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ON THE TAUTOLOGICAL RING OF MODULI SPACES OF RIEMANN SURFACES.

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Abstract

This thesis "On the Tautological Ring of Moduli Spaces of Riemann Surfaces." gives an overview of known results about the relations on the tautological ring of Moduli spaces. First, we introduce the Moduli Space of Stable Curves as an Orbifold of a etale groupoid, a generalization of complex manifold and orbit space of a group. Then we define the Tautological Ring on it as a subring of the Cohomology Ring. Finally, we present the work of Pandharipande, Pixton and Dvonkine in [10], they discovered a set of relations on the Tautological Ring that is, up to date, the largest known. This set is obtained by Cohomological Field Theories, a tool to compute and glue cohomology classes in a Tautological way, and a group action on CohFTs. Using Teleman's characterization of Cohomological Field Theories in a specific case, they manage to deduce an explicit formula for a suitable modification of Witten's 3-spin CohFT. This turns out to vanish non trivially, providing a set of relations.

Davide Accadia

To my Father,
May the earth rest lightly on you.

‘Ultimately, man should not ask what the meaning of his life is, but rather must recognize that it is he who is asked. In a word, each man is questioned by life; and he can only answer to life by answering for his own life; to life he can only respond by being responsible.’

(Viktor E. Frankl, Man’s Search for Meaning)

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Introduction

The objective of this thesis is to present some modern advancements in the study of Moduli Spaces of curves. The main focus is on the Tautological Ring of Moduli spaces, a subring of the Cohomology ring that seems to be more intuitively understandable, than the whole cohomology ring.

The first chapter is dedicated to the construction of the Deligne-Mumford compactification and its natural orbifold structure. This chapter follows the construction by Robbin and Solomon, we will limit ourselves to the definition, and overall structure of the object. Concluding with the presentation of the concept of DM convergence that helps us grasp the topology on this space.

In the second chapter the Tautological Ring is introduced, and by giving some examples of tautological and non-tautological classes, and presenting a plethora of other results and conjectures we aim to achieve a basic perception of the direction and of the extent of what is known and what questions arise during the study of these natural objects.

In the last chapter we review the advancements by Pandharipande, Pixton and Dvonkine in a paper from 2015. In this paper they mainly use and manipulate Cohomological Field Theories. By definition a CohFT is a family of cohomology classes that is closed by the pushforward of glueing maps. With some insight in the physical motivations behind the Moduli Spaces and Witten's class, they were able to find the largest set of Tautological relations known until now. Two group actions are introduced: the T action and the R action, defining CohFTs as a summation over a set of graphs of some modification of the original CohFT. These two actions can be then combined to form an action with stronger properties, the unit preserving R action or $R.$ action, that allows us to use Givental-Teleman's classification of semisimple CohFTs. Witten's shifted class, is a class defined starting from Witten's class, and it is very similar to the result of T applied on Witten's class. Givental-Teleman theorem provides us with a formula for Witten's shifted class as the image of a specific $R.$ action on a very simple CohFT. The formulas we get are non trivial even for degrees of Witten's shifted class that should be vanishing, these formulas constitute the set of relations discovered in this paper. These relations generalise the previously set of known relations, the Faber-Zagier relations. It is proven that the approach of this paper cannot yield a greater set of relations.

Chapter 1

The moduli space of curves

The scope of this chapter is to define the Deligne-Mumford Moduli Space.

Definition 1.1. (Riemann Surface with n marked points.)
A marked Riemann Surface is a triple (Σ, s_*, j) where s_* is a sequence of n distinct points on Σ so:

$$s_* \in \Sigma^n \setminus \Delta.$$

Where Δ is the set of n-uples, with at least one repetition, also called the "Fat diagonal".

Remark. Since all Riemann Surface of genus g are diffeomorphic, we can define the moduli space of curves of genus g as

$$\mathcal{M}_g = \mathcal{J}(\Sigma)/\mathrm{Diff}(\Sigma)$$

This is independent on the substrate Σ as any diffeomorphism induce a bijection on the complex structures and a group isomorphism on Diff.

Now we wish to define the moduli space of stable curves, we will start by giving an overview of orbifold structures.

Definition 1.2. (Groupoid)
A Groupoid is a category in which every morphism is an isomorphism.

Remark. (Notation)
Let B be the set of objects of a groupoid and let Γ be the set of morphisms, then we will denote the set of morphisms from a and b in B by $\Gamma_{a,b}$ and by $\Gamma_a := \Gamma_{a,a}$ the automorphism group of $a \in B$. Define the source and target maps $s, t : \Gamma \to B$ associating to a morphism $a \to b$: a and b respectively. The inversion map $i : \Gamma \to \Gamma$ that sends a morphism in its inverse. The identity section $e : B \to \Gamma$ associating to each object its identity morphism. Lastly, the multiplication map $m : \Gamma \times_{s,t} \Gamma \to \Gamma$ from the set of pairs of composable morphisms, associating the composition to each couple.

Definition 1.3. (Lie Groupoid)
A Lie Groupoid (B, Γ) is a groupoid in which both B and Γ are smooth manifolds, the structure maps are smooth and the source map is a submersion.

Remark. Since the inversion map is a diffeomorphism and $t = s \circ i$ it follows that t is a submersion too, and that $\Gamma \times_{s,t} \Gamma$ is a submanifold of $\Gamma \times \Gamma$, so it makes sense to say that m is smooth.

Definition 1.4. (Homomorphism of Lie Groupoids)
Let $(B, \Gamma), (B', \Gamma')$ be two Lie Groupoids (B, Γ), a morphism from the first to the second is a functor consisting of two smooth maps: one on objects $\iota : B \to B'$ and one on morphisms $\iota : \Gamma \to \Gamma'$ satisfying

$$s' \circ \iota = \iota \circ s, \quad t' \circ \iota = \iota \circ t, \quad e' \circ \iota = \iota \circ e, \quad i' \circ \iota = \iota \circ i \quad \text{and} \quad m'(\iota \times \iota) = \iota \circ m.$$

Definition 1.5. (Lie Groupoid)
A Lie Groupoid (B, Γ) is said to be proper if:

$$s \times t : \Gamma \to B \times B$$

is proper. I.e. the preimage of a compact set through that map is a compact set.

Definition 1.6. (Etale groupoid)
An Etale groupoid is a Lie Groupoid for which the maps s and t are local diffeomorphisms.

Remark. Proper Etale Groupoids are stable, the fact that s is a local diffeomorphism tells us that $\Gamma_{a,a}$ is sparse. And the fact that $s \times t$ is proper tells us that $\Gamma_{a,a}$ is compact, so it is finite.

Definition 1.7. An homomorphism of etale groupoids is called a refinement if it satisfies the following:

- The map induced on the orbit spaces is a bijection $\iota_* : B/\Gamma \to B'/\Gamma'$
- For any $a, b \in B$, ι restricts to a bijection $\Gamma_{a,b} \to \Gamma'_{\iota(a),\iota(b)}$
- The map on object is a local diffeomorphism. Since s and s' are local diffeomorphisms the map on morphisms is one too.

Two proper etale groupoids are called equivalent if they have a common proper refinement.

Definition 1.8. (Orbifold Structure)
Let $(\mathcal{B}, \mathcal{G})$ be a groupoid, and (B, Γ) be a proper etale groupoid, an orbifold structure on $(\mathcal{B}, \mathcal{G})$ us a functor $\sigma : (B, \Gamma) \to (\mathcal{B}, \mathcal{G})$ such that:

- The map induced on the orbit spaces is a bijection $\sigma_* : B/\Gamma \to \mathcal{B}/\mathcal{G}$,
- For any $a, b \in B$, σ restricts to a bijection $\Gamma_{a,b} \to \mathcal{G}_{\iota(a),\iota(b)}$.

Definition 1.9. (Refinement of Orbifold structures.)
A refinement of orbifold structures is just a refinement of the proper etale groupoids, such that the diagram commutes.

$$\begin{tikzcd} (B, \Gamma) \arrow[rr, "\sigma"] \arrow[dr, rightsquigarrow, "\iota"'] & & (\mathcal{B}, \mathcal{G}) \\ & (B', \Gamma') \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

Two orbifolds are said to be equivalent if they have a common refinement.

Definition 1.10. An orbifold is a groupoid equipped with an orbifold structure.

Remark. (Investigating the etale property)
Let B, Γ be a stable etale groupoid. Let $f : a \to b$ be a morphism, then there exist a neighborhood U of a, a neighborhood V of b, and a neighborhood N of f, such that, U and N are diffeomorphic through the source map s while V and N are diffeomorphic through the target map t. This just follows from the groupoid being etale. The diffeomorphism $\phi : t \circ s^{-1} : U \to V$ extends f in the sense that $\phi(a) = b$

In the case $a = b$ we may also that pick a smaller neighborhood for a so to choose it independently of f and N_f, we may restrict it further to have N_f be pairwise disjoint.

A non trivial result is that this defines an homomorphism $\Gamma_a \to \mathrm{Diff}(U)$
Proof. So we want to prove

$$\phi_h = \phi_g \circ \phi_f.$$

The setting is that these three morphism have been defined on the same small neighborhood of a, U. Let $x \in U$, then define $f' = s^{-1}(\phi_g(x)) \in P_f$ and $g' = s^{-1}(x)$ Now f' and g' are composable morphisms, by continuity on m we may pick smaller neighborhood so that $h' = m(f', g')$ is in the neighborhood of h. But $h' : x \to \phi_f \circ \phi_g(x)$ so $\phi_h(x) = t(h')$. The easiest way to walk through this proof is to keep in mind the neighborhoods and just use that s and t are diffeomorphisms.

Proposition 1.11. (Equivalent condition for Proper Etale Groupoid)
Let (B, Γ) be a stable etale groupoid and $a, b \in B$, U, V and N_f neighbourhoods of a, b and $f \in \Gamma_{a,b}$ pairwise disjoint, with the diffeomorphism property as descrived above. Then the gropoid is proper if and only if the neighborhoods can be choosen so that:

$$(s \times t)^{-1}(U \times V) = \bigcup_{f \in \Gamma_{a,b}} N_f$$

This condition is informally stated as, morphisms close to morphisms from $a \to b$ there are exactly all morphisms with base point close to a and target point close to b.

Definition 1.12. (Quotient Topology of the orbit space)
We endow the orbit space of an etale groupoid (B, Γ) with the quotient topology. The quotient map is an homeomorphism by definition. Since the refinement of etale groupoids comes with an homeomophism ι_* of the orbit spaces, this topology is independent of the representative of the class of equivalent etale groupoids. This allows us to define the Orbifold topology on any orbifold's orbit space, $(\mathcal{B}/\mathcal{G})$.

Proposition 1.13. If (B, Γ) is a proper etale groupoid, the quotient topology on B/Γ is Hausdorff.
Proof. We want to show that any two points have disjoint neighborhoods, distinct points in B/Γ have to be images of two points in B with no morphisms between them. And a neighborhood in B/Γ are two sets $u + \Gamma$ with u close to a. So having two disjoint neighborhood is the same as having two neighborhoods of a and b in B so that there are no morphisms between any element of the first and the second. Stated as this, the hausdoff property for this space is a special case of the equivalent property to being proper for an etale groupoid.

Example 1.14. (Manifolds are Orbifolds)
Any manifold can be endowed in a rather natural way with an orbifold structure.

The base groupoid $(\mathcal{B}, \mathcal{G})$, where $\mathcal{B} = M$ and morphisms are just the identity morphisms of each object. The Etale groupoid is given by any countable cover of M, in particular $B = \bigsqcup_\alpha U_\alpha$, and $\Gamma = \bigsqcup_{\alpha,\beta} U_\alpha \cap U_\beta$, in the sense that two copies of the same point provide a morphism sending one into the other. The orbit space is interpretable as making the cover charts come together, effectively giving a copy of M. In fact, the morphism sending an element of U_α to itself on the manifold induces a bijection of the orbit spaces, and of course restricts to a bijection on morphisms, as each intersection in general gives only one morphism for each overlapping. A refinement of covers gives a refinement of the two etale groupoids in the obvious way, so every such orbifold structure is equivalent.

Definition 1.15. (Riemann Surface)
A Riemann surface is an oriented smooth manifold of real dimension two, closed, i.e. compact and without boundary. with a smooth complex structure

$$j : T\Sigma \to T\Sigma.$$

We may identify the surface with the complex structure.

Definition 1.16. (Nodal and marked Surface)
A nodal structure on Σ is a set of couples

$$\nu = \Big\{\{y_1, y_2\}, \dots, \{y_{2k-1}, y_{2k}\}\Big\},$$

this set can be seen as an equivalence relation with which we will quotient the surface. A point marking on Σ is a sequence $r_* = (r_1, \dots, r_n)$, this points will just be marked, so that morphisms of marked Riemann Surfaces will have to respect this structure.

Definition 1.17. (Signature or dual graph)
To each marked nodal surface we may associate a graph, to each connected component of $\Sigma \setminus \nu$ is associated a vertex labeled with the genus of the closure of the component, each node corresponds to an edge between the two components, and finally we may label each vertex with the set of indices of marked points. In the later chapters we will define this object more precisely, as it will prove useful to study Cohomological Field Theories.

Definition 1.18. (Betti Numbers and graph genus)
Defining the homology of a graph Γ as a K cell complexes, we can define the Betti Numbers as

$$h_i = rkH^i(K).$$

In the definition, K is obtained glueing substituting one dimensional cells to edges and zero dimensional cells at vertices. The Betti number h_0 is the number of connected components of the graph and h_1 is the number of independent cycles. These are useful to define the arithmetic genus of a nodal riemann surface, $g = h_1 + \sum_v g_v$.

Remark. (Automorphisms of Marked Nodal Surfaces)
We say that a surface is stable if its automorphism group is finite. Two

surfaces can be diffeomorphic if and only if they have the same signature. Of course for automophisms we require that they are biholomophic, so to respect the complex structure too.

- For genus zero, morphisms are mobius transformations, so fixing three points gives a unique morphism.
- We can see a genus one surface with one marked point as a quotient between the complex plane (line) and a lattice. Since lattices have at most six automorphisms, any torii with one marked point has finite automorphism group. Of course with no marked point there are infinite rotations.
- For genus greater than one Hurwitz theorem states that the number of automorphisms is at most 84(g-1)

Definition 1.19. (Node)
Let $\pi : P \to A$ be an holomorphic map, with $\dim_{\mathbb{C}}(P) = \dim_{\mathbb{C}}(A) + 1$, then for each regular point, the holomorphic implicit function theorem gives us charts so that π writes as:

$$(z, t_1, \dots, t_n) \to (t_1, \dots, t_n),$$

equivalently, the germ of the π at the point is isomorphic to the germ at zero of that same map

$$\mathbb{C}^{n+1} \to \mathbb{C}^n.$$

We call a critical point Nodal if the germ of the family π is isomorphic to

$$(x, y, t_2, \dots, t_{n+1}) \to (xy, t_2, \dots, t_n).$$

Definition 1.20. (Nodal Families)
Let A and P be connected complex manifolds such that $\dim_{\mathbb{C}}(P) = \dim_{\mathbb{C}}(A) + 1$, we call a nodal family an holomorphic map $\pi : P \to A$ such that:

- π is proper
- Every critical point of π is nodal, and the intersection of each fiber with the set of critical points C_π is finite.
- Every regular fiber is a compact Riemann Surface

Definition 1.21. (Desingularization)
Our intention is to interpret the critical fibers of our family as nodal Riemann surfaces, to do so we may define the desingularisation of a fiber. The desingularization of a fiber P_a is a map $u : \Sigma \to P$ where Σ is a (non necessarily connected) compact Riemann Surface such that:

- $u^{-1}(C_\pi)$ is finite
- u restricted to $\Sigma \setminus u^{-1}(C_\pi)$ is a bijection.

The restriction of u is holomorphic in each point, it is proper and invertible, so it is a local diffeomorphism in each compact neighborhood, and being globally bijective it is actually a diffeomorphism. This is why we asked for the function to be proper.

Proposition 1.22. (Canonical Desingularization)

- Every fiber of a critical point admits a desingulatization.
- Let u_1 and u_2 be two desingularizations of the same fiber, then
$$u_2^{-1} \circ u_1 : \Sigma_1 \setminus u_1^{-1}(C_\pi) \to \Sigma_2 \setminus u_2^{-1}(C_\pi)$$
extends to an isomorphism $\Sigma_1 \to \Sigma_2$
- Desingularizations of fibers of critical values are immersions and the preimage of a critical point is composed of two distinct elements.
- It is possible to define a canonical desingularization

Proof. Let us fix a critical value $a \in A$ and P_a its fiber, define the surface
$$\Sigma$$
as the disjoint union of $P_a \setminus C_\pi$ with two copies of $P_a \cap C_\pi$, basically taking the nodal surface and breaking it up. By the definition of nodes we know that the neighborhood of each nodal point in P_a intersects P_a in two disks intersecting at the point. We define $u : \Sigma \to P$ as the identity on non nodal points, and so that sends the two copies of the node to the node This map is holomorphic, at every smooth point as it is the identity, while around nodal points we can use the two disks as coordinates, to see the map as sending two lines identified at a point, to two lines intersecting at a point. This is then a desingularization The second point follows from the removable singularity theorem. The third point follows from the fact that the map of point one is an immersion as the maps $x \to (x, 0)$ and $y \to (0, y)$ are immersions, and thanks to the second point, if one desingularization is an immersion, all of them are.

Definition 1.23. Let $\pi_A : P \to A$ and $\pi_B : Q \to B$ be two nodal families.

Then a fiber morphism is a bijective map $f : P_a \to Q_b$ such that for one desingularization of P_a $u : \Sigma \to P$, $f \cdot u : \Sigma \to Q$ is a desingularization of Q_b. From the previous proposition, this would be the case for any desingularization of P_a.

Definition 1.24. (Morphism of Nodal Families)
Let $\pi_A : P \to A$ and $\pi_B : Q \to B$ two nodal families, then we define any commutative diagram

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d, "\pi_A"'] & Q \arrow[d, "\pi_B"] \\ A \arrow[r, "\phi"'] & B \end{tikzcd}$$

Proposition 1.25. The arithmetic genus of a fiber is locally constant.
Proof. There is a continuous deformation between any two close enough fibers, being the family an holomorphic map.

Definition 1.26. (Marked nodal family)
A marked nodal family is a couple (π, R^*) where π is a nodal family and $R_* = (R_1, \dots, R_n)$ is a sequence of complex submanifolds of P pairwise disjoint and such that the restriction of π to R_i maps diffeomorphically onto A, so definition a section of the fibration π, each surface marks exactly one point per fiber, we are requiring it to be a diffeomorphism so that the surfaces don't contain any critical point. So the marking of the family induces a unambiguous marking of the fibers, hence, of the desingularizations. For marked nodal families, all instances of morphisms are required to be preserving the marking structure.

Definition 1.27. We may define the type of a fiber as the type (g, n) of any of its desingularizations, and so we say a family has type (g, n) if each fiber has type (g, n). We may define the stability of a family similarly, if each fiber is stable then the family is called stable.

Remark. Stability is a local condition but it is not extendable to closure. As it depends only on genus and number of marked points.

Definition 1.28. A nodal unfolding is a triple (π, S_*, b), where (π, S_*) is a marked nodal family, and $b \in B$ is called the base point, Q_b is called the central fiber. We will also say that π_B is an unfolding of the marked nodal surface induced by any desingularization of Q_b.

Definition 1.29. An nodal unfolding π is said to be universal if for any nodal unfolding (π_A, R_*, a) any fiber isomorphism of the central fibers $P_a \to Q_b$ is extendable uniquely to the germ of a morphism $(\Phi, \phi) : \pi_A \to \pi_b$ such that $\Phi(R_i) \subset S_i$.

Definition 1.30. An unfolding $(\pi : Q \to B, S_*, b)$ is said to be infinitesimally universal if the linearized cauchy operator $D_{u,b} : \mathcal{X}_{u,b} \to \mathcal{Y}_u$ is bijective.

$$\mathcal{X}_{u,b} = \Big\{ (\hat{u}, \hat{b}) \in \Omega^0(\Sigma, u^*TQ \times T_bB \text{ such that}$$

$$d\pi(u)\hat{u} = \hat{b}, \quad \hat{u}(s_i) \in T_{u(s_i)}S_i, \text{ and if } u(z_1) = u(z_2) \text{ then } \hat{u}(z_1) = \hat{u}(z_2) \Big\}$$

$$\mathcal{Y}_u = \Big\{ \eta \in \Omega^{0,1}(\Sigma, u^*TQ) | d\pi(u)\eta = 0 \Big\}$$

We may now state the main results, bringing us to the definition of the orbifold structure of the Moduli Space.

Proposition 1.31. An unfolding (π, S_*, b) is universal if and only if it is infinitesimally universal.

Proposition 1.32. If (π_B, S_*, b) is infinitesimally universal unfolding, then every pseudomorphism to π_b is a morphism.

Proposition 1.33. A marked nodal Riemann Surface admits an infinitesimally universal unfolding if and only if it is stable.

Definition 1.34. (Deligne Mumford Moduli space)
Let $\overline{\mathcal{B}}_{g,n}$ be the groupoid of whose objects are marked nodal Riemann surfaces of type (g, n) and whose morphisms are the isomorphisms of marked nodal Riemann surfaces. The Deligne- Mumford moduli space is the orbit space of this groupoid.

Definition 1.35. (Universal marked nodal family)
A marked nodal family satisfying:

- (π_B, S_*, b) is a universal unfolding $\forall b$.
- Every stable marked nodal Riemann Surface of type (g, n) is the domain of a desingularization of at least one fiber of π_B.
- The topology of B admits a countable basis.

Proposition 1.36. For any (g, n) in the stable range, there is a universal marked nodal family.
Proof. This is a corollary to the previous results, the only thing to notice is that it is possible to cover the Moduli space by a countable number of sets. First notice that each stratum is separable, in the groupoid topology, and there are a finite number or strata.

Definition 1.37. (Orbifold structure on the DM Moduli Space)
Let $\pi_B : Q \to B, S_*)$ be a universal marked nodal family. Define the associated groupoid,

$$(B, \Gamma, s, t, e, i, m),$$

where:

- Γ is the set of triples (a, f, b) with $a, b \in B$ and f fiber isomorphism between Q_a and Q_b.
- The structure maps are defined in the obvious way

There is a natural functor $B \to \overline{\mathcal{B}}_{g,n}$ sending b to a desingularization of its fiber, if we then quotient, we get the desired functor.

Proposition 1.38. (Unique complex manifold structure)
There is a unique complex manifold structure on Γ, such that $(B, \Gamma$ is a complex etale Lie groupois with maps s, t, e, i, m.
Proof. Existence: Since the unfolding is universal each morphism $f : a \to b$ extends uniquely to the germ of a morphism of unfoldings.

$$\begin{tikzcd} Q|_{\pi^{-1}(U)} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & Q|_{\pi^{-1}(U)} \arrow[d, "\pi"] \\ U \arrow[r, "\phi"'] & V \end{tikzcd}$$

This defines a fiber isomorphism for each $a \in U$ as:

$$\begin{tikzcd} Q_a \arrow[r, "\Phi_a"] \arrow[d, "\pi"'] & Q_{\phi(a)} \arrow[d, "\pi"] \\ a \in U \arrow[r, "\phi"'] & \phi(a) \in V \end{tikzcd}$$

Effectively defining charts

$$\iota_\Phi : U \to \Gamma, \qquad a \to (a, \Phi_a, \phi_a).$$

since the germ is unique, change of charts are identity maps. The structure maps are holomorphic:

- $s \circ \iota_\Phi = id$

- $t \circ \iota_\Phi = \phi$
- $e(a) = (a, id, a) = \iota_{id}(a) \quad \forall a \in B$
- $\iota^{-1}_{\Phi^{-1}} \circ i \circ \iota_\Phi(a) = \iota^{-1}_{\Phi^{-1}} \circ i(a, f, \phi(a)) = \iota^{-1}_{\Phi^{-1}}(\phi(a), f^{-1}, a) = \phi(a)$
- $m(\iota_\Psi \circ \phi, \Phi) = \iota_{\Psi \circ \Phi}$

Now to prove uniqueness, let Γ' and Γ'' be two copies of Γ with two complex manifold structures satisfying the hypothesis. Then since by the etale property s is a local diffeomorphism, for each point in Γ there are neighborhoods $U \subset \Gamma'$ and $U \subset \Gamma''$ diffeomorphic to a neighborhood in B, so the structures are the same.

Proposition 1.39. The Etale Groupoid (B, Γ) is proper.

Proposition 1.40. (Uniqueness of orbifold structure on DM moduli space.) The orbifold structure on $\overline{\mathcal{M}}_{g,n}$ is independent of the universal marked nodal family used to define it.

Proof. A morphism between universal families induces a refinement of the associated etale groupoid

$$(a_0, f_0, b_0) \to (\phi(a_0), \Phi_{b_0} \circ f_0 \circ \Phi^{-1}_{a_0}, \phi(b_0)).$$

Let $\pi_0 : Q_0 \to B_0$ and $\pi_1 : Q_1 \to P_1$ be universal families then there is a universal family π and morphisms $\pi \to \pi_0$ and $\pi \to \pi_1$. To define π consider a family of neighborhoods $U_b \quad b \in B$, we can define germs of morphisms

$$\Phi_b : Q_0|_{U_b} \to Q_1$$

by property of being universal of π_1. Then, define $\pi : \sqcup Q_0|_{U_b} \to \sqcup U_b$, then there is the inclusion morphism from $\pi \to \pi_0$ and the morphism $\sqcup_b \Phi_b : \pi \to \pi_1$.

The next definitions and results, show us a way to interpret the topology of the moduli spaces of curves.

Definition 1.41. (Deformations)
Let (Σ, ν) be a compact Nodal Riemann Surface, $\gamma \subset \Sigma$ be a disjoint set of embedded circles,, call Σ_γ the surface with boundary obtained by cutting Σ along γ.The condition of being embedded tells us that γ does not intersect the set of nodal points. We can hence define the suture map $\sigma : \Sigma_\gamma \to \Sigma$, which maps the interior of Σ_γ bijectively onto $\Sigma \setminus \gamma$ and sends the boundary of Σ_γ, onto γ in the obvious way, constituting a two to one correspondence.

Now, let (Σ, ν) and (Σ', ν') be two nodal Riemann Surfaces, then a $(\nu' - \nu)-$deformation is a map $\phi : \Sigma' \setminus \gamma' \to \Sigma$ such that :

- $\gamma' \subset \Sigma' \setminus \nu'$
- $\phi_* \nu' \subset \nu$
- $\phi : \sigma' \setminus \gamma' \to \Sigma \setminus \gamma$ is a diffeomorphism. Here γ is the set of nodes on Σ that are not image of nodes in Σ' through ϕ.
- $\phi \circ \sigma|_{int(\Sigma'_{\gamma'})}$ extends to a continuous surjective map $\Sigma'_\gamma \to \Sigma$
- The preimage of ν through the $\phi \circ \sigma$ map is a component of the boundary of $\Sigma'_{\gamma'}$ and actually the preimage of two points that glue to a node are two components of the boundary relative to the same circle in γ'.

We can interpret this as a way of smoothing a nodal surface at nodes, cutting a neighhood around a node and glueing the two ends together, actually it is more correct to think about it as just untying the knot around the node.

Definition 1.42. (Monotypical convergence)
A sequence of $(\nu_k - \nu)$ deformations $\phi : k(\Sigma_k \setminus \gamma_k, \nu_k) \to (\Sigma.\nu)$ is called monotypical if $\phi_k)_*(\nu_k)$ does not depend on k, i.e we are untying the same knots. While we say that the sequence of Riemann Surfaces converge monotypically if there is a monotypical sequence of $\nu_k - \nu)$ deformations such that:

- The sequences of images of marked points converge to the marked points on Σ.
- The pushforward of the complex structure, converges to $j|_{\Sigma\gamma}$ in the C^∞ topology.

Definition 1.43. (DM convergence)
A sequece of marked nodal Riemann Surfaces of type (g, n) is said to DM-converge to (Σ, j, s, ν) is after discarding finitely many terms, the sequence is the dijoint union of finitely many sequences converging monotypically to (Σ, s, ν, j).

Proposition 1.44. (The topology makes this orbifold compact)
What is proven is that every sequence of stable marked nodal Riemann Surfaces admits a DM-convergent subsequence.

Chapter 2

Tautological Ring

The scope of this second chapter is to give an overview of the current results for tautological rings of Moduli spaces.

On orbifolds we may define Homology and Cohomology groups of the underlying topological space.

2.1 Tautological ring and classes

Definition 2.1. The moduli space of rational tail surfaces $\mathcal{M}^{rt}_{g,n}$, is the moduli space of surfaces that have a component of genus g. The moduli space of compact type surfaces $\mathcal{M}^{ct}_{g,n}$, is the moduli space of surfaces whose structure graph is a tree, so the arithmetic genus and the total genus are the same. We may see the moduli space of smooth stable curves $\mathcal{M}_{g,n}$ as the space of stable curves with signature graph a single vertex.

Proposition 2.2. (Poincaré duality) [15]
Poincaré duality holds on for Homology and cohomology groups of smooth compact complex orbifolds.

Definition 2.3. (Glueing and forgetting)
Some maps are intuitively simple to define, the forgetting map p associates to a surface of type $(g, n+1)$ the surface of type (g, n) obtained by forgetting the $n+1$th marking, and collapsing any unstable component to a point.

$$p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

While the glueing maps are:

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$

and

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}.$$

The first sends respectively, a couple of surfaces to the surface obtained by glueing the last marked point of each surface together. The second sends a surface to the surface obtained by glueing the last two marked points together, increasing the arithmetic genus by one.

Definition 2.4. (Universal Curve)
The universal family over is the groupoid domain of the universal marked nodal family used in chapter one basically it consists of the union of all the curves, this space is endowed with the compatible orbifold structure as the moduli space.

Proposition 2.5. The forgetting map and the universal family are isomorphic.
Proof. We wish to prove that there is a morphism

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{C}}_{g,n},$$

simply send the curve with $n+1$ marked point to its position on the curve after forgetting that point, on the position on which it was.

Remark. (Vector bundle over an orbifold)
The hands on way to treat orbifolds, it to consider them as manifolds with a group action defined on them. As we have seen the charts aren't so easily defined as their definition is not as simple as for manifolds. Nevertheless we may define vector bundles on the chart together with a linear lifting of the group action on it, to define vector bundles on orbifolds.

Definition 2.6. (Tautological Ring)
The tautological ring R^* is the minimal subring of the cohomology ring

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

containing the identity and closed under pushforward by forgetting and glueing maps.

Remark. The tautological ring may be defined as a subgroup of the Chow ring or the Cohomology ring, since most results carry over between the two, they are thought as equivalent, even if there is no strong evidence for the fact.

Definition 2.7. (Cotangent line bundle and extension on nodal points)
On the universal curve we can define the cotangent line bundle to each surface at each smooth point. On nodes, consider the chart that uses the two disks on the Riemann Surface as coordinate lines. Then the bundle is generated by the sections

$$\frac{dx}{x}, \frac{dy}{y},$$

since

$$\frac{d(xy)}{xy} = \frac{dx}{dy} + \frac{dy}{y} = 0,$$

on each surface, the line bundle can be extended to every point of the plane.

Definition 2.8. (Chern Classes)
Let L an holomorphic line bundle over a complex manifold, pick a non zero section of the line bundle and let Z be the set of zeroes, and P the set of poles, then the first Chern class $Z - P$ is a well defined divisor. There are higher chern classes associated to complex bundles, but they are harder to define.

Remark. The Chern class operator sends the tensor product of line bundles to the sum of the divisors.

Definition 2.9. (Cotangent line class)
We can push forward the cotangent line bundle through any section, defining a line bundle on the moduli space. We define the ψ_i class as the first Chern classes of the pullback of the cotangent line bundle by the ith marked point section.

Definition 2.10. The κ classes in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ are defined by:

$$\kappa_i = p_*(\psi_n^{i+1})$$

where p is the map forgetting the last point.

Proposition 2.11. (Hodge Bundle)
By the Riemann-Roch theorem, each Riemann surface of genus g admits a g vector space of abelian differentials, this defines a bundle over the moduli space, we call it the hodge bundle. This is the bundle of the holomphic sections of the relative cotangent line bundle.

Proposition 2.12. Let L be a complex line bundle the bundle $L \bigotimes L^*$ is trivial and so is its Chern class.

Proposition 2.13. We have the relation:

$$\psi_i = -p_* D_i^2$$

Proof. Squaring a divisor is the same as taking the normal line bundle relative to the divisor at the marked points, this amounts to taking the tangent bundle to the curve at the marked point. Since the forgetful map and the universal curve are isomorphic this amounts to taking the pushforward by the ith section of that bundle, this is the definition of ψ_i. The sign minus is amounts for the difference of taking the tangent instead of the cotangent.

Definition 2.14. in $\overline{\mathcal{M}}_{0,n}$ we denote by $\delta_{i|jk}$ the divisor of curves for which there is a node between the component containing the marked point i and the component containing the marked points $j, k.$, and by

$$[\delta_{i|jk}] \in \overline{\mathcal{M}}_{0,n},$$

its Poincaré dual class.

Proposition 2.15. (The ψ classes are tautological)
On $\overline{\mathcal{M}}_{0,n}$, we have:

$$\psi_i = [\delta_{i|jk}] \qquad \forall k, j.$$

Proof. Let us take one meromorphic section of the cotangent bundle then we will pushforward by the section π_i, of the ith marked point. For each stable surface, we will define the meromorphic form then the global one is the union of all of them. The graph relative to a genus zero stable surface is a tree, and the genus of each vertex is zero. So, there is a unique path between the component containing j and the one containing k. There is a unique section of the cotangent line bundle with poles at j and k of residues -1 and 1 and with poles on each node between j and k. This section vanishes on any other component, so it will give us a non zero section of the bundle relative to ψ_i if and only if the point marked i is not separated from j and k by one node. One can prove the zero on the divisor is simple, giving us the equality.

Proposition 2.16. [11] [14]
The tautological ring of $\mathcal{M}_g$ is generated by $\kappa_1, \dots \kappa_{\lfloor \frac{g}{3} \rfloor}$

Theorem 2.17. (Graber-Panharipande)
$R^*(\overline{\mathcal{M}}_{g,n})$ is additively generated by classes of the form $i_{\Gamma *}($ monomials of classes ψ and $\kappa)$

Corollary 2.18. $R^*(\overline{\mathcal{M}}_{g,n})$ is closed under pull-backs by glueing and forgetful maps.

Proposition 2.19. [12]
There are some non tautological classes, an example is provided in this paper for the first time, before only existence arguments were made.

Definition 2.20. (Stable range)
If n referring to the number of marked points, we say that (g,n) is in stable range if $2g-2+n>0$ this condition is equivalent to the curves being stable. While, if n is referring to a cohomology degree, we say that k is in the stable range if $g-1-2k>0$, this is due to the first developments of the next stability condition, there are then results improving the stable range for k, improving the next results.

Proposition 2.21. (Stability for high genus)
For $g-1-2k>0$ there are isomorphisms,

$$H^k(\mathcal{M}_g,\mathbb{Q}) \to H^k(\mathcal{M}_{g+1},\mathbb{Q}) \to H^k(\mathcal{M}_{g+2},\mathbb{Q}) \to \dots$$

Proposition 2.22. (Mumford's conjecture)
For k in the stable range, the homomorphism sending x_i to κ_i defined between:

$$\mathbb{Q}[x_1,x_2,\dots] \to H^*(\mathcal{M}_g,\mathbb{Q})$$

is an isomorphism up to degree $2k$.

Proposition 2.23. For k in the stable range, the homomorphism sending y_i to ψ_i defined between:

$$H^*(\mathcal{M}_g,\mathbb{Q})[y_1,\dots,y_n] \to H^*(\mathcal{M}_{g,n},\mathbb{Q})$$

is an isomorphism up to degree $2k$.

Definition 2.24. λ_i is the i-th chern class of the hodge bundle.

Proposition 2.25. (Kontsevich Theorem)
Define F_g as the series

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{k_1,\dots,k_n}(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\dots\psi_n^{k_n})t_{k_1}\dots t_{k_n}.$$

Then the function

$$F=\sum_{g=0}^{\infty}F_g\lambda^{2g-2},$$

satisfies:

$$(2n+1)\Big(\frac{\partial^3}{\partial t_n\partial t_0^2}F\Big)=\Big(\frac{\partial^2}{\partial t_{n-1}\partial t_0}F\Big)\Big(\frac{\partial^3}{\partial t_0^3}F\Big)+2\Big(\frac{\partial^3}{\partial t_{n-1}\partial t_0^2}F\Big)\Big(\frac{\partial^2}{\partial t_0^2}F\Big)+\frac{1}{4}\frac{\partial^5}{\partial t_{n-1}\partial t_0^4}F.$$

Together with the string and dilaton equation and base cases conditions, the series is completely determined, effectively reducing all intersection numbers between ψ classes recursively.

Definition 2.26. (Perfect pairing)
A perfect pairing of $\mathbb{Q}$ vector spaces is a bilinear map

$$(.,.) : V \times W \to \mathbb{Q}$$

such that $v \to (v,.)$ is an isomorphism $V \to W^*$ similarly $w \to (.,w)$ is an isomorphism.

Remark. (Faber's conjectures)
The following three statements are referred to as Faber's conjectures:

- $R^d(\mathcal{M}_g) = 0 \quad \forall d > g-2$
- $R^{g-2}(\mathcal{M}_g) \simeq \mathbb{Q}$
- $R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g)$ is a perfect pairing.

This has been proven for g<24.

Remark. (Gorestein Conjectures)
The first point of Faber Zagier is similar to Gorenstein Conjectures

- $R^*(M^{rt}_{g,n})$ is Gorenstein with socle in degree $g-2+n-\delta_{g,0}$
- $R^*(M^{ct}_{g,n})$ is Gorenstein with socle in degree $2g-3+n$
- $R^*(\overline{\mathcal{M}}_{g,n})$ is Gorenstein with socle in degree $3g-3+n$

Also we know that

- $R^*(M^{rt}_{g,n})$ is one dimensional in degree $g-2+n-\delta_{g,0}$ and vanishes for higher degrees
- $R^*(M^{ct}_{g,n})$ is one dimensional in degree $2g-3+n$ and vanishes for higher degrees
- $R^*(\overline{\mathcal{M}}_{g,n})$ is one dimensional in degree $3g-3+n$ and vanishes for higher degrees

Theorem 2.27. (Petersen-Tommasi) [13]
The rings $R^*(\overline{\mathcal{M}}_{2,20})$ and $R^*(\mathcal{M}^{ct}_{2,n})$ are not Gorenstein.

Definition 2.28. A local Gorenstein ring is a commutative Noetherian local ring R with finite injective dimension as an R-module A Gorenstein ring is a commutative Noetherian ring such that each localization at a prime ideal is a local Gorenstein ring.

Definition 2.29. (Stable Graphs)
A Stable graph is a sixtuple

$$\Gamma = (V, H, L, g : V \to \mathbb{Z}_{\geq 0}, v : H \to V, \iota : H \to H),$$

where V is the set of vertices, g is the genus function associating to each vertex a non-negative integer, H is the half-edge set, v is the map attaching each half-edge to a vertex, ι is an involution i.e $\iota^2 = id$ associating each half-edge with it's complementary, forming an edge, or itself, representing a leg. We may call L the set of legs, and E the set of 2-cycles of ι i.e. edges. Since we want to consider connected surfaces, we impose that (V, E) form a connected graph, and to keep the represented curve stable we must ask

$$2g(v) - 2 + n(v)$$

where $n(v)$ is the number of half-edges attached to v.

The genus of the graph is defined as the sum of the genera on each vertex, plus the number of independent cycles of the graph (V, E).

Remark. We can easily visualise how these pieces come together to form a graph. Each vertex v represents a smooth curve of genus $g(v)$, each edge a node between the two curves, and each leg a marking on the surface.

Remark. The boundary of $\overline{\mathcal{M}}_{g,n}$ is the image of attaching maps of any kind, there is a natural stratification of this space by stable graphs, as to each curve a stable graph can be assigned, and the curve can be seen as the attachment of stable curves, by some composition of attaching maps.

A boundary stratum of $\overline{\mathcal{M}}_{g,n}$ determines a stable graph of genus g and with n legs. There is a canonical morphism

$$\xi_\Gamma : \overline{\mathcal{M}_\Gamma} \to \overline{\mathcal{M}}_{g,n}$$

where

$$\overline{\mathcal{M}}_\Gamma = \prod_v \overline{\mathcal{M}}_{g(v),n(v)}$$

and the family of stable curves over $\overline{\mathcal{M}_\Gamma}$ is constructed by glueing the families of $\overline{\mathcal{M}}_{g(v),n(v)}$ along the corresponding half-edges.

Remark. Every tori is isomorphic to the quotient of the complex plane by a lattice, notice that automorphisms of the complex plane that preserve the origin and the lattice are homotheties. Then another genus one stable curve is obtained by taking the sphere with three marked points and glueing two together.

Proposition 2.30. There is a correspondence between modular forms of weight k and holomorphic sections of the line bundle $\mathcal{L}^{\bigotimes k}$ on $\overline{\mathcal{M}}_{1,1}$.

Proposition 2.31. On $\overline{\mathcal{M}}_{1,1}$ we have

$$ch_1(\mathcal{L}) = \frac{1}{24}.$$

Proof. Pick the modular form of order 6, E_6. It has a zero at i, the stabilizer of that point in the modular fundamental domain is 4. So the first chern class of $\mathcal{L}^{\bigotimes 6}$ is $\frac{1}{4}$, hence the first chern class of $\mathcal{L}$ is $\frac{1}{24}$.

2.2 Strata Algebra

Definition 2.32. (Basic class)
A basic class is a product of monomials in κ classes at each vertex and powers of ψ classes at each half edge,

$$\gamma = \prod_{v \in V} \prod_{i>0} \kappa_i[v]^{x_i[v]} \cdot \prod_{h \in H} \psi_h^{y[h]} \in H^*(\overline{\mathcal{M}}_\Gamma, \mathbb{Q})$$

where $\kappa_i[v] = \pi_*(\psi_{n+1}^{i+1}) \in H^{2i}(\overline{\mathcal{M}}_{g(v),n(v)}, \mathbb{Q})$ is the ith kappa class on the moduli space relative to the vertex. If we group the terms by edges we get a class of degree

$$\sum_i x_i[v] + \sum_{h \in H[v]} y[h]$$

which trivially vanishes if the expression amounts to $3g(v)-3+n(v)$ or more.

Definition 2.33. (Strata Algebra)
Consider the $\mathbb{Q}-$vector space with basis given by pairs (Γ, γ) where Γ a stable graph of type $\{g, n\}$ and γ is a basic class on $\overline{\mathcal{M}}_\Gamma$. Up to isomorphism there are only a finite amount of such pairs. Let us call this space $\mathbf{S}_{g,n}$

On it a product is defined by intersection theory, the details can be found in reference [12]

$$[\Gamma_1, \gamma_1] \cdot [\Gamma_2, \gamma_2] = \sum_\Gamma [\Gamma, \gamma_1 \gamma_2 \epsilon_\Gamma],$$

where Γ is a graph contractible to Γ_1 and to Γ_2 and

$$\epsilon_\Gamma = \prod_{e \in E_1 \cap E_2} -(\psi'_e + \psi''_e)$$

is the Fulton Excess class, ψ are as usual the cotangent line classes corresponding to the two half edges of the edge e.

Remark. The important case where Γ_2 has only one edge will be useful in the computations of general pullbacks by attaching maps. In this case we have Γ is either Γ_1 and in this case we have an excess class to account for, and in the other case E_1 and E_2 are disjoint.

Remark. [8]
These strata algebras should be viewed as generalisations to $\overline{\mathcal{M}}_{g,n}$ of the formal polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, \dots]$ that surjects onto the tautological ring of $\mathcal{M}_{g,n}$ by tautological relation we mean an element of the kernel of the natural surjection $q : \mathbf{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Remark. As we have said, Faber-Zagier family is a set of tautological relations on the moduli spaces of non marked curves conjectured to hold by Faber and Zagier around 2000, and proven to hold in [7]. The set presented at the end of this chapter extends the Faber-Zagier relations, to marked surfaces.

2.3 B_0 and B_1 Series

The B series appear in the study of the Faber-Zagier relations as solutions of a differential equation, while in the study of the P relations, they will be the solution of the recursive equation for the R matrix. We will need the following results to verify that the R matrix we find satisfies the symplectic condition to define the action.

Remark.

$$B_0 = \sum_m \frac{(6m)!}{(2m)!(3m)!}(-T)^m \quad B_1 = \sum_m \frac{1+6k}{1-6k}\frac{(6m)!}{(2m)!(3m)!}(-T)^m$$

We want to prove

$$B_0(T)B_1(-T) + B_0(-T)B_1(T) = 2$$

It is easy to see that this is equivalent to

$$B_0^{\text{even}}(T)B_1^{\text{even}}(T) + B_0^{\text{odd}}(T)B_1^{\text{odd}}(T) = 1,$$

We may expand the left hand side as follows:

$$\sum_{k,k+j=n}^{n}\Big(\frac{1+6k}{1-6k}\frac{6k!}{2k!3k!}\frac{6j!}{2j!3j!}(-1)^k+\frac{1+6k}{1-6k}\frac{6k!}{2k!3k!}\frac{6j!}{2j!3j!}(-1)^j\Big)T^n$$

if n is odd the terms simplify, if n is even, there is a correspondence and we deduce:

$$B_0^{\text{even}}(T)B_1^{\text{even}}(T)+B_0^{\text{odd}}(T)B_1^{\text{odd}}(T)=1$$

It is evident for $T=0$ and so it suffices to prove that the derivative of the left hand side is zero.

We will use the property of a series of being a generalised hyper-geometric, so we can expect it to solve a differential equation, that is basically giving recursive relation between the coefficients. Let

$$A(z)=B_0(z/288)\qquad B(z)=B_1(z/288)$$

Proposition 2.34.

$$A(z)\quad\text{solves}\quad 3z^2\frac{d^2A}{dz^2}+(6z-2)\frac{dA}{dz}+\frac{5}{12}A=0$$

Proof.

$$A=\sum_{m\geq 0}a_m(\frac{z}{288})^m$$

$$\frac{dA}{dz}=\sum_{m\geq 0}\frac{m+1}{288}a_{m+1}(\frac{z}{288})^m$$

$$\frac{d^2A}{dz^2}=\sum_{m\geq 0}\frac{(m+1)(m+2)}{288^2}a_{m+2}(\frac{z}{288})^m$$

$$3z^2\frac{d^2A}{dz^2}=\sum_{m\geq 2}3m(m-1)a_m(\frac{z}{288})^m$$

$$6z\frac{dA}{dz}=\sum_{m\geq 1}6ma_m(\frac{z}{288})^m$$

Plugging the values in the differential equation:

$$(3m(m-1)+6m+\frac{5}{12})a_m=\frac{2(m+1)}{288}a_{m+1}$$

$$\frac{a_{m+1}}{a_m}=12\frac{(6m+1)(6m+5)}{m+1}$$

Substituting a_m and a_{m+1}:

$$\frac{(6m+6)(6m+5)(6m+4)(6m+3)(6m+2)(6m+1))a_m}{(3m+3)(3m+2)(3m+1)(2m+2)(2m+1)a_m} = 12\frac{(6m+1)(6m+5)}{m+1}.$$

Proposition 2.35.

$$B(z) = 3z^2\frac{dA}{dz} + (\frac{z}{2} - 1)A$$

Proof.

$$3z^2\frac{dA}{dz} = \sum_{m\geq 2} 3\cdot 288\cdot(m-1)a_{m-1}(\frac{z}{288})^m$$

$$\frac{z}{2}A = \sum_{m\geq 1}\frac{288}{2}a_{m-1}(\frac{z}{288})^m$$

$$b_m = (3\cdot 288\cdot(m-1) + \frac{288}{2})a_{m-1} - a_m$$

Now substituting:

$$a_{m-1} = a_m\frac{(3m+3)(3m+2)(3m+1)(2m+2)(2m+1)}{(6m+6)(6m+5)(6m+4)(6m+3)(6m+2)(6m+1)},$$

we obtain precisely

$$b_m = \frac{6m+1}{6m-1}a_m$$

Proposition 2.36.

$$\frac{d}{dz}(B_0^{\mathrm{even}}(T)B_1^{\mathrm{even}}(T) + B_0^{\mathrm{odd}}(T)B_1^{\mathrm{odd}}(z)) = 0$$

Proof. Separating the terms relative to odd and even powers is easy to see that the first result is equivalent to

$$\begin{cases} 3z^2\frac{d^2A^{\mathrm{even}}}{dz^2} + 6z\frac{dA^{\mathrm{even}}}{dz} - 2\frac{dA^{\mathrm{odd}}}{dz} + \frac{5}{12}A^{\mathrm{even}} = 0 \\ 3z^2\frac{d^2A^{\mathrm{odd}}}{dz^2} + 6z\frac{dA^{\mathrm{odd}}}{dz} - 2\frac{dA^{\mathrm{even}}}{dz} + \frac{5}{12}A^{\mathrm{odd}} = 0 \end{cases}$$

And the second result is equivalent to

$$\begin{cases} B^{\mathrm{even}} = 3z^2\frac{dA^{\mathrm{odd}}}{dz} + \frac{z}{2}A^{\mathrm{odd}} - A^{\mathrm{even}} \\ B^{\mathrm{odd}} = 3z^2\frac{dA^{\mathrm{even}}}{dz} + \frac{z}{2}A^{\mathrm{even}} - A^{\mathrm{odd}} \end{cases}$$

We can directly differentiate

$$A^{\mathrm{even}}(T)B^{\mathrm{even}}(T) - A^{\mathrm{odd}}(T)B^{\mathrm{odd}}$$

We get

$$\frac{d}{dz}\Big(A^{\text{even}}(T)(3z^2\frac{dA^{\text{odd}}}{dz}+\frac{z}{2}A^{\text{odd}}-A^{\text{even}})-A^{\text{odd}}(3z^2\frac{dA^{\text{even}}}{dz}+\frac{z}{2}A^{\text{even}}-A^{\text{odd}})\Big) =$$

$$= 3z^2(A^{\text{even}}\frac{d^2A^{\text{odd}}}{dz^2} - A^{\text{odd}}\frac{d^2A^{\text{even}}}{dz^2})$$

$$+6z(A^{\text{even}}\frac{dA^{\text{odd}}}{dz} - A^{\text{odd}}\frac{dA^{\text{even}}}{dz})$$

$$-2(A^{\text{even}}\frac{dA^{\text{even}}}{dz} - A^{\text{odd}}\frac{dA^{\text{odd}}}{dz}) =$$

$$= \frac{5}{12}(A^{\text{even}}A^{\text{odd}} - A^{\text{odd}}A^{\text{even}}) = 0$$

2.4 $\mathcal{R}$ relations

Definition 2.37. Let's define a set of strata classes,

$$\mathcal{R}^d_{g,A} \in S^d_{g,n},$$

with $A \in 0,1^n$, (g,n) in the stable range, and d a positive integer greater than Witten's 3-spin class degree.

Define

$$\kappa(f) = \sum_{m\geq 0} \frac{1}{m!} p_{m*}\Big(f(\psi_{n+1}\dots f(\psi_{n+m}\Big) \in H^*(\overline{\mathcal{M}}_{g,n})$$

- At each vertex place $\kappa_v = \kappa(T - TB_0(\zeta_v T))$
- At each leg place $B_l = \zeta_v^{a_l} B_{a_l}(\zeta_v\psi_l)$
- And at each edge place

$$\Delta_e = \frac{\zeta' + \zeta'' - B_0(\zeta'\psi')\zeta'' B_1(\zeta''\psi'')B_1(\zeta'\psi')\zeta' B_0(\zeta''\psi'')}{\psi' + \psi''},$$

As we will see later, this expression makes sense as a power series and it is equivalent to an R-matrix action, because the series B_0 and B_1 form a $2x2$ matrix satisfying the symplectic condition.

$$\mathcal{R}^d_{g,A} = \sum_{\Gamma\in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|}\frac{1}{2^{h^1(\Gamma)}}\Big(\Gamma, \big(\prod \kappa_v \prod B_l \prod \Delta_e\big)_{\prod_v \zeta_v^{g(v)-1}}\Big)$$

where the subscript signals that we are considering the coefficient of the monomial relative to the subscript term only.

The procedure to define these relations is very similar to the definition of R matrix action. In fact we will obtain these quantities as the result of applying R to a specific starting Cohomological Field Theory.

Chapter 3

Describing relations in the tautological ring.

In this chapter we will present a technique developed in [10] to determine a family of tautological relations in cohomology. To do so, some methods of handling cohomology classes in a tautological way have to be introduced.

3.1 Cohomological Field Theories

Let V be a $\mathbb{Q}$ -vector space, equipped with a non-degenerate symmetric two form η and let $\mathbf{1} \in V$ be a distinguished element, the unit vector.

Definition 3.1. (CohFTs and CohFTs with unit)
A Cohomological Field Theory with unit is a family $(\Omega_{g,n})_{g,n}$ for any g and n in the stable range, with

$$\Omega_{g,n} \in H^*(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n},$$

respecting the following three axioms:

- Each $\Omega_{g,n}$ is S_n invariant.
- The pull-back $q^*(\Omega_{g,n})$ is equal to the contraction of $\Omega_{g-1,n+2}$, and the pull-back $r^*(\Omega_{g,n})$ is equal to the contraction of $\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}$, both by the bivector $\sum \eta^{jk} e_j \otimes e_k$.
- For any vectors $v_1, \dots, v_n$ we have:

  $$\Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbf{1}) = p^*(\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n)) \quad \text{and} \quad \Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbf{1}) = \eta(v_1, v_2).$$

The group S_n acts on $H^*(\overline{\mathcal{M}}_{g,n})$ by permuting the marked points, and on $(V^*)^{\otimes n}$ by permuting the n covectors. We will say that a family $(\Omega_{g,n})_{g,n}$ is a CohFT if it satisfies the first two axioms, the unit vector is irrelevant in this context.

We will now introduce two group actions, transforming CohFTs. These actions will work together to define an action on CohFTs with unit.

Definition 3.2. (Translation action)
Let Ω be a CohFT on the vector space V and let $T \in z^2V[[z]]$ then the translation of Ω by T, $T\Omega$ is defined by:

$$(T\Omega)_{g,n}(v_1\otimes\cdots\otimes v_n) = \sum_{m\geq 0}\frac{1}{m!}p_{m*}\Omega_{g,n+m}(v_1\otimes\cdots\otimes v_n\otimes T(\psi_{n+1})\otimes\cdots\otimes T(\psi_{n+m}))$$

Where p_m is the map forgetting the last m points. By the vanishing of T_0 and T_1, the degree of contributions increases until the class vanishes. Therefore, the sum is finite.

Remark. There are two points to be made, regarding this definition. The first is that we evaluate the CohFT on a series of classes, and we mean it as follows:

$$\Omega_{g,n}(\dots,T(\psi_i),\dots) = \sum_{k\geq 2}\psi_i^k\Omega_{g,n}(\dots,T_k,\dots).$$

The second is that, we can interpret these terms as classes on a family of stable graphs, and then we sum these contributions. Each term is the class given by a stable graphs with one vertex and $n+m$ legs. Where the first n legs carry vectors and the last m legs carry ψ classes, that pushed back give rise to κ classes. The first legs are referred to as main legs as they are not forgotten in the pushforward, while the other m legs are referred to as κ legs.

Proposition 3.3. Let Ω be a CohFT, then the translation $T\Omega$ as defined above is a CohFT.
Proof. We have to verify two properties: the first, about S_n invariance, is trivial as S_n acts on vectors on both sides and Ω is a CohFT.

About the second. We are pullbacking through r each class in the sum. In graph terms, this is the same as considering any one edge graph, with the m κ legs placed on each vertex. Since we will be forgetting the κ legs, the order doesn't matter, so each distribution of m_1 legs to the first vertex and m_2 to the second is counted $\binom{m}{m_1,m_2}$ times.

$$(T\Omega)_{g_1,n_1}(v_1\otimes\cdots\otimes v_{n_1}) = \sum_{m_1\geq 0}\frac{1}{m!}p_*\Omega_{g,n_1+m_1}\Big(v_1\otimes\cdots\otimes v_{n_1}\otimes T(\psi_{n_1+1})\otimes\cdots\otimes T(\psi_{n_1+m_1})\Big)$$

Simplifying and splitting the binomial, we get precisely:

$$\frac{1}{m_1!}p_*\Omega_{g,n+m_1}(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m_1}))\eta$$

$$\frac{1}{m_2!}p_*\Omega_{g,n+m_2}(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m_2})).$$

Proposition 3.4. The T action above defined is an abelian group action.
Proof.

$$(T_a+T_b)\Omega_{g,n} = \sum_{m\geq 1} \frac{1}{m!}p_*\Omega_{g,n+m}(v_1\otimes\cdots\otimes v_n\otimes(T_a+T_b)(\psi_{n+1})\otimes\cdots\otimes(T_a+T_b)(\psi_{n+m}))$$

Reordering the terms, to keep T_a contributions first, we get:

$$\sum_{m\geq 1} \frac{1}{m!}p_*\frac{m!}{m_1!m_2!}\Omega_{g,n+m}\Big(v_1 \otimes \cdots \otimes v_n \otimes (T_a)(\psi_{n+1}) \otimes \dots$$

$$\cdots \otimes (T_a)(\psi_{n+m_1}) \otimes (T_b)(\psi_{m_1+1}) \otimes \cdots \otimes (T_b)(\psi_{n+m})\Big)$$

Precisely the action of the two series, subsequently.

Definition 3.5. Given an element $R \in End(V)[[z]]$ we say it is satisfying the symplectic condition if

$$R(z) \cdot R^*(-z) = 1,$$

where R^* it's the adjoint with respect to η. Expressing R in its matrix form $R(t^j e_j) = R^k_j t^j e_k$ we can write the symplectic condition in coordinates,

$$\sum_{l,s,k} R^j_l(z)\eta^{ls}R^k_s(-z)\eta_{ku} = \delta^j_u,$$

equivalently

$$\sum_{l,s} R^j_l(z)\eta^{ls}R^k_s(-z) = \eta^{jk}$$

we can conclude:

$$\frac{\eta^{-1} - R^{-1}(z)\eta^{-1}R^{-1}(w)^t}{z+w}$$

is a well defined power series.

Of course the inverse of R satisfies the symplectic condition too. So, to get a left-group action, it is preferable to define the R action using the inverse instead.

$$\sum_{j,k} \frac{\eta^{jk} - \sum_{l,s}(R^{-1})^j_l(z)\eta^{ls}(R^{-1})^k_s(w)^t}{z+w} e_j \otimes e_k \in V^{\otimes 2}[[z,w]].$$

Definition 3.6. (R action)
Given an element of $End(V)[[z]]$ satisfying the symplectic condition and a CohFT Ω we may define the CohFT $R\Omega$.

We will sum over the set of stable graph of genus g with n legs contributions $Cont_\Gamma \in \Omega_{g,n} \in H^*(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$.

$Cont_\Gamma$ is defined as follows:

- To each vertex we will assign the CohFT element relative to that genus and marked points number,
- To each edge are assigned two cohomology classes attached to a bivector

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g',n'}, \mathbb{Q}) \otimes H^*(\overline{\mathcal{M}}_{g'',n''}, \mathbb{Q}) \otimes (V^2)^{\otimes n}.$$

  Namely:
$$\frac{\eta^{-1} - R^{-1}(\psi'_e)\eta^{-1}R^{-1}(\psi''_e)^t}{\psi'_e + \psi''_e}.$$

- To each leg we assign an element of $H^*(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes End(V)$, namely: $R^{-1}(\psi_l)$.

Proposition 3.7. $R\Omega$ is indeed a CohFT.
Proof. The symmetry under the action of S_n is obvious.

The pullback by r is essentially the pullback of

$$\xi_\Phi : \overline{\mathcal{M}}_\Phi \to \overline{\mathcal{M}}_{g,n}$$

As we have seen the strata product of,

$$[\Gamma_1, \gamma_1] \cdot [\Gamma_2, \gamma_2]$$

is the sum over all Γ, graphs that are contractible to Γ_1 and Γ_2. In this case Γ_1 is the graph relative to the R-contribution, and γ_2 is one edge. There are only two cases for Γ, either the single edge is in the first graph and we have to account for an excess class, or it isn't.
In the first case γ_2 is given by the the R-action edge contribution, and we get an excess class term resulting in a contribution

$$R^{-1}(\psi'_e)\eta^{-1}R^{-1}(\psi''_e)^t - \eta^{-1}$$

placed in the middle.
In the second case, the edge has no class attached to it, the CohFTs rule tells us that to consider the whole graph class is to multiply the two classes with η in the middle. The resulting contribution is given by placing

$$R^{-1}(\psi'_e)\eta^{-1}R^{-1}(\psi''_e)^t$$

at the special edge of Γ that is exactly what is required in the CohFT pullback property. The R^{-1} term is the leg term before glueing the two graphs, η is the way of glueing for CohFTs.

Proposition 3.8. The above defined R-action is indeed an left-group action.

Proof. Let R_a and R_b be two matrices satisfying the symplectic condition, to express $R_a R_b \Omega$ as a sum of graph contributions we can reason as this. The action of R_a will give a sum over stable graphs, with the usual contributions at legs and edges but where at each vertex $R_b \Omega_{g_v,n_v}$ is placed. By linearity we can place then at each vertex a graph of type g_v, n_v, having a leg for each half edge of the original vertex, some of this legs are glued to for edges in the bigger graph, and on them the $R_b^{-1}(\psi_l)$ is of course placed. We ended up with a sum over "big" graphs of a sum over "small graphs", we can just sum over all graphs of type g, n with two kinds of edges, "big" and "small". To each leg, contributions from both actions will be attached, resulting in $R_a^{-1}(\psi_l) R_b^{-1}(\psi_l)$. The contribution for an edge depends on whether it is an edge of the "big" or "small" graph. On "small" edges we have

$$\frac{\eta^{-1} - R_b^{-1}(\psi'_e)\eta^{-1} R_b^{-1}(\psi''_e)^t}{\psi'_e + \psi''_e},$$

while on "big" edges are actually glued from two legs, so while

$$\frac{\eta^{-1} - R_a^{-1}(\psi'_e)\eta^{-1} R_a^{-1}(\psi''_e)^t}{\psi'_e + \psi''_e}$$

is placed in the center of the edge, $R_b^{-1}(\psi')$ and $R_b^{-1}(\psi'')$ are placed on the extremities, resulting in

$$\frac{R_b^{-1}(\psi')\eta^{-1} R_b^{-1}(\psi'') - R_b^{-1}(\psi') R_a^{-1}(\psi'_e)\eta^{-1} R_a^{-1}(\psi''_e)^t R_b^{-1}(\psi'')^t}{\psi'_e + \psi''_e},$$

since each edge in each graph will present itself in both forms, the resulting contribution for the sum ignoring this distinction will be:

$$\frac{\eta^{-1} - R_b^{-1}(\psi') R_a^{-1}(\psi'_e)\eta^{-1} R_a^{-1}(\psi''_e)^t R_b^{-1}(\psi'')^t}{\psi'_e + \psi''_e}.$$

So the resulting class is exactly the same as for the action of $R_a R_b$.

Proposition 3.9. Let $R(z) \in zEnd(V)[[z]]$ satisfying the symplectic condition. Let $T_a, T_b \in z^2 V[[z]]$ satisfying $T_a(z) = R(z) T_b(z)$, then:

$$T_a R\Omega = R T_b \Omega$$

for any CohFT Ω.
Proof. Consider first $T_a R\Omega$ so we get terms of the kind

$$R\Omega_{g,n+m}(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m})).$$

The graph analogue of this is a sum over stable graphs with n main legs and m κ-legs, the result is a sum over all such stable graphs with usual R contributions except for the κ-legs, those have

$$R^{-1}T_a(\psi) = T_b(\psi)$$

attached to it.
In the second case RT_b is then very similiar, thinking in graph terms, this is the sum over all stable graphs with n legs, to which we then attach m other legs, so the sum is over all graphs that remain stable when we remove the last m legs. We can ignore this issue, by noticing that if the graph is of that kind the contribution is vanishing by degree reasons. In particular if a vertex is of genus zero, and has attached to it less than 3 main legs and h κ-legs the moduli space relative to the vertex is of dimension less than m while the degree of the class is at least $2m$.

Proposition 3.10. (Commutativity of multiple forgettings.)
Consider the next commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+k+m} \arrow[r, "P_k"] \arrow[d, "P_m"'] & \overline{\mathcal{M}}_{g,n+m} \arrow[d, "p_m"] \\
\overline{\mathcal{M}}_{g,n+k} \arrow[r, "p_k"'] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

Then $(p_k)^*(p_m)_* = (P_m)_*(P_k)^*$
Proof. This is a well known result. To provide a complete proof of the statement, in the case of orbifolds, could be much harder than using birational equivalence ad in [10].

Proposition 3.11. Let p be the map forgetting the last point, then the equation below holds in $H^*(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q})$

$$p^*(\psi_h^d) = \psi_h^d - \Delta_{h,n+1}p^*(\psi_h^{d-1})$$

Where p is the map forgetting a point.
Proof. We will prove it by induction on d, so consider first the base case

$d = 1$. $p^*(\psi_h) = \psi_h - \Delta_{h,n+1}$. Where $\Delta_{h,n+1}$ is the divisor on the moduli space of surfaces that become unstable when removing the $n+1$th point, i.e. the surfaces with the marked points h and $n+1$ on the same genus zero component. We expect two components, the line bundle and one that lives in $\Delta_{h,n+1}$. Since the other terms are of codimension one we know $p^*(\psi_h) = \psi_h + \alpha \cdot \Delta_{h,n+1}$. We can see $\Delta_{h,n+1}$ as the divisor of nodes of these collapsing maps in the universal curve $\mathcal{C}_{g,n}$. Consider the map

$$r : \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+1},$$

this map is a section of the universal curve. When we pullback our expressions through this map we get

$$\psi_h = p_* \alpha D_i^2.$$

Where, D_i is the divisor of ith marked points on $\overline{\mathcal{M}}_{g,n}$. Clearly, $r^* p^* = id$ and so, also $r^* \psi_i = 0$ because ψ has never vanishing sections on Δ_i, while $r^* \Delta_i$ is basically $p_* D_i^2$ as when we pull back we have to intersect with the image of r that is exactly Δ_i, then using the isomorphism of families, it is the same as $p_* D_i^2$ From which we deduce $\alpha = -1$.

The inductive step is easy,

$$p^*(\psi_h^d) = (\psi_h - \Delta_{h,n+1})^d = \psi_h^d - \Delta_{h,n+1}(\psi_h - \Delta_{h,n+1})^{d-1}$$

as ψ_h times Δ_i is trivial.

Remark. From this result we learn that, when pulling back ψ classes though p we get two components. One is the class itself and behaves well when pushing forward, the other shows an unstable behaviour and lives on the divisor that collapses when forgetting, in this cases we also lose a power of ψ. This distinct behaviour will make itself very evident in the next theorem.

Definition 3.12. (Unit preserving R-matrix action)
Let $(\Omega)_{g,n}$ be a CohFT with unit $\mathbf{1} \in V$, and let $R(z)$ be an R-matrix satisfying the symplectic condition, finally let

$$T_a(z) = z \cdot [R(\mathbf{1}) - \mathbf{1}](z), T_b(z) = z \cdot [\mathbf{1} - R^{-1}(\mathbf{1})](z) \in z^2 V[[z]].$$

We will denote by:

$$R.\Omega := RT_b\Omega,$$

the unit preserving R-matrix action on Ω.

Remark. Since $T_a = RT_b$, $RT_b\Omega$ and $T_aR\Omega$ define the same CohFT.

Proposition 3.13. (Preserves unit)
Let $(\Omega)_{g,n}$ be a CohFT with unit $\mathbf{1} \in V$, and $R(z)$ be an R-matrix satisfying the symplectic condition, then $R.\Omega$ is a CohFT with unit.
Proof. We have to verify the pullback properties

$$R.\Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbf{1}) = p^*(R.\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n))$$

and

$$\Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbf{1}) = \eta(v_1, v_2),$$

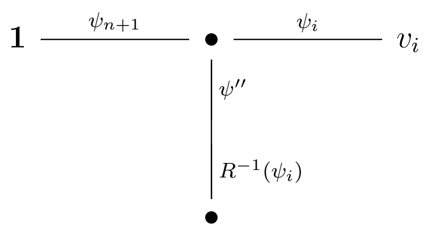
as before we will study the graph pullbacks of the graph contributions.

As we have seen, the pullback of ψ classes gives two contributions, one arises from curves with tame behaviour and the other from curves that collapses when forgetting a point.

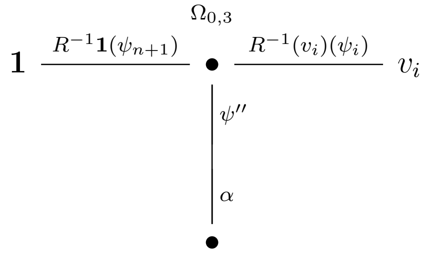
So fix a stable graph Γ, there are four ways of attaching a leg to Γ

- Just attaching an extra leg to a vertex.
- Picking a main leg, and extending the graph with a genus zero vertex with the $n+1$ marked leg, with attached the (1) vector.
- The same as the previous but on a κ leg.
- Placing a genus zero vertex between the two half-egdes of an edge.

Each of these give a contribution that we can deduce from the rules of pullbacks we have seen. The ψ class pullback presents itself in two terms, one is the one we obtain in the first case, i.e. the stable after forgetting stable graphs. The second term is the one present on the other three cases. In this cases, we will account for the loss of a *psi* factor. The first case is trivial, we simply have the starting class multiplied by ψ_{n+1} with attached the vector $\mathbf{1}$. Second case: This is the first case in which we will have to rearrange the cohomology classes on the graph to make it assume the form we wish.



Since on $\overline{\mathcal{M}}_{g,n}$ every cohomology class is the trivial, we can substitute $\psi_{n+1} = R^{-1}(\psi_{n+1})$ and $\psi_i = R^{-1}(\psi_i)$, on the vertex we should place $\eta(v_i, v_i'') = \Omega_{0,3}(v_i \otimes v_i'' \otimes \mathbf{1})$ we will account for it in the computation of the term α.



We originally just had the term $R^{-1}(\psi_i)$,then pushing it forwards, taking into account only the terms dependent on ψ_i and dropping a power of $p^*(\psi_i)$ as required by the previous proposition, we get:

$$\frac{[v_i - R^{-1}(v_i)](p^*(\psi_i)}{p^*(\psi_i)}.$$

Consider that $\psi'' = 0$ and we are short of a η^{-1} term from the previous manipulation we can write

$$\alpha = \frac{\eta^{-1} - R^{-1}(p^*(\psi_i))\eta^{-1}R^{-1}(\psi'')^t}{p^*(\psi_i) + \psi''}.$$

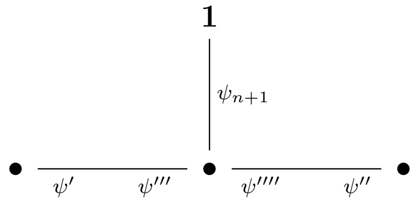
What we get is exactly the graph contribution relative to the unit preserving R-action.

The third case is similiar to the second, attach a vertex to the κ leg, and pushforward by , but we will use the commutativity of the pushforwards by forgetting maps. Before the pullback on the κ leg, we have $T_b(\psi_i)$. pulling back we lose a power of ψ_i getting:

$$-\frac{T_b(\psi_i)}{\psi_i} = [R(\mathbf{1}) - \mathbf{1}](\psi_i).$$

Now forgetting the ith leg, by the fact that forgetting of a point is isomorphic to the universal curve morphism, the class becomes $R(\mathbf{1}) - \mathbf{1}](\psi_{n+1})$, and the graph is the same as in the first case, which contributed ψ_{n+1}. Summing up the two contributions we get $R^{-1}(\mathbf{1}(\psi_{n+1}$, exactly the R. contribution for a leg.

The fourth case: more precisely, we will place the vertex at the end of one half-edge and then glue the other, so for each edge we get two contributions relative to two pullbacks, we will consider the two together as they simplify.



So, before the pullback we have the standard edge insertion,

$$\frac{\eta^{-1} - R^{-1}(\psi')\eta^{-1}R^{-1}(\psi'')^t}{\psi' + \psi''}.$$

Pulling back we get rid of the ψ' free terms and divide by ψ':

$$-\frac{1}{\psi'}\Big[\frac{\eta^{-1} - R^{-1}(\psi')\eta^{-1}R^{-1}(\psi'')^t}{\psi' + \psi''} - \frac{\eta^{-1} - \eta^{-1}R^{-1}(\psi'')^t}{\psi''}\Big].$$

The two contributions add up to

$$\frac{\eta^{-1} - \eta^{-1}R^{-1}(\psi'')^t - R^{-1}(\psi')\eta^{-1} + R^{-1}(\psi')\eta^{-1}R^{-1}(\psi'')^t}{\psi'\psi''},$$

that is equal to

$$\frac{\eta^{-1} - R^{-1}(\psi')\eta^{-1}}{\psi'}\eta\frac{\eta^{-1} - \eta^{-1}R^{-1}(\psi'')^t}{\psi''}.$$

Precisely the product of the two new edges contribution and of the new vertex contribution.

This concludes the proof.

Proposition 3.14. The unit preserving R-action is a left group action.
Proof. We wish to apply Proposition 3.9, to manipulate the formula $R_a.R_b.(\Omega) = R_a(z(\mathbf{1} - R_a^{-1}))R_b(z(\mathbf{1} - R_b^{-1}))$ in fact notice we know that $T_aR_b = R_bT'_a$ in terms of action, if

$$T'_a = z(\mathbf{1} - R_a^{-1})R_B^{-1},$$

as a series.
Therefore,

$$R_a(z(\mathbf{1} - R_a^{-1}))R_b(z(\mathbf{1} - R_b^{-1})) = R_aR_b(R_b^{-1}(z(\mathbf{1} - R_a^{-1})))(z(\mathbf{1} - R_b^{-1})).$$

Here we can just compose the two T actions, getting the action of the sum of the series:

$$R_aR_b(z(R_b^{-1}(\mathbf{1}) - R_b^{-1}R_a^{-1}(\mathbf{1}) + (\mathbf{1} - R_b^{-1}(\mathbf{1})) = (R_aR_b)(z(\mathbf{1} - (R_aR_b)^{-1}(\mathbf{1}))).$$

3.2 Witten's class

Definition 3.15. (Frobenius Manifold)
A Frobenius Manifold is a smooth manifold equipped with:

- A Flat pseudo-Riemannian metric.
- A function F whose third covariant derivatives F_{abc} are structure constants $(a \bullet b, c)$ of a Frobenius Algebra structure.
- The vector field of unities $\mathbf{1}$ of the $\bullet$ product is covariantly constant and preserve multiplication and the metric.

Definition 3.16. (Euler Field)
On a Frobenius Manifold an Euler Field is a vector field such that, the $\bullet$ product, the unity vector field $\mathbf{1}$ and the norm η are eigenvectors of the Lie Derivative $\mathcal{L}_E$, of eigenvalues 0, -1 and $2-\delta$ respectively. Where δ is called conformal dimension. A Frobenius structure with an Euler Field is called conformal, and allows us to introduce some notion of homogeneity.

Remark. From now on we will assume the vector space V relative to CohFTs is a Frobenius Manifold, defined though the CohFT's genus zero sector.

Definition 3.17. (Action of an Euler Field on a CohFT)
Let Ω be a CohFT, and let E be the Euler Field on the Frobenius Manifold V

$$E = \sum_i (\alpha_i t^i + \beta_i) \frac{\partial}{\partial t^i},$$

then we define the action

$$(E.\Omega)_{g,n}(\partial_{i_1} \otimes \cdots \otimes \partial_{i_n}) =$$

$$= \big(deg + \sum_{i=1}^{n} \alpha_i\big)\Omega_{g,n}(\partial_{i_1} \otimes \cdots \otimes \partial_{i_n}) + p_*\Omega_{g,n+1}(\partial_{i_1} \otimes \cdots \otimes \partial_{i_n} \otimes \sum_i \beta_i \partial_i),$$

where p is the map forgetting the last marked point, and

$$deg : H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \to H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

is the function multiplying the class by k.

Definition 3.18. We say a CohFT is homogeneous if $E.\Omega = [(g-1)\delta + n]\Omega$.

Definition 3.19. (Semisimple Frobenius Manifolds)
A Frobenius manifold is said to be semisimple if the algebras $(T_tH, \bullet)$ are semisimple. Since these algebras are finitely dimensional this means that the algebras are cartesian product of two simple subalgebras.

Remark. We can now introduce the main Frobenius Manifold we will study. Let V_r be an $r-1$ dimensional vector space over $\mathbb{Q}$, $e_0, \dots, e_{r-2}$ a basis of V_r. We can define on V_r a non-degenerate bilinear form by

$$\eta_{a,b} = \eta(e_a, e_b) = \delta_{a+b,r-2},$$

with $\mathbf{1} = e_0$ the unit vector. On this space, we will define a Frobenius Manifold structure using Witten's r-spin theory.

Remark. Witten's r-spin theory provides us with cohomology classes parameterized by indices $a_1, \dots, a_n \in \{0, \dots, r-2\}$ as $W_{g,n}(a_1, \dots, a_n) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and we can then define a CohFTs by

$$W_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) = W_{g,n}(a_1, \dots, a_n).$$

The complex degree of the class is given by

$$D_{g,n}(a_1, \dots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r},$$

in the case that D is not an integer, the class is vanishing.

The class satisfies

$$W_{0,3}(a_1, a_2, a_3) = \begin{cases} 1 & a_1 + a_2 + a_3 = r-2 \\ 0 & otherwise \end{cases}$$

$$W_{0,4}(1, 1, r-2, r-2) = \frac{1}{r}[point] \in H^2(\overline{\mathcal{M}}_{0,4}, \mathbb{Q}).$$

The primary genus zero Gromov-Witten Potential of Witten's r-spin class is:

$$F(t^0, \dots, t^{r-2}) = \sum_{n \geq 3} \sum_{a_1, \dots, a_n} \int_{\overline{\mathcal{M}}_{g,n}} W_{0,n}(a_1, \dots, a_n) \frac{t^{a_1}, \dots, t^{a_n}}{n!}$$

In the case $r = 3$, it becomes

$$F = \frac{1}{2}xy^2 + \frac{1}{72}y^4.$$

Defining the quantum product on the tangent space:

$$\partial_i \bullet \partial_j = \sum_{k,l} \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k} \eta^{kl} \partial_l.$$

In the case $r = 3$ Algebra defined by this product is semisimple only outside of $\{y = 0\}$, this is why we will define the translated Witten's class.

The Euler Field we will use is

$$E = \sum_{i=0}^{r-2} (1 - \frac{i}{r}) t^i \frac{\partial}{\partial t^i}, \qquad \delta = \frac{r-2}{r},$$

resulting in conformal dimension δ.

$$E = x\frac{\partial}{\partial x} + \frac{2}{3} y \frac{\partial}{\partial y}$$

$$L_E(\partial_x) = -\partial_x \quad L_E(\partial_y) = -\frac{2}{3}\partial_y.$$

Definition 3.20. The shifted Witten's class is:

$$W^{\tau}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{m \geq 0} \frac{1}{m!} (p_m)_* W^{\tau}_{g,n}(v_1 \otimes \cdots \otimes v_n \otimes \tau \otimes \cdots \otimes \tau),$$

where τ appears m times.

Proposition 3.21. The shifted Witten's class is a CohFT with unit.
Proof Witten's class is a Cohft with unit. To study the pullback of W^{τ} we will inspect the pullback of each term

$$\frac{1}{m!} p_{m*}(R_{g,n}(v_1 \otimes \cdots \otimes v_n \otimes \tau \otimes \cdots \otimes \tau)),$$

though the natural morphism of strata from a single edge with legs to a single vertex with legs. Since we will be forgetting the κ legs, we can group all the contributions in the pull back by the number of κ legs for each vertex, so there are $\binom{m}{m_1 m_2}$ copies of the class.

$$r* \frac{1}{m!} p_{m*}(W_{g,n}(v_1 \otimes \cdots \otimes v \otimes \tau \otimes \cdots \otimes \tau)) = \binom{m}{m_1 m_2} \frac{1}{m!} p_{m_1 *}(W_{g_1, n_1 + m_1}) \eta p_{m_2 *}(W_{g_2, n_2 + m_2}).$$

While the unit property just derives from the commutativity of forgetful maps, and the fact that Witten's class itself is a CohFT with unity.

Remark. An important remark is that, in the above summation, for $m = 0$ we get Witten's class, and for $m > 0$ the terms added have degree at most

$$D_{g,n} - \frac{2m}{r}.$$

This means that we're only lower degree terms, so the classes of degree greater than $D_{g,n}$ will be relations.

The primary genus zero potential for Witten's shifted class is

$$F^\tau(\hat{t}) = F(\tau + \hat{t}) - (\text{terms of degree} < 3)$$

Definition 3.22. The shifted grading operator is defined as

$$\mu(v) = [E, v] + (1 - \frac{\delta}{2})v, \quad \mu(\partial_x) = -\frac{1}{6}\partial_x, \quad \mu(\partial_y) = \frac{1}{6}\partial_y,$$

to compute the commutator, we extend the vector to a flat vector field.

Remark. Introducing a new frame to simplify the computations,

$$\phi = \frac{y}{4}, \quad \partial_x = \partial_0 \phi^{\frac{1}{4}}, \quad \partial_y = \phi^{-\frac{1}{4}} \partial_y.$$

Using the formula

$$\eta(v_1 \bullet v_2, v_3) = W_{0,3}(v_1 \otimes v_2 \otimes v_3),$$

we can write

$$W_{0,3}(0, 1, y) = \begin{cases} 1 & y = 0 \\ 0 & y = 1 \end{cases} \qquad W_{0,3}(1, 1, y) = \begin{cases} 0 & y = 0 \\ 1 & y = 1 \end{cases}$$

Deducing

$$e_0 \bullet e_0 = e_0 \quad e_0 \bullet e_1 = e_1 \quad e_1 \bullet e_1 = e_0.$$

In the alternative frame:

$$\hat{\partial}_0 \bullet \hat{\partial}_0 = \phi^{\frac{1}{4}} \hat{\partial}_0, \quad \hat{\partial}_0 \bullet \hat{\partial}_1 = \phi^{\frac{1}{4}} \hat{\partial}_1, \quad \hat{\partial}_1 \bullet \hat{\partial}_1 = \phi^{\frac{1}{4}} \hat{\partial}_0.$$

Remark. The shift degree operator requires us extend a certain vector to a flat vector field, for $\hat{\partial}_x$ we can just extend the vector to the flat vector field $c \cdot \partial_x$, where x depends on the value of ϕ at the base point of the vector.

Proposition 3.23. Witten's class is homogeneous with respect to the Euler Field we fixed.
Proof Since $\beta_i = 0$, we just have to check the coefficient $deg + \sum \alpha_i$,

$$\frac{(r-2)(g-1) + \sum a_i}{r} + \sum_i (1 - \frac{a_i}{r}) = (g-1)\delta + n.$$

Proposition 3.24. The shifted Witten's class W^τ is homogeneous, with Euler Field:

$$E = \sum_a ((1 - \frac{a}{r})\hat{t}^a + (1 - \frac{a\tau_a}{r})\frac{\partial}{\partial \hat{t}^a}.$$

Proof Let's see the case $\tau = u\partial_k$, for a fixed k, first. Denote

$$W_{g,n+m}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_b \otimes \cdots \otimes \partial_b)$$

by just $W_{g,n+m}$. The degree of $p_{m*}W_{g,n+m}$ is $D_{g,n+m} - m$, as forgetting a point decreases the dimension of the space by one.

$$(E.W^\tau)_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) =$$

$$\frac{u^m}{m!}(\sum_{m\geq 0} \frac{(r-2)(g-1) + \sum a_i + mb}{r} - m + \sum (1 - \frac{a_i}{r})p_{m*}W_{g,n+m}$$

$$+ \sum_{m\geq 0} \frac{u^m}{m!} u(1 - \frac{b}{r})p_{m+1*}W_{g,n+m+1}$$

The coefficient of the second term, in the right hand side, is simply β_i. Rearranging the terms,

$$((g-1)\delta + n)\sum_{m\geq 0} \frac{u^m}{m!}(p_{m*})W_{g,n+m}$$

$$- \sum_{m\geq 1} \frac{u^m}{(m-1)!}(1 - \frac{b}{r})(p_{m*})W_{g,n+m}$$

$$+ \sum_{m\geq 0} \frac{u^{m+1}}{m!}(1 - \frac{b}{r})(p_{m+1*})W_{g,n+m+1}$$

The last two terms cancel out.

Now to prove the general case we will do the same steps. The key to make the terms cancel out is that $\beta_i = \alpha_i \cdot \tau^i$.

Let

$$\tau = \sum_i \tau_i \partial_i$$

$$(E.W^\tau)_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) =$$

$$\sum_{m\geq 0}\sum_{\bar{b}} \frac{1}{m!}\Big(\frac{(r-2)(g-1) + \sum a_i + \sum b_i}{r} - n + \sum_i (1 - \frac{a_i}{r}) + \Big)p_{m*}W^{\bar{b}}_{g,n+m}$$

$$+ p_{m+1*}W^{\bar{b}}_{g,n+m+1}$$

Where

$$W^{\bar{b}}_{g,n+m} = W_{g,n+m}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \tau_{b_1}\partial_{b_1} \otimes \cdots \otimes \tau_{b_n}\partial_{b_m})$$

$$W^{\bar{b}}_{g,n+m+1} = W_{g,n+m+1}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \tau_{b_1}\partial_{b_1} \otimes \cdots \otimes \tau_{b_n}\partial_{b_m} \otimes \sum_j \beta_j \partial_j)$$

Simplifying:

$$\begin{aligned} &= ((g-1)\delta + n) \sum_{m\geq 0} \sum_{\bar{b}} \frac{1}{m!}(p_{m*}) W^{\bar{b}}_{g,n+m} \\ &\quad - \sum_{m\geq 1} \sum_{\bar{b}} \sum_{j} \frac{1}{m!}(1 - \frac{b_j}{r})(p_{m*}) W^{\bar{b}}_{g,n+m} \\ &\quad + \sum_{m\geq 0} \sum_{\bar{b}} \frac{1}{m!}(p_{m+1*}) W^{\bar{b}}_{g,n+m+1}. \end{aligned}$$

The last two series cancel each other out, simply the terms relative to $\overline{b}$ of lenght $m+1$ in the second term, simplify with the terms relative to $\overline{b}$ of lenght m in the third term. The third term series, once broken into n components by linearity of the $n+m+1$ entry, presents every possible combination of ∂_{b_i} with a $\beta_{b_{m+1}}$ term instead of a $\tau_{b_{m+1}}$ coefficient. The second series presents every combination with a β_{b_i} and τ every where else. So now by the first property of CohFTs we can move these b_i terms in last position, gaining a factor m, that balances the $m!$ becoming $m+1$ of the shift.

Proposition 3.25. Topological sector of Witten's 3-spin.

$$\omega_{g,n}(\partial_0^{\otimes n_0} \otimes \partial_1^{\otimes n_1}) = 2^g \psi^{\frac{2g-2+n}{4}} \delta^{odd}_{g+n_1},$$

where ω is the Topological Field Theory of Witten's 3-spin theory.
Proof Since we're working with degree zero classes, via the pullback property of CohFTs, we can pullback our class to a Moduli space of the kind

$$\prod \overline{\mathcal{M}}_{0,3},$$

i.e. the surfaces are represented by rational curves glued together at nodes, forming a stable curve of the same genus and number of legs. In terms of stable graphs this means that our class is product of $W_{0,3}$ terms with η in the middle.
The vectors in the argument are distributed on special points, and at nodes, we must have a 0 term and a 1 term to have η be non vanishing.
Let us start by noticing that $W_{0,3}(i,j,k)$ is different than zero if and only

if the number of 1s in the argument is odd, and in that case it equals $\psi^{\frac{1}{4}}$. Such placement of vectors is possible if and only if $n_1 + g$ is odd, this is easily proven by starting out from a genus zero surface with three marked points, and noticing that any glueing we can do to it maintaining the non-vanishing of the class preserves this property. In this case, we would have a graph of genus g, this means it has g independent cycles, this cycles can be "inverted" independently, hence provide us with the 2^g term. By inverting a cycle we mean to take each node and swap the 0 and 1 term. The last thing missing is to realise that there are $2g - 2 + n$ components, this is done by breaking the g cycles, getting $2g + n$ marked points, again by recursion we can see that this is 2 more than the number of components.

Proposition 3.26. (Givental-Teleman Theorem)
Let $\Omega_{0,n}$ be a genus 0 homogeneous, semisimple CohFT with unit. Then the following hold:

- There exists a unique homogeneous CohFT with unit $\Omega_{g,n}$ extending $\Omega_{0,n}$ to higher genus.
- The extended CohFT $\Omega_{g,n}$ is obtained by an R-Matrix action on the topological sector of $\Omega_{0,n}$ determined by $\Omega_{0,3}$
- The R-matrix is uniquely specified by the $\Omega_{0,n}$ determined by $\Omega_{0,3}$.

Remark. In the following computations we will need to have the quantum product by E operator in matrix representation. In the hat frame it assumes the form:

$$\xi = \begin{bmatrix} x & 2\phi^{\frac{3}{2}} \\ 2\phi^{\frac{3}{2}} & x. \end{bmatrix}$$

Proposition 3.27. (Teleman's recursive formula for R-matrix)
At a semisimple point of a conformal Frobenius manifold, starting with $R_0 = 1$

$$[R_{m+1}, \xi] = (m + \mu)R_m$$

Proof. Substituting our expressions for ξ and μ in the formula, we get:

$$\left[\begin{bmatrix} a_{m+1} & b_{m+1} \\ c_{m+1} & d_{m+1} \end{bmatrix}, \begin{bmatrix} x & 2\phi^{\frac{3}{2}} \\ 2\phi^{\frac{3}{2}} & x \end{bmatrix} \right] = \frac{1}{6} \begin{bmatrix} 6m-1 & 0 \\ 0 & 6m-1 \end{bmatrix} \begin{bmatrix} a_m & b_m \\ c_m & d_m \end{bmatrix}$$

The only solutions are

Proposition 3.28. The R-matrix is

$$R(z) = \begin{bmatrix} B_1^{\text{even}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) & -B_1^{\text{odd}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) \\ -B_0^{\text{odd}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) & B_0^{\text{even}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) \end{bmatrix}$$

By using the relation between B_0 and B_1 discussed in the previous chapter we get:

$$R^{-1}(z) = \begin{bmatrix} B_0^{\text{even}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) & B_1^{\text{odd}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) \\ B_0^{\text{odd}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) & B_1^{\text{even}}\left(\frac{z}{1728\phi^{\frac{3}{2}}}\right) \end{bmatrix}$$

This also shows, that the matrix respects the symplectic condition.

Proposition 3.29. Witten's class explicit formula.

$$W^{\tau}_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = 2^g \sum_d \frac{\phi^{\frac{3}{2}(D-d)}}{1728^d} q(\mathcal{R}^d_{b,(a_1,\dots,a_n)})$$

Proof. Teleman's theorem tells us that $W^{\tau} = R.\omega$, as we have seen in the second chapter, the $\mathcal{R}$ relations have a very similar structure, we have to verify that coefficients are exactly the same. Let us verify it for each tensor product of ∂_x and ∂_y, $R.\omega(\partial_0^{\otimes n_0} \otimes \partial_1^{\otimes n_1})$, now we will match graph by graph each contribution. The contributions of the R-matrix action are

- To each leg $R^{-1}(\psi_l)$
- To each vertex $W^{\tau}_{g(v),n(v)}$
- To each edge
$$\frac{\eta^{-1} - R^{-1}(\psi'_e)\eta^{-1}R^{-1}(\psi''_e)^t}{\psi'_e + \psi''_e}$$
- Then considering all possible ways of adding κ legs with

The terms in the $\mathcal{R}$ contribution are:

- At each vertex place $\kappa_v = \kappa(T - TB_0(\zeta_v T))$
- At each leg place $B_l = \zeta_v^{a_l} B_{a_l}(\zeta_v \psi_l)$
- And at each edge place
$$\Delta_e = \frac{\zeta' + \zeta'' - B_0(\zeta'\psi')\zeta'' B_1(\zeta''\psi'') B_1(\zeta'\psi')\zeta' B_0(\zeta''\psi'')}{\psi' + \psi''},$$

The vertex term of the $\mathcal{R}$ is exactly the sum of the κ leg contributions of the T action of R.

The auxiliary variables ζ_v serve to check the parity condition for ∂_y at each vertex. The form of the T series is

$$T(z) = z \cdot [\partial_x - R^{-1}(\partial_x)](z),$$

as $\mathbf{1} = \partial_x$.

- Powers of ϕ
- Powers of 1728
- Powers of 2
- ζ_v

Consider powers of ϕ first, since we used the frame $(\hat{\partial}_x, \hat{\partial}_y)$ to write the R matrix in the argument of R when we take it out we get:

$$\phi^{\frac{n_1 - n_0}{4}}.$$

Then the R matrix coefficients contain ϕ terms, these show in R_m^{-1} that gives a $\psi^{-\frac{3m}{2}}$ factor, and similarly for $T(z)$. So, the coefficients of R contribute a power $-3d/2$, where d is the degree of the monomial as a cohomology class. Then at each vertex ω contributes a power $(g_v - 2 + n_v)/4$ as we have seen, these sum to $(2g-2+n)/4$, as, the term $h^1(\Gamma)$ we are missing in the sum of genera, we recover it in the sum of valencies. Finally, for each κ leg we get a $\phi^{-\frac{1}{4}}$ from the change of variables, but also a $\phi^{\frac{1}{4}}$ term in the ω contribution, by increasing the valency of the vertex by one. The resulting power is

$$\frac{n_1 - n_0}{4} - \frac{3d}{2} + \frac{2g - 2 + n}{n} = \frac{3}{2}(D - d)$$

As for the ψ terms in the R_m coefficients, we obtain a factor 1728^{-d}.

At each vertex, ω contributes a power g_v, while in the definition of $\mathcal{R}$ relations we account for a power $-h^1(\gamma)$. In the end the difference is precisely 2^g, as required in the statement.

Now we wish to check that the parity condition for the non-vanishing of ω.

$$g_v + n_1 \text{is odd at each vertex.}$$

is equivalent to the condition

$$g_v - 1 + m \text{is even at each vertex.}$$

where m is the power of ζ_v in the $\mathcal{R}$ relation expression. A ζ_v factor comes out from:

- The terms B_0^{odd} and B_1^{odd}, while, since $\zeta_v^2 = 1$ the even components, don't contribute. So, every edge insertion, whereas the ∂ side is towards the vertex, accounts for a power one.
- Every leg marked by ∂_y

The last thing one should notice is that the edge contribution for $\mathcal{R}$ relations, is the sum of the two contributions of the R action, when the bivector is oriented in a way or the other, and marking it with ζ_v if ∂_y is pointing at v.

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