Università degli studi di Padova

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Corso di Laurea Magistrale in Matematica

Tesi di Laurea Magistrale

BATHYMETRY RECONSTRUCTION VIA A TIME-DEPENDENT INTRINSIC SHALLOW WATER MODEL

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Anno Accademico 2020-2021

ABSTRACT

This thesis is based on the works [4] and [15]. We firstly give a presentation of the state of the art of the bathymetry reconstruction problem. Secondly we introduce, in a geometrical setting, the Shallow water model, known for its many applications (dynamics of the athmosphere, geophysical phenomena and more). Furthermore, we use the SW model to derive a novel intrinsic model for the bathymetry reconstruction. More specifically, we find a second order approximation of the Navier-Stokes equations based on the SW model. Finally, employing the Discontinuous Galerkin method, we perform the first steps towards the experimental validation of our model.

Acknowledgements

I would like to express my profound gratefulness to my advisor, prof. Mario Putti, for his contagious positiveness and suggestions that have made this work possible. Thanks for giving me the opportunity to face challenges I would never have thought to be able to overcome.

My sincere gratitude to Elena Bachini, for the constant support through the development of this thesis. Her patience and constructive advices have been very important to me.

I must express my deep appreciation to prof. Vadym Aizinger, for the possibility (teared apart by the Sars-CoV-2 situation) of joining him in Bayreuth and his availability, despite everything.

Many thanks to Hennes Hajduk, for his help and discussions.

Finally, to my family, friends and colleagues that have supported (and beared) me through these years of my career as a student at the University of Padua.

Contents

Introduction

In this thesis we are going to derive a novel intrinsic Shallow water model for the bathymetry reconstruction of e.g. rivers. Elena Bachini Ph.D. thesis [4] has provided a solid terrain and has been fundamental for the development of this work: the present has its roots there.
How can the bottom of a river be reconstructed from its surface data? This question seems to have no answer at a first sight. Surprizingly, it is possible to translate this problem into a mathematical language and to give raise to mathematical-physical models, which complexity is proportional to their distance to the real phaenomenon. In choosing such a model, it is necessary to do a compromise between its fidelity and the ease of dealing with it theoretically. The Shallow water model does exactly this: it furnish a good approximation of the Navier-Stokes equations in the hyphothesis of a thin fluid layer and large wave-length. Its ease of use and versatility lead the SW model itself to a wide applicability: dynamics of the atmosphere (tsunami prediction, hurricane modeling), geophysical phaenomena (dam breaks, debris flows, river flows, avalanches) as well as oceanografic modelling and many others.
Dealing with these complex models one may need to take into consideration also the shape of for example a river bed. The evolution of the Mathematics of the past century brought to light ,among the others, the theory of Differential Geometry: this provides a useful toolbox in dealing with surfaces. Our aim is to employ this theory in the study of the bathymetry reconstruction.

This thesis is structured in the following way. In the first chapter, *Preliminaries*, can be found some basic results such as topics in Continuum Mechanics and the theory of Surface PDEs. Here we give a presentation of geometrical objects such as atlas, transition maps, coordinate curves. Moreover we introduce the notion of regular surface, the Local Coordinate System (LCS) and the Global Coordinate System (GCS), important concepts that will be enconutered in the main following chapters: the LCS will allow us to do explicit computations on a regular surface. Identifying points in the fluid domain from the perspective of the LCS set-up requires to work with the so called tubular neighborhood, also presented in this chapter. Furthermore, is given the definition of the differential operators acting on function, vector fields and tensors defined on a regular surface. Finally the well-known Green's formula and the divergence theorem are recalled: they will become handy in building the numerical formulation of our bathymetry model.

In chapter 2, the Shallow water equations are introduced and derived, starting from the Incompressible Navier-Stokes equations written with respect to the LCS, and thus called *curvilinear NS equations*. Restrictions on the dimension of the system and its simplification are possible thanks to the employment of the Kinematic Boundary Conditions, which are a very natural constraint that we impose on the system itself, and some algebraic manipulations. This process, that involves the normal depth-integration of the system's equations, brings the system dimentionality back from 4 to 3. Under the assumption of a thin fluid layer, i.e. the Shallow water hyphothesis, some approximations lead us to a second order approximation of the Navier-Stokes equations. For each point in the bottom surface, trying to solve this system means seeking for the height of the water, measured along the normal attached to the bottom itself aswell as the depth-averaged velocity vector first two components. In this problem formulation, the bottom surface is a known, fixed in time, data.

Chapter 3 contains the Shallow water model treated in [15], that will be compared with ours in the *Bathymetry reconstruction* chapter, further on.

Chapter 4, *Bathymetry reconstruction with intrinsic geometry*, forms the core of the present thesis. Here the perspective is overturned and we are interested in the opposite problem of finding the shape of the bottom surface, given the top one. Thus, we derive a new set of equations starting from the definition of a tensor that includes in itself both the Continuity equation terms and the ones related to the Momentum equations. The simplest situation is found assuming that the bottom is not eroding, i.e. it doesn't vary in time, nevertheless this model also considers the case of a time-dependent bottom surface. Theorically, with little modifications, our inverse model formulation can also be used to study the direct problem of finding the top surface of a river given a bottom that does depend on time. Finally, we will briefly compare the non-intrinsic direct and inverse models in [15] with the intrinsic ones derived in [4] and in the present thesis.

In the chapter *Numerical set-up for the bathymetry reconstruction model* we will derive the fully discrete Discontinuous Galerking formulation. The paper of G. Dziuk and C. Elliot [14] will furnish a concrete help to arrive at the final result.

Chapter 1

Preliminaries

In the present chapter, we are going to recall some theorical results and notions that will be useful in the future development of the thesis. In the first part, we start from a Continuum Mechanics setup that involves spatial and material coordinates. Then we present the well-known Reynold's (or transport) theorem, the relation between the Continuity equation and isochoric motion. Finally, we also recall the Momentum equations and Cauchy's Tetrahedron theorem.
Furthermore, the second part involves the presentation of our geometric setting, at the basis of Surface PDEs.

1.1 Continuum Mechanics: overview

In this section, we recall basic concepts and results of Continuum Mechanics that are at the very basis of the Navier-Stokes equations, see [10]. Let $\mathcal{B}$ the set of all particles at a certain time such that the external normal on $\partial\mathcal{B}$ exists almost everywhere. For each particle, we call x the space coordinate, $\mathbf{X}$ the material coordinate (particle label) i.e. the initial position of the particle, and $x - \mathbf{X} =: \mathbf{d}$ the displacement of $\mathbf{X}$.

Definition 1.1. *A deformation of $\mathcal{B}$ is a smooth map*

$$x : \mathbb{R}^3 \supseteq \mathcal{B} \ni \mathbf{X} \longmapsto x(\mathbf{X}) \in \mathbb{R}^3$$

such that

a) $x(\cdot)$ *is a diffeomorphism on* $x(\mathcal{B})$

b) $\frac{\partial x_i}{\partial X_L}(X) = F_{iL}(X)$ *displacement gradient has* $det(F(X)) > 0\,\forall x \in \mathcal{B}$.

Definition 1.2. *The motion of* $\mathcal{B}$ *in* $[0,T]$ *is a map*

$$x:\, \mathcal{B}\times[0,T] \ni (\mathbf{X},t) \longmapsto x(\mathbf{X},t) = x \in \mathbb{R}^3$$

such that $\mathbf{X} \longmapsto x(\mathbf{X},t)$ *is a deformation* $\forall t$.

Let $\Gamma = \{(t,x) \subset \mathbb{R}^4 \,\big|\, t \in [0,T], x \in \mathcal{B}_t = x(\mathcal{B},t)\}$ be the space-time trajectory, we can write a scalar field Φ in material or spatial coordinates using the map x and its inverse: $\Phi_m(\mathbf{X},t) = \Phi(x(\mathbf{X},t)) = \Phi\circ x$ and $\Phi(x,t) = \Phi_m(x^{-1}(x,t)) = \Phi_m \circ x^{-1}$. Note that we use the subscript m to underline the reference to the material coordinates.

Theorem 1.1. *(Transport thm. (or Reynold's thm.))*
Denote with $\dot{\Phi}$ *the total derivative in time of* Φ *and let* $F(x)$ *be the jacobian matrix of the coordinate transformation* x. *Using the theorem of change of variables and divergence theorem, the following algebraic manipulations hold:*

$$\begin{aligned}
&\frac{d}{dt}\int_{x(\mathcal{B},t)} \Phi(t,\mathbf{x})\,dx = \frac{d}{dt}\int_{\mathcal{B}} \Phi(x(\mathbf{X},t),t)\,detF(t,\mathbf{X})\,dX = \\
&= \int_{\mathcal{B}} \left[\frac{d}{dt}(\Phi_m)\,detF + \Phi_m\,detF(\nabla\cdot\mathbf{u})_m\right] dX = \int_{x(\mathcal{B},t)} \left[\left(\frac{d}{dt}\Phi_m\right) + \Phi\nabla\cdot\mathbf{u}\right] dx = \\
&= \int_{\mathcal{B}_t} \left[\dot{\Phi} + \Phi\nabla\cdot\mathbf{u}\right] dx = \int_{\mathcal{B}_t} (\Phi' + \nabla_x\Phi\,\mathbf{u} + \Phi\nabla\cdot\mathbf{u})\,dx = \int_{\mathcal{B}_t} [\Phi' + \nabla\cdot(\Phi\mathbf{u})]\,dx = \\
&= \int_{\mathcal{B}_t} \Phi'\,dx + \int_{\partial\mathcal{B}_t} \Phi\,\mathbf{u}\cdot\mathbf{n}\,d_x\sigma\,.
\end{aligned}$$

This series of algebraic manipulations is very important for many results in Continuum Mechanics. For example, recalling that

$$\frac{\partial}{\partial t} detF(t,\mathbf{X}) = \det F(t,\mathbf{X})(\nabla\cdot\mathbf{u})_m\,,$$

we can write the rate of change of volume of the body $\mathcal{B}_t$ ($\Phi \equiv 1$) as:

$$\frac{d}{dt}vol(\mathcal{B}_t)\frac{d}{dt}\int_{\mathcal{B}_t} dx = \frac{d}{dt}\int_{\mathcal{B}} detF\,dX =$$

$$= \int_{\mathcal{B}} \det F(\nabla \cdot \mathbf{u})_m \, dX = \int_{\mathcal{B}_t} \nabla \cdot \mathbf{u} \, dx = \int_{\partial \mathcal{B}_t} \mathbf{u} \cdot \mathbf{n} \, d_x\sigma \,. \tag{1.1}$$

From this computations we can see that the motion is *isochoric*, i.e. vol($\mathcal{B}_t$)=vol($\mathcal{B}$) $\forall\, t$, if and only if $\nabla \cdot \mathbf{u} = 0$. This is exactly the incompressibility condition we will use in the (incompressible) Navier-Stokes equations. An equivalent condition, which can be derived employing a simple change of variables, would be to impose det$F$=1.
If we now assume that the *Principle of Mass Conservation* holds, i.e. $\forall \mathcal{P} \subseteq \mathcal{B}$, $\forall t$, we ask that $m(\mathcal{P}_t) = m(\mathcal{P})$, we can derived the following proposition.

Proposition 1.1. *(Continuity Equation for mass density)*
Let $\rho : \Gamma \to [0, +\infty)$ be the mass density written in spatial coordinates, using theorem (1.1) we have

$$0 = \frac{d}{dt} \int_{\mathcal{P}_t} \rho \, dx = \int_{\mathcal{P}_t} (\dot{\rho} + \rho \nabla \cdot \mathbf{u}) \, dx = \int_{\mathcal{P}_t} [\rho' + \nabla \cdot (\rho \mathbf{u})] \, dx \qquad \forall\, \mathcal{P}_t \in \mathcal{B}_t$$

$$\Rightarrow \dot{\rho} + \rho \nabla \cdot \mathbf{u} = \rho' + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.2}$$

Observe that this is just a way of writing the Continuity Equation: we chose it because it is often employed in fluid dynamics.
Conservation laws of physical quantities (mass, charge, etc.) can be seen as a special case of balance laws, which are more general and can be used to describe more complex phaenomena. Consider $\Psi(x, t)$ scalar function, we can write the associated balance law as

$$\frac{d}{dt} \int_{\mathcal{P}_t} \Psi \, dx = \int_{\mathcal{P}_t} \big[\Psi' + \nabla \cdot (\Psi \mathbf{u})\big] \, dx = \int_{\mathcal{P}_t} r \, dx - \int_{\partial \mathcal{P}_t} \Phi^{nc} \cdot \mathbf{n} \, d_x\sigma \qquad \forall\, \mathcal{P}_t$$

$$\Rightarrow \Psi' + \nabla \cdot \big(\Psi \mathbf{u} + \Phi^{nc}\big) = r \qquad \textit{(Balance Law)} \tag{1.3}$$

where r is the source (or production) term, Φ^{nc} is the non-convective flux, $\mathbf{n}$ is the external normal of $\partial \mathcal{P}_t$. The minus sign on the right hand side is justified because we think that the convective flux gives a positive contribution when it enters the body (when $\Phi^{nc} \cdot \mathbf{n} < 0$). Setting $\Phi^{nc} := -c \nabla_x \Psi$, with $c > 0$ diffusion coefficient, we can write $\Psi' + \nabla \cdot (\Psi \mathbf{u}) = r + c \Delta \Psi$.

Definition 1.3. *(Tensor product)*

Given $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ vectors, we call $\mathbf{v} \otimes \mathbf{w} \in M_{n\times n}(\mathbb{R})$ the tensor product between $\mathbf{v}$ and $\mathbf{w}$. For every $u \in \mathbb{R}^n$ we have:

$$(\mathbf{v} \otimes \mathbf{w})\mathbf{u} := \mathbf{v}(\mathbf{w} \cdot \mathbf{u}) \in \mathbb{R}^n .$$

Moreover, the element $(\mathbf{v} \otimes \mathbf{w})_{ij}$ is given by the intersection of row i and column j, i.e. using the definition :

$$(\mathbf{e}_i)^T(\mathbf{v} \otimes \mathbf{w})\mathbf{e}_j = \mathbf{e}_i \cdot (\mathbf{v}(\mathbf{w} \cdot e_j)) = v_i w_j ,$$

so $(\mathbf{v} \otimes \mathbf{w})_{ij} = v_i w_j \ \forall i, j = 1, \ldots, n$.
We are ready now to write the *Balance Law* for vector fields. Let $\Psi(t, \mathbf{x}) \in \mathbb{R}^3$ be a vector field, then we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{P}_t} \Psi_i \, dx &= \int_{\mathcal{P}_t} \left[\Psi_i' + \nabla \cdot (\Psi_i \mathbf{u})\right] dx = \int_{\mathcal{P}_t} \left[\Psi_i' + \sum_j \frac{\partial}{\partial x_j}(\Psi_i \mathbf{u}_j)\right] dx = \\ &= \int_{\mathcal{P}_t} \left[\Psi_i' + \sum_j \frac{\partial}{\partial x_j}(\Psi \otimes \mathbf{u})_{ij}\right] dx = \int_{\mathcal{P}_t} \left[\Psi_i' + div(\Psi \otimes \mathbf{u})_i\right] dx , \end{aligned} \tag{1.4}$$

where $div\mathbb{T}$ denotes the divergence of a tensor: $(div\mathbb{T})_i = \sum_j \partial/\partial x_j \, \mathbb{T}_{ij}$. This is the left hand side of the *Balance Law* for vector fields in integral form; the right hand side is totally analogous to the scalar field case seen previously.

Forces acting on a body. We distinguish the forces acting on the continuous body $\mathcal{B}_t$ in external and internal forces. They can be of the following types:

- Volume forces (external): they depend on the volume of the body, for example gravitational or elecromagnetical forces.
- Surface forces (external): they act on the boundary of the body, for example pressure forces.
- Close-contact forces (internal): they may depend on the deformation of the body, so they are unknown, and are given by the reciprocal action between internal points of $\mathcal{P}$ that are in contact.

Note that in the case of a particulate body, while we assume it continuous, all the forces are of distant action type. The definitions of the momentum $Q(t)$ and the angular momentum $M_O(t)$ for a region $\mathcal{P} \in \mathcal{B}$ are:

$$Q(t) = \int_{\mathcal{P}_t} \mu e \quad \textit{and} \quad M_O(t) = \int_{\mathcal{P}_t} x \times \mu e.$$

In order to recall fundamental tool for describing internal contact forces we need *Conservation of Momentum* and *Conservation of Angular Momentum* equations:

$$\frac{d}{dt} Q = \int_{\mathcal{P}_t} \mu \dot{e} \, d_x v = \int_{\mathcal{P}_t} b(x,t) \, d_x v + \int_{\partial \mathcal{P}_t} s(x,t,n(x)) \, d_x \sigma \,, \tag{1.5}$$

$$\frac{d}{dt} M_O(t) = \int_{\mathcal{P}_t} x \times \mu \dot{e} \, d_x v = \int_{\mathcal{P}_t} x \times b(x,t) d_x v + \int_{\partial \mathcal{P}_t} x \times s(x,t,n(x)) \, d_x \sigma \,, \tag{1.6}$$

where $s(x,t,n) = dR/d\sigma$ is the internal superficial density of contact forces acting on an infinitesimal area $d\sigma$, and $b, \Sigma : \Gamma \longrightarrow \mathbb{R}^3$ are given functions and $\mathcal{P}$ is an internal part of body $\mathcal{B}$. These balance equations's validity is postulated in Continuum Mechanics, because they are not invariant with respect to a rigid transformation of the reference frame.

Theorem 1.2. *(Cauchy Tetrahedron)*
Let (b,s) be the solicitation along the motion x of $\mathcal{B}$. Balance equations (1.5),(1.6) are satisfied along the motion of $\mathcal{B}$ if and only if there exists a smooth tensorial field $T : \Gamma \longrightarrow Lin$ called Cauchy stress tensor such that, for every $(x,t) \in \Gamma$ it holds

a) $\forall n \in \mathbb{S}^2, s(x,t,n) = T(x,t)n \,,$

b) $\mu(x,t)\dot{e}(x,t) = divT(x,t) + b(x,t) \,,$

c) $T(x,t) \in Sym \,.$

Condition b) *is the local form of Momentum Balance Law.*

Observation. The coupling of the Continuity equation and the Balance Laws forms a system of 7 equations. Nonetheless, in the NS system we will consider only 4 unknowns and drop te Angular Momentum's Balance Law. Then, in the Shallow

Water model we will reduce to 3 unknowns, assuming the standard SW hyphothesis (see Chapter 3). Finally, we would like to remark that further analysis is possible in order to reduce the system, thanks to the theory of *Constitutive Equations*, although we will not enter in details.

1.2 Surface PDEs

In this section, we are going to recall some geometrical concepts, such as atlas, transition maps, coordinate curves. Most importantly, we are going to encounter the notion of Global Coordinate System (GCS) and Local Coordinate System (LCS) aswell as the one of regular region. The first two will be encountered many times in the following chapters; in particular, the second will be modified in the chapter about the bathymetry reconstruction, while the latter will be important in the numerical approximation of a regular surface.
This thesis finds its fundations in the work of Elena Bachini's Ph.D. thesis [4], so we will give a general overview of [4] initial main points, presenting them step by step. So this section starts with the notion of tubular neighborhood and the definition of the intrinsic differential operators acting on functions, vector fields and tensors defined on a regular surface. Then, two well-known results in PDEs theory are recalled: the divergence theorem and the Green's formula.

1.2.1 Geometric setting

We introduce now some of the main geometrical objects we will deal with. For example, the notion of regular surface, central in this thesis: a subset of $\mathbb{R}^3$ in which, for every point, there exists a local parametrization with domain $\mathbb{R}^2$ that respects the topology and that has differential of maximum rank. Formally, the definition reads:

Definition 1.4. *A $\mathcal{C}^k$ regular surface is a subset $\mathcal{S} \subset \mathbb{R}^3$ such that for every point $\mathbf{p} \in \mathcal{S}$ there exists a neighborhood $V \subset \mathbb{R}^3$ and a map $\phi_{\mathbf{p}} : U \to V \cap \mathcal{S}$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap \mathcal{S} \subset \mathbb{R}^3$ such that:*

i) *$\phi_{\mathbf{p}}(U) \subseteq \mathcal{S}$ is an open neighborhood of $\mathbf{p} \in \mathcal{S}$ (i.e. there exists V open neighborhood of $\mathbf{p}$, $V \subset \mathbb{R}^3$ such that $\phi_{\mathbf{p}}(U) = V \cap S$);*

ii) *$\phi_{\mathbf{p}}$ is a homeomorphism with its image;*

iii) *the differential $d\phi_{\mathbf{p}} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $\mathbf{q} \in U$ (or equivalently it has a maximum rank).*

The map $\phi_{\mathbf{p}}$ is called *parametrization* and is an important tool to be intrinsic to the surface. It defines a *system of coordinates* centered in $\mathbf{p}$. Following this terminology, the neighborhood $V \cap \mathcal{S}$ of $\mathbf{p}$ in $\mathcal{S}$ is called a *coordinate neighborhood*, while the coordinates $(x^1_{\mathbf{p}}, x^2_{\mathbf{p}}) = \phi^{-1}_{\mathbf{p}}$ are called *local coordinates* of $\mathbf{p}$. Recall also that $\phi^{-1}_{\mathbf{p}}$ is called the *local chart* in $\mathbf{p}$. If we fix a canonical basis vector $\mathbf{e}_j$ of $\mathbb{R}^2$, the curve $\lambda \mapsto \phi_{\mathbf{p}}(\mathbf{o} + \lambda \mathbf{e}_j)$ is naturally defined and called the j-th *coordinate curve* through the point $\mathbf{p} = \phi_{\mathbf{p}}(\mathbf{o})$. This is simply the projection of the straight $\mathbb{R}^2$ line $\lambda \mapsto \mathbf{o} + \lambda \mathbf{e}_j$ onto the surface $\mathcal{S} \subseteq \mathbb{R}^3$ through the parametrization $\phi_{\mathbf{p}}$. Given two points $\mathbf{p}$ and $\mathbf{q}$ on $\mathcal{S}$, as well as their local parametrizations $\phi_{\mathbf{p}}, \phi_{\mathbf{q}}$, if $U_{\mathbf{p}} \cap U_{\mathbf{q}} \neq \emptyset$, we must require that the *transition* map $\phi_{\mathbf{p}} \circ \phi^{-1}_{\mathbf{q}}$ is a $\mathcal{C}^k$ diffeomorphism, so that the local parametrizations are *compatible*. Ultimately, a family $\{\phi_\alpha\}_{\alpha \in A}$ of compatible local parametrizations $\phi_\alpha : U_\alpha \to \mathcal{S}$ that fully covers all the surface $\mathcal{S}$, i.e. $\mathcal{S} = \bigcup_\alpha \phi_\alpha(U_\alpha)$ is called an *atlas* for the surface $\mathcal{S} \subset \mathbb{R}^3$. Among many possible examples of regular surfaces, will be useful for us to consider the graph of a smooth scalar function.

Example 1.2.1. *Let U open subset of $\mathbb{R}^2$, $f : U \mapsto \mathbb{R}$ an arbitrary smooth function. The graph of f is the set of points of $\mathbb{R}^3$ given by*

$$\mathit{Graph}(f) := \{(x^1, x^2, f(x^1, x^2)) | (x^1, x^2) \in U\} \,,$$

and is a regular surface. In fact, we can check that the conditions in the definition above are satisfied by the map $\phi : U \to \mathbb{R}^3$ given by $\phi(\mathbf{x}) = (x^1, x^2, f(x^1, x^2))$, which is a single local parametrization. Condition i) is satisfied because ϕ is continuous. The inverse of ϕ is simply the restriction of the Graph(f) to the projection on the first two coordinates, which is a continuous function, so item ii) is satisfied aswell. Finally,

$$\mathbf{J}\phi(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x^1}(\mathbf{x}) & \frac{\partial f}{\partial x^2}(\mathbf{x}) \end{bmatrix}$$

has rank maximum rank (2) everywhere, so the third condition is satisfied.

Let us recall also the following definition of *critical* and *regular* points, that will be useful in the next proposition.

Definition 1.5. *Let* $V \subset \mathbb{R}^3$ *be an open set,* $f : V \to \mathbb{R}$ *a* $\mathcal{C}^\infty$ *function. If the differential* $df : \mathbb{R}^3 \to \mathbb{R}$ *is surjective in* $\mathbf{p}$ *then* $\mathbf{p}$ *is a regular point for* f*. On the countrary, if the differential is not surjective we say that* $\mathbf{p}$ *is a critical point for* f*. If* $\mathbf{p}$ *is a critical point,* $f(\mathbf{p})$ *is a critical value, otherwise in an analogous way we say that* $f(\mathbf{p})$ *is a regular value, if* $\mathbf{p}$ *is a regular point.*

Proposition 1.2. *Let* $V \subseteq \mathbb{R}^3$ *be an open set and* f *a smooth function in* V*. If* q *is a regular value of* f*, then every connected component of the level set* $f^{-1}(q) = \{\mathbf{p} \in V | f(\mathbf{p}) = q\}$ *is a regular surface.*

This result is very well known in differential geometry and can be proved using the implicit function theorem. The following proposition says that every regular surface can be seen as the graph of a differentiable function, at least locally.

Proposition 1.3. *If* $\mathcal{S} \subset \mathbb{R}^3$ *is a regular surface and mathbfp* $\in \mathcal{S}$*, then there exists a local parametrization* $\phi : U \to \mathcal{S}$ *in* $\mathbf{p}$ *that takes one of the following forms:*

$$\phi(x^1, x^2) = \begin{cases} (x^1, x^2, f(x^1, x^2)), & or \\ (x^1, f(x^1, x^2), x^2), & or \\ (f(x^1, x^2), x^1, x^2), \end{cases}$$

for a certain function $f \in \mathcal{C}^\infty(U)$*. Local parametrizations allow us also to naturally extend the concepts of continuity and differentiabilty on regular surfaces. In fact, through the parametrization, one can bring back these concepts to the more natural setting* $\mathbb{R}^2$*, as shown in the following definition.*

Definition 1.6. *Let* $\mathcal{S} \subset \mathbb{R}^3$ *be a regular surface with a point* $\mathbf{p} \in \mathcal{S}$*. A function* $f : \mathcal{S} \to \mathbb{R}$ *is of class* $\mathcal{C}^\infty$*, or smooth, at* $\mathbf{p}$ *if there exists a local parametrization* $\phi : U \to \mathcal{S}$ *at* $\mathbf{p}$ *such that* $f \circ \phi : U \to \mathbb{R}$ *is of class* $\mathcal{C}^\infty$ *in a neighborhood of* $\phi^{-1}(\mathbf{p}) \subset \mathbb{R}^2$*.*

The smoothness of a function is a property that does not depend on the local parametrization, thanks to the fact that we assumed the transition maps to be $\mathcal{C}^\infty$, as the following theorem states.

Theorem 1.3. *Let $\mathcal{S}$ be a surface and $\phi : U \to \mathcal{S}$, $\psi : V \to \mathcal{S}$ two local parametrizations, with their intersection $W = \phi(U) \cap \phi(V) \neq \emptyset$. Then, the map $\phi^{-1} \circ \psi\Big|_{\psi^{-1}(W)}$: $\psi^{-1}(W) \to \phi^{-1}(W)$ is a diffeomorphism.*

Consider now the tangent space $T_{\mathbf{p}}\mathcal{S}$ os a surface: it is a 2-dim vector space independent from the parametrization. An important result is

Proposition 1.4. *Let $V \subseteq \mathbb{R}^3$ an open set, and $q \in \mathbb{R}$ a regular value of a function $g \in \mathcal{C}^\infty(V)$. If $\mathcal{S}$ is a connected component of $g^{-1}(q)$ and $\mathbf{p} \in \mathcal{S}$, the tangent plane $T_{\mathbf{p}}\mathcal{S}$ is the subspace of $\mathbb{R}^3$ orthogonal to $\nabla g(\mathbf{p})$.*

Let us consider $\mathbf{p} \in \mathcal{S}$, if $\mathcal{S} = \mathit{Graph}(f)$ with $\phi : U \ni \mathbf{x} \longmapsto \phi(\mathbf{x}) = (x^1, x^2, f(x^1, x^2)) \in \mathcal{S}$ and $\mathbf{p} = \phi(\mathbf{0})$, we can define a level function $g = x^3 - f(x^1, x^2)$ and $\mathcal{S} = g^{-1}(0)$. Basically, $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, g(\mathbf{x}) = 0\}$, with $g : \mathbb{R}^3 \to \mathbb{R}$ defined above. Consider now the set of all curves $\gamma(t) : \mathbb{R} \to \mathbb{R}^3$ such that $\gamma(t) \subseteq \mathbf{S}\ \forall t$ and let $\mathbf{p} = \gamma(t_0)$, with $\mathbf{p}, t_0$ fixed. Since γ is in $\mathbf{S}$, it holds that $g\left(\gamma(t)\right) = 0\ \forall t$, but this implies that $\nabla g(\mathbf{p}) \cdot \dot{\gamma}(t_0) = 0$. Recall that the tangent space $T_{\mathbf{p}}\mathcal{S}$ can be defined as the set of all vectors $\dot{\gamma}(t_0)$ with γ curve passing through $\mathbf{p}$ at $t = t_0$. This means that the gradient vector

$$\nabla g(\mathbf{p}) = \begin{pmatrix} \partial_{1|\mathbf{p}} g \\ \partial_{2|\mathbf{p}} g \\ \partial_{3|\mathbf{p}} g \end{pmatrix}$$

is orthogonal to the tangent plane $T_{\mathbf{p}}\mathcal{S}$. Furthermore, $T_{\mathbf{p}}\mathcal{S}$ can be seen as the (local) linear approximation of the regular surface, by means of a Taylor expansion, i.e., fixed $\mathbf{q} \in \phi(U)$:

$$g(\mathbf{q}) = g(\mathbf{p}) + (\mathbf{q} - \mathbf{p}) \cdot \nabla g(\mathbf{p}) + \mathcal{O}\big((\mathbf{q} - \mathbf{p})^2\big).$$

Note that $\mathbf{p}$ and $\mathbf{q}$ belongs to $\mathcal{S}$, so $0 = g(\mathbf{p}) = g(\mathbf{q})$. Intuitively, the idea is that dividing everything by $\parallel (\mathbf{q} - \mathbf{p}) \parallel$, and taking the limit as $\mathbf{q}$ approaches $\mathbf{p}$, we obtain

$$0 = \lim_{\mathbf{q} \to \mathbf{p}} \frac{(\mathbf{q} - \mathbf{p})}{\parallel (\mathbf{q} - \mathbf{p}) \parallel} \cdot \nabla g(\mathbf{p}),$$

finding once again the orthogonality property presented before. In this sense we will think about the tangent space as the linear approximation of a neighborhood W of $\mathbf{p}$

in $\mathcal{S}$:

$$W = T_{\mathbf{p}}\mathcal{S} + \mathcal{O}\big(diam(W)^2\big).$$

The differential $d\phi_{\mathbf{x}_0}$ establish a relation between subsets of the plane $\mathbb{R}^2$ and the surface. Explicitly the following proposition holds:

Proposition 1.5. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface, $\mathbf{p} \in \mathcal{S}$, and $\phi : U \to \mathcal{S}$ a local parametrization at $\mathbf{p}$ with $\phi(\mathbf{x}_0) = \mathbf{p}$, $\mathbf{x}_0 \in U$. Then, the differential $d\phi_{\mathbf{x}_0}$ is an isomorphism between $\mathbb{R}^2$ and $T_{\mathbf{p}}\mathcal{S}$.*

We recall also that the local parametrization *enduces* a natural basis for the tangent plane. In fact, considering $\{\mathbf{e}_1, \mathbf{e}_2\}$ the canonical basis of $\mathbb{R}^2$, we can define:

$$\partial_i\Big|_{\mathbf{p}} = \frac{\partial}{\partial x^i}\Big|_{\mathbf{p}} = d\phi_0(\mathbf{e}_i) = \frac{\partial \phi}{\partial x^i}(\mathbf{o}) \qquad i = 1, 2\,.$$

Clearly, the set $\{\partial_1|_{\mathbf{p}}, \partial_2|_{\mathbf{p}}\}$ identyfies a basis for the tangent plane $T_{\mathbf{p}}\mathcal{S}$ and thus it is called the basis *induced* by the local parametrization.

We are interested in two ways of describing a (curved) surface:

1) Embedded approach (as an exterior observer). Using the "straight" coordinate system of $\mathbb{R}^3$ every point can be written as a linear combination of the canonical basis vectors $\mathbb{R}^3 \ni \mathbf{p} = x^1\mathbf{e_1} + x^2\mathbf{e_2} + x^3\mathbf{e_3}$. We call this system *Global Coordinate System* (GCS).

2) Intrinsic approach (as an observer "on" the surface). We describe every point using the local curvilinear system in the coordinates $(s^1_{\mathbf{p}}, s^2_{\mathbf{p}})$ that is derived from the local parametrization. Note that the definition of the local curvature system depends on the chosen parametrization: different parametrizations provide different basis.

It is crucial in our analysis to consider intrinsic geometric objects, since their properties are not influenced on the specific reference frame or the space in which they are immersed: for this reason we will focus on the local curvilinear system and in particular on the concept of *metric*. The latter takes its concrete form in the *first fundamental form*, which contains all the infomation needed to practically compute lenghts of tangent vectors to the surface, areas of regions of the surface and so on.

Definition 1.7. *The first fundamental form* $\mathbf{I_p}$ *is the positive definite quadratic form associated with the scalar product:*

$$\mathbf{I_p} : T_\mathbf{p}\mathcal{S} \to \mathbb{R}\,, \qquad \mathbf{I_p}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle_\mathbf{p} \geq 0\,.$$

Scalar product $\langle \cdot, \cdot \rangle_\mathbf{p}$ *information is the same as knowing the first fundamental form* $\mathbf{I_p}$*. In fact, given two vectors* $\mathbf{v}, \mathbf{w}$ *it holds that*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}\big(\langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{w}, \mathbf{w} \rangle\big)$$

which is equivalent of writing

$$\langle \mathbf{v}, \mathbf{w} \rangle_\mathbf{p} = \frac{1}{2}\left[\mathbf{I_p}(\mathbf{v}+\mathbf{w}) - \mathbf{I_p}(\mathbf{v}) - \mathbf{I_p}(\mathbf{w})\right]\,.$$

Consider a local parametrization $\phi : U \to \mathcal{S}$ at a point $\mathbf{p} \in \mathcal{S}$; thus we can use the induced basis of $T_{\mathbf{p}\mathcal{S}}$ to write in coordinates the scalar product as

$$\langle \mathbf{v}, \mathbf{w} \rangle_\mathbf{p} = v^1 w^1 \langle \partial_1, \partial_2 \rangle_\mathbf{p} + (v^1 w^2 + v^2 w^1)\langle \partial_1, \partial_2 \rangle_\mathbf{p} + v^2 w^2 \langle \partial_2, \partial_2 \rangle_\mathbf{p}\,.$$

Definition 1.8. *The metric coefficients of* $\mathcal{S}$ *with respect to* ϕ *are the functions* $E, F, G : U \to \mathbb{R}$ *given by*

$$E(\mathbf{x}) = \langle \partial_1, \partial_2 \rangle\,, \quad F(\mathbf{x}) = \langle \partial_1, \partial_2 \rangle\,, \quad G(\mathbf{x}) = \langle \partial_2, \partial_2 \rangle_\mathbf{p}\,,$$

for all $\mathbf{x} \in U$*.*

These coefficients, in the case of a regular surface, are $\mathcal{C}^\infty$ functions and they contain all the information related to the first fundamental form, in fact $\mathbf{I_p}(\mathbf{v})$ can be written as

$$\begin{aligned}\mathbf{I_p}(\mathbf{v}) &= E(\mathbf{x})(v^1)^2 + 2F(\mathbf{x})v^1 v^2 + G(\mathbf{x})(v^2)^2 = \\ &= \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} E(\mathbf{x}) & F(\mathbf{x}) \\ F(\mathbf{x}) & G(\mathbf{x}) \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \mathbf{v}^T \mathcal{G} \mathbf{v} = \langle \mathbf{v}, \mathbf{v} \rangle_\mathcal{G}\,,\end{aligned}$$

for all $\mathbf{p} = \phi(\mathbf{x}) \in \phi(U)$ and $\mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$. All the important geometric quantities such as lenghts of curves, areas of regions etc. can be computed using the first fundamental form. In fact the knowledge of $I_{\mathbf{p}}(\cdot)$ is equivalent to that of a scalar product operation, as we have seen previously, that induces a norm that we can use to compute distances.

Regular regions. We now introduce the notions of regular region and partition of a region $\mathcal{R} \subseteq \mathcal{S}$, that will be useful also later in the numerical section.

Definition 1.9. *A regular region $R \subseteq \mathcal{S}$ is a connected compact subset of $\mathcal{S}$ obtained as the closure of its interior R° and whose boundary is parametrized by finitely many curvilinear polygons with disjoint supports. If $\mathcal{S}$ is compact, then $R = \mathcal{S}$ is a regular region without boundary.*

Definition 1.10. *Let $R \subseteq \mathcal{S}$ be a regular region of a surface $\mathcal{S}$. A partition of R is a finite family $\mathcal{R} = \{R_1, \dots, R_n\}$ of regular regions contained in R, such that $R = \cup_i R_i$ and the intersection of two regions is contained in their boundaries intersection: $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$, for $i, j = 1, \dots, n$ and $i \neq j$. The diameter $diam R$ of a partition is the maximum of the diameters of the elements of $\mathcal{R}$.*
A pointed partition of R is a pair $(\mathcal{R}, \mathbf{P})$ given by a partition $\mathcal{R}$ of R and a n-tuple $\mathbf{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ of points of R such that $\mathbf{p}_i \in R_i$, for $i = 1, \dots, n$.

Consider a set of points in the region, with the tangent planes associated. The area of a region $\mathcal{R}$, intuitively, can be computed summing over all the infinitesimal areas of the projections on the affine tangent planes of every regular region contained in the region $\mathcal{R}$ itself.

Definition 1.11. *Let $R \subseteq \mathcal{S}$ be a regular partition of a regular surface $\mathcal{S}$ and $(\mathcal{R}, \mathbf{P})$ a pointed partition of R. For all $R_i \in \mathcal{R}$, denote by $\pi_i(R_i)$ the orthogonal projection of R_i on the affine tangent plane $\mathbf{p}_i + T_{\mathbf{p}_i}\mathcal{S}$. The area of the pointed partition is defined as:*

$$Area(\mathcal{R}, \mathbf{P}) = \sum_i Area(\pi_i(R_i))\,.$$

The region R is said to be rectifiable if the limit

$$\mathcal{A}_R = \lim_{diam R \to 0} Area(\mathcal{R}, \mathbf{P})$$

exists and is finite. In this case, the limit is the area of R.

Theorem 1.4. *Let $R \subseteq \mathcal{S}$ be a regular region contained in the image of a local parametrization $\phi : U \to \mathcal{S}$ of a surface $\mathcal{S}$. Then, R is rectifiable and its area is*

$$\mathcal{A}_R = \int_{\phi^{-1}(R)} \sqrt{EG - F^2}\, d\mathbf{x}\,.$$

Let's recall also the following

Lemma 1.1. *Given a local parametrization $\phi : U \to \mathcal{S}$ of a surface $\mathcal{S}$, then:*

$$\| \, \partial_1 \wedge \partial_2 \, \| = \sqrt{EG - F^2}\,,$$

where the simbol wedge $\wedge$ *denotes the vector product in $\mathbb{R}^3$. Moreover, if $\psi : V \to \mathcal{S}$ is another local parametrization with $W = \psi(V) \cap \phi(U) \neq \emptyset$, and if $f = \psi^{-1} \circ \phi_{phi(U)^{-1}}$, then*

$$\partial_1 \wedge \partial_2\big|_{\phi(\mathbf{x})} = \det(\mathbf{J}f)(\mathbf{x})\tilde{\partial}_1 \wedge \tilde{\partial}_2\big|_{\psi\circ f(\mathbf{x})}$$

for all $\mathbf{x} \in \phi^{-1}(W)$, where $\{\tilde{\partial}_1, \tilde{\partial}_2\}$ is the basis induced by ψ.

It is possible to show that this lemma ensures that the integral does not depend on the local parametrization, so that we can define the integral of a function f over a surface R. This definition make use of the local chart ϕ^{-1}, i.e. the inverse of the parametrization ϕ. Indeed, the idea is to transport every calculation to the domain $U \subset \mathbb{R}^2$: in a more general setting, we speak about the *pull-back* and the *push forward.* We have

Definition 1.12. *Let $R \subseteq \mathcal{S}$ be a regular region contained in the image of a local parametrization $\phi : U \to \mathcal{S}$ of a regular surface $\mathcal{S}$, and $f : R \to \mathbb{R}$ a continuous function. The integral of f on R is given by*

$$\int_R f = \int_{\phi^{-1}(R)} (f \circ \phi)\sqrt{EG - F^2}\, d\mathbf{x}\,.$$

Without entering too much in details, let's just recall the Stokes theorem for differential forms:

Theorem 1.5. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface with smooth boundary $\partial\mathcal{S}$ and w a $1-$ differential form with compact support on $\mathcal{S}$. Then:*

$$\int_{\partial\mathcal{S}} w = \int_{\mathcal{S}} dw \,.$$

1.2.2 PDEs on surfaces

Given a local parametrization $\phi : U \to \mathcal{S}$ of $\mathcal{S}$ centered at $\mathbf{p}$, with local coordinates $(s^1_{\mathbf{p}}, s^2_{\mathbf{p}})$, and the induced reference basis vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, $\in T_{\mathbf{p}}\mathcal{S}$. We can define the associated metric $\mathcal{G}_{\mathcal{S}}$, which corresponds to the first fundamental form of $\mathcal{S}$ at $\mathbf{p}$:

$$\mathcal{G}_{\mathcal{S}} := \begin{pmatrix} \langle \mathbf{t}_1, \mathbf{t}_1 \rangle & \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \\ \langle \mathbf{t}_2, \mathbf{t}_1 \rangle & \langle \mathbf{t}_2, \mathbf{t}_2 \rangle \end{pmatrix} .$$

Note that we use the subscript to indicate that the first fundamental form is related to the surface $\mathcal{S}$ and to distinguish it from the 3×3 metric tensor that we will define in the next paragraphs.
What we need is to fix a three-dimensional local system of curvilinear coordinates spanning a neighborhood $\mathcal{N}_{\mathbf{p}} \subset \mathbb{R}^3$ of a point $\mathbf{p}$ belonging to the surface. The notion of *tubular neighorhood* $\mathcal{N}_{\mathbf{p}}$ is introduced through the next proposition with the aim of finding a proper region in which our (LCS) will live.

Proposition 1.6. *Let $\mathcal{S}$ be a regular surface and $\phi : U \to \mathcal{S}$ a local parametrization centered at $\mathbf{p} \in \mathcal{S}$. Then there exists a neighborhood $W \subset \phi(U)$ of $\mathbf{p} \in \mathcal{S}$ and a number $\epsilon > 0$ such that the segments of the normal lines passing through points $\mathbf{p} \in W$, centered at $\mathbf{q}$ and with length 2ϵ, are disjoint.*

The *tubular neighborhood* $\mathcal{N}_{\mathbf{p}}$ of W is just the union of all the segments with lenghts 2ϵ of the normal lines passing through points $\mathbf{q} \in W$. All points in the local neighborhood $\mathcal{N}_{\mathbf{p}}$ can be described using a 3-dim reference frame, called the *Local Curvilinear System (LCS)* formed by the local basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ of the tangent plane $T_{\mathbf{p}}\mathcal{S}$ extended with a vector $\mathbf{t}_3$. The coordinates associated to this reference frame will be denoted by (s^1, s^2, s^3). A result allows us to say that, for every $\mathbf{p} \in \mathcal{N}_{\mathbf{p}}$, the 3-dim coordinate transformation $\mathbf{\Phi}_{\mathbf{p}} : \mathbb{R}^3 \ni \mathbf{x}_{\mathbf{p}} \longmapsto \mathbf{s}_{\mathbf{p}} \in \mathbb{R}^3$ that goes from the GCS to the LCS is a diffeomorphism, when restricted in the tubular neighborhood of $\mathbf{p}$.

In the following sections, we could be more specific making the distinction between physical and contravariant components.

Remark. Every point $\mathbf{q} \in \mathcal{N}_{\mathbf{p}}$ can be expressed in the LCS in the following way. Consider the line passing through q parallel to $\mathbf{t}_3$, in parametric form in the GCS. Then, if we indicate with $r = \gamma(\bar{\lambda})$ the intersection $\gamma \cap \mathcal{S}$, the local coordinates of $\mathbf{q}$ will result

$$(s^1(\mathbf{q}),\, s^2(\mathbf{q}),\, s^3(\mathbf{q})) = (x^1(r),\, x^2(r),\, \bar{\lambda}).$$

One of the main tools in our further theoretical development is the following proposition.

Proposition 1.7. *Let (s^1, s^2) be the curvilinear coordinates on $\mathcal{S}$ and $\mathcal{G}_{\mathcal{S}}$ the associated metric tensor. Let $f : \mathcal{S} \longrightarrow \mathbb{R}$ a scalar differentiable function on $\mathcal{S}$, $X : \mathcal{S} \longrightarrow \mathbb{R}^2$ a vector field on $\mathcal{S}$ and $\mathbb{T} : \Omega \longrightarrow \mathbb{R}^{3\times 3}$ a rank-2 contravariant tensor given by $\mathbb{T} = \{\tau^{ij}\}$. Then, the intrinsic differential operators on $\mathcal{S}$ expressed in the local curvilinear coordinate system are given by the following expressions:*

- *the intrinsic gradient of f is:*

$$\nabla_{\mathcal{G}} f = {\mathcal{G}_{\mathcal{S}}}^{-1} \nabla f = g^{ij} \frac{\partial f}{\partial s^i}; \tag{1.7}$$

- *the intrinsic divergence of X is:*

$$\nabla_{\mathcal{G}} \cdot X = \frac{1}{\sqrt{det(\mathcal{G}_{\mathcal{S}})}} \nabla \cdot \left(\sqrt{det(\mathcal{G}_{\mathcal{S}})} X \right) = \frac{\partial X^i}{\partial s^i} + \Gamma^i_{ij} X^j, \tag{1.8}$$

- *the j-th component of the divergence of $\mathbb{T}$ is:*

$$(\nabla_{\mathcal{G}} \cdot \mathbb{T})^j = \nabla_{\mathcal{G}_i} \tau^{ij} = \frac{1}{\sqrt{det(\mathcal{G}_{\mathcal{S}})}} \frac{\partial}{\partial s^i} \left(\sqrt{det(\mathcal{G}_{\mathcal{S}})} \tau^{ij} \right) + \Gamma^j_{ik} \tau^{ik}, \tag{1.9}$$

where $\nabla_{\mathcal{G}} \cdot \tau^{(\cdot j)}$ identifies the divergence of the j-th column of $\mathbb{T}$, and Γ^k_{ij} denote the Christoffel symbols;

- *the intrinsic Laplace-Beltrami operator of f is:*

$$\Delta_{\mathcal{G}} f = \nabla_{\mathcal{G}} \cdot \nabla_{\mathcal{G}} f = \frac{1}{\sqrt{det(\mathcal{G}_{\mathcal{S}})}} \frac{\partial}{\partial s^i} \left(\sqrt{det(\mathcal{G}_{\mathcal{S}})} g^{ij} \frac{\partial f}{\partial s^j} \right). \tag{1.10}$$

We recall here two important classical results involved in the study of PDEs, i.e. the *divergence theorem* and the well known *Green's formula*, stated in intrinsic form. This is done using the definitions of the intrinsic differential operators of proposition 1.7.

Lemma 1.2. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface with smooth boundary $\partial\mathcal{S}$ and X be continuously differentiable vector field. Then:*

$$\int_{\mathcal{S}} \nabla_{\mathcal{G}} \cdot X \, d\mathbf{s} = \int_{\partial\mathcal{S}} \langle X, \mu \rangle \, d\sigma$$

where $\mu : \mathcal{S} \to \mathbb{R}^2$ denotes the vector tangent to $\mathcal{S}$ and normal to $\partial\mathcal{S}$ with components written with respect to the local reference frame (i.e. $\mu = \mu^1\partial_1 + \mu^2\partial_2$), and $d\mathbf{s}$ and $d\sigma$ are the surface area measure and the curve length measure, respectively.

Another familiar result is the so called *Green's formula*, that reads:

Lemma 1.3. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface with smooth boundary $\partial\mathcal{S}$ and $f, g \in \mathcal{C}^2(\bar{\mathcal{S}})$ be continuously differentiable functions over $\bar{\mathcal{S}}$. Then it holds:*

$$\int_{\mathcal{S}} \langle \nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} g \rangle_{\mathcal{G}} \, d\mathbf{s} = -\int_{\mathcal{S}} \Delta_{\mathcal{G}} f \, g \, d\mathbf{s} + \int_{\partial\mathcal{S}} \langle \nabla_{\mathcal{G}} f, \mu \rangle_{\mathcal{G}} g \, d\sigma$$

where $\mu : \mathcal{S} \to \mathbb{R}^2$ denotes the vector tangent to $\mathcal{S}$ and normal to $\partial\mathcal{S}$ with components written with respect to the local reference frame, and $d\mathbf{s}$ and $d\sigma$ are the surface area measure and the curve length measure, respectively.

Chapter 2

Shallow water equations

Here starts the procedure for the derivation of the Shallow water system: we re-write the Incompressible Navier-Stokes equations in the LCS, we describe the free top surface and the bottom in terms of zero's of functions and derive the Kinematic Boundary conditions. Then we integrate along the normal direction all the equations, starting from the Continuity equation and the Momentum equations. Moreover, with the information given by the integrated equations and some algebraic manipulation, we find the hydrostatic pressure condition that is useful to reduce the dimension of the system from 4 equations to 3, yielding to te final Intrinsic Shallow Water system.

2.1 Intrinsic shallow water equations

Shallow Water models are 2-dim models for fluid dynamics, characterized by a strong simplification of the Navier-Stokes equations after a process of depth average along a specific direction. This decreases the complexity and the numerical cost of three dimentional models at large scales. The main assumptions governing Shallow Water models is that the fluid waves have an amplitude which is negligible with respect to wave lenght. Many physical phaenomena can be studied using this type of tecnique, such as meteorologic, atmospheric, or oceanografic ones, aswell as avalanches, debris fows, landslides and others. In all these application one must take into account a general topography, such as mountain landscapes or, in our case, the bottom of a river. A rigorous investigation is required to derive the equations that take into account the geometric setting.

A new geometrically intrinsic formulation of the SWE will be derived here on general topography, called Intrinsic Shallow Water Equations (ISWE).

2.1.1 Incompressible Navier-Stokes equations

Consider an open domain $\Omega \subset \mathbb{R}^3$, the Navier-Stokes system reads

$$\begin{aligned} &\nabla \cdot \vec{u} = 0 \\ &\frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\vec{u} \otimes \vec{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \mathbb{T} + \vec{g}, \end{aligned} \tag{2.1}$$

where $\vec{u} : \Omega \times [0, t_f] \to \mathbb{R}^3$ is the fluid velocity, ρ its density assumed constant, $p : \Omega \times [0, t_f] \to \mathbb{R}$ is the fluid pressure, $\mathbb{T} : \Omega \to \mathbb{R}^{3\times 3}$ the deviatoric stress tensor, and $\vec{g}$ the gravity acceleration. The relation $\nabla \cdot \vec{u} \otimes \vec{u} = \vec{u} \cdot \nabla \vec{u} + \vec{u} \nabla \cdot \vec{u}$ holds. For clarity, we explicit (2.1) in a general coordinate system y^1, y^2, y^3 by writing:

$$\begin{pmatrix} \partial_{y^1} \\ \partial_{y^2} \\ \partial_{y^3} \end{pmatrix} \cdot \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = 0\,, \tag{2.2}$$

while the following holds for $i = 1, 2, 3$

$$\frac{\partial u^i}{\partial t} + \begin{pmatrix} \partial_{y^1} \\ \partial_{y^2} \\ \partial_{y^3} \end{pmatrix} \cdot \begin{pmatrix} u^1 u^i \\ u^2 u^i \\ u^3 u^i \end{pmatrix} = -\frac{1}{\rho}\partial_{y^i} p + \frac{1}{\rho} \begin{pmatrix} \partial_{y^1} \\ \partial_{y^2} \\ \partial_{y^3} \end{pmatrix} \cdot \begin{pmatrix} \tau_{1i} \\ \tau_{2i} \\ \tau_{3i} \end{pmatrix} + g^i\,. \tag{2.3}$$

Our hyphothesis is that the boundary $\partial\Omega$ is smooth and it is given by the union of the bottom surface, the free surface (the "top") and the lateral surfaces: $\partial\Omega = \mathcal{S}_{\mathcal{B}} \cup \mathcal{S}_{\mathcal{F}} \cup \mathcal{S}_{\mathcal{L}}$. Smoothness assumption is exploited to simplify the analysis of the problem and identify the surfaces introduced above as graphs of some functions. We describe the bottom surface as the graph of $\mathcal{B} : U \times \mathbb{R} \to \mathbb{R}$, $U \subset \mathbb{R}^2$ open and with respect to a global cartesian coordinate system x^1, x^2, x^3 (GCS), with x^3 aligned with

$\vec{g}$ (but having opposite sign), we write

$$\mathcal{S}_{\mathcal{B}} := \left\{(x^1,\, x^2,\, x^3,\, t) \in \mathbb{R}^3 \times \mathbb{R} \;\; such\ that\ x^3 = \mathcal{B}(x^1,\, x^2)\right\}.$$

We can write $\mathcal{S}_{\mathcal{B}} := F_{\mathcal{B}}^{-1}(0)$, where $F_{\mathcal{B}}(x^1, x^2, x^3) := x^3 - \mathcal{B}(x^1, x^2)$. The fluid free surface $\mathcal{S}_{\hat{\mathcal{F}}}$ can be defined using the function $\hat{\mathcal{F}} : U \times [0, t_f] \longrightarrow \mathbb{R}$ in the same way. Then we build the (LCS) requiring the two conditions:

- the first two coordinates run along the bottom surface $\mathcal{S}_{\mathcal{B}}$, their tangent vectors belong at each point $\mathbf{p} \in \mathcal{S}_{\mathcal{B}}$ to the tangent plane $T_{\mathbf{p}}\mathcal{S}_{\mathcal{B}}$;
- the third coordinate crosses the surface orthogonally so a vector tangent to $\mathcal{S}_{\mathcal{B}}$ is everywhere orthogonal to $\mathbf{N}$, the surface normal vector.

The construction of the induced reference frame, which is not orthonormal, is done via Gram-Schmidt orthogonalization of the Monge parametrization. Neglecting the normal direction, the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ form the final orthogonal coordinate system. In practice, we have that

$$\hat{\mathbf{t}}_i(\mathbf{p}) = d\Phi_{\mathbf{p}}(\mathbf{e}_i) = \left(\frac{\partial x^1}{\partial s^i}, \frac{\partial x^2}{\partial s^i}, \frac{\partial x^3}{\partial s^i}\right), \quad i = 1, 2, \tag{2.4}$$

where $d\Phi_{\mathbf{p}}$ is the Jacobian matrix of the coordinate transformation. Then we obtain $\mathbf{t}_1$ and $\mathbf{t}_2$, the orthogonal frame on $T_{\mathbf{p}}\mathcal{S}_{\mathcal{B}}$, with Gram-Schmidt, while the last vector $\mathbf{t}_3$ is chosen to be orthogonal to the previous two and unitary, i.e. $\| \mathbf{t}_3(\mathbf{p}) \| = 1$.

Remark. Vectors $\mathbf{t}_1$ and $\mathbf{t}_1$ cannot be normalized, since they carry an important information: the curvature of $\mathcal{S}_{\mathcal{B}}$. A normalization would imply zero curvature.

The associated metric tensor is accordingly the diagonal matrix

$$\mathcal{G} := \begin{pmatrix} \| \mathbf{t}_1(\mathbf{p}) \|^2 & 0 & 0 \\ 0 & \| \mathbf{t}_2(\mathbf{p}) \|^2 & 0 \\ 0 & 0 & \| \mathbf{t}_3(\mathbf{p}) \|^2 \end{pmatrix} = \begin{pmatrix} h_{(1)}^2 & 0 & 0 \\ 0 & h_{(2)}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

The last step before the derivation of the new (ISWE) formulation consists in the geometric definition of the differental operators that appear in the Navier-Stokes equations: the gradient of a scalar function $\nabla_{\mathcal{G}} f$, the divergence of a vector field, and the divergence of a tensor field. Now that we have defined the metric tensor, recalling proposition (1.7), we have:

- the gradient of a scalar function f is:

$$\nabla_{\mathcal{G}} f = \left(\frac{1}{h_{(1)}^2} \frac{\partial f}{\partial s^1}, \frac{1}{h_{(2)}^2} \frac{\partial f}{\partial s^2}, \frac{\partial f}{\partial s^3} \right); \tag{2.6}$$

- the divergence of a contravariant vector field $\vec{u} = u^1 \mathbf{t}_1 + u^2 \mathbf{t}_2 + u^3 \mathbf{t}_3$ is:

$$\nabla_{\mathcal{G}} \cdot \vec{u} = \frac{1}{h_{(1)} h_{(2)}} \left(\frac{\partial (h_{(1)} h_{(2)} u^1)}{\partial s^1} + \frac{\partial (h_{(1)} h_{(2)} u^2)}{\partial s^2} + \frac{\partial (h_{(1)} h_{(2)} u^3)}{\partial s^3} \right); \tag{2.7}$$

- the j-th component of the divergence of a 3×3 rank-2 contravariant tensor $\mathbb{T} = \{\tau^{ij}\}$ is:

$$\begin{aligned}(\nabla_{\mathcal{G}} \cdot \mathbb{T})^j = \nabla_{\mathcal{G}} \cdot \tau^{(\cdot j)} &+ \frac{1}{h_{(j)}} \left(2\tau^{1j} \frac{\partial h_{(j)}}{\partial s^1} - \tau^{11} \frac{h_{(1)}}{h_{(j)}} \frac{\partial h_{(1)}}{\partial s^j} \right) + \\ &+ \frac{1}{h_{(j)}} \left(2\tau^{2j} \frac{\partial h_{(j)}}{\partial s^2} - \tau^{22} \frac{h_{(2)}}{h_{(j)}} \frac{\partial h_{(2)}}{\partial s^j} \right)\end{aligned} \tag{2.8}$$

where $\nabla_{\mathcal{G}} \cdot \tau^{(\cdot j)}$ identifies the divergence of the j-th column of $\mathbb{T}$. Since in some cases we will be interested only on what happens on the bottom surface, we will reduce our system to a 2-dim local system. For this reason, we indicate with $\mathcal{G}_{\mathcal{S}_{\mathcal{B}}}$ the reduced metric tensor, and the same notation will apply to all the operators. Moreover, note that in the following we will use Einstein summation convenction.

2.1.2 Curvilinear Navier-Stokes equations.

System (2.1) can be written using LCS as:

$$\nabla_{\mathcal{G}} \cdot \vec{u} = 0 \tag{2.9}$$

$$\frac{\partial \vec{u}}{\partial t} + \nabla_{\mathcal{G}} \cdot (\vec{u} \otimes \vec{u}) = -\frac{1}{\rho} \nabla_{\mathcal{G}}\, p + \frac{1}{\rho} \nabla_{\mathcal{G}} \cdot \mathbb{T} + \vec{g}. \tag{2.10}$$

Observe that, for example, in the GCS the representation of the vector field $\vec{g}$ evaluated in an arbitrary point of $\mathbb{R}^3$ is $(0, 0, -g)^T$, i.e. in components we can write $\vec{g}\Big|_{\mathbf{P}} =$

$0\partial x^1\big|_{\mathbf{P}} + 0\partial x^2\big|_{\mathbf{P}} - g\partial x^3\big|_{\mathbf{P}}$. In order to write this in the LCS, we only need to apply the chain rule:

$$-g\left(\frac{\partial x^3}{\partial s^1}\partial s^1 + \frac{\partial x^3}{\partial s^2}\partial s^2 + \frac{\partial x^3}{\partial s^3}\partial s^3\right) .$$

We can now explicit the same quantity in terms of the differential of ϕ, which in our setting is represented by the jacobian of ϕ. Note that vector $\vec{g}$ is contravariant: coefficients have upper indeces, and for this fact, we have to left multiply the vector to the Jacobian (or, equivalently, right multiply the vector with the transposed Jacobian):

$$d\phi(\vec{g}) = (\vec{g})^T J_\phi = [0,0,-g]\begin{bmatrix} \frac{\partial x^1}{\partial s^1} & \frac{\partial x^1}{\partial s^2} & \frac{\partial x^1}{\partial s^3} \\ \frac{\partial x^2}{\partial s^1} & \frac{\partial x^2}{\partial s^2} & \frac{\partial x^2}{\partial s^3} \\ \frac{\partial x^3}{\partial s^1} & \frac{\partial x^3}{\partial s^2} & \frac{\partial x^3}{\partial s^3} \end{bmatrix} = -g\left(\frac{\partial x^3}{\partial s^1}, \frac{\partial x^3}{\partial s^2}, \frac{\partial x^3}{\partial s^3}\right) =$$

$$= -g\left(\frac{\partial x^3}{\partial s^1}ds^1 + \frac{\partial x^3}{\partial s^2}ds^2 + \frac{\partial x^3}{\partial s^3}ds^3\right) .$$

We have just found a covector, i.e. the expression of the differential of ϕ. At this point we have to *raise* the indeces with $\mathcal{G}^{-1}$, the inverse of the metric, using the isomorphism $TM \simeq T^*M$. Recalling that $\mathcal{G}^{-1} = g^{ij}\partial_i \otimes \partial_j$, where $\partial_i = \frac{\partial}{\partial x^i}$, we obtain $\vec{g}_{LCS} = -g\nabla_{\mathcal{G}} x^3$, in fact:

$$\left[g^{ij}\partial_i \otimes \partial_j\right](-g)\frac{\partial x^3}{\partial s^k}ds^k \otimes \partial_j(\cdot) = (-g)g^{ij}\delta_i^k\frac{\partial x^3}{\partial s^k}\partial_j = (-g)g^{ij}\frac{\partial x^3}{\partial s^i}\partial_j = -g\nabla_{\mathcal{G}} x^3 ,$$

when in the last step we employed that the matrix associated to $\mathcal{G}$ is diagonal:

$$(-g)g^{ij}\frac{\partial x^3}{\partial s^i}\partial_j = \delta_j^i(-g)g^{ij}\frac{\partial x^3}{\partial s^i}\partial_j = (-g)g^{ii}\frac{\partial x^3}{\partial s^i}\partial_i .$$

From an algebraic point of view, we built a linear transformation $l : GCS \longrightarrow LCS$ that sends the canonical basis $\mathcal{E}_{\mathbb{R}^3} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the local basis $\mathcal{S}_{\mathbb{R}^3} = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$. This linear transformation is exactly $d\Phi$ with the associated matrix $\alpha_{\mathcal{S}\mathcal{E}}(l) \equiv J_\Phi$. Recalling the expression (2.4), we can see that for example in the first column of matrix J_Φ we find exactly the coordinates of $d\Phi(\mathbf{e}_1) \equiv l(\mathbf{e}_1)$ with respect to the arrival basis of $\mathcal{S}$.

Surfaces description. Through the eyes of an observer in the Local Coordinate

System (LCS), bottom and free surfaces can be written:

$$\mathcal{S}_{\mathcal{B}} := \left\{(s^1, s^2, s^3) \in \mathbb{R}^3 \textit{ such that } s^3 = \mathcal{B}(s^1, s^2) \equiv 0\right\},$$
$$\mathcal{S}_{\mathcal{F}} := \left\{(s^1, s^2, s^3, t) \in \mathbb{R}^3 \times [0, t_f] \textit{ such that } s^3 = \mathcal{F}(s^1, s^2, t) \equiv \eta(s^1, s^2, t)\right\},$$

where $\eta(s^1, s^2, t) = \mathcal{F}(s^1, s^2, t) - \mathcal{B}(s^1, s^2)$ denotes the fluid depth along direction s^3. Our hypothesis are that

- the bottom is impermeable and not eroding, i.e. it is a fixed, time invariant, surface;
- the fluid surface is a function of time.

Recall now that we can express regular surfaces as the graph of some function: in our case, the bottom is $F_{\mathcal{B}} = s^3 - \mathcal{B}(s^1, s^2)$ and the free surface $F_{\mathcal{F}} = s^3 - \mathcal{F}(s^1, s^2, t)$. Observe that the function $F_{\mathcal{F}} = F_{\mathcal{F}}(\mathbf{s}(t), t)$ depends directly on time, while $F_{\mathcal{B}} = F_{\mathcal{B}}(\mathbf{s}(t))$ does not.

The so called *Kinematic Boundary Conditions* hold:

$$\frac{dF_{\mathcal{M}}}{dt} = \frac{\partial F_{\mathcal{M}}}{\partial t} + \vec{u} \cdot \nabla_{\mathcal{G}} F_{\mathcal{M}} \bigg|_{\mathcal{M}} = 0,$$

where $\mathcal{M} = \mathcal{B}$ or $\mathcal{F}$.

Remark. *Here the scalar product is meant in terms of the metric $\mathcal{G}$, i.e. $\cdot \equiv \cdot_{\mathcal{G}}$.*

Focusing on the bottom $\mathcal{B}$ and on the free surface $\mathcal{F}$, and recalling the expression of the intrinsic gradient of a scalar function (eq.(1.7)) we obtain:

$$\frac{dF_{\mathcal{B}}}{dt} = \vec{u}\bigg|_{\mathcal{B}} \cdot \nabla_{\mathcal{G}} F_{\mathcal{B}} = \vec{u} \cdot \nabla_{\mathcal{G}} F_{\mathcal{B}} \bigg|_{s^3=0} = 0, \qquad (2.11)$$

$$\frac{dF_{\mathcal{F}}}{dt} = -\frac{\partial \eta}{\partial t} + \vec{u} \cdot \nabla_{\mathcal{G}} F_{\eta} \bigg|_{s^3=\eta} = -\frac{\partial \eta}{\partial t} - \left(u^1 \frac{\partial \eta}{\partial s^1} + u^2 \frac{\partial \eta}{\partial s^2} - u^3\right) \bigg|_{s^3=\eta} = 0. \qquad (2.12)$$

Observe that $\nabla_{\mathcal{G}} F_{\eta} \Big|_{s^3=\eta} = -\nabla \mathcal{F} \Big|_{s^3=\eta}$. Furthermore, we will assume that the external actions on the fluid surface are negligible. This implies that at the fluid-air interface we are imposing a zero-stress boundary condition:

$$\mathbb{T}_{\mathcal{F}} \cdot \mathbf{N}_{\mathcal{F}} = 0, \quad \mathbf{N}_{\mathcal{F}} = \frac{\nabla \mathcal{F}}{\| \nabla \mathcal{F} \|}, \tag{2.13}$$

where $\mathbf{N}_{\mathcal{F}}$ is the unit normal vector on the free surface $\mathcal{F}$. On the other hand, the bed boundary condition reads

$$\mathbb{T}_{\mathcal{B}} \cdot \mathbf{N}_{\mathcal{B}} = \mathbf{f}_{\mathcal{B}} = \tau_b^1 \mathbf{t}_1 + \tau_b^2 \mathbf{t}_2 + p_{\mathcal{B}} \mathbf{t}_3, \tag{2.14}$$

where $p_{\mathcal{B}}$ indicates the bottom pressure.

Depth integration. We finally employ the principal idea of Shallow Water models, performing a depth integration along s^3 direction locally normal to the bottom surface, spanning a region between the terrain and the free surfaces: $s^3 \in [0,\, \eta(s^1, s^2, t)] \equiv [\mathcal{B}(s^1, s^2),\, \mathcal{F}(s^1, s^2, t)]$. We assume η small enough to be in the region where ϕ is invertible, i.e. in the previously defined tubular neighborhood. The first equation of the system of curvilinear Navier-Stokes equations, eq. (2.9), after depth integration along s^3, yields to the continuity equation, while the second eq. (2.10) yields to the momentum equation. These two, coupled together form the *normally integrated Navier-Stokes equations*:

$$\frac{\partial \eta}{\partial t} + \nabla_{\mathcal{G}} \cdot \int_0^\eta \underline{\vec{u}} = 0\,, \qquad (1 - dim) \tag{2.15}$$

$$\frac{\partial}{\partial t} \int_0^\eta \vec{u} + \nabla_{\mathcal{G}} \cdot \int_0^\eta \vec{u} \otimes \vec{u} = -\frac{1}{\rho} \int_0^\eta \nabla_{\mathcal{G}} p - g \int_0^\eta \nabla_{\mathcal{G}} x^3 + \frac{1}{\rho} \nabla_{\mathcal{G}} \cdot \int_0^\eta \mathbb{T} + \frac{1}{\rho} \mathbb{T}_{\mathcal{B}} \cdot \mathbf{N}_{\mathcal{B}} \tag{2.16}$$

where $\underline{\vec{u}} := [u^1, u^2]^T$ and the curvilinear divergence operator $\nabla_{\mathcal{G}} \cdot$ is adapted to the two-dimensional setting.

Considerations on length scales. It can be shown that the classical SW hypothesis (fluid depth smaller than the characteristic wavelength) is equivalent to assume a small normal velocity. Exploiting eq. (2.9) and using the regularity of the bottom surface, we have that:

$$W_0 \sim \max\left\{\epsilon, H_0\frac{\partial h_{(1)}}{\partial s^1}, H_0\frac{\partial h_{(2)}}{\partial s^1}, H_0\frac{\partial h_{(1)}}{\partial s^2}, H_0\frac{\partial h_{(2)}}{\partial s^2}\right\} V_0 = \epsilon_{\mathcal{G}} V_0\,,$$

where W_0 is the scaling of the s^3- velocity, i.e. $u^3 \sim W_0$, $\epsilon = H_0/L_0 \ll 1$ is the ratio between the depth and the lenght (this corresponds to the classical SW assumption). Note that we assume to have no information on the order of magnitude of the derivatives of these metric coefficients. This result allows us to define a *geometric* aspect ratio $\epsilon_{\mathcal{G}}$ that depends on the global length scale parameter ϵ as well as the information on the local curvatures, given by the derivatives of the metric coeffients. Hence, we can see that the latter ones are of the order of $1/L_0$, with the assumption $\epsilon_{\mathcal{G}} \ll 1$.

SW Approximation. We apply now the SW approximation to reduce the system. We consider $u^3 = \epsilon_{\mathcal{G}} u^i$, $i = 1, 2$, $\epsilon_{\mathcal{G}} \ll 1$, then also that the component of $\vec{u}(\mathbf{s}, t)$ and the stress tensor $\mathbb{T}(\mathbf{s}, t)$ can be expanded in the following way:

$$u^i = u^i_{(0)} + \epsilon_{\mathcal{G}} u^i_{(1)} + \epsilon^2_{\mathcal{G}} u^i_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right) \qquad i = 1, 2\,, \tag{2.17}$$

$$u^3 = \epsilon_{\mathcal{G}} u^3_{(1)} + \epsilon^2_{G} u^3_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right)\,, \tag{2.18}$$

$$\tau^{ij} = \tau^{ij}_0 + \epsilon_{\mathcal{G}} \tau^{ij}_{(1)} + \epsilon^2_{\mathcal{G}} \tau^{ij}_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right) \qquad i, j = 1, 2, 3\,. \tag{2.19}$$

Note: these assumptions will be totally analogous to the one of the next chapter: we write them also here for clearness.

Neglecting the details, we are finally able to state the following theorem.

Theorem 2.1. *The intrinsic shallow water equations, written with respect to the LCS are given by:*

$$\frac{\partial \eta}{\partial t} + \nabla_{\mathcal{G}} \cdot \vec{q} = 0\,, \tag{2.20}$$

$$\begin{aligned}\frac{\partial \vec{q}}{\partial t} + \nabla_{\mathcal{G}} \cdot \left(\frac{1}{\eta}\left(\vec{q} \otimes \vec{q}\right) + \left(\frac{g\eta^2}{2}\frac{\partial x^3}{\partial s^3}\right)\mathcal{G}^{-1}_{sw}\right) \\ + \frac{g\eta^2}{2}\nabla_{\mathcal{G}}\left(\frac{\partial x^3}{\partial s^3}\right) + g\eta\nabla_{\mathcal{G}}(x^3) - \frac{1}{\rho}\left(\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw} + \mathbf{f}_{\mathcal{B}}\right) = 0.\end{aligned} \tag{2.21}$$

They provide an approximation of order $\mathcal{O}\left(\epsilon^2_{\mathcal{G}}\right)$ of the Navier-Stokes equations, under *the assumption of thin fluid layer,* $\eta = \mathcal{O}(\epsilon_{\mathcal{G}})$.

Notation. $\left(\vec{U}(s^1, s^2, t), \mathbf{T}\right)$ are the depth-averaged velocity vector and tensor $\mathbb{T}$: we can also split the velocity vector and the stress tensor

$$\vec{u} = \vec{U} + \tilde{u}, \;\; \textit{where}\; \vec{U}(s^1, s^2, t) = \frac{1}{\eta}\int_0^\eta \vec{u}(\mathbf{s}, t)\, ds^3\,, \quad \int_0^\eta \tilde{u}(\mathbf{s}, t)\, ds^3 = 0\,, \tag{2.22}$$

$$\mathbb{T} = \mathbf{T} + \tilde{\tau}, \;\; \textit{where}\; \mathbf{T}(s^1, s^2, t) = \frac{1}{\eta}\int_0^\eta \mathbb{T}(\mathbf{s}, t)\, ds^3\,, \quad \int_0^\eta \tilde{\tau}(\mathbf{s}, t)\, ds^3 = 0\,. \tag{2.23}$$

Moreover $\vec{q} := [\eta U^1, \eta U^2]$ will be the vector containing our main unknowns. The tensor $\mathbf{T}_{sw}$ is the principal 2-minors of $\mathbf{T}$. Vector $\mathbf{f}_{\mathcal{B}} = [\tau_b^1, \tau_b^2]^T$ contains bed friction information.

Main steps of the proof of Theorem (2.1). Let us recall momentum equation, seen previously:

$$\frac{\partial}{\partial t}\int_0^\eta \vec{u} + \nabla_{\mathcal{G}} \cdot \int_0^\eta \vec{u} \otimes \vec{u} = -\frac{1}{\rho}\int_0^\eta \nabla_{\mathcal{G}} p - g\int_0^\eta \nabla_{\mathcal{G}} x^3 + \frac{1}{\rho}\nabla_{\mathcal{G}} \cdot \int_0^\eta \mathbb{T} + \frac{1}{\rho}\mathbb{T}_{\mathcal{B}} \cdot \mathbf{N}_{\mathcal{B}} \tag{2.18}$$

Here we choose to skip writing all the details: the computations are very similar to the ones that we will find in the following chapter and can be found in [4].

Integrating the third component of the momentum equation and employing some approximations, one reaches the following

$$\frac{1}{\rho}\int_0^\eta \frac{\partial p}{\partial s^3} + g\int_0^\eta \frac{\partial x^3}{\partial s^3} = \mathcal{O}(\epsilon_{\mathcal{G}})\;.$$

This expression can be further manipulated as follows. Observe that we are neglecting the effects of surface tension and wind on the free surface, for this reason we can set $p\Big|_{s^3=\eta} = 0$, to find by direct integration:

$$p\Big|_0 = \rho g \eta \frac{\partial x^3}{\partial s^3} + \mathcal{O}(\epsilon_{\mathcal{G}})\,. \tag{2.24}$$

Notice also that $\partial x^3/\partial s^3$ is constant in s^3, since the direction s^3 is assumed rectilinear. We would like to remark that $\eta\partial x^3/\partial s^3$, evaluated at a point $\mathbf{P} \in \mathcal{S}_{\mathcal{B}}$ is exactly the *vertical* height of the water measured above point $\mathbf{P}$ itself (see [8]). The expression

(2.24) tells us that, in a first order approximation, the fluid pressure varies linearly along the s^3 direction. Furthermore, neglecting terms of order $(\epsilon_{\mathcal{G}})$, here we can find the hydrostatic pressure condition applied to the bottom of the surface, along its normal direction: the pressure on a point in $\mathcal{S}_{\mathcal{B}}$ depends only on the weight of the water above it. Now the idea is to simplify also the s^1 and s^2 components of the Momentum's equation with our approximations and use the pressure condition.

In the end we arrive at a form of the Momentum equation intrinsic to the bottom surface:

$$\frac{\partial \vec{q}}{\partial t} + \nabla_{\mathcal{G}} \cdot \left(\frac{1}{\eta} (\vec{q} \otimes \vec{q}) + \left(\frac{g\eta^2}{2} \frac{\partial x^3}{\partial s^3} \right) \mathcal{G}_{sw}^{-1} \right) + \frac{g\eta^2}{2} \nabla_{\mathcal{G}} \left(\frac{\partial x^3}{\partial s^3} \right) + g\eta \nabla_{\mathcal{G}}(x^3) - \frac{1}{\rho} \left(\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw} + \mathbf{f}_{\mathcal{B}} \right) = 0.$$

Remark. Some mathematical properties of the model can be proven, like invariance under rotation, the existence of an energy equation and well-balanceness (preserved lake-at-rest steady state): see [4] for details.

2.1.3 Balance law formulation of ISWE

System (2.20),(2.21) can be written in a more compact form which will be useful in the next sections:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathrm{div}_{\mathcal{G}} \underline{\underline{F}}(s, \mathbf{U}) + \mathbf{S}(s, \mathbf{U}) = 0, \tag{2.25}$$

where $U = [\eta, \eta U^1, \eta U^2]^T = [\eta, q^1, q^2]^T$ is the conservative variable, $\eta : \Gamma \times [0, t_f] \longrightarrow \mathbb{R}$, *and* $\mathbf{q} = [q^1, q^2], \mathbf{q} : \Gamma \times [0, t_f] \longrightarrow \mathbb{R}^2$.

The flux function is given by

$$\underline{\underline{F}}(\mathbf{s}, \mathbf{U}) = \begin{bmatrix} q^1 & q^2 \\ \frac{(q^1)^2}{\eta} + \frac{g\eta^2}{2h_{(1)}^2} \frac{\partial x^3}{\partial s^3} & \frac{q^1 q^2}{\eta} \\ \frac{q^1 q^2}{\eta} & \frac{(q^2)^2}{\eta} + \frac{g\eta^2}{2h_{(2)}^2} \frac{\partial x^3}{\partial s^3} \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{F}}^{\eta} \\ \\ \underline{\underline{F}}^{\mathbf{q}} \\ \\ \end{bmatrix}. \tag{2.26}$$

We will see that also in the bathimetry formulation, this term will have the same form. $\underline{\underline{F}}$ depends on $\mathbf{s}$ because of the presence of metric coefficients and bottom slope $\partial x^3/\partial s^3$. We write $\mathrm{div}_{\mathcal{G}}$ to have a more compact notation: $\mathrm{div}_{\mathcal{G}} := [\nabla^{\eta}_{\mathcal{G}}\cdot, \nabla^{\mathbf{q}}_{\mathcal{G}}\cdot]^T$. Finally, the source function $\mathbf{S}$ reads

$$\mathbf{S}(\mathbf{s},\eta) = \begin{bmatrix} 0 \\ \dfrac{g\eta^2}{2h^2_{(1)}}\dfrac{\partial}{\partial s^1}\left(\dfrac{\partial x^3}{\partial s^3}\right) + \dfrac{g\eta}{h^2_{(1)}}\dfrac{\partial x^3}{\partial s^1} - \dfrac{1}{\rho}[\nabla_{\mathcal{G}}\cdot \mathbf{T}_{sw}]^{(1,\cdot)} - \dfrac{\tau^1_b}{\rho} \\ \dfrac{g\eta^2}{2h^2_{(2)}}\dfrac{\partial}{\partial s^2}\left(\dfrac{\partial x^3}{\partial s^3}\right) + \dfrac{g\eta}{h^2_{(2)}}\dfrac{\partial x^3}{\partial s^2} - \dfrac{1}{\rho}[\nabla_{\mathcal{G}}\cdot \mathbf{T}_{sw}]^{(2,\cdot)} - \dfrac{\tau^2_b}{\rho} \end{bmatrix} = \begin{bmatrix} S^{\eta} \\ \\ \mathbf{S}^{\mathbf{q}} \\ \\ \end{bmatrix}. \qquad (2.27)$$

Note the presence of the bottom slope and its derivatives, of the metric terms, the two-dimensional averaged stress tensor $\mathbf{T}_{sw}$, the bottom friction parameter τ_b and the conserved variable η. Both the flux and the source terms are uniformly continuous with respect to $\mathbf{s}$ thanks to the regularity assumption on the bottom surface.

2.2 Intrinsic Finite Volume Scheme

Starting from a surface triangulation we solve the ISWE system by means of Finite Volume scheme. In order to obtain the standard FV scheme, we test eq. (2.25) with a piece-wise constant (in space and time) function v_i for all regions $R_i \in \mathcal{R}(\Gamma)$:

$$\int_{R_i} \frac{\partial \mathbf{U}}{\partial t} v_i + \int_{R_i} \mathrm{div}_{\mathcal{G}}\, \underline{\underline{F}}(s,\mathbf{U}) v_i + \int_{R_i} \mathbf{S}(s,\mathbf{U}) v_i = 0. \qquad (2.28)$$

Note that, for every i, the function v_i is constant in the region R_i, so dividing every term by $\mathcal{A}_{R_i}$ we can write

$$\frac{d}{dt}\left(\frac{1}{\mathcal{A}_{R_i}}\int_{R_i} \mathbf{U}\right) + \frac{1}{\mathcal{A}_{R_i}}\int_{R_i} \mathrm{div}_{\mathcal{G}}\, \underline{\underline{F}}(s,\mathbf{U}) + \frac{1}{\mathcal{A}_{R_i}}\int_{R_i} \mathbf{S}(s,\mathbf{U}) = 0.$$

Now if we define the following cell-averaged quantities, in $\mathcal{R}(\Gamma)$ as

$$\mathbf{U}_i = \frac{1}{\mathcal{A}_{R_i}}\int_{R_i} \mathbf{U}\, ds\,, \qquad \mathbf{F}_{ij} = \frac{1}{l_{ij}}\int_{\sigma_{ij}} \langle \underline{\underline{F}}, \nu_{ij}\rangle_{\mathcal{G}}\, d\sigma\,, \qquad \mathbf{S}_i = \frac{1}{\mathcal{A}_{R_i}}\int_{R_i} \mathbf{S}\, d\mathbf{s}, \qquad (2.29)$$

we can apply divergence theorem, obtaining

$$\frac{d}{dt}\mathbf{U}_i + \frac{1}{\mathcal{A}_{R_i}} \sum_{j=1}^{N_{\sigma_i}} l_{ij}\mathbf{F}_{ij}(\mathbf{U}) + \mathbf{S}_i(\eta) = 0\,,$$

observing that $\partial R_i = \bigcup_{j=1}^{N_{\sigma(i)}} \sigma_{ij}$ where $N_{\sigma(i)}$ is the number of edges of the region R_i. Explicit integration with respect to $t \in [t^k, t^{k+1}]$ yields finally to

$$\mathbf{U}_i^{k+1} = \mathbf{U}_i^k - \frac{1}{\mathcal{A}_{R_i}} \sum_{j=1}^{N_{\sigma(i)}} l_{ij} \int_{t^k}^{t^{k+1}} \mathbf{F}_{ij}(\mathbf{U})\, dt - \int_{t^k}^{t^{k+1}} \mathbf{S}_i(\eta)\, dt. \tag{2.30}$$

The main idea here to reach the approximation of the finite volumes scheme is to mantain the exclusive use of geometrically intrinsic quantities, and approximate the flux function using a Riemann solver tecnique.

Chapter 3

State of the art of the bathymetry reconstruction problem

In this chapter, we will go through the main steps of the two-dimensional SWE model described in [15], and based on the works [16],[17],[18]. For simplicity we will keep the original notation of [15] and only later compare the results to our model.

3.1 Set up and notation

Consider a flow domain $\Omega(t) \in \mathbb{R}^3$ at a certain time $t \in (t_0, t_{end})$ having a moving free surface ζ, and a bathymetry function z_b. Both of the variables are expressed by means of the canonical coordinate system of $\mathbb{R}^3$ and they depend only on their position in the x, y-plane, i.e. they are independent of the vertical coordinate. To model the physical phaenomenon given by the flow (for example of water), we consider the incompressible Navier-Stokes equations. The system reads:

$$\rho \partial_t \mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \mu\rho\Delta\mathbf{v} + \nabla p = \rho\mathbf{F}\,, \tag{3.1}$$

$$\nabla \cdot \mathbf{v} = 0\,, \tag{3.2}$$

where $\mathbf{v} = [u\ v\ w]^T$ is the velocity vector, p is the pressure, μ is the constant kinematic viscosity coefficient (or diffusivity) and $\mathbf{F}$ denotes the body forces acting on $\Omega(t)$. The second equation is the incompressible condition while the first, even if slightly different from our first equation of system (2.1), is equivalent to the momentum equation. In fact, neglecting the viscosity term, $\mu\rho\Delta\mathbf{v}$ can be incorporated into our term $\frac{1}{\rho}\nabla \cdot \mathbb{T}$. Note that $\mathbf{F}$ includes the Coriolis force f_c, gravity force and the remaining forces represented by term $\hat{\mathbf{f}}$, so

$$\mathbf{F} = -f_c \mathbf{e}_3 \times \mathbf{v} + \begin{bmatrix} \hat{\mathbf{f}} \\ -g \end{bmatrix} .$$

As we have seen in this thesis, we have the assumption

$$\frac{\textit{domain's depth}}{\textit{domain's length}} \ll 1$$

that ensures that we are dealing with a shallow water model. This hyphothesis implies that the vertical velocity magnitude is much less relevant then its horizontal components. At this point, the hydrostatic pressure assumption is brought into our model: as we saw, it states that the vertical component of the pressure gradient at a certain point in our domain depends only on the weight of the fluid column above it. Translated into mathematical language, this can be written as

$$\frac{\partial p(\mathbf{x})}{\partial z} = -g\,\rho\,. \tag{3.3}$$

In this construction, we have a cartesian coordinate system's origin that is set on the zero surface level with its third coordinate pointing upwards, as the negative sign above reminds.

Hydrostatic pressure assumption. The next steps play the role of bringing the pressure gradient term in equation (3.1) on the right hand side. In fact, by integrating along the vertical direction equation (3.3), from a certain vertical height z of the point $\mathbf{x} = (x, y, z)$ to the point $\mathbf{x}_f = (x, y, \xi)$ on the top free surface of vertical height ξ, we have

$$p(\xi(t,x,y)) - p(t,x,y,z) = \int_z^{\xi(t,x,y)} \underbrace{\frac{\partial p}{\partial \xi}(t,x,y,\zeta)}_{=-g\rho}\, d\zeta$$

so

$$p(t,x,y,z) = p_a(t,x,y) + g\rho(\xi(t,x,y) - z) \tag{3.4}$$

where $p_a(t,x,y)$ is the atmospheric pressure, which coincides with $p(\xi(t,x,y))$. From this expression we can easily compute the components of the pressure gradient ∇p:

$$\frac{\partial p}{\partial x} = \frac{\partial p_a}{\partial x} + g\rho\frac{\partial \xi}{\partial x}, \qquad \textit{and} \qquad \frac{\partial p}{\partial y} = \frac{\partial_a}{\partial y} + g\rho\frac{\partial \xi}{\partial y}\,.$$

The terms $\frac{\partial p_a}{\partial x}$ and $\frac{\partial p_a}{\partial y}$ are given data, thus will join the right hand side of equation (3.1), while the terms $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \xi}{\partial y}$, that form the gradient of the water height ξ, are unknown and we will deal with them with the Kinematic Boundary Conditions analysis.

3.1.1 Kinematic Boundary Conditions.

The top and bottom boundaries of our domain, at time t, are described by the functions $z_b(t,x,y)$ and $\xi(t,x,y)$. Note that z_b is also depending on time, thus taking care of the effects of bottom erosion that may be of interest in many applications.
Consider a point $\mathbf{P}$ of coordinates (x,y,z), it belongs to

- the *top surface* where $z = \xi$ if $\xi(t,x,y) - z = 0\,,$
- the *bottom surface*, where $z = z_b$, if $z_b(t,x,y) - z = 0\,.$

If we compute the material derivative of these equations, we obtain exactly the boundary conditions that we need:

$$\frac{\partial \xi}{\partial t} + u\Big|_{\xi}\frac{\partial \xi}{\partial x} + v\Big|_{\xi}\frac{\partial \xi}{\partial y} - w\Big|_{\xi} = 0, \qquad \frac{\partial z_b}{\partial t} + u\Big|_{z_b}\frac{\partial z_b}{\partial x} + v\Big|_{z_b}\frac{\partial z_b}{\partial y} - w\Big|_{z_b} = 0\,. \tag{3.5}$$

Using the notation

$$\nabla_{x,y} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}\,,$$

and recalling that

$$\nabla_{x,y} p = \nabla p_a + g\rho\nabla\xi\,,$$

we manipulate equation (3.1) dividing by ρ and introducing in the first two components of the momentum equation the expression of $\nabla_{x,y}p$ written above. We get the system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + g\nabla_{x,y}\xi + f_c[\mathbf{e}_3 \times \mathbf{v}]^{1,2} = \mathbf{f}\,, \tag{3.6}$$

$$\nabla\cdot\mathbf{v} = 0\,. \tag{3.7}$$

The unknowns here are the velocity term $\mathbf{v}$ and ξ, aswell as the bathymetry z_b. The term $\mathbf{f}$ is given by

$$\mathbf{f} = \hat{\mathbf{f}} - \frac{1}{\rho}\nabla_{x,y}p_a\,.$$

3.1.2 Integration along the height of the water.

At this point, integration along the height of the water is performed employing the Kinematic Boundary Conditions. The height of the water is defined as

$$H := \xi - z_b\,,$$

and the depth-averaged components of the horizontal velocity as

$$\bar{u} := \frac{1}{H}\int_{z_b}^{\xi} u\,dz \qquad\qquad \bar{v} := \frac{1}{H}\int_{z_b}^{\xi} v\,dz\,.$$

Note that from now on we change notation into: $(u_1, u_2) := (u, v)$ and $(\bar{u}_1, \bar{u}_2) := (\bar{u}, \bar{v})$.

Finally, the nonlinear advective term is rewritten using the continuity equation

$$(\mathbf{v}\cdot\nabla)\mathbf{u} = (\mathbf{v}\cdot\nabla + \nabla\cdot\mathbf{v})\cdot\mathbf{u} = \nabla\cdot(\mathbf{u}\mathbf{v}^T)\,,$$

that reminds of our term $\nabla\cdot\vec{u}\otimes\vec{u}$. Terms $\tau_{b,i}$ with $i = 1, 2$ are modelled by

$$\tau_{b,i} := \frac{\eta}{\mu}\sqrt{u^2+v^2}\,u_i\,.$$

In conclusion, considerations on the shallow water assumption brings to some simplification of the advective terms, leading to the following system of depth-integrated equations:

$$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{u}H) = 0\,, \tag{3.8}$$

$$\frac{\partial (uH)}{\partial t} + \nabla \cdot (u\mathbf{u}H - \mu\nabla(uH)) + gH\frac{\partial \xi}{\partial x} + \eta\sqrt{u^2+v^2}u - f_c vH = Hf_1\,, \tag{3.9}$$

$$\frac{\partial (vH)}{\partial t} + \nabla \cdot (v\mathbf{u}H - \mu\nabla(vH)) + gH\frac{\partial \xi}{\partial y} + \eta\sqrt{u^2+v^2}v - f_c uH = Hf_2\,. \tag{3.10}$$

with unknowns given by the averaged velocity vector $\mathbf{u} = [u \quad v]^T$, and the water height H.

3.1.3 Conservative form.

With an appropriate change of unknowns, i.e. using the momentum instead of the velocity, as seen in this thesis, everything can be written in a more suitable way. Indeed the old main variables (H, u, v) given by the height of the water and the first two components of the velocity vector $\mathbf{u}$ are replaced by (H, uH, vH), with the following notation

$$\mathbf{c}^T := \left[H\,(\mathbf{u}H)^T\right]^T =: [H\,U\,V]^T\,.$$

Due to this change of variable, we need to rewrite also the advective terms, that will now read

$$\nabla \cdot (u\mathbf{u}H) = \nabla \cdot \left(\frac{uH\mathbf{u}H}{H}\right)\,, \qquad \nabla \cdot (v\mathbf{u}H) = \nabla \cdot \left(\frac{vH\mathbf{u}H}{H}\right)\,,$$

and the bottom friction term

$$t_{bf} := \eta\frac{\sqrt{(uH)^2+(vH)^2}}{H^2}$$

then

$$\eta\sqrt{u^2+v^2}u_i = \tau_{bf} u_i H\,, i \in \{1,2\}\,.$$

3.2 Forward and inverse problems

In this setup, what really determines the difference between the forward and inverse problem is the primary unknown:

- for the forward problem the primary unknown is the surface elevation ξ. This unknown can be related to the parameter z_b and the height H via the simple relation

$$\xi = H + z_b\,.$$

- for the inverse problem the surface elevation ξ is a given data, while the bathymetry z_b becomes the unknown, computed by

$$z_b = \xi - H\,.$$

Analogous relations can be written for the gravitational term $gH\nabla\xi$. We have

$$gH\nabla\xi = gH\nabla(H + z_b) = \frac{g}{2}\nabla(H^2) + gH\nabla z_b\,. \tag{3.11}$$

3.2.1 Forward system

In the forward problem the z_b term is given, and so its gradient, so the last term in eq. (3.11) can be incorporated into the source of our main equation, while the term $\frac{g}{2}\nabla H^2$ is part of the advective term. The following system of equations can be written as:

$$\partial_t \mathbf{c} + \nabla \cdot \left(\mathcal{A}^f(\mathbf{c}) - \mathcal{B}(\nabla \mathbf{c}^{2,3})\right) = \mathbf{Z}^f(\mathbf{c}) \qquad \textit{in the domain } (t_0, t_{end}) \times \Omega \tag{3.12}$$

where

$$\mathcal{A}^f\left(\begin{bmatrix}H\\U\\V\end{bmatrix}\right)=\begin{bmatrix}U & V\\ \frac{U^2}{H}+\frac{g}{2}H^2 & \frac{UV}{H}\\ \frac{UV}{H} & \frac{V^2}{H}+\frac{g}{2}H^2\end{bmatrix}\,,\quad \mathcal{B}\left(\nabla\mathbf{c}^{2,3}\right)=\mu\begin{bmatrix}0^T\\ \nabla\mathbf{c}^{2,3}\end{bmatrix}\,,$$

$$\mathbf{Z}^f\left(\begin{bmatrix}H\\U\\V\end{bmatrix}\right)=\begin{bmatrix}0\\ Hf_1-gH\partial_x z_b-\tau_{bf}U+f_cV\\ Hf_2-gH\partial y z_b-\tau_{bf}V-f_cU\end{bmatrix}\,.$$

3.2.2 Inverse system

For the inverse problem the gravitational term $gH\nabla\xi$ is kept now as it is, since ξ is not an unknown anymore in this setting, and thus this term becomes part of the source. So, the problem in a conservation form reads

$$\partial_t\mathbf{c}+\nabla\cdot\left(\mathcal{A}^i(\mathbf{c})-\mathcal{B}(\nabla\mathbf{c}^{2,3})\right)=\mathbf{Z}^i(\mathbf{c})\qquad \textit{in the domain } (t_0,t_{end})\times\Omega\,, \tag{3.13}$$

with

$$\mathcal{A}^i\left(\begin{bmatrix}H\\U\\V\end{bmatrix}\right)=\begin{bmatrix}U & V\\ \frac{U^2}{H} & \frac{UV}{H}\\ \frac{UV}{H} & \frac{V^2}{H}\end{bmatrix}\,,\quad \mathcal{B}\left(\nabla\mathbf{c}^{2,3}\right)=\mu\begin{bmatrix}0^T\\ \nabla\mathbf{c}^{2,3}\end{bmatrix}\,,$$

$$\mathbf{Z}^f\left(\begin{bmatrix}H\\U\\V\end{bmatrix}\right)=\begin{bmatrix}0\\ Hf_1-gH\partial_x\xi-\tau_{bf}U+f_cV\\ Hf_2-gH\partial y\xi-\tau_{bf}V-f_cU\end{bmatrix}\,.$$

Many boundary conditions are used for both of the problems on the boundary domain $\partial\Omega=\partial\Omega_F\cup\partial\Omega_L\cup\partial\Omega_0\cup\partial\Omega_R\cup\partial\Omega_S$ (disjoint union). The following are taken into consideration in [15]:

flow boundaries on $\partial\Omega_F$: $\quad \mathbf{u}H=(\mathbf{u}H)_D\,,$

land boundaries on $\partial\Omega_L$: $\quad \mathbf{u}H\cdot\nu=\mathbf{0},\qquad \nabla(\mathbf{u}H)\nu=\mathbf{0}$

outflow boundaries on $\partial\Omega_O$: $\quad \nabla(\mathbf{u}H)\nu=\mathbf{0}\,,$

river boundaries on $\partial\Omega_R$: $\quad H=H_D\,,\qquad \mathbf{u}H=(\mathbf{u}H)_D\,,$

sea boundaries on $\partial\Omega_S$: $\quad H=H_D\qquad \nabla(\mathbf{u}H)\nu=\mathbf{0}\,.$

3.2.3 Regrouping of the forward and inverse formulations

Inviscid ($\mu = 0$) and viscous problems ($\mu > 0$) are treated separately. For $\mu = 0$, the system becomes of the first order (it means that we find only first derivatives of the unknown velocity) and can be written in a form that summarizes both forward and inverse formulations. In fact, the inviscid forward problem is distinguished from the inverse introducing a simple variable ζ, that will make appear or disappear some terms: $\zeta \in \{0, 1\}$ with $\zeta = 1$ for the forward problem and $\zeta = 0$ for the inverse. The general form reads

$$\partial_t \mathbf{c} + \nabla \cdot \mathcal{A}^{\zeta}(\mathbf{c}) = \mathbf{Z}^{\zeta}(\mathbf{c}) \tag{3.14}$$

where we have that

$$\mathcal{A}^{\zeta}(\mathbf{c}) = \begin{bmatrix} c_2 & c_3 \\ \dfrac{(c_2)^2}{c_1} + \zeta \dfrac{g}{2}(c_1)^2 & \dfrac{c_2 c_3}{c_1} \\ \dfrac{c_3 c_2}{c_1} & \dfrac{(c_3)^2}{c_1} + \zeta \dfrac{g}{2}(c_1)^2 \end{bmatrix},$$

$$\mathbf{Z}^{\zeta}(\mathbf{c}) = \begin{bmatrix} 0 \\ c_1 f_1 - g c_1 \partial_x(\zeta z_b + (1-\zeta)\xi) - \tau_{bf} c_2 + f_c c_3 \\ c_1 f_2 - g c_1 \partial_y(\zeta z_b + (1-\zeta)\xi) - \tau_{bf} c_c + f_c c_2 \end{bmatrix}.$$

For the numerical solution, the FESTUNG code is used and modified in [15].

Chapter 4

Bathymetry reconstruction with intrinsic geometry

In this chapter, we will try to reformulate the problem changing to our point of view. We want to extend the (geometric) ISWE model of chapter 2 to be time-dependent, and directly compatible with the idea presented in the previous chapter. Following the idea of integrating along the normal direction with respect to the surface, we need here to introduce a new time dependent-LCS, being the normal to the surface changing every time. We derive a second order approximation of the NS equations intrinsic to the top surface and finally compare our results with the ones in [15].

4.1 The Local time-dependent Coordinate System (LCS-t)

Recall the incrompressible and homogeneous (ρ constant) Navier-Stokes equations:

$$\begin{aligned} &\nabla \cdot \vec{u} = 0 \\ &\frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\vec{u} \otimes \vec{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \mathbb{T} + \vec{g} \end{aligned} \tag{2.1}$$

What we want to do is to express these equations in the new reference frame LCS-t. To this purpouse, we are going to consider the space-time setting, following the idea presented in [6], which seems suitable for our interest. We will introduce then a four dimensional time-dependent metric, that can be viewed as an extension of the metric we used in chapter 2. Our aim is to derive the SWE in the curvilinear time dependent coordinate system. As shown before, the direct and inverse problems derive directly from the SWE, and differ only in the unknown function. For simplicity, we can think about the inverse problem as a direct one, just having a moving bottom. We will consider as $\mathcal{S}_{\mathcal{B}}$ the *bottom* of the river, while we will call $\mathcal{S}_{\mathcal{F}}$ the *top* of the river. Here we introduce a new Local Coordinate System attached to the top having two axes spanning the tangent space of the surface and the third one orthonormal with respect to the others. In the new Local time-dependent Coordinate System (LCS-t), at time $\bar{t}$, we describe bottom and free surfaces as

$$\mathcal{S}_{\mathcal{B}} := \left\{(s^1, s^2, s^3, \bar{t}) \in \mathbb{R}^3 \times [0, t_f] \;\; \textit{such that } s^3 = \mathcal{B}(s^1, s^2, \bar{t}) \equiv \eta(s^1, s^2, \bar{t})\right\},$$
$$\mathcal{S}_{\mathcal{F}} := \left\{(s^1, s^2, s^3, \bar{t}) \in \mathbb{R}^3 \times [0, t_f] \;\; \textit{such that } s^3 = \mathcal{F}(s^1, s^2, \bar{t}) \equiv 0\right\},$$

where $\eta(s^1, s^2, \bar{t}) = \mathcal{B}(s^1, s^2, \bar{t}) - \mathcal{F}(s^1, s^2, \bar{t})$ denotes once again the fluid depth along direction s^3. Notice that, despite being the real river bottom fixed and not eroding in the GCS, using general curvilinear coordinates it could happen that its representation changes in time. In fact, as it will be more clear in the following sections, in the LCS located to the top surface we can represent the bottom as a function of the height measured in the direction orthogonal to the top surface itself, i.e. along s^3. In particular, its representation will indeed change in time, and we represent it by a time-dependent function. It will be clear in the following sections that the Kinematic Boundary Conditions will cancel out in the process of depth integration, no matter the time-dependancy. Following the steps of the direct problem, we define $\mathbf{S}_{\mathcal{B}}$ as the set of points that belongs to the pre-image of 0 according to the function $F_{\mathcal{B}} = s^3 - \mathcal{B}(s^1, s^2, t)$ and analogously $F_{\mathcal{F}} = s^3 - \mathcal{F}(s^1, s^2, t)$. Differently from the direct problem, we have now that both functions $F_{\mathcal{F}} = F_{\mathcal{F}}(\mathbf{s}(t), t)$ and $F_{\mathcal{B}} = F_{\mathcal{B}}(\mathbf{s}(t), t)$ depend directly on time and once again $\mathbf{s} = \mathbf{s}(t)$ describes the position of a moving point, or fluid particle.

Monge Parametrization. The construction of a LCS-t vector basis is analogous to the previous one: we consider a particular parametrization, called the *Monge* *parametrization*, defined by

$$\begin{aligned}\phi : & I \times U \subseteq I \times \mathbb{R}^2 \longrightarrow I \times \mathbb{R}^3 \\ & (t, s^1, s^2) \longmapsto (t, x^1(s^1, s^2), x^2(s^1, s^2), x^3(s^1, s^2, t)) := (t, s^1, s^2, \mathcal{F}(s^1, s^2, t))\end{aligned}$$

The basis for the tangent plane can be computed in the following way.

$$\frac{\partial \phi}{\partial s^1} = \left(\frac{\partial t}{\partial s^1}, \frac{\partial x^1(s^1, s^2)}{\partial s^1}, \frac{\partial x^2(s^1, s^2)}{\partial s^1}, \frac{\partial x^3(s^1, s^2)}{\partial s^1} \right) = (0, 1, 0, \mathcal{F}_{s^1}) \, ,$$

and

$$\frac{\partial \phi}{\partial s^2} = \left(\frac{\partial t}{\partial s^2}, \frac{\partial x^1(s^1, s^2)}{\partial s^2}, \frac{\partial x^2(s^1, s^2)}{\partial s^2}, \frac{\partial x^3(s^1, s^2)}{\partial s^2} \right) = (0, 0, 1, \mathcal{F}_{s^2}) \, ,$$

where $\mathcal{F}_{s^i} = \partial \mathcal{F} / \partial s^i$, for $i = 1, 2$. Finally, we can compute

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \left(\frac{\partial t}{\partial t}, \frac{\partial x^1(s^1, s^2)}{\partial t}, \frac{\partial x^2(s^1, s^2)}{\partial t}, \frac{\partial x^3(s^1, s^2)}{\partial t} \right) = \left(1, 0, 0, \frac{d\mathcal{F}}{dt} \right) = \\ &= (1, 0, 0, \mathcal{F}') \; . \end{aligned} \tag{4.1}$$

This last vector will play an important role in the evolving surface finite element method formulation. Starting from the induced vectors

$$\hat{\mathbf{t}}_i(\mathbf{p}, t) = d\phi_p(e_i) = \left(0, \frac{\partial x^1}{\partial s^i}, \frac{\partial x^2}{\partial s^i}, \frac{\partial x^3}{\partial s^i} \right) \qquad i = 1, 2 \, ,$$

where $d\phi$ is the Jacobian matrix of the coordinate transformation, we can now orthogonalize vector $\hat{\mathbf{t}}_2$ with respect to $\hat{\mathbf{t}}_1$ via Gram-Schmidt, obtaining the desired $\mathbf{t}_1, \mathbf{t}_2$ on $T_{\mathbf{p}} \mathcal{S}_{\mathcal{F}}$. Then, we complete the frame with vectors $\mathbf{t}_3$ and $\mathbf{t}_0$, imposing the orthogonality condition with respect to previous ones, joint with $\parallel \mathbf{t}_3(\mathbf{p}, t) \parallel = \parallel \mathbf{t}_0(\mathbf{p}) \parallel = 1$. In a fixed point $\mathbf{p}, t$ of $\mathcal{S}_{\mathcal{F}} \times I$, our LCS-t takes the explicit form

$$\mathbf{t}_0 = (1, 0, 0, 0) \, , \tag{4.2}$$

$$\mathbf{t}_1(\mathbf{p}, t) = (0, 1, 0, \mathcal{F}_{s^1}) \; , \tag{4.3}$$

$$\mathbf{t}_2(\mathbf{p}, t) = \hat{\mathbf{t}}_2 - \frac{\hat{\mathbf{t}}_2 \cdot \mathbf{t}_1}{\parallel \mathbf{t}_1 \parallel^2} \mathbf{t}_1 = \left(0, -\frac{\mathcal{F}_{s^1} \mathcal{F}_{s^2}}{1 + (\mathcal{F}_{s^1})^2}, 1, \frac{\mathcal{F}_{s^2}}{1 + (\mathcal{F}_{s^1})^2} \right) \, , \tag{4.4}$$

$$\mathbf{t}_3(\mathbf{p},t) = \mathbf{N}(\mathbf{p},t) = \frac{\mathbf{t}_1(\mathbf{p},t) \wedge \mathbf{t}_2(\mathbf{p},t)}{\| \mathbf{t}_1(\mathbf{p},t) \| \| \mathbf{t}_2(\mathbf{p},t) \|} = \frac{(0, -\mathcal{F}_{s^1}, -\mathcal{F}_{s^2}, 1)}{\| \mathbf{t}_1(\mathbf{p},t) \| \| \mathbf{t}_2(\mathbf{p},t) \|} . \tag{4.5}$$

Remark. If we look at the components of vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, we can see that the projection of the first vector on the $< x, y >$ plane is simply $(1,0)$, while the projection of the latter is in general not parallel to x axe. This can be used to think about the form of the coordinate lines.

It is easy to compute the norm of the vectors, for example

$$h^2_{(1)}(t) :=\| \mathbf{t}_1(\mathbf{p},t) \|^2 = 1 + \mathcal{F}_{s^1} ,$$

$$h^2_{(2)}(t) :=\| \mathbf{t}_2(\mathbf{p},t) \|^2 = \frac{1 + \mathcal{F}^2_{s^1} + \mathcal{F}^2_{s^2}}{1 + \mathcal{F}^2_{s^1}} .$$

Let us now define the metric $\mathbb{G} : \{\mathcal{F} \times I\} \times \{\mathcal{F} \times I\} \longrightarrow \mathbb{R}$ as

$$\mathbb{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & h^2_{(1)}(t) & 0 & 0 \\ 0 & 0 & h^2_{(2)}(t) & 0 \\ 0 & 0 & 0 & 1_s \end{pmatrix} , \tag{4.6}$$

so that $\mathbb{G}(\mathbf{v},\mathbf{v}) = \langle \mathbf{v},\mathbf{v} \rangle_{\mathbb{G}} = dt^2 + g_{ij}dx^i dx^j$, where $g_{ij} = \mathcal{G}_{ij}$ is the same metric we used for the direct problem, with the difference that here the terms depends also on the time: $\mathbf{t}_i(\mathbf{p}) = \mathbf{t}_i(\mathbf{p},t)$, $i = 1,2$ and so $g_{ii} = g_{ii}(t)$ with $i = 1,2$.

4.2 Kinematic Boundary Conditions

As seen in the previous chapters we have to couple the Navier-Stokes equations with some Kinematic Boundary Conditions. Recalling that $\mathcal{S}_{\mathcal{F}}$ is now the top surface of the river, and $\mathcal{S}_{\mathcal{B}}$ the bottom surface, we can describe them, in the GCS, as

$$\mathcal{S}_{\mathcal{F}} := \left\{ (x^1, x^2, x^3, t) \in \mathbb{R}^3 \times [0, t_f] \ \textit{such that}\ x^3 = \mathcal{F}(x^1, x^2, t) \equiv 0 \right\} ,$$

$$\mathcal{S}_{\mathcal{B}} := \left\{ (x^1, x^2, x^3, t) \in \mathbb{R}^3 \times [0, t_f] \ \textit{such that}\ x^3 = \mathcal{B}(x^1, x^2, t) \equiv \eta(x^1, x^2, t) \right\} .$$

If we define $F_{\mathcal{M}} = s^3 - \mathcal{M}(s^1, s^2, t)$, with $\mathcal{M} = \mathcal{F}$ *or* $\mathcal{B}$, the following holds

$$\frac{dF_{\mathcal{M}}}{dt} = \frac{\partial F_{\mathcal{M}}}{\partial t} + \vec{u} \cdot \nabla_{\mathcal{G}} F_{\mathcal{M}} \bigg|_{\mathcal{M}} = 0\,.$$

Remark. *Here scalar product is meant in terms of the metric* $\mathbb{G}$ *restricted to the space generate by* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$*, that we call* $\mathcal{G}$*, i.e.* $\cdot \equiv \cdot_{\mathcal{G}}$.

Focusing on the top surface $\mathcal{F}$, and then on the bottom $\mathcal{B}$, and recalling the expression of the intrinsic gradient of a scalar function (eq.(1.7)), we obtain:

$$\frac{dF_{\mathcal{F}}}{dt} = \frac{\partial F_{\mathcal{F}}}{\partial t} + \vec{u} \cdot \nabla_{\mathcal{G}} F_{\mathcal{F}} \bigg|_{s^3=0} = -\frac{\partial \mathcal{F}}{\partial t} - \left(u^1 \frac{\partial \mathcal{F}}{\partial s^1} + u^2 \frac{\partial \mathcal{F}}{\partial s^2} - u^3 \right) \bigg|_{s^3=0} = 0\,, \qquad (4.7)$$

$$\frac{dF_{\mathcal{B}}}{dt} = \frac{\partial F_{\mathcal{B}}}{\partial t} + \vec{u} \cdot \nabla_{\mathcal{G}} F_{\eta} \bigg|_{s^3=\eta} = -\frac{\partial \eta}{\partial t} - \left(u^1 \frac{\partial \eta}{\partial s^1} + u^2 \frac{\partial \eta}{\partial s^2} - u^3 \right) \bigg|_{s^3=\eta} = 0\,. \qquad (4.8)$$

Observe that $\nabla_{\mathcal{G}} F_{\mathcal{F}} \Big|_{s^3=0} = -\nabla \mathcal{F} \Big|_{s^3=0}$ and $\nabla_{\mathcal{G}} F_{\eta} \Big|_{s^3=\eta} = -\nabla \mathcal{B} \Big|_{s^3=\eta}$. Now we will make the assumption that external actions on the fluid surface $\mathcal{F}$ are negligible. This implies, as for the direct problem, that at the fluid-air interface we are imposing a zero-stress boundary equation:

$$\mathbb{T}_{\mathcal{F}} \cdot \mathbf{N}_{\mathcal{F}} = 0, \quad \mathbf{N}_F = \frac{\nabla \mathcal{F}}{\parallel \nabla \mathcal{F} \parallel}, \qquad (4.9)$$

where $\mathbf{N}_{\mathcal{F}}$ is the unit normal vector on the free surface $\mathcal{F}$. On the other hand, the bed boundary condition reads

$$\mathbb{T}_{\mathcal{B}} \cdot \mathbf{N}_{\mathcal{B}} = \mathbf{f}_{\mathcal{B}} = \tau_b^1 \mathbf{t}_1 + \tau_b^2 \mathbf{t}_2 + p_{\mathcal{B}} \mathbf{t}_3, \qquad (4.10)$$

where $p_{\mathcal{B}}$ indicates the real-bottom pressure.

4.3 Curvilinear Navier-Stokes equations (LCS-t)

The first step in the developments of ISWE will be to re-write NS with respect to the LCS-t.

4.3.1 Main ideas

We start presenting the main ideas that brought to the development of the time-dependent ISWE.

In order to re-write eqns. (2.1) in our new LCS-t, we follow the idea in [6]. In the particular case of a scalar quantity u, the integral form of our equations reads

$$\frac{d}{dt}\int_{\mathcal{P}} u\, dV + \int_{\partial\mathcal{P}} g(f(u), n_{\partial\mathcal{P}})\, dV_{\partial\mathcal{P}} \qquad \textit{for a fixed } \mathcal{P} \subset \Omega. \tag{4.11}$$

Note that $u : \Omega \times I \longrightarrow \mathbb{R}$ is the conserved quantity transported according to the flux function $f(\cdot)$: it undergoes to compression and rarefaction due to the time-dependence of $g(\cdot)$. Observe that in the case of our Navier-Stokes equations, we will have to modify this because we work with a vector quantity and not a scalar.

Now we have to take care of the fact that the element of volume dV does depend on time. Thus, we can write

$$\frac{d}{dt}\int_{\mathcal{P}} u\, dV = \int_{\mathcal{P}} \partial_t u dV + \int_{\mathcal{P}} u \partial_t dV\,,$$

where $dV = \sqrt{\det(g_{ij})}dr$, in positive oriented coordinates $r \in \mathbb{R}^d$, with $i, j = 1, 2, 3$. Furthermore, we have that

$$\partial_t dV = \partial_t \sqrt{\det(g_{ij})}dr = \frac{1}{2}\frac{1}{\sqrt{\det(g_{ij})}} \det(g_{ij}) C_1^1(g^{ik}\partial_t g_{kl})\, dr = \frac{1}{2} g^{ij} \partial_t g_{ij}\, dV\,.$$

With $C_1^1(g^{ik}\partial_t g_{kl})$ we indicate the contraction tensor operation, i.e. the trace of the matrix $\mathcal{G}^{-1}\partial_t\mathcal{G}$. Since eqn. (4.11) holds for an arbitrary region $\mathcal{P}$, the following holds:

$$\partial_t u + \lambda u + div f(u) = 0 \quad in\; \Omega \times [0, t_f]\,, \tag{4.12}$$

with

$$\lambda = \frac{1}{2} g^{ij}\partial_t g_{ij} = \frac{1}{2}\left(\frac{1}{h_{(1)}^2} 2h_{(1)}\partial_t h_{(1)} + \frac{1}{h_{(2)}^2} 2h_{(2)}\partial_t h_{(2)}\right) = \frac{\partial_t h_{(1)}}{h_{(1)}} + \frac{\partial_t h_{(2)}}{h_{(2)}}\,. \tag{4.13}$$

Observe that this can also be written as

$$DivF(u) = 0\,, \tag{4.14}$$

where, $F(u) := u\partial_t + f(u)$ is a vector field on $\Omega \times I$ and Div is the divergence operator with respect to the spacetime metric $dt^2 + g_{ij}\,dx^i\,dx^j$ (see [6]). Let us write it in vectorial form

$$F(u) = \begin{bmatrix} u \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f(u)^1 \\ f(u)^2 \\ f(u)^3 \end{bmatrix}.$$

If we apply the formula (1.8) to $u\partial_t$ and notice that $\det(\mathcal{G}) = \det(\mathbb{G})$, we have

$$Div_{\mathbb{G}}(u\partial_t) = \frac{1}{\sqrt{\det \mathcal{G}}}\left[\frac{\partial}{\partial t}\left(\sqrt{\det \mathcal{G}} u\right)\right] = \frac{\partial}{\partial t} u + \lambda u\,.$$

At this point it is clear that if we want to write equations (2.1) in our time-dependent Local Coordinate System LCS-t attached to a surface, in an analogous way we need to write an additional term, similar to λu. A very intuitive explanation of why the equations change form can be found by the curious reader in [9], in the case of the heat equation.

4.3.2 Continuity equation

Let $\rho = \rho(t, x)$ be the density. Following the idea just presented, we derive the Continuity equation in the LCS-t as follows:

$$\nabla_{\mathbb{G}} \cdot \begin{bmatrix} \rho \\ \rho u^1 \\ \rho u^2 \\ \rho u^3 \end{bmatrix} = 0\,, \tag{4.15}$$

which gives

$$0 = \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial t}\left(\sqrt{det(g)}\rho\right) + \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial s^i}\left(\sqrt{det(g)}\rho u^i\right) = \qquad (i = 1, 2, 3)$$
$$= \partial_t \rho + \rho\lambda + \nabla_{\mathcal{G}} \cdot (\rho u) \tag{4.16}$$

In the case of constant density, we have

$$\lambda + \nabla_{\mathcal{G}} \cdot u = 0\,. \tag{4.17}$$

4.3.3 Momentum equation(s)

We want now to derive the three components Momentum equation using formula (1.9) applied to the non-symmetric 4×4 contravariant tensor

$$\mathbb{F} := \begin{pmatrix} 0 & u^1 & u^2 & u^3 \\ 0 & f^{11} & f^{12} & f^{13} \\ 0 & f^{21} & f^{22} & f^{23} \\ 0 & f^{31} & f^{32} & f^{33} \end{pmatrix}, \tag{4.18}$$

where the f^{ij} terms incorporate all the terms that appear in the Navier-Stokes equations, which we can write in tensor form:

$$\begin{pmatrix} f^{11} & f^{12} & f^{13} \\ f^{21} & f^{22} & f^{23} \\ f^{31} & f^{32} & f^{33} \end{pmatrix} := \vec{u}\otimes\vec{u} + \frac{1}{\rho}\begin{pmatrix} \frac{p}{h^2_{(1)}} & 0 & 0 \\ 0 & \frac{p}{h^2_{(2)}} & 0 \\ 0 & 0 & p \end{pmatrix} - \begin{pmatrix} \frac{g^1}{h^2_{(1)}} & 0 & 0 \\ 0 & \frac{g^2}{h^2_{(2)}} & 0 \\ 0 & 0 & g^3 \end{pmatrix} - \frac{1}{\rho}\begin{pmatrix} \tau^{11} & \tau^{12} & \tau^{13} \\ \tau^{21} & \tau^{22} & \tau^{23} \\ \tau^{31} & \tau^{32} & \tau^{33} \end{pmatrix}. \tag{4.19}$$

In fact it can be proved that, for example, the term involving the pressure provides the same result, once we apply to it the tensor divergence $\nabla_{\mathcal{G}}$. In other words, this is just the equivalent tensorial way of writing $\nabla_{\mathcal{G}} p$. Let us call $\mathbb{P} := p\mathcal{G}^{-1}$; using formula (1.9) and keeping in mind that $\mathbb{P}$ is symmetric, we have:

$$(\nabla_{\mathcal{G}} \cdot \mathbb{P})^j = \frac{1}{h_{(1)}h_{(2)}}\frac{\partial\left(p\sqrt{det(g)}g^{jj}\right)}{\partial s^j} + \frac{1}{h_{(j)}}\left(2\mathbb{P}^{jj}\frac{\partial h_{(j)}}{\partial s^j} - \mathbb{P}^{jj}\frac{h_{(j)}}{h_{(j)}}\frac{\partial h_{(j)}}{\partial s^j} - \mathbb{P}^{\bar{j}\bar{j}}\frac{h_{(\bar{j})}}{h_{(j)}}\frac{\partial h_{(\bar{j})}}{\partial s^j}\right) =$$

$$(\bar{j}\ \textit{is defined as:} \qquad j=1 \Rightarrow \bar{j}=2; \quad j=2 \Rightarrow \bar{j}=1)$$

$$= \frac{1}{h_{(1)}h_{(2)}} \frac{\partial\left(p\sqrt{det(g)}g^{jj}\right)}{\partial s^j} + \frac{1}{h_{(j)}}\left(pg^{jj}\frac{\partial h_{(j)}}{\partial s^j}\right) - p\frac{1}{h^2_{(\bar{j})}}\frac{h_{(\bar{j})}}{h_{(j)}}\frac{\partial h_{(\bar{j})}}{\partial s^j}\,. \tag{4.20}$$

So for $j=1$, other components being analogous, this expression yields

$$\begin{aligned}
(\nabla_{\mathcal{G}}\cdot\mathbb{P})^1 &= \frac{1}{h_{(1)}h_{(2)}}\frac{\partial\left(\frac{h_{(1)}h_{(2)}}{h^2_{(1)}}p\right)}{\partial s^1} + \frac{1}{h_{(1)}}\left(p\frac{1}{h^2_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\right) - p\frac{1}{h^2_{(1)}h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1} = \\
&= \frac{1}{h^2_{(1)}}\frac{\partial p}{\partial s^1} + \frac{1}{h_{(1)}h_{(2)}}p\frac{\partial h_{(2)}}{\partial s^1}\frac{h_{(1)}}{h^2_{(1)}} - \frac{1}{h_{(1)}h_{(2)}}p\frac{h_{(2)}}{h^2_{(1)}}\frac{\partial h_{(1)}}{\partial s^1} + \frac{1}{h^3_{(1)}}\frac{\partial h_{(1)}}{\partial s^1} - p\frac{1}{h^2_{(1)}h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1} = \\
&= \frac{1}{h^2_{(1)}}\frac{\partial p}{\partial s^1} = (\nabla_{\mathcal{G}}p)^1\,,
\end{aligned} \tag{4.21}$$

as we expected (see formula (1.7)). In the same way, we can compute $\nabla_{\mathbb{G}}\cdot\mathbb{F}$, being careful on the definitions of the divergence of a generic tensor ($\mathbb{F}$ is not symmetric). Note that the first column of $\mathbb{F}$ is zero, and has no physical meaning: its purpouse is just to form a *square* tensor. The explicit computation of div$\mathbb{F}$ can be found in Appendix.

Finally, the $j-$th component of the Momentum equation is given by $(\nabla_{\mathbb{G}}\cdot\mathbb{F})^j = 0\,, j\in\{1,2,3\}$. Direct computations yield

$$\partial_t\vec{u} + \lambda\vec{u} + \vec{u}\circ\vec{h} + \nabla_{\mathcal{G}}\cdot(\vec{u}\otimes\vec{u}) + \nabla_{\mathcal{G}}\cdot\left(\frac{\mathbb{P}}{\rho} - \mathbb{H}\right) - \frac{1}{\rho}\nabla_{\mathcal{G}}\cdot\mathbb{T} = 0\,, \tag{4.22}$$

$$\textit{with} \quad \vec{h} := \begin{bmatrix} \frac{\partial_t h_{(1)}}{h_{(1)}} \\ \frac{\partial_t h_{(2)}}{h_{(2)}} \\ 0 \end{bmatrix}, \qquad \mathbb{P} := p\mathcal{G}^{-1}\,, \qquad \mathbb{H} := gH\mathcal{G}^{-1}\,.$$

Remark. It is reasonable to treat the Continuity equation and the Momentum equations separately, since they describe two different physical phaenomena.

4.4 The normally integrated Navier-Stokes equations

We are now able to perform depth integration along the top surface normal.

Continuity equation. We apply Leibnitz rule (we are assuming enough regularity of the bottom and free surfaces, as well as of $\vec{u}$), and substituting the Kinematic boundary conditions (4.7), (4.8) we derive:

$$\begin{aligned}
\int_0^\eta \nabla_{\mathcal{G}} \cdot \vec{u} + \int_0^\eta \lambda &= \int_0^\eta \frac{1}{h_{(1)}h_{(2)}} \left(\frac{\partial (h_{(1)}h_{(2)}u^1)}{\partial s^1} + \frac{\partial (h_{(1)}h_{(2)}u^2)}{\partial s^2} + \frac{\partial (h_{(1)}h_{(2)}u^3)}{\partial s^3} \right) + \lambda\eta = \\
&= \frac{1}{h_{(1)}h_{(2)}} \frac{\partial}{\partial s^1} \int_0^\eta h_{(1)}h_{(2)}u^1 + \frac{1}{h_{(1)}h_{(2)}} \frac{\partial}{\partial s^2} \int_0^\eta h_{(1)}h_{(2)}u^2 + \lambda\eta + \\
&+ u^3\Big|_{s^3=\eta} - u^1 \frac{\partial \mathcal{B}}{\partial s^1}\Big|_{s^3=\eta} - u^2 \frac{\partial \mathcal{B}}{\partial s^2}\Big|_{s^3=\eta} \\
&- u^3\Big|_{s^3=0} + u^1 \frac{\partial \mathcal{F}}{\partial s^1}\Big|_{s^3=0} + u^2 \frac{\partial \mathcal{F}}{\partial s^2}\Big|_{s^3=0} = \\
&= \frac{\partial \eta}{\partial t} - \frac{\partial \mathcal{F}}{\partial t} + \lambda\eta + \nabla_{\mathcal{G}} \cdot \int_0^\eta \underline{\vec{u}}\,, \qquad (4.23)
\end{aligned}$$

where $\underline{\vec{u}} = [u^1, u^2]^T$ and the operator $\nabla_{\mathcal{G}}\cdot$ is adapted to the two dimensional setting. Observe that η and $\vec{u}$ are unknown, while the rest is given.

Momentum equation. An equivalent form of equation (4.22) is

$$\frac{\partial \vec{u}}{\partial t} + \nabla_{\mathcal{G}} \cdot (\vec{u} \otimes \vec{u}) + \vec{u}\lambda + \vec{u} \circ \vec{h} = -\frac{1}{\rho}\nabla_{\mathcal{G}} p + \vec{g} + \frac{1}{\rho}\nabla_{\mathcal{G}} \cdot \mathbb{T} \qquad (4.24)$$

where $\circ$ denotes the component-wise product (Hadamard product).

We can now perform depth integration along the normal direction s^3 of the top surface, which gives

$$\int_0^\eta \frac{\partial \vec{u}}{\partial t} + \int_0^\eta \nabla_{\mathcal{G}} \cdot (\vec{u} \otimes \vec{u}) + \int_0^\eta \vec{u}\lambda + \int_0^\eta \vec{u} \circ \vec{h} = -\frac{1}{\rho} \int_0^\eta \nabla_{\mathcal{G}} p - g \int_0^\eta \nabla_{\mathcal{G}} x^3 + \frac{1}{\rho} \int_0^\eta \nabla_{\mathcal{G}} \cdot \mathbb{T}\,. \qquad (4.25)$$

- Employing Leibnitz rule to the LHS we find

$$\frac{\partial}{\partial t} \int_0^\eta \vec{u} - \vec{u}\frac{\partial \eta}{\partial t} + \vec{u}\frac{\partial \mathcal{F}}{\partial t} + \nabla_{\mathcal{G}} \cdot \int_0^\eta \vec{u} \otimes \vec{u} - (\vec{u} \otimes \vec{u})\nabla_{\mathcal{G}} \mathcal{B}\Big|_{s^3=\eta} +$$

$$+ (\vec{u} \otimes \vec{u}) \nabla_{\mathcal{G}} \mathcal{F} \Big|_{s^3=0} + \int_0^\eta \vec{u}\lambda + \int_0^\eta \vec{u} \circ \vec{h} \,,$$

that reduces to

$$\frac{\partial}{\partial t} \int_0^\eta \vec{u} + \nabla_{\mathcal{G}} \cdot \int_0^\eta \vec{u} \otimes \vec{u} + \int_0^\eta \vec{u}\lambda + \int_0^\eta \vec{u} \circ \vec{h} \,,$$

with the Kinematic BCs.

- For the RHS, we have

$$\begin{aligned} -\frac{1}{\rho} \int_0^\eta \nabla_{\mathcal{G}} p - g \int_0^\eta \nabla_{\mathcal{G}} x^3 + \frac{1}{\rho} \nabla_{\mathcal{G}} \cdot \int_0^\eta \mathbb{T} - \frac{\mathbb{T}}{\rho} \nabla_{\mathcal{G}} \mathcal{B} \Big|_{s^3=\eta} + \frac{\mathbb{T}}{\rho} \nabla_{\mathcal{G}} \mathcal{F} \Big|_{s^3=0} = \\ = -\frac{1}{\rho} \int_0^\eta \nabla_{\mathcal{G}} p - g \int_0^\eta \nabla_{\mathcal{G}} x^3 + \frac{1}{\rho} \nabla_{\mathcal{G}} \cdot \int_0^\eta \mathbb{T} - \frac{1}{\rho} \mathbb{T}_{\mathcal{B}} \cdot \mathbf{N}_{\mathcal{B}} \,, \end{aligned}$$

where we used aswell Kinematic BCs. Observe that the form of the RHS is very similar to the one found in chapter 2, i.e. for the time-independent problem.

SW Approximation. As in chapter 2, we postulate $u^3 = \epsilon_{\mathcal{G}} u^i$, $i = 1, 2$, $\epsilon_{\mathcal{G}} \ll 1$ and employ an expansion of the scalar components of the velocity vector $\vec{u}(\mathbf{s}, t)$:

$$u^i = u^i_{(0)} + \epsilon_{\mathcal{G}} u^i_{(1)} + \epsilon^2_{\mathcal{G}} u^i_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right) \qquad i = 1, 2 \,, \tag{4.26}$$

$$u^3 = \epsilon_{\mathcal{G}} u^3_{(1)} + \epsilon^2_{G} u^3_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right) \tag{4.27}$$

and the stress tensor $\mathbb{T}(\mathbf{s}, t)$:

$$\tau^{ij} = \tau^{ij}_0 + \epsilon_{\mathcal{G}} \tau^{ij}_{(1)} + \epsilon^2_{\mathcal{G}} \tau^{ij}_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right) \qquad i, j = 1, 2, 3 \tag{4.28}$$

and again, in particular, we assume that the terms containing u^3 can be expanded in the following way:

$$\tau^{3i} = \epsilon_{\mathcal{G}} \tau^{3i}_{(1)} + \epsilon^2_{\mathcal{G}} \tau^{3i}_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right) \quad \textit{for } i = 1, 2 \quad \textit{and } \tau^{33} = \epsilon^2_{\mathcal{G}} \tau^{33}_{(2)} + \mathcal{O}\left(\epsilon^3_{\mathcal{G}}\right).$$

At this point, we split the velocity vector and stress tensor as

$$\vec{u} = \vec{U} + \tilde{u}, \; \textit{where } \vec{U}(s^1, s^2, t) = \frac{1}{\eta} \int_0^\eta \vec{u}(\mathbf{s}, t) \, ds^3 \,, \quad \int_0^\eta \tilde{u}(\mathbf{s}, t) \, ds^3 = 0 \,, \tag{4.29}$$

$$\mathbb{T} = \mathbf{T} + \tilde{\tau},\ \textit{where}\ \mathbf{T}(s^1, s^2, t) = \frac{1}{\eta}\int_0^\eta \mathbb{T}(\mathbf{s}, t)\, ds^3\,, \qquad \int_0^\eta \tilde{\tau}(\mathbf{s}, t)\, ds^3 = 0\,. \tag{4.30}$$

Basically, we have switched from the variables $\vec{u}$ and $\mathbb{T}$ to their normal depth-averages: $\vec{U}$ and $\mathbf{T}$, and define

$\vec{q} = [\eta U^1, \eta U^2]$ denotes the depth-averaged velocity vector. The tensor

$$\mathbf{T}_{sw} = \eta \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix}$$

is the principal 2-minors of $\mathbf{T}$, and vector $\mathbf{f}_{\mathcal{B}} = [\tau_b^1, \tau_b^2]^T$ is the vector field that contains bed friction information.

After having seen the approximations that we are going to employ and the notation set-up, we are going to derive a second order approximation of the Navier-Stokes equations. The first step is to recall the Continuity and Momentum depth-integrated equations; coupled together they form the *normally integrated Navier-Stokes equations*:

$$\frac{\partial \eta}{\partial t} - \frac{\partial \mathcal{F}}{\partial t} + \lambda\eta + \nabla_{\mathcal{G}} \cdot \int_0^\eta \underline{\vec{u}} = 0\,, \tag{4.31}$$

$$\frac{\partial}{\partial t}\int_0^\eta \vec{u} + \nabla_{\mathcal{G}} \cdot \int_0^\eta \vec{u}\otimes\vec{u} + \int_0^\eta \vec{u}\lambda + \int_0^\eta \vec{u}\circ\vec{h} =$$

$$= -\frac{1}{\rho}\int_0^\eta \nabla_{\mathcal{G}} p - g\int_0^\eta \nabla_{\mathcal{G}} x^3 + \frac{1}{\rho}\nabla_{\mathcal{G}} \cdot \int_0^\eta \mathbb{T} - \frac{1}{\rho}\mathbb{T}_{\mathcal{B}} \cdot \mathbf{N}_{\mathcal{B}} \tag{4.32}$$

4.4.1 Hydrostatic pressure condition

The first step is re-writing the momentum equations component wise. Using the operators defined in eqs. (2.6) to (2.8) and recalling that the terms $\partial h_{(1)}/\partial s^3$, $\partial h_{(2)}/\partial s^3$ vanish and $h_{(3)}$ is a constant, noticing also that $\int_0^\eta \vec{u}\otimes\vec{u}$ is a tensor, the third momentum equation reads:

$$\frac{\partial}{\partial t}\int_0^\eta u^3 + \frac{1}{h_{(1)}h_{(2)}}\left(\frac{\partial}{\partial s^1}\int_0^\eta h_{(1)}h_{(2)}u^1u^3 + \frac{\partial}{\partial s^2}\int_0^\eta h_{(1)}h_{(2)}u^2u^3 + \frac{\partial}{\partial s^3}\int_0^\eta h_{(1)}h_{(2)}\left(u^3\right)^2\right) + \int_0^\eta u^3\lambda =$$

(where we used that the third component of $\vec{u} \circ \vec{h}$ is simply zero)

$$= -\frac{1}{\rho}\int_0^\eta \frac{\partial p}{\partial s^3} - g\int_0^\eta \frac{\partial x^3}{\partial s^3} + \frac{1}{\rho h_{(1)}h_{(2)}}\left(\frac{\partial}{\partial s^1}\int_0^\eta h_{(1)}h_{(2)}\tau^{13} + \frac{\partial}{\partial s^2}\int_0^\eta h_{(1)}h_{(2)}\tau^{23} + \frac{\partial}{\partial s^3}\int_0^\eta h_{(1)}h_{(2)}\tau^{33}\right) + \frac{p_{\mathcal{B}}}{\rho}$$

$$Note: \quad \vec{u} = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}, \quad \vec{u}\otimes\vec{u} = \begin{bmatrix} u^1u^1 & u^1u^2 & u^1u^3 \\ u^2u^1 & u^2u^2 & u^2u^3 \\ u^3u^1 & u^3u^2 & u^3u^3 \end{bmatrix}, \quad \mathbb{T}_{\mathcal{B}}\cdot\mathbf{N}_{\mathcal{B}} = \mathbf{f}_{\mathcal{B}} = \tau_b^1\mathbf{t}_1 + \tau_b^2\mathbf{t}_2 + p_{\mathcal{B}}\mathbf{t}_3\,,$$

$$\nabla_{\mathcal{G}}p = \left(\frac{1}{h_{(1)}^2}\frac{\partial p}{\partial s^1}, \frac{1}{h_{(2)}^2}\frac{\partial p}{\partial s^2}, \frac{\partial p}{\partial s^3}\right), \qquad \mathbb{T} = \begin{bmatrix} \tau^{11} & \tau^{12} & \tau^{13} \\ \tau^{21} & \tau^{22} & \tau^{23} \\ \tau^{31} & \tau^{32} & \tau^{33} \end{bmatrix}.$$

Notice that $\vec{u}\otimes\vec{u}$ and $\mathbb{T}$ are simmetric tensors. Before we perform the next step, recall that:

$$\int_0^\eta \vec{u}\otimes\vec{u} = \eta\vec{U}\otimes\vec{U} + \int_0^\eta \tilde{u}\otimes\tilde{u} =: \clubsuit\,,$$

$$\textit{while} \quad U^3 = \epsilon_{\mathcal{G}}U^3_{(1)} + \mathcal{O}\left(\epsilon_{\mathcal{G}}^2\right) \quad \textit{and} \quad U^i = U^i_{(0)} + \epsilon_{\mathcal{G}}U^i_{(1)} + \mathcal{O}\left(\epsilon_{\mathcal{G}}^2\right)\,, \quad i = 1, 2.$$

Observe that $\clubsuit$ is a 3×3 tensor and that we are interested in its third column, $\clubsuit^{(\cdot 3)}$, and we neglect the $\int \tilde{u}\otimes\tilde{u}$ contribution. Employing the above given approximations, we have for example that

$$\clubsuit^{(13)} = \eta U^1U^3 = \eta\left(U^1_{(0)} + \epsilon_{\mathcal{G}}U^1_{(1)} + \mathcal{O}\left(\epsilon_{\mathcal{G}}^2\right)\right)\left(\epsilon_{\mathcal{G}}U^3_{(1)} + \mathcal{O}\left(\epsilon_{\mathcal{G}}^2\right)\right) =$$
$$= \eta\epsilon_{\mathcal{G}}U^1_{(0)}U^3_{(1)} + \mathcal{O}\left(\epsilon_{\mathcal{G}}^2\right).$$

Now, if we introduce the expanded velocity and tensor components using the approach mentioned in eq.(4.26) and recalled right above here, we obtain

$$\frac{1}{\rho}\int_0^\eta \frac{\partial p}{\partial s^3} + g\int_0^\eta \frac{\partial x^3}{\partial s^3} +$$
$$+\epsilon_{\mathcal{G}}\left[\frac{\partial}{\partial t}\left(\eta U^3_{(1)}\right) + \frac{1}{h_{(1)}h_{(2)}}\left(\frac{\partial}{\partial s^1}\left(\eta U^1_{(0)}U^3_{(1)}h_{(1)}h_{(2)}\right) + \frac{\partial}{\partial s^2}\left(\eta U^2_{(0)}U^3_{(1)}h_{(1)}h_{(2)}\right)\right) + \eta U^3_{(1)}\lambda\right.$$

$$-\frac{1}{\rho h_{(1)}h_{(2)}}\left(\frac{\partial}{\partial s^1}\int_0^\eta \tau^{31}_{(1)}h_{(1)}h_{(2)}+\frac{\partial}{\partial s^2}\int_0^\eta \tau^{32}_{(1)}h_{(1)}h_{(2)}\right)+\frac{p_{\mathcal{B},(1)}}{\rho}\Bigg]+\mathcal{O}\left(\epsilon_{\mathcal{G}}^2\right)=0\,, \tag{4.33}$$

where $p_{\mathcal{B},(1)}$ is a first order approximation of the s^3-component of the shear stress $\mathbb{T}_{\mathcal{B}}\cdot\mathbf{N}_{\mathcal{B}}$ thus assumed to be proportional to $\epsilon_{\mathcal{G}}$.
Indeed, the approximation of the third component of the new terms

$$\int_0^\eta \vec{u}\lambda+\int_0^\eta \vec{u}\circ\vec{h} \tag{4.34}$$

is given by (recall that the third component of the right term is simply zero):

$$\eta U^3\lambda=\eta\left[\epsilon_{\mathcal{G}}U^3_{(1)}+\mathcal{O}\big(\epsilon_{\mathcal{G}}^2\big)\right]\lambda=\epsilon_{\mathcal{G}}\left[\eta U^3_{(1)}\lambda\right]+\mathcal{O}\big(\epsilon_{\mathcal{G}}^2\big)$$

and thus is a second order term, in the hyphothesis that η is small, that does not contribute to a change in form of eq. (4.35).

Let us go back to equation (4.33). The zero-order terms (i.e. that are proportional to $\epsilon^0_{\mathcal{G}}$) are:

$$\frac{1}{\rho}\int_0^\eta\frac{\partial p}{\partial s^3}+g\int_0^\eta\frac{\partial x^3}{\partial s^3}=\mathcal{O}\left(\epsilon_{\mathcal{G}}\right)\,.$$

We neglect the effects of wind and surface tension to set $p\Big|_0=p_{atm}=0$ and we employ the approximations introduced in section 2.1.2 to get:

$$p\Big|_\eta=-\rho g\eta\frac{\partial x^3}{\partial s^3}+\mathcal{O}(\epsilon_{\mathcal{G}})\,. \tag{4.35}$$

Notice that $\partial x^3/\partial s^3$ is constant in s^3, since the direction s^3 is assumed rectilinear.

Remark. In this case, expression (4.35) has a slightly different meaning then the one found in chapter 2. In fact, in the present situation we are performing a measurement starting from a point **P** on the top surface. In the LCS-t attached to the point **P**, the latter has an associated height $\eta=0$, while if we follow s^3 direction for η steps we encounter another point **Q** belonging to the bottom surface. Nevertheless, in this case we have that expression $\eta\partial x^3/\partial s^3$ is not the vertical water below point **P**, but instead is a portion of the water above point **Q**. If we call H_Q the height of the water above point **Q**, we assume that $H_Q - \eta\partial x^3/\partial s^3 = \mathcal{O}(\epsilon_{\mathcal{G}})$.

4.4.2 System reduction

It is exactly the pressure condition that allows us to reduce dimentionally our ISWE system from four to three equations, as in the classical SWE derivation. In fact, our idea is to transport the information we just obtained from the third component of the momentum equation, which has just led us to the hydrostatic pressure condition (4.35), to the other two components of the momentum equation itself.

Putting our attention now on the s^1-component of eq. (4.32), the other (s^2-component) being analogous, we write (using (2.8) with $j = 1$):

$$\frac{\partial}{\partial t}\int_0^\eta u^1 + \frac{1}{h_{(1)}h_{(2)}}\left[\frac{\partial}{\partial s^1}\int_0^\eta h_{(1)}h_{(2)}\left(u^1\right)^2 + \frac{\partial}{\partial s^2}\int_0^\eta h_{(1)}h_{(2)}u^1u^2 + \frac{\partial}{\partial s^3}\int_0^\eta h_{(1)}h_{(2)}u^3u^1\right]$$
$$+\int_0^\eta u^1\lambda + \int_0^\eta u^1\frac{\partial_t h_{(1)}}{h_{(1)}} + \int_0^\eta \frac{\left(u^1\right)^2}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1} + 2\int_0^\eta \frac{u^1u^2}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2} - \int_0^\eta \left(u^2\right)^2\frac{h_{(2)}}{h_{(1)}^2}\frac{\partial h_{(2)}}{\partial s^1} =$$
$$= -\frac{1}{\rho}\int_0^\eta \frac{1}{h_{(1)}^2}\frac{\partial p}{\partial s^1} - g\int_0^\eta \frac{1}{h_{(1)}^2}\frac{\partial x^3}{\partial s^1} + \frac{1}{\rho h_{(1)}h_{(2)}}\left[\frac{\partial}{\partial s^1}\int_0^\eta h_{(1)}h_{(2)}\tau^{11} + \frac{\partial}{\partial s^2}\int_0^\eta h_{(1)}h_{(2)}\tau^{12}\right.$$
$$\left.+\frac{\partial}{\partial s^3}\int_0^\eta h_{(1)}h_{(2)}\tau^{13}\right] + \frac{1}{\rho}\left(\int_0^\eta \frac{\tau^{11}}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1} + 2\int_0^\eta \frac{\tau^{21}}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2} - \int_0^\eta \tau^{22}\frac{h_{(2)}}{h_{(1)}}\frac{\partial h_{(2)}}{\partial s^1}\right) + \frac{\tau_b^1}{\rho}.$$

Now we employ expansions seen above focusing on the left-hand side of the previous equation:

$$\frac{\partial \eta U^1_{(0)}}{\partial t} + \frac{1}{h_{(1)}h_{(2)}}\frac{\partial}{\partial s^1}\left(\eta\left(U^1_{(0)}\right)^2 h_{(1)}h_{(2)}\right) + \frac{1}{h_{(1)}h_{(2)}}\frac{\partial}{\partial s^2}\left(\eta U^1_{(0)}U^2_{(0)}h_{(1)}h_{(2)}\right)$$
$$+\eta U^1_{(0)}\left(2\frac{\partial_t h_{(1)}}{h_{(1)}} + \frac{\partial_t h_{(2)}}{h_{(2)}}\right) +$$
$$+\eta\left(U^1_{(0)}\right)^2\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1} + 2\eta U^1_{(0)}U^2_{(0)}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2} - \eta\left(U^2_{(0)}\right)^2\frac{h_{(2)}}{h_{(1)}}\frac{\partial h_{(2)}}{\partial s^1} + \mathcal{O}(\epsilon_{\mathcal{G}}),$$

and in particular the new terms (4.34) produce:

$$\int_0^\eta u^1\lambda + \int_0^\eta u^1\frac{\partial_t h_{(1)}}{h_{(1)}} = \int_0^\eta 2u^1\frac{\partial_t h_{(1)}}{h_{(1)}} + \int_0^\eta u^1\frac{\partial_t h_{(2)}}{h_{(2)}} = 2\eta U^1\frac{\partial_t h_{(1)}}{h_{(1)}} + \eta U^1\frac{\partial_t h_{(2)}}{h_{(2)}} =$$

$$= \eta U^1 \left(2\frac{\partial_t h_{(1)}}{h_{(1)}} + \frac{\partial h_{(2)}}{h_{(2)}} \right) = \eta \left(U^1_{(0)} + \epsilon_{\mathcal{G}} U^1_{(1)} + \mathcal{O}(\epsilon^2_{\mathcal{G}}) \right) =$$

$$= \eta U^1_{(0)} \left(2\frac{\partial_t h_{(1)}}{h_{(1)}} + \frac{\partial_t h_{(2)}}{h_{(2)}} \right) + \mathcal{O}(\epsilon_{\mathcal{G}}) \,. \tag{4.36}$$

Then on the right-hand side:

$$- \frac{\eta}{\rho h^2_{(1)}} \frac{\partial p}{\partial s^1} - \frac{\eta g}{h^2_{(1)}} \frac{\partial x^3}{\partial s^1} + \frac{1}{h_{(1)} h_{(2)}} \left[\frac{\partial}{\partial s^1} \int_0^\eta \tau^{11}_{(0)} h_{(1)} h_{(2)} + \frac{\partial}{\partial s^2} \int_0^\eta \tau^{12}_{(0)} h_{(1)} h_{(2)} \right]$$
$$+ \frac{1}{\rho h_{(1)}} \frac{\partial h_{(1)}}{\partial s^1} \int_0^\eta \tau^{11}_{(0)} + \frac{2}{\rho h_{(1)}} \frac{\partial h_{(1)}}{\partial s^2} \int_0^\eta \tau^{12}_{(0)} - \frac{h_{(2)}}{\rho h^2_{(1)}} \frac{\partial h_{(2)}}{\partial s^1} \int_0^\eta \tau^{22}_{(0)} + \frac{\tau^1_{b,(0)}}{\rho} + \mathcal{O}(\epsilon_{\mathcal{G}}).$$

Substituting in the right-hand side term the value of the pressure $p\Big|_\eta$ found in equation (4.35) we have

$$+ \frac{\eta}{h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\eta g \frac{\partial x^3}{\partial s^3} \right) - \frac{\eta g}{h^2_{(1)}} \frac{\partial x^3}{\partial s^1} + \frac{1}{\rho} \left(\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw,(0)} \right)^1 + \frac{\tau^1_{b,(0)}}{\rho} + \mathcal{O}(\epsilon_{\mathcal{G}}). \tag{4.37}$$

With the idea to write the blue term in an equivalent way, recalling that g is a constant, observe that (if we assume η and $\partial s^3 / \partial s^3$ to be differentiable functions):

$$\frac{\eta}{h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\eta g \frac{\partial x^3}{\partial s^3} \right) = \frac{\eta g}{h^2_{(1)}} \frac{\partial \eta}{\partial s^1} \frac{\partial x^3}{\partial s^3} + \frac{g \eta^2}{h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\frac{\partial x^3}{\partial s^3} \right) , \qquad \textit{while}$$
$$\frac{g}{h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\frac{\eta^2}{2} \frac{\partial x^3}{\partial s^3} \right) = \frac{\eta g}{h^2_{(1)}} \frac{\partial \eta}{\partial s^1} \frac{\partial x^3}{\partial s^3} + \frac{g \eta^2}{2 h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\frac{\partial x^3}{\partial s^3} \right) ;$$

so we can write, finally:

$$\frac{\eta}{h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\eta g \frac{\partial x^3}{\partial s^3} \right) = \frac{g}{h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\frac{\eta^2}{2} \frac{\partial x^3}{\partial s^3} \right) + \frac{g \eta^2}{2 h^2_{(1)}} \frac{\partial}{\partial s^1} \left(\frac{\partial x^3}{\partial s^3} \right) .$$

Finally, after substitution of (4.35) in the RHS and denoting with $\vec{q} := \left[\eta U^1_{(0)}, \eta U^2_{(0)} \right]$, we can write the Momentum equation in compact form intrinsic to the top surface:

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \lambda + \vec{q} \circ [h^1, h^2]^T + \nabla_{\mathcal{G}} \cdot \left(\frac{1}{\eta} (\vec{q} \otimes \vec{q}) + \left(\frac{g \eta^2}{2} \frac{\partial x^3}{\partial s^3} \right) \mathcal{G}^{-1}_{sw} \right)$$
$$+ \frac{g \eta^2}{2} \nabla_{\mathcal{G}} \left(\frac{\partial x^3}{\partial s^3} \right) + g \eta \nabla_{\mathcal{G}} (x^3) - \frac{1}{\rho} \nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw} + \frac{\mathbf{f}_{\mathcal{B}}}{\rho} = 0 \,. \tag{4.38}$$

Collecting the Continuity equation and the Momentum equations in a system, we have the following theorem.

Theorem 4.1. *The intrinsic shallow water equations, written with respect to the LCS-t are given by*

$$\frac{\partial \eta}{\partial t} - \frac{\partial \mathcal{F}}{\partial t} + \lambda\eta + \nabla_{\mathcal{G}} \cdot \vec{q} = 0\,, \tag{4.39}$$

$$\begin{aligned} \frac{\partial \vec{q}}{\partial t} + \vec{q}\lambda + \vec{q} \circ [h^1, h^2]^T + \nabla_{\mathcal{G}} \cdot \left(\frac{1}{\eta}(\vec{q} \otimes \vec{q}) + \left(\frac{g\eta^2}{2} \frac{\partial x^3}{\partial s^3} \right) \mathcal{G}_{sw}^{-1} \right) \\ + \frac{g\eta^2}{2} \nabla_{\mathcal{G}} \left(\frac{\partial x^3}{\partial s^3} \right) + g\eta \nabla_{\mathcal{G}}(x^3) - \frac{1}{\rho} \nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw} + \frac{\mathbf{f}_{\mathcal{B}}}{\rho} = 0\,. \end{aligned} \tag{4.40}$$

Under the hyphothesis of thin fluid layer $\eta = \mathcal{O}(\epsilon_{\mathcal{G}})$, they are an approximation of order $\mathcal{O}(\epsilon_{\mathcal{G}}^2)$ of the Navier-Stokes equations.

4.4.3 Balance law formulation of time-dependent ISWE

The final step before proceeding with the numerical approximations is to write everything as a compact balance law that will be very useful later on:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathrm{div}_{\mathcal{G}}\, \underline{\underline{F}}(\mathbf{s}, \mathbf{U}) + \mathbf{S}(\mathbf{s}, \mathbf{U}) = 0\,. \tag{4.41}$$

The conserved quantity is $\mathbf{U} = [\eta, \eta U^1, \eta U^2]^T = [\eta, q^1, q^2]^T$, where $\eta : \Gamma \times [0, t_f] \longrightarrow \mathbb{R}$ and $\mathbf{q} = [q^1, q^2]$, $\mathbf{q} : \Gamma \times [0, t_f] \longrightarrow \mathbb{R}^2$. The flux function is the same:

$$\underline{\underline{F}}(\mathbf{s}, \mathbf{U}) = \begin{bmatrix} q^1 & q^2 \\ \dfrac{(q^1)^2}{\eta} + \dfrac{g\eta^2}{2h_{(1)}^2} \dfrac{\partial x^3}{\partial s^3} & \dfrac{q^1 q^2}{\eta} \\ \dfrac{q^1 q^2}{\eta} & \dfrac{(q^2)^2}{\eta} + \dfrac{g\eta^2}{2h_{(2)}^2} \dfrac{\partial x^3}{\partial s^3} \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{F}}^{\eta} \\ \\ \underline{\underline{F}}^{\mathbf{q}} \end{bmatrix}. \tag{4.42}$$

Recall that $\mathrm{div}_{\mathcal{G}} = [\nabla^{\eta}_{\mathcal{G}}\cdot, \nabla^{\mathbf{q}}_{\mathcal{G}}\cdot]^T$, while the term $\mathbf{S}(\mathbf{s},\eta)$ is given by

$$\mathbf{S}(\mathbf{s},\mathbf{U}) = \begin{bmatrix} -\dfrac{\partial \mathcal{F}}{\partial t} + \left(\dfrac{\partial_t h_{(1)}}{h_{(1)}} + \dfrac{\partial_t h_{(2)}}{h_{(2)}}\right)\eta \\ q^1\left(2\dfrac{\partial_t h_{(1)}}{h_{(1)}} + \dfrac{\partial_t h_{(2)}}{h_{(2)}}\right) + \dfrac{g\eta^2}{2h_{(1)}^2}\dfrac{\partial}{\partial s^1}\left(\dfrac{\partial x^3}{\partial s^3}\right) + \dfrac{g\eta}{h_{(1)}^2}\dfrac{\partial x^3}{\partial s^1} - \dfrac{1}{\rho}[\nabla_{\mathcal{G}}\cdot \mathbf{T}_{sw}]^{(1,\cdot)} + \dfrac{\tau_b^1}{\rho} \\ q^2\left(\dfrac{\partial_t h_{(1)}}{h_{(1)}} + 2\dfrac{\partial_t h_{(2)}}{h_{(2)}}\right) + \dfrac{g\eta^2}{2h_{(2)}^2}\dfrac{\partial}{\partial s^2}\left(\dfrac{\partial x^3}{\partial s^3}\right) + \dfrac{g\eta}{h_{(2)}^2}\dfrac{\partial x^3}{\partial s^2} - \dfrac{1}{\rho}[\nabla_{\mathcal{G}}\cdot \mathbf{T}_{sw}]^{(2,\cdot)} + \dfrac{\tau_b^2}{\rho} \end{bmatrix} = \begin{bmatrix} S^{\eta} \\ \\ \mathbf{S}^{\mathbf{q}} \\ \\ \end{bmatrix}. \tag{4.43}$$

4.5 Comparison of the intrinsic and standard approaches

In this section, we are going to compare our intrinsic formulation of the direct and inverse problem to the *standard* setting in [15]. If written in a compact form, equations (4.44) and (4.41) can be synthesized, as we saw, in the following form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathrm{div}_{\mathcal{G}}\, \underline{\underline{F}}(s,\mathbf{U}) + \mathbf{S}(s,\mathbf{U}) = 0\,, \tag{4.44}$$

where we recall that $\mathbf{U} = [\eta, \eta U^1, \eta U^2]^T = [\eta, q^1, q^2]^T$, $\eta : \Gamma \times [0,t_f] \longrightarrow \mathbb{R}$, *and* $\mathbf{q} = [q^1, q^2]\,, \mathbf{q} : \Gamma \times [0,t_f] \longrightarrow \mathbb{R}^2$.

Let us recall some variables' definitions of [15]: ζ is the vertical height of the water, z_b is the bottom height and H denotes the vertical depth of water

$$H := \xi - z_b\,.$$

Furthermore, $\mathbf{v} = [u,v,w]$ is the fluid velocity and $\bar{u}, \bar{v}$ are the averaged velocity first two components, respectively given by

$$\bar{u} = \frac{1}{H}\int_{z_b}^{\xi} u\, dz \qquad \bar{v} = \frac{1}{H}\int_{z_b}^{\xi} v\, dz\,.$$

The balance laws for the problem in [15] can be written, as seen in chapter 3 in the form

$$\partial_t \mathbf{c} + \nabla\cdot\mathcal{A}^{\zeta}(\mathbf{c}) = \mathbf{Z}^{\zeta}(\mathbf{c})$$

with $\mathbf{c} = [H,\, \bar{u}H\,, \bar{v}H] = [H\,U\,V]$, while $\mathcal{A}^\zeta$ and $\mathbf{Z}^\zeta$ denote the flux term and the source term for the inviscid ($\mu = 0$) forward ($\zeta = 1$) and backward ($\zeta = 0$) problems. Now, for the purpouse of this section, i.e. being able to confront different models, let us try to write everything with the notation used in this thesis. Thus the idea is to replace H with η, $\bar{u}$ with U^1, $\bar{v}$ with U^2, c^2 with q^1, c^3 with q^2 and so on in the equations from chapter 3. Notice that in the inverse problem formulation, $\zeta = 0$, $\mathbf{Z}^\zeta$ is such that the variable ξ (the vertical height of water), that was an unknown of the direct problem now becomes a known data and in our notation corresponds to $\mathcal{F}$, while the bottom z_b could be written as $\mathcal{F} - \eta$ in our notation. Regrouping everything, we have

model of chapter 3	(*intrinsic*) model, chapter 2/ 4
ξ	$\mathcal{F}$
$\bar{u}, \bar{v}$	U^1, U^2
H	η
$\mathbf{c} = [H, uH, vH]$	$\mathbf{U} = [\eta, q^1, q^2]$

The following table contains the flux and source terms of both the direct and inverse problems formulation, governed by the conservation law (4.44).

$$\frac{\partial \mathbf{U}}{\partial t} + \mathrm{div}_{\mathcal{G}}\, \underline{\underline{F}}(\mathbf{s},\mathbf{U}) + \mathbf{S}(\mathbf{s},\mathbf{U}) = 0 \qquad F: \text{forward } \textit{problem}, \quad I: \text{inverse } \textit{problem}$$

	standard setting	*Intrinsic* setting
F	$\underline{\underline{F}}(\mathbf{s},\mathbf{U}) = \begin{bmatrix} q^1 & q^2 \\ \frac{(q^1)^2}{\eta} + \frac{g\eta^2}{2} & \frac{q^1 q^2}{\eta} \\ \frac{q^1 q^2}{\eta} & \frac{(q^2)^2}{\eta} + \frac{g\eta^2}{2} \end{bmatrix}$ $\mathbf{S}(\mathbf{s},\eta) = \begin{bmatrix} 0 \\ -g\eta \frac{\partial}{\partial s^1}(\mathcal{F} - \eta) + \eta f_1 - \tau_{bf} q^1 + f_c q^2 \\ -g\eta \frac{\partial}{\partial s^2}(\mathcal{F} - \eta) + \eta f_2 - \tau_{bf} q^2 - f_c q^1 \end{bmatrix}$	$\underline{\underline{F}}(\mathbf{s},\mathbf{U}) = \begin{bmatrix} q^1 & q^2 \\ \frac{(q^1)^2}{\eta} + \frac{g\eta^2}{2h_{(1)}^2}\frac{\partial x^3}{\partial s^3} & \frac{q^1 q^2}{\eta} \\ \frac{q^1 q^2}{\eta} & \frac{(q^2)^2}{\eta} + \frac{g\eta^2}{2h_{(2)}^2}\frac{\partial x^3}{\partial s^3} \end{bmatrix}$ $\mathbf{S}(\mathbf{s},\eta) = \begin{bmatrix} 0 \\ \frac{g\eta^2}{2h_{(1)}^2}\frac{\partial}{\partial s^1}\left(\frac{\partial x^3}{\partial s^3}\right) + \frac{g\eta}{h_{(1)}^2}\frac{\partial x^3}{\partial s^1} - \frac{1}{\rho}[\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw}]^{(1,\cdot)} - \frac{\tau_b^1}{\rho} \\ \frac{g\eta^2}{2h_{(2)}^2}\frac{\partial}{\partial s^2}\left(\frac{\partial x^3}{\partial s^3}\right) + \frac{g\eta}{h_{(2)}^2}\frac{\partial x^3}{\partial s^2} - \frac{1}{\rho}[\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw}]^{(2,\cdot)} - \frac{\tau_b^2}{\rho} \end{bmatrix}$
I	$\underline{\underline{F}}(\mathbf{s},\mathbf{U}) = \begin{bmatrix} q^1 & q^2 \\ \frac{(q^1)^2}{\eta} & \frac{q^1 q^2}{\eta} \\ \frac{q^1 q^2}{\eta} & \frac{(q^2)^2}{\eta} \end{bmatrix}$ $\mathbf{S}(\mathbf{s},\eta) = \begin{bmatrix} 0 \\ -g\eta \frac{\partial}{\partial s^1}\mathcal{F} + \eta f_1 - \tau_{bf} q^1 + f_c q^2 \\ -g\eta \frac{\partial}{\partial s^2}\mathcal{F} + \eta f_2 - \tau_{bf} q^2 - f_c q^1 \end{bmatrix}$	$\underline{\underline{F}}(\mathbf{s},\mathbf{U}) = \begin{bmatrix} q^1 & q^2 \\ \frac{(q^1)^2}{\eta} + \frac{g\eta^2}{2h_{(1)}^2}\frac{\partial x^3}{\partial s^3} & \frac{q^1 q^2}{\eta} \\ \frac{q^1 q^2}{\eta} & \frac{(q^2)^2}{\eta} + \frac{g\eta^2}{2h_{(2)}^2}\frac{\partial x^3}{\partial s^3} \end{bmatrix}$ $S(\mathbf{s},\mathbf{U}) = \begin{bmatrix} -\frac{\partial \mathcal{F}}{\partial t} + \left(\frac{\partial_t h_{(1)}}{h_{(1)}} + \frac{\partial_t h_{(2)}}{h_{(2)}}\right)\eta \\ q^1\left(2\frac{\partial_t h_{(1)}}{h_{(1)}} + \frac{\partial_t h_{(2)}}{h_{(2)}}\right) + \frac{g\eta^2}{2h_{(1)}^2}\frac{\partial}{\partial s^1}\left(\frac{\partial x^3}{\partial s^3}\right) + \frac{g\eta}{h_{(1)}^2}\frac{\partial x^3}{\partial s^1} - \frac{1}{\rho}[\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw}]^{(1,\cdot)} + \frac{\tau_b^1}{\rho} \\ q^2\left(\frac{\partial_t h_{(1)}}{h_{(1)}} + 2\frac{\partial_t h_{(2)}}{h_{(2)}}\right) + \frac{g\eta^2}{2h_{(2)}^2}\frac{\partial}{\partial s^2}\left(\frac{\partial x^3}{\partial s^3}\right) + \frac{g\eta}{h_{(2)}^2}\frac{\partial x^3}{\partial s^2} - \frac{1}{\rho}[\nabla_{\mathcal{G}} \cdot \mathbf{T}_{sw}]^{(2,\cdot)} + \frac{\tau_b^2}{\rho} \end{bmatrix}$

- For the *forward* problem

Focusing on the flux terms, we can notice that the *intrinsic* setting formulation, if applied to the standard setting in which $\frac{\partial x^3}{\partial s^3} = 1$ and the metric coefficients are $h^2_{(1)} = h^2_{(2)} = 1$, exactly gives the flux term on the left. Furthermore, if we look at the source terms, note the similarity between

$$-g\eta \frac{\partial}{\partial s^1} \underbrace{(\mathcal{F} - \eta)}_{z_b} \qquad \textit{and} \qquad \frac{\eta g}{h^2_{(1)}} \frac{\partial x^3}{\partial s^1} \, .$$

In the latter term, recall that the partial derivative $\frac{\partial x^3}{\partial s^1}$ is being evaluated for $\eta = 0$, i.e. on the surface bottom, in fact this term was born from the hydrostatic pressure condition. Also the source term of the *intrinsic* scenario reduces to its left term in the table, in the case of a *standard* setting, since for $i = 1, 2$

$$\frac{g\eta^2}{2h^2_{(1)}} \frac{\partial}{\partial s^i} \left(\frac{\partial x^3}{\partial s^3} \right)$$

vanishes because x^3 and s^3 directions are parallell.

- For the *inverse* problem If the normal to the top surface is vertical, or in the procedure of the Shallow water depth-integration we are integrating along the vertical direction our equations, denoting with $b := \xi - H \equiv \mathcal{F} - \eta \dfrac{\partial x^3}{\partial s^3}$, the term ∇x^3 can be written as

$$\nabla x^3 = \nabla b = \nabla \mathcal{F} - \frac{\partial x^3}{\partial s^3} \nabla \eta - \eta \nabla \left(\frac{\partial x^3}{\partial s^3} \right) .$$

Remark. The bottom right square of the table contains the flux terms of the bathymetry reconstruction. The problem to solve in this case has a time-dependent top surface and a bottom fixed in time. Nevertheless, our model is so general that can be theoretically used also in the case of an eroding bottom. On the countrary, in analogous way, it is possible to see it as a direct model having the bottom surface that is time dependent (as a known data) with the aim of determining the top surface (unknown), also time-dependent. To find the formulation of the latter, one should be careful about the sign of the terms arasing from the hydrostatic pressure condition, since the roles of $p\big|_{s^3=0}$ and $p\big|_{s^3=\eta}$ will be clearly inverted. These two final cases, that look possible in theory, do present many difficulties at a numerical level and could be very problematic to deal with.

Chapter 5

Numerical set-up for the bathymetry reconstruction model

In this chapter we are going to derive the fully-discrete Discontinuous Galerkin (DG) formulation of our model. To do so, we firstly have to approximate the top surface and build the basis functions that will be employed to *test* our equations: this will be done following the work of [4]. Notice that in this case our surface is time-dependent, so the construction has to be done for every time-step. Consider a fixed time $\hat{t} \in [0, T]$. The introduction of the time parameter, apparently harmless, will introduce some issues of linking one solution, computed at the time $\hat{t}$ and associated to the basis space $\mathcal{V}_{\hat{t}}$, to the solution at the time-step $\hat{t} + 1$, because $\mathcal{V}_{\hat{t}} \neq \mathcal{V}_{\hat{t}+1}$. The paper of G. Dziuk and C.Elliot [14] will be very helpful in traducing this problem to a form that is more direct to solve and requires much less numerical effort. Finally, through this manipulation we will be able to write the fully-discrete DG formulation.

5.1 Surface triangulation

Recall that at every time t we have a parametrization $\mathcal{F}(x_1, x_2, t)$ of the top surface that allowed us to build the time-dependent Local Coordinate System. The construction of the mesh surface follows the same steps of [4]. The basic ideas are that we have $\Gamma \subseteq \mathcal{S}$ a regular region which can be divided into small triangles such that

- $\Gamma = \bigcup_i T_i$;

- the intersection of two arbitrary triangles is either empty, or consists of vertices, or is a side;
- every vertex at the boundary of our region R, i.e. a partition of Γ, is a vertex of at least one triangle of the triangulation $\mathcal{T}(\Gamma)$.

We have that $\mathcal{T}(\Gamma) = \bigcup_{i=1}^{N_T} T_i = \bar{\Gamma}$ and $\sigma_{ij} = T_i \bigcap T_j$ is an internal geodesic edge. Furthermore, we will denote by $\mathcal{T}_h(\Gamma)$ the approximate triangulation that form the piece-wise linear surface, made up by the union of flat 2-dim triangles. An important assumption is the constrain

$$\frac{r_T}{h_T} \geq \rho \qquad \forall T \in \mathcal{T}_h(\Gamma)\,,$$

which ensures the non degeneration of the mesh elements. The *orthogonal projection* along the surface normal direction $\mathbf{N}(pr(\mathbf{q}))$ maps a point $\mathbf{q} \in T_h$ to $pr(\mathbf{q}) \in T \subset \mathcal{T}(\Gamma)$.

Proposition 5.1. *Given the triangulations $\mathcal{T}(\Gamma)$, $\mathcal{T}_h(\Gamma)$ and the map pr, the following estimates hold:*

- *the distance between the approximate triangulation and the surface satisfies:*

$$\max_{\mathbf{q} \in \mathcal{T}_h(\Gamma)} \left| \overrightarrow{pr(\mathbf{q})\mathbf{q}} \right| \leq C h^2\,;$$

- *the ratio δ_h between the area measures $d\mathbf{s}$ and $d\mathbf{x}$ of the surface and its approximation, defined by $d\mathbf{s} = \delta_h \, d\mathbf{x}$, satisfies:*

$$\| \, 1 - \delta_h \, \|_{L^\infty(\mathcal{T}_h(\Gamma))} \leq C h^2\,.$$

We assume all the relevant information, for example the tangent plane, to be known at the vertices of the triangulation, while we use interpolated information at quadrature points, i.e. the point where we perform computations.

5.2 Basis functions

For simplicity, we work with first order affine functions on each cell of our domain. At a fixed time $\hat{t}$, the basis functions $\varphi_1, \varphi_2, \varphi_3$ span $\mathcal{V}_h^\Gamma$ satisfying the interpolation property are defined by:

$$\varphi_j \in \mathcal{V}_h^\Gamma, \quad \varphi_j(\mathbf{p}_i) = \delta_{ij} \quad i, j = 1, 2, 3\,,$$

where $\mathbf{p_i} \in \Gamma$are the vertices (the nodes) of the cell, for each cell of $\mathcal{T}(\Gamma)$. We need to distinguish between global and local basis functions. Let us fix a point $\mathbf{x} \in T$ of global coordinates $\mathbf{x}(\mathbf{p})$, we define the affine function $\tilde{\phi}_j$ as a function in $\mathbb{R}^3$, expressed in global coordinates as

$$\tilde{\varphi}_j(\mathbf{x}) = \tilde{a} + \tilde{b}x^1 + \tilde{c}x^2 + \tilde{d}x^3\,,$$

where the coefficients $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ can be determined by solving 4-dim linear systems. For example, in the case of φ_1 the constrains to impose are

$$\varphi_1(\mathbf{p}_1) = 1\,, \varphi_1(\mathbf{p}_2) = 0\,, \varphi_1(\mathbf{p}_3) = 0\,, \varphi_1(\mathbf{q}) = 0\,,$$

with $\mathbf{q} = \mathbf{p}_1 + \mathbf{t}_3(\mathbf{p}_1)$ and $\mathbf{t}_3(\mathbf{p}_1)$ the unitary normal to the surface in $\mathbf{p}_1$. Recalling that $\Phi : \text{LCS} \longmapsto \text{GCS}$, the composition of $\tilde{\phi}_j$ with the latter gives the basis function in the local coordinates:

$$\varphi_j(s^1, s^2) = \tilde{\varphi}_j \circ \phi(s^1, s^2)\,.$$

Observe that only the tangent plane $T_{\mathbf{p}_j}\Gamma$ information is really necessary, in fact assuming that $T \subset \phi_{\mathbf{p}_j}(U)$ for some U open set of $\mathbb{R}^2$ we can approximate linearly the surface: $\phi_{\mathbf{p}_j}(T) = T_{\mathbf{p}_j}\Gamma + \mathcal{O}(h^2)$. The local basis functions are obtained, accordingly, by neglecting higher order terms.

5.3 Evolving surface DG

Starting from the balance law equation found in the previous chapter,

$$\frac{\partial \mathbf{U}}{\partial t} + \operatorname{div}_{\mathcal{G}} \underline{\underline{F}}(\mathbf{s}, \mathbf{U}) + \mathbf{S}(\mathbf{s}, \mathbf{U}) = 0\,, \tag{5.1}$$

we know that $\mathbf{U} = [\eta, q^1, q^2]$, now we want to test equation (5.1) with $v_h \in \mathcal{V}_h(t)$ and integrate in space, over every single cell $T(t)$ of the triangulation $\mathcal{T}(\Gamma(t))$:

$$\int_{T(t)} \frac{\partial \mathbf{U}}{\partial t} v_h \, d\mathbf{s} + \int_{T(t)} \operatorname{div}_{\mathcal{G}} \underline{\underline{F}}(\mathbf{s}, \mathbf{U}) v_h \, d\mathbf{s} + \int_{T(t)} \mathbf{S}(\mathbf{s}, \mathbf{U}) v_h \, d\mathbf{s} = 0. \tag{5.2}$$

Application of the divergence theorem yields

$$\int_{T(t)} \frac{\partial \mathbf{U}}{\partial t} v_h \, d\mathbf{s} + \int_{\partial T(t)} \mathbf{F}^{\nu}_{*}(\mathbf{s}, \mathbf{U}) v_h \, d\sigma - \int_{T(t)} \langle \underline{\underline{F}}(\mathbf{s}, \mathbf{U}), \nabla_{\mathcal{G}} v_h \rangle_{\mathcal{G}} \, d\mathbf{s} + \int_{T(t)} \mathbf{S}(s, \mathbf{U}) v_h \, d\mathbf{s} = 0\,, \tag{5.3}$$

for every $v_h \in \mathcal{V}_h^{\Gamma}(t)$, with $\mathbf{F}^{\nu}_{*}(\mathbf{s}, \mathbf{U})$ the numerical flux at the cell boundary.

5.3.1 Towards the time discretization

We recall the definition of *appropriate* time derivative, following [14]. Given the parametrization

$$\begin{aligned} \phi : & I \times U \subseteq I \times \mathbb{R}^2 \longrightarrow I \times \mathbb{R}^3 \\ & (t, s^1, s^2) \longmapsto (t, x^1(s^1, s^2), x^2(s^1, s^2), x^3(s^1, s^2, t)) := (t, s^1, s^2, \mathcal{F}(s^1, s^2, t)) \end{aligned}$$

we denote with $\Gamma(t)$ the surface at time t and we assume the function $\varphi(\cdot, t) : U \longmapsto \Gamma(t) \in ([0, T], \mathcal{C}^2(U))$, and we say that the velocity of $\Gamma(t)$ is given by

$$v(\phi(\cdot, t), t) = \frac{\partial \phi}{\partial t}(\cdot, t)\,. \tag{5.4}$$

The appropriate time derivative of a function

$$f : \bigcup_{t \in [0,T]} \Gamma(t) \times \{t\} \longrightarrow \mathbb{R}$$

is defined as

$$\partial^\bullet f = \frac{\partial f}{\partial t} + v \cdot \nabla f \,. \tag{5.5}$$

At this point, we discretize the time interval $[0, T]$ in N parts setting $\tau = \dfrac{T}{N}$. So, we have $N+1$ time steps $t^n = n\tau$, for $n \in \{0, \dots, N\}$. Let's focus for a moment on the first term

$$\int_{T(t)} \frac{\partial \mathbf{U}}{\partial t} \varphi_h \, d\mathbf{s} \,.$$

If we use first order affine functions $\varphi_k \,, k \in \{1, 2, 3\}$ and write for each cell $T(t^n)$ an approximation of $\mathbf{U}_h$ in terms of the basis functions,

$$\mathbf{U}_h(\mathbf{s}, t) = \sum_{i=1}^{3} \mathbf{U}_{h,i}(t^n) \varphi_i(\mathbf{s}, t^n) \tag{5.6}$$

we have that the first term becomes now

$$\sum_{i=1}^{3} \int_{T(t^n)} \frac{\partial \mathbf{U}_{h,i}}{\partial t} \varphi_i^n \varphi_k^n \, d\mathbf{s} = \sum_{i=1}^{3} \int_{T(t^n)} \frac{\mathbf{U}_{h,i}^{n+1} - \mathbf{U}_{h,i}^{n}}{\tau} \varphi_i^n \varphi_k^n \, d\mathbf{s} \,, \tag{5.7}$$

where the superscript n in φ_i^n denotes that the basis function φ is relative to time t^n. From the last integral, computed at the $n-th$ time step, it clearly emerges that, roughly speaking, one would need to express the solution $\mathbf{U}_h^{n+1}$ as a linear combination of t^{n+1} basis functions living in the space $\mathcal{V}_h^{\Gamma}(t^{n+1})$, to be able to perform the next time step computation, and so on. For this reason, we follow the approach of G. Dziuk, C.Elliot [14]. In this way, by manipulating that integral, the problem of the connection between solutions at successive time steps can be traduced into the presence of additional terms that are easier to treat numerically. We need to recall the definition of the mass matrix

$$m(\varphi(\cdot, t), \psi(\cdot, t)) = \int_{\Gamma(t)} \varphi(\cdot, t) \psi(\cdot, t) \, dA \,, \tag{5.8}$$

and a couple of results.

Lemma 5.1. *(transport property of the basis functions)*
The basis functions satisfy the transport property

$$\partial_h^\bullet \varphi = 0 \,.$$

Lemma 5.2. *Let $\Gamma_h(t)$ be an evolving admissible triangulation with material velocity V_h. Then it holds*

$$\frac{d}{dt}\int_{\Gamma_h(t)} f\, dA_h = \int_{\Gamma_h(t)} \partial_h^{\bullet} f + f\nabla_{\Gamma_h}\cdot V_h\, dA_h\,,$$

where ∇_{Γ_h} is the intrinsic surface derivative. For a function $\varphi \in \mathcal{V}_h(t), W_h \in \mathcal{V}_h(t)$,

$$\frac{d}{dt}m_h(\varphi, W_h) = m_h(\partial_h^{\bullet}\varphi, W_h) + m_h(\varphi, \partial_h^{\bullet} W_h) + g_h(V_h;\varphi, W_h)\,,$$

with

$$g_h(V_h;\varphi, W_h) = \int_{\Gamma(t)} \varphi(x,t)\psi(x,t)\nabla_{\Gamma}\cdot v(x,t)\, dA(x)\,.$$

Notice that $\nabla_\Gamma \cdot v(x,t)$ is equivalent to our term λ, i.e. the time derivative of the determinant of the metric tensor. We can now go back to the first term of equation (5.3): the idea is to add and subtract the terms g_h and

$$\int_{T(t)} v\cdot\nabla\mathbf{U}_h\varphi$$

to get a more suitable form of our equation. Explicitly we have

$$\begin{aligned}\int_{T(t)} \partial_t\mathbf{U}_h\varphi_h\, d\mathbf{s} &= \int_{T(t)} \partial_t\mathbf{U}_h\varphi_h\, d\mathbf{s} + v\cdot\nabla\mathbf{U}_h\varphi_h\, d\mathbf{s} + g_h - \int_{T(t)} v\cdot\nabla\mathbf{U}_h\varphi_h\, d\mathbf{s} - g_h = \\ &= m(\varphi_h, \partial^{\bullet}\mathbf{U}_h) + g_h - \int_{T(t)} v\cdot\nabla\mathbf{U}_h\varphi_h\, d\mathbf{s} - g_h = \\ &= \frac{d}{dt}m_h(\varphi_h, \mathbf{U}_h) - \int_{T(t)} v\cdot\nabla\mathbf{U}_h\varphi_h\, d\mathbf{s} - g_h\,. \end{aligned} \tag{5.9}$$

Finally, the DG problem becomes

DG Problem . Find $\mathbf{U}_h \in \mathcal{V}_h^{\Gamma}$ such that

$$\begin{aligned}&\frac{d}{dt}\int_{T(t)} \varphi_h\mathbf{U}_h\, d\mathbf{s} - \int_{T(t)} v\cdot\nabla\mathbf{U}_h\varphi_h\, d\mathbf{s} - \lambda\int_{T(t)}\varphi_h\mathbf{U}_h\, d\mathbf{s} = \\ &= -\int_{\partial T(t)} \mathbf{F}_*^{\nu}(\mathbf{s},\mathbf{U}_h)\varphi_h\, d\sigma + \int_{T(t)} \langle \underline{\underline{F}}(\mathbf{s},\mathbf{U}_h), \nabla_{\mathcal{G}}\varphi_h\rangle_{\mathcal{G}}\, d\mathbf{s} - \int_{T(t)} \mathbf{S}(s,\mathbf{U}_h)\varphi_h\, d\mathbf{s} = 0\,,\end{aligned} \tag{5.10}$$

where φ_h belongs to $\mathcal{V}_h^{\Gamma}(t)$.

Notice that $-\lambda\mathbf{U}$ can be incorporated into $\mathbf{S}$, cancelling out some terms. Indeed, if we define a new $\mathbf{S}$ as

$$\mathbf{S}(\mathbf{s},\mathbf{U}) = \begin{bmatrix} -\dfrac{\partial\mathcal{F}}{\partial t} \\ q^1\dfrac{\partial_t h_{(1)}}{h_{(1)}} + \dfrac{g\eta^2}{2h_{(1)}^2}\dfrac{\partial}{\partial s^1}\left(\dfrac{\partial x^3}{\partial s^3}\right) + \dfrac{g\eta}{h_{(1)}^2}\dfrac{\partial x^3}{\partial s^1} - \dfrac{1}{\rho}[\nabla_{\mathcal{G}}\cdot\mathbf{T}_{sw}]^{(1,\cdot)} + \dfrac{\tau_b^1}{\rho} \\ q^2\dfrac{\partial_t h_{(2)}}{h_{(2)}} + \dfrac{g\eta^2}{2h_{(2)}^2}\dfrac{\partial}{\partial s^2}\left(\dfrac{\partial x^3}{\partial s^3}\right) + \dfrac{g\eta}{h_{(2)}^2}\dfrac{\partial x^3}{\partial s^2} - \dfrac{1}{\rho}[\nabla_{\mathcal{G}}\cdot\mathbf{T}_{sw}]^{(2,\cdot)} + \dfrac{\tau_b^2}{\rho} \end{bmatrix} = \begin{bmatrix} S^\eta \\ \\ \mathbf{S}^{\mathbf{q}} \\ \\ \end{bmatrix}, \tag{5.11}$$

we get a new

DG Problem . Find $\mathbf{U}_h \in \mathcal{V}_h^\Gamma$ such that

$$\begin{aligned} &\frac{d}{dt}\int_{T(t)} \varphi_h \mathbf{U}_h \, d\mathbf{s} - \int_{T(t)} v\cdot\nabla\mathbf{U}_h\varphi_h \, d\mathbf{s} = \\ &= -\int_{\partial T(t)} \mathbf{F}_*^\nu(\mathbf{s},\mathbf{U}_h)\varphi_h \, d\sigma + \int_{T(t)} \langle \underline{\underline{F}}(\mathbf{s},\mathbf{U}_h), \nabla_{\mathcal{G}}\varphi_h\rangle_{\mathcal{G}} \, d\mathbf{s} - \int_{T(t)} \mathbf{S}(s,\mathbf{U}_h)\varphi_h \, d\mathbf{s} = 0\,, \end{aligned} \tag{5.12}$$

where φ_h belongs to $\mathcal{V}_h^\Gamma(t)$.

5.3.2 Fully-discrete DG formulation

We can now substitute $\mathbf{U}$ with the approximate solution $\mathbf{U}_h$, that we recall is defined for each cell $T \in \mathcal{T}(\Gamma)$ as

$$\mathbf{U}_h(\mathbf{s},t) = \sum_{i=1}^{3} \mathbf{U}_{h,i}(t^n)\varphi_i(\mathbf{s},t^n)$$

where $\mathbf{U}_{h,i}(t)$ are the values of the numerical solution at the nodes at time t. So, we obtain the so called semi-discrete formulation of our problem:

Semi-discrete DG Problem. Find $\mathbf{U}_h \in \mathcal{V}_h^\Gamma$ such that

$$\sum_i \left(\frac{d}{dt}\int_T \mathbf{U}_{h,i}\varphi_i\varphi_k\right) - \mathbf{U}_{h,i}\int_{T(t)} v\cdot\nabla_{\mathcal{G}}\varphi_h\varphi_k \, d\mathbf{s} =$$

$$= -\int_{\partial T(t)} \mathbf{F}^{\nu}_{*}(\mathbf{s}, \mathbf{U}_h)\varphi_k \, d\sigma + \int_{T(t)} \langle \underline{\underline{F}}(\mathbf{s}, \mathbf{U}_h), \nabla_{\mathcal{G}}\varphi_k \rangle_{\mathcal{G}} \, d\mathbf{s} - \int_{T(t)} \mathbf{S}(s, \mathbf{U}_h)\varphi_k \, d\mathbf{s} = 0 \,, \tag{5.13}$$

where φ_h belongs to $\mathcal{V}_h^{\Gamma}(t)$ and for simplicity we suppose that v is already expressed in terms of $\mathcal{V}_h^{\Gamma}$; instead, given v written analitically, one could use the L^2-projection of v on the function space $\mathcal{V}_h^{\Gamma}$.

Furthermore, for each $i, k = 1, 2, 3$ we define the local mass matrix, the advection matrix and the right-hand side vector:

$$\mathbf{M}_{ik} = \int_T \varphi_i \varphi_k \, d\mathbf{s} \,,$$
$$\mathbf{A}_{ik} = \int_T v \cdot \nabla_{\mathcal{G}}\varphi_i \varphi_k \, d\mathbf{s} \,,$$
$$\mathbf{R}_k = -\int_{\partial T(t)} \mathbf{F}^{\nu}_{*}(\mathbf{s}, \mathbf{U}_h)\varphi_k \, d\sigma + \int_{T(t)} \langle \underline{\underline{F}}(\mathbf{s}, \mathbf{U}_h), \nabla_{\mathcal{G}}\varphi_k \rangle_{\mathcal{G}} \, d\mathbf{s} - \int_{T(t)} \mathbf{S}(s, \mathbf{U}_h)\varphi_k \, d\mathbf{s} = 0 \,.$$

Recall that we divided the time interval $[0, T]$ in N parts setting $\tau = \dfrac{T}{N}$ and the time steps are denoted with $t^n = n\tau$, for $n \in \{0, \ldots, N\}$. Thus, we have the following simple version of the fully discrete DG approximation:

$$\partial_\tau(\mathbf{M}^n \mathbf{u}^n) - \mathbf{A}^{n+1}\mathbf{u}^{n+1} = \mathbf{R}^{n+1} \,.$$

Equivalently, expliciting the ∂_τ derivative, we have

$$(\mathbf{M}^{n+1} - \tau\mathbf{A}^{n+1})\mathbf{u}^{n+1} - \tau\mathbf{R}^{n+1} = \mathbf{M}^n\mathbf{u}^n \,, \tag{5.14}$$

where $\mathbf{u} \equiv \mathbf{U}_{h,i}$ are the solution coefficients.

Conclusions

Through this thesis, we have derived a new intrinsic Shallow water model for the bathymetry reconstruction of e.g. rivers. At the very basis, an application of the tensorial calculus has given birth to the Continuity and Momentum equations: those form the Navier-Stokes equations, written with respect to a local reference frame situated on the top surface. Shallow water equations are derived from the NS equations after depth integration, following the directions of the local normals attached to the top surface itself. Finally, a numerical set-up of the model has been built, using the Discontinuous Galerkin (DG) method.

Future work

Efforts have been made of trying to derive the same equations in a different way. One possible approach would be starting from a Lagrangian function of the NS equations, although this idea seemed troublesome and beyond the purpose of this thesis. Search in this direction seem possible and interesting: in general, something that would provide a parallel way of deriving the same model equations, or showing their validy through numerical experiments.

First steps have been attempted for the latter, the idea constisted in modifying the bathymetry reconstruction part of the FESTUNG code, basically somehow inserting the geometrical information arising from our model. See for example [12], FESTUNG code is a robust code that can handle multiple boundary conditions. Nevertheless this would have required a constistent amount of work, which was not archievable in the time restrictions of this thesis. The work of [15] treated the non-intrinsic bathymetry model and gave raise to the version of the FESTUNG code about the bathymetry reconstruction itself.

Both theoretically and numerically, the geometric nature of our model increases a lot the complexity of the model itself thus keeping a lot of questions opened. The future starting point remains to test numerically our model with experiments, thus proceeding with further analysis of its properties.

Appendix

Let us provide some computations for the Christoffel symbols used in formula (2.8) and in the chapter 4.3.3. Further details or proofs can be found in [7], or [11].

.1 Check of formula (2.8)

Recall that

$$(\nabla_{\mathcal{G}} \cdot \mathbb{T})^j = \nabla_{\mathcal{G}} \cdot \tau^{(\cdot j)} + \Gamma^j_{ik}\tau^{ik} .$$

In general, for $i, j, k = 1, 2, 3$, Γ^j_{ik} forms 27 different coefficients, but the symmetry and orthogonality of $\mathcal{G}$ reduce them by a good amount. Using the Levi-Civita connection, we have

$$\Gamma^j_{ik} = \frac{g^{jl}}{2}\left(\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l}\right) ,$$

so, noticing that $l = j$ otherwise we have zero terms,

$$\Gamma^j_{ik} = \frac{g^{jj}}{2}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right) .$$

- If $i \neq j \neq k$, $\Gamma^j_{ik} = 0$ because $g_{kj} = g_{ik} = 0$; we have 6 components of this type.
- If $k = j$:

$$\Gamma^j_{ij} = \frac{1}{2h^2_{(j)}}\left(\frac{\partial g_{jj}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^j}\right) = \frac{1}{2h^2_{(j)}}2h_j\frac{\partial h_j}{\partial x^i} = \frac{1}{h_{(j)}}\frac{\partial h_{(j)}}{\partial x^i}$$

so

$$\Gamma^j_{ij}\tau^{ij} = \frac{1}{h_{(j)}}\left(\frac{\partial h_{(j)}}{\partial x^1}\tau^{1j} + \frac{\partial h_{(j)}}{\partial s^2}\tau^{2j} + \frac{\partial h_{(j)}}{\partial s^3}\tau^{3j}\right)$$

and recalling that $\dfrac{\partial h_{(j)}}{\partial s^3}\tau^{3j} = 0 \; \forall j$ we have

$$\Gamma^j_{ij}\tau^{ij} = \frac{1}{h_{(j)}}\left(\frac{\partial h_{(j)}}{\partial x^1}\tau^{1j} + \frac{\partial h_{(j)}}{\partial s^2}\tau^{2j}\right) .$$

Analogous result for $i = j$.

- If $k = i$, (but $\neq j$):

$$\frac{1}{2h^2_{(j)}}\left(\underbrace{\frac{\partial g_{ij}}{\partial x^i}}_{0} + \underbrace{\frac{\partial g_{ij}}{\partial x^i}}_{0} - \frac{\partial g_{ii}}{\partial x^i}\right) = \frac{1}{h^2_{(j)}} h_{(i)}\frac{\partial h_{(i)}}{\partial x^j} ,$$

so

$$\frac{1}{h_{(j)}}\left(-\frac{h_{(i)}}{h_{(j)}}\frac{\partial h_{(i)}}{\partial x^j}\right)\tau^{ii} = \frac{1}{h_{(j)}}\left(-\frac{h_{(1)}}{h_{(j)}}\frac{\partial h_{(1)}}{\partial x^j}\tau^{11} - \frac{h_{(2)}}{h_{(j)}}\frac{\partial h_{(2)}}{\partial x^j}\tau^{22} - \frac{h_{(3)}}{h_{(j)}}\underbrace{\frac{\partial h_{(3)}}{\partial x^j}}_{0}\tau^{33}\right) .$$

We have 6 components of this type.

.2 Computations with the asymmetric tensor $\mathbb{F}$

As we said in chapter 4.3.3, the formula (2.8) works only for symmetric tensors. Thus, we need to do the computations starting from the more general (1.9). Recall that the 4×4 contravariant tensor $\mathbb{F}$ is defined as

$$\mathbb{F} := \begin{pmatrix} 0 & u^1 & u^2 & u^3 \\ 0 & f^{11} & f^{12} & f^{13} \\ 0 & f^{21} & f^{22} & f^{23} \\ 0 & f^{31} & f^{32} & f^{33} \end{pmatrix}, \qquad \textit{and} \qquad \mathcal{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & h^2_{(1)} & 0 & 0 \\ 0 & 0 & h^2_{(2)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Nevertheless, what we only need to see is how the new terms $\Gamma^j_{ik}\mathcal{F}^{ik} \quad i = 0, 1, 2, 3;\ j = 0, 1, 2, 3\ k = 0, 1, 2, 3$ look like. First of all, notice that for $k = 0$ we have that $\mathcal{F}^{\cdot 0} = 0$, so we can consider only $k = 1, 2, 3$. Let's see cases $j = 0$ and $j = 1$, other being analogous.

- For $j = 0$, we have that

$$\Gamma^0_{ik} = \frac{g^{00}}{2}\left(\frac{\partial g_{k0}}{\partial x^i} + \frac{\partial g_{0i}}{\partial x^k} - \frac{\partial g_{ik}}{\partial t}\right) = \frac{1}{2}\left(\underbrace{\frac{\partial g_{00}}{\partial x^0}}_{0} + \underbrace{\frac{\partial g_{00}}{\partial x^0}}_{0} - \frac{\partial g_{ik}}{\partial t}\right) =$$
$$= -\frac{1}{2}\frac{\partial g_{ik}}{\partial t} = -\frac{1}{2}\frac{\partial g_{ii}}{\partial t} \quad (i = k, \textit{the rest is } 0)$$

  So we get

$$\Gamma^0_{ii}\mathcal{F}^{ii} = -\frac{1}{2}\frac{\partial h^2_{(1)}}{\partial t}\mathcal{F}^{11} - \frac{1}{2}\frac{\partial h^2_{(2)}}{\partial t}\mathcal{F}^{22}.$$

- For $j = 1$ instead the Christoffel symbols associated with the computations of the first component of the tensor divergence $(\nabla_{\mathcal{G}} \cdot \mathbb{T})$ are

$$\Gamma^1_{ik} = \frac{g^{11}}{2}\left(\frac{\partial g_{k1}}{\partial x^i} + \frac{\partial g_{1i}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^1}\right),$$

  so that multiplying them by $\mathcal{F}^{ik}$ we obtain (remember that $k \in \{1, 2, 3\}$)

$$\Gamma^1_{ik}\mathcal{F}^{ik} = \Gamma^1_{i1}\mathcal{F}^{i1} + \Gamma^1_{i2}\mathcal{F}^{i2} + \Gamma^1_{i3}\mathcal{F}^{i3} =$$
$$= \Gamma^1_{01}\mathcal{F}^{01} + \Gamma^1_{11}\mathcal{F}^{11} + \Gamma^1_{21}\mathcal{F}^{21} + \Gamma^1_{31}\mathcal{F}^{31} +$$

$$+\underbrace{\Gamma^1_{02}}_{0}\mathcal{F}^{02}+\Gamma^1_{12}\mathcal{F}^{12}+\Gamma^1_{22}\mathcal{F}^{22}+\underbrace{\Gamma^1_{32}}_{0}\mathcal{F}^{32}+$$

$$+\Gamma^1_{03}\mathcal{F}^{03}+\Gamma^1_{13}\mathcal{F}^{13}+\underbrace{\Gamma^1_{23}}_{0}\mathcal{F}^{23}+\Gamma^1_{33}\mathcal{F}^{33}=$$

$$=\frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x^0}\mathcal{F}^{01}+\frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x^1}\mathcal{F}^{11}+2\frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x^2}\mathcal{F}^{12}-\frac{1}{2}g^{11}\frac{\partial g_{22}}{\partial x^1}\mathcal{F}^{22}=$$
$$=\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial t}\mathcal{F}^{01}+\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\mathcal{F}^{11}+\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\mathcal{F}^{12}-\frac{h_{(2)}}{h^2_{(1)}}\frac{\partial h_{(2)}}{\partial s^1}\mathcal{F}^{22}=$$
$$=\frac{1}{h_{(1)}}\left(\mathcal{F}^{01}\frac{\partial h_{(1)}}{\partial t}\right)+\frac{1}{h_{(1)}}\left(\mathcal{F}^{11}\frac{\partial h_{(1)}}{\partial s^1}\right)+\frac{1}{h_{(1)}}\left(2\mathcal{F}^{12}\frac{\partial h_{(1)}}{\partial s^2}-\mathcal{F}^{22}\frac{h_{(2)}}{h_{(1)}}\frac{\partial h_{(2)}}{\partial s^1}\right). \tag{16}$$

As expected, the first term of the last row is not multiplied by a factor of 2: this is due to the asymmetry of tensor $\mathbb{F}$ (its first column is made of zeros).

That being said, we have that the first component of the divergence of tensor $\mathbb{F}$ is given by

$$(\nabla_{\mathcal{G}}\cdot\mathbb{F})^1=\nabla_{\mathcal{G}}\cdot\begin{bmatrix}u^1\\f^{11}\\f^{21}\\f^{31}\end{bmatrix}+\frac{1}{h_{(1)}}\left(u^1\frac{\partial h_{(1)}}{\partial t}-f^{00}\frac{h_{(0)}}{h_{(1)}}\frac{\partial h_{(0)}}{\partial s^1}\right)+\frac{1}{h_{(1)}}\left(2f^{11}\frac{\partial h_{(1)}}{\partial s^1}-f^{11}\frac{h_{(1)}}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\right)+$$
$$+\frac{1}{h_{(1)}}\left(2f^{21}\frac{\partial h_{(1)}}{\partial s^2}-f^{22}\frac{h_{(2)}}{h_{(1)}}\frac{\partial h_{(2)}}{\partial s^1}\right).$$

Observe that the first term can be written as

$$\nabla_{\mathcal{G}}\cdot\begin{bmatrix}u^1\\f^{11}\\f^{21}\\f^{31}\end{bmatrix}=\frac{1}{\sqrt{det\mathcal{G}}}\left[\frac{\partial}{\partial t}\left(\sqrt{\det\mathcal{G}}u^1\right)+\frac{\partial}{\partial s^i}\left(\sqrt{\det\mathcal{G}}f^{i1}\right)\right].$$

This is employed to derive the Momentum's equations.

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