ALGANT Master Thesis in Mathematics

A variation on the FC-property on profinite groups

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Introduction

In this thesis we discuss a variation of the FC-property and its impact on the structure of a profinite group. The starting point is this outcome by P. Shumyatsky [12, Theorem 1.5]: if for each element of a profinite group there are finitely many commutators of a given length starting with such element, then the corresponding term of the lower central series of the group is finite. In particular, the group is finite-by-nilpotent.

The property asked for each element of the group in the previous statement can be seen as a generalization of the finite conjugacy class property. An element of a group G with a finite conjugacy class is called an FC-element. More generally, an element $g \in G$ is called an FC_k-element if the cardinality $|g|_k$ of the set

$$X_k(g) = \{[g, x_1, \dots, x_k] \mid x_1, \dots, x_k \in G\}$$

is finite. Note that $|g|_1$ coincides with the cardinality of the conjugacy class of g, so an element g is an FC-element if and only if $|g|_1$ is finite. Thus, an FC-element is an FC_1-element and viceversa. For this reason, we say that the FC_k-property does generalize the finite conjugacy class property.

Using the definitions above, we can rewrite the above result in the following way: if G is a profinite group in which $|g|_k$ is finite for all $g \in G$, then $\gamma_{k+1}(G)$ is finite.

It is remarkable to mention that this result is somehow linked to the question of conciseness of group words in a certain group class. If $\omega = \omega(x_1, \dots, x_k)$ is a group word, *id est* an element of the free group on $x_1, \dots, x_k$, our interest lies in the set of all ω-values in a group G and the verbal subgroup generated by it, which are denoted as

$$G_\omega = \{\omega(x_1, \dots, x_k) \mid x_1, \dots, x_k \in G\} \text{ and } \omega(G) = \langle G_\omega \rangle$$

respectively. In case of topological groups $\omega(G)$ is used to denote the closed subgroup generated by all the ω-values in G.

Let ω be a group word and $\mathcal{C}$ a class of groups. We say that ω is concise in $\mathcal{C}$, if for every $G \in \mathcal{C}$ the condition that G_ω is finite implies that $\omega(G)$ is also finite. Recall that if G is a profinite group, its subgroups with cardinality less than the continuum are actually finite. Then the following question arises in a natural way: is it true that for each profinite group G in $\mathcal{C}$, if G_ω has order less than $2^{\aleph_0}$ then $\omega(G)$ is still finite? Whenever this is the case, the word ω is called *strongly concise* in the class of profinite groups $\mathcal{C}$.

However, as some of you have possibly noticed, the problem studied in [12] does not actually deal with group words. In fact, the expression $[g, x_1, \ldots, x_k]$ contains a "constant term" which depends on the choice of some element in the group, so it is not a proper group word by definition. That is why we introduce the notion of a *generalized group word*. Whereas a word in a specific group G can be seen as a map $\omega\colon G^k \to G$ that represents an element of the free group on $x_1, \ldots, x_k$, a generalized word in a group can be defined as a map $\omega\colon G^k \to G$ such that

$$(x_1, \ldots, x_k) \mapsto \prod_{j=1}^{s} \alpha_j(x_{i_j})^{\varepsilon_j},$$

with $i_1, \ldots, i_s \in \{1, \ldots, k\}$, $\alpha_1, \ldots, \alpha_s \in \mathrm{Aut}(G)$ and $\varepsilon_1, \ldots, \varepsilon_s \in \{\pm 1\}$. So given $g \in G$, we can write $[g, x_1] = x_1^{-1} x_1^g$ as a generalized group word and the same holds for the long commutator $[g, x_1, \ldots, x_k]$. In this way, we can see $X_k(g)$ as the set of values of a generalized word.

In this thesis we will prove that in a profinite group G, if the set of the values of the generalized group word $[g, x_1 \ldots, x_k]$ takes less than the cardinality of the continuum of values for some $g \in G$, then this set is actually finite. This allows to generalize [12, Theorem 1.5] in the following way: if G is a profinite group in which $|g|_k$ is less than the cardinality of the continuum for all $g \in G$, then $\gamma_{k+1}(G)$ is finite. A similar result holds even if we do not have a uniform bound on the length of the commutators. Namely, we prove that: if G is a profinite group in which for each element $g \in G$ there exists some natural number k_g such that $|g|_{k_g}$ is less than the continuum, then G is finite-by-nilpotent.

The thesis starts by recalling some fundamental and handy tools in group theory that will be useful later. In addition, the remaining part of Chapter 1 is devoted to describe the FC and BFC-properties. In particular, we do prove B. H. Neumman's theorem. Having [12] as the main guideline, in Chapter 2 the FC_k-property is described, which generalizes as we know the FC-property. Moreover, we will go through some crucial outcomes and proofs that are given in [12]. Finally, in Chapter 3, with the help of some techniques that are used in conciseness and strong conciseness related questions, we will prove our main, above mentioned, results.

Notation

$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \ldots\}$
$\subseteq_o$, $\subseteq_c$	open/closed subset
$\leq_o$, $\leq_c$	open/closed subgroup
$\trianglelefteq_o$, $\trianglelefteq_c$	open/closed normal subgroup
$(a, b, c, \ldots)$-bounded	upper-bounded by some algebraic expression depending on the parameters $a, b, c, \ldots$
fg	composition of maps $g \circ f$ where $f \colon X \to Y$ and $g \colon Y \to Z$ are maps between sets
$f(X)$	image of the set X under the map $f \colon X \to Y$
$X^{\mathbb{N}}$	space of sequences with elements from the set X and subscripts in $\mathbb{N}$
$\aleph_0$	countable infinity, cardinality of the natural numbers
2^X	cardinality of the set of functions from a set X to $\{0, 1\}$, cardinality of the power set of X
$2^{\aleph_0}$	cardinality of the continuum
$\lvert X \rvert$	cardinality of the set X
$\lvert G \colon H \rvert$	index of the subgroup H in the group G
$C_G(x)$	centralizer of the element x in the group G
$C_G(X)$	centralizer of the set X in the group G
$Z(G)$	center of the group G
$\mathrm{Tor}(G)$	subset formed by the torsion elements of the group G
x^g	element x conjugated by g that is the element $g^{-1}xg$ of a group
x^G	conjugacy class of the element x in the group G
X^G	subset of the conjugates of the elements of the subset X in the group G
$\mathrm{ncl}_G(X)$	normal closure of the subset X in the group G i.e. the subgroup $\langle X^G \rangle$
$\mathrm{core}_G(H)$	normal core of H in the group G i.e. the subgroup $\bigcap_{g \in G} H^g$
$[x, y]$	commutator word $x^{-1}x^y = x^{-1}y^{-1}xy = (y^{-1})^x y$ where x and y belong to a group

$[x_1, \ldots, x_k]$	element of a group G inductively defined as $[[x_1, \ldots, x_{k-1}], x_k]$ with $x_1, \ldots, x_k \in G$
$[H, K]$	subgroup of a group G defined as $\langle [h, k] \mid h \in H, k \in K \rangle$ where H and K are subgroups of G
$[H_1, \ldots, H_k]$	subgroup of a group G inductively defined as $[[H_1, \ldots, H_{k-1}], H_k]$ where $H_1, \ldots, H_k \leq G$
$[x,_k y]$	commutator word $[x, y, \ldots, y]$ with k ocurrences of y
$[H,_k L]$	subgroup of a group G defined as $[H, L, \ldots, L]$ where L is repeated k times and H and L are subgroups of G
G'	commutator/derived subgroup $[G, G]$ of the group G
$\gamma_k(G)$	k-th term of the lower central series of G
$Z_k(G)$	k-th term of the upper central series of G
$X \times Y$	cartasian product between sets X and Y
$X_1 \times \ldots \times X_k$	cartasian product among sets $X_1, \ldots, X_k$
X^k	cartesian product of the set X with itself k times
$\prod_{i \in I} X_i$	cartesian product of the family of sets $\{X_i\}_{i \in I}$
$x = y \pmod N$	the element xy^{-1} of the group G belongs to the normal subgroup N of G
$\max\{X\}$	maximum element of the totally ordered set X

Chapter 1

Preliminary results

1.1 Main definitions and results

We start by recalling some definitions and outcomes that we will use throughout this thesis. Firstly, we introduce the notion of a profinite group, and in order to do so we know there are several approaches we can make. The simplest one would be the one concerning only topological properties.

Definition 1.1.1. A topological group is said to be *profinite* if it is totally disconnected (consequently Hausdorff) and compact. More generally, a topological space is said to be a *profinite space* if it is totally disconnected, Hausdorff and compact.

Nonetheless, there is also another constructive definition for a profinite group which uses inverse limits.

Definition 1.1.2. A *profinite group* is a topological group that is isomorphic to the inverse limit of an inverse system of discrete finite groups.

Whenever a subgroup of a profinite group is considered, we will assume it is a closed subgroup, unless otherwise stated. Similarly, maps between topological spaces will always be continuous.

Definition 1.1.3. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be group properties. A group G is said to be $\mathcal{P}_1$-by-$\mathcal{P}_2$ if there exists a normal subgroup N of G such that N is $\mathcal{P}_1$ and the quotient G/N is $\mathcal{P}_2$. In particular, G is finite-by-nilpotent if there exists a finite normal subgroup N of G such that G/N is nilpotent.

It will be of great importance to recall the next property.

Proposition 1.1.4. *Let G be a compact topological group. Then, open subgroups of G have finite index in G. In particular, open subgroups of profinite groups have finite index.*

The topology of a profinite group is easily described using cosets of the group.

Theorem 1.1.5. *Let G be a profinite group. Then, any open subset of G is a union of cosets of open normal subgroups of G. In other words, the set*

$$\{gN \mid g \in G,\, N \trianglelefteq_o G\}$$

is a basis for the profinite group G.

Next we will recall that profinite groups satisfy a really strong property, which is indeed the main motivation to questions such as the one we make in this work.

Proposition 1.1.6. *A profinite group is either finite or uncountable.*

Proof. By contradiction, we assume $G = \{g_n\}_{n\in\mathbb{N}}$ is an infinite profinite countable group. Then the subset $G\setminus\{g_1\}$ is open because G is in particular Hausdorff, and since a basis for the topology in a profinite space is given by the clopen subsets, there exists some non-empty clopen $U_1 \subseteq G \setminus \{g_1\}$. Observe that any non-empty clopen is infinite since it contains some coset gN with $g \in G$ and $N \trianglelefteq_o G$, which is already infinite because open subgroups have finite index and we are assuming that the group is infinite. Therefore, the subset $U_1 \setminus \{g_2\}$ is non-empty and also open since we can see it as the intersection of the open subsets U_1 and $G \setminus \{g_2\}$. Using this recursive argument, we construct a descending chain of non-empty clopen subsets $\{U_n\}_{n\in\mathbb{N}}$ with empty intersection. By compactness there exists some $n \in \mathbb{N}$ such that $U_n = \emptyset$ which leads us to a contradiction. □

The following theorems are well-known results in group theory that will be really useful later.

Theorem 1.1.7. *(Schur's theorem) Let G be a group. If the center of G has finite index in G, say n, then the derivated subgroup of G has n-bounded finite order.*

Theorem 1.1.8. *(Dietzmann's theorem) Let G be a group and $X \subseteq G$ a finite torsion subset that is closed under conjugation. Then, the subgroup generated by X is finite. What is more, if X has cardinality n and m is the maximum order that an element in G achieves, then $\langle X\rangle$ has (n, m)-bounded order.*

Theorem 1.1.9. *(Baire category theorem) Let G be a profinite group and $\{C_n\}_{n\in\mathbb{N}}$ a family of countably many closed subsets of G. If the given family of closed subsets covers the whole group G, then there exists some $n \in \mathbb{N}$ such that C_n has non-empty interior.*

We continue by recalling a definition that is needed in the next lemma.

Definition 1.1.10. A continuous map $\varphi\colon X \to Y$ between topological spaces is called *nowhere locally constant* if for any non-empty open subset $U \subseteq X$, the map φ restricted to U is not constant.

This last lemma will be essential in the following chapter.

Lemma 1.1.11. *[4, Proposition 2.1] Let $\varphi\colon X \to Y$ be a continuous map between non-empty profinite spaces that is nowhere locally constant. Then the image of φ has cardinality at least the continuum.*

Proof. Let U be a non-empty clopen subset of X. Then there exist a finite discrete space Z_U and a continuous map $\theta_U : Y \to Z_U$ such that the composition $\varphi\theta_U$ is not constant on U. Choose now two distinct non-empty fibers U_1 and U_2 of the map $\varphi\theta_U$ restricted to U. By construction, U_1 and U_2 are non-empty clopen subsets of X contained in U and the intersection between $\varphi(U_1)$ and $\varphi(U_2)$ is clearly empty.

We consider a non-empty clopen subset $A \subseteq X$ such as $A = X$. Then using the construction above, for each sequence $\mathbf{i} = (i_1, i_2, \ldots)$ in $\{1, 2\}$ we can consider the descending chain of non-empty clopen subsets of X

$$A_{i_1} \supseteq (A_{i_1})_{i_2} \supseteq ((A_{i_1})_{i_2})_{i_3} \supseteq \ldots$$

and define the subset

$$A_{\mathbf{i}} = \bigcap_{n\in\mathbb{N}} (\cdots((A_{i_1})_{i_2})_{i_3}\cdots)_{i_n} \subseteq X$$

which is closed. By compactness, each $A_{\mathbf{i}}$ is non-empty and we can choose $a_{\mathbf{i}} \in A_{\mathbf{i}}$ for every sequence $\mathbf{i} \in \{1, 2\}^{\mathbb{N}}$. Note that the image under φ of $a_{\mathbf{i}}$ and $a_{\mathbf{j}}$ is distinct whenever $\mathbf{i} \neq \mathbf{j}$. This implies that the subset

$$B = \{a_{\mathbf{i}} \mid \mathbf{i} \in \{1, 2\}^{\mathbb{N}}\} \subseteq X$$

is mapped injectively into Y under φ and therefore the image of φ has cardinality at least the cardinality of $\varphi(B)$ that is exactly $2^{\aleph_0}$, the continuum. □

The previous lemma is also useful to enhance Proposition 1.1.6, where it was shown that a profinite group is either finite or uncountable. Indeed, an infinite profinite group is not only uncountable but actually its cardinality is at least the continuum. If G is an infinite profinite group it is sufficient to consider the nowhere locally constant continuous map $\mathrm{id}\colon G \to G$ to conclude that G has cardinality at least $2^{\aleph_0}$.

Corollary 1.1.12. *A profinite group is either finite or has cardinality at least the continuum.*

1.2 FC-groups

In the following sections we introduce two new interesting classes of groups, namely the FC and BFC classes of groups. These kind of groups are groups with restrictions in the cardinality of their conjugacy classes as we next see. Most of the content we give is well-known in group theory and can be found for instance in [9].

Definition 1.2.1. An element of a group $g \in G$ is called an *FC-element* if g has finitely many conjugates in G, in other words if the index of the centralizer of g in G is finite.

Definition 1.2.2. The set of FC-elements of a group G, called *FC-center* of G, will be denoted as $FC(G)$. In case the FC-center of G covers the whole group, meaning that every conjugacy class of the group is finite, we will call G an *FC-group.*

Remark 1.2.3. Note that the FC-center of a group can be seen as the generalization of the notion of the center of a group. Indeed, an element lies in the center if and only if its conjugacy class is formed uniquely by the element, so clearly the class is finite and the element belongs to the FC-center.

Examples 1.2.4.

(i) Finite groups are comprehensibly FC-groups.

(ii) All abelian groups are FC-groups.

(iii) Since the center of a group is contained trivially in the centralizer of any element in the group, groups whose center has finite index belong to the class of FC-groups.

(iv) The class of FC-groups is closed with respect to forming subgroups, images and direct products.

Note that we will be using the next fact repeatedly in several proofs.

Proposition 1.2.5. *Let G be a group and H and K subgroups of G of finite index. Then, the index of the subgroup $H \cap K$ in G is finite and*

$$|G\colon H \cap K| \leq |G\colon H| \cdot |G\colon K|.$$

More generally, if $H_1, \ldots, H_k$ are subgroups of finite index in G, then the index of the subgroup $\cap_{i=1}^{k} H_i$ in G is finite and

$$\left| G\colon \bigcap_{i=1}^{k} H_i \right| \leq \prod_{i=1}^{k} |G\colon H_i|.$$

The set of FC-elements of a group is not just a subset of the group as we next see.

Proposition 1.2.6. *For any group G, the subset $FC(G)$ is a characteristic subgroup of G.*

Proof. To begin with, it is clear $FC(G)$ is non-empty because it contains the identity element. Let g_1 and g_2 be two elements of G whose centralizers have finite index in G. From Proposition 1.2.5 the subgroup $C_G(g_1) \cap C_G(g_2)$ has finite index in G and since $C_G(g_1) \cap C_G(g_2)$ is contained in $C_G(g_1 g_2^{-1})$, we deduce that $C_G(g_1 g_2^{-1})$ has finite index in G too, so $g_1 g_2^{-1} \in FC(G)$. Then, $FC(G)$ is a subgroup of G. On the other hand, if $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ then $C_G(\alpha(g)) = \alpha(C_G(g))$. In fact, for $x \in G$ we have

$$x\alpha(g) = \alpha(g)x \iff \alpha^{-1}(x)g = g\alpha^{-1}(x) \iff \alpha^{-1}(x) \in C_G(g).$$

Since automorphisms of G preserve the indeces of their subgroups, the index of $\alpha(C_G(g))$ in G is finite and therefore so is the index of $C_G(\alpha(g))$ in G, i.e. $\alpha(g) \in FC(G)$. □

Whenever a group is an FC-group, the group over its center has desirable properties.

Proposition 1.2.7. *If G is an FC-group, then $G/Z(G)$ is residually finite and torsion.*

Proof. Recall that a group is residually finite if it is isomorphic to a subgroup of the direct product of a family of finite groups. For this reason, we construct a homomorphism

$$\begin{aligned} \varphi\colon G/Z(G) &\to \prod_{g \in G} G/\operatorname{core}_G(C_G(g)) \\ xZ(G) &\mapsto (x \operatorname{core}_G(C_G(g)))_{g \in G} \end{aligned}$$

that is easily checked to be well-defined because $Z(G) \subseteq \operatorname{core}_G(C_G(g))$ for all $g \in G$. Moreover, if $xZ(G)$ and $yZ(G)$ are distinct elements in $G/Z(G)$, we have that $xy^{-1} \notin Z(G)$ thus there exists some $g \in G$ such that $xy^{-1}g \neq gxy^{-1}$ so $xy^{-1} \notin C_G(g)$. In particular, $xy^{-1} \notin \operatorname{core}_G(C_G(g))$ neither therefore $x \operatorname{core}_G(C_G(g)) \neq y \operatorname{core}_G(C_G(g))$. Then, φ is injective. By assumption G is an FC-group which ensures that each $G/\operatorname{core}_G(C_G(g))$ is finite, thus $G/Z(G)$ is residually finite.

It remains to check that $G/Z(G)$ is torsion, i.e. that each element $gZ(G) \in G/Z(G)$ has finite order. Let $x \in G$ and choose a transversal T of $C_G(x)$ in G. Since G is an FC-group, T is a finite subset. We define H as the subgroup obtained by intersecting the centralizers of the elements of T in G. As the intersection is finite and G is an FC-group by Proposition 1.2.5, H has finite index in G. What is more, we can consider the G-action in the finite set of left cosets of H given by left multiplication, so we get that

$G/\operatorname{core}_G(H)$ is isomorphic to some subgroup of the symmetric group of order the index of H in G. Hence the normal core of H in G has finite index as well. Consequently, there exists some $m \in \mathbb{N}$ such that $x^m \in \operatorname{core}_G(H)$. Observe now that $G = \langle T, C_G(x) \rangle$ and $x^m \in C_G(t)$ for all $t \in T$, implying that $x^m \in Z(G)$, so $xZ(G)$ has finite order in $G/Z(G)$. □

In the case of torsion groups, it is possible to characterize FC-groups as the locally finite normal groups.

Definition 1.2.8. A group is said to be *locally finite and normal* if every finite subset is contained in a finite normal subgroup.

Proposition 1.2.9. *A torsion group is an FC-group if and only if it is locally finite and normal.*

Proof. Assume G is a torsion FC-group. Let $F \subseteq G$ be a finite subset. We can then consider the subset

$$E = \{x^g \mid x \in F \text{ and } g \in G\}$$

which is by construction closed under conjugation, it is finite because G is an FC-group and it does trivially contain F. By Dietzmann's theorem E is contained in a finite normal subgroup of G, so is E. This proves that G is locally finite and normal.

Now let us suppose G is locally finite and normal. If $g \in G$, then there exists some finite normal subgroup N of G containing the element g. Since N is normal it contains all the conjugates of g and therefore the conjugacy class of g is finite. So G is an FC-group. □

Remark 1.2.10. Note that a locally finite and normal group is torsion and hence FC from Proposition 1.2.9. Indeed, any element of such a group belongs to a finite subgroup so it necessarily has finite order.

Example 1.2.11. As a consequence of Remark 1.2.10, the direct product of a family of finite groups is an FC-group.

The class of FC-groups is rather interesting and enforces the derived subgroup to be torsion as we shall see.

Proposition 1.2.12. *If G is an FC-group, then G' is torsion. Besides, the elements of finite order of G form a fully-invariant subgroup of G containing G'.*

Proof. First of all, the group $G/Z(G)$ is an epimorphic image of an FC-group so it is also an FC-group. Moreover, by Proposition 1.2.7 this quotient is torsion too so we can use Proposition 1.2.9 and deduce $G/Z(G)$ is a locally finite and normal group. Let $\mathcal{F}$ be the family of finitely generated subgroups of G. Then it is straightforward that

$$G' = \bigcup_{H \in \mathcal{F}} H'$$

holds. For this reason, to verify that G' is torsion it is enough to check that H' is torsion for every $H \in \mathcal{F}$. Let us fix $H = \langle x_1, \dots, x_n \rangle \in \mathcal{F}$ and set $Z = Z(G)$. Then HZ is a subgroup of G, because Z is normal in G, and the subgroup HZ/Z of G/Z is clearly generated by the cosets $x_1Z, \dots, x_nZ$. At this point, using that G over the center is locally finite and normal, we deduce that $\{x_1Z, \dots, x_nZ\}$ is contained in a finite normal subgroup, which must also contain the smallest subgroup containing the subset, i.e. it contains the subgroup $HZ/Z = \langle x_1Z, \dots, x_nZ \rangle$. Then $|H \colon H \cap Z| = |HZ \colon Z|$ is finite. In particular, we know that $H \cap Z \subseteq Z(H)$ thus the index $|H \colon Z(H)|$ is finite as well and by Schur's theorem the derived subgroup H' is finite and therefore torsion as claimed.

Let x and y be two torsion elements of G whose orders are n and m respectively. Then $xy^{-1}G'$ has order dividing nm in the abelianization G/G', so there exists some natural number $l \in \mathbb{N}$ such that $(xy^{-1})^l \in G'$ and thus xy^{-1} is also torsion. Thus $\mathrm{Tor}(G)$ is a subgroup. Then, $\mathrm{Tor}(G)$ is a fully-invariant subgroup of G, since the order of the image of an element under a homomorphism divides the order of the corresponding element. □

We are now able to characterize the class of FC-groups.

Proposition 1.2.13. *A group is an FC-group if and only if it is isomorphic to some subgroup of a direct product of a torsion-free abelian group and a locally finite and normal group.*

Proof. Suppose G is an FC-group. In the one hand, from Proposition 1.2.12 we know that $G' \leq \mathrm{Tor}(G) \trianglelefteq G$, then $G/\mathrm{Tor}(G)$ is trivially torsion-free and abelian. On the other hand, by Zorn's Lemma there exists some maximal torsion-free subgroup M of the center of G and we claim that G/M is a locally finite and normal group. Firstly, $Z(G)/M$ is clearly torsion, as well as the quotient group $G/Z(G)$ due to Proposition 1.2.7. This implies that G/M is torsion. Note that G/M is an FC-group and Proposition 1.2.9 ensures us that it is locally finite and normal. Finally, we can construct the homomorphism from G to the direct product $G/\mathrm{Tor}(G) \times G/M$ that maps an element x to $(x\,\mathrm{Tor}(G), xM)$ and check it is an embedding. In fact, if x and y are elements of G, their images are equal if and only if $xy^{-1} \in \mathrm{Tor}(G) \cap M$. However, this last intersection is trivial because one is torsion while the other is torsion-free, implying that the homomorphism is injective.

Conversely, assume G is a subgroup of $G_1 \times G_2$ where G_1 is a torsion-free abelian group and G_2 is a locally finite and normal group. G_1 is abelian so it is also an FC-group. In addition, by Remark 1.2.10 we know G_2 is an FC-group too. The result follows because FC-groups are closed under forming direct products and subgroups. □

Although we have been able to characterize an FC-group, it is not completely satisfactory because subgroups of direct products are difficult to handle. Next we will see that there is a specific kind of FC-group that admits a precise description.

1.3 BFC-groups and the BFC-theorem

Definition 1.3.1. A group is called a *BFC-group* if the size of the conjugacy classes of the group is bounded, i.e. if there exists a natural number $d \in \mathbb{N}$ such that the conjugacy class of any element has at most d elements.

Theorem 1.3.2. *(B. H. Neumaan's theorem) A group is a BFC-group if and only if its derived subgroup is finite.*

Proof. Let us assume G is a BFC-group, meaning that

$$d = \max\{|x^G| \mid x \in G\} < \infty.$$

Then there exists some element $a \in G$ with $|a^G| = |G\colon C_G(a)| = d$. Let $T = \{t_1, \dots, t_d\}$ be a transversal of $C_G(a)$ in G. In this case we can explictly write which is the conjugacy class of a in G because if $1 \leq i, j \leq d$ then

$$a^{t_i} \neq a^{t_j} \iff t_i t_j^{-1} \notin C_G(a) \iff t_i C_G(a) \neq t_j C_G(a),$$

so $a^G = \{a^{t_1}, \dots, a^{t_d}\}$. The group G is BFC thus in particular the centralizer of each $t \in T$ in G has finite index and therefore $C = \cap_{i=1}^{d} C_G(t_i)$ has finite index in G. Let $S = \{s_1, \dots, s_k\}$ be a transversal of C in G and define the normal subgroup $N = \mathrm{ncl}_G(\langle a, s_1, \dots, s_k\rangle)$. Again N is trivially an FC-group and it is generated by the conjugates of finitely many elements, so it is finitely generated. Moreover, applying Proposition 1.2.7, we get that the FC-group $N/Z(N)$ is torsion, so by Proposition 1.2.9 the quotient $N/Z(N)$ is locally finite and normal. It is clear $N/Z(N)$ is finitely generated, then it must be finite, so by Schur's theorem N' is finite. Since Proposition 1.2.12 ensures that $N' \leq \mathrm{Tor}(N) \leq N$, we can consider the subgroup $\mathrm{Tor}(N)/N'$ of the finitely generated abelian group N/N', therefore $\mathrm{Tor}(N)/N'$ is abelian, torsion and finitely generated. The Structure Theorem for Finitely Generated Abelian groups implies $\mathrm{Tor}(N)/N'$ is finite and as a consequence $|\mathrm{Tor}(N)| = |\mathrm{Tor}(N)\colon N'||N'| < \infty$. In the one hand, we claim that $C' \subseteq N$ holds. If $x \in C$, then $(xa)^{t_i} = x^{t_i} a^{t_i} = x a^{t_i}$ for all $i \in \{1, \dots, d\}$ and we can enlist without repetitions

$$(xa)^G = \{xa^{t_1}, \dots, xa^{t_d}\}.$$

Therefore if $y \in G$, there exists some $i \in \{1, \ldots, d\}$ such that $(xa)^y = xa^{t_i}$, or equivalenty $x^y = xa^{t_i}a^{y^{-1}}$. This implies that $[x, y] = x^{-1}x^y = x^{-1}xa^{t_i}a^{y^{-1}} = a^{t_i}a^{y^{-1}} \in N$ so that $C' \subseteq N$. On the other hand, we prove that $G' \subseteq NC'$ holds too. In order to do so, note that $G = NC$ is satisfied thus if $x_1, x_2 \in G$,

$$\begin{aligned}[x_1, x_2] &= [n_1c_1, n_2c_2] = [n_1, n_2c_2]^{c_1}[c_1, n_2c_2] \\ &= [n_1, c_2]^{c_1}[n_1, n_2]^{c_2c_1}[c_1, c_2][c_1, n_2]^{c_2}\end{aligned}$$

for some $n_1, n_2 \in N$ and $c_1, c_2 \in C$. As the subgroup N is normal in G, the commutator $[x_1, x_2]$ belongs to NC' and consequently $G' \subseteq NC'$. But as we have seen $C' \subseteq N$ so $NC' \subseteq N$ and we have $G' \subseteq N$. We know by Proposition 1.2.12 that G' is torsion then it is contained in the finite subgroup $\mathrm{Tor}(N)$, which implies that G' itself is also finite.

Let now G be a group whose derived subgroup G' is finite and denote by d its order. If x is an element of G, then

$$|x^G| = |\{x^{-1}x^g \mid g \in G\}| = |\{[x, g] \mid g \in G\}| \leq |G'| = d < \infty$$

so G is a BFC-group. □

It is clear by definition that a BFC-group is an FC-group, but is there any chance that the converse also holds? The answer lies beneath.

Example 1.3.3. Let A be a finite non-abelian simple group and $x \in A$ a non-trivial element of A. Let G be the direct product of infinitely countably many copies of A. Elements in G are tuples with infinitely countably many entries and only finitely many of them with a non-trivial element of A. For this reason, each element can only have finitely many conjugates so G is an FC-group. However, we can find elements whose conjugacy classes have order as large as we desire. Indeed, if we denote the infinite tuple with the element x in the first n entries and the trivial element in the rest of the coordinates by

$$\mathbf{x}_n = (x, \ldots, x, 1, \ldots) \in G,$$

it is straightforward that $|\mathbf{x}_n^G| = |x^A|^n$ with $|x^A| \geq 2$. Therefore there is no bound for the size of the conjugacy classes of G, meaning that G is not a BFC-group.

Nonetheless, in our case of interest, *id est* when considering a profinite group, an FC-group is automatically a BFC-group.

Proposition 1.3.4. *[11, Lemma 2.6] A profinite FC-group is necessarily a BFC-group. So its derived subgroup is finite.*

Proof. For each $n \in \mathbb{N}$ we define the subset $\Delta_n = \{x \in G \mid |x^G| \leq n\}$, which is closed by Lemma 2.4.8 when $k = 1$ as we shall see in the following chapter. Since G is an FC-group, all conjugacy classes have finite order so the union of all Δ_n where n ranges over $\mathbb{N}$, is the entire group G. By Baire category theorem we deduce that there exists some $n \in \mathbb{N}$ such that Δ_n has non-empty interior. Equivalently, there exist some element $g \in G$ and open normal subgroup H of G such that the coset gH is contained in Δ_n, i.e. the conjugacy class of any element of the form gh with $h \in H$ has cardinality less or equal than n. In particular, g and its inverse g^{-1} have conjugacy classes with cardinality less or equal than n. Then,

$$|h^G| = |(g^{-1}gh)^G| \leq |(g^{-1})^G| \cdot |(gh)^G| \leq n \cdot n = n^2,$$

for all $h \in H$. We have found a bound for the cardinality of the conjugacy classes of the elements of H, in other words H is a BFC-group. Let T be a (finite) transversal of H in G and m the maximum order of the conjugacy classes of the elements in T, which is clearly finite. As any element of G can be written as a product th with $t \in T$ and $h \in H$ we have that

$$|x^G| = |(th)^G| \leq |t^G| \cdot |h^G| \leq m \cdot n^2 < \infty$$

for all $x \in G$. Therefore G is a BFC-group. □

Chapter 2

A variation of the BFC-theorem

In the following chapters k, n and l will denote positive integers unless otherwise stated.

Following the research paper [12], we will see some of the tools used to prove that a profinite FC_k-group is finite-by-nilpotent.

2.1 FC_k-property

Definition 2.1.1. Let G be a group. Then the following elements of G inductively defined as

$$\begin{cases} \gamma_1(x_1) = x_1, \\ \gamma_2(x_1, x_2) = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \\ \gamma_k(x_1, \ldots, x_k) = [x_1, \ldots, x_k] = [\gamma_{k-1}(x_1, \ldots, x_{k-1}), x_k] \quad \text{for } k \geq 2 \end{cases}$$

with $x_1, \ldots, x_k \in G$ are called *γ_k-values* of G. Moreover, the set of γ_{k+1}-values of G starting with the fixed element g is denoted as

$$X_k(g) = \{[g, x_1, \ldots, x_k] \mid x_1, \ldots, x_k \in G\},$$

and its cardinal is denoted as $|g|_k$.

Definition 2.1.2. Let G be a group. We define the *k-th FC-center* of G to be the set

$$FC_k(G) = \{g \in G \mid |g|_k < \infty\}.$$

Definition 2.1.3. A group G is said to be an *FC_k-group* if it is equal to its k-th FC-center.

2.2 Probabilistic results and Engel theory

Definition 2.2.1. Let G be a finite group and K a subgroup of G. The *commuting probability* of K in G, denoted as $Pr(K,G)$, is defined as the probability that a random element of K commutes with a random element of G, namely

$$Pr(K,G) = \frac{|\{(x,y) \in K \times G \mid xy = yx\}|}{|K||G|}.$$

Proposition 2.2.2. *Let G be a finite group and K a subgroup of G. If $|x^G| \leq n$ for every $x \in K$, then $Pr(K,G)$ is greater or equal than $1/n$.*

Proof. It is inmediate that

$$Pr(K,G) = \frac{1}{|K|}\sum_{x\in K}\frac{|C_G(x)|}{|G|} = \frac{1}{|K|}\sum_{x\in K}\frac{1}{|x^G|} \geq \frac{1}{|K|}\sum_{x\in K}\frac{1}{n} = \frac{1}{n}$$

so we are done. □

The following theorem states that under the assumption that the commuting probability of a subgroup K of a finite group G is positive, then G has "large" subgroups (with bounded index) that "almost" commute with G (commutator subgroups with bounded order).

Theorem 2.2.3. *[5] Let G be a finite group and K a subgroup of G. If $Pr(K,G) \geq \epsilon > 0$, then there exist a normal subgroup T of G and a subgroup B of K such that the indeces $|G\colon T|$ and $|K\colon B|$ and the order of the commutator subgroup $[T,B]$ are ϵ-bounded.*

Definition 2.2.4. An element x of a group G is called (left) *l-Engel* if $[y,_l x] = 1$ for every $y \in G$. The group G is said to be l-Engel if so are the elements of G.

Lemma 2.2.5. *[12, Lemma 2.4] Let G be a metabelian group and suppose a and b are l-Engel elements of G. Then, any element that belongs to the subgroup generated by the elements a and b is $(2l+1)$-Engel.*

2.3 BFC_k-groups are finite-by-nilpotent

The main result of [12] is the following.

Theorem 2.3.1. *[12, Theorem 1.1] Let G be a group such that $|x|_k \leq n$ for all $x \in G$. Then, $\gamma_{k+1}(G)$ has finite (k,n)-bounded order.*

The proof of Theorem 2.3.1 is rather complicated. For this reason, in this section we will only show the main steps towards its proof, which are given in Theorem 2.3.3. But prior to stateting that result, we need to give this lemma whose proof is quite straightforward.

Lemma 2.3.2. *[12, Lemma 2.2] Let G be a group. Then*

(i) $|\gamma_k(G)| \leq n$ if and only if $|\gamma_k(H)| \leq n$ for any finitely generated subgroup H of G.

(ii) In case G is residually finite, $|\gamma_k(G)| \leq n$ if and only if $|\gamma_k(Q)| \leq n$ for any finite quotient Q of G.

Theorem 2.3.3. *[12, Theorem 3.1] Let G be a group in which the cardinality of the conjugacy classes of the γ_k-values is bounded by n. Then, G has a nilpotent subgroup of (k, n)-bounded index and (k, n)-bounded class.*

Proof. If $k = 1$, by B. H. Neumann's theorem it follows that G has a class-two nilpotent normal subgroup of n-bounded index. If $k = 2$, as a consequence of the main result of [6], G has a class-four nilpotent normal subgroup of n-bounded index. We will assume that $k \geq 3$ then.

It was shown in [3] that in this situation the derived group $\gamma_k(G)'$ has n-bounded order. For this reason, we can assume $\gamma_k(G)'$ is trivial so we work under the hypothesis that $\gamma_k(G)$ is abelian, and consequently G is abelian-by-nilpotent. By the first statement of Lemma 2.3.2, we can assume that our group G is finitely generated. Moreover, Hall's classical theorem [8] proves that finitely generated abelian-by-nilpotent groups are residually finite. Now we can apply the second statement of Lemma 2.3.2 so it suffices to check the result for finite quotients of finitely generated subgroups of G. Without loss of generalization we can therefore suppose that G is finite and also that $\gamma_k(G)$ is abelian.

In order to give some taste of the techniques used in this kind of proofs, we will only prove the core case in which G is a metabelian p-group. The rest of the cases are reductions to this base case. Then there exists a maximal abelian normal subgroup N of G that contains G'. Set $\mathbf{G} = G \times \ldots \times G$ where there are $k-1$ ocurrences of G. For each $\mathbf{g} = (g_1, \ldots, g_{k-1}) \in \mathbf{G}$ we define $N_{\mathbf{g}} = [N, g_1, \ldots, g_{k-1}] = \{[x, g_1, \ldots, g_{k-1}] \mid x \in N\}$. We fix $\mathbf{g} = (g_1, \ldots, g_{k-1}) \in \mathbf{G}$ and prove that $N_{\mathbf{g}}$ is a subgroup of G. Clearly, $N_{\mathbf{g}}$ is non-empty for any $\mathbf{g} \in \mathbf{G}$. Since N is normal in G, we have that $[N, G] \subseteq N$. Thus, any element of the form $[n_1, g]$ with $n_1 \in N$ and $g \in G$ commutes with any element of N due to its abelianity. If $n_1, n_2 \in N$, then

$$\begin{aligned}
[n_1n_2, g_1, \ldots, g_{k-1}] &= [[n_1, g_1]^{n_2}[n_2, g_1], g_2, \ldots, g_{k-1}] \\
&= [[n_1, g_1][n_2, g_1], g_2, \ldots, g_{k-1}] \\
&= [[n_1, g_1, g_2]^{[n_2, g_1]}[n_2, g_1, g_2], g_3, \ldots, g_{k-1}] \\
&= [[n_1, g_1, g_2][n_2, g_1, g_2], g_3, \ldots, g_{k-1}].
\end{aligned}$$

By repeating this argument inductively we prove that

$$[n_1n_2, g_1, \ldots, g_{k-1}] = [n_1, g_1, \ldots, g_{k-1}][n_2, g_1, \ldots, g_{k-1}] \qquad (2.1)$$

holds so the operation between elements from $N_{\mathbf{g}}$ lies in $N_{\mathbf{g}}$. Furthermore, equality (2.1) yields

$$[n_1, g_1, \ldots, g_{k-1}]^{-1} = [n_1^{-1}, g_1, \ldots, g_{k-1}],$$

which implies that the inverse of an element of $N_{\mathbf{g}}$ belongs to $N_{\mathbf{g}}$. Hence, $N_{\mathbf{g}} \leq G$. What is more, $N_{\mathbf{g}}$ is also normal in G. Indeed, if $x \in N$ and $g \in G$ then

$$[x, g_1]^g = [x^g, g_1^g] = [x^g, h_1 g_1] = [x^g, g_1][x^g, h_1]^{g_1}$$

where $h_1 = [g, g_1^{-1}] \in G' \leq N$. But note that N is normal in G so $x^g \in N$ and by abelianity of N we get that $[x^g, h_1] = 1$. Using an inductive argument, we easily prove that

$$[x, g_1, \ldots, g_{k-1}]^g = [x^g, g_1, \ldots, g_{k-1}] \in N_{\mathbf{g}},$$

meaning that $N_{\mathbf{g}} \trianglelefteq G$ as claimed. Plainly, elements from $N_{\mathbf{g}}$ are by definition γ_k-values, which implies by assumption that $|x^G| \leq n$ for all $x \in N_{\mathbf{g}}$. In this situation, Proposition 2.2.2 states that $Pr(N_{\mathbf{g}}, G) \geq 1/n > 0$ and thus there exist a normal subgroup $T_{\mathbf{g}}$ of G and a subgroup $B_{\mathbf{g}}$ of $N_{\mathbf{g}}$ such that the indeces $|G\colon T_{\mathbf{g}}|$ and $|N_{\mathbf{g}}\colon B_{\mathbf{g}}|$ and the order of the commutator subgroup $[T_{\mathbf{g}}, B_{\mathbf{g}}]$ are n-bounded, due to Theorem 2.2.3. Normality of $T_{\mathbf{g}}$ in G implies that $[T_{\mathbf{g}}, B_{\mathbf{g}}]$ is contained in $T_{\mathbf{g}}$. What is more, the subgroup $[T_{\mathbf{g}}, B_{\mathbf{g}}]$ is normal in $T_{\mathbf{g}}$ since

$$[b, t_1]^{t_2} = [b, t_2]^{-1}[b, t_1 t_2] \in [B_{\mathbf{g}}, T_{\mathbf{g}}] = [T_{\mathbf{g}}, B_{\mathbf{g}}]$$

for all $b \in B_{\mathbf{g}}$ and $t_1, t_2 \in T_{\mathbf{g}}$. Denote as d the index of $T_{\mathbf{g}}$ in G that is n-bounded. Then, G can be written as the (disjoint) union of some right cosets $Ty_1, \ldots, Ty_d$ for some $y_1, \ldots, y_d \in G$ so that $H_{\mathbf{g}} = \mathrm{ncl}_G([T_{\mathbf{g}}, B_{\mathbf{g}}])$ is

$$\langle [T_{\mathbf{g}}, B_{\mathbf{g}}]^{Ty_1}, \ldots, [T_{\mathbf{g}}, B_{\mathbf{g}}]^{Ty_d} \rangle = \langle [T_{\mathbf{g}}, B_{\mathbf{g}}]^{y_1}, \ldots, [T_{\mathbf{g}}, B_{\mathbf{g}}]^{y_d} \rangle = \prod_{i=1}^{d} [T_{\mathbf{g}}, B_{\mathbf{g}}]^{y_i}$$

because $[T_{\mathbf{g}}, B_{\mathbf{g}}]$ and each $[T_{\mathbf{g}}, B_{\mathbf{g}}]^{y_i}$ are normal in T. It is clear now that the order of the normal closure of $[T_{\mathbf{g}}, B_{\mathbf{g}}]$ in G

$$|H_{\mathbf{g}}| \leq \prod_{i=1}^{d} |[T_{\mathbf{g}}, B_{\mathbf{g}}]^{y_i}| \leq |[T_{\mathbf{g}}, B_{\mathbf{g}}]|^d$$

has n-bounded order. It is convenient to emphasize that the bounds coming from Theorem 2.2.3 do not depend on the choice of $\mathbf{g} \in \mathbf{G}$.

It is well-known that finite p-groups are nilpotent, so is G. Thus, there exists an n-bounded number e such that $H_{\mathbf{g}}$ is contained in $Z_e(G)$ for all $\mathbf{g} \in \mathbf{G}$. Indeed, nilpotency implies that any normal subgroup of G has non-empty intersection with the center of G, so in particular $|H_{\mathbf{g}} \cap Z(G)| \geq 2$. For this reason,

$$\left|\frac{H_{\mathbf{g}}Z(G)}{Z(G)}\right| = \frac{|H_{\mathbf{g}}|}{|H_{\mathbf{g}} \cap Z(G)|} \leq \frac{|H_{\mathbf{g}}|}{2}$$

holds. Using inductively the same argument with the normal subgroup $H_{\mathbf{g}}Z_{i-1}(G)/Z_{i-1}(G)$ of the nilpotent group $G/Z_{i-1}(G)$ we get that

$$\left|\frac{H_{\mathbf{g}}Z_i(G)}{Z_i(G)}\right| = \left|\frac{H_{\mathbf{g}}Z_i(G)/Z_{i-1}(G)}{Z_i(G)/Z_{i-1}(G)}\right| \leq \frac{|H_{\mathbf{g}}|}{2^i}$$

for all $i < \log_2 |H_{\mathbf{g}}|$. Then, there exists an n-bounded number e such that $H_{\mathbf{g}}Z_e(G)/Z_e(G)$ is trivial and thus the claim is proved. As the bounds do not depend on the choice of $\mathbf{g} \in \mathbf{G}$, we can choose the n-bounded number e such that $H_{\mathbf{g}}$ is contained in $Z_e(G)$ for all $\mathbf{g} \in \mathbf{G}$. Pass to the quotient $G/Z_e(G)$ so we can assume $[T_{\mathbf{g}}, B_{\mathbf{g}}] = 1$ for all $\mathbf{g} \in \mathbf{G}$. Since the index $|N_{\mathbf{g}} \colon B_{\mathbf{g}}|$ is n-bounded, there exist n-boundedly many elements $x_1, \ldots, x_s \in N_{\mathbf{g}}$ such that $N_{\mathbf{g}} = \langle x_1, \ldots, x_s, B_{\mathbf{g}} \rangle$. For each $\mathbf{g} \in \mathbf{G}$ we define $C_{\mathbf{g}} = C_G(N_{\mathbf{g}})$ and note that $C_{\mathbf{g}}$ can be also written as $C_G(x_1) \cap \ldots \cap C_G(x_s) \cap C_G(B_{\mathbf{g}})$. By assumption each $[T_{\mathbf{g}}, B_{\mathbf{g}}]$ is trivial so elements of $T_{\mathbf{g}}$ commute with elements of $B_{\mathbf{g}}$, which implies that $T_{\mathbf{g}}$ is contained in $C_G(B_{\mathbf{g}})$ for all $\mathbf{g} \in \mathbf{G}$. Recall $T_{\mathbf{g}}$ has n-bounded index and that the conjugacy classes of the elements $x_1, \ldots, x_s \in N_{\mathbf{g}}$ have order less or equal than n, in other words, each $C_G(x_i)$ has index less or equal than n. Therefore, the subgroup $C_G(x_1) \cap \ldots \cap C_G(x_s) \cap T_{\mathbf{g}}$ has n-bounded index by Proposition 1.2.5, so it has $C_{\mathbf{g}}$ too. Moreover, normality of $N_{\mathbf{g}}$ in G implies normality of $C_{\mathbf{g}}$ in G.

For each $g \in G$ and $i \geq 1$ we define the subgroups $N_{i,g} = [N,_i g] = [N, g, \ldots, g]$ and $C_{i,g} = C_G(N_{i,g})$. Note that these subgroups are a special case of $N_{\mathbf{g}}$ and $C_{\mathbf{g}}$, whenever we consider a constant tuple $\mathbf{g} = (g, \ldots, g) \in \mathbf{G}$, so the previous statements are still true for them. At this point observe that we have two chains

$$N_{1,g} \geq N_{2,g} \geq \ldots \text{ and } C_{1,g} \leq C_{2,g} \leq \ldots$$

for each $g \in G$. If $i \geq 1$ let us denote $\beta_i = \max_{g \in G}\{|G \colon C_{i,g}|\}$. It is clear by construction that $\beta_i \leq \beta_j$ whenever $i \geq j$. In particular, β_{k-1} is the maximum index of the subgroups $C_{k-1,g}$ with $g \in G$, which are as we have proved n-bounded, thus β_{k-1} is n-bounded as well. Observe that

$$\beta_{k-1} \geq \beta_k \geq \ldots \geq \beta_i \geq \ldots$$

is a decreasing sequence of positive integers numbers so it can have at most $\beta_{k-1} - 1$ jumps, meaning that $\beta_i \neq \beta_{i+1}$ for at most $\beta_{k-1} - 1$ values of $i \geq k-1$. With this setup it is possible to find some number $u \geq k-1$ such that $\beta_u = \beta_{2u}$. In fact, we can proceed as follows: if $\beta_{k-1} = \beta_{2(k-1)}$ we do have the number $u = k-1$. Otherwise, $\beta_{k-1} \neq \beta_{2(k-1)}$ and we compare

$\beta_{2(k-1)}$ with $\beta_{2^2(k-1)}$. If they are equal we set $u = 2(k-1)$ and if that is not the case, we continue with this process. As there can be at most $\beta_{k-1} - 1$ jumps we are sure that this process terminates at some point and the number u is found before $2^{\beta_{k-1}}(k-1)$. Then, there exists a (k,n)-bounded number u such that $k-1 \leq u$ and $\beta_u = \beta_{2u}$. Let $g \in G$ be an element such that $C_{2u,g}$ has index β_{2u} in G. As $C_{u,g} \subseteq C_{2u,g}$ holds, maximality of the index β_u implies that $C_{u,g}$ and $C_{2u,g}$ need to be equal. It is trivial then that $C_{u,g} = C_{i,g}$ for $i \in \{u, u+1, \ldots, 2u\}$. We claim that whenever $h \in C_{u,g}$, then

$$N_{i,g} = [N_{u,g},_{i-u}\, g] = [N_{u,g},_{i-u}\, gh] \tag{2.2}$$

holds for $i \in \{u, u+1, \ldots, 2u\}$. While the first equality is straightforward by definition, we prove the second one by induction on i. The base case is trivial so we assume it is true for i and prove it for $i+1$. In the one hand, the induction hypothesis yields to

$$N_{i+1,g} = [N_{u,g},_{i+1-u}\, g] = [[N_{u,g},_{i-u}\, g], g] = [[N_{u,g},_{i-u}\, gh], g].$$

On the other hand, we note that $N_{u,g}$ is a normal subgroup of G (using the analogous proof for $N_{\mathbf{g}}$ in G) and therefore for any $x \in N_{u,g}$ we have that

$$[[x,_{i-u}\, gh], g] = [x,_{i-u}\, gh]^{-1}[x,_{i-u}\, gh]^g = [x,_{i-u}\, gh]^{-1}[x,_{i-u}\, gh]^{gh}$$

because h commutes with any element of $N_{u,g}$ by election. This concludes the proof of (2.2). In particular, it is true that

$$N_{2u,g} = [N_{u,g},_u\, gh]$$

holds. This last equality and the fact that $N_{u,g}$ is contained in N imply that the subgroup $C_{2u,g}$ contains $C_{u,gh}$, meaning that $C_{u,gh} \leq C_{2u,g} = C_{u,g}$. By definition β_u is the largest index of a subgroup of the type $C_{u,x}$ with $x \in G$, therefore $C_{u,g} = C_{u,gh}$ for all $h \in C_{u,g}$.

The aim at this point is to prove that the subgroup $D = C_{u,g}$ is the subgroup of G we are looking for, i.e. that D is a nilpotent subgroup of G of (k,n)-bounded index and class. As $N_{u,g}$ is normal in G, we have that $D = C_G(N_{u,g})$ is normal in G too. Thus, D is a normal subgroup of G whose index is β_u, which is n-bounded because $\beta_u \leq \beta_{k-1}$. It remains to prove D is nilpotent of (k,n)-bounded class.

Set $\overline{G} = G/Z(D)$. Observe that D centralizes $N_{u,gh}$ for any $h \in D$, so $Z(D)$ contains $N_{u,gh}$ for all $h \in D$. If $y \in G$ and $h \in D$, then $[y, gh] \in G' \leq N$ so that $[y,_{u+k}\, gh]$ belongs to the subgroup $[N,_{u+k-1}\, gh]$. Nonetheless, the elements in $[N,_{u+k-1}\, gh] = [[N,_u\, gh],_{k-1}\, gh]$ are trivial modulo $Z(D)$, meaning that $ghZ(D)$ is $(u+k)$-Engel in $\overline{G}$. By Lemma 2.2.5, the element $hZ(D) = (gZ(D))^{-1}ghZ(D)$ is l-Engel in $\overline{G}$ for $l = 2u+2k+1$ and any $h \in D$. Therefore, for any choice of $y \in G$ and $h \in D$ we conclude that $[y,_l\, h] \in Z(D)$ so that $[y,_{l+1} h] = 1$, and thus D is m-Engel for $m = l + 1 = 2u + 2k + 2$. For $\mathbf{g} \in \mathbf{G}$ we denote as $\overline{D}$ the image of D in $G/C_{\mathbf{g}}$, more explicitly the quotient $DC_{\mathbf{g}}/C_{\mathbf{g}}$. Clearly, $\overline{D}$ acts on $N_{\mathbf{g}}$ by the action given by $gC_{\mathbf{g}} \cdot x = x^g$ where $g \in G$ and $x \in N_{\mathbf{g}}$. Consider the semidirect product $\widetilde{G} = N_{\mathbf{g}} \rtimes \overline{D}$ with respect to the above group action. Plainly, $N_{\mathbf{g}}$ is a normal subgroup of $\widetilde{G}$ whose index

$$|\widetilde{G} \colon N_{\mathbf{g}}| = |N_{\mathbf{g}}\overline{D} \colon N_{\mathbf{g}}| = |\overline{D} \colon N_{\mathbf{g}} \cap \overline{D}| = |\overline{D}| \leq |G \colon C_{\mathbf{g}}|$$

is n-bounded. Besides, $\widetilde{G}$ is also metabelian because it has the abelian normal subgroup $N_{\mathbf{g}}$ and the quotient $\widetilde{G}/N_{\mathbf{g}} \cong \overline{D}$ is abelian. In fact, $\overline{D}$ is a subgroup of $G/C_{\mathbf{g}}$ which is abelian due to the facts that $G' \leq N \leq C_{\mathbf{g}}$ and $G/C_{\mathbf{g}} \cong (G/G')/(C_{\mathbf{g}}/G')$. By construction, any element of $\widetilde{G}$ is a product between an element of $N_{\mathbf{g}}$ and another one of $\overline{D}$, which implies that for any $x \in \widetilde{G}$ there exist $a \in N_{\mathbf{g}}$ and $b \in \overline{D}$ such that $x \in \langle a, b \rangle$. In the one hand, all elements of $N_{\mathbf{g}}$ are m-Engel in $\widetilde{G}$ since its abelianity and normality in $\widetilde{G}$ imply that

$$[x, a, a] = [[x, a], a] = 1$$

holds for any $x \in \widetilde{G}$ and $a \in N_{\mathbf{g}}$. On the other hand, D is m-Engel in G so $[x,_m y]$ is trivial for any $x \in N_{\mathbf{g}}$ and $y \in D$. Observe that the element $[x,_m y]$ is a finite product of the form

$$\prod_i (x^{\varepsilon_i})^{y^{\alpha_i}}$$

for some $\varepsilon_i \in \{\pm 1\}$ and $\alpha_i \in \mathbb{Z}$, so by the way the action in $\widetilde{G}$ is defined, the element $[x,_m yC_{\mathbf{g}}]$ is just represented by $[x,_m y]$. Therefore, $\overline{D}$ is m-Engel as well in $\widetilde{G}$ and by Lemma 2.2.5 $\widetilde{G}$ itself is $(2m+1)$-Engel. Now we are in a position where we know that $\widetilde{G}$ is nilpotent of (k, n)-bounded class, say c. In particular,

$$[x, y_1C_{\mathbf{g}}, \dots, y_cC_{\mathbf{g}}] = 1 \text{ for all } x \in N_{\mathbf{g}} \text{ and } y_1, \dots, y_c \in D.$$

But as before, $[x, y_1C_{\mathbf{g}}, \dots, y_cC_{\mathbf{g}}]$ is represented by $[x, y_1, \dots, y_c]$, meaning that $N_{\mathbf{g}} \leq Z_c(D)$ for each $\mathbf{g} \in \mathbf{G}$. In particular, since $G' \leq N$ it is true that $[G', g_1, \dots, g_{k-1}] \leq Z_c(D)$ for any choice of $g_1, \dots, g_{k-1} \in G$ and consequently $\gamma_{k+1}(G) \leq Z_c(D)$. Thus, the quotient $D/Z_c(D)$ is nilpotent of class at most k so that it is possible to glue the upper central series of D up to $Z_c(D)$ with the series of subgroups that correspond to the upper central series of $D/Z_c(D)$ in D. It is clear the obtained series is central in D so the subgroup D is nilpotent of class at most $k + c$, which is (k, n)-bounded. □

2.4 The set $FC_k(G)$

Observe that although the following lemma is taken from [12], the third bound is different. In fact, there is no clear indication that the bound given in [12] is correct.

Lemma 2.4.1. *[12, Lemma 2.5] Let G be a group and a an element of G such that $|a|_k \leq n$. Then,*

(i) $|G\colon C_G(b)| \leq n$ for any $b \in X_{k-1}(a)$,

(ii) $|G\colon C_G(d)| \leq n^2$ for any $d \in X_k(a)$ and

(iii) $|G\colon C_G(X_k(a))| \leq n^{2n}$ and $|X_k(a)^G| \leq n^3$.

Proof. (i) By assumption the set $X_k(a) = \{x^{-1}x^y \mid x \in X_{k-1}(a),\ y \in G\}$ has at most n elements thus

$$|G\colon C_G(b)| = |b^G| = |b|_1 \leq |X_k(a)| \leq n$$

for any $b \in X_{k-1}(a)$.

(ii) It is clear by definition that an element $d \in X_k(a)$ is the product of the inverse of an element in $X_{k-1}(a)$ and a conjugate of such element so

$$|G\colon C_G(d)| = |d^G| \leq |(b^{-1})^G| \cdot |(b^g)^G| = |b^G| \cdot |b^G| \leq n \cdot n = n^2$$

where $b \in X_{k-1}(a)$ and $g \in G$.

(iii) By the previous item we know that elements in $X_k(a)$ have at most n^2 conjugates so

$$|G\colon C_G(X_k(a))| \leq \prod_{x \in X_k(a)} |G\colon C_G(x)| \leq (n^2)^n = n^{2n}$$

and also

$$|X_k(a)^G| \leq \sum_{x \in X_k(a)} |x^G| \leq n \cdot n^2 = n^3.$$

□

The next lemma is a particular case of Lemma 3.1.2, which we will see in Chapter 3.

Lemma 2.4.2. *Let G be a group and $a, b, g_1, \ldots, g_k \in G$. Then, there exist elements $a_0 \in a^G$ and $u_1, \ldots, u_k, v_1, \ldots, v_k \in G$ such that*

$$[ab, g_1, \ldots, g_k] = [a_0, u_1, \ldots, u_k][b, v_1, \ldots, v_k].$$

Proof. We proceed by induction on k. The base case $k = 1$ is satisfied since $[ab, g_1] = [a, g_1]^b[b, g_1] = [a^b, g_1^b][b, g_1]$. We assume the statement is true for $k - 1$. Then,

$$\begin{aligned}[ab, g_1, \ldots, g_k] &= [[ab, g_1, \ldots, g_{k-1}], g_k] \\ &= [[a_0, u_1, \ldots, u_{k-1}][b, v_1, \ldots, v_{k-1}], g_k]\end{aligned}$$

for some $a_0 \in a^G$ and $u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1} \in G$ and using the well-known identity again

$$[ab, g_1, \ldots, g_k] = [a_0, u_1, \ldots, u_{k-1}, g_k]^{[b,g_1,\ldots,g_{k-1}]}[b, g_1, \ldots, g_{k-1}, g_k]$$

we get the result. □

Lemma 2.4.3. *[12, Lemma 5.1] Let G be a group and $a, b \in FC_k(G)$. Then, $|a^{-1}|_k = |a|_k$ and $|ab|_k \leq |a|_k^3|b|_k$.*

Proof. If $x, y \in G$, then

$$[y^{-1}, x] = yx^{-1}y^{-1}xyy^{-1} = y(y^{-1}x^{-1}yx)^{-1}y^{-1} = ([y, x]^{-1})^{y^{-1}}. \tag{2.3}$$

We claim that for any $x_1, \ldots, x_k \in G$ there exist elements $t_1, \ldots, t_k \in G$ such that

$$[a^{-1}, x_1, \ldots, x_k] = [a, t_1, \ldots, t_k]^{-1}. \tag{2.4}$$

The base case $k = 1$ is trivially true due to (2.3). Assume it is true for the case equal k. Let $x_1, \ldots, x_{k+1} \in G$. Using the induction hypothesis

$$[a^{-1}, x_1, \ldots, x_k, x_{k+1}] = [[a^{-1}, x_1, \ldots, x_k], x_{k+1}] = [[a, t_1, \ldots, t_k]^{-1}, x_{k+1}]$$

for some $t_1, \ldots, t_k \in G$ and we can apply again (2.3) to get that

$$[a^{-1}, x_1, \ldots, x_k, x_{k+1}] = [[a, t_1, \ldots, t_k], x_{k+1}^{[a,t_1,\ldots,t_k]^{-1}}]^{-1}$$

so we let $t_{k+1} = x_{k+1}^{[a,t_1,\ldots,t_k]^{-1}}$ and we are done. From statement (2.4) we deduce that the containment $X_k(a^{-1}) \subseteq X_k(a)^{-1}$ holds so $|X_k(a^{-1})| \leq |X_k(a)^{-1}|$. But if we apply this last inequality to the element a^{-1} we also get that $|X_k(a)| \leq |X_k(a^{-1})^{-1}| = |X_k(a^{-1})|$ holds, so the equality $|a^{-1}|_k = |a|_k$ is proved.

By Lemma 2.4.2 if $g_1, \ldots, g_k \in G$, then

$$[ab, g_1, \ldots, g_k] = [a_0, u_1, \ldots, u_k][b, v_1, \ldots, v_k]$$

for some $a_0 \in a^G$ and $u_1, \ldots, u_k, v_1, \ldots, v_k \in G$. As a consequence of Lemma 2.4.1, the factor $[a_1, u_1, \ldots, u_k]$ above can take at most $|a|_k^3$ values, while by hypothesis $[b, v_1, \ldots, v_k]$ can take at most $|b|_k$ values. □

Lemma 2.4.4. *[12, Lemma 5.2] Let G be a group and S a generating set of G. Then*

$$\gamma_{k+1}(G) = \langle X_k(g)^G \mid g \in S \rangle.$$

Proof. Containment $\supseteq$ follows from the definition of $\gamma_{k+1}(G)$ and its normality.

Let $N = \langle X_k(g)^G \mid g \in S \rangle$. Note that N is clearly normal and $[g, x_1, \dots, x_k] = 1 \pmod N$ for any $g \in S$ and $x_1, \dots, x_k \in G$. Consequently, gN belongs to $Z_k(G/N)$ for all $g \in S$, so $G/N = Z_k(G/N)$ and therefore G/N is nilpotent of class at most k. Then, $\gamma_{k+1}(G) \subseteq N$. □

Lemma 2.4.5. *[12, Lemma 5.3] Let G be a finitely generated group, namely by the elements $a_1, \dots, a_r \in G$, and N an abelian normal subgroup of G. If $|a_i|_k \leq n$ for all $i \in \{1, \dots, r\}$, then $[N,_k G]$ has finite (k, n, r)-bounded order.*

Proof. We start by working on the case $k = 1$. In this case G is generated by r elements whose centralizers have index at most n, since $|g|_1 = |g^G| = |G\colon C_G(g)|$ for any $g \in G$. Observe that the equality $Z(G) = \bigcap_{i=1}^r C_G(a_i)$ and assumptions of the lemma imply that

$$|G\colon Z(G)| \leq \prod_{i=1}^{r} |G\colon C_G(a_i)| \leq n^r.$$

Schur's theorem ensures G' has finite (n, r)-bounded order so $[N, G] \subseteq [G, G] = G'$ is finite. Assume now $k \geq 2$. We set $A = \{a_1, \dots, a_r\}$ and we denote the cartesian product of k copies of A as $\mathbf{A}$. For each $\mathbf{b} = (b_1, \dots, b_k) \in \mathbf{A}$ we define the set $N_{\mathbf{b}} = [N, b_1, \dots, b_k] = \{[x, b_1, \dots, b_k] \mid x \in N\}$. As in the proof of Theorem 2.3.3, each $N_{\mathbf{b}}$ is a subgroup of G. Since N is an abelian normal subgroup of G, then $[n_1, b] = n_1^{-1} n_1^b = n_1^b n_1^{-1} = [b, n_1^{-1}]$ is true for any $n_1 \in N$ and $b \in A$, which implies that

$$N_{\mathbf{b}} = [b_1, N, b_2, \dots, b_k] = \{[b_1, x, b_2, \dots, b_k] \mid x \in N\} \subseteq X_k(b_1) \qquad (2.5)$$

has order at most n. We consider the cartesian product $N_0 = \prod_{\mathbf{b} \in \mathbf{A}} \mathrm{ncl}_G(N_{\mathbf{b}})$. Note here that $N_{\mathbf{b}}$ is contained in N and that N is normal in G so $\mathrm{ncl}_G(N_{\mathbf{b}})$ is also contained in N. Since N is abelian, $\mathrm{ncl}_G(N_{\mathbf{b}})$ commutes with $\mathrm{ncl}_G(N_{\mathbf{c}})$ for all $\mathbf{b}, \mathbf{c} \in \mathbf{A}$, so N_0 is a subgroup of N. Moreover, $\mathrm{ncl}_G(N_{\mathbf{b}})$ is equal to

$$\prod_{g \in G} N_{\mathbf{b}}^g = \langle x_1^{g_1} \cdots x_s^{g_s} \mid s \geq 0,\ x_1, \dots, x_s \in N_{\mathbf{b}},\ g_1, \dots, g_s \in G \rangle,$$

which means that $\mathrm{ncl}_G(N_{\mathbf{b}})$ is an abelian subgroup generated by at most n^3 elements due to Lemma 2.4.1. If $x \in N_{\mathbf{b}}$, then x has order at most n because $N_{\mathbf{b}}$ is a finite group of order less or equal than n, thus any generator of $\mathrm{ncl}_G(N_{\mathbf{b}})$ has order at most n too. For this reason, it follows that the order of $\mathrm{ncl}_G(N_{\mathbf{b}})$ is at most n^{n^3} and consequently the order of N_0 is no more than $(n^{n^3})^{r^k} = n^{n^3 r^k}$, i.e. N_0 has (k, n, r)-bounded order. Since A generates G and by the way N_0 has been constructed, any element of the form $[g, x_1, \ldots, x_k]$ with $g \in N$ and $x_1, \ldots, x_k \in G$ is trivial modulo N_0. Therefore, $[N,_k G]$ is contained in N_0 and $[N,_k G]$ has (k, n, r)-bounded order. □

Theorem 2.4.6. *[12, Theorem 1.4] Let G be a finitely generated group, namely by the elements $a_1, \ldots, a_r \in G$. If $|a_i|_k \leq n$ for all $i \in \{1, \ldots, r\}$, then $\gamma_{k+1}(G)$ has finite (k, n, r)-bounded order.*

Proof. By Lemma 2.4.4 we know $N = \gamma_{k+1}(G) = \langle X_k(a_i)^G \mid i \in \{1, \ldots, r\}\rangle$ holds so it follows that $C_G(N) = \cap_{i=1}^{r} C_G(X_k(a_i)^G)$. By Lemma 2.4.1 the size of the conjugates of the sets $X_k(a_i)$ with $i \in \{1, \ldots, r\}$ is finite and bounded by n^3 and also

$$|G \colon C_G(X_k(a_i)^G)| \leq \prod_{x \in X_k(a)^G} |G \colon C_G(x)| \leq (n^2)^{n^3} = n^{2n^3}$$

so consequently

$$|G \colon C_G(N)| \leq \prod_{i=1}^{r} |G \colon C_G(X_k(a_i)^G)| \leq (n^{2n^3})^r = n^{2rn^3}.$$

Then $C_G(N)$ has finite (k, n, r)-bounded index in G. Obviously, we can write the center of N as the intersection between N and the centralizer of N in G, so by the third isomorphism theorem the index $|N \colon Z(N)| = |N \colon N \cap C_G(N)| = |NC_G(N) \colon C_G(N)| \leq |G \colon C_G(N)|$ is (k, n, r)-bounded and Schur's theorem implies N' has finite (k, n, r)-bounded order. It follows we can work modulo N' and assume N is abelian, so we are under the assumptions of Lemma 2.4.5 and conclude $[N,_k G]$ has finite (k, n, r)-bounded order. In the same way, we pass to the quotient $G/[N,_k G]$ and without loss of generalization assume $[N,_k G] = 1$. Recalling the definition of N, we note $1 = [N,_k G] = \gamma_{2k+1}(G)$ so G is nilpotent of class at most $2k$. By induction on the nilpotency class of G, it is possible to prove that $NZ(G)/Z(G)$ has finite (k, n, r)-bounded order. Indeed if G has class at most k, the subgroup N is trivial and the statement is obvious. Otherwise if the class of G is $l > k$, the class of the quotient $G/Z(G)$ is $l - 1$ so by induction hypothesis $\gamma_{k+1}(G/Z(G)) = NZ(G)/Z(G)$ has (k, n, r)-bounded order. As it is proved in [7], in this situation $\gamma_{k+2}(G)$ has finite (k, n, r)-bounded order and therefore we can factor it out. Then the assumption $1 = \gamma_{k+2}(G) = [N, G]$ implies we can assume that N is contained in the center of G. It is straightforward then that N is generated by the bounded finite set $\cup_{i=1}^{r} X_k(a_i)$. Dietzmann's theorem shows it suffices to prove that

$\cup_{i=1}^r X_k(a_i)$ is torsion in order to conclude that N has finite (k, n, r)-bounded order. Let $a \in \{a_1, \dots, a_r\}$ and $g_1, \dots, g_k \in G$. Our claim is that

$$[a, g_1, \dots, g_k]^i = [a, g_1, \dots, g_k^i] \in X_k(a)$$

holds for any $i \geq 1$, which shows that $X_k(a)$ is closed under taking powers of its elements and therefore any element of $X_k(a)$ has order at most n. Arguing by induction on i, the base case is trivial so we assume it is true for $i-1$ and we try to prove for i. Using the identity $[x, yz] = [x, z][x, y][x, y, z]$ for any $x, y, z \in G$ we have

$$[a, g_1 \dots, g_k^i] = [a, g_1, \dots, g_k][a, g_1, \dots, g_k^{i-1}][a, g_1, \dots, g_k^{i-1}, g_k],$$

but since $\gamma_{k+2}(G)$ is trivial the last term above vanishes and the induction hypothesis proves the claim. □

Theorem 2.4.7. *[12, Theorem 1.3] Let G be a group. Then, the set $FC_k(G)$ is a subgroup of G and $\gamma_{k+1}(FC_k(G))$ is locally finite and normal.*

Proof. Let $T = FC_k(G)$. It follows directly from Lemma 2.4.3 that T is a subgroup of G. In order to prove that $\gamma_{k+1}(T)$ is locally normal, we need to find a finite normal subgroup N of $\gamma_{k+1}(T)$ for each finite subset E contained in $\gamma_{k+1}(T)$ such that N contains E. Choose a finite subset $E \subseteq \gamma_{k+1}(T)$. It is evident there exist elements $a_1, \dots, a_r \in T$ such that $E \subseteq \gamma_{k+1}(\langle a_1, \dots, a_r \rangle)$ by finiteness of E. If we set $A = \langle a_1, \dots, a_r \rangle$, Theorem 2.4.6 shows $\gamma_{k+1}(A)$ is finite and Lemma 2.4.4 gives a useful description of $\gamma_{k+1}(A)$, namely

$$\gamma_{k+1}(A) = \langle X_k(a_i)^A \mid i \in \{1, \dots, r\} \rangle.$$

By Lemma 2.4.1 the subgroup $\gamma_{k+1}(A)$ is generated by elements whose centralizers have finite index in G and therefore they have finitely many conjugates in G. Consider the normal closure of $\gamma_{k+1}(A)$ in G that can be written as

$$N = \mathrm{ncl}_G(\gamma_{k+1}(A)) = \langle X_k(a_i)^G \mid i \in \{1, \dots, r\} \rangle.$$

Observe that the set $\cup_{i=1}^r X_k(a_i)^G$ is finite, closed under conjugation and torsion because the elements in $\cup_{i=1}^r X_k(a_i)$ are contained in the finite subgroup $\gamma_{k+1}(A)$. By Dietzmann's theorem, N is finite so E is contained in a finite normal subgroup of G. □

Lemma 2.4.8. *Let G be a profinite group. Then the subset*

$$\Delta_n = \{x \in G \mid |x|_k \leq n\}$$

is closed.

Proof. Firstly, we fix some $l \in \mathbb{N}$ and define the subset

$$Y = \{(x_1, \dots, x_l) \in G^l \mid x_i = 1 \text{ for some } 1 \leq i \leq l\} \subseteq G^l.$$

If π_i denotes the i-th canonical projection map from G^l to G, we can write Y as the union of all the kernels of the projection maps $\pi_1, \dots, \pi_l$, which implies that Y is closed. On the other hand, for any tuple $\mathbf{y} = (y_{11}, \dots, y_{1k}, \dots, y_{(n+1)1}, \dots, y_{(n+1)k}) \in G^{k(n+1)}$ we consider the continuous map $\varphi_{\mathbf{y}} \colon G \to G^l$ with $l = \binom{n+1}{2}$ such that $x \in G$ is mapped to the vector

$$([x, y_{11}, \dots, y_{1k}][x, y_{21}, \dots, y_{2k}]^{-1}, \dots,$$
$$[x, y_{n1}, \dots, y_{nk}][x, y_{(n+1)1}, \dots, y_{(n+1)k}]^{-1}).$$

Note that this map is continuous since it does only use the continuous group operations. Our claim now is that

$$\Delta_n = \bigcap_{\mathbf{y} \in G^{k(n+1)}} \varphi_{\mathbf{y}}^{-1}(Y)$$

holds. Observe first that if we pick an element $x \in \Delta_n$, there are at most n different γ_{k+1}-values starting with the element x, so if we consider any vector $\mathbf{y} \in G^{k(n+1)}$ we have that $\varphi_{\mathbf{y}}(x)$ must be in Y. Therefore, x lies in the intersection of all $\varphi_{\mathbf{y}}^{-1}(Y)$ with $\mathbf{y} \in G^{k(n+1)}$. For the other inclusion, take some element x in the intersection above. By contradiction, if $|x|_k \geq n+1$ we could consider $n+1$ different commutators $[x, y_{11}, \dots, y_{1k}], \dots, [x, y_{(n+1)1}, \dots, y_{(n+1)k}]$ with $y_{11}, \dots, y_{(n+1)k} \in G$, so that for $\mathbf{y} = (y_{11}, \dots, y_{(n+1)k}) \in G^{k(n+1)}$ we would have $\varphi_{\mathbf{y}}(x) \notin Y$ which is a contradiction. Thus, Δ_n is an intersection of preimages of a closed subset under a continuous map, hence it is closed as desired. □

It could be natural to think that the FC_k-center of a profinite group is also closed, but this is not true in general, not even for the FC-center.

Example 2.4.9. Let $\{S_n\}_{n \in \mathbb{N}}$ be a family of non-abelian finite simple groups and consider its cartesian product $G = \prod_{n \in \mathbb{N}} S_n$. Plainly, an element of G lies in the FC-center of G if and only if it has finitely many non-trivial entries, which is the same as saying that the FC-center of G equals the direct product of the family $\{S_n\}_{n \in \mathbb{N}}$. In addition, if $FC(G)$ was closed, for any $x \notin FC(G)$ it would exist an element $g \in G$ and an open normal subgroup N of G such that $x \in gN$ and $gN \cap FC(G) = \emptyset$. Since each group S_n is simple, the only option for N is either the trivial subgroup or G itself. However, the trivial subgroup has not finite index because G is an infinite group so it is not open, and the whole group has not empty intersection with $FC(G)$. Thus, $FC(G)$ is not closed in this case.

2.5 Profinite FC_k-groups are finite-by-nilpotent

Theorem 2.5.1. *[12, Theorem 1.5] Let G be a profinite group. If G is an FC_k-group, then $\gamma_{k+1}(G)$ is finite.*

Proof. From Lemma 2.4.8 we have an ascending chain of closed subsets

$$\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3 \subseteq \ldots \subseteq \Delta_n \subseteq \ldots$$

whose union is the entire group G. With this setting, we can invoke Baire category theorem so that there exists some $n \in \mathbb{N}$ such that Δ_n has non-empty interior. By definition of the profinite topology, there exist some element $g \in G$ and an open normal subgroup N of G such that gN is contained in Δ_n. Moreover, it is clear that any element of N can be written as a product $x^{-1}y$ where $x, y \in gN$. Thus, Lemma 2.4.3 shows that $|z|_k \leq n^4$ for any $z \in N$. Let $T = \{t_1, \ldots, t_r\}$ be a transversal of N in G, and denote as m the maximum of $|t_1|_k, \ldots, |t_r|_k$, which is finite because G is an FC_k-group. As any element x of G is a product of an element of T and an element of N, we can apply Lemma 2.4.3 again and get that $|x|_k \leq m^3 n^4$, so by Theorem 2.3.1 $\gamma_{k+1}(G)$ has finite order. □

It is a straightforward observation that the converse of the theorem holds as well. Actually, a more general statement holds.

Proposition 2.5.2. *Let G be a finite-by-nilpotent group. Then, there exists some $k \in \mathbb{N}$ such that $|g|_k$ is finite for all $g \in G$.*

Proof. By assumption there exists a finite normal subgroup N of G such that the quotient G/N is nilpotent, say of nilpotency class k. Thus, we have that $[g, g_1, \ldots, g_k] \in N$ for all $g, g_1, \ldots, g_k \in G$ so finiteness of N implies that $|g|_k$ is finite for all $g \in G$. □

Chapter 3

From finite to less than the continuum

3.1 Set of values of the generalized word $[g, x_1, \ldots, x_k]$

We introduce useful notation that we will be using in this chapter.

Notation 3.1.1. Let G be a group, $g \in G$, $A_1, \ldots, A_k, B_1, \ldots, B_k \subseteq G$ and $I \subseteq \{1, \ldots, k\}$. Let $(a_i)_{i \in I} \in G^I$ and $(b_i)_{i \in \bar{I}} \in G^{\bar{I}}$ where $\bar{I} = \{1, \ldots, n\} \setminus I$ and G^I stands for the cartesian product of $|I|$ copies of G. We will use the following notation:

$$X_k(A_1, \ldots, A_k) = \{[a_1, \ldots, a_k] \mid a_1 \in A_1, \ldots, a_k \in A_k\}.$$

$$X_k(g, A_1, \ldots, A_k) = \{[g, a_1, \ldots, a_k] \mid a_1 \in A_1, \ldots, a_k \in A_k\}.$$

$$\omega_{k,I}(a_i; b_i) = [x_1, \ldots, x_k],\ x_i = a_i \text{ if } i \in I \text{ and } x_i = b_i \text{ otherwise.}$$

$$\omega_{k,I}(g, a_i; b_i) = [g, x_1, \ldots, x_k],\ x_i = a_i \text{ if } i \in I \text{ and } x_i = b_i \text{ otherwise.}$$

$$X_{k,I}(A_i; B_i) = \{\omega_{k,I}(a_i; b_i) \mid a_i \in A_i \text{ if } i \in I,\ b_i \in B_i \text{ if } i \in \bar{I}\}.$$

$$X_{k,I}(g, A_i; B_i) = \{\omega_{k,I}(g, a_i; b_i) \mid a_i \in A_i \text{ if } i \in I,\ b_i \in B_i \text{ if } i \in \bar{I}\}.$$

In some cases, whenever the length of the commutators is clear due to the context, we will drop the subscript k.

We adjust the statement and proof of [1, Lemma 2.4] to give the following lemma.

Lemma 3.1.2. *Let G be a group, H a normal subgroup of G, $g_1, \ldots, g_k \in G$, $h \in H$ and some fixed $s \in \{1, \ldots, k\}$. Then there exist some elements $b_1, \ldots, b_k \in H$ such that*

$$[g_1, \ldots, g_{s-1}, g_s h, g_{s+1}, \ldots, g_k] = [g_1^{b_1}, \ldots, g_k^{b_k}][g_1, \ldots, g_{s-1}, h, g_{s+1}, \ldots, g_k].$$

Proof. We proceed by induction on the length k of the commutator. The base case is clear since it suffices to consider the value $b_1 = 1$. Assume it is true for the commutator length $n-1$. We will cover two possible cases. If $s \neq k$, then by induction hypothesis there exist some elements $a_1, \ldots, a_{k-1} \in H$ such that $[g_1, \ldots, g_{s-1}, g_s h, g_{s+1}, \ldots, g_k]$ is equal to

$$[[g_1^{b_1}, \ldots, g_{k-1}^{b_{k-1}}][g_1, \ldots, g_{s-1}, h, g_{s+1}, \ldots, g_{k-1}], g_k]$$

and therefore we can rewrite it as

$$[g_1^{b_1}, \ldots, g_{k-1}^{b_{k-1}}, g_k]^{[g_1, \ldots, g_{s-1}, h, g_{s+1}, \ldots, g_{k-1}]}[g_1, \ldots, g_{s-1}, h, g_{s+1}, \ldots, g_k].$$

It is enough to note that since h belongs to the normal subgroup H then $[g_1, \ldots, g_{s-1}, h, g_{s+1}, \ldots, g_{k-1}]$ lies in H too. Now if $s = k$, then

$$\begin{aligned}[g_1, \ldots, g_{k-1}, g_k h] &= [g_1, \ldots, g_{k-1}, h][g_1, \ldots, g_{k-1}, g_k]^h \\ &= [g_1, \ldots, g_{k-1}, g_k]^{h[g_1, \ldots, g_{k-1}, h]^{-1}}[g_1, \ldots, g_{k-1}, h]\end{aligned}$$

so that again $h[g_1, \ldots, g_{k-1}, h]^{-1} \in H$ and the result is proved. □

Using [2, Lemma 2.5] as a reference, we can prove the next lemma.

Lemma 3.1.3. *Let G be a group, $A_1, \ldots, A_k$ and H normal subgroups of G, V a subgroup of G, $g \in G$ and $a_i \in A_i$ for $i \in \{1, \ldots, k\}$. Assume that*

$$X_k(a_1(H \cap A_1), \ldots, a_k(H \cap A_k)) \subseteq gV$$

holds. Then, for any proper subset I of $\{1, \ldots, k\}$ we have that

$$X_{k,I}(a_i(H \cap A_i); H \cap A_i) \subseteq V.$$

Proof. The proof is done by induction on $k - |I|$, so we start by assuming that $I = \{1, \ldots, k\} \setminus \{j\}$ for some index $j \in \{1, \ldots, k\}$. In order to shorten the notation, we will write H_i instead of $H \cap A_i$ when $i \in \{1, \ldots, k\}$. Let us consider the elements $g_1, \ldots, g_k$ where $g_i \in a_i H_i$ when $i \neq j$ and $g_j \in H_j$. By Lemma 3.1.2, there exist elements $b_1, \ldots, b_k \in H_j$ such that

$$\omega_{k,I}(g_i; a_i g_i) = \omega_{k,I}(g_i^{b_i}; a_i^{b_i})[g_1, \ldots, g_k]. \tag{3.1}$$

Observe that the element $\omega_{k,I}(g_i; a_i g_i)$ is by construction in the subset $X_k(a_1 H_1, \ldots, a_k H_k)$, but so is the element $\omega_{k,I}(g_i^{b_i}; a_i^{b_i})$. In fact, for $i \neq j$ there exists some $h_i \in H_i$ such that

$$g_i^{b_i} = (a_i h_i)^{b_i} = a_i[a_i, b_i]h_i^{b_i} \in a_i H_i,$$

and on the other hand

$$a_j^{b_j} = a_j[a_j, b_j] \in a_j H_j.$$

Since $X_k(a_1H_1, \ldots, a_kH_k) \subseteq gV$ holds by assumption, equality (3.1) gives

$$[g_1, \ldots, g_k] = \omega_{k,I}(g_i^{b_i}; a_i^{b_i})^{-1}\omega_{k,I}(g_i; a_ig_i) \in (gV)^{-1}(gV) = V$$

as we wanted. Now assume that $k - |I| \geq 2$ and let $I^* = I \cup \{j\}$ for some index $j \in \{1, \ldots, k\} \setminus I$. Consider the elements $g_1, \ldots, g_k$ where $g_i \in a_iH_i$ when $i \in I$ and $g_i \in H_i$ otherwise. Evidently, the element $[g_1, \ldots, g_{j-1}, a_jg_j, g_{j+1}, \ldots, g_k]$ belongs to the subset $X_{k,I^*}(a_iH_i; H_i)$ by construction. It is possible again to use Lemma 3.1.2 so that there are some elements $b_1, \ldots, b_k \in H_j$ such that

$$\omega_{\{1,\ldots,k\}\setminus\{j\}}(g_i; a_ig_i) = \omega_{\{1,\ldots,k\}\setminus\{j\}}(g_i^{b_i}; a_i^{b_i})[g_1, \ldots, g_k]. \tag{3.2}$$

Note that the element $\omega_{\{1,\ldots,k\}\setminus\{j\}}(g_i^{b_i}; a_i^{b_i})$ lies in $X_{k,I^*}(a_iH_i; H_i)$ since if $i \in I$ there exists some $h_i \in H_i$ such that

$$g_i^{b_i} = (a_ih_i)^{b_i} = a_i[a_i, b_i]h_i^{b_i} \in a_iH_i,$$

if $i \notin I^*$ we have that $g_i^{b_i} \in H_i$ and finally

$$a_j^{b_j} = a_j[a_j, b_j] \in a_jH_j.$$

By the induction hypothesis we have that $X_{k,I^*}(a_iH_i; H_i)$ is contained in the subgroup V, so from equality (3.2) we conclude that $[g_1, \ldots, g_k] \in V$ and the proof is complete. □

It is necessary to prove the following lemmas in order to prove our desired theorem afterwards.

Lemma 3.1.4. *Let G be a group. If H is a normal subgroup of G, V a subgroup of G and $x, g, g_1, \ldots, g_k \in G$ such that*

$$X_k(g, g_1H, \ldots, g_kH) \subseteq xV,$$

then

$$X_{k,I}(g, g_iH; H) \subseteq V \text{ for all } I \subseteq \{1, \ldots, k\}$$

and in particular,

$$X_k(g, H, \ldots, H) \subseteq V.$$

Proof. This lemma is a consequence of Lemma 3.1.3 when considering $A_1 = 1$ and $A_i = G$ for all $i \in \{2, \ldots, k+1\}$. □

Lemma 3.1.5. *Let G be a group and $I \subseteq \{1, \ldots, k\}$. If $A_1, \ldots, A_k$ and H are normal subgroups of G and $g \in G$ such that*

$$X_{k,J}(g, A_i; A_i \cap H) = 1 \text{ for all } J \subsetneq I,$$

then for any $(g_i)_{i\in I} \in \prod_{i\in I} A_i$ and $(h_i)_{i=1}^k \in \prod_{i=1}^k (A_i \cap H)$ we obtain that

$$\omega_I(g, g_ih_i; h_i) = \omega_I(g, g_i; h_i).$$

Proof. Let $s \in I$, $(g_i)_{i \in I} \in \prod_{i \in I} A_i$ and $(h_i)_{i=1}^k \in \prod_{i=1}^k (A_i \cap H)$. By Lemma 3.1.2 there exist some elements $b_0, \dots, b_k \in A_s$ such that

$$\begin{aligned}
\omega_I(g, g_i h_i; h_i) &= [g, c_1, \dots, g_s h_s, \dots, c_k] \\
&= [g, c_1, \dots, h_s^{g_s^{-1}} g_s, \dots, c_k] \\
&= [g^{b_0}, c_1^{b_1}, \dots, h_s^{g_s^{-1} b_s}, \dots, c_k^{b_k}][g, c_1, \dots, g_s, \dots, c_k] \\
&= [g, c_1^{b_1 b_0^{-1}}, \dots, h_s^{g_s^{-1} b_s b_0^{-1}}, \dots, c_k^{b_k b_0^{-1}}]^{b_0}[g, c_1, \dots, g_s, \dots, c_k],
\end{aligned}$$

where $c_i \in G$ denotes the corresponding element in the position $i \in \{1, \dots, k\}$ of the commutator. Note that in the first commutator of the last equality, in a position $i \in I \setminus \{s\}$ we have an element of A_i conjugated by some element, which gives back an element of A_i by normality of A_i in G. In a position $i \notin I$ we have an element of A_i again by normality of A_i, but it also belongs to H due to its normality. Finally, in the chosen position $s \in I$ we have an element of $A_s \cap H$ conjugated by another element so similarly it lies in $A_s \cap H$. This means that the commutator is in $X_{k, I \setminus \{s\}}(g, A_i; A_i \cap H)$ which by assumption is trivial so

$$\omega_I(g, g_i h_i; h_i) = [g, c_1, \dots, g_s, \dots, c_k].$$

Repeating the argument for each $i \in I$ we manage to remove every h_i whenever $i \in I$ and we are done. □

Lemma 3.1.6. *Let G be a profinite group. Let H be an open normal subgroup of G and V a normal subgroup of G such that there exist*

(i) *an element $g \in G$ such that $|g|_k < 2^{\aleph_0}$ and*

(ii) *a subset $I \subseteq \{1, \dots, k\}$ such that $X_{k,J}(g, G; H) \subseteq V$ for all $J \subsetneq I$.*

Then for any fixed $\boldsymbol{g} = (g_i)_{i \in I} \in G^I$ there exists an open normal subgroup $U_{\boldsymbol{g}}$ of H such that

$$X_{k,I}(g, g_i; U_{\boldsymbol{g}}) \subseteq V.$$

Proof. We consider any $\mathbf{g} = (g_i)_{i \in I} \in G^I$ and define the continuous map

$$\begin{aligned}
\varphi_{\mathbf{g}} \colon H^k &\to G \\
(h_i)_{i=1}^k &\mapsto \omega_I(g, g_i h_i; h_i).
\end{aligned}$$

By Lemma 1.1.11, this map must be locally constant since

$$|\varphi_{\mathbf{g}}(H^k)| = |X_{k,I}(g, gH; H)| \le |g|_k < 2^{\aleph_0},$$

meaning that there exists some non-empty open $\widetilde{U} \subseteq H^k$ in which $\varphi_{\mathbf{g}}$ is constant. In addition, H is open in G thus H is a profinite group endowed with the profinite topology inherited from G so H^k inherits the profinite topology coming from G^k. Therefore the non-empty open $\widetilde{U}$ of H^k contains a basic open which is of the form $b_1U_1 \times \ldots \times b_kU_k$ for some $b_1, \ldots, b_k \in G$ and $U_1, \ldots, U_k \trianglelefteq_o G$. In particular, $\varphi_{\mathbf{g}}$ is constant in the basic open $b_1U \times \ldots \times b_kU$ where $U = \cap_{i=1}^k U_i \trianglelefteq_o G$, meaning that

$$X_{k,I}(g, g_ib_iU; b_iU) \subseteq \omega_I(g, g_ib_i; b_i) \cdot 1.$$

But we are now able to use Lemma 3.1.4 with the trivial subgroup, the normal subgroup U in G and the elements $\omega_I(g, g_ib_i; b_i), g$ and g_ib_i when $i \in I$ and b_i otherwise, which implies that $X_I(g, g_ib_iU; U) = 1$. On the other hand, we can rewrite the second condition stated in the hypothesis of the lemma modulo the normal subgroup V so we have

$$X_{k,J}(g, G; H) = 1 \pmod V \text{ for all } J \subsetneq I.$$

Therefore, using Lemma 3.1.5 in the quotient group G/V with the normal subgroups $A_i = G/V$ for $i \in \{1, \ldots, k\}$ and HV/V, we get

$$\omega_I(g, x_ih_i; h_i) = \omega_I(g, x_i; h_i) \pmod V \text{ for all } x_i \in G, h_i \in HV,$$

for $i \in \{1, \ldots, k\}$. Specifically, since $b_iU \subseteq H$ for $i \in \{1, \ldots, k\}$,

$$1 = X_I(g, g_ib_iU; U) = X_I(g, g_i; U) \pmod V$$

and therefore

$$X_{k,I}(g, g_i; U_{\mathbf{g}}) \subseteq V$$

with $U_{\mathbf{g}} = U$ as claimed. □

At this point, we are ready to prove this theorem.

Theorem 3.1.7. *Let G be a profinite group and $g \in G$ an element such that $|g|_k < 2^{\aleph_0}$. Then $|g|_k$ is actually finite.*

Proof. We start by constructing a continuous map

$$\begin{aligned} \varphi_g \colon G^k &\to G \\ (g_i)_{i=1}^k &\mapsto [g, g_1, \ldots, g_k], \end{aligned}$$

which is locally constant by Lemma 1.1.11 because $|\varphi(G^k)| = |g|_k < 2^{\aleph_0}$. Arguing as in the proof above, there exist some $H \trianglelefteq_o G$ and $g_1, \ldots, g_k \in G$ such that φ_g is constant on $g_1H \times \ldots \times g_kH$, which means that

$$[g, g_1h_1, \ldots, g_kh_k] = [g, g_1, \ldots, g_k] \text{ for all } h_1, \ldots, h_k \in H.$$

Thus, applying Lemma 3.1.4 we have that $X_k(g, H, \ldots, H) = 1$. We claim the following: for any subset $I \subsetneq \{1, \ldots, k\}$ there exists some open normal subgroup $U_I \trianglelefteq_o G$ such that $X_{k,I}(g, G; U_I) = 1$. If this is the case, we can consider the open normal subgroup

$$W = \bigcap_{I \subsetneq \{1,\ldots,k\}} U_I$$

and we will have

$$X_{k,I}(g, G; W) = 1 \text{ for all } I \subsetneq \{1, \ldots, k\}.$$

Then all the conditions asked in Lemma 3.1.5 are satisfied so

$$[g, g_1 w_1, \ldots, g_k w_k] = [g, g_1, \ldots, g_k]$$

holds for any $(g_i)_{i=1}^k \in G^k$ and $(w_i)_{i=1}^k \in W^k$. In particular, if T is a left transversal of W in G, which is finite because open subgroups have finite index in profinite groups, then

$$\begin{aligned} X_k(g) &= \{[g, g_1, \ldots, g_k] \mid g_1, \ldots, g_k \in G\} \\ &= \{[g, t_1 w_1, \ldots, t_k w_k] \mid t_1, \ldots, t_k \in T,\ w_1, \ldots, w_k \in W\} \\ &= \{[g, t_1, \ldots, t_k] \mid t_1, \ldots, t_k \in T\} \end{aligned}$$

is finite. So it suffices to prove the claim.

By induction on the cardinal of the subset I, we need to prove that there exists $U_I \trianglelefteq_o G$ such that $X_{k,I}(g, G; U_I) = 1$. In case $I = \emptyset$, we have already seen that $X_k(g, H, \ldots, H) = 1$ so it suffices to take $U_I = H$. Let us assume now that the subset I is non-empty. Using the induction hypothesis, for each $J \subsetneq I$ there exists some $U_J \trianglelefteq_o G$ such that $X_{k,J}(g, G; U_J) = 1$. Then we can consider

$$U = \bigcap_{J \subsetneq I} U_J$$

which is again an open normal subgroup of G such that

$$X_{k,J}(g, G; U) = 1 \text{ for all } J \subsetneq I. \tag{3.3}$$

Let R be a (finite) left transversal of U in G. If we pick any $\mathbf{r} = (r_i)_{i \in I} \in R^I$, we can apply Lemma 3.1.6 so there exists some open normal subgroup $U_{\mathbf{r}}$ of U where $X_{k,I}(g, r_i; U_{\mathbf{r}}) = 1$. Observe that $U_{\mathbf{r}}$ is also open in G so there exist some $x_{\mathbf{r}} \in G$ and $N_{\mathbf{r}} \trianglelefteq_o G$ such that $x_{\mathbf{r}} N_{\mathbf{r}} \subseteq U_{\mathbf{r}}$. In particular, $x_{\mathbf{r}} \in U_{\mathbf{r}}$ hence $N_{\mathbf{r}} \subseteq U_{\mathbf{r}}$. We now set

$$U_I = \bigcap_{\mathbf{r} \in R^I} N_{\mathbf{r}} \trianglelefteq_o G$$

and check that it is exactly what we needed. It is clear by construction that $X_I(g, r_i; U_I) = 1$ for any $\mathbf{r} = (r_i)_{i \in I} \in R^I$. From (3.3) using Lemma 3.1.5 we can deduce that

$$\omega_I(g, r_i u_i; u_i) = \omega_I(g, r_i; u_i)$$

for any choice of $(r_i)_{i \in I} \in R^I$ and $(u_i)_{i=1}^k \in U^k$, and therefore

$$X_I(g, r_i U; U_I) = X_I(g, r_i; U_I) = 1$$

holds for any $(r_i)_{i \in I} \in R^I$. So we are done since all the left cosets $r_i U$ with $r_i \in R$ cover the whole group G, hence $X_I(g, G; U_I) = 1$. □

3.2 Profinite (generalized) FC_k-groups

Overall we can put everything together and get one of our main results.

Theorem 3.2.1. *Let G be a profinite group such that $|g|_k < 2^{\aleph_0}$ for all $g \in G$. Then, $\gamma_{k+1}(G)$ is finite.*

Proof. From Theorem 3.1.7 it follows that $|g|_k$ is finite for all $g \in G$ so by Theorem 2.5.1 we conclude that $\gamma_{k+1}(G)$ has finite order. □

In order to analyze the case where there is no uniform bound on the length of the commutators, we need a couple of results on the $FC_k(G)$ subset.

Proposition 3.2.2. *Let G be a group. Then, $FC_k(G)$ is contained in $FC_{k+1}(G)$.*

Proof. Let g be an element of $FC_k(G)$. Then $|g|_k$ is finite, i.e. there are only finitely many elements of G that can be written as commutators of length $k+1$ starting with the fixed element g. Note that

$$[g, g_1, \ldots, g_{k+1}] = [g, g_1, \ldots, g_k]^{-1}[g, g_1, \ldots, g_k]^{g_{k+1}} \tag{3.4}$$

holds for any $g_1, \ldots, g_{k+1} \in G$. In the one hand, since the set $X_k(g)$ is finite by hypothesis, the set $X_k(g)^{-1}$ is finite as well. On the other hand, from Lemma 2.4.1 it follows that an element of the form $[g, g_1, \ldots, g_k]$ with $g_1, \ldots, g_k \in G$ has finitely many conjugates. Therefore equality (3.4) implies there are only finitely many elements in $X_{k+1}(g)$. □

Corollary 3.2.3. *Let G be a group. Then, $FC_{k_1}(G)$ is contained in $FC_{k_2}(G)$ whenever k_1 and k_2 are positive integers such that $k_1 \leq k_2$.*

Proof. Assume $k_1 \leq k_2$. By Proposition 3.2.2 we can inductively form the chain

$$FC_{k_1}(G) \subseteq FC_{k_1+1}(G) \subseteq \ldots \subseteq FC_{k_2}(G),$$

and thus $FC_{k_1}(G) \subseteq FC_{k_2}(G)$. □

Theorem 3.2.4. *Let G be a profinite group. Assume that for each element $g \in G$ there exists a natural number $k_g \in \mathbb{N}$ such that $|g|_{k_g} < 2^{\aleph_0}$. Then, G is finite-by-nilpotent.*

Proof. For natural numbers k and n we can define the set

$$\Delta_{k,n} = \{x \in G \mid |x|_k \leq n\},$$

which is closed by Proposition 2.4.8. Moreover, using Theorem 3.1.7 we in fact notice that for each $g \in G$ there exists some $k_g \in \mathbb{N}$ such that $|g|_{k_g}$ is not only less than the continuum but finite. Thus, the entire G can be covered with the closed subsets $\Delta_{k,n}$ where k and n range among the natural numbers. Under these circumstances Baire category theorem ensures there exist natural numbers k and n such that the interior of $\Delta_{k,n}$ is non-empty. Therefore, there exist an element $g \in G$ and an open normal subgroup $H \trianglelefteq_o G$ such that the left coset gH is contained in $\Delta_{k,n}$, which means that $|gh|_k \leq n$ for all $h \in H$. In particular, $|g|_k \leq n$. Furthermore, if $h \in H$ Lemma 2.4.3 yields the following bound:

$$|h|_k = |g^{-1}gh|_k \leq |g^{-1}|_k^3 |gh|_k = |g|_k^3 |gh|_k \leq n^3 \cdot n = n^4,$$

which actually proves that H is contained in $FC_k(G)$. Let $T = \{t_1, \ldots, t_r\}$ be a (finite) transversal of H in G. Then there exist some natural numbers $k_s \in \mathbb{N}$ with $s \in \{1, \ldots, r\}$ such that $|t_s|_{k_s}$ is finite for $s \in \{1, \ldots, r\}$. We set $q = \max\{k, k_1, \ldots, k_r\}$ and apply Corollary 3.2.3 to deduce that $t_1, \ldots, t_r$ and the elements of H are contained in $FC_q(G)$. Since $G = \langle t_1, \ldots, t_r, H\rangle$, we deduce that $G \subseteq FC_q(G)$ so it is clear that these elements generate the whole group so $G = FC_q(G)$. This means that $|x|_q$ is finite for all $x \in G$, so by Theorem 3.2.1 the group G is finite-by-nilpotent. □

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