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Tilting objects in functor category

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Introduction

Category theory has played a central role in mathematical research during the last few decades, mainly because of its level of abstractness which allows to apply its tools and techniques to most of the classical fields of algebra and further develop the knowledge of algebraic structures.

Because of its properties, one of the most studied and useful categories is the category of modules over a ring R: so much so that, for instance, it has been proved by Gabriel and Popescu that any Grothendieck category can be embedded, as a Giraud subcategory, in a category of modules over a certain ring. This is just an example of how crucial it is to deeply understand the properties of this category.

Over the last years, several techniques have been developed to do so. One of the approaches, proposed by M. Auslander in the environment of representation theory, is to consider the category of (contravariant additive) functors from the category of finitely presented modules over a ring R to the category of abelian groups: as proved in Chapter 2 of this thesis, this is a Grothendieck category in which the category of modules itself turns out to be a Giraud subcategory, via the so-called Yoneda embedding.

Another fundamental tool, also risen from representation theory, has been the notion of tilting module and, more generally, tilting object in a certain category. Tilting theory allows, for instance, to see the relations between a category of modules over a ring or an algebra and the category of modules over the endomorphism ring of a tilting object in the starting category, and this correspondence can be generalized to a tilting theory for abelian categories.

The aim of this work is to investigate both approaches and to see how they can be related, by applying the abstract tilting theory to the category of functors mentioned above: in particular, it is shown that some properties can be proved when a tilting object in the functor category is seen (or "localized") in its Giraud subcategory R-Mod.

In the first Chapter of this thesis Grothendieck categories and Giraud subcategories are introduced, together with some of their basic properties and interrelations. The third and fourth sections of the Chapter are devoted to introducing torsion theories and seeing how they can be transferred from a Grothendieck category to its Giraud subcategory and vice versa; a similar correspondence holds when considering Serre subcategories.

In Chapter 2 the functor category is introduced, first in the general setting $(\mathcal{B}^{op}, Ab)$ where $\mathcal{B}$ is a small preadditive category, and then in the special case of the category $((R\text{-mod})^{op}, Ab)$. For instance, it is shown that this is a Grothendieck category of which R-Mod is a Giraud subcategory, and it is shown how the finitely generated and presented objects behave in these categories. In particular, the second section of the Chapter deals with the relation of torsion pairs introduced above and applied to this specific setting, where it is proved that a less strong hypothesis is required in order to achieve such correspondence. The third section introduces the notion of pure-injective envelope in an abstract category, and proves the existence of such objects in a general functor category $(\mathcal{B}^{op}, Ab)$; this is a strong result that allows to characterize flat cotorsion objects in the category $((R\text{-mod})^{op}, Ab)$, which turn out to be precisely the embedded pure-injective modules of R-Mod.

Finally, the third Chapter of the thesis is devoted to introducing Tilting theory and showing how the main result, namely the Tilting Theorem, can be proved in the general setting of abelian categories. Furthermore, sections two and three show the results that emerged during the drafting of this thesis when considering a tilting object $V \in ((R\text{-mod})^{op}, Ab)$ (both in the case when it is small and when it isn't) and localizing it to the category of modules: it is shown that the module $V(R)$ shares some properties with a tilting module and two equivalent characterizations are given for it to actually be a tilting module.

Chapter 1

Preliminaries

1.1 Grothendieck categories

Grothendieck categories will be the starting point of this work. In this section we will give the definition and state and prove some main results, following [1].

Definition 1.1.1. *An abelian category* $\mathcal{C}$ *is said to be*

(**C1**) *if it has arbitrary coproducts and the functor* $\coprod$ *preserves monomorphisms;*

(**C2**) *if it has arbitrary products and coproducts and, for every family of objects* $\{A_i : i \in I\}$*, the canonical morphism* $\delta : \coprod_{i \in I} A_i \to \prod_{i \in I} A_i$ *is a monomorphism;*

(**C3**) *if it is cocomplete and, for any directed family of subobjects* $\{A_i : i \in I\}$ *of an object* A *and any subobject* B *of* A *one has*

$$(\sum_{i \in I} A_i) \cap B = \sum_{i \in I} (A_i \cap B)$$

In fact, condition (**C3**) can be expressed via some properties of the direct limit functor.

Proposition 1.1.1. *Let* $\mathcal{C}$ *be a cocomplete abelian category. Then the following are equivalent:*

i) $\mathcal{C}$ is a (**C3**) *category;*

ii) given any directed family of subobjects $\{A_i : i \in I\}$ of an object A, their direct limit $\varinjlim_{i\in I} A_i$ is a subobject of A;

iii) given any directed family of subobjects $\{A_i : i \in I\}$ of an object A, their direct limit $\varinjlim_{i\in I} A_i$ coincides with their sum $\sum_{i\in I} A_i$;

iv) given any directed set I, the associated functor $\varinjlim_{i\in I}$ is (left) exact.

Proposition 1.1.2. *Every* (**C2**) *category is* (**C1**), *and every* (**C3**) *category is* (**C2**).

Proof. The first statement follows from the commutativity of the diagram

$$\begin{tikzcd}
\coprod_{i\in I} A_i \arrow[r, "\coprod_{i\in I} f_i"] \arrow[d, hook, "\delta_A"] & \coprod_{i\in I} B_i \arrow[d, hook, "\delta_B"] \\
\prod_{i\in I} A_i \arrow[r, "\prod_{i\in I} f_i"] & \prod_{i\in I} B_i
\end{tikzcd}$$

For the second statement, given a family of objects $\{A_i : i \in I\}$ in $\mathcal{C}$, we can take as a system of indexes the set $\Phi(I)$ of all finite parts F of I. Then the family

$$\coprod_{i\in F} A_i \hookrightarrow \prod_{i\in I} A_i$$

is a directed system of subobjects and

$$\varinjlim_{F\in\Phi(I)} \coprod_{i\in F} A_i = \coprod_{i\in I} A_i$$

□

Definition 1.1.2. *An abelian* (**C3**) *category is said to be a **Grothendieck category** if it has a family of generators.*

We will use the notation $(U, A) := \mathrm{Hom}_{\mathcal{C}}(U, A)$ when it doesn't cause ambiguity.

Proposition 1.1.3. *Any Grothendieck category $\mathcal{C}$ is locally small and any of its objects is the direct limit of its finitely generated subobjects*

Proof. First we want to show that $\mathcal{C}$ is locally small, i.e. that for every object in $A \in \mathcal{C}$ its family of subobjects is a set. We will denote the family of generators of $\mathcal{C}$ as $\{U_i : i \in I\}$. Given $A \in \mathcal{C}$ we want to show an injection

$$\{\text{ subobjects of } A\} \hookrightarrow \prod_{i \in I} \mathcal{P}(U_i, A)$$

by setting

$$(\beta : B \hookrightarrow A) \mapsto \langle \beta \rangle = \prod_{i \in I} \operatorname{Im}(U_i, \beta)$$

where $\operatorname{Im}(U_i, \beta)$ is the set of all morphisms $U_i \to A$ that factor through β:

$$\begin{tikzcd}
U_i \arrow[rr] \arrow[dr, dashed] & & A \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Such a map is well defined because, if $[\beta] = [\beta']$, it is easy to see that $\langle \beta \rangle = \langle \beta' \rangle$. We want to show that it is injective. Let $[\beta] \neq [\beta']$ be two different subobjects of A and let us consider their intersection

$$\begin{tikzcd}
B \cap B' \arrow[r, hook, "\gamma"] \arrow[d, hook, "\gamma'"] & B \arrow[d, hook, "\beta"] \\
B' \arrow[r, hook, "\beta'"] & A
\end{tikzcd}$$

Since the two subobjects are different and $\mathcal{C}$ is abelian, at least one of γ, γ' must not be an epi: let us assume that γ is not epi, so that there must be a diagram

$$\begin{tikzcd}
B \cap B' \arrow[r, hook, "\gamma"] & B \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & C
\end{tikzcd}$$

with $\alpha_1 \neq \alpha_2$ and $\alpha_1 \gamma = \alpha_2 \gamma$. Since the objects U_i generate $\mathcal{C}$, there must be $i \in I$ and a diagram

$$\begin{tikzcd}
U_i \arrow[r, "u"] & B \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & C
\end{tikzcd}$$

with $\alpha_1 u \neq \alpha_2 u$. Then u can not factor through γ, so in the pullback diagram

$$\begin{tikzcd}
U_i \arrow[drr, bend left, "u"] \arrow[dr, dashed, "\nexists v"] \arrow[ddr, bend right, "\nexists u'"'] & & \\
& B \cap B' \arrow[r, hook, "\gamma"] \arrow[d, hook, "\gamma'"] & B \arrow[d, hook, "\beta"] \\
& B' \arrow[r, hook, "\beta'"] & A
\end{tikzcd}$$

there can't exist a morphism v making the upper triangle commutative: this proves that there can't be a morphism $u' : U_i \to B'$ such that $\beta' u' = \beta u$. Then

$$\beta u \in \operatorname{Im}(U_i, \beta) \setminus \operatorname{Im}(U_i, \beta')$$

which proves that $\operatorname{Im}(U_i, \beta) \neq \operatorname{Im}(U_i, \beta')$, and so $\langle \beta \rangle \neq \langle \beta' \rangle$.

For the second part of the statement, notice that given $A \in \mathcal{C}$ the family of finitely generated subobjects of A is directed, so that by Proposition 1.1.1 it admits a direct limit $L \leq A$ in $\mathcal{C}$. If by contradiction we assumed $L \neq A$ then we would have the following commutative diagram with exact row

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & A \arrow[r, "\pi \neq 0"] & A/L \arrow[r] & 0 \\
& & U \arrow[u, "\exists u"] \arrow[ur, "\neq 0"'] & &
\end{tikzcd}$$

where U is a generator and $u : U \to A$ a morphism such that $\pi u \neq 0$. Then u does not factor through L, even though $\operatorname{Im} u$ is a finitely generated subobject of A, which is a contradiction. □

Definition 1.1.3. *Let A be an object in an abelian category $\mathcal{C}$ and let A_0 be a subobject of A. A_0 is said to be* ***essential*** *in A (and A is said an* ***essential extension*** *of A_0) if $A_0 \cap A' \neq 0$ for every subobject $A' \neq 0$ of A.*

Proposition 1.1.4. *Let $\mathcal{C}$ be an abelian category and $A_0 \overset{i}{\hookrightarrow} A$ a subobject of $A \in \mathcal{C}$. Then A_0 is essential in A if and only if every morphism $f : A \to B$ is injective whenever fi is injective.*

Proof. Clearly $\operatorname{Ker} fi = \operatorname{Ker} f \cap A_0$, so that if A_0 is essential in A then the property is true.

Viceversa, assuming A_0 not essential in A, there must be a subobject $0 \neq A' \leq A$ such that $A_0 \cap A' = 0$. Considering the composition $A_0 \overset{i}{\hookrightarrow} A \overset{\pi}{\twoheadrightarrow} A/A'$ we get that Ker $\pi i = A_0 \cap$ Ker $\pi = A_0 \cap A' = 0$, whereas Ker $\pi = A' \neq 0$. $\square$

Proposition 1.1.5. *Let $\mathcal{C}$ be a* (**C3**) *locally small abelian category (e.g. a Grothendieck category) and let $A \hookrightarrow E$ be a monomorphism in $\mathcal{C}$. Then the following are equivalent:*

i) A is essential in E and E is an injective object;

ii) A is essential in E and E is a maximal essential extension of A;

iii) E is a minimal injective extension of A.

Moreover, such an extension is unique up to isomorphisms that induce the identity on A.

Definition 1.1.4. *An extension E of A satisfying the equivalent conditions of Proposition 1.1.4 is called an* ***injective envelope*** *of A and is usually denoted as $E(A)$.*
We will say that the category $\mathcal{C}$ ***has injective envelopes*** *if the object $E(A)$ exists for every $A \in \mathcal{C}$.*
We will say that $\mathcal{C}$ ***has enough injectives*** *if every object A in $\mathcal{C}$ is a subobject of an injective object E in $\mathcal{C}$.*

Our goal is to prove that every Grothendieck category has injective envelopes. First, let $\mathcal{C}$ be an abelian category with a generator U: we can consider the ring $R = \mathrm{Hom}_{\mathcal{C}}(U,U)$ of endomorphisms of U and thus obtain the left exact functor $\mathrm{Hom}_{\mathcal{C}}(U,-) : \mathcal{C} \to$ Mod-R (indeed, for every $A \in \mathcal{C}$ the abelian group $\mathrm{Hom}_{\mathcal{C}}(U,A)$ can be endowed with the structure of right R-module by setting $f \cdot r := f \circ r$ for $f \in \mathrm{Hom}_{\mathcal{C}}(U,A)$ and $r \in R$).

Proposition 1.1.6. *In an abelian category $\mathcal{C}$ with a generator U, if $A_0 \overset{i}{\hookrightarrow} A$ is an essential monomorphism in $\mathcal{C}$, then the monomorphism $Hom_{\mathcal{C}}(U,A_0) \overset{(U,i)}{\hookrightarrow} Hom_{\mathcal{C}}(U,A)$ is essential in Mod-R.*

Proof. We need to show that for every $0 \neq f \in (U,A)$ there exists $r \in (U,U)$

and $0 \neq f_0 \in (U, A_0)$ such that $fr = if_0$:

$$\begin{tikzcd}
U \arrow[r, "\exists r"] \arrow[d, "\exists f_0 \neq 0"'] & U \arrow[d, "\forall f \neq 0"] \\
A_0 \arrow[r, hook, "i"'] & A
\end{tikzcd}$$

Let $I = \mathrm{Im} f$: since $f \neq 0$, I is a non-zero subobject of A. Since A_0 is essential in A it must be $A_0 \cap I \neq 0$. Thus we get the commutative pullback diagram

$$\begin{tikzcd}
U \arrow[r, "\exists"] \arrow[dr, "\neq 0"] & PB \arrow[r, hook] \arrow[d, two heads, "\neq 0"] & U \arrow[d, two heads] \arrow[dd, bend left, "f \neq 0"] \\
 & A_0 \cap I \arrow[r, hook] \arrow[d, hook] & I \arrow[d, hook] \\
 & A_0 \arrow[r, hook, "i"'] & A
\end{tikzcd}$$

which is what we wanted. □

Theorem 1.1.1. *Every Grothendieck category $\mathcal{C}$ has injective envelopes.*

Proof. Let U be a generator in $\mathcal{C}$, R its endomorphisms ring and let A be any object in $\mathcal{C}$. Since the category Mod-R has enough injectives for every ring R, there exists an injective object E in Mod-R and a monomorphism $(U, A) \overset{e}{\hookrightarrow} E$. Let $\mathcal{D}$ be the class

$$\mathcal{D} = \{(B, i, f) | B \in \mathcal{C}, i : A \overset{ess}{\hookrightarrow} B, f : (U, B) \to E, f(U, i) = e\}$$

Then every triple (B, i, f) in $\mathcal{D}$ makes the following diagram commutative:

$$\begin{tikzcd}
(U, A) \arrow[rr, hook, "{(U,i)}"] \arrow[dr, hook, "e"'] & & (U, B) \arrow[dl, hook, "f"] \\
 & E &
\end{tikzcd}$$

where f is a monomorphism because e is, and (U, i) is an essential monomorphism. We can now introduce a relation in $\mathcal{D}$ by setting $(B, i, f) \leq (B', i', f')$ if there exists a monomorphism $v : B \hookrightarrow B'$ such that both diagrams

$$\begin{tikzcd}
 & A \arrow[dl, hook', "i"'] \arrow[dr, hook, "i'"] & \\
B \arrow[rr, hook, "v"'] & & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
(U, B) \arrow[rr, hook, "{(U,v)}"] \arrow[dr, hook, "f"'] & & (U, B') \arrow[dl, hook', "f'"] \\
 & E &
\end{tikzcd}$$

are commutative. One can easily prove that if $(B, i, f) \leq (B', i', f')$ and $(B', i', f') \leq (B, i, f)$ then v is an isomorphism. Moreover, the induced equivalence classes form a partially ordered set, being $\mathcal{C}$ locally small and being the morphisms f actually monomorphisms. Since the direct limit functor is exact in $\mathcal{C}$, one can prove that every chain of this partially ordered set has an upper bound. By the Zorn Lemma, it must admit a maximal element (B^*, i^*, f^*). Then $A \xhookrightarrow[ess]{i^*} B^*$ is the requested injective envelope. □

Proposition 1.1.7. *Let $\mathcal{C}$ be a complete or cocomplete abelian category with a generator U and enough injectives. Then $\mathcal{C}$ has an injective cogenerator.*

Proof. We will assume $\mathcal{C}$ to be complete. Since it is abelian and locally small (by Prop. 1.1.3), the subobjects of U form a set. We can then consider the object $\prod\limits_{J \leq U} U/J$ in $\mathcal{C}$ and let E be an injective object containing it. We want to prove that E is an injective cogenerator: to do so, it is enough to check that for any non-zero $A \in \mathcal{C}$ there exists a non-zero morphism $f : A \to E$. Let $\phi : U \to A$ be a non-zero morphism (it exists since A is non-zero), and let $J_0 := \mathrm{Ker}\phi \neq U$. We then have a non-zero monomorphism $U/J_0 \hookrightarrow A$ and, since E in injective, there must be a non-zero morphism f making the following diagram commutative:

$$\begin{tikzcd}
U/J_0 \arrow[r, hook] \arrow[d, hook] & \prod\limits_{J \leq U} U/J \arrow[r, hook] & E \\
A \arrow[urr, "f"'] & &
\end{tikzcd}$$

□

1.2 Giraud subcategories

In this section we define the notion of reflective and Giraud subcategories of a Grothendieck category and state some of their properties.

Let $\mathcal{G}$ be a complete Grothendieck category and let $\mathcal{C}$ be a full subcategory of $\mathcal{G}$. This means that there is a natural inclusion functor $i : \mathcal{C} \to \mathcal{G}$ which is a fully faithful embedding.

Definition 1.2.1. *$\mathcal{C}$ is said to be a* ***reflective subcategory*** *of $\mathcal{G}$ if there exists a functor $l : \mathcal{G} \to \mathcal{C}$ making (l, i) an adjoint pair. Such a functor is called* ***localization functor****.*

This means that, if $\mathcal{C}$ is a reflective subcategory of $\mathcal{G}$, we have a natural isomorphism

$$\varphi_{A,C} : (lA, C)_{\mathcal{C}} \overset{\cong}{\to} (A, iC)_{\mathcal{G}}$$

for every $A \in \mathcal{G}$ and $C \in \mathcal{C}$ (we will sometimes identify $C = i(C)$).
The adjoint pair brings along the unit and counit of the adjunction, namely $\sigma_A : A \to ilA$ and $\zeta_C : liC \to C$ for $A \in \mathcal{G}$ and $C \in \mathcal{C}$, and these solve the following universal problem: given a morphism $\alpha : A \to C$, with $C \in \mathcal{C}$, there exists a unique $\bar{\alpha} : lA \to C$ making the following diagram commutative:

$$\begin{tikzcd}
A \arrow[r, "\sigma_A"] \arrow[dr, "\alpha"'] & lA \arrow[d, "\bar{\alpha}"] \\
& C
\end{tikzcd}$$

From this it is clear that ζ_C is an isomorphism for every $C \in \mathcal{C}$, i.e. $C \cong liC$.

Proposition 1.2.1. *If there exists $\alpha : ilA \to A$ such that $\alpha\sigma_A = id_A$, then σ_A is an isomorphism*

Proof. $\sigma_A : A \to ilA$ has the two factorizations $\sigma_A = \mathrm{id}_{ilA} \circ \sigma_A = (\sigma_A \circ \alpha) \circ \sigma_A$ over ilA, so the unicity of the factorization implies $\sigma_A \circ \alpha = \mathrm{id}_{ilA}$. Hence σ_A is an isomorphism. □

Proposition 1.2.2. *The reflective subcategory $\mathcal{C}$ is complete and cocomplete.*

Proof. First note that $\mathcal{C}$ is preadditive, since it is a full subcategory of $\mathcal{G}$.
Let $\mathcal{I}$ be a small category and $G : \mathcal{I} \to \mathcal{C}$ a functor. Then $\varprojlim iG$ exists and we denote it by B, with canonical morphisms $\pi_i : B \to G(i)$. Since $G(i) \in \mathcal{C}$, there exist $\bar{\pi}_i : ilB \to G(i)$ such that $\bar{\pi}_i \circ \sigma_B = \pi_i$, $\forall i$. The family $(\bar{\pi}_i)_{i \in \mathcal{I}}$ is compatible with the morphisms in $\mathcal{I}$, for if $\lambda : i \to j$ in $\mathcal{I}$, then

$$G(\lambda)\bar{\pi}_i\sigma_B = G(\lambda)\pi_i = \pi_j = \bar{\pi}_j\sigma_B$$

and hence $G(\lambda)\bar{\pi}_i = \bar{\pi}_j$. Consequently there is induced a morphism $\beta : ilB \to B$ such that $\pi_i\beta = \bar{\pi}_i$ $\forall i$. Then $\pi_i\beta\sigma_B = \bar{\pi}_i\sigma_B = \pi_i$ $\forall i$, so $\beta\sigma_B = \mathrm{id}_B$. It follows from the previous Proposition that σ_B is an isomorphism, so that lB is a limit for G in $\mathcal{C}$.
To prove that $\varinjlim G$ exists in $\mathcal{C}$ is easier, because the left adjoint functor l preserves colimits and we therefore have $l(\varinjlim iG) = \varinjlim liG = \varinjlim G$, since $li \cong \mathrm{id}$. □

This last result shows that, in the above situation, limits in $\mathcal{C}$ are computed in $\mathcal{G}$, whereas colimits in $\mathcal{C}$ are computed in $\mathcal{G}$ and then localized to $\mathcal{C}$:

$$\varprojlim_{\mathcal{C}} G \cong \varprojlim_{\mathcal{G}} iG \quad \textit{and} \quad \varinjlim_{\mathcal{C}} G \cong l(\varinjlim_{\mathcal{G}} iG)$$

Proposition 1.2.3. *If $\mathcal{C}$ is reflective in $\mathcal{G}$ and the localization functor preserves kernels, then $\mathcal{C}$ is an abelian category with exact direct limits and a generator.*

Proof. We have proved that $\mathcal{C}$ is preadditive and has limits and colimits. To show that it is abelian, it only remains to prove that if α is a morphism in $\mathcal{C}$ then Coker(Ker α) $\cong$ Ker(Coker α), and this is true because

$$\mathrm{Coker}_{\mathcal{C}}\ (\mathrm{Ker}_{\mathcal{C}}\alpha) \cong l(\mathrm{Coker}_{\mathcal{G}}\ (\mathrm{Ker}_{\mathcal{G}}\alpha)) \overset{(1)}{\cong} l(\mathrm{Ker}_{\mathcal{G}}\ (\mathrm{Coker}_{\mathcal{G}}\alpha)) \overset{(2)}{\cong}$$
$$\overset{(2)}{\cong} \mathrm{Ker}_{\mathcal{C}}(l\ \mathrm{Coker}_{\mathcal{G}}\alpha) \cong \mathrm{Ker}_{\mathcal{C}}\ (\mathrm{Coker}_{\mathcal{C}}\alpha)$$

where (1) holds since $\mathcal{G}$ is abelian and (2) holds since l is (left) exact.
Next we show that direct limits are exact. Let $\mathcal{I}$ be a directed small category and $G, G' : \mathcal{I} \to \mathcal{C}$ two functors with a monomorphism $G \to G'$. The induced morphism $\varinjlim iG \to \varinjlim iG'$ is a monomorphism in $\mathcal{G}$, and since l preserves monomorphisms, it follows that $\varinjlim G \to \varinjlim G'$ is a monomorphism in $\mathcal{C}$. Finally, it is easy to see that if U is a generator for $\mathcal{G}$, then lU is a generator for $\mathcal{C}$. □

Definition 1.2.2. *A reflective subcategory of $\mathcal{G}$ is called a **Giraud subcategory** if the localization functor is (left) exact.*

The previous result shows that if $\mathcal{C}$ is a Giraud subcategory of $\mathcal{G}$ then it is a Grothendieck category, even though its abelian structure is not the one inherited by $\mathcal{G}$, since the inclusion functor i is not generally right exact (we recall that kernels in $\mathcal{C}$ are the ones inherited by $\mathcal{G}$, whereas cokernels in $\mathcal{C}$ are computed in $\mathcal{G}$ and then localized).

Proposition 1.2.4. *Let $\mathcal{C}$ be a Giraud subcategory of $\mathcal{G}$. An object $E \in \mathcal{C}$ is injective in $\mathcal{C}$ if and only if it is injective as an object in $\mathcal{G}$.*

Proof. The inclusion functor i is left exact, so it's easy to see that every object in $\mathcal{C}$ that is injective in $\mathcal{G}$ is also injective in $\mathcal{C}$.

Viceversa, let $E \in \mathcal{C}$ be an injective object: we want to show that it is injective as an object in $\mathcal{G}$. Let $\alpha : A \to A'$ be a monomorphism in $\mathcal{G}$ and let $f \in (A, iE)_{\mathcal{G}}$. We get the commutative diagram in $\mathcal{G}$

$$\begin{tikzcd}
A \arrow[rr, hook, "\alpha"] \arrow[dr, "\sigma_A"] \arrow[dd, "f"'] & & A' \arrow[dr, "\sigma_{A'}"] & \\
& ilA \arrow[rr, hook, "il(\alpha)"] \arrow[dl, "\bar{f}"] & & ilA' \arrow[dlll, bend left, "\exists g"] \\
iE & & &
\end{tikzcd}$$

where $il(\alpha)$ is mono since i, l are left exact, and the morphism g exists since E is injective in $\mathcal{C}$ and makes the lower triangle commutative. So we have $(g\sigma_{A'})\alpha = f$, thus proving the injectivity of iE (i.e. of E as an object in $\mathcal{G}$). □

1.3 Torsion theory

In this section we axiomatize the concept of torsion and show some basic results. We will assume our category $\mathcal{C}$ to be abelian, complete, cocomplete and locally small.

Definition 1.3.1. *A **preradical** of $\mathcal{C}$ is a subfunctor r of the identity functor of $\mathcal{C}$: to any object $C \in \mathcal{C}$ it assigns a subobject $r(C)$ such that every morphism $C \to D$ induces a morphism $r(C) \to r(D)$ by restriction.*

If r_1 and r_2 are preradicals, one defines preradicals r_1r_2 and $r_1 : r_2$ as follows:

$$r_1r_2(C) = r_1(r_2(C)),$$

$$(r_1 : r_2)(C)/r_1(C) = r_2(C/r_1(C)).$$

Definition 1.3.2. *A preradical r is called **idempotent** if $rr = r$ and is called a **radical** if $r : r = r$.*

Lemma 1.3.1. *If r is a radical and $D \leq r(C)$, then $r(C/D) = r(C)/D$.*

Proof. The canonical morphism $C \to C/D$ induces $r(C) \to r(C/D)$ with kernel D, so $r(C)/D \leq r(C/D)$. On the other hand, the canonical morphism $\alpha : C/D \to C/r(C)$ induces the zero morphism on $r(C/D)$, so $r(C/D) \leq \mathrm{Ker}\alpha = r(C)/D$. □

To a preradical r one can associate two classes of objects of $\mathcal{C}$, namely

$$\mathcal{T}_r = \{C | r(C) = C\},$$

$$\mathcal{F}_r = \{C | r(C) = 0\}.$$

Proposition 1.3.1. *$\mathcal{T}_r$ is closed under quotient objects and coproducts, while $\mathcal{F}_r$ is closed under subobjects and products.*

Proof. It is easy to see that $\mathcal{T}_r$ is closed under quotients. Let $(C_i)_I$ be an arbitrary family of objects in $\mathcal{T}_r$. Since $r(C_i) = C_i$, the image of each canonical monomorphism $C_i \to \bigoplus\limits_I C_i$ is contained in $r(\bigoplus\limits_I C_i)$, and it follows from the definition of coproduct that $r(\bigoplus\limits_I C_i) = \bigoplus\limits_I C_i$. The corresponding results for $\mathcal{F}_r$ follow by duality. □

It follows from the last result that if $C \in \mathcal{T}_r$ and $D \in \mathcal{F}_r$ then $\mathrm{Hom}_{\mathcal{C}}(C, D) = 0$.

Definition 1.3.3. *A class of objects of $\mathcal{C}$ is called a* ***pretorsion class*** *if it is closed under quotients and coproducts. It is called a* ***pretorsion-free class*** *if it is closed under subobjects and products.*

Let $\mathcal{T}$ be a pretorsion class. If C is an arbitrary object of $\mathcal{T}$ and we denote by $t(C)$ the sum of all subobjects of C belonging to $\mathcal{T}$ then clearly $t(C) \in \mathcal{T}$. Hence every object C contains a largest subobject $t(C)$ belonging to $\mathcal{T}$. In this way $\mathcal{T}$ gives rise to a preradical t of $\mathcal{C}$, and t is clearly idempotent. Combining this procedure with the previous assignment $r \mapsto \mathcal{T}_r$, restricted to idempotent r, we obtain:

Proposition 1.3.2. *There is a bijective correspondence between idempotent preradicals of $\mathcal{C}$ and pretorsion classes of objects of $\mathcal{C}$. Dually, there is a bijective correspondence between radicals of $\mathcal{C}$ and pretorsion-free classes of objects of $\mathcal{C}$.*

Proposition 1.3.3. *The following assertions are equivalent for a preradical r:*

(a) r is a left exact functor.

(b) If $D \leq C$, then $r(D) = r(C) \cap D$.

(c) r is idempotent and $\mathcal{T}_r$ is closed under subobjects.

Proof. (a) $\Leftrightarrow$ (b) : Since the kernel of the morphism $r(C) \to r(D/C)$, induced from $C \to D/C$, is equal to $r(C) \cap D$, it is clear that (b) is equivalent to left exactness of r.
(b) $\Rightarrow$ (c) : By applying (b) to $r(C) \leq C$ one sees that r is idempotent. It is obvious that $\mathcal{T}_r$ is closed under subobjects.
(c) $\Rightarrow$ (b) : The inclusions $r(D) \leq r(C) \cap D \leq D$ are trivial. On the other hand, $r(C) \cap D$ belongs to $\mathcal{T}_r$ as a subobject of $r(C)$, and r idempotent implies $r(C) \cap D = r(D)$. □

Definition 1.3.4. *A pretorsion class is called* ***hereditary*** *if it is closed under subobjects.*

The last two propositions then lead to the following result:

Proposition 1.3.4. *There is a bijective correspondence between left exact preradicals and hereditary pretorsion classes.*

We will now begin our discussion on Torsion theory, starting with the following

Definition 1.3.5. *A* ***torsion theory*** *for $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that*

(i) $Hom_{\mathcal{C}}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$.
(ii) If $Hom_{\mathcal{C}}(C, F) = 0$ for all $F \in \mathcal{F}$ then $C \in \mathcal{T}$.
(iii) If $Hom_{\mathcal{C}}(T, C) = 0$ for all $T \in \mathcal{T}$ then $C \in \mathcal{F}$.

If this is the case, $\mathcal{T}$ is called a ***torsion class*** *and its objects are* ***torsion objects****, whereas $\mathcal{F}$ is called a* ***torsion-free class*** *and its objects are* ***torsion-free objects****.*

Any class of objects $\mathcal{S}$ can *generate* a torsion theory by setting

$$\mathcal{F} := \{F | Hom(C,F) = 0 \ \ for\ all\ C \in \mathcal{S}\},$$

$$\mathcal{T} := \{T | Hom(T,F) = 0 \ \ for\ all\ F \in \mathcal{F}\}.$$

Clearly this pair is a torsion theory, and $\mathcal{T}$ is the smallest torsion class containing $\mathcal{S}$. Dually, the class $\mathcal{S}$ can *cogenerate* a torsion theory $(\mathcal{T},\mathcal{F})$ such that $\mathcal{F}$ is the smallest torsion-free class containing $\mathcal{S}$.

Proposition 1.3.5. *The following properties of a class $\mathcal{T}$ of objects are equivalent:*

(a) $\mathcal{T}$ is a torsion class for some torsion theory.
(b) $\mathcal{T}$ is closed under quotients, coproducts and extensions.

Proof. Suppose $(\mathcal{T},\mathcal{F})$ is a torsion theory. $\mathcal{T}$ is obviously closed under quotients, and it is closed under coproducts because $\mathrm{Hom}(\bigoplus T_i, F) \cong \prod \mathrm{Hom}(T_i, F)$. Let $0 \to C' \to C \to C'' \to 0$ be exact with $C', C'' \in \mathcal{T}$. If F is torsion-free and there is a morphism $\alpha : C \to F$ then α is zero on C', so it factors over C''. But also $\mathrm{Hom}(C'', F) = 0$, so $\alpha = 0$ and $C \in \mathcal{T}$.
Conversely, assume that $\mathcal{T}$ is closed under quotients, coproducts and extensions. Let $(\mathcal{T}', \mathcal{F})$ be the torsion theory generated by $\mathcal{T}$. We want to show that $\mathcal{T} = \mathcal{T}'$, so suppose $\mathrm{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$. Since $\mathcal{T}$ is a pretorsion class, there is a largest subobject T of C belonging to $\mathcal{T}$. To show that $T = C$ it suffices to show that $C/T \in \mathcal{F}$. Suppose we have $\alpha : T'' \to C/T$ for some $T'' \in \mathcal{T}$. The image of α also belongs to $\mathcal{T}$, and if $\alpha \neq 0$ then we would get a subobject of C which strictly contains T and belongs to $\mathcal{T}$ (since $\mathcal{T}$ is closed under extensions). This would contradict the maximality of T, and so we must have $\alpha = 0$ and $C/T \in \mathcal{F}$. □

By duality one also has:

Proposition 1.3.6. *The following properties of a class $\mathcal{F}$ of objects are equivalent:*

(a) $\mathcal{F}$ is a torsion-free class for some torsion theory.
(b) $\mathcal{F}$ is closed under subobjects, products and extensions.

If $(\mathcal{T},\mathcal{F})$ is a torsion theory, then $\mathcal{T}$ is in particular a pretorsion class, so every object $C \in \mathcal{C}$ contains a largest subobject $t(C)$ belonging to $\mathcal{T}$,

called the *torsion subobject* of C. An object C is torsion-free if and only if $t(C) = 0$, because $C \in \mathcal{F}$ if and only if $\operatorname{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$. The idempotent preradical t is actually a radical, as is easily seen from the fact that $\mathcal{T}$ is closed under extensions.
Conversely, if t is an idempotent radical of $\mathcal{C}$, then one obtains a torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ with

$$\mathcal{T}_t := \{C | t(C) = C\},$$

$$\mathcal{F}_t := \{C | t(C) = 0\}.$$

The Proposition 1.3.2 now specializes to:

Proposition 1.3.7. *There is a bijective correspondence between torsion theories and idempotent radicals of $\mathcal{C}$.*

Definition 1.3.6. *A torsion theory $(\mathcal{T}, \mathcal{F})$ is called* ***hereditary*** *if the class $\mathcal{T}$ is closed under subobjects.*

Recalling Proposition 1.3.3, we conclude that this occurs if and only if the associated radical t is left exact, thus obtaining the following

Proposition 1.3.8. *There is a bijective correspondence between hereditary torsion theories and left exact radicals.*

For this last result we will assume $\mathcal{C}$ to be a Grothendieck category.

Proposition 1.3.9. *A torsion theory $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.*

Proof. If t is left exact and $F \in \mathcal{F}$ then $t(E(F)) \cap F = t(F) = 0$, which implies $E(F) \in \mathcal{F}$ since F is essential in $E(F)$.
Suppose conversely that $\mathcal{F}$ is closed under injective envelopes. If $T \in \mathcal{T}$ and $C \leq T$ then there is a morphism $\beta : T \to E(C/t(C))$ such that the diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\alpha"] & T \arrow[d, "\beta"] \\
C/t(C) \arrow[r] & E(C/t(C))
\end{tikzcd}$$

commutes. But $E(C/t(C))$ is torsion free, so $\beta = 0$. This implies $\alpha = 0$ and hence $C = t(C) \in \mathcal{T}$. □

We will now give an example of torsion theory linked to our previous talk about Giraud subcategories. More specifically, if $\mathcal{C}$ is a Giraud subcategory of a Grothendieck category $\mathcal{G}$, $i : \mathcal{C} \to \mathcal{G}$ is the inclusion functor and $l : \mathcal{G} \to \mathcal{C}$ its left adjoint, we can consider the classes

$$\mathcal{T} := \{B \in \mathcal{G} |\ l(B) = 0\},$$

$$\mathcal{F} := \{B \in \mathcal{G} |\ \sigma_B : B \to ilB \ \ is \ \ mono\}.$$

Proposition 1.3.10. *$(\mathcal{T}, \mathcal{F})$, as defined above, is a hereditary torsion theory.*

Proof. From the exactness of l follows immediately that $\mathcal{T}$ is closed under subobjects, quotients and extensions. Since l has a right adjoint it preserves coproducts, so that $\mathcal{T}$ is also closed under coproducts. $\mathcal{T}$ is thus a hereditary torsion class.
Clearly $\mathrm{Hom}(T, C) = 0$ for every $T \in \mathcal{T}, C \in \mathcal{C}$, and it follows that $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}, F \in \mathcal{F}$. Conversely, if B is an object such that $\mathrm{Hom}(T, B) = 0$ for all $T \in \mathcal{T}$ then $B \in \mathcal{F}$ because the kernel of $B \to ilB$ belongs to $\mathcal{T}$. □

1.4 Moving torsion theories through Giraud subcategories

in this section we will address the problem of "moving" torsion theories between categories; more specifically, given a Grothendieck category $\mathcal{G}$ and a Giraud subcategory $\mathcal{C}$, we want to investigate the way torsion theories on $\mathcal{G}$ reflect on $\mathcal{C}$ and, conversely, how torsion theories in $\mathcal{C}$ can be suitably extended to torsion theories on $\mathcal{G}$. We will refer to [4] for the main results.

The setting is the usual one: we have $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "i"] & \mathcal{G} \arrow[l, bend left, "l"] \end{tikzcd}$ with $\mathcal{G}$ Grothendieck category and $\mathcal{C}$ a Giraud subcategory, i the inclusion functor and l the localizing functor (which is exact).

Proposition 1.4.1. *Let $\mathcal{T}$ be a torsion class in $\mathcal{C}$. Then the class*

$$l^{\leftarrow}(\mathcal{T}) = \{D \in \mathcal{G} \mid l(D) \in \mathcal{T}\}$$

is a torsion class in $\mathcal{G}$.

Proof. Clearly the class $l^{\leftarrow}(\mathcal{T})$ is closed under quotients and coproducts, because so is $\mathcal{T}$ and l preserves arbitrary colimits. We need to check that $l^{\leftarrow}(\mathcal{T})$ is closed under extensions. Let $0 \to D' \to D \to D'' \to 0$ be a short exact sequence in $\mathcal{G}$ with $D', D'' \in l^{\leftarrow}(\mathcal{T})$. By applying the functor l we get the exact sequence in $\mathcal{C} : 0 \to l(D') \to l(D) \to l(D'') \to 0$ where $l(D'), l(D'') \in \mathcal{T}$ by definition, so by closure under extensions we get $l(D) \in \mathcal{T}$ and so $D \in l^{\leftarrow}(\mathcal{T})$. □

As seen at the end of the previous section, the kernel of the localization functor and the class $\{B \in \mathcal{G} |\ \sigma_B : B \to ilB \ \ is \ \ mono\}$ form a torsion theory in $\mathcal{G}$. We will denote this torsion pair as $(\mathcal{S}, \mathcal{S}^{\perp})$, where $\mathcal{S} := \text{Ker}\ l$.

Proposition 1.4.2. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathcal{C}$. Then the pair $(\mathcal{T}', \mathcal{F}')$:*

$$\mathcal{T}' := l^{\leftarrow}(\mathcal{T}) = \{X \in \mathcal{G} \mid l(X) \in \mathcal{T}\},$$

$$\mathcal{F}' := l^{\leftarrow}(\mathcal{F}) \cap \mathcal{S}^{\perp} = \{Y \in \mathcal{G} \mid Y \in \mathcal{S}^{\perp} \ \ and \ \ l(Y) \in \mathcal{F}\}$$

defines a torsion theory on $\mathcal{G}$ such that $i(\mathcal{T}) \subseteq \mathcal{T}', i(\mathcal{F}) \subseteq \mathcal{F}', l(\mathcal{T}') = \mathcal{T}, l(\mathcal{F}') = \mathcal{F}$.

Proof. For any $T \in \mathcal{T}$ we have $liT \cong T$, which proves that $i(\mathfrak{T}) \subseteq \mathfrak{T}'$. Moreover, given $F \in \mathcal{F}$, it is clear that $i(F) \in \mathcal{S}^{\perp}$ and $li(F) \cong F \in \mathcal{F}$, hence $i(\mathcal{F}) \subseteq \mathcal{F}'$. We deduce that $\mathcal{T} = li(\mathcal{T}) \subseteq l(\mathcal{T}') \subseteq \mathcal{T}$ and $\mathcal{F} = li(\mathcal{F}) \subseteq l(\mathcal{F}') \subseteq \mathcal{F}$, which proves that $l(\mathcal{T}') = \mathcal{T}$ and $l(\mathcal{F}') = \mathcal{F}$. Let us show that $(\mathcal{T}', \mathcal{F}')$ is a torsion theory on $\mathcal{G}$.
Given $X \in \mathcal{T}'$ and $Y \in \mathcal{F}'$,

$$\text{Hom}_{\mathcal{G}}(X, Y) \hookrightarrow \text{Hom}_{\mathcal{G}}(X, ilY) \cong \text{Hom}_{\mathcal{C}}(lX, lY) = 0$$

where the first inclusion holds since $Y \in \mathcal{F}' \subseteq \mathcal{S}^{\perp}$.
It remains to prove that for any $D \in \mathcal{G}$ there exists a short exact sequence

$$0 \to X \to D \to Y \to 0$$

with $X \in \mathcal{T}'$ and $Y \in \mathcal{F}'$. Given $D \in \mathcal{G}$ there exist $T \in \mathcal{T}, F \in \mathcal{F}$ such that the sequence

$$0 \to T \to l(D) \to F \to 0$$

is exact in $\mathcal{C}$. Applying the functor i and denoting by X the pullback of the diagram

$$\begin{tikzcd}
iT \arrow[r] & ilD \\
 & D \arrow[u, "\sigma_D"]
\end{tikzcd}$$

we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & iT \arrow[r] & ilD \arrow[r] & iF & \\
0 \arrow[r] & X \arrow[r] \arrow[u] & D \arrow[r] \arrow[u, "\sigma_D"] & D/X \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

where the map $D/X \hookrightarrow iF$ is a monomorphism since the first square is cartesian.
Let us apply the functor l to the above diagram, remembering that it is exact, so that it preserves pullbacks and exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & lD \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & lX \arrow[r] \arrow[u, "\cong"] & lD \arrow[r] \arrow[u, "id_{lD}"] & l(D/X) \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

where the first row is exact since $li \cong \mathrm{id}_{\mathcal{C}}$, $lX \cong T \in \mathcal{T}$ since it is the pullback of the diagram

$$\begin{tikzcd}
T \arrow[r] & lD \\
 & lD \arrow[u, "id_{lD}"']
\end{tikzcd}$$

which proves that $X \in \mathcal{T}'$ and so $l(D/X) \cong F \in \mathcal{F}$; also, since $D/X \hookrightarrow iF \cong il(D/X)$, we get $D/X \in \mathcal{S}^{\perp}$, thus $D/X \in \mathcal{F}'$. □

We will now consider the opposite direction:

Proposition 1.4.3. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory in $\mathcal{G}$ and let*

$$l(\mathcal{X}) := \{T \in \mathcal{C} \mid T \cong lX,\ \exists X \in \mathcal{X}\},$$

$$l(\mathcal{Y}) := \{F \in \mathcal{C} \mid F \cong lY,\ \exists Y \in \mathcal{Y}\}.$$

Then $(l(\mathcal{X}), l(\mathcal{Y}))$ defines a torsion theory on $\mathcal{C}$ if and only if $il(\mathcal{Y}) \subseteq \mathcal{Y}$. If this is the case, $i^{\leftarrow}(\mathcal{Y}) = l(\mathcal{Y})$.

Proof. First let's assume that $il(\mathcal{Y}) \subseteq \mathcal{Y}$. Since $li \cong \mathrm{id}_{\mathcal{C}}$, one immediately has that $i^{\leftarrow}(\mathcal{Y}) = l(\mathcal{Y})$ and this is a torsion-free class in $\mathcal{C}$. Given $T \in l(\mathcal{X})$ (i.e. $T \cong lX$ with $X \in \mathcal{X}$) and $F \in i^{\leftarrow}(\mathcal{Y})$, one has $\mathrm{Hom}_{\mathcal{C}}(T, F) = \mathrm{Hom}_{\mathcal{C}}(lX, F) \cong \mathrm{Hom}_{\mathcal{G}}(X, iF) = 0$, since $iF \in \mathcal{Y}$ by definition of $i^{\leftarrow}(\mathcal{Y})$. Now let $C \in \mathcal{C}$. There exist $X \in \mathcal{X}, Y \in \mathcal{Y}$ and a short exact sequence in $\mathcal{G}$

$$0 \to X \to iC \to Y \to 0.$$

Applying the exact functor l to this sequence we get the sequence in $\mathcal{C}$

$$0 \to lX \to C \to lY \to 0$$

where $lX \in l(\mathcal{X})$ and $lY \in l(\mathcal{Y})$, which proves that $(l(\mathcal{X}), l(\mathcal{Y}))$ is a torsion pair on $\mathcal{C}$.
Conversely, if $(l(\mathcal{X}), l(\mathcal{Y}))$ is a torsion theory on $\mathcal{C}$ then for every $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ one has

$$0 = \mathrm{Hom}_{\mathcal{C}}(lX, lY) \cong \mathrm{Hom}_{\mathcal{G}}(X, ilY),$$

therefore $ilY \in \mathcal{Y}$. □

The last two propositions lead to the following result:

Theorem 1.4.1. *Let $\mathcal{G}$ be a Grothendieck category with a Giraud subcategory $\mathcal{C}$. There is a bijective correspondence between torsion theories $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{G}$ satisfying $il(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$ and torsion theories $(\mathcal{T}, \mathcal{F})$ on $\mathcal{C}$.*

Proof. From one side, taking a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$, the torsion pair $(\mathcal{T}', \mathcal{F}')$ defined in Proposition 1.4.2 satisfies $il(\mathcal{F}') \subseteq \mathcal{F}'$ and one can easily verify that $(l(\mathcal{T}'), l(\mathcal{F}')) = (\mathcal{T}, \mathcal{F})$.
On the other hand, given a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{G}$ satisfying $il(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$, its corresponding torsion pair on $\mathcal{C}$ is, by Proposition 1.4.3, $(l(\mathcal{X}), l(\mathcal{Y}))$, for which it is clear that $l(\mathcal{Y})' = l^{\leftarrow}(l(\mathcal{Y})) \cap \mathcal{S}^{\perp} = \mathcal{Y}$ (since $\mathcal{Y} \subseteq \mathcal{S}^{\perp}$), and so $(\mathcal{X}, \mathcal{Y}) = (l(\mathcal{X})', l(\mathcal{Y})')$. □

As a final result for this section, we want to show a bijective correspondence between another important class of subcategories.

Definition 1.4.1. *Let $\mathcal{G}$ be an abelian category. A full subcategory $\mathcal{S}$ of $\mathcal{G}$ is called a* ***Serre subcategory*** *if, for every short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{G}$ we have that $X \in \mathcal{S}$ if and only if $X', X'' \in \mathcal{S}$.*

In other words, a full subcategory $\mathcal{S}$ of $\mathcal{G}$ is a Serre subcategory if it is closed under subobjects, quotients and extensions.

Theorem 1.4.2. *Let $\mathcal{G}$ be a Grothendieck category with a Giraud subcategory $\mathcal{C}$. There is a one to one correspondence between Serre classes $\mathcal{X}$ in $\mathcal{C}$ and Serre classes $\mathcal{Y}$ in $\mathcal{G}$ satisfying $\mathcal{Y} \supseteq \mathcal{S}$, where $\mathcal{S} := Ker\ l$.*

Proof. First we want to show that, under these assumptions, the class $\mathcal{Y}$ satisfies $il(\mathcal{Y}) \subseteq \mathcal{Y}$.
Indeed, let $Y \in \mathcal{Y}$. We have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & Y \arrow[rr, "\eta_Y"] \arrow[dr, two heads] & & ilY \arrow[r] & C \arrow[r] & 0 \\
& & & D \arrow[ur, hook] & & &
\end{tikzcd}$$

where $K = \mathrm{Ker}(\eta_Y)$, $C = \mathrm{Coker}(\eta_Y)$ and D is given by the epi - mono factorization of η_Y.
Now since $Y \in \mathcal{Y}$ and this is a Serre class, also $K, D \in \mathcal{Y}$; moreover, since $C \in \mathcal{S} \subseteq \mathcal{Y}$, it must be $il(Y) \in \mathcal{Y}$ as wanted.
Given a Serre class $\mathcal{X}$ in $\mathcal{C}$, we want to prove that $l^{\leftarrow}(\mathcal{X})$ is a Serre class in $\mathcal{G}$. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{G}$. Then, applying l, we get the exact sequence in $\mathcal{C}$

$$0 \to lA \to lB \to lC \to 0$$

where, of course, $lB \in \mathcal{X} \Leftrightarrow lA, lC \in \mathcal{X}$, i.e. $B \in l^{\leftarrow}(\mathcal{X}) \Leftrightarrow A, C \in l^{\leftarrow}(\mathcal{X})$.
On the other hand, given a Serre class $\mathcal{Y}$ in $\mathcal{G}$ such that $\mathcal{Y} \supseteq \mathcal{S}$, we want to prove that $l(\mathcal{Y})$ is a Serre class in $\mathcal{C}$: let $0 \to A \to lY \to C \to 0$ be a short exact sequence in $\mathcal{C}$ with $Y \in \mathcal{Y}$. Applying the left exact functor i we get

$$\begin{tikzcd}
0 \arrow[r] & iA \arrow[r] & ilY \arrow[rr] \arrow[dr, two heads] & & iC \\
& & & D \arrow[ur, hook] &
\end{tikzcd}$$

Since $ilY \in \mathcal{Y}$ and this is a Serre class, also $iA, D \in \mathcal{Y}$. Then $l(iA) \cong A \in l(\mathcal{Y})$. Also, applying l to this last diagram, we get that $C \cong lD \in \mathcal{Y}$.
Let us consider now a short exact sequence in $\mathcal{C}$

$$0 \to lY \to A \to lY' \to 0$$

with $Y, Y' \in \mathscr{Y}$. Applying i we get

$$\begin{tikzcd}
0 \arrow[r] & ilY \arrow[r] & iA \arrow[rr] \arrow[dr, two heads] & & ilY' \\
& & & D \arrow[ur, hook] &
\end{tikzcd}$$

Since $ilY' \in \mathscr{Y}$, also $D \in \mathscr{Y}$; then, since $ilY, D \in \mathscr{Y}$ and $\mathscr{Y}$ is closed under extensions, also $iA \in \mathscr{Y}$, which means that $A \cong liA \in l(\mathscr{Y})$. $\square$

Chapter 2

The Functor category

2.1 Definition and basic properties

Throughout this chapter we will focus on the category $\mathcal{C} = ((R\text{-mod})^{op}, Ab)$ of additive contravariant functors from the category of finitely presented left modules over a ring R to the category of abelian groups. That is to say, the objects in $\mathcal{C}$ are the additive contravariant functors $F : R\text{-mod} \to Ab$ and the morphisms between $F, G \in \mathcal{C}$ are the natural transformations $Nat(F, G)$. We start by recalling the famous

Lemma 2.1.1 (Yoneda Lemma)**.** *Let $\mathcal{B}$ be a small preadditive category. For $B \in \mathcal{B}$, let $(-, B)$ denote the functor $Hom_{\mathcal{B}}(-, B) : \mathcal{B}^{op} \to Ab$. Then for every object $B \in \mathcal{B}$ and every additive functor $T : \mathcal{B}^{op} \to Ab$ there is a natural isomorphism*

$$Nat((-, B), T) \cong T(B)$$

which is natural in both B and T.

Proof. See [10], Proposition 7.3. □

If one applies the Yoneda Lemma to $T = (-, B')$ one gets a natural isomorphism $Nat((-, B), (-, B')) \cong \mathrm{Hom}_{\mathcal{B}}(B, B')$. The functor $B \mapsto (-, B)$ is therefore a full embedding of $\mathcal{B}$ in $(\mathcal{B}^{op}, Ab)$.

Definition 2.1.1. *A functor in $(\mathcal{B}^{op}, Ab)$ is called **representable** if it is of the form $(-, B)$ for some $B \in \mathcal{B}$.*

The last observation leads to

Proposition 2.1.1. *A small preadditive category $\mathcal{B}$ is equivalent to the full subcategory of $(\mathcal{B}^{op}, Ab)$ consisting of representable functors.*

By definition, a sequence $0 \to T' \to T \to T'' \to 0$ in $(\mathcal{B}^{op}, Ab)$ is exact if and only if the sequence in $Ab : 0 \to T'(B) \to T(B) \to T''(B) \to 0$ is exact for every $B \in \mathcal{B}$. Also, existing limits and colimits in $(\mathcal{B}^{op}, Ab)$ are defined object-wise.

Proposition 2.1.2. *The family $((-, B))_{B \in \mathcal{B}}$ is a family of projective generators for $(\mathcal{B}^{op}, Ab)$.*

Proof. Let $F \in (\mathcal{B}^{op}, Ab)$. For each class of isomorphisms of the objects of $\mathcal{B}$ we choose a representative B. Then we define the map $\bigoplus\limits_B (-, B)^{(F(B))} \to F$ where, for $b \in F(B)$, the b-th component is given by $f_b : (-, B) \to F$, f_b being the map associated to b via the Yoneda correspondence. This map is clearly surjective, so that the class $((-, B))_{B \in \mathcal{B}}$ generates $(\mathcal{B}^{op}, Ab)$.
To see that each $(-, B)$ is projective, assume to have a diagram

$$\begin{tikzcd}
(-, B) \arrow[dr, "f"] & \\
F \arrow[r, two heads, "g"] & G
\end{tikzcd}$$

then the map f corresponds to some $b \in G(B)$ via Yoneda; choosing $a \in F(B)$ such that $g(a) = b$ (g is surjective) we then get that a corresponds to some $h : (-, B) \to F$ via Yoneda, and h is the required morphism that makes the above diagram commutative. ☐

Proposition 2.1.3. *The category $(\mathcal{B}^{op}, Ab)$ is a Grothendieck category.*

Proof. Since colimits in $(\mathcal{B}^{op}, Ab)$ are computed object-wise, the exactness of direct limits in Ab implies exactness of direct limits in $(\mathcal{B}^{op}, Ab)$. Finally, the objects in $\mathcal{B}$ form a family of projective generators for $(\mathcal{B}^{op}, Ab)$. ☐

From now on the category $\mathcal{B}$ will be the skeletally small category of finitely presented left modules over a ring R, denoted as R-mod.
As we said before, limits and colimits in $((R\text{-mod})^{op}, Ab)$ are computed object-wise: if $A \in R$-mod, then

$$(\varinjlim F_i)(A) := \varinjlim F_i(A)$$

where the direct limit on the right is taken in the category of abelian groups.

Definition 2.1.2. *An object $G \in ((R\text{-mod})^{op}, Ab)$ is called* ***flat*** *if it is isomorphic to a direct limit of representable functors, $G \cong \varinjlim(-, A_i)$.*

The (full) subcategory of $((R\text{-mod})^{op}, Ab)$ of flat functors is denoted by $\mathrm{Flat}((R\text{-mod})^{op}, Ab)$. We recall the following characterization of finitely presented modules (actually, of finitely presented objects in general):

Proposition 2.1.4. *Let $\mathcal{C}$ be a locally finitely generated category. An object $C \in \mathcal{C}$ is finitely presented if and only if the functor $Hom_{\mathcal{C}}(C, -)$ preserves direct limits.*

Proof. See [10], Proposition V.3.4. □

Thus, for a flat object G in $((R\text{-mod})^{op}, Ab)$, we have

$$G \cong \varinjlim(-, A_i) \cong (-, \varinjlim A_i) = (-, M)$$

where we call $M := \varinjlim A_i$. This fact leads to the following

Proposition 2.1.5. *The functor R-Mod $\to$ $((R\text{-mod})^{op}, Ab)$ given by $M \mapsto (-, M)$ is a full and faithful left exact functor. It yields an equivalence between the category R-Mod of R-modules and the category Flat($(R\text{-mod})^{op}, Ab$).*

In general, the category R-mod is not abelian. However, it has cokernels, so we may call a contravariant functor $F \in ((R\text{-mod})^{op}, Ab)$ **left exact** if it takes cokernels to kernels. Since every every flat functor in $((R\text{-mod})^{op}, Ab)$ is isomorphic to a functor of the form $(-, M)$, it is left exact. As the following proposition states, the converse is also true.

Observation. Given any $F \in ((R\text{-mod})^{op}, Ab)$, we can evaluate F at R, since it is trivially a finitely presented R-module, thus obtaining the abelian group $F(R)$. $F(R)$ can be actually given the structure of left R-module in the following way: given $r \in R$, let $\bar{r} : R \to R$ be the right multiplication by r, i.e. $\bar{r}(x) = xr$, for $x \in R$. Given $a \in F(R)$, we define the scalar product on $F(R)$ as $r \cdot a := F(\bar{r})(a)$. One can easily check that this definition gives $F(R)$ the structure of left R-module.

Proposition 2.1.6. *A functor in $((R\text{-mod})^{op}, Ab)$ is flat if and only if it is left exact.*

Proof. We only need to prove that every left exact functor is flat. Let $F \in ((R\text{-mod})^{op}, Ab)$ be left exact; we want to prove that $F \cong (-, F(R))$.
First, there is an obvious isomorphism $\alpha(R) : F(R) \to (R, F(R))$ of abelian groups; it induces, for every finitely generated free module R^n an isomorphism $\alpha(R^n) : F(R^n) \to (R, F(R^n))$. Given $A \in R$-mod and a presentation

$$R^m \to R^n \to A \to 0$$

we get, applying the functors F and $(-, F(R))$:

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] \arrow[d, dashed, "\alpha(A)"] & F(R^n) \arrow[r] \arrow[d, "\cong"] & F(R^m) \arrow[d, "\cong"] \\
0 \arrow[r] & (A, F(R)) \arrow[r] & (R^n, F(R)) \arrow[r] & (R^m, F(R))
\end{tikzcd}$$

where the isomorphism $\alpha(A)$ is induced by left exactness. □

This observation, together with a routine diagram chase, may be used to prove the following.

Proposition 2.1.7. *If the sequence in* $((R\text{-mod})^{op}, Ab)$

$$0 \to (-, M) \to F \to (-, K) \to 0$$

is exact, then F *is flat.*

From now on we will denote by i the embedding functor

$$i : R\text{-Mod} \to ((R\text{-mod})^{op}, Ab)$$

$$M \mapsto (-, M)$$

Observation. For every $M \in R$-Mod there is a natural isomorphism $(R, M) \cong M$: indeed, given $f \in (R, M)$, one associates $f(1_R) \in M$; conversely, given $m \in M$, one associates $f_m : R \to M$ defining $f_m(1_R) := m$. These two maps (associations) are clearly the inverse of each other.

Definition 2.1.3. *The functor* $-_R : ((R\text{-mod})^{op}, Ab) \to R\text{-Mod}$ *defined by* $F \mapsto F(R)$ *is called the* ***R-evaluation functor***.

Clearly, the R-evaluation functor is exact: given $0 \to F \to G \to H \to 0$, the sequence $0 \to F(R) \to G(R) \to H(R) \to 0$ is exact by definition.
Moreover, by the above observation we easily get that $(-_R) \circ i \cong \mathrm{id}_{R-Mod}$: given $M \in R$-Mod, $((-_R) \circ i)(M) = (R, M) \cong M$; also, given any morphism $f : M \to N$ in R-Mod, one immediately gets the commutative diagram

$$\begin{tikzcd}
(R, M) \arrow[r, "((-_R)\circ i)(f)"] \arrow[d, "\cong"] & (R, N) \arrow[d, "\cong"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

We now want to show that the category R-Mod can actually be seen as a Giraud subcategory of $((R\text{-mod})^{op}, Ab)$, as defined in Chapter 1. In order to do so, we need the following

Lemma 2.1.2. *For every $F \in ((R\text{-mod})^{op}, Ab)$ there is a natural transformation of functors $\eta_F : F \to (-, F(R))$ such that $(\eta_F)_R = id_{F(R)}$.*

Proof. We need to define, for every $A \in R$-mod, the action of $(\eta_F)_A : F(A) \to (A, F(R))$ as a homomorphism of abelian groups, i.e. we need to define a map $(\eta_F)_A(x) : A \to F(R)$ for every $x \in F(A)$. Given $a \in A$, let $\varphi_a : R \to A$ be the map $1_R \mapsto a$. Applying F we get the map $F(\varphi_a) : F(A) \to F(R)$; finally, we define $(\eta_F)_A(x)(a) := F(\varphi_a)(x)$.
For the second statement, recall by the last observation that $(\eta_F)_R(x)$, as an element of $(R, F(R))$, is associated via the natural isomorphism to $(\eta_F)_R(x)(1_R) = F(\varphi_{1_R})(x) = F(id_R)(x) = id_{F(R)}(x) = x$, so that $(\eta_F)_R = \mathrm{id}_{F(R)}$ □

Proposition 2.1.8. *The pair $\langle -_R, i \rangle$ is an adjoint pair of functors.*

Proof. Given $F \in ((R\text{-mod})^{op}, Ab)$ and $M \in R$-Mod, we need to exhibit an isomorphism

$$\mathrm{Hom}_R(F(R), M) \cong \mathrm{Hom}_{\mathcal{C}}(F, iM)$$

which is natural in both F and M.
In order to do so, we will define two morphisms between these two abelian groups and show that they are inverse to each other. First, let's define

$$\Phi : \mathrm{Hom}_R(F(R), M) \to \mathrm{Hom}_{\mathcal{C}}(F, iM)$$

as follows: given $\alpha \in (F(R), M)$, consider $i\alpha$ in the functor category and then compose it with η_F as defined above: we thus set $\Phi(\alpha) := i\alpha \circ \eta_F$. This is clearly an homomorphism of abelian groups.
Conversely, let's define

$$\Psi : \mathrm{Hom}_{\mathcal{C}}(F, iM) \to \mathrm{Hom}_R(F(R), M)$$

as $\Psi(f) := f_R$, for $f \in \mathrm{Hom}_{\mathcal{C}}(F, iM)$ (keeping in mind the isomorphism $(R, M) \cong M$).
On one hand we have, for $\alpha \in \mathrm{Hom}_R(F(R), M)$,

$$\Psi(\Phi(\alpha)) = \Psi(i\alpha \circ \eta_F) = \alpha \circ (\eta_F)_R = \alpha$$

by the definition of η_F and the fact that $(-_R) \circ i = \mathrm{id}_{R-Mod}$.
On the other hand, given $f \in \mathrm{Hom}_{\mathcal{C}}(F, iM)$, we have

$$\Phi(\Psi(f)) = \Psi(f_R) = if_R \circ \eta_F.$$

This is a transformation of functors $F \to iM$, and we want to show that it is equal to f. In order to do so, let A be a finitely presented R-module: we need to prove that, for every $x \in F(A)$, $f_A(x) = (if_R \circ \eta_F)_A(x)$: these are two maps $A \to F(R)$, so we need to prove that their action is equal on every $a \in A$.
By definition, $(if_R \circ \eta_F)_A(x)(a) = (if_R)_A((\eta_F)_A(x)(a)) = (f_R \circ (\eta_F)_A(x))(a) = f_R((\eta_F)_A(x)(a)) = f_R(F(\varphi_a)(x)) = (\star)$, where, as above, $\varphi_a : R \to A$ is the map $1_R \mapsto a$. Since $f : F \to iM$ is a natural transformation of functors, we have the commutative diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(\varphi_a)"] \arrow[d, "f_A"'] & F(R) \arrow[d, "f_R"] \\
(A, M) \arrow[r, "iM(\varphi_a)"'] & (R, M) \cong M
\end{tikzcd}$$

so that $(\star) = f_R(F(\varphi_a)(x)) = iM(\varphi_a) \circ f_A(x) = f_A(x) \circ \varphi_a$. This is indeed the map corresponding, via the isomorphism $(R, M) \cong M$, exactly to $(f_A(x) \circ \varphi_a)(1_R) = f_A(x)(a)$, as we wanted. $\square$

We have finally proved the following

Theorem 2.1.1. *R-Mod is a Giraud subcategory of $((R\text{-mod})^{op}, Ab)$ via the embedding functor*

$$i : R\text{-}Mod \to ((R\text{-}mod)^{op}, Ab)$$

$$M \mapsto (-, M)$$

and the localizing functor

$$-_R : ((R\text{-}mod)^{op}, Ab) \to R\text{-}Mod$$

$$F \mapsto F(R)$$

In accordance with the notation used in Chapter 1, we will denote the R-evaluating functor $-_R$ as l.
We then have the following setting:

$$\begin{tikzcd} R - Mod \arrow[r, bend left, "i"] & ((R - mod)^{op}, Ab) \arrow[l, bend left, "l"] \end{tikzcd}$$

with $li \cong \mathrm{id}_{R-Mod}$ and the unity of the adjunction $\eta_F : F \to ilF$ as described above. All the results showed in Chapter 1 can thus be applied to the particular Grothendieck category $((R\text{-mod})^{op}, Ab)$ and its Giraud category R-Mod.

2.2 Correspondence of torsion classes

In this short section we will show how the one to one correspondence between torsion classes of a Grothendieck category and a Giraud subcategory requires a less strong hypothesis when applied to the context of the category $((R\text{-mod})^{op}, Ab)$. First of all, we need to show that the correspondence still holds when dealing with hereditary torsion pairs.

Lemma 2.2.1. *Let $\mathcal{G}$ be a Grothendieck category with a Giraud subcategory $\mathcal{C}$. There is a one to one correspondence between hereditary torsion pairs $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$ and hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$ satisfying the condition $il(\mathcal{F}) \subseteq \mathcal{F} \subseteq \mathcal{S}^{\perp}$ where $\mathcal{S} := Ker(l)$.*

Proof. The correspondence between torsion pairs has already been proven in Theorem 1.4.1. What's left to prove is that the correspondence still holds when the torsion classes are hereditary.
Let $\mathcal{X}$ be an hereditary torsion class in $\mathcal{C}$; we want $l^{\leftarrow}(\mathcal{X})$ to be hereditary in $\mathcal{G}$: given any $D \in l^{\leftarrow}(\mathcal{X})$ and any subobject $D' \hookrightarrow D$ then, applying the exact functor l, we get $lD' \hookrightarrow lD$ in $\mathcal{C}$, where $lD \in \mathcal{X}$ and $\mathcal{X}$ is hereditary, so that $lD' \in \mathcal{X}$ and $D' \in l^{\leftarrow}(\mathcal{X})$.
On the other hand, given an hereditary torsion class $\mathcal{T}$ in $\mathcal{G}$ satisfying the above condition, we want $l(\mathcal{T})$ to be hereditary in $\mathcal{C}$. Notice that the condition $\mathcal{F} \subseteq \mathcal{S}^{\perp}$ implies that $\mathcal{T} \supseteq \mathcal{S}$: indeed, $\mathcal{T} =^{\perp} \mathcal{F} \supseteq^{\perp} (\mathcal{S}^{\perp}) \supseteq \mathcal{S}$.
Now let $T \in \mathcal{T}$, $M \in \mathcal{C}$ and $\epsilon : M \hookrightarrow lT$.
Since $M \cong l(iM)$, it suffices to prove that $iM \in \mathcal{T}$ in order to have that $M \in l(\mathcal{T})$.
Applying the functor i we get the short exact sequence (with $c := \operatorname{coker}(i\epsilon)$)

$$\begin{tikzcd}
 & & T \arrow[d, "\eta_T"] & & \\
0 \arrow[r] & iM \arrow[r, hook, "i\epsilon"'] & ilT \arrow[r, two heads, "c"'] & C \arrow[r] & 0
\end{tikzcd}$$

which can be completed to a pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & PB \arrow[r, hook, "\bar{i\epsilon}"] \arrow[d, "\bar{\eta_T}"] & T \arrow[r, "\bar{c}"] \arrow[d, "\eta_T"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & iM \arrow[r, hook, "i\epsilon"'] & ilT \arrow[r, two heads, "c"'] & C \arrow[r] & 0
\end{tikzcd}$$

We can now consider the kernel and cokernel of $\eta_T, \bar{\eta_T}$ and get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, hook] & K' \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r] & PB \arrow[r, hook, "\bar{i\epsilon}"] \arrow[d, "\bar{\eta_T}"] & T \arrow[r, "\bar{c}"] \arrow[d, "\eta_T"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & iM \arrow[r, hook, "i\epsilon"'] \arrow[d, two heads] & ilT \arrow[r, two heads, "c"'] \arrow[d, two heads] & C \arrow[r] \arrow[d] & 0 \\
 & D \arrow[r] & D' \arrow[r] & 0 &
\end{tikzcd}$$

By the Snake Lemma, we get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & K' \arrow[r] & 0 \arrow[r] & D \arrow[r] & D' \arrow[r] & 0
\end{tikzcd}$$

which shows that $K \cong K'$ and $D \cong D'$. Now we focus on the first column of the diagram: we have

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook] & PB \arrow[rr, "\bar{\eta_T}"] \arrow[dr, two heads] & & iM \arrow[r, two heads] & D \arrow[r] & 0 \\
& & & G \arrow[ur, hook] & & &
\end{tikzcd}$$

where we're considering the epi - mono factorization of $\bar{\eta_T}$.
We have that $PB \in \mathcal{T}$ which is a hereditary torsion class, so that also K and G must be in $\mathcal{T}$. Also, $D \in \mathcal{T}$ since $D \cong D'$ and $D' \in \mathcal{S} \subseteq \mathcal{T}$ (because $l(\eta_T)$ is an isomorphism, so $l(K') = l(D') = 0$). Since $\mathcal{T}$ is closed under extensions, it must be $iM \in \mathcal{T}$. □

If we specialize this setting to the case of $\mathcal{G} = ((R\text{-mod})^{op}, Ab)$ and $\mathcal{C} = R$-Mod, the condition $\mathcal{F} \subseteq \mathcal{S}^{\perp}$ automatically implies that $il(\mathcal{F}) \subseteq \mathcal{F}$ when dealing with a hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$. Thus the following

Theorem 2.2.1. *There is a one to one correspondence between hereditary torsion pairs* $(\mathcal{X}, \mathcal{Y})$ *in R-Mod and hereditary torsion pairs* $(\mathcal{T}, \mathcal{F})$ *in* $((R\text{-mod})^{op}, Ab)$ *satisfying* $\mathcal{T} \supseteq \mathcal{S}$*, where* $\mathcal{S} = Ker(l)$*.*

Proof. Since the condition $\mathcal{T} \supseteq \mathcal{S}$ is equivalent to $\mathcal{F} \subseteq \mathcal{S}^{\perp}$, we only need to verify that this condition implies $il(\mathcal{F}) \subseteq \mathcal{F}$ and then apply the previous Lemma.
This means that, given any $F \in \mathcal{F}$, we want $ilF \in \mathcal{F}$.
Let T $\in \mathcal{T}$ and assume there is a morphism $\alpha : T \to ilF$; we get the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\eta_F"] & ilF \arrow[r, "c"] & C \arrow[r] & 0 \\
0 \arrow[r] & PB \arrow[r, "\bar{\eta_F}"'] \arrow[u, "\bar{\alpha}"] & T \arrow[r, "\bar{c}"'] \arrow[u, "\alpha"] & C \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $c = \mathrm{coker}(\eta_F)$ and $C \in \mathcal{S}$ because, applying l to the first row, we get an isomorphism $l(\eta_F) : lF \to lilF \cong lF$. Also, η_F is a monomorphism because $\mathcal{F} \subseteq \mathcal{S}^{\perp}$ by the hypothesis.
So we have $PB \hookrightarrow T$ and $\mathcal{T}$ is hereditary, which implies $PB \in \mathcal{T}$.
Since $F \in \mathcal{F}$, it must be $\bar{\alpha} = 0$. Then $0 = \eta_F \circ \bar{\alpha} = \alpha \circ \overline{\eta_F}$ and so, since $\bar{c} = \mathrm{coker}(\overline{\eta_F})$, $\exists! \phi : C \to ilF$ such that $\alpha = \phi \circ \bar{c}$.

So we have the commutative diagram

$$\begin{tikzcd}
ilF & \\
T \arrow[u, "\alpha"] \arrow[r, "\bar{c}"'] & C \arrow[ul, "\phi"']
\end{tikzcd}$$

which, applying l, gives

$$\begin{tikzcd}
lF & \\
lT \arrow[u, "\alpha_R"] \arrow[r, "0"'] & 0 \arrow[ul, "0"']
\end{tikzcd}$$

(recall that $l(C) = 0$ since $\mathrm{coker}(\eta_F) \in \mathrm{Ker}(l)$).
This means that $l(\alpha) = \alpha_R = 0$.
Now every $M \in R$-mod has a presentation $R^m \to R^n \xrightarrow{g} M \to 0$ which yields the commutative diagram

$$\begin{tikzcd}
T(M) \arrow[r, "T(g)"] \arrow[d, "\alpha_M"'] & T(R^n) \arrow[d, "\alpha_{R^n}"] \\
ilF(M) \arrow[r, hook, "ilF(g)"'] & ilF(R^n)
\end{tikzcd}$$

Notice that $ilF(g)$ is mono because the functor $ilF = (-, F(R))$ is contravariant and left exact.
Then it suffices to prove that $\alpha_{R^n} = 0$ to conclude that $\alpha_M = 0$ for any arbitrary finitely presented R-module M, which means that $\alpha = 0$.
We recall that the functors in the category $\mathcal{G}$ are additive, so they preserve finite products and coproducts. This means that for every $F \in \mathcal{G}$ there is an isomorphism $\Phi_F : F(R)^n \to F(R^n)$ defined by $\Phi_F((x_i)_i) := \sum\limits_i F(\epsilon_i)(x_i)$ with inverse map given by $\Psi_F : F(R^n) \to F(R)^n$, $\Psi_F(x) := (F(\pi_i)(x))_i$, where we denote with ϵ_i, π_i the i-th embeddings and projections $R \overset{\epsilon_i}{\hookrightarrow} R^n \overset{\pi_i}{\twoheadrightarrow} R$.
Assume we have a natural trasformation $\sigma : F \to G$, with $F, G \in \mathcal{G}$, such

that $\sigma_R = 0$. We get the diagram

$$\begin{tikzcd}
F(R^n) \arrow[d, "\sigma_{R^n}"'] & F(R)^n \arrow[l, "\Phi_F", "\simeq"'] \arrow[d, dashed, "\beta"] \\
G(R^n) \arrow[r, "\simeq", "\Psi_G"'] & G(R^n)
\end{tikzcd}$$

where β is defined as $\beta := \Psi_G \circ \sigma_{R^n} \circ \Phi_F$.
Now given any element $((x_i)_i) \in F(R)^n$ we get
$\beta((x_i)_i) = (\Psi_G\circ\sigma_{R^n}\circ\Phi_F)((x_i)_i) = (\Psi_G\circ\sigma_{R^n})(\sum\limits_i F(\epsilon_i)(x_i)) = \Psi_G(\sum\limits_i \sigma_{R^n}(F(\epsilon_i)(x_i))) =$
$(G(\pi_j)(\sum\limits_i(\sigma_{R^n} \circ F(\epsilon_i))(x_i))_j \overset{\star}{=} (G(\pi_j)(\sum\limits_i(G(\epsilon_i) \circ \sigma_R)(x_i))_j =$
$= (G(\pi_j)(\sum\limits_i G(\epsilon_i)(\sigma_R(x_i))))_j = 0$
where the equality $(\star)$ holds since we have the natural commutative diagram

$$\begin{tikzcd}
F(R) \arrow[r, "F(\epsilon_i)"] \arrow[d, "\sigma_R"'] & F(R^n) \arrow[d, "\sigma_{R^n}"] \\
G(R) \arrow[r, "G(\epsilon_i)"'] & G(R^n)
\end{tikzcd}$$

So $\beta = 0$ and Ψ_G, Φ_F are isomorphisms, which means that it must be $\sigma_{R^n} = 0$. If we apply this argument to our situation $T \overset{\alpha}{\to} ilF$ we get that $\alpha_{R^n} = 0$ as required. □

2.3 Pure-injective envelopes

The aim of this section is to define and investigate the notion of pure-injective object and pure-injective envelope in a locally finitely presented category, thus generalizing the notion of pure-injective modules as defined, for example, in [11]. As we will see, the use of the functor category will play a fundamental role in proving existence of such objects. A fully detailed description can be found in [8]. We begin with recalling some definitions and notations introduced in Section 1 of this Chapter.
Let $\mathcal{C}$ be an additive category with direct limits. As we have already seen, an object $A \in \mathcal{C}$ is called **finitely presented** if the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ preserves direct limits. We will denote the subcategory of finitely presented objects of $\mathcal{C}$ as $\mathrm{fp}(\mathcal{C})$.

Definition 2.3.1. *The category $\mathcal{C}$ is said to be* ***locally finitely presented*** *if $fp(\mathcal{C})$ is skeletally small and every object in $\mathcal{C}$ is the direct limit of finitely presented objects.*

Definition 2.3.2. *A preadditive category $\mathcal{C}$ is said to have* ***split idempotents*** *if, for every $M \in \mathcal{C}$, each idempotent $e = e^2 \in End(M)$ has a kernel and the canonical map $Ker(e) \bigoplus Ker(1 - e) \to M$ is an isomorphism.*

Given a small additive category $\mathcal{B}$ with split idempotents, one can associate the category of additive contravariant functors from $\mathcal{B}$ to the category of abelian groups, denoted, as in Section 1, by $(\mathcal{B}^{op}, Ab)$. As seen in Proposition 2.1.3, this is a Grothendieck category, and it is also a locally finitely presented category (see [Lazard]). We recall that, in such a category, the finitely generated projective objects are the representable functors $(-, B)$ with $B \in \mathcal{B}$.

Proposition 2.3.1. *A functor $F \in (\mathcal{B}^{op}, Ab)$ is finitely presented if and only if there exists a presentation by finitely generated projective objects, i.e. an exact sequence of the form*

$$(-, A) \to (-, B) \to F \to 0$$

with $A, B \in \mathcal{B}$.

Again, we will call a functor in $(\mathcal{B}^{op}, Ab)$ **flat** if it is a direct limit of representable objects, and we will denote the subcategory of $(\mathcal{B}^{op}, Ab)$ of flat functors as $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$.

Definition 2.3.3. *Let $\mathcal{C}$ be a locally finitely presented additive category and $\mathcal{B} = fp(\mathcal{C})$. A sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{C}$ is called ***pure-exact*** *if the corresponding sequence in $(\mathcal{B}^{op}, Ab)$*

$$0 \to (-, X) \to (-, Y) \to (-, Z) \to 0$$

is exact.

The following result can be proved as in the category of modules over a ring.

Lemma 2.3.1. *Let $\mathcal{B}$ be a small additive category with split idempotents and*

$$\Sigma : 0 \to X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \to 0$$

a short exact sequence in $(\mathcal{B}^{op}, Ab)$. If Z is a flat functor, then

(i) the sequence Σ is a pure-exact sequence in $(\mathcal{B}^{op}, Ab)$;
(ii) the functor X is flat if and only Y is flat.

A morphism $\nu : Y \to Z$ in a locally finitely presented additive category $\mathcal{C}$ is called a **pure-epimorphism** if it is part of a pure-exact sequence Σ as above; similarly, a morphism $\mu : X \to Y$ in $\mathcal{C}$ is called a **pure-monomorphism** if it is part of a pure-exact sequence Σ as above.

Definition 2.3.4. *Let $\mathcal{C} = Flat(\mathcal{B}^{op}, Ab)$. An object $M \in \mathcal{C}$ is called **pure-injective** if the functor $Hom_{\mathcal{C}}(-, M)$ is exact when applied to any pure-exact sequence in $\mathcal{C}$.*

As can be easily seen, this is equivalent to the condition that every pure exact sequence

$$0 \to M \to Y \to Z \to 0$$

is split.

Definition 2.3.5. *A functor $C \in (\mathcal{B}^{op}, Ab)$ is called **cotorsion** if $Ext^1(Z, C) = 0$ for every $Z \in Flat(\mathcal{B}^{op}, Ab)$, where $Ext^1(Z, C)$ is computed in the Grothendieck category $(\mathcal{B}^{op}, Ab)$.*

Observation. Every pure-injective object $M \in (\mathcal{B}^{op}, Ab)$ is cotorsion. Indeed, if a short exact sequence as above is given and Z is flat then the sequence is pure exact, and therefore split.

Lemma 2.3.2. *Let $\mathcal{B}$ be a small additive category with split idempotents and $\mathcal{C} = Flat(\mathcal{B}^{op}, Ab)$. An object $M \in \mathcal{C}$ is pure-injective if and only if it is cotorsion when considered as an object of $(\mathcal{B}^{op}, Ab)$.*

Proof. Let $M \in \mathcal{C}$ be a pure-injective object and consider a short exact sequence $\Sigma : 0 \to M \to Y \to Z \to 0$ in $(\mathcal{B}^{op}, Ab)$ with Z flat. By Lemma 2.3.1 (ii) the functor Y is flat, so that Σ is a pure-exact sequence in Flat$(\mathcal{B}^{op}, Ab)$. Since M is pure-injective, Σ is split and then $\mathrm{Ext}^1(Z, M) = 0$.
Conversely, suppose that $M \in (\mathcal{B}^{op}, Ab)$ is a flat cotorsion functor and let $\Sigma : 0 \to X \to Y \to Z \to 0$ be a pure-exact sequence in $\mathcal{C}$. This is a sequence in $(\mathcal{B}^{op}, Ab)$, so we may consider part of the long exact sequence associated to the functor $(-, A)$ applied to Σ,

$$0 \to (Z, M) \to (Y, M) \to (X, M) \to \mathrm{Ext}^1(Z, M).$$

By the hypothesis, $\mathrm{Ext}^1(Z, M) = 0$, so that M is pure-injective. □

Let $\mathcal{C}$ be an additive category and $\mathcal{X}$ a full additive subcategory. Given an object $C \in \mathcal{C}$, a morphism $\eta : C \to X$ to an object X in $\mathcal{X}$ is called a **$\mathcal{X}$-preenvelope** of C if it has the property that any morphism $\delta : C \to Y$ with $Y \in \mathcal{X}$ factors through η according to the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "\delta"] & X \arrow[dl, dashed] \\ Y & \end{tikzcd}$$

A $\mathcal{X}$-preenvelope $\eta : C \to X$ of $C \in \mathcal{C}$ is called a **$\mathcal{X}$-envelope** provided that the only solutions to the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "\eta"] & X \arrow[dl, dashed, "\rho"] \\ X & \end{tikzcd}$$

are automorphisms $\rho : X \to X$. This latter property ensures that a $\mathcal{X}$-envelope of C is unique up to isomorphisms over C. The notions of **$\mathcal{X}$-precover** and **$\mathcal{X}$-cover** are defined dually.

Lemma 2.3.3 (Wakamatsu's Lemma)**.** *Let $\mathcal{G}$ be a Grothendieck category and $\mathcal{X} \subseteq \mathcal{G}$ a full additive subcategory that is closed under extensions. If $G \in \mathcal{G}$ and $\varphi : G \to X$ is a $\mathcal{X}$-envelope, then the cokernel $D = Coker\varphi$ has the property that $Ext^1(D, X') = 0$ for every $X' \in \mathcal{X}$.*

Proof. See [12]. □

Lemma 2.3.4. *Let $\mathcal{G}$ be a Grothendieck category with a flat generator and $X \in \mathcal{G}$ an object which admits a flat cover. If $Ext^1(X, C) = 0$ for every cotorsion object of $\mathcal{G}$, then X is flat.*

Proof. Let $\gamma : FC(X) \to X$ be a flat cover of X. Since $\mathcal{G}$ has a flat generator, the morphism γ is an epimorphism. By the dual to Wakamatsu's Lemma, the kernel $K = \mathrm{Ker}\ \gamma$ is cotorsion,

$$0 \to K \to FC(X) \xrightarrow{\gamma} X \to 0$$

By the hypothesis, the sequence is split, so that X is flat. □

We now have all the tools required to prove the main result:

Theorem 2.3.1. *Let $\mathcal{C}$ be a locally finitely presented additive category and X an object of $\mathcal{C}$. There exists a pure-injective envelope $\eta : X \to PE(X)$ which is a pure-monomorphism.*

Proof. Let $\mathcal{B} = \mathrm{fp}(\mathcal{C})$, so that $\mathcal{C} \cong \mathrm{Flat}(\mathcal{B}^{op}, Ab)$. By Theorem 2.7 of [6], every object of a Grothendieck category with a flat generator admits a flat cover. This applies to the category $(\mathcal{B}^{op}, Ab)$ which has a projective generator $\coprod_{A \in \mathcal{B}} \mathrm{Hom}_{\mathcal{B}}(-, A)$. Following the proof of [12], Theorem 3.4.6, one can prove that every object $G \in (\mathcal{B}^{op}, Ab)$ admits a cotorsion envelope $\eta : G \to CE(G)$. Let us note that the cotorsion envelope of G is a monomorphism with a flat cokernel:
the injective envelope of G in $(\mathcal{B}^{op}, Ab)$ is a monomorphism $\delta : G \to E(G)$. Since $E(G)$ is cotorsion, the injective envelope of G factors through the cotorsion envelope. The cotorsion envelope is therefore a monomorphism. Consider the cokernel $Z = \mathrm{Coker}\ \eta$ of the cotorsion envelope,

$$0 \to G \xrightarrow{\eta} CE(G) \to Z \to 0.$$

By Wakamatsu's Lemma, $\mathrm{Ext}^1(Z, C) = 0$ for every cotorsion object of $(\mathcal{B}^{op}, Ab)$. By Lemma 2.3.4, the functor Z is flat.
Now let $Z \in \mathrm{Flat}(\mathcal{B}^{op}, Ab)$ and consider the cotorsion envelope $\eta : Z \to CE(Z)$ in $(\mathcal{B}^{op}, Ab)$. Since the cokernel of the cotorsion envelope is flat, Lemma 2.3.1(ii) implies that the cotorsion envelope $CE(Z)$ is flat and that the morphism $\eta : Z \to CE(Z)$ is a pure-monomorphism in $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$. Let us verify that η is the pure-injective envelope of Z in $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$. By Lemma 2.3.2, $CE(Z)$ is pure-injective in $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$. If $\mu : Z \to M$ is a morphism to a pure-injective object in $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$ then, again by Lemma 2.3.2, M is a cotorsion object of $(\mathcal{B}^{op}, Ab)$, so the morphism μ factors through the cotorsion envelope $\eta : Z \to CE(Z)$. It follows that η is a pure-injective preenvelope of Z in $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$. But since it is the cotorsion envelope of Z, any endomorphism $\rho : CE(Z) \to CE(Z)$ over Z is necessarily an automorphism. Thus $\eta : Z \to CE(Z)$ is the pure-injective envelope of Z in $\mathrm{Flat}(\mathcal{B}^{op}, Ab)$. □

We will end this section by proving a result in the case when $\mathcal{B}$ is the category R-mod of finitely presented left modules.

Theorem 2.3.2. *A flat functor* $(-,M) \in ((R\text{-}mod)^{op}, Ab)$ *is cotorsion if and only if* M *is a pure-injective module. If* M *is an* R*-module and* $m : M \to PE(M)$ *is the pure-injective envelope, then*

$$(-,m) : (-,M) \to (-,PE(M))$$

is the cotorsion envelope of the flat functor $(-,M)$*.*

Proof. Suppose that $(-,M)$ is a cotorsion object, and consider the pure-injective envelope $m : M \to PE(M)$. The short exact sequence

$$0 \to M \overset{m}{\to} PE(M) \to PE(M)/M \to 0$$

is pure-exact, so the corresponding sequence in $((R\text{-mod})^{op}, Ab)$

$$0 \to (-,M) \to (-,PE(M)) \to (-,PE(M)/M) \to 0$$

is exact. As $(-,PE(M)/M)$ is flat, the sequence splits and $M = PE(M)$ is pure-injective.
Conversely, suppose that M is a pure-injective module and consider an extension

$$0 \to (-,M) \overset{\mu}{\to} G \overset{\nu}{\to} (-,Z) \to 0$$

of $(-,M)$ by a flat functor $(-,Z)$. By Proposition 2.1.6, G is isomorphic to the flat functor $(-,G(R))$. Replacing G with $(-,G(R))$ we get that $\mu = (-,f)$ and $\nu = (-,g)$ where

$$0 \to M \overset{f}{\to} G(R) \overset{g}{\to} Z \to 0$$

is a pure exact sequence. As M is pure-injective, the sequence splits.
To prove the second statement, let M be a left R-module. The transformation of functors $(-,m) : (-,M) \to (-,PE(M))$ is a monomorphism into a cotorsion functor $(-,PE(M))$ whose cokernel is the flat object $(-,PE(M)/M)$. Thus the morphism $(-,m)$ is a cotorsion preenvelope. To see that it is a cotorsion envelope, just note that any endomorphism of $(-,PE(M))$ over $(-,M)$ is of the form $(-,g)$ with $g : PE(M) \to PE(M)$ an endomorphism over M. As $PE(M)$ is the pure-injective envelope of M, the endomorphism g must be an automorphism. $\square$

Chapter 3

Tilting theory

3.1 The Tilting theorem

Throughout this section, $\mathcal{A}$ will denote an abelian category and V an object of $\mathcal{A}$ such that $V^{(\alpha)}$ exists in $\mathcal{A}$ for every cardinal α.

Proposition 3.1.1. *Let $R = End_{\mathcal{A}}(V)$, $H_V = Hom_{\mathcal{A}}(V, -) : \mathcal{A} \to$ Mod-R. Then H_V has a left adjoint additive functor T_V : Mod-$R \to \mathcal{A}$ such that $T_V(R) = V$. Let $\sigma : 1_{Mod-R} \to H_V T_V$ and $\rho : T_V H_V \to 1_{\mathcal{A}}$ be respectively the unit and counit of the adjunction $\langle T_V, H_V \rangle$. Let us define*

$$Tr_V : \mathcal{A} \to \mathcal{A} \text{ by } Tr_V(M) = \sum\{Imf | f \in Hom_{\mathcal{A}}(V, M)\}$$

$$Ann_V : \text{Mod-}R \to \text{Mod-}R \text{ by } Ann_V(N) = \sum\{L |\ L \overset{i}{\hookrightarrow} N,\ T_V(i) = 0\}$$

and

$$GenV = \{M \in \mathcal{A} |\ Tr_V(M) = M\},$$
$$FaithV = \{N \in \text{Mod-}R |\ Ann_V(N) = 0\}$$

Then:

a) The canonical inclusion $Tr_V(M) \hookrightarrow M$ induces a natural isomorphism $H_V(Tr_V(M)) \cong H_V(M)$, and the canonical projection $N \twoheadrightarrow N/Ann_V(N)$ induces a natural isomorphism $T_V(N) \cong T_V(N/Ann_V(N))$.

b) Tr_V is an idempotent preradical, and Ann_V is a radical.

c) $Tr_V(M) = Im\ \rho_M$, *and* $Ann_V(N) = Ker\ \sigma_N$.

d) $T_V(Mod\text{-}R) \subseteq GenV$, *and* $H_V(\mathcal{A}) \subseteq FaithV$.

e) GenV is closed under (existing) coproducts and factors in $\mathcal{A}$*, and FaithV is closed under products and submodules in Mod-R.*

Proof. The proof of the existence of the functor T_V can be found in [9].

a) The first part is clear, since $\mathrm{Im}(f) \subseteq \mathrm{Tr}_V(M)$ for any $f \in (VM)$. Now let $\mathrm{Ann}_V(N) = \sum\{L_\lambda |\ \lambda \in \Lambda\}$, with $j_\lambda : L_\lambda \hookrightarrow \mathrm{Ann}_V(N)$, for each $i_\lambda : L_\lambda \hookrightarrow N$ with $T_V(i_\lambda) = 0$. Applying the functor T_V to the commutative diagram

$$\begin{tikzcd}
 & \bigoplus L_\lambda \arrow[dl, "\oplus j_\lambda"'] \arrow[dr, "\oplus i_\lambda"] & \\
Ann_V(N) \arrow[rr, "i"] & & N
\end{tikzcd}$$

we immediately obtain $T_V(i) = 0$, since $\oplus j_\lambda$ is an epimorphism and T_V is right exact and commutes with direct sums. Therefore, if we apply T_V to the exact sequence

$$0 \to \mathrm{Ann}_V(N) \xrightarrow{i} N \xrightarrow{\pi} N/\mathrm{Ann}_V(N) \to 0$$

we obtain the sequence

$$T_V\ (\mathrm{Ann}_V(N)) \xrightarrow{0} T_V(N) \xrightarrow{T_V(\pi)} T_V(N/\mathrm{Ann}_V(N)) \to 0$$

which shows that $T_V(\pi)$ is an isomorphism.

b) The first part is clear. Moreover, using the right exactness of T_V and, for $i_\lambda : L_\lambda \hookrightarrow M$ and $f : M \to N$, using the commutative diagram

$$\begin{tikzcd}
L_\lambda \arrow[rr, "fi_\lambda"] \arrow[dr] & & N \\
 & f(L_\lambda) \arrow[ur] &
\end{tikzcd}$$

it becomes clear that Ann_V is a preradical. Let us prove that it is a radical, i.e. $\mathrm{Ann}_V(N/\mathrm{Ann}_V(N)) = 0$. Let $\mathrm{Ann}_V(N) \leq L \leq N$ such that $T_V(i) = 0$ for

$i : L/\mathrm{Ann}_V(N) \hookrightarrow N/\mathrm{Ann}_V(N)$. Applying T_V to the commutative diagram

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[d, two heads, "\pi_L"] & N \arrow[d, "\pi_N"] \\ L/Ann_V(N) \arrow[r, "i"] & N/Ann_V(N) \end{tikzcd}$$

we have $0 = T_V(i)T_V(\pi_L) = T_V(i\pi_L) = T_V(\pi_N j) = T_V(\pi_N)T_V(j)$, so that $T_V(j) = 0$ since $T_V(\pi_N)$ is an isomorphism by a). This gives $L \leq \mathrm{Ann}_V(N)$, as desired.

c) Im ρ_M is a factor of $T_V H_V(M)$, and $T_V H_V(M) \in T_V$ (Mod-R) $= T_V$ (Gen$R$) $\subseteq$ Gen$T_V(R)$ $=$ Gen$V$ since T_V is right exact and preserves coproducts. Therefore Im $\rho_M \in$ Gen$V$, i.e. Im $\rho_M \subseteq \mathrm{Tr}_V(M)$. Conversely, let $V^{(\alpha)} \overset{\varphi}{\to} M$ be a morphism such that Im $\varphi = \mathrm{Tr}_V(M)$. In the commutative diagram

$$\begin{tikzcd} V^{(\alpha)} \arrow[r, "\varphi"] & M \\ T_V H_V(V^{(\alpha)}) \arrow[u, "\rho_{V^{(\alpha)}}"] \arrow[r, "T_V H_V(\varphi)"] & T_V H_V(M) \arrow[u, "\rho_M"'] \end{tikzcd}$$

$\rho_{V^{(\alpha)}}$ is epi-split by adjointness, since $V^{(\alpha)} = T_V(R^{(\alpha)})$. Thus $\mathrm{Tr}_V(M) =$ Im $\varphi \leq$ Im ρ_M, and so they are equal.

Now let $N \in$ Mod-R. From the commutative diagram

$$\begin{tikzcd} Ann_V(N) \arrow[r, "i"] \arrow[d, "\sigma_{Ann_V(N)}"] & N \arrow[d, "\sigma_N"] \\ H_V T_V(Ann_V(N)) \arrow[r, "H_V T_V(i)"] & H_V T_V(N) \end{tikzcd}$$

since $T_V(i) = 0$ (as in the proof of a)), we see that $\sigma_N i = 0$, i.e. $\mathrm{Ann}_V(N) \leq$ Ker σ_N. Conversely, if i :Ker $\sigma_N \hookrightarrow N$ is the canonical inclusion, then $\sigma_N i = 0$, so that $T_V(\sigma_N)T_V(i) = 0$ and so $T_V(i) = 0$, since $T_V(\sigma_N)$ is mono-split by adjointness. This proves that Ker $\sigma_N \leq \mathrm{Ann}_V(N)$, and so they are equal.

d) By c), it follows that $M \in$ Gen$V$ if and only if ρ_M is epi, and $N \in$ Faith$V$ if and only if σ_N is mono. Since by adjointness ρ_M is epi-spit for any $M \in T_V$ (Mod-R), and σ_N is mono-split for any $N \in H_V(\mathcal{A})$, d) follows.

e) follows from b) thanks to [10], Proposition VI.1.4. □

The last proposition suggests that Gen$V \subseteq \mathcal{A}$ and Faith$V \subseteq$ Mod-R are the largest full subcategories between which the adjoint pair $\langle T_V, H_V\rangle$ can induce an equivalence.

Definition 3.1.1. *$V \in \mathcal{A}$ is called a* ***$*$-object*** *if $\langle T_V, H_V\rangle$ induces an equivalence*

$$H_V : GenV \rightleftarrows FaithV : T_V$$

Note that Gen$V$ is closed under factors and coproducts, and Faith$V$ is closed under submodules and direct products. These properties, together with the equivalence, characterize $*$-objects, as shown by the following

Theorem 3.1.1. *Let $\mathcal{A}$ be a cocomplete abelian category, and let R be a ring. Let $\mathcal{G} \subseteq \mathcal{A}$ be a full subcategory closed under factors and coproducts, and let $\mathcal{F} \subseteq$ Mod-R be a full subcategory closed under submodules and direct products, and assume there is a category equivalence*

$$H : \mathcal{G} \rightleftarrows \mathcal{F} : T$$

Let $\bar{R} = R/\mathbf{r}(\mathcal{F})$. Then $\bar{R}$ is in $\mathcal{F}$, and setting $V = T(\bar{R})$ we have natural isomorphisms $H \cong H_V$ and $T \cong T_V$ and equalities $\mathcal{G} = GenV$ and $\mathcal{F} = FaithV$. In particular, V is a $*$-object in $\mathcal{A}$ and $\bar{R} \cong End_{\mathcal{A}}(V)$.

Proof. Since $\mathcal{F}$ is closed under submodules and products, $\bar{R}$ is in $\mathcal{F}$. For any $M \in \mathcal{G}$ we have $H(M) \cong \mathrm{Hom}_R(\bar{R}, H(M)) \cong \mathrm{Hom}_{\mathcal{A}}(V, M)$ canonically in Mod-R. Moreover, $\mathrm{End}_{\mathcal{A}}(V) \cong \mathrm{End}_R(\bar{R}) \cong \bar{R}$ canonically. Given any $N \in \mathcal{F} \subseteq$ Mod-$\bar{R}$, from the exact sequence $\bar{R}^{(\alpha)} \to N \to 0$ we obtain the exact sequence $V^{(\alpha)} \to T_V(N) \to 0$ which gives $T_V(N) \in \mathcal{G}$, since $\mathcal{G}$ is closed under coproducts and factors. Therefore $T \cong T_V$, as both functors are left adjoint to $H \cong H_V$. From statement c) in Proposition 3.1.1, we derive the inclusions $\mathcal{G} \subseteq$ Gen$V$ and $\mathcal{F} \subseteq$ Faith$V$. On the other hand, $V \in \mathcal{G}$ and the closure properties of $\mathcal{G}$ immediately give Gen$V \subseteq \mathcal{G}$. Moreover, if $N \in$ Faith$V$, then from statements b) and c) in Proposition 3.1.1 we derive $N \overset{\sigma_N}{\hookrightarrow} H_V T_V(N) \in H_V(\mathrm{Gen}V) = H_V(\mathcal{G}) = H(\mathcal{G}) \subseteq \mathcal{F}$, hence $N \in \mathcal{F}$ by the closure properties of $\mathcal{F}$. This shows that Faith$V \subseteq \mathcal{F}$. □

Lemma 3.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $\mathcal{G} \subseteq \mathcal{A}$ and $\mathcal{F} \subseteq \mathcal{B}$ be full subcategories each one of which is either closed under subobjects or factors. Let $\langle T, H\rangle$ be an adjoint pair of additive functors $\mathcal{G} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{F}$, with unit $\sigma : 1 \to HT$ and counit $\rho : TH \to 1$. Then* *a) If ρ_M is an isomorphism for all $M \in \mathcal{G}$, then T preserves the exactness of short exact sequences with objects in $H(\mathcal{G})$;*

b) If σ_N is an isomorphism for all $N \in \mathcal{F}$, then H preserves the exactness of short exact sequences with objects in $T(\mathcal{F})$.

Proof. We will only prove a), as the proof of b) is dual. Let $0 \to L \xrightarrow{f} L' \to L'' \to 0$ be an exact sequence with $L, L', L'' \in H(\mathcal{G})$. Since T is right exact we get the commutative diagram with exact row

$$\begin{tikzcd}
T(L) \arrow[rr, "T(f)"] \arrow[dr, two heads, "g"] & & T(L') \arrow[r] & T(L'') \arrow[r] & 0 \\
& ImT(f) \arrow[ur, hook, "h"] & & &
\end{tikzcd}$$

where we have decomposed $T(f)$. Since the sequence in $\mathcal{G}$ $0 \to \mathrm{Im}T(f) \xrightarrow{h} T(L') \to T(L'') \to 0$ is exact, we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "H(g)\circ\sigma_L"] & L' \arrow[r] \arrow[d, "\sigma_{L'}"] & L'' \arrow[r] \arrow[d, "\sigma_{L''}"] & 0 \\
0 \arrow[r] & H(ImT(f)) \arrow[r, "H(h)"] & HT(L') \arrow[r] & HT(L'') &
\end{tikzcd}$$

where $\sigma_L, \sigma_{L'}, \sigma_{L''}$ are isomorphisms, since $H(\rho_M) \circ \sigma_{H(M)} = 1_{H(M)}$ for every $M \in \mathcal{G}$ (by the triangular property of the adjunction). Hence $H(g)$, and then $TH(g)$, are isomorphisms. From the commutative diagram in $\mathcal{G}$:

$$\begin{tikzcd}
T(L) \arrow[r, "g"] & ImT(f) \\
THT(L) \arrow[u, "\rho_{T(L)}", "\cong"'] \arrow[r, "TH(g)", "\cong"'] & TH(ImT(f)) \arrow[u, "\rho_{ImT(f)}", "\cong"']
\end{tikzcd}$$

it follows that g is an isomorphism, which means that $T(f)$ is a monomorphism. □

Let $\mathcal{A}$ be an abelian category and $V \in \mathcal{A}$ such that $\exists V^{(\alpha)} \in \mathcal{A}$ for any cardinal α. We shall denote by Gen$V$ the full subcategory of $\mathcal{A}$ generated by V and by $\overline{GenV}$ the closure of Gen$V$ under subobjects: $\overline{GenV}$ is the smallest exact abelian subcategory of $\mathcal{A}$ containing Gen$V$. Moreover we let Pres$V$ denote the full subcategory of Gen$V$ consisting of the objects in $\mathcal{A}$ presented by V, i.e.

$$\text{Pres}V = \{M \in \mathcal{A} |\ \exists \text{ exact sequence } V^{(\beta)} \to V^{(\alpha)} \to M \to 0\}.$$

Finally, let $R = \text{End}_{\mathcal{A}}(V)$ and

$$V^{\perp} = \text{Ker Ext}^1(V, -), \qquad V_{\perp} = \text{Ker Hom}_{\mathcal{A}}(V, -)$$

Proposition 3.1.2. *Let* $V \in \mathcal{A}$.

a) If $GenV \subseteq V^{\perp}$ *then* Tr_V *is a radical. In particular,* $(GenV, V_{\perp})$ *is a torsion theory in* $\mathcal{A}$.

b) If $GenV = V^{\perp}$*, then* $GenV = PresV$.

c) If $\overline{GenV} = \mathcal{A}$*, then the equality* $GenV = V^{\perp}$ *is equivalent to the following conditions:*

- *i)* $projdim V \leq 1$.
- *ii)* $Ext^1(V, V^{(\alpha)}) = 0$ *for any cardinal* α.
- *iii) if* $M \in \mathcal{A}$ *and* $Hom_{\mathcal{A}}(V, M) = 0 = Ext^1(V, M)$*, then* $M = 0$.

Proof. a) Let $M \in \mathcal{A}$ and consider the canonical sequence

$$0 \to Tr_V(M) \to M \to M/Tr_V(M) \to 0.$$

We obtain the exact sequence

$$0 \to H_V(Tr_V(M)) \overset{\cong}{\to} H_V(M) \to H_V(M/Tr_V(M)) \to \text{Ext}^1(V, Tr_V(M)) = 0$$

which shows that $H_V(M/Tr_V(M)) = 0$, i.e. $\text{Tr}_V(M/Tr_V(M)) = 0$. This and Proposition 3.1.1 b) prove that Tr_V is an idempotent radical. This shows that for any $M \in \mathcal{A}$, $\text{Tr}_V(M)$ is the unique subobject of M such that $\text{Tr}_V(M) \in$ Gen$V$ and $M/Tr_V(M) \in V_{\perp}$, and so (Gen$V, V_{\perp}$) is a torsion theory in $\mathcal{A}$.

b) Let $M \in$ Gen$V$ and $\alpha = \text{Hom}_{\mathcal{A}}(V, M)$. Then we have the exact sequence

$$0 \to K \to V^{(\alpha)} \overset{\varphi}{\to} M \to 0$$

and

$$H_V(V^{(\alpha)}) \overset{H_V(\varphi)}{\to} H_V(M) \to \text{Ext}^1(V, K) \to 0$$

where the morphism $H_V(\varphi)$ is an epimorphism by construction. Therefore $\mathrm{Ext}^1(V,K)=0$, so by assumption $K\in \mathrm{Gen}V$. This proves that $M\in \mathrm{Pres}V$.

c) Let $\overline{GenV}=\mathcal{A}$ and $\mathrm{Gen}V=V^{\perp}$. Let us prove i), showing that $\mathrm{Ext}^2(V,M)=0$ for any $M\in\mathcal{A}$. Indeed, given a representative of an element in $\mathrm{Ext}^2(V,M)$, say

$$(\epsilon): 0\to M\to E_1\overset{f}{\to} E_2\to V\to 0$$

let $I=\mathrm{Im}f$. Embedding E_1 in a suitable object $X\in\mathrm{Gen}V$, we get a push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, hook] & I \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & X \arrow[r] & P' \arrow[r] & 0
\end{tikzcd}$$

where X, and so P', are in $\mathrm{Gen}V$. Then we have a second push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, hook] & E_2 \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P' \arrow[r] & P'' \arrow[r] & V \arrow[r] & 0
\end{tikzcd}$$

By glueing these two diagrams together we derive a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r, "f"] \arrow[d, hook] & E_2 \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & X \arrow[r, "g"] & P'' \arrow[r, "\pi"] & V \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{Im}g=P'\in V^{\perp}$. Then π is epi-split, and so $\epsilon\sim 0$. This proves i). Condition ii) is contained in the hypothesis, and condition iii) follows from a).

Conversely, let us assume that conditions i), ii) and iii) hold. The first condition assures that $V^{\perp}$ is closed under factors. Therefore, using the second condition we immediately see that $\mathrm{Gen}V\subseteq V^{\perp}$. in order to prove the opposite inclusion, given any $M\in V^{\perp}$, from the exact sequence

$$0\to Tr_V(M)\to M\to M/Tr_V(M)\to 0$$

and using condition i) we obtain the exact sequence

$$\begin{aligned}&0 \to \mathrm{Hom}_{\mathcal{A}}(V, Tr_V(M)) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(V, M) \to \mathrm{Hom}_{\mathcal{A}}(V, M/Tr_V(M)) \to \\ &\to \mathrm{Ext}^1(V, Tr_V(M)) = 0 = \mathrm{Ext}^1(V, M) \to \mathrm{Ext}^1(V, M/Tr_V(M)) \to 0.\end{aligned}$$

Hence $\mathrm{Hom}_{\mathcal{A}}(V, M/Tr_V(M)) = 0 = \mathrm{Ext}^1(V, M/Tr_V(M))$. Now condition iii) gives $M/Tr_V(M) = 0$, i.e. $M = Tr_V(M) \in \mathrm{Gen}V$. This proves that $V^{\perp} \subseteq \mathrm{Gen}V$. □

Observation. If $\mathcal{A}$ has enough injectives, then $\overline{GenV} = \mathcal{A}$ whenever $\mathrm{Gen}V = V^{\perp}$. Indeed, if $\mathcal{A}$ has enough injectives then every object of $\mathcal{A}$ embeds in an injective object which, by definition, belongs to $V^{\perp} = \mathrm{Gen}V$.

Definition 3.1.2. *$V \in \mathcal{A}$ is a **(self-)small** if the functor $Hom_{\mathcal{A}}(V, -)$ commutes with arbitrary direct sums (of copies of V).*

Definition 3.1.3. *An object V in an abelian category $\mathcal{A}$ such that $\exists V^{(\alpha)} \in \mathcal{A}$ for any cardinal α is called a **tilting object** if*

- *i) V is self-small;*
- *ii) $GenV = V^{\perp}$;*
- *iii) $\overline{GenV} = \mathcal{A}$.*

From last Proposition we see that to a tilting object $V \in \mathcal{A}$ is naturally associated a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, namely $\mathcal{T} = V^{\perp}$ and $\mathcal{F} = V_{\perp}$.

Theorem 3.1.2. *Let $\mathcal{A}$ be an abelian category such that $\exists V^{(\alpha)} \in \mathcal{A}$ for any cardinal α. Then the following are equivalent:*

- *a) V is a $*$-object;*
- *b) V is a tilting object in $\overline{GenV}$;*
- *c) ρ is monic in $\mathcal{A}$ and σ is epic in Mod-R;*
- *d) V is selfsmall, $GenV = PresV$ and H_V preserves short exact sequences in $\mathcal{A}$ with all terms in $GenV$;*
- *e) V is selfsmall and for any short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$ with M (and N) in $GenV$, the sequence $0 \to H_V(L) \to H_V(M) \to H_V(N) \to 0$ is exact if and only if $L \in GenV$.*

Proof. a) $\Longrightarrow$ b): V is selfsmall since $H_V(V^{(\alpha)}) = H_V T_V(R^{(\alpha)}) \cong R^{(\alpha)} = H_V(V)^{(\alpha)}$ canonically. We can assume that $\mathcal{A} = \overline{GenV}$. In order to prove that Gen$V \subseteq V^{\perp}$, given any $M \in$ Gen$V$, we show that any s.e.s $0 \to M \to X \overset{\pi}{\to} V \to 0$ in $\mathcal{A}$ splits: let $X \overset{i}{\hookrightarrow} L$ be a fixed embedding with $L \in$ Gen$V$, and let us consider the push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r, "\pi"] \arrow[d, hook, "i"] & V \arrow[r] \arrow[d, hook, "j"] & 0 \\
0 \arrow[r] & M \arrow[r] & L \arrow[r, "p"] & P \arrow[r] & 0
\end{tikzcd}$$

where the second row is in Gen$V$. Then we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_V(M) \arrow[r] \arrow[d, equal] & H_V(X) \arrow[r, "H_V(\pi)"] \arrow[d] & H_V(V) \arrow[r, "\delta"] \arrow[d] & Ext^1(V,M) \arrow[d, equal] \\
0 \arrow[r] & H_V(M) \arrow[r] & H_V(L) \arrow[r, "H_V(p)"] & H_V(P) \arrow[r, "\gamma"] & Ext^1(V,M)
\end{tikzcd}$$

Since the morphism $H_V(p)$ is epic (because of Lemma 3.1.1), we see that $\gamma = 0$, so that $\delta = 0$ too. This shows that $H_V(\pi)$ is epic, so that the initial sequence splits.
Conversely, let us prove that $V^{\perp} \subseteq$ Gen$V$. Given $M \in V^{\perp}$, let

$$0 \to M \to X_0 \overset{\varphi}{\to} X_1 \to 0$$

be a fixed exact sequence with X_0 (and X_1) in Gen$V$. Since $\mathrm{Ext}^1(V, M) = 0$, by assumption, $H_V(\varphi)$ is epic. Therefore we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & X_0 \arrow[r, "\varphi"] & X_1 \arrow[r] & 0 \\
... \arrow[r] & T_V H_V(M) \arrow[r] \arrow[u, "\rho_M"] & T_V H_V(X_0) \arrow[r, "T_V H_V(\varphi)"] \arrow[u, "\rho_{X_0}", "\cong"'] & T_V H_V(X_1) \arrow[r] \arrow[u, "\rho_{X_1}", "\cong"'] & 0
\end{tikzcd}$$

which shows that ρ_M is epic, i.e. $M \in$ Gen$V$.

b) $\Longrightarrow$ e): Assume that $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathcal{A}$ with M (and N) in Gen$V$. Then, since by assumption Gen$V = V^{\perp}$, the sequence $0 \to H_V(L) \to H_V(M) \to H_V(N) \to \mathrm{Ext}^1(V, L) \to 0$ is exact, and so $\mathrm{Ext}^1(V, L) = 0$ if and only if $L \in$ Gen$V$.

e) $\implies$ d): Let $M \in \mathrm{Gen}V$, and let $\alpha = H_V(M)$. Then there is a short exact sequence $0 \to K \to V^{(\alpha)} \overset{\varphi}{\to} M \to 0$ such that $H_V(\varphi)$ is epic. By hypothesis, we must have $K \in \mathrm{Gen}V$. This shows that $M \in \mathrm{Pres}V$.

d) $\implies$ c): Let $N \in$ Mod-R and let $R^{(\beta)} \to R^{(\alpha)} \overset{\varphi}{\to} N \to 0$ be exact. Since H_V is exact on $\mathrm{Gen}V$ by assumption, it preserves the exactness of the sequence $0 \to K \to T_V(R^{(\alpha)}) \overset{T_V(\varphi)}{\to} T_V(N) \to 0$. Thus we have a commutative diagram with exact rows

$$\begin{tikzcd}
R^{(\alpha)} \arrow[r, "\varphi"] \arrow[d, "\sigma_{R^{(\alpha)}}", "\cong"'] & N \arrow[r] \arrow[d, "\sigma_N"] & 0 \\
H_V T_V(R^{(\alpha)}) \arrow[r, "H_V T_V(\varphi)"] & H_V T_V(N) \arrow[r] & 0
\end{tikzcd}$$

where $\sigma_{R^{(\alpha)}}$ is an isomorphism since V is selfsmall. This proves that σ_N is epic for any $N \in$ Mod-R. In order to prove that ρ is monic in $\mathcal{A}$, thanks to statement a) in Proposition 3.1.1, it is sufficient to prove that ρ is monic in $\mathrm{Gen}V = \mathrm{Pres}V$. Moreover, we see that ρ is monic in $T_V(\text{Mod-}R)$, since by adjunction $\rho_{T_V(-)} \circ T_V(\sigma_-) = 1_{T_V(-)}$ and $T_V(\sigma_-)$ is an isomorphism since we have already proved that σ_-, and so $T_V(\sigma_-)$, is an epimorphism in Mod-R. Therefore, it remains to be proved that $\mathrm{Pres}V \subseteq T_V(\text{Mod-}R)$. Let $M \in \mathrm{Pres}V$ and let $V^{(\beta)} \to V^{(\alpha)} \overset{\varphi}{\to} M \to 0$ be exact. Applying T_V to the exact sequence (where $c = \mathrm{coker}\ H_V(\varphi)$)

$$H_V(V^{(\beta)}) \to H_V(V^{(\alpha)}) \overset{c}{\to} C \to 0$$

we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
T_V H_V(V^{(\beta)}) \arrow[r] \arrow[d, "\rho_{V^{(\beta)}}", "\cong"'] & T_V H_V(V^{(\alpha)}) \arrow[r] \arrow[d, "\rho_{V^{(\alpha)}}", "\cong"'] & T_V(C) \arrow[r] & 0 \\
V^{(\beta)} \arrow[r] & V^{(\alpha)} \arrow[r, "\varphi"] & M \arrow[r] & 0
\end{tikzcd}$$

which proves that $M \cong T_V(C) \in T_V(\text{Mod-}R)$.

c) $\implies$ a): This is an immediate consequence of Proposition 3.1.1. □

We will now state and prove the main result of this section, namely the Tilting theorem. First we need the following

Lemma 3.1.2. *Let $V \in \mathcal{A}$ be a tilting object, $R = End_{\mathcal{A}}(V)$, and let $T_V^{(i)}$, $i \geq 1$, be the i-th left derived functor of T_V. Then*

- *a) Faith$V$ = Ker $T_V^{(i)}$;*
- *b) $T_V^{(i)} = 0$ for all $i \geq 2$;*
- *c) Ann_V is an idempotent radical;*
- *d) ($KerT_V$, $KerT'_V$) is a torsion theory in Mod-R;*
- *e) for any $N \in$ Mod-R the canonical inclusion $Ann_V \hookrightarrow N$ induces a natural isomorphism $T'_V(Ann_V(N)) \cong T'_V(N)$.*

Proof. a) If $N \in \mathrm{Faith}V$, then by d) and e) of Proposition 3.1.1 there is an exact sequence in Faith$V$

$$0 \to K \to R^{(\alpha)} \to N \to 0$$

On the one hand we have the exact sequence

$$0 \to T'_V(N) \to T_V(K) \to T_V(R^{(\alpha)}) \to T_V(N) \to 0,$$

on the other hand, thanks to Theorem 3.1.2, we know that V is a $*$-object, and so by Lemma 3.1.1 a) the functor T_V preserves exact sequences in Faith$V$. Thus $T'_V(N) = 0$, and the inclusion Faith$V \subseteq \mathrm{Ker}T'_V$ is proved. Conversely, for any $N \in \mathrm{Ker}T'_V$ we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\cong"] & R^{(\alpha)} \arrow[r] \arrow[d, "\cong"] & N \arrow[r] \arrow[d, "\sigma_N"] & 0 \\
0 \arrow[r] & H_V T_V(K) \arrow[r] & H_V T_V(R^{(\alpha)}) \arrow[r] & H_V T_V(N) \arrow[r] & 0
\end{tikzcd}$$

where the first two vertical canonical maps are isomorphisms thanks to Theorem 3.1.2. This shows that σ_N is monic, so that $N \in \mathrm{Faith}V$ by Proposition 3.1.1 c).

b) Given any $N \in$ Mod-R and a short exact sequence

$$0 \to K \to R^{(\alpha)} \to N \to 0$$

since $K \in \mathrm{Faith}V = \mathrm{Ker}T'_V$, we see by induction that $T^{(i+1)}(N) \cong T^{(i)}(K)$ is zero for any $i \geq 1$.

c) We have already remarked in b) of Proposition 3.1.1 that Ann_V is a radical. Since by a) Faith$V = \mathrm{Ker}T'_V$ is obviosly closed under extensions, we can conclude that the associated radical Ann_V is idempotent.

d) Thanks to c) we see that $(\mathcal{T},\ \mathrm{Ker}T'_V)$ is a torsion theory, where $\mathcal{T} = \{N \in \text{Mod-}R \mid \ \mathrm{Ann}_V(N) = N\}$. It remains to be proved that $\mathcal{T} = \mathrm{Ker}T_V$. First, let $N \in \mathcal{T}$. Then by Proposition 3.1.1 a) we have $T_V(N) \cong T_V(N/Ann_V(N)) = T_V(0) = 0$. Conversely, if $N \in \mathrm{Ker}T_V$, then for any embedding $L \hookrightarrow N$ we have $T_V(i) = 0$, which proves that $\mathrm{Ann}_V(N) = N$, i.e. $N \in \mathcal{T}$.

e) Given any $N \in \text{Mod-}R$ and the associated canonical exact sequence

$$0 \to \mathrm{Ann}_V(N) \to N \to N/\mathrm{Ann}_V(N) \to 0,$$

by a), b) and d) we see that $T'_V(\mathrm{Ann}_V(N)) \cong T'_V(N)$ canonically. □

Theorem 3.1.3. *Let V be a tilting object in an abelian category $\mathcal{A}$, $R = End_{\mathcal{A}}(V)$, $H_V = Hom_{\mathcal{A}}(V,-), H'_V = Ext^1(V,-), T_V$ the left adjoint to H_V, and T'_V the first left derived functor of T_V. Set*

$$\mathcal{T} = KerH'_V,\ \mathcal{F} = KerH_V,\ \mathcal{X} = KerT_V,\ \mathcal{Y} = KerT'_V.$$

Then:

a) $(\mathcal{T},\mathcal{F})$ is a torsion theory in $\mathcal{A}$ with $\mathcal{T} = GenV$, and $(\mathcal{X},\mathcal{Y})$ is a torsion theory in Mod-R with $\mathcal{Y} = FaithV$;

b) the functors $H_V|_{\mathcal{T}}, T_V|_{\mathcal{Y}}, H'_V|_{\mathcal{F}}, T'_V|_{\mathcal{X}}$ are exact, and they induce a pair of category equivalences $\mathcal{T} \underset{T_V}{\overset{H_V}{\rightleftarrows}} \mathcal{Y}$ and $\mathcal{F} \underset{T'_V}{\overset{H'_V}{\rightleftarrows}} \mathcal{X}$;

c) $T_V H'_V = 0 = T'_V H_V$ and $H_V T'_V = 0 = H'_V T_V$;

d) there are natural transformations θ and η that, together with the adjoint transformations ρ and σ, yield the exact sequences

$$0 \to T_V H_V(M) \overset{\rho_M}{\to} M \overset{\eta_M}{\to} T'_V H'_V(M) \to 0$$

and

$$0 \to H'_V T'_V(N) \overset{\theta_N}{\to} N \overset{\sigma_N}{\to} H_V T_V(N) \to 0$$

for each $M \in \mathcal{A}$ and for each $N \in$ Mod-R.

Proof. Statement a) is contained in Proposition 3.1.2 and Lemma 3.1.2. The first part of b) regarding the exactness of the four restricted functors and the existence of the first equivalence is an immediate consequence of Theorem

3.1.2, Lemma 3.1.1, Proposition 3.1.2 c) and Lemma 3.1.2 b). Moreover, part of d) is contained in Theorem 3.1.2 and Proposition 3.1.1.

In order to prove c), we start with an arbitrary object $M \in \mathcal{A}$ and a fixed associated short exact sequence

$$(*) : 0 \to M \to X_0 \to X_1 \to 0$$

with X_0 and X_1 in Gen$V = \mathcal{T}$. Applying H_V we get

$$H_V(X_0) \to H_V(X_1) \to H'_V(M) \to H'_V(X_0) = 0.$$

Applying T_V we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] & 0 & \\
T_V H_V(X_0) \arrow[r] \arrow[u, "\rho_{X_0}", "\cong"'] & T_V H_V(X_1) \arrow[r] \arrow[u, "\rho_{X_1}", "\cong"'] & T_V H'_V(M) \arrow[r] & 0
\end{tikzcd}$$

which shows that $T_V H'_V(M) = 0$. Moreover, thanks to Proposition 3.1.1 d) and Lemma 3.1.2 a), we have $H_V(M) \in$ Faith$V =$ KerT'_V, and so $T'_V H_V(M) = 0$.

On the other hand, for any $N \in$ Mod-R let us consider an exact sequence of the form

$$(**) : 0 \to K \to R^{(\alpha)} \to N \to 0.$$

Note that both $R^{(\alpha)}$ and the submodule K are in Faith$V$. Applying H_V to the exact sequence $0 \to T'_V(R^{(\alpha)}) \to T'_V(N) \to T_V(K) \to T_V(R^{(\alpha)})$, we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & K \arrow[r] \arrow[d, "\cong"', "\sigma_K"] & R^{(\alpha)} \arrow[d, "\cong"', "\sigma_{R^{(\alpha)}}"] \\
0 \arrow[r] & H_V T'_V(N) \arrow[r] & H_V T_V(K) \arrow[r] & H_V T_V(R^{(\alpha)})
\end{tikzcd}$$

which shows that $H_V T'_V(N) = 0$. Finally, by Proposition 3.1.1 d) and the hypothesis, we have $T(N) \in$ Gen$V = \mathcal{T} =$ KerH'_V, therefore $H'_V T_V(N) = 0$. This completes the proof of c).

In order to prove the second half of b), first we remark that the inclusion Im$H'_V \subseteq \mathcal{X}$ follows from $T_V H'_V = 0$ and, similarly, the inclusion Im$T'_V \subseteq \mathcal{F}$ follows from $H_V T'_V = 0$.

Next, let $M \in \mathcal{F}$. Applying H_V to the exact sequence $(*)$, we obtain the exact sequence $0 \to H_V(X_0) \to H_V(X_1) \to H'_V(M) \to M \to 0$ and, applying T_V to this, we obtain the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\eta_M"] & X_0 \arrow[r] & X_1 \arrow[r] & 0 \\
0 \arrow[r] & T'_V H'_V(M) \arrow[r] & T_V H_V(X_0) \arrow[r] \arrow[u, "\rho_{X_0}", "\cong"'] & T_V H_V(X_1) \arrow[r] \arrow[u, "\rho_{X_1}", "\cong"'] & 0
\end{tikzcd}$$

where η_M is the unique isomorphism making the diagram commutative. Similarly, given any $N \in \mathcal{X}$ and an exact sequence of the form $(**)$, we define $\theta_N : H'_V T'_V(N) \to N$ as the unique isomorphism making the diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\sigma_K", "\cong"'] & R^{(\alpha)} \arrow[r] \arrow[d, "\sigma_{R^{(\alpha)}}", "\cong"'] & N \arrow[r] & 0 \\
0 \arrow[r] & H_V T_V(K) \arrow[r] & H_V T_V(R^{(\alpha)}) \arrow[r] & H'_V T'_V(N) \arrow[r] \arrow[u, "\theta_N"] & 0
\end{tikzcd}$$

commutative. It can be shown that θ_N does not depend on the choice of $(**)$, and that $(\eta_M)_{M \in \mathcal{F}}$ and $(\theta_N)_{N \in \mathcal{X}}$ are natural maps. This proves that $\mathcal{F} \underset{T'_V}{\overset{H'_V}{\rightleftarrows}} \mathcal{X}$ is an equivalence.

To complete the proof of d), we first recall that Lemma 3.1.2 e) states that for any $N \in$ Mod-R the canonical inclusion $\mathrm{Ann}_V(N) \hookrightarrow N$ induces a natural isomorphism $T'_V(\mathrm{Ann}_V(N)) \cong T'_V(N)$. Then, since from Proposition 3.1.2 c) we have projdim$V \leq 1$, we can similarly prove that for any $M \in \mathcal{A}$ the canonical projection $M \twoheadrightarrow M/Tr_V(M)$ induces a natural isomorphism $H'_V(M) \cong H'_V(M/Tr_V(M))$. Because of this, we can extend the definitions of η and θ to a pair of natural morphisms defined in $\mathcal{A}$ and Mod-R respectively, making the diagrams

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\eta_M"] & M/Tr_V(M) \arrow[r] \arrow[d, "\eta_{M/Tr_V(M)}", "\cong"'] & 0 \\
T'_V H'_V(M) \arrow[r, "\cong"] & T'_V H'_V(M/Tr_V(M)) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & Ann_V(N) \arrow[r] & N \\
 & H'_V T'_V(Ann_V(N)) \arrow[r, "\cong"] \arrow[u, "\theta_{Ann_V(N)}", "\cong"'] & H'_V T'_V(N) \arrow[u, "\theta_N"]
\end{tikzcd}$$

commutative for any $M \in \mathcal{A}$ and $N \in$ Mod-R. Thus we see that η_M is epic, $\mathrm{Ker}(\eta_M) = \mathrm{Tr}_V(M)$, θ_N is monic and $\mathrm{Im}(\theta_N) = \mathrm{Ann}_V(N)$. Applying Proposition 3.1.1 c), we complete the proof of d). □

Of course, one can elaborate a tilting theory in the case when the category $\mathcal{A}$ is a Grothendieck category and obtain a Tilting Theorem as the one above (see [2]). If this is the case, we get the following

Proposition 3.1.3. *Let $\mathcal{A}$ be a Grothendieck category and $V \in \mathcal{A}$ a tilting object. Then the functor $Hom_{\mathcal{A}}(V, -)$ preserves direct limits in $\mathcal{A}$.*

Proof. Let $(M_\lambda, f_{\lambda\mu})$ be a direct system in $\mathcal{A}$. As the functor direct limit is exact in $\mathcal{A}$, from the exact sequences

$$0 \to Tr_V(M_\lambda) \to M_\lambda \to M_\lambda / Tr_V(M_\lambda) \to 0$$

we get the exact sequence

$$0 \to \varinjlim Tr_V(M_\lambda) \to \varinjlim M_\lambda \to \varinjlim M_\lambda / Tr_V(M_\lambda) \to 0.$$

As $\varinjlim Tr_V(M_\lambda) \in \mathrm{Gen}V = V^\perp$, we get the exact sequence

$$(1): \ 0 \to H_V(\varinjlim Tr_V(M_\lambda)) \to H_V(\varinjlim M_\lambda) \to H_V(\varinjlim M_\lambda / Tr_V(M_\lambda)) \to 0.$$

From Theorem 3.1.3 d) we have $M_\lambda / Tr_V(M_\lambda) \cong T'_V H'_V(M_\lambda)$, therefore we obtain

$$H_V(\varinjlim M_\lambda / Tr_V(M_\lambda)) \cong H_V(\varinjlim T'_V H'_V(M_\lambda)) \cong H_V T'_V(\varinjlim H'_V(M_\lambda)) = 0$$

because $H_V T'_V = 0$ by Theorem 3.1.3 c). Combining this with (1) we get

$$(2): \ H_V(\varinjlim M_\lambda) \cong H_V(\varinjlim Tr_V(M_\lambda))$$

canonically. Now, from Theorem 3.1.3 d), we have $\mathrm{Tr}_V(M_\lambda) \cong T_V H_V(M_\lambda)$ and so we obtain canonical isomorphisms

$$(3): \ H_V(\varinjlim Tr_V(M_\lambda)) \cong H_V(\varinjlim T_V H_V(M_\lambda)) \cong$$
$$\cong H_V T_V(\varinjlim H_V(M_\lambda)) \cong \varinjlim H_V(M_\lambda),$$

where the last isomorphism follows from Theorem 3.1.3 b) and from the fact that $\varinjlim H_V(M_\lambda) \in \mathrm{Ker}\ T'_V$. Combining (2) with (3), we get the result. □

Corollary 3.1.1. *Let $\mathcal{A}$ be a Grothendieck locally finitely generated category. Then any tilting object of $\mathcal{A}$ is finitely presented.*

Proof. This is an immediate consequence of last Proposition and Proposition 2.1.4. □

3.2 Tilting objects in functor category

In this section we will apply the notions introduced above to the functor category studied in Chapter 2. In particular, we will show what properties hold for a tilting functor when localized to the category R-Mod.

Let $V \in \mathcal{G} = ((R\text{-mod})^{op}, Ab))$ be a tilting functor and let $\mathcal{S} = \operatorname{Ker}(l)$ be the kernel of the localization functor. Then we have the following:

Lemma 3.2.1. *If we assume that $GenV \supseteq \mathcal{S}$ then the following properties hold:*

(i) The class $V(R)_\perp$ is torsion free;

(ii) An R-module $M \in Gen(V(R)) \Leftrightarrow iM \in GenV$;

(iii) V(R) generates the injective modules;

(iv) $il(GenV) \subseteq GenV$;

(v) $Pres(V(R)) = Gen(V(R))$;

(vi) V(R) is finitely presented;

(vii) ilV is a direct summand of V^n for some $n \in \mathbb{N}$;

(viii) $l(GenV) = Gen(V(R))$ and, assuming $il(V_\perp) \subseteq V_\perp$, $l(V_\perp) = V(R)_\perp$. In particular, $(Gen(V(R)), V(R)_\perp)$ is a torsion pair.

Proof. (i) This is actually always true. The class $V(R)_\perp$ is clearly closed under subobjects and products, so we only need to prove closure under extensions: let $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be a s.e.s. in R-Mod with $M, L \in V(R)_\perp$ and let $\alpha : V(R) \to N$ be a morphism. Since $g \circ \alpha = 0$, as $L \in V(R)_\perp$, there exists a unique $\beta : V(R) \to M$ such that $f \circ \beta = \alpha$. But it must be $\beta = 0$, as $M \in V(R)_\perp$, and so $\alpha = 0$.

(ii) One implication is obvious: indeed, if $iM \in \text{Gen}V$ then $M \in \text{Gen}(V(R))$ since the functor l is exact and colimit-preserving.
Conversely, let $M \in \text{Gen}(V(R))$. This means that $\exists V(R)^{(\alpha)} \twoheadrightarrow M$ and, without loss of generality, we can assume that $\alpha = \operatorname{Hom}_R(V(R), M)$, so that the

epimorphism above is given by the diagram $(*)$

$$\begin{tikzcd}
V(R)^{(\alpha)} \arrow[r, two heads, "\triangledown\sigma"] & M \\
& V(R) \arrow[u, "\sigma"] \arrow[ul, hook, "\epsilon_\sigma"]
\end{tikzcd}$$

By the adjunction $\mathrm{Hom}_R(V(R), M) \cong \mathrm{Hom}_{\mathcal{G}}(V, iM)$, we get a similar diagram $(**)$ in $\mathcal{G}$,

$$\begin{tikzcd}
V^{(\alpha)} \arrow[r, "\triangledown\hat{\sigma}"] & iM \\
& V \arrow[u, "\hat{\sigma}"] \arrow[ul, hook, "\epsilon_{\hat{\sigma}}"]
\end{tikzcd}$$

Let's consider $c = \mathrm{coker}(\triangledown\hat{\sigma})$ and the diagram

$$\begin{tikzcd}
V^{(\alpha)} \arrow[rr, "\triangledown\hat{\sigma}"] \arrow[dr, two heads] & & iM \arrow[r, "c"] & C \arrow[r] & 0 \\
& K \arrow[ur, hook] & & &
\end{tikzcd}$$

Since applying l to diagram $(**)$ gives diagram $(*)$, it must be $C(R) = 0$, so $C \in \mathcal{S} \subseteq \mathrm{Gen}V$; also, $K \in \mathrm{Gen}V$ by construction. Since $\mathrm{Gen}V$ is closed under extensions, it must be $iM \in \mathrm{Gen}V$.

(iii) Given an injective module E, by (ii) we have $E \in \mathrm{Gen}(V(R)) \Leftrightarrow iE \in \mathrm{Gen}V$, and this is true because iE is still injective in $\mathcal{G}$ by Proposition 1.2.4 and $\mathrm{Gen}V = V^{\perp}$ contains all the injectives.

(iv) Given $G \in \mathrm{Gen}V$ then, by property (ii), $ilG \in \mathrm{Gen}V \Leftrightarrow lG \in \mathrm{Gen}(V(R))$, which is obviously true.

(v) The inclusion $(\subseteq)$ is obvious; on the other hand, given $M \in \mathrm{Gen}(V(R))$, then $iM \in \mathrm{Gen}V$ by point (ii). Since $\mathrm{Gen}V = \mathrm{Pres}V$, there is a presentation

$$0 \to K \to V^{(\alpha)} \to iM \to 0$$

with $K \in \mathrm{Gen}V$. Then, applying l, we get a presentation

$$0 \to K(R) \to V(R)^{(\alpha)} \to M \to 0$$

with $K(R) \in \mathrm{Gen}(V(R))$.

(vi) We want to prove that $\mathrm{Hom}_R(lV, -)$ commutes with arbitrary direct limits: given any directed set of R-modules $\{M_j \mid j \in J\}$,

$$\mathrm{Hom}_R(lV, \varinjlim M_j) \underset{(1)}{\cong} \mathrm{Hom}_{\mathcal{G}}(V, i(\varinjlim M_j)) = \mathrm{Hom}_{\mathcal{G}}(V, (-, \varinjlim M_j)) \underset{(2)}{\cong}$$

$$\cong \mathrm{Hom}_{\mathcal{G}}(V, \varinjlim(-, M_j)) \underset{(3)}{\cong} \varinjlim \mathrm{Hom}_{\mathcal{G}}(V, i(M_j)) \underset{(4)}{\cong} \varinjlim \mathrm{Hom}_R(lV, M_j)$$

where equalities (1) and (4) are given by the adjunction, equality (2) holds since the functor $(-, \varinjlim M_j)$ is evaluated on finitely presented modules, and equality (3) holds since the functor $\mathrm{Hom}_{\mathcal{G}}(V, -)$ preserves direct limits (by Corollary 3.1.1).

(*vii*) By the previous point, ilV is a finitely generated projective functor (as seen in Chapter 2, Section 2), hence it is finitely presented. Also, by point (*iv*), $ilV \in \mathrm{Gen}V$. This means that there exists a splitting short exact sequence $0 \to K \to V^{(\alpha)} \to ilV \to 0$. In particular, there exists a splitting monomorphism $f : ilV \hookrightarrow V^{(\alpha)}$. Now, since ilV is finitely presented, $(ilV, V^{(\alpha)}) \cong (ilV, V)^{(\alpha)}$; let $\bar{f}$ be the image of f under this isomorphism. Then $\bar{f} = (f_j)_j$ with $f_j : ilV \to V$ and $f_j \neq 0$ only for a finite number of $j \in \alpha$. Let n be the number of non-zero components of $\bar{f}$: then we get the commutative diagram

$$\begin{tikzcd}
ilV \arrow[rr, hook, "f"] \arrow[dr, "f^*"'] & & V^{(\alpha)} \\
& V^n \arrow[ur, hook, "\epsilon"'] &
\end{tikzcd}$$

f^* being the diagonal morphism of $(f_j)_{j=1,\dots,n}$.
Since f is a splitting monomorphism, so is f^*, so that ilV is a direct summand of V^n.

(*viii*) The first equality follows immediately from property (*ii*). Let $F \in V(R)_\perp$: since $\mathrm{Hom}_R(V(R), F) \cong \mathrm{Hom}_{\mathcal{G}}(V, iF)$, we get that $iF \in V_\perp$, and so $F = liF \in l(V_\perp)$. Conversely, let $F \in V_\perp$: by the hypothesis, $ilF \in V_\perp$, so that $0 = \mathrm{Hom}_{\mathcal{G}}(V, ilF) \cong \mathrm{Hom}_R(V(R), F(R))$. This means that $F(R) \in V(R)_\perp$. By Proposition 1.4.3, $(l(\mathrm{Gen}V), l(V_\perp)) = (\mathrm{Gen}(V(R)), V(R)_\perp)$ is a torsion pair in R-Mod. □

The previous Lemma shows that the module $V(R)$ shares some properties with a tilting module. Actually, the following result holds:

Theorem 3.2.1. *Assume that* $\mathcal{G} \supseteq \mathcal{S}$ *and* $il(V_\perp) \subseteq (V_\perp)$. *Then the following are equivalent:*

a) V(R) is a tilting module

- *b) Every short exact sequence with terms in Gen(V(R)) is pure*
- *c) Every sequence* $0 \to M \to E(M) \to E(M)/M \to 0$ *, with* $M \in$ *Gen(V(R)) and E(M) its injective envelope, is pure. In other words, every module in Gen(V(R)) is absolutely pure.*

Proof. a) ⇒ b): Let $0 \to M \to N \to L \to 0$ be a s.e.s. in R-Mod with $M, N, L \in \mathrm{Gen}(V(R))$. This gives the following diagram in $\mathcal{G}$

$$\begin{tikzcd}
 & & & & & & 0 \\
 & & & & & iL/C \arrow[ur] & \\
0 \arrow[r] & iM \arrow[r] & iN \arrow[rr, "f"] \arrow[dr, two heads] & & iL \arrow[ur, two heads] & & \\
 & & & C \arrow[ur, hook] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Now we apply the functor $\mathrm{Hom}_{\mathcal{G}}(V, -)$ to this diagram:

$$\begin{tikzcd}
 & & & & & & 0 \\
 & & & & & (V, iL/C) \arrow[ur] & \\
0 \arrow[r] & (V, iM) \arrow[r] & (V, iN) \arrow[rr, "\hat{f}"] \arrow[dr, two heads] & & (V, iL) \arrow[ur, two heads] & & \\
 & & & (V, C) \arrow[ur, hook] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where we have exactness on the lower line and the diagonal one since iM and C are in $\mathrm{Gen}V = V^{\perp}$. But, looking at the horizontal line, we notice that

$(V, iM) \cong (V(R), M)$ (and similarly for N and L), so this sequence can actually be obtained applying the functor $\mathrm{Hom}_R(V(R), -)$ to our initial short exact sequence; this means that $\hat{f}$ is actually an epimorphism, since $M \in \mathrm{Gen}(V(R))$ and $V(R)$ is a tilting module. Then it must be $(V, iL) \cong (V, C)$, i.e. $(V, iL/C) = 0$. But then the functor iL/C is both in $\mathrm{Gen}V$ (since iL is) and in $V_\perp$, so it must be $iL/C = 0$, i.e. $C = iL$, which means that the sequence $0 \to iM \to iN \to iL \to 0$ is exact.

b) $\Rightarrow$ c): This is clear since $iE(M) \in \mathrm{Gen}V = V^\perp$ (it is still injective), so $E(M)$ (and then $E(M)/M$) are in $\mathrm{Gen}(V(R))$.

c) $\Rightarrow$ a) : We want to prove that $V(R)$ is a tilting module according to the definition given in Section 1. We already know by point (*vi*) of Lemma 3.2.1 that $V(R)$ is (self)small; moreover, once we prove $\mathrm{Gen}(V(R)) = V(R)^\perp$, the condition $\overline{Gen(lV)} = R$-Mod will immediately follow, since the category R-Mod has enough injectives (by a previous Observation). Thus we only need to prove $\mathrm{Gen}(V(R)) = V(R)^\perp$.

$\mathrm{Gen}(V(R)) \subseteq V(R)^\perp$:
Let $M \in \mathrm{Gen}(V(R))$ and consider its injective envelope $E = E(M)$

$$0 \to M \overset{\alpha}{\to} E \overset{\beta}{\to} E/M \to 0.$$

Then we get the exact sequence

$$0 \to (lV, M) \overset{\bar{\alpha}}{\to} (lV, E) \overset{\bar{\beta}}{\to} (lV, E/M) \to \mathrm{Ext}^1(lV, M) \to 0$$

which shows that $\mathrm{Ext}^1(lV, M) = {}^{(lV, E/M)}\!\diagup_{\bar{\beta}(lV, E)}$.
We also have the commutative square given by the adjunction

$$\begin{tikzcd}
(lV, E) \arrow[r, "\bar{\beta}"] \arrow[d, equal, "\cong"'] & (lV, E/M) \arrow[d, equal, "\cong"] \\
(V, iE) \arrow[r, "i\bar{\beta}"] & (V, i(E/M))
\end{tikzcd}$$

so that $\mathrm{Ext}^1(lV, M) = {}^{(lV, E/M)}\!\diagup_{\bar{\beta}(lV, E)} \cong {}^{(V, i(E/M))}\!\diagup_{i\bar{\beta}(V, iE)}$.
By the hypothesis, the sequence $0 \to iM \overset{i\alpha}{\to} iE \overset{\imath\beta}{\to} i(E/M) \to 0$ is exact, so

that, since $iM \in \mathrm{Gen}V = V^{\perp}$, the sequence

$$0 \longrightarrow (V, iM) \xrightarrow{\bar{i\alpha}} (V, iE) \xrightarrow{\bar{i\beta}} (V, i(E/M)) \longrightarrow 0$$

is also exact, which means that $0 = {}^{(V, i(E/M))}\!\big/_{\bar{i\beta}(V, iE)} \cong \mathrm{Ext}^1(lV, M)$. Then $M \in V(R)^{\perp}$.

$V(R)^{\perp} \subseteq \mathrm{Gen}(V(R))$:
First we prove that projdim$(V(R)) \leq 1$: let $M \in R$-Mod and consider its injective envelope $0 \to M \to E \to E/M \to 0$. We get the long exact sequence

$$0 \longrightarrow (lV, M) \longrightarrow (lV, E) \longrightarrow (lV, E/M) \longrightarrow Ext^1(lV, M) \ \to$$

$$\to \ Ext^1(lV, E) \longrightarrow Ext^1(lV, E/M) \longrightarrow Ext^2(lV, M) \longrightarrow Ext^2(lV, E)$$

By property (ii) of Lemma 3.2.1, since $iE \in \mathrm{Gen}V$ (it is injective), E (and so E/M) must be in $\mathrm{Gen}(V(R)) \subseteq V(R)^{\perp}$. This means that $\mathrm{Ext}^1(lV, E/M) = 0 = \mathrm{Ext}^2(lV, E)$ and so $\mathrm{Ext}^2(lV, M) = 0$.

Now let $M \in V(R)^{\perp}$ and consider the torsion and torsion-free part of the functor iM given by the torsion theory $(\mathrm{Gen}V,\ V_{\perp})$ in $\mathcal{G}$,

$$0 \to T \to iM \to F \to 0$$

Applying l we get $0 \to lT \to M \to lF \to 0$ where $lT \in \mathrm{Gen}(V(R))$ and $lF \in V(R)_{\perp}$ (by property $(viii)$ of Lemma 3.2.1). From this we get the sequence

$$0 \longrightarrow (lV, lT) \longrightarrow (lV, M) \longrightarrow (lV, lF) \longrightarrow Ext^1(lV, lT) \ \longrightarrow$$

$$\longrightarrow \ Ext^1(lV, M) \longrightarrow Ext^1(lV, lF) \longrightarrow Ext^2(lV, lT) \longrightarrow ...$$

where $\mathrm{Ext}^1(lV, M) = 0$ by the hypothesis and $\mathrm{Ext}^2(lV, lT) = 0$ because projdim$(lV) \leq 1$. Then $\mathrm{Ext}^1(lV, lF) = 0$, i.e. $lF \in V(R)^{\perp}$.
So we have an R-module $F(R)$ that belongs both to $V(R)^{\perp}$ and $V(R)_{\perp}$. We

want to prove that $F(R) = 0$ so that $M \cong T(R) \in \mathrm{Gen}V(R)$.
The functor iR is in $\mathcal{G} = \overline{GenV}$, so that we can find a short exact sequence $0 \to iR \to G \to C \to 0$ where G (so also C) is in $\mathrm{Gen}V$. This gives the short exact sequence in R-Mod: $0 \to R \to lG \to lC \to 0$, where lG, lC are in $\mathrm{Gen}(lV)$. Applying the functor $\mathrm{Hom}_R(-, lF)$ yields the long exact sequence

$$0 \longrightarrow (lC, lF) \longrightarrow (lG, lF) \longrightarrow (R, lF) \longrightarrow$$

$$\longrightarrow Ext^1(lC, lF) \longrightarrow Ext^1(lG, lF) \longrightarrow 0$$

where $(lG, lF) = 0$ because $lG \in \mathrm{Gen}(lV)$ and $lF \in (lV)_\perp$.
Now we want to prove that $\mathrm{Ext}^1(lC, lF) = 0$:
since $C \in \mathrm{Gen}V = \mathrm{Pres}V$, we can find a short exact sequence in $\mathcal{G}$

$$0 \to K \to V^{(\beta)} \to C \to 0$$

with K in $\mathrm{Gen}V$. Applying the functor l and then $\mathrm{Hom}_R(-, lF)$ to this sequence, we get

$$0 \longrightarrow (lC, lF) \longrightarrow (lV^{(\beta)}, lF) \longrightarrow (lK, lF) \longrightarrow Ext^1(lC, lF) \longrightarrow$$

$$\longrightarrow Ext^1(lV^{(\beta)}, lF) \longrightarrow Ext^1(lK, lF) \longrightarrow ...$$

where $(lK, lF) = 0$ because $lK \in \mathrm{Gen}(lV)$ and $lF \in (lV)_\perp$, and also $\mathrm{Ext}^1(lV^{(\beta)}, lF) \cong \mathrm{Ext}^1(lV, lF)^\beta = 0$.
Then $\mathrm{Ext}^1(lC, lF) = 0$, which implies that $0 = (R, lF) \cong lF$, as we wanted.
□

It is important to point out that the majority of the results obtained in this section do not actually require V to be a small object in $\mathcal{G}$. In the case when V is small, and thus finitely presented, something more can be said.

3.3 The finitely presented case

As remarked at the end of the last section, it isn't crucial to require V to be small in order to prove the main result. We will now assume V to be finitely presented. Keeping in mind Proposition 3.1.2, we know that V is a tilting object in $\mathcal{G}$ if and only if the following conditions are satisfied:

a) V is finitely presented and $\operatorname{projdim} V \leq 1$;

b) $\operatorname{Ext}^1(V, V^{(X)}) = 0$ for every set X;

c) If $\operatorname{Hom}(V, G) = 0 = \operatorname{Ext}^1(V, G)$ for $G \in \mathcal{G}$, then $G = 0$.

First of all, since we are assuming that V is finitely presented and that $\operatorname{projdim} V \leq 1$, we get, keeping in mind that the finitely generated projective objects in $\mathcal{G}$ are precisely the representable objects, that there must be an exact sequence in $\mathcal{G}$

$$(*) : 0 \to (-, F_1) \overset{i\alpha}{\to} (-, F_0) \to V \to 0$$

where F_0, F_1 are finitely presented modules and

$$0 \to F_1 \overset{\alpha}{\to} F_0 \overset{c}{\to} V(R) \to 0$$

is exact in R-mod. For $G \in \mathcal{G}$, applying $(-, G)$ to the sequence $(*)$, we get by the Yoneda Lemma

$$\begin{tikzcd}
0 \arrow[r] & (V,G) \arrow[r] & ((-,F_0),G) \arrow[r, "{(i\alpha,G)}"] \arrow[d, "\cong"'] & ((-,F_1),G) \arrow[r] \arrow[d, "\cong"] & Ext^1(V,G) \arrow[r] & 0 \\
& & G(F_0) \arrow[r, "G(\alpha)"] & G(F_1) & &
\end{tikzcd}$$

This diagram shows that condition c) is equivalent to the following:
if $0 \neq G \in \mathcal{G}$, then $G(\alpha)$ is not an isomorphism.

The sequence $(*)$ can also show an equivalent condition for condition b): indeed, if we apply the functor $(-, V^{(X)})$ to $(*)$, we get

$$\begin{tikzcd}
0 \arrow[r] & (V,V^{(X)}) \arrow[r] \arrow[d, "\cong"] & ((-,F_0),V^{(X)}) \arrow[r, "{(i\alpha,V^{(X)})}"] \arrow[d, "\cong"'] & ((-,F_1),V^{(X)}) \arrow[r] \arrow[d, "\cong"] & Ext^1(V,V^{(X)}) \arrow[r] & 0 \\
& (V,V)^{(X)} & V^{(X)}(F_0) \arrow[r, "V^{(X)}(\alpha)"] & V^{(X)}(F_1) & &
\end{tikzcd}$$

which shows that condition b) is equivalent to requiring that $V^{(X)}(\alpha)$ is an epimorphism for every set X.

Finally, we have the following commutative diagram with exact columns in *Ab* given by the functor V and the representable functors $(-, F_0)$, $(-, F_1)$:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & V(V(R)) \arrow[u] \arrow[r, "V(c)"] & V(F_0) \arrow[u] \arrow[r, "V(\alpha)"] & V(F_1) \arrow[u] & \\
0 \arrow[r] & (V(R), F_0) \arrow[u] \arrow[r] & (F_0, F_0) \arrow[u] \arrow[r, "(\alpha, F_0)"] & (F_1, F_0) \arrow[u, "f"'] \arrow[r] & Ext^1(V(R), F_0) \\
0 \arrow[r] & (V(R), F_1) \arrow[u] \arrow[r] & (F_0, F_1) \arrow[u] \arrow[r] & (F_1, F_1) \arrow[u] \arrow[r] & Ext^1(V(R), F_1) \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

which shows that $V(\alpha)$ is an epimorphism if and only if $f \circ (\alpha, F_0)$ is an epimorphism; thus a sufficient condition for $V(\alpha)$ to be an epimorphism is that for every $\sigma : F_1 \to F_0$ there exists $\varphi : F_0 \to F_0$ such that the diagram

$$\begin{tikzcd}
F_1 \arrow[r, hook, "\alpha"] \arrow[d, "\sigma"] & F_0 \arrow[dl, dashed, "\phi"] \\
F_0 &
\end{tikzcd}$$

is commutative.

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