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Quantitative Estimates for the Singular Strata of Minimizing Harmonic Maps

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Introduction

The context. The problem of minimizing the Dirichlet energy functional with a given boundary condition is one of the most classical and widely known problems of the calculus of variations: for a fixed open domain $\Omega \subset \mathbb{R}^m$ and a map u_0 defined on $\partial\Omega$, one can investigate existence, uniqueness and properties of minimizers for

$$\mathcal{E}(u) \doteq \int_\Omega |\nabla u(x)|^2 \, dx;$$

the minimization is done among functions satisfying $u = u_0$. One first task to carry out is to identify a suitable space for the functions u: the most natural choice turns out to be the Sobolev space $W^{1,2}(\Omega, \mathbb{R})$, consisting of L^2 functions with distributional derivatives in L^2. In this setting, one can prove that:

- A minimum exists, through the direct method of calculus of variations; *i.e.*, by exploiting the Weierstrass Theorem and the lower semicontinuity of the functional $\mathcal{E}$;
- The minimum is unique, by exploiting the convexity of the energy functional.

A complete treatise of this topic can be found, for example, in [Dac14, Section 3.2]. Moreover, it is easy to see that the Dirichlet minimizer satisfies the Laplace Equation $\Delta u = 0$ in the weak sense: this is done by considering the family of variations

$$t \longmapsto u_t \doteq u + t\varphi$$

for a smooth map φ, and then imposing

$$\frac{d}{dt}\mathcal{E}(u_t) = 0.$$

Then, thanks to the Weyl Lemma (see [Dac14, Section 4.3]), this implies that the minimizers of the energy functional are actually smooth (or coincide a.e. with smooth functions), and satisfy $\Delta u = 0$ in the classical sense. Finally, one can notice that considering functions that take values in $\mathbb{R}^N$ doesn't really produce remarkable changes: the definition of the Dirichlet energy requires now the Hilbert-Schmidt norm of ∇u, but then the same techniques can be applied to show existence, uniqueness and analiticity of minimizers.

The regularity problem, however, changes radically if we constrain our mappings to take values in some n-dimensional manifold $\mathcal{N}$ embedded in $\mathbb{R}^N$. In this case, when we try to deduce the validity of a Euler-Lagrange equation, we are not allowed anymore to consider all possible variations: we need to restrict to maps that still take values in $\mathcal{N}$. In particular, two types of variations are examined:

- **External variations**: we consider, for $t \in (-\varepsilon, \varepsilon)$ and a test function $\varphi \in C_0^\infty(\Omega, R^N)$, the family

$$t \longmapsto u_t = \pi_{\mathcal{N}} \circ (u + t\varphi),$$

  where $\pi_{\mathcal{N}}$ is the projection on the manifold $\mathcal{N}$. Critical points of $t \mapsto \mathcal{E}(u_t)$ are called **weakly harmonic maps** and satisfy the Euler Lagrange equation

$$\Delta u + A(u)\,(\nabla u, \nabla u) = 0 \quad \text{in } \Omega \tag{ELwh}$$

  in the weak sense, where A is the second fundamental form of $\mathcal{N}$.

- **Internal variations**: here variations of the type

$$t \longmapsto u \circ \phi_t$$

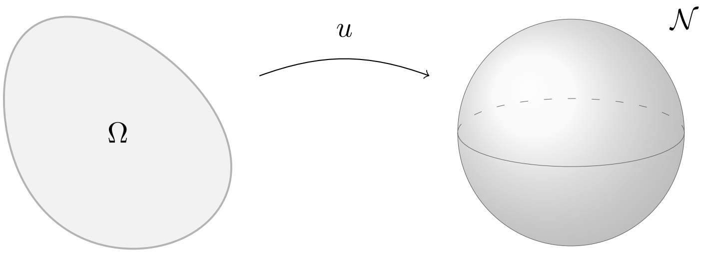
  are taken in consideration, where ϕ_t is a family of diffeomorphisms of the domain fixing the boundary. Critical points of $t \mapsto \mathcal{E}(u_t)$ which are also weakly harmonic are called **stationary harmonic maps**, and satisfy the Euler Lagrange equation

$$\operatorname{div}\left[2\langle \nabla_h u, \nabla u\rangle - |\nabla u|^2 e_h\right] = 0 \quad \text{for all } h = 1, \ldots, m. \tag{ELsh}$$

  in the weak sense.

The first part of this thesis (Chapter 1) presents these concepts in a more well-structured way, and introduces two fundamental tools for their analysis: the Monotonicity Formula and the ε-Regularity Theorem.

The results. At this point, we are interested to find out if something can be said about regularity of these particular harmonic maps. Unfortunately, weak harmonicity alone is not enough for obtaining general results: actually, as proved by Rivière in [Riv95], one can even find weakly harmonic maps into the sphere $\mathbb{S}^n$ that are everywhere discontinuous. On the contrary, for stationary maps and energy minimizing maps the situation is way better: results in this direction are the main content of the well-known paper of



Schoen and Uhlenbeck [SU82] and of the more recent articles [CN13] and [NV17]. The work of this thesis aims at a profound study of the arguments developed in the latter two articles. Notice however that results as strong as in the unconstrained case are not achievable: we'll see that the map defined almost everywhere as

$$\begin{aligned} \mathbb{R}^N \times \mathbb{R}^h &\longrightarrow \mathbb{S}^{N-1} \\ (x, y) &\longmapsto \frac{x}{|x|} \end{aligned}$$

is energy minimizing in $B_1^N(0) \times B_1^h(0)$ when $N \geq 3$ and $h \geq 0$, but it clearly has a h-dimensional subspace of singularities.

What one hopes to achieve is some valid upper bound on the dimension and the measure of the *singular set*

$$\mathcal{S}(u) \doteq \{x \in \Omega \mid u \text{ is not continuous at } x\}\,.$$

In the already mentioned work of Schoen and Uhlenbeck, the estimate $\dim_{\mathscr{H}}(\mathcal{S}(u)) \leq m - 3$ is obtained, where we denote by $\dim_{\mathscr{H}}$ the Hausdorff dimension of a set; in some sense, the situation can not be worse than in the example we have just presented. The paper of Cheeger and Naber [CN13] shows more: actually, a Minkowski-type estimate can be achieved.

Theorem 1 (Cheeger, Naber 2013)**.** *Assume u is an energy minimizing map from $B_2(0) \subset \mathbb{R}^m$ to $\mathcal{N}$ with total energy bounded by Λ. For any $\delta > 0$ there exists a constant $C_{CN}(m, \mathcal{N}, \Lambda, \delta)$ such that for all $0 < r < 1$*

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)\right) \leq C_{CN} r^{3-\delta}.$$

As a consequence, the Minkowski dimension of $\mathcal{S}(u)$ is at most $m - 3$.

The proof of this theorem is the main content of Chapter 2. Exploiting techniques that are far more advanced, a substantial improvement of this result has been reached by Naber and Valtorta in [NV17]: not only can we remove the term δ from the estimate, but we also can obtain some strong information about the structure of $\mathcal{S}(u)$:

Theorem 2 (Naber, Valtorta 2017)**.** *Assume u is an energy minimizing map from $B_2(0) \subset \mathbb{R}^m$ to $\mathcal{N}$ with total energy bounded by Λ. There exists a constant $C_{NV}(m, \mathcal{N}, \Lambda)$ such that for all $0 < r < 1$*

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)\right) \leq C_{NV} r^3.$$

In particular, the $(m-3)$-dimensional Minkowski content (and thus the Hausdorff measure) of $\mathcal{S}(u)$ is at most C_{NV}. Moreover, $\mathcal{S}(u)$ is $(m-3)$-rectifiable.

In Chapter 3, we first present all the tools necessary to the proof of this theorem, and then develop the complete proof.

The tools. The approach introduced in [SU82] suggests to *stratify* the singular set as follows: the k^{th} stratum $\mathcal{S}^k(u)$ of $\mathcal{S}(u)$ contains by definition the points in which u is locally at most k-symmetric; here with k-symmetry we mean homogeneity and translational invariance with respect to to a k-subspace. What is actually done in the more recent papers – and in this thesis – is to consider a *quantitative* stratification of the singular set, based on a notion of almost-symmetry. For $\eta > 0$ and $r > 0$ we say that u is (η, r, k)-symmetric at x if u is η-close to a k-symmetric map in $B_r(x)$ (in a L^2 sense); that is, for a k symmetric map g,

$$\fint_{B_r(x)} |g - u|^2 < \eta.$$

Then, fixing η and r, one can define (informally):

$$\mathcal{S}^k_{\eta,r}(u) \doteq \left\{ x \in \Omega \;\middle|\; \begin{array}{c} \text{for all } r \leq s \leq 1,\ u \text{ is } \textit{at most} \\ (\eta, s, k)\text{-symmetric at } x \end{array} \right\}.$$

The actual result proved by Naber and Valtorta is the following:

Theorem 3. *There exists a constant $C'_{NV}(m, \mathcal{N}, \Lambda, \eta)$ such that: if u is stationary harmonic and has energy bounded by Λ, then for all $0 < r < 1$*

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) \cap B_1(0)\right) \leq C'_{NV} r^{m-k}.$$

Moreover $\bigcap_r \mathcal{S}^k_{\eta,r}(u)$ is k-rectifiable.

An essential notion to be introduced is that of *normalized energy* of a map u: it is widely used both for proving Theorem 3 and for obtaining the results about $\mathcal{S}(u)$ as corollaries of Theorem 3. For $x \in \Omega$ and $r > 0$ we define

$$\theta(x, r) \doteq r^{2-m} \int_{B_r(x)} |\nabla u|^2 .$$

Then, two very important facts can be proved:

- **Monotonicity Formula**: for all $x \in \Omega$ and almost all r,

$$\frac{d}{dr}\theta(x,r) = 2r^{2-m} \int_{\partial B_r(x)} \left|\partial_{r_x(y)} u(y)\right|^2 \, d\sigma(y).$$

  As an easy consequence, homogeneity of u at x can be characterized in terms of $\theta(x,\cdot)$. Since k-symmetry is implied by homogeneity at $k+1$ linearly independent points, also k-symmetry is closely related to θ. Finally, this will imply that almost-k-symmetry can be deduced by a "pinching condition" of the type

$$\theta(x_i, r) - \theta(x_i, s) < \varepsilon,$$

  valid for $k+1$ points x_i and with ε small enough. This will be crucial for the proof of Theorem 3.

- **ε-Regularity**: a fundamental result of Schoen and Uhlenbeck assures that, if u is minimizing and $\theta(x,r) < \varepsilon_0$ for a suitable $\varepsilon_0 > 0$, then u is smooth in a neighborhood of x. Developing this argument, one can show that, if u is minimizing and η is small enough,

$$\mathcal{S}(u) \subset \bigcap_{r>0} \mathcal{S}^{m-3}_{\eta,r}(u);$$

  and this, together with Theorem 3, readily implies Theorems 1 and 2.

These key results are the building bricks of the proofs we perform in this thesis. Other more advanced techniques are then presented in Chapter 3, such as a suitable form of the Reifenberg Theorem and some results concerning the L^2-approximation of measures with k-planes.

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Chapter 1

Preliminaries on harmonic maps

Notation. The main objects of our study will be mappings that take values in a smooth manifold. From now on, $\mathcal{N}$ will be a compact n-dimensional Riemannian manifold with no boundary. By a very well known result of Nash (see [Nas54]), any such manifold admits an isometric embedding into a Euclidean space $\mathbb{R}^N$ for some N: accordingly, we'll always assume without loss of generality that $\mathcal{N}$ is embedded in $\mathbb{R}^N$.

Such restrictions are not required for the domain of our mappings, thus we could assume it to be a smooth Riemannian manifold $\mathcal{M}$, possibly non-compact and with a boundary. However, since most of the concepts and results we'll give are *local*, we'll usually choose as a domain an open bounded set $\Omega \subset \mathbb{R}^m$ with smooth boundary (and frequently the m-dimensional ball $B_2(0)$); we'll state explicitly when other choices are done. Moreover, since the results will be valid only for $m \geq 3$, from now on this will be the only case we consider; for lower dimensions, other techniques are available (see [HW08, Paragraph 4.2]).

Our aim is to study maps which minimize an energy functional under appropriate perturbations. These will be what we call harmonic maps; precise definition are given in the following sections. We'll consider maps in the Sobolev space $W^{1,2}(\Omega, \mathcal{N})$, where we define

$$W^{1,2}(\Omega, \mathcal{N}) \doteq \left\{ u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right) \mid u(x) \in \mathcal{N} \text{ for almost all } x \in \Omega \right\}.$$

Notice that this is *not* a vector space in general. Moreover, we already give the definition of the energy functional we want to minimize:

Definition 1.1 (Energy functional)**.** For every map $u \in W^{1,2}(\Omega, \mathcal{N})$, we define the functional

$$\mathcal{E}(u) \doteq \int_\Omega |\nabla u|^2 \, dx.$$

Remark. By classical results on reflexive spaces, we have that any bounded sequence of functions in $W^{1,2}(\Omega, \mathbb{R}^N)$ has a subsequence that converges weakly in $W^{1,2}(\Omega, \mathbb{R}^N)$. Moreover, the Rellich-Kondrashov Embedding Theorem assures that in our setting (dimension $m \leq 3$ and Sobolev exponent $p = 2$) the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^N)$ is compactly embedded in $L^2(\Omega, \mathbb{R}^N)$. Thus any bounded sequence of functions in $W^{1,2}(\Omega, \mathbb{R}^N)$ has a subsequence that converges strongly in $L^2(\Omega, \mathbb{R}^N)$; a further subsequence then converges almost everywhere: if the original sequence was in $W^{1,2}(\Omega, \mathcal{N})$, also the limit map will belong to this set. Finally, notice that by the compactness assumption on $\mathcal{N}$ any map in $W^{1,2}(\Omega, \mathcal{N})$ has L^∞-norm (and thus L^2-norm) uniformly bounded by a constant. Thus we can state explicitly a first useful result about the mappings we're working with.

Theorem 1.1. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of maps in $W^{1,2}(\Omega, \mathcal{N})$ with $\mathcal{E}(u_j) \leq \Lambda$, where $\Lambda > 0$ is a constant. Then $\{u_j\}_{j\in\mathbb{N}}$ admits a subsequence that converges weakly in $W^{1,2}\Big(\Omega, \mathbb{R}^N\Big)$, strongly in $L^2\Big(\Omega, \mathbb{R}^N\Big)$ and almost everywhere to a map $\bar{u} \in W^{1,2}(\Omega, \mathcal{N})$.*

For all the classical results we exploited (and for a rigorous definition of the Sobolev spaces involved), we refer to the book [Dac14]. As in the previous theorem, from now on Λ will always represent an upper bound for the energy functional.

Remark. It should be clear that, in this context and with this notation, ∇u is *not* referring to a *vector* depending on x, but instead to a “matrix”. Moreover, the norm we are using on the space of matrices is the Hilbert-Schmidt norm: accordingly, $|\nabla u|^2$ is actually the sum

$$|\nabla u(x)|^2 = \sum_{i=1}^{m} \sum_{\alpha=1}^{N} \left(\frac{\partial u^\alpha}{\partial x^i}(x) \right)^2 .$$

1.1 External variation

As a first type of perturbations for the energy functional, we consider the most classical and natural ones, namely variations in the target space: given a map $u \in W^{1,2}(\Omega, \mathcal{N})$, we'd like to consider $t \mapsto u + t\varphi$ for some smooth function φ. Unfortunately, this family of mappings does not belong, in general, to $W^{1,2}(\Omega, \mathcal{N})$: we have to work around this problem to get a precise definition.

Define, for any $\varrho > 0$ and for any set $S \subset \mathbb{R}^N$, the ϱ-neighborhood (or ϱ-fattening) of S:

$$\mathcal{B}_\varrho(S) = \Big\{ y \in \mathbb{R}^N \mid \operatorname{dist}(y, S) < \varrho \Big\} .$$

By classical results on tubular neighborhoods of smooth and compact submanifolds of $\mathbb{R}^N$ (see for example [Hir94, Section 4.5]), there exists a constant

$\bar{\varrho} > 0$ with the following property: for any $y \in \mathcal{B}_{\bar{\varrho}}(\mathcal{N})$ there exists a unique point $\pi_{\mathcal{N}}(y) \in \mathcal{N}$ such that

$$|y - \pi_{\mathcal{N}}(y)| = \operatorname{dist}(y, \mathcal{N}).$$

We call *nearest point projection* the map $\pi_{\mathcal{N}} : \mathcal{B}_{\bar{\varrho}}(\mathcal{N}) \to \mathcal{N}$. Choosing $\bar{\varrho}$ sufficiently small, $\pi_{\mathcal{N}}$ is a C^∞ function between manifolds. Consider again a map $u \in W^{1,2}(\Omega, \mathcal{N})$ and a smooth and compactly supported function $\varphi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right)$; define, for $t \in \mathbb{R}$ (sufficiently small), the map $w_t \doteq u + t\varphi$. Clearly, w_t does not, in general, map Ω to $\mathcal{N}$; however, for $|t|$ small, $w_t(\Omega) \subset \mathcal{B}_{\bar{\varrho}}(\mathcal{N})$. In particular, for $|t|$ small the composition $u_t \doteq \pi_{\mathcal{N}} \circ (u + t\varphi)$ is well defined and in $W^{1,2}(\Omega, \mathcal{N})$. So the upcoming definition makes sense:

Definition 1.2 (Weakly harmonic maps)**.** A map $u \in W^{1,2}(\Omega, \mathcal{N})$ is called **weakly harmonic** if for any test function $\varphi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right)$ the following holds:

$$\frac{d}{dt}\mathcal{E}(u_t)\bigg|_{t=0} = \frac{d}{dt}\int_\Omega \left|\nabla\left(\pi_{\mathcal{N}} \circ (u + t\varphi)\right)(x)\right|^2\, dx\bigg|_{t=0} = 0.$$

The Euler-Lagrange equation. Developing the integrand that appears in the above definition, we get:

$$\begin{aligned}
&\left|\nabla\left(\pi_{\mathcal{N}} \circ (u + t\varphi)\right)(x)\right|^2 = \\
&\quad= \sum_{i=1}^m \left|\nabla\pi_{\mathcal{N}}[u + t\varphi]\left(\nabla_i(u + t\varphi)\right)\right|^2 = \\
&\quad= \sum_{i=1}^m \left|\nabla\pi_{\mathcal{N}}[u + t\varphi]\left(\nabla_i u\right)\right|^2 + \\
&\qquad\qquad + 2t\Big\langle \nabla\pi_{\mathcal{N}}[u + t\varphi]\left(\nabla_i u\right), \nabla\pi_{\mathcal{N}}[u + t\varphi]\left(\nabla_i \varphi\right) \Big\rangle + \\
&\qquad\qquad\qquad + t^2\left|\nabla\pi_{\mathcal{N}}[u + t\varphi]\left(\nabla_i\varphi\right)\right|^2
\end{aligned}$$

Then derivating in the variable t and evaluating at $t = 0$ we find, after taking the derivative inside the integral:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u_t)\bigg|_{t=0} = \sum_{i=1}^m \int_\Omega 2\,\Big\langle &\operatorname{Hess}\pi_{\mathcal{N}}(u(x))\left(\nabla_i u, \varphi\right) + \\
&+ \nabla\pi_{\mathcal{N}}\left(\nabla_i\varphi\right), \nabla\pi_{\mathcal{N}}\left(\nabla_i u\right) \Big\rangle\, dx. \qquad (1.1)
\end{aligned}$$

Here $\nabla\pi_{\mathcal{N}}$ is intended to be computed at the point $u(x)$, while $\nabla_i u, \varphi$ and $\nabla_i\varphi$ are computed at x. To obtain a cleaner expression for this integral, we need a definition and a lemma; we refer to [Mos05, Section 3.1] for the details.

Definition 1.3 (Second fundamental form). We define the **second fundamental form** of the submanifold $\mathcal{N}$ of $\mathbb{R}^N$ as the unique section A of the tensor space $T^\star\mathcal{N} \otimes T^\star\mathcal{N} \otimes (T\mathcal{N})^\perp$ such that

$$\langle \nu, A(x)(X,Y)\rangle = \langle \nabla_X \nu, Y\rangle$$

for every $x \in \mathcal{N}$, every tangent vector fields $X, Y \in \mathcal{T}(\mathcal{N})$ and every normal section $\nu : \mathcal{N} \to T(\mathbb{R}^N)\big|_{\mathcal{N}}$ (*i.e.*, a section satisfying $\nu(y) \in (T_y\mathcal{N})^\perp$ for any $y \in \mathcal{N}$). Here ∇_X is the covariant derivative in $\mathbb{R}^N$ in direction X.

Notice that, by the orthogonality of ν and Y, the second fundamental form also satisfies

$$\langle \nu, A(x)(X,Y)\rangle = X\langle \nu, Y\rangle - \langle \nu, \nabla_X Y\rangle = -\langle \nu, \nabla_X Y\rangle .$$

Lemma 1.2 (Derivatives of $\pi_{\mathcal{N}}$). *The following assertions hold:*

(i) Consider $\pi_{\mathcal{N}} : B_{\bar{\varrho}}(\mathcal{N}) \to \mathcal{N}$ and a point x belonging to the manifold $\mathcal{N}$. Then the first derivative $\mathrm{d}\,\pi_{\mathcal{N}}(x)$ coincides with the orthonormal projection on the tangent space $T_x\mathcal{N}$.

(ii) For every $x \in \mathcal{N}$, every couple of tangent vector fields X, Y and every normal section ν, the following identities hold:

$$\operatorname{Hess}\pi_{\mathcal{N}}(x)\,(X,Y) = -A(x)(X,Y) \tag{1.2}$$

$$\langle Y, \operatorname{Hess}\pi_{\mathcal{N}}(x)\,(X,\nu)\rangle = -\langle \nu, A(x)\,(X,Y)\rangle . \tag{1.3}$$

As a consequence, at any point A is a symmetric bilinear form.

Now consider again Equation (1.1). Notice that for any i the vector $\nabla_i u(x)$ belongs to the tangent space $T_{u(x)}\mathcal{N}$, so by the previous lemma $\nabla\pi_{\mathcal{N}}(u(x))[\nabla_i u(x)] = \nabla_i u(x)$. Moreover, since $\nabla_i u(x)$ is orthogonal to $\left(T_{u(x)}\mathcal{N}\right)^\perp$, we have:

$$\begin{aligned}\langle \nabla\pi_{\mathcal{N}}(u(x))[\nabla_i\varphi(x)], \nabla_i u(x)\rangle &= \left\langle \pi_{T_{u(x)}\mathcal{N}}[\nabla_i\varphi(x)], \nabla_i u(x)\right\rangle = \\ &= \left\langle \pi_{T_{u(x)}\mathcal{N}}[\nabla_i\varphi] + \pi_{\left(T_{u(x)}\mathcal{N}\right)^\perp}[\nabla_i\varphi], \nabla_i u(x)\right\rangle = \\ &= \langle \nabla_i\varphi(x), \nabla_i u(x)\rangle .\end{aligned}$$

In view of this observation, the equation $\frac{d}{dt}\mathcal{E}(u_t)\Big|_{t=0} = 0$ becomes (removing an irrelevant coefficient 2):

$$\sum_{i=1}^m \int_\Omega \langle \operatorname{Hess}\pi_{\mathcal{N}}(u(x))\,(\nabla_i u, \varphi)\,, \nabla_i u\rangle + \langle \nabla_i\varphi, \nabla_i u\rangle\; dx = 0. \tag{1.4}$$

We decompose φ as into its perpendicular and tangent parts with respect to $T_{u(x)}\mathcal{N}$: that is, $\varphi = \varphi^{\perp} + \varphi^{\top}$, where

$$\varphi^{\top}(x) = \mathrm{d}\,\pi_{\mathcal{N}}(u(x))[\varphi(x)] \qquad \text{and} \qquad \varphi^{\perp}(x) = \varphi(x) - \varphi^{\top}(x).$$

Then we have, by Lemma 1.2 (recall that $A(X,Y)$ is orthogonal to $\mathcal{N}$ whenever X and Y are tangent vector fields):

$$\begin{aligned}
&\left\langle \operatorname{Hess}\pi_{\mathcal{N}}(u(x))\left(\nabla_i u, \varphi^{\top}\right), \nabla_i u\right\rangle = 0\\
&\left\langle \operatorname{Hess}\pi_{\mathcal{N}}(u(x))\left(\nabla_i u, \varphi^{\perp}\right), \nabla_i u\right\rangle = -\left\langle \varphi^{\perp}, A(u(x))\left(\nabla_i u(x), \nabla_i u(x)\right)\right\rangle,
\end{aligned}$$

and the last term of the second equation is also equal to

$$-\left\langle \varphi, A(u(x))\left(\nabla_i u(x), \nabla_i u(x)\right)\right\rangle,$$

again by orthogonality. Then finally, Equation (1.4) becomes

$$\sum_{i=1}^{m}\int_{\Omega}\left\langle \nabla_i\varphi, \nabla_i u\right\rangle - \left\langle \varphi, A(u(x))\left(\nabla_i u(x), \nabla_i u(x)\right)\right\rangle\, dx = 0, \tag{1.5}$$

or, more compactly,

$$\int_{\Omega}\left\langle \nabla\varphi, \nabla u\right\rangle - \left\langle \varphi, A(u(x))\left(\nabla u(x), \nabla u(x)\right)\right\rangle\, dx = 0. \tag{1.6}$$

This in turn is equivalent to the differential equation

$$\Delta u = A(u)(\nabla u, \nabla u) \tag{1.7}$$

in the distributional sense.

Second Fundamental Form of the Sphere. In order to give an explicit example, we perform here the computation in the case $\mathcal{N} = \mathbb{S}^{N-1}$, where $\mathbb{S}^{N-1}$ is the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$:

$$\mathbb{S}^{N-1} \doteq \left\{ x \in \mathbb{R}^N \,\middle|\, |x| = 1 \right\}.$$

The fastest way of doing this is to use Equation (1.2), thus we primarily need to calculate the Hessian of the projection map onto $\mathbb{S}^{N-1}$. One needs to be careful on what the Hessian represents: it is actually a bilinear map with domain $\mathbb{R}^N \times \mathbb{R}^N$, and it also *takes values in* $\mathbb{R}^N$; in particular, it takes the form

$$\operatorname{Hess}\pi_{\mathbb{S}^{N-1}}(x) = \left\{ a_{ij}^{k}(x) \right\}_{1\le i,j,k\le N}.$$

We have that, for any $x \in \mathbb{R}^N \setminus \{0\}$:

$$\pi_{\mathbb{S}^{N-1}}(x) = \frac{x}{|x|}.$$

A very straightforward algebraic computation carries, for all $1 \leq i, j, k \leq N$:

$$\frac{\partial}{\partial x^j}\frac{x^k}{|x|} = \frac{\delta_j^k}{|x|} - \frac{x_j x^k}{|x|^3}$$

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{x^k}{|x|} = \frac{3x_i x_j x^k}{|x|^5} - \frac{x_i\delta_j^k + x_j\delta_i^k + x^k\delta_{ij}}{|x|^3}.$$

Now assume that x is a point on $\mathbb{S}^{N-1}$ and $X, Y \in \mathcal{T}\left(\mathbb{S}^{N-1}\right)$ are two tangent vector fields. Then we have that (using the Einstein summation convention on repeated indices):

$$\begin{aligned}(\operatorname{Hess}\pi_{\mathbb{S}^{N-1}}(x)\,(X(x), Y(x)))^k &= \\ &= 3x_i X^i x_j Y^j x^k - \left(x_i X^i Y^k + x_j Y^j X^k + X^i Y^i x^k\right) \\ &= 3\langle x, X\rangle\langle x, Y\rangle x^k - \left(\langle x, X\rangle Y^k + \langle x, Y\rangle X^k + \langle X, Y\rangle x^k\right).\end{aligned}$$

But X and Y are tangent to the sphere, hence $X(x)$ and $Y(x)$ are orthogonal to x. Then we obtain the following expression for the second fundamental form:

$$A(x)(X,Y)^k = -(\operatorname{Hess}\pi_{\mathbb{S}^{N-1}}(x)\,(X(x), Y(x)))^k = \langle X(x), Y(x)\rangle x^k,$$

which we can rewrite as

$$A(x)(X,Y) = \langle X(x), Y(x)\rangle x. \tag{1.8}$$

1.2 Internal variation

We consider here a second class of perturbations: this time, we pre-compose the map $u \in W^{1,2}(\Omega, \mathcal{N})$ with a suitable family of diffeomorphisms of the domain. In the following definition, we ask that both this new condition and weak harmonicity are satisfied; the reason is that in general none of the two conditions implies the other.

Let again Ω be a smooth bounded open set in $\mathbb{R}^m$, and let $u : \Omega \longrightarrow \mathcal{N}$ be a map in $W^{1,2}(\Omega, \mathcal{N})$.

Definition 1.4 (Stationary harmonic maps)**.** We say that u is **stationary** if:

(i) u is weakly harmonic;

(ii) Let $\Phi = \{\phi_t\}_{t\in I}$ be any smooth family of diffeomorphisms of Ω, with I open interval containing 0; assume that $\phi_0 \equiv \mathrm{id}_\Omega$, and that there exists a compact set $K \subset \Omega$ such that $\phi_t\big|_{\Omega\setminus K} = \mathrm{id}_{\Omega\setminus K}$ for any $t \in I$; then

$$\frac{d}{dt}\mathcal{E}\left(u\circ\phi_t\right)\Big|_{t=0} = \frac{d}{dt}\int_\Omega |\nabla(u\circ\phi_t)(x)|^2 dx\Big|_{t=0} = 0. \tag{1.9}$$

As in the section dedicated to external variations, we attempt to write the latter condition in a more manageable manner, in order to obtain another weak Euler-Lagrange equation. Consider then a smooth and compactly supported vector field $X : \Omega \to \mathbb{R}^m$. Its flux defines a family of diffeomorphisms as in the Definition 1.4; conversely, any such family of diffeomorphisms can be seen as the flux of a compactly supported vector field, the *infinitesimal generator* vector field (see for example [Arn92, Paragraph 1.4.4]).

Notice that, by the elementary chain rule, the following computation holds:

$$|\nabla(u\circ\phi_t)(x)|^2 = \sum_{i=1}^m\sum_{\alpha=1}^N\left(\frac{\partial}{\partial x^i}(u^\alpha\circ\phi_t)(x)\right)^2 = \\ = \sum_{i=1}^m\sum_{\alpha=1}^N\left(\sum_{k=1}^m\frac{\partial u^\alpha}{\partial x^k}(\phi_t(x))\frac{\partial\phi_t^k}{\partial x^i}(x)\right)^2$$

Now by smoothness of Φ we can expand $t\mapsto\phi_t(x)$ as a Taylor polynomial:

$$\phi_t(x) = x + tX(x) + O(t^2);$$

so in particular for any $1\le i,k\le m$ we have:

$$\frac{\partial\phi_t^k}{\partial x^i}(x) = \delta_i^k + t\frac{\partial X^k}{\partial x^i}(x) + O(t^2).$$

Thus we have the following equalities (recall that i and k are summed from 1 to m, while α is summed from 1 to N):

$$\int_\Omega|\nabla(u\circ\phi_t)(x)|^2dx = \sum_{i,\alpha}\int_\Omega\left(\sum_{k=1}^m\frac{\partial u^\alpha}{\partial x^k}(\phi_t(x))\frac{\partial\phi_t^k}{\partial x^i}(x)\right)^2 dx = \\ = \sum_{i,\alpha}\int_\Omega\left(\sum_k\frac{\partial u^\alpha}{\partial x^k}(\phi_t(x))\left(\delta_i^k + t\frac{\partial X^k}{\partial x^i}(x) + O(t^2)\right)\right)^2 dx = \\ = \sum_{i,\alpha}\int_\Omega\left(\frac{\partial u^\alpha}{\partial x^i}(\phi_t(x)) + t\sum_k\frac{\partial u^\alpha}{\partial x^k}(\phi_t(x))\frac{\partial X^k}{\partial x^i}(x) + O(t^2)\right)^2 dx = \\ = \sum_{i,\alpha}\int_\Omega\left[\left(\frac{\partial u^\alpha}{\partial x^i}(\phi_t(x))\right)^2 + \right. \\ \left. + 2t\left(\frac{\partial u^\alpha}{\partial x^i}(\phi_t(x))\sum_k\frac{\partial u^\alpha}{\partial x^k}(\phi_t(x))\frac{\partial X^k}{\partial x^i}(x)\right) + O(t^2)\right] dx.$$

Consider the inverse diffeomorphism $\phi_t^{-1} = \phi_{-t}$ and apply the associated change of variable $x = \phi_{-t}(y)$ to the above integral; we have to compute the following determinant:

$$\begin{aligned}\det \mathbf{Jac}\, \phi_{-t}(y) &= \det(\mathbb{1}_m - t\frac{\partial X}{\partial x}(y) + O(t^2)) = \\ &= 1 - t\,\mathrm{Tr}\left(\frac{\partial X}{\partial x}(y)\right) + O(t^2) = 1 - t\,\mathrm{div}\, X(y) + O(t^2),\end{aligned}$$

where the equality

$$\det(\mathbb{1} + tA) = 1 + t\,\mathrm{Tr}(A) + O(t^2)$$

follows from elementary linear algebra computations. Then we get:

$$\begin{aligned}&\int_\Omega |\nabla(u\circ\phi_t)(x)|^2 dx = \\ &= \sum_{i,\alpha} \int_{\phi_{-t}^{\leftarrow}(\Omega)} \left[\left(\frac{\partial u^\alpha}{\partial x^i}(y)\right)^2 + 2t\left(\frac{\partial u^\alpha}{\partial x^i}(y) \sum_k \frac{\partial u^\alpha}{\partial x^k}(y)\frac{\partial X^k}{\partial x^i}(\phi_{-t}(y))\right) + \right. \\ &\qquad \left. + O(t^2)\right] \cdot \left|1 - t\,\mathrm{div}\, X(y) + O(t^2)\right| dy.\end{aligned}$$

Observe now that $1 - t\,\mathrm{div}\, X(y) + O(t^2) > 0$ if t is small, so the absolute value can be ignored; moreover, by smoothness of ϕ_{-t},

$$\phi_{-t}(y) = y - tX(y) + O(t^2),$$

hence

$$\frac{\partial X^k}{\partial x^i}(\phi_{-t}(y)) = \frac{\partial X^k}{\partial x^i}(y) + O(t)$$

by the smoothness of X. So the previous integral becomes

$$\begin{aligned}\sum_{i,\alpha}\int_\Omega \Bigg[&\left(\frac{\partial u^\alpha}{\partial x^i}(y)\right)^2 + \\ &+ 2t\left(\frac{\partial u^\alpha}{\partial x^i}(y)\sum_k \frac{\partial u^\alpha}{\partial x^k}(y)\frac{\partial X^k}{\partial x^i}(y)\right) - \\ &- t\left(\frac{\partial u^\alpha}{\partial x^i}(y)\right)^2 \sum_{k=1}^m \frac{\partial X^k}{\partial x^k}(y) + O(t^2)\Bigg] dy.\end{aligned}$$

Observe that

$$\sum_{i,\alpha,k}\left(\frac{\partial u^\alpha}{\partial x^i}(y)\right)^2 \frac{\partial X^k}{\partial x^k}(y) = \sum_{i,\alpha,k}\sum_{j=1}^m \left(\frac{\partial u^\alpha}{\partial x^j}(y)\right)^2 \delta_{ik}\frac{\partial X^k}{\partial x^i}(y)$$

By Equation (1.9), the following holds:

$$\sum_{i,k=1}^m \int_\Omega \left[2\sum_{\alpha=1}^N \frac{\partial u^\alpha}{\partial x^i}(y)\frac{\partial u^\alpha}{\partial x^k}(y) - \sum_{\alpha=1}^N\sum_{j=1}^m \left(\frac{\partial u^\alpha}{\partial x^j}(y)\right)^2 \delta_{ik}\right]\frac{\partial X^k}{\partial x^i}(y)dy = 0.$$

Thus for any $X \in C_0^\infty(\Omega, \mathbb{R}^m)$ the following integral equation holds:

$$\int_\Omega \sum_{i,k=1}^m \left[2\langle \nabla_i u, \nabla_k u\rangle - |\nabla u|^2 \delta_{ik}\right] \frac{\partial X^k}{\partial x^i} dx = 0, \tag{1.10}$$

where we are using the notation $\nabla_i u$ for $\frac{\partial u}{\partial x^i}$. Now what we have obtained characterizes completely stationary harmonic maps, thanks to the equivalence between the formulation in terms of diffeomorphisms and the one in terms of vector fields. Define, for every $h = 1, \ldots, m$, the following vector field:

$$S_h(x) = 2\langle \nabla u(x), \nabla_h u(x)\rangle - |\nabla u(x)|^2 e_h,$$

where $\{e_h\}_{h=1}^m$ is the canonical basis of $\mathbb{R}^m$. Then the previous integral equation is completely equivalent to the condition $\operatorname{div} S_h = 0$ in Ω for all h (in the distributional sense). Explicitly:

$$\operatorname{div}\left[2\langle \nabla_h u, \nabla u\rangle - |\nabla u|^2 e_h\right] = 0 \quad \text{for all } h = 1, \ldots, m$$

in distributional sense.

Computation of the divergence. For a map $F \in L^1(\Omega, \mathbb{R}^m)$ the distributional divergence $\operatorname{div} F \in \mathcal{D}'(\Omega)$ is defined as follows: for any test function $\psi \in C_0^\infty(\Omega, \mathbb{R})$,

$$\langle \operatorname{div} F, \psi\rangle_{\mathcal{D}', \mathcal{D}} \doteq -\int_\Omega \langle F(x), \nabla\psi(x)\rangle\, dx = -\sum_{i=1}^m \int_\Omega F^i \frac{\partial \psi}{\partial x^i}\, dx.$$

Assume that Equation (1.10) holds for all $X \in C_0^\infty(\Omega, \mathbb{R}^m)$, and let $\psi \in C_0^\infty(\Omega, \mathbb{R})$ be a test function; then for all $h = 1, \ldots, m$ the vector field $X \doteq \psi e_h$ is smooth and compactly supported, with

$$\frac{\partial\left(\psi e_h\right)^k}{\partial x_i} = \frac{\partial \psi}{\partial x_i} \delta_h^k.$$

Thus, fixed h, Equation (1.10) implies that for any test function ψ

$$\int_\Omega \sum_i^m \left[2\langle \nabla_i u, \nabla_h u\rangle - |\nabla u|^2 \delta_{ih}\right] \frac{\partial \psi}{\partial x^i} dx = 0,$$

which is exactly the definition of $\operatorname{div} S_h = 0$. The opposite implication is trivial: given a vector field $X \in C_0^\infty(\Omega, \mathbb{R}^m)$, all its components are test functions, so one can apply the definition m times and sum the obtained equalities.

1.3 Energy minimizing maps

Summarizing content of the last two sections, we have seen that a map u in the Sobolev space $W^{1,2}(\Omega, \mathcal{N})$ is said to be:

- **Weakly harmonic** if it's a critical point of the energy functional under external variations: for any $\varphi \in C_0^\infty(\Omega, \mathcal{N})$,
$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}\left(\pi_{\mathcal{N}} \circ (u + t\varphi)\right) = 0; \tag{WH}$$
this happens if and only if u satisfies the Euler-Lagrange equation
$$\Delta u + A(u)\left(\nabla u, \nabla u\right) = 0 \quad \text{in } \Omega \tag{ELwh}$$
in the weak sense.

- **Stationary harmonic** if it is weakly harmonic and if it's a critical point of the energy functional under internal variations: for any smooth family of diffeomorphisms $\{\phi_t\}_{t\in I}$ of Ω onto itself, with I open interval containing 0 and $\phi_0 \equiv \mathrm{id}_\Omega$,
$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}\left(u \circ \phi_t\right) = 0. \tag{SH}$$
This is equivalent to requiring that for any vector field $X \in C_0^\infty(\Omega, \mathbb{R}^m)$ the following identity holds:
$$\int_\Omega \sum_{i,k=1}^m \left[2\langle \nabla_i u, \nabla_k u\rangle - |\nabla u|^2 \delta_{ik}\right] \frac{\partial X^k}{\partial x^i} dx = 0. \tag{SH2}$$
or, again equivalently, that the Euler-Lagrange equation
$$\operatorname{div}\left[2\langle \nabla_h u, \nabla u\rangle - |\nabla u|^2 e_h\right] = 0 \quad \text{for all } h = 1, \dots, m. \tag{ELsh}$$
is satisfied in the weak sense.

Finally, we give a precise definition of what we mean with "energy minimizing map": these maps are critical points both for internal and for external variations, and will be the class of maps we'll mainly analyze.

Definition 1.5. We say that $u \in W^{1,2}(\Omega, \mathcal{N})$ is a **minimizing harmonic map** if for any $v \in W^{1,2}(\Omega, \mathcal{N})$ such that $u \equiv v$ outside a compact set of Ω we have
$$\mathcal{E}(u) \le \mathcal{E}(v).$$

Clearly, if u is minimizing, then it is a minimizer for both external and internal variations: thus, the following inclusions hold:
$$\{\text{minimizing maps}\} \subset \{\text{stationary maps}\} \subset \{\text{weakly harmonic maps}\}.$$

Proposition 1.3 (Existence of minimizing maps)**.** *Let Ω be an open bounded set with smooth boundary, and $\mathcal{N}$ a compact Riemannian manifold with no boundary. Let $u \in W^{1,2}(\Omega, \mathcal{N})$. There exists a map $\hat{u} \in W^{1,2}(\Omega, \mathcal{N})$ which is energy minimizing and coincides with u outside a compact set of Ω.*

Proof. The set

$$\mathcal{V} \doteq \left\{ v \in W^{1,2}(\Omega, \mathcal{N}) \;\middle|\; \begin{array}{c} v \equiv u \text{ outside a compact set of } \Omega, \\ \text{and } \mathcal{E}(v) \leq \mathcal{E}(u) \end{array} \right\}$$

is bounded in $W^{1,2}(\Omega, \mathcal{N})$; indeed, $\mathcal{N}$ is compact and so for some $R > 0$ we have $|v(x)| \leq R$ for all $x \in \Omega$ and for all $v \in W^{1,2}(\Omega, \mathcal{N})$.

Given a sequence $v_i \in \mathcal{V}$ which minimizes the energy, by Theorem 1.1 there exists a subsequence that converges to a certain $\hat{u} \in W^{1,2}\big(\Omega, \mathbb{R}^N\big)$ weakly in $W^{1,2}\big(\Omega, \mathbb{R}^N\big)$, strongly in $L^2\big(\Omega, \mathbb{R}^N\big)$ and almost everywhere. Thus $\hat{u} \in W^{1,2}(\Omega, \mathcal{N})$; moreover, by lower semicontinuity of the energy with respect to the weak $W^{1,2}$ topology we have

$$\mathcal{E}(\hat{u}) \leq \liminf_{i \to \infty} \mathcal{E}(v_i),$$

which proves that $\hat{u}$ is energy minimizing. □

For minimizing maps, a very strong compactness result holds, and it will play a central role in several of the proofs appearing in the next chapters: we refer for a proof to [Sim96, pp. 32–35], where a Lemma of Luckhaus is exploited to show that the weak $W^{1,2}$ limit of a sequence of minimizing maps is in fact a strong $W^{1,2}$ limit (up to subsequences) and it is a minimizing map itself.

Theorem 1.4 (Compactness)**.** *Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence of minimizing harmonic maps in $W^{1,2}(\Omega, \mathcal{N})$ with bounded energy:*

$$\int_\Omega |\nabla u_i|^2 \, dx \leq \Lambda < \infty \quad \forall i \in \mathbb{N}.$$

Then there exists a subsequence $\{u_{i_k}\}_{k \in \mathbb{N}}$ and a minimizing harmonic map $\bar{u} \in W^{1,2}(\Omega, \mathcal{N})$ such that $u_{i_k} \to \bar{u}$ strongly in $W^{1,2}(\Omega, \mathcal{N})$.

Remark. Notice that the classical theorems stated at the beginning of this chapter only give *weak* convergence, and say nothing about the harmonicity of the limit map.

1.4 A monotonicity formula

We now introduce the notion of *normalized energy*, together with a slightly modified version; as we'll see, this will be a central and ubiquitous tool in the analysis of harmonic maps. Let $u : \Omega \subset \mathbb{R}^m \longrightarrow \mathcal{N} \subset \mathbb{R}^N$ be a *stationary* harmonic map.

Definition 1.6 (Normalized energy)**.** For all $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, we define the **normalized energy** as the function

$$\theta(x, r) \doteq r^{2-m} \int_{B_r(x)} |\nabla u(y)|^2 \, dy.$$

When needed, we'll specify the dependence on the map u by writing θ^u or $\theta[u](x, r)$.

The reason for the presence of the term r^{2-m} will be clear in a moment: it assures a scale invariance property of θ. Now fix a function $\psi \in C^\infty([0, \infty))$ with support contained in $[0, 1]$, and assume ψ is non-increasing (equivalently, $\psi' \leq 0$).

Definition 1.7 (Modified normalized energy)**.** We define, for all $x \in \Omega$ and for all $r < \operatorname{dist}(x, \partial\Omega)$, the **modified normalized energy**:

$$\theta_\psi(x, r) \doteq r^{2-m} \int_\Omega \psi\left(\frac{|y - x|}{r}\right) |\nabla u(y)|^2 dy;$$

notice that we could equivalently integrate over the whole $\mathbb{R}^m$, or over $B_r(x)$, since ψ is supported in $[0, 1]$. Here again, the dependence on u will be indicated with θ_ψ^u or $\theta_\psi[u](x, r)$ if necessary.

Notice that, if we drop the smoothness assumption on ψ, the function θ is simply θ_ψ with $\psi = \chi_{[0,1]}$. While the simple normalized energy θ is more common in literature, in Chapters 2 and 3 we'll mainly use the modified one, since its regularity properties make it more manageable in computations. However, in most part of our work they would be interchangeable, and the results proved in this section for the modified energy are actually valid for both.

As a first result, we prove a scale invariance property of θ_ψ, which justifies the term r^{2-m} in the normalized energy. We begin with a definition.

Definition 1.8. Let $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$; let $u : \Omega \to \mathcal{N}$ be an arbitrary map. We define the *rescaled map* $\tilde{u} = T_{x,r}u$ from $B\left(0, \frac{1}{r} \operatorname{dist}(x, \partial\Omega)\right)$ to $\mathcal{N}$ as

$$\tilde{u}(y) = T_{x,r}u(y) \doteq u(x + ry).$$

It's easy to see that $T_{x,r}u$ is weakly (resp. stationary, resp. minimizing) harmonic whenever u is weakly (resp. stationary, resp. minimizing) harmonic.

Lemma 1.5 (Scale and translation invariance)**.** *For any $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$,*

$$\theta_\psi^u(x, r) = \theta_\psi^{\tilde{u}}(0, 1).$$

Proof. Call $\lambda = \lambda_{x,r}$ the diffeomorphism $\lambda(y) = x + ry$ from $B_1(0)$ to $B_r(x)$. Notice that

$$\nabla\tilde{u}(y) = \nabla(u \circ \lambda)(y) = \nabla u\,[\lambda(y)] \cdot \nabla\lambda(y) = r\nabla u(x + ry),$$

and in particular $|\nabla\tilde{u}(y)|^2 = r^2|\nabla u(x + ry)|^2$. Now we apply the change of variables induced by the diffeomorphism $\lambda^{-1}(z) = \frac{z-x}{r}$ to the integral defining $\theta_\psi^{\tilde{u}}$; a factor r^{-m} appears as the Jacobian determinant of λ^{-1}:

$$\begin{aligned}\theta_\psi^{\tilde{u}}(0,1) &= \int_{B_1(0)} \psi(|y|)\,|\nabla\tilde{u}(y)|^2\,dy = r^2\int_{B_1(0)} \psi(|y|)\,|\nabla u(x + ry)|^2\,dy = \\ &= r^2\int_{B_r(x)} \psi\left(\frac{|z-x|}{r}\right)|\nabla u(z)|^2 r^{-m}\,dz = \theta_\psi^u(x,r)\,.\end{aligned}$$

□

Observe that in the previous proof the regularity of ψ has not been used, so this is a completely valid proof for the simple normalized energy as well.

Remark. With very straightforward computations one can see that, more generally, if we also have $w \in \Omega$ and $\tau > 0$ small enough, then

$$\theta_\psi^u(x + \tau w, r\tau) = \theta_\psi^{\tilde{u}}(w,\tau).$$

The main feature of θ_ψ is its monotonicity: indeed, we are going to prove that (with an appropriate choice of ψ) for any x, $\theta_\psi(x,\cdot)$ is non-decreasing. Also, for x, y in Ω, denote with $r_x(y)$ the unit vector in the direction going from x to y:

$$r_x(y) \doteq \frac{y-x}{|y-x|},$$

and let $\partial_{r_x(y)}u(y)$ be the directional derivative along $r_x(y)$:

$$\partial_{r_x(y)}u(y) \doteq \left\langle \nabla u(y), \frac{y-x}{|y-x|}\right\rangle.$$

Theorem 1.6 (Monotonicity formula)**.** *Let $u \in W^{1,2}(\Omega,\mathcal{N})$ be a stationary harmonic map, and $\psi \in C_0^\infty([0,\infty))$ with $\operatorname{spt}(\psi) \subset [0,1]$. Fix $x \in \Omega$ and $0 < r < \operatorname{dist}(x,\partial\Omega)$. Then $\theta_\psi(x,\cdot)$ has a derivative at r and the following equality holds:*

$$\frac{d}{dr}\theta_\psi(x,r) = -2r^{-m}\int_\Omega |y-x|\psi'\left(\frac{|y-x|}{r}\right)|\partial_{r_x(y)}u(y)|^2 dy. \tag{MF}$$

Proof. We'll proceed in two steps.

Step 1. Assume first that $\overline{B_1(0)} \subset \Omega$, $x = 0$, $r = 1$; the general case will then follow by scale invariance. We have to prove the following identity:

$$\left.\frac{d}{dr}\theta_\psi(0,r)\right|_{r=1} = -2\int_\Omega \psi'(|y|)\,|y||\partial_{\frac{y}{|y|}}u(y)|^2 dy. \tag{1.11}$$

Consider the following vector field:

$$Y(y) = \psi(|y|)\, y \in C_0^\infty(\Omega, \mathbb{R}^m).$$

A simple computation gives, for $1 \le i, j \le m$,

$$\frac{\partial Y^j}{\partial y^i} = \psi'(|y|)\frac{y_i y_j}{|y|} + \psi(|y|)\,\delta_{ij}.$$

Then, with this choice of Y, Equation (SH2) reads:

$$\begin{aligned}
0 &= \int_\Omega \sum_{i,j=1}^m \left[2\langle \nabla_i u, \nabla_j u\rangle - |\nabla u|^2 \delta_{ij}\right]\left[\psi'(|y|)\frac{y_i y_j}{|y|} + \psi(|y|)\,\delta_{ij}\right] dy \\
&= \int_\Omega \left[\frac{2}{|y|}\psi'(|y|)\sum_{i,j=1}^m \langle y_i \nabla_i u, y_j \nabla_j u\rangle + \right. \\
&\qquad + 2\psi(|y|)\,|\nabla u|^2 - |y|\psi'(|y|)\,|\nabla u|^2 - \\
&\qquad\qquad \left. - m\psi(|y|)\,|\nabla u|^2\right] dy.
\end{aligned}$$

Notice that

$$\sum_{i,j=1}^m \langle y_i \nabla_i u, y_j \nabla_j u\rangle = |y|^2 \left\langle \sum_{i=1}^m \frac{y_i}{|y|}\nabla_i u, \sum_{j=1}^m \frac{y_j}{|y|}\nabla_j u\right\rangle = |y|^2 |\partial_{\frac{y}{|y|}} u(y)|^2,$$

so we obtain

$$\begin{aligned}
(2-m)\int_\Omega \psi(|y|)\,|\nabla u|^2\, dx - \int_\Omega |y|\psi'(|y|)\,|\nabla u|^2\, dy = \\
= -2\int_\Omega |y|\psi'(|y|)\,|\partial_{\frac{y}{|y|}} u(y)|^2\, dy.
\end{aligned}$$

Thus it suffices to show that the left hand side is the derivative of $\theta_\psi(0,\cdot)$ at 1, and this is an elementary computation; notice that, by a classical result of real analysis (see for example [Fol99, Theorem 2.27]), we're allowed to change the order of differentiation and integration, since $u \in W^{1,2}(\Omega, \mathcal{N})$ and $\psi'\left(\frac{|y|}{r}\right)\frac{|y|}{r^2}$ is bounded for any $r \ge r_0$:

$$\begin{aligned}
\frac{d}{dr}\theta_\psi(0,r) = (2-m)r^{1-m}\int_\Omega \psi\left(\frac{|y|}{r}\right)|\nabla u(y)|^2 dy + \\
+ r^{2-m}\int_\Omega \psi'\left(\frac{|y|}{r}\right)\left(-\frac{|y|}{r^2}\right)|\nabla u|^2\, dy.
\end{aligned}$$

Taking $r = 1$, this is exactly what we were looking for.

Step 2. Consider now the general case: arbitrarily fix $x \in \Omega$ and $\bar{r} > 0$ so that $\overline{B_{\bar{r}}(x)} \subset \Omega$. By scale invariance, we know that

$$\theta_\psi[u](x,r) = \theta_\psi\left[T_{x,r}u\right](0,1)$$

for all r in a neighborhood of $\bar{r}$. Hence in particular

$$\frac{d}{dr}\theta_\psi[u](x,r)\Big|_{r=\bar{r}} = \frac{d}{dr}\theta_\psi\left[T_{x,r}u\right](0,1)\Big|_{r=\bar{r}}.$$

Notice that by Step 1 we have information about the following quantity:

$$\frac{d}{ds}\theta_\psi\left[T_{x,\bar{r}}u\right](0,s)\Big|_{s=1},$$

which is not directly the information we seek, but is really close. Indeed, using the (already seen) fact that

$$\left|\nabla\left(u\circ\lambda_{x,\bar{r}}\right)(y)\right|^2 = \bar{r}^2\left|\nabla u[x+\bar{r}y]\right|^2$$

and a change of variable $y = \frac{s}{r}w$ we get:

$$\begin{aligned}
&\frac{d}{ds}\theta_\psi\left[T_{x,\bar{r}}u\right](0,s)\Big|_{s=1} = \\
&\quad = \bar{r}\frac{d}{ds}\theta_\psi\left[T_{x,\bar{r}}u\right]\left(0,\frac{s}{\bar{r}}\right)\Big|_{s=\bar{r}} = \\
&\quad = \bar{r}\frac{d}{ds}\left[\left(\frac{s}{\bar{r}}\right)^{2-m}\int_\Omega \psi\left(\frac{\bar{r}}{s}|y|\right)\left|\nabla\left(u\circ\lambda_{x,\bar{r}}\right)(y)\right|^2\,dy\right]\Big|_{s=\bar{r}} = \\
&\quad = \bar{r}\frac{d}{ds}\left[\left(\frac{s}{\bar{r}}\right)^{2-m}\int_\Omega \psi\left(|w|\right)\frac{\bar{r}^2}{s^2}s^2\left|\nabla u[x+sw]\right|^2\left(\frac{s}{\bar{r}}\right)^m\,dw\right]\Big|_{s=\bar{r}} = \\
&\quad = \bar{r}\frac{d}{ds}\int_\Omega \psi(|w|)\left|\nabla\left(u\circ\lambda_{x,s}\right)(w)\right|^2\,dw\Big|_{s=\bar{r}}.
\end{aligned}$$

This last term is now clearly equal to $\bar{r}\frac{d}{dr}\theta_\psi\left[T_{x,r}u\right](0,1)\Big|_{r=\bar{r}}$. Thus by Step 1 we obtain:

$$\frac{d}{dr}\theta_\psi[u](x,r)\Big|_{r=\bar{r}} = -\frac{2}{\bar{r}}\int_\Omega |y|\psi'\left(|y|\right)\left|\partial_{\frac{y}{|y|}}T_{x,\bar{r}}u(y)\right|^2\,dy;$$

Now notice that

$$\left|\partial_{\frac{y}{|y|}}T_{x,\bar{r}}u(y)\right|^2 = \bar{r}^2\left\langle\nabla u[x+ry],\frac{y}{|y|}\right\rangle^2;$$

performing the change of variable $y = \frac{z-x}{\bar{r}}$ inside the integral, we finally obtain

$$\begin{aligned}
&\frac{d}{dr}\theta_\psi[u](x,r)\Big|_{r=\bar{r}} = \\
&\qquad = -\frac{2}{\bar{r}}\int_\Omega \frac{|y-x|}{\bar{r}}\psi'\left(\frac{|y-x|}{\bar{r}}\right)\bar{r}^2\left\langle\nabla u[z],\frac{z-x}{|z-x|}\right\rangle^2\bar{r}^{-m}\,dz,
\end{aligned}$$

which is, up to little adjustment, the desired result. □

Corollary 1.7. *Let* $u \in W^{1,2}(\Omega, \mathcal{N})$ *be a stationary harmonic map, and* $\psi \in C_0^\infty([0,\infty))$ *with* $\operatorname{spt}(\psi) \subset [0,1]$*.*

(i) If $x \in \Omega$ *and* $0 < s < r < \operatorname{dist}(x, \partial\Omega)$*, then*

$$\theta_\psi(x,r) - \theta_\psi(x,s) = \\ = 2\int_\Omega \left(\Psi\left(\frac{|y-x|}{r}\right) - \Psi\left(\frac{|y-x|}{s}\right)\right) |y-x|^{2-m} \left|\partial_{r_x(y)} u(y)\right|^2 dy, \tag{1.12}$$

where $\Psi(t)$ *is a primitive of* $t^{m-2}\psi'(t)$*.*

(ii) If ψ *is non-increasing, then* θ_ψ *is non-decreasing.*

(iii) If ψ *is non-increasing, then the following function is well defined:*

$$\theta_\psi(x) \doteq \theta_\psi(x,0) = \lim_{r\to 0} \theta_\psi(x,r).$$

Proof. The equality in (i) follows by integrating the equality

$$\frac{d}{dr}\theta_\psi(x,r) = 2\int_\Omega \left(\frac{|y-x|}{r}\right)^{m-2} \psi'\left(\frac{|y-x|}{r}\right) \cdot \\ \cdot \left(-\frac{|y-x|}{r^2}\right) |y-x|^{2-m} \left|\partial_{r_x(y)} u(y)\right|^2 dy. \tag{MF}$$

from s to r and changing the order of the integrals via the Tonelli Theorem. Assertion (ii) is a trivial consequence of the fact that $\psi' \le 0$, while (iii) follows by the fact that $\theta_\psi(x,\cdot)$ is non-decreasing and bounded from below. □

Remark (Choice of ψ). In order to efficiently exploit the Monotonicity Formula and the previous corollary, we'll always choose ψ strictly decreasing in $(0,1)$, so that $\psi' < 0$.

Another rather natural and elegant choice would be to require that $\psi^{(2n+1)}(0) = 0$ for any $n \ge 0$, in order to have the map $h \to \psi(|h|)$ infinitely differentiable: in Lemma 1.8 we'll explore a bit the consequences that this choice would have on the modified normalized energy. However, for technical reasons we will *not* assume this condition to hold, since we'll need a different and incompatible hypothesis during the course of Chapter 3.

We collect in the following lemma a couple of simple but useful facts about the behavior of θ_ψ. Note that the Monotonicity Formula gives the derivability of $\theta_\psi(x,\cdot)$ as a function of r with x *fixed*.

Lemma 1.8. *The following facts hold true:*

(i) *Let u be stationary harmonic, $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$. The map*

$$\begin{aligned} \theta_\psi(\cdot, r) \colon \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > r\} &\longrightarrow \mathbb{R} \\ x &\longmapsto \theta_\psi(x, r) \end{aligned}$$

is continuous. If $\psi^{(2n+1)}(0) = 0$ for any $n \geq 0$, then $\theta_\psi(\cdot, r)$ is differentiable.

(ii) *The map*

$$\begin{aligned} \theta_\psi(\cdot) \equiv \theta_\psi(\cdot, 0) \colon \Omega &\longrightarrow \mathbb{R} \\ x &\longmapsto \theta_\psi(x) \end{aligned}$$

is upper semicontinuous.

(iii) *Let $x \in \Omega$, $0 < r < \operatorname{dist}(\partial\Omega)$. If $u_i \to \bar{u}$* strongly *in $W^{1,2}(\Omega, \mathcal{N})$, then $\lim_{i\to\infty} \theta_\psi^{u_i}(x, r) = \theta_\psi^{\bar{u}}(x, r)$.*

(iv) *Let $x \in \Omega$, $0 < r < \operatorname{dist}(\partial\Omega)$. If $u_i \to \bar{u}$* weakly *in $W^{1,2}(\Omega, \mathcal{N})$, then $\liminf_{i\to\infty} \theta_\psi^{u_i}(x, r) \geq \theta_\psi^{\bar{u}}(x, r)$.*

Proof. Assume $x_i \to x$ for $i \to \infty$. Clearly $\psi\left(\frac{|\cdot - x_i|}{r}\right) |\nabla u|^2$ is dominated by $C\,|\nabla u|^2$, where C is a constant. Then by the Dominated Convergence Theorem

$$\lim_{i\to\infty} r^{2-m} \int_\Omega \psi\left(\frac{|y - x_i|}{r}\right) |\nabla u(y)|^2 \, dy = r^{2-m} \int_\Omega \psi\left(\frac{|y - x|}{r}\right) |\nabla u(y)|^2 \, dy,$$

which is assertion (i).

To show (ii) one needs to prove that, for any λ fixed, the set

$$\{x \in \Omega \mid \theta_\psi(x) < \lambda\}$$

is open in Ω. By definition of $\theta_\psi(x)$, $\theta_\psi(x) < \lambda$ is true if and only if $\theta_\psi(x, r) < \lambda$ for some $r > 0$. Then

$$\{x \in \Omega \mid \theta_\psi(x) < \lambda\} = \bigcup_{r>0} \{x \in \Omega \mid \theta_\psi(x, r) < \lambda\}$$

is a union of open sets (by continuity of $\theta_\psi(\cdot, r)$), so it is an open set.

Part (iii) is a immediate consequence of the strong convergence: indeed, if $u_i \xrightarrow{W^{1,2}} \bar{u}$, then

$$\lim_{i\to\infty} \int_\Omega |\nabla u_i(y) - \nabla \bar{u}(y)|^2 \, dy = 0,$$

and this of course implies the convergence of the normalized energies.

As a consequence, the functional $u \to \theta^u_\psi(x,r)$ (for x and r fixed) is continuous from $W^{1,2}(\Omega,\mathcal{N})$-strong to $\mathbb{R}$; moreover, it is easily seen to be a convex operator. By a very well known result (see for example [Bre11, Section 3.3]), this implies that the same operator is lower semicontinuous for the weak topology, which is (iv). □

As a consequence of the Monotonicity Formula, one can prove an analogous result for the classical normalized energy, for example by approximating the function $\chi_{[0,1]}$ with a sequence of smooth functions ψ. For a complete proof, we refer to [Sim96, Section 2.4].

Corollary 1.9. *Let $u \in W^{1,2}(\Omega,\mathcal{N})$ be a stationary harmonic map and let $x \in \Omega$. Then we have:*

1. *For almost all $r \in (0, \operatorname{dist}(x,\partial\Omega))$,*

$$\frac{d}{dr}\theta(x,r) = 2r^{2-m}\int_{\partial B_r(x)} \left|\partial_{r_x(y)}u(y)\right|^2 \, d\sigma(y);$$

2. *For all $0 < s < r < \operatorname{dist}(x,\partial\Omega)$,*

$$\theta(x,r) - \theta(x,s) = 2\int_{B_r(x)\setminus B_s(x)} |x-y|^{2-m}\left|\partial_{r_x(y)}u(y)\right|^2 \, dy.$$

1.5 Regularity scale and ε-regularity

We present in this section a fundamental result due to Schoen and Uhlenbeck (see [SU82]); it is a first important application of the Monotonicity formula, and it's known in the literature as the ε-regularity Theorem; we then state a couple of immediate consequences. These results will be useful for the explicit example we describe in Section 1.6, and will appear again in Section 2.5, where they'll play a primary role.

Firstly, the ε-regularity Theorem is stated in its basic version, as found in the original paper by Schoen and Uhlenbeck; we keep the formulation in terms of θ, instead of θ_ψ, to ease the subsequent computations:

Theorem 1.10 (ε-regularity)**.** *There exists a constant $\varepsilon_1 = \varepsilon_1(m,\mathcal{N})$ such that the following holds. Let $u \in W^{1,2}(\Omega,\mathcal{N})$ be a minimizing harmonic map, and let $x \in \Omega$, $0 < r < \operatorname{dist}(x,\partial\Omega)$. If $\theta(x,r) < \varepsilon_1$, then u is Hölder continuous in $B_{\frac{r}{2}}(x)$.*

The following lemma permits to rewrite the ε-regularity Theorem in a stronger form (Corollary 1.12); for the a proof of the lemma, we refer to [Sch84, Theorems 3.1 and 4.1]

Lemma 1.11. *Let $u \in W^{1,2}(\Omega,\mathcal{N})$, $x \in \Omega$ and $0 < r < \operatorname{dist}(x,\partial\Omega)$.*

1. *If $u \in C^{0,\alpha}(\Omega, \mathcal{N})$ for some $0 < \alpha \leq 1$, and u is a weakly harmonic map, then $u \in C^{\infty}(\Omega, \mathcal{N})$.*

2. *Assume u is a C^2 solution of Equation (ELwh) in Ω. Then there exist $\varepsilon_2 = \varepsilon_2(m, \mathcal{N})$ and $C_1 = C_1(m, \mathcal{N})$ such that the following holds: if*
$$\theta(x, r) \leq \varepsilon_2,$$
then
$$\sup_{y \in B_{\frac{r}{2}}(x)} |\nabla u(y)|^2 \leq C_1 \theta(x, r).$$

We give now a couple of important definitions: basically, we are introducing a first notion of the "singular set" of a harmonic map; in the next chapters, we'll explore this idea in a more quantitative way. We need first to define the *regularity scale* of a map at a given point.

Definition 1.9. Let $\Omega \subset \mathbb{R}^m$ be a open set, and let $u : \Omega \to \mathcal{N}$ be a measurable map. For any $x \in \Omega$ we define the **regularity scale** of u at x as:
$$r_u(x) \doteq \max \left\{ 0 \leq r \leq \operatorname{dist}(x, \partial\Omega) \;\middle|\; \begin{array}{c} u \text{ is Lipschitz in } B_r(x); \\ r \sup_{B_r(x)} |\nabla u| \leq 1 \end{array} \right\},$$
with the agreement that $r_u(x)$ is zero if u is not Lipschitz in any neighborhood of x.

Remark. By the definition, it is evident that r_u has a good behavior with respect to scaling transformation, in the following sense: given $x \in \Omega$, $0 < \varrho < \operatorname{dist}(x, \partial\Omega)$ and $u : \Omega \to \mathcal{N}$, the regularity scale of $T_{x,\varrho} u$ at 0 is given by $\frac{1}{\varrho} r_u(x)$. Indeed:

- If u is Lipschitz in $B_r(x)$, then $T_{x,\varrho} u$ is trivially Lipschitz in $B_{\frac{r}{\varrho}}(0)$;
- The following identity holds:
$$\begin{aligned} \frac{r}{\varrho} \sup_{B_{\frac{r}{\varrho}}(0)} |\nabla T_{x,\varrho} u| &= \frac{r}{\varrho} \sup_{B_{\frac{r}{\varrho}}(0)} \varrho \, |\nabla u(x + \varrho y)| = \\ &= r \sup_{B_r(x)} |\nabla u(y)| \,. \end{aligned}$$

Now we are ready to define the following sets of singular points, where the level of singularity is measured through the regularity scale:

Definition 1.10. Let $u \colon \Omega \to \mathcal{N}$ be measurable, and $r \geq 0$ be a positive number. We denote with $\mathcal{Z}_r(u)$ the set
$$\mathcal{Z}_r(u) \doteq \{x \in \Omega \mid r_u(x) \leq r\} \,;$$
moreover, we define the **singular set** of u as follows:
$$\mathcal{S}(u) = \mathcal{Z}_0(u) \doteq \{x \in \Omega \mid r_u(x) = 0\} \,.$$

Thanks to Lemma 1.11, the ε-regularity Theorem 1.10 can be restated in terms of r_u:

Corollary 1.12 (ε-regularity, strong form)**.** *There exists a constant $\varepsilon_3 = \varepsilon_3(m, \mathcal{N})$ such that the following holds. Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map, and let $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$. If $\theta(x, r) < \varepsilon_3$, then u is C^∞ in $B_{\frac{r}{2}}(x)$; moreover, the regularity scale of u at x is at least $\frac{r}{4}$:*

$$r_u(x) \geq \frac{r}{4}.$$

A first rough estimate on the dimension of the singular set can be derived directly from the ε-regularity results. We refer to Appendix A for the definition of Hausdorff measure and Minkowski content.

Proposition 1.13. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a minimizing harmonic map. Then*

$$\mathscr{H}^{m-2}(\mathcal{S}(u) \cap B_1(0)) = \mathscr{M}^{m-2}(\mathcal{S}(u) \cap B_1(0)) = 0$$

Remark. For simplicity, we are assuming that the domain Ω of our map is the ball $B_2(0)$; it is clear that then the result can be extended to any open bounded regular domain, provided that we take the Hausdorff measure (and Minkowski content) of a set which is (uniformly) far from the boundary.

Proof. Consider, for $r > 0$ small enough, the set $\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)$, and define the family of closed balls

$$\mathcal{F}_r \doteq \left\{ \overline{B_r(x)} \;\middle|\; x \in \mathcal{S}(u) \cap B_1(0) \right\}.$$

Since it is a cover of $\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)$ with $\sup_{B \in \mathcal{F}_r} \operatorname{diam} B < \infty$, by the Vitali Covering Theorem (see [EG15, Theorem 1.24]) there exists a *finite* subfamily $\mathcal{G}_r$ made of *disjoint* balls such that

$$\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0) \subset \bigcup_{B_r(x) \in \mathcal{G}_r} B_{5r}(x).$$

We denote by $\{x_i\}_i$ the centers of the balls in $\mathcal{G}_r$; notice however that they depend on r. Then we have, indicating with $|\mathcal{G}_r|$ the cardinality of $\mathcal{G}_r$:

$$\frac{\operatorname{Vol}\left(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)\right)}{r^2} \leq \frac{|\mathcal{G}_r|\,(5r)^m}{r^2} = 5^m\,|\mathcal{G}_r|\,r^{m-2}. \tag{1.13}$$

Since the balls belonging to $\mathcal{G}_r$ are centered in $\mathcal{S}(u)$, by the ε-regularity Theorem for all $B_r(x_i) \in \mathcal{G}_r$ we have that:

$$\int_{B_r(x_i)} |\nabla u(y)|^2 \, dy = r^{m-2}\theta(x_i, r) \geq r^{m-2}\varepsilon_3.$$

In particular,

$$r^{m-2} \leq \frac{1}{\varepsilon_3} \int_{B_r(x_i)} |\nabla u(y)| \, dy; \tag{1.14}$$

this is true for all the balls in $\mathcal{G}_r$, which are disjoint, thus by summing all these equalities we get:

$$|\mathcal{G}_r| \, r^{m-2} \leq \frac{1}{\varepsilon_3} \int_{\bigcup_{\mathcal{G}_r} B_r(x_i)} |\nabla u(y)| \, dy; \tag{1.15}$$

Now consider for a moment the following weaker consequence:

$$|\mathcal{G}_r| \, r^{m-2} \leq \frac{\Lambda}{\varepsilon_3}; \tag{1.16}$$

combining Equations (1.13) and (1.16) we obtain:

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)\right) \leq \frac{5^m \Lambda}{\varepsilon_3} r^2, \tag{1.17}$$

so in particular the volume at the left hand side converges to 0 as r tends to 0. If we now combine Equation (1.13) with the stronger inequality Equation (1.14) we obtain:

$$\begin{aligned} \frac{\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)\right)}{r^2} &\leq \frac{5^m}{\varepsilon_3} \int_{\bigcup_{\mathcal{G}_r} B_r(x_i)} |\nabla u(y)|^2 \, dy = \\ &= \frac{5^m}{\varepsilon_3} \int_{\Omega} |\nabla u(y)|^2 \chi_{\bigcup_{\mathcal{G}_r} B_r(x_i)} \, dy. \end{aligned}$$

By the Dominated Convergence Theorem, the last term converges to 0 as $r \to 0$: indeed, the characteristic function appearing in the integral is less than or equal to the characteristic function of $\mathcal{B}_r(\mathcal{S}(u))$; as we have just seen, the latter converges to 0 in L^1 as r tends to 0. Thus, for any sequence $r_i \to 0$, we can find a subsequence such that the corresponding integrand converges to 0; by Dominated convergence, then, the subsequence of the integrals converges to 0: this is enough to reach the conclusion. This convergence result tells us precisely that $\mathscr{M}^{m-2}(\mathcal{S}(u) \cap B_1(0)) = 0$.

The very same estimates work for the Hausdorff measure: by definition of the Hausdorff premeasure $\mathscr{H}^{m-2}_{5r}$, we have

$$\begin{aligned} \mathscr{H}^{m-2}_{5r}(\mathcal{S}(u) \cap B_1(0)) &\leq \omega_{m-2} \, |\mathcal{G}_r| \, (5r)^{m-2} \leq \\ &\leq \frac{5^{m-2}\omega_{m-2}}{\varepsilon_3} \int_{\Omega} |\nabla u(y)|^2 \chi_{\bigcup_{\mathcal{G}_r} B_r(x_i)} \, dy, \end{aligned}$$

which gives us $\mathscr{H}^{m-2}(\mathcal{S}(u) \cap B_1(0)) = 0$ for $r \to 0$. □

1.6 An explicit example

Consider, for $N \geq 3$ and $h \geq 0$ natural numbers, the set

$$\Omega \doteq B_1^N(0) \times B_1^h(0),$$

where the superscript on the ball indicates the dimension of the ambient space; on this set, consider the maps defined (almost everywhere) as follows:

$$\begin{aligned} p = p_{N,h} \colon \Omega &\longrightarrow \mathbb{S}^{N-1} \subset \mathbb{R}^N \\ (x,y) &\longmapsto \frac{x}{|x|}; \end{aligned}$$

here as always $\mathbb{S}^{N-1}$ is the $N-1$-dimensional sphere in $\mathbb{R}^N$. We'd like to show that all these maps are minimizing harmonic.

As a first observation, notice that by the computations already done in Section 1.1 we have, for all $x \neq 0$, $1 \leq j \leq N$, $1 \leq l \leq h$, $1 \leq k \leq N$:

$$\frac{\partial}{\partial x^j} p_{N,h}(x,y)^k = \frac{\delta_j^k}{|x|} - \frac{x_j x^k}{|x|^3}, \qquad \frac{\partial}{\partial y^l} p_{N,h}(x,y)^k = 0;$$

$$\frac{\partial^2}{\partial (x^j)^2} p_{N,h}(x,y) = \frac{3x_j^2 x^k}{|x|^5} - \frac{2x_j\delta_j^k + x^k}{|x|^3}, \qquad \frac{\partial^2}{\partial (y^l)^2} p_{N,h}(x,y) = 0.$$

Then elaborating these formulae we get:

$$\begin{aligned} |\nabla p_{N,h}(x,y)|^2 &= \frac{1}{|x|^2} \sum_{1\leq j,k\leq N} \left(\delta_j^k - 2\delta_j^k \frac{x_j x^k}{|x|^2} + \frac{x_j^2 (x^k)^2}{|x|^4} \right) \\ &= \frac{1}{|x|^2} \left(n - 2\sum_{j=1}^N \frac{x_j^2}{|x|^2} + \sum_{1\leq j,k\leq N} \frac{x_j^2 (x^k)^2}{|x|^4} \right) = \frac{N-1}{|x|^2}; \\ \Delta p_{N,h}(x,y)^k &= \frac{3x^k}{|x|^5} \sum_{j=1}^N x_j^2 - \frac{2}{|x|^3} \sum_{j=1}^N x_j \delta_j^k - N\frac{x^k}{|x^3|} = \frac{1-N}{|x|^3} x^k; \\ \Delta p_{N,h}(x,y) &= \frac{1-N}{|x|^3} x. \end{aligned}$$

This tells us, in first place, that for $N \geq 3$ the maps $p_{N,h}$ are actually in $W^{1,2}\big(\Omega, \mathbb{S}^{N-1}\big)$. Recalling the formula (1.8) for the second fundamental form for the sphere, we get that

$$\Delta p_{N,h} + A(x,y)\left(\nabla p_{N,h}, \nabla p_{N,h}\right) = \Delta p_{N,h} + \left(|\nabla p_{N,h}|^2\right) \frac{x}{|x|} = 0;$$

so outside of the subspace $\{0\} \times B_1^k(0)$ the Equation (ELwh) is satisfied in the classical sense. With a little effort, one could prove that actually $p_{N,h}$ is a weak solution of the equation in Ω, and thus weakly harmonic; instead we will prove directly that it is energy minimizing, following the technique introduced in [Lin87].

Case $h = 0$: Lin's approach. We start by analyzing the map $p_N \doteq p_{N,0}$. Notice that by the computations we just made, the explicit value of the energy of p_N is available:

$$\begin{aligned}\mathcal{E}(p_N) &= \int_{B_1^N(0)} |\nabla p_N(x)|^2 \, dx = \int_{B_1^N(0)} \frac{N-1}{|x|^2} \, dx = \\ &= (N-1) \int_0^1 \int_{\partial B_1^N(0)} \frac{1}{r^2} d\sigma \, dr = (N-1) \left|\mathbb{S}^{N-1}\right| \int_0^1 r^{N-3} \, dr = \\ &= \frac{N-1}{N-2} \left|\mathbb{S}^{N-1}\right| \doteq K(N).\end{aligned}$$

So our goal is to show that if $u \in W^{1,2}\left(B_1^N(0), \mathbb{S}^{N-1}\right)$ is a map with $u(x) = p_N(x)$ in a neighborhood of $\partial B_1^N(0)$, then its energy is greater than or equal to $K(N)$. The upcoming lemma goes in this direction. Recall that we denote by $\mathrm{Tr}(A)$ the trace of the matrix A.

Lemma 1.14. *Let $u \in W^{1,2}\left(B_1^N(0), \mathbb{S}^{N-1}\right)$ be a map such that $u(x) = p_N(x)$ for all x in a neighborhood of $\mathbb{S}^{N-1}$. The following identity holds:*

$$\frac{1}{N-2} \int_{B_1^N(0)} \left[(\operatorname{div} u(x))^2 - \mathrm{Tr}\left[(\nabla u(x))^2\right]\right] \, dx = \frac{N-1}{N-2} \left|\mathbb{S}^{N-1}\right|. \tag{1.18}$$

Sketch of Proof. The following relations are satisfied (in all the summations the indices run from 1 to N):

$$\begin{aligned}\operatorname{div}\left((\operatorname{div} u)\, u\right) &= \sum_i \frac{\partial}{\partial x^i} \left(\sum_j u^i \left(\frac{\partial u^j}{\partial x^j}\right)\right) = \\ &= \sum_{i,j} \left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} u^j\right) u^i + \sum_{i,j} \frac{\partial u^i}{\partial x^i} \frac{\partial u^j}{\partial x^j}; \\ \operatorname{div}\left((\nabla u)\, u\right) &= \sum_j \frac{\partial}{\partial x^j} \left(\sum_i \left(\frac{\partial u^j}{\partial x^i}\right) u^i\right) = \\ &= \sum_{i,j} \left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} u^j\right) u^i + \sum_{i,j} \frac{\partial u^j}{\partial x^i} \frac{\partial u^i}{\partial x^j}.\end{aligned}$$

Subtracting side by side, we get:

$$\begin{aligned}\operatorname{div}\left[(\operatorname{div} u)\, u - (\nabla u)\, u\right] &= \sum_i \sum_j \frac{\partial u^i}{\partial x^i} \frac{\partial u^j}{\partial x^j} - \sum_i \sum_j \frac{\partial u^j}{\partial x^i} \frac{\partial u^i}{\partial x^j} = \\ &= (\operatorname{div} u(x))^2 - \mathrm{Tr}\left[(\nabla u(x))^2\right].\end{aligned}$$

Now we apply an appropriate version of the Divergence Theorem (see for example [Wil13, Section 6.3]), and we observe that both $u(x)$ and the normal unit vector $\nu(x)$ to $\mathbb{S}^{N-1}$ equal x on $\mathbb{S}^{N-1}$; moreover $u \equiv p_N$ in a neighborhood of the sphere. We obtain:

$$\int_{B_1^N(0)} \left[(\operatorname{div} u(x))^2 - \operatorname{Tr}\left[(\nabla u(x))^2\right]\right] dx =$$

$$= \int_{\mathbb{S}^{N-1}} [(\operatorname{div} u)\, u - (\nabla u)\, u] \cdot \nu \, d\sigma =$$

$$= \int_{\mathbb{S}^{N-1}} [(\operatorname{div} p_N(x)) \langle u(x), \nu(x)\rangle - \langle \nu(x), \nabla p_N(x) x\rangle] ;$$

but $\langle u(x), \nu(x)\rangle = 1$ on the sphere, and $\operatorname{div} p_N(x)$ simply equals $N-1$; moreover,

$$\langle \nu(x), \nabla p_N(x) x\rangle = 0$$

because p_N is homogeneous. This carries the result immediately. □

Theorem 1.15. *The map p_N is energy minimizing in $W^{1,2}\left(B_1^N(0), \mathbb{S}^{N-1}\right)$.*

Proof. By Proposition 1.3, there must exist a minimizing map v which coincides with p_N outside a compact. By Proposition 1.13, the singular set of v has $\mathscr{H}^{m-2}$-measure zero (eventually extending v a bit outside the ball), so it is C^1 at $\mathscr{L}^m$-almost every point of Ω.

Let $x \in \Omega$ be a point in which v is differentiable. We claim that

$$|\nabla v(x)|^2 \geq \frac{1}{N-2} \left[(\operatorname{div} v(x))^2 - \operatorname{Tr}\left[(\nabla v(x))^2\right]\right] ;$$

then our statement follows easily by integrating this inequality on $B_1(0)$ and using Lemma 1.14.

To prove the claim, define the following quantity:

$$f_v(x) \doteq |\nabla v(x)|^2 - \frac{1}{N-2} \left[(\operatorname{div} v(x))^2 - \operatorname{Tr}\left[(\nabla v(x))^2\right]\right] ;$$

we have to show that $f_v(x) \geq 0$. Now f_v is stable for rotations, that is: for any rotation Q of $\mathbb{R}^m$ set $\hat{v} \doteq Q^T \circ v \circ Q$; then we have $f_{\hat{v}} = f_v$. So we can assume without loss of generality that $v(x) = e_n = (0, \dots, 0, 1)$. In particular, the last component v^n must have derivatives equal to 0 at x, since v_n has a maximum at x and v is C^1 at that point. Hence the following facts are true:

- By the inequality
$$\left(\sum_{j=1}^{M} 1 \cdot a_j\right)^2 \leq M \sum_{j=1}^{M} a_j^2,$$
which is true for all a_j by Cauchy-Schwarz, we get:
$$(\operatorname{div} v(x))^2 = \left(\sum_{j=1}^{N-1} \frac{\partial v^j}{\partial x^j}(x)\right)^2 \leq (N-1) \sum_{j=1}^{N-1} \left(\frac{\partial v^j}{\partial x^j}(x)\right)^2 .$$

- By definition of the trace, and by the elementary inequality

$$2a_{ij}a_{ji} \geq -\left(a_{ij}^2 + a_{ji}^2\right),$$

we get:

$$\begin{aligned}\mathrm{Tr}\left[(\nabla v(x))^2\right] &= \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial v^j}{\partial x^i}(x)\frac{\partial v^i}{\partial x^j}(x) = \\ &= \sum_{j=1}^{N-1}\left(\frac{\partial v^j}{\partial x^j}(x)\right)^2 + 2\sum_{1\leq i<j\leq N} \frac{\partial v^j}{\partial x^i}(x)\frac{\partial v^i}{\partial x^j}(x) \geq \\ &\geq \sum_{j=1}^{N-1}\left(\frac{\partial v^j}{\partial x^j}(x)\right)^2 - \sum_{i\neq j}\left(\frac{\partial v^j}{\partial x^i}(x)\right)^2.\end{aligned}$$

- By definition of the Hilbert-Schmidt norm,

$$|\nabla v(x)|^2 = \sum_{j=1}^{N-1}\left(\frac{\partial v^j}{\partial x^j}(x)\right)^2 + \sum_{i\neq j}\left(\frac{\partial v^j}{\partial x^i}(x)\right)^2.$$

Combining these three relations, we get:

$$(N-2)f_v(x) \geq (N-3)\sum_{i\neq j}\left(\frac{\partial v^j}{\partial x^i}(x)\right)^2,$$

which is clearly greater than or equal to 0 whenever $N \geq 3$. This proves the theorem. □

Case $h > 0$. To prove that $p_{N,h}$ is actually energy minimizing also for $h > 0$, it suffices to prove the following more general result.

Proposition 1.16. *Let $\Omega_1 \subset \mathbb{R}^l$ and $\Omega_2 \subset \mathbb{R}^h$ be open bounded regular sets and $\mathcal{N}$ a n-dimensional manifold in $\mathbb{R}^N$ (with the usual hypotheses); let $u \in W^{1,2}(\Omega_1, \mathcal{N})$ be energy minimizing. Define the map $\hat{u} \in W^{1,2}(\Omega_1 \times \Omega_2, \mathcal{N})$ as $\hat{u}(x, y) \doteq u(x)$. Then $\hat{u}$ is energy minimizing.*

Proof. Assume by contradiction there exists $v \in W^{1,2}(\Omega_1 \times \Omega_2, \mathcal{N})$ which coincides with $\hat{u}$ outside a compact set, and such that $\mathcal{E}(v) < \mathcal{E}(\hat{u})$. It is clear that

$$|\nabla v(x,y)|^2 = |\nabla_x v(x,y)|^2 + |\nabla_y v(x,y)|^2 \geq |\nabla_x v(x,y)|^2,$$

where we denote $\nabla_x v(x,y)$ the first l columns of $\nabla v(x,y)$ and with $\nabla_y v(x,y)$ the last h columns; instead, $|\nabla \hat{u}(x,y)|^2 = |\nabla_x \hat{u}(x,y)|^2$. Then we have:

$$\int_{\Omega_1\times\Omega_2} |\nabla_x v(x,y)|^2 \leq \mathcal{E}(v) < \mathcal{E}(\hat{u}) \leq \int_{\Omega_1\times\Omega_2} |\nabla_x \hat{u}(x,y)|^2,$$

thus by Tonelli Theorem

$$\int_{\Omega_2}\int_{\Omega_1}\left[|\nabla_x v(x,y)|^2 - |\nabla u(x)|^2\right]\,dx\,dy < 0.$$

In particular, there exists a subset U of Ω_2 of positive $\mathscr{L}^h$-measure such that for all $y \in U$

$$\int_{\Omega_1}|\nabla_x v(x,y)|^2\,dx < \int_{\Omega_1}|\nabla u(x)|^2\,dx = \mathcal{E}(u);$$

for any of this y, the map $x \mapsto v(x,y)$ coincides with u outside of a compact set and has energy strictly smaller than the energy of u, contradicting the minimality of u. □

Remark. One thing we can observe from this explicit example is the following: the map $p_{N,h}$ has a h-dimensional subspace of singularities (the subspace $x_1 = \cdots = x_N = 0$), where the function is not continuous. Calling $m = N + h$ the total dimension of the domain, we have found minimizing harmonic maps such that

$$\mathscr{H}^{m-N}(\mathcal{S}(u)) = \mathscr{M}^{m-N}(\mathcal{S}(u)) > 0.$$

In particular, choosing $N = 3$, we have that the dimension of the singular set (Hausdorff or Minkowski) is $m - 3$. This means that

$$\dim \mathcal{S}(u) \leq m - 3$$

is the best result we can hope to achieve for a minimizing harmonic map: to prove this, indeed, will be the overall goal of this work.

Chapter 2

Quantitative Stratification and Dimension Estimates

This chapter is devoted to the study of the singularities of minimizing harmonic maps, via the definition of a *stratification* for the singular set: assume that a minimizing map $u \in W^{1,2}(\Omega, \mathcal{N})$ is given, with $\Omega \subset \mathbb{R}^m$; for any subdimension $k \leq m$, we will examine the set of points at which u has *at most* k independent directions of "almost-symmetry". Following this procedure for k ranging from 0 to m, we obtain a decomposition of the singular set in a sequence of increasing layers, called *strata*.

Following the work of Cheeger and Naber [CN13], we'll give an estimate on the Minkowski dimension of all these *strata*. Precise definitions of Hausdorff measure and Minkowski content are given in Appendix A. As a consequence of these results, we'll be able to produce an estimate on the singular set we introduced in Definition 1.10: one final result of this chapter will be that for any $\delta > 0$ there exists a constant C such that

$$\operatorname{Vol}\left(\mathcal{B}_r(\mathcal{S}(u))\right) \leq C r^{3-\delta};$$

and this will follow easily once we have clarified how the stratification is made and once we have proved the main results for the layers of the stratification.

In this chapter, Ω will still be a bounded open subset of $\mathbb{R}^m$, with regular boundary; for convenience, we'll also assume it connected.

2.1 Symmetry and almost-symmetry

We begin with the definition of a class of "model maps", satisfying some strong symmetry properties. Our definitions of regularity for harmonic maps will rest on these concepts: indeed, we'll quantify the k-symmetry of a harmonic map observing how close it is (in a L^2 sense) to one of these model maps. As always, $\mathcal{N}$ is a compact n-dimensional manifold without boundary embedded in $\mathbb{R}^N$.

Definition 2.1. Let $h : \mathbb{R}^m \to \mathcal{N}$ be a measurable map and $0 \leq k \leq m$ an integer. We say that h is k**-symmetric** (at the point $y \in \mathbb{R}^m$) if:

(i) h is homogeneous with respect to y: *i.e.*, for every $\lambda > 0$ and $z \in \mathbb{R}^m$

$$h(y + \lambda z) = h(y + z);$$

(ii) There exists a k-plane $V \subseteq \mathbb{R}^m$ such that h is V-invariant: *i.e.*, for every point $z \in \mathbb{R}^m$ and vector $v \in V$,

$$h(z + v) = h(z).$$

Given an open subset $A \subset \mathbb{R}^m$, we say that a measurable map $h : A \to \mathcal{N}$ is k-symmetric in A at the point $y \in \mathbb{R}^m$ if there exists a map $\tilde{h} : \mathbb{R}^m \to \mathcal{N}$ which is k-symmetric at y and such that $\tilde{h}\Big|_A \equiv h$. Also, we still say that a map h is k-symmetric if it coincides almost everywhere with a k-symmetric map.

Remark. If h is m-symmetric, then it is trivially constant, since it needs to be invariant for the whole $\mathbb{R}^m$. If h is 0-symmetric, we're only saying it is homogeneous.

An immediate geometric consequence of the definition is the following sufficient condition:

Lemma 2.1. *Let $h : \mathbb{R}^m \to \mathcal{N}$ be a measurable function. Let $0 \leq k \leq m$ be an integer. If there exist $k + 1$ points $\{x_i\}_{i=0}^k$ in general position (*i.e.*, spanning a k-dimensional affine subspace) such that h is homogeneous with respect to all of them, then h is k-symmetric and has*

$$\operatorname{span}\{x_1 - x_0, \ldots, x_k - x_0\}$$

as an invariant subspace.

Proof. Assume first that $k = 1$, $x_0 = 0$ and $x_1 = e_1$. Given a third point $p \in \mathbb{R}^m$, we want to show that $h(p + \alpha e_1) = h(p)$ for any $\alpha \in \mathbb{R}$. If p belongs to the straight line $\operatorname{span}\{e_1\}$, the result is trivial. In the case $p \notin \operatorname{span}\{e_1\}$, we'll show more: indeed, h is constant on the whole half plane

$$\pi^+ \doteq \{\alpha_1 e_1 + \alpha_2 e_2 \mid \alpha_1 \in \mathbb{R},\, \alpha_2 > 0\}$$

defined by the straight line $\operatorname{span}\{e_1\}$ and containing p. Up to a linear transformation, we can assume that $p = e_2$. By the homogeneity with respect to x_0 we know that $h(\beta e_2) = h(e_2)$ for any $\beta > 0$; call $\hat{h} \doteq h(e_2)$. Since h is also constant along the half lines originating from e_1, we have that $h(y) = \hat{h}$ for all $y = \alpha_1 e_1 + \alpha_2 e_2$ in π^+ such that $\alpha_1 < 1$. Now exploiting again the fact that h is homogeneous with respect to the origin (and considering the points of π^+ in the stripe $0 < \alpha_1 < 1$) we get that $h(x)$ must equal $\hat{h}$ also in $\pi^+ \cap \{\alpha_1 > 0\}$; thus h is constant in π^+.

We have then discovered that $h(p + \alpha_i(x_0 - x_i)) = h(p)$ for all $p \in \mathbb{R}^m$ and all $\alpha_i \in \mathbb{R}$. But this trivially implies that

$$h\left(p + \sum_{i=1}^{k} \alpha_i(x_i - x_0)\right) = h(p) \quad \forall p \in \mathbb{R}^m,\ \forall(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k,$$

simply by iterating the previous process. □

Remark. The converse is also clearly true: if a map h is k-symmetric, then one can easily find $k+1$ points in general position such that h is homogeneous with respect to all of them, just by translating the known homogeneous point in the k directions of invariance.

For stationary harmonic maps, homogeneity has a very powerful characterization in terms of the (modified) normalized energy. A slight modification of this easy result will be a key point in the next sections. Again, ψ is now chosen to be strictly decreasing in $]0, 1[$.

Proposition 2.2. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a stationary harmonic map, and let $\psi \in C_0^\infty([0, \infty), \mathbb{R})$ be a function with* $\operatorname{spt}\psi \subset [0, 1]$ *and $\psi' < 0$ in $(0, 1)$. Assume $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$. Then the following are equivalent:*

(a) u is homogeneous with respect to x in $B_r(x)$;

(b) There exists $0 < s < r$ such that $\theta_\psi(x, r) - \theta_\psi(x, s) = 0$;

Proof. If u is homogeneous with respect to x, then clearly $\partial_{r_x(y)} u(y) = 0$ for any y in the mentioned ball; by the Monotonicity Formula (MF), this readily implies that $\frac{d}{ds}\theta_\psi(x, s) = 0$ for *all* s, and this in particular tells us that $\theta_\psi(x, \cdot)$ is constant.

Conversely, assume that $\theta_\psi(x, r) - \theta_\psi(x, s) = 0$ for one s. By monotonicity, $\theta_\psi(x, \cdot)$ must be constant in the whole interval $[s, r]$. Then, again by (MF), we have

$$-2r^{-m} \int_\Omega |y - x| \psi'\left(\frac{|y - x|}{r}\right) |\partial_{r_x(y)} u(y)|^2 dy = 0;$$

and now r is fixed and positive; $|y - x|\psi'\left(\frac{|y-x|}{r}\right)$ is strictly negative for all $y \in B_r(x) \setminus \{x\}$. Thus $\partial_{r_x(y)} u(y)$ is 0 for all such y: this means exactly that u is radially constant. □

As a corollary of Lemma 2.1 and Proposition 2.2, we obtain a sufficient condition for k-symmetry in the case of stationary harmonic maps, again by exploiting the modified normalized energy.

Corollary 2.3. *Let* $0 < s < r$*; let* $u \in W^{1,2}(\Omega, \mathcal{N})$ *be a stationary harmonic map, and take* $\psi \in C_0^\infty([0,\infty), \mathbb{R})$ *as before. Let* $0 \le k \le m$ *be an integer.* ***If*** *there exist* $k+1$ *points* $\{x_i\}_{i=0}^k$ *such that:*

- $x_i \in B_{\frac{r}{2}}(x_0) \subset \Omega$ *for any* $i = 1, \dots, k$*;*
- $\{x_i\}_{i=0}^k$ *span a* k*-dimensional affine subspace (*i.e., $\{x_i\}_{i=0}^k$ *are in general position);*
- *For all* $i = 0, \dots, k$,

$$\theta_\psi(x_i, r) - \theta_\psi(x_i, s) = 0;$$

then u *is* k*-symmetric in* $B_{\frac{r}{2}}(x_0)$ *at the point* x_0*. The converse is also true.*

2.1.1 Tangent maps

Now fix a map $u \in W^{1,2}(\Omega, \mathcal{N})$, and remember the Definition 1.8 of rescaled map from Chapter 1:

$$T_{x,r}u(y) = u(x + ry).$$

Definition 2.2 (Tangent maps)**.** We say that a map $g : B_1(0) \to \mathcal{N}$ is a **tangent map** of u at the point $x \in \Omega$ if there exists a sequence $\{r_i\}_{i\in\mathbb{N}}$ converging to zero such that

$$\lim_{i\to\infty} \int_{B_1(0)} |g(y) - T_{x,r_i}u(y)|^2 \, dy = 0.$$

Roughly speaking, we are zooming the map u around the point x and looking for a suitable local approximation. Taking inspiration from this, we'll soon give a precise definition of "almost-symmetry".

Proposition 2.4 (Tangent maps of harmonic maps)**.** *Let* $u \in W^{1,2}(\Omega, \mathcal{N})$ *be a minimizing harmonic map with energy bounded by* Λ*. Then, for any* $x \in \Omega$*:*

1. u *has at least a tangent map* g *at* x*;*
2. *Any tangent map of* u *at* x *is* 0*-symmetric.*
3. *Every tangent map is also energy minimizing.*

Proof. As we know, the maps $T_{x,r}u\big|_{B_1(0)}$ are stationary for all $0 < r < 1$, and

$$\mathcal{E}\left(T_{x,r}u\big|_{B_1(0)}\right) = \theta^u(x, r) \le \Lambda.$$

Thus by compactness the sequence of maps $\left\{T_{x,j^{-1}}u\right\}_{j\in\mathbb{N}}$ has a subsequence that converges weakly in $W^{1,2}(\Omega, \mathcal{N})$ and strongly in L^2 to a map $g \in W^{1,2}(B_1(0), \mathcal{N})$. But then by definition g is a tangent map of u at x.

If $h : B_1(0) \to \mathcal{N}$ is any other tangent map at x, there exists a sequence $\{r_i\} \to 0$ such that

$$T_{x,r_i}u|_{B_1(0)} \xrightarrow{L^2} h.$$

But then again there exists a subsequence of $\{T_{x,r_i}u\}$ (not relabeled) that converges $W^{1,2}(B_1(0),\mathcal{N})$-strongly to an energy minimizing map, and by the uniqueness of the limit h itself must be energy minimizing. Moreover, using strong convergence, scale invariance, and the monotonicity of $\theta_\psi(x,\cdot)$, we have:

$$\begin{aligned}\theta^h_\psi(0,1) - \theta^h_\psi\left(0,\frac{1}{2}\right) &= \\ &= \lim_{i\to\infty}\left[\theta_\psi\left[T_{x,r_i}u|_{B_1(0)}\right](0,1) - \theta_\psi\left[T_{x,r_i}u|_{B_1(0)}\right]\left(0,\frac{1}{2}\right)\right] = \\ &= \lim_{i\to\infty}\left[\theta^u_\psi(x,r_i) - \theta_\psi\left(x,\frac{1}{2}r_i\right)\right] = 0;\end{aligned}$$

so h is homogeneous by Proposition 2.2. □

Remark. It's not difficult to see that assertions 1 and 2 of Proposition 2.4 are also true for stationary harmonic maps; however, the compactness argument needed to prove 2 is slightly more complicated in that case.

Remark. Tangent maps might not be unique; in fact, uniqueness results are known only in some special cases (see [Sim96, Section 3.10]).

2.1.2 Almost symmetry

As already said, we need to state in a clear and quantitative way what it means for a map to be "nearly symmetric". We do so in the following definition.

Definition 2.3. Let $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$; fix $\eta > 0$ and an integer $0 \le k \le m$. We say that $u \in W^{1,2}(\Omega,\mathcal{N})$ is **(η, r, k)-symmetric at x** if there exists a map $h : \mathbb{R}^m \to \mathcal{N}$ such that:

1. h is k-symmetric at the origin;
2. We have
$$\int_{B_1(0)} |h(y) - T_{x,r}u(y)|^2 \, dy < \eta.$$

We'll also use the expression **(η, r)-homogeneous** to denote a $(\eta, r, 0)$-symmetric map.

Remark. It is clear that we could equivalently require the existence of a map h such that h is k symmetric *at* x and

$$\frac{1}{r^m}\int_{B_r(x)} |h(y) - u(y)|^2 \, dy < \eta.$$

Just as in Corollary 2.3, also (η, r, k)-symmetry can be characterized through the function θ_ψ. This is heuristically clear, since the derivative of θ_ψ depends on the radial derivative of u in a very explicit way: hence, whenever $\theta_\psi(x, r) - \theta_\psi(x, s)$ is small enough (with $x \in \Omega$ and $r > s$), u must be arbitrarily "close" to a 0-symmetric map. Moreover, in view of Corollary 2.3, we can expect that if we have $k+1$ points of almost-0-symmetry, we can gain the almost-k-symmetry condition. For this purpose, a definition of *effective linear independence* is needed:

Definition 2.4. Given $k+1$ points $\{x_i\}_{i=0}^k$ in $\mathbb{R}^m$ (with $0 \le k \le m$), and $\lambda > 0$, we say that $\{x_i\}_i$ are in λ-general position if for all $j = 1, \dots, k$

$$\operatorname{dist}\left(x_j, x_0 + \operatorname{span}\left\{x_1 - x_0, \dots, x_{j-1} - x_0\right\}\right) \ge \lambda.$$

When the condition is met, we also say that $\{x_1 - x_0, \dots, x_k - x_0\}$ are λ-linearly independent.

Before going on, we need a preliminary lemma: effective linear independence passes to the limit.

Lemma 2.5. *For any $j \in \mathbb{N}$, let $\{x_{ij}\}_{i=0}^k$ be $k+1$ points of $\mathbb{R}^m$ in λ-general position, with $\lambda > 0$. Assume that $x_{ij} \xrightarrow{j\to\infty} \bar{x}_i$ for all $i = 0, \dots, k$. Then $\{\bar{x}_i\}_{i=0}^k$ are still in λ-general position.*

Proof. By continuity of the distance function, we clearly have that

$$\lim_{j\to\infty} \operatorname{dist}(x_{1j}, x_{0j}) = \operatorname{dist}(\bar{x}_1, \bar{x}_0),$$

so in particular the last term is still greater than or equal to λ.

By induction, assume that $\{\bar{x}_i\}_{i=1}^{h-1}$ are in λ-general position. Let

$$V_h^j \doteq x_{0j} + \operatorname{span}\left\{x_{1j} - x_{0j}, \dots, x_{h-1,j} - x_{0j}\right\}$$

and let A_h^j be the matrix whose columns are the vectors

$$\left\{x_{1j} - x_{0j}, \dots, x_{h-1,j} - x_{0j}\right\}.$$

By an elementary linear algebra fact (see for example [Str80, Section 4.2]), the orthogonal projection of a point p on V_h^j is given by

$$\pi_{V_h^j}(p) = x_{0j} + A_h^j\left((A_h^j)^T A_h^j\right)^{-1}(A_h^j)^T(p - x_{0j}).$$

To prove the result we need to compute the limit

$$\lim_{j\to\infty} \operatorname{dist}\left(x_{hj}, \pi_{V_h^j}(x_{hj})\right);$$

but now the distance is continuous, and we have written $\pi_{V_h^j}(x_{hj})$ in terms of continuous functions of the points $\{x_{0j}, \dots, x_{hj}\}$. This means precisely that the map we are taking the limit on is a continuous function of those same points, so the statement follows. □

Remark. It is easy to see that, on the contrary, simple linear independence does not pass to the limit: this is the main reason why we need the new concept. Indeed, without the effectiveness condition, a sequence of sets of independent points can even collapse to a single point.

Now, the key fact about the new notion of effective linear independence is that, for a set of points which "effectively" span a k-dimensional subspace, a pinching condition on the normalized energy of a map guarantees that the map is almost k-symmetric. We now assume that the function u is *minimizing*, in order to apply the Compactness Theorem 1.4.

Proposition 2.6. *Fix the following constants:*

$\Lambda > 0:$	*a bound for the energy,*
$\tau > 0:$	*controlling the effective linear independence,*
$0 < \gamma < 1:$	*appearing as a "ratio of radii" in the condition on* θ_ψ,
$\eta > 0:$	*giving the desired parameter for almost-symmetry.*

There exists an $\varepsilon_4 = \varepsilon_4(m, \mathcal{N}, \Lambda, \tau, \gamma, \eta)$ *such that the following implication holds: let* $x \in \Omega$ *and* $0 < r < \frac{1}{2}\operatorname{dist}(\partial\Omega)$, *and let* u *be a* minimizing *harmonic map with energy bounded by* Λ; ***if*** *there exist* $k+1$ *points* $\{x_i\}_{i=0}^k$ *such that*

(i) $x_i \in B_{\frac{r}{2}}(x)$ *for all* $i = 0, \dots, k$;

(ii) $\{x_i\}_{i=0}^k$ *are* $(r\tau)$*-linearly independent;*

(iii) $\theta_\psi(x_i, r) - \theta_\psi(x_i, \gamma r) < \varepsilon_4$ *for all* $i = 0, \dots, k$,

then u *is* $(\eta, \frac{1}{2}r, k)$*-symmetric at* x.

Proof of Proposition 2.6. Without loss of generality, we can assume $x = 0$ and $r = 1$; the general case follows by the application of this particular result to the function $\tilde{u}(y) = u(x + ry)$, and by using the scale invariance of the normalized energy:

$$\theta_\psi^{\tilde{u}}(\tilde{x}, \tilde{r}) = \theta_\psi^u(x + r\tilde{x}, r\tilde{r}).$$

By contradiction, assume that there exist $\{u_j\}_{j\in\mathbb{N}}$ minimizing harmonic maps with energy bounded by Λ and, for every $i = 0, \dots, k$, a sequence of points $\{x_{ij}\}_{j\in\mathbb{N}}$ in $B_{\frac{1}{2}}(0)$ such that

- $\{x_{ij}\}_{i=0}^k$ are τ-linearly independent for all j;
- $\theta_\psi(x_{ij}, 1) - \theta_\psi(x_{ij}, \gamma) < \frac{1}{j}$ for all i, j;
- u_j is not $(\eta, \frac{1}{2}, k)$-symmetric at 0.

By the Compactness Theorem 1.4 for minimizing maps in $W^{1,2}(\Omega,\mathcal{N})$, up to a subsequence we have $u_j \to \bar{u}$ strongly in $W^{1,2}(\Omega,\mathcal{N})$, where $\bar{u}$ is a minimizing harmonic map. Also, up to subsequences, for all $i = 0,\dots,k$ there exists $\bar{x}_i$ in $\overline{B_{\frac{1}{2}}(0)}$ such that $x_{ij} \to \bar{x}_i$ for $j \to \infty$.

Now τ-linear independence passes to limit by Lemma 2.5, so $\{\bar{x}_i\}_{i=0}^m$ are still in τ-general position. Let's show that the following equality holds for all ϱ (such that all quantities are well defined):

$$\theta_\psi^{\bar{u}}(\bar{x}_i,\varrho) = \lim_{j\to\infty} \theta_\psi^{u_j}(x_{ij},\varrho).$$

Indeed, for every $\delta > 0$ there exists $\hat{\jmath}$ such that

$$\bigcap_{j\geq\hat{\jmath}} B_\varrho(x_{ij}) \supset B_{\varrho-\delta}(\bar{x}_i) \qquad \text{and} \qquad \bigcup_{j\geq\hat{\jmath}} B_\varrho(x_{ij}) \subset B_{\varrho+\delta}(\bar{x}_i).$$

Then for $j \geq \hat{\jmath}$ the following inequalities hold

$$\int_{B_{\varrho-\delta}(\bar{x}_i)} |\nabla u_j|^2 dx \leq \int_{B_\varrho(x_{ij})} |\nabla u_j|^2 dx \leq \int_{B_{\varrho+\delta}(\bar{x}_i)} |\nabla u_j|^2 dx.$$

Moreover, by the convergence in $W^{1,2}(\Omega,\mathcal{N})$, for $j \to \infty$

$$\int_{B_{\varrho-\delta}(\bar{x}_i)} |\nabla u_j|^2 dx \longrightarrow \int_{B_{\varrho-\delta}(\bar{x}_i)} |\nabla \bar{u}|^2 dx$$
$$\int_{B_{\varrho+\delta}(\bar{x}_i)} |\nabla u_j|^2 dx \longrightarrow \int_{B_{\varrho+\delta}(\bar{x}_i)} |\nabla \bar{u}|^2 dx,$$

so we obtain that for any $\delta > 0$

$$\int_{B_{\varrho-\delta}(\bar{x}_i)} |\nabla \bar{u}|^2 dx \leq \liminf_{j\to\infty} \int_{B_\varrho(x_{ij})} |\nabla u_j|^2 dx \leq$$
$$\leq \limsup_{j\to\infty} \int_{B_\varrho(x_{ij})} |\nabla u_j|^2 dx \leq \int_{B_{\varrho+\delta}(\bar{x}_i)} |\nabla \bar{u}|^2 dx.$$

But then taking the limit for $\delta \to 0$ and using the Dominated Convergence Theorem we prove the claim.

Then for every $\bar{x}_i$ we have

$$\theta_\psi^{\bar{u}}(\bar{x}_i,1) - \theta_\psi^{\bar{u}}(\bar{x}_i,\gamma) = \lim_{j\to\infty} \left[\theta_\psi^{u_j}(x_{ij},1) - \theta_\psi^{u_j}(x_{ij},\gamma)\right] = 0,$$

so in particular $\bar{u}$ is k-symmetric by Corollary 2.3; and the u_j's are converging to $\bar{u}$ in $L^2(\Omega,\mathcal{N})$-norm, which says exactly that u_j is $(\eta,\frac{1}{2},k)$-symmetric for some j. □

We find useful to introduce the following definition:

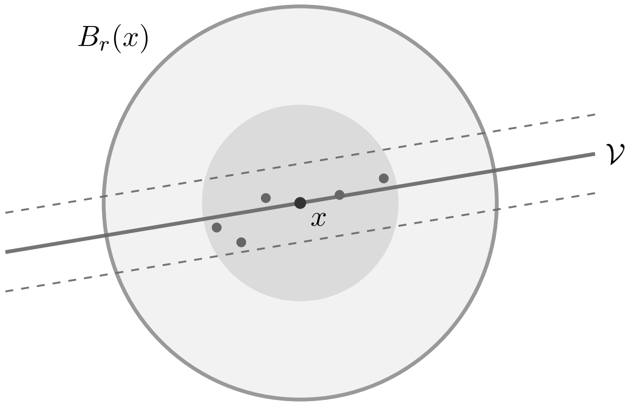


Figure 2.1: The set of "pinched" points $\mathcal{C}_{\varepsilon_5,\gamma}(x,r)$ lies inside the fattening of a k-subspace, if u is not close to be $(k+1)$-symmetric (see Corollary 2.7).

Definition 2.5. Let x, r, γ be as in the statement of Proposition 2.6, and let $\varepsilon > 0$. We define the set

$$\mathcal{C}(x,r) = \mathcal{C}_{\varepsilon,\gamma}(x,r) \doteq \left\{ y \in B_{\frac{r}{2}}(x) \mid \theta_\psi(y,r) - \theta_\psi(y,\gamma r) < \varepsilon \right\}.$$

In view of this definition, Proposition 2.6 can be restated as:

Corollary 2.7. *Fix Λ, γ, η as before, and $\varrho > 0$. There exists an $\varepsilon_5 = \varepsilon_5(m,\mathcal{N},\Lambda,\varrho,\gamma,\eta)$ such that: **if** u is a minimizing harmonic map, $x \in \Omega$, $0 < r < \frac{1}{2}\operatorname{dist}(x,\partial\Omega)$, and x is such that u is not $(\eta, \frac{1}{2}r, k+1)$-symmetric at x, **then** there exists a k-dimensional affine subspace $\mathcal{V} = \mathcal{V}(u,x,r)$ of $\mathbb{R}^m$ such that the following inclusion holds.*

$$\mathcal{C}_{\varepsilon_5,\gamma}(x,r) = \left\{ y \in B_{\frac{1}{2}r}(x) \mid \theta(y,r) - \theta(y,\gamma r) < \varepsilon_5 \right\} \subset B_{\frac{1}{2}\varrho r}(\mathcal{V}).$$

2.1.3 L^2-Limits of invariant maps

During the course of this chapter, we'll also need a result that guarantees that k-symmetry passes to the limit. The proof is elementary but a bit technical.

Proposition 2.8. *Let $g \in L^2(B_1(0),\mathcal{N})$, let $1 \le k \le m$ be an integer, and let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of k-symmetric maps (at the origin). If g is the strong-$L^2(B_1(0),\mathcal{N})$ limit of the h_i's, then g is k-symmetric (at the origin).*

Proof. Without loss of generality, we can assume that $h_i \to g$ almost everywhere, upon selecting a suitable subsequence. Also, we can clearly think of the h_i's to be defined in all $\mathbb{R}^m$. We begin by proving homogeneity, and then examine the k-invariance.

STEP 1. We are assuming that all the maps h_i are homogeneous with respect to the origin, that is

$$h_i(\lambda x) = h_i(x)$$

for all $\lambda > 0$ and $x \in B_1(0)$. For $x \in B_1(0)$ outside a set K of measure zero,

$$g(x) = \lim_{i\to\infty} h_i(x).$$

Now, take a direction $v \in \mathbb{S}^{m-1}$; if $\lambda_1 \neq \lambda_2$ and both $\lambda_1 v$, $\lambda_2 v$ belong to $B_1(0) \setminus K$, then necessarily $g(\lambda_1 v) = g(\lambda_2 v)$. If there exists such a λ (*i.e.*, such that $\lambda v \in B_1(0) \setminus K$), then we can redefine g to equal $g(\lambda v)$ in the half-line generating from 0 with direction v. If such a λ doesn't exist, we simply redefine g to equal an arbitrary value in that half line. The modified map is homogeneous, and is still the almost everywhere limit of the h_i, so coincides almost everywhere with g.

STEP 2. Assume that h_i is invariant with respect to the subspace

$$V_i = \operatorname{span}\{v_{1i}, \dots, v_{ki}\},$$

where the v_{ji}'s form a orthonormal basis of V_i. Upon taking a subsequence, we can assume that for each $j = 1, \dots, k$ there exists $\bar{v}_j$ such that

$$\lim_{i\to\infty} v_{ji} = \bar{v}_j;$$

moreover, by a simple linear algebra argument (analogous to the one portrayed in Lemma 2.5), the vectors $\bar{v}_1, \dots, \bar{v}_k$ are still linearly independent. Define then the subspace

$$\bar{V} \doteq \operatorname{span}\{\bar{v}_1, \dots, \bar{v}_k\},$$

and let $R_i \in \mathcal{O}(\mathbb{R}^m)$ be the rotation that brings V_i to $\bar{V}$. It is clear that for any $\delta > 0$ there exists $\hat{\imath} \in \mathbb{N}$ such that for all $i \geq \hat{\imath}$ and for all $x \in B_2(0)$ we have

$$|x - R_i x| < \delta.$$

We first prove the following claim: g is also the L^2-limit of the sequence of maps $\{h_i \circ R_i\}_{i\in\mathbb{N}}$. In fact we have:

$$\|h_i \circ R_i - g\|_{L^2} \leq \|h_i \circ R_i - g \circ R_i\|_{L^2} + \|g \circ R_i - g\|_{L^2};$$

with a simple change of variables, the first term of the right hand side clearly equals $\|h_i - g\|_{L^2}$, and so tends to zero as $i \to \infty$. Concerning the second term, we observe the following: for any $\varepsilon > 0$, by classical density arguments (see [Bre11, Section 4.4]) there exists a continuous function $\phi \in C^0_0(B_1(0), \mathbb{R}^N)$ with compact support such that $\|g - \phi\|_{L^2} < \varepsilon$; and now we have:

$$\|g \circ R_i - g\|_{L^2} \leq \|g \circ R_i - \phi \circ R_i\|_{L^2} + \|\phi \circ R_i - \phi\|_{L^2} + \|\phi - g\|_{L^2}.$$

The first and the third terms are smaller than ε by our definition of ϕ (again, by the use of a change of variable); the second one is less than ε as well for i big enough: in fact, ϕ is continuous in a compact set and thus uniformly continuous, and as we have already observed $|x - R_i x|$ can be made arbitrarily small. This concludes the proof of our claim.

STEP 3. In particular, we can assume without loss of generality that the situation is the following:

$$g = \lim_{i\to\infty} h_i \text{ in } L^2(B_2(0), \mathcal{N}),$$
$$\exists V \ k\text{-subspace such that } h_i \text{ is } V\text{-invariant for all } i.$$

Now we know that h_i converges pointwise to g outside a set H of measure zero; if $x, y \in B_1(0) \setminus H$ and $x - y \in V$, then necessarily $g(x) = g(y)$, since $h_i(x) = h_i(y)$ for all i. If for some $x \in V^\perp$ we have that $x + V \subset H$, we redefine g to be constant on $x + V$; if $x + V^\perp$ contains a point of $B_1(0) \setminus H$ we redefine g so that it takes that value in all $x + V$; in the end, we have obtained a k-invariant map by modifying g on a null set. With some further effort, the modifications operated in STEP 1 and STEP 3 can be made in such a way that the map is both homogeneous and V-invariant. □

2.2 Stratifications

Finally, we define the singular strata, that will soon become the main objects of our study: our overall goal will be to estimate their dimension and clarify their structure. For the sake of simplicity, we now choose a slightly less general setting: from now on, Ω will be a ball centered at the origin. In particular, in this very first definition, we assume u to be defined on the ball of radius 2, and the strata will be consequently defined as subsets of the unit ball.

Definition 2.6 (Stratifications)**.** Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a stationary harmonic map. Let $k \in \{0, \dots, m\}$.

1. For any $\eta > 0$ and any (sufficiently small) $r > 0$ we say that $x \in B_1(0)$ belongs to the $k^{\mathbf{th}}$ (η, r)**-stratum** $\mathcal{S}^k_{\eta,r}(u)$ if the following holds:

$$\max\Big\{0 \le j \le m \,\Big|\, u \text{ is } (\eta, s, j)\text{-symmetric at } x \text{ for some } r \le s < 1\Big\} \le k;$$

In other words,

$$\mathcal{S}^k_{\eta,r}(u) \doteq \Big\{ x \in B_1(0) \,\Big|\, \int_{B_1(0)} |h(y) - T_{x,s}u(y)|^2 \, dy \geq \eta \quad \text{for all } r \leq s < 1 \text{ and all } (k+1)\text{-symmetric maps } h \Big\}.$$

2. For any $\eta > 0$ we say that $x \in B_1(0)$ belongs to the $k^{\mathbf{th}}$ **η-stratum** $\mathcal{S}^k_\eta(u)$ if the following holds:

$$\max \Big\{ 0 \leq j \leq m \,\Big|\, u \text{ is } (\eta, r, j)\text{-symmetric at } x \quad \text{for some } 0 < r < 1 \Big\} \leq k;$$

In other words,

$$\mathcal{S}^k_\eta(u) \doteq \Big\{ x \in B_1(0) \,\Big|\, \int_{B_1(0)} |h(y) - T_{x,r}u(y)|^2 \, dy \geq \eta \quad \text{for all } 0 < r < 1 \text{ and all } (k+1)\text{-symmetric maps } h \Big\}.$$

3. Finally, we define the $k^{\mathbf{th}}$ **stratum** $\mathcal{S}^k(u)$ as

$$\mathcal{S}^k(u) \doteq \{ x \in B_1(0) \mid \text{no tangent map at } x \text{ is } (k+1)\text{-symmetric} \}.$$

The following properties are immediate consequences of the definitions.

Lemma 2.9. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a stationary harmonic map. Then these properties hold:*

(i) Monotonicities: *If $\eta' \geq \eta$, $r' \leq r$, $k' \leq k$ then*

$$\mathcal{S}^{k'}_{\eta',r'}(u) \subseteq \mathcal{S}^k_{\eta,r}(u).$$

(ii) Relations between parameters: *The following set equalities hold for all k and η:*

$$\mathcal{S}^k_\eta(u) = \bigcap_{r>0} \mathcal{S}^k_{\eta,r}(u)$$

$$\mathcal{S}^k(u) = \bigcup_{\eta>0} \mathcal{S}^k_\eta(u).$$

(iii) Trivial stratum: *For any η and r,*

$$\mathcal{S}^m_{\eta,r}(u) = \mathcal{S}^m_\eta(u) = \mathcal{S}^m(u) = B_1(0).$$

Proof. The only non-trivial fact is the identity

$$\mathcal{S}^k(u) = \bigcup_{\eta>0} \mathcal{S}^k_\eta(u).$$

The inclusion "$\supset$" is easily done: let $x \in \bigcup_{\eta>0} \mathcal{S}^k_\eta(u)$; if by contradiction x doesn't belong to the singular stratum $\mathcal{S}^k(u)$, then there exists a $(k+1)$-symmetric tangent map h at x. By definition, then, there exists a sequence $r_i \to 0$ such that

$$\int_{B_1(0)} |h(y) - T_{x,r_i}u(y)|^2 \, dy \xrightarrow{i\to\infty} 0.$$

So x does not belong to any of the $\mathcal{S}^k_\eta(u)$, $\eta > 0$, which contradicts our assumption.

Let now $x \in \mathcal{S}^k(u)$, and assume that x does not belong to any of the $\mathcal{S}^k_\eta(u)$. Take a sequence $\{\eta_i\}_{i\in\mathbb{N}}$ approaching zero; for any i, there exist $r_i > 0$ and a $(k+1)$-symmetric map h_i such that

$$\int_{B_1(0)} |h_i(y) - T_{x,r_i}u(y)|^2 \, dy < \eta_i.$$

Now assume first that r_i tends to 0; by one of the usual compactness arguments there exists a subsequence of $\{T_{x,r_i}u\}_i$ (not relabeled) which converges weakly in $W^{1,2}(\Omega,\mathcal{N})$ and strongly in $L^2(\Omega,\mathcal{N})$ to a map $g \in W^{1,2}(\Omega,\mathcal{N})$, which needs to be homogeneous by Proposition 2.4. By triangle inequality, we also have

$$\|h_i - g\|_{L^2} \le \|h_i - T_{x,r_i}u\|_{L^2} + \|T_{x,r_i}u - g\|_{L^2} \xrightarrow{i\to\infty} 0,$$

so g is the L^2-limit of $(k+1)$-symmetric maps. Using Proposition 2.8, we can see that g needs to be $(k+1)$-symmetric as well: thus, x does not belong to $\mathcal{S}^k(u)$.

The case in which r_i is greater than a certain $\bar{r}$ for infinite i is even easier: for any i, using the change of variable $y = \frac{r_i}{\bar{r}} z$ and exploiting the homogeneity of h we get

$$\int_{B_1(0)} |h_i(y) - T_{x,\bar{r}}u(y)|^2 \, dy = \left(\frac{r_i}{\bar{r}}\right)^m \int_{B_{\frac{\bar{r}}{r_i}}(0)} |h_i(z) - T_{x,r_i}u(z)|^2 \, dz \le$$

$$\le \bar{r}^{-m} \int_{B_1(0)} |h_i(z) - T_{x,r_i}u(z)|^2 \, dz < \bar{r}^{-m}\eta_i.$$

So the h_i's are converging in strong-L^2 to the map $T_{x,\bar{r}}u$: in particular, $T_{x,\bar{r}}u$ itself is $(k+1)$-symmetric. □

We now state the main result of this chapter, due to Cheeger and Naber and first proved in [CN13]. Recall that, for any subset $S \subset \mathbb{R}^m$ and any $r > 0$, we indicate with $\mathcal{B}_r(S)$ the r-fattening of S:

$$\mathcal{B}_r(S) \doteq \{x \in \mathbb{R}^m \mid \operatorname{dist}(x,S) < r\} = \bigcup_{x\in S} B_r(x).$$

Theorem 2.10. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ. Then for all (sufficiently small) $\eta > 0$ and $\delta > 0$ there exists a constant $C_2 = C_2(m, \mathcal{N}, \Lambda, \eta, \delta)$ such that for all $0 < r < 1$ the following estimate holds true:*

$$\operatorname{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) \cap B_1(0)\right) \leq C_2 r^{m-k-\delta}. \tag{CN}$$

2.2.1 Explicit example

As a preliminary exercise, we can try to identify the singular strata of the map $p_N(x) = \frac{x}{|x|}$ from Section 1.6, at least qualitatively. We assume the map is defined on $B_2(0)$ and study the behavior in $B_1(0)$.

Observation 1. First of all, the origin 0 must belong to all the singular strata $\mathcal{S}^k_{\eta,r}(p_N)$ (for η small enough). In fact, $T_{0,r}p_N \equiv p_N$ for all r by homogeneity: that means that 0 belongs to a certain stratum $\mathcal{S}^k_{\eta,r}(p_N)$ if and only if it belongs to $\mathcal{S}^k_{\eta,s}(p_N)$ for every $0 < s \leq 1$. Now assume p_N is η-close to a 1-invariant map h (in a L^2 sense) in $B_1(0)$; by rotational invariance of p_N, we can assume that h is invariant with respect to the subspace span $\{e_N\}$, so

$$h(x_1, \ldots, x_N) = \tilde{h}(x_1, \ldots, x_{N-1}).$$

Now we have

$$\int_{B_1(0)} \left| h(x) - \frac{x}{|x|} \right|^2 dx \geq$$
$$\geq \int_{B_{\frac{1}{2}}(0) \cap \{x_n = 0\}} dx_1 \cdots dx_{N-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \tilde{h}(x_1, \ldots, x_{N-1}) - \frac{x}{|x|} \right|^2 dx_N, \tag{2.1}$$

and very simple computations show that the inner integral is greater than a universal constant, depending only on N and not on the map h. Thus, for all η sufficiently small, p_N is not close to be 1-invariant at any scale r (nor, *a fortiori*, k-invariant for bigger k).

Computations. We give here an extremely rough estimate: for any $(N-1)$-tuple $(x_1, \ldots, x_{N-1})$, the term $\tilde{h}(x_1, \ldots, x_{N-1})$ assumes a fixed value on the sphere, say $(\xi_1, \ldots, \xi_N)$; assume for example $\xi_N \geq 0$ (the symmetric case is analogous). Now we have, fixing $(x_1, \ldots, x_{N-1})$ in $B_{\frac{1}{2}}(0) \cap \{x_n = 0\}$ and $-\frac{1}{2} \leq x_N \leq \frac{1}{2}$:

$$\left| \tilde{h}(x_1, \ldots, x_{N-1}) - \frac{x}{|x|} \right|^2 = \sum_{i=1}^N \left(\xi_i - \frac{x_i}{|x|} \right)^2 \geq \left(\xi_N - \frac{x_N}{|x|} \right)^2;$$

as a consequence, the inner integral from the previous identity (2.1) can be bounded from below as follows:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \tilde{h}(x_1, \ldots, x_{N-1}) - \frac{x}{|x|} \right|^2 dx_N \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\xi_N - \frac{x_N}{|x|} \right)^2 dx_N \geq$$
$$\geq \int_{-\frac{1}{2}}^{-\frac{1}{4}} \left(\xi_N - \frac{x_N}{|x|} \right)^2 dx_N.$$

But now for $\xi_N \geq 0$, $x_N \leq -\frac{1}{4}$ and $|x| \leq 2^{-\frac{1}{2}}$ we have

$$\xi_N - \frac{x_N}{|x|} \geq 2^{-\frac{3}{2}};$$

so in particular

$$\int_{B_1(0)} \left| h(x) - \frac{x}{|x|} \right|^2 dx \geq \int_{B_{\frac{1}{2}}(0) \cap \{x_n = 0\}} dx_1 \cdots dx_{N-1} \int_{-\frac{1}{2}}^{-\frac{1}{4}} 2^{-\frac{3}{2}}\, dx_N =$$
$$= \frac{1}{4} 2^{-\frac{3}{2}} \omega_{N-1} \left(\frac{1}{2} \right)^{N-1} = 2^{-\frac{5}{2}-N} \omega_{N-1} \doteq \Xi_N.$$

For all η below this value, 0 surely belongs to any stratum $\mathcal{S}^k_{\eta,r}(p_N)$.

Observation 2. Secondly, again by rotational invariance, it is clear that a point $y \in B_1(0)$ belongs to a singular stratum if and only if the whole "shell" $\partial B_{|y|}(0)$ is contained in that stratum.

Observation 3. Notice also that for all $x \neq 0$, $0 < r < |x|$, $y \in B_1(0)$ and for all $\lambda > 1$ we have

$$T_{x,r} p_N(y) = \frac{x + ry}{|x + ry|} = \frac{\frac{x}{\lambda} + \frac{r}{\lambda} y}{|\frac{x}{\lambda} + \frac{r}{\lambda} y|} = T_{\frac{x}{\lambda}, \frac{r}{\lambda}} u(y);$$

so we have for all λ:

$$x \in \mathcal{S}^k_{\eta,r}(p_N) \quad \Leftrightarrow \quad \frac{x}{\lambda} \in \mathcal{S}^k_{\eta,\frac{r}{\lambda}}(p_N).$$

In particular, by the monotonicity properties from Lemma 2.9, if $x \in \mathcal{S}^k_{\eta,r}(p_N)$ then $\frac{x}{\lambda}$ belongs to the same stratum $\mathcal{S}^k_{\eta,r}(p_N)$ for all $\lambda > 1$: together with Observation 2, this tells us that all the singular strata are balls centered at the origin.

Observation 4. Now fix r for a moment. If $|y| < \frac{r}{2}$, then the origin is contained in $B_r(y)$ and away from its boundary; thus y is likely to belong to all $\mathcal{S}^0_{\eta,r}(p_N)$ (for η small), since $T_{y,s}$ won't be 1-invariant for any $s \geq r$. In particular $B_{\frac{r}{2}}(0) \subset \mathcal{S}^k_{\eta,r}(p_N)$.

Computations. More precisely, we can argue as follows: if h is a 1-invariant map and $s \geq r$, then we have:

$$\int_{B_1(0)} |h(z) - T_{y,s} p_N(z)|^2\, dz = \int_{B_1(0)} \left| h(z) - \frac{y + sz}{|y + sz|} \right|^2 dz =$$
$$= \int_{B_1\left(\frac{y}{s}\right)} \left| h\Big(w - \frac{y}{s}\Big) - \frac{sw}{|sw|} \right|^2 dw,$$

where the change of variables $w = z + \frac{y}{s}$ has been made; moreover, since $\left|\frac{y}{s}\right| \le \left|\frac{y}{r}\right| < \frac{1}{2}$, by elementary geometric properties $B_{\frac{1}{2}}(0) \subset B_1\left(\frac{y}{s}\right)$, and so:

$$\int_{B_1(0)} |h(z) - T_{y,s}p_N(z)|^2 \, dz \ge \int_{B_{\frac{1}{2}}(0)} \left| h\left(w - \frac{y}{s}\right) - \frac{w}{|w|} \right|^2 dw.$$

Now the map $w \mapsto h\left(w - \frac{y}{s}\right)$ is still 1-invariant; by an argument analogous to the one depicted in Observation 1, it is clear that this last integral is greater than or equal to a universal constant.

Observation 5. On the contrary, if we fix $\eta > 0$ small and the radius $\frac{1}{2}$, we can see that for some $\tilde{r} = \tilde{r}(\eta)$ the points of the shell $\partial B_{\frac{1}{2}}(0)$ will eventually fall out of $\mathcal{S}^0_{\eta,\tilde{r}}(p_N)$, since the map is there smooth (and analogously for other k): fixed $y \in \partial B_{\frac{1}{2}}(0)$, it suffices to consider the constant (thus m-symmetric) map $z \mapsto \frac{y}{|y|}$; and it's easy to see that the integral

$$\int_{B_1(0)} \left| \frac{y}{|y|} - \frac{y + rz}{|y + rz|} \right|^2 dz$$

converges to zero as r tends to 0. Then by Observation 3 we have, for all $\lambda > 1$,

$$\mathcal{S}^0_{\eta,\frac{\tilde{r}}{\lambda}}(p_N) \subset B_{\frac{1}{2\lambda}}(0).$$

In other words, for all $r \le \tilde{r}$

$$\mathcal{S}^0_{\eta,r}(p_N) \subset B_{\frac{1}{2}\frac{r}{\tilde{r}}}(0).$$

Thus we can conclude that given η and k, for any $r > 0$ small the stratum $\mathcal{S}^k_{\eta,r}(p_N)$ is a ball $B_{\alpha r}(0)$ with $\frac{1}{2} \le \alpha \le \tilde{\alpha}(\eta, k)$. Thus $\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(p_N)\right)$ is a ball with radius between $\frac{r}{2}$ and $C(\eta, k)r$, and so there exist constants a_1, a_2 such that:

$$a_1 \le \frac{\operatorname{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(p_N)\right)\right)}{r^m} \le a_2.$$

2.3 Some useful covering lemmas

We present here the proof of a couple of preliminary lemmas which we'll use to "build" the proof of Theorem 2.10. They all descend from elementary geometric properties, so their proofs do not carry essential information.

Covering Lemma 1. *Fix a $\bar{\varrho} > 0$. There exists a constant $c_1 = c_1(m, \bar{\varrho})$ such that for all $r > 0$, $x \in \mathbb{R}^m$ and $\varrho \le \bar{\varrho}$, the ball $B_r(x) \subset \mathbb{R}^m$ can be covered by $c_1\varrho^{-m}$ balls of radius ϱr.*

Proof. We proceed in three steps.

STEP 1. First of all we notice that it is completely equivalent to prove that for all $\varrho \leq \bar{\varrho}$ the unit ball $B_1(0)$ can be covered with $c_1 \varrho^{-m}$ balls of radius ϱ: applying the usual transformation $\lambda_{x,r}$ introduced in Lemma 1.5 we obtain directly the general statement.

STEP 2. Now, instead of covering the unit ball, we consider for a moment the cube $[-1,1]^m$ and cover it with a controlled number of balls centered in the cube itself. Define ξ the integer number

$$\xi \doteq \left\lceil \frac{\sqrt{m}}{\varrho} \right\rceil,$$

where $\lceil \alpha \rceil$ is the smallest integer greater than α, and notice that

$$\frac{\sqrt{m}}{\varrho} \leq \xi \leq \frac{\sqrt{m}}{\varrho} + 1 \leq \frac{\sqrt{m} + \bar{\varrho}}{\varrho}.$$

Consider the grid of points of type

$$\left(\frac{i_1}{\xi}, \ldots, \frac{i_m}{\xi} \right),$$

where for all $h = 1, \ldots, m$ we have

$$i_h \in \{-\xi, \ldots, 1, 0, 1, \ldots, \xi\}.$$

The family of balls centered at points of the grid and with radius ϱ is a covering of the cube: indeed, if $y \in [-1,1]^m$, there exist $i_1, \ldots, i_m$ such that

$$\left| y_h - \frac{i_h}{\xi} \right| \leq \frac{1}{2\xi} \qquad \text{for all } h = 1, \ldots, m;$$

so in particular

$$\left| y - \left(\frac{i_1}{\xi}, \ldots, \frac{i_m}{\xi} \right) \right| \leq \sqrt{m} \frac{1}{2\xi} \leq \frac{\varrho}{2} < \varrho.$$

The number of balls in this covering coincides with the number of points of the grid, which equals $(2\xi + 1)^m$; and the following estimate holds:

$$(2\xi + 1)^m \leq \left(2 \frac{\sqrt{m} + \bar{\varrho}}{\varrho} + \frac{\bar{\varrho}}{\varrho} \right)^m \leq \left(2\sqrt{m} + 3\bar{\varrho} \right)^m \varrho^{-m},$$

so the conclusion follows.

STEP 3. In particular, we can find a covering $\mathscr{C}$ of the cube made with $2^m C \varrho^{-m}$ balls of radius $\frac{\varrho}{2}$. We construct a covering of the unit ball in the following way:

- If $B_{\frac{\varrho}{2}}(y) \in \mathscr{C}$ is centered in the unit ball, we take $B_\varrho(y)$.

- If $B \in \mathscr{C}$ is not centered in the unit ball but intersects it non-trivially, we take $B_\varrho(y')$ with y' any point in the intersection of B and the unit ball. Notice that $B_\varrho(y') \supset B$.

Then the new family is made of balls of radius ϱ centered in $B_1(0)$, in a number which is at most $2^m C \varrho^{-m}$; and by construction it covers the unit ball, so the conclusion is reached. □

Covering Lemma 2. *Fix a $\bar{\varrho} > 0$ and $0 \le k \le m$. There exists a constant $c_2 = c_2(m, k, \bar{\varrho})$ such that the following holds: assume V is an affine k-dimensional subspace of $\mathbb{R}^m$, and $B_r(x) \subset \mathbb{R}^m$ is a ball of radius $r > 0$; if $0 < \varrho \le \bar{\varrho}$, then $\mathcal{B}_{\varrho r}(V) \cap B_r(x)$ can be covered by $c_2 \varrho^{-k}$ balls of radius ϱr.*

Sketch of Proof. One way of proving this is just to trace back the proof of Covering Lemma 1 and adapt it a bit: without loss of generality, we can assume that $x = 0$ and $r = 1$; we can assume that V is a linear subspace (otherwise we can cover $\mathcal{B}_\varrho(V)$ with a smaller number of balls), and up to rotations of the space we can assume it is the subspace

$$V = \operatorname{span}\{e_1, \dots, e_k\} = \{x \in \mathbb{R}^m \mid x_{k+1} = \cdots = x_m = 0\}.$$

Moreover, we can perform the proof for the cube $[-1, 1]^m$ and then use the same trick as in Step 3 of Covering Lemma 1.

We consider the same ξ and the same grid of points; let $\tilde{\xi} \doteq \lceil \sqrt{m} + \bar{\varrho} \rceil$. Now $\mathcal{B}_\varrho(V)$ is covered by the balls of radius ϱ centered at the points of the grid of type

$$\left(\frac{i_1}{\xi}, \dots, \frac{i_m}{\xi}\right) \quad \text{with} \quad \begin{cases} i_1, \dots, i_k \in \{-\xi, \dots, \xi\} \\ i_{k+1}, \dots, i_m \in \left\{-\tilde{\xi}, \dots, \tilde{\xi}\right\}, \end{cases}$$

since

$$\frac{\tilde{\xi}}{\xi} \ge \frac{\sqrt{m} + \bar{\varrho}}{\frac{\sqrt{m}+\bar{\varrho}}{\varrho}} = \varrho.$$

The cardinality of this covering is

$$\begin{aligned}(2\xi + 1)^k \left(2\tilde{\xi} + 1\right)^{m-k} &\le \left(2\sqrt{m} + \bar{\varrho}\right)^k \varrho^{-k} \left(3\lceil \sqrt{m} + \bar{\varrho} \rceil\right)^{m-k} = \\ &= C(m, k, \bar{\varrho}) \varrho^{-k}.\end{aligned}$$

□

Irrelevant parameters. Clearly, the parameter $\bar{\varrho}$ will not play a relevant role in the future; thus, we can simply choose $\bar{\varrho} = 1$ and drop the dependence on $\bar{\varrho}$ in the constants c_1 and c_2. In this case, the numbers $c_1 \varrho^{-m}$ and $c_2 \varrho^{-k}$ are greater than or equal to 1 for any ϱ. Also, we can forget about the dependence of c_2 on k by taking

$$c_2'(m) \doteq \sup_{0 \le k \le m} c_2(m, k).$$

Since it will be of independent interest in the next section, we state here a more precise version of Step 3 of Covering Lemma 1.

Covering Lemma 3. *There exists a constant $\xi_1 = \xi_1(m)$ such that the following holds. Let $A \subset \mathbb{R}^m$ be a bounded set, $\varrho > 0$ and $\{B_\varrho(x_i)\}_{i=1,\ldots,L}$ a finite covering of A with balls of radius ϱ. Let $S \subset A$ be a subset of A. Then S can be covered with $\xi_1 L$ balls of radius ϱ* with centers in *S.*

Proof. We only need to show that for any $i = 1, \ldots, L$ the set $B_\varrho(x_i) \cap S$ can be covered with a fixed number of ϱ-balls centered in $B_\varrho(x_i) \cap S$ itself; equivalently, we can simply show that any subset $\tilde{S}$ of the unit ball can be covered with a fixed number of balls of radius 1 centered at $\tilde{S}$. Consider the family

$$\mathcal{F} \doteq \left\{ \overline{B_{\frac{1}{10}}(y)} \;\middle|\; y \in \tilde{S} \right\}.$$

By the Vitali Covering Theorem (see [EG15, Theorem 1.24]), there exists a subfamily $\mathcal{G}$ of $\mathcal{F}$ made of disjoint balls such that

$$\tilde{S} \subset \bigcup_{B \in \mathcal{F}} B \subset \bigcup \left\{ \overline{B_{\frac{1}{2}}(y)} \;\middle|\; \overline{B_{\frac{1}{10}}(y)} \in \mathcal{G} \right\};$$

in particular,

$$\tilde{S} \subset \bigcup \left\{ B_1(y) \;\middle|\; \overline{B_{\frac{1}{10}}(y)} \in \mathcal{G} \right\}.$$

But the number of balls in $\mathcal{G}$ is at most the number of disjoint balls of radius $\frac{1}{10}$ fitting in the ball $B_2(0)$, which is a finite constant depending only on the dimension m of the ambient space: this proves the result. □

Covering Lemma 4. *Let $A \subset \mathbb{R}^m$ be a bounded set, $\varrho > 0$ and $\{B_{x_i}(\varrho)\}_{i=1}^L$ a finite covering of A with balls of radius ϱ. Then*

$$\mathcal{B}_\varrho(A) \subset \bigcup_{i=1}^{L} B_{2\varrho}(x_i).$$

In particular,

$$\mathrm{Vol}\left(\mathcal{B}_\varrho(A)\right) \leq \sum_{i=1}^{L} \omega_m (2\varrho)^m = L\xi_2(m)\varrho^m.$$

Proof. Assume

$$A \subset \bigcup_{i=1}^{L} B_\varrho(x_i).$$

If $y \in \mathcal{B}_\varrho(A)$, then there exist $x \in A$ such that $|y - x| < \varrho$ and i such that $|x - x_i| < \varrho$. Then by the triangle inequality $|y - x_i| < 2\varrho$, and this proves the statement. □

2.4 Proof of the Main Theorem

The proof of Theorem 2.10 is carried out in several steps and relies on a covering argument; more specifically, we first prove that, if we fix $\varrho > 0$ and $s \in \mathbb{N}$, then the stratum $\mathcal{S}^k_{\eta,\varrho^s}(u)$ is contained in the union of a controlled number of balls of radius ϱ^{s+1}.

In order to accomplish this task, we introduce here some notation we will need.

Notation. Let u, Λ and η be as in the statement of Theorem 2.10, and fix a $0 < \varrho < \frac{1}{4}$; we'll give ϱ a precise value during the course of the proof.
1. For any $j \in \mathbb{N}_{\geq 1}$, define the subset $\mathcal{G}_j$ of $B_1(0)$ as

$$\mathcal{G}_j \doteq \left\{ x \in B_1(0) \mid \theta_\psi(x, 2\varrho^j) - \theta_\psi(x, \frac{1}{2}\varrho^j) < \varepsilon_5 \right\},$$

where ε_5 is the constant given by Corollary 2.7, depending (in this case) on $m, \mathcal{N}, \Lambda, \varrho,\ \eta$ (notice that we fixed $\gamma = \frac{1}{4}$). The letter $\mathcal{G}$ stands for "good"; the points in $\mathcal{G}_j$ are indeed those which behave well at scale ϱ^j, in the following sense: if $z \in \mathcal{S}^k_{\eta,\varrho^j}(u)$, then we know that $\mathcal{G}_j \cap B_{\varrho^j}(z)$ is contained in the fattening of a k-plane. Thus we have some very strong information about good points lying near the stratum. The particular choice of coefficients in front of the ϱ^j's, as well as the upper bound $\frac{1}{4}$ given on ϱ, are driven by some technicalities appearing in the near future, but do not carry any substantial information.
2. For all $x \in B_1(0)$, we construct a sequence $V(x) = \{V(x)[j]\}_{j\geq 1}$ of zeros and ones following this rule:

$$V(x)[j] = \begin{cases} 0 & \text{if } x \in \mathcal{G}_j \\ 1 & \text{if } x \notin \mathcal{G}_j \end{cases},$$

meaning that $V(x)[j]$ is 0 if and only if x is "good" at scale j.
3. Moreover, given $s \in \mathbb{N}$, we'll indicate with $V(x)\big|_s$ the s-dimensional vector

$$V(x)\big|_s = (V(x)[1], \ldots, V(x)[s]),$$

and with $\mathbf{T}_s = \{0,1\}^s$ the set of all the s-tuples consisting of zeros and ones. $V(x)\big|_s$ carries information about the goodness of x at the scales $\varrho, \varrho^2, \ldots, \varrho^s$. This will be important, because at some point we'll fix an integer s and consider $\mathcal{S}^k_{\eta,\varrho^s}(u)$: thus we are not interested in what happens at smaller scales than ϱ^s.
4. Also, for a sequence $\{T[j]\}_{j\geq 1}$ of zeros and ones denote

$$|T| = \sum_{j=1}^{\infty} T[j] \in \mathbb{N} \cup \{\infty\}.$$

5. Finally, given a vector $T \in \mathbf{T}_s$, we define

$$\mathcal{F}^k_{\eta,r}(T) \doteq \left\{ x \in \mathcal{S}^k_{\eta,r} \colon V(x)\big|_s = T \right\},$$

where we are setting $\mathcal{S}^k_{\eta,r} \doteq \mathcal{S}^k_{\eta,r}(u) \cap B_1(0)$. Basically, we are splitting $\mathcal{S}^k_{\eta,r}$ in 2^s pieces, each one of which is represented by an s-sequence of zeros and ones; all the points in $\mathcal{F}^k_{\eta,r}(T)$ "behave in the same way" at all scales $\varrho, \varrho^2, \dots, \varrho^s$.

Our approach will be to cover $\mathcal{F}^k_{\eta,\varrho^s}(T)$ for each T with a controlled number of balls; joining all these coverings, this will give a cover of the whole $\mathcal{S}^k_{\eta,\varrho^s}$. In order for this strategy to be effective, we also need to estimate how many sequences $T \in \mathbf{T}_s$ define a *non-empty* set $\mathcal{F}^k_{\eta,\varrho^s}(T)$. The following result provides this information.

Lemma 2.11. *For all $x \in B_1(0)$,*

$$|V(x)| \le \frac{\Lambda}{\varepsilon_5} \doteq K(m, \mathcal{N}, \Lambda, \varrho, \eta).$$

Proof. In fact, by monotonicity, $\theta_\psi(x, 2\varrho) - \theta_\psi(x, 0) \le \theta_\psi(x, 1) \le \Lambda$; moreover, again using monotonicity,

$$\begin{aligned}\theta_\psi(x, 2\varrho) - \theta_\psi(x, 0) &= \sum_{j=1}^{\infty} \left(\theta_\psi(x, 2\varrho^j) - \theta_\psi\left(x, 2\varrho^{j+1}\right)\right) \ge \\ &\ge \sum_{j=1}^{\infty} \left(\theta_\psi\left(x, 2\varrho^j\right) - \theta_\psi\left(x, \frac{1}{2}\varrho^j\right)\right) \ge \varepsilon_5 |V(x)|.\end{aligned}$$

Here we have used the fact that $\varrho < \frac{1}{4}$. □

Remark. As a consequence, if $|T| > K(m, \mathcal{N}, \Lambda, \varrho, \eta)$ for a sequence $T \in \mathbf{T}_s$, then $\mathcal{F}^k_{\eta,\varrho^s}(T) = \varnothing$: the disjoint union

$$\mathcal{S}^k_{\eta,\varrho^s}(u) = \dot{\bigcup_{T \in \mathbf{T}_s}} \mathcal{F}^k_{\eta,\varrho^s}(T)$$

actually involves a number of items which is, in general, smaller than 2^s. A precise computation is performed in the proof of Theorem 2.10.

The following covering Lemma states rigorously our strategy.

Covering Lemma 5. *Assume $\varrho \in \left]0, \frac{1}{4}\right[$ is a fixed number. Let also $s \in \mathbb{N}_{\ge 1}$ be fixed; for any $T \in \mathbf{T}_s$ there exists a covering of $\mathcal{F}^k_{\eta,\varrho^s}(T)$ of the type*

$$\mathcal{F}^k_{\eta,\varrho^s}(T) \subset \bigcup_{B \in \mathcal{U}(T)} B,$$

where $\mathcal{U}(T)$ *is a collection of at most*

$$\xi_1(m)^{s+1}\left(c_1(m)\varrho^{-m}\right)^{|T|+1}\left(c_2(m)\varrho^{-k}\right)^{s-|T|} \tag{2.2}$$

balls of radius ϱ^{s+1} with centers in $\mathcal{F}^k_{\eta,\varrho^s}$*. Here the constants* c_1*,* c_2 *and* ξ_1 *are the same as in Covering Lemmas 1, 2 and 3.*

Interpretation of the bound (2.2). As we'll see, the fundamental fact to point out about this upper bound is that it has the form $C_3\left(c_3(m)\varrho^{-k}\right)^s$, where $c_3 = \xi_1 c_2$ and C_3 is a constant which *does not depend on s*.

We decided however to write it in this more precise form, in order to separate the pieces coming from different parts of the construction: the "zero-*th*" step will produce a term $(c_1(m)\varrho^{-m})$, while the i-th step with $i \geq 1$ will produce either a factor $(c_1(m)\varrho^{-m})$ or $\left(c_2(m)\varrho^{-k}\right)$, depending on the i-th component of T. Since at any step we also are using the Covering Lemma 3, a term ξ_1 appears.

Proof. Let s and T be fixed. We proceed by inductively refining an initial cover of $\mathcal{F}^k_{\eta,\varrho^s}$; the induction is done on $0 \leq i \leq s$. Recall that the following inclusions hold:

$$\mathcal{F}^k_{\eta,\varrho^s} \subset \mathcal{S}^k_{\eta,\varrho^s}(u) \subset \mathcal{S}^k_{\eta,\varrho^{s-1}}(u) \subset \cdots \subset \mathcal{S}^k_{\eta,\varrho}(u)$$

Step 0. By the Covering Lemma 3, we can simply cover $\mathcal{F}^k_{\eta,\varrho^s} = \mathcal{F}^k_{\eta,\varrho^s}(T)$ with $\xi_1 c_1 \varrho^{-m}$ balls of radius ϱ centered in $\mathcal{F}^k_{\eta,\varrho^s}$. We call

$$\mathcal{U}_1 = \{B_{1,j}\}_{j\in J_1}$$

this covering.

Step 1. Take a ball $B \in \mathcal{U}_1$. It is a ball of radius ϱ centered (in particular) in $\mathcal{S}^k_{\eta,\varrho}(u)$. Now we split two cases:

- If $T[1] = 0$, then by definition of $\mathcal{F}^k_{\eta,\varrho^s}(T)$ we have:

$$\mathcal{F}^k_{\eta,\varrho^s} \cap B \subset \mathcal{S}^k_{\eta,\varrho}(u) \cap \left\{y \in B \mid \theta_\psi\left(y, 2\varrho\right) - \theta_\psi\left(y, \frac{1}{2}\varrho\right) < \varepsilon_5\right\},$$

  which, applying Corollary 2.7 with $r = 2\varrho$, is contained in $\mathcal{B}_{\varrho^2}(\mathcal{V}) \cap B$ for some k-dimensional affine subspace $\mathcal{V}$. Then for any ball $B \in \mathcal{U}_1$, $\mathcal{F}^k_{\eta,\varrho^s} \cap B$ can be covered by $\xi_1 c_2 \varrho^{-k}$ balls of radius ϱ^2 centered in $\mathcal{F}^k_{\eta,\varrho^s} \cap B$ by the Covering Lemmas 2 and 3. In particular, the whole $\mathcal{F}^k_{\eta,\varrho^s}$ can be covered by

$$(\xi_1 c_1 \varrho^{-m})\left(\xi_1 c_2 \varrho^{-k}\right)$$

  balls of radius ϱ^2 centered in $\mathcal{F}^k_{\eta,\varrho^s}$.

- If instead $T[1] = 1$, then for every ball $B \in \mathcal{U}_1$ we simply cover $\mathcal{F}^k_{\eta,\varrho^s} \cap B$ with $\xi_1 c_1 \varrho^{-m}$ balls of radius ϱ^2 by the use of Covering Lemmas 1 and 3, so that $\mathcal{F}^k_{\eta,\varrho^s}$ is covered by
$$\left(\xi_1 c_1 \varrho^{-m}\right)^2$$
balls of radius ϱ^2 centered in $\mathcal{F}^k_{\eta,\varrho^s}$.

In any case, we have obtained a covering $\mathcal{U}_2 = \{B_{2,j}\}_{j\in J_2}$ of $\mathcal{F}^k_{\eta,\varrho^s}$ with a controlled number of balls of radius ϱ^2 centered at $\mathcal{F}^k_{\eta,\varrho^s}$.

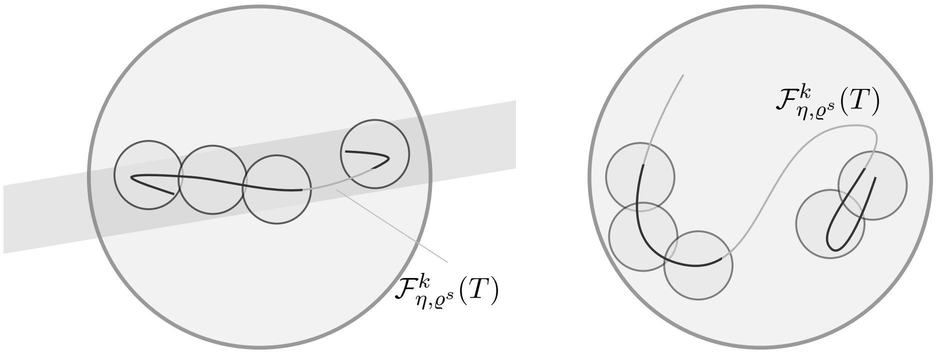
STEP 2. Now assume $2 \leq i \leq s$, and assume we have a covering $\mathcal{U}_i$ of $\mathcal{F}^k_{\eta,\varrho^s}$ made of a controlled number L of balls of radius ϱ^i centered at $\mathcal{F}^k_{\eta,\varrho^s}$. Consider a $B \in \mathcal{U}_i$; again, two cases (visually represented in Figure 2.2):

- If $T[i] = 0$, then
$$\mathcal{F}^k_{\eta,\varrho^s} \cap B \subset \mathcal{S}^k_{\eta,\varrho^i}(u) \cap \left\{ y \in B \mid \theta_\psi\left(y, 2\varrho^i\right) - \theta_\psi\left(y, \frac{1}{2}\varrho^i\right) < \varepsilon_5 \right\},$$
which is contained in $\mathcal{B}_{\varrho^{i+1}}(\mathcal{V})$ for some affine subspace $\mathcal{V}$; so $\mathcal{F}^k_{\eta,\varrho^s} \cap B$ can be covered by $\xi_1 c_2 \varrho^{-k}$ balls of radius ϱ^{i+1}. By repeating the procedure for all balls in $\mathcal{U}_i$ we obtain a covering $\mathcal{U}_{i+1}$ of $\mathcal{F}^k_{\eta,\varrho^s}$ with balls of radius ϱ^{i+1} centered in $\mathcal{F}^k_{\eta,\varrho^s}$; moreover, the cardinality of the family $\mathcal{U}_{i+1}$ is controlled by $L\left(\xi_1 c_2 \varrho^{-k}\right)$.
- If $T[i] = 1$, then for every ball $B \in \mathcal{U}_i$ we cover $\mathcal{F}^k_{\eta,\varrho^s} \cap B$ with $\xi_1 c_1 \varrho^{-m}$ balls of radius ϱ^{i+1} by the use of Covering Lemmas 1 and 3. Again, $\mathcal{F}^k_{\eta,\varrho^s}$ is thus covered by
$$L\left(\xi_1 c_1 \varrho^{-m}\right)$$
balls of radius ϱ^{i+1} centered in $\mathcal{F}^k_{\eta,\varrho^s}$.

Now counting how many times the choice fell on each case we obtain exactly the estimate given in Equation (2.2). ☐

We are now ready to prove Theorem 2.10: as we have anticipated, we'll first fix a value for ϱ, and prove the result for all $\mathcal{S}^k_{\eta,\varrho^s}(u)$ with $s \in \mathbb{N}$; then we'll extend the result to all r.

Proof of Theorem 2.10. Define
$$\varrho = \varrho(m, \delta) \doteq (\xi_1 c_2)^{-\frac{2}{\delta}};$$
notice that if δ is sufficiently small, then ϱ is arbitrarily small as well. We begin by proving the theorem for $r = \varrho^s$ for all $s \in \mathbb{N}$.



(a) Case $T[i] = 0$ **(b)** Case $T[i] = 1$

Figure 2.2: The inductive step: if $T[i] = 0$, we cover $\mathcal{F}^k_{\eta,\varrho^s} \cap B$ with $c\varrho^{-k}$ balls of radius ϱ^{i+1}; if $T[i] = 1$, we cover it with $c\varrho^{-m}$ balls of radius ϱ^{i+1}.

STEP 1. Fix $s \in \mathbb{N}$. $\mathcal{S}^k_{\eta,\varrho^s}(u)$ can be partitioned as

$$\mathcal{S}^k_{\eta,\varrho^s}(u) = \dot{\bigcup_{T\in\mathbf{T}_s}} \mathcal{F}^k_{\eta,\varrho^s}(T); \tag{2.3}$$

notice, however, that if $|T| > K$ (where K is the constant introduced in Lemma 2.11), then $\mathcal{F}^k_{\eta,\varrho^s} = \varnothing$. As a first consequence, the union in Equation (2.3) is not taken on all 2^s sequences of $\mathbf{T}_s$, but only on

$$\binom{s}{0} + \binom{s}{1} + \cdots + \binom{s}{\min\{s, K\}},$$

since the ones can appear at most K times. This number is clearly less than or equal to Ks^K, since for $i \leq K$ we have

$$\binom{s}{i} = \frac{s(s-1)\cdots(s-i+1)}{i!} \leq s^i \leq s^K.$$

Furthermore, Ks^K (seen as a function of s) is elementarily bounded by $C_4\,(\xi_1 c_2)^s$, (*i.e.* $C_4\varrho^{-\frac{\delta}{2}s}$) for some constant $C_4 = C_4(m, \mathcal{N}, \Lambda, \eta)$.

STEP 2. Now for any "good" sequence T (*i.e.*, such that $|T| \leq K$) we know by Covering Lemma 5 that $\mathcal{F}^k_{\eta,\varrho^s}(T)$ is covered by a number of balls of radius ϱ^{s+1} which is at most

$$\xi_1(m)^{s+1}\left(c_1(m)\varrho^{-m}\right)^{|T|+1}\left(c_2(m)\varrho^{-k}\right)^{s-|T|} \leq$$
$$\leq \left[\xi_1 c_1^{K+1}\left(\xi_1 c_1\right)^{-\frac{2}{\delta}m(K+1)}\right]\left[\xi_1 c_2\left(\xi_1 c_2\right)^{-\frac{2}{\delta}k}\right]^s.$$

This can be written as $C_5(m, \mathcal{N}, \Lambda, \delta, \eta)\,(\varrho^s)^{-\frac{\delta}{2}-k}$. Considering what we learnt in STEP 1, we can conclude that $\mathcal{S}^k_{\eta,\varrho^s}(u)$ is covered by

$$C_4\varrho^{-\frac{\delta}{2}s}C_5\,(\varrho^s)^{-\frac{\delta}{2}-k} = C_6\,(\varrho^s)^{-\delta-k}$$

balls of radius ϱ^{s+1} (which of course implies that it's covered by the same number of balls of radius ϱ^s). This means that, exploiting the Covering Lemma 4,

$$\mathrm{Vol}\left(\mathcal{B}_{\varrho^s}\left(\mathcal{S}^k_{\eta,\varrho^s}(u)\right) \cap B_1(0)\right) \leq C_6 \left(\varrho^s\right)^{-\delta-k} \xi_2 \left(\varrho^s\right)^m = C_7 \left(\varrho^s\right)^{m-k-\delta},$$

with $C_7 = C_7(m, \mathcal{N}, \lambda, \delta, \eta)$, which is what we were proving.

STEP 3. Finally, let $r > 0$ be arbitrary (but small enough that all quantities are defined); there exists $s \in \mathbb{N}$ such that $\varrho^{s+1} < r \leq \varrho^s$. Then $\mathcal{S}^k_{\eta,r}(u) \subset \mathcal{S}^k_{\eta,\varrho^s}(u)$; hence, just as in STEP 2, $\mathcal{S}^k_{\eta,r}(u)$ is covered by a union of $C_6 \left(\varrho^s\right)^{-\delta-k}$ balls of radius ϱ^{s+1} (and thus the same number of balls of radius r). Thus the following holds true:

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) \cap B_1(0)\right) \leq C_6 \left(\varrho^s\right)^{-\delta-k} \xi_2 r^m = C_7 r^{m-k-\delta} \left(\frac{\varrho^s}{r}\right)^{-\delta-k};$$

since $\frac{\varrho^s}{r} < \frac{1}{\varrho}$, this is less than or equal to $C_8 r^{m-k-\delta}$, where again $C_8 = C_8(m, \mathcal{N}, \lambda, \delta, \eta)$. □

2.5 Further estimates for the singular set

In this section, we state some important consequences of Theorem 2.10 concerning the regularity of energy minimizing maps; this is still done following [CN13]. These results are expressed in terms of the *regularity scale* of a map u, which was defined in Section 1.5.

Again, for the sake of simplicity, we assume that our working set Ω is in fact the ball $B_2(0)$ (or, alternatively, that Ω contains this ball).

Theorem 2.12. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ. For all $\delta > 0$ there exists a constant $C_9 = C_9(m, \mathcal{N}, \Lambda, \delta)$ such that the following estimate holds for all $0 < r < 1$:*

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{Z}_r(u)) \cap B_1(0)\right) \leq C_9 r^{3-\delta}. \tag{2.4}$$

As a consequence, the Minkowski dimension of the singular set $\mathcal{S}(u)$ is at most $m-3$.

In order to prove this theorem, we need a series of sublemmas of independent interest. The first one is a result about homogeneity: it states that, whenever a map is minimizing, $(m-2)$-symmetry already implies that the map is constant:

Sublemma 2.12.1. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be an energy minimizing map. If u is $(m-2)$-symmetric at a point $x \in B_2(0)$, then it is constant.*

Proof. Assume without loss of generality that $x = 0$. By assumption, u is homogeneous with respect to the origin. If u is not constant, there exist two points $x_1 \neq x_2$ with $|x_i| = 1$ such that $y_1 = u(x_1) \neq u(x_2) = y_2$; then by homogeneity every neighborhood of the origin contains two points whose images are respectively y_1 and y_2. Hence, u cannot be continuous at 0, so $0 \in \mathcal{S}(u)$.

Now by $(m-2)$-invariance there is a $(m-2)$-subspace of singular points; so in particular $\mathscr{H}^{m-2}(\mathcal{S}(u)) > 0$. However, since u is minimizing, this is not possible, otherwise Proposition 1.13 would be contradicted. Thus, u needs to be constant. □

The second sublemma is a quantitative version of the previous one:

Sublemma 2.12.2. *Let $\varepsilon > 0$, and fix $\Lambda > 0$. There exists a constant $\delta_1 = \delta_1(m, \mathcal{N}, \Lambda, \varepsilon)$ such that the following holds:* ***if*** *$u \in W^{1,2}(B_2(0), \mathcal{N})$ is a energy minimizing map with energy bounded by Λ, and u is $(\delta_1, r, m-2)$-symmetric at a point x – with $x \in B_2(0)$ and $0 < r \leq \operatorname{dist}(x, \partial B_2(0))$ –* ***then*** *u is (ε, r, m)-symmetric at the origin. In particular,*

$$\mathcal{S}^{m-1}_{\varepsilon,r}(u) \subset \mathcal{S}^{m-3}_{\delta_1,r}(u).$$

Proof. By scale invariance, we only need to prove the result for $x = 0$, $r = 1$. Assume by contradiction that for some $\varepsilon > 0$ there exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ of energy minimizing maps in $W^{1,2}(B_2(0), \mathcal{N})$ such that:

(a) u_i is $(\frac{1}{i}, 1, m-2)$-symmetric at the origin: *i.e.*, there exists a $(m-2)$-symmetric map h_i such that

$$\int_{B_1(0)} |h_i - u_i|^2 \, dx < \frac{1}{i};$$

(b) u_i is not $(\varepsilon, 1, m)$-symmetric: that is, for all constants $\xi \in \mathcal{N}$,

$$\int_{B_1(0)} |u_i - \xi|^2 \, dx > \varepsilon.$$

By the usual Compactness Theorem 1.4, we can assume that the u_i's converge (in $W^{1,2}$-strong) to a energy minimizing map $\bar{u}$ with energy bounded by Λ. By (a) and the triangle inequality, also the h_i converge in L^2-strong to $\bar{u}$: $\bar{u}$ is the L^2-limit of $(m-2)$-symmetric maps. Then by Proposition 2.8 $\bar{u}$ is itself $(m-2)$-symmetric; since $\bar{u}$ is also minimizing, by Sublemma 2.12.1 it is constant. As a consequence, u_i is converging in L^2 to a constant map, thus contradicting assumption (b). □

We now use a compactness argument to show the following: if a map is minimizing and close to a constant map, then its energy is small.

Sublemma 2.12.3. *For any $\varepsilon > 0$ there exists a $\delta_2 = \delta_2(m, \mathcal{N}, \Lambda, \varepsilon)$ such that the following holds:* ***if*** *$u \in W^{1,2}(\Omega, \mathcal{N})$ is energy minimizing and its energy is bounded by Λ, $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$, and there exists $\xi \in \mathcal{N}$ such that*

$$\int_{B_r(x)} |u_i - \xi|^2 \, dx < \delta_2,$$

then *$\theta(x, r) < \varepsilon$.*

Proof. As usual, we assume that $x = 0$ and $r = 1$ by scale invariance. By contradiction, assume $\{u_i\}_{i\in\mathbb{N}}$ is a sequence of energy minimizing maps and $\{\xi_i\}_{i\in\mathbb{N}}$ a sequence of points of $\mathcal{N}$ such that:

$$\int_{B_1(0)} |u_i - \xi_i|^2 \, dx < \frac{1}{i},$$
$$\theta^{u_i}(0, 1) \geq \varepsilon$$

for a $\varepsilon > 0$. By compactness (for minimizing maps, and for points of $\mathcal{N}$) we can assume that u_i converges to a $\bar{u}$ in $W^{1,2}(\Omega, \mathcal{N})$ strongly, and ξ converges to a point $\bar{\xi} \in \mathcal{N}$. Now for all $i \in \mathbb{N}$

$$\left\|\bar{u} - \bar{\xi}\right\|_{L^2} \leq \|\bar{u} - u_i\|_{L^2} + \|u_i - \xi_i\|_{L^2} + \left\|\xi_i - \bar{\xi}\right\|_{L^2};$$

and the right hand side converges to 0 as $i \to \infty$. So $\bar{u}$ is almost everywhere constant and equal to $\bar{\xi}$; however, by the strong convergence,

$$\theta^{\bar{u}}(0, 1) = \lim_{i\to\infty} \theta^{u_i}(0, 1) \geq \varepsilon,$$

which is a contradiction. □

Finally, we prove a quantitative version of the ε-regularity.

Sublemma 2.12.4. *Let $\Lambda > 0$. There exists an $\varepsilon_6 = \varepsilon_6(m, \mathcal{N}, \Lambda)$ such that the following holds: let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ; let $x \in B_2(0)$ and $0 < r \leq \frac{1}{4} \operatorname{dist}(x, \partial B_2(0))$.* ***If*** *$u$ is $(\varepsilon_6, 4r, m-2)$-symmetric at a point $x \in B_2(0)$,* ***then*** *$r_u(x) \geq r$. As a consequence,*

$$\mathcal{Z}_r(u) \subset \mathcal{S}^{m-3}_{\varepsilon_6, 4r}(u).$$

Proof. By the strong version of the ε-regularity (Corollary 1.12) we know that there exists an $\varepsilon_3 = \varepsilon_3(m, \mathcal{N})$ such that if the normalized energy $\theta(x, 4r)$ is less than ε_3 then the conclusion holds. By Sublemma 2.12.3, that normalized energy is less than ε_3 whenever

$$\int_{B_{4r}(x)} |u_i - \xi|^2 \, dx < \delta_2(m, \mathcal{N}, \Lambda, \varepsilon_3),$$

and by Sublemma 2.12.2 this is true if u is $(\delta_1, 4r, m-2)$-symmetric at x, where $\delta_1 = \delta_1(m, \mathcal{N}, \Lambda, \delta_2)$. Thus, defining ε_6 as this value of δ_1, ε_6 only depends on m, $\mathcal{N}$ and Λ and the statement is proved. □

Now we are in a position to prove the main theorem of this section, Theorem 2.12. Recall that, for a given $\delta > 0$, we need to prove the existence of a constant C depending on δ such that

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{Z}_r(u)) \cap B_1(0)\right) \le C_9 r^{3-\delta}.$$

Proof of Theorem 2.12. By Sublemma 2.12.4, we have that

$$\mathcal{B}_r(\mathcal{Z}_r(u)) \cap B_1(0) \subset \mathcal{B}_r\left(\mathcal{S}^{m-3}_{\varepsilon_6,4r}(u)\right) \cap B_1(0);$$

now for any fixed δ our main Theorem 2.10 ensures the existence of a constant $C_2(m, \mathcal{N}, \Lambda, \delta)$ such that

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^{m-3}_{\varepsilon_6,4r}(u)\right) \cap B_1(0)\right) \le C_2(4r)^{3-\delta}.$$

Combining this relations, the result is immediately proved. □

From Theorem 2.12, we can also deduce the following corollary: for a suitable p, the L^p-norm of the gradient of u is bounded by a constant.

Corollary 2.13. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ. For all $0 < p < 3$, there exists a constant $C_{10} = C_{10}(m, \mathcal{N}, \Lambda, p)$ such that the following bound holds (independently of u):*

$$\int_{B_1(0)} |\nabla u(x)|^p \, dx \le \int_{B_1(0)} r_u^{-p}(x)\, dx \le C_{10}. \tag{2.5}$$

Proof. The first inequality is clear: for any fixed point $x \in B_1(0)$, by the definition of regularity scale, either $r_u(x) = 0$ or $|\nabla u(x)| \le r_u(x)^{-1}$; however, as a consequence of Proposition 1.13, the latter condition is verified outside a set of null $\mathscr{H}^{m-2}$-measure, so

$$\int_{B_1(0)} |\nabla u(x)|^p \, dx \le \int_{B_1(0)} r_u^{-p}(x)\, dx.$$

As for the second one, by Tonelli Theorem we have that

$$\begin{aligned}\int_{B_1(0)} r_u^{-p}(x)\, dx &= \int_0^\infty \mathrm{Vol}\left(\{x \in B_1(0) \mid r_u^{-p}(x) \ge s\}\right) ds\\ &= \int_0^\infty \mathrm{Vol}\left(\left\{x \in B_1(0) \,\middle|\, r_u(x) \le s^{-\frac{1}{p}}\right\}\right) ds =\\ &= \int_0^\infty \mathrm{Vol}\left(\mathcal{Z}_{s^{-\frac{1}{p}}}(u) \cap B_1(0)\right) ds.\end{aligned}$$

While the convergence of this integral at the first extremum is trivial, since

$$\mathrm{Vol}\left(\mathcal{Z}_{s^{-\frac{1}{p}}}(u) \cap B_1(0)\right) \le \mathrm{Vol}\, B_1(0) = 1,$$

the convergence at infinity is assured by Theorem 2.12. Indeed, for $0 < p < 3$ fixed, pick $0 < \delta < 3 - p$, so that $\alpha \doteq \frac{3-\delta}{p} > 1$; then we have

$$\int_0^\infty \mathrm{Vol}\left(\mathcal{Z}_{s^{-\frac{1}{p}}}(u) \cap B_1(0)\right) ds \leq \int_0^1 1\, ds + C_9 \int_1^\infty s^{-\frac{3-\delta}{p}}\, ds \leq$$
$$\leq 1 + C_9(m, \mathcal{N}, \Lambda, \delta) K(p, \delta) \doteq$$
$$\doteq C_{10}(m, \mathcal{N}, \Lambda, p).$$

□

Remark. Notice that this last result is sharp, in the sense that we can not obtain the same result for $p = 3$. This is easily seen for the map $p_3(x) = \frac{x}{|x|}$ from $B_1^3(0)$ to $\mathbb{S}^2$. We know that p_3 is energy minimizing; however, as we computed in Section 1.6,

$$|\nabla p_3(x)|^2 = \frac{2}{|x|^2} \qquad \Longrightarrow \qquad |\nabla p_3(x)|^3 \approx \frac{1}{|x|^3};$$

and the right hand side in the last term is not integrable in $B_1^3(0)$.

Chapter 3

Bounds on the Minkowski Content

The main aim of this chapter is to improve the results of Chapter 2 on the Minkowski content of the singular strata, also gaining information on their k-rectifiability. The techniques involved are considerably more advanced than the ones we used in Chapter 2: we're going to prove the main results by exploiting a suitable form of the Reifenberg Theorem and some sharp estimates on the normalized energy θ_ψ, as done in [NV17]. More precisely, we will prove the following:

Theorem 3.1. *Let $u \in W^{1,2}(B_2(0), \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ.*

(i) For any $\eta > 0$ there exists a constant $C_1 = C_1(m, \mathcal{N}, \Lambda, \eta) > 0$ such that:

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) \cap B_1(0)\right) \leq C_1 r^{m-k}$$

for any $k = 0, \dots, m$ and for all $0 < r \leq 1$;

(ii) For any $\eta > 0$ there exists a constant $C_2 = C_2(m, \mathcal{N}, \Lambda, \eta) > 0$ such that:

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta}(u)\right) \cap B_1(0)\right) \leq C_2 r^{m-k}$$

for any $k = 0, \dots, m$ and for all $0 < r \leq 1$; moreover, the η-stratum $\mathcal{S}^k_\eta(u)$ is k-rectifiable.

(iii) The stratum $\mathcal{S}^k(u)$ is k-rectifiable.

Remark. Notice that, since we are assuming that u is minimizing, the only relevant subdimensions are $k \leq m-3$: in Sublemma 2.12.2 we have proved that all the higher subdimensions can be reconducted to this case.

The definition of rectifiability we are using can be found in Appendix A. We restrict again to the case $\Omega \supset B_2(0)$, for convenience; actually, we'll soon need other conditions on Ω: it must be large enough to contain all the objects we're going to define. Recall that by definition of the singular strata we are assuming that they are contained in the unit ball $B_1(0)$.

The following corollary is a straightforward consequence of Theorem 3.1:

Corollary 3.2. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ. Then the effective η-stratum $\mathcal{S}^k_\eta(u)$ has Minkowski dimension less than or equal to k, and its upper Minkowski content is bounded by C_2.*

We need, first of all, to introduce some new notions that will play a central role during the course of the chapter.

3.1 Basic notions

To begin, for any given measure μ we define a quantity β^k_μ which indicates how close its support is to be included in an affine subspace (at scale r); this was first introduced by Jones in [Jon90] and then further developed in [Paj02]. As we will see, when this tool is applied to some $\mathscr{H}^k \llcorner S$ (the restriction of a Hausdorff measure to a given set), it represents an indicator for the rectifiability of the set. Recall that, if μ is a Borel measure on the set Ω, its support is defined as the complement of the union of all the open sets $U \subset \Omega$ such that $\mu(U) = 0$.

Notation. If $k = 0, \ldots, m$ is an integer, we denote:

$$\mathbf{G}^k(\mathbb{R}^m) = \{V \mid V \text{ is a } k\text{-dimensional vector subspace of } \mathbb{R}^m\};$$
$$\mathbf{H}^k(\mathbb{R}^m) = \{W \mid W \text{ is a } k\text{-dimensional affine subspace of } \mathbb{R}^m\}.$$

Definition 3.1 (Jones' numbers)**.** Let $\Omega \subset \mathbb{R}^m$ be a regular bounded open set, and let μ be a non-negative Radon measure on Ω. Let $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$. Fix an integer $k = \{0, \ldots, m\}$ and a real number $1 \le p < \infty$. We define the p^{th} k-dimensional Jones' number as

$$\beta^k_{\mu,p}(x,r) \doteq \left(\inf \left\{ \int_{B_r(x)} \left(\frac{\operatorname{dist}(y,V)}{r} \right)^p \frac{d\mu(y)}{r^k} \;\middle|\; V \in \mathbf{H}^k(\mathbb{R}^m) \right\} \right)^{\frac{1}{p}}.$$

When $\mu = \mathscr{H}^k \llcorner S$, with S a $\mathscr{H}^k$-measurable subset of $B_1(0)$, we write $\beta^k_{S,p}(x,r)$ instead of $\beta^k_{\mathscr{H}^k \llcorner S,p}(x,r)$.

Notation. In the future, we'll only need Jones' numbers with $p = 2$. For this reason, to slightly simplify the notation, we also denote

$$D^k_\mu(x,r) \doteq \beta^k_{\mu,2}(x,r)^2,$$

and analogously for $D^k_S(x,r)$.

Observation. It's easy to see that the inf is actually a min, with the usual compactness argument: indeed, let

$$V_j = x_j + \operatorname{span}\{v_{1j}, \ldots, v_{kj}\}$$

be a minimizing sequence of k-subspaces, with $\{v_{1j}, \ldots, v_{kj}\}$ orthonormal vectors. Then up to subsequences x_j converges to a point $\bar{x}$ and each v_{ij} converges to a $\bar{v}_i$, and $\{\bar{v}_1, \ldots, \bar{v}_k\}$ is still an orthonormal set of vectors. Call $\bar{V} \doteq \bar{x} + \operatorname{span}\{\bar{v}_1, \ldots \bar{v}_k\}$. Exploiting the continuity of the distance function (like in Lemma 2.5), it is straightforward to see that for any $y \in B_r(x)$

$$\lim_{j\to\infty} \operatorname{dist}(y, V_k) = \operatorname{dist}(y, V).$$

Since the functions $y \mapsto \operatorname{dist}(y, V_k)$ are bounded in $B_r(x)$ by the constant $2r$ (provided we choose V_k intersecting $B_r(x)$, which we have to do), we can apply the Dominated Convergence Theorem to show that

$$\lim_{j\to\infty} \int_{B_r(x)} \left(\frac{\operatorname{dist}(y, V_j)}{r}\right)^p \frac{d\mu(y)}{r^k} = \int_{B_r(x)} \left(\frac{\operatorname{dist}(y, \bar{V})}{r}\right)^p \frac{d\mu(y)}{r^k};$$

this proves that $\bar{V}$ is minimizing. Notice, however, that the minimizing subspace is not necessarily unique (for example, considering rotational-invariant measures).

Example. When L is a k-plane and $\mu = \mathscr{H}^k \llcorner (L \cap B_1(0))$, then $\beta^k_{\mu,p}(0,1)$ is clearly 0, and the minimum is achieved by the subspace L itself.

Furthermore, we give a name to the quantity $\theta_\psi(x, \sigma r) - \theta_\psi(x, r)$, which we already used in Chapter 2 and will be even more useful in the near future. For technical reasons, we change a bit the assumptions on ψ, being aware that this does not affect the substance of the main results:

Important Remark. For the remainder of this chapter, we assume the following hypotheses on the test function ψ that appears in the definition of the modified normalized energy θ_ψ:

- ψ is supported in $[0, 3)$ and infinitely many times differentiable in $[0, \infty)$;
- $\psi' < 0$ in $[0, 3)$;
- There exists a constant $\xi > 0$ such that

$$-\psi'(t) \geq \xi \quad \text{for all } t \in [0, 2]. \tag{3.1}$$

As a consequence, given $x \in \Omega$ we'll only define $\theta_\psi(x, r)$ for $r < \frac{1}{3}\operatorname{dist}(x, \partial\Omega)$.

Definition 3.2. Let $\sigma > 1$. Let $x \in \Omega$ and $0 < r < \frac{1}{3\sigma}\operatorname{dist}(x, \partial\Omega)$. Assume u is a stationary harmonic map in $W^{1,2}(\Omega, \mathcal{N})$. Then we define:

$$P_{u,\sigma}(x, r) = \theta_\psi(x, \sigma r) - \theta_\psi(x, r).$$

We'll omit the subscript u if there's no ambiguity.

As we'll see shortly, both $\beta^k_{\mu,p}$ and $P_{u,\sigma}$ have some nice scale invariance properties. Recall that we denote with $\lambda = \lambda_{x,r}$ the diffeomorphism

$$\begin{aligned}\lambda_{x,r}\colon \mathbb{R}^m &\longrightarrow \mathbb{R}^m\\ y &\longmapsto x + ry.\end{aligned}$$

Definition 3.3 (Rescaled measure)**.** Let μ be a Radon measure on a set $\Omega \subset \mathbb{R}^m$; let $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$. On the set

$$\frac{\Omega - x}{r} \doteq \lambda^{\leftarrow}_{x,r}(\Omega) = \{y \in \mathbb{R}^m \mid x + ry \in \Omega\}$$

we define the measure $\tilde{\mu} = T_{x,r}\mu$ by imposing

$$\tilde{\mu}(A) \doteq \mu(\lambda_{x,r}(A)) = \mu(x + rA)$$

for any measurable set A such that $\lambda_{x,r}(A) \subset \Omega$.

It is then clear that for any μ-measurable function h defined on Ω, $h \circ \lambda_{x,r}$ is $\tilde{\mu}$-measurable and

$$\int_{\lambda^{\leftarrow}_{x,r}(\Omega)} h \circ \lambda_{x,r}(y) d\tilde{\mu}(y) = \int_\Omega h(z) d\mu(z).$$

Lemma 3.3 (Scale invariance)**.** *Let $x \in \Omega$, fix $\sigma > 1$ and $0 < r < \frac{1}{3\sigma}\operatorname{dist}(x, \partial\Omega)$. The following relations hold true:*

- *For any μ Radon measure on Ω, and any $k = \{0, \dots, m\}$, $1 \le p < \infty$,*

$$\beta^k_{\mu,p}(x, r)^p = r^{-k} \beta^k_{T_{x,r}\mu,p}(0, 1)^p.$$

- *For any $u \in W^{1,2}(\Omega, \mathcal{N})$,*

$$P_{u,\sigma}(x, r) = P_{T_{x,r}u,\sigma}(0, 1).$$

Proof. The second property is a trivial consequence of the scale invariance property of θ_ψ.

Now for any $V \in \mathbf{H}^k(\mathbb{R}^m)$, let $V_x \doteq x + V$. The map $V \mapsto V_x$ is obviously a bijection from $\mathbf{H}^k(\mathbb{R}^m)$ to itself. For each $y \in B_1(0)$ and each $V \in \mathbf{H}^k(\mathbb{R}^m)$ we have

$$\operatorname{dist}(y, V) = r^{-1} \operatorname{dist}(\lambda_{x,r}(y), V_x)$$

by elementary geometric properties. Then for any $V \in \mathbf{H}^k(\mathbb{R}^m)$ we have

$$\begin{aligned}\int_{B_1(0)} (\operatorname{dist}(y, V))^p \, d\tilde{\mu}(y) &= \int_{B_1(0)} \left(\frac{\operatorname{dist}(\lambda_{x,r}(y), V_x)}{r}\right)^p d\tilde{\mu}(y) =\\ &= \int_{B_r(x)} \left(\frac{\operatorname{dist}(y, V_x)}{r}\right)^p d\mu(y).\end{aligned}$$

Since taking the inf on $V \in \mathbf{H}^k(\mathbb{R}^m)$ is the same as taking the inf on the V_x, the assertion follows easily. □

3.1.1 Reifenberg Theorems

Before going on, we show how Jones' numbers will be applied, in order to give a motivation for the next section. A proof of the following two theorems can be found in [NV17]; other information can be found in [ENV16] and [ENV18], while similar arguments are developed in [DT12], [Tor95] and [Miś18]. In the first of the theorems, we associate a "discrete" measure μ to a collection of balls, and we give a sufficient condition on D^k_μ in order to have a uniform estimate on the radii of the balls. The second theorem gives a sufficient condition on D^k_S that guarantees the rectifiability of a given set S. In particular, this describes a path to follow in order to obtain the rectifiability of the singular strata: we'll need to estimate the Jones' numbers associated to the restriction measure $\mathscr{H}^k \llcorner \mathcal{S}^k_{\eta,r}(u)$.

Notice that we now assume that the ball $B_3(0)$ (in $\mathbb{R}^m$) is contained in our working set Ω, in order to give a sense to all the terms involved.

Theorem 3.4 (Rectifiable Reifenberg, 1)**.** *There exist two constants $\delta_{Rf1} = \delta_{Rf1}(m) > 0$ and $C_{Rf1} = C_{Rf1}(m) > 0$ such that the following holds. Let $\mathcal{C} \subset B_1(0) \subset \mathbb{R}^m$, and let $\mathcal{F} \doteq \{B_{r_x}(x)\}_{x\in\mathcal{C}}$ be a collection of* disjoint *balls with centers in $\mathcal{C}$ such that $B_{r_x}(x) \subset B_3(0)$ for any x. Define the k-dimensional measure associated to the collection $\mathcal{F}$ as*

$$\mu = \sum_{x\in\mathcal{C}} \omega_k r_x^k \delta_x,$$

where δ_x is the Dirac measure at x. Assume that for any ball $B_r(x)$ contained in $B_2(0)$ we have

$$\int_{B_r(x)} \left(\int_0^r D^k_\mu(y,s) \frac{ds}{s} \right) d\mu(y) < \delta_{Rf1} r^k. \tag{RR1}$$

Then the radii r_x have the uniform packing estimate

$$\sum_{x\in\mathcal{C}} r_x^k < C_{Rf1}.$$

Theorem 3.5 (Rectifiable Reifenberg, 2)**.** *There exist two constants $\delta_{Rf2} = \delta_{Rf2}(m) > 0$ and $C_{Rf2} = C_{Rf2}(m) > 0$ such that the following holds. Let $S \subset B_3(0)$ be a $\mathscr{H}^k$-measurable subset with the following property: for any $x \in B_1(0)$ and $r \le 1$, we have*

$$\int_{S\cap B_r(x)} \left(\int_0^r D^k_S(y,s) \frac{ds}{s} \right) d\mathscr{H}^k(y) < \delta_{Rf2} r^k, \tag{RR2}$$

where $D^k_S = D^k_{\mathscr{H}^k \llcorner S}$. Then:

(i) $S \cap B_1(0)$ is k-rectifiable;

(ii) For any $x \in S$ and $r > 0$ such that $B_r(x) \subset B_1(0)$ we have

$$\mathscr{H}^k(S \cap B_r(x)) \leq C_{Rf2} r^k.$$

Example. In order to see that the assumption (RR2) makes sense, we try to estimate the Jones' number $D^k_S(y, s)$ and the inner integral of Equation (RR2) in a particularly simple case: when S is the graph of a C^2 function f from $\mathbb{R}^k$ to $\mathbb{R}^{m-k}$, with $1 \leq k < m$; explicitly, S is a set of type

$$S \doteq \left\{ (z, f(z)) \;\middle|\; z \in \mathbb{R}^k \right\} \subset \mathbb{R}^m.$$

Fix $y_0 = (z_0, f(z_0))$ for some $z_0 \in \mathbb{R}^k$. It is intuitively clear that a reasonable estimate on $D^k_S(y, s)$ can be obtained by taking as a k-subspace the tangent space to S at y; this can be described as

$$V_0 \doteq \left\{ (z, f(z_0) + \langle \nabla f(z_0), z - z_0 \rangle) \;\middle|\; z \in \mathbb{R}^k \right\}.$$

By definition of the measure $\mathscr{H}^k \llcorner S$, we are only interested in computing the distance of points of S from the tangent space; for some $y = (z, f(z))$ this is surely less than or equal to

$$|f(z) - f(z_0) - \langle \nabla f(z_0), z - z_0 \rangle| \, ;$$

but now since f is C^2 we're allowed to use its Taylor expansion of order two, which implies

$$\operatorname{dist}(y, V_0) \leq \left\| \nabla^2 f \right\|_\infty |z - z_0|^2 \, .$$

Thus we obtain:

$$D^k_S(y_0, s) \leq s^{-k-2} \int_{S \cap B^k(z_0, s)} \left\| \nabla^2 f \right\|^2_\infty |\pi_k(y) - z_0|^4 \, d\mathscr{H}^k(y),$$

where π_k is the projection on the first k coordinates. Now $|\pi_k(y) - z_0|$ can be simply bounded by s; thus we have

$$D^k_S(y_0, S) \leq \left\| \nabla^2 f \right\|^2_\infty s^{2-k} \mathscr{H}^k(S).$$

By the Area Formula, for which we refer to [DeL08, Proposition 4.3], we have that

$$\mathscr{H}^k(S) = \int_{\pi_k(B_s(y_0))} \mathbf{Jac}\, f(x) \, dx, \tag{3.2}$$

so, in particular, for a constant $C_3(k)$ we have:

$$\begin{aligned} D^k_S(y_0, s) &\leq C(k) \left\| \nabla f \right\|_\infty \left\| \nabla^2 f \right\|^2_\infty s^{2-k} \operatorname{Vol}\left(B^k_s(z_0) \right) \leq \\ &\leq C_3(k) \left\| \nabla f \right\|_\infty \left\| \nabla^2 f \right\|^2_\infty s^2. \end{aligned}$$

As an easy consequence, for some $C_4(k)$,

$$\int_0^r D_S^k(y_0,s)\frac{ds}{s} \leq C_3(k)\,\|\nabla f\|_\infty \left\|\nabla^2 f\right\|_\infty^2 \int_0^r s\,ds \leq$$
$$\leq C_4(k)\,\|\nabla f\|_\infty \left\|\nabla^2 f\right\|_\infty^2 \frac{r^2}{2}.$$

Now, to obtain the expression at the left hand side of Equation (RR2), we need to compute the integral over $S\cap B_r(x)$ of the term we've just estimated. Exploiting again Equation (3.2), we find that

$$\int_{S\cap B_r(x)} \left(\int_0^r D_S^k(y,s)\frac{ds}{s}\right) d\mathscr{H}^k(y) \leq C_4(k)\,\|\nabla f\|_\infty \left\|\nabla^2 f\right\|_\infty^2 \frac{r^{2+k}}{2}.$$

This not only ensures that the rate of convergence is at least the one needed for the application of Theorem 3.5, but also for r small we can get a constant as small as we want.

3.2 Best approximating planes

In this section we prove that the numbers D_μ^k can be bounded using the quantity $P_{u,\sigma}$ defined in Definition 3.2. In particular, we don't need to directly compute D_μ^k to apply the Reifenberg Theorems 3.4 and 3.5.

We first introduce a couple of tools which will turn out to be useful in describing a measure and its "best linear approximation".

Definition 3.4. Let μ be a measure with support in $B_1(0)$. We define:

- the **center of mass** of μ as the point $x_{cm} \in B_1(0)$ such that

$$x_{cm}^\mu = x_{cm} = \fint_{B_1(0)} x\,d\mu(x);$$

- the **second moment** of μ as the bilinear form Q such that for all $v, w \in \mathbb{R}^m$

$$Q^\mu(v,w) = Q(v,w) = \int_{B_1(0)} \left[(x - x_{cm})\cdot v\right]\left[(x - x_{cm})\cdot w\right]\,d\mu(x).$$

Since Q is symmetric and positive-definite, by the Spectral Theorem the associated matrix (which we still denote by Q) admits an orthonormal basis of eigenvectors, with non-negative eigenvalues. We denote with $\lambda_1(\mu),\ldots,\lambda_m(\mu)$ the eigenvalues of Q in decreasing order, and with $v_1(\mu),\ldots,v_m(\mu)$ the respective eigenvectors (pairwise orthogonal and of norm 1); that is:

$$\lambda_k v_k = \int_{B_1(0)} \left[(x - x_{cm})\cdot v_k\right](x - x_{cm})\,d\mu(x);$$
$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_m(\mu).$$

Since Q is symmetric, the eigenvalues λ_k and the eigenvectors v_k have a very practical variational characterization, known as the *Courant-Fisher characterization*: we state it for general bilinear forms and then we adapt it to our case. The proof is a simple linear algebra computation, which can be found for example in [KM97, Section 2.10].

Proposition 3.6. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a symmetric linear operator, and define the associated quadratic form $q(v) \doteq \langle v, f(v) \rangle$. Let $\lambda_1 \geq \cdots \geq \lambda_m$ be the eigenvalues of f, and let $\{v_1, \dots, v_m\}$ be a set of orthonormal eigenvectors. Then*

$$\lambda_1 = \max \{q(v) \mid |v| = 1\}, \qquad \lambda_m = \min \{q(v) \mid |v| = 1\};$$

and for all $k = 2, \dots, m-1$ ve have:

$$\begin{aligned} \lambda_k &= \max \{q(v) \mid |v| = 1,\ \textit{and}\ \langle v, v_j \rangle = 0\ \textit{for all}\ j < k\}, \\ \lambda_k &= \min \{q(v) \mid |v| = 1,\ \textit{and}\ \langle v, v_j \rangle = 0\ \textit{for all}\ j > k\}. \end{aligned}$$

Moreover, for all k,

$$\lambda_k = \min_{L \in \mathbf{G}^{m-k+1}(\mathbb{R}^m)} \{\max \{q(v) \mid v \in L,\ |v| = 1\}\}.$$

Corollary 3.7. *Let μ be a measure with support in $B_1(0)$. The following statements hold:*

1. *$\lambda_1(\mu)$ satisfies*

$$\lambda_1(\mu) = \max \left\{ \int_{B_1(0)} \langle x - x_{cm}, v \rangle^2 \, d\mu(x) \,\middle|\, |v| = 1 \right\},$$

   and v_1 is any of the unit vectors achieving the maximum;

2. *For any $k = 2, \dots, m$, $\lambda_k(\mu)$ satisfies*

$$\lambda_k(\mu) = \max \left\{ \int_{B_1(0)} \langle x - x_{cm}, v \rangle^2 \, d\mu(x) \,\middle|\, |v| = 1\ \textit{and}\ \langle v, v_j \rangle = 0\ \forall j < k \right\},$$

   and v_k is any of the unit vectors achieving the maximum;

3. *The dual statements with the minimum hold true;*

4. *For all k,*

$$\lambda_k = \min_{L \in \mathbf{G}^{m-k+1}(\mathbb{R}^m)} \left\{ \max_{\substack{v \in L \\ |v|=1}} \left\{ \int_{B_1(0)} \langle x - x_{cm}, v \rangle^2 \, d\mu(x) \right\} \right\}.$$

The previous corollary hides a really crucial fact: heuristically, the line $x_{cm} + \operatorname{span}\{v_1\}$ is the 1-dimensional subspace where μ is mostly concentrated, and λ_1 is an index of the dispersion along this line; similarly, the subsequent eigenvectors $v_2, \dots, v_m$ represent directions of decreasing concentration for the measure μ. With this in mind, for any $k = 0, \dots, m$ define the following affine subspace:

$$V_k = V_k^{\mu} \doteq x_{cm}^{\mu} + \operatorname{span}\{v_1(\mu), \dots, v_k(\mu)\}\,;$$

we make the previous heuristic concrete in the following lemma.

Lemma 3.8. *Let μ be a probability measure on $B_1(0)$, and let $1 \le k \le m-1$ be an integer. The functional*

$$\begin{aligned} \mathcal{I}\colon \mathbf{H}^k(\mathbb{R}^m) &\longrightarrow [0,\infty) \\ L &\longmapsto \int_{B_1(0)} \operatorname{dist}^2(y, L)\; d\mu(y) \end{aligned}$$

attains its minimum at V_k^{μ}. Moreover, the following identity holds:

$$\int_{B_1(0)} \operatorname{dist}^2(y, V_k^{\mu})\; d\mu(y) = \lambda_{k+1}(\mu) + \cdots \lambda_m(\mu). \tag{3.3}$$

Proof. First of all, we prove Equation (3.3). The m-tuple $\{v_1, \dots, v_m\}$ is an orthonormal basis for $\mathbb{R}^m$; for $y \in B_1(0)$, we can write $y = \sum_j \langle y, v_j\rangle\, v_j$, and then

$$\operatorname{dist}^2(y, V_k) = \sum_{j=k+1}^{m} \langle y - x_{cm}, v_j\rangle^2\,,$$

hence

$$\begin{aligned} \int_{B_1(0)} \operatorname{dist}^2(y, V_k)\, d\mu(y) &= \sum_{j=k+1}^{m} \int_{B_1(0)} \langle y - x_{cm}, v_j\rangle^2\; d\mu(y) = \\ &= \sum_{j=k+1}^{m} \langle \lambda_j v_j, v_j\rangle = \sum_{j=k+1}^{m} \lambda_j, \end{aligned}$$

which is our statement.

We can assume, without loss of generality, that μ has its center of mass at the origin. Otherwise, we apply a translation and the only thing that changes is that now μ is supported in $B_2(0)$; since there's no substantial modification, we go on thinking of μ as supported in $B_1(0)$ in order to simplify computations.

First of all we show that the minimum of $\mathcal{I}$ must be a subspace passing through the origin. Assume not, and let $L = x_0 + L_0$ be a subspace reaching the minimum. Let $\{\zeta_1, \dots, \zeta_k\}$ be an orthonormal basis for L_0 and complete it to an orthonormal basis $\{\zeta_1, \dots, \zeta_m\}$ of $\mathbb{R}^m$. We have, for any $\in B_1(0)$:

$$\begin{aligned}\operatorname{dist}^2(y, L_0) &= \sum_{j=k+1}^{m} \langle y, \zeta_j \rangle^2 \\ \operatorname{dist}^2(y, L) &= \sum_{j=k+1}^{m} \langle y - x_0, \zeta_j \rangle^2\end{aligned}$$

so in particular

$$\begin{aligned}\int_{B_1(0)} \operatorname{dist}^2(y, L)\, d\mu(y) = \int_{B_1(0)} &\operatorname{dist}^2(y, L_0)\, d\mu(y) - \\ &- 2 \sum_{j=k+1}^{m} \langle x_0, \zeta_j \rangle \left\langle \int_{B_1(0)} y_j\, d\mu(y), \zeta_j \right\rangle + \\ &+ \mu\left(B_1(0)\right) \sum_{j=k+1}^{m} \langle x_0, \zeta_j \rangle^2 .\end{aligned}$$

Here the second term in the right hand side is zero, since $\int_{B_1(0)} y_j\, d\mu(y) = 0$ by definition of center of mass. The third term is strictly greater than zero if we assume $x_0 \notin L_0$. This implies that $\mathcal{I}(L_0) > \mathcal{I}(L)$, contradicting our assumption.

So we found out that the minimizing subspace contains the center of mass – in our case, the origin. We now prove by induction on the codimension $m-k$ that V_k is the minimum of $\mathcal{I}$. First of all, consider V_{m-1}; by Statement 3 of Corollary 3.7 we have for all $w \in \mathbb{R}^m$ with $|w| = 1$:

$$\begin{aligned}\int_{B_1(0)} \operatorname{dist}^2(y, V_{m-1})\, d\mu(y) &= \int_{B_1(0)} \langle y, v_m \rangle^2\, d\mu(y) = \lambda_m \\ &\leq \int_{B_1(0)} \langle y, w \rangle^2\, d\mu(y) = \int_{B_1(0)} \operatorname{dist}^2(y, w^\perp)\, d\mu(y)\end{aligned}$$

and this clearly proves the statement for $k = m - 1$.

Assume we have proved that for any k-subspace W_k

$$\lambda_{k+1} + \cdots + \lambda_m \leq \int_{B_1(0)} \operatorname{dist}^2(y, W_k)\, d\mu(y);$$

let W_{k-1} have dimension $k - 1$. For any orthonormal basis $\{w_k, \dots, w_m\}$ of $W_{k-1}^\perp$, we have

$$\int_{B_1(0)} \operatorname{dist}^2(y, W_{k-1})\, d\mu(y) = \sum_{j=k}^{m} \int_{B_1(0)} \langle y, w_j \rangle^2\, d\mu(y).$$

Now, since $W_{k-1}^\perp$ is $(m-k+1)$-dimensional, by Statement 4 of Corollary 3.7 there exists $w \in W_{k-1}^\perp$ with $|w| = 1$ such that

$$\lambda_k \leq \int_{B_1(0)} \langle y, w \rangle^2\, d\mu(y);$$

then we can build an orthonormal basis for $W_{k-1}^{\perp}$ having w as the first vector; then we have

$$\begin{aligned}\int_{B_1(0)} &\operatorname{dist}^2(y, W_{k-1})\, d\mu(y) \\ &= \int_{B_1(0)} \langle y, w\rangle^2 \, d\mu(y) + \int_{B_1(0)} \operatorname{dist}^2(y, W_{k-1} \oplus \langle w\rangle)\, d\mu(y) \leq \\ &\leq \lambda_k + \lambda_{k+1} + \cdots + \lambda_m,\end{aligned}$$

which is what we wanted to prove. □

We now state the most important theorem of this section. This is the context: assume that the minimizing harmonic map u is *not* almost $(k+1)$-symmetric at x, and assume this is not due to a lack of 0-homogeneity; in this case, we already know from Chapter 2 that the points in which P_σ is small are bound to lie near a k-dimensional subspace. Here we say that, in this same framework, for any measure μ the Jones' numbers D_μ^k are controlled by the behavior of P_σ in a ball around the point x.

Theorem 3.9 (L^2-best approximation). *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be minimizing harmonic with energy bounded by Λ; fix the following constants:*

$\sigma > 1:$	*a scale parameter for θ_ψ;*
$\eta > 0:$	*a parameter for almost $(k+1)$-symmetry;*
$\kappa \geq 1:$	*a coefficient for the radius.*

Let $x \in \Omega$ and $0 < r < \bar{r}$ (where $\bar{r}$ is chosen in such a way that all the appearing terms make sense). There exist two constants $C_5 = C_5(m, \mathcal{N}, \Lambda, \sigma, \eta, \kappa)$ and $\delta_1 = \delta_1(m, \mathcal{N}, \Lambda, \eta, \kappa)$ such that: if u is $(\delta_1, \kappa r, 0)$-symmetric at x but not $(\eta, \kappa r, k+1)$-symmetric, then for any finite measure μ defined on $B_r(x)$ we have

$$D_\mu^k(x, r) \leq C_5 r^{-k} \int_{B_r(x)} P_{u,\sigma}(y, r)\, d\mu(y).$$

Remark. First of all, notice that in the integral in the right hand side we are considering points $y \in B_r(x)$, and for each of them we need to compute $P_{u,\sigma}(y, r)$. This is why we need the assumption that, for example, $0 < r < \frac{1}{4\sigma} \operatorname{dist}(x, \partial\Omega)$: we have to ensure the existence of the various terms. The choice of $\bar{r}$ is also influenced by the presence of the coefficient κ: we need $r < \frac{1}{\kappa} \operatorname{dist}(x, \partial\Omega)$ in order to use $(\eta, \kappa r)$-symmetry. In this regard, we point out that the presence of the coefficient κ in the statement is only justified by the use we will make of this theorem in Section 3.4: we will need it in this form, although no important role is played by this extra constant.

Observe that by the scale invariance properties from Lemma 3.3 we only need to prove the result for $\Omega \supset B_{4\sigma}(0)$, $x = 0$, $r = 1$; moreover, the inequality does not change if we substitute μ with a multiple of it (and is void for $\mu = 0$), thus we can assume μ is a probability measure. So these will be our assumptions, and we'll show

$$D^k_\mu(0,1) \le C_5 \int_{B_1(0)} P_{u,\sigma}(y,1)\, d\mu(y).$$

The proof of this result rests on a number of sublemmas, that we prove in the first place. First of all, we we want to obtain a quantitative relation between the quantity $P_{u,\sigma}$ and the almost 0-symmetry of u. This sublemma is again in the same spirit of the results in Chapter 2; no measure μ is involved.

Sublemma 3.9.1. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a stationary harmonic map, and let $\sigma > 1$. Assume that $B_{4\sigma}(0) \subset \Omega$. There exists a constant C_6 depending only on m and σ such that for any $x \in B_1(0)$*

$$\int_{B_2(x)} |\langle \nabla u(y), x-y\rangle|^2\, dy \le C_6 P_{u,\sigma}(x,1);$$

in particular, we also have:

$$\int_{B_1(0)} |\langle \nabla u(y), x-y\rangle|^2\, dy \le C_7 P_{u,\sigma}(x,1)$$

for $C_7 = C_7(m,\sigma)$.

Proof. Let $x \in B_1(0)$. By definition, we have that $P_\sigma(x,1) = \theta_\psi(x,\sigma) - \theta_\psi(x,1)$, so by the Monotonicity Formula (MF) and the Tonelli Theorem we have (see also Corollary 1.7):

$$\begin{aligned} P_\sigma(x,1) &= -2\int_\Omega \int_1^\sigma s^{-m}\psi'\left(\frac{|x-y|}{s}\right)|x-y|\left|\partial_{r_x(y)}u(y)\right|^2 ds\, dy = \\ &= -2\int_\Omega \frac{\langle \nabla u(y), x-y\rangle^2}{|x-y|}\int_1^\sigma s^{-m}\psi'\left(\frac{|x-y|}{s}\right) ds\, dy. \end{aligned}$$

For a better visualization, we apply the change of variable $t = \frac{|x-y|}{s}$ in the inner integral, obtaining:

$$\begin{aligned} P_\sigma(x,1) &= -2\int_\Omega \frac{\langle \nabla u(y), x-y\rangle^2}{|x-y|}\int_{\frac{|x-y|}{\sigma}}^{|x-y|} |x-y|^{-m}\, t^m \psi'(t)\frac{|x-y|}{t^2}\, dt\, dy = \\ &= 2\int_\Omega |x-y|^{-m}\langle \nabla u(y), x-y\rangle^2 \int_{\frac{|x-y|}{\sigma}}^{|x-y|} t^{m-2}\left(-\psi'(t)\right) dt\, dy. \end{aligned}$$

By the positivity of the integrand, this is greater than or equal to the same integral with domain $B_2(x)$. But then $|x-y| \le 2$: this means that $-\psi'(t)$ in the inner integral can be bounded from below by ξ (see the Important

Remark at the beginning of Section 3.1). Thus we get, computing the inner integrand:

$$P_\sigma(x,1) \geq 2\xi \int_{B_2(x)} |x-y|^{-m} \langle \nabla u(y), x-y\rangle^2 \frac{1-\sigma^{-(m-1)}}{m-1} |x-y|^{m-1}\, dy =$$
$$= \frac{2\xi\left(1-\sigma^{-(m-1)}\right)}{m-1} \int_{B_2(x)} \frac{\langle \nabla u(y), x-y\rangle^2}{|x-y|}\, dy.$$

Since all the balls $B_2(x)$ for $x \in B_1(0)$ contain $B_1(0)$, and here $|x-y|^{-1} \geq \frac{1}{2}$, we obtain:

$$\int_{B_1(0)} \langle \nabla u(y), x-y\rangle^2 \, dy \leq \frac{m-1}{\xi(1-\sigma^{1-m})} P_\sigma(x,1),$$

which is what we needed. □

Now given μ we show that we can control the energy of u along each eigenvector $v_k(\mu)$ by knowing the quantity P_σ in the unit ball.

Sublemma 3.9.2. *Let $\sigma > 1$ and assume $\Omega \supset B_{4\sigma}(0)$. Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a stationary harmonic map with energy bounded by Λ. Let μ be a probability measure on the ball $B_1(0)$, and let $\lambda_1, \dots, \lambda_m$ and $v_1, \dots, v_m$ be the eigenvalues and eigenvectors defined after Definition 3.4. There exists a constant $C_8 = C_8(m, \mathcal{N}) > 0$ such that the following holds for any $j = 1, \dots, m$:*

$$\lambda_j \int_{B_1(0)} |\langle \nabla u(y), v_j\rangle|^2 \, dx \leq C_8 \int_{B_1(0)} P_\sigma(x,1)\, d\mu(x).$$

Notice that the integral at the left hand side is taken with respect to the Lebesgue measure, whereas the one at the right hand side is taken with respect to the measure μ.

Proof. Assume for a moment that $x_{cm} = 0$. By definition of eigenvalues, we have

$$\lambda_j v_j = \int_{B_1(0)} \langle x, v_j\rangle\, x \, d\mu(x). \tag{3.4}$$

Fix $y \in B_1(0)$. Multiplying by $\nabla u(y)$ both sides of the previous equality, we obtain

$$\lambda_j \langle \nabla u(y), v_j\rangle = \int_{B_1(0)} \langle x, v_j\rangle \langle \nabla u(y), x\rangle \, d\mu(x);$$

moreover, since $x_{cm} = 0$, we have

$$\int_{B_1(0)} \langle x, v_j\rangle \langle \nabla u(y), y\rangle \, d\mu(x) = \langle \nabla u(y), y\rangle \left\langle \int_{B_1(0)} x\, d\mu(x), v_j \right\rangle = 0.$$

This means that we can write

$$\lambda_j \langle \nabla u(y), v_j \rangle = \int_{B_1(0)} \langle x, v_j \rangle \langle \nabla u(y), x - y \rangle \, d\mu(x).$$

We take the squares of both sides and apply Hölder inequality:

$$\begin{aligned} \lambda_j^2 \langle \nabla u(y), v_j \rangle^2 &\leq \left(\int_{B_1(0)} \langle x, v_j \rangle^2 \, d\mu(x) \right) \left(\int_{B_1(0)} \langle \nabla u(y), x - y \rangle^2 \, d\mu(x) \right) = \\ &= \lambda_j \int_{B_1(0)} \langle \nabla u(y), x - y \rangle^2 \, d\mu(x). \end{aligned} \tag{3.5}$$

Now all the eigenvalues are non-negative, and if $\lambda_j = 0$ the statement is trivial, so we can divide by λ_j; this holds for all $y \in B_1(0)$, thus we can then integrate both sides on $B_1(0)$ with respect to the variable y. At this point we get, also using Tonelli's Theorem:

$$\lambda_j \int_{B_1(0)} \langle \nabla u(y), v_j \rangle^2 \, dy \leq \int_{B_1(0)} \int_{B_1(0)} \langle \nabla u(y), x - y \rangle^2 \, dy \, d\mu(x).$$

But now the inner integral can be estimated using Sublemma 3.9.1, thus obtaining

$$\lambda_j \int_{B_1(0)} \langle \nabla u(y), v_j \rangle^2 \, dy \leq C \int_{B_1(0)} P_\sigma(x, 1) \, d\mu(x),$$

which was our exact statement.

Now drop the hypothesis that x_{cm} is zero. By tracing back the proof of Sublemma 3.9.1, we see that we can also obtain the estimate

$$\int_{B_2(0)} |\langle \nabla u(y), x - y \rangle|^2 \, dy \leq C_9 P_{u,\sigma}(x, 1)$$

for $x \in B_2(0)$ (instead of $B_1(0)$) and integrating y in $B_2(0)$ (instead of $B_1(0)$); the only difference is that the estimate is obtained with a different constant $C_9(m, \sigma)$ – and actually under a stronger assumption on ψ. Now the measure $\tilde{\mu} \doteq T_{x_{cm},1}\mu$ is supported in $B_2(0)$ and is "centered" at 0; as it is clear by the definition, eigenvalues and eigenvectors of μ and $\tilde{\mu}$ coincide; moreover, the map

$$\tilde{u}(x) = T_{x_{cm},1} u(x) = u(x_{cm} + x)$$

is well defined on $B_2(0)$. So we proceed like this: we reproduce the argument we've just made, from Equation (3.4) to Equation (3.5); but we do it taking y in $B_2(0)$, considering the map $\tilde{u}$ instead of u, and computing the integrals (with respect to $\tilde{\mu}$) over $B_2(0)$. Going on with the argument, we find that

$$\lambda_j \int_{B_2(0)} \langle \nabla \tilde{u}(y), v_j \rangle^2 \, dy \leq C \int_{B_2(0)} P_{\tilde{u},\sigma}(x, 1) \, d\tilde{\mu}(x). \tag{3.6}$$

Now applying the change of variables $z = x - x_{cm}$ in the right hand side we obtain the integral

$$\int_{B_2(x_{cm})} P_{u,\sigma}(x,1)\, d\mu(x);$$

however, the measure μ is actually supported in $B_1(0)$ by hypothesis, and $B_1(0) \subset B_2(x_{cm})$, so the previous integral is *equal* to

$$\int_{B_1(0)} P_{u,\sigma}(x,1)\, d\mu(x).$$

Moreover, with the same change of variables, the left hand side of Equation (3.6) becomes

$$\lambda_j \int_{B_2(x_{cm})} \langle \nabla u(y), v_j \rangle^2 \; dy;$$

this, in turn is greater than or equal to the same integral taken over $B_1(0)$. Combining these relations, we find the desired inequality. □

The last sublemma again exploits only tools from Chapter 2. Its content is intuitively very clear: we already know the notion of almost k-symmetry contains information on both almost-homogeneity and almost-k-invariance; thus, if an almost-homogeneous map u is *not* almost-k-symmetric, then for any k-subspace L the quantity

$$\int \langle \nabla u(y), L \rangle^2$$

must be far from zero. Here we are using the notation $\langle \nabla u(y), L \rangle^2$ for $\sum_{i=1}^{k} \langle \nabla u(y), v_i \rangle$, with $\{v_i\}_{i=1}^{k}$ an orthonormal basis of L. The statement is actually more precise:

Sublemma 3.9.3. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map. Fix a parameter η for almost-symmetry and a radius $0 < \varsigma < 1$. There exists a constant $\delta_2 = \delta_2(m, \mathcal{N}, \Lambda, \eta, \varsigma)$ such that: for $1 \le k \le m$ and $B_{2r}(x) \subset \Omega$,* ***if*** *u is minimizing, $(\delta_2, r, 0)$-symmetric at x but* not *(η, r, k)-symmetric* ***then***

$$r^{2-m} \int_{B_{\varsigma r}(x)} \langle \nabla u(y), L \rangle^2 \, dy \ge \delta_2$$

for all the k-subspaces $L \in \mathbf{G}^k(\mathbb{R}^m)$.

Proof. As we always do in these circumstances, we assume $x = 0$ and $r = 1$, and argue by contradiction exploiting compactness. Assume there exist a parameter η, a sequence of minimizing harmonic maps $\{u_i\}_{i\in\mathbb{N}}$, and a sequence of k-subspaces $\{L_i\}_{i\in\mathbb{N}}$ such that:

- u is $(\frac{1}{i}, 1, 0)$-symmetric at 0;
- u is *not* $(\eta, 1, k)$-symmetric at 0;
- The following holds:

$$\int_{B_\varsigma(0)} \langle \nabla u_i(y), L_i\rangle^2 \, dy < \frac{1}{i}.$$

Clearly, up to taking $u_i \circ R_i$ for some rotations R_i, we can assume that $L_i = L$ for all i. Now by the usual Compactness Theorem 1.4 u_i converges to a minimizing harmonic map u in the strong $W^{1,2}$ sense; this clearly implies that

$$\lim_{i\to\infty} \int_{B_\varsigma(0)} \langle \nabla u_i(y), L\rangle^2 \, dy = \int_{B_\varsigma(0)} \langle \nabla u(y), L\rangle^2 \, dy = 0;$$

indeed, if $\{v_j\}_{j=1}^k$ is an orthonormal basis for L, then:

$$\int_{B_\varsigma(0)} \langle \nabla (u-u_i)(y), L_i\rangle^2 \, dy = \sum_{j=1}^k \int_{B_\varsigma(0)} |\langle \nabla (u-u_i)(y), v_i\rangle|^2 \, dy \leq$$
$$\leq \sum_{j=1}^k |v_j|^2 \int_{B_\varsigma(0)} |\nabla u - \nabla u_i|^2 \, dy \longrightarrow 0.$$

Now by the usual argument involving the normalized energy θ_ψ, the map u is homogeneous; moreover, since

$$\int_{B_\varsigma(0)} \langle \nabla u(y), L\rangle^2 \, dy = 0,$$

the map u is L-invariant in $B_\varsigma(0)$. So by "translating" the origin along L we find $k+1$ points of homogeneity in general position; by applying Corollary 2.3, we can conclude that u is k-symmetric with respect to the origin in the whole $B_1(0)$. But the maps u_i are converging in L^2-strong to u: this contradicts the fact that for some η they're not $(\eta, 1, k)$-symmetric at 0. □

Finally, the proof of Theorem 3.9 is a consequence of the previous sublemmas. Remember that we are assuming that μ is a probability measure, $x = 0$ and $r = 1$.

Proof of Theorem 3.9. By definition of Jones' numbers and by the identity (3.3) of Lemma 3.8, $D^k_\mu(0,1)$ equals the sum $\lambda_{k+1}(\mu) + \cdots + \lambda_m(\mu)$. However, since the eigenvalues are ordered decreasingly, we only need to estimate λ_{k+1}: indeed,

$$\lambda_{k+1}(\mu) + \cdots + \lambda_m(\mu) \leq (m-k)\lambda_{k+1}(\mu).$$

By Sublemma 3.9.2 we know that for all j we have:

$$\lambda_j \int_{B_1(0)} |\langle \nabla u(y), v_j \rangle|^2 \, dx \leq C_8 \int_{B_1(0)} P_\sigma(x,1) \, d\mu(x),$$

so in particular, again exploiting the decreasing order of the eigenvalues,

$$\begin{aligned} \lambda_{k+1} \int_{B_1(0)} |\langle \nabla u(y), V_{k+1} \rangle|^2 \, dx &= \sum_{j=1}^{k+1} \lambda_{k+1} \int_{B_1(0)} |\langle \nabla u(y), v_j \rangle|^2 \, dx \leq \\ &= \sum_{j=1}^{k+1} \lambda_j \int_{B_1(0)} |\langle \nabla u(y), v_j \rangle|^2 \, dx \leq \\ &\leq (k+1) C_8 \int_{B_1(0)} P_\sigma(x,1) \, d\mu(x). \end{aligned}$$

But now we are assuming that u is $(\eta, \kappa r, k+1)$-symmetric: if we choose $\delta_1 = \delta_2(m, \mathcal{N}, \Lambda, \eta, \frac{1}{\kappa})$, by Sublemma 3.9.3 we have

$$\int_{B_1(0)} |\langle \nabla u(y), V_{k+1} \rangle|^2 \, dx \geq \delta_1;$$

as a consequence,

$$(m-k)\lambda_{k+1} \leq \frac{(m-k)(k+1) C_8(m, \mathcal{N})}{\delta_1(m, \mathcal{N}\Lambda, \eta, \kappa)} \int_{B_1(0)} P_\sigma(x,1) \, d\mu(x).$$

This is exactly our statement. □

3.3 A few technical lemmas

We present in this section a couple of technical results that will turn out to be useful in the main proofs of this chapter. In the following statements, Ω is assumed to contain $B_2(0)$, and to be big enough to give sense to all the quantities involved; Λ is a bound for the total energy.

In the first lemma, we give an alternative characterization of almost symmetry: indeed, it is heuristically clear that almost k-invariance can be achieved if the quantity

$$\int |\langle \nabla u(z), L \rangle|^2 \, dz$$

is sufficiently small for some k-subspace L; we have seen something about this in Sublemma 3.9.3. Here we show that this is enough not only for almost k-invariance but also for almost-homogeneity.

Lemma 3.10. *Fix a constant $\eta > 0$, the parameter controlling almost symmetry. There exist a constant $\delta_3 = \delta_3(m, \mathcal{N}, \Lambda, \eta)$ and a "scale factor"*

$\bar{r} = \bar{r}(m, \mathcal{N}, \Lambda, \eta)$ such that the following holds: let u be a minimizing harmonic map, $x \in \Omega$, $r > 0$. ***If*** *the condition*

$$r^{2-m} \int_{B_r(x)} |\langle \nabla u(z), L\rangle|^2 \, dz < \delta_3,$$

holds for a $(k+1)$-subspace $L \in \mathbf{H}^{k+1}(\mathbb{R}^m)$, ***then*** *we have:*

$$\mathcal{S}^k_{\eta,\bar{r}r}(u) \cap B_{\frac{1}{2}r}(x) = \varnothing.$$

Moreover, we can choose $\bar{r}$ to be $\delta_3^{\frac{1}{2(m-2)}}$.

Proof. Assume as always that $x = 0$ and $r = 1$. By contradiction, take a sequence of minimizing harmonic maps $\{u_j\}_{j\in\mathbb{N}}$ such that

$$\int_{B_1(0)} |\langle \nabla u_j(z), L\rangle|^2 \, dz < \frac{1}{j}$$

for some $(k+1)$-subspace L (a priori depending on j, but then fixed by the use of rotations); call $\bar{r}_j \doteq j^{-\frac{1}{2(m-2)}}$ and assume there exists a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ belonging to

$$\mathcal{S}^k_{\eta,\bar{r}_j}(u_j) \cap B_{\frac{1}{2}}(0).$$

Step 1. Call $C_{10}(m) \doteq 2(\log 2)(m-2)$. For any $y \in B_1(0)$, any u minimizing harmonic map and any $j \in \mathbb{N}$ there exists a radius $r_{y,j}(u) \geq \bar{r}_j$ such that

$$\theta_\psi(y, r_{y,j}) - \theta_\psi\left(y, \frac{1}{2} r_{y,j}\right) < \frac{C_{10}(m)\Lambda}{\log j}.$$

Indeed, if this is not true for some y, u and j, then

$$\theta_\psi(y, 2^{-i}) - \theta_\psi(y, 2^{-(i+1)}) \geq \frac{C_{10}(m)\Lambda}{\log j}$$

for all i such that $2^{-i} \geq j^{-\frac{1}{2(m-2)}}$, *i.e.*, for all i such that

$$i \leq \frac{1}{2(m-2)} \frac{\log j}{\log 2} = \frac{\log j}{C_{10}(m)};$$

in particular, we have:

$$\sum_{i=0}^{\left\lfloor \frac{\log j}{C_{10}(m)} \right\rfloor} \left(\theta_\psi(y, 2^{-i}) - \theta_\psi(y, 2^{-(i+1)})\right) \geq \left(\left\lfloor \frac{\log j}{C_{10}(m)} \right\rfloor + 1\right) \frac{C_{10}(m)\Lambda}{\log j},$$

thus

$$\Lambda \geq \theta_\psi(y, 1) = \sum_{i=0}^{\infty} \left(\theta_\psi(y, 2^{-i}) - \theta_\psi(y, 2^{-(i+1)})\right) > \Lambda,$$

which can not be true.

Step 2. Consider our contradicting sequence, and for all j call $r_j \doteq r_{x_j,j}(u_j) \geq \bar{r}_j$. Moreover, call w_j the map

$$w_j \doteq T_{x_j,r_j} u_j.$$

Then, by the fact that $r_j \geq \bar{r}_j$, we have for all j:

$$\int_{B_1(0)} |\langle \nabla w_j(z), L\rangle|^2 \, dz = r_j^{2-m} \int_{B_{r_j}(x_j)} |\langle \nabla u_j(z), L\rangle|^2 \, dz \leq$$
$$\leq \left(j^{-\frac{1}{2(m-2)}}\right)^{2-m} \frac{1}{j} = \frac{1}{\sqrt{j}} \to 0.$$

Since the w_j's are again minimizing, up to subsequences they converge in $W^{1,2}$ to a minimizing harmonic map $\bar{w}$; as a consequence of the previous estimate and of the strong $W_{1,2}$ limit, $\bar{w}$ is L-invariant.

Step 3. Moreover, for all j we have the following:

$$\theta_\psi^{w_j}(0,1) - \theta_\psi^{w_j}\left(0, \frac{1}{2}\right) = \theta_\psi^{u_j}(x_j, r_j) - \theta_\psi^{u_j}\left(x_j, \frac{1}{2} r_j\right) < \frac{C_{10}(m)\Lambda}{\log j} \to 0.$$

Again by the convergence in strong $W^{1,2}$, and exploiting Proposition 2.2, we then find that $\bar{w}$ is homogeneous: together with Step 2, this tells us that $\bar{w}$ is $(k+1)$-symmetric. But the maps w_j are converging in L^2 to $\bar{w}$, so all of them for j big enough must be $(\eta, 1, k+1)$-symmetric at 0; and this implies that the corresponding u_j are $(\eta, r_j, k+1)$-symmetric at x_j with $r_j \geq \bar{r}_j$, contradicting the assumption that x_j belongs to $\mathcal{S}^k_{\eta,\bar{r}_j}(u_j)$. □

The following lemma makes use of a slight modification of the set $\mathcal{C}(x,r)$ (introduced in Section 2.1.2), which contains the points near x satisfying a pinching condition on the normalized energy: here we show that if this set effectively spans a k-subspace V, then we gain some information on a singular stratum $\mathcal{S}^k_{\eta,\delta r}(u)$ (*cf.* Corollary 2.7). Precisely, the singular stratum must lie inside a fattening of V.

With a slight abuse of terminology, we say that a set S spans τ-*effectively* a subspace $V \in \mathbf{H}^k(\mathbb{R}^m)$ if there exist $k+1$ points of S in τ-general position that span V.

Lemma 3.11. *Fix $\varrho > 0$ and $\eta > 0$. There exists a constant $\xi_3 = \xi_3(m, \mathcal{N}, \Lambda, \varrho, \eta)$ such that the following implication holds true. Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ, let $x \in B_1(0)$ and $0 < r < 1$. Fix an $\varepsilon > 0$, and consider*

$$\mathcal{K}_{\varepsilon,\varrho}(x,r) \doteq \{y \in B_{2r}(x) \mid \theta_\psi(y,r) - \theta_\psi(y, \varrho r) < \varepsilon\}.$$

If $\varepsilon \leq \xi_3$ and $\mathcal{K}_{\varepsilon,\varrho}(x,r)$ spans an affine k-subspace V in a ϱr-effective way, then

$$\mathcal{S}^k_{\eta,\delta r}(u) \cap B_r(x) \subset \mathcal{B}_{2\varrho r}(V)$$

for all $\delta \leq \xi_3$.

Remark. For purely technical reasons, we need to define $\mathcal{K}_{\varepsilon,\varrho}(x,r)$ as the set of points *in the* $2r$*-ball* that satisfy the pinching condition; the definition of $\mathcal{C}$ from Corollary 2.7 involved the $\frac{r}{2}$-ball.

Proof. We can assume without loss of generality that $B_r(x)$ is the unit ball $B_1(0)$. By assumption, we can find $k+1$ points $\{y_0,\dots,y_k\}$ in $\mathcal{K}_{\varepsilon,\varrho}(x,r)$ that span ϱ-effectively the k-subspace V. Consider a point w external to the tubular neighborhood $\mathcal{B}_{2\varrho}(V)$. By Lemma 3.10, we only need to find a $(k+1)$-subspace L and a $\tau \leq \varrho$ such that the quantity

$$\tau^{2-m}\int_{B_\tau(w)} |\langle \nabla u(z), L\rangle|^2\, dz$$

is small enough.

STEP 1. By an argument similar to the one portrayed in Sublemma 3.9.1, we find that the quantity

$$\int_{B_\varrho(w)} |\langle \nabla u(z), V\rangle|^2\, dz$$

can be made arbitrarily small just by taking a sufficiently small ξ_3. Indeed, for all $i=0,\dots,k$, the ball $B_\varrho(w)$ is contained in $B_4(y_i)$; and we can apply (an appropriate version) of Sublemma 3.9.1 to find out that

$$\begin{aligned}\int_{B_\varrho(w)} |\langle \nabla u(z), z-y_i\rangle|^2\, dz &\leq \int_{B_4(y_i)} |\langle \nabla u(z), z-y_i\rangle|^2\, dz \leq \\ &\leq C_{11}(m,\varrho)\,[\theta_\psi(y_i,1)-\theta_\psi(y_i,\varrho)]\end{aligned}$$

for some constant C_{11}. Now for any vector v of norm 1 such that v is contained in the linear subspace associated to V, we have that

$$v = \sum_{i=1}^k \alpha_i (y_i - y_0),$$

where the α_i's are bounded by a constant $c_4(m,\varrho)$. Thus we have, by triangle inequality and for a new constant $C_{12}(m,\varrho)$,

$$\begin{aligned}&\int_{B_\varrho(w)} |\langle \nabla u(z), v\rangle|^z\, dz = \\ &\quad = \int_{B_\varrho(w)} \left|\sum_{i=1}^k \alpha_i \left\langle \nabla u(z), y_i - z\right\rangle + \left(\sum_{i=1}^k \alpha_i\right)\left\langle \nabla u(z), z-y_0\right\rangle\right| dz \leq \\ &\quad \leq C_{13}\sum_{i=0}^k (\theta_\psi(y_i,1)-\theta_\psi(y_i,\varrho)) \leq C_{12}\varepsilon.\end{aligned}$$

By summing the inequalities obtained for the vectors of an orthonormal basis, we get exactly that there exists $C_{14}(\varrho,m)$ such that

$$\int_{B_\varrho(w)} |\langle \nabla u(z), V\rangle|^2\, dx \leq C_{14}\varepsilon. \tag{3.7}$$

STEP 2. As a second step we show that we can also make the quantity

$$\int \left| \left\langle \nabla u(z), \frac{w - \pi_V(w)}{|w - \pi_V(w)|} \right\rangle \right|^2 dz$$

as small as we want, so we gain another direction along which the energy is controlled. To begin with, we consider for any point $z \in B_\varrho(w)$ the projection $\pi_V(z)$: it can be written as

$$\pi_V(z) = y_0 + \sum_{i=1}^{k} \beta_i(z)(y_i - y_0),$$

with β_i bounded by $c_4(m, \varrho)$; thus we have

$$z - \pi_V(z) = \left(1 - \sum_{i=1}^{k} \beta_i(z)\right)(z - y_0) + \sum_{i=1}^{k} \beta_i(z)(z - y_i).$$

In particular, by the application of the same technique as in STEP 1, we get that

$$\int_{B_\varrho(w)} |\langle \nabla u(z), z - \pi_V(z)\rangle|^2 \, dz \le C_{15}\varepsilon$$

for a constant $C_{15}(m)$, and thus (since $|z - \pi_V(z)| \ge \varrho$),

$$\int_{B_\varrho(w)} \left| \left\langle \nabla u(z), \frac{z - \pi_V(z)}{|z - \pi_V(z)|} \right\rangle \right|^2 dz \le C_{15}\varrho^2\varepsilon.$$

Now, by the triangle inequality (and simple algebraic properties) and setting $h(z) \doteq \frac{z-\pi_V(z)}{|z-\pi_V(z)|}$, we can estimate, for any $\tau \le \varrho$,

$$\begin{aligned}
&\int_{B_\tau(w)} |\langle \nabla u(z), h(w)\rangle|^2 \, dz \le \\
&\qquad \le 2\int_{B_\tau(w)} |\langle \nabla u(z), h(z)\rangle|^2 \, dz + 2\int_{B_\tau(w)} |\langle \nabla u(z), h(z) - h(w)\rangle|^2 \, dz \le \\
&\qquad \le 2\int_{B_\tau(w)} |\langle \nabla u(z), h(z)\rangle|^2 \, dz + 2\int_{B_\tau(w)} |\nabla u(z)|^2 \, |h(z) - h(w)|^2 \, dz.
\end{aligned}$$

The first addend is what we've just estimated with $C_{15}\varrho^2 = \varepsilon C_{16}(m, \varrho)\varepsilon$; concerning the second one, we observe what follows. Any $z \in B_\varrho(w)$ can be written as

$$z = w + v_\top + v_\perp,$$

where $v_\top$ belongs to the linear subspace associated to V and $v_\perp \in V^\perp$. Clearly, $\pi_V(z) = \pi_V(w) + v_\top$, so

$$h(z) = \frac{w - \pi_V(w) + v_\perp}{|w - \pi_V(w) + v_\perp|};$$

if z is such that the second equality in

$$|z-\pi_V(z)| = |w-\pi_V(w)+v_\perp| = |w-\pi_V(w)|$$

holds, then we have

$$|h(z)-h(w)| = \frac{|v_\perp|}{|w-\pi_V(w)|} \leq \frac{|z-w|}{\varrho}.$$

Otherwise, we set $\tilde{z} \doteq \pi_V(z) + |w-\pi_V(w)|\,(z-\pi_V(z))$; by easy geometric properties we can see that $|\tilde{z}-w| \leq c(m)\,|z-w|$, so in general

$$|h(z)-h(w)| \leq C_{17}(m,\varrho)\,|z-w|\,.$$

This tells us in particular that there exists a constant $C_{18}(m,\varrho,\Lambda)$ such that

$$\int_{B_\tau(w)} |\nabla u(z)|^2\,|h(z)-h(w)|^2\,dz \leq C_{17}^2(m,\varrho)\tau^2 \int_{B_\tau(w)} |\nabla u(z)|^2\,dz \leq \\ \leq C_{18}(m,\varrho,\Lambda)\tau^m,$$

since $\tau^{2-m}\int_{B_w(\tau)} |\nabla u|^2$ is bounded.

Step 3. By putting together the information we got from the first two steps, we obtain the following: define $W \doteq \tilde{V} \oplus h(x)$, where $\tilde{V}$ is the linear subspace associated to V; for any $\tau \leq \varrho$, we have

$$\tau^{2-m}\int_{B_\tau(w)} |\langle\nabla u(z), W\rangle|^2\,dx \leq \\ \leq 2(C_{14}(\varrho,m)+2C_{16}(\varrho,m))\tau^{2-m}\varepsilon + 4C_{18}(m,\varrho,\Lambda)\tau^2.$$

Thus, recalling the role of δ_3 in Lemma 3.10, we choose τ such that the second term is smaller than $\frac{1}{2}\delta_3$; after this, we choose ξ_3 (and so ε) such that also the first term is less than $\frac{1}{2}\delta_3$. Thus we can apply Lemma 3.10 to the ball $B_\tau(w)$: the information we get is even stronger, but what is important to us is that w does not belong to $\mathcal{S}^k_{\eta,\bar{r}}(u)$, where $\bar{r}$ is the radius introduced in Lemma 3.10. □

The upcoming lemma says the following: if we have a set of points that satisfy a suitable pinching condition on θ_ψ, and they effctively span a k-subspace L, then all the points of L inherit a (possibly weaker) pinching condition.

Definition 3.5. Let $u \in W^{1,2}(\Omega,\mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ; let $x \in \Omega$, $r > 0$; fix the constants $\delta > 0$ (controlling the pinching condition) and $0 < \varrho < 1$ (controlling the radius). Let $E \leq \Lambda$ be any number such that $\theta_\psi(y,r) \leq E$ for all $y \in B_r(x)$. We call $\mathcal{U}(x,r)$ the set

$$\mathcal{U}^{u,E}_{\delta,\varrho}(x,r) \doteq \{y \in B_r(x) \mid \theta_\psi(y,\varrho r) > E-\delta\}\,.$$

Lemma 3.12. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map with energy bounded by a constant Λ, and let $x \in \Omega$, $r > 0$. Fix a radius $0 < \varrho < 1$ and a constant $\gamma > 0$. Let $E \leq \Lambda$ be any number such that $\theta_\psi(y, r) \leq E$ for all $y \in B_r(x)$. There exists a constant $\delta_4 = \delta_4(m, \mathcal{N}, \Lambda, \varrho, \gamma)$ independent of u, x and r such that the following holds.* ***If*** *the set*

$$\mathcal{U}(x, r) \doteq \{y \in B_r(x) \mid \theta_\psi(y, \varrho r) > E - \delta_4\}$$

ϱr-effectively spans a k-dimensional subspace L, ***then*** *for all the points z in $L \cap B_{2r}(x)$ we have the following lower bound on the normalized energy at scale ϱ:*

$$\theta_\psi(z, \varrho r) > E - \gamma.$$

Remark. The set $\mathcal{U}(x, r)$ is clearly contained in

$$\{y \in B_r(x) \mid \theta_\psi(y, r) - \theta_\psi(y, \varrho r) < \delta_4\};$$

up to some technical changes in the appearing constants, this is again the set $\mathcal{C}_{\delta_4, \varrho}(x, r)$ as defined in Section 2.1.2. Analogously, the condition $\theta_\psi(z, \varrho r) > E - \gamma$ that we have obtained on L implies that, for all $z \in L \cap B_r(x)$,

$$\theta_\psi(z, r) - \theta_\psi(z, \varrho r) < \gamma.$$

Proof. First of all, apply the usual strategy: by scale invariance, we can reduce the problem to $x = 0$, $r = 1$. Then, we argue by contradiction; we find a constant η and the following sequences of items:

- $\{u_j\}_{j \in \mathbb{N}}$, a sequence of minimizing harmonic maps with Λ-bounded energy and $\theta_\psi(y, 1) \leq E$ for all $y \in B_1(0)$; up to subsequences, it converges to a minimizing harmonic map $\bar{u}$ in $W^{1,2}$.

- A sequence of families of points $\{y_{0j}, \ldots, y_{kj}\}_{j \in \mathbb{N}}$ in $B_1(0)$; for any of these points we have

$$\theta_\psi^{u_j}(y_{ij}, \varrho) > E - \frac{1}{j},$$

  and each family spans ϱ-effectively a k-subspace L_j; up to rotations, we can assume $L_j = L$ for all j. Without loss of generality, we can assume that each sequence of points $\{y_{ij}\}_{j \in \mathbb{N}}$ converges to a $\bar{y}_i$ in $\overline{B_1(0)}$; then $\{\bar{y}_1, \ldots, \bar{y}_k\}$ still ϱ-effectively spans L.

- A sequence of points $z_j \in L \cap B_2(0)$ such that

$$\theta_\psi^{u_i}(z_j, \varrho) < E - \gamma,$$

  and such that the sequence converges to a point $\bar{z} \in L \cap \overline{B_2(0)}$.

But then by the fact that the energy of u_i converges to the energy of $\bar{u}$ we have that

$$\theta^{\bar{u}}_{\psi}(\bar{y}_i, \varrho) = E,$$
$$\theta^{\bar{u}}_{\psi}(\bar{z}, \varrho) \leq E - \gamma.$$

The first line together with the fact that $\theta^{\bar{u}}_{\psi}(y, 1) \leq E$, says that for all $1 \in \{0, \dots, k\}$

$$\theta^{\bar{u}}_{\psi}(\bar{y}_i, 1) - \theta^{\bar{u}}_{\psi}(\bar{y}_i, \varrho) = 0,$$

and so $\bar{u}$ is $(k+1)$-symmetric and has L as an invariant space; but $\bar{z}$ belongs to L, so in particular $\theta^{\bar{u}}_{\psi}(\bar{z}, \varrho) = E$, which contradicts the second line. □

Finally, we prove that if an appropriate pinching condition is satisfied at two different points, then the lack of almost symmetry at one point transfers to a lack of almost symmetry at the other.

Lemma 3.13. *Fix the following constants:*

$\eta > 0:$	*a parameter for almost-symmetry;*
$0 < \gamma < 1:$	*controlling the ratio of radii on pinching conditions;*
$\sigma > 0:$	*a "lower bound" for radius.*

There exists a constant $\delta_5 = \delta_5(m, \mathcal{N}, \Lambda, \eta, \gamma, \sigma)$ such that the following implication holds: let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map (with Ω wide enough), x and y points in Ω, $r > 0$; ***if*** *the following conditions are satisfied:*

(i) $|x - y| < \frac{1}{2}r$;

(ii) $\theta_{\psi}(x, r) - \theta_{\psi}(x, \gamma r) < \delta_5$;

(iii) $\theta_{\psi}(y, r) - \theta_{\psi}(y, \gamma r) < \delta_5$;

(iv) For some $\sigma r \leq s \leq 2$, u is not $(\eta, s, k+1)$-symmetric at y;

then *u is not $(\frac{\eta}{2}, s, k+1)$-symmetric at y.*

Proof. Without loss of generality, we can assume that $r = 1$ and $x = 0$; otherwise, we can as usual consider $\tilde{u} = T_{x,r}u$, by esploiting the usual scale invariance:

$$\theta^{u}_{\psi}(y, r\tau) = \theta^{\tilde{u}}_{\psi}(\frac{x - y}{r}, \tau).$$

By contradiction, assume there exist a sequence $\{u_i\}_{i\in\mathbb{N}}$ of minimizing maps, a sequence of radii $\{s_i\}_{i\in\mathbb{N}} \subset [\sigma, 2]$ and a sequence $\{y_i\}_{i\in\mathbb{N}} \subset B_{\frac{1}{2}}(0)$ such that for all i:

- $\theta^{u_i}_{\psi}(0, 1) - \theta^{u_i}_{\psi}(0, \gamma) < \frac{1}{i}$;

- $\theta_\psi^{u_i}(y_i, 1) - \theta_\psi^{u_i}(y_i, \gamma) < \frac{1}{i}$;
- u_i is not $(\eta, s_i, k+1)$-symmetric at y_i;
- There exists a $(k+1)$-symmetric map h_i such that

$$\int_{B_1(0)} |h_i - T_{0,s_i} u_i|^2 \, dx \leq \frac{\eta}{2}. \tag{3.8}$$

By compactness, we can assume $u_i \to \bar{u}$ in strong $W^{1,2}$ (with $\bar{u}$ minimizing), $y_i \to \bar{y}$ in $\overline{B_{\frac{1}{2}}(0)}$, and $s_i \to \bar{s}$ in $[\sigma, 2]$. In particular, $T_{0,s_i} u_i$ converges to $T_{0,\bar{s}}\bar{u}$ and $T_{y_i,s_i} u_i$ converges to $T_{\bar{y},\bar{s}}\bar{u}$ in $L^2(B_1(0); \mathbb{R}^m)$. Observe the following:

- By the usual argument, involving strong convergence and Proposition 2.2 (and a little care about the convergence of the y_i's), we find that $\bar{u}$ is homogeneous with respect to both 0 and $\bar{y}$ in the respective balls of radius 1;
- The lack of almost-symmetry at y_i is preserved by the limit: indeed, for any $(k+1)$-symmetric map h, the difference $h - T_{y_i,s_i} u_i$ converges in L^2 to $h - T_{\bar{y},\bar{s}}\bar{u}$, and thus:

$$\eta \leq \lim_{i \to \infty} \int_{B_1(0)} |h - T_{y_i,s_i} u_i|^2 \, dx = \int_{B_1(0)} |h - T_{\bar{y},\bar{s}}\bar{u}|^2 \, dx.$$

  This means that $\bar{u}$ is not $(\eta, \bar{s}, k+1)$-symmetric at $\bar{y}$.

Notice, however, that for i large enough we have that

$$\int_{B_1(0)} |T_{0,s_i} u_i - T_{0,\bar{s}}\bar{u}|^2 \, dx$$

is arbitrarily small; hence, by triangle inequality and Equation (3.8), we also have

$$\int_{B_1(0)} |h_i - T_{0,\bar{s}}\bar{u}|^2 \, dx \leq \frac{3}{4}\eta.$$

This means that $\bar{u}$ is $(\frac{3}{4}\eta, \bar{s}, k+1)$-symmetric at 0, since each h_i is $(k+1)$-symmetric. This is already a contradiction if $\bar{y}$ happens to be 0. On the other side, even if $\bar{y} \neq 0$, then we find that $\bar{u}$ is invariant along the direction joining 0 to $\bar{y}$ (by Lemma 2.1); then in particular $T_{0,\bar{s}}\bar{u}$ coincides with $T_{\bar{y},\bar{s}}\bar{u}$, and therefore $\bar{u}$ is $(\eta, \bar{s}, k+1)$-symmetric at 0 if and only if it is $(\eta, \bar{s}, k+1)$-symmetric at $\bar{y}$. This gives us the desired contradiction. □

3.4 Covering arguments

In this section we develop an inductive covering argument, which will be the core of the proof of the main result of this chapter, Theorem 3.1. Since this is the overall goal, we first prove that we don't actually need to prove the Theorem 3.1 for all $0 < r \leq 1$: it is enough to prove it for $\varrho^{\hat{\jmath}}$ with $\hat{\jmath} \in \mathbb{N}$, for some $0 < \varrho << 1$, in the exact same way we did in Chapter 2. During the course of the proof we'll give a precise value to ϱ, depending only on m.

Claim. *Fix Λ and η and assume we have proved the following: there exists a $C(m, \mathcal{N}, \Lambda, \eta)$ such that for any minimizing harmonic map u, any $0 \leq k \leq m$ and any $j \in \mathbb{N}$*

$$\operatorname{Vol}\left(\mathcal{B}_{\varrho^j}\left(\mathcal{S}^k_{\eta,\varrho^j}(u)\right) \cap B_1(0)\right) \leq C\left(\varrho^j\right)^{m-k},$$

with $0 < \varrho < 1$ a fixed value. Then there exists a constant $\tilde{C}(m, \mathcal{N}, \Lambda, \eta, \varrho)$ such that for any minimizing harmonic map u, any $0 \leq k \leq m$ and any $0 < r \leq 1$

$$\operatorname{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) B_1(0)\right) \leq \tilde{C} r^{m-k}.$$

Proof. For any r there exists $j \in \mathbb{N}$ such that

$$\varrho^{j+1} \leq r \leq \varrho^j.$$

In particular, $1 \leq \frac{\varrho^j}{r} \leq \varrho^{-1}$. Now by the monotonicity properties of the singular strata and by elementary geometric properties

$$\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) \subset \mathcal{B}_{\varrho^j}\left(\mathcal{S}^k_{\eta,\varrho^j}(u)\right);$$

thus we have

$$\operatorname{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) B_1(0)\right) \leq C\left(\varrho^j\right)^{m-k} = \\ = C\left(\frac{\varrho^j}{r}\right)^{m-k} r^{m-k} \leq \frac{C}{\varrho^{m-k}} r^{m-k}.$$

□

3.4.1 First Covering Lemma

Remembering what we learned from the Covering Lemma 4 of Chapter 2, we realize that if we want to estimate the volume of $\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right)$ we need to find a covering of $\mathcal{S}^k_{\eta,r}(u)$ with balls of radius r. This result is achieved through several re-coverings of the singular stratum, described in the next sections. We begin by covering $\mathcal{S}^k_{\eta,r}(u)$ (for any $r = \varrho^j$) with a first "rough" collection of balls, having the following property:

- Some of the balls will have radius r, so they will be good candidates for a final covering;
- On the other balls, we have a nice “energy drop” condition: when we fix one of these balls B, at all points of B except those lying near a $(k-1)$-subspace the energy has fallen below a “uniformly controlled” threshold.

Moreover, we obtain an effective control on the sum $\sum r_x^k$ of the k^{th}-powers of the radii. The geometric construction of the covering makes intensive use of the technical lemmas from Section 3.3, but does not involve further effort; the most challenging part of the argument (Sublemma 3.14.2) is to achieve the above mentioned estimate on $\sum r_x^k$; here the first Reifenberg Theorem and the L^2-best approximation Theorem are exploited.

Lemma 3.14. *Fix the following parameters:*

$\Lambda > 0:$	*a bound for the energy;*
$\eta > 0:$	*the “closeness parameter” for the stratum;*
$0 < \varrho < \dfrac{1}{100}:$	*a constant that will assume a precise value in Lemma 3.15.*

Also, assume that $\varrho = 5^{-\varkappa}$ *for an integer* $\varkappa$*. There exist* ***two constants*** $\delta_6 = \delta_6(m, \mathcal{N}, \Lambda, \eta, \varrho)$ *and* $C_I = C_I(m)$ *such that the following holds.*

- *Let* $u \in W^{1,2}(B_4(0), \mathcal{N})$ *be a minimizing harmonic map with energy bounded by* Λ*;*
- *Let* $0 \le k \le m$ *be an integer number; fix an integer* $\hat{\jmath} \in \mathbb{N}$*; call* $r = \varrho^{\hat{\jmath}}$*; let* $\delta < \delta_6$ *be a constant;*
- *Assume that* $E \le \Lambda$ *is an arbitrary number such that* $\theta_\psi(y, 1) \le E$ *for all* $y \in B_1(0)$.

Then *any subset* $\mathcal{S}$ *of the singular stratum* $\mathcal{S}^k_{\eta,\delta r}(u) \cap B_1(0)$ *admits a* finite *covering of the type*

$$\mathcal{S} \subset \bigcup_{x \in \mathcal{C}} B_{r_x}(x),$$

where:

1. *All the radii satisfy* $r_x \ge r$*, and* $r_x = \varrho^h$ *for some* $0 \le h \le \hat{\jmath}$*;*
2. *The radii are controlled by*
$$\sum_{x \in \mathcal{C}} r_x^k \le C_I; \tag{3.9}$$
3. *For any center* $x \in \mathcal{C}$*, one of the following two options is satisfied:*

(A) $r_x = r$;

(B) *There exists a $(k-1)$-dimensional affine subspace L_x such that the set of points*

$$\Upsilon_x \doteq \left\{ y \in \mathcal{S} \cap B_{2r_x}(x) \;\middle|\; \theta_\psi\left(\frac{\varrho}{10} r_x\right) > E - \delta \right\}$$

is contained in $\mathcal{B}_{\frac{\varrho}{5} r_x}(L_x) \cap B_{2r_x}(x)$.

We point out that the constants δ_6 and C_I do not depend on u and r, but the covering clearly does; in the upcoming proofs and sublemmas, $r = \varrho^{\hat{\jmath}}$ will be a fixed radius. Moreover, at this stage ϱ is simply a constant "small enough": just by convenience, we are imposing it to be less then 100^{-1}, even if so far some less strict condition would be enough; again for convenience, we are assuming it to be a negative power of 5 (the application of this assumption will be clear during the course of the proof).

We begin by building inductively a sequence of intermediate coverings; the last step will provide the covering for Lemma 3.14 (but we'll need to work considerably more to obtain the radii estimates). At any stage of this (finite) induction, we classify as "bad balls" those which satisfy condition (B) of Lemma 3.14; these balls will remain untouched by the subsequent steps, and so they will still appear as bad balls until the end. Thus, they will be part of the final covering $\mathcal{C}$ of Lemma 3.14. On the contrary, at any step the covering gets refined on the other balls: we call them good balls.

To have the statement more clear, recall that δ_5 is the constant coming from Lemma 3.13, which assures that almost symmetry "spreads uniformly", in some sense.

Sublemma 3.14.1 (Intermediate coverings)**.** *Assume the setting is the same of Lemma 3.14; also, fix an arbitrary $0 < \gamma < \delta_5$. There exists a $\delta_7(m, \mathcal{N}, \Lambda, \eta, \varrho, \gamma)$ such that for all $\delta < \delta_7$ we have: for any $j \in \mathbb{N}$, $1 \le j \le \hat{\jmath}$, there exist two finite sets of centers $\mathcal{C}_b^j$ and $\mathcal{C}_g^j$, and a collection of radii $r_{x;j}$ associated to the centers, such that the following properties hold:*

1. *The balls centered in $\mathcal{C}^j \doteq \mathcal{C}_b^j \cup \mathcal{C}_g^j$ with radii $r_{x;j}$ form a covering of $\mathcal{S}$:*

$$\mathcal{S} \subset \bigcup_{x \in \mathcal{C}_b^j} B_{r_{x;j}}(x) \cup \bigcup_{x \in \mathcal{C}_g^j} B_{r_{x;j}}(x);$$

2. Bad balls*: if $x \in \mathcal{C}_b^j$, then $r_{x;j} = \varrho^h$ for some $h \le j$ (so in particular $r_{x;j} \ge \varrho^j$); moreover, there exists a $(k-1)$-dimensional affine subspace $L_{x;j}$ such that the set of points*

$$\Upsilon_{x;j}^\delta = \Upsilon_{x;j} \doteq \left\{ y \in \mathcal{S} \cap B_{2r_{x;j}}(x) \;\middle|\; \theta_\psi\left(\frac{\varrho}{10} r_{x;j}\right) > E - \delta \right\}$$

is contained in $\mathcal{B}_{\frac{\varrho}{5} r_x}(L_x)$;

3. Good balls*: if $x \in \mathcal{C}_g^j$, then $r_{x;j} = \varrho^j$; moreover, the set $\Upsilon^\delta_{x;j}$ spans a k-dimensional subspace $V_{x;j}$ in a $\frac{\varrho r_{x;j}}{10}$-effective way;*

4. *For any pair of centers $x \neq y \in \mathcal{C}^j$, we have*
$$B_{\frac{1}{5}r_{x;j}}(x) \cap B_{\frac{1}{5}r_{y;j}}(y) = \varnothing;$$

5. *For all $x \in \mathcal{C}^j$ with $j \geq 1$, the pinching condition $\theta_\psi(x, \frac{1}{10}r_{x;j}) \geq E - \gamma$ is satisfied;*

6. *For all $x \in \mathcal{C}^j$ with $j \geq 1$, and for all $r_{x;j} \leq s \leq 1$, u is not $(\frac{\eta}{2}, s, k+1)$-symmetric at x.*

Proof. We prove the sublemma by induction, with a preliminary "anomalous" step.

Step 0. Let $j = 0$. Consider, for any $\delta > 0$, the set
$$\Upsilon^\delta_{0;0} \doteq \left\{ y \in \mathcal{S} \cap B_2(0) \;\middle|\; \theta_\psi\left(y, \frac{\varrho}{10}\right) > E - \delta \right\}.$$

If $\Upsilon^\delta_{0;0}$ is contained in $\mathcal{B}_{\frac{\varrho}{5}}(L_{0,0})$ for some $(k-1)$-dimensional subspace $L_{0,0}$, then we define $\mathcal{C}_b^0 = \{0\}$, $r_{0,} = 0$, $\mathcal{C}_g^0 = \varnothing$, so that $\mathcal{S} \cap B_1(0)$ is covered by the unit ball: $B_1(0)$ gets classified as a bad ball, and it will remain the only ball in the covering until the end of the inductive process (*i.e.*, in the subsequent steps nothing is done). This cover doesn't actually satisfy conditions 5 and 6, but it is irrelevant, since in this case Lemma 3.14 is trivial.
Otherwise, we necessarily have that $\Upsilon^\delta_{0;0}$ spans $\frac{\varrho}{5}$-effectively an affine k-subspace $V_{0;0}$. In this case, the covering can be refined; the unit ball is a good ball (but it doesn't satisfy the required conditions).
Step 1. Let $j = 1$. As we've just seen, the only case in which there's something we can do is when $\Upsilon^\delta_{0;0}$ spans $\frac{\varrho}{5}$-effectively an affine k-subspace $V_{0;0}$. As a trivial consequence, $V_{0;0}$ is also effectively spanned by $\Upsilon^\delta_{0;0}$ with "effectiveness" $\frac{\varrho}{10}$ instead of $\frac{\varrho}{5}$; that is: there exist $k+1$ points $\{y_{1,1}, \ldots, y_{1,k+1}\}$ points of $\Upsilon^\delta_{0;0} \cap V_{0;0}$ in $\frac{\varrho}{10}$-general position. Now $\Upsilon^\delta_{0;0}$ is a subset of the set $\mathcal{K}_{\delta,\frac{\varrho}{10}}(0,0)$ defined in Lemma 3.11: if we choose $\delta_7 \leq \xi_3$, that lemma can be applied. This tells us that, since $r \leq 1$, we have:
$$\mathcal{S}^k_{\eta,\delta r}(u) \cap B_1(0) \subset \mathcal{B}_{\frac{\varrho}{5}}(V_{0;0}).$$

Now by the Vitali Covering Theorem we can easily find a covering of $\mathcal{B}_{\frac{\varrho}{5}}(V_{0;0}) \cap B_1(0)$ with balls of radius ϱ and set of centers $\mathcal{C}_g^1 \subset V_{0;0}$, with the property that for any couple of centers $x \neq y$ the balls $B_{\frac{\varrho}{5}}(x)$ and $B_{\frac{\varrho}{5}}(y)$ are disjoint. Moreover, by Lemma 3.12 we have that all the points of $V_{0;0} \cap B_2(0)$ (and in particular the centers $\mathcal{C}_g^1$) satisfy
$$\theta_\psi\left(x, \frac{\varrho}{10}\right) > E - \gamma,$$

provided again we choose $\delta_7 \leq \delta_4$.
Furthermore, any point of $V_{0;0} \cap B_1(0)$ has distance less than 2 from a $y_{1,j} \in \Upsilon^{\delta}_{0;0} \cap V_{0;0}$. Let x be a center in $\mathcal{C}^1_g$; at x the pinching condition is satisfied with parameter γ; and in the ball $B_4(x)$ there exists a point of $\mathcal{S}^k_{\eta,\delta r}(u)$ satisfying the (stronger) pinching condition with δ. If δ is small, upon some adjustment of the involved radii we can apply Lemma 3.13 to find out that u is not $(\frac{\eta}{2}, s, k+1)$-symmetric at x for $s > \varrho$.

STEP 2. Now assume we have the covering for some $j \in \mathbb{N}$, $1 \leq j \leq \hat{\jmath}$. It is clear that the bad balls centered in $\mathcal{C}^j_b$ can still appear as bad balls for the $(j+1)^{\text{th}}$ step, so all we have to do is to refine the part of the covering made of good balls. So consider the set

$$\mathcal{S}_j \doteq \mathcal{S} \setminus \bigcup_{x \in \mathcal{C}^j_b} B_{r_x}(x).$$

To cover it, we need to find a re-covering of the balls $B_{\varrho^j}(x)$ with $x \in \mathcal{C}^j_g$. Consider one of these x: observing how the covering is defined, we can see that also in this case the set $\Upsilon^{\delta}_{x;j}$ spans a k-dimensional subspace $V_{x;j}$ in a $\frac{1}{5}\varrho^{j+1}$-effective way. Thus the set

$$\mathcal{K}_{\delta,\varrho}(x, \varrho^j) = \left\{ y \in B_{2\varrho^j}(x) \;\middle|\; \theta_\psi\left(y, \varrho^j\right) - \theta_\psi\left(y, \varrho^{j+1}\right) < \delta \right\}$$

$\frac{\varrho^{j+1}}{10}$-effectively spans $V_{x;j}$, and this (thanks again to Lemma 3.11) implies that

$$\mathcal{S}^k_{\eta,\delta r}(u) \cap B_{2\varrho^j}(x) \subset \mathcal{B}_{\frac{1}{5}\varrho^{j+1}}(V_{x;j}),$$

since we have

$$\delta_7 \leq \min\left\{\xi_3, \delta_4\right\}.$$

In particular, the set $\mathcal{S}_j$ is covered by $\mathcal{B}_{\frac{1}{5}\varrho^{j+1}}(A_j)$, where A_j is the set

$$A_j \doteq \left(\bigcup_{x \in \mathcal{C}^j_g} B_{\varrho^j}(x) \cap V_x \right) \setminus \left(\bigcup_{x \in \mathcal{C}^j_b} B_{\frac{1}{2} r_{x;j}}(x) \right);$$

this is actually the union of pieces of k-planes, and we are defining it in such a way that all the centers of the bad balls are sufficiently far, in order to avoid disjointness problems with the balls we're going to define. Just as in STEP 1, all the points of A_j satisfy

$$\theta_\psi\left(x, \frac{1}{10}\varrho^{j+1}\right) > E - \gamma,$$

by the definition of A_j and by Lemma 3.12. This in particular means that

$$\theta_\psi\left(x, \frac{1}{10}\varrho^{j}\right) - \theta_\psi\left(x, \frac{1}{10}\varrho^{j+1}\right) \leq \gamma;$$

moreover, for any point of A_{j+1} there exists an element of $\mathcal{S}^k_{\eta,\delta r}(u)$ that is distant less than $2\varrho^j$ from it: for any $s > \varrho \cdot \varrho^j$, u is not $(\frac{\eta}{2}, s, k+1)$-symmetric at the points of A_{j+1}, and the smallness condition required on δ is the same as in Step 1, since we are applying Lemma 3.13 *with the same initial constants* but at a different scale. At this point, the only thing left to do is to cover $\mathcal{S}_j$ with balls of radius ϱ^{j+1} such that the corresponding balls of radius $\frac{1}{5}\varrho^{j+1}$ are disjoint. Then, we subdivide them in bad balls (whose centers we define $\mathcal{C}^{j+1}_{b,new}$) and good balls, depending on the behavior of $\Upsilon^\delta_{x;j+1}$ – bad if it is contained in the enlargement of a $(k-1)$-dimensional subspace, otherwise good. Finally, we define $\mathcal{C}^{j+1}_g$ as the centers of the good balls just defined, and $\mathcal{C}^{j+1}_b$ as the union of the old bad centers and the bad ones we are adding in this step:

$$\mathcal{C}^{j+1}_b \doteq \mathcal{C}^j_b \cup \mathcal{C}^{j+1}_{b,new}.$$

□

If we assume that $r = \varrho^{\hat{\jmath}}$ for a certain $\hat{\jmath}$, then all the claims of Lemma 3.14 are proved except for the estimate on the radii. This will be the next goal, and here's where the first Reifenberg Theorem is exploited: under the assumption that $r = \varrho^{\hat{\jmath}}$, we exploit Theorem 3.4 to obtain the desired estimate.

Sublemma 3.14.2. *Let the setting be the same of the previous sublemma, with $0 < \gamma < \delta_5$, and let $\delta < \delta_7$. Assume $r = \varrho^{\hat{\jmath}}$ for a fixed $\hat{\jmath} \in \mathbb{N}$, and consider the covering $\mathcal{C}^{\hat{\jmath}} = \mathcal{C}^{\hat{\jmath}}_b \cup \mathcal{C}^{\hat{\jmath}}_g$ given by Sublemma 3.14.1. If γ is chosen small enough, there exists a constant C_{19} depending only on m such that*

$$\sum_{x\in\mathcal{C}} r^k_{x;\hat{\jmath}} \leq C_{19}.$$

Proof. For simplicity, we drop everywhere the indication of $\hat{\jmath}$, since it is fixed. The first Reifenberg Theorem (Theorem 3.4) provides exactly the estimate we need, once we verify its assumption (RR1); for this reason, our aim will be to show that for any ball $B_\tau(w)$ contained in $B_2(0)$ the following bound holds:

$$\int_{B_\tau(w)} \left(\int_0^\tau D^k_\mu(y,s) \frac{ds}{s} \right) d\mu(y) < \delta_{Rf1} \tau^k. \tag{3.10}$$

Here μ is the measure defined by

$$\mu \doteq \sum_{x\in\mathcal{C}^{\hat{\jmath}}} \omega_k r^k_x \delta_x.$$

Now define also the following sets of centers: for any $h \in \{0, \ldots, \hat{\jmath}\}$,

$$\mathcal{C}_h = \mathcal{C}^{\hat{\jmath}}_h \doteq \left\{ x \in \mathcal{C}^{\hat{\jmath}} \,\middle|\, r_x \leq \varrho^{\hat{\jmath}-h} \right\},$$

together with the associated measures

$$\mu_h \doteq \sum_{x \in \mathcal{C}^{\hat{\jmath}}_h} \omega_k r_x^k \delta_x.$$

Basically, we are initially considering only the centers whose radius is r (for $h = 0$), and then adding larger balls a few at a time until we reach the complete covering $\mathcal{C}^{\hat{\jmath}}$; so what we can do is to prove by induction on h that the condition (3.10) is satisfied for all the measures μ_h: the last step $h = \hat{\jmath}$ will give us the needed result. The general strategy will be the following:

1. For any h, we find an appropriate way to apply the Best Approximation Theorem 3.9 to obtain an estimate of the type

$$D^k_\mu(y, s) \leq C_{20} s^{-k} \int_{B_s(y)} \hat{P}_{u,5}(z, s)\, d\mu(z),$$

   where $\hat{P}_{u,\sigma}$ is a modification of $P_{u,\sigma}$ and where the constant C_{20} depends on m, $\mathcal{N}$, Λ and η. Notice that we are now fixing $\sigma = 5$.

2. We show that for any τ (small enough) and for $x \in \mathcal{C}$ the following inequality holds:

$$\int_0^\tau \hat{P}_{u,5}(x, s) \frac{ds}{s} \leq c_5(\varrho)\gamma;$$

3. If for some h we dispose of an estimate

$$\mu_h(B_{\varrho^{\hat{\jmath}-\ell}}(w)) \leq C(\varrho^{\hat{\jmath}-\ell})^k,$$

   valid for all $w \in B_2(0)$ and for all $\ell = 0, \ldots, h$, then this inequality together with the first two steps allows us to apply the Reifenberg Theorem 3.4 to μ_{h+1}, provided γ is small;

4. We show that for μ_0 the estimate

$$\mu_0(B_{\varrho^{\hat{\jmath}}}(w)) \leq c_6(m)(\varrho^{\hat{\jmath}})^k$$

   holds; so the step 3 can be applied with $h = 0$, and this gives a condition on μ_1;

5. If we manage to apply the step 3 with $h = \hat{\jmath} - 1$, then we are finished since the condition given by the Reifenberg Theorem is the one we need. Otherwise, applying the step 3 to some smaller h, the acquired result leads to an estimate

$$\mu_{h+1}(B_{\varrho^{\hat{\jmath}-(h+1)}}(w)) \leq c_7 \left(\varrho^{\hat{\jmath}-(h+1)}\right)^k,$$

   with c_7 independent of h, so we can iterate the process.

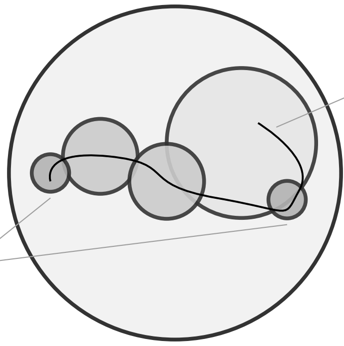


Figure 3.1: A visual representation of the balls associated to the sequence of centers $\{\mathcal{C}_h\}_{h=0,\dots\hat{\jmath}}$: at the step $h=0$ we only have the smaller and darker balls, and then we add the remaining ones increasing the radius.

PART 1. Fix a $h = 0,\dots,\hat{\jmath}$. Observe first that, if we pick a center $x \in \mathcal{C}_h$ and a $\frac{1}{10}r_x \le \tilde{s} \le \frac{1}{5}$, then we have $\theta_\psi(x, 5\tilde{s}) \le E$ by definition of E, and $\theta_\psi(x,\tilde{s}) \ge E-\gamma$ by the property 5 of the covering (from Sublemma 3.14.1). In particular,

$$P_{u,5}(x,\tilde{s}) = \theta_\psi(x,5\tilde{s}) - \theta_\psi(x,\tilde{s}) < \gamma. \tag{3.11}$$

Consider the following quantity

$$\hat{P}_{u,5}(x,s) = \hat{P}(x,s) \doteq \begin{cases} P_{u,5}(x,s) & \text{if } s > \dfrac{r_x}{5} \\ 0 & \text{if } s \le \dfrac{r_x}{5} \end{cases}.$$

If $s \le \frac{r_x}{5}$, both $D^k_{\mu_h}(x,s)$ and $\hat{P}(x,s)$ are zero: the latter by definition, the former because x is the only center contained in $B_s(x)$, and so any k-subspace containing x realizes the minimum 0 in the definition of $D^k_{\mu_h}$. If instead $s > \frac{r_x}{5}$, then u is not $(\frac{\eta}{2}, 5s, k+1)$-symmetric at x by property 6 of the covering. However, provided γ is small enough, by (3.11) (with $\tilde{s} = 5s$) and Proposition 2.6, u is $(\delta_2, 5s, 0)$-symmetric at x, where $\delta_2(m,\mathcal{N},\Lambda,\frac{\eta}{2},5)$ is the constant produced by Theorem 3.9. Thus Theorem 3.9 can be applied: we obtain, for $x \in \mathcal{C}_h$ and $C_{20}(m,\mathcal{N},\Lambda,\eta)$,

$$D^k_{\mu_h}(x,s) \le C_{20} s^{-k} \int_{B_s(x)} \hat{P}_{u,5}(z,s)\, d\mu_h(z). \tag{3.12}$$

PART 2. Consider $\varrho^{\hat{\jmath}} < \tau \le \frac{1}{5}$ and pick a center $x \in \mathcal{C}$ with $r_x = \varrho^\ell$, $\ell \ge 1$. We have the following estimate:

$$\int_0^\tau \hat{P}_{u,5}(x,s)\frac{ds}{s} \le \int_{\frac{1}{5}\varrho^\ell}^\tau \hat{P}_{u,5}(x,s)\frac{ds}{s} + \int_\tau^{\frac{1}{5}} \hat{P}_{u,5}(x,s)\frac{ds}{s}.$$

Now the right hand side equals

$$\int_{5^{-\varkappa\ell-1}}^{5^{-1}} \hat{P}_{u,5}(x,s)\frac{ds}{s}; \tag{3.13}$$

for this expression we have the following estimate:

$$\begin{aligned}\int_{5^{-\varkappa\ell-1}}^{5^{-1}} \hat{P}_{u,5}(x,s)\frac{ds}{s} &\leq \sum_{j=1}^{-\varkappa\ell} \int_{5^{-(j+1)}}^{5^{-j}} \frac{\theta_\psi(x,5s)-\theta_\psi(x,s)}{s}\,ds \leq \\ &\leq \sum_{j=1}^{-\varkappa\ell} \int_{5^{-(j+1)}}^{5^{-j}} \frac{\theta_\psi(x,5^{-j+1})-\theta_\psi(x,5^{-j-1})}{5^{-j-1}}\,ds \leq \\ &\leq \left(5^{-1}-1\right)\sum_{j=1}^{-\varkappa\ell}\left[\theta_\psi\left(x,5^{-j+1}\right)-\theta_\psi\left(x,5^{-j-1}\right)\right].\end{aligned}$$

Since the last is a telescopic sum, we obtain:

$$\begin{aligned}\int_{5^{-\varkappa\ell-1}}^{5^{-1}} \hat{P}_{u,5}(x,s)\frac{ds}{s} \leq C\Big[&\left(\theta_\psi(x,1)-\theta_\psi\left(x,5^{-\varkappa\ell}\right)\right)+\\ &+\left(\theta_\psi\left(x,\frac{1}{5}\right)-\theta_\psi\left(x,\frac{1}{5}5^{-\varkappa\ell}\right)\right)\Big];\end{aligned}$$

and this is smaller than $c_5\gamma$ by the same argument which led to Equation (3.11).

Part 3. Now assume that for a certain $h = 0,\dots,\hat{\jmath}-1$ we have, for all $w \in B_2(0)$ and all $\ell = 0,\dots,h$:

$$\mu_h(B_{\varrho^{\hat{\jmath}-\ell}}(w)) \leq C(\varrho^{\hat{\jmath}-\ell})^k, \tag{3.14}$$

where $C = C(m)$. Consider the measure μ_{h+1}: by definition, we are taking the measure μ_h and adding the contributions given by the (larger) balls of radius $\varrho^{\hat{\jmath}-(h+1)}$. Consider a point $w \in B_2(0)$ and a radius $\tau \leq \min\left\{\frac{1}{2}\varrho^{\hat{\jmath}-(h+1)}, \frac{1}{5}\right\}$. The following inequality holds, by Part 1 and Tonelli Theorem:

$$\begin{aligned}&\int_{B_\tau(w)}\left(\int_0^\tau D^k_{\mu_{h+1}}(y,s)\frac{ds}{s}\right)d\mu_{h+1}(y) \leq\\ &\quad\leq \int_{B_\tau(w)}\left(\int_0^\tau\left(C_{20}s^{-k}\int_{B_s(y)}\hat{P}_{u,5}(z,s)\,d\mu_{h+1}(z)\right)\frac{ds}{s}\right)d\mu_{h+1}(y) =\\ &\quad= \int_0^\tau C_{20}s^{-k}\left(\int_{B_\tau(w)}\left(\int_{B_s(y)}\hat{P}_{u,5}(z,s)\,d\mu_{h+1}(z)\right)d\mu_{h+1}(y)\right)\frac{ds}{s};\end{aligned}$$

remember that μ_{h+1} is supported in $\mathcal{C}_{h+1}$, so the behaviour of the inner integrand outside of $\mathcal{C}_{h+1}$ is irrelevant. Now notice that by the triangle inequality

$$|z-w| \leq |z-y| + |y-w|,$$

so the set

$$\{(y,z) \in \mathbb{R}^m \times \mathbb{R}^m \mid y \in B_\tau(w), z \in B_s(y)\}$$

is contained in

$$\{(y,z) \in \mathbb{R}^m \times \mathbb{R}^m \mid z \in B_{\tau+s}(w), y \in B_s(z)\}\,.$$

Using again Tonelli Theorem, we also switch the two integrals in μ_{h+1}, thus getting:

$$\begin{aligned}
&\int_{B_\tau(w)} \left(\int_0^\tau D^k_{\mu_{h+1}}(y,s) \frac{ds}{s} \right) d\mu_{h+1}(y) \leq \\
&\quad \leq \int_0^\tau C_{20} s^{-k} \left(\int_{B_{\tau+s}(w)} \hat{P}_{u,5}(z,s) \left(\int_{B_s(z)} d\mu_{h+1}(y) \right) d\mu_{h+1}(z) \right) \frac{ds}{s} \leq \\
&\quad \leq \int_0^\tau C_{20} s^{-k} \int_{B_{\tau+s}(w)} \hat{P}_{u,5}(z,s) \mu_{h+1}(B_s(z))\, d\mu_{h+1}(z) \, \frac{ds}{s}.
\end{aligned}$$

Hence we need to estimate $\mu_{h+1}(B_s(z))$ for $z \in \mathcal{C}_{h+1}$; actually there's no need to do it for $s < \frac{\varrho^{\hat{\jmath}}}{5}$, since in that case the term $\hat{P}$ will be 0 for all the relevant z. Now, we consider separately three cases:

- Assume $\varrho^{\hat{\jmath}-\ell+1} \leq s \leq \varrho^{\hat{\jmath}-\ell}$ for some $\ell = 1,\dots,h$; then $B_s(z) \subset B_{\varrho^{\hat{\jmath}-\ell}}(z)$. In this last ball the only point of $\mathcal{C}_{h+1} \setminus \mathcal{C}_h$ can be z himself: indeed, since by assumption $\varrho < \frac{1}{5}$, for any $x \in \mathcal{C}_{h+1} \setminus \mathcal{C}_h$ we have:

$$B_{\varrho^{\hat{\jmath}-\ell}}(x) \subset B_{\varrho^{\hat{\jmath}-h}}(x) \subset B_{\frac{1}{5}\varrho^{\hat{\jmath}-(h+1)}}(x),$$

  and the right hand side is disjoint from any other ball of the covering. Moreover, if z belongs to $\mathcal{C}_{h+1} \setminus \mathcal{C}_h$, then $\hat{P}(z, \varrho^{\hat{\jmath}-\ell}) = 0$ because $\varrho^{\hat{\jmath}-\ell} < \frac{r_z}{5}$, so we don't need to consider this case. Thus, in case $z \in \mathcal{C}_h$ we have:

$$\begin{aligned}
\mu_{h+1}(B_{\varrho^{\hat{\jmath}-\ell}}(z)) \leq \sum_{x \in \mathcal{C}_h \cap B_{\varrho^{\hat{\jmath}-\ell}}(z)} \omega_k r_x^k = \\
= \mu_h(B_{\varrho^{\hat{\jmath}-\ell}}(z)) \leq C(\varrho^{\hat{\jmath}-\ell})^k
\end{aligned}$$

  by Equation (3.14), and C is *the same constant of the assumed estimate* (this will be fundamental for proceeding with the inductive argument). Moreover,

$$s^{-k}\mu_{h+1}(B_s(z)) \leq C \frac{(\varrho^{\hat{\jmath}-\ell})^k}{(\varrho^{\hat{\jmath}-\ell+1})^k} \leq C(m,\varrho)\varrho^{-k} \doteq C_{21}(m,\varrho).$$

- Assume instead $\varrho^{\hat{\jmath}-h} \leq s \leq \varrho^{\hat{\jmath}-(h+1)}$. Then we estimate:

$$\begin{aligned}
&\mu_{h+1}(B_s(z)) \leq \mu_{h+1}(B_{\varrho^{\hat{\jmath}-(h+1)}}(z)) \leq \\
&\quad \leq \sum_{x \in \mathcal{C}_h \cap B_{\varrho^{\hat{\jmath}-(h+1)}}(z)} \omega_k r_x^k + \sum_{\substack{x \in B_{\varrho^{\hat{\jmath}-(h+1)}}(z) \\ x \in \mathcal{C}_{h+1} \setminus \mathcal{C}_h}} \omega_k \left(\varrho^{\hat{\jmath}-(h+1)} \right)^k
\end{aligned}$$

The first term in the sum is actually $\mu_h(B_{\varrho^{\hat{\jmath}-(h+1)}}(z))$; since the ball $B_{\varrho^{\hat{\jmath}-(h+1)}}(z)$ can be covered by a controlled number $c_8(m)$ of balls of radius $\varrho^{\hat{\jmath}-h}$, we can bound that term with

$$c_8 C(\varrho^{\hat{\jmath}-h})^k$$

by the hypothesis (3.14). On the other hand, the second term can be bounded by

$$c_9(m)\omega_k\left(\varrho^{\hat{\jmath}-(h+1)}\right)^k,$$

since the balls $B_{\frac{1}{5}\varrho^{\hat{\jmath}-(h+1)}}(x)$ centered in $\mathcal{C}$ are disjoint. So we get:

$$\mu_{h+1}(B_s(z)) \leq c_{10}(m,\varrho)\left(\varrho^{\hat{\jmath}-(h+1)}\right)^k$$
$$s^{-k}\mu_{h+1}(B_s(z)) \leq C_{22}(m,\varrho);$$

notice however that the constant in the first inequality is not the same C as in our hypothesis: this prevents us from using only this simple argument for the induction, and forces us to exploit Reifenberg.

- Finally, if $\frac{\varrho^{\hat{\jmath}}}{5} \leq \varrho^{\hat{\jmath}}$, we use the very same argument as in the first case, and get the same estimate, only with a different constant.

So, switching back to the integral we were evaluating, and swapping again the remaining integrals using the Tonelli Theorem, we find

$$\int_{B_\tau(w)} \left(\int_0^\tau D^k_{\mu_{h+1}}(y,s)\frac{ds}{s}\right) d\mu_{h+1}(y) \leq \\ C(m,\varrho)\int_{B_{2\tau}(w)} \left(\int_0^\tau \hat{P}_{5,u}(z,s)\frac{ds}{s}\right) d\mu_{h+1}(z);$$

and here we can apply what we discovered in Part 2, so we get for a $C_{23}(m,\varrho)$:

$$\int_{B_\tau(w)} \left(\int_0^\tau D^k_{\mu_{h+1}}(y,s)\frac{ds}{s}\right) d\mu_{h+1}(y) \leq \\ \leq C_{23}(m,\varrho)c_5(\varrho)\mu_{h+1}(B_{2\tau}(w))\gamma.$$

Now, if $\tau < \frac{1}{5}\varrho^{\hat{\jmath}}$ or if $B_{2\tau}(w)$ contains a point of $\mathcal{C}_{h+1}\setminus\mathcal{C}_h$ we can argue as with the first case of the analysis of s and see that the initial integral is 0; otherwise, with the same computations we just did for s we get that

$$\mu_{h+1}(B_{2\tau}(w)) \leq C_{24}(m,\varrho)\tau^k.$$

Choosing γ smaller than

$$\frac{\delta_{Rf1}}{C_{23}C_{24}c_5},$$

which depends only on m and ϱ, we get that

$$\int_{B_\tau(w)} \left(\int_0^\tau D^k_{\mu_{h+1}}(y,s) \frac{ds}{s} \right) d\mu_{h+1}(y) \leq \delta_{Rf1} \tau^k \tag{3.15}$$

for all $0 < \tau \leq \min\left\{\frac{1}{2}\varrho^{\hat{\jmath}-(h+1)}, \frac{1}{5}\right\}$. This is almost what we needed to apply the Reifenberg Theorem: in Part 5 we show how to exploit this information anyway; before, we show that actually for $h = 0$ the assumption is verified.

Part 4. For $h = 0$, the inequality is actually trivial: in fact, μ_0 only considers the balls centered in $\mathcal{C}$ with radius $\varrho^{\hat{\jmath}}$; for an arbitrary ball $B = B_{\varrho^{\hat{\jmath}}}(w)$, the number of centers $\mathcal{C}_{\hat{\jmath}}$ contained in B is bounded by a dimensional constant $c_{11}(m)$, since the corresponding balls with radius $\frac{1}{5}\varrho^{\hat{\jmath}}$ are disjoint; hence:

$$\mu_0(B_{\varrho^{\hat{\jmath}}}(w)) \leq c(m)\left(\varrho^{\hat{\jmath}}\right)^k.$$

Part 5. Now consider the conclusions of Part 3. Define

$$\tilde{\tau} \doteq \min\left\{\frac{1}{2}\varrho^{\hat{\jmath}-(h+1)}, \frac{1}{5}\right\},$$

and fix a ball $B_{\tilde{\tau}}(\tilde{w})$ in $B_2(0)$. For all the balls

$$B_\tau(w) \subset B_{\tilde{\tau}}(\tilde{w})$$

we have found the estimate (3.15); now consider the transformation

$$\begin{aligned} \lambda_{\tilde{w},\tilde{\tau}} : B_1(0) &\longrightarrow B_{\tilde{\tau}}(\tilde{w}) \\ x &\longmapsto \tilde{w} + \tilde{\tau}x \end{aligned}$$

and the measure on $B_1(0)$

$$\tilde{\mu} \doteq \tilde{\tau}^{-k} T_{\tilde{w},\tilde{\tau}} \mu_{h+1};$$

notice that this measure differs from the rescaled measure we defined in Definition 3.3 for the term $\tilde{\tau}^{-k}$, thus we need to correct the properties we used in that context. Applying the transformations

$$x = \lambda^{-1}_{\tilde{w},\tilde{\tau}}(y) = \frac{y - \tilde{w}}{\tilde{\tau}}, \qquad \tilde{s} = \frac{s}{\tilde{\tau}}$$

to the left hand side of Equation (3.15), and exploiting the scale invariance properties of the Jones' numbers, we find the following:

$$\begin{aligned} &\int_{B_\tau(w)} \left(\int_0^\tau D^k_{\mu_{h+1}}(y,s) \frac{ds}{s} \right) d\mu_{h+1}(y) = \\ &\quad = \int_{B_{\frac{\tau}{\tilde{\tau}}}(w-\tilde{w})} \tilde{\tau}^k \left(\int_0^{\frac{\tau}{\tilde{\tau}}} D^k_{\mu_{h+1}}(\tilde{w}+\tilde{\tau}x, \tilde{\tau}\tilde{s}) \frac{d\tilde{s}}{\tilde{s}} \right) d\tilde{\mu}(x) \\ &\quad = \int_{B_{\frac{\tau}{\tilde{\tau}}}(w-\tilde{w})} \tilde{\tau}^k \left(\int_0^{\frac{\tau}{\tilde{\tau}}} D^k_{\tilde{\mu}}(x, \tilde{s}) \frac{d\tilde{s}}{\tilde{s}} \right) d\tilde{\mu}(x); \end{aligned}$$

the term $\tilde{\tau}^k$ comes from the change of variables, while no further terms are produced when we rescale the number $D^k_\cdot$. In particular, we have found that for all the balls $B_\tau(w) \subset B_{\tilde{\tau}}(\tilde{w})$ we have

$$\int_{B_{\frac{\tau}{\tilde{\tau}}}(w-\tilde{w})} \left(\int_0^{\frac{\tau}{\tilde{\tau}}} D^k_{\tilde{\mu}}(x,\tilde{s}) \frac{d\tilde{s}}{\tilde{s}} \right) d\tilde{\mu}(x) \leq \delta_{Rf1} \left(\frac{\tau}{\tilde{\tau}} \right)^k ;$$

equivalently, for all the balls $B_\sigma(y) \subset B_1(0)$ we have:

$$\int_{B_\sigma(y)} \left(\int_0^\sigma D^k_{\tilde{\mu}}(x,s) \frac{ds}{s} \right) d\tilde{\mu}(x) \leq \delta_{Rf1} \sigma^k .$$

This means that we can apply the first Reifenberg Theorem to the measure $\tilde{\mu}$, which has the form

$$\tilde{\mu} = \tilde{\tau}^{-k} \sum_{x \in \mathcal{C}_{h+1} \cap B_{\tilde{\tau}}(\tilde{w})} \omega_k r_x^k \delta_{\frac{x-\tilde{w}}{\tilde{\tau}}} ;$$

therefore we get

$$\sum_{x \in \mathcal{C}_{h+1} \cap B_{\tilde{\tau}}(\tilde{w})} \left(\frac{r_x}{\tilde{\tau}} \right)^k \leq C_{Rf1}(m).$$

Now if $h+1 < \hat{\jmath}$, then we cover any ball of radius $\varrho^{\hat{\jmath}-(h+1)}$ with a fixed (dimensional) number of balls of radius $\tilde{\tau}$, thus obtaining

$$\mu_{h+1} \left(B_{\varrho^{\hat{\jmath}-(h+1)}}(w) \right) \leq C(m) \tilde{\tau}^k = C(m) \left(\varrho^{\hat{\jmath}-(h+1)} \right)^k ,$$

which is what we need to go on with the induction (we also need the same information for $\ell < h+1$,but we have already obtained it in Part 3). If instead $h+1 = \hat{\jmath}$, then – up to covering the unit ball with a number of balls of radius $\frac{1}{5}$ which depends only on m – the estimate we found carries exactly the information we needed:

$$\sum_{x \in \mathcal{C}} r_x^k \leq C(m).$$

□

3.4.2 Second Covering Lemma

The following lemma improves what we have obtained in Lemma 3.14. Remember that we still had a “free” parameter ϱ to be chosen appropriately.

As we have observed, we can keep the “good balls” produced by Lemma 3.14 (or, more precisely, the last inductive step of Sublemma 3.14.1), since they are of the right size for our purpose. However, “bad balls” from Covering Lemma 1 need to be refined: the energy drop condition we obtained there is still too weak to provide useful information. In the following lemma, other balls of the right size are produced, while the remaining balls satisfy a much stronger energy drop condition.

Lemma 3.15. *Fix the following parameters:*

$\Lambda > 0:$	*a bound for the energy;*
$\eta > 0:$	*the "closeness parameter" for the stratum.*

There exist two constants $\delta_8 = \delta_8(m, \mathcal{N}, \Lambda, \eta)$ and $C_{II} = C_{II}(m)$ such that the following holds. Let $u \in W^{1,2}(B_4(0), \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ, and let $0 \le k \le m$ be an integer number. Assume that $E \le \Lambda$ is an arbitrary number such that $\theta_\psi(y,1) \le E$ for all $y \in B_1(0)$. For all fixed $\delta < \delta_8$, any subset $\mathcal{S}$ of the singular stratum $\mathcal{S}^k_{\eta,\delta r}(u) \cap B_1(0)$ admits a finite *covering of the type*

$$\mathcal{S} \subset \bigcup_{x \in \mathcal{C}} B_{r_x}(x),$$

where:

1. *All the radii satisfy $r_x \ge r$;*
2. *The radii are controlled by*
$$\sum_{x \in \mathcal{C}} r_x^k \le C_{II}; \tag{3.16}$$
3. *For any center $x \in \mathcal{C}$, one of the following two options is satisfied:*
   (A) $r_x = r$;
   (B) We have the following uniform energy drop: for all $y \in B_{r_x}(x) \cap \mathcal{S}$,
$$\theta_\psi\left(y, \frac{1}{10} r_x\right) \le E - \delta.$$

Also in this case, we first need to produce inductively a sequence of intermediate coverings.

Sublemma 3.15.1 (Intermediate coverings)**.** *Let the assumptions be the same as in Lemma 3.15. There exist $\delta_9 = \delta_9(m, \mathcal{N}, \Lambda, \eta)$ and $\bar{\varrho} = \bar{\varrho}(m)$ such that for all $\delta < \delta_9$ and $\varrho < \bar{\varrho}$ we have: for any $\hat{\jmath} \in \mathbb{N}$ and $r = \varrho^{\hat{\jmath}}$, and for all $1 \le j \le \hat{\jmath}$ there exist three finite sets of centers $\tilde{\mathcal{C}}^j_b$, $\tilde{\mathcal{C}}^j_r$ and $\tilde{\mathcal{C}}^j_f$, and a collection of radii $r_{x;j}$ associated to the centers, such that the following properties hold:*

1. *The balls centered in $\tilde{\mathcal{C}}^j \doteq \tilde{\mathcal{C}}^j_b \cup \tilde{\mathcal{C}}^j_r \cup \tilde{\mathcal{C}}^j_f$ with radii $r_{x;j}$ form a covering of $\mathcal{S}$:*
$$\mathcal{S} \subset \bigcup_{x \in \tilde{\mathcal{C}}^j_b} B_{r_{x;j}}(x) \cup \bigcup_{x \in \tilde{\mathcal{C}}^j_r} B_{r_{x;j}}(x) \cup \bigcup_{x \in \tilde{\mathcal{C}}^j_f} B_{r_{x;j}}(x)$$
2. r-balls*: if $x \in \tilde{\mathcal{C}}^j_r$, then $r_{x;j} = r$.*

3. Final balls*: if* $x \in \tilde{\mathcal{C}}_f^j$*, then* $r_{x;j} > r$ *and*

$$\theta_\psi\left(z, \frac{r_{x;j}}{10}\right) \leq E - \delta \qquad \text{for all } z \in B_{r_x}(x). \tag{3.17}$$

4. Bad balls*: if* $x \in \tilde{\mathcal{C}}_b^j$*, then* $r < r_{x;j} \leq \varrho^j$ *and condition* (3.17) *is not satisfied.*

5. *There exists a constant* $C_{25} = C_{25}(m)$ *such that*

$$\sum_{x \in \tilde{\mathcal{C}}_r^j \cup \tilde{\mathcal{C}}_f^j} r_{x;j}^k \leq C_{25} \left(\sum_{i=1}^j 2^{-i} \right), \qquad \sum_{x \in \tilde{\mathcal{C}}_b^j} r_{x;j}^k \leq 2^{-j}.$$

The key idea here is to apply Lemma 3.14 at every step of the induction: on one side, new balls of radius r are produced; on the other side, *each one* of the other balls of the covering can be split in a (large) portion where we have a uniform energy drop, and a smaller one that we'll need to re-cover. Notice that the r-balls and final balls that we gain at every step still remain good and final at any successive step, without being modified anymore; what we need to do is to refine bad balls.

Proof. Again, we use induction to prove the result; both the first step and the induction step rely on Lemma 3.14. Remember that we are assuming $r = \varrho^{\hat{j}}$.

Step 1. Consider the covering centered in $\mathcal{C}$ given by Lemma 3.14 for a fixed ϱ. Two classes of balls can be distinguished:

- A first class is made of the balls with radius r (we call $\mathcal{C}_r$ their centers): at the end of Step 1 we'll collect their centers in $\tilde{\mathcal{C}}_r^1$, together with some other balls coming from the following refinement.
- The second class is made of balls with radius bigger than r (we call $\mathcal{C}_+$ their centers) and has the property that, for any of its centers x, the set Υ_x is contained in $\mathcal{B}_{\frac{\varrho}{5} r_x}(L_x)$ for some $(k-1)$-subspace L_x.

Choose, for a fixed $x \in \mathcal{C}_+$, two sets of centers $\tilde{\mathcal{C}}_f^{1;x}$ and $\tilde{\mathcal{C}}_b^{1;x}$ such that:

$$B_{r_x}(x) \cap \mathcal{B}_{\frac{\varrho}{5} r_x}(L_x) \subset \bigcup_{y \in \tilde{\mathcal{C}}_b^{1;x}} B_{\varrho r_x}(y)$$

$$B_{r_x}(x) \setminus \mathcal{B}_{\frac{\varrho}{5} r_x}(L_x) \subset \bigcup_{y \in \tilde{\mathcal{C}}_f^{1;x}} B_{\varrho r_x}(y)$$

$$B_{\frac{\varrho}{5} r_x}(y) \cap B_{\frac{\varrho}{5} r_x}(z) = \varnothing \qquad \text{for all } y \neq z \in \tilde{\mathcal{C}}_f^{1;x} \cup \tilde{\mathcal{C}}_b^{1;x}.$$

So now we can already set

$$\tilde{\mathcal{C}}_f^1 \doteq \bigcup_{x \in \mathcal{C}_+} \tilde{\mathcal{C}}_f^{1;x},$$

since the corresponding balls satisfy

$$\theta_\psi\left(z, \frac{r_x}{10}\right) \leq E - \delta \qquad \text{for all } z \in B_{r_x}(x);$$

moreover, we separate the balls that have reached the radius r and those which still have a bigger radius:

$$\tilde{\mathcal{C}}_b^1 \doteq \bigcup_{\substack{x \in \mathcal{C}_+ \\ \varrho r_x > r}} \tilde{\mathcal{C}}_b^{1;x}$$

$$\tilde{\mathcal{C}}_r^1 \doteq \mathcal{C}_r \cup \bigcup_{\substack{x \in \mathcal{C}_+ \\ \varrho r_x = r}} \tilde{\mathcal{C}}_b^{1;x}.$$

The only thing left to prove in STEP 1 is that the given estimates hold. Notice that $r_{y;1}$ is still r for the balls coming from $\mathcal{C}_r$, and is now ϱr_x for the balls in $\tilde{\mathcal{C}}_f^{1;x}$, $\tilde{\mathcal{C}}_r^{1;x}$ and $\tilde{\mathcal{C}}_b^{1;x}$. We have the following information:

- Final balls: here we simply know that for any $x \in \mathcal{C}_+$ we have used $c(m)\varrho^{-m}$ balls of radius ϱr_x to cover the part of $B_{r_x}(x)$ lying outside of $\mathcal{B}_{\frac{\varrho}{5} r_x}(L_x)$. Thus we have:

$$\sum_{y \in \tilde{\mathcal{C}}_f^{1;x}} r_{y;1}^k \leq c(m)\varrho^{-m}(\varrho r_x)^k = C_f(m,\varrho) r_x^k.$$

- Bad balls: here the situation is better, since we only needed to cover the enlargement of a $(k-1)$-subspace (for all $x \in \mathcal{C}_+$): the needed number of balls of radius ϱr_x is $c(m)\varrho^{-(k-1)}$, by the classical covering results we developed in Section 2.3:

$$\sum_{y \in \tilde{\mathcal{C}}_b^{1;x}} r_{y;1}^k \leq c(m)\varrho^{-(k-1)}(\varrho r_x)^k = C_{26}\varrho r_x^k$$

  with $C_{26} = C_{26}(m)$.

- For the "new" r-balls, the same estimate as the one for bad balls holds.

As a consequence, for the whole family of bad balls we have, by the estimates of Lemma 3.14:

$$\sum_{y \in \tilde{\mathcal{C}}_b^1} r_{y;1}^k \leq C_{26}(m)\varrho \sum_{x \in \mathcal{C}_+} r_x^k \leq C_{26}(m) C_I(m)\varrho;$$

clearly then, choosing

$$\bar{\varrho}(m) \leq \frac{1}{2C_{26}(m)C_I(m)}$$

we have

$$\sum_{y \in \tilde{\mathcal{C}}_b^1} r_{y;1}^k \leq \frac{1}{2}.$$

Moreover, we can also estimate:

$$\sum_{y\in\tilde{\mathcal{C}}_r^1\cup\tilde{\mathcal{C}}_f^1} r_{y;1}^k \leq \sum_{x\in\mathcal{C}_r} r^k + \sum_{x\in\tilde{\mathcal{C}}_f^1} r_x^k + C_{26}\varrho r^k,$$

where the last term comes from the new r-balls, and hence:

$$\sum_{y\in\tilde{\mathcal{C}}_r^1\cup\tilde{\mathcal{C}}_f^1} r_{y;1}^k \leq C_I(m) + C_f(m,\varrho(m)) + \frac{1}{2} \doteq \frac{1}{2}C_{25}.$$

This ends the proof of Step 1.

Step 2. Assume that for some j we have found sets of centers $\tilde{\mathcal{C}}_b^j$, $\tilde{\mathcal{C}}_r^j$ and $\tilde{\mathcal{C}}_f^j$ with the given properties. As we already observed, r-balls and final balls will maintain their status in the $(j+1)^{\text{th}}$ step as well. Consider instead a bad ball $B_{r_{x;j}}(x)$. Through the usual transformation $\lambda_{x,r_{x;j}}^{-1}$, one can dilate the ball until it becomes the unit ball; here the situation is the same as in Lemma 3.14, only with all the radii rescaled with $r_{x;j}$; applying the lemma and then going back to the original bad ball with $\lambda_{x,r_{x;j}}$, we find a covering of $\mathcal{S}\cap B_{r_{x;j}}(x)$ with centers in points $\tilde{\mathcal{C}}^{j;x}$ and radii $r_{y;j}$, and with the following properties:

(i) $r_{y;j} \geq r$;

(ii) The estimate

$$\sum_{x\in\tilde{\mathcal{C}}^{j;x}} r_{y;j}^k \leq C_I(m) r_{x;j}^k \tag{3.18}$$

holds;

(iii) Either $r_{y;j} = r$, or the set

$$\Upsilon_{y;j}^\delta = \left\{ z\in\mathcal{S}\cap B_{2r_{y;j}}(y) \;\middle|\; \theta_\psi\left(z,\frac{\varrho}{10}r_{y;j}\right)\right\}$$

is contained in $\mathcal{B}_{\frac{\varrho}{5}r_{y;j}}(L_{y;j})$ for some $(k-1)$-subspace $L_{y;j}$.

Now the balls $B_{r_{y;j}}(y)$ with $r_{y;j} = r$ can be collected together with the r-balls we already have; we call $\hat{\mathcal{C}}_r^{j;x}$ this new family of centers. On the other balls (call $\tilde{\mathcal{C}}_+^{j;x}$ their centers), we reproduce the same procedure as in Step 1: fix $y\in\tilde{\mathcal{C}}_+^{j;x}$; choose two sets of centers $\tilde{\mathcal{C}}_f^{j+1;y}$ and $\tilde{\mathcal{C}}_b^{j+1;y}$ such that:

$$B_{r_{y;j}}(y)\cap\mathcal{B}_{\frac{\varrho}{5}r_{y;j}}(L_{y;j}) \subset \bigcup_{z\in\tilde{\mathcal{C}}_b^{j+1;y}} B_{\varrho r_{y;j}}(z)$$

$$B_{r_{y;j}}(y)\setminus\mathcal{B}_{\frac{\varrho}{5}r_{y;j}}(L_{y;j}) \subset \bigcup_{z\in\tilde{\mathcal{C}}_f^{j+1;y}} B_{\varrho r_{y;j}}(z)$$

$$B_{\frac{\varrho}{5}r_{y;j}}(z)\cap B_{\frac{\varrho}{5}r_{y;j}}(w) = \varnothing \qquad \text{for all } z\neq w\in\tilde{\mathcal{C}}_f^{j+1;y}\cup\tilde{\mathcal{C}}_b^{j+1;y}.$$

Now:

- All the points in the balls centered in $\tilde{\mathcal{C}}_f^{j+1;y}$ satisfy the energy drop condition, so we can define

$$\hat{\mathcal{C}}_f^{j+1} \doteq \bigcup_{x\in\tilde{\mathcal{C}}_b^j} \bigcup_{y\in\tilde{\mathcal{C}}_+^{j;x}} \tilde{\mathcal{C}}_f^{j+1;y}$$

and then join this set with the previous final balls to obtain

$$\tilde{\mathcal{C}}_f^{j+1} \doteq \hat{\mathcal{C}}_f^{j+1} \cup \tilde{\mathcal{C}}_f^j.$$

Notice that for $\tilde{\mathcal{C}}_f^{j+1;y}$ we have:

$$\sum_{z\in\tilde{\mathcal{C}}_f^{j+1;y}} r_{z;j+1}^k \leq c(m)\varrho^{-m}(\varrho r_{y;j})^k = C_f(m,\varrho) r_{y;j}^k;$$

so in particular for $\hat{\mathcal{C}}_f^{j+1}$:

$$\begin{aligned}\sum_{z\in\hat{\mathcal{C}}_f^{j+1}} r_{z;j+1}^k &\leq \sum_{x\in\tilde{\mathcal{C}}_b^j} \sum_{y\in\tilde{\mathcal{C}}_+^{j;x}} C_f(m,\varrho) r_{y;j}^k \leq \\ &\leq \sum_{x\in\tilde{\mathcal{C}}_b^j} C_f(m,\varrho) C_I r_{x;j}^k \leq C_f(m,\varrho) C_I 2^{-j}\end{aligned}$$

- Similarly, we have three types of r-balls: the ones coming from the previous steps; the ones produced by the application of Lemma 3.14 to bad balls (which we called $\hat{\mathcal{C}}_r^j$); and the balls centered in $\tilde{\mathcal{C}}_b^{j+1;y}$ with $\varrho r_{y;j} = r$. We call $\tilde{\mathcal{C}}_r^{j+1}$ the union of these families:

$$\tilde{\mathcal{C}}_r^{j+1} \doteq \tilde{\mathcal{C}}_r^j \cup \left(\bigcup_{x\in\tilde{\mathcal{C}}_b^j} \hat{\mathcal{C}}_r^{j;x} \right) \cup \left(\bigcup_{x\in\tilde{\mathcal{C}}_b^j} \bigcup_{\substack{y\in\tilde{\mathcal{C}}_+^{j;x} \\ \varrho r_{y;j}=r}} \tilde{\mathcal{C}}_b^{y+1;j} \right).$$

Observe that we have, for $x \in \tilde{\mathcal{C}}_b^j$ and $y \in \tilde{\mathcal{C}}_+^{j;x}$:

$$\sum_{z\in\tilde{\mathcal{C}}_b^{j+1;y}} r_z^k \leq c(m)\varrho^{-k}\varrho^k r_{y;j}^k \leq C_{26}\varrho r_{y;j}^k;$$

moreover, for $x \in \tilde{\mathcal{C}}_b^j$, we have trivially:

$$\sum_{y\in\hat{\mathcal{C}}_g^{j;x}} r_{y;j}^k \leq C_I(m) r_{x;j}^k.$$

- Finally, the new bad balls are what remains of the last re-covering:

$$\tilde{\mathcal{C}}_b^{j+1} \doteq \bigcup_{x\in\tilde{\mathcal{C}}_b^j} \bigcup_{\substack{y\in\tilde{\mathcal{C}}_+^{j;x} \\ \varrho r_{y;j}>r}} \tilde{\mathcal{C}}_b^{j+1;y}.$$

As for r-balls, for fixed $x \in \tilde{\mathcal{C}}_b^j$ and $y \in \tilde{\mathcal{C}}_+^{j;x}$, we have

$$\sum_{z \in \tilde{\mathcal{C}}_b^{j+1;y}} r_z^k \leq c(m)\varrho^{-k}\varrho^k r_{y;j}^k \leq C_{26}\varrho r_{y;j}^k.$$

So now what remains to do is to complete the estimates on the radii. For what concerns bad balls, we can use the same argument as in Step 1, the inductive assumption and the upper bound (3.18):

$$\begin{aligned}\sum_{z \in \tilde{\mathcal{C}}_b^{j+1}} r_{z;j+1}^k \leq \sum_{x \in \tilde{\mathcal{C}}_b^j} \sum_{\substack{y \in \tilde{\mathcal{C}}_+^{j;x} \\ \varrho r_{y;j} > r}} C_{26}\varrho r_{y;j}^k \leq \\ \leq \sum_{x \in \tilde{\mathcal{C}}_b^j} C_{26} C_I \varrho r_{x;j}^k \leq (C_{26} C_I \varrho) 2^{-j} \leq 2^{-j-1},\end{aligned}$$

where the last estimate is a consequence of the choice of ϱ. Clearly the same estimate works for the last category of r-balls. For the totality of r- and final balls we have:

$$\sum_{z \in \tilde{\mathcal{C}}_r^{j+1} \cup \tilde{\mathcal{C}}_f^{j+1}} r_{z;j+1}^k \leq \sum_{x \in \tilde{\mathcal{C}}_r^j \cup \tilde{\mathcal{C}}_f^j} r_{x;y}^k + C_f C_I 2^{-j} + C_I 2^{-j} + 2^{-j-1},$$

where the second addend comes from new final balls, the third one comes from (new) r-balls of the second type, and the last addend comes from (new) r-balls of the third type. By consequently redefining the constant C_{25}, we have proved Sublemma 3.15.1. □

Proof of Lemma 3.15. Now the lemma is a trivial consequence of Sublemma 3.15.1: indeed, at the j^{th} step of the inductive argument, the intermediate covering contains balls of radius at most ϱ^j, except for the final balls $\tilde{\mathcal{C}}_f^j$. However we are assuming without loss of generality that $r = \varrho^{\hat{\jmath}}$ for some $\hat{\jmath} \in \mathbb{N}$ and for a ϱ smaller than the $\bar{\varrho}(m)$ fixed in the sublemma: so at the $\hat{\jmath}^{\text{th}}$ step there will only be r- and final balls. □

3.5 Proof of Main Theorem and consequences

With the tools we have developed so far, we are now able to prove the main result of this chapter, that is Theorem 3.1. We begin with the first statement.

3.5.1 Proof of the first part

Recall that we want to prove the following assertion:

Claim. *For any $\eta > 0$, there exists a constant $C_1(m, \mathcal{N}, \Lambda, \eta)$ such that the following holds: for all minimizing harmonic maps $u \in W^{1,2}(\Omega, \mathcal{N})$ with energy bounded by Λ, all $0 \leq k \leq m$ and all $r > 0$ (smaller than an upper radius r_0), we have:*

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,r}(u)\right) \cap B_1(0)\right) \leq C_1 r^{m-k}. \tag{NV}$$

Also recall that, by what we have proved at the beginning of Section 3.4, we can implicitly assume that r is of the form $\varrho^{\hat{\jmath}}$ and use the covering lemmas.

Actually, since we are making use of Lemma 3.15 (and thus indirectly of Lemma 3.11), we only manage to prove directly that

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,\delta r}(u)\right) \cap B_1(0)\right) \leq C_1 r^{m-k}$$

for some small δ depending only on m, $\mathcal{N}$, Λ and η, and for all $r \leq 1$. This, however, is enough to obtain a satisfactory result: for $r \leq \delta^{-1}$, we obtain exactly what we needed, only with a constant which is $C_1\delta^{k-m}$.

We prove one last inductive covering lemma:

Lemma 3.16. *Fix $\delta < \delta_8$. Assume that $E \leq \Lambda$ is an arbitrary number such that $\theta_\psi(y, 1) \leq E$ for all $y \in B_1(0)$. For any number $i \in \mathbb{N}$ there exists a covering $\{B_{r_x}(x)\}_{x \in \mathcal{D}_i}$ of the set $\mathcal{S} \doteq \mathcal{S}^k_{\eta,\delta r}(u) \cap B_1(0)$ with the following properties:*

(i) The radii r_x satisfy

$$\sum_{x \in \mathcal{D}_i} r_x^k \leq (c_{12}(m) C_{II}(m))^i$$

for some new dimensional constant $c_{12}(m)$ and the old constant C_{II} coming from Lemma 3.15;

(ii) For any center $x \in \mathcal{D}_i$, one of the following two options is verified:

(A) $r_x \leq r$;

(B) For all $y \in \mathcal{S} \cap B_{2r_x}(x)$, we have

$$\theta_\psi(y, r_x) \leq E - i\delta.$$

Proof. The case $i = 0$ is trivially true: the only ball needed in the covering is the unit ball $B_1(0)$. Assume then that the statement is true for some $i \geq 0$, and consider a ball $B_{r_x}(x)$ centered at a point $x \in \mathcal{D}_i$. By applying the transformation λ^{-1}_{x,r_x} to this ball, we find ourself in the setting needed for Lemma 3.15, where the ball $B_{r_x}(x)$ we were considering has been enlarged to the unit ball. Hence we apply the lemma, and then switch back to the original setting through the map λ_{x,r_x}: this gives us a covering of $\mathcal{S} \cap B_{r_x}(x)$ made of balls $\{B_{r_y}(y)\}_{y \in \mathcal{D}^x_i}$ with the following properties:

- The radii r_y satisfy
$$\sum_{y\in\mathcal{D}_i^x} r_y^k \leq C_{II} r_x^k$$
- Either $r_y = r$, or
$$\theta_\psi\left(z, \frac{1}{10} r_y\right) \leq (E - i\delta) - \delta$$
for all $z \in B_{r_y}(y)$.

Notice that in the second property we are also using the inductive assumption, and Lemma 3.15 has been used with the parameter $E - \delta i$ instead of E. Now we are almost finished, since we only need to re-cover the balls $B_{r_y}(y)$ with smaller balls of radius $\frac{1}{10} r_y$ (and the number of them is bounded by a dimensional constant): thus we have, for all these new balls $\{B_{r_{\tilde{y}}}(\tilde{y})\}_{\tilde{y}\in\tilde{\mathcal{D}}_i^x}$, that

- The radii $r_{\tilde{y}}$ satisfy
$$\sum_{\tilde{y}\in\mathcal{D}_i^x} r_{\tilde{y}}^k \leq c(m) \sum_{y\in\mathcal{D}_i^x} \left(\frac{1}{10} r_y\right)^k \leq c_{12}(m) C_{II}(m) r_x^k$$
- Either $r_{\tilde{y}} \leq r$, or $\theta_\psi(z, r_z) \leq E - (i+1)\delta$ for all $z \in B_{r_{\tilde{y}}}(\tilde{y})$.

Thus we can define
$$\mathcal{D}^{i+1} \doteq \bigcup_{x\in\mathcal{D}^i} \tilde{\mathcal{D}}_i^x;$$
then by summing all the r_x^k coming from the various refinements of the balls in the original covering $\{B_{r_x}(x)\}_{x\in\mathcal{D}_i}$, we get:
$$\sum_{z\in\mathcal{D}^{i+1}} r_z^k \leq c_{12}(m) C_{II}(m) \sum_{x\in\mathcal{D}_i} r_x^k \leq (c_{12}(m) C_{II}(m))^{i+1}.$$
The statement is proved. □

Now we have everything we need to prove the main theorem:

Proof of Theorem 3.1. Consider the information given by the previous Lemma 3.16 for the integer
$$\hat{\imath} \doteq \left\lfloor \frac{E}{\delta} \right\rfloor + 1.$$
We obtain a covering $\{B_{r_x}(x)\}_{x\in\mathcal{D}}$ of $\mathcal{S}^k_{\eta,\delta r}(u) \cap B_1(0)$ with the property that, for a new constant C_{27},
$$\sum_{x\in\mathcal{D}} r_x^k \leq C_{27}(m, \hat{\imath}) = C_{27}(m, \mathcal{N}, \Lambda, \eta);$$

moreover, the option (B) cannot be verified, since $E - \hat{\imath}\delta < 0$: this means that all the balls of the covering have radius smaller than or equal to r. In particular, using for example Covering Lemma 4 from Chapter 2, this implies the covering estimate

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_{\eta,\delta r}(u)\right) \cap B_1(0)\right) \leq \sum_{x\in\mathcal{D}} \mathrm{Vol}\left(B_{2r_x}(x)\right) \leq$$
$$\leq 2^m r^{m-k} \sum_{x\in\mathcal{D}} r_x^k = 2^m C_{27}(m, \mathcal{N}, \Lambda, \eta) r^{m-k}.$$

□

Remark. Notice that from the Claim we just proved, another piece of Theorem 3.1 follows trivially: indeed, the stratum $\mathcal{S}^k_\eta(u)$ is contained in all the strata $\mathcal{S}^k_{\eta,\delta r}(u)$ (being their intersection): thus the estimate

$$\mathrm{Vol}\left(\mathcal{B}_r\left(\mathcal{S}^k_\eta(u)\right) \cap B_1(0)\right) \leq C r^{m-k}$$

is an immediate consequence.

3.5.2 Proof of the second part

The second part of Theorem 3.1 states the following:

Claim. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be minimizing harmonic. For any $\eta > 0$ and any $0 \leq k \leq m$, the stratum $\mathcal{S}^k_\eta(u)$ is k-rectifiable.*

As we'll see shortly, the result follows easily from this lemma.

Lemma 3.17. *Let $\mathcal{S} \subset \mathcal{S}^k_\eta(u) \cap B_1(0)$ be a $\mathscr{H}^k$-measurable subset. There exist a universal constant $0 < \kappa < 1$ and a further $\mathscr{H}^k$-measurable subset $\mathcal{R} \subset \mathcal{S}$ with the following properties:*

1. *$\mathscr{H}^k(\mathcal{R}) \leq \kappa \mathscr{H}^k(\mathcal{S})$;*
2. *The set $\mathcal{S} \setminus \mathcal{R}$ is k-rectifiable.*

Before proving this lemma, which requires some effort, we show how it is applied to prove the Claim.

Proof of the Claim. By induction, for any $j \in \mathbb{N}$ there exists a $\mathscr{H}^k$-measurable set $\mathcal{R}_j \subset \mathcal{S}^k_\eta(u)$ such that:

- $\mathscr{H}^k(\mathcal{R}_j) \leq \kappa^j \mathscr{H}^k \mathcal{S}^k_\eta(u)$;
- The set $\mathcal{S} \setminus \mathcal{R}_j$ is k-rectifiable.

This is easily proved: the step $j = 1$ comes from the application of Lemma 3.17 to the stratum $\mathcal{S}^k_\eta(u)$, while the $(j+1)^{\text{th}}$ step descends from the application of the same lemma to $\mathcal{S}^k_\eta(u) \setminus \mathcal{R}_j$. Now we can define

$$\tilde{\mathcal{R}} \doteq \bigcap_{j\in\mathbb{N}} \mathcal{R}_j$$
$$\tilde{\mathcal{S}} \doteq \mathcal{S}^k_\eta(u) \setminus \tilde{\mathcal{R}} = \bigcup_{j\in\mathbb{N}} \left(\mathcal{S}^k_\eta(u) \setminus \mathcal{R}_j\right).$$

Here $\mathcal{R}$ has $\mathscr{H}^k$-measure zero; and $\tilde{S}$ is the countable union of sets, each of which is countable union of Lipschitz k-graphs; therefore $\tilde{S}$ itself is a countable union of Lipschitz k-graphs. This means precisely that $\mathcal{S}^k_\eta(u)$ is k-rectifiable. □

Now we turn to prove Lemma 3.17.

Proof. We can assume that $\mathscr{H}^k(\mathcal{S}) > 0$, otherwise the statement is trivial.
Step 1. Consider the following map: for $x \in B_1(0)$ and $0 < r < 1$,

$$f_r(x) \doteq \theta_\psi(x, r) - \theta_\psi(x, 0).$$

As we know, as r tends to 0, the map f_r converges pointwise (and decreasingly) to the constant function $f_0 \equiv 0$; moreover, all the maps f_r are bounded by the constant map E, which is integrable with respect to the measure $\mathscr{H}^k \llcorner \mathcal{S}$. Now fix a $\delta > 0$. By the Dominated Convergence Theorem, there exists a $\bar{r} > 0$ depending on δ such that

$$\int_{\mathcal{S}} f_{4\bar{r}}(x)\, d\mathscr{H}^k(x) \leq \delta^2 \mathscr{H}^k(\mathcal{S}). \tag{3.19}$$

Consider the following sets:

$$F_\delta \doteq \{x \in \mathcal{S} \mid f_{4\bar{r}}(x) > \delta\}$$
$$G_\delta \doteq \{x \in \mathcal{S} \mid f_{4\bar{r}}(x) \leq \delta\} = \mathcal{S} \setminus F_\delta;$$

observe that, since $f_{4\bar{r}}$ is nonnegative, we have:

$$\int_{\mathcal{S}} f_{4\bar{r}}(x)\, d\mathscr{H}^k(x) = \int_{F_\delta} f_{4\bar{r}}(x)\, d\mathscr{H}^k(x) + \int_{G_\delta} f_{4\bar{r}}(x)\, d\mathscr{H}^k(x) \geq$$
$$\geq \int_{F_\delta} f_{4\bar{r}}(x)\, d\mathscr{H}^k(x) \geq \delta \mathscr{H}^k(F_\delta);$$

this, combined with Equation (3.19), gives

$$\mathscr{H}^k(F_\delta) \leq \delta \mathscr{H}^k(\mathcal{S}).$$

We claim that, for δ sufficiently small, the set G_δ is k-rectifiable; if we manage to show this, then the lemma is proved. In order to prove this claim, we conider a *finite* covering $\{B_{\bar{r}}(x_i)\}_{i=1}^{L}$ of G_δ made with balls of the fixed radius $\bar{r}$. It is sufficient to show that for δ small $G_\delta \cap B_{\bar{r}}(x_i)$ is rectifiable for any i: our main aim will be now to check the applicability of the second Reifenberg Theorem (Theorem 3.5), that gives exactly that result.

STEP 2. Fix a $i \in \{1, \dots, L\}$, and apply the usual transformation $\lambda^{-1}_{x_i,\bar{r}}$ to the ball $B_{\bar{r}}(x_i)$. We set

$$\tilde{u} = T_{x_i,\bar{r}}u, \qquad \tilde{G}_\delta = \lambda^{\leftarrow}_{x_i,r}(G_\delta) \cap B_1(0).$$

Also, we define μ_δ he measure $\mathscr{H}^k \llcorner \tilde{G}_\delta$ on the unit ball $B_1(0)$. Notice that for any $x \in \tilde{G}_\delta$ we have:

$$\theta^{\tilde{u}}_\psi(x, 4) - \theta^{\tilde{u}}_\psi(x, 0) \le \delta,$$

by the definition of G_δ and the usual scale invariance properties of θ_ψ. This means that by choosing δ small we can apply Proposition 2.6 and get that, for any point $x \in \tilde{G}_\delta$ and any $0 < s \le 1$, $\tilde{u}$ is $(\delta_1, s, 0)$-symmetric at x, where δ_1 is the constant produced by Theorem 3.9. On the other hand, since G_δ was a subset of $\mathcal{S}^k_\eta(u)$, u was not $(\eta, \bar{r}s, k+1)$-symmetric at the points of G_δ for any $0 < s \le 1$; thus, for any point $x \in \tilde{G}_\delta$ and $0 < s \le 1$, $\tilde{u}$ is not $(\eta, s, k+1)$-symmetric at x. This is what we need to apply Theorem 3.9 on any ball $B_s(x)$; and we apply it to the finite measure $\mu_\delta = \mathscr{H}^k \llcorner \tilde{G}_\delta$. We obtain that, for any $x \in \tilde{G}_\delta$ and any $0 < s \le 1$,

$$D^k_{\tilde{G}_\delta}(x, s) \le C_5 s^{-k} \int_{B_s(x)} P_{\tilde{u},\sigma}(y, s)\, d\mu_\delta(y)$$

This goes in the direction we need, since we are trying to check if Equation (RR2) is satisfied. Following what we did in the proof of Sublemma 3.14.2, we first fix $w \in B_1(0)$ and $r \le 1$; for all $0 < s \le r$ we compute:

$$\int_{B_r(w)} D^k_{\tilde{G}_\delta}(x, s)\, d\mu_\delta(x) \le C_5 s^{-k} \int_{B_r(w)} \left(\int_{B_s(x)} P_{\tilde{u},\sigma}(y, s)\, d\mu_\delta(y) \right) d\mu_\delta(x),$$

for a $1 < \sigma \le 2$ that we can leave implicit. Observe that we are allowed to do this since μ_δ is supported in $\tilde{G}_\delta$. As we have already noticed in Sublemma 3.14.2, if $|x - w| < r$ and $|y - x| < s$, then $|y - w| < r + s$: thus we can estimate

$$\int_{B_r(w)} D^k_{\tilde{G}_\delta}(x, s)\, d\mu_\delta(x) \le C_5 s^{-k} \int_{B_{r+s}(w)} \int_{B_s(y)} P_{\tilde{u},\sigma}(y, s)\, d\mu_\delta(x)\, d\mu_\delta(y) \le$$
$$\le C_5 s^{-k} \int_{B_{r+s}(w)} P_{\tilde{u},\sigma}(y, s) \mathscr{H}^k \left(\tilde{G}_\delta \cap B_s(y) \right) d\mu_\delta(y).$$

But now we can exploit the uniform volume estimates given by the first part of Theorem 3.1 (appropriately rescaled); we get the following uniform *a priori* upper bound:

$$\mathscr{H}^k\Big(\lambda^{\leftarrow}_{x_i,\bar{r}}\Big(\mathcal{S}^k_\eta(u)\Big)\cap B_s(y)\Big) = \bar{r}^{-k}\mathscr{H}^k\Big(\mathcal{S}^k_\eta(u)\cap\lambda_{x_i,\bar{r}}(B_s(y))\Big)\le \tag{3.20}$$

$$\le C_2\bar{r}^{-k}(\bar{r}s)^k = C_2 s^k; \tag{3.21}$$

notice that thanks to this *a priori* estimate it is not necessary to reproduce the induction argument of Sublemma 3.14.2. Plugging this information in the previous inequality we get:

$$\int_{B_r(w)} D^k_{\tilde{G}_\delta}(x,s)\,d\mu_\delta(x)\le C_2C_5\int_{B_{r+s}(w)} P_{\tilde{u},\sigma}(y,s)\,d\mu_\delta(y).$$

In order to check the validity of Equation (RR2), we now consider the left hand side of that inequality: applying Tonelli Theorem (twice), we find:

$$\begin{aligned}\int_{B_r(w)}\left(\int_0^r D^k_{\tilde{G}_\delta}(x,s)\,\frac{ds}{s}\right)d\mu_\delta(x) &= \int_0^r\left(\int_{B_r(w)} D^k_{\tilde{G}_\delta}(x,s)\,d\mu_\delta(x)\right)\frac{ds}{s}\le\\ &\le C_2C_5\int_0^r\left(\int_{B_{2r}(w)} P_{\tilde{u},\sigma}(y,s)\,d\mu_\delta(y)\right)\frac{ds}{s}=\\ &= C_2C_5\int_{B_{2r}(w)}\left(\int_0^r P_{\tilde{u},\sigma}(y,s)\,\frac{ds}{s}\right)d\mu_\delta(y).\end{aligned}$$

Consider for a moment the inner integral; r can simply be bounded by 1. We use basically the same trick we exploited in Part 2 of Sublemma 3.14.2:

$$\begin{aligned}\int_0^1 P_{\tilde{u},\sigma}(y,s)\,\frac{ds}{s} &= \sum_{j=0}^{\infty}\int_{\sigma^{-(j+1)}}^{\sigma^{-j}}\frac{\theta^{\tilde{u}}_\psi(y,\sigma s)-\theta^{\tilde{u}}_\psi(y,s)}{s}ds\le\\ &\le\sum_{j=0}^{\infty}\int_{\sigma^{-(j+1)}}^{\sigma^{-j}}\frac{\theta^{\tilde{u}}_\psi(y,\sigma^{-j+1})-\theta^{\tilde{u}}_\psi(y,\sigma^{-j-1})}{\sigma^{-j-1}}ds\le\\ &\le C(\sigma)\sum_{j=0}^{\infty}\Big[\theta^{\tilde{u}}_\psi(y,\sigma^{-j+1})-\theta^{\tilde{u}}_\psi(y,\sigma^{-j-1})\Big]\le\\ &\le C(\sigma)\Big[\Big(\theta^{\tilde{u}}_\psi(y,\sigma)-\theta^{\tilde{u}}_\psi(y,0)\Big)+\Big(\theta^{\tilde{u}}_\psi(y,\sigma)-\theta^{\tilde{u}}_\psi(y,0)\Big)\Big]\le\\ &\le C(\sigma)\delta.\end{aligned}$$

Therefore we can insert this piece of information in the previous integral; fixing for example $\sigma = 2$ and using again the upper bound (3.20) on the measure of the singular stratum, we find:

$$\begin{aligned}\int_{B_r(w)}\left(\int_0^r D^k_{\tilde{G}_\delta}(x,s)\,\frac{ds}{s}\right)d\mu_\delta(x) &\le C_{28}\mu_\delta(B_{2r}(w))\delta\le\\ &\le C_{29}\delta r^k\end{aligned}$$

for a constant $C_{29}(m, \mathcal{N}, \Lambda, \eta)$. Taking

$$\delta < \frac{\delta_{Rf2}}{C_{29}},$$

we get exactly the hypothesis needed for the second Reifenberg Theorem: thus $\tilde{G}_\delta$ is k-rectifiable, and tracing back the steps of the proof this proves the k-rectifiability of G_δ.

□

3.5.3 Consequences

A couple of very easy consequences can be drawn from Theorem 3.1, in the same way we did in Section 2.5: indeed, in that section (precisely in Sublemma 2.12.4) we proved the existence of an ε_6 with the property that, for any $r > 0$ small enough,

$$\mathcal{Z}_r(u) \subset \mathcal{S}^{m-3}_{\varepsilon_6, 4r}(u),$$

where we recall that $\mathcal{Z}_r(u)$ is the set of points where the regularity scale is smaller than r. Then, with the very same computation we performed in Theorem 2.12, and exploiting the fact that we have improved the estimate, it's trivial to see that the following holds:

Theorem 3.18. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ. There exists a constant $C_{30} = C_{30}(m, \mathcal{N}, \Lambda)$ such that the following estimate holds for all $0 < r < 1$:*

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{Z}_r(u)) \cap B_1(0)\right) \leq C_{30} r^3. \tag{3.22}$$

In particular, for the singular set $\mathcal{S}(u)$,

$$\mathrm{Vol}\left(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)\right) \leq C_{30} r^3. \tag{3.23}$$

Moreover, we can slightly improve Corollary 2.13: in that occasion, we had proved that for any $0 < p < 3$ both ∇u and r_u^{-1} were uniformly bounded in the space L^p; we showed, however, that the same statement for $p = 3$ is false. Instead, thanks to Theorem 3.1, we can immediately see the following:

Corollary 3.19. *Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map with energy bounded by Λ. There exists a constant $C_{31} = C_{31}(m, \mathcal{N}, \Lambda)$ such that for any $0 < r < 1$ the following inequalities hold:*

$$\mathrm{Vol}\left(\left\{x \in B_1(0) \;\middle|\; |\nabla u(x)| > \frac{1}{r}\right\}\right) \leq \mathrm{Vol}\left(\{x \in B_1(0) \mid r_u < r\}\right) \leq$$
$$\leq C_{31} r^3.$$

In other words, both ∇u and r_u^{-1} are uniformly bounded in the space $L^3_{weak}(B_1(0))$.

Proof. The first inequality is just a rewriting of what we already showed in Corollary 2.13, while the second one follows again from Sublemma 2.12.4. Moreover, by replacing r with $\frac{1}{s}$, the two inequalities imply that

$$\|\nabla u\|_{L^3_{weak}} \leq \left\|r_u^{-1}\right\|_{L^3_{weak}} \leq C_{31},$$

just by definition of the space of weak-L^3 functions. Observe that the estimate we've got is valid only for $s \geq 1$; however, the needed estimates for $0 < s < 1$ are trivially true. □

Appendix A

Basic notions of Measure Theory

We collect here a couple of measure theoretical definitions that are needed in this work. For a complete picture of these topics, we refer to the classical books [Fol99], [EG15] and [Mat95].

Recall that, for a set $S \subset \mathbb{R}^m$ and a positive number $\varrho > 0$ we denote with $\mathcal{B}_\varrho(S)$ the ϱ-neighborhood of S:

$$\mathcal{B}_\varrho(S) \doteq \{y \in \mathbb{R}^m \mid \operatorname{dist}(y, S) < \varrho\}.$$

Definition A.1 (Hausdorff measure)**.** Let $S \subset \mathbb{R}^m$ be an arbitrary set, and $s \geq 0$ a real number. We define the **s-dimensional Hausdorff** (outer) **measure** of S as

$$\mathscr{H}^s(S) \doteq \lim_{\delta \downarrow 0} \mathscr{H}^s_\delta(S),$$

where for any $\delta > 0$

$$\mathscr{H}^s_\delta(S) \doteq \inf \left\{ \omega_s \sum_{i=0}^{\infty} \left(\frac{\operatorname{diam} S_i}{2} \right)^s \;\middle|\; \begin{array}{l} S_i \subset \mathbb{R}^m \\ \operatorname{diam} S_i \leq \delta \\ S \subset \bigcup_i S_i \end{array} \right\}.$$

Here ω_s is the number

$$\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2} + 1)}$$

(and Γ is the classical Euler function); if s is an integer, ω_s is the measure of the s-dimensional ball.

Intuitively, the Hausdorff measure describes the "s-dimensional volume" of objects in $\mathbb{R}^m$, and it's the standard way of accomplishing this task. A second notion for the same purpose is available:

Definition A.2 (Minkowski content)**.** Let $S \subset \mathbb{R}^m$ be an arbitrary set, and let $s \geq 0$. We define the lower and upper **s-dimensional Minkowski content** of S respectively as:

$$\mathscr{M}^s_*(S) = \liminf_{\varrho \to 0} \frac{\mathscr{L}^m\left(\mathcal{B}_\varrho(S)\right)}{\omega_{m-s}\varrho^{m-s}},$$
$$\mathscr{M}^{*s}(S) = \limsup_{\varrho \to 0} \frac{\mathscr{L}^m\left(\mathcal{B}_\varrho(S)\right)}{\omega_{m-s}\varrho^{m-s}}.$$

If both quantities coincide, we denote their common value with $\mathscr{M}^s(S)$.

Remark. Notice that, contrary to the Hausdorff measure, the Minkowski content is *not* a measure in the classical sense, since it is not countably additive. This can be deduced, for example, from the following observation: the Minkowski content of a set and the one of its closure are the same; so for example the 1-dimensional Minkowski content of all the sets $[0,1]$, $[0,1] \cap \mathbb{Q}$ and $[0,1] \setminus \mathbb{Q}$ (in $\mathbb{R}^1$) is 1, although the disjoint union of the last two of them equals the first of them.

Remark. By elementary geometric considerations, it is easy to see that for any set S and for any dimension s the inequality

$$\mathscr{H}^s(S) \leq C \mathscr{M}^s_*(S)$$

holds, for some constant $C(m,s)$; we refer to [Mat95, Paragraph 5.5] for the precise computation.

The opposite inequality is, in general, false: a classical example is the set

$$\left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} \cup \{0\}\,;$$

its 1-dimensional Hausdorff measure is 0 (since it's countable), but it has positive 1-dimensional (lower) Minkowski content.

For the definition of rectifiability, we follow the book [DeL08].

Definition A.3 (Rectifiability)**.** Let $A \subset \mathbb{R}^m$ be a $\mathscr{H}^k$-measurable set. We say that A is k-sectifiable if there exists a countable family $\{\Gamma_i\}_{i \in \mathbb{N}}$ of k-dimensional Lipschitz graphs such that

$$\mathscr{H}^k\left(A \setminus \bigcup_{i \in \mathbb{N}} \Gamma_i\right) = 0.$$

Remark. A very famous result, which can be found in the classical textbook of Federer (see [Fed69, Theorem 3.2.39]) states the following: if A is a closed k-rectifiable subset of $\mathbb{R}^m$, then

$$\mathscr{H}^k(S) = \mathscr{M}^k_*(S) = \mathscr{M}^{*k}(S).$$

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