follows. Section 2 illustrates the relationship between symmetries and topological defects, showing how the new shift in perspective allows us to describe symmetries in terms of topological operators rather than conserved currents. This new theoretical framework is tied to the notion of generalized global symmetries, which extends the concept of symmetry to higher-form, higher-group and non-invertible symmetries. Throughout this section we provide a brief review of generalized global symmetries, starting from ordinary global symmetries and their topological operators, and progressively introducing higher-form symmetries, spontaneous symmetry breaking, screening, gauging, 't Hooft anomalies, higher-group symmetries, discrete symmetries and BF theory, and finally non-invertible symmetries. Throughout this exposition, we will repeatedly refer to Maxwell theory as an enlightening example where many notions related to generalized symmetries can be successfully applied. Section 3 collects the string-theoretic background needed for the rest of the thesis: the bosonic string and its quantization, the superstring and its low-energy effective actions, the worldvolume actions of

Dp -branes, dimensional compactifications and reductions, and the Hanany-Witten effect. Section 4 reviews the AdS/CFT correspondence, from the conformal symmetry of the boundary theory and the geometry of anti-de Sitter space to the precise statement of the duality and the holographic dictionary. The final part of this section then turns to the interplay between symmetries and holography, which forms the conceptual backbone of the original work: we discuss how the generalized symmetries of a boundary theory can be read off from the bulk, first for discrete symmetries and then for continuous, non-abelian ones. Section 5 is devoted to introduce 6-dimensional superconformal field theories (SCFT's), which represent our working stage for the computations carried on in the next Section. After recalling the basic properties of 6-dimensional superconformal field theories, we describe how the $(1,0)$ theories of interest are engineered from the NS5-D6-D8 brane systems of Hanany and Zafaroni, and we introduce the family of massive type IIA $AdS_7 \times M_3$ backgrounds that provide their holographic duals, together with the universal consistent truncation to seven-dimensional gauged supergravity used throughout the rest of the work. Finally, Section 6 contains the original results of this thesis: after introducing the class of type IIA $AdS_7 \times M_3$ backgrounds, we construct the charged and symmetry operators associated with the non-abelian R-symmetry and discuss how their linking realizes the charge measurement process at the level of the underlying string and brane configuration.

To conclude, Section 7 succinctly summarizes our results and proposes some ideas to extend our finding to other string theory backgrounds.

2 Symmetries and topological defects

Symmetries are a cornerstone of physics, offering a powerful framework for understanding the behavior of physical systems. In recent years, the concept of generalized global symmetries has emerged as a crucial tool in the study of quantum field theories and string theory [34]. Under this broader perspective, the very notion of symmetry has expanded well beyond the group-theoretic framework traditionally associated with it, paving the way for higher-form symmetries, higher-group symmetries and, rather surprisingly, non-invertible symmetries.

Such shift in perspective has drastically changed of we conceive symmetries. Rather than looking at them as an action on fields, we can identify them as topological defects supported on p -codimensional cycles Σ_{d-p} . Their topological nature is guaranteed by the fact that correlation functions with topological symmetry operators insertions are insensitive to the any smooth deformation of Σ_{d-p} .

This new approach to symmetries buys us new methods to study the symmetry structure of a given QFT; in particular, as we will extensively show in Sections 4 and 6, string theory and the celebrated *AdS/CFT* correspondence allow us to retrieve, from a top down perspective, the symmetry data of a theory, even when one does not know its exact lagrangian, as it is often the case for 6-dimensional SCFT's, see Section 5.

In this section we give an overview of generalized global symmetries, their mathematical structure, and their implications for our understanding of fundamental physics.

2.1 Review of Ordinary Global Symmetries

We start with a brief review of ordinary global symmetries in quantum field theory. A global symmetry is a transformation that leaves the action of a theory invariant and does not depend on spacetime coordinates. In particular, given an action $S[\phi]$ for a set of fields ϕ in d dimensions, a global symmetry is a transformation $\phi \rightarrow \phi'$ such that

$$S[\phi'] = S[\phi]. \quad (2.1.1)$$

The composition of two global symmetries is also a global symmetry, and the set of all global symmetries forms a group, known as the global symmetry group of the theory. Global symmetries can be continuous or discrete: loosely speaking, the parameters identifying the transformation vary over a continuous or a discrete set, respectively. A standard result in quantum field theory is Noether's theorem, [54], which states that for every continuous global symmetry, there is a corresponding conserved current and charge. Let $S[\phi] = \int d^d x \mathcal{L}(\phi)$ and suppose that under an infinitesimal symmetry transformation the fields undergo the change

$$\phi(x) \rightarrow \phi(x) + \epsilon \delta\phi(x), \quad (2.1.2)$$

where $\delta\phi(x)$ is a small variation of the fields and ϵ is a small parameter which characterizes the transformation. To find the associated conserved current [35], we first promote the parameter ϵ to a function of spacetime, $\epsilon(x)$. The transformation is then no longer a symmetry of the action, and we expect the variation of the action to be proportional to the derivative of $\epsilon(x)$, i.e.

$$\delta S = \int d^d x j^\mu \partial_\mu \epsilon(x), \quad (2.1.3)$$

where j^μ is the Noether current associated with the symmetry. Regarding the transformation (2.1.2) as an arbitrary variation of the fields, we may employ the equations of motion to set $\delta S = 0$. Using the fact that $\epsilon(x)$ is an arbitrary function and integrating by parts, we then infer that the Noether current is conserved on shell,

$$\partial_\mu j^\mu = 0 \quad (2.1.4)$$

From the conservation of the current, we can define a conserved charge Q as the integral of the time component of the current over a spatial slice,

$$Q = \int d^{d-1}x j^0. \quad (2.1.5)$$

From (2.1.4) and employing the divergence theorem, it is readily shown that

$$\frac{dQ}{dt} = 0 \quad (2.1.6)$$

In classical field theory, the conserved charge Q generates the symmetry transformation on the fields via the Poisson bracket,

$$\delta\phi(x) = \{Q, \phi(x)\}. \quad (2.1.7)$$

In the quantum setting an analogous relation holds, with the Poisson bracket replaced by the commutator. To see this [46][22], consider a local operator $\mathcal{O}(x)$ in a given representation R of the global symmetry group, i.e.

$$\delta\mathcal{O}(x) = T_R^a(g)\mathcal{O}(x) \quad (2.1.8)$$

where T_R^a are the generators of the symmetry group in the representation R and δ is the variation of the operator under the symmetry transformation g . Its correlation functions with other operators are given by the path integral

$$\langle \mathcal{O}(x) \cdots \rangle = \int \mathcal{D}\phi e^{iS[\phi]} \mathcal{O}(x) \cdots \quad (2.1.9)$$

where $\cdots$ denotes the insertion of other operators at different points. Under a symmetry transformation g , the correlation functions must be invariant, i.e.

$$\langle \delta\mathcal{O}(x) \cdots \rangle = 0 \quad (2.1.10)$$

On the other hand, we can also express the variation of the expectation value of the operator $\mathcal{O}(x)$ under the symmetry transformation g as¹

$$\begin{aligned} \langle \delta\mathcal{O}(x) \cdots \rangle &= \int \mathcal{D}\phi e^{iS[\phi]} [(\delta\mathcal{O}(x) + i\mathcal{O}(x)\delta S[\phi]) + \mathcal{O}(x)\delta(\cdots)] \\ &= \int \mathcal{D}\phi e^{iS[\phi]} \left[\left(T_R^a(g)\mathcal{O}(x) + i\mathcal{O}(x) \int d^d y j^\mu(y) \partial_\mu \epsilon(y) \right) + \mathcal{O}(x)\delta(\cdots) \right]. \end{aligned} \quad (2.1.11)$$

where in the second line we made use of (2.1.3) (after integrating by parts). Then from the arbitrariness of $\cdots$ and $\epsilon(y)$, we obtain the famous Ward-Takahashi identity for global symmetries,

$$\langle \partial_\mu j^\mu(x) \mathcal{O}(y) \cdots \rangle = \delta^d(x-y) T_R^a(g) \langle \mathcal{O}(y) \cdots \rangle \quad (2.1.12)$$

Integrating the above expression over a spatial slice, we can also express the Ward-Takahashi identity in terms of the conserved charge Q ,

$$\langle [Q, \mathcal{O}(y)] \cdots \rangle = T_R^a(g) \langle \mathcal{O}(y) \cdots \rangle \quad (2.1.13)$$

thus obtaining the quantum counterpart of (2.1.7) for $\mathcal{O}(x) = \phi(x)$.

¹Here we are assuming that no quantum anomalies appear.

2.1.1 Ordinary symmetries and topological operators

In order to establish the connection between ordinary global symmetries and topology, we now rephrase some of the results of the previous section in a more geometric way. First, we can rewrite the conservation law (2.1.4) in terms of differential forms as in [45],

$$d \star j = 0, \quad (2.1.14)$$

where $\star$ is the Hodge star operator and j is a one form obtained by lowering the index of the Noether current. We can then define a topological operator $U_g(\Sigma)$ associated with the symmetry transformation g as the exponential of the integral of the Noether current over a codimension-1 manifold Σ ,

$$U_g(\Sigma) = \exp \left(i\lambda \int_{\Sigma} \star j \right). \quad (2.1.15)$$

It is easy to see that the operator $U_g(\Sigma)$ is topological, i.e. its correlation functions do not depend on the shape of Σ , but only on its homology class. Indeed, consider a small deformation of Σ to Σ' , such that the two manifolds are homologically equivalent, i.e. $\Sigma' = \Sigma \cup \partial M$ for some manifold M (see Figure1). Then, we have

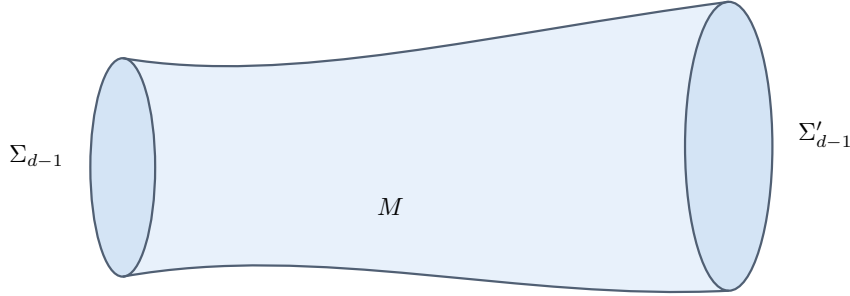


Figure 1: The surfaces Σ_{d-1} and Σ'_{d-1} cobound the region M , hence define the same homology class.

$$U_g(\Sigma') = \exp \left(i\lambda \int_{\Sigma'} \star j \right) = \exp \left(i\lambda \int_{\Sigma} \star j + i\lambda \int_{\partial M} \star j \right) = U_g(\Sigma) \exp \left(i\lambda \int_M d \star j \right) = U_g(\Sigma), \quad (2.1.16)$$

where in the last step we made use of the conservation law (2.1.14) and Stokes' theorem. The topological operators $U_g(\Sigma)$ satisfy the following group multiplication law

$$U_g(\Sigma)U_h(\Sigma) = U_{gh}(\Sigma) \quad (2.1.17)$$

which can be verified by considering the Ward identity in the special case where the operator $\mathcal{O}(y)$ is identified with the conserved current $j^\nu(y)$, i.e.

$$\langle \partial_\mu j_a^\mu(x) j_b^\nu(y) \cdots \rangle = \delta^d(x-y) f_{ab}^c \langle j_c^\nu(y) \cdots \rangle \quad (2.1.18)$$

where f_{ab}^c are the structure constants of the symmetry group². We now turn to how the operator $U_g(\Sigma)$ acts on local operators. Consider a local operator $\mathcal{O}(x)$ in a given representation R of the global symmetry group, and let Σ be a codimension-1 manifold and consider a deformation which brings Σ to Σ' intersecting the point x where the operator $\mathcal{O}$ is inserted (see Figure 2). Using Stokes's

²Notice that (2.1.18) is just a restatement of the fact that conserved currents belong to the adjoint representation of the symmetry group

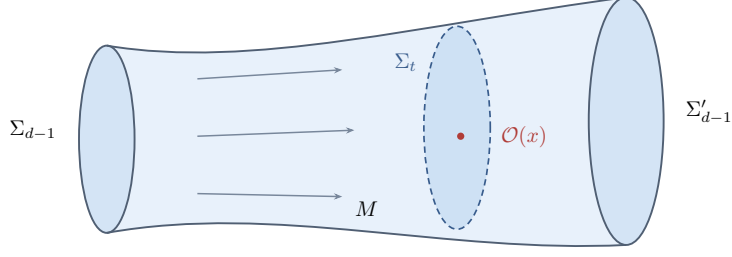


Figure 2: Deformation of the codimension-1 manifold Σ into Σ' sweeping across the operator insertion $\mathcal{O}(x)$, with M the region bounded by $\Sigma \cup \Sigma'$.

theorem and the Ward identity (2.1.18), we then have

$$\begin{aligned}
 \langle U_g(\Sigma) \mathcal{O}(x) U_{g^{-1}}(\Sigma') \rangle &= \left\langle \exp \left(i \lambda^a \int_M d \star j_a \right) \mathcal{O}(x) \right\rangle \\
 &= \left\langle \sum_{n=0}^{+\infty} \frac{(i \lambda^a \int_M d \star j_a)^n}{n!} \mathcal{O}(x) \right\rangle \\
 &= \left\langle \sum_{n=0}^{+\infty} \frac{(i \lambda^a \int_M \partial_\mu j_a^\mu(y) d^d y)^n}{n!} \mathcal{O}(x) \right\rangle \\
 &= \left\langle \sum_{n=0}^{+\infty} \frac{(i \lambda_a \int_M \delta^d(x-y) T_R^a(g))^n}{n!} \mathcal{O}(x) \right\rangle \\
 &= \langle \exp(i \lambda_a T_R^a(g)) \mathcal{O}(x) \rangle
 \end{aligned} \tag{2.1.19}$$

where M is a d -dimensional manifold whose boundary is $\Sigma \cup \Sigma'$ (with the orientation of Σ' opposite to that of Σ) and the group element g is related to the parameters λ_a by $g = \exp(i \lambda_a T^a)$. Equation (2.1.19) can be rephrased in the following way

$$U_g(\Sigma) \mathcal{O}(x) = \exp(i \lambda_a T_R^a(g)) \cdot \mathcal{O}(x) U_g(\Sigma') \tag{2.1.20}$$

where we have omitted the brackets denoting the vacuum expectation value for simplicity. Hence, the symmetry operator acts on local operators by linking with their support.

The interpretation of the symmetry operator $U_g(\Sigma)$ is subject to the choice of the submanifold Σ on which it is defined. In particular, if Σ is chosen to be the whole space at a given fixed time, then the operator $U_g(\Sigma)$ can be identified with the conserved unitary operator acting on the Hilbert space $\mathcal{H}$. However, if Σ is localized in one of the spatial directions and extends in time, then $U_g(\Sigma)$ is a defect which modifies the quantization conditions resulting in a g -twisted Hilbert space $\mathcal{H}_g$. For a more general Σ the interpretation is more subtle; however, since we will deal with the euclideanized version of the theory, the words operator and defect will be used interchangeably, [34] [60] [46].

Finally, anticipating the discussion of the next section, we will define ordinary global symmetries as 0-form symmetries.

2.2 Higher form symmetries

This new paradigm shift provides powerful generalizations, for instance, the notion of higher-form symmetries, which extends the concept of global symmetries to transformations that act on operators supported on higher-dimensional submanifolds. Following [19], a p form symmetry is defined by a set of invertible and topological operators U defined on a codimension- $(p+1)$ submanifold. Furthermore,

the above operators realize a group structure, denoted by $G^{(p)}$, meaning that each element g of the group is associated with a topological operator $U_g(\Sigma_{d-p-1})$ and the group multiplication law describes the composition of operators, i.e.

$$U_g(\Sigma_{d-p-1})U_h(\Sigma_{d-p-1}) = U_{gh}(\Sigma_{d-p-1}) \quad (2.2.1)$$

The above equation is often referred to as the fusion rule of the symmetry operators. In the case of a continuous p -form symmetry, a higher form generalization of Noether's theorem ensures the existence of a co-closed $(p+1)$ -form current, j_{p+1} , which allows us to give an explicit realization of the symmetry operator, mimicking the construction given in the previous section. Explicitly

$$U_g(\Sigma_{d-p-1}) = \exp \left(i\lambda \int_{\Sigma_{d-p-1}} \star j_{p+1} \right). \quad (2.2.2)$$

One of the key aspects of higher-form symmetries is that they always consist of abelian groups, at least when $p \geq 1$. We can easily show this by considering the fusion of two symmetry operators $U_g(\Sigma_{d-p-1})$ and $U_h(\Sigma'_{d-p-1})$; due to their topological nature, we can always perform a deformation which reverses the order of the two operators without intersecting the two manifolds over which they are defined, see Figure 3. Thus, we cannot have a sensible definition of ordering of operators, leaving the only

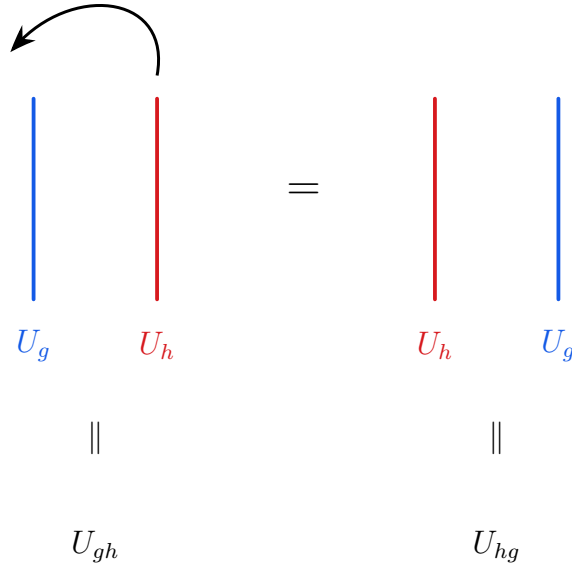


Figure 3: We can move the operator U_h (in red) to the left of U_g (in blue) in a way that the two do not intersect each other at any point. Hence, there is no sensible ordering, making the symmetry group necessarily abelian for $p \geq 1$.

possibility of an abelian fusion rule, i.e.

$$U_g(\Sigma_{d-p-1})U_h(\Sigma'_{d-p-1}) = U_h(\Sigma'_{d-p-1})U_g(\Sigma_{d-p-1}).$$

We now specify how a p -form symmetry acts on operators; firstly, we notice that a p -form symmetry operator $U_g(\Sigma_{d-p-1})$ can only link with operators supported on submanifolds of dimension at least p . This can be proved by a simple dimensional argument; consider a $q < p$ operator $\mathcal{O}(M_q)$ supported on a q -dimensional submanifold M_q and a p -form symmetry operator $U_g(\Sigma_{d-p-1})$ defined on a $(d-p-1)$ -dimensional manifold Σ_{d-p-1} , then one can always deform Σ_{d-p-1} to a manifold Σ'_{d-p-1} in such a way that it never intersects M_q . Therefore, a p -form symmetry can only act on operators supported on submanifolds of dimension at least p . Thus, the first case that we are led to consider is the

action of a p -form symmetry on a p -dimensional operator $\mathcal{O}(M_p)$. We proceed by showing that p -dimensional operators are representations of the p -form symmetry group. Indeed, let us consider a symmetry operator $U_g(\Sigma_{d-p-1})$ and an *irreducible* p -dimensional operator $\mathcal{O}(M_p)$ ³, and imagine deforming Σ_{d-p-1} past M_p ; this leaves behind a topological operator $\mathcal{O}(x)$, being x the intersection point of Σ_{d-p-1} and M_p (see Figure 4):

$$U_g(\Sigma_{d-p-1})\mathcal{O}(M_p) = \mathcal{O}(x)\mathcal{O}(M_p)U_g(\Sigma'_{d-p-1}) \quad (2.2.3)$$

Since $\mathcal{O}(M_p)$ is irreducible, we must have $\mathcal{O}(x) = \rho(g) \cdot \mathbb{I}$, where $\rho(g)$ is a complex number which

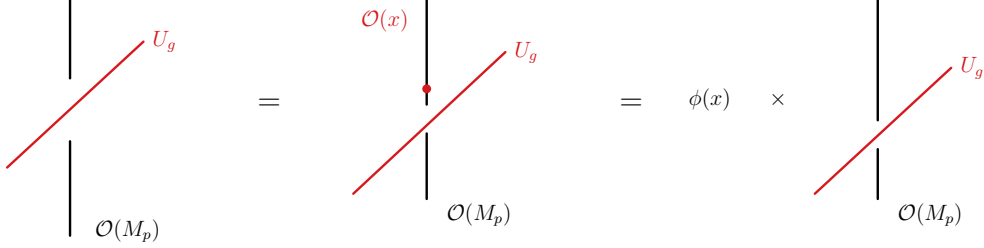


Figure 4: Moving the topological symmetry operator U_g across the irreducible p -dimensional operator $\mathcal{O}(M_p)$ produces a local operator $\mathcal{O}(x)$, being x the intersection point between the topological operator and the charged operator.

depends on the group element g and $\mathbb{I}$ is the identity operator. So we have

$$U_g(\Sigma_{d-p-1})\mathcal{O}(M_p) = \rho(g) \cdot \mathcal{O}(M_p)U_g(\Sigma'_{d-p-1}) \quad (2.2.4)$$

and using the fusion rule of the symmetry operators, we must have

$$\rho(g) \cdot \rho(h) = \rho(gh),$$

which means that ρ furnishes a 1-dimensional representation of the p -form symmetry group. In particular, this implies that

$$\rho(g) \in U(1) \subset \mathbb{C}.$$

Since the irreps of any abelian group are one-dimensional, we can conclude that p -dimensional operators are representations of the p -form symmetry group. Equivalently, (2.2.4) can be recast as

$$U_g(S_{d-p-1})\mathcal{O}(M_p) = \rho(g)\mathcal{O}(M_p), \quad (2.2.5)$$

where S_{d-p-1} is a $(d-p-1)$ -dimensional sphere which links with M_p at the point x and on the R.H.S. we have shrunk Σ' to a point (see Figure 5). For a continuous symmetry the Ward identity (2.1.12) gives the local statement

$$d \star j_{p+1}(x)\mathcal{O}(M_p) = \delta^{d-p}(M_p) q \mathcal{O}(M_p), \quad (2.2.6)$$

where $\delta^{d-p}(M_p)$ is a delta function with support on the p -dimensional submanifold M_p and q is the charge of the operator $\mathcal{O}(M_p)$ under the p -form symmetry⁴. Then, the action of the symmetry operator can be realized as in (2.1.19).

A p -form symmetry can also act on operators of dimension greater than p . In that case the action is richer and need not reduce to a phase; we will not need this case here, and refer the reader to [18] for a full treatment.

³By which we mean that there are no topological operators which can be inserted at a point $x \in M_p$ other than a multiple of the identity

⁴Observe that p -form symmetries are always abelian, so the generators T_R appearing in 2.1.13 are always c-numbers

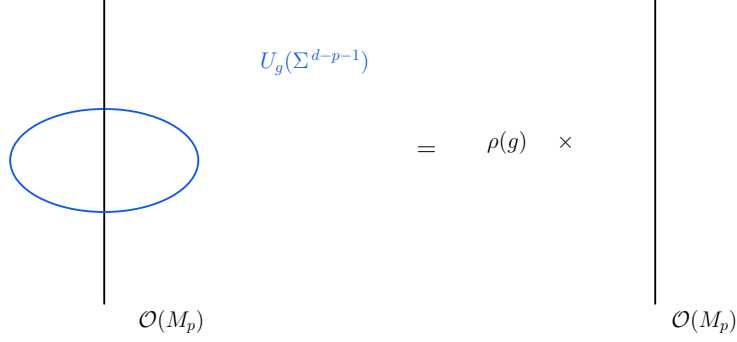


Figure 5: Action by linking of the $p \geq 1$ -form symmetry operator U_g on the p -dimensional operator $\mathcal{O}(M_p)$.

2.2.1 Examples of higher form symmetries: Maxwell theory

To get acquainted with the concept of higher form symmetries, we now discuss a simple example: Maxwell theory in four dimensions without matter. We recall that the action of Maxwell theory is given by

$$S = \frac{1}{2e^2} \int f \wedge \star f, \quad (2.2.7)$$

where $f = da$ is the field strength of the gauge field a and e is the coupling constant. The equations of motion derived from the action (2.2.7) are given by

$$d \star f = 0, \quad (2.2.8)$$

along with the Bianchi identity

$$df = 0. \quad (2.2.9)$$

From the equations of motion (2.2.8) and the Bianchi identity (2.2.9), we can identify two conserved currents, which are given by $j_e = \frac{1}{e^2} f$ and $j_m = -\frac{1}{2\pi} \star f$. The current j_e is associated with the electric 1-form symmetry, while the current j_m is associated with the magnetic 1-form symmetry. Indeed, we can construct the corresponding symmetry operators as follows

$$U_e(\Sigma_2) = \exp \left(i \lambda_e \int_{\Sigma_2} \star j_e \right) = \exp \left(i \frac{\lambda_e}{e^2} \int_{\Sigma_2} \star f \right), \quad (2.2.10)$$

and

$$U_m(\Sigma_2) = \exp \left(i \lambda_m \int_{\Sigma_2} \star j_m \right) = \exp \left(-i \frac{\lambda_m}{2\pi} \int_{\Sigma_2} f \right), \quad (2.2.11)$$

where λ_e and λ_m are the parameters associated with the electric and magnetic 1-form symmetries, respectively⁵. Due to charge quantization, the parameters λ_e and λ_m are periodic, with period 2π , which means that the electric and magnetic 1-form symmetries are indeed $U(1)$ symmetries, denoted by $U(1)_e^{(1)}$ and $U(1)_m^{(1)}$, respectively⁶. We now identify the charged operators under the electric and magnetic 1-form symmetries. Starting with the electric 1-form symmetry, we can easily see that the charged operators are given by Wilson lines, which are defined as

$$W_q(C) = \exp \left(i q \int_C a \right) = \exp \left(i q \int_{M^4} \delta(C) \wedge a \right), \quad (2.2.12)$$

⁵Differently from the notation used in the previous section, the subscript m and e do not refer to some group elements, but merely stands of "magnetic" and "electric", respectively.

⁶The presence of the factors $\frac{1}{e^2}$ and $\frac{1}{2\pi}$ in the currents is indeed related to the quantization of the electric and magnetic charges.

where $\delta(C)$ is the dual form to the curve C and $q \in \mathbb{Z}$ is the electric charge of the Wilson line. In order to grasp the physical meaning behind the above operator, we can study how the equation of motion (2.2.8) is modified in the presence of a Wilson line. Indeed, we can consider the path integral of Maxwell theory with the insertion of a Wilson line $W_q(C)$, which is given by

$$\langle W_q(C) \rangle = \int \mathcal{D}a e^{iS[a]} W_q(C). \quad (2.2.13)$$

Now, we can perform a change of variable in the path integral by shifting the gauge field a by a small amount δa , i.e. $a \rightarrow a + \delta a$; this leads to

$$\begin{aligned} \langle W_q(C) \rangle &= \int \mathcal{D}a e^{iS[a]} W_q(C) \\ &= \int \mathcal{D}a e^{iS[a+\delta a]} W_q(C) e^{iq \int \delta(C) \wedge \delta a} \\ &= \int \mathcal{D}a e^{iS[a]} e^{i \int \delta a \wedge \left(\frac{1}{e^2} d \star f - q \delta(C) \right)} W_q(C). \end{aligned} \quad (2.2.14)$$

Since the change of variable is just a relabeling of the integration variable, we must have that the L.H.S equals the R.H.S of the equation (2.2.14), which implies

$$\frac{1}{e^2} d \star f = q \delta(C). \quad (2.2.15)$$

Thus, the insertion of the Wilson line $W_q(C)$ can be reinterpreted as a heavy probe particle of electric charge q moving along the curve C . From equation (2.2.15) we can infer the action of the electric 1-form symmetry on the Wilson line $W_q(C)$ in a way similar to that of (2.1.19), in particular we have

$$U_e(\Sigma_2) W_q(C) = e^{i\lambda_e q} W_q(C) U_e(\Sigma'_2), \quad (2.2.16)$$

where Σ_2 is a 2-dimensional manifold which links with the curve C and Σ'_2 is a homotopic deformation of Σ_2 which does not link with C . To identify the operators charged under the magnetic symmetry it might be useful to employ a dual description of Maxwell theory, which can be retrieved in the following way, [46] [35]. Take the action:

$$S = \frac{1}{2e^2} \int f \wedge \star f + \frac{1}{2\pi} \int f \wedge d\tilde{a}, \quad (2.2.17)$$

where $\tilde{a}$ is a 1-form gauge field⁷ and f is an unconstrained 2-form field. The corresponding equations of motion for $\tilde{a}$ coming from the action (2.2.17) imply that f is closed, and from the quantization condition on $\tilde{a}$ we can also infer that f is quantized, i.e. $\int f \in 2\pi\mathbb{Z}$; thus, the above action is physically equivalent to the original Maxwell action (2.2.7). Now, integrating out the field f in the path integral, we can obtain a dual description of Maxwell theory in terms of the gauge field $\tilde{a}$; in particular, the equation of motion for f derived from the action (2.2.17) is given by

$$\frac{1}{e^2} \star f + \frac{1}{2\pi} d\tilde{a} = 0, \quad (2.2.18)$$

which can be solved for f as

$$f = \frac{e^2}{2\pi} \star d\tilde{a}. \quad (2.2.19)$$

Substituting the above expression for f back into the action (2.2.17), we obtain

$$S = \frac{e^2}{8\pi^2} \int d\tilde{a} \wedge \star d\tilde{a}. \quad (2.2.20)$$

⁷Namely, $\tilde{a} \sim \tilde{a} + d\theta$ such that $\theta \sim \theta + 2\pi$ and $\int \frac{\tilde{a}}{2\pi} \in \mathbb{Z}$.

In particular, defining

$$\tilde{f} = d\tilde{a} \quad (2.2.21)$$

and

$$\tilde{e}^2 = \frac{4\pi^2}{e^2}, \quad (2.2.22)$$

we can see that the dual description of Maxwell theory is again a Maxwell theory, with $f \rightarrow \tilde{f}$ and $e \rightarrow \tilde{e}$. Being a Maxwell theory, the dual description also admits an electric and a magnetic 1-form symmetry. In particular, focusing, say, on the electric 1-form symmetry, we have (cfr. (2.2.10) and (2.2.11))

$$\tilde{U}_e(\Sigma_2) = \exp\left(i\tilde{\lambda}_e \int_{\Sigma_2} \star \tilde{j}_e\right) = \exp\left(i\frac{\tilde{\lambda}_e}{\tilde{e}^2} \int_{\Sigma_2} \star \tilde{f}\right) = \exp\left(-i\frac{\tilde{\lambda}_e}{2\pi} \int_{\Sigma_2} f\right) = U_m(\Sigma_2), \quad (2.2.23)$$

where we have made use of equations (2.2.19), (2.2.21) and (2.2.22). An analogous relation holds between $\tilde{U}_m$ and U_e . Thus, in the dual Maxwell theory the role of the electric and magnetic 1-form symmetries is interchanged. As a consequence, the charged operators under the electric 1-form symmetry in the dual description are given by the Wilson lines of the gauge field $\tilde{a}$ (called 't Hooft lines in the original description):

$$T_m(C) = \exp\left(im \int_C \tilde{a}\right) = \exp\left(im \int_{M^4} \delta(C) \wedge \tilde{a}\right), \quad (2.2.24)$$

where $\delta(C)$ is the dual form to the curve C and $m \in \mathbb{Z}$ is the magnetic charge of the 't Hooft line. This time the physical interpretation of the 't Hooft line is that of a heavy probe particle of magnetic charge m . The action of the magnetic 1-form symmetry on the 't Hooft line is identical to that of the electric 1-form symmetry on the Wilson line (see equation (2.2.16)).

2.2.2 Spontaneous symmetry breaking

It is well known that in quantum field theory, symmetries can be spontaneously broken, meaning that the vacuum state of the theory does not respect the symmetry of the action. This phenomenon has profound implications, leading to the emergence of massless excitations known as Goldstone bosons. The same holds also in the case of higher-form symmetries, where the spontaneous breaking of a p -form symmetry leads to the appearance of massless p -form fields, which can be interpreted as "Goldstone bosons" associated with the broken symmetry, [48], [51], [34]. The proof of the above statement is a straightforward generalization of the one usually given for ordinary symmetries, and we will omit it here, for the sake of brevity. We will, instead, focus on the consequences which arise from the spontaneous breaking of a p -form symmetry.

We recall that in the case of ordinary symmetries, the spontaneous breaking of a symmetry is characterized by the non-vanishing of the vacuum expectation value (vev) of a charged operator $\mathcal{O}(x)$, or equivalently, by the non-vanishing of its two-point correlation function at large distances, i.e.

$$\langle \mathcal{O}(x) \rangle \neq 0 \quad \text{or} \quad \lim_{|x-y| \rightarrow +\infty} \langle \mathcal{O}(x) \mathcal{O}(y) \rangle \neq 0. \quad (2.2.25)$$

As concerns higher form symmetries, mimicking the above definition, we can say that a p -form symmetry is spontaneously broken if there exists a charged p -dimensional operator $\mathcal{O}(M_p)$ whose asymptotic correlation function does not vanish, i.e.

$$\lim_{R \rightarrow +\infty} \langle \mathcal{O}(S^p(R)) \rangle \neq 0, \quad (2.2.26)$$

where $S^p(R)$ is a p -dimensional surface of radius R .

We will now restrict our attention to line operators; we can see that the spontaneous breaking of a 1-form symmetry is characterized by the non-vanishing of the vacuum expectation value of a Wilson line or a 't Hooft line. In general, the vev of a line operator $\mathcal{O}$ supported on a curve C can show two different behaviors, depending on the details of the theory. In particular, it could exhibit what is known as an area law, meaning that the vev of the operator decays exponentially with the area $A(C)$ of the surface bounded by the curve C , i.e.

$$\langle \mathcal{O}(C) \rangle = e^{-T \cdot A(C)}, \quad (2.2.27)$$

or it could follow a perimeter law, meaning that the vev of the operator decays exponentially with the length $L(C)$ of the curve C , i.e.

$$\langle \mathcal{O}(C) \rangle = e^{-T' \cdot L(C)}, \quad (2.2.28)$$

where T and T' are constants which depend on the details of the theory. The distinction between them is physical, but only after renormalization: a line operator carries a divergence proportional to its length, and adding the corresponding counterterm defines a renormalized operator

$$\mathcal{O}_{Ren}(S_{Ren}^1) = \mathcal{O}(S_{Ren}^1) \cdot \exp \left(\int_{S_{Ren}^1} \text{vol}_{S_{Ren}^1} \right). \quad (2.2.29)$$

This new regularized operator $\mathcal{O}_{Ren}$ has a non-vanishing vev, thus the symmetry is spontaneously broken. In contrast, an operator following (2.2.27) even after renormalization has a vanishing vev, thus the symmetry is unbroken. So, in conclusion, the fate of a 1-form symmetry depends on the behavior of the vev of the line operators, which can be used as an order parameter to study the symmetry breaking. Moreover, as we showed before, line operators correspond to the insertion of heavy probe particles in the theory; specifically, if one considers a line operator on the particular curve C shown in figure 6, then the vev of the operator can be interpreted as the amplitude for the creation of a particle-antiparticle pair, which propagate along the two branches of the curve C and then annihilate each other, [42]. Such amplitude is related to the potential energy $V(R)$ of the particle-

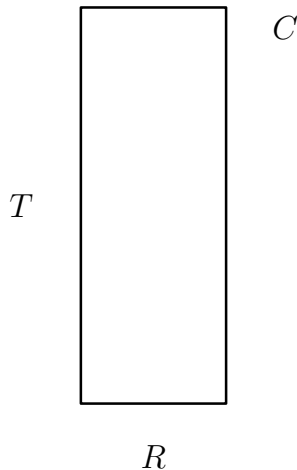


Figure 6: The Wilson Loop is inserted along the path C , where R is the displacement between the particle-antiparticle pair and T is the time between the pair creation and annihilation.

antiparticle pair, which is a function of the distance R between the two particles, by the following relation

$$\langle \mathcal{O}(C) \rangle = e^{-V(R)T}, \quad (2.2.30)$$

where T is the time interval during which the particles propagate. Thus, the two different behaviors described above can be rephrased in terms of the potential energy $V(R)$ of the particle-antiparticle

pair; in particular, the area law (2.2.27) corresponds to a linearly growing potential

$$V(R) \sim R,$$

which is a signature of confinement, while the perimeter law (2.2.28) corresponds to a constant potential

$$V(R) \sim \text{const},$$

which is a signature of deconfinement.

So, we can conclude that the spontaneous breaking of a 1-form symmetry is related to the confinement/deconfinement of the charged particles under the symmetry.

2.2.3 Spontaneous symmetry breaking in Maxwell theory

Pure Maxwell theory in four dimensions is again a simple, yet instructive, example of a theory with 1-form symmetry breaking. In Coulomb phase, the vev of a Wilson line (and by electric-magnetic duality, also of a 't Hooft line) is such that

$$V(R) \sim \frac{1}{R},$$

which, just like the perimeter law, signals a spontaneous breaking of the electric and magnetic 1-form symmetries. Starting from the considerations made above, we should thus expect the existence of a massless 1-form field: indeed, the photon is precisely this Goldstone boson. Actually, given that both the electric and magnetic 1-form symmetries are spontaneously broken, we could erroneously conclude that there should be two massless 1-form fields; however, as we are going to prove, the existence of the massless photon is enough to account for the breaking of both symmetries. In fact, as argued by [34], one could compactify the theory on a circle, thus obtaining a three-dimensional theory; depending on whether the original four-dimensional symmetry operators wrap the compactified direction or not, we would obtain either a 0-form or a 1-form symmetry in three dimensions from both the electric and magnetic original 1-form symmetries. The resulting 0-form symmetries are spontaneously broken⁸, leading to the existence of two massless scalar fields, which, upon uplifting to four dimensions, rearrange themselves into the two polarizations of a single massless 1-form field, the photon.

Of course the above discussion is valid only in the absence of matter; indeed, if we introduce electrically charged matter, then the electric 1-form symmetry is explicitly broken⁹, while the magnetic 1-form symmetry remains unbroken. However, rather counterintuitively, especially if compared to the case of ordinary symmetries, the photon remains massless, even though it should now be treated as pseudo-Goldstone boson. This somewhat surprising result can be inferred by looking at what happens to the theory in the low energy limit: as shown by [26], when one explicitly breaks a Maxwell-like 1-form symmetry that is spontaneously broken, the low energy effective theory, obtained by integrating out the massive matter degrees of freedom, recovers the original 1-form symmetry, which is then spontaneously broken, again leading to a massless photon¹⁰.

2.2.4 Screening of extended operator in higher form symmetries

As anticipated in the previous section, adding electrically charged matter to Maxwell theory (more generally to a non abelian gauge theory) explicitly breaks the $U(1)_e$ symmetry. Actually, as we are

⁸While the 3d 1-form symmetries remain unbroken.

⁹We will state this more deeply in the following section, but it is readily shown simply by looking at the RHS of (2.2.8), which is no more 0 in presence of electrically charged matter.

¹⁰Crucially, the EFT obtained by integrating out the massive modes is not itself sufficient to show that the $U(1)_e$ symmetry is emergent in the IR; the authors of [26] clearly show this with a counterexample.

going to see, the above statement is a bit too strong; in fact, part of the original 1-form continuous symmetry could still survive, though only a discrete subgroup of it. Moreover, we could also consider adding magnetically charged matter, which would explicitly break the $U(1)_m$ symmetry, or, more generally, dyonic matter, which would explicitly break both the $U(1)_e$ and $U(1)_m$ symmetries. In order to study the general fate of a p -form symmetry of a gauge theory in presence of matter, it is useful to introduce the notion of screening of extended operators.

We say that an extended $p \geq 1$ operator $\mathcal{O}(M_p)$ is screened to another extended p operator $\mathcal{O}'(M_p)$ if there exists a local $p-1$ operator $\mathcal{O}_{p-1}(M_{p-1})$ such that $\mathcal{O}_{p-1}(M_{p-1})$ can be inserted between $\mathcal{O}(M_p)$ and $\mathcal{O}'(M_p)$. In particular, whenever $\mathcal{O}'(M_p)$ is the identity operator, we say that $\mathcal{O}(M_p)$ is totally screened.

Another useful notion is that of Pontryagin dual group. As mentioned in section 2.2, a p -form symmetry group $G^{(p)}$ is always abelian, and thus its irreducible 1d representations could be identified with group homomorphisms

$$\phi : G^{(p)} \rightarrow U(1).$$

The set of all such homomorphisms, equipped with the group multiplication of complex numbers, forms a group called the Pontryagin dual group of $G^{(p)}$, which we will denote by $\hat{G}^{(p)}$. Then the charges of p -dimensional operators under the p -form symmetry group $G^{(p)}$ are just elements of the Pontryagin dual group $\hat{G}^{(p)}$.

Starting from these premises, we can now describe the fate of a p -form symmetry in presence of matter. First, we notice that each pair of screened operators $\mathcal{O}(M_p)$ and $\mathcal{O}'(M_p)$ necessarily have the same charge under the p -form symmetry group $G^{(p)}$ (see figure 7), e.g.

$$Q(\mathcal{O}) = Q(\mathcal{O}') \in \hat{G}^{(p)}.$$

This allows us to define an equivalence relation on the set of p -dimensional operators, by saying that

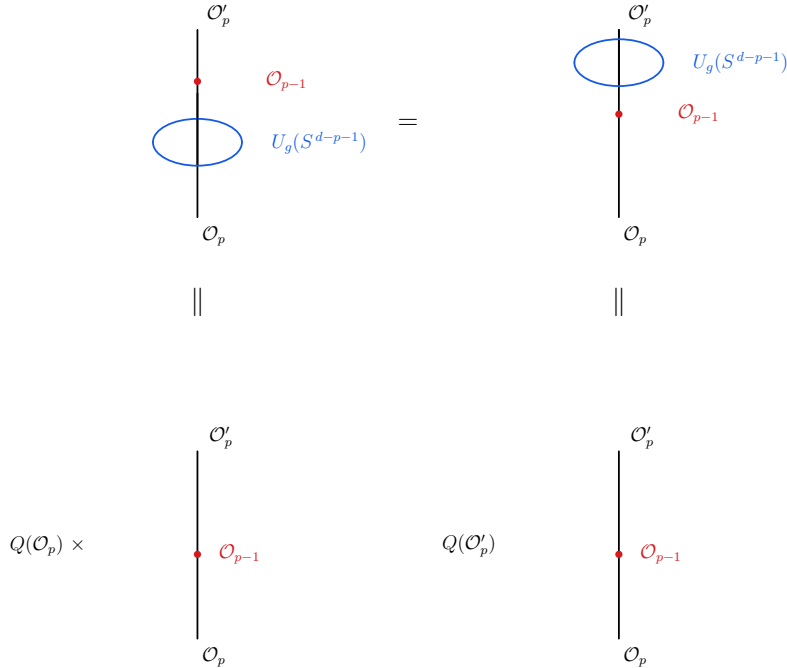


Figure 7: The two configurations at the top are related by topological move, hence they are equivalent. On the left, we measure the charge of $\mathcal{O}_p$, while on the right the charge of $\mathcal{O}'_p$; it follows that their charges are equal.

two operators are equivalent if they are screened to each other. Let D_p be the set of equivalence

classes of p -dimensional operators and suppose that it acquires the structure of a group from the OPE of the operators. Then, we can define a p -form symmetry group $G^{(p)}$ such that its charges distinguish different elements of D_p , namely

$$G^{(p)} = \hat{D}_p. \quad (2.2.31)$$

Consequently, the charges of the p -form symmetry group $G^{(p)}$ are given by

$$\hat{G}^{(p)} = \hat{\hat{D}}_p = D_p, \quad (2.2.32)$$

where we have made use of the fact that the Pontryagin dual of a Pontryagin dual group is isomorphic to the original group.

2.2.5 Screening in Maxwell theory

Once again, Maxwell theory provides an instructive playground for the arguments of the previous section. In fact, let us study what happens to the electric and magnetic 1-form symmetries of Maxwell theory in presence of a single matter field ϕ of electric charge $q \in \mathbb{Z}$. The first striking fact is that Wilson Lines of charge q can "end", meaning that they can be screened to the identity operator by the insertion of a field ϕ at the end of the line, see figure 8. In fact, this configuration is gauge invariant,

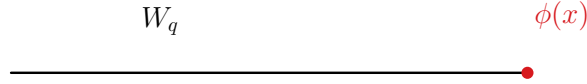


Figure 8: The matter field ϕ is inserted at the endpoint x of the Wilson line. Notice that this configuration is gauge invariant, hence well defined.

since the gauge transformation of the Wilson line is exactly canceled by the gauge transformation of the matter field ϕ . More in detail, after a gauge transformation like

$$a \rightarrow a - d\theta,$$

the field ϕ transforms as

$$\phi(x) \rightarrow e^{iq\theta} \phi(x).$$

Consider a Wilson line $W_q(C)$ of charge q supported on a curve C which ends at a point x , then under the gauge transformation we have

$$W_q(C) = \exp \left(iq \int_{-\infty}^x a \right) \rightarrow W_q(C) \cdot \exp \left(-iq \int_{-\infty}^x d\theta \right) = W_q(C) \cdot e^{-iq\theta(x)}. \quad (2.2.33)$$

So the combination

$$W_q(C)\phi(x)$$

is gauge invariant. Following the reasonings of the previous section, we can conclude that the Wilson line $W_q(C)$, being completely screened, has no charge under the electric 1-form symmetry, which is thus explicitly broken by the presence of the matter field ϕ . This can also be understood pictorially by looking at figure 9. In general, one can add n powers of the field ϕ , resulting in the fact that Wilson lines of charge nq are also totally screened. However, this does not mean that every Wilson line is totally screened; take for example a Wilson line of (integer) charge $p \neq q$; it is well known that for any pair of integer numbers p and q there exist two integers, say m and n , such that

$$p = nq + m. \quad (2.2.34)$$

Then, the combination

$$W_p(C)\phi^n(x)W_m(C') \quad (2.2.35)$$

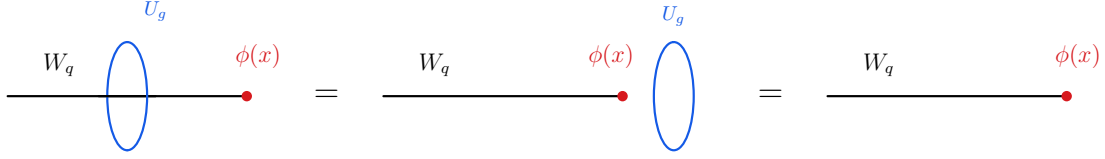


Figure 9: At the left the symmetry operator U_g links with the screened Wilson line W_q ; then, it is moved to the right and made to shrink to a point. The net effect reveals that the Wilson line possesses no charge.

with C a line ending in x and C' another line starting from x , is gauge invariant. Then, it is evident that the set D_p of equivalence classes is simply given by $\mathbb{Z}_q$. Using the general result given in (2.2.31), we have that the explicitly unbroken symmetry reduces to

$$G^{(1)} = \hat{\mathbb{Z}}_q = \mathbb{Z}_q, \quad (2.2.36)$$

where in the last equation we have made use of the fact that the Pontryagin dual of a finite group coincides with the group itself. So, to conclude, we see that the original $U(1)_e$ symmetry is explicitly broken down to $\mathbb{Z}_q$, while $U(1)_m$ remains unbroken. Of course, we could repeat the same steps for the case of magnetically charged matter, or, more generally, for dyonic charged matter. Such details, which we will not cover, are found in [19].

2.2.6 Gauging of continuous higher form symmetries

Another interesting aspect concerning higher form symmetries is the notion of gauging. We recall that when discussing ordinary symmetries, we are allowed to couple the theory to an external background gauge field through the conserved current,

$$\int A_1 \wedge \star j_1. \quad (2.2.37)$$

From the fact that j_1 is conserved (e.g. $d \star j_1 = 0$), we immediately deduce that such coupling is well defined, i.e. is invariant under gauge transformations of the type:

$$A_1 \rightarrow A_1 + d\lambda. \quad (2.2.38)$$

Of course, we can readily generalize this result to p -form symmetries in the following way

$$\int A_{p+1} \wedge \star j_{p+1}. \quad (2.2.39)$$

In a similar way as above, from the fact that j_{p+1} is co-closed, we may conclude that the following transformations

$$A_{p+1} \rightarrow A_{p+1} + d\lambda_p \quad (2.2.40)$$

leaves the action invariant¹¹.

So far the background field is a fixed source. To gauge the symmetry, we promote it to a dynamical field and sum over its gauge inequivalent configurations. Concretely, let $\mathcal{T}$ a theory with partition function $\mathcal{Z}_{\mathcal{T}}$ which enjoys a p -form global symmetry $G^{(p)}$, then the partition function of the new theory obtained by gauging $G^{(p)}$, denoted by $\mathcal{T}/G^{(p)}$, is given by

$$\mathcal{Z}_{\mathcal{T}/G^{(p)}} \sim \int \mathcal{D}[A_{p+1}] \mathcal{Z}_{\mathcal{T}}[A_{p+1}], \quad (2.2.41)$$

where $\mathcal{Z}_{\mathcal{T}}[A_{p+1}]$ is the partition function of $\mathcal{T}$ coupled to the background gauge field A_{p+1} . As we are going to show, it is not always true that the gauged theory is a well-defined theory.

¹¹The field A_{p+1} is a $p+1$ gauge field. It is defined recursively as a local $p+1$ form subject to the redundancy $A_{p+1} \sim A_{p+1} + d\lambda_p$, with λ_p a p form gauge field with flux properly quantized $\int \frac{dA_{p+1}}{2\pi} \in \mathbb{Z}$.

2.2.7 Anomaly

Background fields can be useful to probe anomalies in quantum field theories. Let $\mathcal{Z}$ be the partition function of a given theory featuring a p -form symmetry; if we couple the theory with a $p+1$ gauge field A_{p+1} , the partition function becomes a functional of A_{p+1} , e.g.

$$\mathcal{Z}[A_{p+1}] = \int \mathcal{D}\phi^i \exp(-S[\phi, A_{p+1}]), \quad (2.2.42)$$

where $\{\phi^i\}$ denotes the set of all fields (not necessarily scalars) belonging to the original, ungauged, theory. We will say that the theory is affected by an (t'Hooft) anomaly if

$$\mathcal{Z}[A_{p+1} + d\lambda_p] \neq \mathcal{Z}[A_{p+1}]. \quad (2.2.43)$$

In particular, we can write

$$\mathcal{Z}[A_{p+1} + d\lambda_p] = e^{-i \int \mathcal{A}[A_{p+1}, \lambda_p]} \mathcal{Z}[A_{p+1}], \quad (2.2.44)$$

where $\mathcal{A}$ is such that $\mathcal{A}[A_{p+1}, 0] = 0$. It should be noted that, in the literature, the term *anomaly* may appear with a different meaning and refers to the concept of ABJ anomaly. The latter is related to the fact that, upon quantization, a classical global symmetry may fail to be a symmetry of the quantum theory. ABJ anomalies have been extensively studied through the so called "triangle diagrams" in perturbation theory and the Fujikawa regularization of the path integral, see [2], [14] and [32]. Evidently, whenever 't Hooft anomalies are present, there is no way to gauge the global p -form symmetry in a sensible way, thus such anomalies represent an obstruction to gauging.

A useful description of t'Hooft anomalies relies on the concept of *anomaly inflow*. We start by writing the exterior derivative of the anomaly phase $\mathcal{A}[A_{p+1}, \lambda_p]$ as the gauge variation of a $(d+1)$ -form $\hat{\mathcal{A}}$ depending only on the $p+1$ gauge field A_{p+1} ,

$$d\mathcal{A}[A_{p+1}, \lambda_p] = \delta_\lambda \hat{\mathcal{A}}[A_{p+1}], \quad (2.2.45)$$

then we modify the partition function in the following way

$$\hat{\mathcal{Z}}[A_{p+1}] = \mathcal{Z}[A_{p+1}] e^{i \int_{N_{d+1}} \hat{\mathcal{A}}[A_{p+1}]}, \quad (2.2.46)$$

being $\partial N_{d+1} = M_d$, where M_d is the spacetime over which our original theory is defined. It is easy to see that the new partition function is gauge invariant; in fact

$$\begin{aligned} \hat{\mathcal{Z}}[A_{p+1} + d\lambda_p] &= \mathcal{Z}[A_{p+1} + d\lambda_p] e^{i \int_{N_{d+1}} \hat{\mathcal{A}}[A_{p+1} + d\lambda_p]} \\ &= \mathcal{Z}[A_{p+1}] e^{-i \int \mathcal{A}[A_{p+1}, \lambda_p]} e^{i \int_{N_{d+1}} \hat{\mathcal{A}}[A_{p+1}] + \delta_\lambda \hat{\mathcal{A}}[A_{p+1}]} \\ &= \mathcal{Z}[A_{p+1}] e^{i \int_{N_{d+1}} \hat{\mathcal{A}}[A_{p+1}]} = \hat{\mathcal{Z}}[A_{p+1}], \end{aligned} \quad (2.2.47)$$

where in the third equality we have made use of (2.2.45) and Stoke's theorem.

2.2.8 Anomalies in Maxwell Theory

We recall from section 2.2.1 that pure Maxwell theory features two 1-form symmetries with currents as below,

$$j_e = \frac{1}{e^2} f, \quad j_m = \frac{1}{2\pi} \star f. \quad (2.2.48)$$

We can couple the theory to two background gauge fields B_e and B_m as follows

$$\begin{aligned} S[B_e, B_m] &= \frac{1}{2e^2} \int f \wedge \star f + \int B_e \wedge \star j_e + \int B_m \wedge \star j_m. \\ &= \frac{1}{2e^2} \int f \wedge \star f + \frac{1}{e^2} \int B_e \wedge f + \frac{1}{2\pi} \int B_m \wedge f. \end{aligned} \quad (2.2.49)$$

Moreover, we can also add any local counterterm depending only on the background gauge fields; in particular we can add the following counterterm

$$S_{ct}[B_e] = \frac{1}{2e^2} \int B_e \wedge \star B_e. \quad (2.2.50)$$

Then, our action becomes

$$\begin{aligned} S[B_e, B_m] &= \frac{1}{2e^2} \int f \wedge \star f + \frac{1}{e^2} \int B_e \wedge f + \frac{1}{2\pi} \int B_m \wedge f + S_{ct}[B_e] \\ &= \frac{1}{2e^2} \int (f + B_e) \wedge \star (f + B_e) + \frac{1}{2\pi} \int B_m \wedge f. \end{aligned} \quad (2.2.51)$$

Now, suppose we perform a gauge transformation on the magnetic background gauge field as follows,

$$B_m \rightarrow B_m + d\lambda_m, \quad (2.2.52)$$

then we can clearly see that the action is invariant under such transformation; indeed, since both a and λ_m are gauge fields, we have that their fluxes are quantized, meaning that

$$\delta_{\lambda_m} S = \frac{1}{2\pi} \int f \wedge d\lambda_m \in 2\pi\mathbb{Z}. \quad (2.2.53)$$

Let us now see what happens if we try to perform a similar variation concerning the electric background gauge field,

$$B_e \rightarrow B_e + d\lambda_e. \quad (2.2.54)$$

In this case, the first term is unaffected if we consider a shift in the path integral of a as $a \rightarrow a - \lambda_e$ ¹². However, as regards the second term, it shifts the action by

$$\delta_{\lambda_e} S = -\frac{1}{2\pi} \int d\lambda_e \wedge B_m. \quad (2.2.55)$$

Such phase is not quantized, since fluxes of B_m need not be quantized, thus the action is not invariant under the above transformation. Therefore, we can conclude that whenever the magnetic 1-form symmetry is turned on, the electric 1-form symmetry is anomalous. Of course, we could have made any another choice of local counterterms, for example we could also add the following piece

$$S'_{ct}[B_m] = \frac{1}{2\pi} \int B_e \wedge \star B_m. \quad (2.2.56)$$

This choice leads to the following action

$$S[B_e, B_m] = \frac{1}{2e^2} \int (f + B_e) \wedge \star (f + B_e) + \frac{1}{2\pi} \int (f + B_e) \wedge B_m. \quad (2.2.57)$$

¹²This is an allowed shift since λ_e is a gauge field.

Now, it is evident that the action is invariant under the gauge transformation of the electric background gauge field; however, the magnetic 1-form symmetry is no more anomalous-free. Indeed, under the gauge transformation of the magnetic background gauge field, we have that the action shifts by

$$\delta_{\lambda_m} S = \frac{1}{2\pi} \int f \wedge d\lambda_m + \frac{1}{2\pi} \int B_e \wedge d\lambda_m, \quad (2.2.58)$$

where the first term is the innocuous contribution that we have already seen above, while the second one is a genuine anomaly term, since the flux of B_e is not quantized.

To summarize, we can say that the electric and magnetic 1-form symmetries of Maxwell theory are mutually anomalous, meaning that whenever one of them is turned on, the other one is anomalous. This shows a particular property of mixed 't Hooft anomalies, which is that local counterterms can be added to the action in order to shift the anomaly from one symmetry to another, but they cannot be removed completely.

Finally, we can employ the anomaly inflow mechanism to "get rid" of the anomaly, by adding a 5d term in the action,

$$S_{5d} = \frac{1}{2\pi} \int_{N_5} B_e \wedge dB_m, \quad \partial N_5 = M_4, \quad (2.2.59)$$

which is such that its gauge variation cancels the anomaly of the 4d action, as

$$\delta_{\lambda_e} S_{5d} = \frac{1}{2\pi} \int_{N_5} d\lambda_e \wedge dB_m = \frac{1}{2\pi} \int_{M_4} d\lambda_e \wedge B_m. \quad (2.2.60)$$

Thus, we see that the variation of 5d term exactly cancels the anomaly of the 4d action.

2.3 Higher continuous groups

In this section we introduce the notion of higher groups. For brevity, we will not give a rigorous definition, focusing instead on their physical meaning and their relation to higher form symmetries. Intuitively, a higher group is the structure that arises when several higher form symmetries of different degrees are combined in a non-trivial way. More specifically, following [21], suppose we have a theory that enjoys n higher form symmetries of degrees

$$p_1 < p_2 < \cdots < p_n.$$

To each of these symmetries we can associate a background gauge field A_{p_i+1} , which is a $p_i + 1$ form gauge field. We say that the theory has a $(p_n + 1)$ higher group symmetry $\mathbb{G}^{p_n+1}$ if the background gauge fields A_{p_i+1} are not independent, but they are related to each other by a non-trivial transformation law:

$$A_{p_i+1} \rightarrow A_{p_i+1} + d\Lambda_{p_i} + \sum_{k < p} \Lambda_{p_k} \wedge \alpha_k^{(i)}(\{A_{p_k+1}\}) + \text{Schwinger terms} \quad (2.3.1)$$

where the $\alpha_k^{(i)}(\{A_{p_k+1}\})$ are $(p_i - p_k + 1)$ -forms and the Schwinger terms are non-linear contributions of the gauge transformation parameters Λ_{p_k} , restricted to depend only on the parameters of lower degree $p_k < p_i$. This implies that the higher group symmetries possess a nested structure, of the type

$$\mathbb{G}^{p_n+1} \subset \mathbb{G}_{p_1}^{p_n+1} \subset \cdots \subset \mathbb{G}_{p_{n-1}}^{p_n+1} \subset \mathbb{G}_{p_n}^{p_n+1}, \quad (2.3.2)$$

where $\mathbb{G}_{p_i}^{p_n+1}$ is the $(p_n + 1)$ higher group whose lowest degree symmetry is p_i .

We now briefly show what physical constraints are imposed by the existence of a higher group symmetry. First, we look at the symmetry breaking pattern of higher group symmetries. Suppose that the

highest degree symmetry p_n of the higher group is spontaneously broken; then, from equation (2.3.1), the lower degree symmetries p_i with $i < n$ are spontaneously broken as well, since the transformation law of the corresponding background gauge fields A_{p_i+1} is non-trivial. This means that the entire higher group symmetry is spontaneously broken. From a simple generalization of the above argument, we can see that the breaking of the higher form symmetry of degree p_i implies the breaking of all the higher form symmetries of degree p_j with $j < i$.

This conclusion carries an interesting physical consequence. Consider starting from a UV theory with no higher form symmetries which, upon flowing to the IR, develops an emergent higher group symmetry. Since the higher group symmetry must obey the nested structure (2.3.2), denoting by E_i the energy scale at which the p_i higher form symmetry emerges, we must have that

$$E_1 \lesssim E_2 \lesssim \cdots \lesssim E_n, \quad (2.3.3)$$

meaning that the higher form symmetries of lower degree emerge at lower energy scales, while the higher form symmetries of higher degree emerge at higher energy scales.

This is a remarkable result: the emergence of higher form symmetries is not arbitrary, but is constrained by the higher group structure. In particular, it is a hierarchical process, in which higher degree symmetries emerge first, followed by those of lower degree.

2.4 Discrete symmetries

Even though we have only shown explicit examples of continuous higher form and higher group symmetries, the concept of generalized symmetries is not limited to the continuous case; indeed, when we defined p -form symmetries in Section 2.2, we did not specify whether the symmetry group $G^{(p)}$ was continuous or discrete. This means that most of the results we have shown so far are also valid for discrete symmetries, with the only exception of the Goldstone theorem, which, as for ordinary symmetries, demands the symmetry group to be continuous.

The contrast with continuous symmetries concerns the explicit realization of the symmetry operators themselves: while a continuous p -form symmetry provides a conserved $(p+1)$ -form current from which the operators can be built explicitly, discrete symmetries do not come equipped with such currents, so realizing their operators is trickier.

A precise mathematical description of discrete symmetries would require us to introduce discrete gauge theories, as reviewed in [30]. However, in order to avoid delving too much into the mathematical details, we will follow a more physical approach, which somewhat hides the mathematical technicalities, but still captures the main physical ideas behind discrete symmetries, [22].

For instance, to keep the discussion simple, we will focus our attention to the group $\mathbb{Z}_n$, one of the simplest discrete groups. What makes $\mathbb{Z}_n$ special is that it can be realized as a subgroup of $U(1)$, which allows us to embed the discrete symmetry into a continuous one. A $\mathbb{Z}_n$ connection is restricted to have holonomies valued in $\mathbb{Z}_n$

$$\exp\left(i \oint_{\gamma} A\right) = e^{\frac{2\pi i k}{n}}, \quad k \in \mathbb{Z}_n, \quad (2.4.1)$$

In particular, whenever γ is contractible, we have that the field strength $F = dA$ is trivial, meaning that the $\mathbb{Z}_n$ gauge field is (locally) flat. This fact can be understood by simply noticing that a continuous deformation of the contractible loop γ should correspond to a continuous change of the holonomy, which is not possible since the holonomy is valued in a discrete group and so it would be forced to "jump" discontinuously. In particular, for contractible loops, we can shrink γ to a point,

thus obtaining that the holonomy is trivial. Moreover, if $\gamma = \partial D_2$ then we can use Stoke's theorem to write

$$\oint_{\gamma} A = \int_{D_2} F = 0, \quad (2.4.2)$$

which shows that the field strength F is (locally) trivial.

The above discrete gauge field A can be described by an associated $U(1)$ gauge field $\hat{A}$, called integral lift, which is required to possess the same holonomies as A , namely

$$\exp \left(i \oint_{\gamma} A \right) = \exp \left(i \oint_{\gamma} \hat{A} \right) \quad (2.4.3)$$

which implies

$$\oint_{\gamma} \frac{A}{2\pi} = \oint_{\gamma} \frac{\hat{A}}{2\pi} \bmod \mathbb{Z}. \quad (2.4.4)$$

Thus, the choice of lift is far from being unique; such choices are related to each other by large gauge transformations of the form

$$\hat{A} \rightarrow \hat{A} + \Lambda, \quad \oint_{\gamma} \Lambda \in 2\pi\mathbb{Z}. \quad (2.4.5)$$

Finally, given the fact that $U(1)$ gauge field need not be flat, we can define the "field strength" of the discrete gauge field A as

$$\frac{F}{2\pi} := \frac{d\hat{A}}{2\pi} \bmod n. \quad (2.4.6)$$

This definition is related (up to some normalization constant) to the so called Bockstein map, an operation in discrete cohomology which serves as definition of "field strength" for the discrete gauge fields.

2.4.1 BF theory

One the most known examples of discrete gauge theories in physics is the so called BF theory. BF theory is one of the earliest examples of a topological quantum field theory and its action was first proposed by [43] in the context of quantum gravity.

Consider the following theory in a d -dimensional spacetime,

$$S = \frac{n}{2\pi} \int B_{d-p-1} \wedge dA_p, \quad (2.4.7)$$

where B_{d-p-1} is a $(d-p-1)$ -form and A_p is a p -form. The action enjoys a gauge symmetry given by

$$A_p \rightarrow A_p + d\Lambda_{p-1}, \quad B_{d-p-1} \rightarrow B_{d-p-1} + d\Lambda_{d-p-2}, \quad (2.4.8)$$

where

$$\oint \frac{d\Lambda_q}{2\pi} \in \mathbb{Z}, \quad \text{for } q = (p-1) \text{ or } (d-p-2). \quad (2.4.9)$$

Moreover, since no Hodge star operator is present in the action, the theory is independent of any metric, thus it is topological. Finally, the equation of motion, given by varying with respect to A_p and B_{d-p-1} , are

$$\frac{n}{2\pi} dB_{d-p-1} = 0, \quad \frac{n}{2\pi} dA_p = 0. \quad (2.4.10)$$

Up to now, there is no reason to believe that this theory is really a $\mathbb{Z}_n$ gauge theory, but it can be proven as follows. First we will prove that the path integral, together with the quantization condition of the $U(1)$ fields strength

$$\int \frac{dA_p}{2\pi} \in \mathbb{Z} \quad \int \frac{dB_{d-p-1}}{2\pi} \in \mathbb{Z}, \quad (2.4.11)$$

forces the holonomies to be in $\mathbb{Z}_n$. In the path integral, the field configuration of A_p may be labeled by the integer quantity $k = \int \frac{dA_p}{2\pi}$. Thus, the path integral is something like

$$\begin{aligned} \mathcal{Z} &\sim \int \mathcal{D}B \mathcal{D}A \, e^{\frac{in}{2\pi} \int B_{d-p-1} \wedge dA_p} \\ &= \mathcal{N} \sum_{k \in \mathbb{Z}} \int \mathcal{D}B \, e^{ink \int B_{d-p-1}} = 2\pi \mathcal{N} \sum_{m \in \mathbb{Z}} \delta \left(n \int B_{d-p-1} - 2\pi m \right), \end{aligned} \quad (2.4.12)$$

where $\mathcal{N}$ is some constant independent of B_{d-p-1} coming from the trivial sector of A_p , and in the last equation we have made use (of a generalization) of

$$\sum_{k \in \mathbb{Z}} e^{ik\theta} = 2\pi \sum_{m \in \mathbb{Z}} \delta(\theta - 2\pi m). \quad (2.4.13)$$

Therefore, the only relevant term for the path integral are those for which B_{d-p-1} has discrete holonomies. Therefore, BF theory is indeed a discrete gauge theory.

Now we observe that the theory possesses $\mathbb{Z}_n^{(p)}$ and $\mathbb{Z}_n^{(d-p-1)}$ global symmetries which are realized by the following transformations:

$$\mathbb{Z}_n^{(p)} : \quad A_p \rightarrow A_p + \frac{1}{n} \epsilon_p, \quad \int \epsilon_p = 2\pi \mathbb{Z}, \quad (2.4.14)$$

$$\mathbb{Z}_n^{(d-p-1)} : \quad B_{d-p-1} \rightarrow B_{d-p-1} + \frac{1}{n} \epsilon_{d-p-1}, \quad \int \epsilon_{d-p-1} = 2\pi \mathbb{Z}. \quad (2.4.15)$$

Indeed, by inspecting the equation of motion (2.4.10), we can argue that there exist two conserved "currents", namely

$$\star j_{d-p} = \frac{n}{2\pi} A_p, \quad \star j_{p+1} = \frac{n}{2\pi} B_{d-p-1}. \quad (2.4.16)$$

Even though such currents are not well defined (they transform under gauge transformations (2.4.8)), the operators

$$U_{\frac{2\pi k}{n}}(\Sigma) = \exp \left(\frac{2\pi k}{n} \int_{\Sigma} \frac{n}{2\pi} A_p \right) = \exp \left(ik \int_{\Sigma} A_p \right), \quad (2.4.17)$$

$$\tilde{U}_{\frac{2\pi k}{n}}(\tilde{\Sigma}) = \exp \left(\frac{2\pi k}{n} \int_{\tilde{\Sigma}} \frac{n}{2\pi} B_{d-p-1} \right) = \exp \left(ik \int_{\tilde{\Sigma}} B_{d-p-1} \right), \quad (2.4.18)$$

are still topological (from (2.4.10)) and gauge invariant.

Now we will argue that the corresponding charged operators are the Wilson lines of A_p and B_{d-p-1} :

$$W_m(\Gamma) = \exp \left(im \int_{\Gamma} B_{d-p-1} \right), \quad (2.4.19)$$

$$W_m(\tilde{\Gamma}) = \exp \left(im \int_{\tilde{\Gamma}} A_p \right). \quad (2.4.20)$$

Indeed, taking into account the symmetry transformations (2.4.14) and (2.4.15), we see that the Wilson lines are $\mathbb{Z}_n$ charged:

$$W_m(\Gamma) \rightarrow e^{i\frac{m}{n}} W_m(\Gamma) \quad (2.4.21)$$

$$W_m(\tilde{\Gamma}) \rightarrow e^{i\frac{m}{n}} W_m(\tilde{\Gamma}). \quad (2.4.22)$$

Finally, in order to describe how the symmetry operators act on the charged operators we notice that the insertion of, say, $W_m(\Gamma)$ is equivalent to shifting the action by

$$\delta S = im \int A_p \wedge \delta(\Gamma), \quad (2.4.23)$$

being $\delta(\Gamma)$ the Poincaré dual to the manifold Γ . In turn, the flux of B_{d-p-1} gets sourced,

$$\frac{dB_{d-p-1}}{2\pi} = \frac{m}{n} \delta(\Gamma). \quad (2.4.24)$$

Ultimately, this means (see equation (2.1.20) and (2.2.6))

$$U_{\frac{2\pi k}{n}}(\Sigma) W_m(\Gamma) = e^{\frac{2\pi i k m}{n}} W_m(\Gamma) U_{\frac{2\pi k}{n}}(\Sigma') \quad (2.4.25)$$

where we have supposed that Σ links with Γ and Σ' is a continuous deformation of Σ which does not link with Γ . An analogous treatment holds for the charged operator $W_m(\tilde{\Gamma})$ and the symmetry operator $\tilde{U}_{\frac{2\pi k}{n}}(\tilde{\Sigma})$.

2.5 Non-invertible symmetries

Finally, we would like to end this section by mentioning a further generalization coming from the paradigm "symmetries as topological operators", that is the notion of non-invertible symmetries. Up to now, we have only dealt with symmetries described by invertible operators, which are associated to group structures (or higher group structures). However, the topological nature of the symmetry operators admits a more general description, which is not necessarily associated to the notion of a group. In turn, this leads to the novel concept of non-invertible symmetries, namely symmetries whose operators are not tied to a group structure, but rather to a more general algebraic structure, which is that of category. In the following we will omit the mathematical details of category theory, and we will try to convey the main physical ideas behind the non-invertible symmetries. To begin with, consider a set $\mathcal{C}$ (more properly a "class") such that any element ("object") $x \in \mathcal{C}$ corresponds to a topological operator U_x . Then, in sharp contrast with the symmetries discussed so far, the fusion of two operators U_{x_1} and U_{x_2} does not necessarily lead to the operator $U_{x_1 x_2}$ associated to the product of the two elements x_1 and x_2 (such product is not even defined in general), but rather to a more general form like:

$$U_{x_1} \times U_{x_2} = \begin{cases} \sum_i U_{x_i} \\ \mathcal{T} \otimes U_{x_1 x_2} \\ \dots \end{cases}, \quad (2.5.1)$$

where $\mathcal{T}$ is the partition function of a topological quantum field theory (TQFT).

Non-invertible symmetries are mostly understood in $(1+1)$ -dimension, where the categorical structure behind them is under control. For example, they have been successfully employed in studying the Ising model at the critical temperature exploiting the so called Kramers Wannier self-duality¹³, [27]. They also provide a new perspective on the neutral pion decay, as discussed, for instance, in [46, 60].

¹³The Kramers-Wannier duality is a self-duality only at the critical temperature, otherwise it is just a map between high temperature and low temperature phase, [46].

3 Fundamentals of String Theory

String theory is a vast and rich topic in modern theoretical physics; given the breadth of notions and facts surrounding it, we will simply outline the main ideas of string theory, concerning ourselves only with those aspects that will play a major role in the later sections of this thesis work; such as the AdS/CFT correspondence, 10-dimensional supergravity, and the holographic realization of symmetry defects.

Born as a candidate framework for a consistent theory of quantum gravity, string theory posits that the fundamental degrees of freedom are not point particles but one-dimensional extended objects, whose vibrational modes reproduce the spectrum of particles and fields observed at low energies. Beyond its gravitational content, the theory naturally accommodates gauge interactions and, through its web of dualities and extended objects—most notably Dp -branes—provides a remarkably fertile setting in which seemingly distinct physical systems are related to one another.

Among these developments, one of the most far-reaching is the realization that certain string backgrounds admit an equivalent description in terms of a lower-dimensional field theory living on their boundary. This is the essence of the AdS/CFT correspondence, reviewed in Section 4, which underlies the holographic constructions at the heart of this work. The material collected in the present section is therefore intended not only as a self-contained review, but also as the necessary groundwork for the holographic realization of symmetry defects discussed later on.

What follows is based on the works of [13], [63], [56] and [57].

3.1 Bosonic string

The central idea of string theory is that, at the fundamental level, matter is not made of point-like particles but of extended one-dimensional objects: strings.

Our first goal is thus to write down a suitable action describing the dynamics of such strings; the simplest one we can conceive leads to bosonic string theory. The route we adopt to write the action of a one-dimensional string moving in a D -dimensional spacetime mimics the familiar construction for ordinary point-like particles. Indeed, the relativistic action of a (classical) particle is

$$S = -mc^2 \int_{\gamma} d\tau \sqrt{-g_{\mu\nu}(X(\tau)) \dot{X}^{\mu}(\tau) \dot{X}^{\nu}(\tau)}, \quad (3.1.1)$$

where γ is the worldline swept by the particle as it moves in spacetime, and τ is an unphysical parameter along the curve γ . The coordinates $X^{\mu}(\tau)$ can be regarded geometrically as an embedding of the one-dimensional worldline γ into the D -dimensional spacetime, and the action (3.1.1) is proportional to the "length" of γ . The least action principle then tells us that particles move so as to extremize their worldline length. The same logic generalizes to any object extending in p spatial dimensions: the one-dimensional worldline is promoted to a $(p+1)$ -dimensional worldvolume. Focusing on strings, let $X^{\mu}(\tau, \sigma)$ be the embedding of the two-dimensional worldsheet into spacetime; then, in analogy with (3.1.1), the action of a string is (in natural units)

$$S = -T \int d\tau d\sigma \sqrt{-\det [g_{\mu\nu}(X(\tau, \sigma)) \partial_{\alpha} X^{\mu}(\tau, \sigma) \partial_{\beta} X^{\nu}(\tau, \sigma)]}, \quad (3.1.2)$$

where τ and σ are worldsheet coordinates and T is the tension (energy per unit length) of the string. Here σ is a spacelike coordinate on the worldsheet and τ a timelike one; for a general p -extended object one has p spatial coordinates and a single timelike one. The determinant is taken with respect to the worldsheet coordinates, labelled by the indices α and β . Equation (3.1.2) is known as *Nambu-Goto*

action¹⁴. The square root in (3.1.2) is somewhat problematic, especially in the quantum theory. For this reason it is often useful to rewrite it in a different form: we introduce an auxiliary worldsheet metric $h_{\alpha\beta}(\tau, \sigma)$ and consider the action (with $h := \det(h_{\alpha\beta})$)

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X, \quad (3.1.3)$$

where

$$X \cdot X := g_{\mu\nu} X^\mu X^\nu, \quad h^{\alpha\beta} h_{\beta\gamma} = \delta_\gamma^\alpha, \quad (3.1.4)$$

and $d^2\sigma$ is shorthand for $d\sigma d\tau$.

It is easy to show that (3.1.3) is equivalent to (3.1.2). Varying the action (3.1.3) with respect to the auxiliary field $h_{\alpha\beta}$, we get

$$\partial_\alpha X \cdot \partial_\beta X = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X. \quad (3.1.5)$$

Taking minus the square root of the determinant of both sides of this equation, we get

$$\sqrt{-\det(\partial_\alpha X \cdot \partial_\beta X)} = \sqrt{-h} \frac{1}{2} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X. \quad (3.1.6)$$

Hence (3.1.3) reduces to (3.1.2) once $h_{\alpha\beta}$ is set on shell. Action (3.1.3) is known as the Polyakov action. The Polyakov action enjoys several symmetries:

- Poincaré Invariance:

$$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu. \quad (3.1.7)$$

- Worldsheet reparametrization invariance:

$$\sigma^\alpha \rightarrow \sigma'^\alpha(\sigma), \quad h_{\alpha\beta}(\sigma) = \frac{\partial \sigma'^\gamma}{\partial \sigma^\alpha} \frac{\partial \sigma'^\delta}{\partial \sigma^\beta} h'_{\gamma\delta}(\sigma'). \quad (3.1.8)$$

- Weyl invariance:

$$h_{\alpha\beta} \rightarrow e^{\phi(\sigma)} h_{\alpha\beta}, \quad \delta X^\mu = 0. \quad (3.1.9)$$

The first is a true global symmetry, while the last two are gauge symmetries. We also notice that the embedding coordinates X^μ behave as scalars under worldsheet reparametrization, namely:

$$X'^\mu(\sigma') = X^\mu(\sigma). \quad (3.1.10)$$

Gauge symmetries are particularly useful when it comes to find an explicit solution to the equation of motion coming from (3.1.3). In fact, exploiting such symmetries we are allowed to choose a particular set of coordinates on the worldsheet for which the Polyakov action becomes simpler to handle. In addition, it can be shown, [13], that in flat Minkowski spacetime we can reduce (3.1.3) to the simpler gauge-fixed action:

$$S = \frac{T}{2} \int \dot{X}^2 - X'^2, \quad (3.1.11)$$

where

$$\dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu := \frac{\partial X^\mu}{\partial \sigma}. \quad (3.1.12)$$

¹⁴Actually, it is a gauge-fixed version of (3.1.2) which is referred to as *Nambu-Goto* action.

Then, the corresponding equations of motion are

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0. \quad (3.1.13)$$

Furthermore, we observe that in order to properly define the above variational problem, we should also supply the above equations of motion with some boundary conditions. As we will see, the choice of boundary conditions will have non trivial physical implications. Suppose that the spacelike variable σ is chosen to belong to the interval $[0, \pi]$ (we can always do this.); then, upon varying (3.1.11) with respect to X^μ , we also get a boundary term which is

$$-T \int d\tau X'^\mu \delta X_\mu \big|_{\sigma=\pi} - X'^\mu \delta X_\mu \big|_{\sigma=0}. \quad (3.1.14)$$

Such boundary term can be removed by choosing for each X^μ one of the following boundary conditions:

- Closed string condition:

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau). \quad (3.1.15)$$

- Open string with Neumann boundary conditions:

$$X'^\mu = 0 \quad \text{at } \sigma = 0, \pi. \quad (3.1.16)$$

- Open string with Dirichlet boundary conditions:

$$X^\mu(\sigma = 0, \tau) = X_0^\mu, \quad X^\mu(\sigma = \pi, \tau) = X_\pi^\mu, \quad (3.1.17)$$

where X_0^μ and X_π^μ are constant.

The first condition makes the string close on itself, while the remaining two b.c. imply that the string is open. In particular, the Neumann boundary condition means that no momentum flows to the end of the string, while the Dirichlet condition fixes the ends of the string to be fixed. Pictorially, we can visualize the difference between the two open string boundary condition by looking at Figure 10. It is worth noting that Dirichlet boundary conditions breaks Poincaré invariance, however this does

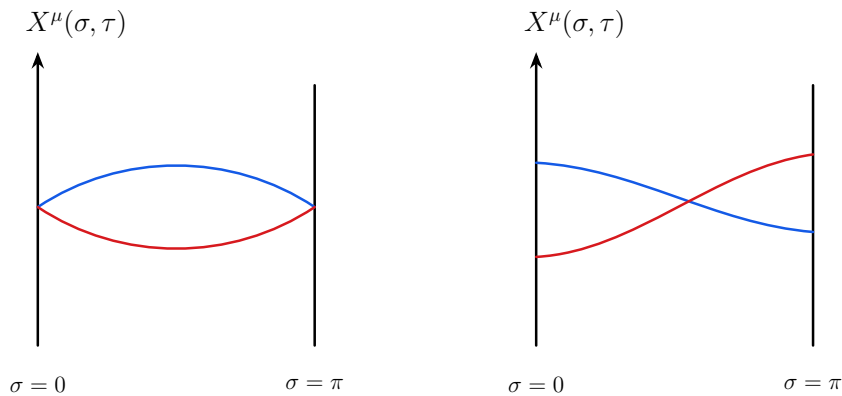


Figure 10: On the left: Dirichlet boundary condition along the μ direction; on the right: Neumann boundary condition along μ direction. The red and blue line represent the same string at different times.

not imply that such condition is unphysical, far from it. Indeed, as we will see later, the constant X_0^μ and X_π^μ are the position (in the μ direction) occupied by two p -dimensional extended objects known as Dp branes.

3.1.1 Quantization of the closed bosonic string

In order to give a full quantum treatment of the closed bosonic string, we first need to find a classical mode expansion of the solutions (3.1.13) fulfilling condition (3.1.15). Without showing all the details (which can be found in [63], for example), below we simply write down the solution of the closed string problem:

$$X^\mu = X_L^\mu + X_R^\mu, \quad (3.1.18)$$

where

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau - \sigma) + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)}, \quad (3.1.19)$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau + \sigma) + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}. \quad (3.1.20)$$

Here, we have introduced the string length scale l_s which is related to the tension T of the string as below

$$T = \frac{1}{2\pi\alpha'}, \quad \alpha' = \frac{l_s^2}{2}, \quad (3.1.21)$$

where α' is usually referred to as Regge slope. Furthermore, the real parameters x^μ and p^μ found in (3.1.19) and (3.1.20) correspond to the center of mass and total momentum of the string, respectively. Finally, the modes α_n^μ and $\tilde{\alpha}_n^\mu$ are complex numbers which must obey the following constraints

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \quad \text{and} \quad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*. \quad (3.1.22)$$

The above simply follow by imposing that X^μ is real.

In order to quantize our theory, we need to define the canonically conjugate momentum P^μ and, after promoting X^μ and P^μ to operator, we must impose the following commutation relations

$$[X^\mu(\sigma, \tau), P^\nu(\sigma', \tau)] = -i\eta^{\mu\nu}\delta(\sigma - \sigma'), \quad \text{zero otherwise.} \quad (3.1.23)$$

In turn, this implies that

$$[x^\mu, p_\nu] = i\delta_\nu^\mu, \quad (3.1.24)$$

while the Fourier modes α_m^μ and $\tilde{\alpha}_m^\mu$ (which are now operators) obey the following

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \quad (3.1.25)$$

and zero otherwise. By rescaling the operators α as

$$a_m^\mu = \frac{1}{\sqrt{m}}\alpha_m^\mu, \quad a_m^{\dagger\mu} = \frac{1}{\sqrt{m}}\alpha_{-m}^\mu, \quad \text{for } m > 0, \quad (3.1.26)$$

and similarly for $\tilde{\alpha}$, we see that equation (3.1.25) reduces to the algebra of the harmonic oscillator, with a slight but significant distinction. Indeed, whenever $\mu = \nu = 0$ we may end up with something like

$$[a_m^0, a_m^0] = -1 \quad (3.1.27)$$

The presence of the minus sign can be catastrophic, since as readily seen, it leads to the existence of negative norm states, i.e. states with negative probability. Such states are not physically sensible and should not correspond to any physical degree of freedom; thus quantizing such theory is not straightforward. In the canonical approach we could follow two distinct routes to quantize the theory; either we follow a covariant approach, reminiscent of the Gupta-Bleuler quantization of electrodynamics, or

we could choose a suitable gauge, the light-cone gauge, in which only the true degrees of freedom are manifest.

Loosely speaking, the first approach imposes some constraint on the Hilbert space of states which identify (up to some equivalence classes) the real physical states of the theory. The advantage of such approach is that Poincaré invariance is manifest throughout the whole procedure; the drawback is that the spectrum of the theory is not as evident as in the light-cone gauge quantization. On the other hand, the light-cone quantization method has the disadvantage of not being manifestly Poincaré invariant; at the end of the computations we need to make sure that no quantum anomaly, capable of breaking Poincaré covariance, has emerged.

Of course, both approach are equivalent and should bring the same results when carried out correctly; in the following we briefly explain how light-cone gauge quantization works, while we defer to [37] for the details of the covariant approach.

As already mentioned, light-cone gauge exploits the gauge freedom (reparametrization invariance) of the theory in order to get rid of unphysical degrees of freedom. In fact, equation (3.1.11) has not completely fixed the gauge choice; meaning that we can still perform an additional gauge transformation. First, introduce the spacetime light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm x^{D-1}), \quad (3.1.28)$$

then, using the residual gauge freedom, we can set

$$X^+ = x^+ + l_s^2 p^+ \tilde{\tau} \quad (3.1.29)$$

where $\tilde{\tau}$ and $\tilde{\sigma}$ are the gauge transformed worldsheet coordinates. As regards X^- , it is completely fixed by the Virasoro constraint coming from the vanishing of the stress energy tensor. The striking fact is that the oscillator modes α_n^- and $\tilde{\alpha}_n^-$ of X^- are completely determined in terms of the transverse oscillator modes α_m^i and $\tilde{\alpha}_m^i$, with $i = 1, \dots, (D-2)$. Since only the spatial modes survives, it is evident that our theory no longer hosts negative norm states. Thus, we've managed to decouple the unphysical modes from the theory. However, there is still a question that we should ask, that is whether the quantum theory still respects Poincaré invariance. To answer such question, we should inspect if the Lorentz generators $J^{\mu\nu}$ still obey the Lorentz algebra; the only non-trivial check is

$$[J^{i-}, J^{j-}] = 0. \quad (3.1.30)$$

After a lengthy computation, it can be shown that equation (3.1.30) holds provided that the spacetime dimension $D = 26$ ¹⁵. In the end, we come up with a rather remarkable fact: in order to get a consistent quantum bosonic string theory, we need to consider a 26-dimensional spacetime¹⁶.

3.1.2 Spectrum of the closed string

The quantization procedure sketched in the previous section also furnishes the mass of each string state; it reads

$$M^2 = \frac{4}{\alpha'} (N - 1) = \frac{4}{\alpha'} (\tilde{N} - 1), \quad (3.1.31)$$

where N and $\tilde{N}$ are the right and left movers number level defined as

$$N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i, \quad \tilde{N} = \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i. \quad (3.1.32)$$

¹⁵It also imposes the value of the normal ordering constant a appearing in the zero mode of the stress energy tensor to be equal to 1.

¹⁶Although we have explicitly made reference only to the closed string, the same conclusions also apply to the open string.

In the closed string we have the level matching condition

$$N = \tilde{N}. \quad (3.1.33)$$

Now, starting from the ground state $|0; p\rangle$ (defined as being the state annihilated by all the lowering operators a which is also an eigenstate of the operator p with eigenvalue p), we can build any excitation by applying the raising operators $a^\dagger$ and then read its mass from equation (3.1.31). Here we will focus only on the states with $N = 0$ and $N = 1$.

- **Ground state tachyon**

If we apply formula (3.1.31) to the ground state (for which $N=0$), we get a negative mass; the ground state is a tachyon, meaning that we are expanding starting from an unstable maximum of the potential. So far there are no convincing arguments that could suggest the presence of a real minimum of the potential; indeed we are not even sure if bosonic string theory makes sense at all. However, once we take into account fermions, we will see that the tachyon problem disappears.

- **N=1 level**

The first excited states can be built by applying raising operator $a_1^\dagger$ to the ground state; since the level matching condition must hold, we also need to apply $\tilde{a}_1^\dagger$. Thus, the first excited states are¹⁷

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0; p\rangle, \quad \text{with } i, j = 1, \dots, D-2. \quad (3.1.34)$$

In this case, (3.1.31) tells us that such states are massless; moreover, since $\tilde{\alpha}_{-1}^j$ and α_{-1}^j both behave as vector under $SO(24)$, we have that (3.1.34) is a *reducible* representation of $SO(24)$. This fact should not come as a surprise; in fact, from the Wigner classification we know that massless particles arrange themselves in irreducible representations of the little group. In $D = 26$, the little group is just $SO(24)$. Finally, to get the irreducible representations out of (3.1.34), the usual procedure of symmetrization/anti-symmetrization gives us all the irreducible representations. We end up with a particle in the singlet representation which we call the Dilaton; a particle in the antisymmetric representation which is referred to as Kalb-Ramond 2-form and finally we get a particle in the traceless symmetric representation. The latter is a spin 2 particle, which, by an old argument originally due to Weinberg and Feynman, can be identified with the graviton.

3.1.3 Quantization of the open string

As far as the open string is concerned, much of the previous section carries over with minor changes. We recall that in this case we are free to choose among Dirichlet or Neumann boundary conditions, see equations (3.1), (3.1). Let us impose Neumann boundary conditions for the first $p+1$ spacetime coordinates, that is to say

$$X'^a|_{\sigma=0,\pi} = 0, \quad \text{for } a = 0, \dots, p, \quad (3.1.35)$$

and Dirichlet boundary conditions for the remaining ones,

$$X^I|_{\sigma=0,\pi} = X_{0,\pi}^I, \quad \text{for } I = p+1, \dots, D-1. \quad (3.1.36)$$

Such a choice of boundary conditions breaks the Lorentz group,

$$SO(1, D-1) \rightarrow SO(1, p) \times SO(D-p-1), \quad (3.1.37)$$

where, for the moment, we did not specify the value of the spacetime dimension D . As already anticipated above, the ends of the strings are fixed to lie in a $(p+1)$ hypersurface in spacetime

¹⁷Recall that $a^\dagger \sim \alpha_{-m}$ with $m > 0$.

(actually two hypersurfaces when $X_0^I \neq X_\pi^I$); such hypersurface is commonly referred to as a Dp brane, a new, dynamical p dimensional extended object. From now, after introducing the spacetime light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^p), \quad (3.1.38)$$

the light-cone quantization procedure follows the same steps as before; the only difference is that, thanks to the open string boundary conditions (either Neumann or Dirichlet), only one set of the Fourier mode, say α , is independent. This means that when quantizing the open string we should only care about left movers or right movers, but not both. Also in this case, in order to have a sensible quantization of the theory, we are led to consider $D = 26$ and $a = 1$. The mass formula that we get following all the steps of the quantization is

$$M^2 = \frac{1}{\alpha'} \left(\sum_{i=1}^{p-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{I=p+1}^{D-1} \sum_{n>0} \alpha_{-n}^I \alpha_n^I \right). \quad (3.1.39)$$

As already said, no $\tilde{\alpha}$ appears. At this point, we can study the spectrum of the open string, or, at least, of the first excited levels.

3.1.4 Spectrum of open string

- **Ground state**

Like the closed string, the ground state $|0; p\rangle$ (defined in the same manner as before) possesses a negative mass, hence it is a tachyon.

- **N=1 level**

Making use of (3.1.39), we immediately see that the first excited states are massless. They fall into two distinct groups; excitations longitudinal to the brane, namely

$$\alpha_{-1}^a |0; p\rangle, \quad \text{for } a = 1, \dots, p-1, \quad (3.1.40)$$

and transverse excitations like

$$\alpha_{-1}^I |0; p\rangle, \quad \text{for } I = p+1, \dots, D-1. \quad (3.1.41)$$

In the former case, we have massless spin 1 particles on the worldvolume of the Dp brane; they are photon-like excitations. In the latter, we have a set of scalars on the worldvolume of the Dp brane.

3.2 Superstring theory

There are many reasons not to be satisfied with bosonic string theory. Aside from the conceptual issues brought by the existence of the ground state tachyon, we also see that all the excitations correspond to bosonic particles; the theory cannot explain the existence of fermions in our universe. In order to overcome the above difficulties, two approaches have been developed: the so called RNS formalism and the GS formalism. Here we will briefly mention some aspects of the former one. However, regardless of the method adopted, they are equivalent and both will turn out to require supersymmetry.

The idea of the RNS formalism is simply to enlarge (3.1.3) by adding a fermion term; then, the total action in conformal gauge reads (setting $T = \frac{1}{\pi}$):

$$S = -\frac{1}{2\pi} \int d^2\sigma \left(\partial^\alpha X_\mu \partial_\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right), \quad (3.2.1)$$

Here, ψ^μ are D Majorana fields that transform as vectors under the D -dimensional Lorentz group $SO(D-1, 1)$, while ρ^α are a two dimensional representation of the Dirac algebra

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}, \quad \alpha, \beta = 0, 1. \quad (3.2.2)$$

It could be shown, [13], that (3.2.1) is invariant under global worldsheet supersymmetry

$$\delta X^\mu = \bar{\epsilon} \psi^\mu, \quad (3.2.3)$$

$$\delta \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon, \quad (3.2.4)$$

where ϵ is a constant infinitesimal Majorana spinor. That is why the fermionic string theory is also known as superstring.

From here, light-cone quantization of the superstring proceed mostly as the bosonic case, with the small caveat that now we must impose anti-commutation relations for the fermionic fields. This time the quantization procedure forces us to impose $D = 10$. The only aspect that we would like to stress about the superstring quantization concerns the boundary condition for the fermionic sector. Upon varying the action (3.2.1) with respect to ψ^μ , one sees that the boundary contribution goes like (suppressing Lorentz indices)

$$\delta S \sim \int d\tau (\psi_+ \delta \psi_+ - \psi_- \delta \psi_-) |_{\sigma=\pi} - (\psi_+ \delta \psi_+ - \psi_- \delta \psi_-) |_{\sigma=0}, \quad (3.2.5)$$

where ψ_+ and ψ_- are the upper and lower component of ψ . A suitable choice of boundary conditions allow us to get rid of (3.2.5).

For the open string, we demand that both term of (3.2.5) are zero independently. This is achieved by imposing:

$$\psi_+^\mu = \pm \psi_-^\mu, \quad \text{at } \sigma = 0, \pi. \quad (3.2.6)$$

Since the overall sign is unphysical, we can choose, without loss of generality that

$$\psi_+^\mu |_{\sigma=0} = \psi_-^\mu |_{\sigma=0}. \quad (3.2.7)$$

Then, the relative sign at the other end of the string becomes meaningful. There two possible cases:

- **Ramond boundary condition**

$$\psi_+^\mu |_{\sigma=\pi} = \psi_-^\mu |_{\sigma=\pi}. \quad (3.2.8)$$

- **Neveu-Schwarz boundary condition**

$$\psi_+^\mu |_{\sigma=\pi} = -\psi_-^\mu |_{\sigma=\pi}. \quad (3.2.9)$$

Finally, as far as the closed string is concerned, we are free to impose periodic or anti-periodic boundary conditions, namely

$$\psi_\pm^\mu(\sigma) = \pm \psi_\pm^\mu(\sigma + \pi). \quad (3.2.10)$$

In particular, periodic (R) and anti-periodic(NS) boundary conditions can be imposed on left movers ψ_-^μ and right movers ψ_+^μ , separately.

3.2.1 Spectrum of the superstring

The superstring spectrum obtained after quantization contains, in addition, a tachyon and further states that spoil spacetime supersymmetry. To recover a supersymmetric spectrum, these states must be truncated away — a procedure known as the GSO projection. One introduces a G-parity operator in each of the NS and R sectors; the projection then retains only the NS states of positive G-parity, while in the R sector one is free to keep either positive or negative G-parity.

In view of the next sections, devoted to the supergravity theories that arise as low-energy limits of string theory, we briefly describe the massless spectrum of the closed superstring. Recall that the closed superstring has four distinct sectors, according to the boundary conditions imposed on the left- and right-moving fermions: NS-NS, NS-R, R-NS, and R-R. The GSO projection then leaves a choice: one may impose the same G-parity in the two R sectors, or opposite G-parities. The two choices yield distinct theories — type IIB when the R sectors share the same G-parity, type IIA when they have opposite parity. In the end, the massless spectrum of the two theories is the following.

- **Type IIA theory**

From the NS-NS sector we get a 2-form gauge field, a singlet (the dilaton) and a rank 2 traceless symmetric tensor (the graviton). The NS-R and R-NS sectors both contain a gravitino and a dilatino; the gravitinos in the two sectors have opposite chirality. Finally, the R-R sector is made of a one form and a three form gauge fields.

- **Type IIB theory**

The NS-NS sector of type IIB is identical to that of type IIA; the same also holds for the NS-R and R-NS sectors, though this time gravitinos share the same chirality. The R-R sector hosts a zero form, a two form and a four form gauge fields.

3.3 Low energy effective theories

Up to now, we have always assumed to be working with a string living in a flat, empty spacetime. Now, we could ask what happens if the string sits in a background where the fields that we have found above are turned on. Or said another way: does the string backreact to the presence of the fields coming from the excitation of the string itself? The answer is positive.

In the bosonic case, we can see that the string naturally couples to the massless bosonic field as follows:

$$S = -\frac{1}{4\pi\alpha'} \int d^\sigma \sqrt{-h} h^{\alpha\beta} \left[G^{\mu\nu}(X) \partial_\alpha X_\mu \partial_\beta X_\nu + i B^{\mu\nu}(X) \epsilon^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\nu + \alpha' \Phi(X) R^{(2)} \right], \quad (3.3.1)$$

where $G^{\mu\nu}$, $B_{\mu\nu}$ and Φ are the graviton, the Kalb-Ramond 2-form and the dilaton, respectively. Here $R^{(2)}$ stands for the worldsheet Ricci scalar, not to be confused with the spacetime one. There is an important fact that should be stressed here: the dilaton coupling found in (3.3.1) breaks Weyl invariance. At first, this could sound unsettling since Weyl invariance was a gauge symmetry of our theory, hence breaking it would make the theory no more self-consistent. Actually there is a way out. We can treat (3.3.1) as an interacting 2-dimensional field theory where the fields B , G and Φ may be thought of as coupling constants; it is known that in QFT the coupling constants change with the energy scale and their evolution is governed by the beta function, one for each coupling. We can then look at the vev of the trace of the stress-energy tensor T^α_α as a measure of the breaking of Weyl invariance. As shown by [24], we have

$$\langle T^\alpha_\alpha \rangle = -\frac{1}{2\alpha'} \beta_{\mu\nu}(G) h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{i}{2\alpha'} \beta_{\mu\nu}(B) \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \beta(\Phi) R^{(2)}, \quad (3.3.2)$$

where the beta functions are, at one loop,

$$\beta_{\mu\nu}(G) = \alpha' \mathcal{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda k} H_\nu{}^{\lambda k}, \quad (3.3.3)$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu}, \quad (3.3.4)$$

$$\beta(\Phi) = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}. \quad (3.3.5)$$

Here $\mathcal{R}_{\mu\nu}$ is the spacetime Ricci tensor and $H_3 = dB_2$.

We can now demand Weyl invariance by simply imposing

$$\beta_{\mu\nu}(G) = 0, \quad \beta_{\mu\nu}(B) = 0, \quad \beta(\Phi) = 0. \quad (3.3.6)$$

Equations (3.3.6) can be interpreted as equations of motion for the closed string fields. Moreover, such equations can be derived by varying the following action:

$$S = \frac{1}{2\kappa_0^2} \int d^{26} X \sqrt{-G} e^{-2\phi} \left(\mathcal{R} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi \right). \quad (3.3.7)$$

The above is referred to as low energy effective action of the bosonic string, where low energy reflects the fact that it has been derived considering up to one loop corrections to the beta functions.

3.3.1 Supergravity theories

A similar story also holds for the superstring. In this case, the low energy effective actions must describe the dynamics of Φ , $B_{\mu\nu}$, $G_{\mu\nu}$ as well as the additional massless bosonic and fermionic fields. Then, the total action can be split up into three distinct pieces:

$$S = S_1 + S_2 + S_{fermi}. \quad (3.3.8)$$

Here, S_1 is the same as (3.3.6) but in ten dimensions rather than 26¹⁸, while S_2 and S_{fermi} depend on the particular superstring theory we are dealing with. In particular, S_2 takes into account the bosonic fields, while S_{fermi} the fermionic ones. For the symmetry operators construction that we will explore in section 6, we will only need to write down the bosonic actions.

- **Type IIA theory**

The bosonic action of type IIA theory is (in differential form notation):

$$S_2 = -\frac{1}{4\kappa_0^2} \int d^{10} X \left[\sqrt{-G} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) + B_2 \wedge F_4 \wedge F_4 \right]. \quad (3.3.9)$$

Here $F_2 = dC_1$ and $F_4 = dC_3$ where C_1 and C_3 are the Ramond-Ramond 1-form and 3-form gauge fields, while $\tilde{F}_4 = F_4 - C_1 \wedge H_3$.

- **Type IIB theory**

The bosonic action of type IIB theory is instead:

$$S_2 = -\frac{1}{4\kappa_0^2} \int d^{10} X \left[\sqrt{-G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) + C_4 \wedge H_3 \wedge F_3 \right]. \quad (3.3.10)$$

This time, $F_1 = dC_0$, $F_3 = dC_2$ and $F_5 = dC_4$ where C_1 and C_3 are the Ramond-Ramond 0-form, 2-form and 4-form gauge fields, while $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$. Action (3.3.10) must also be supplied by the following self-duality condition

$$F_5 = \star F_5. \quad (3.3.11)$$

¹⁸This is strictly true only for type II A and type II B theories, otherwise it is slightly different.

3.3.2 Low energy actions for Dp branes

The previous analysis was carried out in the closed string setting. We now show how it can be extended to the open string. Suppose we turn on the $U(1)$ gauge field A_a living on the worldvolume of the Dp brane; then, the action of the open string splits in two pieces

$$S = S_{Neumann} + S_{Dirichlet}. \quad (3.3.12)$$

The first term describes the fluctuation parallel to the Dp brane on which the string ends and it reads

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\alpha X^a \partial_\alpha X_a + i \int_{\partial\mathcal{M}} d\tau A^a \dot{X}_a, \quad a = 0, \dots, p. \quad (3.3.13)$$

Instead, $S_{Dirichlet}$ is made up fields obeying Dirichlet boundary conditions for which $\dot{X}|_{\partial\mathcal{M}} = 0$, hence while it keeps the same bulk term of (3.3.13) with $a \rightarrow I = p+1, \dots, D-1$, it lacks the boundary contribution. Therefore, the Dirichlet part of the action does not play any role in the calculation of the beta function, since it does not couple to the background field A_a .

Now, the vanishing of the beta function give us the following equation of motion, [1],

$$\partial_b F_{ac} \left[\frac{1}{1 - 4\pi^2 \alpha' F^2} \right]^{ab} = 0, \quad (3.3.14)$$

where

$$F_{ab} = \frac{\partial A_b}{\partial X^a} - \frac{\partial A_a}{\partial X^b}. \quad (3.3.15)$$

Equation (3.3.14) can be derived by the Born-Infeld action:

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})} \quad (3.3.16)$$

where ξ are worldvolume coordinates on the brane and T_p is its tension. Action (3.3.16) describes the low energy dynamics of a Dp brane in the presence of a $U(1)$ gauge field A_a on its worldvolume; it is valid as long the gradient of the field strength F is small.

Above, we restricted to the case in which only the worldvolume gauge field A was turned on, but we could also study what happens when the worldvolume scalar fields ϕ^I are present. This time, (3.3.16) is replaced by the Dirac-Born-Infeld action, DBI for short,

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})}, \quad (3.3.17)$$

with

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu}. \quad (3.3.18)$$

Finally, we could also include the brane backreaction to the closed string fields, Φ , B and G ; this leads to

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab})}, \quad (3.3.19)$$

where

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}, \quad (3.3.20)$$

and

$$B_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}. \quad (3.3.21)$$

Up to now we have described the low-energy actions of Dp-branes in the bosonic theory; we now turn to the coupling of such a brane in the superstring setting. Its action consists of two pieces: one is the same as (3.3.19), while the other describes how the Dp brane interacts with the RR fields. In particular, a Dp brane couples naturally to the $(p+1)$ -form RR gauge potential through

$$\mu_p \int C_{p+1}, \quad (3.3.22)$$

C_{p+1} denotes the pullback of the potential onto the worldvolume. Equation (3.3.22) is the higher-dimensional generalization of the electric coupling of a pointlike particle; indeed, by Gauss law we rewrite it as

$$\mu_p = \int_{S^{D-p-2}} \star F_{p+2}, \quad (3.3.23)$$

where F_{p+2} is the field strength associated with C_{p+1} , and S^{D-p-2} is a $(D-p-2)$ -dimensional sphere which surrounds the Dp brane¹⁹. Just as in electromagnetism, we may also ask about magnetic sources. The magnetic charge is measured by the flux of F_{p+2} ,

$$\int_{S^{p+2}} F_{p+2}. \quad (3.3.24)$$

Here, the $(p+2)$ -dimensional sphere can surround a $(D-p-4)$ brane which is therefore the magnetic dual of the original Dp brane. It is convenient to introduce redundant fields potential in our theory by defining

$$\star F_{p+2} := F_{D-p-2} = dC_{D-p-3}, \quad (3.3.25)$$

so that the magnetically dual branes appear as electric sources for the dual potentials. However, equation (3.3.22) is not the full story. In order to ensure invariance under T-duality²⁰, the full coupling of a Dp brane is the Wess-Zumino term

$$\mu_p \int \left(\sum_q C_q \wedge e^{2\pi\alpha' F_2 + B_2} \right)_{p+1}, \quad (3.3.26)$$

where only the differential forms with degree equal to $p+1$ survive in the worldvolume integral. The sum over the gauge potentials C_q runs over odd or even q according to whether we are in type IIA or type IIB theory.

3.4 Compactifications and Reductions

As mentioned above, superstring theory forces the spacetime to have 10 dimensions, 6 more than the usual 4 dimensions we are accustomed to. This might sound like an invalidation of the theory, but such a conclusion is too harsh. In this section we provide some arguments that make the extra dimensions more plausible.

The main idea is that these additional dimensions are "small" and compactified, so that a low energy observer is not able to resolve them. In order to make such statement clearer, we will consider the

¹⁹Here, we are keeping the spacetime dimension unspecified; for the superstring we recall that $D = 10$.

²⁰T-duality is the statement that string theory on a circle of radius R is physically identical to string theory on a circle of radius $\frac{\alpha'}{R}$. More on this can be found in [57] and [13]

simplest compactification for the bosonic string theory. We assume that the 26-dimensional spacetime can be realized as

$$\mathcal{M}^{26} = \mathbb{R}^{1,24} \times S^1. \quad (3.4.1)$$

The S^1 has radius R so that

$$X^{25} \sim X^{25} + 2\pi R. \quad (3.4.2)$$

At length scale $\gg R$, the dynamics of the internal compactified space S^1 is essentially frozen; therefore, we will assume that all the fields depend only on the external coordinates belonging to $\mathbb{R}^{1,24}$. Consider for example the Einstein metric $G_{\mu\nu}$; it can be decomposed in terms of 25-dimensional fields $\tilde{G}_{\mu\nu}$, A_μ and ϕ ,

$$ds^2 = \tilde{G}_{\mu\nu} dX^\mu dX^\nu + e^{2\sigma} (dX^{25} + A_\mu dX^\mu)^2. \quad (3.4.3)$$

The vector field A_μ is indeed a gauge field and it is related to the isometry of the S^1 . In fact, the 26-dimensional theory is invariant under diffeomorphism

$$X^M \rightarrow X^M + V^M(X), \quad M = 0, \dots, 25. \quad (3.4.4)$$

We recall that under (3.4.4), the original 26-dimensional metric G_{MN} is subject to an infinitesimal variation given by

$$\delta G_{MN} = \mathcal{L}_V G_{MN} = V^P \partial_P G_{MN} + \partial_M V^P G_{PN} + \partial_N V^P G_{MP}, \quad (3.4.5)$$

where $\mathcal{L}_V$ denotes the Lie derivative along the direction of V . Suppose that V lies only on the 25th direction and it is independent of X^{25} ; then, (3.4.5) becomes

$$\delta G_{\mu 25} = \partial_\mu V^{25} G_{25 25} = e^{2\sigma} \partial_\mu V^{25}, \quad (3.4.6)$$

which means

$$\delta A_\mu = \partial_\mu V^{25}. \quad (3.4.7)$$

Now, if we plug (3.4.3) into equation (3.3.7) we get²¹

$$S \supset \frac{2\pi R}{\kappa^2} \int d^{25} X \sqrt{-\tilde{G}} e^\sigma \left(\mathcal{R}^{25} - \frac{1}{2} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \sigma \partial^\mu \sigma \right), \quad (3.4.8)$$

being $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thus, free gravity in 26 dimensions can be recast as gravity coupled to a gauge field A_μ and a massless scalar field σ in one lower dimension.

Let us now relax the above assumptions, and allow again an explicit dependence on the internal compactified direction. This will have a striking physical consequence. It is easier to work with the scalar field Φ ; we can expand it in Fourier modes on the circle:

$$\Phi(X^\mu, X^{25}) = \sum_{n=-\infty}^{+\infty} \Phi_n(X^\mu) e^{\frac{inX^{25}}{R}} \quad (3.4.9)$$

In this way, the kinetic term of Φ in (3.3.7) becomes

$$S \supset 2\pi R \int d^{25} X \sum_{n=-\infty}^{+\infty} \left(\partial_\mu \Phi_n \partial^\mu \Phi_{-n} + \frac{n^2}{R^2} |\Phi_n|^2 \right). \quad (3.4.10)$$

²¹Actually, (3.3.7) is written in what is known as "String Frame"; since our metric ansatz (3.4.3) is in the "Einstein frame", before plugging in the ansatz we should first change frame in (3.3.7).

This means that a scalar field in 26 dimensions becomes an infinite tower of massive states for $n \neq 0$, with mass

$$M^2 = \frac{n^2}{R^2}. \quad (3.4.11)$$

The compactification procedure sketched above for the bosonic string theory is known as Kaluza-Klein reduction. Of course, it could be generalized to arbitrary spacetime and employed in the superstring case. The relevant features that we would like to stress are the following: firstly, isometries of the internal space are mapped to gauge fields in the compactified theory (like A_μ above); secondly, the mode expansion of the higher dimensional fields usually furnishes a massive tower of states in the compactified theory.

3.4.1 Type IIA theory from eleven dimensional supergravity

Type IIA theory features even dimensional Dp branes; let us focus on $D0$ branes for the moment. It can be shown that the tension (or better, the mass) of a $D0$ brane is, [62],

$$m_{D0} = \frac{1}{l_s g_s}, \quad (3.4.12)$$

where g_s is the string coupling constant and is related to the vev of the dilaton Φ . Suppose to consider a state made of k $D0$ branes; the mass of this state is k times $1/l_s g_s$. This somehow resembles the Kaluza-Klein tower of massive states that we found in the previous example, see equation (3.4.11). Pushing further the analogy, we could introduce the radius of the eleven dimension

$$L_{10} = l_s g_s. \quad (3.4.13)$$

This could imply that at strong coupling, when L_{10} is much larger than the Planck scale, type IIA theory admits an eleven dimensional interpretation. Indeed, consider the following action in 11 dimensions²²

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}X \sqrt{-G_{11}} \left(\mathcal{R}_{11} - \frac{1}{2}|G_4|^2 - \frac{1}{6} \int A_3 \wedge G_4 \wedge G_4 \right), \quad (3.4.14)$$

where $G_4 = dA_3$. Plugging in (3.4.14) the ansatz

$$ds_{11}^2 = e^{-\frac{2}{3}\Phi} \left[ds_{10}^2 + e^{2\Phi} (dx^{10} - C_1)^2 \right], \quad (3.4.15)$$

$$A_3 = C_3 - B_2 \wedge dx^{10}, \quad (3.4.16)$$

we retrieve equation (3.3.9). Since (3.4.14) is not renormalizable, it should be considered as an effective action of an unknown UV theory. Such theory has received the name of M-theory.

Moreover, we would like discuss how the M-theory branes are mapped to type IIA branes. First, from the bosonic piece of the low energy M-theory action, we may argue that only two stable branes exist in M-theory: the electrical charged brane under the gauge field A_3 known as M2 brane, and its magnetic dual, the $M5$ brane. So, the idea is to recover type II branes starting from these two. The uplift from type IIA to M-theory is concisely shown in table 1.

It is worth pointing out that table 1 leaves out $D6$ and $D8$ branes. This is because their uplift to eleven dimensions is not so simple. As for the $D6$, one can show that its uplift is a pure geometrical solution in M-theory, [62], while $D8$ has no clear uplift to M-theory at all.

²²It is just the bosonic content.

Type IIA object	Couples to	M-theory origin	Relation to circle
D0-brane	C_1	KK momentum mode	momentum $1/L_{10}$ along z
Fundamental string (F1)	B_2	M2-brane	wrapped on z
D2-brane	C_3	M2-brane	transverse to z
D4-brane	C_5	M5-brane	wrapped on z
NS5-brane	B_6	M5-brane	transverse to z

Table 1: Uplift of type IIA branes to M-theory on a circle in the $x^{10} \equiv z$ direction of radius L_{10} . Objects wrapped on the circle descend from an M-brane with one leg along z ; unwrapped (transverse) objects keep all legs in the ten non-compact directions.

Finally, one more ingredient that we need to introduce for our further discussions about symmetry operators in holography, is the massive deformation of type IIA theory, known as Romans theory. This massive deformation is obtained by turning on a non-zero constant flux for the RR 0-form F_0 , and was first introduced by [58] from a supergravity perspective. Its stringy origin was later clarified by Polchinski, who showed that the Romans mass is sourced by $D8$ branes. More specifically, the $D8$ branes are domain walls for the Romans mass, and the flux of F_0 jumps by one unit when crossing a $D8$ brane.

Crucially, there is no known uplift of Romans theory to M-theory, and the $D8$ brane is not a solution of eleven-dimensional supergravity.

3.5 Hanany-Witten Effect

Due to its relevance throughout the remainder of the thesis, the last string-related topic we discuss is the Hanany–Witten effect [39]. The original setup involves a $D5$ -brane and an $NS5$ -brane in type IIB theory, with the worldvolume configurations shown in table 2.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	×	×	×	×	×	×				
D5	×	×	×					×	×	×

Table 2: Brane configuration for the Hanany–Witten setup in type IIB. A $\times$ marks a direction along the brane’s worldvolume; a blank marks a transverse direction. Both branes share the common worldvolume $x^{0,1,2}$.

When we move the two branes along their common transverse direction, x^6 , they will eventually cross. Both branes share the common worldvolume directions $x^{0,1,2}$, and each is pointlike in x^6 ; their remaining worldvolume directions are mutually transverse. The statement of the Hanany-Witten effect is that after the branes cross, a $D3$ -brane stretching between them is created. To argue for this claim, let X_D and X_{NS} be the worldvolumes of the $D5$ and $NS5$ respectively. Take the type IIB background to be $\mathbb{R}^3 \times M_7$, with

$$X_{NS} = R^3 \times Y_{NS}, \quad X_D = R^3 \times Y_D,$$

where Y_D and Y_{NS} are compact, homologically trivial 3-dimensional manifolds in M^7 . Recalling that the $D5$ and $NS5$ are the magnetic sources for C_2 and B_2 respectively, and writing $H = dB_2$, $F_3 = dC_2$

locally, we define the linking number by

$$\text{Link}(Y_{NS}, Y_D) = \int_{Y_D} \frac{H}{2\pi}, \quad \text{or equivalently} \quad \text{Link}(Y_{NS}, Y_D) = - \int_{Y_{NS}} \frac{F_3}{2\pi} \quad (3.5.1)$$

On the worldvolume of the D5-brane, B_2 is not gauge invariant: it mixes with the field strength F_2 of the worldvolume $U(1)$ gauge field to form the gauge-invariant combination²³

$$\mathcal{F}_2 = B_2 - F_2. \quad (3.5.2)$$

When no other branes are present, $d\mathcal{F}_2 = 0$, and hence $H|_{Y_D} = dF_2$ is exact on Y_D . Since Y_D is closed, Stokes' theorem then forces $\int_{Y_D} H = 0$: if the NS5 and D5 begin in a separated configuration their mutual linking number vanishes. Naively this would seem to obstruct their crossing, since crossing changes the linking number by one unit and there is nothing to compensate it. The resolution is that a D3-brane is created when the two branes cross, restoring conservation of the linking number. Let the D3 have worldvolume $\mathbb{R}^3 \times C$, with C a curve in M_7 , and let p be a point at which C ends on Y_D . The endpoint of the D3 acts as a magnetic source for the worldvolume gauge field on the D5, modifying the closure of

$$dF_2 = \pm \delta(p), \quad (3.5.3)$$

where the " $\pm$ " sign reflects whether the D3 begins or ends on the D5 worldvolume. Including k such D3-branes, the restriction of H to Y_D becomes

$$H = d\mathcal{F}_2 - \sum_{\alpha}^k \epsilon_{\alpha} \delta p_{\alpha}, \quad (3.5.4)$$

with $\epsilon_{\alpha} = \pm 1$ the (orientation-dependent) sign of the α -th endpoint. Inserting this into (3.5.1), the vanishing of the total linking number becomes the statement that the linking number of Y_D (with respect to all NS5-branes), *plus* the net number of D3-branes ending on it, sums to zero. Thus a nonzero linking number generated by the crossing is exactly cancelled by a created D3-brane: when the initially separated D5 and NS5 are pushed through each other, a D3-brane connecting the two—stretched along x^6 —is created.

We have explicitly proved the statement in the simplifying hypothesis that Y_D and Y_{NS} were compact manifolds in M^7 . Though, one can reach the same conclusions even when we relax the above hypothesis and take $M^7 = \mathbb{R}^7$ and Y_D and Y_{NS} copies of $\mathbb{R}^3 \subset \mathbb{R}^7$.

Finally, as shown in [5], it is possible to generalize the Hanany-Witten effect in other string theory setups which include type IIA and M-theory.

²³Apart from normalization factors, this is the same gauge-invariant combination that appears in (3.3.19).

4 *AdS/CFT duality and Symmetries*

In view of the central role that holography plays in this thesis, mainly concerning the construction of R-symmetry defects, we devote this section to a brief review of the AdS/CFT correspondence and its relation to symmetries.

Among the most remarkable developments in modern theoretical physics is the realization that certain quantum field theories admit an equivalent description in terms of a gravitational theory living in one additional spacetime dimension. This idea, known as the AdS/CFT correspondence or, more broadly, as holography, was first proposed by Maldacena [50] and has since become one of the central tools for probing the strongly coupled regime of gauge theories.

Since its original formulation the correspondence has been extended well beyond the maximally supersymmetric case, to setups with reduced supersymmetry, to non-conformal theories, and to backgrounds of the form $AdS_{d+1} \times M$ for various internal manifolds M . Of particular relevance for what follows is the way global symmetries are realized holographically: a global symmetry of the boundary theory corresponds to a gauge symmetry in the bulk, and the generalized symmetries discussed in Section 2 are no exception. As we shall see, the topological operators generating such symmetries admit a natural description in terms of branes wrapping cycles in the internal geometry, which is the setting in which the R-symmetry defects of this thesis will be constructed.

In this section we outline the main ideas underlying the AdS/CFT correspondence. Our aim is not to give a complete account, for which we refer to [3, 66], but rather to convey how the duality works and in what sense the two descriptions can be regarded as equivalent. We will focus throughout on the standard example of type IIB string theory on $AdS_5 \times S^5$ and its dual $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, which is both the original setting in which the correspondence was formulated and the one in which it is best understood.

Additionally, at the end of this section we will review how holography can be used to probe the symmetry structure of a CFT using suitable constructions in string theory. In particular, we will bring two examples, taken from [5] and [11].

4.1 Conformal Field Theories-CFT

We begin by introducing the field theory side of the story, starting from the definition of conformal field theories (CFT). Actually, we have already met CFTs in the previous sections dedicated to string theory, where we argued that the 2-dimensional sigma model of the worldsheet theory was a conformal field theory.

In general, as the name suggests, a CFT is a field theory which is invariant under conformal transformations. The conformal group is the set of transformations which leaves the metric invariant up to some positive scale factor,

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x). \quad (4.1.1)$$

At the infinitesimal level, assuming that

$$x^\mu \rightarrow x^\mu + v^\mu(x) \quad (4.1.2)$$

and

$$\Omega(x) = 1 + \frac{1}{2}\omega(x), \quad (4.1.3)$$

conformal transformations of Minkowski space are subject to the following condition:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\eta_{\mu\nu}. \quad (4.1.4)$$

Whenever the spacetime dimensions are $d \neq 2$, the set of solutions of (4.1.4) is given by a small set of functions at most quadratic in x . More precisely we have

$$\delta x_\mu = a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + (b_\mu x^2 - 2x_\mu(bx)), \quad (4.1.5)$$

where a , λ , b_μ are constants and $\omega_{\mu\nu}$ is a constant antisymmetric tensor. The first two terms are simply the Poincaré transformations of the d -dimensional spacetime; as expected, Poincaré transformations are a subgroup of the larger group of conformal transformations. The third term is related to dilatations, and the corresponding finite transformation is

$$x^\mu \rightarrow \lambda x^\mu. \quad (4.1.6)$$

Finally, the last term is associated with finite special conformal transformations,

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2(bx) + (bx)^2}. \quad (4.1.7)$$

The number of corresponding generators is

$$\frac{(d+1)(d+2)}{2}. \quad (4.1.8)$$

There is also a discrete symmetry which acts as a conformal transformation,

$$x^\mu \rightarrow \frac{x^\mu}{x^2}. \quad (4.1.9)$$

Adding (4.1.9), one obtains the full (non connected) conformal group. For $d = 2$, (4.1.4) admits infinite solutions. Indeed, this is the case for the sigma model theory we previously dealt with. For the remainder, we will only take into account the case $d \neq 2$.

We will denote by $J_{\mu\nu}$ the generators of Lorentz transformations, P_μ the generators of translations, D the one related to scaling transformations and K_μ for special conformal transformations. The above generators close the following algebra:

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= i\eta_{\mu\rho}J_{\nu\sigma} \pm \text{permutations} \\ [J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\ [J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\ [J_{\mu\nu}, D] &= 0 \\ [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, P_\nu] &= -2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D \end{aligned} \quad (4.1.10)$$

Interestingly, if one packages all the generators into

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\ -\frac{K_\nu - P_\nu}{2} & 0 & D \\ \frac{K_\nu + P_\nu}{2} & -D & 0 \end{pmatrix} \quad M, N = 1, \dots, d+2 \quad (4.1.11)$$

and computes the algebra of J_{MN} , then one can see that they are the generators of spacetime rotations in $d+2$ dimension with signature $(2, d)$. This implies that the conformal group is homomorphic to $SO(2, D)^{24}$. In the special case of $d = 4$, the conformal group is $SO(2, 4)$ which admits $SO(2) \times (SU(2) \times SU(2))$ as compact subgroups. $SO(2)$ is related to the dilation generators D , while $SU(2) \times SU(2)$ corresponds to the $J_{\mu\nu}$ Lorentz generators. Then, any field in the CFT is labelled by the eigenvalue Δ of D and the pair (j_1, j_2) . To conclude, we observe that the conformal algebra (4.1.10) can be enlarged to accomodate supersymmetry; more on this will be discussed in Section 5.

²⁴Actually this is strictly true at the level of algebras.

4.2 *AdS* spacetime

Let us now turn to the other side of the correspondence: *AdS* spacetime. AdS_{d+1} is the maximally symmetric solution of Einstein equations in $d + 1$ dimensions with negative cosmological constant. It can also be described as the hyperboloid of equation

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^d X_i^2 = R^2, \quad (4.2.1)$$

embedded in flat $\mathbb{R}^{2,d}$ with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^d dX_i^2. \quad (4.2.2)$$

By construction, the space has $SO(2, d)$ symmetry, just like the conformal group. A set of global coordinates is given by

$$\begin{aligned} X_0 &= R \cosh \rho \cos \tau \\ X_{d+1} &= R \cosh \rho \sin \tau \\ X_i &= R \sinh \rho \Omega_i, \quad \left(i = 1, \dots, d, \sum_i \Omega_i^2 = 1 \right). \end{aligned} \quad (4.2.3)$$

To avoid closed timelike curves deriving from the periodicity of $\tau \in [0, 2\pi]$, we can consider the universal cover of *AdS*, with $-\infty < \tau < +\infty$.

In what follows, it is useful to introduce a more convenient set of coordinates,

$$\begin{aligned} X_0 &= \frac{1}{2u} (1 + u^2(R^2 + \vec{x}^2 - t^2)) \\ X_{d+1} &= Rut \\ X_i &= Rux^i, \quad i = 1, \dots, d-1 \\ X_d &= \frac{1}{2u} (1 - u^2(R^2 - \vec{x}^2 + t^2)) \end{aligned} \quad (4.2.4)$$

The *AdS* metric in this coordinate, known as Poincaré coordinates, is

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(dx_\mu dx^\mu) \right). \quad (4.2.5)$$

In this set of coordinates, each slice of $u = \text{constant}$ is a d -dimensional Minkowski spacetime, hence the name. This set of coordinates is not global, and the singularity at $u = 0$ is merely a coordinate singularity, since it can be eliminated by an appropriate choice of coordinates (as, for example, in (4.2.3)). The plane at $u = \infty$ is the boundary²⁵ of AdS_{d+1} .

When adopting (4.2.4), the subgroup $SO(1, 1) \times SO(1, d-1)$ of the full isometry group $SO(2, d)$ of AdS_{d+1} is manifest, while the $SO(2) \times SO(4)$ subgroup is manifest when (4.2.3) is used instead.

4.3 The correspondence

The most celebrated example of the holographic correspondence relates type IIB string theory on the $AdS_5 \times S^5$ geometry to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, a four-dimensional conformal gauge

²⁵Actually, the conformal boundary.

theory living on the boundary of the anti-de Sitter space. It emerges naturally from the near-horizon limit of a stack of N D3-branes, which can be described either through the open strings ending on the branes, giving the gauge theory, or through the curved geometry they source, giving the gravitational background. The equivalence of these two descriptions is the content of the duality. Its most striking feature is that it relates weak and strong coupling: when the gauge theory is strongly coupled and hard to treat perturbatively, the dual gravitational description becomes weakly coupled and tractable, offering a remarkable window into regimes that would otherwise be inaccessible. More specifically, take N coincident D3 branes and study the system at low energies, below the string scale l_s ; then, only the massless modes can be excited. We can analyze the system from two different perspectives. Referring to section 3.3, we know that we can use low energy effective actions to describe the dynamics of the massless excitations of the closed strings, the open strings and their reciprocal interactions. The complete action is

$$S = S_{bulk} + S_{branes} + S_{int}. \quad (4.3.1)$$

The first term is (3.3.10), supplemented by the corresponding fermionic terms and higher derivative corrections, whereas the second one is an improved version of the low-energy Dp -brane action encountered in Section 3.3.2, now specialized to $p = 3$, plus some higher derivative corrections. Although the action of a single D3-brane is essentially the abelian Dirac-Born-Infeld theory (plus the topological Wess-Zumino term (3.3.26)) already discussed, the situation changes when N D3-branes are made to coincide: the open strings stretching between different branes give rise to additional light degrees of freedom, and the worldvolume gauge symmetry is enhanced from $U(1)^N$ to $U(N)$, [57]. In the low-energy limit the resulting theory is precisely $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$ (or $SU(N)$, once the decoupled overall $U(1)$ is removed), which is the gauge theory appearing on one side of the correspondence. Finally, S_{int} describes the interactions between the bulk modes and the brane ones.

We are interested in the low energy limit of our theory; in order to see more clearly what happens in this regime, it is convenient to perform the limiting procedure in the following way:

$$\begin{aligned} l_s \rightarrow 0, \quad (\alpha' \rightarrow 0) \\ g_s, N \text{ fixed.} \end{aligned} \quad (4.3.2)$$

In this limit, $\kappa \sim g_s \alpha'^2 \rightarrow 0$, hence all terms with positive powers of κ are neglected. Specifically, S_{int} , being proportional to κ or higher powers, is totally discarded, while, as anticipated, S_{branes} reduces to $\mathcal{N} = 4$ Super conformal Yang-Mills theory while S_{bulk} becomes free supergravity. Summing up, the low energy theory is composed of two decoupled systems:

$$\mathcal{N} = 4 \text{ SCFT} \oplus \text{Free supergravity}.$$

Now, we can adopt a different point of view to study the low energy limit of the same system.

Branes can be seen as solitons sourcing the closed string fields massless modes; it was shown in [44] that we can find a D3 brane solution of type IIB supergravity given by

$$\begin{aligned} ds^2 &= f^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \\ F_5 &= (1 + \star)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1}, \\ f &= 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s \alpha' N. \end{aligned} \quad (4.3.3)$$

From (4.3.3) we can deduce that the energy E_p of a particle measured by a static observer at constant position r and the energy E of the same particle measured by an observer at infinity are related by

$$E = f^{-1/4} E_p. \quad (4.3.4)$$

This means that objects closer and closer to $r = 0$ have less and less energy as seen by the observer at infinity. Then, from the point of view of an observer at infinity we have two kinds of low energy

excitations: massless particles propagating in the bulk with long wavelength $\gg R$ and excitations of any kind brought closer and closer to $r = 0$. We will argue that these two kinds of excitations are decoupled. Indeed, the bulk excitations decouple from the near horizon geometry ($r \rightarrow 0$) because the wavelength of the particle is much larger than the gravitational size of the brane (which is of order $\sim R$); on the other hand, excitations that live near $r = 0$ find it harder and harder to climb the gravitational potential and escape to the asymptotic region. In the end, we are led to the two decoupled systems

Near horizon geometry $\oplus$ Free supergravity.

By comparison with the previous findings, it is natural to identify the near horizon geometry with the superconformal Yang Mills theory mentioned above.

The near horizon limit of (4.3.3) is

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right], \quad (4.3.5)$$

where $U := r/\alpha'$ and in the limiting process we kept r/α' fixed. The geometry of (4.3.5) is that of $AdS_5 \times S^5$. We can finally formulate the famous conjecture, that is $U(N)$ super-Yang-Mills theory in $3 + 1$ dimensions is the same as (or dual to) type IIB superstring theory on $AdS_5 \times S^5$. Moreover, one could see, [66], that the strong coupling regime with large N of the CFT is mapped to the weakly coupled regime of gravity in the near horizon limit, and vice versa.

4.4 The holographic dictionary

Having described the physical insights that led us to formulate the AdS/CFT correspondence in type IIB theory, we conclude this section by illustrating how physical observables of the CFT are actually computed from the dual AdS gravity description.

As suggested by Witten, [64], since the bulk isometries $SO(2,4)$ of AdS_5 act conformally on its boundary, it is reasonable to infer that the four-dimensional spacetime in which the CFT lives is dual to the (conformal) boundary of AdS . The precise dynamical statement of duality is the famous GKP-Witten formula, [64], [38], which realizes this map at the level of generating functionals. One takes the boundary value of each bulk field to be the source $\phi_0(x)$ that couples to the dual operator $\mathcal{O}$ in the CFT action, and equates the two partition functions:

$$\left\langle \exp \int d^4x \phi_0(x) \mathcal{O}(x) \right\rangle_{\text{CFT}} = Z_{\text{string}}[\phi(x, z)|_{z \rightarrow 0} \rightarrow \phi_0(x)], \quad (4.4.1)$$

The left side is the CFT generating functional of connected correlators of $\mathcal{O}$; the right side is the full string (in the supergravity limit, gravitational) partition function in AdS , evaluated with Dirichlet-type boundary conditions that impose the source ϕ_0 as the leading near-boundary data of ϕ . In the regime of large N and large 't Hooft coupling mentioned earlier, where the bulk is weakly curved and classical, the right side reduces to $e^{-S_{\text{on-shell}}[\phi_0]}$ with $S_{\text{on-shell}}$ the supergravity action evaluated on the classical solution determined by ϕ_0 . CFT correlation functions are then obtained by functional differentiation with respect to the sources,

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta^n S_{\text{on-shell}}}{\delta \phi_0(x_1) \cdots \delta \phi_0(x_n)} \Big|_{\phi_0=0}. \quad (4.4.2)$$

Finally, in order to identify which bulk AdS field ϕ is mapped to the CFT operator $\mathcal{O}$, one can resort to symmetry arguments. The idea is that ϕ and $\mathcal{O}$ should share the same $SO(2, d)$ quantum numbers. In particular, whenever the CFT operator is a conserved quantity associated to a global symmetry G , the most natural coupling comes by covariantizing the action by adding appropriate

AdS gauge fields. Thus, a remarkable aspect of the correspondence is that bulk gauge symmetries are mapped to global symmetries of the CFT. For instance, in the type IIB duality described earlier, the $SO(6) \sim SU(4)$ isometries of the internal manifold S^5 descend to gauge symmetries of the effective AdS_5 theory obtained after compactifications; by the holographic duality they are translated to the global $SU(4)_R$ symmetries of the dual CFT.

4.5 Symmetry defect operators and Holography

A growing body of work has established holography as a powerful tool for describing the symmetry structure of a low-energy theory. In [5, 7], for instance, it was shown how the symmetry operators of theories holographically dual to a given string-theory setup can be constructed systematically from probe-brane configurations. This holographic method is well understood for discrete symmetries, and only recently attempts have been made to extend it to the continuous case. The authors of [16] proposed a way to encode abelian continuous symmetries holographically, identifying non-BPS Dp branes as candidates for the holographic duals of the symmetry operators. More precisely, they showed that even after the non-BPS brane has decayed it leaves behind a $U(1)$ parameter, which suggests that such branes may indeed realize continuous symmetries. Building on this observation, and appealing to a regularization procedure borrowed from QFT, the authors of [25] argued that such non-BPS $D(p-1)$ branes originate from the decay of a $Dp/\overline{Dp}$ system, thereby furnishing a way to regulate the symmetry operator.

Non-abelian symmetries were first addressed in this context by [11] which, together with [10], points towards a systematic approach to symmetries in holography. The key insight is that the brane/anti-brane pair mentioned above can be realized as a single BPS U-shaped brane hanging from the conformal boundary of AdS_{d+1} , while the charged operator is dual to a Dp brane whose external direction coincides with the radial direction of AdS_{d+1} . These works also supply a string-theoretic mechanism, based on the Hanany-Witten effect, that accounts for how charge measurement is realized in the underlying string setup.

In what follows we illustrate a few explicit examples of top-down holographic realizations of defect operators, drawing on the constructions of [5] and [11] for the abelian discrete case and the continuous non-abelian case, respectively.

4.5.1 Holography for discrete abelian symmetries

The linchpin behind the construction of [5] is that branes, in a precise topological limit, can reproduce the topological subsector of the original gravity action in AdS_{d+1} , which in turn encodes the symmetry structure of the dual CFT. The reduced topological field theory is tied to the notion of Symmetry Topological Field Theory (SymTFT), which has recently emerged as a powerful tool for analyzing the symmetry structure of quantum field theories, especially for discrete symmetries,[19].

Assuming that the symmetry group $\mathcal{S}$ factorizes into abelian discrete symmetries,

$$\mathcal{S} = \prod_p G^{(p)},$$

with $G^p = \prod_i \mathbb{Z}_{n_i^p}$, then, denoting with B_{p+1}^i the corresponding background gauge fields, the topological subsector is expected to be captured by the following $(d+1)$ -dimensional action:

$$S = \int_{M_{d+1}} \sum_p \sum_{ij} n_{ij}^p b_{p+1}^i \wedge \hat{d} b_{d-p-1}^j + \mathcal{A}(\{b_{p+1}^i\}), \quad (4.5.1)$$

where the first terms are BF-couplings between the (now dynamical) gauge fields b and their duals $\hat{b}$, and $\mathcal{A}$ is a possible anomaly term. Note that we adopt here the continuum description introduced in Section 2 to represent the finite-group cocycles associated with the discrete symmetry.

The top-down reconstruction of the this topological subsector of bulk gravity proceeds as follows. Start from a string theory whose holographic solution is

$$M_{D+1} = AdS_{d+1} \times X_n,$$

with $D = 9$ for type II theories and $D = 10$ for M-theory. The goal is to reduce over the internal compact manifold X_n , retaining only the topological couplings. This “topological limit” corresponds to a low-energy regime in which only the topological couplings survive and higher-derivative terms are negligible.

The topological $(D + 1)$ -dimensional bulk action can be recast as a $(D + 2)$ -dimensional theory depending only on the field strengths $F^{(i)}$ rather than on the gauge potentials. It usually takes the form

$$S_{D+2} = \int_{M_{D+2}} \left[\frac{1}{2} \kappa_{ij} F^{(i)} \wedge dF^{(j)} - \kappa_{ij} F^{(i)} \wedge J^{(j)} \right], \quad (4.5.2)$$

where the $F^{(i)}$ are the $(D + 2)$ -dimensional fluxes descending from the original $(D + 1)$ -dimensional theory, while the $J^{(i)}$ are built from the $(D + 1)$ -dimensional brane source insertions. After compactification on the internal space one obtains a $(d + 2)$ -dimensional theory, namely action (4.5.1), formulated on a one-dimension-higher auxiliary manifold.

One may further enrich (4.5.2) by adding Chern-Simons terms $\text{CS}(\{F^{(i)}\})$, polynomial in the fluxes $F^{(i)}$. The closed form $\text{CS}(\{F^{(i)}\})$ is related to the physical $(D + 1)$ -dimensional Chern-Simons terms through the descent procedure,

$$S_{D+1} \supset \int_{M_{D+1}} I_{D+1}^{(0)}, \quad dI_{D+1}^{(0)} = \text{CS}(\{F^{(i)}\}). \quad (4.5.3)$$

As an example, the topological sector of type IIB theory is captured by the auxiliary eleven-dimensional action

$$S_{IIB} = \int_{M_{11}} F_1 \wedge dF_9 - F_3 \wedge dF_7 + \frac{1}{2} F_5 \wedge dF_5 + H_3 \wedge dH_7 + H_3(F_1 \wedge F_7 - F_3 \wedge F_5), \quad (4.5.4)$$

where we have introduced the Hodge-dual fields F_7 , F_9 and H_7 . This action must be supplemented by a source term for each flux; the corresponding Bianchi identities in the presence of sources, which we omit for brevity, are collected in Appendix A of [5]. To carry out the compactification on the internal space, we expand the flux content on representatives of the non-trivial cohomology of X_n . The BF-terms of the reduced $(d + 2)$ -dimensional auxiliary action can arise either from the expansion of the Chern-Simons terms or from the $F^{(i)} \wedge dF^{(j)}$ terms in (4.5.2).

Let us now put these general ideas into practice. Consider the Klebanov-Strassler solution [47], dual to $4d \mathcal{N} = 1$ $SU(M)$ SYM. This is the back-reacted configuration of N D3 branes probing the resolved conifold, i.e. the cone $C(T^{1,1})$ over the Sasaki-Einstein space $T^{1,1}$, together with M D5 branes wrapping the $S^2 \subset T^{1,1}$. The relevant flux quantization is

$$\int_{S^3} F_3 = M \quad (4.5.5)$$

The expansion ansatz are given by

$$\begin{aligned} F^{(1)} &= F_3 = M \text{vol}_{S^3} + f^{(1,1)} \text{vol}_{S^2} + f^{(1,2)}, \\ F^{(2)} &= H_3 = f^{(2)}, \\ F^{(3)} &= F_5 = F_{\text{bkg}}^{(3)} + f^{(3,1)} \wedge \text{vol}_{S^2} + f^{(3,2)} \wedge \text{vol}_{S^3}, \end{aligned} \quad (4.5.6)$$

where the volume forms are normalized as

$$\int_{S^3} \text{vol}_{S^3} = 1, \quad \int_{S^2} \text{vol}_{S^2} = 1, \quad \int_{T^{1,1}} \text{vol}_{S^3} \wedge \text{vol}_{S^2} = 1, \quad (4.5.7)$$

the $f^{(i)}$ are differential forms with legs only along the external space, and $F_{\text{bkg}}^{(3)}$ denotes a non-vanishing background value of the $F^{(3)}$ flux. Inserting (4.5.6) into the Chern-Simons term $H_3 \wedge F_3 \wedge F_5$ of (4.5.4) yields the $(d+2)$ -dimensional BF term

$$S_{d+2} = \int_{M_{d+2}} M f^{(2)} \wedge f^{(3,1)}, \quad (4.5.8)$$

with $d = 4$. Had we included source terms for the fluxes,

$$\begin{aligned} dH_7 &= J^{(2)} : \quad J^{(2)} = j^{(2,1)} \wedge \text{vol}_{T^{1,1}} + \dots \\ dF_5 &= J^{(3)} : \quad J^{(3)} = j^{(3,1)} \wedge \text{vol}_{S^3} + \dots \end{aligned} \quad (4.5.9)$$

then we would obtain

$$S_{d+2}^{\text{BF+sources}} = \int_{M_{d+2}} \left[M f^{(2)} \wedge f^{(3,1)} - j^{(2,1)} \wedge f^{(2)} - j^{(3,1)} \wedge f^{(3,1)} \right]. \quad (4.5.10)$$

Finally, regarding equation (4.5.1) as a SymTFT, one should also add condensation defects to (4.5.1); these are realized by coupling the SymTFT to lower-dimensional Dijkgraaf-Witten type theories,

$$S_{\text{SymTFT}} \supset n_p \int_{M_{d+1}} b_{p+1} \wedge d\hat{b}_{d-p-1} + \sum_{k \geq 1} \int_{M_{d-k}} \left(b_{p+1} \wedge a_{d-k-p-1} + n_p a_{d-k-p-1} \wedge d\hat{a}_p \right), \quad (4.5.11)$$

This admits a stringy interpretation in terms of $Dp/\overline{Dp}$ brane pairs: adding the action for a brane/anti-brane pair (again described by an auxiliary one-dimension-higher action) and performing the reduction above reproduces the extra terms in (4.5.11)²⁶.

4.5.2 Continuous Non-abelian symmetries from branes

To conclude, we discuss how [11] constructs an explicit string-theory realization of topological symmetry operators for continuous, non-abelian symmetries in holography, a case that had remained largely unexplored.

The setup is type IIB string theory on $AdS_5 \times S^5$, dual to $4d$ $\mathcal{N} = 4$ super Yang-Mills, the same Maldacena conjecture that we already met. The relevant symmetry is the global $SO(6)$ R-symmetry of the dual SCFT, inherited from the isometry group of the internal S^5 .

We first present the general features of the construction of [11], later expanded in [10], and then specialize to the $AdS_5 \times S^5$ case treated in [11]. In Section 6 we apply the same construction to propose a holographic realization of the symmetry structure in the more exotic $AdS_7 \times M_3$ solutions of type IIA theory, introduced in Section 5.

4.5.2.1 Low energy description

We start from a low-energy theory on AdS_{d+1} , obtained by reducing the bulk string theory over the internal manifold. The $(d+1)$ -dimensional action reads²⁷

$$S_{\text{eff}} \supset \int_{AdS_{d+1}} \left[-\frac{1}{2} \tau_{IJ} F^I \wedge \star F^J + \text{CS}_{d+1}(A, dA) \right], \quad (4.5.12)$$

²⁶As before, only the terms relevant to the topological limit are retained.

²⁷The truncation retains only the bosonic content of the original string theory. It is not a consistent truncation, but it suffices for our purposes.

where τ_{IJ} is a set of constants and CS_{d+1} is the $(d+1)$ -dimensional Chern-Simons term, whose precise form will not be needed below.

Identifying the radial coordinate of AdS_{d+1} with Euclidean time, we analyse the low-energy effective dynamics in the Hamiltonian framework. Varying the action with respect to the time components of A^I gives the Gauss-law constraint $G_I = 0$, with

$$G_I = \tau_{IJ} D \star F^J + G_I^{CS}. \quad (4.5.13)$$

Here G_I is a d -form evaluated on a constant-time slice of AdS_{d+1} , and G_I^{CS} is the Chern-Simons contribution to the constraint. The symbol D denotes the gauge-covariant derivative,

$$D \star F^I = d \star F^I - f_{JK}{}^I A^J \wedge \star F^K. \quad (4.5.14)$$

Evaluating (4.5.13) at the conformal boundary of AdS_{d+1} and imposing Dirichlet boundary conditions on the gauge fields A^I , the constraint reduces to

$$G_I = \tau_{IJ} d \star F^J, \quad (4.5.15)$$

where G_I^{CS} has been dropped since it is polynomial in A^I , which vanishes at the boundary. The bulk gauge transformations generated by (4.5.13) then act as global symmetries of the dual theory, with conserved current G_I . We can now write the boundary symmetry operators of this global symmetry, in analogy with Section 2. Let θ^I be a set of constant parameters and $\mathcal{B}$ a d -chain on the boundary of AdS_{d+1} with $\partial\mathcal{B} = M_{d-1}$; using the exactness of G_I at the conformal boundary, we obtain

$$U(\theta, M_{d-1}) = \exp \left(\theta^I \int_{M_{d-1}} \tau_{IJ} \star F^J \right). \quad (4.5.16)$$

4.5.2.2 Regularization by thickening

As explained in Section 2, measuring a charged operator amounts to linking it with a suitable symmetry operator. However, as stressed at the beginning of Section 6, this description breaks down for continuous global symmetries, owing to divergences that must be regulated. As we shall see, this regularization is precisely what motivates the holographic construction of the symmetry operator in [10]. Following [12], we use a simple example to illustrate how such divergences are handled. Consider the action of a complex scalar field ϕ ,

$$S[\phi] = \int d^d x (\partial^\mu \phi^\dagger \partial_\mu \phi - V(|\phi|)). \quad (4.5.17)$$

This action is invariant under the $U(1)$ symmetry $\phi \rightarrow e^{i\alpha} \phi$, whose associated conserved current is

$$j = i(\phi^\dagger d\phi - \phi d\phi^\dagger), \quad (4.5.18)$$

with corresponding symmetry operator

$$U(\alpha, \Sigma) = \exp \left(i\alpha \int_\Sigma \star j \right). \quad (4.5.19)$$

Take Σ to be, for simplicity, the hypersurface at constant $t = 0$. We wish to compute the correlation function

$$\langle \phi(t_1, \vec{x}_1) \cdots \phi(t_{i-1}, \vec{x}_{i-1}) e^{i\alpha \int_{t=0} d^{d-1}x j_0} \phi(t_i, \vec{x}_i) \rangle, \quad (4.5.20)$$

with $t_1 < \cdots < t_{i-1} < 0 < t_i$. The field redefinition

$$\phi = e^{i\alpha\Theta(t)} \phi' \quad (4.5.21)$$

removes the insertion of $U(\alpha, \Sigma)$ and correctly assigns a factor $e^{i\alpha}$ to every operator ϕ inserted at $t > 0$. The variation of the kinetic term, however, generates a divergent contribution proportional to $[\delta(t)]^2$, so a suitable regularization of (4.5.21) is required. To this end we introduce the one-parameter family of functions

$$f_\lambda(t) = \frac{1}{1 + e^{-\frac{t}{\lambda}}}, \quad (4.5.22)$$

and note that

$$\lim_{\lambda \rightarrow 0} f_\lambda(t) = \Theta(t). \quad (4.5.23)$$

We regulate the divergence by thickening the symmetry operator,

$$e^{i\alpha \int_{t=0} d^d x j_0} \rightarrow e^{i \int d^d x (\alpha f'_\lambda(t) j_0 - \alpha^2 f'_\lambda(t) |\phi|^2)}. \quad (4.5.24)$$

The regularized field redefinition

$$\phi = e^{i\alpha f_\lambda(t)} \phi' \quad (4.5.25)$$

now removes the defect and assigns each field its correct phase, with no divergent terms. Taking $\lambda \rightarrow 0$ then recovers the action of the original topological operator. One might still question the topological nature of the regularized operator (4.5.24). Inserting it, however, is equivalent to switching on the flat background gauge field

$$A = df_\lambda(t) = f'_\lambda(t) dt. \quad (4.5.26)$$

In this background the divergence of the stress-energy tensor reads

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} \left(i(D_\mu \phi)^\dagger \phi - i\phi^\dagger D_\mu \phi \right). \quad (4.5.27)$$

Since $F^{\mu\nu} = 0$ for (4.5.26), the stress-energy tensor is conserved and (4.5.24) is therefore topological. This also clarifies the role of the “seagull” term $|\phi|^2 A^2$ in (4.5.24): it is exactly the remnant of the covariantization of the kinetic term of the complex scalar. The lesson of this simple example is that, to handle the divergences properly, the symmetry operator must be regulated, and this can be done by “thickening” the operator itself—a remark that lies at the heart of its holographic realization.

4.5.2.3 Hanging branes as symmetry operators

In light of these arguments, [10] proposed that the thickening regularization points to a family of d -dimensional objects whose “zero-width limit” reproduces the unregulated $(d-1)$ -dimensional symmetry operator (4.5.16). Concretely, the symmetry operators are conjectured to correspond to U-shaped branes hanging from the conformal boundary of AdS_{d+1} . As shown in [25], such a U-shaped hanging brane can be viewed as a $Dp/\overline{Dp}$ pair: in the zero-width limit the two branes undergo tachyon condensation and decay into a non-BPS $D(p-1)$ brane on the boundary, which is identified with the unregulated symmetry operator. A similar construction appears in [29], where the continuous symmetry operators are identified with particular fluxbrane configurations wrapping cycles at the asymptotic boundary of the internal manifold; Appendix A of [29] shows that these configurations can be realized as soliton backgrounds of a D-brane/anti-D-brane pair, in agreement with the symmetry-operator construction of [10, 11].

The detailed form of the probe Dp brane realizing the regulated symmetry operator is theory-dependent, but its general structure can be described. Its support wraps the submanifold $\gamma_1 \times M_{d-1} \times X_{p-d+1}$, where $M_{d-1} \subset \partial AdS_{d+1}$, γ_1 is a small arc dipping into the bulk of AdS_{d+1} and joining two boundary points, and X_{p-d+1} is the internal manifold wrapped by the brane.

4.5.2.4 Charged operators

Turning to the charged operators, the effective AdS_{d+1} action (4.5.12) admits Wilson-line operators transforming in a representation of the gauge group. Such Wilson lines can be defined as the parallel transport of the gauge field along a curve γ in AdS_{d+1} ; for our purposes, however, it is more convenient to use an equivalent description in which the Wilson line is a quantum-mechanical model supported on M_1 and coupled to the bulk gauge field A . Its action is, [10],

$$S_{WL} = \int_{M_1} \langle q, \chi^{-1} d_A \chi \rangle, \quad \text{with } \chi^{-1} d_A \chi = \chi^{-1} d\chi + \chi^{-1} A \chi. \quad (4.5.28)$$

Here χ is a G -valued function localized on M_1 , $\chi^{-1} d_A \chi$ takes values in the Lie algebra $\mathfrak{g}$ of G , and q is an element of the dual algebra $\mathfrak{g}^*$; the bracket $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

For $G = SO(n+1)$ the action (4.5.28) simplifies considerably:

$$S_{WL} = \frac{1}{2} \int_{M_1} q_{ab} (\chi^{-1} d_A \chi)^{ab}, \quad (4.5.29)$$

where $a, b = 1, \dots, n+1$ are fundamental $SO(n+1)$ indices and χ is an $SO(n+1)$ matrix, i.e. $\chi^{-1} = \chi^T$. The Lie algebra $\mathfrak{g}$ of $SO(n+1)$ consists of antisymmetric matrices, and since the Cartan-Killing metric induces the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$, q_{ab} may likewise be taken to be antisymmetric. The Wilson line is mapped to a charged operator of the dual CFT by imposing Dirichlet boundary conditions on the bulk gauge field A at the conformal boundary of AdS_{d+1} ; see Figure 11. The key point is

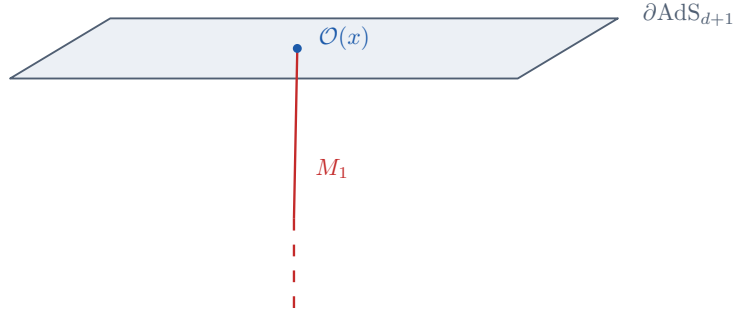


Figure 11: The Wilson line is mapped to a charged operator $\mathcal{O}(x)$ of the dual CFT, sitting at the conformal boundary of AdS_{d+1} and supported on the line M_1 extending into the bulk.

that a suitable probe-brane configuration in the bulk can reproduce the action (4.5.29). Specifically, whenever the internal space contains an S^n factor from which the gauge group $SO(n+1)$ descends, the action (4.5.29) is recovered holographically from the topological Wess-Zumino couplings of a brane wrapping a maximal $S^{n-2} \subset S^n$ and extending along the radial direction of AdS_{d+1} . One must also account for the fluctuations of the brane along the S^n directions, encoded by scalar fields Y^i on the worldvolume subject to the constraint $Y^i Y^i = 1$. Their profile is

$$Y^a(\tau, \sigma^i) = \chi^a_b(\tau) Y^b_{(0)}(\sigma^i), \quad (4.5.30)$$

where (τ, σ^i) are worldvolume coordinates, with τ parametrizing the radial direction of AdS_{d+1} and σ^i the wrapped S^{n-2} . The functions $Y^a_{(0)}$ describe the brane wrapping²⁸ the S^{n-2} at fixed τ , while $\chi(\tau)$ is $SO(n+1)$ -valued, i.e. $\chi \chi^T = 1$.

²⁸One could equally consider configurations in which the brane wraps the $(n-2)$ -sphere w times; we set $w = 1$ throughout.

4.5.2.5 R-symmetries in $AdS_5 \times S^5$

Finally, following the reasoning above, we collect the results of [11] in the specific setting of $AdS_5 \times S^5$ type IIB theory.

For the charged operators, [11] show that a D3 brane wrapping the radial direction $M_1 \subset AdS_5$ and a maximal 3-sphere $S^3 \subset S^5$ correctly reproduces the expected low-energy action (4.5.29) of an operator charged under $SO(6)_R$. The internal S^3 is obtained by setting two of the six y^i coordinates parametrizing S^5 to zero: denoting by Σ^{ab} the maximal 3-sphere at $y^a = 0 = y^b$ with $a \neq b$, the general internal cycle wrapped by the D3 is

$$\Sigma = \frac{1}{2} n_{ab} \Sigma^{ab}, \quad (4.5.31)$$

with $n_{ab} = -n_{ba}$ a set of integers and $\Sigma^{ab} = -\Sigma^{ba}$ understood in the chain sense.

For the symmetry operators, the relevant string configuration is a bound state of a D5 brane and a five-dimensional Kaluza-Klein (KK) monopole; the need for the KK monopole will be explained shortly. According to the general construction, the D5 wraps $\gamma_1 \times M_3 \times \Sigma_2$, with $M_3 \subset AdS_5$ and $\Sigma_2 \subset S^5$. To define Σ_2 it is convenient to write S^5 as a Hopf fibration over $\mathbb{CP}^2$, introducing complex coordinates

$$zw_1 = y_1 + iy_2, \quad zw_2 = y_3 + iy_4, \quad z = y_5 + iy_6, \quad (4.5.32)$$

with $z = e^{i\psi}$. The S^5 metric then takes the form

$$ds_{S^5}^2 = ds_{\mathbb{CP}^2}^2 + (d\psi + \mathcal{A})^2, \quad (4.5.33)$$

where the first term is the Fubini-Study metric on $\mathbb{CP}^2$ (parametrized by w_1 and w_2) and the connection one-form $\mathcal{A}$ satisfies

$$d\mathcal{A} = 2J, \quad (4.5.34)$$

with J the Kähler form on $\mathbb{CP}^2$. Since $\mathbb{CP}^2$ has non-trivial homology in degree 2, we may define Σ_2 as

$$\Sigma_2 = \text{PD} \left[\frac{d\mathcal{A}}{2\pi} \right]_{\mathbb{CP}^2}. \quad (4.5.35)$$

When uplifted to the total space S^5 , Σ_2 becomes a trivial cycle, as expected since S^n has no non-trivial homology in degrees below n . After some computation, the D5 configuration above yields the symmetry operator

$$U_{D5}(\alpha_{D5}, m_{ab}, M_3) = \exp \left(\frac{\alpha_{D5} m_{ab}}{2g_{\text{flux}}^2} \int_{M_3} \star_5 F^{ab} \right), \quad (4.5.36)$$

where $\star_5 F^{ab}$ is the AdS_5 Hodge dual of the $SO(6)$ gauge field strength and

$$\alpha_{D5} = \int_{\gamma_1} a_1, \quad (4.5.37)$$

with a_1 the $U(1)$ worldvolume gauge potential on the D5. The integer coefficients m_{ab} reflect the particular way the Hopf fibration is realized, i.e. how the y^a coordinates in (4.5.32) are permuted, and g_{flux} is a numerical constant.

The five-dimensional KK monopole shares its worldvolume with the D5, and it can be shown [11] that it gives rise to the symmetry operator

$$U_{KK}(\alpha_{KK}, m_{ab}, M_3) = \exp \left(\frac{\alpha_{KK} m_{ab}}{2g_{E,H}^2} \int_{M_3} \star_5 F^{ab} \right), \quad (4.5.38)$$

where $g_{E,H}$ is another numerical constant. The role of the KK monopole becomes clear upon combining (4.5.36) and (4.5.38), which gives

$$U_{D5-KK}(\alpha, m_{ab}, M_3) = \exp \left(\frac{\alpha m_{ab}}{2g^2} \int_{M_3} \star_5 F^{ab} \right), \quad (4.5.39)$$

where g is the effective coupling constant of the low-energy AdS_5 action; this reproduces the correct normalization²⁹.

4.5.2.6 Charge measurement

Finally, [11] showed that the holographic counterpart of charge measurement is provided by an F1 string via the Hanany-Witten effect, generated when the D3 and D5 branes cross each other.

This can be seen in the following way. The worldsheet of the F1 string consist of a semidisk bounded by the arc γ_1 and th conformal boundary of AdS_5 . Now, if we focus on the topological terms of the F1 string action, we recall that F1 couples with the $U(1)$ worldvolume gauge field a_1 of the D5 brane on which it ends. The relevant term is

$$S_{F1} \supset \int_{\partial W_2} a_1, \quad (4.5.40)$$

where W_2 is the worldsheet of the F1 string. The boundary of the F1 worldsheet is a segment on the conformal boundary and the arc γ_1 ; however the gauge field a_1 is supported only on the latter, so the exponentiated action reduces to

$$\exp \left(i \Sigma_3 \cdot \Sigma_2 \int_{\gamma_1} a_1 \right) = \exp (i \alpha \Sigma_3 \cdot \Sigma_2), \quad (4.5.41)$$

where we have recalled that $\alpha = \int_{\gamma_1} a_1$ and we added the internal intersection number $\Sigma_3 \cdot \Sigma_2$ between the 3-cycle Σ_3 wrapped by the D3 and the 2-cycle Σ_2 wrapped by the D5.

It can be proved that, [11], the exponential above reduces to

$$\exp \left(\frac{1}{2} \theta^{ab} N q_{ab} \right), \quad (4.5.42)$$

thus different choices of the symmetry operator (corresponding to different θ^{ab}) can be used to probe the R-charge of the D3 branes.

²⁹This requires assuming $\alpha \equiv \alpha_{D5} = 2\alpha_{KK}$.

5 $6d$ -SCFT's and their holographic realizations

In the previous section we introduced the notion of CFT, reviewed some elementary facts about conformal invariance, and highlighted the correspondence between the AdS isometry group in $d + 1$ dimensions and conformal invariance in d dimensions. This ultimately led us to the formulation of the AdS/CFT duality, of which the one proposed by Maldacena [50] is an example. More precisely, Maldacena's conjecture, reviewed in Section 4.3, relates a supersymmetric conformal field theory (SCFT) to type IIB theory on $AdS_5 \times S^5$, which also enjoys supersymmetry. This is not the only instance in physics where SCFT's play a major role: they are ubiquitous in high-energy physics as well as in condensed matter physics. It is not surprising, then, that SCFT's have been intensively studied in recent years.

Among all SCFT's, perhaps the most intriguing are those living in six dimensions. Indeed, as shown by [52], the superconformal algebra can be constructed only when the spacetime dimension d is less than or equal to six; consequently, 6-dimensional SCFT's are the highest-dimensional theories in which conformal invariance and supersymmetry can coexist. For a long time it was believed that such theories could not exist; however, as pointed out by [59], the so-called “tensionless string theories” engineered from string constructions in [61, 65] turned out to be genuine examples of 6-dimensional SCFT's.

One of the most challenging aspects in the study of 6-dimensional SCFT's is that, to date, no Lagrangian description of any of these theories is known. This is due to several exotic features shared by such theories, such as the fact that the coupling constants of marginal operators are expected to diverge at the non-trivial conformal fixed point [59], as well as the presence of tensor multiplets, which makes it difficult to write down a Lorentz-covariant Lagrangian.

Despite these difficulties, many string-theory constructions have proved successful in building 6-dimensional SCFT's from a top-down perspective. As far as our thesis work is concerned, in this section we will briefly show how the authors of [40, 23] engineered 6d SCFT's from a system of NS5-D6-D8 branes in type IIA theory. In particular, as conjectured by [33] building on the work of [4], these 6d SCFT's should be dual to massive AdS_7 solutions of type IIA theory. Before diving into the stringy realizations of such theories, however, we will briefly illustrate some basic properties common to any 6d SCFT, following the review [41].

5.1 Basic properties of 6d SCFT's

The conformal algebra (4.1.10) is no longer sufficient when supersymmetry is present; we must supplement it with $\mathcal{N}$ fermionic supercharge operators Q and $\mathcal{N}$ superconformal operators S , alongside the generators R of the symmetry that rotates the supercharges. Roughly speaking, the Q 's can be thought of as the “square root” of the translation generators P ,

$$\{Q, Q\} \sim P, \quad (5.1.1)$$

while the S 's are the “square root” of the generators K of the special conformal transformations,

$$\{S, S\} \sim K. \quad (5.1.2)$$

Since it is not relevant for the remainder of our discussion, we omit the full superconformal algebra. We simply state that in six dimensions the superconformal algebra is $\mathfrak{osp}(6, 2|\mathcal{N})$, with bosonic subalgebra

$$\mathfrak{osp}(6, 2|\mathcal{N}) \supset \mathfrak{so}(6, 2) \times \mathfrak{sp}(\mathcal{N}), \quad (5.1.3)$$

where $\mathfrak{so}(6, 2)$ is just the conformal algebra in six dimensions and $\mathfrak{sp}(\mathcal{N})$ is the algebra of the R-symmetry.

The Q 's and the S 's defined above transform as spinors of positive and negative chirality, respectively, under the $\mathfrak{so}(5, 1)$ Lorentz algebra, and combine into a single spinor of positive chirality of $\mathfrak{so}(6, 2)$. In addition, they belong to the fundamental representation of the R-symmetry algebra $\mathfrak{sp}(\mathcal{N})$.

In six dimensions, the number $\mathcal{N}$ of supercharges is constrained to satisfy

$$\mathcal{N} \leq 2,$$

hence SCFT's in six dimensions come equipped with either $\mathcal{N} = 1$ or $\mathcal{N} = 2$ and are denoted by $(1, 0)$ and $(2, 0)$ theories, respectively. This nomenclature stems from the fact that in six dimensions one can impose a pseudo-Majorana condition together with a chirality condition, which yields pseudo-Majorana-Weyl spinors:

$$Q_{\alpha, i} = \Omega_{ij} C_{\alpha\beta} Q^{\beta, j\dagger}. \quad (5.1.4)$$

In the above relation, $C_{\alpha\beta}$ is a suitable charge-conjugation matrix built from the Clifford algebra of $\mathfrak{so}(5, 1)$, and Ω_{ij} plays an analogous role in the R-symmetry algebra.

It is useful to recall that in 6-dimensional $(1, 0)$ theories (not necessarily conformally invariant) the low-spin massless states that appear after string compactifications are

- **The gravity multiplet**, which has one graviton of spin $(1, 0)$, two gravitinos of spin $(\frac{1}{2}, 1)$ and one additional field of spin $(0, 1)$, a chiral two-form with self-dual field strength.
- **The tensor multiplet**, which has one tensor field 2-form B (with anti-self-dual field strength H) of spin $(1, 0)$, two fermions of spin $(\frac{1}{2}, 0)$, and one scalar of spin $(0, 0)$.
- **The vector multiplet**, which has one vector field of spin $(\frac{1}{2}, \frac{1}{2})$ and two fermions of spin $(0, \frac{1}{2})$.
- **The hypermultiplet**, which has two fermions of spin $(\frac{1}{2}, 0)$ and four scalars of spin $(0, 0)$.

This implies that the only low-spin degrees of freedom of the 6-dimensional $(1, 0)$ SCFT's are those of the tensor multiplet, the vector multiplet and the hypermultiplet, whereas the gravity multiplet is discarded, since 6-dimensional SCFT's are non-gravitational theories.

In the $\mathcal{N} = 2$ case, the only low-spin massless states belong to the $(2, 0)$ tensor multiplet, which is made up of five scalars transforming in the $\mathbf{5}$ of the $\mathfrak{sp}(2) \simeq \mathfrak{so}(5)$ R-symmetry. Moreover, any $\mathcal{N} = 2$ supermultiplet can be decomposed in terms of $\mathcal{N} = 1$ supermultiplets: the $(2, 0)$ tensor multiplet can be regarded as a combination of a $(1, 0)$ tensor multiplet and a $(1, 0)$ hypermultiplet.

As mentioned at the beginning of the section, interacting field theories involving tensor multiplets can hardly be cast in a Lagrangian description; for instance, if one tries to write a kinetic term for the two-form tensor field B , the usual term

$$\mathcal{L} = -\frac{1}{2} dB \wedge \star dB \quad (5.1.5)$$

is immediately seen to vanish, owing to the anti-self-duality relation $\star dB = -dB$.

Among the available low-spin multiplets in $(1, 0)$ theories, the only ones containing scalar fields are the tensor multiplet and the hypermultiplet. The moduli space therefore splits into two branches: the tensor branch, in which the scalar fields of the tensor multiplet acquire a non-vanishing vacuum expectation value (VEV), and the Higgs branch, in which the scalar fields of the hypermultiplet acquire a VEV.

One crucial aspect worth mentioning is the role played by anomalies in realizing consistent 6-dimensional SCFT's. Suppose that at least one of the scalar fields has been given a VEV; then a mass scale is introduced and the theory ceases to be conformally invariant. We have thus moved away from the

conformal fixed point and obtained a new theory, which is still a chiral theory in an even-dimensional spacetime. Theories of this kind are generically plagued by chiral anomalies. Such anomalies are encoded in the four-point functions $\langle j_{\alpha_1}^{\mu_1} j_{\alpha_2}^{\mu_2} j_{\alpha_3}^{\mu_3} j_{\alpha_4}^{\mu_4} \rangle$, where $j_{\alpha_i}^{\mu_i}$ is the current associated with the symmetry α_i (these four-point functions are the 6-dimensional analogue of the three-point functions encountered when studying chiral anomalies in 4d). Depending on whether the currents in the four-point function belong to the same symmetry or to distinct ones, we speak of pure and mixed anomalies, respectively. The anomalies of the non-conformal theory are related to those of the 6d SCFT, and thus impose constraints on the superconformal theory. All of these anomalies are encoded in the so-called anomaly polynomial, which is an 8-form I_8 constructed from the field strengths F_α associated with the symmetries α . The anomaly polynomial I_8 is related to the anomalous variation of the six-dimensional action I_6 by the descent procedure, which reads

$$I_8 = dI_7, \quad \delta I_7 = dI_6. \quad (5.1.6)$$

anomalies receive both one loop contributions as well as contributions from the variation of the two-form potentials associated with the tensor multiplets. In six dimensions this is often referred to as the Green-Schwarz-Sagnotti-West mechanism. Thus we may write the anomaly polynomial as

$$I_8 = I_8^{1\text{-loop}} + I_8^{\text{GS}}. \quad (5.1.7)$$

In particular, it can be seen, [41], that the Green-Schwarz contribution to the anomaly polynomial is

$$I_8^{\text{GS}} = \frac{1}{2} A^{ij} I_i I_j, \quad (5.1.8)$$

where A^{ij} is negative definite symmetric matrix³⁰ and I_i is a 4-form constructed from the field strengths of the symmetries of the theory. In particular, the Green-Schwarz contribution to the anomaly polynomial always contains a term of the form

$$\left(\sum_{ij} c_{ij} \text{Tr}(F_i^2) \right)^2, \quad (5.1.9)$$

where F_i is the field strength of the i -th symmetry, with i and j running over both gauge and global symmetries. Typically, whenever an anomaly involves a gauge symmetry, it spoils the consistency of the theory and therefore must be cancelled.

We conclude this brief overview of 6-dimensional SCFT's by pointing out that the BPS states of such theories correspond to tensionless strings. First, we note that the objects charged under the two-form B are strings. This can be seen as follows. Whenever electric sources of the B field are present, the corresponding field strength H satisfies

$$d \star H = \star J \quad (5.1.10)$$

where J is the 2-form current localized on the worldvolume swept out by the charged object. Hence, such an object extends along a 1-dimensional space, i.e. a string. Moreover, given the aforementioned self-duality relation, the same string is also a magnetic source for the B field. Finally, if one includes a central charge Z in the superconformal algebra, the states charged under Z are precisely these strings.

5.1.1 Type IIA top down construction of (1,0) theories

In this section we explore one of the many attempts to engineer 6d SCFT's from string theory. We review the so-called Hanany-Zaffaroni setup [40, 23], which can be regarded as a higher-dimensional T-dual analogue of a similar setup studied in [39].

³⁰Depending on the conventions adopted, it may be a positive definite symmetric matrix

Consider, in flat type IIA theory, N NS5 branes wrapping directions (012345), each located at a fixed position x_i^6 along the sixth direction, together with a stack of r_i D6 branes wrapping directions (0123456) and stretching between two NS5 branes along the sixth direction. In addition, we place a stack of n_i D8 branes wrapping directions (012345789) at a point along x^6 located on the worldvolume of the D6 branes connecting the i -th and the $(i+1)$ -th NS5 brane. See Figure 12.

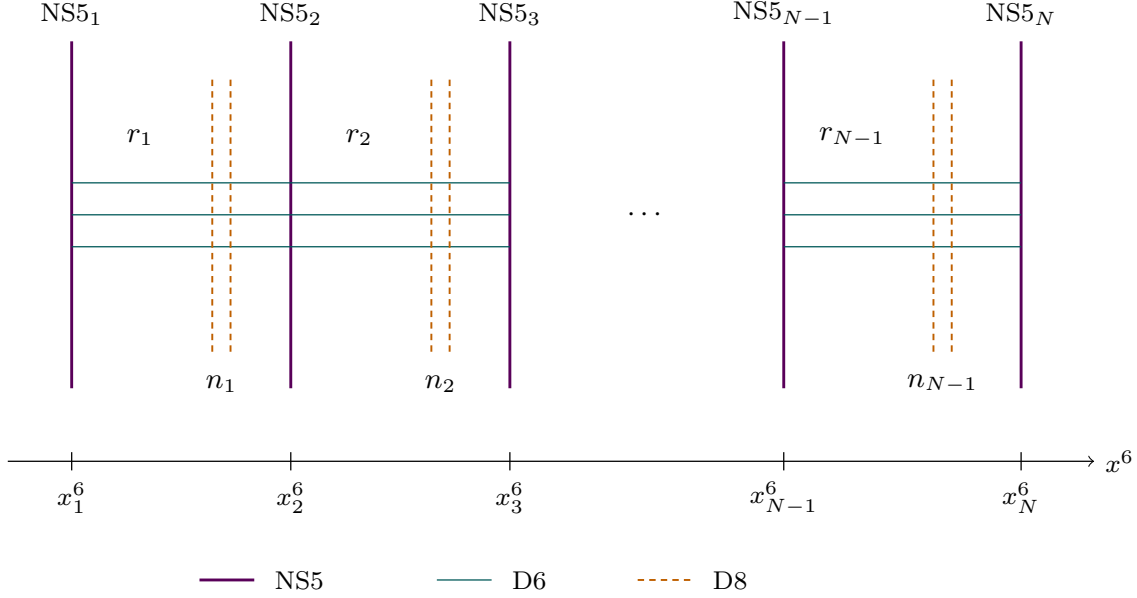


Figure 12: Pictorial representation of the Hanany-Zaffaroni setup in type IIA theory. For illustration purposes the stacks of D6 and D8 branes are represented by displaced lines, in the actual string construction they are all coincident.

Because of the branes, the original Lorentz symmetry $SO(1, 9)$ is broken to $SO(1, 5) \times SO(3)$; this is precisely the bosonic content of the algebra of supersymmetric $(1, 0)$ theories in six dimensions. The first factor is the Lorentz symmetry in six dimensions, while $SO(3)$ is identified with the $Sp(1) \simeq SU(2)$ R-symmetry³¹.

We now focus on the effective theory living on the common (012345) directions of each stack of D6 branes, obtained by performing a Kaluza-Klein reduction on the compact x^6 direction. The above analysis suggests that this effective theory is indeed a six-dimensional $(1, 0)$ supersymmetric theory. More specifically, for each interval between the i -th and the $(i+1)$ -th NS5 brane, the low-spin content consists of

- **Vector multiplets**, provided by the stack of D6 branes, which give rise to an $SU(r_i)$ gauge theory in each interval.
- **Bifundamental hypermultiplets**, i.e. pairs of hypermultiplets in the fundamental and anti-fundamental representations of the gauge groups $SU(r_i)$ and $SU(r_{i+1})$. They arise from the oscillations of strings localized on each NS5 brane, whose ends terminate on two adjacent D6 branes³².
- **Fundamental hypermultiplets**, which originate from the oscillations of strings stretched between D6 and D8 branes. They transform in the fundamental/anti-fundamental representation of the gauge symmetry $SU(r_i)$ and in the anti-fundamental/fundamental representation of the

³¹Notice the difference with equation (5.1.3): there, since the theory was also conformal, the Lorentz group was enhanced to the conformal group.

³²Strictly speaking, the vector multiplet would lead to a $U(r_i)$ gauge symmetry. However, through a Green-Schwarz mechanism, at low energies all the $U(1)$ factors become massive, so the gauge groups are just $SU(r_i)$.

global symmetry $U(n_i)$, respectively. The global $U(n_i)$ arises from the (frozen) vector multiplet on the worldvolume of each D8 stack.

- **Tensor multiplets.** The NS5 branes provide a $(2,0)$ tensor multiplet, but one of them (the one parametrizing the centre-of-mass motion of the system in the directions transverse to the NS5 branes) is decoupled. As stressed above, the $(2,0)$ tensor multiplet decomposes into a $(1,0)$ tensor multiplet and a $(1,0)$ hypermultiplet. The four scalars ϕ^i , $i = 6, 7, 8, 9$ of this multiplet correspond to the four transverse directions of the NS5 branes, x^i , $i = 6, 7, 8, 9$. In particular, x^6 belongs to the $(1,0)$ tensor multiplet, while the remaining ones belong to the $(1,0)$ hypermultiplet.

In general, the parameters associated with infinitely extended branes are frozen from the point of view of the six-dimensional observer. This is why, for example, the vector multiplet of the D8 branes realizes a global symmetry rather than a gauge one. At first sight this seems to contradict the fact that the bifundamental multiplets arising from NS5 oscillations are dynamical in our theory. The resolution is that, although infinitely extended, the NS5 branes wrap the very same six dimensions in which the supersymmetric theory is defined; hence, oscillations of the NS5 worldvolumes are dynamical fields for the six-dimensional observer.

The matter content of the gauge theory is succinctly summarized by a quiver diagram, such as the one shown in Figure 13; for this reason, such theories are also called quiver gauge theories. Moreover, since the associated quiver diagrams are linear chains, these theories are known as linear quiver gauge theories.

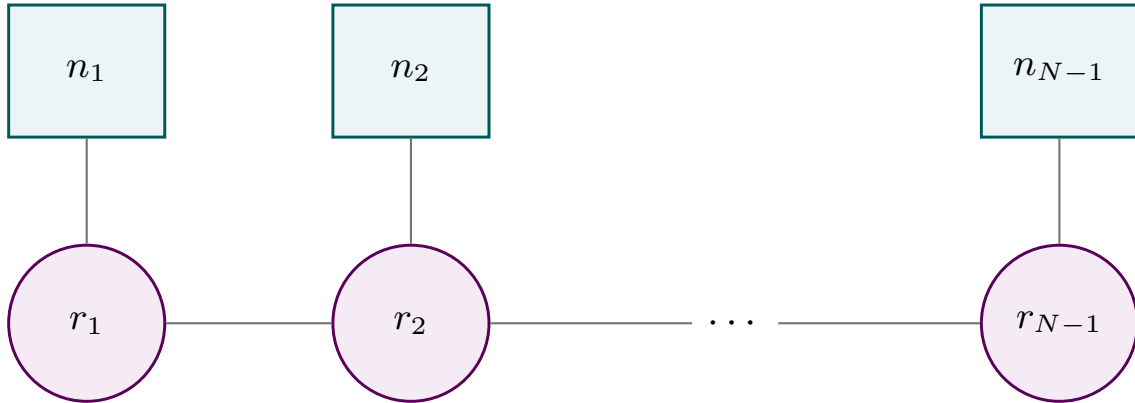


Figure 13: Quiver diagram representing the $(1,0)$ supersymmetric theory obtained by Kaluza-Klein reduction on the (012345) directions of the D6's worldvolumes. Squared boxes represent flavour groups $SU(n_i)$, whereas circle boxes represent gauge groups $SU(r_i)$. In addition, the lines connecting two consecutive gauge groups represent the hypermultiplets in the bifundamental representation, while the lines connecting each gauge group to the corresponding flavour group stand for the fundamental hypermultiplets.

It can be shown [39] that the gauge coupling constant of each gauge group $SU(r_i)$ obeys the relation

$$\frac{1}{g_i^2} = \frac{x_{i+1}^6 - x_i^6}{g_s l_s^3} = \phi_{i+1}^6 - \phi_i^6, \quad (5.1.11)$$

where g_s is the string coupling and l_s is the string scale.

In general, the matter content of $(1,0)$ supersymmetric theories is heavily constrained by anomalies; it can be shown [28] that anomaly cancellation requires

$$n_i = 2r_i - r_{i+1} - r_{i-1}. \quad (5.1.12)$$

This can be derived from an anomaly matching argument. In particular, we recall from the earlier discussion that the anomaly polynomial is made up of two pieces: the one-loop contributions and

the Green-Schwarz contribution. It is shown in [28] that the one-loop contribution to the anomaly polynomial is

$$\begin{aligned}
I^{\text{tot}} = \frac{1}{24} & \left[\sum_{i=1}^{N-1} \left[(-2r_i + r_{i-1} + r_{i+1} + f_i) \left(\text{tr} F_i^4 + \frac{p_1}{2} \text{tr} F_i^2 \right) - 12r_i c_2 \text{tr} F_i^2 \right] \right. \\
& - 3 \sum_{i,j} C_{ij} \text{tr} F_i^2 \text{tr} F_j^2 + \left(2(N-1) - \sum_i r_i^2 \right) \left(c_2^2 + \frac{1}{2} c_2 p_1 \right) \\
& + \frac{N-1}{240} (23p_1^2 - 116p_2) \\
& \left. + \frac{7p_1^2 - 4p_2}{240} \left(N-1 + \frac{1}{2} \sum_i r_i (-2r_i + r_{i-1} + r_{i+1} + 2f_i) \right) \right].
\end{aligned} \tag{5.1.13}$$

with

$$C_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}. \tag{5.1.14}$$

The term proportional to $\left(\text{tr} F_i^4 + \frac{p_1}{2} \text{tr} F_i^2 \right)$ can be removed by applying condition (5.1.12). The pieces proportional to $C_{ij} \text{tr} F_i^2 \text{tr} F_j^2$ and $r_i c_2 \text{tr} F_i^2$ can be recast as

$$-\frac{1}{8} C_{ij} I_i I_j + \frac{1}{2} C_{ij}^{-1} r_i r_j c_2^2, \quad I_i \equiv \text{tr} F_i^2 + 2c_2 C_{ij}^{-1} r_j. \tag{5.1.15}$$

Of the above equation, only the first term is proportional to the gauge field strengths. By comparison with (5.1.8), we see that this term can be cancelled by the Green-Schwarz contribution to the anomaly polynomial. The second term appearing in (5.1.15) as well as the surviving pieces of (5.1.13) then yield the physical anomaly of the theory, related only to the global symmetries.

Finally, we note that the tensor branch of these gauge theories is parametrized by the scalar ϕ^6 , i.e. by the relative distances between the NS5 branes along the x^6 direction. Collapsing all the NS5 branes yields a highly non-trivial conformal theory; from (5.1.11) we see that the gauge coupling becomes very large at the conformal fixed point. We have thus succeeded in realizing a non-trivial 6-dimensional (1, 0) SCFT.

5.2 Holographic duals

The existence of six-dimensional superconformal theories can be probed holographically by looking for supersymmetric AdS_7 vacua of ten- and eleven-dimensional supergravity. In M-theory the answer is classical and rigid: the only possibility is the Freund–Rubin solution $AdS_7 \times S^4$ and its orbifolds, dual to the (2, 0) theories. The remaining options are therefore $AdS_7 \times M_3$ backgrounds of type IIB, or of type IIA with nonvanishing Romans mass F_0 , which does not lift to eleven dimensions. Both cases were settled by [4].

5.2.1 The pure spinor approach

The strategy is an adaptation of the generalised complex geometry methods developed for $\text{Mink}_4 \times M_6$ and $AdS_4 \times M_6$ vacua [36]. One starts from the system of [49], equivalent to supersymmetry for $\text{Mink}_6 \times M_4$ solutions and written in terms of a pair of pure spinors $\phi^{1,2}$ defining an $SU(2) \times SU(2)$ structure on M_4 . An AdS_7 vacuum is then obtained by regarding AdS_7 as a warped product of Mink_6 with a line,

$$ds_{AdS_7}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\text{Mink}_6}^2, \quad e^{A_4} = \rho e^A, \quad ds_{M_4}^2 = \frac{e^{2A}}{\rho^2} d\rho^2 + ds_{M_3}^2, \tag{5.2.1}$$

so that any $AdS_7 \times M_3$ solution is a particular $Mink_6 \times M_4$ solution. Decomposing the four-dimensional pure spinors along $d\rho$ yields a closed system of equations for the three-dimensional bilinears $\psi^1 = \chi_1 \otimes \chi_2^\dagger$ and $\psi^2 = \chi_1 \otimes \chi_2^{c\dagger}$, built out of two unit-norm spinors on M_3 .

The decisive simplification is that spinorial geometry in three dimensions is elementary enough that the algebraic constraints on $\psi^{1,2}$ can be solved explicitly: the two spinors are parametrised by a single spinor χ and an angle ψ , and the resulting bilinears are expressed through a vielbein $\{e_a\}$ and two further angles $\theta_{1,2}$. Two consequences follow immediately. First, the zero-form parts of the differential system reduce in type IIB to $\cos\psi \cos\theta_1 = \cos\psi \sin\theta_1 = \sin\psi e^{i\theta_2} = 0$, which admits no solution: type IIB is excluded at once, without any further analysis. Second, and unusually, in type IIA the one-form parts of the system determine the vielbein *itself*, rather than its exterior derivative, in terms of derivatives of the parametrising functions. The local form of the metric and of the fluxes is therefore fixed without imposing any Ansatz on the internal geometry, and one finds that M_3 is necessarily an S^2 fibration over a one-dimensional base — as expected, since the $SU(2)$ isometry of the fibre is the R-symmetry of the dual $(1,0)$ theory.

Introducing $x \equiv \cos\psi \sin\theta_1$, supersymmetry forces the dilaton and x to be functionally dependent on the warping, $\phi = \phi(A)$ and $x = x(A)$, and reduces the entire problem to two coupled ODEs, namely $F_0 = \text{const}$ together with the relation $x dx = (1 + x^2)d\phi - (5 + x^2)dA$. Using the proper distance r along the base as a coordinate, the internal metric and the ODE system read

$$ds_{M_3}^2 = dr^2 + \frac{1}{16}e^{2A} (1 - x^2) ds_{S^2}^2, \quad (5.2.2)$$

$$\begin{aligned} \frac{d\phi}{dr} &= \frac{1}{4} \frac{e^A}{\sqrt{(1-x^2)}} \left(12x + (2x^2 - 5)F_0 e^{A+\phi} \right), \\ \frac{dx}{dr} &= -\frac{1}{2} e^{-A} \sqrt{1-x^2} \left(4 + xF_0 e^{A+\phi} \right), \\ \frac{dA}{dr} &= \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x^2}} \left(4x - F_0 e^{A+\phi} \right). \end{aligned} \quad (5.2.3)$$

All remaining fields are algebraically determined; in particular

$$F_2 = \frac{1}{16} \sqrt{1-x^2} e^{A-\phi} \left(x e^{A+\phi} F_0 - 4 \right) \text{vol}_{S^2}, \quad H = - \left(6e^{-A} + xF_0 e^\phi \right) \text{vol}_3. \quad (5.2.4)$$

Bianchi identities and equations of motion are then automatically satisfied. A reformulation in which all fields are generated by a single function of a rescaled coordinate z , obeying a third-order equation sourced by F_0 , was introduced later [28] and will be used in Section 6.

Compactness is analysed directly on (5.2.2). Since $\partial_r(xe^{3A-\phi}) \leq 0$, the coordinate r cannot be periodically identified, so the only option is to shrink the S^2 at two endpoints, giving $M_3 \cong S^3$. Regularity at the poles requires

$$\left\{ x = 1, e^{A+\phi} = \frac{4}{F_0} \right\} \text{ at } r_N, \quad \left\{ x = -1, e^{A+\phi} = -\frac{4}{F_0} \right\} \text{ at } r_S, \quad (5.2.5)$$

which is manifestly impossible for $F_0 = 0$: massless solutions necessarily develop D6-brane (or O6-plane) singularities at the poles, and indeed the unique $F_0 = 0$ background is the reduction of $AdS_7 \times S^4/\mathbb{Z}_k$, with k D6-branes and k anti-D6-branes at the two poles. Genuinely regular, compact solutions therefore exist only in massive type IIA, and require D8-brane sources. Across a D8 stack the fields are continuous while their derivatives jump; supersymmetry then fixes the position of the stack through a condition on $q \equiv \frac{1}{4}\sqrt{1-x^2}e^{A-\phi}$ in terms of the flux integers on the two sides. The D8-branes carry worldvolume flux and are thus D8/D6 bound states, stabilised at their calibrated position by the balance between the DBI tension and the Wess-Zumino coupling to the RR flux. [4] constructed such solutions numerically for one and two D8 stacks and conjectured, correctly, that arbitrarily many stacks are possible.

5.2.2 Field theory interpretation and anomaly match

The holographically dual field theories to such AdS solutions were identified by [33] as the linear $(1, 0)$ quivers engineered by the NS5–D6–D8 systems of [40, 23]. The dictionary is direct: the interval swept by r is the holographic image of the direction x^6 along which the D6-branes are suspended; the quantised Romans mass $n_0 = 2\pi F_0$ counts the D8-branes, so that its jumps across a stack reproduce the number of flavours attached to a node; the two-form flux integer n_2 counts the D6-branes, hence the rank of the corresponding gauge group; and the number of NS5-branes, i.e. the number of tensor multiplets, is the quantised H -flux $N = \frac{1}{4\pi^2} \int_{S^3} H$, which is precisely the winding number of the periodic B -field variable used to enforce flux quantisation. In this identification the supergravity statement that F_0 is piecewise constant, with n_2 varying linearly in between, is the continuum version of the gauge anomaly cancellation condition (5.1.12) obeyed by the ranks of the quiver. Supergravity is reliable when the quiver is long, so that the discrete sequence of ranks is well approximated by a smooth profile.

The correspondence was tested quantitatively by [28]. On the field theory side the anomaly polynomial of a linear quiver can be computed exactly, by summing the one-loop contributions of vector, hyper and tensor multiplets over the nodes and adding the Green–Schwarz term required to cancel the reducible anomalies; the coefficient controlling the number of degrees of freedom grows cubically with the ranks, in analogy with the N^3 behaviour of the $(2, 0)$ theory. In the limit of a long quiver the sum over nodes becomes an integral over the interval, and the result reproduces the holographic free energy computed from the on-shell action of the corresponding AdS_7 background, node by node in the continuum sense rather than merely at leading order. This provides a rather sharp check of the whole construction.

Finally, every solution in the class admits a consistent truncation to minimal gauged supergravity in seven dimensions [55], so that any seven-dimensional solution uplifts on an arbitrary M_3 of the family. This is what makes the class a practical starting point for the study of probe branes and defects, to which we turn in the next Section.

6 R-symmetries from holography in $AdS_7 \times M_3$ type IIA theory

This section is devoted to the main work of our thesis. Having introduced the basic notions of the AdS/CFT duality as well as the interplay between topological defects and holography in Section 4, we can finally apply such tools to study the R -symmetry structure of the (1,0) 6-dimensional SCFT's dual to $AdS_7 \times M_3$ type IIA solutions reviewed in Section 5.

More specifically, adopting a similar strategy to the one developed by [11, 10] and reviewed in Section 4, we will provide some evidence in favor of the fact that D0 branes and F1 strings are holographically dual to R -charged operators, whereas the symmetry operator is encoded in a particular D8 brane configuration.

6.1 Universal consistent truncation

To begin with, we recall from Section 5 that type IIA theory admits an infinite family of AdS_7 solutions, classified in [4]. Each solution features an internal compact space M_3 realized as an S^2 fibration over an interval. With a suitable choice of coordinates on the interval, the 10-dimensional metric is given by (5.2.2), while the dilaton ϕ , the warping factor A and the supersymmetry parameter x obey the system of ODEs (5.2.3). The explicit form of the RR 2-form flux and the NS-NS 3-form is given in (5.2.4).

The precise 6-dimensional dual SCFT depends on the particular solution of (5.2.3), which encodes the quiver data of the corresponding Hanany-Zaffaroni setup. However, as proven in [55], all these solutions share a common 7-dimensional minimal gauged supergravity sector, whose Lagrangian is

$$\begin{aligned} \mathcal{L} = & R - \frac{1}{2} * d\varphi \wedge d\varphi - V(\varphi) * 1 \\ & - \frac{1}{2} e^{\frac{4}{\sqrt{10}}\varphi} * \mathcal{F}_4 \wedge \mathcal{F}_4 - \frac{1}{2} e^{-\frac{2}{\sqrt{10}}\varphi} * \mathcal{F}_2^i \wedge \mathcal{F}_2^i + \frac{1}{2} \mathcal{F}_2^i \wedge \mathcal{F}_2^i \wedge A_3 - h \mathcal{F}_4 \wedge A_3, \end{aligned} \quad (6.1.1)$$

where $\mathcal{F}_2^i$ is the field strength of the $SU(2)$ gauge field $\mathcal{A}^i$,

$$\mathcal{F}_2^i = d\mathcal{A}^i - \frac{1}{2} g \epsilon^{ijk} \mathcal{A}^j \wedge \mathcal{A}^k. \quad (6.1.2)$$

The $SU(2)$ gauge symmetry arises from the isometry of the S^2 fiber of the internal space M_3 ; as in the $AdS_5 \times S^5$ case, this gauge symmetry is mapped to the global R -symmetry of the holographically dual 6-dimensional SCFT.

The Kaluza-Klein reduction ansatz that realizes the compactification from $AdS_7 \times M_3$ to minimal 7-dimensional gauged supergravity (6.1.1) was provided in [55]; for our purposes it is useful to report here the reduction ansatz for the metric and fluxes. Unlike [55], we are interested only in the unperturbed vacuum solutions, so in our expressions we set $X = 1$ ($\implies w = 1$).

The ten dimensional metric is

$$ds_{10}^2 = \frac{\ell}{8} g^2 e^{2A} ds_{AdS_7}^2 + \ell ds_{M_3}^2, \quad ds_{M_3}^2 = dr^2 + \frac{1-x^2}{16} e^{2A} Ds_{S^2}^2. \quad (6.1.3)$$

Here, $Ds_{S^2}^2$ is the covariantized S^2 metric,

$$Ds_{S^2}^2 = Dy^i Dy^i, \quad Dy^i = dy^i + \epsilon^{ijk} y^j g \mathcal{A}^k, \quad (6.1.4)$$

where y^i are cartesian coordinates which parametrize S^2 ,

$$y^i y^i = 1. \quad (6.1.5)$$

The Neveu-Schwarz potential B is

$$\ell^{-1} B = \frac{1}{16} e^{2A} x \sqrt{1-x^2} \text{vol}_2 - e^A dr \wedge \left(m - \frac{1}{2} g y^i A^i \right), \quad (6.1.6)$$

where $\text{vol}_2 = \frac{1}{2} \epsilon^{ijk} y^i Dy^j \wedge Dy^k$, and m is defined by $dm = -\frac{1}{2} \text{vol}_{S^2}$, with vol_{S^2} the usual (non-covariant) volume form on S^2 . From (6.1.6) we obtain the field strength $H = dB$,

$$\ell^{-1} H = -6 e^{-A} \text{vol}_{M_3} - \ell F_0 e^\phi x \text{vol}_{M_3} - \frac{1}{4} e^A dr \wedge y^i g \mathcal{F}_2^i - \frac{1}{16} e^{2A} x \sqrt{1-x^2} g \mathcal{F}_2^i \wedge Dy^i \quad (6.1.7)$$

Finally, we will also need the reduction formulas for the Ramond-Ramond gauge fields,

$$F_2 = -q(\text{vol}_2 + y^i g \mathcal{F}_2^i) + \frac{1}{16} \ell F_0 e^{2A} x \sqrt{1-x^2} \text{vol}_2, \quad (6.1.8)$$

$$\ell^{-1} F_4 = -\frac{q}{16} e^{2A} \sqrt{1-x^2} y^i g \mathcal{F}_2^i \wedge \text{vol}_2 - \frac{q}{4} e^A dr \wedge \epsilon^{ijk} g \mathcal{F}_2^i y^j Dy^k, \quad (6.1.9)$$

where $q = \frac{1}{4} e^{A-\phi} \sqrt{1-x^2}$. In (6.1.9) we have kept only the terms containing $\mathcal{F}_2^i$, since the remaining pieces are not relevant for our discussion.

6.1.1 The z coordinate

To simplify our computations, it is convenient to change the parametrization of the internal interval of M_3 from the r coordinate to the z coordinate introduced in [28]. For the remainder of the thesis, we will carry out our calculations in z coordinate, however Appendix B contains the massless $F_0 = 0$ case studied by means of the original r coordinate.

To begin with, [8] showed that the system of ODEs (5.2.3) can be reduced to a single differential equation for a new function $\beta(y)$,

$$(q')^2 = \frac{2}{9} F_0 \quad q \equiv -\frac{4y\sqrt{\beta}}{\beta'} \quad (6.1.10)$$

where prime denotes differentiation with respect to the variable y (not to be confused with y^i), defined by,

$$dr = \left(\frac{3}{4} \right)^2 \frac{e^{2A}}{\sqrt{\beta}} dy. \quad (6.1.11)$$

In terms of β , the functions A , ϕ and x are

$$\begin{aligned} e^A &= \frac{2}{3} \left(-\frac{\beta'}{y} \right)^{\frac{1}{4}} \\ e^\phi &= \frac{(-\beta'/y)^{\frac{5}{4}}}{12\sqrt{4\beta - y\beta'}} \\ x^2 &= \frac{-y\beta'}{4\beta - y\beta'}. \end{aligned} \quad (6.1.12)$$

Reference [28] then defines the z coordinate by

$$dy = 9\pi q dz. \quad (6.1.13)$$

In this coordinate the internal interval corresponds to $z \in [0, N]$. From (6.1.13) and (6.1.10), one finds

$$y = -\frac{\dot{\alpha}}{18\pi}, \quad q = -\frac{\ddot{\alpha}}{162\pi^2} \quad (6.1.14)$$

where $\alpha \equiv \sqrt{\beta}$ and the dot denotes differentiation with respect to z . The function α is piecewise smooth on each interval $z \in [i, i+1]$, $i = 0, \dots, N-1$, and its explicit expression was found in [28] to be

$$\alpha(z) = \alpha_i - 2(9\pi)y_i(z-i) - \frac{(9\pi)^2}{2}r_i(z-i)^2 - \frac{(9\pi)^2}{6}(r_{i+1}-r_i)(z-i)^3, \quad z \in [i, i+1], \quad (6.1.15)$$

where y_i and α_i are integration constants.

With this new function, the value of the piecewise constant function F_0 is expressed as

$$F_0(z) = -\frac{1}{162\pi^3} \ddot{\alpha}(z) = \frac{1}{2\pi}(r_{i+1}-r_i), \quad z \in [i, i+1]. \quad (6.1.16)$$

The last term can be read as a discrete derivative of the gauge-group rank along the quiver; hence regions where the rank remains constant correspond to massless regions.

Using equations (6.1.14), (6.1.13), (6.1.11) and (6.1.12), the ansatz formulas for the fluxes become

$$F_2 = \frac{\ddot{\alpha}}{162\pi^2} (\text{vol}_2 + y_i g \mathcal{F}_2^i) + \ell F_0 \frac{\pi \dot{\alpha} \alpha}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} \text{vol}_2 \quad (6.1.17)$$

$$\ell^{-1} F_4 = \frac{\ddot{\alpha}}{162\pi^2} \frac{\pi \alpha \dot{\alpha}}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} y_i g \mathcal{F}_2^i \wedge \text{vol}_2 + \frac{\ddot{\alpha}}{162\pi} dz \wedge \epsilon_{ijk} g \mathcal{F}_2^i \wedge y^j D y^k + \dots \quad (6.1.18)$$

$$\ell^{-1} B = \frac{\pi \alpha \dot{\alpha}}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} \text{vol}_2 - 2\pi dz \wedge \left(m - \frac{1}{2} y_i g \mathcal{A}^i \right) \quad (6.1.19)$$

$$\ell^{-1} H = \pi \left(-1 + \partial_z \left[\frac{\alpha \dot{\alpha}}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} \right] \right) dz \wedge \text{vol}_2 - \pi dz \wedge y_i g \mathcal{F}_2^i - \frac{\pi \alpha \dot{\alpha}}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} g \mathcal{F}_2^i \wedge D y^i. \quad (6.1.20)$$

while the metric is

$$ds_{10}^2 = \ell \pi \sqrt{2} g^2 \sqrt{-\frac{\alpha}{\ddot{\alpha}}} ds_{AdS_7}^2 + \ell \pi \sqrt{2} \sqrt{-\frac{\ddot{\alpha}}{\alpha}} \left(dz^2 + \frac{\alpha^2}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} D s_{S^2}^2 \right). \quad (6.1.21)$$

6.1.2 Computations of Hodge duals Ramond-Ramond fields

Before showing how the charged and symmetry operators are actually built, it is useful to compute explicitly the Hodge-dual Ramond-Ramond gauge fields. Following the democratic formulation [17] employed in [55], we have

$$F_8 = \star_{10} F_2, \quad F_6 = -\star_{10} F_4. \quad (6.1.22)$$

For convenience, it is useful to define

$$L_1 = \ell^{\frac{1}{2}} \pi^{1/2} 2^{1/4} \left(-\frac{\ddot{\alpha}}{\alpha} \right)^{1/4}, \quad L_2 = \ell^{\frac{1}{2}} \pi^{1/2} 2^{1/4} \left(-\frac{\ddot{\alpha}}{\alpha} \right)^{1/4} \left(\frac{\alpha^2}{\dot{\alpha}^2 - 2\alpha \ddot{\alpha}} D s_{S^2}^2 \right)^{1/2}, \quad (6.1.23)$$

which are the radii of the internal interval I and of S^2 , respectively.

Using formulas (A.0.13) and (A.0.17) from Appendix A, we immediately find

$$F_8 \supset \frac{\ddot{\alpha}}{162\pi^2} y_i g \star_{10} \mathcal{F}_2^i = \frac{\ddot{\alpha}}{162\pi^2} g y_i L_1 L_2^2 \star_7 \mathcal{F}_2^i \wedge dz \wedge \text{vol}_2 \quad (6.1.24)$$

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$$F_6 \supset -\ell \frac{\ddot{\alpha}}{162\pi^2} \frac{\pi\alpha\dot{\alpha}}{\dot{\alpha} - 2\alpha\ddot{\alpha}} y_i g L_1 L_2^{-2} \star_7 \mathcal{F}_2^i \wedge dz - \ell \frac{\ddot{\alpha}}{162\pi} g \epsilon_{ijk} L_1^{-1} \star_7 \mathcal{F}_2^i \wedge y^i \star_2 Dy^k \quad (6.1.25)$$

where again we have kept only the terms containing $\mathcal{F}_2^i$. The symbol $\star_2$ denotes the Hodge dual with respect to the unit-radius S^2 metric. Using equation (3.13) of [9],

$$\star_2 Dy^i = \epsilon^{ijk} y^j Dy^k, \quad (6.1.26)$$

we can recast (6.1.25) as

$$F_6 \supset -\ell \frac{\ddot{\alpha}}{162\pi^2} \frac{\pi\alpha\dot{\alpha}}{\dot{\alpha} - 2\alpha\ddot{\alpha}} y_i g L_1 L_2^{-2} \star_7 \mathcal{F}_2^i \wedge dz - \ell \frac{\ddot{\alpha}}{162\pi} g L_1^{-1} \star_7 \mathcal{F}_2^i \wedge Dy^i, \quad (6.1.27)$$

where we used that on the unit 2-sphere $y^i y^i = 1$ and $y^m Dy^m = 0$.

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In this section we derive the brane configurations that realize the charged operators of the $SU(2)$ R-symmetry of the 6-dimensional SCFT dual to type IIA theory on $AdS_7 \times M_3$. We initially follow closely the procedure of [10], although our analysis will contain a new ingredient absent from [10].

For dimensional reasons, the only candidate brane that can holographically represent an R-charged operator in the dual theory is a $D0$ brane wrapping a single point in the S^2 factor of the internal space M_3 and sitting at position z in the radial direction of AdS_7 . Strictly speaking, according to the general construction, the probe brane should wrap $S^0 \subset S^2$, i.e. a pair of points in S^2 rather than a single point. However, as explained in [10], wrapping either S^0 or a single point merely corresponds to building representations of $SO(3)$ or of $SU(2)$, respectively. Since we are interested in $SU(2)$ representations, we consider a single point in S^2 .

The topological Wess-Zumino coupling of a $D0$ brane is

$$S_{D0} \supset \int_{M_1} C_1, \quad (6.2.1)$$

however, it is easier to write this action by inflow, using the relation $dC_1 = F_2 - BF_0 \equiv \tilde{F}_2$, so that

$$S_{D0} \supset \int_{N_2} \tilde{F}_2, \quad (6.2.2)$$

with $\partial N_2 = M_1$. Upon using (6.1.17) and (6.1.19), we find

$$\tilde{F}_2 = \frac{\ddot{\alpha}}{162\pi^2} (\text{vol}_2 + y_i g \mathcal{F}_2^i) \quad (6.2.3)$$

The other ingredient we need is the pull-back of $\tilde{F}_2$ onto the worldvolume of the $D0$ brane. Recall that the embedding of the brane in the S^2 fiber is encoded by equation (4.5.30), where this time the Y_0^a are constants. Denoting by $P[\omega]$ the worldvolume pull-back of a form ω , the covariant derivative appearing in (6.1.4) pulls back to

$$\begin{aligned} P[y^i] &= \chi^{ij} Y_0^j, \\ P[Dy^i] &= d\chi^{ij} Y_0^j + gP[\mathcal{A}^{ik}] \chi^{kj} Y_0^j = d_A \chi^{ij} Y_0^j, \end{aligned} \quad (6.2.4)$$

where we have defined

$$d_A \chi^{ij} = d\chi^{ij} + gP[\mathcal{A}^{ik}] \chi^{kj}. \quad (6.2.5)$$

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To make contact with the notation of [10], we have rewritten the gauge field as $\mathcal{A}^i = \frac{1}{2}\epsilon^{ijk}\mathcal{A}^{jk}$, i.e. we have switched from the adjoint to the fundamental representation. The covariant derivative in (6.1.4) thus reads

$$Dy^i = dy^i + g\mathcal{A}^{ij}y^j, \quad (6.2.6)$$

whose pull-back is indeed (6.2.4). It is also convenient to define

$$\Theta^{ij} = (\chi^{-1}d_A\chi)^{ij}. \quad (6.2.7)$$

Since $\chi \in SO(3)$ (so that $\chi^T = \chi^{-1}$) and $\mathcal{A}^{ij} = -\mathcal{A}^{ji}$, we immediately have $\Theta^{ij} = -\Theta^{ji}$. Now, from (6.2.3), the pull-back of $\tilde{F}_2$ onto the worldvolume of the $D0$ brane is (writing $\mathcal{F}_2^i$ as $\frac{1}{2}\epsilon^{ijk}\mathcal{F}_2^{jk}$)

$$P[\tilde{F}_2] = \frac{\ddot{\alpha}}{162\pi^2} \left(\frac{1}{2}\epsilon^{ijk}P[y^iDy^j \wedge Dy^k] + \frac{g}{2}\epsilon^{ijk}P[y^i\mathcal{F}_2^{jk}] \right) \quad (6.2.8)$$

We start with the first term of (6.2.8) which, upon using (6.2.4), reads

$$\begin{aligned} \frac{1}{2}\epsilon^{ijk}P[y^iDy^j \wedge Dy^k] &= \frac{1}{2}\epsilon^{ijk}\chi^{ia}Y_{(0)}^a d_A\chi^{jb}Y_{(0)}^b \wedge d_A\chi^{kc}Y_{(0)}^c \\ &= \frac{1}{2}\epsilon^{ijk}Y_{(0)}^a Y_{(0)}^b Y_{(0)}^c \chi^{ia}\chi^{jl}\chi^{km}(\chi^{-1}d_A\chi)^{lb} \wedge (\chi^{-1}d_A\chi)^{mc} \\ &= \frac{1}{2}\epsilon^{alm}Y_{(0)}^a Y_{(0)}^b Y_{(0)}^c (\chi^{-1}d_A\chi)^{lb} \wedge (\chi^{-1}d_A\chi)^{mc} \\ &= \frac{1}{2}\epsilon^{alm}Y_{(0)}^a Y_{(0)}^b Y_{(0)}^c \Theta^{lb} \wedge \Theta^{mc}, \end{aligned} \quad (6.2.9)$$

where in the third line we used $\det(\chi) = 1$. Since the indices run from 1 to 3, the antisymmetrization over four indices vanishes identically:

$$\epsilon^{[alm}Y_{(0)}^{b]} = 0, \quad \implies \quad \epsilon^{alm}Y_{(0)}^b - \epsilon^{blm}Y_{(0)}^a - \epsilon^{abm}Y_{(0)}^l - \epsilon^{alb}Y_{(0)}^m = 0, \quad (6.2.10)$$

hence

$$\epsilon^{alm}Y_{(0)}^b = \epsilon^{blm}Y_{(0)}^a + \epsilon^{abm}Y_{(0)}^l + \epsilon^{alb}Y_{(0)}^m. \quad (6.2.11)$$

Plugging this into (6.2.9), we obtain

$$\begin{aligned} \frac{1}{2}\epsilon^{alm}Y_{(0)}^a Y_{(0)}^b Y_{(0)}^c \Theta^{lb} \wedge \Theta^{mc} &= \frac{1}{2}Y_{(0)}^a Y_{(0)}^a Y_{(0)}^c \epsilon^{blm}\Theta^{lb} \wedge \Theta^{mc} \\ &\quad + \frac{1}{2}Y_{(0)}^a Y_{(0)}^l Y_{(0)}^c \epsilon^{abm}\Theta^{lb} \wedge \Theta^{mc} \\ &\quad + \frac{1}{2}Y_{(0)}^a Y_{(0)}^m Y_{(0)}^c \epsilon^{alb}\Theta^{lb} \wedge \Theta^{mc}. \end{aligned} \quad (6.2.12)$$

The term in the third line vanishes, since Θ^{mc} is antisymmetric in m, c while $Y_{(0)}^m Y_{(0)}^c$ is symmetric. As for the term in the first line, using $Y_{(0)}^a Y_{(0)}^a = 1$, the skew-symmetry of Θ and the skew-symmetry of the wedge product, it can be cast as

$$-\frac{1}{2}\epsilon^{mlb}Y_{(0)}^c \Theta^{cm} \wedge \Theta^{lb}. \quad (6.2.13)$$

Finally, exchanging b and l in the first Θ , the term in the second line becomes

$$-\frac{1}{2}\epsilon^{abm}Y_{(0)}^a Y_{(0)}^l Y_{(0)}^c \Theta^{bl} \wedge \Theta^{mc}, \quad (6.2.14)$$

which, up to a relabelling of indices, equals the left-hand side with a minus sign. Altogether, we find

$$\frac{1}{2}\epsilon^{ijk}P[y^iDy^j \wedge Dy^k] = -\frac{1}{4}\epsilon^{mlb}Y_{(0)}^c \Theta^{cm} \wedge \Theta^{lb}. \quad (6.2.15)$$

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We now turn to the second term in (6.2.8), which reads

$$\frac{g}{2} y^i \epsilon^{ijk} \mathcal{F}_2^{jk} = \frac{g}{2} \chi^{ia} Y_{(0)}^a \epsilon^{ijk} \mathcal{F}_2^{jk} = \frac{g}{2} Y_{(0)}^a \epsilon^{ijk} \chi^{ia} \chi^{je} \chi^{kf} (\chi^{-1} \mathcal{F}_2 \chi)^{ef} = \frac{g}{2} \epsilon^{aef} Y_{(0)}^a (\chi^{-1} \mathcal{F}_2 \chi)^{ef}. \quad (6.2.16)$$

Hence, the worldvolume pull-back of $\tilde{F}_2$ becomes

$$P[\tilde{F}_2] = \frac{\ddot{\alpha}}{162\pi^2} \left(-\frac{1}{4} \epsilon^{ijk} Y_{(0)}^l \Theta^{li} \wedge \Theta^{jk} + \frac{g}{2} \epsilon^{ijk} Y_{(0)}^i (\chi^{-1} \mathcal{F}_2 \chi)^{jk} \right) \quad (6.2.17)$$

It is convenient to write $Y_0^i = \frac{1}{2} \epsilon^{ijk} Y_0^{jk}$; then, using the identity

$$\epsilon^{ijk} \epsilon^{lmn} = \begin{vmatrix} \delta^{il} & \delta^{im} & \delta^{in} \\ \delta^{jl} & \delta^{jm} & \delta^{jn} \\ \delta^{kl} & \delta^{km} & \delta^{kn} \end{vmatrix}, \quad (6.2.18)$$

expression (6.2.17) becomes

$$P[\tilde{F}_2] = \frac{1}{2} \frac{\ddot{\alpha}}{162\pi^2} Y_{(0)}^{ij} (g(\chi^{-1} \mathcal{F}_2 \chi)^{ij} - (\Theta \wedge \Theta)^{ij}). \quad (6.2.19)$$

We now observe that (omitting indices)

$$d\Theta = d(\chi^{-1} d_A \chi) = d\chi^{-1} \wedge d\chi + g d\chi^{-1} \wedge \mathcal{A} \chi - g \chi^{-1} d\mathcal{A} \chi, \quad (6.2.20)$$

moreover

$$(\chi^{-1} d_A \chi) \wedge (\chi^{-1} d_A \chi) = -d\chi^{-1} \wedge d\chi - g d\chi^{-1} \wedge \mathcal{A} \chi + g \chi^{-1} \mathcal{A} \wedge d\chi + g^2 \chi^{-1} \mathcal{A} \wedge \mathcal{A} \chi, \quad (6.2.21)$$

where we used $d\chi \chi^{-1} = -\chi d\chi^{-1}$. Since

$$\mathcal{F}_2 = d\mathcal{A} + g\mathcal{A} \wedge \mathcal{A}, \quad (6.2.22)$$

comparing (6.2.20), (6.2.21) and (6.2.22) yields the identity

$$d(\chi^{-1} d_A \chi)^{ij} = (g(\chi^{-1} \mathcal{F}_2 \chi) - (\chi^{-1} d_A \chi)^2)^{ij}. \quad (6.2.23)$$

The worldvolume pull-back of $\tilde{F}_2$ therefore takes the final form

$$P[\tilde{F}_2] = \frac{1}{2} \frac{\ddot{\alpha}}{162\pi^2} Y_{(0)}^{ij} d(\chi^{-1} d_A \chi)^{ij}, \quad (6.2.24)$$

and the $D0$ -brane action becomes

$$S_{D0} \supset \int_{N_2} P[\tilde{F}_2] = \int_{N_2} \frac{1}{2} \frac{\ddot{\alpha}}{162\pi^2} Y_{(0)}^{ij} d(\chi^{-1} d_A \chi)^{ij} = \frac{1}{2} q^{ij} \int_{M_1} (\chi^{-1} d_A \chi)^{ij}, \quad (6.2.25)$$

with

$$q^{ij} = \frac{\ddot{\alpha}}{162\pi^2} Y_{(0)}^{ij} = -\frac{r_i + (z-i)(r_{i+1} - r_i)}{2} Y_{(0)}^{ij}. \quad (6.2.26)$$

where the index i is such that $z \in [i, i+1]$. This has exactly the same functional form as (4.5.29), confirming that this configuration describes the correct charged operator under $SU(2)_R$.

Unlike the cases studied in [10], however, this is not the end of the story: as we now show, F1 string configurations can also holographically describe operators charged under the R-symmetry. Consider an F1 string whose worldvolume wraps the radial direction M_1 of AdS_7 and extends between two consecutive stacks of D8 branes located at $z = j_b$ and $z = j_{b+1}$. This time it is more convenient to describe the F1 action directly in terms of the B field potential,

$$S_{F1} = -\frac{1}{2\pi} \int_{M_1 \times [j_b, j_{b+1}]} B + \int_{M_1} a_{b+1} - a_b, \quad (6.2.27)$$

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where we have added a boundary term describing the oscillations of the string endpoints on the two D8 stacks at $z = j_b$ and $z = j_{b+1}$. As shown in [15], the D8 branes usually carry D6 charge dissolved on them; this is needed to balance any discontinuity across z of the Page charge $\tilde{F}_2$ (see equations (3.15) and (3.16) of [15]). However, the Page charge depends on the gauge choice for the B field; adopting the ansatz (6.1.19), we see that it is continuous across the entire z interval,

$$\int_{S^2} \frac{\tilde{F}_2}{2\pi} = \int_{S^2} \frac{1}{2\pi} \frac{\ddot{\alpha}}{162\pi^2} \text{vol}_2 = -(r_i + (z - i)(r_{i+1} - r_i)), \quad (6.2.28)$$

which is continuous in z . This means that, with this particular gauge choice, the D8 sources need not carry D6 charge; hence the second term in (6.2.27) vanishes.

Let us look more closely at the non-zero terms of (6.2.27). The only contributions from the B field (6.1.19) are those with a leg along z , so that

$$S_{F1} = \int_{M_1 \times [j_b, j_{b+1}]} P \left[\left(m - \frac{1}{2} y_i g \mathcal{A}^i \right) \wedge dz \right], \quad (6.2.29)$$

where P again denotes the worldvolume pull-back. The position of the F1 string along the S^2 fiber is parametrized exactly as in the D0-brane configuration; therefore,

$$P[y_i \mathcal{A}^i] = \chi^{ij} Y_{(0)}^j \mathcal{A}^i. \quad (6.2.30)$$

Introducing $\mathcal{A}^i = \frac{1}{2} \epsilon^{iab} \mathcal{A}^{ab}$, this becomes

$$\begin{aligned} P[y_i \mathcal{A}^i] &= \frac{1}{2} \epsilon^{iab} \chi^{ij} Y_{(0)}^j \mathcal{A}^{ab} \\ &= \frac{1}{2} \epsilon^{iab} Y_{(0)}^j \chi^{ij} \chi^{am} \chi^{bn} (\chi^{-1})^{mk} (\chi^{-1})^{nl} \mathcal{A}^{kl} \\ &= \frac{1}{2} \epsilon^{jmn} Y_{(0)}^j (\chi^{-1})^{mk} (\chi^{-1})^{nl} \mathcal{A}^{kl} \\ &= \frac{1}{2} Y_{(0)}^{mn} (\chi^{-1})^{mk} \mathcal{A}^{kl} \chi^{ln} = \frac{1}{2} Y_{(0)}^{mn} (\chi^{-1} \mathcal{A} \chi)^{mn}, \end{aligned} \quad (6.2.31)$$

where in the second line we wrote $\mathcal{A}^{ab} = (\chi \chi^{-1})^{ak} (\chi \chi^{-1})^{bl} \mathcal{A}^{kl}$, in the third line we used $\det \chi = 1$, and in the fourth line we defined $Y_{(0)}^{ij} = \epsilon^{ijk} Y_{(0)}^k$ and recalled that $\chi^{-1} = \chi^T$.

It is now useful to compute the pull-back of m ; it is easier to inspect first the pull-back of its exterior derivative,

$$P[dm] = P \left[-\frac{1}{2} \text{vol}_{S^2} \right]. \quad (6.2.32)$$

This is almost identical to (6.2.15); we simply replace the covariant derivative d_A by the ordinary exterior derivative d . Taking the $-\frac{1}{2}$ prefactor into account, we have

$$P[dm] = P \left[-\frac{1}{2} \text{vol}_{S^2} \right] = \frac{1}{8} \epsilon^{ijk} Y_{(0)}^l (\chi^{-1} d\chi)^{li} \wedge (\chi^{-1} d\chi)^{jk} = \frac{1}{4} Y_{(0)}^{ij} [(\chi^{-1} d\chi) \wedge (\chi^{-1} d\chi)]^{ij}. \quad (6.2.33)$$

Moreover, making use of the fact that $d(\chi \chi^{-1}) = 0$, we notice that

$$[(\chi^{-1} d\chi) \wedge (\chi^{-1} d\chi)]^{ij} = -[(d\chi^{-1} \chi) \wedge (\chi^{-1} d\chi)]^{ij} = -(d\chi^{-1} \wedge d\chi)^{ij} = -d(\chi^{-1} \wedge d\chi)^{ij}. \quad (6.2.34)$$

Therefore

$$P[dm] = -\frac{1}{4} Y_{(0)}^{ij} d(\chi^{-1} d\chi)^{ij}, \quad (6.2.35)$$

and since the pull-back commutes with the exterior derivative, we conclude that

$$P[m] = -\frac{1}{4} Y_{(0)}^{ij} (\chi^{-1} d\chi)^{ij}, \quad (6.2.36)$$

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up to total-derivative terms.

Inserting (6.2.36) and (6.2.31) into (6.2.29), the F1 string action becomes

$$S_{F1} = -\frac{1}{4}Y_{(0)}^{ij} \int_{j_b}^{j_{b+1}} dz \int_{M_1} (\chi^{-1} d\chi + \chi^{-1} g \mathcal{A} \chi)^{ij} = \frac{1}{2} q^{ij} \int_{M_1} (\chi^{-1} d_{\mathcal{A}} \chi)^{ij}, \quad (6.2.37)$$

with

$$q^{ij} = -\frac{1}{2}(j_{b+1} - j_b) Y_{(0)}^{ij}. \quad (6.2.38)$$

This shows that F1 strings holographically realize charged operators.

That both D0 branes and F1 strings are charged under the R-symmetry should not come as a surprise. According to [15], each 6d SCFT dual to a type IIA solution on $AdS_7 \times M_3$ contains two types of R-charged operators, string mesons and baryons, which in the string-theory description are dual to F1 strings and D0 branes, respectively.

Reassuringly, the authors of [15] have shown that the Cartan $U(1)$ R-charge of an F1 string stretched between two consecutive stacks of D8 branes at $z = j_b$ and $z = j_{b+1}$ is

$$Q_R(F1) = \frac{j_{b+1} - j_b}{2}, \quad (6.2.39)$$

in agreement with (6.2.38).

If instead we compare the D0 R-charge computed in [15] with our equation (6.2.26), we find a discrepancy. This should not worry us: the D0 charge depends explicitly on the gauge choice for the field B . This subtlety was absent in the F1 case, since any gauge choice for B would ultimately produce a non-zero boundary term in the F1 action, coming from the D6 charge dissolved in the D8 sources, and one would recover the same R-charge as in (6.2.38).

We can trace this ambiguity in the D0 R-charge to the fact that, in general, such states are non-BPS, as pointed out in [15]. To render them BPS, we attach F1 strings to them: if the D0 lies to the left of the massless region, we add $2\pi F_0$ strings whose left ends attach to the D0 brane and whose right ends attach to the stack of D8 branes at position $z = i_p$, where $[i_p, i_{p+1}]$ is the integer interval on which the gauge-group rank is constant and maximal, denoted $r_{\max}$. If the D0 lies to the right of the massless region, the strings are attached in the opposite way.

Take, for instance, the first case, with the D0 at position $z \in [i, i+1]$. From equation (6.2.37), the $U(1)$ R-charge of the $2\pi F_0$ F1 strings stretching from the D0 brane to the D8 brane is

$$\begin{aligned} -\frac{1}{2} \int_z^{i_p} 2\pi F_0 dz &= -\frac{1}{2}(r_{i+1} - r_i) \int_z^{i+1} dz - \frac{1}{2}(r_{i+2} - r_{i+1}) \int_{i+1}^{i+2} dz - \\ &\quad - \cdots - \frac{1}{2}(r_{\max} - r_{i+l-1}) \int_{i+l-1}^{i_p=i+l} dz, \end{aligned} \quad (6.2.40)$$

where we assumed that i_p lies l positions away from the D0 brane. Summing the first two terms of (6.2.40) gives

$$\frac{1}{2}(r_{i+2} - r_i - (z - i)(r_{i+1} - r_i)), \quad (6.2.41)$$

while, from the third and fourth terms onward, each successive pair sums to

$$\frac{1}{2}(r_{i+k} - r_{i+k-1}), \quad k = 3, \dots, l. \quad (6.2.42)$$

6.3 Symmetry operators

Hence

$$\begin{aligned} \frac{1}{2} \int_z^{i_p} 2\pi F_0 dz &= \\ &= - \left[\frac{1}{2} (r_{i+2} - r_i - (z-i)(r_{i+1} - r_i)) \right] - \frac{1}{2} [r_{i+3} - r_{i+2}] - \cdots - \frac{1}{2} [r_{\max} - r_{i+l-1}] = \\ &= - \frac{r_{\max} - r_i - (z-i)(r_{i+1} - r_i)}{2}. \end{aligned} \quad (6.2.43)$$

Adding this to (6.2.26), the $U(1)$ R-charge of the D0-F1 bound state is (up to an overall sign)

$$Q_R(F1 + D0) = \frac{r_{\max}}{2}, \quad (6.2.44)$$

just as in [15]. An entirely analogous treatment holds when the D0 brane lies to the right of the massless region.

Whenever the D0 is placed in a region where $F_0 = 0$, its R-charge is independent of B and hence gauge-invariant. Evaluating (6.2.26) in the massless region, where $r_i = r_{i+1} = r_{\max}$, we again find

$$Q_R(D0) = \frac{r_{\max}}{2}. \quad (6.2.45)$$

This concludes our discussion of holographically R-charged operators; we now turn to the study of symmetry operators.

6.3 Symmetry operators

In this section we investigate the possible symmetry operators for the $SU(2)_R$ symmetry.

We argue that the brane configuration implementing the $SU(2)_R$ symmetry in the SCFT dual to type IIA theory on $AdS_7 \times M_3$ is a $D8$ brane. Following the general construction of Section 4, the worldvolume wrapped by the $D8$ brane is a subspace $\Sigma_9 = \gamma_1 \times N_5 \times M_3$, with $N_5 \subset \partial AdS_7$, M_3 the entire internal manifold, and γ_1 an arc penetrating in the bulk which connects two points in the conformal boundary. As we did for the charged operators, we again introduce an auxiliary higher-dimensional manifold $M_{10} = N_2 \times N_5 \times M_3$ with $\partial N_2 = \gamma_1$ to describe the $D8$ -brane action by inflow. The Wess-Zumino couplings then read

$$S_{D8} = \int_{M_{10}} F_8 \wedge 2\pi f_2 + \frac{1}{2} F_6 \wedge (2\pi f_2 + B)^2 + \cdots, \quad (6.3.1)$$

where we have omitted terms that do not contain $\star_7 \mathcal{F}_2^i$, as they are irrelevant for our purposes. We also assume that the $U(1)$ worldvolume gauge field strength f_2 decomposes as

$$f_2 = da_1 + f_2^{M_3}, \quad (6.3.2)$$

where a_1 is chosen to saturate the integration over γ_1 ,

$$\int_{\gamma_1} a_1 = \beta, \quad (6.3.3)$$

and $f_2^{M_3}$ is a 2-form with purely internal legs. We will show that, regardless of the specific form of $f_2^{M_3}$, several terms in (6.3.1) vanish.

Starting with the term $\sim \int F_8 \wedge f_2$, this vanishes since, from (6.1.24), it is proportional to

$$\sim \int_{S^2} y^i \text{vol}_{S^2} = 0. \quad (6.3.4)$$

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We next consider the contributions from $\sim \int F_6 \wedge (2\pi f_2 + B)^2$. Expanding the square gives three terms, two of which again vanish. Indeed, take

$$\int F_6 \wedge B \wedge f_2. \quad (6.3.5)$$

Using the ansätze (6.1.25) and (6.1.6), the only terms containing $\star_7 \mathcal{F}_2^i$ that arise from (6.3.5) are proportional to

$$\sim \int_{S^2} y^i \text{vol}_{S^2}, \quad (6.3.6)$$

and

$$\sim \int_{S^2} dy^i \wedge m. \quad (6.3.7)$$

The first vanishes, and (6.3.7) vanishes as well:

$$\int_{S^2} dy^i \wedge m = \int_{S^2} y^i dm = -\frac{1}{2} \int_{S^2} y^i \text{vol}_{S^2} = 0, \quad (6.3.8)$$

where in the first equality we used Stokes' theorem together with $\partial S^2 = \emptyset$.

Next, consider

$$\sim \int F_6 \wedge B \wedge B. \quad (6.3.9)$$

Clearly, no non-zero term containing $\star_7 \mathcal{F}_2^i$ can arise from (6.3.9).

We are thus left with

$$S_{D8} \supset \frac{1}{2(2\pi)^6} \int_{M_{10}} F_6 \wedge f_2 \wedge f_2. \quad (6.3.10)$$

Given the structure (6.1.25), we turn on the following internal $f_2^{M_3}$ field strength

$$f_2^{M_3} = K(z) y^j v^j \text{vol}_{S^2} + R(z) v^i \epsilon_{ijk} y^j dy^k \wedge dz, \quad (6.3.11)$$

with v^j a fixed direction on the unit sphere. The functions $K(z)$ and $R(z)$ must be chosen in a way to ensure that the field strength $f_2^{M_3}$ is closed; in this regard, note that

$$df_2^{M_3} = K'(z) y^i v^i \text{vol}_{S^2} + R(z) v^i \epsilon_{ijk} dy^i \wedge dy^k \wedge dz. \quad (6.3.12)$$

Since $v^i \epsilon_{ijk} dy^j \wedge dy^k$ is a top-dimensional form on S^2 , it must be proportional to vol_{S^2} ; more precisely,

$$v^i \epsilon_{ijk} dy^j \wedge dy^k = 2 (v_1 dy^2 \wedge dy^3 + v_2 dy^3 \wedge dy^1 + v_3 dy^1 \wedge dy^2). \quad (6.3.13)$$

Using $y^i y^i = 1$ and $y^i dy^i = 0$, we obtain the identities

$$\begin{aligned} y^1 \text{vol}_{S^2} &= dy^2 \wedge dy^3 \\ y^2 \text{vol}_{S^2} &= dy^3 \wedge dy^1 \\ y^3 \text{vol}_{S^2} &= dy^1 \wedge dy^2. \end{aligned} \quad (6.3.14)$$

For instance, the first relation follows from

$$\begin{aligned} y^1 \text{vol}_{S^2} &= y^1 (y_1 dy^2 \wedge dy^3 + y_2 dy^3 \wedge dy^1 + y_3 dy^1 \wedge dy^2) \\ &= y^1 \left(y_1 dy^2 \wedge dy^3 + y^2 dy^3 \wedge \left(-\frac{y^2 dy^2 + y^3 dy^3}{y^1} \right) + y^3 \left(-\frac{y^2 dy^2 + y^3 dy^3}{y^1} \right) \wedge dy^2 \right) \\ &= y^1 \left(y_1 dy^2 \wedge dy^3 + \frac{(y^2)^2}{y^1} dy^2 \wedge dy^3 + \frac{(y^3)^2}{y^1} dy^2 \wedge dy^3 \right) \\ &= dy^3 \wedge dy^2 \end{aligned} \quad (6.3.15)$$

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The others follow analogously. One can then write

$$\begin{aligned}
 v^i \epsilon_{ijk} dy^j \wedge dy^k &= \\
 &= 2 (v^1 dy^2 \wedge dy^3 + v^2 dy^3 \wedge dy^1 + v^3 dy^1 \wedge dy^2) \\
 &= 2 (v^1 y^1 + v^2 y^2 + v^3 y^3) \text{vol}_{S^2} \\
 &= 2(v^i y^i) \text{vol}_{S^2}.
 \end{aligned} \tag{6.3.16}$$

Hence, equation (6.3.12) can be written as

$$df_2^{M_3} = (y^i v^i \text{vol}_{S^2} \wedge dz)(K'(z) + 2R(z)). \tag{6.3.17}$$

Requiring $df_2^{M_3} = 0$ forces K and R to obey

$$R(z) = -\frac{1}{2}K'(z). \tag{6.3.18}$$

Equation (6.3.10) then becomes

$$S_{D8} \supset \frac{1}{(2\pi)^6} \int_{N_2} da_1 \int_{N_5 \times M_3} F_6 \wedge f_2^{M_3} = \frac{\beta}{(2\pi)^6} \int_{N_5 \times M_3} F_6 \wedge f_2^{M_3}. \tag{6.3.19}$$

Before proceeding, we note that, making the L_1 and L_2 factors in (6.1.25) explicit, one has

$$\begin{aligned}
 F_6 &\supset \frac{\ell^{1/2} g}{162 \cdot 2^{1/4} \pi^{3/2}} \alpha^{1/4} (-\ddot{\alpha})^{3/4} (\star_7 \mathcal{F}_2^i) \wedge \left[\frac{\dot{\alpha}}{\alpha} y^i dz + dy^i \right] = \\
 &= \frac{\ell^{1/2} g}{162 \cdot 2^{1/4} \pi^{3/2}} \left(\frac{-\ddot{\alpha}}{\alpha} \right)^{3/4} (\star_7 \mathcal{F}_2^i) \wedge d(\alpha y^i).
 \end{aligned} \tag{6.3.20}$$

Let us now focus on the integration over the internal M_3 space; since $df_2^{M_3} = 0$, we can write it as

$$\int_{M_3} \left(\frac{-\ddot{\alpha}}{\alpha} \right)^{3/4} d(\alpha y^i) \wedge f_2^{M_3} = \int_{M_3} \frac{3}{4} \left(\frac{\ddot{\alpha} \alpha - \ddot{\alpha} \dot{\alpha}}{\alpha} \right) \left(\frac{-\ddot{\alpha}}{\alpha} \right)^{-1/4} y^i dz \wedge f_2^{M_3}, \tag{6.3.21}$$

where we have dropped the total-derivative term $d \left[\left(\frac{-\ddot{\alpha}}{\alpha} \right)^{3/4} \alpha y^i \wedge f_2^{M_3} \right]$. This shows that the only term in (6.3.11) contributing to the integral is the one proportional to vol_{S^2} ³³.

In summary, the action of the D8-brane symmetry operator is

$$S_{D8} \supset \frac{\beta}{(2\pi)^6} T \int_{N_5} \star_7 \mathcal{F}_2^i \int_{S^2} y^i y^j v^j \text{vol}_{S^2} \int_0^N \left(\frac{\ddot{\alpha} \alpha - \ddot{\alpha} \dot{\alpha}}{\alpha} \right) \left(\frac{-\ddot{\alpha}}{\alpha} \right)^{-1/4} K(z) dz, \tag{6.3.22}$$

where $T = \frac{\ell^{1/2} g}{162 \cdot 2^{1/4} \pi^{3/2}}$.

The function $K(z)$ can be any regular function on $z \in [0, N]$ that makes the integral nonzero. Its detailed shape is not fixed, a part from being regular across M_3 . In addition, the profile of the z -function α depends on the specific quiver data; as an example, we focus on the uniform quiver gauge theory of Section 5.1 of [15]. The brane setup is depicted by the quiver diagram in Figure 14. This configuration uplifts to N coincident M5 branes on $\mathbb{C}^2/\mathbb{Z}_k$, whose near-horizon limit is M-theory on $AdS_7 \times S^4/\mathbb{Z}_k$. This is dual to a six-dimensional $\mathcal{N} = (1, 0)$ SCFT living on the worldvolume of the M5 branes [31]. When $k = 1$, the $(1, 0)$ supersymmetry is enhanced to $(2, 0)$ and we recover the celebrated 6d $(2, 0)$ SCFT / $AdS_7 \times S^4$ duality found by Maldacena in [50].

³³The second term in (6.3.11) is of course still needed to preserve the closedness of $f_2^{M_3}$.

6.4 Charge measurement

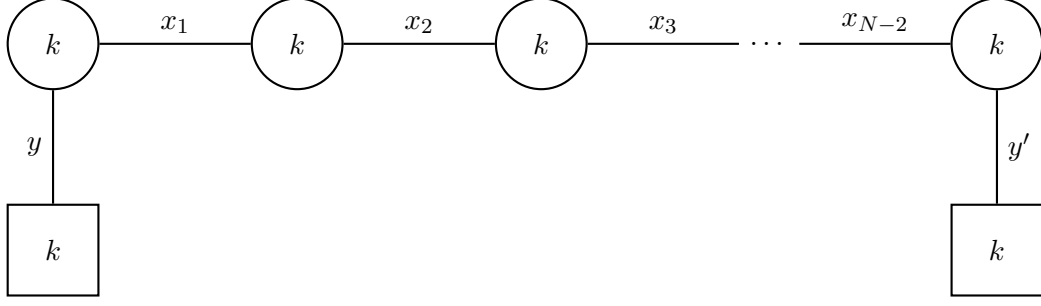


Figure 14: The uniform linear quiver gauge theory representing N M5-branes on $\mathbb{C}^2/\mathbb{Z}_k$.

The expression for α in the uniform-quiver case was computed in the z coordinate in [15] and reads

$$\alpha = \frac{(9\pi)^2}{2} k z (N - z). \quad (6.3.23)$$

From (6.3.23) we readily obtain

$$\dot{\alpha} = \frac{(9\pi)^2}{2} k [N - 2z], \quad \ddot{\alpha}(z) = -(9\pi)^2 k. \quad (6.3.24)$$

Plugging these into the z integral in (6.3.22), we find, after some algebra,

$$\int_0^N dz \frac{(9\pi)^2 k}{2^{1/4}} \frac{N - 2z}{[z(N - z)^{3/4}]} K(z). \quad (6.3.25)$$

Choosing, for instance, $K(z) = [z(N - z)^{7/4}]$, which is regular with regular derivative on the z interval (so that $f_2^{M_3}$ is well defined), the integral reduces to

$$\frac{(9\pi)^2 k}{2^{1/4}} \int_0^N (N - 2z)(N - z) dz = \frac{(9\pi)^2 k}{2^{1/4}} \int_0^N (N^2 - 3Nz + 2z^2) dz = \frac{(9\pi)^2 k}{2^{1/4}} \frac{N^3}{6}. \quad (6.3.26)$$

Finally, using

$$\int_{S^2} y^i y^j \text{vol}_{S^2} = \frac{4\pi}{3} \delta^{ij}, \quad (6.3.27)$$

the D8-brane action reduces to

$$S_{D8} \supset \theta^i \int_{N_5} \star_7 \mathcal{F}_2^i, \quad (6.3.28)$$

with

$$\theta^i = v^i \frac{\beta}{(2\pi)^6} T \frac{4\pi}{3} \frac{(9\pi)^2 k}{2^{1/4}}. \quad (6.3.29)$$

As (6.3.28) shows, a probe D8 brane reproduces an R-symmetry operator holographically. The different choices of the function $K(z)$ as well as the different values of β , give us all the different R-symmetry operators.

6.4 Charge measurement

Finally, we briefly describe how the charge measurement takes place at the string level. The physical channel by which the probe-brane symmetry operator detects the charged operator is the Hanany-Witten effect. Table 3 collects the directions wrapped by the $D0$ and $D8$ branes. By comparison with the Hanany-Witten setup in type IIA theory found in [5], we can infer that the above configuration is a good candidate for a Hanany-Witten transition. Indeed, when the $D0$ worldline is pushed through

	AdS ₇							M ₃		
	z	x^0	x^1	x^2	x^3	x^4	x^5	r	θ	ϕ
D0	×									
D8	×	×	×	×	×	×		×	×	×

Table 3: Brane configuration in type IIA on $AdS_7 \times M_3$. Here, z is the radial direction of AdS_7 , $x^0, \dots, x^5$ are the boundary directions, r spans the internal interval I_r , and θ and ϕ are coordinates on S^2 .

the hanging $D8$ brane, a fundamental $F1$ string is created, signalling that the charge measurement has occurred.

The same mechanism, however, cannot be applied to the R-charged $F1$ string: there is no extended object with which an $F1$ string can undergo a Hanany-Witten transition. One might then wonder how its charge is measured holographically.

Nevertheless, according to [15], the $D0$ and $F1$ charged operators are not entirely unrelated. The operators to which they are dual—baryons and mesons, respectively—are such that products of baryon operators can be expressed in terms of products of string mesons. This relation has a holographic counterpart: an instantonic $D2$ -brane wrapping all of M_3 mediates the process by which $D0$ branes are converted into $F1$ strings, and vice versa. We therefore argue that, after converting $F1$ strings into $D0$ branes via this instantonic $D2$ brane, their R-charge is measured by the same hanging $D8$ brane.

7 Conclusions and Outlook

Our work has shed some light on the R-symmetry structure of the SCFT's dual to type IIA theory in the $AdS_7 \times M_3$ background. Our findings for the charged and symmetry operators match the expectations stated in the conclusions of [10], with the addition of an extra charged operator—the $F1$ string—whose presence agrees with the results of [15].

Moreover, it is known that the massless type IIA theory in $AdS_7 \times M_3$ admits an uplift to M-theory on $AdS_7 \times S^4$ (or an orbifold thereof), whose dual SCFT enjoys an $SO(5)_R$ symmetry descending from the isometry of S^4 [4]. It would therefore be interesting to see whether our construction could help unveil the holographic realization of the R-symmetry in M-theory.

However, $D0$ and $D8$ branes do not have simple uplifts to M-theory. Nevertheless, we suggest that the additional probe operators we have found could provide some insight into the M-theory setup. Indeed, as explained in Section 6, $D0$ branes and $F1$ strings are not entirely independent; rather than trying to uplift the $D0$ configuration, one might extract the M-theory R-charged operator from the $F1$ string. Although exotic, this is consistent with the fact that suitable $M2$ branes could realize R-charged operators holographically in M-theory. More precisely, in the consistent bosonic truncation of M-theory on $AdS_7 \times S^4$ of [20], the field strength of the M-theory 3-form is (up to normalization)

$$G_4 \sim \epsilon^{a_1 \dots a_5} y^{a_1} Dy^{a_2} \wedge \dots \wedge Dy^{a_5} + 2g\epsilon^{a_1 \dots a_5} y^{a_1} F^{a_2 a_3} \wedge Dy^{a_4} \wedge Dy^{a_5} + \dots \quad (7.0.1)$$

where F^{ab} is the field strength of the $SO(5)$ gauge symmetry of the corresponding effective 7-dimensional supergravity. Using the general procedure of [10], one can show that an $M2$ brane wrapping the submanifold $\Sigma_3 = M_1 \times S^2$, with $S^2 \subset S^5$, is a good candidate to realize R-symmetry charged operators. It is therefore plausible that type IIA $F1$ string configurations uplift to $M2$ configurations.

As for the symmetry operator, the situation is more delicate. There is no known way to uplift a $D8$ brane to eleven-dimensional M-theory; moreover, even if such an exotic uplift existed, it would be of little help in understanding how the charge measurement actually takes place in M-theory, since the only known Hanany-Witten transition in M-theory involves an $M2$ brane and an $M5$ brane. We hope to address these questions in a future work.

A Hodge dual formulas

In Section 6.1.2 we need to evaluate the Hodge star operator of differential forms defined on product space of (pseudo-)Riemannian manifolds. Here we will provide some useful formulas to make the computations simpler. First, [53], given a (pseudo-)Riemannian manifold M of dimension n , we define the Hodge dual operator $\star$ as a linear map

$$\star : \Omega^r(M) \rightarrow \Omega^{n-r}(M),$$

where $\Omega^q(M)$ denotes the space of differential q -forms on M . It acts on a coordinate basis vector as

$$\star(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(n-r)!} \epsilon_{\nu_1 \dots \nu_r \nu_{r+1} \dots \nu_n} g^{\nu_1 \mu_1} \dots g^{\nu_r \mu_r} dx^{\nu_{r+1}} \wedge \dots \wedge dx^{\nu_n}. \quad (\text{A.0.1})$$

Then, given an r -form ω ,

$$\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}, \quad (\text{A.0.2})$$

We have,

$$\star \omega = \frac{\sqrt{|g|}}{r!(n-r)!} \omega_{\mu_1 \dots \mu_r} \epsilon^{\mu_1 \dots \mu_r}_{\nu_{r+1} \dots \nu_n} dx^{\nu_{r+1}} \wedge \dots \wedge dx^{\nu_n}. \quad (\text{A.0.3})$$

An important relation involving the Hodge dual is the following [53]: given $\omega, \eta \in \Omega^r(M)$,

$$\omega \wedge \star \eta = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \eta^{\mu_1 \dots \mu_r} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \eta^{\mu_1 \dots \mu_r} \text{vol}_M. \quad (\text{A.0.4})$$

Let us take an n -dimensional manifold $M = M_1 \times M_2$, with M_1 and M_2 two (pseudo-)Riemannian manifolds of dimension n_1 and n_2 , respectively. Then, clearly

$$ds_M^2 = ds_{M_1}^2 + ds_{M_2}^2, \quad (\text{A.0.5})$$

while the volume form of M is

$$\text{vol}_M = \text{vol}_{M_1} \wedge \text{vol}_{M_2}. \quad (\text{A.0.6})$$

In particular, choosing a set of coordinates $\{x^\mu\}_{\mu=1}^{n_1}$ on M_1 and $\{x^\nu\}_{\nu=n_1+1}^n$ on M_2 , we have that the metric tensor of M is block diagonal,

$$g_M = \begin{pmatrix} g_{M_1} & 0 \\ 0 & g_{M_2} \end{pmatrix}, \quad (\text{A.0.7})$$

so that the metric determinant g is simply given by the product $g = g_1 g_2$.

Let us now take $\omega \in \Omega^{p+q}(M)$ such that

$$\omega = \alpha \wedge \beta, \quad \alpha \in \Omega^p(M_1), \quad \beta \in \Omega^q(M_2). \quad (\text{A.0.8})$$

Then, using (A.0.1) we can show that

$$\begin{aligned} \star \omega &= \star(\alpha \wedge \beta) = \frac{1}{p!} \frac{1}{q!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_q} \star(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_q}) \\ &= \frac{1}{p!} \frac{1}{q!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_q} \frac{\sqrt{|g|}}{(n-p-q)!} \epsilon^{\mu_1 \dots \mu_p \nu_1 \dots \nu_q}_{\rho_{p+q+1} \dots \rho_n} dx^{\rho_{p+q+1}} \wedge \dots \wedge dx^{\rho_n} \\ &= \frac{1}{p!} \frac{1}{q!} \alpha^{\mu_1 \dots \mu_p} \beta^{\nu_1 \dots \nu_q} \frac{\sqrt{|g_1|} \sqrt{|g_2|}}{(n-p-q)!} \epsilon_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q \rho_{p+q+1} \dots \rho_n} dx^{\rho_{p+q+1}} \wedge \dots \wedge dx^{\rho_n}. \end{aligned} \quad (\text{A.0.9})$$

where in the last line we used the fact that the metric is block diagonal and $\mu_j \neq \nu_k$ for all j, k . The set of indices $\{\rho_{p+q+1}, \dots, \rho_n\}$ runs over all M indices; we now split it into two sets, one for M_1 and one for M_2 . First, we notice that the non-zero contributions to the Levi-Civita symbol come from terms where $n_1 - p$ indices among the $\{\rho_{p+q+1}, \dots, \rho_n\}$ are taken from M_1 , while the remaining $n_2 - q$ are taken from M_2 . The number of ways we can do this is given by the binomial coefficient

$$\binom{n-p-q}{n_1-p} = \frac{(n-p-q)!}{(n_1-p)!(n_2-q)!}, \quad (\text{A.0.10})$$

where we used $n = n_1 + n_2$. Thus, we can write (A.0.9) as

$$\begin{aligned} \star(\alpha \wedge \beta) &= \frac{1}{p!} \frac{1}{q!} \alpha^{\mu_1 \dots \mu_p} \beta^{\nu_1 \dots \nu_q} \frac{\sqrt{|g_1|} \sqrt{|g_2|}}{(n_1-p)!(n_2-q)!} \epsilon_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q \mu_{p+1} \dots \mu_{n_1} \nu_{q+1} \dots \nu_{n_2}} \\ &\quad \times dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_{n_1}} \wedge dx^{\nu_{q+1}} \wedge \dots \wedge dx^{\nu_{n_2}} \\ &= (-1)^{q(n_1-p)} \frac{1}{p!} \frac{1}{q!} \alpha^{\mu_1 \dots \mu_p} \beta^{\nu_1 \dots \nu_q} \frac{\sqrt{|g_1|} \sqrt{|g_2|}}{(n_1-p)!(n_2-q)!} \epsilon_{\mu_1 \dots \mu_p \mu_{p+1} \dots \mu_{n_1} \nu_1 \dots \nu_q \nu_{q+1} \dots \nu_{n_2}} \\ &\quad \times dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_{n_1}} \wedge dx^{\nu_{q+1}} \wedge \dots \wedge dx^{\nu_{n_2}}, \end{aligned} \quad (\text{A.0.11})$$

where μ_j and ν_k are indices of M_1 and M_2 , respectively.

Finally, since the Levi-Civita symbol features two distinct set of indices, we can split it into two Levi-Civita symbols, one for M_1 and one for M_2 ,

$$\epsilon_{\mu_1 \dots \mu_p \mu_{p+1} \dots \mu_{n_1} \nu_1 \dots \nu_q \nu_{q+1} \dots \nu_{n_2}} = \epsilon_{\mu_1 \dots \mu_p \mu_{p+1} \dots \mu_{n_1}}^{(M_1)} \epsilon_{\nu_1 \dots \nu_q \nu_{q+1} \dots \nu_{n_2}}^{(M_2)}. \quad (\text{A.0.12})$$

In conclusion, we have the following useful formula for the Hodge dual of a product form,

$$\star(\alpha \wedge \beta) = (-1)^{q(n_1-p)} \star_1 \alpha \wedge \star_2 \beta, \quad (\text{A.0.13})$$

where $\star_1$ and $\star_2$ are the Hodge duals on M_1 and M_2 , respectively.

Another formula used in Section 6.1.2 relates the Hodge dual of two conformally related metrics. Let ds^2 and $\tilde{ds}^2$ be two metrics of the n -dimensional manifold M related by a conformal factor Ω ,

$$\tilde{ds}^2 = \Omega^2 ds^2. \quad (\text{A.0.14})$$

Then, the corresponding volume forms are related by

$$\widetilde{\text{vol}}_M = \Omega^n \text{vol}_M. \quad (\text{A.0.15})$$

Denoting by $\tilde{\star}$ the Hodge dual taken with respect to the metric $\tilde{ds}^2$, given any two r -forms ω and η on M , from (A.0.4) we have

$$\begin{aligned} \omega \wedge \tilde{\star} \eta &= \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \eta^{\mu_1 \dots \mu_r} \widetilde{\text{vol}}_M \\ &= \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \eta_{\nu_1 \dots \nu_r} \tilde{g}^{\mu_1 \nu_1} \dots \tilde{g}^{\mu_r \nu_r} \Omega^n \text{vol}_M \\ &= \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \eta_{\nu_1 \dots \nu_r} g^{\mu_1 \nu_1} \dots g^{\mu_r \nu_r} \Omega^{n-2r} \text{vol}_M \\ &= \Omega^{n-2r} \omega \wedge \star \eta. \end{aligned} \quad (\text{A.0.16})$$

Finally, from the arbitrariness of ω , the following relation holds

$$\tilde{\star} \eta = \Omega^{n-2r} \star \eta. \quad (\text{A.0.17})$$

B Massless type IIA symmetry defects in r coordinate

In this appendix we repeat the computation of the charged and symmetry operators carried out in Section 6, this time working in the r coordinate. We restrict throughout to the massless case $F_0 = 0$.

B.1 Massless relations

We recall that massless type IIA theory on $AdS_7 \times M_3$ uplifts to M-theory on $AdS_7 \times S^4$ (or an orbifold thereof), as can be seen by reducing along a suitable isometry of S^4 [4]. In this case the following relations between x , A and ϕ hold,

$$x = \cos(\alpha), \quad x = \sqrt{1 - e^{4(A-A_0)}}, \quad \phi = 3A - \phi_0, \quad (\text{B.1.1})$$

where $\alpha \in [0, \pi]$, and A_0 and ϕ_0 are integration constants related to the overall radius R of the eleven-dimensional metric,

$$A_0 = \frac{1}{2} \log \left(\frac{R^3}{2} \right), \quad \phi_0 = 3 \log(R). \quad (\text{B.1.2})$$

Differently from [4], we are just considering the case $k = 1$, which correspond exactly to $AdS_7 \times S^4$; one can easily generalize to $k \in \mathbb{N}$ and obtain solutions whose uplifts is $AdS_7 \times S^4/\mathbb{Z}_k$.

Using these relations, we can show that the function q appearing in the ansatz for the RR fluxes is identically equal to $1/2$,

$$q = \frac{1}{4} e^{A-\phi} \sqrt{1-x^2} = \frac{1}{4} e^{-2A_0} e^{3A-\phi} = \frac{1}{4} \frac{2}{R^3} R^3 = \frac{1}{2} \quad (\text{B.1.3})$$

B.2 Hodge dual relations

We first rewrite the Hodge-dual relations in the r coordinate. Starting from the ansatz (6.1.3) for the ten-dimensional metric in the r coordinate and using the results of Appendix A, we immediately obtain (recall that $q = 1/2$)

$$F_8 \supset -\frac{1}{2} y^i g R_1 R_2^2 \star_7 \mathcal{F}_2^i \wedge dr \wedge \text{vol}_{S^2} \quad (\text{B.2.1})$$

$$F_6 \supset -\frac{1}{32} x e^{2A} \sqrt{1-x^2} y^i g R_1 R_2^2 \star_7 \mathcal{F}_2^i \wedge dr - \frac{1}{8} e^A \epsilon^{ijk} g R_1^{-1} \star_7 \mathcal{F}_2^i y^j \wedge \star_2 dy^k, \quad (\text{B.2.2})$$

where

$$R_1 = \ell^{1/2}, \quad R_2 = \ell^{1/2} \sqrt{\frac{1-x^2}{16}} e^A. \quad (\text{B.2.3})$$

Using once more the identity employed in (6.1.27), we can rewrite (B.2.2) as

$$F_6 \supset -\frac{1}{32} x e^{2A} \sqrt{1-x^2} y^i g R_1 R_2^2 \star_7 \mathcal{F}_2^i \wedge dr - \frac{1}{8} e^A g R_1^{-1} \star_7 \mathcal{F}_2^i \wedge dy^i. \quad (\text{B.2.4})$$

B.3 Charged Operators

The computation of the D0-brane action closely parallels the one already presented in Section 6. Indeed, in the massless case the RR 2-form flux is simply

$$F_2 = -\frac{1}{2} (\text{vol}_2 + y^i g \mathcal{F}_2^i). \quad (\text{B.3.1})$$

Writing the D0-brane action in terms of an auxiliary two-dimensional manifold, we then have

$$S_{D0} \supset \int_{N_2} F_2 = -\frac{1}{2} \int_{N_2} \text{vol}_2 + y^i g \mathcal{F}_2^i, \quad (\text{B.3.2})$$

with N_2 defined as in (6.2.2). This is identical to (6.2.2) up to an overall factor. Repeating the same steps that led to (6.2.26), we find

$$S_{D0} \supset \frac{1}{2} q^{ij} \int_{M_1} (\chi^{-1} d_A \chi)^{ij}, \quad (\text{B.3.3})$$

with

$$q^{ij} = -\frac{1}{2} Y_{(0)}^{ij}. \quad (\text{B.3.4})$$

For the F1 string, we now adopt the same inflow method used for the D0 brane. Specializing the H field-strength ansatz (6.1.7) to the massless case gives

$$\ell^{-1} H = -6 e^{-A} \text{vol}_{M_3} - \frac{1}{4} e^A dr \wedge y^i g \mathcal{F}_2^i - \frac{1}{16} e^{2A} x \sqrt{1-x^2} g \mathcal{F}_2^i \wedge Dy^i \quad (\text{B.3.5})$$

Recalling that the worldvolume swept by the F1 string is $M_1 \times I_r$, we can write its action as

$$\begin{aligned} S_{F1} &\supset -\frac{1}{2\pi} \int_{N_2 \times I_r} H = \frac{1}{2\pi} \int_{N_2 \times I_r} \left[6\ell e^{-A} \text{vol}_{M_3} + \frac{\ell}{4} e^A dr \wedge y^i g \mathcal{F}_2^i \right] \\ &= \frac{1}{2\pi} \int_{N_2} \int_{I_r} \left[6\ell e^A \frac{1-x^2}{16} dr \wedge \text{vol}_2 + \frac{\ell}{4} e^A dr \wedge y^i g \mathcal{F}_2^i \right]. \end{aligned} \quad (\text{B.3.6})$$

Exploiting the massless relations (B.1.1) together with the second line of (5.2.3), we obtain

$$e^A = (1-x^2)^{1/4} e^{A_0}, \quad dr = -\frac{1}{2} \frac{e^A}{\sqrt{1-x^2}} dx, \quad (\text{B.3.7})$$

from which we have

$$e^A (1-x^2) dr = -\frac{1}{2} e^{2A} (1-x^2) \frac{1}{\sqrt{1-x^2}} dx = -\frac{1}{2} e^{2A_0} (1-x^2) dx. \quad (\text{B.3.8})$$

$$e^A dr = -\frac{1}{2} e^{2A_0} dx. \quad (\text{B.3.9})$$

Finally, since $x = \cos(\alpha)$ with $\alpha \in [0, \pi]$, we have $x \in [1, -1]$, so that

$$\int_{I_r} 6\ell e^A \frac{(1-x^2)}{16} dr = \frac{6}{16} \ell \int_1^{-1} -\frac{1}{2} e^{2A_0} (1-x^2) dx = \frac{\ell}{4} e^{2A_0} \quad (\text{B.3.10})$$

and

$$\int_{I_r} \ell \frac{1}{4} e^A dr = \frac{\ell}{4} \int_1^{-1} -\frac{1}{2} e^{2A_0} dx = \frac{\ell}{4} e^{2A_0}. \quad (\text{B.3.11})$$

Putting everything together, (B.3.6) becomes

$$S_{F1} \supset \frac{\ell}{8\pi} e^{2A_0} \int_{N_2} \text{vol}_2 + y^i g \mathcal{F}_2^i, \quad (\text{B.3.12})$$

exactly as for the D0 brane. Following the same steps as before, we then find

$$S_{F1} \supset \frac{1}{2} q^{ij} \int_{M_1} (\chi^{-1} d_A \chi)^{ij} \quad (\text{B.3.13})$$

with

$$q^{ij} = \frac{\ell}{8\pi} e^{2A_0} Y_{(0)}^{ij}. \quad (\text{B.3.14})$$

We now wish to compare these results with those of [15]. The comparison can still be carried out, although in this coordinate it is somewhat less transparent than in the z coordinate.

We begin with the D0 brane. In the massless case, [15] find that its Cartan $U(1)$ R-charge is

$$Q_R(D0) = \frac{r_{\max}}{2}; \quad (\text{B.3.15})$$

and that

$$r_{\max} = - \int_{S^2} \frac{F_2}{2\pi}. \quad (\text{B.3.16})$$

Inserting our expression for the RR 2-form flux, we obtain

$$r_{\max} = \frac{1}{2} \int_{S^2} \frac{\text{vol}_2}{2\pi} = 1, \quad (\text{B.3.17})$$

so our result for the D0 brane agrees with that of [15].

An analogous treatment holds for the F1 string. According to [15], its R-charge in the massless case is

$$Q_R(F1) = \frac{N}{2}, \quad (\text{B.3.18})$$

where N is given by

$$N = - \frac{1}{4\pi^2} \int_{M_3} H. \quad (\text{B.3.19})$$

In our case

$$- \frac{1}{4\pi^2} \int_{M_3} H = - \frac{1}{4\pi^2} \int_{M_3} -6\ell e^{-A} \text{vol}_{M_3} = - \frac{1}{4\pi^2} \int_{S^2} \text{vol}_{S^2} \int_{I_r} -6\ell e^A \frac{1-x^2}{16} dr = \frac{\ell}{4\pi} e^{2A_0} \quad (\text{B.3.20})$$

Comparing with (B.3.14), we see that the two descriptions again agree.

B.4 Symmetry operators

The computation of the symmetry operator is identical to the one in Section 6. As in equation (6.3.22), the Wess-Zumino term of the D8 brane in the r coordinate reads

$$S_{D8} \supset \frac{\beta}{(2\pi)^6} \int_{N_5} \star_5 \mathcal{F}_2^i \int_{S^2} v^j y^i y^j \text{vol}_{S^2} \int_{I_r} dr \frac{\ell^{-1}}{2} \frac{x}{\sqrt{1-x^2}} K(r), \quad (\text{B.4.1})$$

where the r integral is

$$\int_{I_r} dr \frac{\ell^{-1}}{2} \frac{x}{\sqrt{1-x^2}} K(r) = - \frac{\ell^{-1}}{4} e^{A_0} \int_1^{-1} dx x (1-x^2)^{3/4} K(r(x)), \quad (\text{B.4.2})$$

and we have again used the massless relations. Choosing a regular function $K(r)$, we recover the expected structure of an $SU(2)_R$ symmetry operator.

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