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Tesi di Laurea

Late-time integrated Sachs-Wolfe effect in mimetic Horndeski gravity

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Chapter 1

Introduction

Since its first formulation in 1915, general relativity has given a theoretical description of gravity that is astonishingly compatible with experiments and observations. Amongst its many observational successes, we remember the gravitational redshift ([1]), the gravitational time dilation ([2]), the Shapiro delay ([3]), the deflection of light ([4]) and the gravitational waves ([5]).
Fixing a small set of six parameters through observations, the cosmological model founded on general relativity (usually known as ΛCDM model) provide us with other confirmed predictions, such as the abundance of chemical elements formed during the primordial nucleosynthesis ([6]), the large-scale structure of the universe ([7]) and the existence and properties of the cosmic microwave background (CMB) radiation ([8]).
As it is well known, the ΛCDM model needs the existence of two *dark components* in the universe in order to be consistent with the observations: dark matter and dark energy. Since these two components have never been observed and they also present some theoretical problems([9] [10] [11]), it has become a common research current to try to explain the phenomena that they give origin to by modifying the law of gravitation, without introducing new energy sources ([12] [13]).
Amongst these *modified gravity* theories, the so-called *mimetic scenario* has lately attracted much attention. The first formulation of mimetic gravity ([14]) performs a conformal transformation on the Einstein-Hilbert action of general relativity, using an auxiliary metric and a scalar field. The outcome of this conformal transformation is to switch on a new scalar degree of freedom of gravity, which behaves exactly as a pressureless perfect fluid, thus mimicking a cold dark matter component. In this model, the observed cold dark matter energy density would, in general, be the sum of two unknown amounts of energy density contributions, one coming from hypothetical dark matter particles and the other from the "mimetic" dark matter which is only a gravitational effect. In a subsequent article ([15]) it is shown that, introducing a potential for the scalar field and considering the cosmological solutions, it is possible to reproduce almost any background expansion history for the universe.
The initial mimetic dark matter model is generalized in [16], considering a very general scalar-tensor theory of gravity instead of general relativity: in particular, the results contained in [16] are valid for Horndeski gravity, which is the most general healthy second-order scalar-tensor theory of gravity. The authors also consider generic disformal transformations instead of simple conformal ones: it is shown that Horndeski gravity is invariant under invertible disformal transformations, but when the transformation is non-invertible the resulting theory is a generalization of the original mimetic dark matter model, with new equations of motion. This new theory has been defined *mimetic Horndeski gravity*. Finally they show that there

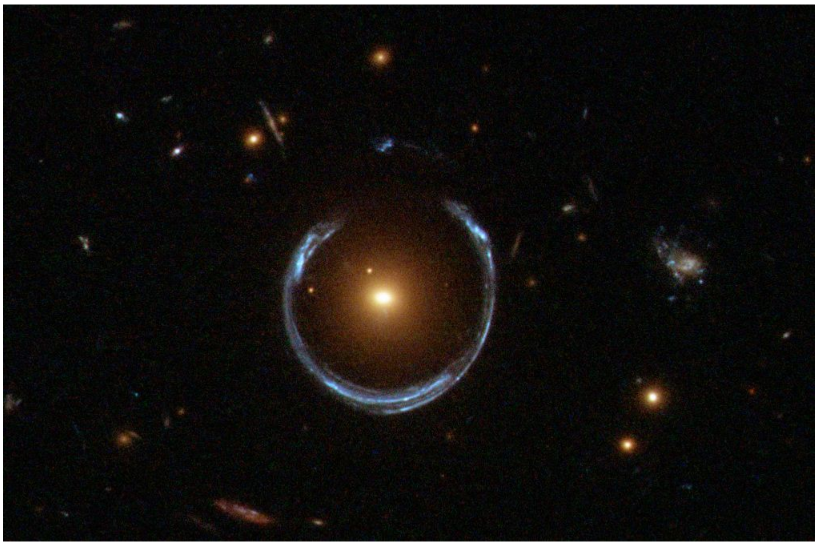

Figure 1.1: A spectacular observational confirmation of general relativity: the mass of a luminous red galaxy gravitationally distorts the light from a much more distant blue galaxy, forming a nearly complete ring. The image was taken by Hubble Space Telescope.

are mimetic Horndeski models that present interesting cosmological features, being able to reproduce a perfect fluid or a ΛCDM background expansion history. The issue of cosmological perturbations in mimetic Horndeski gravity is studied in [17], where the time evolution equations for the scalar perturbations are obtained.
Since mimetic Horndeski gravity has proved to be compatible with the observed background expansion history of the universe, we need other predictions that can be compared with observations in order to decide whether this theory could be a valid alternative to general relativity. The goal of this thesis is to perform an analytical calculation of the late-time *integrated Sachs-Wolfe* (ISW) effect predicted by mimetic Horndeski gravity, and then to compare it with the one predicted by general relativity.
The ISW effect is one of the sources of the anisotropies observed in the CMB radiation ([18]). It is generated by the fact that a photon of the CMB experience a redshift or a blueshift if, during its travel, it falls into a time-varying gravitational potential well: in particular we expect that, at late times, the acceleration in the expansion of the universe causes a decay of the gravitational potential, creating the so-called *late-time ISW effect*. This effect is largely dependent on the time evolution of the gravitational potential, that is different from a theory of gravity to another: different theories of gravity could predict different equations of motion for the metric perturbations, and therefore they could predict different ISW effects.

The thesis is organized as follows.
In Chapter 2 we present a review of the standard ΛCDM model of cosmology, discussing also the hot Big Bang model that predicts the existence of the CMB radiation.
In Chapter 3 we discuss the issue of cosmological perturbations in a general metric theory of gravity, presenting the gauge problem and the different ways to deal with it.
In Chapter 4 we obtain the Boltzmann equation for photons, that describes the evolution of their distribution function in a perturbed universe.
In Chapter 5 we study the perturbations of the temperature field and how they change during the free streaming of the photons from the last scattering surface to us. Here we introduce

the integrated Sachs-Wolfe effect.
In Chapter 6 we review the first formulation of mimetic gravity, considering a conformal transformation of the Einstein-Hilbert action of general relativity.
In Chapter 7 we generalize the mimetic framework to a very general scalar-tensor theory of gravity, considering also more general disformal transformations: here we present mimetic Horndeski gravity.
In Chapter 8 we obtain the equations for the time evolution of the scalar metric perturbations in general relativity and mimetic Horndeski gravity. In the case of mimetic Horndeski we impose a ΛCDM or a perfect fluid dark energy background expansion history, investigating how these constraints change the evolution of perturbations. We also solve the equations in the particular and simpler case of mimetic cubic Horndeski.
In Chapter 9 we perform the analytical calculation of the late-time ISW effect in general relativity and mimetic Horndeski gravity.
In Chapter 10 we present the conclusions of the thesis.

We use a $(-,+,+,+)$ metric signature and we set $c = 1$. Other notation conventions will be defined when needed in the following.

Chapter 2

Standard model of cosmology

2.1 Cosmological principle

Cosmology in its modern formulation is based upon the cosmological principle, which states that every comoving observer sees the universe around himself as homogeneous in space and isotropic on large scales ([19][20]). When we use the expression "comoving observer", we mean an observer that is still with respect to the source of the geometry of the universe: this also implies that we accept a metric theory of gravity, that is a theory in which gravity is a manifestation of the geometrical properties of spacetime.
In a metric theory of gravity the spacetime can be described as a Lorentzian manifold, and the cosmological principle forces the line element to assume a very symmetric form (the Friedmann-Robertson-Walker or FRW metric):

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.1}$$

where t is the cosmic time (operatively, it is the time that would be measured by a comoving clock), $a(t)$ is the scale factor (that is function of the cosmic time only), (r, θ, ϕ) are polar comoving coordinates (such that a comoving observer has r, θ and ϕ fixed) and k is the spatial curvature constant, whose value can be $+1$, 0 or -1 corresponding to a closed, flat, or open universe respectively.
It is sometimes useful to define the *conformal time* as $d\eta = \frac{dt}{a(t)}$: using this conformal time, equation (2.1) becomes

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.2}$$

2.2 General relativity and ΛCDM model

General relativity was the first metric theory of gravity and it was published in its final form in 1915 by Albert Einstein. The model gives us a very simple and elegant law that explains how spacetime is curved by sources of mass or energy: Einstein equations in their most general form are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2.3}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, G is Newton's gravitational constant, $T_{\mu\nu}$ is the stress-energy tensor of the mass-energy in the universe and Λ is the cosmological constant. This cosmological constant can be seen as a form of energy that fills the space in a perfectly homogeneous way, so it can be included in $T_{\mu\nu}$ as a particular energy source.

If we now assume the cosmological principle (so a metric in the form of equation (2.1)) and we assume also that the matter component in the universe is represented by a perfect fluid of energy density $\rho(t)$ and isotropic pressure $p(t)$ (they are both function of time only because of homogeneity of space), then equation (2.3) gives us the Friedmann equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \tag{2.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \tag{2.5}$$

where we defined the Hubble parameter $H(t) = \frac{\dot{a}}{a}$. Here and in the following we use the dot to indicate the derivative with respect to the cosmic time t, while the $'$ will indicate the derivative with respect to the conformal time η.
Using the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ (that can be shown starting from the definition of Einstein tensor), we also obtain the continuity equation

$$\dot{\rho} = -3H(\rho + p) \tag{2.6}$$

Defining the critical density $\rho_c(t) \equiv \frac{3H^2(t)}{8\pi G}$, equation (2.4) can be put in the form

$$\Omega(t) - 1 \equiv \frac{\rho(t)}{\rho_c(t)} - 1 = \frac{k}{a^2(t)H^2(t)} \tag{2.7}$$

This means that the mass-energy density of the cosmic fluid determines the global geometry of the universe: if $\rho(t) > \rho_c(t)$ then $k = 1$ and the universe must be closed, if $\rho(t) < \rho_c(t)$ then $k = -1$ and the universe must be open and finally if $\rho(t) = \rho_c(t)$ then $k = 0$ and the universe must be flat.
The latest results of the Planck satellite ([21]) constrain the current value of ρ to be very close to[1] $\rho_c(t_0) = 1.87847(23) \times 10^{-29} h^2$ g $\cdot$ cm^{-3} (where t_0 represent today's cosmic time), so the universe can be considered flat. But a problem arises when we try to determine what kind of energy sources contributes to such a value of the present energy density $\rho(t_0)$.

First of all let's start with matter: from [21] we know that $\Omega_m(t_0) \equiv \frac{\rho_m(t_0)}{\rho_c(t_0)} = 0.316(14)$, so matter represents little more than the 30% of the total energy budget of the universe. But we know also that baryonic matter (the usual form of matter that we are familiar with) can account for no more than the 5% of $\rho(t_0)$ ([21]): this implies that there is an unknown form of matter, called *dark matter*, that is responsible for the 25% of the total energy of the universe. This dark matter, in order to be compatible with observations, must be neutral, collisionless, non relativistic and stable: it must be also *cold* in the sense that it experienced decoupling from all the other forms of matter when it was already non relativistic.
What about the remaining 70% of the total energy budget? We know that photons cannot represent more than the 0.01% of $\rho(t_0)$, so this 70% must be covered by an unknown form of energy, called *dark energy*. The nature of this dark energy is still a mystery: many models

[1] Since the experimental determination of the present value of the Hubble parameter is really complicated, it is common to parametrize the value of $H(t_0)$ with h: $H(t_0) = h \times \frac{100\text{km/s}}{\text{Mpc}}$

have been proposed, but the most common approach is to consider it as a small cosmological constant Λ that, as we said before, fills the space with a constant value of energy density. It is thought that dark energy is responsible for the currently observed acceleration in the expansion of the universe.

In summary, the cosmological solutions of general relativity's equations need two unknown (and still experimentally undetected) *dark* components in order to satisfy observational constraints: dark matter and dark energy. Considering that the best results in fitting data are given by cold dark matter (CDM) and cosmological constant (Λ) respectively, the ΛCDM model is now called the standard model of cosmology.

2.3 Modified gravity

The two dark components that are the basis of ΛCDM model pose a number of problems: as we said in the previous Section by now no experimental detection of dark matter or dark energy has been performed, but this is not the only challenge.
There are also theoretical problems: for example this model seems to have a "small-scale problem" ([9]), predicting too many dwarf galaxies and too much dark matter in the innermost regions of the galaxies. Moreover the nature of the cosmological constant is still unknown: the modern interpretation of Λ is based on the vacuum energy of quantum field theory, but there is a discrepancy between theoretical predictions and observations of 120 orders of magnitude ([11]).

The presence of these unknown energy components has motivated many studies that try to explain the phenomena that they give origin to by modifying the law of gravitation, without introducing new energy sources: for this reason these theories can be grouped under the label *modified gravity*.
Among these theories, in this thesis we will focus on mimetic Horndeski gravity, which is a particular modification of a scalar-tensor theory of gravity.

2.4 Friedmann Models

Now we try to solve the system of equations (2.4)-(2.6):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \tag{2.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \tag{2.9}$$

$$\dot{\rho} = -3H(\rho + p) \tag{2.10}$$

It can be shown ([24]) that (2.9) can be obtained from (2.8) and (2.10), so we have only two independent equations for the three unknown quantities $a(t)$, $\rho(t)$ and $p(t)$. This means that we need a third equation in order to find a solution for the system: if we assume the cosmic fluid to be barotropic, then the third equation could be taken to be the equation of state of the fluid

$$p = w\rho \tag{2.11}$$

where we assumed a simple linear law with w constant in time.

If we restrict the discussion to a flat FRW metric ($k = 0$) we find the solutions ([24])

$$a(t) \propto t^{\frac{2}{3(1+w)}} \qquad \rho(t) \propto a^{-3(1+w)} \tag{2.12}$$

For example if we have non relativistic matter ($w = 0$) we obtain

$$a(t) \propto t^{\frac{2}{3}} \qquad \rho(t) \propto a^{-3} \qquad p(t) = 0 \tag{2.13}$$

while if we have radiation ($w = 1/3$) we obtain

$$a(t) \propto t^{\frac{1}{2}} \qquad \rho(t) \propto a^{-4} \qquad p(t) = \frac{1}{3}\rho(t) \tag{2.14}$$

If we have only a cosmological constant ($w = -1$) the solution is in the form

$$a(t) \propto e^{Ht} \qquad \rho(t) = const \qquad p(t) = -\rho(t) \tag{2.15}$$

with $H = \sqrt{\frac{\Lambda}{3}}$ constant in time.
If we consider the ΛCDM model, at late times (after radiation has become negligible) the universe experiences a transition from the matter domination to the cosmological constant domination, and the scale factor assumes the form

$$a(t) = a_i \sinh^{\frac{2}{3}}(Ct) \tag{2.16}$$

where $C = \sqrt{\frac{3\Lambda}{4}}$. In fact if $Ct \ll 1$, we can approximate $\sinh(Ct) \simeq Ct$ so

$$a(t) \simeq a_i C^{\frac{2}{3}} t^{\frac{2}{3}} \tag{2.17}$$

that is exactly the time dependence of a cold dark matter dominated universe. Instead if $Ct \gg 1$, we can approximate $\sinh(Ct) \simeq \frac{1}{2}\exp(Ct)$ so

$$a(t) \simeq a_i 2^{-\frac{2}{3}} \exp\left(\sqrt{\frac{\Lambda}{3}} t\right) \tag{2.18}$$

that is the time dependence of a cosmological constant dominated universe.

It is easy to show ([24]) that the Friedmann models with $-\frac{1}{3} < w < 1$ have the property that they present a time in the past where a vanishes and the energy density diverges. This instant is called *Big Bang singularity* and can be taken to be the origin of time ($t = 0$). It is worth noting that the existence of this singularity is a direct consequence of four conditions:

- the validity of cosmological principle
- the validity of general relativity and then of Friedmann equations
- the present (observed) expansion of the universe[2]
- the correctness of the equation of state in the form $p = w\rho$ with $-\frac{1}{3} < w < 1$.

If these four conditions are assumed to be true, then the existence of the Big Bang is inevitable, establishing the foundations for the *hot Big Bang model*: the very early universe was in a hot and dense phase and subsequently it expanded, decreasing in density and falling in temperature.

[2] The expansion of the universe was firstly proposed in 1929 by Hubble, who measured the receding velocity of distant galaxies finding a proportionality relation between this velocity and the distance: $v = H(t_0)d$

2.5 The hot Big Bang model

It is useful to review the most important stages of the chronology of the universe as predicted by the hot Big Bang model.

- Singularity ($t = 0$). As we said above the Big Bang singularity appears when we extrapolate the results of general relativity in a situation where this theory is no longer valid: it seems likely that in a complete quantum theory of gravity there would be no singularity.
- Planck epoch ($t < 10^{-43}$s). It is conjectured that in this epoch a quantum theory of gravity could unify all the four known fundamental interaction (gravitational, nuclear strong, nuclear weak and electromagnetic).
- Grand unification epoch ($t < 10^{-36}$s). It is conjectured that in this epoch gravity becomes distinct from the other three interactions, which are now described by a *grand unification theory* (GUT).
- Electroweak epoch ($t = 10^{-36}$s). The electroweak interaction becomes distinct from the strong interaction.
- Inflation ($t = 10^{-33}\text{s} - 10^{-32}\text{s}$). The universe experiences an exponential expansion by a factor bigger than 10^{26}.
- Quarks epoch ($t = 10^{-12}\text{s} - 10^{-6}\text{s}$). The electromagnetic interaction becomes distinct from the nuclear weak interaction, allowing the elementary particles to acquire mass through Higgs mechanism. Matter is now in the form of a *quark-gluon plasma.*
- Hadron epoch ($t = 10^{-6}\text{s} - 1\text{s}$). The temperature is now sufficiently low to allow quarks to bind, forming hadrons: the first protons and neutrons appear.
- Lepton epoch ($t = 1\text{s} - 10\text{s}$). Hadrons and antihadrons annihilate each other, producing leptons and antileptons. Neutrinos decouple from matter, originating a *cosmic neutrino background.*
- Photon epoch ($t = 10\text{s} - 7 \times 10^4\text{y}$). Most of the leptons and antileptons annihilate each other, causing the universe to be dominated by radiation. In this epoch photons are continuously scattered by charged particles (mostly by electrons), so the universe can be considered a super-hot glowing fog. Between $t = 3$m and $t = 20$m the *primordial nucleosyntesis* arises: helium, deuterium and lithium nuclei are produced through nuclear fusion reactions, starting from protons and neutrons.
- Beginning of matter-dominated era ($t = 7 \times 10^4$y). The energy density of matter dominates radiation and dark energy: the universe's expansion decelerates.
- Recombination ($t = 3.8 \times 10^5$y). Temperature is low enough to allow electrons to combine with protons and nuclei in order to form neutral atoms: photons are no more in thermal equilibrium with matter (since photons' scattering with charged particles becomes rare) and the universe becomes transparent for radiation, allowing photons to propagate freely. This photons constitute the cosmic microwave background (CMB) radiation: the spatial surface from which they start propagating is known as *surface of last scattering.*

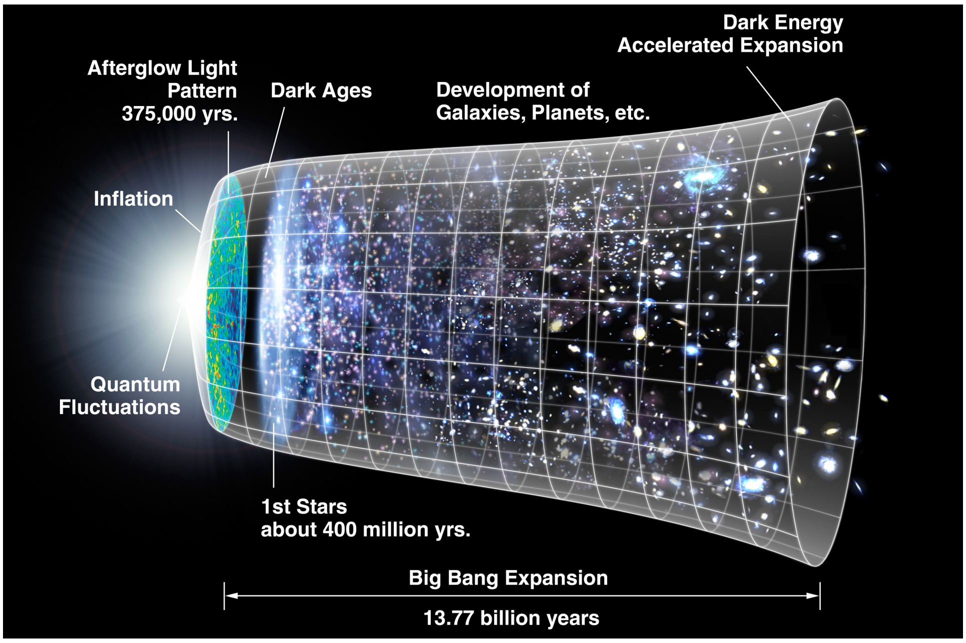

Figure 2.1: Synthetic diagram of the evolution of the observable universe

- Dark ages ($t = 3.8 \times 10^5\mathrm{y} - 1.5 \times 10^8\mathrm{y}$). No stars are already formed and the only light produced is from *spin-flip transition* in excited hydrogen atoms.
- Galaxy formation ($t = 10^9\mathrm{y} - 10^{10}\mathrm{y}$). Galaxy clusters and superclusters begin to form.
- Beginning of dark energy-dominated era ($t = 10^{10}\mathrm{y}$). The matter density falls below dark energy density, causing the universe's expansion to re-accelerate.
- Present time ($t = 1.38 \times 10^{10}\mathrm{y}$).

Chapter 3

Cosmological perturbations

The cosmological principle constrains the universe to be homogeneous and isotropic on *large scales*, but we know that on small scales this cannot be true. We observe that matter is clustered in planets and stars, and that these planets and stars form galaxies and clusters of galaxies, between which very large void regions exist: on small scales the universe is not homogeneous. Furthermore we know that the cosmic microwave background radiation is not perfectly isotropic: if we look at different directions in the sky we measure small fluctuation in the temperature of the CMB, of order $\Delta T/T = 10^{-5}$.
The standard model explains very well many characteristics of the observed universe (like its expansion and cooling or the existence of the CMB), but the examples above indicate that we have to go beyond the cosmological principle in order to study small scale phenomena. To do this we introduce the cosmological perturbation theory.

3.1 The gauge problem

Let's consider a generic tensor T: it could be for example the metric tensor $g_{\mu\nu}$ or a scalar field φ. We can define the perturbation of this tensor as $\Delta T \equiv T - T_0$, where T is the value assumed by the tensor in the physical (pertubed) spacetime (denoted as M) and T_0 is the value assumed by the tensor in the background (unperturbed) spacetime (denoted as M_0).
But we know that two tensors can be compared only if they are calculated in the same point of the spacetime: since T and T_0 live on different spacetimes, in order to define perturbations we must also define a one-to-one correspondence between the points of M and the points of M_0. To choose a particular correspondence means to choose a *gauge* and to change the correspondence means to make a *gauge transformation*.
So we can define, for example, two different gauges ψ and ϕ in such a way that, if P is a point on M_0, we have $\psi(P) = O \in M$ and $\phi(P) = O' \in M$ (see Figure 3.1). It is now clear that we have two different tensors on M corresponding to T_0 calculated in $P \in M_0$: the tensor calculated in O (named T) if we choose the gauge ψ and the tensor calculated in O' (named $\tilde{T}$) if we choose the gauge ϕ. This means that we obtain two different perturbation depending on the gauge: $\Delta T = T - T_0$ for ψ and $\tilde{\Delta T} = \tilde{T} - T_0$ for ϕ.

3.2 Gauge transformations: active and passive approach

When dealing with gauge transformations, two approaches are possible: the active one or the passive one.

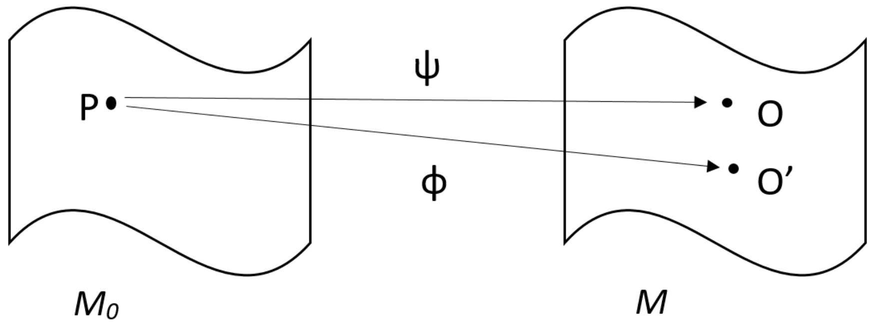

Figure 3.1: The gauge problem

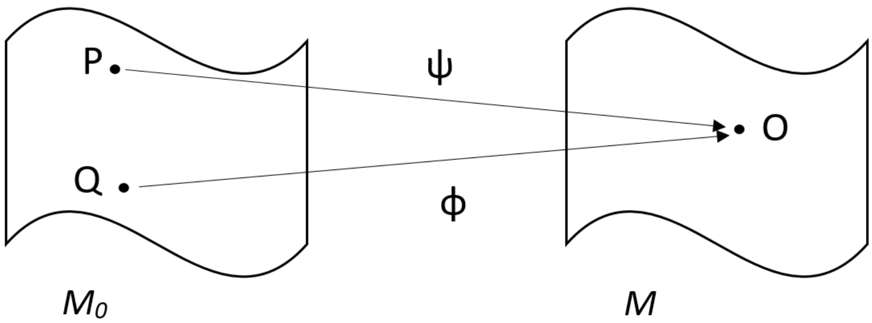

Figure 3.2: Active approach

A change in the choice of the gauge has the direct consequence that the point $O \in M$ will have a different corresponding point on M_0: before the transformation $\psi(P) = O$ so $P = \psi^{-1}(O)$, while after the gauge transformation from ψ to ϕ we will have $\phi(Q) = O$ so $Q = \phi^{-1}(O)$ (see Figure 3.2). This means that we can write $Q = \phi^{-1}(O) = \phi^{-1}(\psi(P))$: we have just built a one-to-one correspondence between points on M_0, that can be defined as

$$\Phi : P(\in M_0) \to Q(\in M_0) = \phi^{-1}(\psi(P)) \tag{3.1}$$

If a coordinate system x^μ has been defined on M_0, then the coordinates of Q are given as a function of the coordinates of P by the law $x^\mu(Q) = \Phi^\mu(x(P))$: in this way a gauge transformation can be regarded as an *active coordinate transformation.*

This active approach allows us to define a practical method to build a gauge transformation Φ:

1. We fix a coordinate system x^μ on M_0
2. We define a vector field $\xi^\mu(x)$ on M_0
3. Introducing a parameter λ we define a *congruence of curves* $x^\mu(\lambda)$ such that $\frac{d}{d\lambda}x^\mu(\lambda) = \xi^\mu$
4. Considering for example a point P such that $x^\mu(P) = x^\mu(\lambda = 0)$, the coordinates $x^\mu(Q)$ of a point Q that is at an infinitesimal distance λ from the point P along the curve $x^\mu(\lambda)$ are
$$x^\mu(Q) = x^\mu(P) + \lambda\xi^\mu(x(P)) \tag{3.2}$$

We can see immediately that equation (3.2) is a infinitesimal coordinate transformation, defined by the vector field $\xi^\mu(x)$.

We can look at equation (3.2) in a different way. The passive view of the gauge transformation considers it an ordinary coordinate transformation:

$$x^\mu(P) = x^\mu(Q) - \lambda\xi^\mu(x(P)) = x^\mu(Q) - \lambda\xi^\mu(x(Q)) + o(\xi^2) \tag{3.3}$$

Now we can introduce a new coordinate system y^μ, such that at first order we get

$$y^\mu(Q) \equiv x^\mu(P) = x^\mu(Q) - \lambda\xi^\mu(x(Q)) \tag{3.4}$$

As we anticipated, equation (3.4) represents an ordinary infinitesimal coordinate transformation.

3.3 Gauge transformations: tensors

We are now able to find how a generic tensor transforms under a gauge transformation. Let's take for example a vector field Z with components Z^μ in the coordinates x^μ: under a gauge transformation Z^μ goes in

$$\tilde{Z}^\mu(x(P)) = Z'^\mu(y(Q)) \tag{3.5}$$

where $Z'^\mu(y(Q)) = \left.\frac{\partial y^\mu}{\partial x^\nu}\right|_{x(Q)} Z^\nu(x(Q))$ is the usual transformation of a vector field Z under a change of coordinates $x \to y(x)$.
Taking an infinitesimal coordinate transformation $y^\mu = x^\mu - \lambda\xi^\mu$ we get

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \lambda\frac{\partial \xi^\mu}{\partial x^\nu} \tag{3.6}$$

so equation (3.5) becomes

$$\tilde{Z}^\mu(x(P)) = Z^\mu(x(Q)) - \lambda\frac{\partial\xi^\mu}{\partial x^\nu}Z^\nu(x(Q)) \tag{3.7}$$

Using now equation (3.2) we get

$$\begin{aligned}\tilde{Z}^\mu(x(P)) &= Z^\mu(x(P)) + \lambda\frac{\partial Z^\mu}{\partial x^\nu}\xi^\nu - \lambda\frac{\partial\xi^\mu}{\partial x^\nu}Z^\nu(x(P)) + o(\xi^2) = \\ &= Z^\mu(x(P)) + \mathcal{L}_\xi Z^\mu\end{aligned} \tag{3.8}$$

taking $\lambda = 1$ and ξ infinitesimal and defining the Lie derivative of the tensor Z along the vector field ξ as $\mathcal{L}_\xi Z$. We notice that (3.8) is the equation of *Lie dragging*.

Repeating this procedure for the other kinds of tensor it can be shown that for a generic tensor T the transformation law under a gauge transformation is

$$\tilde{T} = T + \mathcal{L}_\xi T \tag{3.9}$$

This means that now we can also find how perturbations transform. We know that $\Delta T = T - T_0$ in the first gauge and $\tilde{\Delta T} = \tilde{T} - T_0$ in the second gauge, so $T = T_0 + \Delta T$ and

$\tilde{T} = T_0 + \tilde{\Delta T}$.
Using now equation (3.9) we get

$$\tilde{T} = T + \mathcal{L}_\xi T = T_0 + \Delta T + \mathcal{L}_\xi T \tag{3.10}$$

and so $\tilde{\Delta T} = \Delta T + \mathcal{L}_\xi T$. But the Lie derivative contains terms of first order in ξ, so at first order in ξ we can write:

$$\tilde{\Delta T} = \Delta T + \mathcal{L}_\xi T_0 \tag{3.11}$$

3.4 Cosmological perturbations

As we said in the introduction of this Chapter, we need to consider a physical spacetime with small perturbations with respect to the perfectly homogeneous and isotropic universe described by the cosmological principle. Using the notation of the previous Sections, M_0 will be a flat FRW spacetime

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] = a^2(\eta)[-d\eta^2 + dx^2 + dy^2 + dz^2] \tag{3.12}$$

while M will be the physical universe in which there are small inhomogeneities. The metric tensor $g_{\mu\nu}$ of this perturbed spacetime can be written, using conformal time, as:

$$g_{00}(\eta, \vec{x}) = -a^2(\eta)\left[1 + 2\sum_{r=1}^{\infty}\frac{1}{r!}\Phi^{(r)}(\eta, \vec{x})\right] \tag{3.13}$$

$$g_{0i}(\eta, \vec{x}) = g_{i0}(\eta, \vec{x}) = a^2(\eta)\sum_{r=1}^{\infty}\frac{1}{r!}\omega_i^{(r)}(\eta, \vec{x}) \tag{3.14}$$

$$g_{ij}(\eta, \vec{x}) = a^2(\eta)\left\{\left[1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\Psi^{(r)}(\eta, \vec{x})\right]\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}\chi_{ij}^{(r)}(\eta, \vec{x})\right\} \tag{3.15}$$

where r is the order of the perturbation.
We note that in g_{ij} we have separated the diagonal part (proportional to δ_{ij}) from the off-diagonal part, taking the tensor $\chi_{ij}^{(r)}(\eta, \vec{x})$ to be traceless ($\chi^{(r)i}{}_i(\eta, \vec{x}) = 0$). We note also that spatial indexes are raised and lowered using the Kronecker delta δ_{ij}.

Using the fact that spatial indexes are now "flat indexes", the perturbations of the metric can be decomposed in scalar, vector and tensor components:

- $\Phi^{(r)}$ and $\Psi^{(r)}$ are scalar perturbations.
- $\omega_i^{(r)}$ can be decomposed, using Helmholtz's theorem, in this way ([22]):

$$\omega_i^{(r)} = \partial_i \omega_\parallel^{(r)} + \omega_i^{(r)\perp} \tag{3.16}$$

  $\omega_\parallel^{(r)}$ is a scalar perturbation and $\omega_i^{(r)\perp}$ is a vector perturbation such that $\partial^i \omega_i^{(r)\perp} = 0$ (all vector perturbations are defined to be solenoidal).

- $\chi_{ij}^{(r)}$ can be decomposed in this way ([22]):

$$\chi_{ij}^{(r)} = D_{ij}\chi_\parallel^{(r)} + \partial_i \chi_j^{(r)\perp} + \partial_j \chi_i^{(r)\perp} + \chi_{ij}^{(r)T} \tag{3.17}$$

$\chi_{\|}^{(r)}$ is a scalar perturbation, $\chi_i^{(r)\perp}$ is a vector perturbation (so $\partial^i \chi_i^{(r)\perp} = 0$) and $\chi_{ij}^{(r)T}$ is a tensor perturbation such that $\partial^i \chi_{ij}^{(r)T} = 0$ and $\chi^{(r)T\,i}_{\;\;\;i} = 0$ (all tensor perturbations are defined to be solenoidal and traceless). We have also defined $D_{ij} \equiv \partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2$, in order to keep $D_{ij}\chi_{\|}^{(r)}$ traceless.

This decomposition turns out to be useful because, at first order ($r = 1$) both in general relativity and in its extensions, scalar, vector and tensor perturbations evolve separately: in the time evolution equations there is no coupling between perturbations of different kinds.

We need also to perturb the stress-energy tensor $T_{\mu\nu}$ because, as we said in the previous Sections, the energy density of the universe is not perfectly homogeneous in space. If we consider a perfect fluid, the stress-energy tensor can be written as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} \tag{3.18}$$

where u_μ is the 4-velocity of the fluid element, $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ is a projector on hypersurfaces orthogonal to u_μ ($h_{\mu\nu}u^\nu = 0$), ρ is the energy density of the fluid and p is the isotropic pressure. For a perturbed fluid the quantities in equation (3.18) can be written as

$$\rho(\eta, \vec{x}) = \rho_0(\eta) + \sum_{r=1}^{\infty} \frac{1}{r!}\delta\rho^{(r)}(\eta, \vec{x}) \tag{3.19}$$

$$p(\eta, \vec{x}) = p_0(\eta) + \sum_{r=1}^{\infty} \frac{1}{r!}\delta p^{(r)}(\eta, \vec{x}) \tag{3.20}$$

$$u^\mu(\eta, \vec{x}) = \frac{1}{a(\eta)}\left(\delta_0^\mu + \sum_{r=1}^{\infty} \frac{1}{r!} v^\mu_{(r)}(\eta, \vec{x})\right) \tag{3.21}$$

Some comments are needed here.
First of all, we know that for every fluid there is an equation of state. Its most general form is $p = p(\rho, s)$ where s is the entropy density, so the pressure perturbation can always be decomposed as follows (omitting the arguments of the functions involved):

$$\delta p^{(r)} = \left.\frac{\partial p}{\partial \rho}\right|_{s=const} \delta\rho^{(r)} + \left.\frac{\partial p}{\partial s}\right|_{\rho=const} \delta s^{(r)} = c_s^2 \delta\rho^{(r)} + \delta p^{(r)}_{n.a.} \tag{3.22}$$

where we have defined the adiabatic speed of sound $c_s \equiv \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{s=const}}$ and the non adiabatic pressure perturbation $\delta p^{(r)}_{n.a.} = \left.\frac{\partial p}{\partial s}\right|_{\rho=const} \delta s^{(r)}$.
Considering now the expression for u^μ, we note that the first term is the background 4-velocity of a comoving fluid element (remember that in curved spacetime the constraint $u^\mu u_\mu = -1$ makes the 4-velocity of an unmoving observer to be $u^\mu = \frac{1}{\sqrt{-g_{00}}}\delta_0^\mu$). Furthermore it can be shown that the constraint $u^\mu u_\mu = -1$ makes it possible, for every order r, to express $v^0_{(r)}$ as a function of the metric perturbations only: for example it is easy to show that $v^0_{(1)} = -\Phi^{(1)}$. This implies that we can take the only three independent components of the 4-velocity to be the spatial ones $v^i_{(r)}$: as we did for the metric perturbations we can now decompose these spatial components as

$$v^i_{(r)} = \partial^i v^{\|}_{(r)} + v^{\perp i}_{(r)} \tag{3.23}$$

where $v^{\|}_{(r)}$ is a scalar perturbation and $v^{\perp i}_{(r)}$ is a vector perturbation such that $\partial_i v^{\perp i}_{(r)} = 0$.

3.5 Gauge fixing at first order

As we made clear in Section 3.2, a gauge transformation is completely defined by a vector field $\xi(x)$, so at first order ($r = 1$) it can be identified by the four components $\xi^{\mu}_{(1)} = (\xi^0_{(1)}, \xi^i_{(1)})$. We can define $\xi^0_{(1)} \equiv \alpha$ and decompose $\xi^i_{(1)} = \partial^i \beta + d^i$: α and β are scalar quantities, while d^i is a vector quantity ($\partial_i d^i = 0$).
We are now able to see how first order perturbations change under a gauge transformation defined by α, β and d^i. Using the general law (3.11), indicating with a tilde the perturbations in the new gauge and omitting the superscript (1), we can find for the metric:

$$\tilde{\Phi} = \Phi + \alpha' + \frac{a'}{a}\alpha \tag{3.24}$$

$$\tilde{\omega}_i = \omega_i - \partial_i\alpha + \partial_i\beta' + d'_i \quad \Rightarrow \quad \tilde{\omega}^{\|} = \omega^{\|} - \alpha + \beta' \quad , \quad \tilde{\omega}^{\perp}_i = \omega^{\perp}_i + d'_i \tag{3.25}$$

$$\tilde{\Psi} = \Psi - \frac{1}{3}\nabla^2\beta - \frac{a'}{a}\alpha \tag{3.26}$$

$$\tilde{\chi}_{ij} = \chi_{ij} + 2D_{ij}\beta + \partial_i d_j + \partial_j d_i \;\Rightarrow\; \tilde{\chi}^{\|} = \chi^{\|} + 2\beta \;,\; \tilde{\chi}^{\perp}_i = \chi^{\perp}_i + d_i \;,\; \tilde{\chi}^T_{ij} = \chi^T_{ij} \tag{3.27}$$

where $' \equiv \frac{d}{d\eta}$ and $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$.
For the quantities in the stress-energy tensor we get:

$$\tilde{\delta\rho} = \delta\rho + \rho'_0\alpha \tag{3.28}$$

$$\tilde{v}^0 = v^0 - \alpha' - \frac{a'}{a}\alpha \tag{3.29}$$

$$\tilde{v}^i = v^i - \partial^i\beta' - d^{i\prime} \quad \Rightarrow \quad \tilde{v}^{\|} = v^{\|} - \beta' \quad \text{and} \quad \tilde{v}^{\perp\, i} = v^{\perp\, i} - d^{i\prime} \tag{3.30}$$

In order to fix a gauge we must fix the four components of the vector field ξ: as we said at the beginning of this Section, ξ is completely determined by two scalars (α and β) and a vector (d^i), so the gauge is fixed when we fix a value for any two scalar and one vector perturbations. In fact for example if we fix $\tilde{\chi}^{\|} = 0$, then using (3.27) this condition forces us to choose a gauge transformation with $\beta = -\frac{1}{2}\chi^{\|}$: in a similar way if we fix another scalar and a tensor also α and d^i are constrained, completely defining the gauge transformation.
Here are some examples of gauge choices:

- Poisson gauge: $\omega^{\|} = 0$, $\chi^{\|} = 0$ and $\chi^i_{\perp} = 0$. It is a particular case of the longitudinal gauge ($\omega^{\|} = 0$ and $\chi^{\|} = 0$), also called conformal newtonian because in this gauge the evolution equations in general relativity have direct correspondents in newtonian gravity. It is called also orthogonal zero-shear gauge.

- Synchronous gauge: $\Phi = 0$. If we add the conditions $\omega^{\|} = 0$ and $\omega^i_{\perp} = 0$ we get the synchronous and time-orthogonal gauge: in this gauge all the comoving observers have the same proper time. It can be shown that the synchronous and time-orthogonal gauge has a residual gauge freedom, so another constraint is needed to completely fix the gauge (usually initial conditions are used).

- Comoving gauge: $v^{\|} = 0$ and $v^i_{\perp} = 0$. Usually the third condition is taken to be $\omega^{\|} = 0$, because $T^0_i \propto v^{\|} + \omega^{\|} = 0$ and so the energy flux is null in this gauge.

- Spatially flat gauge: $\Psi = 0$, $\chi^{\|} = 0$ and $\chi^i_{\perp} = 0$. Hypersurfaces at $\eta = \mathit{const}$ are left unperturbed in this gauge.

- Uniform energy density gauge: $\delta\rho = 0$.

3.6 Gauge-invariant perturbations

There are two ways in which the gauge problem can be kept under control.
The first one consists in choosing a gauge and then making all calculations consistently in that gauge: observable quantities should always be gauge invariant, so it doesn't matter the gauge chosen to calculate them. This first approach usually presents simpler calculations, but one should always be careful with any residual gauge invariance that could generate unphysical solutions.
The second approach is to use gauge-invariant perturbations. In fact it is possible to build, starting from the perturbations of the metric and stress-energy tensors, some quantities that remain unchanged under a gauge transformation, avoiding the gauge choice issue.
Looking at the transformation laws (3.24)-(3.30) we can define scalar, vector and tensor gauge-invariant perturbations.

Scalar perturbations
We can build two purely geometric gauge-invariant quantities starting from scalar perturbations ([23],[25]):

$$2\Phi_A = 2\Phi + 2\omega^{\|\prime} + 2\frac{a'}{a}\omega^{\|} - \left(\chi^{\|\prime\prime} + \frac{a'}{a}\chi^{\|\prime}\right) \tag{3.31}$$

$$2\Phi_H = -2\Psi - \frac{1}{3}\nabla^2\chi^{\|} + 2\frac{a'}{a}\omega^{\|} - \frac{a'}{a}\chi^{\|\prime} \tag{3.32}$$

We note that in the Poisson gauge we get $\Phi_A = \Phi$ and $\Phi_H = -\Psi$.
If we use also perturbations of the stress-energy tensor we have three more gauge-invariant quantities:

$$2v_s = 2v^{\|} + \chi^{\|\prime} \tag{3.33}$$

that is the scalar shear amplitude associated to the velocity field of matter,

$$\epsilon_m = \delta\rho + \rho_0'(v^{\|} + \omega^{\|}) \tag{3.34}$$

that is the energy density perturbation in the comoving gauge,

$$\epsilon_g = \delta\rho + \rho_0'(2\omega^{\|} - \chi^{\|\prime}) \tag{3.35}$$

that is the energy density perturbation in the zero-shear gauge.

Vector perturbations
We can build only one purely geometric gauge-invariant quantity starting from vector perturbations:

$$\Psi_i = \omega_i^{\perp} - \chi_i^{\perp\prime} \tag{3.36}$$

that corresponds to the geometric vector shear amplitude.
Using also stress-energy perturbations we get:

$$v_s^i = v_{\perp}^i + \chi_{\perp}^{i\prime} \tag{3.37}$$

that is the shear amplitude built starting from $v_{\perp}^i$,

$$v_c^i = v_{\perp}^i + \omega_{\perp}^i \tag{3.38}$$

that is the vorticity tensor amplitude.

Tensor perturbations
From equation (3.27) we know that $\tilde{\chi}_{ij}^T = \chi_{ij}^T$, so tensor perturbations are automatically gauge-invariant at linear order.

Chapter 4

The Boltzmann equation for photons

The final goal of this thesis is to study a particular source of anisotropy for the CMB in the framework of a theory of modified gravity. However anisotropies in the cosmic distribution of photons are complicated to calculate. The photons' propagation is affected by gravity and by Compton scattering with electrons. The electrons are tightly coupled to protons and they are both affected by gravity. The metric itself is influenced by photons, electrons and protons, plus neutrinos and dark matter. This means that if we want to obtain the photons' distribution, we need to solve also for the other components.
The tool that is going to help us in this quest is Boltzmann equation. As it is known, Boltzmann equation describes the evolution of the distribution function f of a particular species in the universe[1]:

$$\frac{d}{dt}f(\vec{x},\vec{p},t) = \mathbb{C}[f(\vec{x},\vec{p},t)] \tag{4.1}$$

where in the right hand side we have the collisional operator that takes into account all possible collision terms. Instead, it can be shown (see [19]) that the total derivative in the left hand side of equation (4.1) (sometimes called *Liouville operator* and indicated as $\mathbb{L}[f]$) in curved spacetime becomes

$$\frac{d}{dt}f(x^\mu,p^\mu) = p^\alpha\frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma\frac{\partial f}{\partial p^\alpha} \tag{4.2}$$

where $\Gamma^\alpha_{\beta\gamma}$ represent the Christoffel symbols of the metric.

4.1 Metric perturbations and gauge choice

It is easy to show that in a FRW universe the perfect isotropy and homogeneity of space constrain the distribution function f to be function of $x^0 = t$ and $p^0 = E$ (or $|\vec{p}|$) only, and that the Liouville operator becomes

$$\mathbb{L}[f(t,E)] = E\frac{\partial f}{\partial t} - \frac{\dot{a}(t)}{a(t)}|\vec{p}|^2\frac{\partial f}{\partial E} \tag{4.3}$$

But we should not forget what we said above: photons' propagation is influenced by the curvature of spacetime and the curvature of spacetime is a consequence of all kinds of energy

[1] The distribution function $f(\vec{x},\vec{p},t)$ of a particle species is defined in such a way that $dN = \frac{1}{(2\pi)^3}f(\vec{x},\vec{p},t)d^3xd^3p$ is the number of the particles in the infinitesimal volume d^3xd^3p of the phase space, where $\vec{x}$ indicates the particle's position and $\vec{p}$ indicates the particle's momentum

sources, including photons. So we need to consider a metric perturbed by the presence of radiation and matter, and equation (4.3) cannot be used.
Using equations (3.13)-(3.15), the most general perturbed metric can be written at first order in this way:

$$\begin{aligned} ds^2 = &-(1+2\Phi(t,\vec{x}))dt^2 + 2a(t)\omega_i(t,\vec{x})dx^i dt + \\ &+ a^2(t)[(1-2\Psi(t,\vec{x}))\delta_{ij} + \chi_{ij}(t,\vec{x})]dx^i dx^j \end{aligned} \tag{4.4}$$

Remembering now the decomposition of perturbations into scalar, vector and tensor parts, we can rewrite equation (4.4) as

$$\begin{aligned} ds^2 = &-(1+2\Phi)dt^2 + 2a(t)(\partial_i\omega^{\parallel} + \omega_i^{\perp})dx^i dt + \\ &+ a^2(t)\left[(1-2\Psi)\delta_{ij} + D_{ij}\chi^{\parallel} + \partial_i\chi_j^{\perp} + \partial_j\chi_i^{\perp} + \chi_{ij}^T\right]dx^i dx^j \end{aligned} \tag{4.5}$$

where Φ, $\omega^{\parallel}$, Ψ and $\chi^{\parallel}$ are scalar perturbations, $\omega_i^{\perp}$ and $\chi_i^{\perp}$ are vector perturbations and χ_{ij}^T is a tensor perturbation.

We can now choose a particular gauge in order to simplify our calculations, using the gauge invariance of metric theories of gravity discussed in Chapter 3: we will use Poisson gauge, so we take $\omega^{\parallel} = 0$, $\chi^{\parallel} = 0$ and $\chi_i^{\perp} = 0$.
Furthermore, in the following we will consider only scalar perturbations. This choice is motivated by the fact that the scalar perturbations in the metric are the most important between those that couple to radiation perturbations: in fact vector perturbations usually have amplitudes that decrease rapidly in time, while tensor perturbations describe gravitational waves, which have a small influence on photons (see [20]).
Considering both these things, in what follows we will work with a metric in the form

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j \tag{4.6}$$

4.2 Liouville operator for photons

As we have already made clear, the distribution function f will be function of the spacetime point x^μ and the particle's momentum, that can be defined as $P^\mu = \frac{dx^\mu}{d\lambda}$ using an affine parameter λ. Since we are considering photons, we should remember that

$$P^\mu P^\nu g_{\mu\nu} = 0 \tag{4.7}$$

because we are dealing with massless particles. Defining now $p^2 \equiv P^i P^j g_{ij}$, equation (4.7) together with equation (4.6) gives us at first order

$$P^0 = \frac{p}{\sqrt{1+2\Phi}} \simeq p(1-\Phi) \tag{4.8}$$

that allows us to eliminate P^0 from the equations in favour of p, since they are not independent quantities.
Therefore, defining the unit vector $\hat{p}^i$ such that $P^i = p\hat{p}^i$, we can now express the total derivative in the Liouville operator as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} \tag{4.9}$$

Let's now analyze the different terms in the right hand side of equation (4.9).
It is easy to show that the last term vanishes at first order. In fact at zeroth order f is simply the Bose-Einstein distribution, that depends only on p and not on the direction $\hat{p}^i$: so $\frac{\partial f}{\partial \hat{p}^i}$ is a first order term. Also $\frac{d\hat{p}^i}{dt}$ is a first order term, since the direction of the photons changes only in the presence of Φ and Ψ. This means that $\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}$ is a second order term and can be neglected.
Using similar arguments it can be shown (see [20]) that the second term in equation (4.9) reads at first order $\frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a}$, so we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{dp}{dt} \tag{4.10}$$

In order to calculate $\frac{dp}{dt}$ we use the geodesic equation

$$\frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta \tag{4.11}$$

If we now explicit the Christoffel symbols for the perturbed metric and we use equation (4.8), at first order we get (see [20])

$$\frac{1}{p}\frac{dp}{dt} = -H + \frac{\partial \Psi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i} \tag{4.12}$$

Equation (4.12) describes the change in the momentum as a photon moves through a perturbed FRW universe. The first term accounts for the loss of momentum due to the Hubble expansion. Observing that, with our sign conventions, an overdense region has $\Psi < 0$ and $\Phi < 0$, the second term tells us that a photon in a deepening gravitational well ($\frac{\partial \Psi}{\partial t} < 0$) loses energy: it is understandable, as the deepening well makes it more difficult for the photon to emerge, thereby increasing the magnitude of the redshift. Finally, the last term in equation (4.12) describes how a photon traveling into a well ($\hat{p}^i\frac{\partial \Phi}{\partial x^i} < 0$) gets blueshifted because it is being pulled towards the center.
We are now in position to rewrite equation (4.10):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H - \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right] \tag{4.13}$$

To go further we have to expand the photon distribution function f about the zero-order form, that is Bose-Einstein distribution

$$f^{(0)}(p,t) = \frac{1}{e^{p/T(t)} - 1} \tag{4.14}$$

where $T(t)$ is the temperature, function of t only because of homogeneity and isotropy. We do that by writing

$$f(t,\vec{x},p,\hat{p}) = \left\{\exp\left[\frac{p}{T(t)[1+\Theta(\vec{x},\hat{p},t)]}\right] - 1\right\}^{-1} \tag{4.15}$$

where the function $\Theta(\vec{x},\hat{p},t) = \frac{\delta T}{T}$ takes into account the inhomogeneities and the anisotropies in the photon distribution function. We are assuming that this temperature perturbation does not depend on the momentum magnitude p: the reason for this assumption comes from the

fact that, at first order, all the interactions we are going to consider leave the magnitude of the photons' momentum unchanged.
Expanding now the right hand side of equation (4.15) at first order in Θ, we get

$$f(t,\vec{x},p,\hat{p}) \simeq f^{(0)}(p,t) - p\frac{\partial f^{(0)}(p,t)}{\partial p}\Theta(\vec{x},\hat{p},t) \tag{4.16}$$

where $f^{(0)}(p,t)$ is the zeroth order Bose-Einstein distribution defined in equation (4.14) and we used $T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p}$.

If we now look only at zeroth order terms, the full Boltzmann equation becomes

$$\frac{\partial f^{(0)}(p,t)}{\partial t} - Hp\frac{\partial f^{(0)}(p,t)}{\partial p} = 0 \tag{4.17}$$

where we have set the collision term in the right hand side of equation (4.1) to zero. In fact we will see that there is no zeroth order collision term: this is reasonable, since the collision terms will create inhomogeneities and anisotropies that are first order perturbations.
If we now rewrite the time derivative in (4.17) as

$$\frac{\partial f^{(0)}(p,t)}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} = -\frac{1}{T}\frac{dT}{dt}p\frac{\partial f^{(0)}}{\partial p} \tag{4.18}$$

we get

$$\left[-\frac{1}{T}\frac{dT}{dt} - \frac{1}{a}\frac{da}{dt}\right]\frac{\partial f^{(0)}}{\partial p} = 0 \tag{4.19}$$

Since $\frac{\partial f^{(0)}}{\partial p} \neq 0$, we have

$$\frac{dT}{T} = -\frac{da}{a} \quad \Rightarrow \quad T \propto \frac{1}{a} \tag{4.20}$$

We note that equation (4.20) describes the familiar result of how the photon's wavelenght is stretched as the universe expands.

To find a first order equation for Θ we substitute equation (4.16) into equation (4.13), obtaining

$$\begin{aligned} \left.\frac{df}{dt}\right|_{\text{first order}} =& -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\Theta\frac{\partial}{\partial p}\left[\frac{\partial f^{(0)}}{\partial p}p\right] + \\ & -p\frac{\partial f^{(0)}}{\partial p}\left[-\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right] \end{aligned} \tag{4.21}$$

Rewriting the time derivative as a temperature derivative and using again $T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p}$ we finally get

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} - \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right] \tag{4.22}$$

The first two terms in the right hand side of equation (4.22) account for free streaming, while the last two terms account for the effects of gravity.

4.3 Collisional operator for photons

In this Section we analyze the right hand side of the Boltzmann equation (4.1) in the case of photons. Since Compton scattering is the most significant interaction that can alter the photon distribution function, we consider only this process:

$$e^-(\vec{q}) \quad + \quad \gamma(\vec{p}) \quad \longleftrightarrow \quad e^-(\vec{q}\,') \quad + \quad \gamma(\vec{p}\,')$$

The collisional term for this scattering is (see [20])

$$\begin{aligned} \mathbb{C}[f(\vec{p})] = & \frac{1}{p}\int\frac{d^3q}{(2\pi)^3 2E_e(q)}\int\frac{d^3q'}{(2\pi)^3 2E_e(q)}\int\frac{d^3p'}{(2\pi)^3 2E(p)}|\mathcal{M}|^2(2\pi)^4\times \\ & \times\delta^3(\vec{p}+\vec{q}-\vec{p}\,'-\vec{q}\,')\delta(E(p)+E_e(q)-E(p')-E_e(q'))\times \\ & \times[f_e(\vec{q}\,')f(\vec{p}\,')-f_e(\vec{q})f(\vec{p})] \end{aligned} \tag{4.23}$$

where we have neglected Pauli suppression and Bose enhancement factors (at first order it is a valid assumption). We have defined the momentum magnitude $q = |\vec{q}|$ as before, the energy of the electron $E_e(q)$, the energy of the photon $E(p)$, the amplitude for the process $\mathcal{M}$ and the distribution function for the electrons $f_e(\vec{q})$. The delta functions enforce energy and momentum conservation.
In order to go further we have to make some assumptions. First of all we take the electrons to be non relativistic, so we can consider

$$E_e(q) \simeq m_e + \frac{q^2}{2m_e} \tag{4.24}$$

while for photons $E(p) = p$. Moreover, in the epochs we are interested in, the kinetic energy of the electrons is much smaller than the rest mass m_e, so in the denominators of equation (4.23) we can substitute E_e with m_e.
Furthermore, non relativistic Compton scattering is nearly elastic, so at zeroth order $p' \simeq p$ and $q'^2 \simeq q^2$ and we can expand p' and q' around p and q respectively.
We now need to calculate the amplitude for Compton scattering and we will take it to be constant

$$|\mathcal{M}|^2 = 8\pi\sigma_T m_e^2 \tag{4.25}$$

with σ_T being the well-known Thomson cross-section. Equation (4.25) neglects two dependencies:

- Angular dependence. The amplitude squared has an angular dependence proportional to the factor $1 + \cos^2(\hat{p}\cdot\hat{p}')$, but it can be ignored if we accept a 1% inaccuracy.

- Polarization dependence. The amplitude squared has a polarization dependence proportional to the factor $|\hat{\epsilon}\cdot\hat{\epsilon}'|^2$, where $\hat{\epsilon}$ and $\hat{\epsilon}'$ are the polarizations of the incoming and outgoing photons. This means that the CMB will be polarized due to Compton scattering: even if we are not concerned in polarization, the temperature anisotropies are coupled with polarization fields so a small effect will be present, but we will ignore it.

Remembering that electrons are tightly coupled to baryons through Coulomb scattering, we can also consider the bulk velocity of electrons to be nearly equal to the baryonic velocity, so

we can take $\frac{1}{2m_e}\vec{q} \simeq \vec{v_b}$.
Finally, defining the monopole part of the temperature perturbation

$$\Theta_0(t, \vec{x}) \equiv \frac{1}{4\pi}\int d\Omega\Theta(t, \vec{x}, \hat{p}) \tag{4.26}$$

we find

$$\mathbb{C}[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p}n_e\sigma_T[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v_b}] \tag{4.27}$$

where $n_e = \int d^3q\frac{f_e(\vec{q})}{(2\pi)^3}$ is the electron number density and we have written explicitly only the functions' arguments that depend on the direction of the photon.

4.4 Boltzmann equation for photons

In order to obtain Boltzmann equation for photons we have to equal the Liouville operator and the collisional operator: from equations (4.22) and (4.27) we get

$$\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T[\Theta_0 - \Theta + \hat{p}\cdot\vec{v_b}] \tag{4.28}$$

It is convenient to use conformal time $d\eta = \frac{dt}{a}$, so we get

$$\Theta' + \hat{p}^i\frac{\partial\Theta}{\partial x^i} - \Psi' + \hat{p}^i\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T a[\Theta_0 - \Theta + \hat{p}\cdot\vec{v_b}] \tag{4.29}$$

where the $'$ represents as usual the derivative with respect to conformal time.

Since equation (4.29) is a partial differential linear equation, it is useful to use Fourier transforms because the different Fourier modes will evolve independently. Our convention for Fourier transforms will be

$$\Theta(\vec{x}) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k}) \tag{4.30}$$

We define now the cosine of the angle between the wavevector $\vec{k}$ and the photon direction $\hat{p}$ to be

$$\mu \equiv \frac{\vec{k}\cdot\hat{p}}{|\vec{k}|} \tag{4.31}$$

We define also the *optical depth*[2] to be

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta' n_e\sigma_T a \tag{4.32}$$

in such a way that

$$\tau'(\eta) = -n_e\sigma_T a \tag{4.33}$$

Considering these definitions, equation (4.29) in Fourier space becomes

$$\tilde{\Theta}' + ik\mu\tilde{\Theta} - \tilde{\Psi}' + ik\mu\tilde{\Phi} = -\tau'\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v_b}\right] \tag{4.34}$$

[2]It can be considered as a measure of how "difficult" is for a photon to propagate from η to η_0: if τ is large, then a photon will experience many Compton scattering before it can be detected by an observer at present time η_0.

For completeness, if we don't neglect the angular dependence in the amplitude squared of Compton scattering, we get

$$\tilde{\Theta}' + ik\mu\tilde{\Theta} - \tilde{\Psi}' + ik\mu\tilde{\Phi} = -\tau' \left[\tilde{\Theta_0} - \tilde{\Theta} + \mu\tilde{v_b} - \frac{1}{2}\mathcal{P}_2(\mu)\tilde{\Theta_2}\right] \tag{4.35}$$

where $\mathcal{P}_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$ is the second Legendre polynomial and

$$\Theta_2 \equiv \int_{-1}^{1} \frac{d\mu}{2}\mathcal{P}_2(\mu)\Theta \tag{4.36}$$

Chapter 5

Free streaming

The perturbations of the photon distribution function evolve in two completely different ways before and after the epoch of recombination ($t \simeq 3.8 \times 10^5$y). Before recombination the photons are tightly coupled to electrons and protons: all together they can be described as a single "baryon-photon" fluid. After recombination photons free-stream from the surface of last scattering to us today: in this Chapter we are going to see how perturbations change during this free-streaming, modifying the anisotropies in the CMB that we observe.

First of all, we define the multipole moments of the temperature perturbation Θ

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_l(\mu)\Theta(\mu) \tag{5.1}$$

that generalize equations (4.26) and (4.36). If we take two photons separated by a comoving distance k^{-1}, we see them coming from an angular separation

$$\theta \simeq \frac{k^{-1}}{\eta_0 - \eta*} \tag{5.2}$$

because $\eta_0 - \eta*$ is the comoving distance between us and the surface of last scattering, if we define $\eta*$ to be the conformal time of recombination. If we decompose the temperature field into multipole moments, then an angular scale θ will roughly correspond to $1/l$. So, remembering that $\eta* \ll \eta_0$, inhomogeneities on scales k^{-1} will become anisotropies on angular scales $l \simeq k\eta_0$.
In this argument we have implicitly assumed that, between the surface of last scattering and the observer, nothing happens to the photons: this is true for a matter-dominated universe, because the gravitational potentials Φ and Ψ that the photons encounter during the journey remain constant in time. But we must take into account that recombination occurs not too much later than the matter-radiation equivalence, and the non-negligible radiation energy density causes the potential to significantly change in time. Moreover, at late times, the acceleration in the expansion of the universe leads to the decay of the gravitational potentials. The first effect is known as *early-time integrated Sachs-Wolfe effect* and it is usually lumped in with the primordial CMB, since the energy density fluctuations that cause it are in practice undetectable. The second effect is known as *late-time integrated Sachs-Wolfe effect* and will be studied in the framework of mimetic Horndeski gravity in the last Chapter of this thesis.

5.1 Free streaming

In this Section we want to obtain an expression for the multipole moments $\Theta_l(\eta_0)$ at present time in terms of the perturbations of the temperature at the time of recombination $\eta*$ and in terms of the perturbations of the metric.
We start from the Boltzmann equation for photons (4.34), removing the tilde to indicate Fourier transforms: rearranging both sides we get

$$\Theta' + (ik\mu - \tau')\Theta = \Psi' - ik\mu\Phi - \tau'[\Theta_0 + \mu v_b] \tag{5.3}$$

Defining the right hand side to be the source function $\tilde{S} \equiv \Psi' - ik\mu\Phi - \tau'[\Theta_0 + \mu v_b]$ and manipulating the left hand side, we obtain

$$e^{-ik\mu\eta+\tau}\frac{d}{d\eta}[\Theta e^{ik\mu\eta-\tau}] = \tilde{S} \tag{5.4}$$

Multiplying now by $e^{ik\mu\eta-\tau}$ and integrating over η from the initial time η_i to η_0, we have

$$\Theta(\eta_0) = \Theta(\eta_i)e^{ik\mu(\eta_i-\eta_0)}e^{-\tau(\eta_i)} + \int_{\eta_i}^{\eta_0} d\eta\tilde{S}(\eta)e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{5.5}$$

having used the fact that $\tau(\eta_0) = 0$, which follows from the definition of the optical depth (4.32).
If we take the initial time η_i early enough, the optical depth $\tau(\eta_i)$ will be extremely large (because Compton scattering will be very frequent): so we can set the first term in the right hand side of equation (5.5) to zero and set the lower limit in the integral to 0, because any contribution to the integrand from $\eta < \eta_i$ will be negligible, being suppressed by a factor $e^{-\tau(\eta)}$. Therefore we get, writing explicitly all the arguments,

$$\Theta(k,\mu,\eta_0) = \int_0^{\eta_0} d\eta\tilde{S}(k,\mu,\eta)e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{5.6}$$

If $\tilde{S}$ did not depend on μ we could multiply both sides of equation (5.6) by the Legendre polynomial $\mathcal{P}_l(\mu)$ and then integrate over μ. Using the identity

$$\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_l(\mu)e^{ik\mu(\eta-\eta_0)} = \frac{1}{(-i)^l}j_l[k(\eta-\eta_0)] \tag{5.7}$$

where the j_l are the spherical Bessel functions, we would get

$$(-i)^l\Theta_l(k,\eta_0) = \frac{1}{(-i)^l}\int_0^{\eta_0} d\eta\tilde{S}(k,\eta)e^{-\tau(\eta)}j_l[k(\eta-\eta_0)] \tag{5.8}$$

so

$$\Theta_l(k,\eta_0) = (-1)^l\int_0^{\eta_0} d\eta\tilde{S}(k,\eta)e^{-\tau(\eta)}j_l[k(\eta-\eta_0)] \tag{5.9}$$

The μ dependence in $\tilde{S}$ can be treated noting that in equation (5.6) the source function is multiplied by the exponential $e^{ik\mu(\eta-\eta_0)}$: thus everywhere we encounter a factor μ in the explicit expression of $\tilde{S}$ we can replace it with

$$\mu \quad\to\quad \frac{1}{ik}\frac{d}{d\eta} \tag{5.10}$$

In order to show that this method works, we take the $-ik\mu\Phi$ term in $\tilde{S}$:

$$- ik \int_0^{\eta_0} d\eta\mu\Phi e^{ik\mu(\eta-\eta_0)-\tau(\eta)} = - \int_0^{\eta_0} d\eta\Phi e^{-\tau(\eta)} \frac{d}{d\eta} e^{ik\mu(\eta-\eta_0)} \tag{5.11}$$

and if we integrate by parts we get

$$\begin{aligned}-ik \int_0^{\eta_0} d\eta\mu\Phi e^{ik\mu(\eta-\eta_0)-\tau(\eta)} =& - \left[\Phi e^{-\tau(\eta)} e^{ik\mu(\eta-\eta_0)}\right]_0^{\eta_0} + \\ &+ \int_0^{\eta_0} d\eta \, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta} \left[\Phi e^{-\tau(\eta)}\right]\end{aligned} \tag{5.12}$$

The surface terms can be neglected: at $\eta = 0$ there is a $e^{\tau(0)}$ damping, while at $\eta = \eta_0$ there is no angular dependence, so it is an alteration in the monopole that we cannot detect. So if we now define a new source function

$$S(k, \eta) \equiv e^{-\tau} \left[\Psi' - \tau'\Theta_0\right] + \frac{d}{d\eta} \left[e^{-\tau} \left(\Phi - \frac{iv_b\tau'}{k}\right)\right] \tag{5.13}$$

and we use the property of the spherical Bessel functions $j_l(x) = (-1)^l j_l(-x)$, we obtain

$$\Theta_l(k, \eta_0) = \int_0^{\eta_0} d\eta S(k, \eta) j_l[k(\eta_0 - \eta)] \tag{5.14}$$

At this point we can define the *visibility function*

$$g(\eta) \equiv -\tau' e^{-\tau} \tag{5.15}$$

that can be thought as the probability density that a photon last scattered at time η. This function is sharply peaked at $\eta = \eta*$: before recombination τ is large, so g is small, while after recombination $-\tau'$ (the scattering rate) is small, so again g is suppressed. Using the visibility function, the source function in equation (5.13) becomes

$$\begin{aligned}S(k, \eta) =& g(\eta)[\Theta_0(k, \eta) + \Phi(k, \eta)]+ \\ &+ \frac{d}{d\eta} \left(\frac{iv_b(k, \eta)g(\eta)}{k}\right) + \\ &+ e^{-\tau} \left[\Phi'(k, \eta) + \Psi'(k, \eta)\right]\end{aligned} \tag{5.16}$$

In this way equation (5.14) becomes

$$\begin{aligned}\Theta_l(k, \eta_0) =& \int_0^{\eta_0} d\eta g(\eta)[\Theta_0(k, \eta) + \Phi(k, \eta)] j_l[k(\eta_0 - \eta)]+ \\ &+ \int_0^{\eta_0} d\eta \frac{d}{d\eta} \left(\frac{iv_b(k, \eta)g(\eta)}{k}\right) j_l[k(\eta_0 - \eta)]+ \\ &+ \int_0^{\eta_0} d\eta \, e^{-\tau} \left[\Phi'(k, \eta) + \Psi'(k, \eta)\right] j_l[k(\eta_0 - \eta)]\end{aligned} \tag{5.17}$$

If we approximate the $g(\eta)$ function with a delta function peaked at $\eta*$, and then we use integration by parts in the second term of the right hand side of equation (5.17), we have

$$\begin{aligned}\Theta_l(k, \eta_0) \simeq& [\Theta_0(k, \eta*) + \Phi(k, \eta*)] j_l[k(\eta_0 - \eta*)]+ \\ &- \frac{iv_b(\eta*)}{k} \frac{d}{d\eta} j_l[k(\eta_0 - \eta)]\bigg|_{\eta=\eta*} + \\ &+ \int_0^{\eta_0} d\eta \, e^{-\tau} \left[\Phi'(k, \eta) + \Psi'(k, \eta)\right] j_l[k(\eta_0 - \eta)]\end{aligned} \tag{5.18}$$

Finally, using the relation

$$\frac{d}{dx}j_l(x) = j_{l-1}(x) - \frac{l-1}{x}j_l(x) \tag{5.19}$$

we get the well-known formula

$$\begin{aligned}\Theta_l(k,\eta_0) \simeq &[\Theta_0(k,\eta*) + \Phi(k,\eta*)]j_l[k(\eta_0-\eta*)]+\\ &+ iv_b(\eta*)\left(j_{l-1}[k(\eta_0-\eta*)] - \frac{l-1}{k(\eta_0-\eta*)}j_l[k(\eta_0-\eta*)]\right)+\\ &+\int_0^{\eta_0} d\eta\, e^{-\tau}\left[\Phi'(k,\eta)+\Psi'(k,\eta)\right]j_l[k(\eta_0-\eta)]\end{aligned} \tag{5.20}$$

Let's now analyze the three terms in the right hand side of equation (5.20).

- The first term accounts for the *Sachs-Wolfe effect*. It gives a mathematical justification for the statement we made at the beginning of this Chapter, saying that inhomogeneities on scales k^{-1} become anisotropies on angular scales $l \simeq k\eta_0$. The presence of the potential Φ can be explained with the fact that the photons we see today had to travel out of the potentials they were in at the time of recombination: as they emerged from these potential perturbations, their waveleghts were changed (redshifted or blueshifted) and also was changed their energy. Thus the temperature we observe today is $\Theta_0 + \Phi$ calculated at $\eta*$.

- The second term accounts for *Doppler effect*.

- The third term accounts for the *integrated Sachs-Wolfe effect* (ISW effect). As we said before, this term consider the fact that potentials could change in time as the photons travel from the surface of last scattering to us. If the potentials remain constant in time a photon will gain some energy when it enters a potential well, but it will lose the same amount of energy when it leaves the well. Instead if the potentials change in time, the depth of the well could change while the photon is still inside of it: this means that the photon will lose a different amount of energy leaving the well, if compared to the amount of energy it gained falling into the well.

5.2 The C_l coefficients

In the previous Section we have found a way to calculate the multipole moments $\Theta_l(k,\eta)$, but they are rather abstract: we need to relate these quantities to something we use in practice to characterize the anisotropy pattern of the CMB.
First of all we recall that in equation (4.15) we wrote the temperature field in the universe as

$$T(\vec{x},\hat{p},\eta) = T(\eta)[1+\Theta(\vec{x},\hat{p},\eta)] \tag{5.21}$$

Considering that we observe this temperature only on the Earth ($\vec{x}_0$) and now (η_0), the only dependence that we can describe is the one on the direction of the incoming photons $\hat{p}$: observers make maps of the sky, reporting the temperature measured in the different directions. The resolution of the experiments constrains the resolution of the maps: the smaller is the angular resolution of the telescope, the more accurate will be the map (see Figure 5.1 and 5.2).

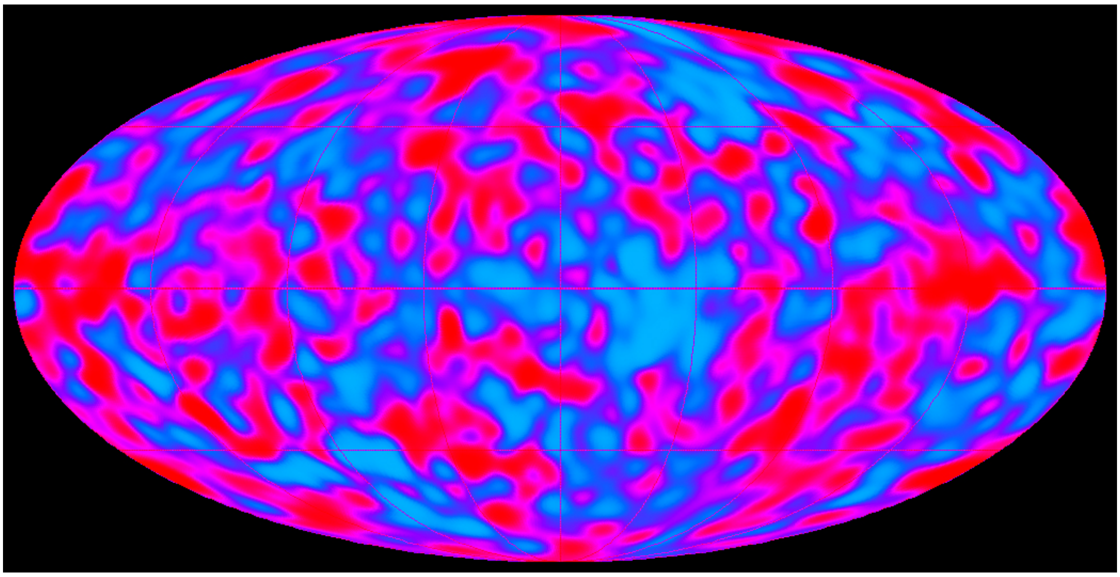

Figure 5.1: The full sky map of the CMB anisotropies, realized using data from the COBE satellite (1992)

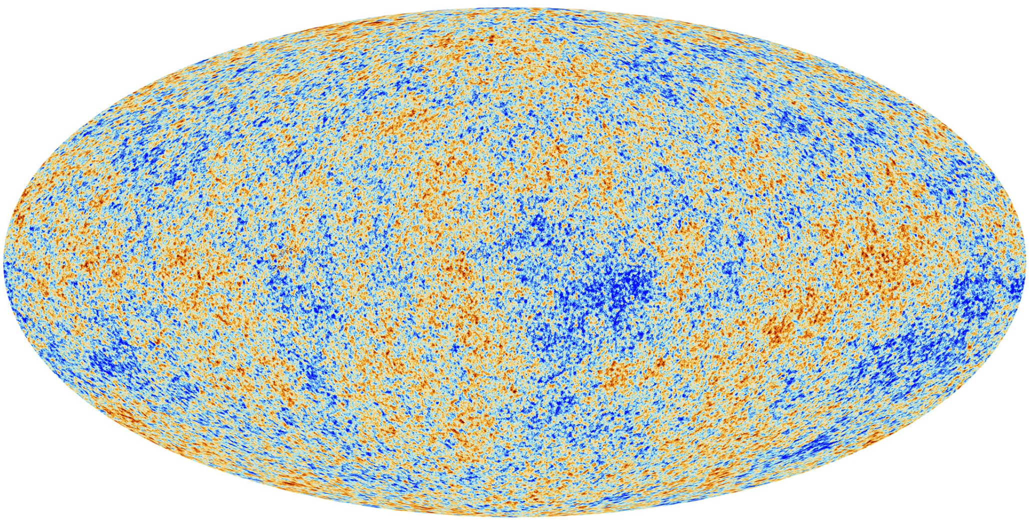

Figure 5.2: The full sky map of the CMB anisotropies, realized using data from the Planck satellite (2013)

We start by expanding the temperature perturbation in terms of spherical harmonics Y_{lm}:

$$\Theta(\vec{x}, \hat{p}, \eta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}(\vec{x}, \eta) Y_{lm}(\hat{p}) \tag{5.22}$$

where we use the $\hat{p}$ dependence, but the classical θ, ϕ dependence in the spherical harmonics can be recovered considering that $\hat{p}$ is a unit vector, so

$$\hat{p}_x = \sin\theta\cos\phi \quad \hat{p}_y = \sin\theta\sin\phi \quad \hat{p}_z = \cos\theta \tag{5.23}$$

In order to explain the relation between the expansion (5.22) and the resolution of the sky maps obtained using satellites, we can start with an example: let's consider an experiment that maps the full sky (4π radians$^2 \simeq 41,000$ degrees^2) with an angular resolution of 7°. Each "pixel" has an area of $(7°)^2$ so we need 840 pixels to cover the full sky: such an experiment has 840 independent pieces of information.
If we want to describe the temperature perturbations in terms of the a_{lm} coefficients, there is some l_{max} above which we have no information. In fact the total number of recoverable coefficients will be 840 so, remembering that for a fixed l there are $2l + 1$ different a_{lm} coefficients, this l_{max} satisfies

$$840 = \sum_{l=0}^{l_{max}} (2l + 1) = (l_{max} + 1)^2 \tag{5.24}$$

from which we obtain $l_{max} = 28$. So an experiment with angular resolution of 7°allows us to write a decomposition of the temperature field, like the one in (5.22), up to $l = 28$.

Now we want to relate the observable a_{lm} coefficients to the abstract multipole moments Θ_l. Using the orthogonality of spherical harmonics ($d\Omega$ is the solid angle spanned by $\hat{p}$)

$$\int d\Omega Y_{lm}(\hat{p}) Y^*_{l'm'}(\hat{p}) = \delta_{ll'}\delta_{mm'} \tag{5.25}$$

we can multiply equation (5.22) by $Y^*_{l'm'}(\hat{p})$ and then integrate over the full solid angle, getting

$$a_{lm}(\vec{x}, \eta) = \int d\Omega Y^*_{lm}(\hat{p}) \Theta(\vec{x}, \hat{p}, \eta) \tag{5.26}$$

Since in the previous Section we have obtained solutions for the Fourier transforms $\Theta_l(k, \eta)$, we rewrite equation (5.26) as

$$a_{lm}(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega Y^*_{lm}(\hat{p}) \Theta(\vec{k}, \hat{p}, \eta) \tag{5.27}$$

A point must be made clear now. As we said above, with an experiment we can determine the a_{lm} coefficient up to a maximum value of l, but these coefficients characterize temperature perturbations that are originated by quantum fluctuations during inflation, so they must be regarded as random variables described by a probability distribution. Therefore their particular values have no significance, but they can give us information about the distribution from which they are drawn.
These distributions have a null mean value, but they have a non-zero variance: we can define the C_l coefficients in such a way that

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'}\delta_{mm'} C_l \tag{5.28}$$

We note that, for a fixed l, each a_{lm} has the same variance: for example for $l = 100$ all 201 a_{100m} are drawn from the same distribution with variance C_{100}. So if we measure, with an experiment, the 201 values of the a_{100m}, we obtain a very good sampling of the distribution for $l = 100$, getting a good handle on the value of C_{100}. On the other hand if we measure the 5 values of the a_{2m}, we don't get very much information about C_2. This means that there is a fundamental uncertainty in the knowledge of the C_l coefficients, called *cosmic variance*:

$$\left(\frac{\Delta C_l}{C_l}\right)_{\text{cosmic variance}} = \sqrt{\frac{2}{2l+1}} \tag{5.29}$$

We are now going to write the C_l coefficients as functions of the Θ_l. Using equation (5.27) we have

$$\begin{aligned} C_l = \langle a_{lm} a^*_{lm} \rangle = &\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega Y^*_{lm}(\hat{p}) \times \\ &\times \int \frac{d^3k'}{(2\pi)^3} e^{-i\vec{k}'\cdot\vec{x}} \int d\Omega' Y_{lm}(\hat{p}') \times \\ &\times \left\langle \Theta(\vec{k},\hat{p}) \Theta^*(\vec{k}',\hat{p}') \right\rangle \end{aligned} \tag{5.30}$$

The average $\left\langle \Theta(\vec{k},\hat{p}) \Theta^*(\vec{k}',\hat{p}') \right\rangle$ is complicated to evaluate, since it depends both on the initial perturbations (generated randomly during inflation) and on their evolution to become anisotropies. In order to overcome this problem we can use the matter perturbations

$$\delta(\vec{x},\eta) \equiv \frac{\delta\rho_m}{\rho_m} \tag{5.31}$$

and write the temperature perturbations as $\Theta = \delta \times \frac{\Theta}{\delta}$: the ratio Θ/δ does not depend on the initial amplitude ([20]) so it can be removed from the averaging over the distribution

$$\left\langle \Theta(\vec{k},\hat{p}) \Theta^*(\vec{k}',\hat{p}') \right\rangle = \left\langle \delta(\vec{k}) \delta^*(\vec{k}') \right\rangle \frac{\Theta(\vec{k},\hat{p})}{\delta(\vec{k})} \frac{\Theta^*(\vec{k}',\hat{p}')}{\delta^*(\vec{k}')} \tag{5.32}$$

Remembering the definition of the matter power spectrum $P(k)$

$$\left\langle \delta(\vec{k}) \delta^*(\vec{k}') \right\rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(k) \tag{5.33}$$

and using the fact that the ratio Θ/δ depends only on the magnitude of $\vec{k}$ and the dot product $\hat{k}\cdot\hat{p}$ (see [20]), we can write

$$C_l = \int \frac{d^3k}{(2\pi)^3} P(k) \int d\Omega Y^*_{lm}(\hat{p}) \frac{\Theta(k,\hat{k}\cdot\hat{p})}{\delta(k)} \int d\Omega' Y_{lm}(\hat{p}') \frac{\Theta^*(k,\hat{k}\cdot\hat{p}')}{\delta^*(k)} \tag{5.34}$$

Now we can invert equation (5.1), obtaining

$$\Theta(k,\hat{k}\cdot\hat{p}) = \sum_{l=0}^{\infty} (-i)^l (2l+1) \mathcal{P}_l(\hat{k}\cdot\hat{p}) \Theta_l(k) \tag{5.35}$$

Substituting this into equation (5.34) we get

$$\begin{aligned} C_l = &\int \frac{d^3k}{(2\pi)^3} P(k) \sum_{l'=0}^{\infty} \sum_{l''=0}^{\infty} (-i)^{l'} (i)^{l''} (2l'+1)(2l''+1) \frac{\Theta_{l'}(k) \Theta^*_{l''}(k)}{|\delta(k)|^2} \times \\ &\times \int d\Omega \mathcal{P}_{l'}(\hat{k}\cdot\hat{p}) Y^*_{lm}(\hat{p}) \int d\Omega' \mathcal{P}_{l''}(\hat{k}\cdot\hat{p}') Y_{lm}(\hat{p}') \end{aligned} \tag{5.36}$$

Using the relations (see [20])

$$\int d\Omega \mathcal{P}_{l'}(\hat{k}\cdot\hat{p}) Y^*_{lm}(\hat{p}) = \frac{4\pi}{2l+1} Y^*_{lm}(\hat{k})\delta_{ll'} \tag{5.37}$$

and

$$\int d\Omega \mathcal{P}_{l'}(\hat{k}\cdot\hat{p}) Y_{lm}(\hat{p}) = \frac{4\pi}{2l+1} Y_{lm}(\hat{k})\delta_{ll'} \tag{5.38}$$

we have

$$\begin{aligned} C_l &= \int \frac{d^3k}{(2\pi)^3} P(k)(2l+1)^2 \left|\frac{\Theta_l(k)}{\delta(k)}\right|^2 \frac{(4\pi)^2}{(2l+1)^2} |Y_{lm}(\hat{k})|^2 = \\ &= (4\pi)^2 \int \frac{d^3k}{(2\pi)^3} P(k) \left|\frac{\Theta_l(k)}{\delta(k)}\right|^2 |Y_{lm}(\hat{k})|^2 = \\ &= \frac{2}{\pi}\int_0^\infty dk k^2 P(k) \left|\frac{\Theta_l(k)}{\delta(k)}\right|^2 \int d\Omega |Y_{lm}(\hat{k})|^2 \end{aligned} \tag{5.39}$$

and remembering the normalization of the spherical harmonics

$$\int d\Omega |Y_{lm}(\hat{k})|^2 = 1 \tag{5.40}$$

we finally obtain

$$C_l = \frac{2}{\pi}\int_0^\infty dk k^2 P(k) \left|\frac{\Theta_l(k)}{\delta(k)}\right|^2 \tag{5.41}$$

Chapter 6

Mimetic gravity

In this Chapter we are going to discuss the first appearance of the mimetic framework into the field of modified gravity. As it is explained in [14], Chamseddine and Mukhanov started from the action of general relativity: they made a simple conformal transformation on the metric $g_{\mu\nu}$, using an auxiliary metric $l_{\mu\nu}$ and a scalar field φ, and they found an extra degree of freedom that can mimic cold dark matter. In a subsequent work ([15]) Chamseddine, Mukhanov and Vikman proposed a small generalization of the first model, adding a potential for the scalar field φ to the action: they showed that by accurately choosing this potential, one can mimic many cosmological models (Friedmann models, quintessence, inflation...). In the following Chapters we will set $8\pi G = 1$.

6.1 Mimetic dark matter

First of all, let's consider a physical metric $g_{\mu\nu}$ that can be rewritten, through a conformal transformation, in terms of an auxiliary metric $l_{\mu\nu}$ and a scalar field φ

$$g_{\mu\nu} = -(l^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi)l_{\mu\nu} \tag{6.1}$$

Remembering the Einstein-Hilbert action of general relativity

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \mathcal{L}_m\right] \tag{6.2}$$

we know that if we perform the variation of this action with respect to the physical metric $g_{\mu\nu}$, we obtain the usual Einstein equations with $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ and $\mathcal{L}_m$ being the lagrangian density of the matter fields:

$$\delta S = \int d^4x \frac{\delta S}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta} = \frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\delta g_{\alpha\beta} \tag{6.3}$$

so

$$\delta S = 0 \quad \forall\delta g_{\alpha\beta} \iff G^{\alpha\beta} = T^{\alpha\beta} \tag{6.4}$$

that is equivalent to equation (2.3) if we set $8\pi G = 1$ and consider the cosmological constant as a particular source of energy already encapsulated in the stress-energy tensor.
But if we now consider equation (6.1), we can express the variation of the physical metric $\delta g_{\alpha\beta}$ in terms of the variations of the auxiliary metric $\delta l_{\alpha\beta}$ and the scalar field $\delta\varphi$:

$$\begin{aligned}\delta g_{\alpha\beta} &= -(l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi)\delta l_{\alpha\beta} - l_{\alpha\beta}\delta(l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi) = \\ &= -(l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi)\delta l_{\alpha\beta} - l_{\alpha\beta}(-l^{\kappa\mu}l^{\lambda\nu}\delta l_{\mu\nu}\partial_\kappa\varphi\partial_\lambda\varphi + 2l^{\kappa\lambda}\partial_\kappa\delta\varphi\partial_\lambda\varphi) = \\ &= -(l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi)\delta l_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta + g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\varphi\partial_\lambda\varphi) - 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\varphi\partial_\lambda\varphi\end{aligned} \tag{6.5}$$

where we used the relations

$$\delta l^{\mu\nu} = -l^{\mu\alpha} l^{\nu\beta} \delta l_{\alpha\beta} \qquad l_{\alpha\beta} = -\frac{1}{l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi} g_{\alpha\beta} \qquad l^{\alpha\beta} = -(l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi) g^{\alpha\beta} \tag{6.6}$$

This means that the variation of the action can be written as

$$\begin{aligned} \delta S = -\frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\times \\ \times\,[(l^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi)\delta l_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta + g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\varphi\partial_\lambda\varphi) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\varphi\partial_\lambda\varphi] \end{aligned} \tag{6.7}$$

If we take the variation with respect to the auxiliary metric $l_{\mu\nu}$ we obtain

$$G^{\mu\nu} = T^{\mu\nu} - (G - T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\varphi\partial_\beta\varphi \tag{6.8}$$

where $G \equiv g_{\mu\nu}G^{\mu\nu}$ and $T \equiv g_{\mu\nu}T^{\mu\nu}$, while if we take the variation with respect to the scalar field φ we get

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}(G - T)g^{\alpha\beta}\partial_\beta\varphi) = 0 \tag{6.9}$$

Furthermore, since

$$g^{\mu\nu} = -\frac{1}{l^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi} l^{\mu\nu} \tag{6.10}$$

the scalar field satisfies the constraint equation

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = -1 \tag{6.11}$$

Thus the gravitational field, in addition to two transverse degrees of freedom describing gravitons, acquires an extra longitudinal degree of freedom shared by the scalar field φ and a conformal factor of the physical metric. To understand what this extra degree of freedom describes we can rewrite equation (6.8) as

$$G^{\mu\nu} = T^{\mu\nu} + \tilde{T}^{\mu\nu} \tag{6.12}$$

where

$$\tilde{T}^{\mu\nu} \equiv -(G - T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\varphi\partial_\beta\varphi \tag{6.13}$$

Comparing this expression with the stress-energy tensor of a perfect fluid (equation (3.18))

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \tag{6.14}$$

if we set $p = 0$ and we make the identifications

$$\rho \equiv -(G - T) \qquad u^\mu \equiv g^{\mu\alpha}\partial_\alpha\varphi \tag{6.15}$$

the stress-energy tensor in equation (6.14) becomes equivalent to $\tilde{T}^{\mu\nu}$ in equation (6.13).
Noticing that the constraint equation (6.11) is equivalent to the normalization condition on the four-velocity $g^{\mu\nu}u_\mu u_\nu = -1$, we conclude that the extra degree of freedom imitates pressureless dust with energy density $-(G - T)$ (*mimetic dark matter*), with the scalar field φ playing the role of the velocity potential.

In order to find how the Einstein equations are modified by the presence of the term $\tilde{T}^{\mu\nu}$, we have to solve equation (6.9) finding $G - T$. Since we are interested in cosmological solutions, we take a flat FRW metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{6.16}$$

Moreover we could take the hypersurfaces of constant time to be the same as the hypersurfaces of constant φ (see [14]), so

$$\varphi(x^\mu) = t \tag{6.17}$$

in such a way that equation (6.11) is satisfied. Now equation (6.9) becomes

$$\frac{d}{dt}[a^3(G-T)] = 0 \tag{6.18}$$

and so we get

$$-(G-T) = \frac{C(x^i)}{a^3} \tag{6.19}$$

where $C(x^i)$ is a constant of integration depending only on spatial coordinates. Remembering that $-(G-T)$ represented the energy density of mimetic dark matter, we have recovered the a^{-3} dependence of the energy density of pressureless dust in an expanding universe: we have found a mimetic dark matter that behaves exactly as dark matter, whose amount is determined by the constant of integration $C(x^i)$.

6.2 Lagrange multiplier

There is an alternative path that can be used to obtain the equations of motion (6.8) and (6.9) without performing a conformal transformation on the physical metric.
As it is explained in [15] and [26], we can take Einstein-Hilbert action of general relativity (equation (6.2)) and then add a non-dynamical scalar field φ with the constraint

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = -1 \tag{6.20}$$

implemented by the Lagrange multiplier λ:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \mathcal{L}_m + \lambda(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 1)\right] \tag{6.21}$$

Taking the variation of this action with respect to the metric $g_{\mu\nu}$ and using the constraint (6.20) we get

$$G_{\mu\nu} = T_{\mu\nu} - 2\lambda\partial_\mu\varphi\partial_\nu\varphi \tag{6.22}$$

while taking the variation with respect to the Lagrange multiplier λ we obtain the constraint equation (6.20).
Taking the trace of equation (6.22) we have

$$G = T + 2\lambda \quad \Longrightarrow \quad \lambda = \frac{1}{2}(G-T) \tag{6.23}$$

and so equation (6.22) becomes

$$G_{\mu\nu} = T_{\mu\nu} - (G-T)\partial_\mu\varphi\partial_\nu\varphi \tag{6.24}$$

that is equivalent to equation (6.8). Finally if we take the variation with respect to the scalar field φ we find

$$\delta S = 0 \quad \forall\delta\varphi \iff \partial_\alpha(2\sqrt{-g}\lambda g^{\alpha\beta}\partial_\beta\varphi) = 0 \tag{6.25}$$

and if we multiply for $\frac{1}{\sqrt{-g}}$ and we use equation (6.23) we obtain

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}(G-T)g^{\alpha\beta}\partial_\beta\varphi) = 0 \tag{6.26}$$

that is equivalent to equation (6.9).

6.3 Cosmology with mimetic matter

One year after the first article ([14]), Chamseddine, Mukhanov and Vikman ([15]) proposed a small modification of the original model, adding a potential for the scalar field φ in the action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \mathcal{L}_m + \lambda(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 1) - V(\varphi)\right] \tag{6.27}$$

Taking the variation of this action with respect to the Lagrange multiplier we obtain the constraint

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = -1 \tag{6.28}$$

while taking the variation of this action with respect to the metric $g_{\mu\nu}$, and using the constraint, we get

$$G_{\mu\nu} = T_{\mu\nu} - 2\lambda\partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}V(\varphi) \tag{6.29}$$

Taking the trace of equation (6.29) we have

$$G = T + 2\lambda + 4V \quad\Longrightarrow\quad \lambda = \frac{1}{2}(G - T - 4V) \tag{6.30}$$

and so equation (6.29) becomes

$$G_{\mu\nu} = T_{\mu\nu} - (G - T - 4V(\varphi))\partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}V(\varphi) \tag{6.31}$$

Finally if we take the variation with respect to the scalar field, using equation (6.30) we obtain

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}(G - T - 4V)g^{\alpha\beta}\partial_\beta\varphi) = -\frac{\partial V}{\partial\varphi} \tag{6.32}$$

We note that equation (6.31) can be rewritten as

$$G_{\mu\nu} = T_{\mu\nu} + \tilde{T}_{\mu\nu} \tag{6.33}$$

with $\tilde{T}_{\mu\nu}$ being the stress-energy tensor of a fluid of energy density $\rho = -(G - T - 3V)$, pressure $p = V$ and velocity potential φ.

In order to find cosmological solutions we take a flat FRW metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{6.34}$$

and we take

$$\varphi(x^\mu) = -t \tag{6.35}$$

in a similar way to what we did above. In this way equation (6.32) becomes

$$\frac{1}{a^3}\frac{d}{dt}[a^3(-\rho - V)] = -\dot{V} \tag{6.36}$$

that can be integrated to give

$$\rho = -V + \frac{1}{a^3}\int dt\, a^3\dot{V} \tag{6.37}$$

Integrating by parts and changing the variable of integration we get

$$\rho = -\frac{3}{a^3}\int da\, a^2 V \tag{6.38}$$

Taking now the 00 component of equation (6.31) and assuming the absence of any kind of matter ($T_{\mu\nu} = 0$), we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho = -\frac{1}{a^3}\int da\, a^2 V \tag{6.39}$$

and differentiating it with respect to time we obtain

$$2\dot{H} + 3H^2 = -V(t) \tag{6.40}$$

Defining a new variable

$$y \equiv a^{3/2} \tag{6.41}$$

equation (6.40) becomes

$$\ddot{y} + \frac{3}{4}V(t)y = 0 \tag{6.42}$$

Choosing in the appropriate way the potential $V(t)$ we can get nearly every possible solution $y(t)$, and so nearly every possible expansion history $a(t)$ for the universe.

6.3.1 Cosmological solutions

First, we take the potential

$$V(\varphi) = -\frac{\alpha}{\varphi^2} = -\frac{\alpha}{t^2} \tag{6.43}$$

with $\alpha \geq -1/3$ being a constant. The general solution of the equation

$$\ddot{y} - \frac{3\alpha}{4t^2}y = 0 \tag{6.44}$$

is

$$y(t) = C_1 t^{\frac{1}{2}(1+\sqrt{1+3\alpha})} + C_2 t^{\frac{1}{2}(1-\sqrt{1+3\alpha})} \tag{6.45}$$

where C_1 and C_2 are constants of integration. Since in a flat universe the scale factor $a(t)$ is defined up to an overall normalization constant, assuming $C_1 \neq 0$ we have the general solution for the scale factor

$$a(t) = t^{\frac{1}{3}(1+\sqrt{1+3\alpha})}\left(1 + At^{-\sqrt{1+3\alpha}}\right)^{2/3} \tag{6.46}$$

with $A = C_2/C_1$ being a constant of integration.
Remembering now equation (6.39), we can find the energy density

$$\rho = 3\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3t^2}\left(1 + \sqrt{1+3\alpha}\frac{1 - At^{-\sqrt{1+3\alpha}}}{1 + At^{-\sqrt{1+3\alpha}}}\right)^2 \tag{6.47}$$

and considering that

$$p = V = -\frac{\alpha}{t^2} \tag{6.48}$$

we obtain an equation of state for the mimetic matter

$$w = \frac{p}{\rho} = -3\alpha\left(1 + \sqrt{1+3\alpha}\frac{1 - At^{-\sqrt{1+3\alpha}}}{1 + At^{-\sqrt{1+3\alpha}}}\right)^{-2} \tag{6.49}$$

This equation of state is clearly dependent on time, but for small and large t it approaches a constant:

- If $\alpha = -1/3$ we get $p = \rho$ (ultra-hard matter) and $a(t) \propto t^{1/3}$.
- If $\alpha = -1/4$ we get $p = \frac{1}{3}\rho$ (ultra-relativistic matter) if $t \to \infty$ and $p = 3\rho$ if $t \to 0$.
- If $\alpha \simeq 0$ we get $p \simeq 0$ (pressureless dust).
- If $\alpha \gg 1$ we get $p = -\rho$ (cosmological constant).

If we take a more general power law potential

$$V(\varphi) = -\alpha\varphi^n = -\alpha t^n \tag{6.50}$$

the solution of equation (6.42) is given in terms of the modified Bessel functions of the first kind

$$y(t) \propto t^{1/2} I_{\frac{1}{n+2}}\left(\frac{\sqrt{-3\alpha}}{n+2} t^{\frac{n+2}{2}}\right) \tag{6.51}$$

If $n < -2$, for large t we get $y(t) \propto t$ and therefore $a(t) \propto t^{2/3}$, so the behavior of a dust dominated universe. If $n > -2$, for large t we get

$$y(t) \propto t^{-n/4} \exp\left(\pm i \frac{\sqrt{-3\alpha}}{n+2} t^{\frac{n+2}{2}}\right) \tag{6.52}$$

In this case the sign of α plays a crucial role: if α is negative, equation (6.52) describes an oscillating universe with singularities, while if α is positive, it describes a universe that undergoes an accelerated expansion.

Chapter 7

Mimetic Horndeski gravity

In this Chapter we are going to extend the mimetic model introduced above. As we have explained, mimetic dark matter first appeared performing a conformal transformation on the physical metric and then taking the variation of the Einstein-Hilbert action of general relativity.
The generalization that we propose follows two directions. First, instead of conformal transformations of the physical metric, we consider more general *disformal* transformations of the type

$$g_{\mu\nu} = A(\varphi, w)l_{\mu\nu} + B(\varphi, w)\partial_\mu\varphi\partial_\nu\varphi \tag{7.1}$$

where

$$w \equiv l^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \tag{7.2}$$

and $A(\varphi, w)$ and $B(\varphi, w)$ are generic functions of their two arguments.
Second, instead of general relativity we consider a very general scalar-tensor theory of gravity, so the Einstein-Hilbert action is substituted by

$$S = \int d^4x\sqrt{-g}\mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1}g_{\mu\nu}, ..., \partial_{\lambda_1}...\partial_{\lambda_p}g_{\mu\nu}, \varphi, \partial_{\lambda_1}\varphi, ..., \partial_{\lambda_1}...\partial_{\lambda_q}\varphi] + \int d^4x\sqrt{-g}\mathcal{L}_m \tag{7.3}$$

with $p, q \geq 2$.

7.1 Horndeski gravity

The idea of using a scalar-tensor theory to describe gravity is one of the most simple generalizations of general relativity. As we said above, general relativity is a metric theory: gravitation is completely described using just the metric $g_{\mu\nu}$, and so Einstein's model can be considered a tensor theory of gravity.
Remembering the cosmological problems we have spoken of at the beginning, physicists have always tried to find alternative theories that could explain the same phenomena without introducing dark components in the universe: one class of these alternative theories is represented by scalar-tensor models. This models are founded on the idea that gravity cannot be described only by the metric tensor $g_{\mu\nu}$, but we also need a scalar field φ that is non-minimally coupled to the metric.
Therefore, in general, we can assume that gravity is described by the action

$$S = \int d^4x\sqrt{-g}\mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1}g_{\mu\nu}, ..., \partial_{\lambda_1}...\partial_{\lambda_p}g_{\mu\nu}, \varphi, \partial_{\lambda_1}\varphi, ..., \partial_{\lambda_1}...\partial_{\lambda_q}\varphi] + \int d^4x\sqrt{-g}\mathcal{L}_m \tag{7.4}$$

with $p, q \geq 2$ and supposing that the matter lagrangian is function only of the metric and the matter fields: $\mathcal{L}_m = \mathcal{L}_m[g_{\mu\nu}, \phi_m]$.

In order to obtain the equations of motion, we calculate the variation of the action with respect to φ, $g_{\mu\nu}$ and ϕ_m:

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}(E^{\mu\nu} - T^{\mu\nu})\delta g_{\mu\nu} + \int d^4x \Omega_\varphi \delta\varphi + \int d^4x \Omega_m \delta\phi_m \tag{7.5}$$

where we defined

$$\Omega_\varphi \equiv \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\varphi} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\varphi} + \sum_{h=1}^{q}(-1)^h \frac{d}{dx^{\lambda_1}}\cdots\frac{d}{dx^{\lambda_h}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_{\lambda_1}...\partial_{\lambda_h}\varphi)} \tag{7.6}$$

$$E^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}} + \sum_{h=1}^{p}(-1)^h \frac{d}{dx^{\lambda_1}}\cdots\frac{d}{dx^{\lambda_h}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_{\lambda_1}...\partial_{\lambda_h} g_{\mu\nu})}\right) \tag{7.7}$$

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} \tag{7.8}$$

$$\Omega_m \equiv \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\phi_m} \tag{7.9}$$

Now it is clear that the equations of motion are

$$E^{\mu\nu} = T^{\mu\nu} \tag{7.10}$$

$$\Omega_\varphi = 0 \tag{7.11}$$

$$\Omega_m = 0 \tag{7.12}$$

General relativity can be considered as a particular scalar-tensor theory: in fact, if we take $\mathcal{L} = \frac{1}{2}R$ we obtain $E^{\mu\nu} = G^{\mu\nu}$ and equation (7.10) becomes the usual Einstein equation $G^{\mu\nu} = T^{\mu\nu}$.

In 1974 Horndeski ([27]) found the lagrangian for the most general 4-D local covariant scalar-tensor theory that can be derived from an action and that has second-order equations of motion for both the metric and the scalar field: this last property in particular guarantees that Horndeski theory is free from higher-derivative ghosts. Horndeski action can be written as

$$S_H = \int d^4x\sqrt{-g}\mathcal{L}_H = \int d^4x\sqrt{-g}\sum_{n=0}^{3}\mathcal{L}_n \tag{7.13}$$

where

$$\mathcal{L}_0 = K(X, \varphi) \tag{7.14}$$

$$\mathcal{L}_1 = -G_3(X, \varphi)\Box\varphi \tag{7.15}$$

$$\mathcal{L}_2 = G_{4,X}(X, \varphi)\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2\right] + RG_4(X, \varphi) \tag{7.16}$$

$$\mathcal{L}_3 = -\frac{1}{6}G_{5,X}(X, \varphi)\left[(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)^2 + 2(\nabla_\mu\nabla_\nu\varphi)^3\right] + G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi G_5(X, \varphi) \tag{7.17}$$

and

$$
\begin{aligned}
&X \equiv -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \\
&(\nabla_\mu \nabla_\nu \varphi)^2 \equiv \nabla_\mu \nabla_\nu \varphi \nabla^\mu \nabla^\nu \varphi \\
&(\nabla_\mu \nabla_\nu \varphi)^3 \equiv \nabla_\mu \nabla_\nu \varphi \nabla^\mu \nabla^\rho \varphi \nabla^\nu \nabla_\rho \varphi
\end{aligned}
\tag{7.18}
$$

while $K(X,\varphi)$, $G_3(X,\varphi)$, $G_4(X,\varphi)$ and $G_5(X,\varphi)$ are free functions of their two arguments and define a particular theory in the Horndeski class. The subscript $_{,X}$ denotes derivative with respect to X.

We note that general relativity is a particular theory inside the Horndeski class: taking

$$
K = G_3 = G_5 = 0 \quad G_4 = \frac{1}{2} \tag{7.19}
$$

we get

$$
S = \frac{1}{2} \int d^4x \sqrt{-g} R \tag{7.20}
$$

that is exactly Einstein-Hilbert action if matter is absent ($\mathcal{L}_m = 0$).

In 2015 Gleyzes, Langlois, Piazza and Vernizzi ([28] [29]) introduced an extension of Horndeski theory, adding the terms

$$
F_4(X,\varphi) \epsilon^{\mu\nu\rho}{}_\sigma \epsilon^{\alpha\beta\gamma\sigma} \nabla_\mu \varphi \nabla_\alpha \varphi \nabla_\nu \nabla_\beta \varphi \nabla_\rho \nabla_\gamma \varphi \tag{7.21}
$$

and

$$
F_5(X,\varphi) \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \nabla_\mu \varphi \nabla_\alpha \varphi \nabla_\nu \nabla_\beta \varphi \nabla_\rho \nabla_\gamma \varphi \nabla_\sigma \nabla_\delta \varphi \tag{7.22}
$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol.
This model (called G^3 theory) possesses third-order equations of motion for the metric, but the true propagating degrees of freedom obey well-behaved second-order equations and are thus free from higher-derivative ghosts ([28]). However, in the following Chapters we will consider only Horndeski theory and not this healthy extension.

7.2 Disformal transformations

As we anticipated at the beginning of this Chapter we will go beyond the simple conformal transformations, considering the more general disformal transformations of the metric. A disformal transformation can be written as

$$
g_{\mu\nu} = A(\varphi, w) l_{\mu\nu} + B(\varphi, w) \partial_\mu \varphi \partial_\nu \varphi \tag{7.23}
$$

where w is defined as

$$
w \equiv l^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \tag{7.24}
$$

$A(\varphi, w)$ and $B(\varphi, w)$ are generic functions of their two arguments, $g_{\mu\nu}$ is the physical metric, $l_{\mu\nu}$ is the auxiliary new metric and φ is the scalar field present also in the action of the scalar-tensor theory.

7.2.1 Non-invertibility of a disformal transformation

Now we want to find out when a generic disformal transformation (7.23) is not invertible. The issue of the non-invertibility of a conformal transformation was first discussed in [30], but here we will consider a more general disformal transformation.
The inverse of the physical metric $g_{\mu\nu}$ can be found to be ([16])

$$g^{\mu\nu} = \frac{1}{A(\varphi, w)} l^{\mu\nu} + \frac{B(\varphi, w)}{B(\varphi, w) g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi - 1} g^{\mu\alpha} g^{\nu\beta}\partial_\alpha\varphi\partial_\beta\varphi \tag{7.25}$$

with the condition that $B(\varphi, w) g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi - 1 \neq 0$.
Defining now

$$\chi \equiv g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \tag{7.26}$$

if we multiply equation (7.25) for $\partial_\mu\varphi\partial_\nu\varphi$ we get

$$\chi = \frac{w}{A(\varphi, w)} + \frac{B(\varphi, w)\chi^2}{B(\varphi, w)\chi - 1} \tag{7.27}$$

and isolating w we obtain

$$w = \frac{A(\varphi, w)\chi}{1 - B(\varphi, w)\chi} = \frac{A(\varphi, w) g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}{1 - B(\varphi, w) g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi} \tag{7.28}$$

Defining

$$G(\varphi, w) \equiv \frac{w(1 - B(\varphi, w) g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi)}{A(\varphi, w)} \tag{7.29}$$

equation (7.28) can be written as

$$G(\varphi, w) = g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \tag{7.30}$$

For a fixed given φ, the inverse function theorem shows that, if $\left.\frac{dG(\varphi,w)}{dw}\right|_{w=w*} \neq 0$, then the inverse function G^{-1} exists in the neighborhood of $w*$. In that case one can write w as a function of $g_{\mu\nu}$ only as $w = G^{-1}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi)$ and then use equation (7.23) to write $l_{\mu\nu}$ as a function of only $g_{\mu\nu}$, obtaining the inverse transformation $l_{\alpha\beta}(g_{\mu\nu})$ we were looking for.
On the other hand, the non-existence of G^{-1} implies that

$$\left.\frac{dG(\varphi, w)}{dw}\right|_{w=w*} = 0 \tag{7.31}$$

that can be solved as

$$G(\varphi, w) = \frac{1}{b(\varphi)} \tag{7.32}$$

If we are in the case of equation (7.32), then the disformal transformation cannot be inverted and, using equation (7.30), we have

$$b(\varphi) = \frac{1}{g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi} \tag{7.33}$$

that can be used with equation (7.28) to find

$$B(\varphi, w) = -\frac{A(\varphi, w)}{w} + b(\varphi) \tag{7.34}$$

In summary, we have found that a disformal transformation of the type given in equation (7.23) is non-invertible if the two free functions $A(\varphi, w)$ and $B(\varphi, w)$ satisfy the relation given by equation (7.34), with $b(\varphi)$ being a free "potential" function.

7.2.2 Disformal transformation method

If we perform a disformal transformation of the type (7.23), the variation of the metric becomes

$$\delta g_{\mu\nu} = A\delta l_{\mu\nu} + l_{\mu\nu}\delta A + \delta B\partial_\mu\varphi\partial_\nu\varphi + B\partial_\mu(\delta\varphi)\partial_\nu\varphi + B\partial_\mu\varphi\partial_\nu(\delta\varphi) \tag{7.35}$$

Taking into account that

$$\begin{aligned} \delta A(\varphi, w \equiv l^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi) &= \frac{\partial A}{\partial\varphi}\delta\varphi + \frac{\partial A}{\partial w}\delta w = \\ &= \frac{\partial A}{\partial\varphi}\delta\varphi + \frac{\partial A}{\partial w}[\delta l^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + 2l^{\alpha\beta}\partial_\alpha(\delta\varphi)\partial_\beta\varphi] \end{aligned} \tag{7.36}$$

and that, using the fact that $l_{\mu\alpha}l^{\alpha\nu} = \delta^\nu_\mu$,

$$\delta l^{\alpha\beta} = -l^{\rho\alpha}l^{\sigma\beta}\delta l_{\rho\sigma} \tag{7.37}$$

the variation written in equation (7.35) becomes

$$\begin{aligned} \delta g_{\mu\nu} =& A\delta l_{\mu\nu} - \left(l_{\mu\nu}\frac{\partial A}{\partial w} + \partial_\mu\varphi\partial_\nu\varphi\frac{\partial B}{\partial w}\right)[l^{\rho\alpha}l^{\sigma\beta}\delta l_{\rho\sigma}\partial_\alpha\varphi\partial_\beta\varphi - 2l^{\alpha\beta}\partial_\alpha(\delta\varphi)\partial_\beta\varphi] + \\ &+ \left(l_{\mu\nu}\frac{\partial A}{\partial\varphi} + \partial_\mu\varphi\partial_\nu\varphi\frac{\partial B}{\partial\varphi}\right)\delta\varphi + B[\partial_\mu\varphi\partial_\nu(\delta\varphi) + \partial_\mu(\delta\varphi)\partial_\nu\varphi] \end{aligned} \tag{7.38}$$

Inserting equation (7.38) into equation (7.5), we can take the variation of the action of a generic scalar-tensor theory with respect to the auxiliary metric $l_{\mu\nu}$, obtaining the equation of motion

$$A(E^{\mu\nu} - T^{\mu\nu}) = \left(\alpha_1\frac{\partial A}{\partial w} + \alpha_2\frac{\partial B}{\partial w}\right)(l^{\mu\rho}\partial_\rho\varphi)(l^{\nu\sigma}\partial_\sigma\varphi) \tag{7.39}$$

while taking the variation with respect to the scalar field φ we have

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_\rho\left\{\sqrt{-g}\partial_\sigma\varphi\left[B(E^{\rho\sigma} - T^{\rho\sigma}) + \left(\alpha_1\frac{\partial A}{\partial w} + \alpha_2\frac{\partial B}{\partial w}\right)l^{\rho\sigma}\right]\right\} - \frac{\Omega_\varphi}{\sqrt{-g}} = \\ = \frac{1}{2}\left(\alpha_1\frac{\partial A}{\partial\varphi} + \alpha_2\frac{\partial B}{\partial\varphi}\right) \end{aligned} \tag{7.40}$$

where we have defined

$$\alpha_1 \equiv (E^{\rho\sigma} - T^{\rho\sigma})l_{\rho\sigma} \qquad \alpha_2 \equiv (E^{\rho\sigma} - T^{\rho\sigma})\partial_\rho\varphi\partial_\sigma\varphi \tag{7.41}$$

The equation of motion for matter is again

$$\Omega_m = 0 \tag{7.42}$$

If we now contract equation (7.39) with $l_{\mu\nu}$ and with $\partial_\mu\varphi\partial_\nu\varphi$, we find respectively

$$\alpha_1\left(A - w\frac{\partial A}{\partial w}\right) - \alpha_2 w\frac{\partial B}{\partial w} = 0 \qquad \alpha_1 w^2\frac{\partial A}{\partial w} - \alpha_2\left(A - w^2\frac{\partial B}{\partial w}\right) = 0 \tag{7.43}$$

This two equations form a linear system of algebraic equations for α_1 and α_2, that can be written as

$$M\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = 0 \qquad \text{where} \qquad M = \begin{pmatrix} A - w\frac{\partial A}{\partial w} & -w\frac{\partial B}{\partial w} \\ w^2\frac{\partial A}{\partial w} & -A + w^2\frac{\partial B}{\partial w}\end{pmatrix} \tag{7.44}$$

and the determinant of the system is

$$\begin{aligned}
\det(M) &= \left(A - w\frac{\partial A}{\partial w}\right)\left(-A + w^2\frac{\partial B}{\partial w}\right) + w^3\frac{\partial A}{\partial w}\frac{\partial B}{\partial w} = \\
&= A\left[-A + w^2\frac{\partial B}{\partial w} + w\frac{\partial A}{\partial w}\right] = \\
&= w^2 A\frac{\partial}{\partial w}\left(B + \frac{A}{w}\right)
\end{aligned} \tag{7.45}$$

The solutions of the system are different depending on whether its determinant is zero or not. If $\det(M) \neq 0$, then the only solution is $\alpha_1 = \alpha_2 = 0$ and the equations of motion (7.39) and (7.40) reduce to

$$E^{\mu\nu} = T^{\mu\nu} \tag{7.46}$$

$$\Omega_\varphi = 0 \tag{7.47}$$

that, in addition to $\Omega_m = 0$, are the same equations of the original theory before doing any disformal transformation. This shows that, if $\det(M) \neq 0$, a generic scalar-tensor theory is invariant under disformal transformations: this result should not be surprising, since we are just doing a well-behaved invertible change of coordinates.

On the other hand, the determinant of the system is zero if

$$\frac{\partial}{\partial w}\left(B + \frac{A}{w}\right) = 0 \tag{7.48}$$

that is satisfied if the free function $B(\varphi, w)$ is in the form

$$B(\varphi, w) = -\frac{A(\varphi, w)}{w} + b(\varphi) \tag{7.49}$$

with $b(\varphi)$ being an integration constant. Substituting this equation into the system (7.44) we obtain

$$\alpha_2 = w\alpha_1 \tag{7.50}$$

and the equations of motion (7.39) and (7.40) become

$$E^{\mu\nu} = T^{\mu\nu} + \frac{\alpha_1}{w}(l^{\mu\rho}\partial_\rho\varphi)(l^{\nu\sigma}\partial_\sigma\varphi) \tag{7.51}$$

$$\frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}b\alpha_1 l^{\rho\sigma}\partial_\sigma\varphi) - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{1}{2}\alpha_1 w\frac{db}{d\varphi} \tag{7.52}$$

Using now the identities $l^{\mu\rho}\partial_\rho\varphi = bwg^{\mu\rho}\partial_\rho\varphi$ and $\alpha_1 = \frac{E-T}{bw}$, where $E - T \equiv g_{\mu\nu}(E^{\mu\nu} - T^{\mu\nu})$, we find ([16])

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1 \tag{7.53}$$

and the equations of motion (7.51) and (7.52) simplify to

$$E^{\mu\nu} = T^{\mu\nu} + (E - T)b\partial^\mu\varphi\partial^\nu\varphi \tag{7.54}$$

$$\frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}(E - T)bg^{\rho\sigma}\partial_\sigma\varphi) - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{1}{2}(E - T)\frac{1}{b}\frac{db}{d\varphi} \tag{7.55}$$

Adding also the matter equation, the full system of equations of motion of this new *mimetic* theory of gravity is

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1 \tag{7.56}$$

$$E^{\mu\nu} = T^{\mu\nu} + (E-T)b\partial^\mu\varphi\partial^\nu\varphi \tag{7.57}$$

$$\frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}(E-T)bg^{\rho\sigma}\partial_\sigma\varphi) - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{1}{2}(E-T)\frac{1}{b}\frac{db}{d\varphi} \tag{7.58}$$

$$\Omega_m = 0 \tag{7.59}$$

in contrast with the equations of motion (7.10)-(7.12) of the original scalar-tensor theory.

We note that condition (7.49) is the same we found in the previous Subsection for a disformal transformation to be non-invertible. This means that if we perform a non-invertible disformal transformation on a general scalar-tensor theory with equations of motion (7.10)-(7.12), we obtain a different theory with different equations of motion (7.56)-(7.59): this new theory is called *mimetic gravity*.

7.3 Mimetic gravity from a Lagrange multiplier

As we have shown in the previous Chapter for the conformal mimetic modification of general relativity, we can obtain the mimetic modification of a general scalar-tensor theory (with action given by equation (7.3)) without performing any disformal transformation or introducing an auxiliary metric.
In order to do that, we simply impose the kinematical constraint $b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1$ on the scalar field φ, introducing a Lagrange multiplier λ in the action:

$$\begin{aligned} S = &\int d^4x\sqrt{-g}\mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1}g_{\mu\nu}, ..., \partial_{\lambda_1}...\partial_{\lambda_p}g_{\mu\nu}, \varphi, \partial_{\lambda_1}\varphi, ..., \partial_{\lambda_1}...\partial_{\lambda_q}\varphi] + \\ &+ \int d^4x\sqrt{-g}\mathcal{L}_m + \int d^4x\sqrt{-g}\lambda(b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 1) \end{aligned} \tag{7.60}$$

with $b(\varphi)$ being a known potential function that defines the theory.
Taking the variation of this action with respect to λ, $g_{\mu\nu}$, φ and ϕ_m we get respectively ([16])

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1 \tag{7.61}$$

$$E^{\mu\nu} = T^{\mu\nu} + 2\lambda b\partial^\mu\varphi\partial^\nu\varphi \tag{7.62}$$

$$\frac{2}{\sqrt{-g}}\partial_\rho(\sqrt{-g}\lambda bg^{\rho\sigma}\partial_\sigma\varphi) - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{\lambda}{b}\frac{db}{d\varphi} \tag{7.63}$$

$$\Omega_m = 0 \tag{7.64}$$

If we now take the trace of equation (7.62), using also equation (7.61) we get

$$\lambda = \frac{E-T}{2} \tag{7.65}$$

and substituting this into equations (7.62)-(7.64) we have the full system of equations of motion

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1 \tag{7.66}$$

$$E^{\mu\nu} = T^{\mu\nu} + (E-T)b\partial^\mu\varphi\partial^\nu\varphi \tag{7.67}$$

$$\frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}(E-T)bg^{\rho\sigma}\partial_\sigma\varphi)-\frac{\Omega_\varphi}{\sqrt{-g}}=\frac{1}{2}(E-T)\frac{1}{b}\frac{db}{d\varphi} \tag{7.68}$$

$$\Omega_m=0 \tag{7.69}$$

These equations are the same as equations (7.56)-(7.59), so we have just shown that mimetic gravity can be obtained from the original theory simply by adding the kinematical constraint $b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi=1$ in the action with a Lagrange multiplier.

7.4 Independent equations of motion

Not all the equations (7.66)-(7.69) are independent. First of all we show that equation (7.68) can be derived from the other equations of motion: remembering that $\nabla_\mu j^\mu=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}j^\mu)$ if j^μ is a vector quantity ([31]), equation (7.68) can be written as

$$\nabla_\mu[(E-T)b\partial^\mu\varphi]-\frac{\Omega_\varphi}{\sqrt{-g}}=\frac{1}{2}(E-T)\frac{1}{b}\frac{db}{d\varphi} \tag{7.70}$$

Taking now the covariant derivative of equation (7.67) we obtain

$$\nabla_\mu E^{\mu\nu}=\nabla_\mu T^{\mu\nu}+\nabla_\mu[(E-T)b(\varphi)\partial^\mu\varphi]\partial^\nu\varphi+(E-T)b(\varphi)\partial^\mu\varphi\nabla_\mu(\partial^\nu\varphi) \tag{7.71}$$

Since we have assumed that S_m can be written as a functional of the matter fields and the metric $g_{\mu\nu}$, using the equation of motion (7.69) together with the Horndeski identity ([27] [16])

$$\sqrt{-g}\nabla_\mu E^{\mu\nu}=\Omega_\varphi\nabla^\nu\varphi \tag{7.72}$$

it can be shown that the stress-energy tensor $T^{\mu\nu}$ obeys a continuity equation $\nabla_\mu T^{\mu\nu}=0$. Furthermore, taking the covariant derivative of equation (7.66) we have

$$\nabla^\nu[b(\varphi)\partial^\mu\varphi\partial_\mu\varphi]=0 \tag{7.73}$$

and, since $\nabla^\nu b(\varphi)=\partial^\nu b(\varphi)$ and $\nabla^\nu\varphi=\partial^\nu\varphi$,

$$\partial^\nu b(\varphi)\partial^\mu\varphi\partial_\mu\varphi+2b(\varphi)\nabla^\nu\nabla^\mu\varphi\nabla_\mu\varphi=0 \tag{7.74}$$

Using now $\partial^\nu b(\varphi)=\frac{db}{d\varphi}\partial^\nu\varphi$ and the fact that covariant derivatives commute on scalars, we get

$$\frac{db}{d\varphi}\partial^\nu\varphi\partial^\mu\varphi\partial_\mu\varphi=-2b(\varphi)\nabla^\mu\nabla^\nu\varphi\nabla_\mu\varphi \tag{7.75}$$

and finally

$$b(\varphi)\partial^\mu\varphi\nabla_\mu(\partial^\nu\varphi)=-\frac{1}{2}\frac{db}{d\varphi}\partial^\nu\varphi\partial^\mu\varphi\partial_\mu\varphi \tag{7.76}$$

Using these results, equation (7.71) becomes

$$\nabla_\mu E^{\mu\nu}=\partial^\nu\varphi\left[\nabla_\mu[(E-T)b(\varphi)\partial^\mu\varphi]-\frac{1}{2}\frac{1}{b(\varphi)}\frac{db(\varphi)}{d\varphi}(E-T)\right] \tag{7.77}$$

and taking into account again the Horndeski identity (7.72), together with the fact that $\partial^\nu\varphi\neq 0$ at least for one index ν, we finally find

$$\nabla_\mu[(E-T)b\partial^\mu\varphi]-\frac{\Omega_\varphi}{\sqrt{-g}}=\frac{1}{2}(E-T)\frac{1}{b}\frac{db}{d\varphi} \tag{7.78}$$

that is exactly the same equation as (7.70). So we have just shown that equation (7.68) can be derived from the other equations of motion.

Now we show that the 00 component of equation (7.67) can be derived from the other components, together with equation (7.66): this constraint equation can be written as

$$b(\varphi)g^{00}(\partial_0\varphi)^2 + 2b(\varphi)g^{0i}\partial_0\varphi\partial_i\varphi + b(\varphi)g^{ij}\partial_i\varphi\partial_j\varphi = 1 \tag{7.79}$$

Multiplying both sides by $E - T = g^{\mu\nu}(E_{\mu\nu} - T_{\mu\nu})$ we have

$$\begin{aligned}&(E-T)b(\varphi)g^{00}(\partial_0\varphi)^2 + 2(E-T)b(\varphi)g^{0i}\partial_0\varphi\partial_i\varphi + (E-T)b(\varphi)g^{ij}\partial_i\varphi\partial_j\varphi = \\ &= g^{00}(E_{00} - T_{00}) + 2g^{0i}(E_{0i} - T_{0i}) + g^{ij}(E_{ij} - T_{ij})\end{aligned} \tag{7.80}$$

The $0i$ and ij components of equation (7.67) are

$$E^{0i} - T^{0i} = (E-T)b(\varphi)\partial^0\varphi\partial^i\varphi \qquad E^{ij} - T^{ij} = (E-T)b(\varphi)\partial^i\varphi\partial^j\varphi \tag{7.81}$$

and substituting them in equation (7.80) we get

$$(E-T)b(\varphi)g^{00}(\partial_0\varphi)^2 = g^{00}(E_{00} - T_{00}) \tag{7.82}$$

Since $g^{00} \neq 0$ we finally find

$$E_{00} - T_{00} = (E-T)b(\varphi)(\partial_0\varphi)^2 \tag{7.83}$$

that is exactly the 00 component of equation (7.67).

These results allow us to conclude that a full list of independent equations of motion for mimetic gravity can be written as

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1 \tag{7.84}$$

$$E^{\mu i} = T^{\mu i} + (E-T)b\partial^\mu\varphi\partial^i\varphi \tag{7.85}$$

$$\Omega_m = 0 \tag{7.86}$$

7.5 Cosmology in mimetic Horndeski gravity

In this Section we are going to consider two particular mimetic Horndeski models, fixing the functions $K(X,\varphi)$, $G_3(X,\varphi)$, $G_4(X,\varphi)$ and $G_5(X,\varphi)$, and we will investigate their cosmological solutions assuming a flat FRW metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{7.87}$$

and considering the absence of matter ($S_m = 0$).

7.5.1 Mimetic canonical scalar field

We first consider the mimetic theory of a scalar field with canonical kinetic term and no potential:

$$K(X,\varphi) = c_2 X \qquad G_3(X,\varphi) = 0 \qquad G_4(X,\varphi) = 1/2 \qquad G_5(X,\varphi) = 0 \tag{7.88}$$

so the gravitational lagrangian of the theory becomes

$$\mathcal{L} = -\frac{1}{2}c_2 g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}R \tag{7.89}$$

We can now calculate the tensor $E_{\mu\nu}$:

$$\begin{aligned} E_{\mu\nu} &\equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = \\ &= \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(-\frac{1}{2}\sqrt{-g}c_2 g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right) + \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\frac{1}{2}\sqrt{-g}R\right) = \\ &= -\frac{1}{\sqrt{-g}}c_2\partial_\alpha\varphi\partial_\beta\varphi\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}g^{\alpha\beta}) + G_{\mu\nu} = \\ &= \frac{1}{2}c_2 g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - c_2\partial_\mu\varphi\partial_\nu\varphi + G_{\mu\nu} \end{aligned} \tag{7.90}$$

where in the last passage we used

$$\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}g^{\alpha\beta}) = \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}g^{\alpha\beta} + \sqrt{-g}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\alpha\beta} + \sqrt{-g}\delta^\alpha_\mu\delta^\beta_\nu \tag{7.91}$$

In a flat FRW universe, and considering also the scalar field φ to be function of time only because of homogeneity of space, we have $g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi = -\dot\varphi^2$ and so

$$E_{00} = \frac{1}{2}c_2\dot\varphi^2 - c_2\dot\varphi^2 + G_{00} = -\frac{1}{2}c_2\dot\varphi^2 + 3H^2 \tag{7.92}$$

$$E_{0i} = E_{i0} = 0 \tag{7.93}$$

$$\begin{aligned} E_{ij} &= -\frac{1}{2}c_2 a^2\delta_{ij}\dot\varphi^2 + G_{ij} = \\ &= -\frac{1}{2}c_2 a^2\delta_{ij}\dot\varphi^2 - 2a^2\left[\frac{\ddot a}{a} + \frac{1}{2}H^2\right]\delta_{ij} = \\ &= -a^2\delta_{ij}\left(2\dot H + 3H^2 + \frac{1}{2}c_2\dot\varphi^2\right) \end{aligned} \tag{7.94}$$

$$E = E_{\mu\nu}g^{\mu\nu} = E_{00}g^{00} + E_{ij}g^{ij} = -12H^2 - 6\dot H - c_2\dot\varphi^2 \tag{7.95}$$

where we have used the expression of $G_{\mu\nu}$ for a flat FRW universe given in [23].
The independent equations of motion (7.84) and (7.85) for this simple model reduce to

$$b(\varphi)\dot\varphi^2 + 1 = 0 \tag{7.96}$$

$$4\dot H + 6H^2 + c_2\dot\varphi^2 = 0 \tag{7.97}$$

and it easy to check ([16]) that they admit the solution

$$a(t) = t^{\frac{2}{3(1+w)}} \qquad \varphi(t) = \pm\sqrt{-\frac{\alpha}{c_2}}\log\frac{t}{t_i} \qquad b(\varphi) = \frac{c_2}{\alpha}t^2 = \frac{c_2}{\alpha}t_i^2 e^{\pm\sqrt{-\frac{c_2}{\alpha}}\varphi} \tag{7.98}$$

where t_i is an integration constant, w is a constant parameter and $\alpha \equiv -\frac{8w}{3(1+w)^2}$. This shows that this simple mimetic model can mimic the background expansion history of a universe dominated by a perfect fluid with constant equation of state w.
By accordingly adjusting the function $b(\varphi)$, one can obtain almost any background expansion history: this is similar to what we had in the previous Chapter, adding a potential to the original mimetic dark matter model.

7.5.2 Mimetic cubic Galileon

The mimetic cubic Galileon model takes

$$K(X,\varphi) = c_2 X \qquad G_3(X,\varphi) = 2\frac{c_3}{\tilde{\Lambda}^3}X \qquad G_4(X,\varphi) = 1/2 \qquad G_5(X,\varphi) = 0 \tag{7.99}$$

and in the following we will set the cutoff scale to be $\tilde{\Lambda} = 1$. The gravitational part of the lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}c_2 g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + c_3 g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Box\varphi + \frac{1}{2}R \tag{7.100}$$

and the tensor $E_{\mu\nu}$ is

$$\begin{aligned} E_{\mu\nu} =& \frac{1}{2}c_2 g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - c_2\partial_\mu\varphi\partial_\nu\varphi + G_{\mu\nu} + \\ &+ \frac{2}{\sqrt{-g}}c_3\partial_\mu\varphi\partial_\nu\varphi[g^{\alpha\beta}(\partial_\alpha\sqrt{-g})\partial_\beta\varphi] + \frac{2}{\sqrt{-g}}c_3(g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi)(\partial_\mu\sqrt{-g})\partial_\nu\varphi + \\ &- \frac{1}{\sqrt{-g}}(g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi)g^{\alpha\beta}\partial_\beta\varphi\partial_\alpha(\sqrt{-g}g_{\mu\nu}) - c_3(g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi)(g^{\alpha\beta}\partial_\beta\partial_\alpha\varphi)g_{\mu\nu} + \\ &+ 2c_3\partial_\mu\varphi\partial_\nu\varphi(g^{\alpha\beta}\partial_\beta\partial_\alpha\varphi) + 2c_3\partial_\mu\partial_\nu\varphi(g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi) \end{aligned} \tag{7.101}$$

In a flat FRW universe, and considering again the scalar field φ to be function of time only, the independent equations of motion for this model become ([16])

$$b(\varphi)\dot{\varphi}^2 + 1 = 0 \tag{7.102}$$

$$4\dot{H} + 6H^2 + \dot{\varphi}^2(c_2 - 4c_3\ddot{\varphi}) = 0 \tag{7.103}$$

We now want to reproduce the expansion history of a universe filled with dark matter and a positive cosmological constant Λ, that is

$$a(t) = a_i \sinh^{\frac{2}{3}}(Ct) \tag{7.104}$$

where $C = \sqrt{3\Lambda/4}$. Integrating once equation (7.103), with $a(t)$ given by equation (7.104), we get

$$4\frac{c_3}{c_2}\left[-\tan^{-1}\left(\pm\sqrt{\frac{3c_2}{8C^2}}\dot{\varphi}\right) \pm \sqrt{\frac{3c_2}{8C^2}}\dot{\varphi}\right] = t \tag{7.105}$$

and inverting it, we can use equation (7.102) to find $b(\varphi)$: this can be done numerically (see Figure 7.1). Choosing the model parameters as $C = c_2 = c_3 = a_i = 1$, the matter-dominated era ends around $t \simeq 1$ and after that, the universe becomes dominated by the cosmological constant. The previous equations can be solved in the limits of small and large time: for $Ct \ll 1$ we have $\dot{\varphi} \propto t^{1/3}$ and $b(\varphi) \propto -\varphi^{-1/2}$, while for $Ct \gg 1$ we have $\dot{\varphi} \propto t$ and $b(\varphi) \propto -\varphi^{-1}$.
This means that, by choosing a function $b(\varphi)$ with these suitable asymptotic limits, we can approximately reproduce the expansion history of a ΛCDM universe.

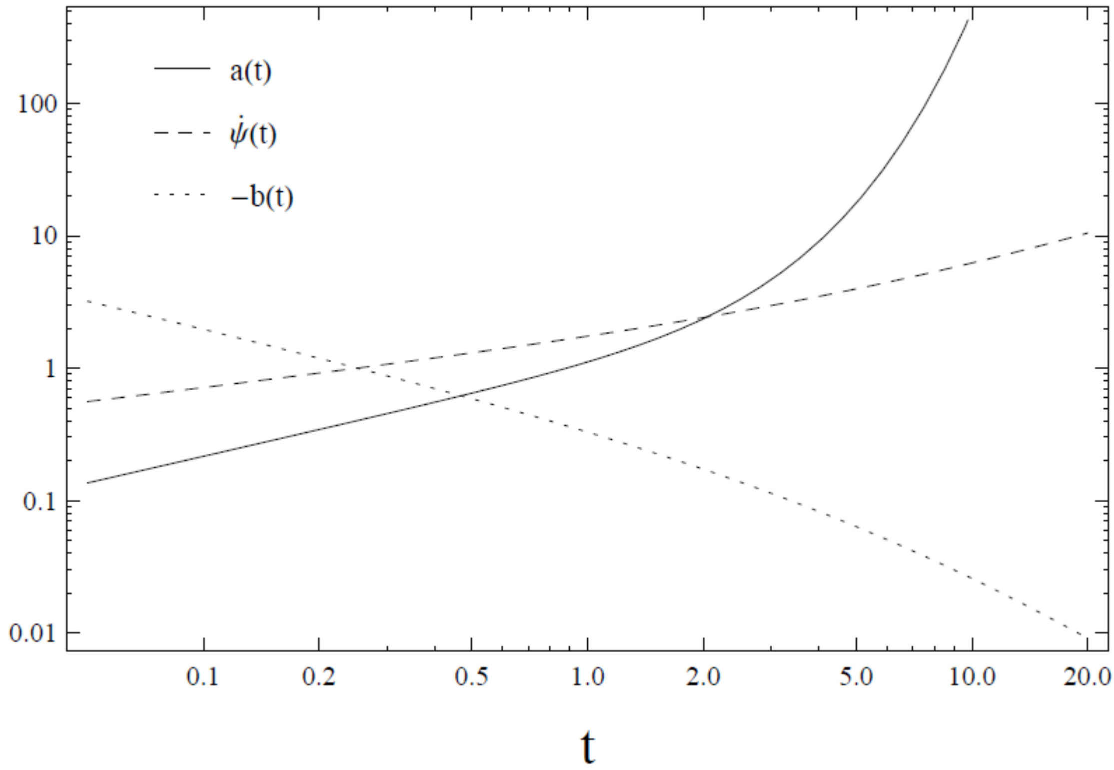

Figure 7.1: Plot of $a(t)$ (solid line), $\dot{\varphi}$ (dashed line) and $-b(t)$ (dotted line) as function of t. The parameters are chosen to be $C = c_2 = c_3 = a_i = 1$. The picture is taken from [16].

Chapter 8

Time evolution of cosmological perturbations

In this Chapter we are going to investigate the evolution in time of the first order cosmological perturbations that we introduced in Chapter 3. This evolution in time is different depending on the theory of gravity chosen: we will start discussing the evolution of perturbations in general relativity, then we will move to Horndeski gravity and mimetic Horndeski gravity.

We consider only scalar perturbations and we choose the Poisson gauge, so we deal with a metric

$$ds^2 = a^2(\eta)[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j] \tag{8.1}$$

while in the presence of matter we consider a perfect fluid with no anisotropic stress:

$$\rho = \rho_0(\eta) + \delta\rho \qquad p = p_0(\eta) + \delta p \qquad u^\mu = \frac{1}{a(\eta)}(\delta^\mu_0 + v^\mu) \tag{8.2}$$

8.1 Cosmological perturbations in general relativity

In order to find the time evolution of cosmological perturbations in general relativity, we have to perturb the Einstein equation $G_{\mu\nu} = T_{\mu\nu}$ considering the perturbations given in equations (8.1) and (8.2). After some calculations we obtain four independent equations ([32])

$$-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi = \frac{1}{2}a^2\delta\rho \tag{8.3}$$

$$\mathcal{H}\Phi + \Psi' = \frac{1}{2}a(\rho_0 + p_0)v^{\|} \tag{8.4}$$

$$(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' + \frac{1}{2}\nabla^2(\Phi - \Psi) = \frac{1}{2}a^2\delta p \tag{8.5}$$

$$\Phi = \Psi \tag{8.6}$$

where $\mathcal{H} \equiv \frac{a'}{a}$. Substituting equation (8.6) into (8.3)-(8.5) we get

$$\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = \frac{1}{2}a^2\delta\rho \tag{8.7}$$

$$(a\Phi)' = \frac{1}{2}a^2(\rho_0 + p_0)v^{\|} \tag{8.8}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{1}{2}a^2\delta p \tag{8.9}$$

We know that $\delta p = c_s^2\delta\rho + \delta p_{n.a.}$, so using equation (8.7) we get

$$\begin{aligned}\frac{1}{2}a^2\delta p &= \frac{1}{2}a^2 c_s^2\delta\rho + \frac{1}{2}a^2\delta p_{n.a.} = \\ &= c_s^2\nabla^2\Phi - 3\mathcal{H}c_s^2\Phi' - 3\mathcal{H}^2 c_s^2\Phi + \frac{1}{2}a^2\delta p_{n.a.}\end{aligned} \tag{8.10}$$

and substituting this into equation (8.9) we obtain

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2]\Phi = \frac{1}{2}a^2\delta p_{n.a.} \tag{8.11}$$

In the following we will consider only adiabatic perturbations, so the gravitational potential Φ obeys the second-order differential equation given by

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2]\Phi = 0 \tag{8.12}$$

In order to solve this equation we need to perform a change of variable, defining

$$u \equiv \exp\left[\frac{3}{2}\int(1 + c_s^2)\mathcal{H}d\eta\right]\Phi = \frac{1}{\sqrt{\rho_0 + p_0}}\Phi \tag{8.13}$$

and

$$\theta \equiv \frac{1}{a}\left(\frac{\rho_0}{\rho_0 + p_0}\right)^{1/2} = \frac{1}{a}\left[\frac{2}{3}\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\right]^{-1/2} \tag{8.14}$$

where we have used $c_s^2 = p_0'/\rho_0'$ and the background (Friedmann) equations

$$\rho_0' = -3\mathcal{H}(\rho_0 + p_0) \qquad \mathcal{H}^2 = \frac{1}{3}a^2\rho_0 \qquad \mathcal{H}^2 - \mathcal{H}' = \frac{1}{2}a^2(\rho_0 + p_0) \tag{8.15}$$

In this new variables equation (8.12) becomes ([33])

$$u'' - c_s^2\nabla^2 u - \frac{\theta''}{\theta}u = 0 \tag{8.16}$$

and using Fourier transforms

$$u'' + c_s^2 k^2 u - \frac{\theta''}{\theta}u = 0 \tag{8.17}$$

Equation (8.17) can be analytically solved only in the small scale and large scale limits. In the large scale limit ($|c_s|k\eta \ll 1$) it becomes

$$u'' - \frac{\theta''}{\theta}u = 0 \tag{8.18}$$

and it has a trivial solution in $u_1 = A\theta$, with A being a constant. The second solution u_2 can be found with the d'Alembert method: assuming that it can be written as $u_2 = vu_1 = Av\theta$ we have

$$u_2' = Av'\theta + Av\theta' \qquad u_2'' = Av''\theta + 2Av'\theta' + Av\theta'' \tag{8.19}$$

and substituting in equation (8.18) we get

$$v''\theta + 2v'\theta' + v\theta'' - \frac{\theta''}{\theta}v\theta = 0 \Longrightarrow v'' + 2\frac{\theta'}{\theta}v' = 0 \tag{8.20}$$

Defining now $z \equiv v'$, equation (8.20) becomes

$$\frac{z'}{z} = -2\frac{\theta'}{\theta} \tag{8.21}$$

that can be easily solved, finding $z = v' = \frac{1}{\theta^2}$ and so $v = \int \frac{d\eta}{\theta^2}$. Therefore the second mode is $u_2 = v\theta = \theta \int \frac{d\eta}{\theta^2}$ and the general solution for equation (8.18) is

$$u = c_1\theta + c_2\theta \int_{\eta_i} \frac{d\eta}{\theta^2} = c_2\theta \int_{\bar{\eta}_i} \frac{d\eta}{\theta^2} \tag{8.22}$$

where c_1 and c_2 are integration constants and in the second passage we have changed the lower limit of integration in order to absorb the c_1 mode.
Using now the definition of θ given in terms of a and $\mathcal{H}$ in equation (8.14), we obtain

$$\begin{aligned} \int \frac{d\eta}{\theta^2} &= \frac{2}{3}\int d\eta a^2 \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) = \frac{2}{3}\left[\int d\eta a^2 + \int d\eta a^2 \left(\frac{1}{\mathcal{H}}\right)'\right] = \\ &= \frac{2}{3}\left[\int d\eta a^2 + \frac{a^2}{\mathcal{H}} - 2\int d\eta a a' \frac{1}{\mathcal{H}}\right] = \frac{2}{3}\left[\frac{a^2}{\mathcal{H}} - \int d\eta a^2\right] \end{aligned} \tag{8.23}$$

where in the third passage we integrated by parts, and the potential Φ is

$$\begin{aligned} \Phi &= (\rho_0 + p_0)^{1/2} u = (\rho_0 + p_0)^{1/2} \cdot c_2\theta \int \frac{d\eta}{\theta^2} = \\ &= (\rho_0 + p_0)^{1/2} \cdot c_2 \frac{1}{a} \frac{\rho_0^{1/2}}{(\rho_0 + p_0)^{1/2}} \cdot \frac{2}{3}\left[\frac{a^2}{\mathcal{H}} - \int d\eta a^2\right] = \\ &= A\frac{\mathcal{H}}{a^2}\left[\frac{a^2}{\mathcal{H}} - \int d\eta a^2\right] = A\left[1 - \frac{\mathcal{H}}{a^2}\int d\eta a^2\right] \end{aligned} \tag{8.24}$$

where we used the definition of θ given in equation (8.14), together with the background equation $\mathcal{H}^2 = \frac{1}{3}a^2\rho_0$ that implies $\rho_0^{1/2} \propto \frac{\mathcal{H}}{a}$. A is again constant in time, and reintroducing the arguments of the functions we have

$$\Phi(k,\eta) = A(k)\left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_i}^{\eta} d\tilde{\eta} a^2(\tilde{\eta})\right] \tag{8.25}$$

In the small scale limit ($|c_s|k\eta \ll 1$) equation (8.17) becomes

$$u'' + c_s^2 k^2 u = 0 \tag{8.26}$$

If c_s is constant in time, equation (8.26) has the solution

$$u = c_1 e^{ikc_s\eta} + c_2 e^{-ikc_s\eta} \tag{8.27}$$

so

$$\Phi(k,\eta) = \rho_0^{1/2}(\eta)\left[c_1 e^{ikc_s\eta} + c_2 e^{-ikc_s\eta}\right] \tag{8.28}$$

while if c_s is real and varies in time more slowly than u, the WKB method gives us

$$u = A\frac{1}{\sqrt{c_s}}\cos\left(k\int_{\eta_i} c_s d\eta\right) \tag{8.29}$$

so

$$\Phi(k,\eta) = A(k)\left(\frac{\rho_0(\eta) + p_0(\eta)}{c_s(\eta)}\right)^{1/2} \cos\left(k\int_{\eta_i}^{\eta} c_s(\tilde{\eta})d\tilde{\eta}\right) \tag{8.30}$$

8.1.1 The case of dark energy

If we consider a universe dominated by a dark energy fluid with a constant equation of state $w < 0$, the evolution of the cosmological perturbations that we have just investigated presents some problems. In fact, in this case we have $c_s^2 = w < 0$, so c_s is pure imaginary. Therefore the small scale solution (8.28) becomes

$$\Phi(k,\eta) = \rho_0^{1/2}(\eta)\left[c_1 e^{k|c_s|\eta} + c_2 e^{-k|c_s|\eta}\right] \tag{8.31}$$

where the exponentially growing mode is completely incompatible with the observations.
In order to solve this problem, we have to consider a more general expression for the pressure perturbation

$$\delta p = c_s^2 \delta\rho + (c_s^2 - c_a^2)\rho_0' v^{\|} \tag{8.32}$$

where $c_a^2 = \frac{p_0'}{\rho_0'}$ is the real adiabatic speed of sound (and if w is constant we have $c_a^2 = w$) while the speed of sound c_s is now a free parameter (see [40]). For ordinary fluids without non-adiabatic perturbations we can take $c_a = c_s$, obtaining the evolution equation (8.12), but for dark energy this cannot be done if we want to avoid a diverging gravitational potential.
The problem for dark energy is solved if we impose a null speed of sound and the absence of non-adiabatic perturbations, getting $\delta p = 0$ and so an evolution equation for Φ

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 0 \tag{8.33}$$

that allows the solution for all scales

$$\Phi(k,\eta) = A(k)\left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_i}^{\eta} d\tilde{\eta}\, a^2(\tilde{\eta})\right] \tag{8.34}$$

8.2 Cosmological perturbations in Horndeski gravity

In this Section we present some results for linear scalar perturbations in Horndeski gravity, since they will be useful also when we consider the mimetic extension. We assume that matter is absent, so $S_m = 0$: we expect this to be a good approximation during the time when the effective energy density of the scalar field is much larger than the other usual components, like radiation or cold dark matter.
The metric is the one in equation (8.1) and the scalar field is expanded as

$$\varphi(\eta, \vec{x}) = \varphi_0(\eta) + \delta\varphi(\eta, \vec{x}) \tag{8.35}$$

where φ_0 denotes the background value of the scalar field and $\delta\varphi$ indicates the perturbation.

The independent equations of motion for Horndeski gravity, if matter is absent, are simply

$$E_{\mu\nu} = 0 \tag{8.36}$$

because Horndeski identity (7.72), together with equation (8.36), implies $\Omega_\varphi = 0$.
At the background level they reduce to $E_{\mu\nu}^{(0)} = 0$, where the superscript (0) denotes a background quantity: the explicit expression of $E_{\mu\nu}^{(0)}$ for a flat FRW background can be found in the Appendix A of [17].
At first order (denoted by the superscript (1)) the tensor $E_{\mu\nu}$ can be written as

$$E_{00}^{(1)} = f_1\Psi' + f_2\delta\varphi' + f_3\Phi + f_4\delta\varphi + f_5\nabla^2\Psi + f_6\nabla^2\delta\varphi \tag{8.37}$$

$$E_{0i}^{(1)} = \partial_i(f_{18}\Psi' + f_{19}\delta\varphi' + f_{20}\delta\varphi + f_{21}\Phi) \tag{8.38}$$

$$\begin{aligned} E_{ij}^{(1)} =&\partial_i\partial_j(f_7\Psi + f_8\delta\varphi + f_9\Phi) + \delta_{ij}(-f_7\nabla^2\Psi - f_8\nabla^2\delta\varphi - f_9\nabla^2\Phi + \\ &+ f_{10}\Psi'' + f_{11}\delta\varphi'' + f_{12}\Psi' + f_{13}\delta\varphi' + f_{14}\Phi' + f_{15}\Psi + f_{16}\delta\varphi + f_{17}\Phi) \end{aligned} \tag{8.39}$$

where the f_i are linear functions of K, G_3, G_4, G_5 and their derivatives evaluated on the background, so they are functions of time only. Their explicit expression, and some useful relations between them, can be found in the Appendix B of [17].
The equations of motion for the first order perturbations are $E_{\mu\nu}^{(1)} = 0$:

$$f_1\Psi' + f_2\delta\varphi' + f_3\Phi + f_4\delta\varphi + f_5\nabla^2\Psi + f_6\nabla^2\delta\varphi = 0 \tag{8.40}$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + f_{20}\delta\varphi + f_{21}\Phi = 0 \tag{8.41}$$

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0 \tag{8.42}$$

$$f_{10}\Psi'' + f_{11}\delta\varphi'' + f_{12}\Psi' + f_{13}\delta\varphi' + f_{14}\Phi' + f_{15}\Psi + f_{16}\delta\varphi + f_{17}\Phi = 0 \tag{8.43}$$

but equation (8.43) can be derived from equations (8.41) and (8.42) (see [17]), using some of the identities in the Appendix B of [17]. So we have three independent equations for the three perturbations Ψ, Φ and $\delta\varphi$

$$f_1\Psi' + f_2\delta\varphi' + f_3\Phi + f_4\delta\varphi + f_5\nabla^2\Psi + f_6\nabla^2\delta\varphi = 0 \tag{8.44}$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + f_{20}\delta\varphi + f_{21}\Phi = 0 \tag{8.45}$$

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0 \tag{8.46}$$

There are two physical implications of equation (8.46). First, the anisotropic stress is in general not zero ($\Psi \neq \Phi$) even if matter is absent. Second, at least one of the perturbations is not a new dynamical degree of freedom, since one of them can be isolated to be a linear function of the other two.

8.3 Cosmological perturbations in mimetic Horndeski gravity

As we have found in the previous Chapter, the independent equations of motion for mimetic Horndeski gravity (if matter is absent) are equations (7.84) and (7.85):

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1 \qquad E_{\mu i} = Eb(\varphi)\partial_\mu\varphi\partial_i\varphi \tag{8.47}$$

At the background level they reduce to

$$-\frac{1}{a^2}b_0(\varphi_0')^2 = 1 \tag{8.48}$$

$$E_{\mu i}^{(0)} = 0 \tag{8.49}$$

where $b_0 \equiv b(\varphi_0)$ and the explicit expressions for $E_{\mu i}^{(0)}$ are the same of the previous Section and can be found in Appendix B of [17].
At first order the equations of motion are

$$2b_0\delta\varphi' + \varphi_0' b_{,\varphi}\delta\varphi - 2b_0\varphi_0'\Phi = 0 \tag{8.50}$$

$$E_{0i}^{(1)} = E^{(0)}b_0\varphi_0'\partial_i\delta\varphi \tag{8.51}$$

$$E_{ij}^{(1)} = 0 \tag{8.52}$$

where $E^{(0)}$ is the zeroth-order trace of the tensor $E_{\mu\nu}$, the subscript $,\varphi$ denotes the derivative with respect to the scalar field φ and $b_{,\varphi} \equiv b_{,\varphi}(\varphi_0)$.
Using the expressions (8.37)-(8.39), equation (8.52) gives us

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0 \tag{8.53}$$

$$f_{10}\Psi'' + f_{11}\delta\varphi'' + f_{12}\Psi' + f_{13}\delta\varphi' + f_{14}\Phi' + f_{15}\Psi + f_{16}\delta\varphi + f_{17}\Phi = 0 \tag{8.54}$$

while, using also the background constraint (8.48), equation (8.51) implies

$$f_{18}\Psi' + f_{19}\delta\varphi' + \left(f_{20} + \frac{a^2E^{(0)}}{\varphi_0'}\right)\delta\varphi + f_{21}\Phi = 0 \tag{8.55}$$

In a similar way to what we did for the simple Horndeski model, it is possible to show that equation (8.54) can be derived by equations (8.50), (8.53) and (8.55) (see [17]). This means that a system of independent equations of motion for first order perturbations is

$$2b_0\delta\varphi' + \varphi_0'b_{,\varphi}\delta\varphi - 2b_0\varphi_0'\Phi = 0 \tag{8.56}$$

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0 \tag{8.57}$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + \left(f_{20} + \frac{a^2E^{(0)}}{\varphi_0'}\right)\delta\varphi + f_{21}\Phi = 0 \tag{8.58}$$

We note that equation (8.57) is the same as in Horndeski gravity: this means that also in the mimetic modification there could be anisotropic stress ($\Psi \neq \Phi$) even if matter is absent, and that one of the three perturbations is not a new dynamical degree of freedom.
We note also that in the system of equations (8.56)-(8.58) there are no spatial derivatives, so we can anticipate that the sound speed for the dynamical scalar degree of freedom will be exactly zero.
Finally, it is important to note that one can set $b(\varphi) = -1$ without loss of generality. In that case we would have $b_{,\varphi} = 0$ and equation (8.56) would become

$$2b_0\delta\varphi' - 2b_0\varphi_0'\Phi = 0 \quad \Longrightarrow \quad \Phi = \frac{\delta\varphi'}{\varphi_0'} \tag{8.59}$$

This is possible because Horndeski theory is form-invariant under a field redefinition of the type $d\tilde{\varphi} \equiv \sqrt{|b(\varphi)|}d\varphi$. Applying this transformation to a mimetic Horndeski model fixed by the functions $K(X,\varphi)$, $G_i(X,\varphi)$ and $b(\varphi)$, we obtain an equivalent mimetic model (again in the Horndeski class) with $\tilde{b}(\tilde{\varphi}) = -1$ and new Horndeski functions $\tilde{K}(\tilde{X},\tilde{\varphi})$ and $\tilde{G}_i(\tilde{X},\tilde{\varphi})$.

Combining equations (8.56)-(8.58) one can find a second order differential equation for the potential Φ (see [17]):

$$\Phi'' + \left[\frac{B_2}{B_3} + \left(\log\frac{B_3}{B_1}\right)' + \mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right]\Phi' + \left[\frac{B_1}{B_3}\varphi_0' + \frac{B_1}{B_3}\left(\frac{B_2}{B_1}\right)' + \frac{B_2}{B_3}\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\right]\Phi = 0 \tag{8.60}$$

where the B_i functions are defined as

$$B_1 = f_{20} + \frac{f_{10}f_8f_7'}{f_7^2} + f_{11}\left(-\mathcal{H} + \frac{\varphi_0''}{\varphi_0'}\right) - \frac{f_{10}}{f_7}\left[f_8' + f_8\left(-\mathcal{H} + \frac{\varphi_0''}{\varphi_0'}\right)\right] + a^2\frac{E^{(0)}}{\varphi_0'} \tag{8.61}$$

$$B_2 = f_{14} + \frac{f_{10} f_9 f_7'}{f_7^2} + f_{11}\varphi_0' - \frac{f_{10}(f_9' + f_8\varphi_0')}{f_7} \tag{8.62}$$

$$B_3 = 2\frac{f_9^2}{f_7} \tag{8.63}$$

In equation (8.60) there is no spatial Laplacian term, so the sound speed of the perturbations is exactly zero as we anticipated.

8.4 Imposing a ΛCDM background expansion history

Equations (8.56)-(8.58) and equation (8.60) are valid in general for a FRW metric. In this Subsection we want to impose a ΛCDM background expansion history for the universe, given by

$$a(t) = a_i \sinh^{\frac{2}{3}}(Ct) \tag{8.64}$$

In fact a very large set of observations coming from different probes (supernovae, clusters of galaxies, baryonic acoustic oscillations and weak gravitational lensing are four examples, see [34]) indicates that the background expansion history of the universe, when radiation becomes negligible, can be described by equation (8.64), even if the correct theory of gravity may not be general relativity. In that case there is no cosmological constant, so the parameter C is not related to Λ and it is fixed directly from observations.

We start by taking $b(\varphi) = -1$, so, using equation (8.64), the first background equation of motion (8.48) becomes

$$(\varphi_0')^2 = a^2 \quad \Longrightarrow \quad \varphi_0' = \pm a = \pm a_i \sinh^{2/3}(Ct) \tag{8.65}$$

and other useful quantities are

$$\mathcal{H} = \frac{a'}{a} = \frac{2}{3} C a_i \sinh^{-1/3}(Ct)\cosh(Ct) \tag{8.66}$$

$$\mathcal{H}' = -\frac{2}{9} C^2 a_i^2 \sinh^{-2/3}(Ct)\cosh^2(Ct) + \frac{2}{3} C^2 a_i^2 \sinh^{4/3}(Ct) \tag{8.67}$$

$$\varphi_0'' = \pm\frac{2}{3} C a_i^2 \sinh^{1/3}(Ct)\cosh(Ct) \tag{8.68}$$

Taking now the explicit expression of $E_{ij}^{(0)}$ given in Appendix A of [17], and substituting the quantities (8.65)-(8.68), the second background equation of motion (8.49) becomes

$$\begin{aligned}
&a_i^2 \sinh^{4/3}(Ct) K - a_i^2 \sinh^{4/3}(Ct) G_{3,\varphi} + \frac{8}{3} C^2 a_i^2 \sinh^{4/3}(Ct) G_4 + \\
&- \frac{8}{3} C^2 a_i^2 \sinh^{4/3}(Ct) G_{4,X} \pm \frac{8}{3} C a_i^2 \sinh^{1/3}(Ct)\cosh(Ct) G_{4,\varphi} + 2a_i^2 \sinh^{4/3}(Ct) G_{4,\varphi\varphi} + \\
&\mp \frac{8}{3} C a_i^2 \sinh^{1/3}(Ct)\cosh(Ct) G_{4,X\varphi} + \frac{4}{3} C^2 a_i^2 \sinh^{4/3}(Ct) G_{5,\varphi} + \\
&\pm \left[\frac{8}{27} C^3 a_i^2 \sinh^{-5/3}(Ct)\cosh^3(Ct) - \frac{8}{9} C^3 a_i^2 \sinh^{1/3}(Ct)\right] G_{5,X} + \\
&- \frac{4}{9} C^2 a_i^2 \sinh^{-2/3}(Ct)\cosh^2(Ct) G_{5,X\varphi} \pm \frac{4}{3} C a_i^2 \sinh^{1/3}(Ct)\cosh(Ct) G_{5,\varphi\varphi} = 0
\end{aligned} \tag{8.69}$$

where the Horndeski functions K, G_i and their derivatives are evaluated on the the background quantities φ_0 and X_0. This means that a mimetic Horndeski model allows a ΛCDM background expansion history only if the Horndeski functions obey the zeroth order constraint

$$\begin{aligned}K(X_0,\varphi_0) =&G_{3,\varphi} - \frac{8}{3}C^2G_4 + \frac{8}{3}C^2G_{4,X} \mp \frac{8}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{4,\varphi}+\\&- 2G_{4,\varphi\varphi} \pm \frac{8}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{4,X\varphi} - \frac{4}{3}C^2G_{5,\varphi}+\\&\mp \frac{8}{27}C^3\left[\sinh^{-3}(Ct)\cosh^3(Ct) - 3\sinh^{-1}(Ct)\cosh(Ct)\right]G_{5,X}+\\&+ \frac{4}{9}C^2\sinh^{-2}(Ct)\cosh^2(Ct)G_{5,X\varphi} \mp \frac{4}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,\varphi\varphi}\end{aligned} \tag{8.70}$$

where again the arguments φ_0 and X_0 of the functions G_i are implicit.

Considering now the evolution of perturbations, it is described by the system of equations (8.57)-(8.59)

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0 \tag{8.71}$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + \left(f_{20} + \frac{a^2E^{(0)}}{\varphi_0'}\right)\delta\varphi + f_{21}\Phi = 0 \tag{8.72}$$

$$\Phi = \frac{\delta\varphi'}{\varphi_0'} \tag{8.73}$$

where equations (8.64) and (8.70) give us

$$f_7 = -2G_4 + G_{5,\varphi} \tag{8.74}$$

$$\begin{aligned}f_8 =&2G_{4,\varphi} \mp \frac{4}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{4,X} - 2G_{4,X\varphi}+\\&\pm \frac{4}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,\varphi} + \left[\frac{2}{9}C^2\sinh^{-2}(Ct)\cosh^2(Ct) - \frac{2}{3}C^2\right]G_{5,X}+\\&\mp \frac{2}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,X\varphi} + G_{5,\varphi\varphi}\end{aligned} \tag{8.75}$$

$$f_9 = 2G_4 - 2G_{4,X} + G_{5,\varphi} \mp \frac{2}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,X} \tag{8.76}$$

$$f_{10} = f_{18} = -4G_4 + 4G_{4,X} - 2G_{5,\varphi} \pm \frac{4}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,X} \tag{8.77}$$

$$\begin{aligned}f_{11} = f_{19} =&- G_{3,X} + 2G_{4,\varphi} \mp \frac{8}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{4,X} + 2G_{4,X\varphi}+\\&\mp \frac{8}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{4,XX} \pm \frac{8}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,\varphi}+\\&- \frac{4}{3}C^2\sinh^{-2}(Ct)\cosh^2(Ct)G_{5,X} \pm \frac{4}{3}C\sinh^{-1}(Ct)\cosh(Ct)G_{5,X\varphi}+\\&- \frac{4}{9}C^2\sinh^{-2}(Ct)\cosh^2(Ct)G_{5,XX}\end{aligned} \tag{8.78}$$

$$f_{20}+\frac{a^2E^{(0)}}{\varphi_0'}=\pm\left[\frac{8}{3}C^2a_i\sinh^{-4/3}(Ct)\cosh^2(Ct)-\frac{8}{3}C^2a_i\sinh^{2/3}(Ct)\right]G_4+$$
$$\pm\left[-\frac{8}{3}C^2a_i\sinh^{-4/3}(Ct)\cosh^2(Ct)+\frac{8}{3}C^2a_i\sinh^{2/3}(Ct)\right]G_{4,X}+$$
$$\pm\left[\frac{4}{3}C^2a_i\sinh^{-4/3}(Ct)\cosh^2(Ct)-\frac{4}{3}C^2a_i\sinh^{2/3}(Ct)\right]G_{5,\varphi}+$$
$$+\left[-\frac{8}{9}C^3a_i\sinh^{-7/3}(Ct)\cosh^3(Ct)+\frac{8}{9}C^3a_i\sinh^{-1/3}(Ct)\cosh(Ct)\right]G_{5,X} \tag{8.79}$$

$$f_{14}=f_{21}=\pm\, a_i\sinh^{2/3}(Ct)G_{3,X}-\frac{8}{3}Ca_i\sinh^{-1/3}(Ct)\cosh(Ct)G_4+$$
$$+\frac{16}{3}Ca_i\sinh^{-1/3}(Ct)\cosh(Ct)G_{4,X}\mp 2a_i\sinh^{2/3}(Ct)G_{4,\varphi}+$$
$$\mp 2a_i\sinh^{2/3}(Ct)G_{4,X\varphi}+\frac{8}{3}Ca_i\sinh^{-1/3}(Ct)\cosh(Ct)G_{4,XX}+$$
$$-4Ca_i\sinh^{-1/3}(Ct)\cosh(Ct)G_{5,\varphi}\pm\frac{20}{9}C^2a_i\sinh^{-4/3}(Ct)\cosh^2(Ct)G_{5,X}+$$
$$-\frac{4}{3}Ca_i\sinh^{-1/3}(Ct)\cosh(Ct)G_{5,X\varphi}\pm\frac{4}{9}C^2a_i\sinh^{-4/3}(Ct)\cosh^2(Ct)G_{5,XX} \tag{8.80}$$

Using equations (8.65), (8.66) and (8.68) we get $\mathcal{H}-\frac{\varphi_0''}{\varphi_0'}=0$, so the second order differential equation for the potential Φ (8.60) becomes

$$\Phi''+\left[\frac{B_2}{B_3}+\left(\log\frac{B_3}{B_1}\right)'\right]\Phi'+\left[\frac{B_1}{B_3}\varphi_0'+\frac{B_1}{B_3}\left(\frac{B_2}{B_1}\right)'\right]\Phi=0 \tag{8.81}$$

where

$$B_1=f_{20}+a^2\frac{E^{(0)}}{\varphi_0'}+\frac{f_{10}f_8f_7'}{f_7^2}-\frac{f_{10}f_8'}{f_7} \tag{8.82}$$

$$B_2=f_{14}+\frac{f_{10}f_9f_7'}{f_7^2}+f_{11}\varphi_0'-\frac{f_{10}(f_9'+f_8\varphi_0')}{f_7} \tag{8.83}$$

$$B_3=2\frac{f_9^2}{f_7} \tag{8.84}$$

and

$$f_7'=\mp 2a_i\sinh^{2/3}(Ct)G_{4,\varphi}\pm a_i\sinh^{2/3}(Ct)G_{5,\varphi\varphi} \tag{8.85}$$

$$
\begin{aligned}
f_8' = &\pm \left[-\frac{4}{3}C^2 a_i \sinh^{2/3}(Ct) + \frac{4}{3}C^2 a_i \sinh^{-4/3}(Ct)\cosh^2(Ct)\right] G_{4,X} + \\
&- \frac{4}{3}Ca_i \sinh^{-1/3}(Ct)\cosh(Ct) G_{4,X\varphi} \pm 2a_i \sinh^{2/3}(Ct) G_{4,\varphi\varphi} \mp 2a_i \sinh^{2/3}(Ct) G_{4,X\varphi\varphi} + \\
&\pm \left[\frac{4}{3}C^2 a_i \sinh^{2/3}(Ct) - \frac{4}{3}C^2 a_i \sinh^{-4/3}(Ct)\cosh^2(Ct)\right] G_{5,\varphi} + \\
&+ \left[\frac{4}{9}C^3 a_i \sinh^{-1/3}(Ct)\cosh(Ct) - \frac{4}{9}C^3 a_i \sinh^{-7/3}(Ct)\cosh^3(Ct)\right] G_{5,X} + \\
&\pm \left[-\frac{4}{3}C^2 a_i \sinh^{2/3}(Ct) + \frac{8}{9}C^2 a_i \sinh^{-4/3}(Ct)\cosh^2(Ct)\right] G_{5,X\varphi} + \\
&+ \frac{4}{3}Ca_i \sinh^{-1/3}(Ct)\cosh(Ct) G_{5,\varphi\varphi} - \frac{2}{3}Ca_i \sinh^{-1/3}(Ct)\cosh(Ct) G_{5,X\varphi\varphi} + \\
&\pm a_i \sinh^{2/3}(Ct) G_{5,\varphi\varphi\varphi}
\end{aligned}
\tag{8.86}
$$

$$
\begin{aligned}
f_9' = &\pm 2a_i \sinh^{2/3}(Ct) G_{4,\varphi} \mp 2a_i \sinh^{2/3}(Ct) G_{4,X\varphi} + \\
&\pm \left[-\frac{2}{3}C^2 a_i \sinh^{2/3}(Ct) + \frac{2}{3}C^2 a_i \sinh^{-4/3}(Ct)\cosh^2(Ct)\right] G_{5,X} + \\
&\pm a_i \sinh^{2/3}(Ct) G_{5,\varphi\varphi} - \frac{2}{3}a_i \sinh^{2/3}(Ct) G_{5,X\varphi}
\end{aligned}
\tag{8.87}
$$

8.4.1 Mimetic cubic Horndeski gravity

Now we investigate the case when $G_4(X,\varphi) = 1/2$ and $G_5(X,\varphi) = 0$, while the other Horndeski functions $K(X,\varphi)$ and $G_3(X,\varphi)$ are kept general. These particular models are grouped under the name of *mimetic cubic Horndeski*, and they include also the models described in Section 7.5, that we showed to have an interesting cosmological behavior: in particular, the mimetic cubic Galileon (7.99) was able to reproduce a ΛCDM background expansion history. In this particular case, the zeroth order constraint on the Horndeski functions (8.70) becomes

$$
K(X_0,\varphi_0) = G_{3,\varphi}(X_0,\varphi_0) - \frac{4}{3}C^2 \tag{8.88}
$$

while the f_i functions (8.74)-(8.80) become

$$
f_7 = -1 \tag{8.89}
$$

$$
f_8 = 0 \tag{8.90}
$$

$$
f_9 = 1 \tag{8.91}
$$

$$
f_{10} = f_{18} = -2 \tag{8.92}
$$

$$
f_{11} = f_{19} = -G_{3,X} \tag{8.93}
$$

$$
f_{20} + \frac{a^2 E^{(0)}}{\varphi_0'} = \pm\frac{1}{2}\left[\frac{8}{3}C^2 a_i \sinh^{-4/3}(Ct)\cosh^2(Ct) - \frac{8}{3}C^2 a_i \sinh^{2/3}(Ct)\right] \tag{8.94}
$$

$$
f_{14} = f_{21} = \pm a_i \sinh^{2/3}(Ct) G_{3,X} - \frac{4}{3}Ca_i \sinh^{-1/3}(Ct)\cosh(Ct) \tag{8.95}
$$

We note that, using equations (8.89)-(8.91), the first equation of motion for the perturbations (8.71) becomes

$$
\Phi = \Psi \tag{8.96}
$$

so in mimetic cubic Horndeski, if we impose a ΛCDM background, there is no effective anisotropic stress if matter is absent. Actually, it is easy to show that equations (8.89)-(8.91) hold in general for every background expansion history, so the absence of a gravitational slip is a general feature of mimetic cubic Horndeski gravity.
Using instead the full set (8.89)-(8.95), the B_i functions (8.82)-(8.84) become

$$B_1 = \pm\frac{4}{3}C^2 a_i \left[\sinh^{-4/3}(Ct)\cosh^2(Ct) - \sinh^{2/3}(Ct)\right] \tag{8.97}$$

$$B_2 = -\frac{4}{3}C a_i \sinh^{-1/3}(Ct)\cosh(Ct) \tag{8.98}$$

$$B_3 = -2 \tag{8.99}$$

and the second order differential equation for the potential Φ (8.81) becomes

$$\Phi'' + 2Ca_i \sinh^{-1/3}(Ct)\cosh(Ct)\Phi' + \frac{4}{3}C^2 a_i^2 \sinh^{4/3}(Ct)\Phi = 0 \tag{8.100}$$

that can be rewritten as

$$\Phi'' + 3\,\mathcal{H}\,\Phi' + \left(\mathcal{H}^2 + \mathcal{H}'\right)\Phi = 0 \tag{8.101}$$

In Section 8.1 we solved the second order differential equation (8.12) for the potential Φ with a change of variable and using the background equations of the theory, to obtain a solution that contained only geometrical quantities. In the case of mimetic Horndeski gravity, the absence of the sound speed in the equation allows us to perform a change of variable that involves only geometrical quantities:

$$Q \equiv \sqrt{-\frac{a}{H_{,N}}}\Phi \tag{8.102}$$

where $N \equiv \log a$ and the subscript $,N$ denotes the derivative with respect to N. Defining

$$\Theta \equiv \frac{H}{\sqrt{-aH_{,N}}} \tag{8.103}$$

equation (8.101) can be written as ([17])

$$Q_{,NN} - \frac{\Theta_{,NN}}{\Theta}Q = 0 \tag{8.104}$$

A trivial solution for this equation is $Q_1 = A\Theta$, with A being a constant. The second solution Q_2 can be found with the d'Alembert method exactly as we did in Section 8.1 for equation (8.18), and the general solution can be written as

$$\begin{aligned} Q &= c_2\Theta\int\frac{dN}{\Theta^2} = c_2\frac{H}{\sqrt{-aH_{,N}}}\int dN\left(-\frac{aH_{,N}}{H^2}\right) = \\ &= c_2\frac{H}{\sqrt{-aH_{,N}}}\int dN a\frac{d}{dN}\left(\frac{1}{H}\right) = c_2\frac{H}{\sqrt{-aH_{,N}}}\left(\frac{a}{H} - \int dN\frac{da}{dN}\frac{1}{H}\right) = \\ &= c_2\sqrt{-\frac{a}{H_{,N}}}\left(1 - \frac{H}{a}\int\frac{da}{H}\right) \end{aligned} \tag{8.105}$$

where in the fourth passage we integrated by parts. Finally, redefining A the multiplication constant, the potential Φ is

$$\begin{aligned}\Phi &= \sqrt{-\frac{H_{,N}}{a}}Q = A\left(1-\frac{H}{a}\int\frac{da}{H}\right) = \\ &= A\left(1-\frac{1}{a^2}\frac{da}{dt}\int da\frac{a}{da/dt}\right) = A\left(1-\frac{\mathcal{H}}{a^2}\int d\eta a^2\right)\end{aligned} \tag{8.106}$$

that is exactly what we found in equation (8.24) for large scales in general relativity. The difference is that, in this case, solution (8.106) for Φ is valid for perturbations on all scales: reintroducing the arguments of the functions it becomes

$$\Phi(k,\eta) = A(k)\left[1-\frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_i}^{\eta} d\tilde{\eta}a^2(\tilde{\eta})\right] \tag{8.107}$$

8.5 Imposing a *perfect fluid dark energy* background expansion history

As we have explained in Section 2.4, the cosmological constant can be interpreted as a homogeneous perfect fluid with equation of state $w = -1$. If we substitute the cosmological constant with a perfect fluid dark energy (PFDE) with a constant generic equation of state w_{DE}, when this dark energy becomes dominant over matter, general relativity predicts a background expansion history given by

$$a(t) = a_i t^{\frac{2}{3(1+w_{DE})}} \tag{8.108}$$

If we take w_{DE} to be close enough to the cosmological constant value -1, equation (8.108) describes a background expansion history that is indistinguishable from the ΛCDM solution when Λ is dominant. Therefore a perfect fluid dark energy with $w_{DE} \simeq -1$ can be used in general relativity as an alternative option instead of a cosmological constant, in order to explain the observed background expansion history.
Considering this, it is interesting to study cosmological perturbations in mimetic Horndeski gravity imposing a background expansion history given by the domination of a perfect fluid with constant generic equation of state $w \neq -1$:

$$a(t) = a_i t^{\frac{2}{3(1+w)}} \tag{8.109}$$

Again we take $b(\varphi) = -1$, so the first background equation of motion (8.48) gives us

$$(\varphi_0')^2 = a^2 \quad \Longrightarrow \quad \varphi_0' = \pm a = \pm a_i t^{\frac{2}{3(1+w)}} \tag{8.110}$$

and the other useful quantities are

$$\mathcal{H} = \frac{a'}{a} = \frac{2a_i}{3(1+w)}t^{-\frac{1+3w}{3(1+w)}} \tag{8.111}$$

$$\mathcal{H}' = -\frac{2(1+3w)a_i^2}{9(1+w)^2}t^{-\frac{2+6w}{3(1+w)}} \tag{8.112}$$

$$\varphi_0'' = \pm\frac{2a_i^2}{3(1+w)}t^{\frac{1-3w}{3(1+w)}} \tag{8.113}$$

Following the same procedure used in the previous Section, the second background equation of motion (8.49) becomes

$$\begin{aligned}
&a_i^2 t^{\frac{4}{3(1+w)}} K - a_i^2 t^{\frac{4}{3(1+w)}} G_{3,\varphi} - \frac{8wa_i^2}{3(1+w)^2} t^{-\frac{2+6w}{3(1+w)}} G_4 + \frac{8wa_i^2}{3(1+w)^2} t^{-\frac{2+6w}{3(1+w)}} G_{4,X} + \\
&\pm \frac{8a_i^2}{3(1+w)} t^{\frac{1-3w}{3(1+w)}} G_{4,\varphi} + 2a_i^2 t^{\frac{4}{3(1+w)}} G_{4,\varphi\varphi} \mp \frac{8a_i^2}{3(1+w)} t^{\frac{1-3w}{3(1+w)}} G_{4,X\varphi} + \\
&- \frac{4wa_i^2}{3(1+w)^2} t^{-\frac{2+6w}{3(1+w)}} G_{5,\varphi} \pm \frac{8(1+3w)a_i^2}{27(1+w)^3} t^{-\frac{5+9w}{3(1+w)}} G_{5,X} - \frac{4a_i^2}{9(1+w)^2} t^{-\frac{2+6w}{3(1+w)}} G_{5,X\varphi} + \\
&\pm \frac{4a_i^2}{3(1+w)} t^{\frac{1-3w}{3(1+w)}} G_{5,\varphi\varphi} = 0
\end{aligned} \tag{8.114}$$

where the Horndeski functions K, G_i and their derivatives are evaluated on the background quantities φ_0 and X_0. Therefore a mimetic Horndeski model allows a perfect fluid background expansion history only if the Horndeski functions obey the zeroth order constraint

$$\begin{aligned}
K(X_0, \varphi_0) =& G_{3,\varphi} + \frac{8w}{3(1+w)^2} t^{-2} G_4 - \frac{8w}{3(1+w)^2} t^{-2} G_{4,X} \mp \frac{8}{3(1+w)} t^{-1} G_{4,\varphi} - 2G_{4,\varphi\varphi} + \\
&\pm \frac{8}{3(1+w)} t^{-1} G_{4,X\varphi} - \frac{4w}{3(1+w)^2} t^{-2} G_{5,\varphi} \mp \frac{8(1+3w)}{27(1+w)^3} t^{-3} G_{5,X} + \\
&+ \frac{4}{9(1+w)^2} t^{-2} G_{5,X\varphi} \mp \frac{4}{3(1+w)} t^{-1} G_{5,\varphi\varphi}
\end{aligned} \tag{8.115}$$

where again the arguments φ_0 and X_0 of the functions G_i are implicit.

As a consequence, if we want to reproduce a universe that is dominated by cold dark matter with EOS $w = 0$ at early times and that, afterward, becomes dominated by a perfect fluid dark energy with EOS w_{DE}, the Horndeski functions have to obey the zeroth order constraint

$$\begin{aligned}
K(X_0, \varphi_0) =& G_{3,\varphi} \mp \frac{8}{3} t^{-1} G_{4,\varphi} - 2G_{4,\varphi\varphi} \pm \frac{8}{3} t^{-1} G_{4,X\varphi} \mp \frac{8}{27} t^{-3} G_{5,X} + \\
&+ \frac{4}{9} t^{-2} G_{5,X\varphi} \mp \frac{4}{3} t^{-1} G_{5,\varphi\varphi}
\end{aligned} \tag{8.116}$$

for small t, and

$$\begin{aligned}
K(X_0, \varphi_0) =& G_{3,\varphi} + \frac{8w_{DE}}{3(1+w_{DE})^2} t^{-2} G_4 - \frac{8w_{DE}}{3(1+w_{DE})^2} t^{-2} G_{4,X} \mp \frac{8}{3(1+w_{DE})} t^{-1} G_{4,\varphi} + \\
&- 2G_{4,\varphi\varphi} \pm \frac{8}{3(1+w_{DE})} t^{-1} G_{4,X\varphi} - \frac{4w_{DE}}{3(1+w_{DE})^2} t^{-2} G_{5,\varphi} + \\
&\mp \frac{8(1+3w_{DE})}{27(1+w_{DE})^3} t^{-3} G_{5,X} + \frac{4}{9(1+w_{DE})^2} t^{-2} G_{5,X\varphi} \mp \frac{4}{3(1+w_{DE})} t^{-1} G_{5,\varphi\varphi}
\end{aligned} \tag{8.117}$$

for large t.

Considering now the evolution of perturbations, it is described also in this case by the system of equation (8.57)-(8.59)

$$f_7 \Psi + f_8 \delta\varphi + f_9 \Phi = 0 \tag{8.118}$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + \left(f_{20} + \frac{a^2E^{(0)}}{\varphi_0'}\right)\delta\varphi + f_{21}\Phi = 0 \tag{8.119}$$

$$\Phi = \frac{\delta\varphi'}{\varphi_0'} \tag{8.120}$$

where equations (8.109) and (8.115) give us

$$f_7 = -2G_4 + G_{5,\varphi} \tag{8.121}$$

$$\begin{aligned} f_8 =& 2G_{4,\varphi} \mp \frac{4}{3(1+w)}t^{-1}G_{4,X} - 2G_{4,X\varphi} \pm \frac{4}{3(1+w)}t^{-1}G_{5,\varphi} + \frac{2(1+3w)}{9(1+w)^2}t^{-2}G_{5,X}+ \\ &\mp \frac{2}{3(1+w)}t^{-1}G_{5,X\varphi} + G_{5,\varphi\varphi} \end{aligned} \tag{8.122}$$

$$f_9 = 2G_4 - 2G_{4,X} + G_{5,\varphi} \mp \frac{2}{3(1+w)}t^{-1}G_{5,X} \tag{8.123}$$

$$f_{10} = f_{18} = -4G_4 + 4G_{4,X} - 2G_{5,\varphi} \pm \frac{4}{3(1+w)}t^{-1}G_{5,X} \tag{8.124}$$

$$\begin{aligned} f_{11} = f_{19} =& -G_{3,X} + 2G_{4,\varphi} \mp \frac{8}{3(1+w)}t^{-1}G_{4,X} + 2G_{4,X\varphi} \mp \frac{8}{3(1+w)}t^{-1}G_{4,XX}+ \\ &\pm \frac{8}{3(1+w)}t^{-1}G_{5,\varphi} - \frac{4}{3(1+w)^2}t^{-2}G_{5,X} \pm \frac{4}{3(1+w)}t^{-1}G_{5,X\varphi}+ \\ &- \frac{4}{9(1+w)^2}t^{-2}G_{5,XX} \end{aligned} \tag{8.125}$$

$$\begin{aligned} f_{20} + \frac{a^2E^{(0)}}{\varphi_0'} =& \pm \frac{8a_i}{3(1+w)}t^{-\frac{4+6w}{3(1+w)}}G_4 \mp \frac{8a_i}{3(1+w)}t^{-\frac{4+6w}{3(1+w)}}G_{4,X}+ \\ &\pm \frac{4a_i}{3(1+w)}t^{-\frac{4+6w}{3(1+w)}}G_{5,\varphi} - \frac{8a_i}{9(1+w)^2}t^{-\frac{7+9w}{3(1+w)}}G_{5,X} \end{aligned} \tag{8.126}$$

$$\begin{aligned} f_{14} = f_{21} =& \pm a_i t^{\frac{2}{3(1+w)}}G_{3,X} - \frac{8a_i}{3(1+w)}t^{-\frac{1+3w}{3(1+w)}}G_4 + \frac{16a_i}{3(1+w)}t^{-\frac{1+3w}{3(1+w)}}G_{4,X}+ \\ &\mp 2a_i t^{\frac{2}{3(1+w)}}G_{4,\varphi} \mp 2a_i t^{\frac{2}{3(1+w)}}G_{4,X\varphi} + \frac{8a_i}{3(1+w)}t^{-\frac{1+3w}{3(1+w)}}G_{4,XX}+ \\ &- \frac{4a_i}{1+w}t^{-\frac{1+3w}{3(1+w)}}G_{5,\varphi} \pm \frac{20a_i}{9(1+w)^2}t^{-\frac{4+6w}{3(1+w)}}G_{5,X} - \frac{4a_i}{3(1+w)}t^{-\frac{1+3w}{3(1+w)}}G_{5,X\varphi}+ \\ &\pm \frac{4a_i}{9(1+w)^2}t^{-\frac{4+6w}{3(1+w)}}G_{5,XX} \end{aligned} \tag{8.127}$$

Using equations (8.110), (8.111) and (8.113) we get $\mathcal{H} - \frac{\varphi_0''}{\varphi_0'} = 0$ again, so the second order differential equation for the potential Φ (8.60) becomes

$$\Phi'' + \left[\frac{B_2}{B_3} + \left(\log\frac{B_3}{B_1}\right)'\right]\Phi' + \left[\frac{B_1}{B_3}\varphi_0' + \frac{B_1}{B_3}\left(\frac{B_2}{B_1}\right)'\right]\Phi = 0 \tag{8.128}$$

where

$$B_1 = f_{20} + a^2\frac{E^{(0)}}{\varphi_0'} + \frac{f_{10}f_8f_7'}{f_7^2} - \frac{f_{10}f_8'}{f_7} \tag{8.129}$$

$$B_2 = f_{14} + \frac{f_{10} f_9 f_7'}{f_7^2} + f_{11}\varphi_0' - \frac{f_{10}(f_9' + f_8\varphi_0')}{f_7} \tag{8.130}$$

$$B_3 = 2\frac{f_9^2}{f_7} \tag{8.131}$$

and

$$f_7' = \mp 2a_i t^{\frac{2}{3(1+w)}} G_{4,\varphi} \pm a_i t^{\frac{2}{3(1+w)}} G_{5,\varphi\varphi} \tag{8.132}$$

$$\begin{aligned} f_8' = &\pm \frac{4a_i}{3(1+w)} t^{-\frac{4+6w}{3(1+w)}} G_{4,X} - \frac{4a_i}{3(1+w)} t^{-\frac{1+3w}{3(1+w)}} G_{4,X\varphi} \pm 2a_i t^{\frac{2}{3(1+w)}} G_{4,\varphi\varphi} + \\ &\mp 2a_i t^{\frac{2}{3(1+w)}} G_{4,X\varphi\varphi} \mp \frac{4a_i}{3(1+w)} t^{-\frac{4+6w}{3(1+w)}} G_{5,\varphi} - \frac{4(1+3w)a_i}{9(1+w)^2} t^{-\frac{7+9w}{3(1+w)}} G_{5,X} + \\ &\pm \frac{4(2+3w)a_i}{9(1+w)^2} t^{-\frac{4+6w}{3(1+w)}} G_{5,X\varphi} + \frac{4a_i}{3(1+w)} t^{-\frac{1+3w}{3(1+w)}} G_{5,\varphi\varphi} - \frac{2a_i}{3(1+w)} t^{-\frac{1+3w}{3(1+w)}} G_{5,X\varphi\varphi} + \\ &\pm a_i t^{\frac{2}{3(1+w)}} G_{5,\varphi\varphi\varphi} \end{aligned} \tag{8.133}$$

$$\begin{aligned} f_9' = &\pm 2a_i t^{\frac{2}{3(1+w)}} G_{4,\varphi} \mp 2a_i t^{\frac{2}{3(1+w)}} G_{4,X\varphi} \pm \frac{2a_i}{3(1+w)} t^{-\frac{4+6w}{3(1+w)}} G_{5,X} + \\ &\pm a_i t^{\frac{2}{3(1+w)}} G_{5,\varphi\varphi} - \frac{2a_i}{3(1+w)} t^{-\frac{1+3w}{3(1+w)}} G_{5,X\varphi} \end{aligned} \tag{8.134}$$

8.5.1 Mimetic cubic Horndeski gravity

Since equation (8.128) is still too complicated to be solved analytically, we investigate the mimetic cubic Horndeski case as we did in the previous Section, taking $G_4 = 1/2$ and $G_5 = 0$. The zeroth order constraint on the Horndeski functions (8.115) becomes

$$K(X_0, \varphi_0) = G_{3,\varphi}(X_0, \varphi_0) + \frac{4w}{3(1+w)^2} t^{-2} \tag{8.135}$$

while the f_i functions (8.121)-(8.127) become

$$f_7 = -1 \tag{8.136}$$

$$f_8 = 0 \tag{8.137}$$

$$f_9 = 1 \tag{8.138}$$

$$f_{10} = f_{18} = -2 \tag{8.139}$$

$$f_{11} = f_{19} = -G_{3,X} \tag{8.140}$$

$$f_{20} + \frac{a^2 E^{(0)}}{\varphi_0'} = \pm \frac{4a_i}{3(1+w)} t^{-\frac{4+6w}{3(1+w)}} \tag{8.141}$$

$$f_{14} = f_{21} = \pm a_i t^{\frac{2}{3(1+w)}} G_{3,X} - \frac{4a_i}{3(1+w)} t^{-\frac{1+3w}{3(1+w)}} \tag{8.142}$$

Taking equations (8.136)-(8.138) the first equation of motion for the perturbations (8.118) becomes

$$\Phi = \Psi \tag{8.143}$$

so, as we anticipated in the previous Section, in mimetic cubic Horndeski there is no effective anisotropic stress if matter is absent, even imposing a perfect fluid background.
In this particular case the B_i functions (8.129)-(8.131) become

$$B_1 = \pm \frac{4a_i}{3(1+w)} t^{-\frac{4+6w}{3(1+w)}} \tag{8.144}$$

$$B_2 = -\frac{4a_i}{3(1+w)} t^{-\frac{1+3w}{3(1+w)}} \tag{8.145}$$

$$B_3 = -2 \tag{8.146}$$

and the second order differential equation for the potential Φ (8.128) becomes

$$\Phi'' + 2a_i t^{-\frac{1+3w}{3(1+w)}} \Phi' = 0 \tag{8.147}$$

Using the definition of conformal time $d\eta = \frac{dt}{a(t)}$, we get

$$\eta = \frac{3(1+w)}{a_i(1+3w)} t^{\frac{1+3w}{3(1+w)}} \tag{8.148}$$

so equation (8.147) in terms of conformal time becomes

$$\Phi'' + \frac{6(1+w)}{1+3w} \frac{1}{\eta} \Phi' = 0 \tag{8.149}$$

Equation (8.149) has the solution

$$\Phi(\eta) = c_1 + c_2 \eta^{-\frac{5+3w}{1+3w}} \tag{8.150}$$

but it is easy to show that this solution can be rewritten (defining new constants $A(k)$ and η_i instead of c_1 and c_2) in the more familiar form

$$\Phi(k,\eta) = A(k) \left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)} \int_{\eta_i}^{\eta} d\tilde{\eta} a^2(\tilde{\eta}) \right] \tag{8.151}$$

where this time the integral in equation (8.151) can be calculated to give equation (8.150).

Chapter 9

Analytical calculations of the integrated Sachs-Wolfe effect

In this Chapter we perform the analytical calculation of the late-time integrated Sachs-Wolfe effect introduced in Chapter 5, considering the time evolution of the potentials in the framework of general relativity and mimetic Horndeski gravity that we discussed in the previous Chapter.

The final goal of the Chapter is to evaluate the contribution of the late-time ISW effect to the C_l coefficient given by equation (5.41):

$$C_l = \frac{2}{\pi} \int_0^\infty dk k^2 P(k) \left| \frac{\Theta_l(k)}{\delta(k)} \right|^2 \tag{9.1}$$

If we consider the power spectrum of the matter fluctuations $P(k)$ to be generated by quantum fluctuations during inflation, it is a well-known result that $P(k) = |\delta(k)|^2$ (see [19], [35] and [36]). This means that equation (9.1) becomes

$$C_l = \frac{2}{\pi} \int_0^\infty dk k^2 |\Theta_l(k)|^2 \tag{9.2}$$

The contribution of the late-time ISW effect to $\Theta_l(k)$ is given by the last line of equation (5.20):

$$\Theta_l^{ISW}(k) = \int_0^{\eta_0} d\eta \, e^{-\tau} \left[\Phi'(k,\eta) + \Psi'(k,\eta)\right] j_l[k(\eta_0 - \eta)] \tag{9.3}$$

We know that before recombination ($\eta*$) Compton scattering is very frequent, so the optical depth τ is extremely large, while after recombination Compton scattering becomes ineffective and the optical depth is nearly zero. Therefore we can approximate the function $e^{-\tau}$ to be 0 for $\eta < \eta*$ and 1 for $\eta > \eta*$, so equation (9.3) becomes

$$\Theta_l^{ISW}(k) = \int_{\eta*}^{\eta_0} d\eta \left[\Phi'(k,\eta) + \Psi'(k,\eta)\right] j_l[k(\eta_0 - \eta)] \tag{9.4}$$

In what follows we will take the time evolution (after recombination) of the potentials Φ and Ψ that we discussed in the previous Chapter and we will use it in equations (9.4) and (9.2), in order to determine the contribution to the CMB's anisotropies of the late-time ISW effect, considering the different theories of gravity that we presented.

9.1 ISW effect in general relativity

9.1.1 ΛCDM background

If we consider the ΛCDM model, we have the domination of the cosmological constant Λ over matter after the time η_Λ when $\rho_\Lambda(\eta_\Lambda) = \rho_m(\eta_\Lambda)$. We know that ρ_Λ is constant in time, while ρ_m decays in time as a^{-3}, so $\rho_m(\eta_\Lambda) = \rho_m(\eta_0)\left(\frac{a(\eta_0)}{a(\eta_\Lambda)}\right)^3$. Therefore the time η_Λ is determined through the scale factor:

$$a(\eta_\Lambda) = a(\eta_0)\left(\frac{\rho_m(\eta_0)}{\rho_\Lambda}\right)^{1/3} = a(\eta_0)\left(\frac{\Omega_m(\eta_0)}{\Omega_\Lambda(\eta_0)}\right)^{1/3} \tag{9.5}$$

The time η_Λ gives a temporal scale for the variation of the potentials Φ and Ψ, so we can distinguish two different cases depending on the scale k of the perturbation.

- If $k \ll \frac{1}{\eta_\Lambda}$ (large scales), then $k(\eta_0 - \eta)$ is small for $\eta* < \eta < \eta_0$ and in equation (9.4) we can approximate $j_l[k(\eta_0 - \eta)] \simeq j_l[k(\eta_0 - \eta_\Lambda)]$, obtaining

$$\begin{aligned}\Theta_l^{ISW}(k) \simeq j_l[k(\eta_0 - \eta_\Lambda)] \int_{\eta*}^{\eta_0} d\eta \left[\Phi'(k,\eta) + \Psi'(k,\eta)\right] = \\ = j_l[k(\eta_0 - \eta_\Lambda)](\Delta\Phi(k) + \Delta\Psi(k))\end{aligned} \tag{9.6}$$

  where $\Delta\Phi(k) \equiv \Phi(k,\eta_0) - \Phi(k,\eta*)$ and $\Delta\Psi(k) \equiv \Psi(k,\eta_0) - \Psi(k,\eta*)$.
  The physical interpretation of this result is the following. Since the wavelength is much bigger than the time scale of the variation of the potentials ($\frac{1}{k} \propto \lambda \gg \eta_\Lambda$), the change in the potentials is so rapid that the photon is influenced only by the difference between the initial and the final values: it receives a *kick*, given by an instantaneous decay of the potentials.

- If $k \gg \frac{1}{\eta_\Lambda}$ (small scales), then $k(\eta_0 - \eta)$ can be really large. Therefore we must consider that $j_l[k(\eta_0 - \eta)]$ oscillates rapidly in this regime, but it has a sharp peak at $\eta_k = \eta_0 - \frac{1}{k}\left(l + \frac{1}{2}\right)$ (see [37] and [38]). So we can approximate the Bessel function as a non-normalized delta function, obtaining

$$\begin{aligned}\Theta_l^{ISW}(k) &\simeq \left[\Phi'(k,\eta_k) + \Psi'(k,\eta_k)\right] \int_{\eta*}^{\eta_0} d\eta j_l[k(\eta_0 - \eta)] = \\ &= -\frac{1}{k}\left[\Phi'(k,\eta_k) + \Psi'(k,\eta_k)\right] \int_{k(\eta_0-\eta*)}^{0} dx\, j_l(x) \simeq \\ &\simeq \frac{1}{k}\left[\Phi'(k,\eta_k) + \Psi'(k,\eta_k)\right] \int_0^\infty dx\, j_l(x) = \\ &= \frac{1}{k}\left[\Phi'(k,\eta_k) + \Psi'(k,\eta_k)\right] I_l\end{aligned} \tag{9.7}$$

  where in the second passage we performed the change of variables $x = k(\eta_0 - \eta)$, in the third passage we substituted $k(\eta_0 - \eta*)$ with ∞ in the integral limit because we are dealing with large k, and in the fourth passage we have defined the integral

$$I_l \equiv \int_0^\infty dx\, j_l(x) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left[\frac{1}{2}(l+1)\right]}{\Gamma\left[\frac{1}{2}(l+2)\right]} \tag{9.8}$$

  Physically, the photon traverses many wavelengths during the potentials' decay, suffering alternating redshifts and blueshifts: the result is a cancellation of the contributions.

Taking into account this distinction, we can write

$$\begin{aligned} C_l^{ISW} &= \frac{2}{\pi}\int_0^{1/\eta_\Lambda} dkk^2 \left|\Theta_l^{ISW}(k)\right|^2 + \frac{2}{\pi}\int_{1/\eta_\Lambda}^{\infty} dkk^2 \left|\Theta_l^{ISW}(k)\right|^2 = \\ &= C_l^{ISW}(k\eta_\Lambda < 1) + C_l^{ISW}(k\eta_\Lambda > 1) \end{aligned} \tag{9.9}$$

in such a way that in order to calculate $C_l^{ISW}(k\eta_\Lambda < 1)$ we will use the $k \ll \frac{1}{\eta_\Lambda}$ approximation, so equation (9.6), while in order to calculate $C_l^{ISW}(k\eta_\Lambda > 1)$ we will use the $k \gg \frac{1}{\eta_\Lambda}$ approximation, so equation (9.7).
Now we must consider the time evolution of the potentials. First of all, since we are considering matter with no anisotropic stress, we have $\Psi(\eta) = \Phi(\eta)$ for every time η. Second, since we are considering only the late-time contribution to ISW effect, we are dealing with matter and cosmological constant domination: this means that we can take the sound speed of the perturbations to be exactly zero, because $c_s^2 = 0$ for both matter and cosmological constant as we have explained in the previous Chapter. Therefore we can take the potential Φ to obey equation (8.18) and so we have a time evolution

$$\Phi(k,\eta) = A(k)\left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_i}^{\eta} a^2(\tilde{\eta})d\tilde{\eta}\right] \tag{9.10}$$

We immediately note that $A(k) = \Phi(k,\eta_i)$, so taking $\eta_i = \eta*$ and using the well-known (see [32] and [39]) relation $\Phi(k,\eta*) = \frac{9}{10}\Phi(k,0)$ (where $\Phi(k,0)$ is the power spectrum of perturbations generated during inflation), we get

$$\Phi(k,\eta) = \frac{9}{10}\Phi(k,0)\left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta*}^{\eta} a^2(\tilde{\eta})d\tilde{\eta}\right] \tag{9.11}$$

for all scales k.

For $k \ll \frac{1}{\eta_\Lambda}$, equation (9.6), together with the fact that $\Psi = \Phi$, gives us

$$\Theta_l^{ISW}(k) = 2j_l[k(\eta_0 - \eta_\Lambda)]\Delta\Phi(k) \tag{9.12}$$

where

$$\Delta\Phi(k) = \Phi(k,\eta_0) - \Phi(k,\eta*) = -\frac{9}{10}\Phi(k,0)\frac{\mathcal{H}(\eta_0)}{a^2(\eta_0)}\int_{\eta*}^{\eta_0} a^2(\tilde{\eta})d\tilde{\eta} \tag{9.13}$$

Therefore

$$\begin{aligned} C_l^{ISW}(k\eta_\Lambda < 1) &= \frac{2}{\pi}\int_0^{1/\eta_\Lambda} dkk^2 \left|\Theta_l^{ISW}(k)\right|^2 = \\ &= \frac{8}{\pi}\int_0^{1/\eta_\Lambda} dkk^2 j_l^2[k(\eta_0 - \eta_\Lambda)]\left|\frac{9}{10}\Phi(k,0)\frac{\mathcal{H}(\eta_0)}{a^2(\eta_0)}\int_{\eta*}^{\eta_0} a^2(\tilde{\eta})d\tilde{\eta}\right|^2 \end{aligned} \tag{9.14}$$

and taking a generic power-law primordial spectrum $k^3|\Phi(k,0)|^2 = Bk^{n-1}$ (with B and n constant), we obtain

$$C_l^{ISW}(k\eta_\Lambda < 1) = \frac{8}{\pi}B\int_0^{1/\eta_\Lambda} \frac{dk}{k}k^{n-1}j_l^2[k(\eta_0 - \eta_\Lambda)]\left|\frac{9}{10}\frac{\mathcal{H}(\eta_0)}{a^2(\eta_0)}\int_{\eta*}^{\eta_0} a^2(\tilde{\eta})d\tilde{\eta}\right|^2 \tag{9.15}$$

For $k \gg \frac{1}{\eta_\Lambda}$, equation (9.7), together with the fact that $\Psi = \Phi$, gives us

$$\Theta_l^{ISW}(k) = \frac{2}{k}\Phi'(k,\eta_k)I_l \tag{9.16}$$

where

$$\Phi'(k,\eta_k) = \frac{9}{10}\Phi(k,0)\,\frac{d}{d\eta}\left[1-\frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta*}^{\eta} a^2(\tilde{\eta})d\tilde{\eta}\right]\Bigg|_{\eta=\eta_k} = \\ = \frac{9}{10}\Phi(k,0)\tilde{\Phi}'(\eta_k) \tag{9.17}$$

having defined $\tilde{\Phi}(\eta) \equiv 1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta*}^{\eta} a^2(\tilde{\eta})d\tilde{\eta}$.
Since $\eta_k = \eta_0 - \frac{1}{k}\left(l+\frac{1}{2}\right)$ is very close to η_0 if k is large, we can expand $\tilde{\Phi}'(k,\eta_k)$ around η_0

$$\tilde{\Phi}'(\eta_k) \simeq \tilde{\Phi}'(\eta_0) + \tilde{\Phi}''(\eta_0)(\eta_k-\eta_0) = \\ = \tilde{\Phi}'(\eta_0) - \frac{1}{k}\left(l+\frac{1}{2}\right)\tilde{\Phi}''(\eta_0) \tag{9.18}$$

so

$$\left|\tilde{\Phi}'(\eta_k)\right|^2 \simeq \left|\tilde{\Phi}'(\eta_0)\right|^2 - \frac{2}{k}\left(l+\frac{1}{2}\right)\tilde{\Phi}'(\eta_0)\tilde{\Phi}''(\eta_0) + \\ + \frac{1}{k^2}\left(l+\frac{1}{2}\right)^2\left|\tilde{\Phi}''(\eta_0)\right|^2 \tag{9.19}$$

Moreover we can assume that the potentials start to change in time for $\eta > \eta_\Lambda$, when the cosmological constant becomes important. Therefore, for the modes $k \gg \frac{1}{\eta_\Lambda}$ we can impose $\eta_k > \eta_\Lambda$ and, remembering the definition of η_k, we can substitute the lower limit of the integral $C_l^{ISW}(k\eta_\Lambda > 1)$ with the quantity $\tilde{k} = \frac{l+1/2}{\eta_0-\eta_\Lambda}$ (see [39]).
So we have

$$C_l^{ISW}(k\eta_\Lambda > 1) = \frac{2}{\pi}\int_{\tilde{k}}^{\infty} dk k^2\left|\Theta_l^{ISW}(k)\right|^2 = \\ = \frac{8}{\pi}\left(\frac{9}{10}\right)^2 I_l^2\int_{\tilde{k}}^{\infty} dk\,|\Phi(k,0)|^2\left|\tilde{\Phi}'(\eta_k)\right|^2 = \\ = \frac{8}{\pi}\left(\frac{9}{10}\right)^2 I_l^2\int_{\tilde{k}}^{\infty} dk\,|\Phi(k,0)|^2\left[\left|\tilde{\Phi}'(\eta_0)\right|^2 + \right. \\ \left. - \frac{2}{k}\left(l+\frac{1}{2}\right)\tilde{\Phi}'(\eta_0)\tilde{\Phi}''(\eta_0) + \frac{1}{k^2}\left(l+\frac{1}{2}\right)^2\left|\tilde{\Phi}''(\eta_0)\right|^2\right] \tag{9.20}$$

and taking a generic power-law primordial spectrum $k^3|\Phi(k,0)|^2 = Bk^{n-1}$ we obtain

$$C_l^{ISW}(k\eta_\Lambda > 1) = \frac{8}{\pi}\left(\frac{9}{10}\right)^2 I_l^2 B\left[\int_{\tilde{k}}^{\infty} dk k^{n-4}\left|\tilde{\Phi}'(\eta_0)\right|^2 + \right. \\ - 2\left(l+\frac{1}{2}\right)\int_{\tilde{k}}^{\infty} dk k^{n-5}\tilde{\Phi}'(\eta_0)\tilde{\Phi}''(\eta_0) + \\ \left. + \left(l+\frac{1}{2}\right)^2\int_{\tilde{k}}^{\infty} dk k^{n-6}\left|\tilde{\Phi}''(\eta_0)\right|^2\right] = \\ = \frac{8}{\pi}\left(\frac{9}{10}\right)^2 I_l^2 B\left[\frac{1}{3-n}\left(\frac{\eta_0-\eta_\Lambda}{l+1/2}\right)^{3-n}\left|\tilde{\Phi}'(\eta_0)\right|^2 + \right. \\ - 2\left(l+\frac{1}{2}\right)\frac{1}{4-n}\left(\frac{\eta_0-\eta_\Lambda}{l+1/2}\right)^{4-n}\tilde{\Phi}'(\eta_0)\tilde{\Phi}''(\eta_0) + \\ \left. + \left(l+\frac{1}{2}\right)^2\frac{1}{5-n}\left(\frac{\eta_0-\eta_\Lambda}{l+1/2}\right)^{5-n}\left|\tilde{\Phi}''(\eta_0)\right|^2\right] \tag{9.21}$$

where the integrals converge if $n < 3$. Using the fact that for $l \gg 1$

$$I_l^2 = \frac{\pi}{4}\left(\frac{\Gamma\left[\frac{1}{2}(l+1)\right]}{\Gamma\left[\frac{1}{2}(l+2)\right]}\right)^2 \propto \frac{1}{l} \tag{9.22}$$

and considering a Harrison-Zeldovich primordial power spectrum ($n = 1$), we find that for large scales (and so $l \gg 1$)

$$C_l^{ISW} \propto \frac{1}{l^3} \tag{9.23}$$

9.1.2 CDM and PFDE background

If we consider a universe that at the time of recombination is dominated by cold dark matter and that at late times becomes dominated by a perfect fluid dark energy with EOS w_{DE}, the temporal scale for the variation of the potentials Φ and Ψ is given by the conformal time $\eta_{1/2}$ at which the effective equation of state of the universe becomes $w = -1/2$: it is easy to show (see [39] and [38]) that in the case of a cosmological constant ($w_{DE} = -1$) we have $\eta_{1/2} = \eta_\Lambda$. Therefore, assuming also that CDM and the PFDE are free from anisotropic stress, so $\Psi = \Phi$, we get the same two cases of the previous Section

$$\Theta_l^{ISW}(k) = 2j_l[k(\eta_0 - \eta_{1/2})]\Delta\Phi(k) \qquad \text{if} \qquad k \ll \frac{1}{\eta_{1/2}} \tag{9.24}$$

$$\Theta_l^{ISW}(k) = \frac{2}{k}\Phi'(k, \eta_k)I_l \qquad \text{if} \qquad k \gg \frac{1}{\eta_{1/2}} \tag{9.25}$$

and so

$$\begin{aligned} C_l^{ISW} &= \frac{2}{\pi}\int_0^{1/\eta_{1/2}} dkk^2 \left|\Theta_l^{ISW}(k)\right|^2 + \frac{2}{\pi}\int_{1/\eta_{1/2}}^{\infty} dkk^2 \left|\Theta_l^{ISW}(k)\right|^2 = \\ &= C_l^{ISW}(k\eta_{1/2} < 1) + C_l^{ISW}(k\eta_{1/2} > 1) \end{aligned} \tag{9.26}$$

Since we are dealing with cold dark matter and dark energy, we can take again the speed of sound to be $c_s^2 = 0$, so the time evolution of the potential is given again by equation (8.34) for all scales k:

$$\Phi(k, \eta) = A(k)\left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_i}^{\eta} a^2(\tilde{\eta})d\tilde{\eta}\right] \tag{9.27}$$

This means that the time evolution of the potential Φ is completely determined by the background expansion history, in the same identical way for every CDM-PFDE universe, also if the PFDE is represented by a cosmological constant. Now we must remember that we have strong observational constraints upon the background expansion history of the universe, and the time evolution of the potential Φ will be the same if this observed expansion history is caused by the domination of a cosmological constant or a generic perfect fluid dark energy with $w_{DE} \neq -1$. Therefore the ΛCDM and CDM-PFDE models predict the same time evolution for Φ as long as the observations cannot distinguish which one describes the right background expansion history, and consequently they predict the same late-time ISW effect.

9.2 ISW effect in mimetic cubic Horndeski gravity

In this Section we consider mimetic cubic Horndeski gravity, since in the particular case when $G_4 = 1/2$ and $G_5 = 0$ we have found the fully analytical solution for the time evolution of the potential Φ to be, for both ΛCDM and CDM-PFDE backgrounds,

$$\Phi(k,\eta) = A(k)\left[1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_i}^{\eta} a^2(\tilde{\eta})d\tilde{\eta}\right] \tag{9.28}$$

Independently on the exact explanation for the origin of it (ΛCDM or CDM-PFDE), if we impose that our theory should reproduce the observed background expansion history, then the time evolution of Φ will be exactly the same in the two cases and it will be identical to what we saw in the previous Section for general relativity. Moreover, the existence of the temporal scale $\eta_{1/2}$ is a direct consequence of the background expansion history, so it is granted also in mimetic cubic Horndeski gravity if we impose ΛCDM or CDM-PFDE background as we did in the previous Chapter.
Remembering also that in mimetic cubic Horndeski we have $\Psi = \Phi$ if matter is absent, we can repeat the same identical procedure of the previous Section to find that mimetic cubic Horndeski predicts the same late-time ISW effect of general relativity.

9.3 ISW in generic mimetic Horndeski gravity

If we consider a generic mimetic Horndeski theory (without fixing $G_4 = 1/2$ and $G_5 = 0$), we know that the time evolution of the potential Φ is described by equation (8.60)

$$\Phi'' + \left[\frac{B_2}{B_3} + \left(\log\frac{B_3}{B_1}\right)' + \mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right]\Phi' + \left[\frac{B_1}{B_3}\varphi_0' + \frac{B_1}{B_3}\left(\frac{B_2}{B_1}\right)' + \frac{B_2}{B_3}\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\right]\Phi = 0 \tag{9.29}$$

and if we impose a ΛCDM or CDM-PFDE background expansion history it becomes

$$\Phi'' + \left[\frac{B_2}{B_3} + \left(\log\frac{B_3}{B_1}\right)'\right]\Phi' + \left[\frac{B_1}{B_3}\varphi_0' + \frac{B_1}{B_3}\left(\frac{B_2}{B_1}\right)'\right]\Phi = 0 \tag{9.30}$$

where the explicit expression of the functions B_i depends on the particular background chosen, as we have shown in Sections 8.4 and 8.5.
Equation (9.30) cannot be solved analytically, so the only option in order to find the time evolution of the potential is to use numerical calculations. It seems legit to suppose that this time evolution could present a temporal scale $\eta_{1/2}$ given by the background expansion history, like in the mimetic cubic Horndeski case, but this hypothesis should be confirmed finding a numerical solution of equation (9.30): if it proves to be correct, it would be possible to use equation (9.9) also in this case, allowing us to perform different calculations for large and small scales k.
Besides of the fact that the time evolution of the potential Φ could be significantly different from general relativity, there is another reason that makes us suppose that mimetic Horndeski gravity could predict a different late-time ISW effect from Einstein's theory: in fact we should remember that, in mimetic Horndeski, first-order perturbations obey equation (8.57)

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0 \tag{9.31}$$

that, in general, implies the existence of a gravitational slip such that $\Psi(\eta) \neq \Phi(\eta)$.
Therefore we expect that only a numerical solution of the full system of equations (8.57)-(8.59) could give us the time evolution of the potentials Φ and Ψ, and finally allow us to calculate the late-time ISW effect in the most general mimetic Horndeski theory.

Chapter 10

Conclusions

In this Thesis we considered the modification of general relativity given by mimetic Horndeski gravity: in fact, we know that the cosmological solutions of Einstein's theory need, in order to be consistent with observations, the existence of dark matter and dark energy. Since these two dark components have never been observed so far, and considering that they present also theoretical problems (such as the *cosmological constant problem* or the *small-scale problem*), many alternative theories have been proposed, trying to explain the same phenomena without introducing new energy sources: mimetic Horndeski gravity is one of these theories.
Mimetic Horndeski gravity is a modification of Horndeski model, which is the most general 4-D local covariant scalar-tensor theory that can be derived from an action and that has second-order equations of motion for both the metric and the scalar field: the mimetic modification can be obtained simply by imposing the constraint $b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1$ on the scalar field φ, with $b(\varphi)$ being a free potential function.
We showed that mimetic Horndeski gravity can easily reproduce the observed background expansion history of the late-time universe, compatible with the ΛCDM or CDM-PFDE models of general relativity: it is sufficient to impose a zero-order constraint on the functions $K(X_0, \varphi_0)$ and $G_i(X_0, \varphi_0)$ that define the particular theory inside the Horndeski class. Given this, it makes sense to investigate other predictions of mimetic Horndeski gravity, and we focused on the late-time integrated Sachs-Wolfe effect.
The late-time ISW effect is a particular source of spectral distortion in the CMB radiation: the photons propagating from the last scattering surface to us encounter time-varying metric perturbations (the potentials Φ and Ψ), and therefore get blue-shifted or red-shifted. Since different theories of gravity predict different time evolution of the potentials, they also predict different late-time ISW effects.
We considered two cases, imposing first a ΛCDM and then a CDM-PFDE background expansion history. Since the equations for the time evolution of the potentials were too complicated to be solved analytically, we restricted the discussion to the case of mimetic cubic Horndeski gravity: what we found is that in this particular case the time evolution of the potentials is the same as in general relativity, since it is determined in the same way by the background expansion history. This means that mimetic cubic Horndeski gravity predicts the same late-time ISW effect of general relativity.
The most general case of mimetic Horndeski gravity could be studied with a numerical solution of the time evolution equations for the potentials. We expect this more general model to predict a late-time ISW effect that is different from general relativity, because it is present a gravitational slip ($\Psi \neq \Phi$) and because the equations for the time evolution of the potentials are different from the ones in Einstein's theory.

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