



UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Corso di Laurea Magistrale in Fisica

Tesi di Laurea

Non-linear sigma models on K3 and Umbral Moonshine

Relatore

Prof. Roberto Volpato

Laureando

Gabriele Sgroi

Anno Accademico 2017/2018

Contents

1	Introduction	1
2	Conformal Field Theory	5
2.1	Aspects of conformal symmetry in 2D	5
2.2	Space of states and correlation functions	9
2.3	Ordering prescriptions	11
2.4	The conformal structure	12
2.4.1	State-field correspondence	14
2.5	Simple examples	15
2.5.1	The free boson	15
2.5.2	The free fermion	17
2.5.3	Kac-Moody algebras	19
2.6	Conformal field theory with toroidal worldsheets	20
2.7	Conformal field theory on orbifolds	21
3	Superconformal field theory	23
3.1	Superconformal algebras	23
3.1.1	$N = 1$	23
3.1.2	$N = 2$	24
3.1.3	$N = 4$	24
3.2	The space of states	24
3.2.1	Spectral flow	26
3.2.2	The elliptic genus	27
4	Non-linear sigma models and string theory	29
4.1	Bosonic strings	29
4.1.1	Bosonic closed string spectrum	30
4.2	Superstrings	32
4.3	Supersymmetric NLSM	33
4.4	Moduli space of NLSM on K3	34
5	Moonshine	37
5.1	Monstrous Moonshine	37
5.2	Moonshine in the elliptic genus of K3	39
6	Umbral twining genera decomposition	45
6.1	Decomposition with $N=4$ characters	45
6.1.1	Models with $L_2(11)$ symmetry and algebra extension	47
6.2	$\mathbb{C}^2/\mathbb{Z}_3$ characters	48
6.2.1	Bosonic characters	50

6.2.2	Fermionic characters	52
6.2.3	Twisted characters	54
6.3	Decomposition with $\mathbb{C}^2/\mathbb{Z}_3$ characters	56
7	Conclusions	59
	Appendices	61
A	Definitions	63
B	Basics of complex geometry	65
C	Roots and Lattices	69
D	Coefficients of $N = 4$ algebra decomposition	71
E	Coefficients of $\mathbb{C}^2/\mathbb{Z}_3$ characters decomposition	75

1. Introduction

As it is well known, the two most successful theories that describe our universe so far, Quantum Field Theory and General Relativity, fail when one tries to combine them together. Despite the efforts of the last half-century we are still far away from an experiment-confirmed theory of quantum gravity. Lacking of the beacon of experimental data, for the moment, the only possibility to advance in the field is to rely on theoretical considerations. This is sometimes misunderstood as a philosophical approach or just a search for beauty in mathematical equations. Despite the motives that lead someone into the study of the subject can be vary, the theoretical approach has underlying “practical” principles at its core. For example, one of the preferred feature of a theory is the dependence on as few free parameters as possible. While there is no strict necessity for a theory to be true to depend on few free parameters, this greatly reduce the need for experimental inputs that, as we pointed out before, at the moment are lacking. Another common misconception is the idea of “beauty” of a theory. We do not want to attempt to give here a precise meaning to it, we will just point out that, generally, beautiful theories are the ones which possess suitable features, such as a high degree of symmetry, that make them treatable through the exploit of simple, but ingenious, principles. Again a theory could be true without possessing these features, but it is tautological that we cannot understand a theory that resists any attempt of being understood. Furthermore, theories with an underlying rich mathematical structure are generally more constrained and thus could give more possibilities to test their validity. We will not go any further into this apology of the formal approach to theoretical physics, we just hope the reader not acquainted with this approach will not be too estranged while reading this work.

String theory (for an introduction see [1] and [2]) is one of the major candidates for a quantum theory of gravity. It has revealed itself to be a consistent theory of quantum gravity and to possess a very rich and complex underlying mathematical structure with the need of few free parameters and some strong constraints on the structure of the theory itself. One of the current major issues of string theory is that it predicts a huge number of possible vacua corresponding to as many physical settings. Recently, since we were not able to explicitly reproduce some particular backgrounds so far, some claims have been made (see [3] and [4]) that only few of the vacua predicted by string theory, or even none of them, is able to reproduce our universe. Apart from that, consistent string theories require supersymmetry to get rid of the presence of tachyons, that would lead to an unstable theory, in the spectrum. Furthermore one of the major constraint in the structure of the theory is that the theory itself predicts the number of dimensions of the space-time in order to be consistent. This results into the prediction of extra dimensions apart from the usual 4 that are observed. Extra dimensions can be consistent with experiments only if they are very small. Thus string theory need a compactification procedure on the target space and a way to obtain it is through a non-linear sigma model, which consist of a 2-dimensional conformal field theory whose scalar fields are the components of coordinates of a compact manifold. Since generally non-linear sigma models which compactify all the six (for superstring theories) extra dimensions are too complicated, a lot of effort has been put to study first the properties of the 4 dimensional case. In particular the most studied models are

compactifications on particular manifolds called Calabi-Yau manifolds. In 4 dimensions there are only two kind of such manifolds: the tori and $K3$ surfaces. Toroidal compactifications are simpler to study but, sometimes due to the triviality of the resulting theory, they are not satisfying. Non-linear sigma models on $K3$ surfaces instead, while still treatable, possess a lot of non-trivial properties which make them more interesting to be investigated. The study of these models has revealed some unexpected connections between two seemingly unrelated areas of Mathematics: modular forms and finite group representations. These kind of surprising connections, which come in different kinds, are generally known as moonshine phenomena (see [5], [6]). The most incredible aspect of these phenomena is that the link between the areas they connect seems to be related to some particular physical settings. It is known, for example, that monstrous moonshine can be understood building an appropriate 2-dimensional conformal field theory (see [7] and [8]). Moreover, the discovery of some of these mathematical relations while studying a Physics setting has once again strengthened the relation between Mathematics and Physics. However, despite the recent efforts and abstract proofs, a clear physical picture giving a natural explanation to the emergence of these moonshine phenomena is still lacking. Analogously, it is still not clear what could be the physical implications of these relations in the study of non-linear sigma model on $K3$ surfaces. The goal of this thesis is to try to give new possible interpretations of these phenomena and to try to exploit them to better understand some non-linear sigma models on $K3$ surfaces. In particular we are interested in the models who admit a subgroup one of the groups predicted by the so-called umbral moonshine conjecture as a symmetry group. These models are not well understood, as most non-trivial non-linear sigma models on $K3$ surfaces, due to the scarcity of information one can extract on them. In the original part of this master thesis work we will find evidence that the symmetry algebra of these models can be extended beyond the standard superconformal algebra coming from the geometric structure of $K3$. We will also investigate a simple model which possess an algebra that constitutes a good candidate for the extension of the aforementioned models.

The work is organized as follows:

In section 2 we present an introduction to conformal field theories in 2 dimensions. After illustrating the main consequences of conformal invariance, we discuss the structure of the space of states, the highest weight representations of the Virasoro algebra and state-field correspondence. We then work with some simple examples of conformal field theories, namely the free boson and fermion and we give some details of theories whose currents generate a Kac-Moody algebra. In the last two subsections we discuss conformal field theory with a torus as world-sheet and conformal field theories with an orbifold target space. The last subsection will be particularly important for the comprehension of section 6.2.

In section 3 we discuss superconformal field theories. We discuss in particular the case of $N = 1, 2, 4$ supersymmetry and we will introduce some important tools that will be particularly important in the following such as the elliptic genus and spectral flow. We will also present briefly the characters of the $N = 4$ superconformal algebra since they will be important in section 5.

In section 4 we briefly introduce a particular kind of two dimensional conformal field theory, which is especially important for string theory known as non-linear sigma models. This will link the concepts explained in the previous sections to string theory. We will focus on non-linear sigma models on a particular kind of manifolds, namely Calabi-Yau manifolds and in particular on those with $K3$ target spaces. The conformal field theory underlying the non-linear sigma models on these spaces possesses supersymmetry which makes them easier to study, especially in the case of $K3$ surfaces where the supersymmetry group is bigger. In the last subsection, we will also briefly discuss the moduli space of non-linear sigma models on $K3$ surfaces and their symmetries since these will play a role in the discussion of the moonshine phenomena.

In section 5 we illustrate the main kind of moonshine of interest for Physics. We will first

discuss monstrous moonshine which is the first (historically) and most studied example of this class of phenomena. We will then illustrate the moonshine phenomena related to the elliptic genus of $K3$ surfaces. In particular we will discuss the discovery of the Mathieu moonshine and then we will embed it in the wider family of umbral moonshine, which actually constitutes of 23 different, but related, moonshines. We will introduce the main mathematical tools used as soon as we will need them, but we will rely on the basic notions of roots and lattices that can be found in appendix C.

In section 6 we expose the original part of the work of this master thesis. The first part of the work consist in the decomposition of the elliptic genus of $K3$ surfaces in terms of irreducible representations of some umbral groups and of the characters of the $N = 4$ superconformal algebra. We collect the result in appendix D. We will subsequently discuss how this decomposition could influence some models in the moduli space of $K3$ non-linear sigma models which admit one of the groups considered as symmetry group. After that, to continue the original part of the work, we compute the characters of the free $\mathbb{C}^2/\mathbb{Z}_3$ orbifold model and we attempt a decomposition of the elliptic genus of $K3$ in terms of these characters and of irreducible representations of the same umbral groups considered previously in order to try to find a possible interpretation of the results of section 6.1.

In the appendices we give the definition of the main mathematical object used throughout the work and we present simple introductory discussions to the basic concepts of complex geometry and lattices. In the last two appendices, as we already said, we collect the coefficients of the decomposition of the elliptic genus of $K3$ surfaces in terms of the irreducible representations of some umbral groups.

2. Conformal Field Theory

In this section we give an introduction to conformal field theories. We will first discuss in an informal way the main consequences of conformal invariance in 2 dimensions and then give a systematic description of conformal field theories. For the latter part we will follow the approach of [9] and [10]. We will define a conformal field theory by giving its space of state and the collection of its correlation functions defined as maps from the state space to the space of suitable functions of the coordinates. It is important to stress that correlation functions will be the fundamental object of our theory: we will assign to each state a field such that the vacuum expectation value of the fields will exactly give the same result as the one given by the correlation functions acting on the corresponding states. This is the opposite way of what one generally does in practice: the usual approach is to obtain the correlation functions from the fields by some method, since they are generally not a priori known. The approach we will follow can then be used as a posteriori justification of the usual results, one can in fact obtain the correlation functions from methods which are not mathematically well-defined (for example the path integral formulation) and then, after having checked that they satisfy some suitable properties, use them to build a well-defined conformal field theory.

After giving the main ingredients we will discuss two simple but important examples, which will be also used in the following sections, the free boson and the free fermion. In the last subsection we will briefly discuss conformal field theories defined on the torus which are important in 1-loop string computations.

2.1 Aspects of conformal symmetry in 2D

We will introduce here the basic concepts and implications of conformal invariance in two dimensions. We will also discuss its main implications both for the classical and the quantum theory. This section is meant to be expository and we will not go through all the calculations and details, we encourage the reader to refer to [11] or [12] or to wait for the following sections for more informations. The reader who is acquainted with the basics of conformal field theory can skip this section.

Let us consider a 2-dimensional space-time Σ with metric tensor $g_{\mu\nu}$. A conformal transformation of the coordinates $z := (z^0, z^1)$ is a diffeomorphism $z \rightarrow z' = w(z)$ such that the metric transforms as

$$g'_{\mu\nu}(z') = \Lambda(z)g_{\mu\nu}(z) . \quad (2.1)$$

Under a generic diffeomorphism the contravariant metric tensor transforms as

$$g'^{\mu\nu}(z') = \frac{\partial w^\mu}{\partial z^\alpha} \frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta}(z) , \quad (2.2)$$

so (2.1) requires

$$\begin{aligned} \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 &= \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \\ \frac{\partial w^0}{\partial z^0} \frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1} \frac{\partial w^1}{\partial z^1} &= 0 \end{aligned} \quad (2.3)$$

which is equivalent to

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \quad (2.4)$$

or to

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} . \quad (2.5)$$

It is then convenient to use complex coordinates

$$\begin{aligned} z &:= z^0 + iz^1 , & \bar{z} &:= z := z^0 - iz^1 , \\ \partial &:= \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) , & \bar{\partial} &:= \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \end{aligned} \quad (2.6)$$

in this way (2.4) and (2.5) are equivalent to the Cauchy-Riemann equations for holomorphic and antiholomorphic functions. Therefore the local conformal transformations in 2 dimension are the set of all (anti)holomorphic maps in some open set. These do not form a group since we have not required them to be invertible nor everywhere well-defined. It turns out (see [11]) that the maps that form such group depend on the topology of Σ . For example, for the Riemann sphere they are

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \frac{SL(2, \mathbb{C})}{\mathbb{Z}_2} , \quad (2.7)$$

where the group operation is the composition of maps and the action of \mathbb{Z}_2 is given by sending $(a, b, c, d) \rightarrow (-a, -b - c, -d)$ which clearly leaves $f(z) = \frac{az+b}{cz+d}$ invariant. We call the group of global conformal transformations the special conformal group.

We have seen that conformal transformations are given maps (anti)holomorphic in some open set. In general they will be meromorphic functions outside these open sets. Considering an infinitesimal transformation we can then perform a Laurent expansion

$$\begin{aligned} z' &= z + \epsilon(z) = z + \sum_{n=-\infty}^{\infty} \epsilon_n (-z^{n+1}) , \\ \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n (-\bar{z}^{n+1}) , \end{aligned} \quad (2.8)$$

where $\epsilon_n, \bar{\epsilon}_n$ are (small) constant. The generators corresponding to a transformation for a particular n are

$$\begin{aligned} l_n &:= -z^{n+1} \partial_z , \\ \bar{l}_n &:= -\bar{z}^{n+1} \partial_{\bar{z}} . \end{aligned} \quad (2.9)$$

The commutators between these generators are easily computed

$$\begin{aligned} [l_m, l_n] &= (m-n) l_{m+n} , \\ [\bar{l}_m, \bar{l}_n] &= (m-n) \bar{l}_{m+n} , \\ [l_m, \bar{l}_n] &= 0 , \end{aligned} \quad (2.10)$$

so the l_n s and \bar{l}_n s generate two copies of the Witt algebra. Notice that the special conformal group of the sphere is generated by l_{-1}, l_0 and l_1 . In particular, recalling (2.7), l_{-1} generates the translations $z \rightarrow z + b$, l_0 generates dilations $z \rightarrow az$ and l_1 generates the special conformal transformations $z \rightarrow \frac{1}{cz+1}$. For later convenience we also introduce a central extension of the Witt algebra, the so called Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0} . \quad (2.11)$$

Given a field $\phi(z, \bar{z})$ of conformal dimensions (h, \bar{h}) we call it primary if under any conformal map $z \rightarrow w(z), \bar{z} \rightarrow \bar{w}(\bar{z})$ it transforms as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz} \right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}} \phi(z, \bar{z}) , \quad (2.12)$$

h and \bar{h} are called the conformal dimensions of the field. A field is quasi-primary if it transforms as (2.12) only under the special conformal group.

If we have a field theory with a conformal invariant action, using Noether's theorem we can build a conserved current j^μ . Considering an infinitesimal conformal transformation $z^\mu \rightarrow z^\mu + \epsilon^\mu(z)$, we can write our current as $j_\mu = T_{\mu\nu}\epsilon^\nu$. The tensor $T_{\mu\nu}$ is symmetric and it is called the stress-energy tensor. Conservation of the current $\partial_\mu j^\mu = 0$, when ϵ is constant, implies $\partial^\mu T_{\mu\nu} = 0$. Considering a general infinitesimal transformation

$$\begin{aligned} 0 = \partial_\mu j^\mu &= \partial^\mu T_{\mu\nu}\epsilon^\nu + T_{\mu\nu}\partial^\mu\epsilon^\nu = \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) = \\ &= \frac{1}{2}T_{\mu\nu}\eta^{\mu\nu}(\partial \cdot \epsilon) = \frac{1}{2}T_\mu{}^\mu(\partial \cdot \epsilon) , \end{aligned} \quad (2.13)$$

where we have used the fact that for an infinitesimal conformal transformation in 2 dimensions $(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) = \eta^{\mu\nu}(\partial \cdot \epsilon)$. Since the previous result must hold for generic infinitesimal conformal transformations it implies that the stress-energy tensor is traceless $T_\mu{}^\mu = 0$.

It is useful to write the stress-energy tensor in terms of complex coordinates. We consider an euclidean metric $\eta_{\mu\nu} = \delta_\nu^\mu$ and use the coordinates given in equation (2.6). The tensor transformation property $T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} T_{\alpha\beta}$ together with the tracelessness discussed before imply that the only non-zero components of the stress-energy tensor are

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) , \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) . \end{aligned} \quad (2.14)$$

Using $\partial^\mu T_{\mu\nu} = 0$ and the tracelessness it is easy to show

$$\begin{aligned} \partial_{\bar{z}} T_{zz} &= 0 , \\ \partial_z T_{\bar{z}\bar{z}} &= 0 , \end{aligned} \quad (2.15)$$

so T_{zz} and $T_{\bar{z}\bar{z}}$ are respectively holomorphic and antiholomorphic. We then simply write them $T(z) := T_{zz}$ and $\bar{T}(\bar{z}) := T_{\bar{z}\bar{z}}$.

The conserved Noether's charge $Q = \int dz^1 j_0$ in complex coordinates reads

$$Q = \frac{1}{2\pi i} \oint_{\mathcal{C}} (dz T(z)\epsilon(z) + d\bar{z} \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})) , \quad (2.16)$$

where the contour \mathcal{C} encircles all the singularities and we always consider contour integrals taken counterclockwise.

In complex coordinates the condition for the conservation of a current $(j_z, j_{\bar{z}})$ is

$$\bar{\partial} j_z + \partial j_{\bar{z}} = 0 , \quad (2.17)$$

so currents $(j_z, 0)$ with j_z holomorphic or $(0, j_{\bar{z}})$ with $j_{\bar{z}}$ antiholomorphic are conserved. In particular starting from $T(z)$, and analogously for $\bar{T}(\bar{z})$, it is possible to build an infinite number of conserved currents by setting $(j_z, j_{\bar{z}}) = (z^n T(z), 0)$. The conserved charges

$$L_n = \oint z^n T(z) dz \quad (2.18)$$

are the modes of the expansion of the stress energy tensor

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} . \quad (2.19)$$

In general conformal field theories can possess other holomorphic (analogously antiholomorphic) fields whose components will be conserved for the reasons we said above. If it is the case, their modes together with their commutation relations (between them and with the modes of the stress-energy tensor) constitute an extension of the Witt (Virasoro in the quantum theory) algebra called the chiral algebra.

We will now discuss the main implications of conformal invariance on the quantum theory. We have seen that, when dealing with 2-dimensional conformal invariance, it is convenient to work with complex numbers and (anti)holomorphic functions. In order to exploit the power of complex analysis it is then convenient to map the space-time of a conformal invariant quantum field theory into the complex plane, this will also lead to the concept of radial quantization.

Let us consider a flat 2-dimensional space-time with euclidean signature. For concreteness we denote the time direction with z^0 and the space direction with z^1 , but the choice of a time direction is arbitrary. We then compactify the space direction on a circle, for simplicity we consider it of unit radius. Thus our space-time is actually a cylinder of infinite length. We can map this cylinder into the complex plane through the map

$$z = e^w = e^{z^0 + iz^1}, \quad (2.20)$$

where we have introduced the complex coordinate $w := z^0 + iz^1$. With this map time translation $z^0 \rightarrow z^0 + a$ correspond to complex dilation $z \rightarrow e^a z$ while space translation $z^1 \rightarrow z^1 + b$ correspond to complex rotations $z \rightarrow e^{ib} z$. Furthermore, the infinite past of the cylinder is mapped into the origin on the complex plane. Thus given a primary quantum field $\phi(z, \bar{z})$ with conformal dimensions (h, \bar{h}) we define an asymptotic in-state $|\phi\rangle$ through

$$|\phi\rangle = \lim_{z, \bar{z} \rightarrow 0} \phi(z, \bar{z}) |0\rangle, \quad (2.21)$$

where $|0\rangle$ is the vacuum state of the theory.

Defining the hermitian of the field $\phi(z, \bar{z})$ to be

$$\phi^\dagger(z, \bar{z}) := z^{-2\bar{h}} \bar{z}^{-2h} \phi\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) \quad (2.22)$$

we can also define an asymptotic out-state $\langle\phi|$ through

$$\langle\phi| = \lim_{z, \bar{z} \rightarrow 0} \langle 0| \phi^\dagger(z, \bar{z}) = \lim_{w, \bar{w} \rightarrow \infty} \langle 0| w^{2h} \bar{w}^{2\bar{h}} \phi(w, \bar{w}). \quad (2.23)$$

A very important feature is that conformal invariance fixes the form of some correlation functions between primary fields. We will just give here the result without derivation (see again [11]). The two-point function between primary fields has the form

$$\langle\phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{(z_1 - z_2)^{2h_1} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}_1}} \delta_{h_1}^{h_2} \delta_{\bar{h}_1}^{\bar{h}_2} \quad (2.24)$$

and in particular it is not null only when the two fields have the same conformal dimensions. The form of 3-point function is

$$\begin{aligned} \langle\phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3)\rangle &= C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times \\ &\quad \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}, \end{aligned} \quad (2.25)$$

where C_{12} and C_{123} depend on the fields and $z_{ij} := z_i - z_j$. Conformal invariance is not enough to fix the form of higher number of points functions.

Correlation function can be obtained by standard quantum field theory methods, but for 2-dimensional conformal field theories it is possible to define the so called Operator Product Expansion (OPE) which can be used to systematically obtain the correlation functions. The main idea behind the OPE is to approximate the product of two fields with a Laurent expansion

$$A(z, \bar{z})B(w, \bar{w}) = \sum_{n=-\infty}^N \sum_{m=-\infty}^M \frac{C_n(w, \bar{w})}{(z-w)^n (\bar{z}-\bar{w})^m} , \quad (2.26)$$

where the C_n are regular operators in the limit $(z, \bar{z}) \rightarrow (w, \bar{w})$. Since the previous expression is an operator equality, it has to hold between all correlation functions. The knowledge of the OPE together with conformal invariance, which fixes the form of 2 and 3 points functions, is enough to obtain all the correlation functions of the theory. Conformal symmetry implies the OPE of a primary field, of conformal dimension h , with the stress-energy tensor to be of the form

$$T(z)\Phi(w, \bar{w}) = \frac{h}{(z-w)^2}\Phi(w, \bar{w}) + \frac{1}{z-w}\partial_w\Phi(w, \bar{w}) + \dots , \quad (2.27)$$

where the dots stand for non-singular terms. An analogous expression holds for the antiholomorphic stress-energy tensor. It is important to point out that the stress-energy tensor is not a primary field, in fact the OPE of the stress-energy tensor with itself reads

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots , \quad (2.28)$$

where the dots denote again non-singular terms. It can be shown that the modes in the Fourier expansion of the stress-energy tensor

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad (2.29)$$

satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \quad (2.30)$$

which, as we already said, is a central extension of the Witt algebra.

Since the Hamiltonian and the momentum are the quantum generators for the time and space translations respectively, since the central extension vanish for L_0 and \bar{L}_0 , we have

$$H = L_0 + \bar{L}_0 \quad \text{and} \quad P = i(L_0 - \bar{L}_0) . \quad (2.31)$$

2.2 Space of states and correlation functions

We will now formalize the properties of 2 dimensional conformal field theory, some of which have already appeared in the previous section. We start by defining the space of state. We want the state of space to be decomposable in terms of the eigenstates of L_0 and \bar{L}_0 so that it will in particular be decomposable in terms of eigenstate of the Hamiltonian and of the momentum. Formally, the space of states of a two dimensional conformal field theory with unique vacuum is a bigraded, infinite dimensional vector space over \mathbb{C}

$$\mathcal{H} = \bigoplus_{h, \bar{h}} V(h, \bar{h}) \quad (2.32)$$

such that each $V(h, \bar{h})$ is finite dimensional, $\dim V(h, \bar{h}) = 0$ if $h < -h_0$ or $\bar{h} < -\bar{h}_0$ for fixed $h_0, \bar{h}_0 \in \mathbb{R}_0^+$, and $V(0, 0) = \mathbb{C}$. The unit element $|0\rangle$ in $V(0, 0) = \mathbb{C}$ is called the vacuum state. The subspaces

$$\mathcal{W} := \bigoplus_h V(h, 0), \quad \bar{\mathcal{W}} := \bigoplus_{\bar{h}} V(0, \bar{h}) \quad (2.33)$$

are called the holomorphic (or chiral) and antiholomorphic (or antichiral) algebras. Furthermore we will assume that \mathcal{H} carries a real structure.

We now formalize the concept of the OPE. We denote with $\mathcal{H}\{z, \bar{z}\}$ the space of functions $f : \mathbb{C} \rightarrow \bar{\mathcal{H}}$ that are real analytic on \mathbb{C}^* and possess the following behaviour around $z = 0$:

$$f(z, \bar{z}) = \sum_{r \in R, n \in \mathbb{Z}} a_{rn} z^{r+n} \bar{z}^r, \quad (2.34)$$

where R is a countable subset of \mathbb{R} , $a_{rn} \in \mathcal{H}$ with only finitely many coefficients a_{rn} for $r+n < 0$ or $r < 0$.

An Operator Product Expansion (OPE) is a map

$$\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}\{z, \bar{z}\}$$

compatible with the gradings and real structure of \mathcal{H} .

Let us denote with $\mathcal{F}(z_1, \dots, z_n)$ the space of maps $f : \mathbb{C}^n \rightarrow \mathbb{C}$ that are real analytic outside partial diagonals and which possess a behaviour like (2.34) in each singularity.

To each state $|\Phi\rangle \in V(h, \bar{h})$ we associate a linear operator $\Phi(z, \bar{z}) : \mathcal{H} \rightarrow \bar{\mathcal{H}}$ such that

$$|\Phi\rangle = \lim_{z, \bar{z} \rightarrow 0} \Phi(z, \bar{z}) |0\rangle \quad (2.35)$$

and the operator associated to $|0\rangle$ is the identity $\mathbb{1}$. These operators are called fields. We will later show that (2.35) together with conformal invariance is enough to completely fix the fields. For every $n \in \mathbb{N}$ we then have a linear map

$$\begin{aligned} F_n : \quad \mathcal{H}^{\otimes n} &\rightarrow \mathcal{F}(z_1, \dots, z_n) \\ |\Phi^1\rangle \otimes \dots \otimes |\Phi^n\rangle &\rightarrow \langle 0 | \Phi^1(z_1, \bar{z}_1) \dots \Phi^n(z_n, \bar{z}_n) |0\rangle. \end{aligned} \quad (2.36)$$

$\langle 0 | \Phi^1(z_1, \bar{z}_1) \dots \Phi^n(z_n, \bar{z}_n) |0\rangle$ is called an n point correlation function (or simply n point function).

We require that our n point functions are such that the following diagram commute

$$\begin{array}{ccc} \mathcal{H}^{\otimes k+1} & \xrightarrow{F_{k+1}} & \mathcal{F}(z_0, \dots, z_k) \\ \text{OPE} \downarrow & & \downarrow \text{Laurent expansion} \\ \mathcal{H}^{\otimes k} \{z, \bar{z}\} & \xrightarrow{F_k} & \mathcal{F}(z_1, \dots, z_k) \end{array}$$

where the OPE is taken with respect of the first two factors and the right vertical arrow is a Laurent expansion around 0 of $z = z_1 - z_0$. With this requirement we can write the OPE between two fields in the usual way as an expansion of the product of two fields

$$A(z, \bar{z})B(w, \bar{w}) = \sum_{n=-\infty}^N \sum_{m=-\infty}^M \frac{C_n(w, \bar{w})}{(z-w)^n (\bar{z}-\bar{w})^m}, \quad (2.37)$$

where the fields $C_n(w, \bar{w})$ are regular at $w = z$ and $\bar{w} = \bar{z}$, $N, M \in \mathbb{N}$ and it is intended that this expression must hold between correlation functions.

Furthermore, we make another assumption on the correlation functions: given a $n + 1$ point function with the above properties, for any fixed i, j , the residues in $z_{ij} := z_i - z_j$ in 0 are n point functions with the same properties.

A vector space $F = \bigoplus_n F_n$ of linear maps $F_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{F}(z_1, \dots, z_n)$ satisfying the above properties is called a representation of an OPE $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}\{z, \bar{z}\}$.

We define a conformal field theory to be a representation of its OPE.

2.3 Ordering prescriptions

Up to now we have not given any prescription on how the fields inserted in (2.36) have to be ordered.

It is well known that in a quantum field theory the fields inserted in a n point function have to be time ordered. In the previous section we have seen that, in a 2d conformal field theory, the complex plane is obtained by a 2 dimensional real euclidean space-time (z_0, z_1) with a compact spatial direction z_1 , mapping $z = e^{z_0 + iz_1}$. Thus points in the original space-time with later time z_0 will correspond to points in the complex plane with larger radius $|z|$. In a 2d conformal field theory defined on the complex plane (i.e. all the variables in correlation functions take values in \mathbb{C}) we will thus assume that the fields in a n point function are radial ordered

$$\mathbf{R}(A(z, \bar{z})B(w, \bar{w})) := \begin{cases} A(z, \bar{z})B(w, \bar{w}) & \text{for } |z| > |w| \\ \epsilon B(w, \bar{w})A(z, \bar{z}) & \text{for } |z| < |w| \end{cases}, \quad (2.38)$$

where we have to include the factor ϵ if the space of state \mathcal{H} is also \mathbb{Z}_2 graded due to the presence of fermions (see section 3.2), and it has values $\epsilon = -1$ if both $A(z)$ and $B(w)$ are odd with respect to the grading (i.e. both are fermions) and $\epsilon = 1$ otherwise. It generalizes easily for n fields by associativity.

To be consistent with the previous requirement, we then choose the fields of our theory to be local, i.e. $\Phi(z, \bar{z})\Psi(w, \bar{w}) = \epsilon\Psi(w, \bar{w})\Phi(z, \bar{z})$, with ϵ defined as above, if $z \neq w$.

In analogy to ordinary quantum field theory we also define normal ordering for later convenience. Its purpose is the usual one of defining finite quantities from divergent ones.

We define normal ordering by means of the OPE of two fields

$$\phi(z, \bar{z})\chi(w, \bar{w}) = \text{singular parts} + \sum_{n,m=0}^{\infty} \frac{(z-w)^n(\bar{z}-\bar{w})^m}{n!m!} \mathbf{N}(\chi \partial^n \bar{\partial}^m \phi)(w, \bar{w}), \quad (2.39)$$

applying $\frac{1}{2\pi i} \oint dz (z-w)^{-1}(\bar{z}-\bar{w})^{-1}$ to both sides of (2.39) we obtain

$$\mathbf{N}(\chi\phi)(w, \bar{w}) = \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{\phi(z, \bar{z})\chi(w, \bar{w})}{(z-w)(\bar{z}-\bar{w})}, \quad (2.40)$$

where $\mathcal{C}(w)$ is a contour encircling w and no other singularities.

We remark that, since formula (2.37) has to hold when the fields are insertions of a n point function, the OPE is defined between radial ordered fields. So (2.37) actually reads:

$$\mathbf{R}(A(z, \bar{z})B(w, \bar{w})) = \sum_{n=-\infty}^N \sum_{m=-\infty}^M \frac{C_n(w, \bar{w})}{(z-w)^n(\bar{z}-\bar{w})^m}. \quad (2.41)$$

In particular (2.40) is intended as

$$\mathbf{N}(\chi\phi)(w, \bar{w}) = \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{\mathbf{R}\phi(z, \bar{z})\chi(w, \bar{w})}{(z-w)(\bar{z}-\bar{w})}. \quad (2.42)$$

In the rest of this work we will not write explicitly the radial ordering symbol, but radial ordering is always assumed between expressions that are intended to be inserted in a n point function.

In the following we will mostly denote normal ordered product with $:\dots:$ instead than with $N(\dots)$.

It is useful, since we will use it later, to give a formula for the component of the normal order (2.40) for chiral fields. If we decompose the fields as

$$\phi(z) = \sum_{n \in \mathbb{Z}} z^{-n-h_\phi} \phi_n \quad \chi(z) = \sum_{n \in \mathbb{Z}} z^{-n-h_\chi} \chi_n \quad (2.43)$$

and

$$N(\chi\phi)(w) = \sum_{n \in \mathbb{Z}} z^{-n-h_\phi-h_\chi} : \chi\phi :_n \quad (2.44)$$

then

$$\begin{aligned} N(\chi\phi)(w) &= \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{R(\phi(z)\chi(w))}{z-w} = \frac{1}{2\pi i} \oint_{|z|>|w|} dz \frac{\phi(z)\chi(w)}{z-w} + \frac{1}{2\pi i} \oint_{|w|>|z|} dz \frac{\epsilon\chi(z)\phi(w)}{z-w} = \\ &= \frac{1}{2\pi i} \oint_{|z|>|w|} dz \sum_{\substack{n,p \\ l \geq 0}} w^{l-p-h_\phi} z^{-n-h_\chi-l-1} \chi_n \phi_p + \\ &= \frac{1}{2\pi i} \oint_{|w|>|z|} dz \sum_{\substack{n,p \\ l \geq 0}} w^{-l-1-p-h_\phi} z^{l-n-h_\chi} \epsilon\phi_p \chi_n = \\ &= \sum_{\substack{p \\ n \leq -h_\chi}} w^{-n-p-h_\chi-h_\phi} \chi_n \phi_p + \epsilon \sum_{\substack{p \\ n > -h_\chi}} w^{-n-p-h_\chi-h_\phi} \phi_p \chi_n, \end{aligned} \quad (2.45)$$

where we have used the definition of radial ordering in the first line and used the geometric series $\frac{1}{z-w} = \sum_{l \geq 0} \frac{w^l}{z^{l+1}}$ in the second. Comparing with the expression above we have

$$:\chi\phi :_n = \sum_{m \leq -h_\chi} \chi_m \phi_{n-m} + \epsilon \sum_{m > -h_\chi} \phi_{n-m} \chi_m. \quad (2.46)$$

With similar calculations it is easy to show also

$$:\chi\partial\phi :_n = \sum_{m \leq -h_\chi} (-h_\phi - m) \chi_m \phi_{n-m} + \epsilon \sum_{m > -h_\chi} (-h_\phi - m) \phi_{n-m} \chi_m. \quad (2.47)$$

2.4 The conformal structure

We now describe, in the formalism of the previous section, the conformal structure of the theory, with particular emphasis to the stress-energy tensor and its properties.

We require that the space of states of a conformal field theory contains a state $|T\rangle \in V(2;0)$ with a real holomorphic associated field $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, whose Fourier components $L_n =$

$\frac{1}{2\pi i} \oint dz z^{n+1} T(z)$ satisfy $(L_m)^\dagger = L_{-m}$ and generate a left-handed Virasoro algebra with central charge c :

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \quad (2.48)$$

and an antiholomorphic field $\bar{T}(\bar{z}) \in V(0, 2)$ with analogous properties generating a right-handed Virasoro algebra commuting with the left-handed one. The field $T(z)$ is called the stress-energy tensor of the theory.

We furthermore require that our space of state \mathcal{H} is the space of a representation of the left and right-handed Virasoro algebras such that if $|\Phi\rangle \in V(h, \bar{h})$ for any other state $|\Psi^i\rangle$ it holds:

$$\begin{aligned} \frac{\partial}{\partial z} \langle 0 | \Phi(z, \bar{z}) \Psi^1(z_1, \bar{z}_1) \cdots \Psi^n(z_n, \bar{z}_n) | 0 \rangle &= \langle 0 | (L_{-1}\Phi)(z, \bar{z}) \Psi^1(z_1, \bar{z}_1) \cdots \Psi^n(z_n, \bar{z}_n) | 0 \rangle , \\ \frac{\partial}{\partial \bar{z}} \langle 0 | \Phi(z, \bar{z}) \Psi^1(z_1, \bar{z}_1) \cdots \Psi^n(z_n, \bar{z}_n) | 0 \rangle &= \langle 0 | (\bar{L}_{-1}\Phi)(z, \bar{z}) \Psi^1(z_1, \bar{z}_1) \cdots \Psi^n(z_n, \bar{z}_n) | 0 \rangle . \end{aligned} \quad (2.49)$$

We have already defined primary fields through their transformation properties under conformal tranformations. We will now define them by state-fields correspondence relating them to certain states in particular representations of the Virasoro algebra. We define a primary state to be a highest weight state of the left and the right handed Virasoro algebras, i.e. $L_m |\Phi\rangle = 0$ and $\bar{L}_m |\Phi\rangle = 0$ for $m > 0$, $L_0 |\Phi\rangle = h |\Phi\rangle$ and $\bar{L}_0 |\Phi\rangle = \bar{h} |\Phi\rangle$. We call a field $\Phi(z, \bar{z}) \in V(h, \bar{h})$ primary if it has the following OPEs

$$\begin{aligned} T(z)\Phi(w, \bar{w}) &= \frac{h}{(z-w)^2} \Phi(w, \bar{w}) + \frac{1}{z-w} \partial_w \Phi(w, \bar{w}) + \dots , \\ \bar{T}(\bar{z})\Phi(w, \bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \Phi(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial_{\bar{w}} \Phi(w, \bar{w}) + \dots , \end{aligned} \quad (2.50)$$

where the dots stand for terms regular in the limit $z \rightarrow w$.

The corresponding states $|\Phi\rangle = \Phi(0, 0) |0\rangle$ are primary, in fact

$$\begin{aligned} L_0 \Phi(0, 0) |0\rangle &= \frac{1}{2\pi i} \oint dz z T(z) \Phi(0, 0) |0\rangle = \\ &= \frac{1}{2\pi i} \oint dz z \left(\frac{h}{z^2} \Phi(0, 0) + \frac{1}{z} \partial_w \Phi(w, 0) \Big|_{w=0} \right) |0\rangle = h |\Phi\rangle , \end{aligned} \quad (2.51)$$

where we have used the residue theorem in the last line. Notice that $T(z)\Phi(0, 0)$ is always correctly radial ordered in the whole complex plane. Furthermore, for $n > 0$

$$\begin{aligned} L_n \Phi(0, 0) |0\rangle &= \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \Phi(0, 0) |0\rangle = \\ &= \frac{1}{2\pi i} \oint dz z^{n+1} \left(\frac{h}{z^2} \Phi(0, 0) + \frac{1}{z} \partial_w \Phi(w, 0) \Big|_{w=0} \right) |0\rangle = 0 \end{aligned} \quad (2.52)$$

because there are no poles for $n > 0$. Similar relations hold for $\bar{T}(\bar{z})$. Viceversa if $|\Phi\rangle$ is primary the corresponding field is primary. In fact, with calculation similar to the ones before, $L_n |\Phi\rangle = 0$ for $n > 0$ implies that the OPE has a pole of order at most 2, $L_0 |\Phi\rangle = h |\Phi\rangle$ fixes the coefficient of the pole of order 2 to be equal to h and (2.49) implies that the coefficient of the order 1 pole is $\partial_w \Phi(w, \bar{w})$. Again, the extension to the right-handed part is straightforward. We say a state $|\phi\rangle$ has conformal dimension (h, \bar{h}) if

$$L_0 |\phi\rangle = h |\phi\rangle \quad \text{and} \quad \bar{L}_0 |\phi\rangle = \bar{h} |\phi\rangle . \quad (2.53)$$

It is useful to remark that, given a state $|\phi\rangle$ of conformal dimension h , the L_n s increase or decrease its conformal dimension depending on the sign of n . In fact if $L_0 |\phi\rangle = h |\phi\rangle$, using the commutation relations (2.48), we have

$$L_0 L_n |\phi\rangle = L_n L_0 |\phi\rangle - n L_n |\phi\rangle = (h - n) L_n |\phi\rangle \quad (2.54)$$

so $L_n |\phi\rangle$ is an eigenstate of L_0 with eigenvalue $h - n$. An analogous result holds obviously for the right-handed Virasoro generators. In general if $|\phi\rangle$ has conformal dimensions (h, \bar{h}) we have that the state

$$L_{n_1} L_{n_2} \cdots L_{n_r} \bar{L}_{\bar{n}_1} \bar{L}_{\bar{n}_2} \cdots \bar{L}_{\bar{n}_s} |\phi\rangle \quad (2.55)$$

has conformal dimensions

$$\left(h - \sum_{i=1}^r n_i, \bar{h} - \sum_{i=1}^s \bar{n}_i \right) . \quad (2.56)$$

We have defined a primary state $|\phi\rangle$ to be a highest weight state of the Virasoro algebra, i.e. $L_m |\phi\rangle = 0$, $\bar{L}_m |\phi\rangle = 0$ for $m > 0$ and $L_0 |\phi\rangle = h |\phi\rangle$, $\bar{L}_0 |\phi\rangle = \bar{h} |\phi\rangle$. Starting from a primary state $|\phi\rangle$ we can build a representation of the left and right handed Virasoro algebra by acting with L_{-m} and \bar{L}_{-m} , $m > 0$. We call the state $L_{-i_1} \cdots L_{-i_n} \bar{L}_{-\bar{i}_1} \bar{L}_{-\bar{i}_n} \cdots |\phi\rangle$, with all the i_j s and \bar{i}_j s greater than zero, a secondary state or a descendant of $|\phi\rangle$. We will now show that any state of a unitary conformal field theory belongs to a highest weight representation. Suppose there is a state that cannot be written as a linear combination of primary and secondary fields of a highest weight representation. Since every state can be written as a linear combination of eigenstates of L_0 , at least one of them must not belong to a highest weight representation. Let us consider the smallest dimension L_0 eigenstate $|\chi\rangle$ (from (2.32) the spectrum of L_0 must be bounded from below) with this property. L_0 eigenstates are orthogonal, so $|\chi\rangle$ is orthogonal to all the states of the highest weight representations. Since the state $|\chi\rangle$ does not belong to a highest weight representation, it is in particular not primary and then the state $L_n |\chi\rangle$, is not null for some $n > 0$. But, because $L_{-m} = L_m^\dagger$, then

$$\langle \chi | L_{-m} L_m | \chi \rangle > 0 . \quad (2.57)$$

Since $|\chi\rangle$ is the lowest dimension state that does not belong to a highest weight representation then $L_n |\chi\rangle$ belongs to a highest weight representation and so does $L_{-n} L_n |\chi\rangle$, but (2.57) is in contradiction to the fact that $|\chi\rangle$ is orthogonal to all the states of a highest weight representation. So all the state of a unitary conformal field theory must belong to a highest weight representation.

2.4.1 State-field correspondence

Up to now we have not really defined the fields of our theory, we have just said that they are local and they must satisfy (2.35). As mentioned earlier this, together with the conformal structure of the theory, will be enough to uniquely define them.

We first notice from (2.49) that, choosing $|\Phi\rangle = |0\rangle$ and since the operator associated to $|0\rangle$ is the identity which is constant, we have

$$(L_{-1} |0\rangle)(z, \bar{z}) = 0$$

in all correlation functions, so we can choose the operator associated to $L_{-1} |0\rangle$ to be identically zero. But then we find

$$L_{-1} |0\rangle = (L_{-1} |0\rangle)(0, 0) |0\rangle = 0 , \quad (2.58)$$

so we can assign to the state $L_{-1} |\Phi\rangle$ the field $[L_{-1}, \Phi(z, \bar{z})]$, in fact

$$[L_{-1}, \Phi(0, 0)] |0\rangle = L_{-1} \Phi(0, 0) |0\rangle - \Phi(0, 0) L_{-1} |0\rangle = L_{-1} \Phi(0, 0) |0\rangle = L_{-1} |\Phi\rangle . \quad (2.59)$$

But then, again from (2.49), we have that

$$[L_{-1}, \Phi(z, \bar{z})] = \frac{\partial}{\partial z} \Phi(z, \bar{z}) \quad (2.60)$$

must hold in all correlation functions. So we can require that, for every state $|\Phi\rangle$, the correspondent field $\Phi(z, \bar{z})$ satisfies, in addition to (2.35), also (2.60).

It is easy to show, using Baker–Campbell–Hausdorff formula, that (2.60) implies

$$e^{\lambda L_{-1}} \Phi(z, \bar{z}) e^{-\lambda L_{-1}} = \Phi(z + \lambda, \bar{z}) . \quad (2.61)$$

Repeating the same argument for the right handed side we obtain the similar relation:

$$e^{\lambda \bar{L}_{-1}} \Phi(z, \bar{z}) e^{-\lambda \bar{L}_{-1}} = \Phi(z, \bar{z} + \lambda) . \quad (2.62)$$

From (2.35) we have to associate to $|\Phi\rangle$ an operator $\Phi(0, 0)$, we then define the field $\Phi(z, \bar{z})$ associated to the state $|\Phi\rangle$ to be

$$\Phi(z, \bar{z}) := e^{zL_{-1} + \bar{z}\bar{L}_{-1}} \Phi(0, 0) e^{-zL_{-1} - \bar{z}\bar{L}_{-1}} . \quad (2.63)$$

In particular it holds

$$\Phi(z, \bar{z}) |0\rangle = e^{zL_{-1} + \bar{z}\bar{L}_{-1}} |\Phi\rangle . \quad (2.64)$$

Let us now show that the field defined in this way is unique.

If $\Psi(z, \bar{z})$ is another field such that $\Psi(z, \bar{z}) |0\rangle = e^{zL_{-1} + \bar{z}\bar{L}_{-1}} |\Phi\rangle$, given another arbitrary state $\chi \in \mathcal{H}$ then

$$\Psi(z, \bar{z}) |\chi\rangle = \Psi(z, \bar{z}) \chi(0, 0) |0\rangle = \epsilon \chi(0, 0) \Psi(z, \bar{z}) |0\rangle = \epsilon \chi(0, 0) e^{zL_{-1} + \bar{z}\bar{L}_{-1}} |\Phi\rangle , \quad (2.65)$$

where $\chi(z, \bar{z})$ is the state associated with $|\chi\rangle$ and we have also used the locality property of the fields. We have that

$$\epsilon \chi(0, 0) e^{zL_{-1} + \bar{z}\bar{L}_{-1}} |\Phi\rangle = \epsilon \chi(0, 0) \Phi(z, \bar{z}) |0\rangle = \Phi(z, \bar{z}) \chi(0, 0) |0\rangle = \Phi(z, \bar{z}) |\chi\rangle , \quad (2.66)$$

where we have used the fact that the grading of $\Phi(z, \bar{z})$ and $\Psi(z, \bar{z})$ must clearly be the same since they have the same action on the vacuum. So $\Phi(z, \bar{z}) |\chi\rangle = \Psi(z, \bar{z}) |\chi\rangle$ on every state and thus the identity holds as an identity between operators $\Phi(z, \bar{z}) = \Psi(z, \bar{z})$.

Given a field $\Phi(z, \bar{z})$ we define its adjoint demanding that

$$\langle \Phi | \Psi \rangle = \lim_{z, \bar{z}, w, \bar{w} \rightarrow 0} \langle 0 | \Phi(z, \bar{z})^\dagger \Psi(w, \bar{w}) | 0 \rangle \quad (2.67)$$

for any other field $\Psi(w, \bar{w})$. In particular for primary fields, to be consistent with (2.24), we define their conjugate to be

$$\Phi^\dagger(z, \bar{z}) := z^{-2\bar{h}} \bar{z}^{-2h} \Phi\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) . \quad (2.68)$$

This coincides with the definition given in the previous section. Considering the mode expansion of a primary field $\Phi(z, \bar{z}) = \sum_{m, n} z^{-n-h} \bar{z}^{-m-\bar{h}} \Phi_{n, \bar{m}}$ the definition above implies $\Phi_{n, \bar{m}}^\dagger = \Phi_{-n, -\bar{m}}$.

2.5 Simple examples

In this section we discuss simple examples of conformal field theory. We start discussing the free boson and the free fermion and then we will discuss more generally systems whose currents form a Kac-Moody algebra. The purpose is to briefly explain the basic properties that we will use in section 6.2. For a more detailed discussion see [11] or [12].

2.5.1 The free boson

A free boson is a real scalar field with respect to conformal transformations, i.e.

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) \quad (2.69)$$

under a conformal transformation $z \rightarrow z'$.

The action of a free boson defined on the complex plane is given by

$$S = \frac{1}{4\pi} \int dz d\bar{z} \partial X(z, \bar{z}) \cdot \bar{\partial} X(z, \bar{z}) . \quad (2.70)$$

The equations of motion are easily obtained

$$\partial \bar{\partial} X(z, \bar{z}) = 0 . \quad (2.71)$$

Defining

$$\begin{aligned} j(z) &= i\partial X(z, \bar{z}) , \\ \bar{j}(\bar{z}) &= i\bar{\partial} X(z, \bar{z}) \end{aligned} \quad (2.72)$$

from (2.71) they are, respectively, holomorphic and antiholomorphic fields and from (2.69) they are primary with conformal dimension $h = 1$ and $\bar{h} = 1$ respectively. The quickest way to quantize the theory is to promote j and \bar{j} to operators and then build the space of states using the state-field correspondence. For brevity, we use the following OPEs as an ansatz

$$\begin{aligned} j(z)j(w) &= \frac{1}{(z-w)^2} + \dots , \\ \bar{j}(\bar{z})\bar{j}(\bar{w}) &= \frac{1}{(\bar{z}-\bar{w})^2} + \dots , \\ j(z)\bar{j}(\bar{w}) &= \dots , \end{aligned} \quad (2.73)$$

where the dots stand for finite terms. It is a simple task to check that these are the correct expressions by computing the two-point function by standard methods (see for example [11])

$$\langle X(z, \bar{z}) X(w, \bar{w}) \rangle = -\frac{1}{2} \log |z - w|^2 \quad (2.74)$$

and taking the derivatives. Expanding $j(z)$ in modes

$$j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1} \quad (2.75)$$

we can compute the commutator

$$\begin{aligned} [j_n, j_m] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^n w^m [j(z), j(w)] = \\ &= \oint \frac{dw}{2\pi i} w^m \left(\oint_{|z| > |w|} \frac{dz}{2\pi i} z^n j(z) j(w) - \oint_{|z| < |w|} \frac{dz}{2\pi i} z^n j(w) j(z) \right) = \\ &= \oint \frac{dw}{2\pi i} w^m \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} z^n \frac{1}{(z-w)^2} = \oint \frac{dw}{2\pi i} n w^{n-1} w^m = n \delta_{n+m, 0} , \end{aligned} \quad (2.76)$$

where we have used the fact that the fields must be radial ordered and $\mathcal{C}(w)$ is a contour encircling w . Analogously

$$\begin{aligned} [\bar{j}_n, \bar{j}_m] &= n \delta_{n+m, 0} , \\ [j_n, \bar{j}_m] &= 0 . \end{aligned} \quad (2.77)$$

The stress-energy tensor is given by

$$T(z) = \frac{1}{2} : j(z) j(w) : , \quad (2.78)$$

it is a simple task to show that it satisfies the right properties.

The operators of the theory are given by linear combinations of products of the j s and the \bar{j} s

thus, defining the vacuum state $|0\rangle$ as the state annihilated by all non-negative modes j_n and \bar{j}_m , by state-field correspondence the space of state will be given by

$$\mathcal{H} = \{\text{Fock space freely generated by } j_{-n}, \bar{j}_{-m} \text{ acting on } |0\rangle, n, m > 0\} . \quad (2.79)$$

From

$$\langle 0 | L_2 L_{-2} | 0 \rangle = \langle 0 | [L_2, L_{-2}] | 0 \rangle = \frac{c}{2} \quad (2.80)$$

after some computations, one obtains that the central charge for a free boson is $c = 1$.

2.5.2 The free fermion

Let us consider the system of two hermitian anticommuting fields $\psi(z, \bar{z})$, $\bar{\psi}(z, \bar{z})$ with the following action

$$S = \frac{1}{4\pi} \int dz d\bar{z} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) . \quad (2.81)$$

The equations of motion

$$\partial \bar{\psi} = 0 \quad \bar{\partial} \psi = 0 \quad (2.82)$$

tell us that ψ is a holomorphic field while $\bar{\psi}$ is antiholomorphic. It is easy to show that conformal invariance of the action implies that they must have conformal dimension $\frac{1}{2}$. There are two different sectors corresponding to different behaviours under a 2π rotation on the complex plane

$$\begin{aligned} \psi(e^{2\pi i} z) &= \psi(z) & \text{Neveu-Schwarz sector (NS) ,} \\ \psi(e^{2\pi i} z) &= -\psi(z) & \text{Ramond sector (R) .} \end{aligned} \quad (2.83)$$

These behaviours lead to different mode expansions

$$\psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}} \quad \begin{cases} r \in \mathbb{Z} + \frac{1}{2} & \text{Neveu-Schwarz sector} \\ r \in \mathbb{Z} & \text{Ramond sector} \end{cases} \quad (2.84)$$

and analogously for the antiholomorphic field.

We quantize our theory by promoting ψ and $\bar{\psi}$ to operators. For brevity we start again from the given formulas for the OPEs

$$\begin{aligned} \psi(z)\psi(w) &= \frac{1}{z-w} + \dots , \\ \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) &= \frac{1}{\bar{z}-\bar{w}} + \dots , \\ \psi(z)\bar{\psi}(\bar{w}) &= \dots , \end{aligned} \quad (2.85)$$

where the dots stand for non-singular terms as usual. It is easy to show that these are the correct OPEs by comparing them with the two point functions that can be found in [11].

We can then compute the anticommutators between the modes

$$\begin{aligned} \{\psi_r, \psi_s\} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^r w^s \{\psi(z), \psi(w)\} = \\ &= \oint \frac{dw}{2\pi i} w^s \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} z^r \psi(z) \psi(w) - \oint_{|z|<|w|} \frac{dz}{2\pi i} z^r (-1) \psi(w) \psi(z) \right) = \\ &= \oint \frac{dw}{2\pi i} w^s \oint_{\mathbb{C}(w)} z^r \mathbf{R}(\psi(z) \psi(w)) = \\ &= \oint \frac{dw}{2\pi i} w^s \oint_{\mathbb{C}(w)} \frac{dz}{2\pi i} z^r \frac{1}{z-w} = \oint \frac{dw}{2\pi i} w^{r-1} w^s = \delta_{r+s,0} , \end{aligned} \quad (2.86)$$

where we have recalled from (2.38) that under radial ordering fermionic fields take a minus sign when they switch. Analogously

$$\begin{aligned}\{\bar{\psi}_r, \bar{\psi}_s\} &= \delta_{r+s,0} , \\ \{\psi_r, \bar{\psi}_s\} &= 0 ,\end{aligned}\tag{2.87}$$

the only difference between the R and the NS sector is given by integer or half-integer indices. The (holomorphic) stress-energy tensor is given by the limit

$$T(z) = \lim_{z \rightarrow w} \frac{1}{2} \left(-\psi(z) \partial_w \psi(w) + \frac{1}{(z-w)^2} \right) \tag{2.88}$$

and an analogous expression holds for the antiholomorphic sector. This expression, which makes the vev of $T(z)$ finite, is needed to ensure the right commutation relations of the Virasoro modes and differs from normal ordering by an additive constant. In the NS sector this just leads to the usual normal ordering

$$T(z) = \frac{1}{2} : \psi(z) \partial \psi(z) : . \tag{2.89}$$

In the Ramond sector

$$\begin{aligned}\langle \psi(z) \psi(w) \rangle &= \sum_{k,q \in \mathbb{Z}} z^{-k-\frac{1}{2}} w^{-q-\frac{1}{2}} \langle \psi_k \psi_q \rangle = \frac{1}{2\sqrt{zw}} + \sum_{k=1}^{\infty} z^{-k-\frac{1}{2}} w^{-k-\frac{1}{2}} = \\ &= \frac{1}{\sqrt{zw}} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{w}{z} \right)^k \right] = \frac{1}{2\sqrt{zw}} \frac{z+w}{z-w} = \frac{1}{2} \frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z-w} ,\end{aligned}\tag{2.90}$$

where we have used the fact $\langle \psi_k \psi_q \rangle = \delta_{n+m,0}$. In fact, observing that

$$\frac{1}{z-w} = \sum_n \frac{w^n}{z^{n+1}} = \sum_m w^{m-\frac{1}{2}} z^{-m-\frac{1}{2}} , \tag{2.91}$$

from the OPE (2.85) we have

$$\sum_{n,m} z^{-n-\frac{1}{2}} w^{-m-\frac{1}{2}} \langle \psi_n \psi_m \rangle = \sum_m w^{m-\frac{1}{2}} z^{-m-\frac{1}{2}} \tag{2.92}$$

which gives the desired result.

Notice that expression (2.90) is consistent with (2.85) since it gives exactly the same expression in the limit $z \rightarrow w$. It is easy to compute

$$\langle \psi(z) \partial \psi(w) \rangle = -\frac{1}{2} \frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{(z-w)^2} + \frac{1}{4} \frac{1}{w^{\frac{3}{2}} z^{\frac{1}{2}}} \tag{2.93}$$

which leads to

$$\langle T(z) \rangle = \frac{1}{16z^2} . \tag{2.94}$$

Thus, since L_0 is the mode multiplied by z^{-2} in the expansion of the stress-energy tensor, in the Ramond sector we have

$$\begin{aligned}L_n &= \frac{1}{2} : \psi \partial \psi :_n \quad n \neq 0 , \\ L_0 &= \frac{1}{2} : \psi \partial \psi :_0 + \frac{1}{16} .\end{aligned}\tag{2.95}$$

The NS sector space of states is built in a manner similar to the one of the free boson. Starting from the vacuum state $|0\rangle$, which is again the state annihilated by all the non-negative modes ψ_r and $\bar{\psi}_s$, we can obtain the space of states by state-field correspondence

$$\mathcal{H}_{NS} = \{ \text{Fock space freely generated by } \psi_{-r}, \bar{\psi}_{-s} \text{ acting on } |0\rangle, r, s > 0 \} . \tag{2.96}$$

The space of states of the Ramond sector is slightly more complicated because of the presence of zero modes ψ_0 and $\bar{\psi}_0$ which do not change the eigenvalue of L_0 : there is a degenerate vacuum. In particular, if one allows the existence of both chiral and antichiral conserved fermion numbers, there are four ground states and we label them with

$$|0\rangle_{L\pm} \otimes |0\rangle_{R\pm} , \quad (2.97)$$

where L and R stand for the chiral sector (corresponding to ψ) and the antichiral sector (corresponding to $\bar{\psi}$) respectively. See [13] for a more detailed discussion. State-field correspondence will work as before but now the fields can be applied to each vacuum separately. So

$$\mathcal{H}_{\mathcal{R}} = \{ \text{Fock space freely generated by } \psi_{-r}, \bar{\psi}_{-s} \text{ acting on } |0\rangle_{L\pm} \otimes |0\rangle_{R\pm}, r, s > 0 \} . \quad (2.98)$$

2.5.3 Kac-Moody algebras

A lot of conformal field theories have chiral algebras that are bigger than just the Virasoro algebra. One example which is particularly interesting is given by theories whose chiral algebra contains a Kac-Moody algebras. We will see in section 3 that, for example, $N = 4$ superconformal field theories possess a chiral algebra containing a Kac-Moody algebra.

We have seen that for a free boson the modes of the current $j = i\partial X(z, \bar{z})$ satisfy the commutation relations

$$[j_n, j_m] = n\delta_{n+m,0} , \quad (2.99)$$

we want now to consider more general systems. An obvious generalization is to consider a system made by currents, i.e. primary fields of conformal dimension $h = 1$, whose components generate, via the commutation relations, a Kac-Moody algebra. Given a finite-dimensional, semi-simple Lie algebra \mathfrak{g} generated by currents $j^i, i = 1, \dots, \dim \mathfrak{g}$, with commutation relations

$$[j^a, j^b] = if^{abc}j^c , \quad (2.100)$$

the Kac-Moody algebra $\hat{\mathfrak{g}}_k$ of level k is defined by the commutation relations

$$[j_m^a, j_m^b] = if^{abc}j_c + km\delta^{ab}\delta_{m+n,0} , \quad (2.101)$$

we have used Einstein convention in which repeated indices are summed over.

We will then consider the system made up by currents with the following OPE

$$j^a(z)j^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w}j^c(w) . \quad (2.102)$$

The commutation relations of the modes of these currents give exactly (2.101), in fact

$$\begin{aligned} [j_n^a, j_m^b] &= \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} z^n w^m [j^a(z), j^b(w)] = \\ &= \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} z^n w^m \left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w}j^c(w) \right) = \\ &= \oint \frac{dw}{2\pi i} w^{n+m} if^{abc}j^c(z) + \oint \frac{dw}{2\pi i} m z^{n+m-1} k\delta^{ab} = if^{abc}j_{n+m}^c(z) + km\delta^{ab}\delta_{n+m,0} . \end{aligned} \quad (2.103)$$

We want to build a stress-energy tensor, whose modes satisfy the Virasoro algebra, such that the currents $j^i(z)$ are primary fields of conformal dimension $h = 1$. The general construction

is called the Sugawara construction (see [12]). Guided by analogy with the free boson, we will just use the ansatz

$$T(z) = C : j^a(z)j^a(z) : . \quad (2.104)$$

Using the Kac-Moody algebra commutation relations, one easily obtains

$$[L_m, j_n^a] = -2Cn(k + C_{\mathfrak{g}})j_{n+m}^a , \quad (2.105)$$

where $C_{\mathfrak{g}}$ is the dual Coxeter number given by

$$f^{bac}f^{bcd} = -2C_{\mathfrak{g}}\delta^{ab} . \quad (2.106)$$

In order for the currents to be primary fields of conformal dimension $h = 1$ we have to choose the normalization of C such that

$$T(z) = \frac{1}{2(k + C_{\mathfrak{g}})} : j^a(z)j^a(z) : , \quad (2.107)$$

with some computations one can show that the modes of this form of the stress-energy tensor also satisfy the Virasoro algebra commutation relations.

The same construction can be repeated analogously with antiholomorphic currents $\vec{j}^i(\bar{z})$ of conformal dimension $\bar{h} = 1$.

It is useful to notice that, from (2.101), the zero modes form a subalgebra isomorphic to \mathfrak{g} . So in particular, systems which possess a chiral algebra containing a Kac-Moody algebra $\hat{\mathfrak{g}}$ admit the group with algebra \mathfrak{g} , generated by the zero modes, as a symmetry group.

2.6 Conformal field theory with toroidal worldsheets

In the previous sections we have discussed conformal field theory defined on the complex plane. For many applications in string theory, it is useful to consider theories defined on the torus. We will make use of what we have defined up to now for conformal field theories on the complex plane since a torus can be defined identifying the points of the complex plane \mathbb{C} .

Given two complex numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ we define an equivalence relation \sim through

$$z \sim z + m\alpha_1 + n\alpha_2 , \quad (2.108)$$

then our torus will simply be

$$\mathbb{T} = \mathbb{C} / \sim . \quad (2.109)$$

The shape of the torus is described by the modular parameter $\tau := \frac{\alpha_1}{\alpha_2}$. However different choice for α_1, α_2 can parametrize the same torus. To see this let us consider the lattice spanned by (α_1, α_2) , the torus is given by identifying opposite “sides” of this lattice. If $\beta_1, \beta_2 \in \mathbb{C}$ span the same lattice then

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \frac{SL(2, \mathbb{Z})}{\mathbb{Z}_2} \quad (2.110)$$

must hold since when inverting (2.110) we want the inverse matrix to have integer entries and $(-\alpha_1, -\alpha_2)$ generates the same lattice as (α_1, α_2) . These matrices form the so called modular group of the torus. We can always choose, through a conformal transformation, $(\alpha_1, \alpha_2) = (1, \tau)$. The action of the modular group on the modular parameter τ is given by

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \frac{SL(2, \mathbb{Z})}{\mathbb{Z}_2} . \quad (2.111)$$

The torus is clearly left invariant by a modular transformation. It can be shown, but it is non-trivial to prove, that the generators of the modular group are the so called S and T transformations

$$\begin{aligned} S : \tau &\rightarrow -\frac{1}{\tau} , \\ T : \tau &\rightarrow \tau + 1 . \end{aligned} \quad (2.112)$$

In the definition of our fields we are now free to chose periodicity conditions with respect to the two cycles of the torus, i.e. $\psi(z + n\alpha_1 + m\alpha_2) = e^{2\pi i(nu+mu)}\psi(z)$ with $u, v \in \{0, \frac{1}{2}\}$.

The partition function of the conformal field theory on the torus is given by

$$Z(\tau) = Tr_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) . \quad (2.113)$$

For consistency, the partition function of a conformal field theory defined on the torus must be modular invariant since we want conformal field theories built on the same torus with equivalent modular parameters to be the same.

2.7 Conformal field theory on orbifolds

In this section we discuss theories in which the scalar fields map the world-sheet into an orbifold

$$O = M/G \quad (2.114)$$

i.e. a manifold M in which we identify points which belong to the same orbit of an action of a discrete group G . This is a particular type of non-linear sigma model, they will be discussed in generality in section 4. We will follow the approach of [14]. The methods of this section will be used when we will compute the characters of the orbifold $\mathbb{C}^2/\mathbb{Z}_3$. Orbifolds can be thought as manifolds with “singularities”, which correspond to the fixed points of the group actions. A very interesting fact is that strings behave well when encountering these singularities and thus string theories with an orbifold target space are well-defined.

The easiest way to build the space of states of an orbifold conformal field theory is to start from the space of states of the parent theory, i.e. the theory with target space M , and to restrict to the states that are left invariant by the action of the group. However this is not the full space of the orbifold theory. In fact to each fixed point one must add $|G| - 1$ twisted sectors \mathcal{H}_h , $h \in G$, which correspond to the fields that satisfy

$$\Phi(e^{2\pi i}(z)) = \pm h \Phi(z) , \quad (2.115)$$

where the sign \pm depends on the boundary condition, Ramond or Neveu-Schwarz, if Φ is a fermionic field. We are denoting with h the action of $h \in G$ on the fields. Each twisted sector is obtained by state-field correspondence acting with twisted fields on the vacuum. Analogously, it is possible to define a twisted ground state $|\tau_h\rangle \in \mathcal{H}_h$ which is obtained acting on the ground state of the untwisted sector with an opportune twist field $|\tau_h\rangle = \tau_h(z, \bar{z}) |0\rangle$ and all the state of the twisted sector will be obtained acting with untwisted fields on this ground state. Adding the twisted sectors can be interpreted as introducing a cut in the world-sheet torus from 0 to $2\pi \sim 0$ such that (2.115) holds. The need to include this sector is evident when considering the partition function of a CFT with a toroidal world-sheet. We use the notation

$$g \square_h := Tr_{\mathcal{H}_h} \left(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) \quad (2.116)$$

that makes clear which boundary condition we are assuming with respect to the two cycles of the torus. In the untwisted sector, the projection operator into G invariant states is

$$P = \frac{1}{|G|} \sum_{g \in G} g , \quad (2.117)$$

thus the partition function restricted to G invariant states is

$$Z(\tau, z) = \frac{1}{|G|} \sum_{g \in G} g \square_{\mathbb{1}}. \quad (2.118)$$

This partition function is not modular invariant. In fact, under a S transformation $\tau \rightarrow -\frac{1}{\tau}$

$$g \square_{\mathbb{1}} \rightarrow \mathbb{1} \square_g \quad (2.119)$$

because a S transformation exchange the torus whose lattice is spanned by (α_1, α_2) with the one whose lattice is spanned by $(-\alpha_2, \alpha_1)$.

To have a modular invariant partition functions is thus necessary to include also the twisted sectors. Notice that from (2.115)

$$g\Phi(e^{2\pi i} z) = (ghg^{-1})g\Phi(z) \quad (2.120)$$

so $\mathcal{H}_h \cong \mathcal{H}_{ghg^{-1}}$. Since we are identifying $|\phi\rangle \in \mathcal{H}_h$ and $|g\phi\rangle \in \mathcal{H}_{ghg^{-1}}$ to avoid overcounting of states if the group is non-abelian, the projection operator in the h twisted sector must be restricted to commuting elements

$$P_h = \frac{1}{|G|} \sum_{\substack{g \in G \\ hg=gh}} g. \quad (2.121)$$

The total partition function including the twisted sectors is then

$$Z(\tau, z) = \frac{1}{|G|} \sum_{\substack{g, h \in G \\ hg=gh}} g \square_h. \quad (2.122)$$

Since a S transformation, $\tau \rightarrow -\frac{1}{\tau}$, sends

$$g \square_h \rightarrow h^{-1} \square_g, \quad (2.123)$$

while a T transformation, $\tau \rightarrow \tau + 1$, sends

$$g \square_h \rightarrow hg \square_h, \quad (2.124)$$

using the fact that S and T transformations generate the modular group, it is an easy calculation to show that (2.122) is modular invariant. In fact, under a general modular transformation $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ we will have

$$g \square_h \rightarrow g^d h^b \square_{g^c h^a}. \quad (2.125)$$

3. Superconformal field theory

In this section we will discuss the basic aspects of superconformal field theory. We will discuss $N = 1, 2, 4$ superconformal algebras and we will introduce some concept that will be important in the following sections such as the elliptic genus and spectral flow. We will also give the formulas of the characters of the $N = 4$ superconformal algebra and give a free field realization of it.

Supersymmetry (for an introduction see [15]) is a symmetry relating bosons and fermions. Roughly speaking, it associates to every boson in the theory a certain number of fermions. Its algebraic structure is realized by anticommutators and, by Coleman and Mandula theorem, supersymmetry is the only non-trivial extension of the Poincaré algebra of a quantum field theory with “realistic” properties. Unfortunately there are no experimental evidences that supersymmetry is realized in nature so far.

The importance of supersymmetry in string theory is related to the fact that supersymmetric string theories do not contain tachyons in their spectrum, we will discuss this in more detail in section 4. Since, as we will see in section 4, the perturbative aspects of string theory are intimately related to conformal field theories, we are interested to study the main properties of theories which are both supersymmetric and conformal. These are known as superconformal field theories and come in different kinds depending on how many supercurrents are present. The number of supercharges is generally denoted with N . We will expose the main aspects of $N = 1, 2, 4$ superconformal field theories which are the most interesting cases to consider for compactification.

3.1 Superconformal algebras

We will enlist here the commutation relations between the generator of $N = 1, 2, 4$ superconformal algebras.

3.1.1 $N = 1$

The $N = 1$ superconformal algebra is generated by the modes of the stress-energy tensor $T(z)$, and of a supercurrent $G(z) = \sum_r G_r z^{r-\frac{3}{2}}$, i.e. a holomorphic hermitian ($G(\bar{z}^\dagger) = G(z)$) field of conformal dimension $\frac{3}{2}$. They satisfy

$$\begin{aligned} [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} , \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3} \left(r^2 - \frac{1}{4}\right) \delta_{r+s,0} . \end{aligned} \tag{3.1}$$

These commutation relations, together with the ones of the Virasoro algebra (2.48), generate a $N = 1$ superconformal algebra.

3.1.2 $N = 2$

A $N = 2$ superconformal algebra is an extension of the $N = 1$ superconformal algebra in which the supercurrent $G(z)$ splits into $G(z) = \frac{1}{\sqrt{2}}(G^+ + G^-)$ such that $(G^+(\bar{z}))^\dagger = G^-(z)$ and their modes satisfy

$$\begin{aligned} [L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right) G_{m+r}^\pm, & [J_m, G_r^\pm] &= \pm G_{m+r}^\pm, \\ \{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, & [L_m, J_n] &= -mJ_{m+n}, \\ \{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0, & [J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0}. \end{aligned} \quad (3.2)$$

We call Q the charge associated with the $U(1)$ current J .

3.1.3 $N = 4$

A $N = 4$ Superconformal algebra, which can only occur when $c = 6k$, is an extension of the $N = 2$ superconformal algebra with 3 currents generating a $\hat{\mathfrak{su}}(2)_k$ Kac-Moody algebra

$$\begin{aligned} [L_m, J_n^3] &= -nJ_{m+n}^3, & [L_m, J_n^\pm] &= -nJ_{m+n}^\pm, \\ [2J_m^3, 2J_n^3] &= 2km\delta_{m+n,0}, & [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm, \\ [J_m^+, J_n^-] &= km\delta_{m+n,0} + 2J_{m+n}^3, & [J_m^\pm, J_n^\pm] &= 0 \end{aligned} \quad (3.3)$$

and 4 supercurrent G^\pm, G'^\pm such that

$$\begin{aligned} \{G_r^+, G_s^-\} &= \{G_r'^+, G_s'^-\} = 2L_{r+s} \pm 2(r-s)J_{r+s}^3 + 2k\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\ \{G_r^\pm, G_s'^\mp\} &= 2(s-r)J_{r+s}^\pm, & \{G_r^\pm, G_s'^\pm\} &= 0, \\ [L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right) G_{m+r}^\pm, & [L_m, G_r'^\pm] &= \left(\frac{m}{2} - r\right) G_{m+r}'^\pm, \\ [J_m^3, G_r^\pm] &= \pm \frac{1}{2} G_{m+r}^\pm, & [J_m^3, G_r'^\pm] &= \mp \frac{1}{2} G_{m+r}'^\pm, \\ [J_m^\pm, G_r^\mp] &= \pm G_{m+r}'^\mp, & [J_m^\pm, G_r^\pm] &= 0, \\ [J_m^\pm, G_r'^\pm] &= \mp G_{m+r}^\pm, & [J_m^\pm, G_r'^\mp] &= 0. \end{aligned} \quad (3.4)$$

In the following we will mostly interested in the case $k = 1$, i.e. $c = 6$. Comparing (3.4) with (3.2) it is easy to notice that the $N = 4$ superconformal algebra contains a $N = 2$ superconformal algebra with current $J = 2J^3$. In particular, the charges measured with respect to the current of the $N = 2$ superconformal algebra are twice the charges measured with respect to J^3 . This has to be carefully taken into account when using the formulas of the following sections. To avoid confusion we will generally denote the charge of J_3 with l .

3.2 The space of states

The space of states of a superconformal field theory contains both bosons and fermions. Thus the representations R, \bar{R} of the chiral algebras $\mathcal{W}, \bar{\mathcal{W}}$ of a superconformal field theory are $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded:

$$\begin{aligned} R &= R^b \oplus R^f, \\ R^i &= R_{NS}^i \oplus R_R^i \quad i = \{b, f\}, \\ \bar{R} &= \bar{R}^b \oplus \bar{R}^f, \\ \bar{R}^i &= \bar{R}_{NS}^i \oplus \bar{R}_R^i \quad i = \{b, f\}. \end{aligned} \quad (3.5)$$

Where the states in R^b and \bar{R}^b are bosonic, while the states in R^f and \bar{R}^f are fermionic and the subscripts stand for the Ramond (R) or Neveu-Schwarz (NS) sector. They correspond to

the periodicity condition chosen for the fields associated to fermionic states

$$\begin{aligned} \psi(e^{2\pi i} z) &= \psi(z) & \text{if } |\psi\rangle \in R_{NS}^f, \\ \psi(e^{2\pi i} z) &= -\psi(z) & \text{if } |\psi\rangle \in R_R^f \end{aligned} \quad (3.6)$$

and analogously for antiholomorphic fields.

The full space then decompose as

$$\mathcal{H} = \mathcal{H}_{NS\ NS} \oplus \mathcal{H}_{NS\ R} \oplus \mathcal{H}_{R\ NS} \oplus \mathcal{H}_{R\ R} \quad (3.7)$$

with

$$\begin{aligned} \mathcal{H}_{NS\ NS} &= \bigoplus_{i,j} R_{NS}^i \otimes \bar{R}_{NS}^j, & \mathcal{H}_{NS\ R} &= \bigoplus_{i,j} R_{NS}^i \otimes \bar{R}_R^j, \\ \mathcal{H}_{R\ NS} &= \bigoplus_{i,j} R_R^i \otimes \bar{R}_{NS}^j, & \mathcal{H}_{R\ R} &= \bigoplus_{i,j} R_R^i \otimes \bar{R}_R^j. \end{aligned} \quad (3.8)$$

For later convenience, we define the holomorphic fermion number operator $(-1)^F$ to be the unitary operator with eigenvalues ± 1 which commutes with holomorphic bosonic fields and all antiholomorphic fields, it anticommutes with holomorphic fermionic fields and it is such that $(-1)^F |0\rangle = |0\rangle$. Exchanging holomorphic and antiholomorphic in the previous definition one obtains the antiholomorphic fermion number operator $(-1)^{\bar{F}}$. The operator F count the number of fermionic state in the holomorphic sector, while \bar{F} count the number of fermionic states in the antiholomorphic sector.

Each chiral algebra of a superconformal field theory is a representation of a superconformal algebra. We will say a theory is (N, \bar{N}) supersymmetric if the (anti)holomorphic sector is the space of a representation of a N (\bar{N} , respectively) superconformal algebra, where if N (\bar{N}) is zero it means that the corresponding sector is non-supersymmetric. We are interested in theories which possess the same supersymmetry in both sectors, and in particular in theories that are at least $N = (2, 2)$ supersymmetric.

In the following we will extensively work with highest weight representations of the $N = (2, 2)$ superconformal algebra. A representation with highest weight state $|\Phi\rangle$, i.e. an eigenstate of G_0 and L_0 annihilated by all the positive modes G_n, L_n for $n > 0$, of conformal dimension h and charge Q is built as a Fock space acting on $|\Phi\rangle$ with creation operators L_{-m}, G_{-r} for $m, r > 0$ on $|\Phi\rangle$. Unitarity imposes constraints on the values of (h, Q) of a primary state $|\Phi\rangle$, in fact in the NS sector

$$\langle \Phi | \{G_{\pm\frac{1}{2}}^+, G_{\mp\frac{1}{2}}^-\} | \Phi \rangle = \langle \Phi | 2L_0 \mp J_0 | \Phi \rangle = 2h \pm Q, \quad (3.9)$$

but, since $(G^+(\bar{z}))^\dagger = G^-(z)$ implies $(G_r^+)^\dagger = G_{-r}^-$, we have

$$\langle \Phi | \{G_{\pm\frac{1}{2}}^+, G_{\mp\frac{1}{2}}^-\} | \Phi \rangle = \|G_{\mp\frac{1}{2}}^- | \Phi \rangle\|^2 + \|G_{\pm\frac{1}{2}}^+ | \Phi \rangle\|^2 \geq 0, \quad (3.10)$$

so $h \geq \frac{|Q|}{2}$. Representations with $h = \pm \frac{Q}{2}$ are called massless, otherwise they are called massive. Since we will need them in section 5, we now give character formulas of the irreducible representations for the $N = (4, 4)$ superconformal algebra. We consider the unitary highest weight representations of the $N = 4$ superconformal algebra. They are built on states which are labelled by their conformal weight h and charge $l = \frac{1}{2}Q$. Their construction is the same of the $N = 2$ highest weight representations, the only difficulty is that the unitary bound is more complicated in the $N = 4$ case. We will work only with holomorphic characters in the Ramond sector. Given a representation on the vector space $V \subset \mathcal{H}_{RR}$ its character is given by

$$ch^V(\tau, z) := Tr_V (-1)^F q^{L_0 - \frac{c}{24}} y^{J_0}. \quad (3.11)$$

The characters of the highest weight representations of the $N = (4, 4)$ superconformal algebra can be found in [5], we report them here since they will be used in what follows

$$\begin{aligned} ch_{h, l=\frac{1}{2}}^{N=4}(\tau, z) &= q^{h-\frac{3}{8}} \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} , \\ ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, z) &= \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \mu(\tau, z) , \end{aligned} \quad (3.12)$$

where

$$\mu(\tau, z) := \frac{-ie^{\pi iz}}{\theta_1(\tau, z)} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi i n z}}{1 - q^n e^{2\pi i z}} . \quad (3.13)$$

In the previous formulas there is a little abuse of notation since the form of character with $(h = \frac{1}{4}, l = \frac{1}{2})$ is not the one for a single representation but rather it involves a sum of them. The correct formula would be

$$ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) = q^{h-\frac{3}{8}} \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} - 2 \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \mu(\tau, z) . \quad (3.14)$$

When evaluated at $z = 0$, the true characters take the following values

$$\begin{aligned} ch_{h=\frac{1}{4}+n, l=\frac{1}{2}}^{N=4}(\tau, 0) &= 0 , \\ ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, 0) &= 1 , \\ ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, 0) &= -2 . \end{aligned} \quad (3.15)$$

In particular the massive character in the limit $n \rightarrow 0$ is 0 when evaluated at $z = 0$, while the true massless character takes value -2 at $z = 0$.

We will mostly use the notation with abuses in the following except when stated otherwise.

We will also present here a free fields realization of a $N = (4, 4)$ superconformal field theory, it will be used in section 6.2. A $N = 4$ superconformal algebra can be realized with 4 free boson currents $j_i = \partial X_i$ and 4 free fermions ψ_i , $i = 1, \dots, 4$. The fields that realize (3.4) are given by

$$\begin{aligned} J^3 &= \frac{1}{2} \left(: \psi_+^{(1)} \psi_-^{(1)} : + : \psi_+^{(2)} \psi_-^{(2)} : \right) , & J^\pm &= \pm : \psi_\pm^{(1)} \psi_\pm^{(2)} : , \\ G^\pm &= \sqrt{2} \left(: \psi_\pm^{(1)} j_\mp^{(1)} : + : \psi_\pm^{(2)} j_\mp^{(2)} : \right) , & G'^\pm &= \sqrt{2} \left(: \psi_\mp^{(1)} j_\mp^{(2)} : - : \psi_\mp^{(1)} j_\mp^{(1)} : \right) , \\ T &= : j_+^{(1)} j_-^{(1)} : + : j_+^{(2)} j_-^{(2)} : + \frac{1}{2} \left(: \partial \psi_+^{(1)} \psi_-^{(1)} : + : \partial \psi_-^{(1)} \psi_+^{(1)} : + : \partial \psi_+^{(2)} \psi_-^{(2)} : + : \partial \psi_-^{(2)} \psi_+^{(2)} : \right) , \end{aligned} \quad (3.16)$$

where we have introduced the complex fields

$$\begin{aligned} \psi_\pm^1 &= \frac{1}{\sqrt{2}} (\psi_1 \pm i\psi_2) , & \psi_\pm^2 &= \frac{1}{\sqrt{2}} (\psi_3 \pm i\psi_4) , \\ j_\pm^1 &= \frac{1}{\sqrt{2}} (j_1 \pm ij_2) , & j_\pm^2 &= \frac{1}{\sqrt{2}} (j_3 \pm ij_4) . \end{aligned} \quad (3.17)$$

3.2.1 Spectral flow

We will now describe an important feature of theories which are at least $N = 2$ superconformal, the spectral flow. It consists of a continuous deformation of the $N = 2$ superconformal algebra generators which leave the commutation relations (3.2) unchanged.

We can describe a one parameter family of $N = (2, 2)$ superconformal algebras, if we allow $r \in \mathbb{R}$ for the modes of G_r^\pm , by sending $(L_n, J_n, G_r^\pm) \rightarrow (L_n^\theta, J_n^\theta, G_{r^\theta}^{\pm})$ with

$$L_n^\theta := L_n + \theta J_n + \frac{c}{6} \theta^2 \delta_{n,0} , \quad J_n^\theta := J_n + \frac{c}{3} \theta \delta_{n,0} , \quad G_{r^\theta}^{\pm} := G_{(r \pm \theta)}^\pm \quad \theta \in \left[-\frac{1}{2}, \frac{1}{2}\right) . \quad (3.18)$$

It is easy to show that these modes satisfy again (3.2). $\theta = -\frac{1}{2}$ gives the representation of the $N = (2, 2)$ superconformal algebra in the Ramond sector. There the constraint due to unitarity,

given by the expectation value of $\{G_0^+, G_0^-\}$, is $h \geq \frac{c}{24}$.

The above transformation is generated by a $U(1)$ operator $U_\theta =: e^{\theta J}$;, when $\theta \in \{-\frac{1}{2}, \frac{1}{2}\}$ it is called spectral flow operator. It is easy to show that the action of a (left-right symmetric) spectral flow on a state with quantum numbers $(h, Q; \bar{h}, \bar{Q})$ is given by

$$(h, Q; \bar{h}, \bar{Q}) \xrightarrow{U_{\pm\frac{1}{2}} \bar{U}_{\pm\frac{1}{2}}} \left(h \pm \frac{Q}{2} + \frac{c}{24}, Q \pm \frac{c}{6}; \bar{h} \pm \frac{\bar{Q}}{2} + \frac{c}{24}, \bar{Q} \pm \frac{c}{6} \right). \quad (3.19)$$

3.2.2 The elliptic genus

We will now introduce the elliptic genus, which will play a crucial role in the following part of this work. For more informations see [16] and [17].

We will give the conformal field theory definition of the elliptic genus. There is also a geometric definition for Calabi-Yau manifolds but we will not worry about that, we just state without proof that the two coincide for non-linear sigma models on Calabi-Yau manifolds.

The (conformal field theoretic) elliptic genus of a unitary $N = (2, 2)$ superconformal field theory (on the torus) is given by:

$$\phi(\tau, z) := \text{Tr}_{\mathcal{H}_{RR}} \left[(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0} (-1)^{\bar{F}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right], \quad (3.20)$$

where $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.

We will now state, without proof, some properties of the elliptic genus.

The elliptic genus is independent of \bar{q} and it is a weak Jacobi form of index $m = \frac{c}{6}$ and weight $w = 0$, i.e. it transforms as:

$$\begin{aligned} \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^w e^{2\pi i m \frac{cz^2}{c\tau+d}} \phi(\tau, z) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \\ \phi(\tau, z + l\tau + l') &= e^{2\pi i m(l^2\tau + 2lz)} \phi(\tau, z) \quad l, l' \in \mathbb{Z} \end{aligned} \quad (3.21)$$

and has the following Fourier expansion:

$$\phi(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} c(n, l) q^n y^l$$

with $c(n, l) = (-1)^w c(n, -l)$. The transformation properties of the elliptic genus can be explicitly verified using the path integral formulation, see for example [18].

The most important property is perhaps the fact that the elliptic genus is constant on each connected component of the moduli space of a $N = (2, 2)$ superconformal field theory.

The elliptic genus for a $K3$ surface was computed in [19] and equals to (see appendix A for the definition of the Jacobi theta functions)

$$\phi_{K3}(\tau, z) = 8 \left[\left(\frac{\theta_2(\tau, z)}{\theta_2(\tau, 0)} \right)^2 + \left(\frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 + \left(\frac{\theta_4(\tau, z)}{\theta_4(\tau, 0)} \right)^2 \right]. \quad (3.22)$$

If the conformal field theory has a finite symmetry group G which commutes with the superconformal symmetry, it is useful to introduce also the twining genus associated to $g \in G$

$$\phi_g(\tau, z) := \text{Tr}_{\mathcal{H}_{RR}} \left[g(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0} (-1)^{\bar{F}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right]. \quad (3.23)$$

The twining genera are conjectured to transform as weak Jacobi forms of index 1 and weight 0, possibly up to a phase, under the group

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 \pmod{N}, d \equiv 1 \pmod{N} \right\}, \quad (3.24)$$

where N is the order of g .

4. Non-linear sigma models and string theory

Now that we have discussed the basic properties of 2-dimensional conformal field theories, we can discuss the main features of their application in string theory. We will not give a pedagogical introduction to string theory, see [1] and [2] for a good introduction to strings and superstrings. We will just try to condense the basic results found there in this section to give an idea of the setting in which this work takes place.

4.1 Bosonic strings

The underlying idea that defines string theory is to consider, as the fundamental elements of the theory, not point-particles but 1-dimensional objects called strings. It is well known that parametrizing the space-time coordinates of a point particle $X^\mu = X^\mu(\tau)$ the locus of the particle's coordinates will be a line called the world-line. A 1-dimensional object, instead, will sweep out a 2-dimensional surface M , sometimes called the world-sheet. We will not use this terminology to avoid confusion with the space of the parameters referred with the same name. In fact we can parametrize the 2-dimensional surface swept out by the strings with two parameters, generally denoted with (τ, σ) , and this is what we will call the world-sheet. We will call the space in which the strings “live” the target space. As in the point-particle case, it is important that the physical properties of our theory, and in particular the action, do not depend on the parametrization. Furthermore, we know that the point-particle action is proportional to the proper time along the world-line. So the simplest Lorentz invariant action with these properties we can write for the strings is proportional to the area of the surface they sweep, the so-called Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\tau d\sigma \sqrt{-h} , \quad (4.1)$$

where α' is a constant with the dimension of a length squared and it related to the magnitude of the string length (or, equivalently, energy). At energy scales much lower than α' the string will behave as a point particle and its behaviour is well approximated by an ordinary quantum field theory. We have denoted with h the determinant of the induced metric $h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$ where $g_{\mu\nu}$ is the target space metric. It is in general more useful to work with the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu , \quad (4.2)$$

we will use the convention that greek indices are always contracted with the target space metric. Here the worldsheet metric $\gamma_{ab} = \gamma_{ab}(\tau, \sigma)$ is a field as the X s. We have denoted with γ its determinant and with γ^{ab} the components of its inverse. It is a simple calculation to show

that when the equation of motion for γ are satisfied the Polyakov action coincides with the Nambu-Goto action. Polyakov action exhibit a new symmetry not present in the Nambu-Goto action, 2-dimensional Weyl invariance

$$\begin{aligned} X'^{\mu}(\tau, \sigma) &= X^{\mu}(\tau, \sigma) , \\ \gamma'_{ab}(\tau, \sigma) &= e^{2\omega(\tau, \sigma)} \gamma_{ab}(\tau, \sigma) , \end{aligned} \quad (4.3)$$

where $\omega(\sigma, \tau)$ is an arbitrary real local parameter.

Defining the stress-energy tensor

$$T^{ab}(\tau, \sigma) = -4\pi \sqrt{-y} \frac{\delta}{\delta \gamma_{ab}} S_P , \quad (4.4)$$

it is easy to show that Weyl invariance implies it is traceless $T_a^a = 0$. To work with the Polyakov action, in particular with the path integral formulation, it is necessary to choose a gauge fixing of the metric with respect to diffeomorphism \times Weyl invariance of the Polyakov action. It is generally useful to work with flat metrics, in 2 dimension is enough to choose Ricci-flat metrics. However there is a residual gauge: working with complex coordinates $z = \tau + i\sigma$ let us consider a holomorphic coordinate transformation

$$z' = f(z) \quad (4.5)$$

with $f(z)$ a holomorphic function, combined with a Weyl transformation. The gauge-fixed metric transforms as

$$ds'^2 = e^{2\omega} |\partial_z f|^{-2} dz' d\bar{z}' . \quad (4.6)$$

In particular, for $\omega = \ln |\partial_z f|$ the metric is invariant. Thus, after the gauge fixing, the action is invariant under the residual gauge of holomorphic transformation of the complex coordinates. Since we have seen in section 2 that the holomorphic functions make up the local conformal group, after gauge fixing, all we have left to deal with is a 2-dimensional conformal field theory on the world-sheet. Thus the perturbative quantum aspects of string theory reduce to the study of an opportune 2-dimensional conformal field theory. This makes connections with what we have seen in the previous sections of this work. Notice that if we fix $\gamma_{ab} = \delta_{ab}$ Polyakov action coincides with the free boson action of section 2.69.

4.1.1 Bosonic closed string spectrum

We will now discuss briefly the spectrum of bosonic closed strings, for more details see [1]. Notice that the worldsheet of a closed string is a cylinder giving thus a natural interpretation to the step that led to radial quantization in section 2. We gauge-fix the worldsheet metric to be the flat Minkowsky metric, by working in light-cone gauge, i.e. we make a change of coordinates with a Lorentz transformation $X^{\mu} \rightarrow X'^{\mu} = (X^+, X^-, X^2, \dots)$ where

$$X^{\pm} = \frac{1}{\sqrt{2}} (X^0 \pm X^1) , \quad (4.7)$$

we will fix the residual gauge imposing $X^+ = \tau$. Choosing opportunely the units, the equations of motion become

$$\partial_a \partial^a X^{\mu} = 0 \quad (4.8)$$

with $a = \tau, \sigma$, with the constraint (coming from the equation of motion of the worldsheet metric which now is fixed)

$$(\partial_{\tau} X + \partial_{\sigma} X)^2 = 0 . \quad (4.9)$$

For closed strings we have $X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma)$ and the general solution of the equation of motion (4.8) for the transverse coordinates is given by

$$X^i(\tau, \sigma) = x^i + \frac{p^i}{p^+} \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^i}{n} e^{-\frac{2\pi i n}{l}(\sigma + \tau)} + \frac{\tilde{\alpha}_n^i}{n} e^{-\frac{2\pi i n}{l}(\sigma - \tau)} \quad (4.10)$$

with

$$\begin{aligned} x^\mu &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma X^\mu(\tau, \sigma) , \\ p^\mu &= \int_0^{2\pi} d\sigma \Pi^\mu , \end{aligned} \quad (4.11)$$

where L is the Lagrangian associated to the Polyakov action and $\Pi^\mu = \frac{\partial L}{\partial(\partial_\tau X^\mu)}$ is the canonical momentum associated to X^μ . The independent degree of freedom of (4.10) are the transverse oscillator α_n^i , $\tilde{\alpha}_n^i$ and the longitudinal center-of-mass variables x^i , p^i , x^- , p^+ . To quantize the theory we can impose the usual equal time commutation relations of canonical quantization

$$\begin{aligned} [X^-(\sigma), \Pi^-(\sigma')] &= i\delta(\sigma - \sigma') , \\ [X^i(\sigma), \Pi^j(\sigma')] &= i\delta^{ij}\delta(\sigma - \sigma') . \end{aligned} \quad (4.12)$$

These imply, after some computations

$$\begin{aligned} [x^-, p^+] &= -i , & [x^i, p^j] &= i\delta^{ij} , \\ [\alpha_m^i, \alpha_n^j] &= m\delta^{ij}\delta_{m, -n} , & [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] &= m\delta^{ij}\delta_{m, -n} . \end{aligned} \quad (4.13)$$

The space of state is built as a Fock space acting with the transverse oscillators on a state $|0, 0, k\rangle$ which is annihilated by all the positive modes α_n^i , $\tilde{\alpha}_n^i$, for $n > 0$ and has momentum $p^\mu |0, 0, k\rangle = k^\mu |0, 0, k\rangle$. A general state will have the form

$$|N, \tilde{N}, k\rangle = \left[\prod_{i=2}^{D-1} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^i)^{N_{i_n}} (\tilde{\alpha}_{-n}^i)^{\tilde{N}_{i_n}}}{\sqrt{n^{N_{i_n}} N_{i_n}! \tilde{n}^{\tilde{N}_{i_n}} \tilde{N}_{i_n}!}} \right] |0, 0, k\rangle . \quad (4.14)$$

As usual in quantum field theory the space of state must be restricted to gauge invariant state. In particular, invariance under σ -translations implies for a physical state $N = \tilde{N}$. Using this, after a normal ordering procedure it is possible to show that the mass of physical states is (see [1])

$$m^2 = \frac{4}{\alpha'} \left(N + \frac{2-D}{24} \right) , \quad (4.15)$$

where D is the dimension of the target space. As we will point out later, to have a consistent bosonic string theory the target space must have dimension $D = 26$. In particular, for $N = \tilde{N} = 0$, the state $|0, 0, k\rangle$ has negative mass squared and it is thus a tachyon. Apart from the fact that tachyons have never been observed, their presence would make the vacuum unstable. At the next level $N = \tilde{N} = 1$, for $D = 26$, we have the massless states

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, 0, k\rangle , \quad (4.16)$$

they transform as 2-vectors under $SO(D-2)$ (notice that light-cone gauge preserves only $SO(D-2)$ out of the full $SO(D-1)$ space-rotations group in D dimensions). This is a reducible representation and it decomposes into a symmetric traceless tensor, an antisymmetric tensor and a scalar. The symmetric traceless tensor can be identified with a graviton, the antisymmetric tensor is called the Kalb-Ramond field, while the scalar is referred as the dilaton.

4.2 Superstrings

The action S_P is not the more general action we can write for string theory, in particular we see the absence of fermionic fields that we need to include if we hope to make any connection with well established physics. We have already studied an example of fermionic conformal field theories in section 2.5.2, so a simple modification to the Polyakov action (4.2) is to add to the action the fermion action (2.81). In general it is more interesting to consider supersymmetric string theory, in fact non-supersymmetric string theory have unsuitable properties like the presence of tachyons in their spectrum. We then add for each free boson a free fermion. The system built in this way possesses $N = 1$ worldsheet superconformal symmetry with the holomorphic supercurrent in the NS sector given by $G(z) = i\psi^\mu \partial X_\mu$. The elimination of the tachyons from the spectrum and the promotion of worldsheet supersymmetry into space-time supersymmetry is achieved through the so called GSO projection. Long story short, this is done by keeping in the spectrum only states with a particular eigenvalue under the chiral fermion number operators $(-1)^{F_L}$, $(-1)^{F_R}$ (they are definite in a similar way to the holomorphic and antiholomorphic fermion number operators discussed in section 3). Different consistent choices of the eigenvalues in the various sectors of the superstring spectrum defines different theories, in particular

$$\begin{array}{l} \text{Type IIA} \\ \text{Type IIB} \end{array} \quad \begin{array}{c} \text{Left sector} \\ \left\{ \begin{array}{cc} \text{NS} & + \\ \text{R} & - \end{array} \right\} \\ \left\{ \begin{array}{cc} \text{NS} & + \\ \text{R} & - \end{array} \right\} \end{array} \quad \begin{array}{c} \text{Right sector} \\ \left\{ \begin{array}{cc} \text{NS} & + \\ \text{R} & + \end{array} \right\} \\ \left\{ \begin{array}{cc} \text{NS} & + \\ \text{R} & - \end{array} \right\} \end{array} ,$$

where the Neveu-Schwarz sector and the Ramond sector depend on the periodicity conditions of the fermions and are defined as in section 3.

Up to now we have not specified the properties of the target space. As we said before, it turns out that in order to have consistent string theories the dimension of the target space cannot be arbitrary but it is fixed by the theory itself. The constraint can be derived by the requirement of the cancellation of the Weyl anomaly, in order to have Weyl invariance preserved also at the quantum level, and this fixes the central charge the conformal field theory must have. Since the central charge is related to the number of bosons and fermions present (each one is a component of field in the target space) this fixes the dimension of the target space itself. Analogously the same constraint can be derived if one uses a non-covariant quantization, for example working in light-cone gauge, and then requires that the final correlation functions are Lorentz covariant. Independently of which method one chooses at the end one obtains that the allowed dimensions for the target space are

$$\begin{aligned} D = 26 & \quad \text{bosonic strings} , \\ D = 10 & \quad \text{superstrings} . \end{aligned} \tag{4.17}$$

We will focus on the supersymmetric case. The appearance of extra dimensions apart from the usual 4 that are observed requires attention in order to not be trivially in conflict with experiments done so far. A possible solution is that the extra dimensions are very small so that their presence was not revealed by the experiments up to now. With this interpretation, the target space of superstrings will then have the form

$$X = M^4 \times Y^6 , \tag{4.18}$$

where M^4 is a 4-dimensional Minkowski space-time and Y^6 is a compact 6-dimensional Riemannian manifold. The conformal field theory corresponding to string with such a target space

will factorize into two conformal field theories, one corresponding to M^4 and one corresponding to Y^6 . One way to study (super)string theory on the latter target space is to consider a (supersymmetric) non-linear sigma model. We will now discuss briefly the main properties of nonlinear sigma models (see also [20]), for a review of the basic notions of complex geometry see appendix B .

4.3 Supersymmetric NLSM

A non-linear sigma model is a 2-dimensional conformal field theory whose scalar (i.e. $\hbar = \hbar = 0$) fields are maps from a 2 dimensional worldsheet Σ (typically a Riemann surface) to a Riemannian manifold X . X is called the target space of the non-linear sigma model.

The action for a supersymmetric non-linear sigma model is given by

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2z (G_{\mu\nu}(X) + B_{\mu\nu}(X)) \partial_z X^\mu \partial_{\bar{z}} X^\nu + G_{\mu\nu}(X) (\psi^\mu D_{\bar{z}} \psi^\nu + \bar{\psi}^\mu D_z \bar{\psi}^\nu) + \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \bar{\psi}^\rho \bar{\psi}^\sigma + \alpha' \Phi(X) R^{(2)} , \quad (4.19)$$

where the X^μ s are bosonic fields from the worldsheet Σ to X , the metric $G_{\mu\nu}$ is the pullback to the worldsheet of the metric on X through the X^μ fields, $R_{i\bar{j}k\bar{l}}$ is the pullback of the Riemann curvature on X , the Kalb-Ramond field $B_{\mu\nu}(\phi)$ is an antisymmetric closed 2-form on the target space pulled back to the worldsheet, the dilaton Φ is a scalar function of the coordinates, $R^{(2)}$ is the Ricci scalar of the worldsheet Σ and the ψ ($\bar{\psi}$ respectively) fermionic fields are sections of the (anti)holomorphic cotangent bundle $T^{*(1,0)}X$ ($T^{*(0,1)}X$ respectively). The covariant derivatives are given by

$$\begin{aligned} D_{\bar{z}} \psi^\nu &= \partial_{\bar{z}} \psi^\nu + \Gamma_{\rho\sigma}^\nu(X) \partial_{\bar{z}} X^\rho \psi^\sigma , \\ D_z \bar{\psi}^\nu &= \partial_z \bar{\psi}^\nu + \Gamma_{\rho\sigma}^\nu(X) \partial_z X^\rho \bar{\psi}^\sigma , \end{aligned} \quad (4.20)$$

where Γ is the Christoffel connection.

We want the previous action to be conformal invariant in order to obtain a consistent string theory. The simplest way to guarantee conformal invariance, although not the only one (see for example [21]), is to fix the dilaton to be constant with a saddle-point approximation and the metric to be Ricci-flat. The role of the dilaton is to fix the coupling constant of the theory. Furthermore, since in 2 dimensions the integral of the Ricci scalar is related to the Euler characteristic by

$$\chi(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} \sqrt{g} R^{(2)} d^2z \quad (4.21)$$

and for a Riemann surface Σ the Euler characteristic takes the constant value $\chi(\Sigma) = 2 - 2g$, the genus g of the worldsheet determines the order of the coupling constant one is working with.

For a fixed order in perturbation theory we can then simply work with the action

$$S = \frac{1}{4\pi} \int_{\Sigma} d^2z (G_{\mu\nu}(X) + B_{\mu\nu}(X)) \partial_z X^\mu \partial_{\bar{z}} X^\nu + G_{\mu\nu}(X) (\psi^\mu D_{\bar{z}} \psi^\nu + \bar{\psi}^\mu D_z \bar{\psi}^\nu) + \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \bar{\psi}^\rho \bar{\psi}^\sigma . \quad (4.22)$$

It is useful to stress that unless both the metric and the Kalb-Ramond field are constant, the previous action is not quadratic in the fields and then contains interactions. The coupling of these interactions depends on the curvature of the target space, in particular they will be weakly coupled for small values of the curvature while for large curvature the non-linear sigma

model is strongly coupled and thus difficult to study.

The nonlinear sigma model with action (4.22) is $N = 1$ supersymmetric. If we want to extend the $N = 1$ supersymmetry we will need to impose geometric constraints on the target space. In fact an infinitesimal supersymmetry transformation will have the form $\delta_\epsilon X^i = \bar{\epsilon} l_j^i \psi^j$ for some $l \in \Gamma(TX \oplus T^*X)$. While for $N = 1$ supersymmetry it is possible to choose a normalization such that $l_j^i = \delta_j^i$, it has been shown in [22] that for $N = 2$ supersymmetry the second l_j^i is an almost complex structure that gives the target space the structure of a Kähler manifold and for $N = 4$ there are 3 almost complex structures which give the target space the structure of a hyperkähler manifold. These results are actually independent of conformal invariance, adding the previous requirements to our theory, which we already said comprehend the Ricci-flatness of the metric, we restrict the target space to be a Calabi-Yau manifold. Actually the Ricci-flatness of the metric is true only at the lowest order in the perturbative expansion, while it could receive corrections at higher orders. However for theories which possess at least $N = 4$ supersymmetries a Ricci-flatness is an exact requirement.

Due to the complexity of 6-dimensional¹ Calabi-Yau manifolds, generally non-linear sigma models on them are too complicated to study their properties in detail. Thus a lot of effort has been put in understanding first the properties of non-linear sigma models on 4-dimensional Calabi-Yau manifolds with the hope to find ideas that could be generalized or that could shed light on the more complicated 6-dimensional case. Furthermore some 6-dimensional Calabi-Yau manifolds are obtained locally by taking the cartesian product of a 4-dimensional Calabi-Yau times a 2-dimensional one, in these cases the extension is trivial since the only 2-dimensional Calabi-Yau manifolds are the tori.

4-dimensional Calabi-Yau manifolds are simpler to study and they come into only two different kinds: the tori and $K3$ surfaces. The latter are the more interesting ones since they exhibit a lot of non-trivial features, while it is still possible to exploit $N = 4$ supersymmetry, thanks to the fact that $K3$ surfaces possess a hyperkähler structure, to simplify the treatment of the non-linear sigma model. Moreover $K3$ surfaces possess some simple orbifold limits (such as $\mathbb{T}^4/\mathbb{Z}_2$ and $\mathbb{T}^4/\mathbb{Z}_3$) in which the non-linear sigma model can be fully solved.

4.4 Moduli space of NLSM on K3

Given the definition and the basic properties of a non-linear sigma model, we will now discuss briefly the moduli space of non-linear sigma models on $K3$ surfaces. The moduli space can be seen as the space of inequivalent conformal field theories, arising from the non-linear sigma model, we can build on $K3$ surfaces. From the action (4.22) we see that the free parameters we can vary are the metric $G_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$. By changing the metric we target different geometries on $K3$ surfaces, it can be shown that the geometric moduli space of Einstein metrics on $K3$ surfaces is 58 dimensional (see [20]). We have said that the Kalb-Ramond field is a closed 2-form, actually we are interested in the cohomology class of B since any exact part will not contribute to (4.22). The dimension of the second cohomology group of a $K3$ surface is 22 (see again [20]). So in total we have 80 parameters spanning the moduli space of $K3$ surfaces.

We will now give the form of the moduli space of non-linear sigma models on $K3$ surfaces. Let $\Gamma^{a,b}$ be the unique self-dual lattice in $a + b$ dimensions with signature (a, b) (see appendix C), $O(a, b)$ be the orthogonal group on $\mathbb{R}^{a,b} \supset \Gamma^{a,b}$, with a metric having signature (a, b) , $O(\Gamma^{a,b}) < O(a, b)$ the subgroup preserving $\Gamma^{a,b}$ and $O^+(a, b)$ the index 2 subgroup of $O(a, b)$ whose maximal compact subgroup is $SO(a) \times O(b)$. It has been shown that $O^+(\Gamma^{4,20})$ is a

¹We will always refer to the real dimension of the manifold.

group of equivalence between different non-linear sigma models. It contains the diffeomorphisms between different $K3$ surfaces and translations of the Kalb-Ramond field of the type $B \rightarrow B + e$ with $e \in H^2(X, \mathbb{Z})$ (since such a translation just shifts the action by $2\pi i$). In addition to these geometric symmetries, it can also contain symmetries that do not correspond to a manifest symmetry of the non-linear sigma model action. This is the case, for example, of mirror symmetry. For a more detailed discussion see [20] and [23].

It has been shown that the moduli space of non-linear sigma models on $K3$ surfaces is given by

$$M = (SO(4) \times O(20)) \backslash O^+(4, 20) / O^+(\Gamma^{4,20}) . \quad (4.23)$$

In particular we see that $(SO(4) \times O(20)) \backslash O^+(4, 20) \subset \mathbb{R}^{4,20} \cong \Gamma^{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$ is the Grassmanian of positive four-planes, i.e. the space of positive four planes contained in $\mathbb{R}^{4,20}$. Thus choosing a point in M is equivalent to choose a four-plane in $\Gamma^{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$. The set of singular four planes, i.e. the planes orthogonal to a root $v \in \Gamma^{4,20}$ with $v^2 = -2$, correspond to the singular limits of non-linear sigma models on $K3$ surfaces.

For later purposes, since they will be interesting when we will introduce umbral moonshine in the next section, we will now discuss briefly the symmetries of non-linear sigma models on $K3$ surfaces which preserve $N = (4, 4)$ supersymmetry and the spectral flow. The classification of these symmetries was initiated in [24] and completed in [25] where singular points were included. In the non-singular case, given a plane Π corresponding to a point in M , the symmetry group G of the non-linear sigma model corresponding to Π is given by

$$G = \text{Stab}(\Pi) , \quad (4.24)$$

where we are denoting with $\text{Stab}(\Pi)$ the largest subgroup of $O(\Gamma^{4,20})$ whose action on $\Gamma^{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$ fixes Π point-wise.

For the non-singular case there is a complete classification of the $\text{Stab}(\Pi)$ s (see [24] and [26]). For some of the groups, it has been found an explicit description of the corresponding non-linear sigma model (generally as an orbifold of \mathbb{T}^4) while for many others the corresponding non-linear sigma model is not well understood. This is the case, for example, of the group $L_2(11)$ which will be considered in section 6, as part of the original work of this master thesis, in an attempt to try to better understand the properties of such models.

5. Moonshine

The term moonshine is used to denote some surprising relationships between representations of finite groups and certain modular forms, traditionally thought as part of independent branches of mathematics. Even more interesting is the fact that these phenomena have deep connections with some physical models. In fact, for example, the construction of the monstrous moonshine module can be understood considering the space of state of an opportune CFT. Furthermore, Mathieu moonshine (which turned out to be a particular case in the more general class of phenomena known as umbral moonshine) was discovered decomposing the elliptic genus of $K3$ into irreducible characters of $N = 4$ superconformal algebra. More generally, the appearance of the umbral moonshine phenomenon in the elliptic genus of $K3$ is of great interest given the role that $K3$ surfaces play in the effort to understand string compactification. Understanding the umbral moonshine phenomenon could lead to a better comprehension of non-linear sigma models on $K3$ surfaces. Conversely, a physical understanding could shed light to the mathematical structure underlying the moonshine phenomena. In fact, despite the efforts of the last few years, a clear and complete physical explanation of the appearance of moonshine in the elliptic genus of $K3$ surfaces is still missing and its discovery could give important insights and new ideas useful in the topic of string compactification.

In this section we will first describe briefly Monstrous moonshine and umbral moonshine following the approach of [27]. The exposition is thought to just discuss the main ideas behind their construction in order to give a the necessary background to understand the context in which the Master thesis work lays.

5.1 Monstrous Moonshine

Monstrous moonshine was the first and most studied example of the moonshine phenomena. The conjecture was initiated by McKay observation (extended by Thompson) that the first coefficients in the expansion of the Klein J -function

$$J(\tau) = \frac{E_4^3(\tau)}{\eta^{24}(\tau)} - 744 = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots \quad (5.1)$$

can be written as particularly simple positive linear combinations of the dimensions of irreducible representations of the Monster group \mathbb{M} , which is the largest sporadic group and has order $246 \cdot 320 \cdot 59 \cdot 76 \cdot 112 \cdot 133 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$. In particular

$$\begin{aligned} 196884 &= \mathbf{1} + \mathbf{196883} , \\ 21493760 &= \mathbf{1} + \mathbf{196883} + \mathbf{21296876} , \\ 864299970 &= 2 \cdot \mathbf{1} + 2 \cdot \mathbf{196883} + \mathbf{21296876} + \mathbf{842609326} , \\ &\dots \end{aligned} \quad (5.2)$$

The importance of the Klein J -function lays in the fact that it is the unique (up to modular transformations) generator of modular functions. Modular functions are meromorphic functions

$f : \mathbb{H} \rightarrow \mathbb{C}$ which grow, as $\tau \rightarrow \infty$, like $e^{2\pi i \tau m}$ and such that $f(\gamma\tau) = f(\tau)$ under a modular transformation

$$\gamma : \tau \rightarrow \frac{a\tau+b}{c\tau+d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) . \quad (5.3)$$

We are denoting with \mathbb{H} the complex upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$. We more generally consider the complex upper-half plane extended by including the cusps $i\infty \cup \mathbb{Q}$. Modular functions form a function field with a unique generator called the Haptmodul. Modular functions are a particular case of modular forms of weight k . The latter are defined as holomorphic functions on \mathbb{H} that, under a modular transformation (5.3) transform as

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) . \quad (5.4)$$

A natural generalization is to consider the transformation properties valid only for some subgroup $\Gamma \subset SL(2, \mathbb{Z})$. We will write f_Γ to indicate that f is modular only under transformations with parameters in Γ .

Monstrous moonshine conjecture states that, for each $g \in \mathbb{M}$, writing

$$J(\tau) = \sum_{n \geq -1} \dim V_n q^n , \quad (5.5)$$

where V_n are the spaces of representations of the monster group, the McKay-Thompson series

$$T_g(\tau) := \sum_{n \geq -1} \text{Tr}_{V_n}(g) q^n \quad (5.6)$$

coincides with the unique Haptmodul J_{Γ_g} with expansion $q^{-1} + O(q)$ at $\tau = i\infty$ for some genus zero subgroup¹ Γ_g such that $\Gamma_0(N) \triangleleft \Gamma_g \leq SL(2, \mathbb{R})$ for some N dividing $|g|\text{gcd}(24, g)$. The group $\Gamma_0(N)$ is defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 \pmod{N} \right\} . \quad (5.7)$$

A first important step toward a proof of the Monstrous moonshine conjecture was obtained by Frenkel, Lepowsky and Meurman in [7] and [8] with the construction of the monster module² V . This was achieved considering the space of states of a particular conformal field theory in 2 dimensions. The starting point to construct V is to consider 24 chiral bosons $X^i(z)$ compactified on the 24-dimensional torus \mathbb{R}^{24}/Λ , where Λ is the Leech lattice (see appendix C). The partition function of this model has the expansion $Z_{\frac{\mathbb{R}^{24}}{\Lambda}}(\tau) = q^{-1} + \dots$. Since the partition function is modular invariant it has to be equal to $J(\tau)$ up to an additive constant. Since the Leech lattice has no root, it is possible to show that

$$Z_{\mathbb{R}^{24}/\Lambda}(\tau) = J(\tau) + 24 . \quad (5.8)$$

In fact it can be shown that the additive constant correspond to the number of currents and, for a torus \mathbb{R}^{24}/L , they are equal to the number of bosons plus the number of roots of L . To eliminate this constant is sufficient to consider a \mathbb{Z}_2 orbifold of this theory acting as $X^i \rightarrow -X^i$. The module V is the direct sum of \mathbb{Z}_2 invariant projections of the untwisted and twisted sectors. The partition function of this model, which can be easily computed with the methods of section

¹We call a subgroup G of $SL(2, \mathbb{R})$ a genus zero subgroup if J_G is a biholomorphic map between \mathbb{H}/G and a genus zero Riemann surface. Notice that, in particular, $SL(2, \mathbb{Z})$ is genus zero since J maps biholomorphically $\mathbb{H}/SL(2, \mathbb{Z})$ into the Riemann sphere $\mathbb{C} \cup \infty$.

²Given a group G , a G module is simply the space of a representation of G .

2.7, is exactly equal to $J(\tau)$. Furthermore the monster group is a symmetry group of this model so V must be the space of a representation of \mathbb{M} (see [27] for a more detailed discussion). To verify the conjecture it is enough to show that $T_g^V(\tau) = \text{Tr}_V(gq^{L_0 - \frac{c}{24}})$ coincides with the corresponding Hauptmodul J_{Γ_g} . For a general group G , using conformal field theory arguments, these functions are invariant under a subgroup Γ_g^1 of $SL(2, \mathbb{Z})$. However this group will be in general smaller than Γ_g and it is not genus 0. Thanks to the identity discovered independently by Zagier, Borcherds and others

$$p^{-1} \prod_{\substack{m>0 \\ n \in \mathbb{Z}}} (1 - p^m q^n)^{a_{mn}} = J(\rho) - J(\tau) , \quad (5.9)$$

where $p = e^{2\pi i \rho}$ and a_i denotes the coefficient of q^i in the expansion of J , all the coefficients of the Hauptmodul can be fixed knowing just a_1, a_2, a_3, a_5 . It was shown by Borcherds in [28] that both T_g^V and J_{Γ_g} satisfy an analogous identity and that their coefficients coincide. The conjecture was thus proven. See again [27] for more details.

5.2 Moonshine in the elliptic genus of $K3$

Some years ago, it was noticed by Eguchi, Ooguri, Tachikawa in [5] that decomposing the elliptic genus of a $K3$ surface in terms of the characters³ of unitary representations of the $N = 4$ superconformal algebra

$$\phi_{K3}(\tau, z) = \dim(H_{00}) ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, z) - \dim(H_0) ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) + \sum_{n=1}^{\infty} \dim(H_n) ch_{h=n+\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) \quad (5.10)$$

the first coefficients can be written in terms of the dimensions of representations of the Mathieu group M_{24} in a surprisingly easy way. In particular

$$\begin{aligned} H_{00} &= \mathbf{23} + \mathbf{1} & H_0 &= 2 \cdot \mathbf{1} , \\ H_1 &= \mathbf{45} + \overline{\mathbf{45}} & H_2 &= \mathbf{231} + \overline{\mathbf{231}} , \\ H_3 &= \mathbf{770} + \overline{\mathbf{770}} & H_4 &= \mathbf{2277} + \overline{\mathbf{2277}} , \\ H_5 &= 2 \cdot \mathbf{5796} & H_6 &= 2 \cdot \mathbf{3520} + 2 \cdot \overline{\mathbf{10395}} , \\ & & & \dots \end{aligned} \quad (5.11)$$

M_{24} is the largest of the Mathieu groups, which are 5 finite simple groups. They are all subgroups of the permutation group of 24 objects and can be defined as automorphism groups of Steiner systems.

The conjecture was later extended to the twining genera (first to only some of them in [29] and [30] and then to all the twining genera in [31])

$$\phi_g(\tau, z) = \text{Tr}_{H_{00}}(g) ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, z) - \text{Tr}_{H_0}(g) ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) + \sum_{n=1}^{\infty} \text{Tr}_{H_n}(g) ch_{h=n+\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) \quad (5.12)$$

conjecturing that the representations in (5.10) are such that each twining genus is invariant under some subgroup of $SL(2, \mathbb{Z})$. The conjecture was verified for a lot of coefficients. It was at last proven to be true in abstract terms in [32]. Mathieu moonshine would have an easy

³We use the same (abuse of) notation as in section 3.1.3, $ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}$ is not the true massless character but a linear combination of characters. Its form is given by (3.12).

interpretation if M_{24} was a symmetry group at some point in the moduli space of non-linear sigma models on $K3$ surfaces. In that case we would have

$$\mathcal{H}_R = \bigoplus_{i,J} h_{iJ} R_i^{M_{24}} \otimes R_J^{N=4} , \quad (5.13)$$

where $R_i^{M_{24}}$ and $R_J^{N=4}$ are respectively representations of M_{24} and $N = 4$ superconformal symmetry, and then (5.10) would be easily understood. Unfortunately this is not the case. In fact the symmetries of non-linear sigma models on $K3$ were classified in [24] and in no point in the moduli space they include M_{24} as a symmetry group.

Mathieu moonshine it is not the only moonshine that appear in the elliptic genus of $K3$ surfaces, actually the Mathieu group is just one of 23 finite groups predicted by umbral moonshine. Umbral moonshine was originally formulated by Cheng, Duncan and Harvey in [33] for some of the umbral groups and mock modular forms and then extended as a connection between Niemeier lattices and mock modular forms by the same authors in [34]. It was related to the elliptic genus of $K3$ surfaces by Cheng and Harrison in [6].

To discuss the main ideas behind umbral moonshine we have to introduce the concepts of mock modular form and mock Jacobi form. Let h be a holomorphic function on \mathbb{H} with at most exponential growth at all cusps $i\infty \cup \mathbb{Q}$. Let $w \in \mathbb{Z} + \frac{1}{2}$, g be a modular form of weight $2 - w$ with Fourier expansion $g(\tau) = \sum_{n \geq 0} c_g(n) q^n$ and $\Gamma \leq SL(2, \mathbb{R})$ a discrete subgroup. Let g^* be defined as

$$g^*(\tau) := \overline{c_g(0)} \frac{(-\text{Im}(\tau))^{1-w}}{w-1} + \sum_{n>0} (-4\pi n)^{w-1} \overline{c_g(n)} q^{-n} \Gamma(1-w, 4\pi n \text{Im}(\tau)) \quad (5.14)$$

with $\Gamma(1-w, x) = \int_x^\infty e^{-t} t^w dt$. We call h a weakly holomorphic mock modular form for Γ if $\hat{h} = h + g^*$ transforms as a holomorphic modular form of weight w for Γ . In this case we call g the shadow of h and \hat{h} its completion.

We already encountered weak Jacobi forms in the definition of the elliptic genus. Skew-holomorphic Jacobi forms (of weight w and index m) are defined in a similar way but now the transformation property under $SL(2, \mathbb{Z})$ reads

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \frac{|c\bar{\tau}+d|}{(c\bar{\tau}+d)(c\tau+d)^w} e^{2\pi i m \frac{cz^2}{c\tau+d}} \phi(\tau, z) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (5.15)$$

and they have the following Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{D, l \in \mathbb{Z} \\ D \equiv l^2 \pmod{4m}}} c(D, l) \bar{q}^{\frac{D}{4m}} q^{\frac{l^2}{4m}} y^l . \quad (5.16)$$

We will use the fact that elliptic functions, i.e. functions $\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ which satisfy

$$\phi(\tau, z + l\tau + l') = e^{2\pi i m(l^2\tau + 2lz)} \phi(\tau, z) \quad \text{for } l, l' \in \mathbb{Z} , \quad (5.17)$$

admit a theta-decomposition

$$\phi(\tau, z) = \sum_{r \pmod{2m}} h_r(\tau) \theta_{m,r}(\tau, z) \quad (5.18)$$

with

$$\theta_{m,r}(\tau, z) := \sum_{l \equiv r \pmod{2m}} q^{\frac{l^2}{4m}} y^l , \quad (5.19)$$

where the h_r are $2m$ smooth functions from \mathbb{H} to \mathbb{C} . From the definition, weak Jacobi forms are a particular case of elliptic functions. An elliptic function $\phi(\tau, z)$ is a weak mock Jacobi form of weight w and index m if it is limited as $\text{Im}\tau \rightarrow \infty$ for every $z \in \mathbb{C}$, all the h_r s in the theta-decomposition are holomorphic and if there exists a skew-holomorphic Jacobi form of weight $3 - w$ and index m , with decomposition $\sigma = \sum_r \bar{g}_r \theta_{m,r}$, such that

$$\hat{\phi}(\tau, z) = \sum_r \left(h_r(\tau) + \frac{1}{\sqrt{2m}} g_r^*(\tau) \right) \theta_{m,r}(\tau, z) \quad (5.20)$$

transforms as a weak Jacobi form of weight w and index m .

We are now ready to discuss the umbral moonshine conjecture. As we said before there are 23 umbral moonshine groups, and they are related to the 23 Niemeier lattices N^X , where X denotes their root systems which uniquely identify them. Umbral groups are constructed quotienting the automorphism group of a Niemeier lattice by the Weyl group of its root system

$$G^X = \text{Aut}(N^X) / \text{Weyl}(X) . \quad (5.21)$$

We enlist here the umbral groups associated to the Niemeier root systems labelled according to their ADE classification

Niemeier root system X	Umbral Group G^X
A_1^{24}	M_{24}
A_2^{12}	$2.M_{12}$
A_3^8	$2.AGL_3(2)$
A_4^6	$GL_2(5)/2$
$A_5^4 D_4$	$GL_2(3)$
A_6^4	$SL_2(3)$
$A_7^2 D_5$	Dih_4
A_8^3	Dih_6
$A_9^2 D_6$	\mathbb{Z}_4
$A_{11} D_7 E_6$	\mathbb{Z}_2
A_{12}^2	\mathbb{Z}_4
$A_{15} D_9$	\mathbb{Z}_2
$A_{17} E_7$	\mathbb{Z}_2
A_{24}	\mathbb{Z}_2
D_4^6	$3.Sym_6$
D_6^4	Sym_4
D_8^3	Sym_3
$D_{10} E_7^2$	\mathbb{Z}_2
D_{12}^2	\mathbb{Z}_2
$D_{16} E_8$	\mathbb{Z}_1
D_{24}	\mathbb{Z}_1
E_6^4	$GL_2(3)$
E_8^3	Sym_3

Notice in particular that the Mathieu group M_{24} is one of the umbral groups.

To each root system it is possible to associate a mock modular form that will be related by the umbral moonshine conjecture to the group G^X . The main idea that allows us to associate these mock modular forms to the umbral groups is the concept of optimal growth. Optimal growth roughly translates into the requirement that the mock modular forms are bounded at all the cusps which are not equivalent to $i\infty$ and that they have the smallest possible growth, compatible with the modular properties, around $i\infty$. We give an insight of how these mock

modular forms are constructed in the simple case of the mock modular forms with respect to the whole $SL(2, \mathbb{Z})$. Given $\psi = \sum_r h_r \theta_{m,r}$ a mock Jacobi form of weight 1 and index m we say it is optimal if

$$h_r(\tau) = O(q^{-\frac{1}{4m}}) \quad (5.22)$$

when $\text{Im} \rightarrow \infty$ for each $r \in \mathbb{Z}/2m$. Let K be a subgroup of the group of the exact divisors⁴ of m , we say that an index m mock Jacobi form is K -symmetric if

$$\psi = \sum_{r \bmod 2m} h_r \theta_{m,r} = \sum_{r \bmod 2m} h_r \theta_{m,a(n)r} \quad n \in K, \quad (5.23)$$

where

$$a(n) = \begin{cases} 1 \bmod \frac{2m}{n} \\ -1 \bmod \frac{2m}{n} \end{cases}. \quad (5.24)$$

Furthermore we say a mock modular has positive integral coefficients if writing

$$\psi = \sum_{1 \leq r \leq m-1} \tilde{h}_r (\theta_{m,r} - \theta_{m,r}) , \quad (5.25)$$

which is possible since odd weight Jacobi form are odd under $z \rightarrow -z$, h_r has the expansion

$$\tilde{h}_r = \begin{cases} -2q^{-\frac{1}{4m}} + \sum_{n \geq 0} c_{r,n} q^{\frac{n}{4m}} & \text{if } r^2 = 1 \bmod 4m \\ \sum_{n \geq 0} c_{r,n} q^{\frac{n}{4m}} & \text{otherwise} \end{cases} \quad (5.26)$$

with $c_{r,n} \in \mathbb{N}$. If we choose the normalization $h_1 = -2q^{-\frac{1}{4m}}(1 + O(q))$ it turns out that there are only 23 optimal K -symmetric mock Jacobi forms with positive integral coefficients of weight 1 under $SL(2, \mathbb{Z})$. They are in one-to-one correspondence with the Niemeier root systems (see [27] for an explanation in terms of ADE classification). In particular writing $\psi^X = \sum_r H_r^X \theta_{m,r}$, where $(H^X)_r$ are vector-valued mock modular forms, the Coxeter number of X is equal to the index of ψ^X and defining the set I^X labelling the independent components of H^X we can write

$$\psi^X = \sum_{r \in I^X} H_r^X \sum_{n \in K} (\theta_{m,a(n)r} - \theta_{m,-a(n)r}) . \quad (5.27)$$

We have considered the case of mock modular forms for $SL(2, \mathbb{Z})$ but umbral moonshine is actually more general. We report here the statement of umbral moonshine conjecture which can be found in [27]:

Conjecture. Let G^X be a umbral group defined in (5.21), $m = \text{Cox}(X)$ the Coxeter number of X and I^X a subset of $\{1, 2, \dots, m-1\}$ as before. Then there exists a bi-graded infinite-dimensional G^X -module

$$K^X = \bigoplus_{r \in I^X} \bigoplus_{\substack{D \leq 0 \\ D \equiv r^2 \bmod 4m}} K_{r,D}^X \quad (5.28)$$

such that for any $g \in G^X$ and for any $r \in I^X$, the graded character (adding $-2q^{-\frac{1}{4m}}$ for $r = 1$) coincides with the component $H_{g,r}^X$ of a vector valued mock modular form H^X

$$H_{g,r}^X = -2q^{-\frac{1}{4m}} \delta_{r,1} + \sum_{\substack{D \leq 0 \\ D \equiv r^2 \bmod 4m}}^{\infty} q^{-\frac{D}{4m}} \text{Tr}_{K_{r,D}^X}(g) . \quad (5.29)$$

⁴We say n is an exact divisor of m if $n|m$ and $(n, \frac{m}{n}) = 1$. The exact divisors form a group with multiplication $n * n' = \frac{nn'}{(n, n')^2}$.

Umbral moonshine conjecture has been proved in [35].

We will now briefly discuss how the previous conjecture is related to the elliptic genus of $K3$. $K3$ surfaces can have at most du Val type singularities, i.e. singularity of the type \mathbb{C}^2/G where G is a finite subgroup of $SU(2)_{\mathbb{C}}$. It was found in [6] that there are 23 ways to split the part corresponding to the singularities in the elliptic genus, each one related to a different Niemeier root system X

$$\phi(\tau, z) = \phi^X(\tau, z) + \frac{\theta_1^2(\tau, z)}{2\eta^6(\tau)} \frac{1}{2\pi i} \frac{\partial \psi^X}{\partial z}(\tau, z) \Big|_{z=0}, \quad (5.30)$$

where ϕ^X is the part of the elliptic genus corresponding to the singularities in the split given by the Niemeier root system X and ψ^X is the optimal mock Jacobi form discussed above corresponding to X . It is interesting to point out that for the root system corresponding to the Mathieu group

$$\phi^{A_1^{24}} = 24 \frac{\theta_1^2(\tau, z)^2}{\eta(\tau)^3} \mu(\tau, z) = 24 ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, z) \quad (5.31)$$

which is exactly the first term in the decomposition in (5.10).

An analogous decomposition is valid in general for the twining genera

$$\phi_g(\tau, z) = \phi_g^X(\tau, z) + \frac{\theta_1^2(\tau, z)}{2\eta^6(\tau)} \frac{1}{2\pi i} \frac{\partial \psi_g^X}{\partial z}(\tau, z) \Big|_{z=0}. \quad (5.32)$$

These, for the case of A_1^{24} , leads to the form of the twining genera (5.12). It is important to stress that some of the previous decompositions are just formal in the sense that there are no known singular limits in which $K3$ surfaces develop the right number of singularities corresponding to the umbral groups. This is the case, for example, of M_{24} since there are no limits in which $K3$ surfaces can develop 24 A_1 -type singularities.

Many of the Umbral groups are not symmetry groups of non-linear sigma model on $K3$, so we cannot understand them with a simple decomposition as in (5.13). It is then not clear which is the underlying physical setting the relation between the elliptic genus of $K3$, the umbral groups and the mock modular forms comes from. However some works (see [36], [37], [38] and [39]) have found some physical settings, mainly using string dualities, in which the umbral groups seem to arise naturally.

6. Umbral twining genera decomposition

In this section we expose the original part of this work. The aim was to consider a decomposition as in (5.13) and exploit the decomposition of the twining genera into the irreducible representations to obtain informations on some non-linear sigma models. In fact, the umbral groups contain subgroups which are actually symmetry models of some non-linear sigma models. We will consider in particular the case of $L_2(11)$ which is a subgroup of both M_{24} and $2.M_{12}$.

6.1 Decomposition with N=4 characters

As the first original part of this work we computed the coefficients in the formal decomposition

$$\mathcal{H}_R = \bigoplus_{i,J} h_{iJ} R_i^G \otimes R_J^{N=4} \quad (6.1)$$

into irreducible representations R_i^G for some of the umbral groups G (from what we have said in section 5, \mathcal{H}_{RR} will not be the Ramond sector of a non-linear sigma model on $K3$ for some of the umbral groups). J is an opportune multi-index labelling the conformal weight and charge of the irreducible representations of the $N = 4$ superconformal algebra. We followed the approach of [31] to obtain the coefficients.

Let G be one of the umbral groups and let us assume a decomposition like (5.10) in which the H_i s are spaces of representation of G . We recall the formula of the twining genera

$$\phi_g(\tau, z) = \text{Tr}_{\mathcal{H}_{RR}}(g q^{L_0 - \frac{c}{24}} y^{J_0} (-1)^F \bar{q}^{\bar{L}_0 - \frac{c}{24}} (-1)^{\bar{F}}) \quad (6.2)$$

whit $g \in G$. Using (6.1) and the analogous of (5.10) the twining genera read

$$\phi_g(\tau, z) = \text{Tr}_{H_{00}}(g) ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, z) - \text{Tr}_{H_0}(g) ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) + \sum_{n=1}^{\infty} \text{Tr}_{H_n}(g) ch_{h=n+\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) . \quad (6.3)$$

We recall, from section 3, that the twining genera are conjectured to transform as weak Jacobi forms of index 1 and weight 0. Every weak Jacobi form of index 1 can be written in terms of the standard Jacobi forms $\chi_{0,1}(\tau, z)$, $\chi_{-2,1}(\tau, z)$, of weight 0 and -2 respectively, given by

$$\begin{aligned} \chi_{0,1}(\tau, z) &= 4 \sum_{i=2}^4 \frac{\theta_i(\tau, z)^2}{\theta_i(\tau, 0)^2} , \\ \chi_{-2,1}(\tau, z) &= -\frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} , \end{aligned} \quad (6.4)$$

where the $\theta_i(\tau, z)$ s are the Jacobi theta functions and $\eta(\tau)$ is the Dedekind eta. In particular for the twining genera we have

$$\phi_g(\tau, z) = \frac{1}{12} \phi_g(\tau, 0) \chi_{0,1}(\tau, z) + F_g(\tau) \chi_{-2,1}(\tau, z) , \quad (6.5)$$

where the F_g s are modular forms of weight two and, for the umbral groups, have been computed in [40]. Furthermore from (6.3) we have

$$\phi_g(\tau, 0) = \text{Tr}_{\mathcal{H}_{00}}(g) . \quad (6.6)$$

If the H_n s have to be spaces of (unitary) representations of a umbral group G they can be decomposed into (the spaces of) its irreducible representations R^G

$$H_n = \bigoplus_i h_{n,i} R_i^G . \quad (6.7)$$

Thus (6.3) becomes

$$\begin{aligned} \phi_{K3}(\tau, z) = & \sum_i h_{00i} \text{Tr}_{R_i^G}(g) ch_{h=\frac{1}{4}, l=0}^{N=4}(\tau, z) - \sum_i h_{0i} \text{Tr}_{R_i^G}(g) ch_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) + \\ & \sum_{n=1}^{\infty} \sum_i h_{ni} \text{Tr}_{R_i^G}(g) ch_{h=n+\frac{1}{4}, l=\frac{1}{2}}^{N=4}(\tau, z) . \end{aligned} \quad (6.8)$$

The characters of finite groups satisfy the orthonormality relations

$$\sum_{[g]} c(g) \overline{\text{Tr}_{R_i^G}(g)} \text{Tr}_{R_j^G}(g) = \delta_{ij} , \quad (6.9)$$

where R_i^G and R_j^G are the space of states of two irreducible representations of G , the sum runs over its conjugacy classes and $c(g)$ is the inverse of the order of the centralizer of G

$$c(g) = \frac{n(g)}{|G|} \quad (6.10)$$

with $n(g)$ the number of elements in the conjugacy class of g and $|G|$ the order of G . We remark that (6.10) is independent from the choice of the representative g in each conjugacy classes since both $c(g)$ and the characters $\text{Tr}_{R_i^G}$ are so. Thus multiplying both sides of (6.8) by $c(g) \overline{\text{Tr}_{R_j^G}(g)}$, summing over the conjugacy classes of $|G|$ and using (6.5) we obtain an equation for the coefficients h_{nj}

$$\sum_{n=1}^{\infty} h_{n,i} q^n = h_{0,i} + q^{\frac{1}{8}} \frac{\eta(\tau)^3}{\theta_1(\tau, z)^2} \left(\sum_g c(g) \overline{\text{Tr}_{R_i}(g)} \phi_g(\tau, z) - h_{00,i} ch_{h=\frac{1}{4}, l=0}^{N=4} \right) . \quad (6.11)$$

We assume for all the umbral groups $h_{0i} = 2\delta_i^1$ which is the condition for G to act trivial on the $N = (4, 4)$ superconformal algebra and on the generators of the spectral flow. We will discuss this condition in more details when we will consider also the antiholomorphic part of the superconformal algebra. Using formula (6.11) and the explicit expressions found in [40] for the F_g s we computed the first coefficients for some umbral groups. Results are given in appendix D.

The appearance of coefficients with negative signs has an interesting interpretation if one also consider the antiholomorphic part of the $N = (4, 4)$ algebra. We consider the formal decomposition

$$\mathcal{H}_R = \bigoplus_{i,J,K} h_{iJK} R_i^G \otimes R_J^{N=4} \otimes R_K^{\overline{N}=4} , \quad (6.12)$$

where $R_K^{\overline{N}=4}$ are (the spaces of) irreducible representations of the antiholomorphic part of the $N = (4, 4)$ algebra and J, K are opportune multi-indices labelling the conformal weight and charge of the irreducible representations. With this decomposition the twining genera read

$$\phi_g(\tau, z) = \sum_{i,J,K} h_{iJK} \text{Tr}_{R_i^G}(g) ch_J^{N=4}(\tau, z) ch_K^{\overline{N}=4}(\bar{\tau}, 0) , \quad (6.13)$$

we have obtained this decomposition by inserting $\bar{y}^{\bar{J}_0}$ in (6.2) and then evaluating it for $\bar{z} = 0$. Now the characters evaluated at $\bar{z} = 0$ have the following values¹

$$\begin{aligned} ch_{\bar{h}=\frac{1}{4}+n, \bar{l}=\frac{1}{2}}^{\bar{N}=4}(\bar{\tau}, 0) &= 0, \\ ch_{\bar{h}=\frac{1}{4}, \bar{l}=0}^{\bar{N}=4}(\bar{\tau}, 0) &= 1, \\ ch_{\bar{h}=\frac{1}{4}, \bar{l}=\frac{1}{2}}^{\bar{N}=4}(\bar{\tau}, 0) &= -2, \end{aligned} \quad (6.14)$$

so the appearance of the massless representations ($\bar{h} = \frac{1}{4}, \bar{l} = \frac{1}{2}$) would give a natural explanation to the negative coefficients that we found in the decompositions. Furthermore, the requirement $h_{0i} = 2\delta_i^1$ that we have previously imposed will now become

$$\begin{aligned} h_{i(\frac{1}{4}, \frac{1}{2})(\frac{1}{4}, \frac{1}{2})} &= \delta_i^1, \\ h_{i(\frac{1}{4}, \frac{1}{2})(\frac{1}{4}, 0)} &= 0, \end{aligned} \quad (6.15)$$

where the index $i = 1$ refers to the one-dimensional representation of the group G . This is consistent with the negative sign appearing in (6.3). We will now give a justification for the need of these constraints. As we have said before we need the umbral groups G to act trivially on the superconformal algebra and on the spectral flow generators. The fields in (3.4) and the ones generated by their OPEs form the vacuum representation of the algebra in the NS - NS sector $R_{h=0, l=0}^{N=4} \otimes R_{\bar{h}=0, \bar{l}=0}^{\bar{N}=4}$. This representation is mapped, under left-right symmetric spectral flow, in the representation $R_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4} \otimes R_{\bar{h}=\frac{1}{4}, \bar{l}=\frac{1}{2}}^{\bar{N}=4}$ in the R - R sector. Since we have required also the spectral flow operators to be G -invariant the fields in this representation are G -invariant too. The only irreducible representation of G in which all the elements can act trivially is the 1-dimensional one, so the fields in the representation $R_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4} \otimes R_{\bar{h}=\frac{1}{4}, \bar{l}=\frac{1}{2}}^{\bar{N}=4}$ in the R - R sector must belong to the 1-dimensional representation of G . Furthermore, the presence of representations of the kind $R_{h=\frac{1}{4}, l=\frac{1}{2}}^{N=4} \otimes R_{\bar{h}=\frac{1}{4}, \bar{l}=0}^{\bar{N}=4}$ in the R - R sector would make the elliptic genus vanish (see [41] for a detailed explanation). Since we know the elliptic genus of a $K3$ surface is not null, we can exclude their presence in the decomposition.

What is particularly interesting of the previous construction is the appearance of irreducible representations of the kind

$$R_{h=n+\frac{1}{4}, l=\frac{1}{2}}^{N=4} \otimes R_{\bar{h}=\frac{1}{4}, \bar{l}=\frac{1}{2}}^{\bar{N}=4}. \quad (6.16)$$

In fact if we map them through spectral flow $U_{-\frac{1}{2}} \bar{U}_{-\frac{1}{2}}$ in the NS sector they become

$$R_{h=n, l=0}^{N=4} \otimes R_{\bar{h}=0, \bar{l}=0}^{\bar{N}=4}, \quad (6.17)$$

but this would constitute evidence of the presence of a holomorphic primary field, of conformal weight n , which is not contained in the $N = (4, 4)$ superconformal algebra. In fact the fields contained in the $N = (4, 4)$ superconformal algebra belong to the irreducible representation $R_{h=0, l=0}^{N=4} \otimes R_{\bar{h}=0, \bar{l}=0}^{\bar{N}=4}$ since they are built acting on the vacuum with opportune generators of the algebra.

6.1.1 Models with $L_2(11)$ symmetry and algebra extension

The group $L_2(11)$ is a subgroup of both the umbral groups M_{24} and $2.M_{12}$. It is particularly interesting because it is actually a symmetry group of some non-linear sigma models on $K3$ surfaces. So far, it is not known what is the action of $L_2(11)$ on these models. It is conjectured

¹The reader should be careful that here we are using the true character formula for $ch_{\bar{h}=\frac{1}{4}, \bar{l}=\frac{1}{2}}^{\bar{N}=4}$.

that it could have the same action of M_{24} or of $2.M_{12}$, since it is a subgroup of both of them, but there are no hints on which could be the correct one and different models could admit different actions.

From the results of appendix D we have that, when $L_2(11)$ acts with the action of $2.M_{12}$, negative coefficients appear in (6.8) while the coefficients are all positive when it acts with the action of M_{24} . This is due to the fact that different group actions lead to different functions $F_g(\tau)$ in (6.5) and obviously different F_g s lead to different coefficients in the decomposition (6.8).

As we have pointed out at the end of the previous section, the appearance of negative signs in the decomposition (6.8) can be explained by the presence of fields not contained in the $N = 4$ superconformal algebra. Thus, if $L_2(11)$ has the same action of $2.M_{12}$, we have a hint that the symmetry group of the models that admit $L_2(11)$ as a symmetry group could be extended beyond the $N = 4$ superconformal symmetry while there is not such a “evidence” when it acts with the action of M_{24} . While different symmetry algebras could be enough to establish which is the action of $L_2(11)$ in the non-linear sigma models on $K3$ which admit it as a symmetry group, we cannot conclude from what we have found in appendix D that they have actually different algebras. In fact even if the coefficients in the decomposition (6.8) are all positive, this does not exclude the presence of irreducible representations like (6.16). Actually positive coefficients in (6.8) require

$$\sum_K h_{i,J,K} ch_K^{\overline{N}=4}(\bar{\tau}, 0) > 0 \quad (6.18)$$

in (6.13), which does not necessarily implies $h_{i,J,(\frac{1}{4},\frac{1}{2})} = 0$.

It is interesting but not clear the fact that, in the results of appendix D, negative signs appear only in the coefficients of the 1-dimensional representation. Furthermore, in the table of the coefficients for the group $L_2(11)$ with the action of $2.M_{12}$ the coefficient of the power q^1 is -2 thus suggesting, but as before not necessarily implying, that the representation for $n = 1$ could simply be

$$R_{h=1+\frac{1}{4},l=\frac{1}{2}}^{\overline{N}=4} \otimes \left(0 \cdot R_{h=n+\frac{1}{4},\bar{l}=\frac{1}{2}}^{\overline{N}=4} \oplus 1 \cdot R_{h=\frac{1}{4},\bar{l}=\frac{1}{2}}^{\overline{N}=4} \oplus 0 \cdot R_{h=\frac{1}{4},\bar{l}=0}^{\overline{N}=4} \right) . \quad (6.19)$$

Mapping this in the NS sector we obtain the presence of primary field of conformal dimension $h = 1$ not contained in the $N = 4$ superconformal algebra. This could be obtained by extending the $\hat{\mathfrak{su}}(2)$ Kac-Moody algebra of the $N = 4$ superconformal algebra to a $\hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{u}}(1)$ Kac-Moody algebra, which contains an additional current. We will investigate this possibility in the next section where we will consider the decomposition in terms of an orbifold model which exactly possesses this current algebra.

6.2 $\mathbb{C}^2/\mathbb{Z}_3$ characters

As a following step of the master thesis work, we computed the characters of 4 free bosons X^i and fermions Ψ^i , $i = 1, 2, 3, 4$, on the $\mathbb{C}^2/\mathbb{Z}_3$ non-linear sigma model orbifold. We were motivated by the fact that, from the results found in appendix D for the group $L_2(11)$, the negative coefficient at the first order in q could be explained extending the $\hat{\mathfrak{su}}(2)$ Kac-Moody algebra with a $\hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{u}}(1)$ Kac-Moody algebra. We will see that this model has exactly the latter current algebra. Furthermore the orbifold $\mathbb{C}^2/\mathbb{Z}_3$ possesses a du Val singularity in the origin and we have seen in section 5.2 that this kind of singularity is involved in the decomposition of the elliptic genus which relates it to umbral moonshine. Thus it is interesting to see what happens when we decompose the elliptic genus in terms of the characters of this orbifold.

In order to compute the characters we start by organizing the fields in complex fields

$$\begin{aligned} \Psi_{\pm}^1 &= \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2), & \Psi_{\pm}^2 &= \frac{1}{\sqrt{2}}(\psi_3 \pm i\psi_4), \\ J_{\pm}^1 &= \frac{1}{\sqrt{2}}(j_1 \pm ij_2), & J_{\pm}^2 &= \frac{1}{\sqrt{2}}(j_3 \pm ij_4), \end{aligned} \quad (6.20)$$

where $j_i = \partial X_i$. We denote the action of \mathbb{Z}_3 with g

$$\begin{aligned} gJ_{\pm}^i g^{-1} &= e^{\frac{2\pi k_{\pm}^i}{3}} J_{\pm}^i, \\ g\Psi_{\pm}^i g^{-1} &= e^{\frac{2\pi k'_{\pm}^i}{3}} \Psi_{\pm}^i. \end{aligned} \quad (6.21)$$

We want the action of the orbifold to preserve the $N = 4$ superconformal symmetry since we want to extend it. So it has to leave invariant the fields in (3.16). This implies the following conditions

$$\begin{aligned} k_{\pm}^i &= -k_{\mp}^i, & k'_{\pm}^i &= -k'_{\mp}^i, \\ k_{\pm}^1 &= -k_{\mp}^2, & k'_{\pm}^1 &= -k'_{\mp}^2, \\ k'_{\pm}^i &= k_{\pm}^i. \end{aligned} \quad (6.22)$$

We are free to choose the action of the orbifold with $k_{+}^1 = 1$. In the following we will use the notation

$$\begin{aligned} \Psi_{+}^i &= \Psi^i, & \Psi_{-}^i &= \bar{\Psi}^i, \\ J_{+}^i &= J^i, & J_{-}^i &= \bar{J}^i. \end{aligned} \quad (6.23)$$

The commutation and anticommutation relations become

$$\begin{aligned} [J_n^i, \bar{J}_m^j] &= n\delta_{n+m,0}\delta_{i,j}, & [J_n^i, J_m^j] &= 0, & [\bar{J}_n^i, \bar{J}_m^j] &= 0, \\ \{\Psi_r^i, \bar{\Psi}_s^j\} &= \delta_{r+s,0}\delta_{i,j}, & \{\Psi_r^i, \Psi_s^j\} &= 0, & \{\bar{\Psi}_r^i, \bar{\Psi}_s^j\} &= 0. \end{aligned} \quad (6.24)$$

One must be careful, however, that neither $\bar{\Psi}$ nor \bar{J} are antiholomorphic, actually they are both holomorphic fields. The action of the orbifolds in this notation is given by

$$\begin{aligned} g\Psi^1 g^{-1} &= e^{\frac{2\pi i}{3}} \Psi^1, & g\bar{\Psi}^1 g^{-1} &= e^{-\frac{2\pi i}{3}} \bar{\Psi}^1, \\ g\Psi^2 g^{-1} &= e^{-\frac{2\pi i}{3}} \Psi^2, & g\bar{\Psi}^2 g^{-1} &= e^{\frac{2\pi i}{3}} \bar{\Psi}^2, \\ gJ^1 g^{-1} &= e^{\frac{2\pi i}{3}} J^1, & g\bar{J}^1 g^{-1} &= e^{-\frac{2\pi i}{3}} \bar{J}^1, \\ gJ^2 g^{-1} &= e^{-\frac{2\pi i}{3}} J^2, & g\bar{J}^2 g^{-1} &= e^{\frac{2\pi i}{3}} \bar{J}^2. \end{aligned} \quad (6.25)$$

The symmetry algebra of 4 free bosons and fermions is actually bigger than the $N = 4$ superconformal algebra of (3.16). In fact, apart from the $\hat{\mathfrak{su}}(2)$ Kac-Moody algebra generated by J^3 , J^+ and J^- , it possesses another $\hat{\mathfrak{su}}(2)$ current algebra generated by

$$A = \frac{1}{2} \left(: \Psi^1 \bar{\Psi}^1 : - : \Psi^1 \bar{\Psi}^1 : \right), \quad A^+ = : \Psi^1 \bar{\Psi}^2 :, \quad A^- = - : \bar{\Psi}^1 \Psi^2 :, \quad (6.26)$$

but the action (6.25) leaves invariant only A so it breaks this $\hat{\mathfrak{su}}(2)$ current algebra into a $\hat{\mathfrak{u}}(1)$. So the $\mathbb{C}^2/\mathbb{Z}_3$ orbifold has the right kind of currents algebra we are interested to study.

The holomorphic untwisted characters in the Ramond sector are given by

$$Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{h,l}(\tau, z) = \frac{1}{3} \sum_{k=0}^2 Tr_{V_h} (g^k e^{\frac{-2\pi i k l}{3}} (-1)^F y^{J_0} q^{L_0 - \frac{c}{24}}), \quad (6.27)$$

where V_h is a highest weight representation of the free bosons and fermions algebra with conformal weight h and $l = 0, \dots, 2$ projects into states of g-eigenvalue l . We recall from section 3.1.3 that $J_0 = 2J_0^3$ in (3.16). We will also use $(-1)^F = e^{i\pi J_0}$. Since the stress-energy tensor is just the sum of the single fermions and bosons, and the fermionic charge the sum of the two complex fermions, the trace factorize over the space of the two different complex bosons and fermions. So we can first compute the character for one complex boson and for one complex fermion. From section 2.5 we know that the central charge for a free real boson is $c = 1$ while for a free real fermion is $c = \frac{1}{2}$, so the central charge of our system will be $c = 6$.

6.2.1 Bosonic characters

We first compute the characters with respect to the boson J^1 , we will drop the apex and just write J in the following. Since the fermion current J^3 in (3.16) acts trivially on the bosons we have to compute

$$Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{h,l}(\tau, z) = \frac{1}{3} \sum_{k=0}^2 Tr_{V_h^{bos}}(g^k e^{\frac{-2\pi i k l}{3}} q^{L_0 - \frac{c}{24}}) . \quad (6.28)$$

A generic state in V_h^{bos} is obtained acting with J_{-n} and \bar{J}_{-n} , $n \geq 0$, on highest weight state of dimension $h = |\alpha|^2$, i.e.

$$\begin{aligned} J_0 |\alpha\rangle &= \alpha |\alpha\rangle , & \bar{J}_0 |\alpha\rangle &= \bar{\alpha} |\alpha\rangle , \\ J_n |\alpha\rangle &= 0 , & \bar{J}_n |\alpha\rangle &= 0 \quad \text{for } n > 0. \end{aligned} \quad (6.29)$$

We want to organize the $|\alpha\rangle$ s in g eigenstates. We have

$$J_0 g |\alpha\rangle = e^{-\frac{2\pi i}{3}} g J_0 |\alpha\rangle = e^{-\frac{2\pi i}{3}} \alpha g |\alpha\rangle . \quad (6.30)$$

We see that $g |\alpha\rangle$ is an eigenstate of J_0 with eigenvalue $e^{-\frac{2\pi i}{3}} \alpha$, so

$$g |\alpha\rangle = |e^{-\frac{2\pi i}{3}} \alpha\rangle . \quad (6.31)$$

The action of g clearly does not change the conformal dimension of $|\alpha\rangle$. We can then build the three eigenstates

$$\begin{aligned} |u_1\rangle &= \frac{1}{3}(|\alpha\rangle + g |\alpha\rangle + g^2 |\alpha\rangle) , \\ |u_2\rangle &= \frac{1}{3}(|\alpha\rangle + e^{-\frac{2\pi i}{3}} g |\alpha\rangle + e^{\frac{2\pi i}{3}} g^2 |\alpha\rangle) , \\ |u_3\rangle &= \frac{1}{3}(|\alpha\rangle + e^{\frac{2\pi i}{3}} g |\alpha\rangle + e^{-\frac{2\pi i}{3}} g^2 |\alpha\rangle) . \end{aligned} \quad (6.32)$$

They have eigenvalues $g |u_j\rangle = e^{\frac{2\pi i}{3}(j-1)} |u_j\rangle$ for $\alpha \neq 0$. Notice that the vacuum state $|0\rangle$ with conformal dimension $h = 0$ (and thus $\alpha = 0$) is invariant. The states $|u_i\rangle$ s depend on α but for brevity we will omit it in the notation, the only case when attention is needed is the one with $\alpha = 0$.

The space V_h is built as a Fock space acting on the $|u_i\rangle$ s with negative current modes, so a general state will have the form

$$|n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_i = J_{-1}^{n_1} \dots J_{-r}^{n_r} \bar{J}_{-1}^{\bar{n}_1} \dots \bar{J}_{-s}^{\bar{n}_s} |u_i\rangle . \quad (6.33)$$

Recalling the form of $T(z)$ from (3.16), for a single boson we will have

$$L_0 = \bar{J}_0 J_0 + \sum_{k>0} \bar{J}_{-k} J_k + \sum_{k>0} J_{-k} \bar{J}_k . \quad (6.34)$$

The 0-modes commute with all the J_n s and \bar{J}_n s while

$$\begin{aligned} [\bar{J}_{-k} J_k + J_{-k} \bar{J}_k, J_{-n}] &= k J_{-n} \delta_{k,n} , \\ [\bar{J}_{-k} J_k + J_{-k} \bar{J}_k, \bar{J}_{-n}] &= k \bar{J}_{-n} \delta_{k,n} , \end{aligned} \quad (6.35)$$

so, with a little of computation we find

$$L_0 |n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_i = \left[|\alpha|^2 + \sum_k k(n_k + \bar{n}_k) \right] |n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_i , \quad (6.36)$$

where we have used the fact that J_k and \bar{J}_k annihilate the $|u_i\rangle$ s for $k > 0$.

We want now to find the action of g on the state $|n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_j$. Using (6.21) we have

$$\begin{aligned} g |n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_j &= g J_{-1}^{n_1} \dots J_{-r}^{n_r} \bar{J}_{-1}^{\bar{n}_1} \dots \bar{J}_{-s}^{\bar{n}_s} |u_j\rangle = \\ &= g J_{-1}^{n_1} g^{-1} g \dots g^{-1} g J_{-r}^{n_r} g^{-1} g \bar{J}_{-1}^{\bar{n}_1} g^{-1} g \dots g^{-1} g \bar{J}_{-s}^{\bar{n}_s} g^{-1} g |u_j\rangle = \\ &= e^{\frac{2\pi i}{3} \sum_k (n_k - \bar{n}_k)} e^{\frac{2\pi i}{3} (j-1)(1-\delta_\alpha)} |n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_j, \end{aligned} \quad (6.37)$$

where we have included the behaviour in the particular case when $\alpha = 0$. The action of g^k on the same state is

$$g^k |n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_j = e^{\frac{2\pi i k}{3} \left[\sum_l (n_l - \bar{n}_l) + (j-1)(1-\delta_\alpha) \right]} |n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_j. \quad (6.38)$$

We are now ready to compute (6.28). For $\alpha \neq 0$ we have

$$\begin{aligned} Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{h,l}(\tau, z) &= \frac{1}{3} \sum_{k=0}^2 Tr_{V_h^{fer}}(g^k e^{\frac{-2\pi i k l}{3}} q^{L_0 - \frac{c}{24}}) = \\ &= \frac{1}{3} \sum_{k=0}^2 \sum_{\{n\}, \{\bar{n}\}} \sum_{i,j=1}^3 i \langle n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s | g^k e^{\frac{-2\pi i k l}{3}} q^{L_0 - \frac{c}{24}} | n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s \rangle_j = \\ &= \frac{1}{3} \sum_{k=0}^2 \sum_{\{n\}, \{\bar{n}\}} \sum_{i,j=1}^3 e^{\frac{-2\pi i k l}{3}} q^{|\alpha|^2 + \sum_l l(n_l + \bar{n}_l) - \frac{1}{12}} \times \\ &\quad \times i \langle n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s | g^k | n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s \rangle_j, \end{aligned} \quad (6.39)$$

where the notation $\{n\}$ indicates the values of all the various indices n_1, \dots, n_s and we have used the fact that for a complex boson $c = 2$ since the stress-energy tensor is the sum of two free real bosons.

Now, since the states $|n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s\rangle_j$ are orthonormal we have

$$\langle n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s | g^k | n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s \rangle_j = e^{\frac{2\pi i k}{3} \left[\sum_l (n_l - \bar{n}_l) + (j-1)(1-\delta_\alpha) \right]} \delta_{i,j}. \quad (6.40)$$

But for $\alpha \neq 0$

$$\frac{1}{3} \sum_{j=1}^3 e^{\frac{2\pi i k}{3} \left[\sum_l (n_l - \bar{n}_l) + (j-1) \right]} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}, \quad (6.41)$$

where the sum is 0 if $\alpha, k \neq 0$ because it appears a sum of roots of the unit. So the only non-null characters with $h \neq 0$ are the ones where the orbifold acts trivially.

For $\alpha = 0$ there is only a ground state so we have instead

$$\begin{aligned} Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{0,l}(\tau, z) &= \frac{1}{3} \sum_{k=0}^2 \sum_{\{n\}, \{\bar{n}\}} \langle n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s | g^k e^{\frac{-2\pi i k l}{3}} q^{L_0 - \frac{c}{24}} | n_1 \dots n_r \bar{n}_1 \dots \bar{n}_s \rangle = \\ &= \frac{1}{3} \sum_{k=0}^2 \sum_{\{n\}, \{\bar{n}\}} e^{\frac{-2\pi i k l}{3}} e^{\frac{2\pi i k}{3} \sum_l (n_l - \bar{n}_l)} q^{\sum_l l(n_l + \bar{n}_l) - \frac{1}{12}}. \end{aligned} \quad (6.42)$$

Summing up, we have two different forms for the characters

$$\begin{aligned} Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{0,l}(\tau, z) &= \frac{1}{3} \sum_{k=0}^2 \sum_{\{n\}, \{\bar{n}\}} e^{\frac{-2\pi i k l}{3}} e^{\frac{2\pi i k}{3} \sum_l (n_l - \bar{n}_l)} q^{\sum_l l(n_l + \bar{n}_l) - \frac{1}{12}} \quad \text{for } h = 0 \quad \text{massless}, \\ Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^h(\tau, z) &= \sum_{\{n\}, \{\bar{n}\}} q^{h + \sum_l l(n_l + \bar{n}_l) - \frac{1}{12}} \quad \text{for } h \neq 0 \quad \text{massive}. \end{aligned} \quad (6.43)$$

Let us compute $Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{0,l}$.

$$\begin{aligned} & \frac{1}{3} \sum_{k=0}^2 \sum_{\{n\}, \{\bar{n}\}} e^{-\frac{2\pi i k l}{3}} e^{\frac{2\pi i k}{3} \sum_l (n_l - \bar{n}_l)} q^{\sum_l l(n_l + \bar{n}_l) - \frac{1}{12}} = \\ & \frac{1}{3} \sum_{k=0}^2 q^{-\frac{1}{12}} \sum_{n_1} \left(e^{\frac{2\pi i k}{3}} q \right)^{n_1} \sum_{\bar{n}_1} \left(e^{-\frac{2\pi i k}{3}} q \right)^{\bar{n}_1} \sum_{n_2} \left(e^{\frac{2\pi i k}{3}} q^2 \right)^{n_2} \sum_{\bar{n}_2} \left(e^{-\frac{2\pi i k}{3}} q^2 \right)^{\bar{n}_2} \dots = \\ & \frac{1}{3} \sum_{k=0}^2 q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1 - e^{\frac{2\pi i k}{3}} q^n} \frac{1}{1 - e^{-\frac{2\pi i k}{3}} q^n} . \end{aligned} \quad (6.44)$$

With similar calculations one can compute $Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{h,l}$. Finally, the characters for a single complex boson are

$$\begin{aligned} Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{0,l}(\tau, z) &= \frac{1}{3} \sum_{k=0}^2 e^{-\frac{2\pi i k l}{3}} q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1 - e^{\frac{2\pi i k}{3}} q^n} \frac{1}{1 - e^{-\frac{2\pi i k}{3}} q^n} \quad \text{for } h = 0 \text{ massless ,} \\ Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^h(\tau, z) &= q^{h - \frac{1}{12}} \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^2 \quad \text{for } h \neq 0 \text{ massive .} \end{aligned} \quad (6.45)$$

In the $\mathbb{C}^2/\mathbb{Z}_3$ orbifold model there are two complex bosons and the action of g has opposite sign between them. Furthermore the total conformal dimension will be the sum of the ones of the two bosons, in the following we will indicate again with h this sum. The characters are obtained multiplying the addends in (6.45), since the trace in (6.27) factorizes over the two bosons, with the exchange $k \rightarrow -k$ because of the definition of the action of g . So the bosonic part of the characters of the free $\mathbb{C}^2/\mathbb{Z}_3$ orbifold will be

$$\begin{aligned} Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^{0,l}(\tau, z) &= \frac{1}{3} \sum_{k=0}^2 e^{-\frac{2\pi i k l}{3}} q^{-\frac{1}{6}} \prod_{n=1}^{\infty} \left(\frac{1}{1 - e^{\frac{2\pi i k}{3}} q^n} \frac{1}{1 - e^{-\frac{2\pi i k}{3}} q^n} \right)^2 \quad \text{for } h = 0 \text{ massless ,} \\ Ch_{\mathbb{C}^2/\mathbb{Z}_3, bos}^h(\tau, z) &= q^{h - \frac{1}{6}} \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^4 \quad \text{for } h \neq 0 \text{ massive .} \end{aligned} \quad (6.46)$$

6.2.2 Fermionic characters

We compute here the fermionic characters

$$Ch_{\mathbb{C}^2/\mathbb{Z}_3, fer}^l(\tau, z) = \frac{1}{3} \sum_{k=0}^2 Tr_{V_h^{fer}}(g^k e^{-\frac{2\pi i k l}{3}} (e^{i\pi} y)^{J_0} q^{L_0 - \frac{c}{24}}) , \quad (6.47)$$

where we have used $(-1)^F = e^{i\pi J_0}$. Recall that since J_0 is the charge with respect to the $N = 2$ supersymmetry it equals $J_0 = 2J_0^3$ in (3.16). As we have seen in section 2.5.2 for a real fermion we have to add the constant $\frac{1}{16}$ to L_0 in the Ramond sector. Our system is made up by 4 real fermions and the stress-energy tensor in (3.16) is just the sum of 4 free fermion stress-energy tensors. So to L_0 coming from (3.16) (which refers to the NS sector) we have to add the constant $\frac{1}{4}$

$$L_0 = \sum_{\substack{k>0 \\ k \in \mathbb{Z}}} k(\bar{\Psi}_{-k}^1 \Psi_k^1 + \bar{\Psi}_{-k}^2 \Psi_k^2) + \frac{1}{4} . \quad (6.48)$$

This is not the only subtlety in the Ramond sector, in fact we have already seen that there are degenerate ground states. We want to build ground states which form a representation of the $\hat{\mathfrak{su}}(2)$ Kac-Moody algebra (3.3). The following commutation relations are easily obtained

$$[2J_0^3, \Psi_n^i] = \Psi_n^i , \quad [2J_0^3, \bar{\Psi}_n^i] = -\bar{\Psi}_n^i . \quad (6.49)$$

So in particular if we consider the state such that $J_0 |Q\rangle = Q |Q\rangle$, application of Ψ_n increases its charge of one unit while $\bar{\Psi}$ decreases it of the same amount. We start from a state $|\eta\rangle$ which is annihilated by all the Ψ_n^i with $n \geq 0$ and by all the $\bar{\Psi}_{-n}^i$ with $n > 0$. We cannot choose a state annihilated also by $\bar{\Psi}_0^i$ because it would be inconsistent with the anticommutation relations. Choosing it to be annihilated by $\bar{\Psi}_0^i$ instead of Ψ_0^i is equivalent. The following states have the same conformal dimension:

$$|s_1\rangle = |\eta\rangle, \quad |s_2\rangle = \bar{\Psi}_0^1 |\eta\rangle, \quad |s_3\rangle = \bar{\Psi}_0^2 |\eta\rangle, \quad |s_4\rangle = \bar{\Psi}_0^1 \bar{\Psi}_0^2 |\eta\rangle. \quad (6.50)$$

Furthermore, recalling the form of raising and lowering operators of the $\hat{\mathfrak{su}}(2)$ Kac-Moody algebra from (3.16)

$$J_0^- |s_1\rangle = |s_4\rangle, \quad J_0^+ |s_4\rangle = |s_1\rangle, \quad J_0^\pm |s_3\rangle = 0, \quad J_0^\pm |s_2\rangle = 0 \quad (6.51)$$

so $|s_1\rangle$ and $|s_2\rangle$ form a $SU(2)$ doublet while $|s_3\rangle$ and $|s_4\rangle$ are two singlets. Then their charge will be

$$\begin{aligned} J_0^3 |s_1\rangle &= \frac{1}{2} |s_1\rangle & J_0^3 |s_2\rangle &= 0, \\ J_0^3 |s_4\rangle &= -\frac{1}{2} |s_4\rangle & J_0^3 |s_3\rangle &= 0. \end{aligned} \quad (6.52)$$

States will be built acting with negative modes on the ground states $|s_i\rangle$. So a general state will have the form

$$|n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j = \prod_{i=1,2} \Psi_{-1}^i \dots \Psi_{-r_i}^i \bar{\Psi}_{-1}^i \dots \bar{\Psi}_{-s_i}^i |s_j\rangle, \quad (6.53)$$

where n_k^i , $i = 1, 2$, indicates the occupation numbers of the i -th fermion and it can be equal to 0 or 1 because if there are two identical modes then they vanish due to the anticommutation relations. Such a state, using the commutation relations (6.49), will have a charge given by

$$2J_0^3 |n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j = \left[\sum_{i,k} (n_k^i - \bar{n}_k^i) + Q_j \right] |n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j, \quad (6.54)$$

where we have called Q_j the $N = 2$ charge of $|s_j\rangle$.

We can choose the action of the orbifold such that $g |s_1\rangle = 0$, using (6.21) we then have

$$\begin{aligned} g |s_1\rangle &= 0, & g |s_2\rangle &= -1, \\ g |s_4\rangle &= 0, & g |s_3\rangle &= 1. \end{aligned} \quad (6.55)$$

The action of g on a general state will be

$$g |n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j = e^{\frac{2\pi i}{3} \left[\sum_k (n_k^1 - \bar{n}_k^1 - n_k^2 + \bar{n}_k^2) + g_j \right]} |n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j \quad (6.56)$$

with the notation $g |s_j\rangle = g_j |s_j\rangle$.

Using the anticommutation relations of (6.24) it is easy to show

$$[\bar{\Psi}_{-k} \Psi_k, \Psi_{-n}] = \Psi_{-k} \delta_{k,n}, \quad [\bar{\Psi}_{-k} \Psi_k, \bar{\Psi}_{-n}] = \bar{\Psi}_{-k} \delta_{k,n} \quad (6.57)$$

and with calculations similar to the ones of the previous section we have

$$L_0 |n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j = \left[\sum_{l,k} k(n_k^l + \bar{n}_k^l) + \frac{1}{4} \right] |n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i\rangle_j. \quad (6.58)$$

We can now compute

$$\begin{aligned}
Tr_{V_h^{fer}}(g^k e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{J_0} q^{L_0 - \frac{c}{24}}) = \\
\sum_{r,s=1}^4 \sum_{\{n^i\}, \{\bar{n}^j\}} r \langle n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i | g^k e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{J_0} q^{L_0 - \frac{1}{12}} | n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i \rangle_s = \\
\sum_{r,s=1}^4 \sum_{\{n^i\}, \{\bar{n}^j\}} e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{\left[\sum_{i,k} (n_k^i - \bar{n}_k^i) + Q_s \right]} q^{\left[\sum_{i,k} k(n_k^i + \bar{n}_k^i) + \frac{1}{4} \right] - \frac{1}{12}} \\
r \langle n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i | g^k | n_1^i \dots n_{r_i}^i \bar{n}_1^i \dots \bar{n}_{s_i}^i \rangle_s = \\
\sum_{s=1}^4 \sum_{\{n^i\}, \{\bar{n}^j\}} e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{\left[\sum_{i,k} (n_k^i - \bar{n}_k^i) + Q_s \right]} q^{\left[\sum_{i,k} k(n_k^i + \bar{n}_k^i) + \frac{1}{4} \right] - \frac{1}{12}} e^{\frac{2\pi i}{3} k \left[\sum_k (n_k^1 - \bar{n}_k^1 - n_k^2 + \bar{n}_k^2) + g_s \right]} = \quad (6.59) \\
\sum_{s=1}^4 e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{Q_s} e^{\frac{2\pi i}{3} g_s} q^{\frac{1}{6}} \times \\
\sum_{\{n^i\}, \{\bar{n}^j\}} \left[e^{i\pi(1+\frac{2}{3}k)} y q \right]^{n_1^1} \left[e^{i\pi(1+\frac{2}{3}k)} y q^2 \right]^{n_2^1} \dots \left[e^{-i\pi(1+\frac{2}{3}k)} y^{-1} q \right]^{\bar{n}_1^1} \left[e^{-i\pi(1+\frac{2}{3}k)} y^{-1} q^2 \right]^{\bar{n}_2^1} \dots \\
\left[e^{i\pi(1-\frac{2}{3}k)} y q \right]^{n_1^2} \left[e^{i\pi(1-\frac{2}{3}k)} y q^2 \right]^{n_2^2} \dots \left[e^{-i\pi(1-\frac{2}{3}k)} y^{-1} q \right]^{\bar{n}_1^2} \left[e^{-i\pi(1-\frac{2}{3}k)} y^{-1} q^2 \right]^{\bar{n}_2^2} \dots
\end{aligned}$$

Remembering that $n_k^i = \{0, 1\}$ the fermionic part of the characters reads

$$\begin{aligned}
Ch_{\mathbb{C}^2/\mathbb{Z}_3, fer}^l(\tau, z) = \frac{1}{3} \sum_{k=0}^2 \sum_{s=1}^4 e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{Q_s} e^{\frac{2\pi i}{3} g_s} q^{\frac{1}{6}} \prod_{n=1}^{\infty} \left(1 - e^{\frac{2\pi i k}{3}} y q^n \right) \left(1 - e^{-\frac{2\pi i k}{3}} y^{-1} q^n \right) \times \\
\left(1 - e^{-\frac{2\pi i k}{3}} y q^n \right) \left(1 - e^{\frac{2\pi i k}{3}} y^{-1} q^n \right) . \quad (6.60)
\end{aligned}$$

Since the trace in (6.27) factorizes over the fermions and the bosons the full untwisted characters are given by

$$\begin{aligned}
Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{0,l}(\tau, z) = \frac{1}{3} \sum_{k=0}^2 \sum_{s=1}^4 e^{\frac{-2\pi i k l}{3}} (e^{i\pi} y)^{Q_s} e^{\frac{2\pi i}{3} g_s} \prod_{n=1}^{\infty} \left(\frac{1}{1 - e^{\frac{2\pi i k}{3}} q^n} \frac{1}{1 - e^{-\frac{2\pi i k}{3}} q^n} \right)^2 \times \\
\left(1 - e^{\frac{2\pi i k}{3}} y q^n \right) \left(1 - e^{-\frac{2\pi i k}{3}} y^{-1} q^n \right) \left(1 - e^{-\frac{2\pi i k}{3}} y q^n \right) \left(1 - e^{\frac{2\pi i k}{3}} y^{-1} q^n \right) \quad (6.61)
\end{aligned}$$

in the massless case, while massive characters read

$$\begin{aligned}
Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{h \neq 0}(\tau, z) = q^h \sum_{s=1}^4 (e^{i\pi} y)^{Q_s} e^{\frac{2\pi i}{3} g_s} \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^4 \times \\
\left(1 - e^{\frac{2\pi i k}{3}} y q^n \right) \left(1 - e^{-\frac{2\pi i k}{3}} y^{-1} q^n \right) \left(1 - e^{-\frac{2\pi i k}{3}} y q^n \right) \left(1 - e^{\frac{2\pi i k}{3}} y^{-1} q^n \right) . \quad (6.62)
\end{aligned}$$

6.2.3 Twisted characters

We also computed the characters of the $\mathbb{C}^2/\mathbb{Z}_3$ in the twisted sectors. There are 2 twisted sectors corresponding to

$$\begin{aligned}
J^1(z) = e^{\frac{2\pi i}{3} k} J^1(z) , \quad \Psi^1(e^{2\pi i} z) = e^{\frac{2\pi i}{3} k} \Psi^1(z) , \\
J^2(z) = e^{-\frac{2\pi i}{3} k} J^2(z) , \quad \Psi^2(e^{2\pi i} z) = e^{-\frac{2\pi i}{3} k} \Psi^2(z) \quad (6.63)
\end{aligned}$$

with $k = 1, 2$. To satisfy these periodicity conditions the modes of both the bosons and fermions will be rational, i.e.

$$\begin{aligned} J^1(z) &= \sum_{n \in \mathbb{Z} - \frac{k}{3}} z^{-n-1} J_n^1, & \Psi^1(e^{2\pi i} z) &= \sum_{n \in \mathbb{Z} - \frac{k}{3}} z^{-n-1} \Psi_n^1, \\ J^2(z) &= \sum_{n \in \mathbb{Z} + \frac{k}{3}} z^{-n-1} J_n^2, & \Psi^2(e^{2\pi i} z) &= \sum_{n \in \mathbb{Z} + \frac{k}{3}} z^{-n-1} \Psi_n^2. \end{aligned} \quad (6.64)$$

In particular there are not 0-modes anymore, so there are not the difficulties with the ground states encountered in the previous sections. In fact, there is a unique ground state and we can choose the action of the orbifold in the twisted sector such that it will leave the ground state invariant. The only difficulty here is to find the constant to be added to the 0-mode of the stress-energy tensor as we did in section 2.5.2 for the Ramond sector of the free fermion. Since the stress-energy tensor is just the sum of 2 complex bosons and 2 complex fermions stress-energy tensors, we can compute the constant to be added to each of them separately and then add them up. We use the prescriptions

$$\begin{aligned} T_{bos}^i(z) &= \lim_{z \rightarrow w} \left(-J^i(w) \bar{J}^i(z) + \frac{1}{(z-w)^2} \right), \\ T_{fer}^i(z) &= \lim_{z \rightarrow w} \left(\frac{1}{2} (\partial \Psi^i(z) \bar{\Psi}(w)^i + \partial \bar{\Psi}^i(w) \Psi^i(z)) + \frac{1}{(z-w)^2} \right). \end{aligned} \quad (6.65)$$

We compute $\langle J^1(z) \bar{J}^1(w) \rangle$ in the k -th twisted sector:

$$\langle J^1(z) \bar{J}^1(w) \rangle = \sum_{\substack{m \in \mathbb{Z} + \frac{k}{3} \\ n \in \mathbb{Z} - \frac{k}{3}}} z^{-n-1} w^{-m-1} \langle J_n \bar{J}_m \rangle = \sum_{n \in \mathbb{N} + \frac{k}{3}} n z^{-n-1} w^{n-1} = \frac{w^{-\frac{k}{3}} z^{\frac{k}{3}-1} (kw - kz + 3z)}{3(w-z)^2}. \quad (6.66)$$

Expanding in $w - z$, inserting in the expression of $T_{bos}^1(z)$ and taking the limit one finds

$$\langle T_{bos}^1(z) \rangle = \frac{1}{9z^2} \quad (6.67)$$

for both values of k . Analogously

$$\langle T_{bos}^2(z) \rangle = \frac{1}{9z^2} \quad (6.68)$$

for both values of k . We next compute

$$\langle \Psi^1(z) \bar{\Psi}^1(w) \rangle = \sum_{\substack{m \in \mathbb{Z} + \frac{k}{3} \\ n \in \mathbb{Z} - \frac{k}{3}}} z^{-n-\frac{1}{2}} w^{-m-\frac{1}{2}} \langle \Psi_n \bar{\Psi}_m \rangle = \sum_{n \in \mathbb{N} + \frac{k}{3}} z^{-n-\frac{1}{2}} w^{n-\frac{1}{2}} = \frac{w^{\frac{k}{3}-\frac{1}{2}} z^{\frac{1}{2}-\frac{k}{3}}}{z-w}. \quad (6.69)$$

Taking the derivatives, expanding and taking the limit, after a little of computation one finds

$$\langle T_{fer}^1(z) \rangle = \frac{1}{72z^2} \quad (6.70)$$

in both the twisted sectors. $\langle T_{fer}^2(z) \rangle$ has the same value.

Summing all up one obtains that in both sector the constant to be added to L_0 is $\frac{1}{4}$.

The characters can be obtained from the untwisted ones by replacing the integer modes with the new non-integer ones and neglecting the effect of the zero modes but taking care of the new constant to be added to L_0 . We have

$$\begin{aligned} Ch_{\mathbb{C}^2/\mathbb{Z}_3, \text{twisted}}^s(\tau, z) &= \frac{1}{3} \sum_{k=0}^2 \prod_{n=0}^{\infty} \left(\frac{1}{1 - e^{\frac{2\pi i k}{3}} q^{n+\frac{s}{3}}} \frac{1}{1 - e^{-\frac{2\pi i k}{3}} q^{n+1-\frac{s}{3}}} \right)^2 \times \\ &\quad \left(1 - y e^{\left(\frac{2\pi i k}{3}\right)} q^{n+\frac{s}{3}} \right) \left(1 - \frac{e^{\left(-\frac{2}{3}\pi i k\right)} q^{n-\frac{s}{3}+1}}{y} \right) \left(1 - y e^{\left(-\frac{2}{3}\pi i k\right)} q^{n-\frac{s}{3}+1} \right) \left(1 - \frac{e^{\left(\frac{2\pi i k}{3}\right)} q^{n+\frac{s}{3}}}{y} \right). \end{aligned} \quad (6.71)$$

6.3 Decomposition with $\mathbb{C}^2/\mathbb{Z}_3$ characters

We want to decompose the elliptic genus in a way similar to (5.10) to investigate if, with the algebra of the $\mathbb{C}^2/\mathbb{Z}_3$ orbifold, the decomposition in irreducible representations admit only positive coefficients.

Since the orbifold $\mathbb{C}^2/\mathbb{Z}_3$ is non-compact, the spectrum of L_0 acquires also a continuous part. We will however only consider the discrete part of the spectrum since this is the interesting part to make connections with the elliptic genus of $K3$. So we consider a decomposition with $h = n \in \mathbb{N}^2$

$$\phi_{K3}(\tau, z) = A_{0,l} \sum_{l=0}^2 Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{0,l}(\tau, z) + \sum_{n=1}^{\infty} A_n Ch_{\mathbb{C}^2/\mathbb{Z}_3}^n(\tau, z) + \sum_{s=1}^2 B_s Ch_{\mathbb{C}^2/\mathbb{Z}_3, \text{twisted}}^s(\tau, z) . \quad (6.72)$$

We computed some coefficients expanding in powers of q and equating order by order with the expansion of the elliptic genus from formula (3.22). To the order considered the expansions of the twisted characters B_s , $s = 1, 2$, resulted equal and it could be obtained from a linear combination of untwisted characters expanded to the same order so we set $B_s = 0$, $s = 1, 2$. Furthermore, also the expansions of $A_{0,1}$ and $A_{0,2}$ were the same so we were able to obtain only their sum that we will indicate with $A_{0,1} := A_{0,1} + A_{0,2}$. We give here the results

$A_{0,0}$	$A_{0,1}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
-2	20	72	216	72	504	432	216	576	1080

If we assume $A_{0,l} = \dim H_{0,l}^G$ and $A_n = \dim H_n^G$ for some umbral group G , we want to find a decomposition in terms of irreducible representation as we did in section 6.1. We write

$$H_{0,l}^G = \bigoplus_j h_{0,l,j} R_j^G \quad H_n^G = \bigoplus_j h_{n,j} R_j^G , \quad (6.73)$$

where R_j^G are irreducible representations of the group G . The twining genera will read

$$\phi_g(\tau, z) = \sum_{l=0}^2 Tr_{H_{0,l}}(g) Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{0,l}(\tau, z) + \sum_{n=1}^{\infty} Tr_{H_n}(g) Ch_{\mathbb{C}^2/\mathbb{Z}_3}^n(\tau, z) . \quad (6.74)$$

Using

$$\sum_g c(g) \overline{Tr_{R_i}(g)} Tr_{R_j}(g) = \delta_{ij} \quad (6.75)$$

and

$$\phi_g(\tau, z) = \frac{\phi_g(\tau, 0)}{12} \chi_{0,1}(\tau, z) + F_g(\tau) \chi_{-2,1}(\tau, z) \quad (6.76)$$

we obtain³

$$\begin{aligned} \sum_g c(g) \overline{Tr_{R_i}(g)} F_g(\tau) \chi_{-2,1}(\tau, z) &= \sum_{l=0}^1 h_{0,l,i} \left(Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{0,l}(\tau, z) - \frac{\chi_{0,1}}{12} Ch_{\mathbb{C}^2/\mathbb{Z}_3}^{0,l}(\tau, 0) \right) + \\ &\quad \sum_{n=1}^{\infty} h_{n,i} \left(Ch_{\mathbb{C}^2/\mathbb{Z}_3}^n(\tau, z) - \frac{\chi_{0,1}}{12} Ch_{\mathbb{C}^2/\mathbb{Z}_3}^n(\tau, 0) \right) . \end{aligned} \quad (6.77)$$

²Notice that here h refers only to the bosonic conformal dimension because we have canceled out the $\frac{1}{4}$ coming from the fermionic conformal dimension. If one consider the total conformal dimension one obtains $h = \frac{1}{4} + n$ as in (5.10)

³Again $h_{0,1,i} := h_{0,1,i} + h_{0,2,i}$ because massless characters with $l = 1$ and $l = 2$ were equal up to the order considered.

Again we set $h_{0,0,i} = -2\delta_{i,1}$. Results are shown in appendix E. Among the coefficients there are again coefficients with negative sign (actually a lot more than before although this could be meaningless since the presence of negative coefficients can be hidden as we explained in the previous section). The cause of the issue could be that the new algebra is too large. In fact if some of the characters are too big, some coefficients must take negative sign to match the expansion of the elliptic genus of $K3$. This, indeed, could happen in our case since we have seen that the massive characters of the orbifold are exactly the same of 4 free bosons and fermions which is the largest supersymmetric algebra we can build in 4 dimensions. If this was not the main issue one in principle could try a decomposition with characters of the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ for some $N > 3$, in fact all these models still have a $\hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{u}}(1)$ current algebra. Another possibility, which would still give the right current algebra, could be to consider more exotic orbifolds with continuous groups like $\mathbb{C}^2/U(1)$.

7. Conclusions

In this thesis we have considered some aspects of non-linear sigma models on $K3$ surfaces, a particular kind of $N = (4, 4)$ superconformal field theories. These models are widely studied in an attempt to understand some features of string compactification which constitute an indispensable ingredient for consistent string theories. In particular we considered the decomposition of the elliptic genus in terms of the characters of the $N = 4$ superconformal algebra, and some abstract aspects of the umbral moonshine conjecture to extract informations on some non-linear sigma models which are not treatable with standard methods. This underlines the importance of the study of non-standard methods and of the mathematical properties of the theory in order to find new ways to extract informations where they seem to be inaccessible with the standard approaches.

We have found evidences that the models which possess $L_2(11)$ as a symmetry group, whose existence is known thanks to the fact that the symmetries of non-linear models on $K3$ surfaces have been classified, could admit a chiral algebra extended beyond the $N = (4, 4)$ superconformal algebra. In particular a plausible extension could contain a $\hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{u}}(1)$ Kac-Moody algebra.

Subsequently, we have investigated the $\mathbb{C}^2/\mathbb{Z}_3$ orbifold model, which seemed to constitute a good candidate, but we have concluded, with the same method used previously, that its algebra do not solve the problem of the appearance of negative coefficients. We have indicated some other candidates which can possess the right extended algebra, namely $\mathbb{C}^2/\mathbb{Z}_N$ models with $N > 3$ (although if the problem of the previous model is that the massive characters are too big, these models would present the same issue) and more exotic orbifolds with a continuous group like $\mathbb{C}^2/U(1)$. Other possible extensions of this work include the search for alternative interpretations of the appearance of negative sign coefficients in umbral groups which are not symmetry groups of non-linear sigma models, as well as the extension of the analysis of the decomposition of the elliptic genus in terms of irreducible representations for the other umbral groups and of the subgroups which are symmetry groups of some non-linear sigma model on $K3$.

Appendices

A. Definitions

In this section we collect some important definitions of standard functions we used throughout the work and that were not defined elsewhere.

For the rest of this section, we set $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$.

The Dedekind eta is a modular function of weight $\frac{1}{2}$. We used the following definition for the Dedekind eta function

$$\eta(t) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) . \quad (\text{A.1})$$

Jacobi theta functions are Jacobi forms (see (3.21)) of weight $\frac{1}{2}$ and index 1. The conventions we used for the Jacobi theta functions are

$$\begin{aligned} \theta_1(\tau, z) &= -iq^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^{n-1}) , \\ \theta_2(\tau, z) &= 2q^{\frac{1}{8}}\cos(\pi z) \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^n) , \\ \theta_3(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^{n-\frac{1}{2}})(1 + y^{-1}q^{n-\frac{1}{2}}) , \\ \theta_4(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^{n-\frac{1}{2}})(1 - y^{-1}q^{n-\frac{1}{2}}) . \end{aligned} \quad (\text{A.2})$$

The Eisenstein series E_{2n} , for $n > 1$, are modular forms of weight $2n$. They are given by

$$E_{2n}(q) = 1 + c_{2n} \sum_{k=1}^{\infty} \frac{k^{2n-1}q^{2k}}{1 - q^{2k}} , \quad (\text{A.3})$$

where $c_{2n} = -\frac{4n}{B_{2n}}$ and B_{2n} are the Bernoulli numbers given by

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}} . \quad (\text{A.4})$$

B. Basics of complex geometry

The purpose of this section is to give an overview of the basic concepts of complex geometry needed to understand the geometrical objects used in this work, it is not thought to be an exhaustive introduction to the subject. We will just expose facts without proofs and we will not discuss them much, for a more exhaustive exposition see [42] and references therein. It is assumed a basic knowledge of differential and Riemannian geometry.

Let M be a $2m$ dimensional real manifold (we will always assume them to be C^∞), we will denote the space of tensor fields of type (k, l) by $\Gamma(\bigotimes^k TM \otimes^l T^*M)$. We define an almost complex structure to be a smooth tensor field $J \in \Gamma(TM \otimes T^*M)$ such that $J^2 = \mathbb{1}$. Given an almost complex structure J his Nijenhuis tensor is defined as

$$N_J(v, w) = [v, w] + J[v, Jw] + J[Jv, w] - [Jv, Jw] . \quad (\text{B.1})$$

Definition. A complex manifold M , of complex dimension m , is a $2m$ real manifold equipped with an almost complex structure J such that his Nijenhuis tensor is vanishing, i.e. $N_J \equiv 0$

This definition is equivalent to the “standard” one which uses local charts from M to \mathbb{C} .

At every point $p \in M$, J associate a linear map $J : T_p M \rightarrow T_p M$. If we complexify the tangent space this map extends naturally to the complexified tangent space $J : T_p M \otimes \mathbb{C} \rightarrow T_p M \otimes \mathbb{C}$. Since $J^2 = -\mathbb{1}$ its eigenvalues in $T_p M \otimes \mathbb{C}$ are $\pm i$. Let $T_p^{1,0} M$ ($T_p^{0,1} M$ respectively) be the eigenspace of eigenvalue i ($-i$), then $T_p M \otimes \mathbb{C} = T_p^{1,0} M \oplus T_p^{0,1} M$ and since this is true for every $p \in M$ it can be extended to the whole tangent bundle TM and we write $T_{\mathbb{C}} M = T^{1,0} M \oplus T^{0,1} M$ and analogously for the cotangent bundle. Let us denote with $\Omega(M)$ the space of k -forms, i.e. the space of smooth sections of $\Lambda^k T^*M$, then the complexified bundles decompose as

$$\Lambda^k T_{\mathbb{C}}^* M = \bigoplus_{j=0}^k \Lambda^{j, k-j} M , \quad (\text{B.2})$$

where $\Lambda^{p,q} M = \Lambda^p T^{*(1,0)} M \wedge \Lambda^q T^{*(0,1)} M$. We denote the space of (p, q) forms as $\Omega^{(p,q)}(M)$. Let $\alpha = \sum_{IJ} f_{IJ} dz^I \wedge d\bar{z}^J \in \Omega^{p,q}$ be a (p, q) form, where I, J are suitable multi-indices, defining the exterior derivatives

$$\begin{aligned} \partial : \Omega^{(p,q)} &\rightarrow \Omega^{(p+1,q)} \\ \alpha &\rightarrow \sum_{I, J, k} \frac{\partial f_{IJ}}{\partial z_k} dz^k \wedge dz^I \wedge d\bar{z}^J \end{aligned}$$

and

$$\begin{aligned} \bar{\partial} : \Omega^{(p,q)} &\rightarrow \Omega^{(p,q+1)} \\ \alpha &\rightarrow \sum_{I, J, k} \frac{\partial f_{IJ}}{\partial \bar{z}_k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J , \end{aligned}$$

the usual exterior differential decomposes as $d = \partial + \bar{\partial}$. It is easy to show that it holds: $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

Definition. The Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}(M)$ of M are defined as

$$H_{\bar{\partial}}^{p,q}(M) = \frac{\text{Ker}(\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M))}{\text{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \rightarrow \Omega^{p,q}(M))}.$$

The definition using ∂ instead of $\bar{\partial}$ is equivalent to this one. We also define the Hodge numbers $h^{p,q} := \dim H_{\bar{\partial}}^{p,q}(M)$.

Let now M be a complex manifold with complex structure J , and g an hermitian metric on M . We call g hermitian if $g(v, w) = g(Jv, Jw)$. Given a hermitian metric g we can define the two-form $\omega(v, w) = g(Jv, w)$, which is called the hermitian form associated to the hermitian metric g . We are now ready to define what a Kähler manifold is.

Definition. Let (M, J) be a m -dimensional complex manifold, g a Riemannian form on M . We say g is a Kähler metric if its associated hermitian two-form is closed, i.e. $d\omega = 0$, and in this case we call ω a Kähler form. A complex manifold equipped with a Kähler metric is called a Kähler manifold.

Although Kähler manifolds are interesting objects we will work with more involved objects, namely Calabi-Yau manifold. To define what a Calabi-Yau manifold is we need another important concept: holonomy.

Definition. Let M be a n -dimensional Riemannian manifold with metric g and an affine connection ∇ . Given a point $p \in M$ let us consider the set of all closed loops $\{\gamma(t) : 0 \leq t \leq 1, \gamma(0) = \gamma(1) = p\}$. Let us consider the linear transformation $P_\gamma : T_p M \rightarrow T_p M$ which takes a vector $V \in T_p M$ and parallel transports it along $\gamma(t)$. We denote the set of all these transformations $Hol_p(M)$ and call it the holonomy group of M at p .

Clearly $Hol_p(M)$ is a subgroup of $GL(n, (R))$ which is the maximal holonomy group possible. If M is connected (we will always work with connected manifolds), since every two points p, q on the manifold are then connected by some curve, parallel transport along that curve defines an isomorphism between $Hol_p(M)$ and $Hol_q(M)$. So the holonomy group $Hol_p(M)$ is actually independent on the base point p and we will simply denote it with $Hol(M)$. If M is a n -dimensional orientable Riemannian manifold and ∇ is a metric connection, i.e. parallel transport preserves the length of vectors, then $Hol(M)$ must be a subgroup of $SO(n)$. It can be shown that if M is a Kähler manifold of real dimension $2m$, then its holonomy group is a subgroup of $U(m)$.

Definition. A Calabi-Yau manifold, of real dimension $2m$, is a Kähler manifold M with holonomy group $Hol(M)$ contained in $SU(m)$.

It can be shown that the previous definition is equivalent to the following one

Definition. A Calabi-Yau manifold is a Ricci-flat Kähler manifold.

Calabi-Yau manifolds admit many other equivalent definitions and possess a lot of interesting properties but we will not discuss them here, see [42] and references therein for an overview. With all this background we are finally ready to define $K3$ surfaces

Definition. A $K3$ surface is Calabi-Yau manifold M of complex dimension 2 with $h^{1,0}(M) = 0$.

Actually there are only two kinds of Calabi-Yau surfaces of complex dimension 2, $K3$ surfaces and the toruses.

$K3$ surfaces possess more properties than general Calabi-Yau manifolds, it can be shown in

fact that they are hyperkähler manifolds.

Definition. A hyperkähler manifold, is a Calabi-Yau manifold M of real dimension $4m$, with holonomy group $Hol(M)$ contained in $Sp(m)$.

The tangent bundle of a hyperkähler manifold possesses a quaternionic structure generated by 3 almost complex structures $I^2 = J^2 = K^2 = -1$.

C. Roots and Lattices

We give in this appendix the basic definitions of roots and lattices mainly used in section 5. This section is intended mostly to fix the notation throughout this work and it is not intended as an exhaustive introduction to the topic. See [11] or [27] and references therein for a basic introduction to the subject. We will follow the approach of [27].

Let V be a finite-dimensional real vector space of dimension r with an inner product $\langle \cdot, \cdot \rangle$. A finite subset $X \subset V$ is called a root system of rank r if

- X spans V ;
- X is closed under reflections i.e. if $\alpha, \beta \in X$ then $\beta - \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in X$;
- if $\alpha \in X$ and $c\alpha \in X$ then $c \in \{-1, 1\}$;
- $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

The elements of a root system are called roots. A root system is called irreducible if there are not two proper orthogonal subsets X_1, X_2 such that $X = X_1 \cup X_2$. It can be proven that the roots of an irreducible root system have at most two possible lengths¹. If all the roots have the same length the root system is called simply laced. Simple roots are a subset of roots f_i such that every other root can be written as a linear combination of simple roots with all positive or all negative coefficients. The subset of simple roots is unique up to the action of the Weyl group $\text{Weyl}(X)$ which is generate by reflections with respect to all roots. An irreducible root system possesses a highest root θ which is defined by the property that the expansion $\theta = \sum_{i=1}^r a_i f_i$ maximizes the sum $\sum_i a_i$. The Coxeter number of X is given by

$$\text{Cox}(X) := 1 + \sum_{i=1}^r a_i . \quad (\text{C.1})$$

To each irreducible root system is possible to associate a connected Dynkin diagram in the following way:

- Each simple root is associated with a node
- Nodes associated to two distinct simple roots f_i, f_j are connected with N_{ij} lines, with

$$N_{ij} = \frac{\langle f_i, f_j \rangle}{\langle f_i, f_i \rangle} \frac{\langle f_j, f_i \rangle}{\langle f_j, f_j \rangle} . \quad (\text{C.2})$$

For simply laced root systems we have $N_{ij} = \{0, 1\}$. In this case the only possible Dynkin diagrams belong to two infinite families A_n, D_n or three exceptional cases E_6, E_7, E_8 , where

¹We call the norm of a root its length.

the subscript denotes the rank of the corresponding root system, corresponding to

$$\begin{array}{ll}
 A & \bullet \cdots \bullet \\
 D & \bullet \cdots \bullet \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array} \\
 E_6 & \bullet \cdots \bullet \begin{array}{c} \bullet \\ | \end{array} \\
 E_7 & \bullet \cdots \bullet \begin{array}{c} \bullet \\ | \end{array} \bullet \\
 E_8 & \bullet \cdots \bullet \begin{array}{c} \bullet \\ | \end{array} \bullet \cdots \bullet
 \end{array} \tag{C.3}$$

A lattice L of rank n is a free abelian group isomorphic to the additive group \mathbb{Z}^n with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. See [43] for an introduction on the subject. A lattice L is positive-definite if, when embedded in \mathbb{R}^n , the bilinear form induces a positive-definite scalar product on \mathbb{R}^n . It is said integral if $\langle a, b \rangle \in \mathbb{Z}$ for every $a, b \in L$ while it is said even if $\langle a, a \rangle \in 2\mathbb{Z}$ for all $a \in L$. We call a lattice L unimodular if it is isomorphic to its dual lattice defined by

$$L^* := \{ \lambda \in L \otimes_{\mathbb{Z}} \mathbb{R} : \langle \lambda, a \rangle \in \mathbb{Z} \forall a \in L \} . \tag{C.4}$$

The elements $a \in L$ such that $\langle a, a \rangle = 2$ are called the roots of L . They form a root system. Even unimodular lattices of signature (a, b) can only occur when $a - b$ is a multiple of 8. There is only one positive definite even unimodular lattice in 8 dimensions, and two of them in 16 dimensions. It was proven by Niemeier that there exists 24 even unimodular positive-definite lattices in 24 dimensions. One of them, the Leech lattice, has no roots. We will call the other 23 lattices with non-trivial root systems Niemeier lattices. It can also be proven that the Niemeier lattices are uniquely identified by their root systems X , called the Niemeier root systems. Niemeier root systems are given by union of simply-laced root systems $X = \cup_i Y_i$ such that they have the same Coxeter number $\text{Cox}(Y_i) = \text{Cox}(Y_j)$ and that the total rank is equal to the rank of the lattice L , i.e. $\sum_i \text{rk}(Y_i) = 24$. A list of the 23 Niemeier lattices is given in section 5.

D. Coefficients of $N = 4$ algebra decomposition

In this section we collect the results of the decomposition of the twining genera in terms of the characters of the $N = 4$ superconformal algebra. We considered the groups: M_{10} , $2.AGL_3(2)$, $2.M_{12}$, $L_2(11)$. The notation $2.M_{12}$ indicates the group such that $2.M_{12}/\mathbb{Z}_2 = M_{12}$.

We will identify, for the various group, the irreducible representations R_i by their dimension as in the following tables

Table D.1: Dimensions of the irreducible representations of the group $L_2(11)$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
d	1	5	5	10	10	11	12	12

Table D.2: Dimensions of the irreducible representations of the group M_{10}

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
d	1	1	9	9	10	10	10	16

Table D.3: Dimensions of the irreducible representations of the group $2.AGL_3(2)$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}
d	1	3	3	6	7	8	7	7	14	21	21	8	8	8	24	24

Table D.4: Dimensions of the irreducible representations of the group $2.M_{12}$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}
d	1	11	11	16	16	45	54	55	55	55	66	99	120	144
	R_{15}	R_{16}	R_{17}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}	R_{25}	R_{26}		
d	176	10	10	12	32	44	44	110	110	120	160	160		

We present here the coefficients of decomposition (6.11). There are 2 tables for the group $L_2(11)$ referring to the decomposition obtained considering the action of $2.M_{12}$ or M_{24} . The columns label the coefficients of the various order of q^n while the rows label the irreducible representations of the group.

Table D.5: Coefficients for the group $L_2(11)$ with the action of $2.M_{12}$

Ri	H_{00}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}
R_1	4	2	-2	0	-2	2	0	38	62	180	342	760	1364	2730	4790	8728
R_2	0	0	1	7	10	41	91	221	467	1029	2008	4006	7502	13960	25037	44479
R_3	0	0	1	7	10	41	91	221	467	1029	2008	4006	7502	13960	25037	44479
R_4	0	0	4	6	34	68	192	424	980	1980	4116	7900	15134	27746	50336	88562
R_5	2	0	2	10	24	82	174	454	934	2042	4036	8012	14976	27952	50074	88910
R_6	0	0	2	8	26	70	202	454	1042	2174	4436	8660	16530	30446	55126	97332
R_7	0	0	0	6	24	78	200	492	1098	2354	4788	9414	17894	33168	59916	106078
R_8	0	0	0	6	24	78	200	492	1098	2354	4788	9414	17894	33168	59916	106078

 Table D.6: Coefficients for the group $L_2(11)$ with the action of M_{24}

Ri	H_{00}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}
R_1	4	2	0	2	2	8	10	52	84	210	386	820	1448	2838	4940	8922
R_2	0	0	0	6	8	38	86	214	456	1014	1986	3976	7460	13906	24962	44382
R_3	0	0	0	6	8	38	86	214	456	1014	1986	3976	7460	13906	24962	44382
R_4	0	0	2	4	30	62	182	410	958	1950	4072	7840	15050	27638	50186	88368
R_5	2	0	0	8	20	76	164	440	912	2012	3992	7952	14892	27844	49924	88716
R_6	0	0	2	8	26	70	202	454	1042	2174	4436	8660	16530	30446	55126	97332
R_7	0	0	2	8	28	84	210	506	1120	2384	4832	9474	17978	33276	60066	106272
R_8	0	0	2	8	28	84	210	506	1120	2384	4832	9474	17978	33276	60066	106272

 Table D.7: Coefficients for the group M_{10}

Ri	H_{00}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}
R_1	4	2	0	0	0	8	14	46	74	192	360	742	1332	2592	4548	8168
R_2	1	0	0	2	2	8	10	46	80	190	356	748	1338	2588	4536	8170
R_3	0	0	0	6	18	60	140	354	762	1650	3306	6534	12332	22914	41244	73130
R_4	1	0	2	6	16	62	144	352	758	1652	3312	6532	12324	22918	41254	73126
R_5	1	0	0	8	20	68	150	400	842	1840	3662	7282	13670	25502	45780	81300
R_6	0	0	2	4	26	58	166	378	874	1794	3724	7196	13786	25350	45982	81036
R_7	0	0	2	4	26	58	166	378	874	1794	3724	7196	13786	25350	45982	81036
R_8	0	0	2	12	32	100	262	614	1372	2920	5892	11582	21986	40660	73414	129900

 Table D.8: Coefficients for the group $2.AGL_3(2)$

Ri	H_{00}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}
R_1	2	-4	2	-10	-18	-32	-62	-96	-162	-238	-354	-490	-690	-836	-1046	-1064
R_2	0	3	0	4	12	22	52	88	176	307	580	988	1770	2990	5176	8598
R_3	0	3	0	4	12	22	52	88	176	307	580	988	1770	2990	5176	8598
R_4	0	0	0	0	0	6	10	38	86	214	436	928	1792	3478	6372	11586
R_5	0	0	0	0	6	10	38	82	196	376	800	1490	2856	5140	9296	16134
R_6	0	0	0	0	2	6	26	54	146	316	684	1356	2686	4994	9230	16432
R_7	0	2	0	6	18	30	64	134	266	496	954	1742	3176	5634	9912	17074
R_8	2	2	0	8	12	36	60	144	246	522	928	1782	3110	5714	9822	17202
R_9	0	0	0	6	10	38	84	202	400	854	1622	3168	5798	10628	18754	32956
R_{10}	0	0	0	0	8	22	74	174	428	906	1920	3774	7334	13612	24898	44152
R_{11}	0	0	0	2	6	28	66	184	412	928	1890	3818	7276	13688	24796	44284
R_{12}	1	0	2	4	14	30	60	134	264	510	994	1850	3384	6116	10794	18752
R_{13}	0	0	2	6	12	30	64	128	264	518	984	1850	3398	6100	10794	18776
R_{14}	0	0	2	6	12	30	64	128	264	518	984	1850	3398	6100	10794	18776
R_{15}	0	0	0	2	10	32	90	224	512	1106	2276	4500	8608	16006	29024	51488
R_{16}	0	0	0	2	10	32	90	224	512	1106	2276	4500	8608	16006	29024	51488

Table D.9: Coefficients for the group $2.M_{12}$

Ri	H_{00}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}
R_1	1	2	-4	-6	-10	-18	-26	-40	-60	-86	-124	-174	-238	-322	-436	-576
R_2	1	0	2	2	4	8	12	18	26	40	64	94	142	228	354	542
R_3	0	0	0	0	0	0	0	2	6	8	16	36	64	116	206	358
R_4	0	0	1	3	4	7	13	19	31	49	74	120	190	296	473	757
R_5	0	0	1	3	4	7	13	19	31	49	74	120	190	296	473	757
R_6	0	0	0	0	0	0	2	6	12	30	62	118	232	428	770	1376
R_7	0	0	0	0	0	0	2	6	12	30	66	132	258	486	894	1598
R_8	0	0	0	0	0	0	0	2	8	22	54	114	232	454	844	1534
R_9	0	0	0	2	6	8	14	28	46	78	134	232	404	688	1168	1982
R_{10}	0	0	0	0	0	2	6	10	22	48	92	172	318	580	1030	1794
R_{11}	0	0	0	0	0	0	2	8	18	40	88	174	332	626	1134	2010
R_{12}	0	0	0	0	0	2	8	16	34	76	148	284	540	984	1766	3116
R_{13}	0	0	0	0	0	0	2	10	26	60	136	282	554	1056	1944	3490
R_{14}	0	0	0	0	2	6	10	24	54	106	214	414	772	1422	2556	4492
R_{15}	0	0	0	0	0	2	8	20	50	112	230	462	890	1656	3012	5348
R_{16}	0	0	2	2	6	8	14	20	38	50	86	120	190	274	442	638
R_{17}	0	0	2	2	6	8	14	20	38	50	86	120	190	274	442	638
R_{18}	1	0	0	2	2	4	4	14	16	36	46	92	122	236	336	592
R_{19}	0	0	0	0	0	0	4	4	12	24	44	88	172	304	552	980
R_{20}	0	0	0	2	2	6	8	18	26	60	92	178	294	538	888	1576
R_{21}	0	0	0	2	2	6	8	18	26	60	92	178	294	538	888	1576
R_{22}	0	0	0	0	2	2	10	16	42	76	170	302	604	1066	1968	3398
R_{23}	0	0	0	0	2	2	10	16	42	76	170	302	604	1066	1968	3398
R_{24}	0	0	0	0	0	4	4	20	32	84	152	332	588	1156	2016	3676
R_{25}	0	0	0	0	0	2	6	18	44	98	204	414	794	1490	2712	4826
R_{26}	0	0	0	0	0	2	6	18	44	98	204	414	794	1490	2712	4826

E. Coefficients of $\mathbb{C}^2/\mathbb{Z}_3$ characters decomposition

We present here the coefficients of decomposition (6.77). There are again 2 tables for the group $L_2(11)$ referring to the decomposition obtained considering the action of $2.M_{12}$ or M_{24} . The coefficient $H_{0,1}$ is actually the sum $H_{0,1} + H_{0,2}$. The columns label the coefficients of the various order of q^n while the rows label the irreducible representations of the group as in appendix D.

Table E.1: Coefficients for the group $L_2(11)$ with the action of $2.M_{12}$

Ri	$H_{0,0}$	$H_{0,1}$	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
R_1	-2	0	0	10	-24	18	26	-50	12	46
R_2	0	0	1	4	-11	16	3	-2	2	34
R_3	0	0	1	4	-11	16	3	-2	2	34
R_4	0	0	4	-6	16	-14	18	18	20	-42
R_5	0	2	0	6	-12	30	-6	6	-12	66
R_6	0	0	2	2	2	2	32	-22	16	2
R_7	0	0	0	6	6	6	-4	12	12	18
R_8	0	0	0	6	6	6	-4	12	12	18

Table E.2: Coefficients for the group $L_2(11)$ with the action of M_{24}

Ri	$H_{0,0}$	$H_{0,1}$	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
R_1	-2	0	2	6	-26	22	28	-46	8	46
R_2	0	0	0	6	-10	14	2	-4	4	34
R_3	0	0	0	6	-10	14	2	-4	4	34
R_4	0	0	2	-2	18	-18	16	14	24	-42
R_5	0	2	-2	10	-10	26	-8	2	-8	66
R_6	0	0	2	2	2	2	32	-22	16	2
R_7	0	0	2	2	4	10	-2	16	8	18
R_8	0	0	2	2	4	10	-2	16	8	18

Table E.3: Coefficients for the group M_{10}

Ri	$H_{0,0}$	$H_{0,1}$	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
R_1	-2	0	2	4	-22	28	22	-74	16	92
R_2	0	1	-1	3	-7	7	4	3	-8	-1
R_3	0	0	0	6	0	6	-10	24	0	22
R_4	0	1	1	1	-5	29	-4	-23	8	69
R_5	0	1	-1	9	-7	13	-6	27	-8	21
R_6	0	0	2	-2	14	-10	12	10	16	-26
R_7	0	0	2	-2	14	-10	12	10	16	-26
R_8	0	0	2	6	-4	14	22	-12	16	30

 Table E.4: Coefficients for the group $2.AGL_3(2)$

Ri	$H_{0,0}$	$H_{0,1}$	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
R_1	-2	-2	0	4	-4	12	0	-32	14	34
R_2	0	0	3	-5	0	1	6	-8	1	11
R_3	0	0	3	-5	0	1	6	-8	1	11
R_4	0	0	0	0	0	6	-8	8	2	6
R_5	0	0	0	0	6	-8	8	-2	0	-22
R_6	0	0	0	0	2	0	8	-14	14	8
R_7	0	0	2	0	0	-14	4	32	0	-24
R_8	0	2	0	4	-18	20	8	-22	0	54
R_9	0	0	0	6	-8	8	0	0	-16	32
R_{10}	0	0	0	0	8	-2	8	-8	16	-8
R_{11}	0	0	0	2	0	10	-8	16	0	8
R_{12}	0	1	1	-1	-1	3	-2	1	8	3
R_{13}	0	0	2	0	-6	4	4	-4	16	4
R_{14}	0	0	2	0	-6	4	4	-4	16	4
R_{15}	0	0	0	2	4	2	4	4	0	6
R_{16}	0	0	0	2	4	2	4	4	0	6

Table E.5: Coefficients for the group $2.M_{12}$

Ri	$H_{0,0}$	$H_{0,1}$	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
R_1	-2	-3	1	7	-5	-3	6	-21	8	19
R_2	0	1	1	-3	-5	11	6	-21	8	21
R_3	0	0	0	0	0	0	0	2	0	-10
R_4	0	0	1	0	-5	0	7	0	2	0
R_5	0	0	1	0	-5	0	7	0	2	0
R_6	0	0	0	0	0	0	2	0	-6	4
R_7	0	0	0	0	0	0	2	0	-6	4
R_8	0	0	0	0	0	0	0	2	2	-2
R_9	0	0	0	2	0	-10	0	16	2	-4
R_{10}	0	0	0	0	0	2	0	-8	2	12
R_{11}	0	0	0	0	0	0	2	2	-6	-4
R_{12}	0	0	0	0	0	2	2	-8	-4	14
R_{13}	0	0	0	0	0	0	2	4	-4	-8
R_{14}	0	0	0	0	2	0	-8	4	12	-6
R_{15}	0	0	0	0	0	2	2	-4	0	2
R_{16}	0	0	2	-4	0	0	0	8	4	-8
R_{17}	0	0	2	-4	0	0	0	8	4	-8
R_{18}	0	1	-1	3	-7	3	10	-11	4	7
R_{19}	0	0	0	0	0	0	4	-8	0	8
R_{20}	0	0	0	2	-4	0	0	4	2	8
R_{21}	0	0	0	2	-4	0	0	4	2	8
R_{22}	0	0	0	0	2	-4	4	-4	4	0
R_{23}	0	0	0	0	2	-4	4	-4	4	0
R_{24}	0	0	0	0	0	4	-8	8	-8	8
R_{25}	0	0	0	0	0	2	0	0	0	-4
R_{26}	0	0	0	0	0	2	0	0	0	-4

Bibliography

- [1] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.
- [2] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.
- [3] Georges Obied, Hiroshi Ooguri, Lev Spodyneiko, and Cumrun Vafa. De Sitter Space and the Swampland. 2018.
- [4] Prateek Agrawal, Georges Obied, Paul J. Steinhardt, and Cumrun Vafa. On the Cosmological Implications of the String Swampland. *Phys. Lett.*, B784:271–276, 2018.
- [5] Tohru Eguchi, Hiroshi Ooguri, and Yuji Tachikawa. Notes on the $k3$ surface and the mathieu group m_{24} . *Experimental Mathematics*, 20(1):91–96, 2011.
- [6] Miranda C. N. Cheng and Sarah Harrison. Umbral moonshine and $k3$ surfaces. *Communications in Mathematical Physics*, 339(1):221–261, Oct 2015.
- [7] I. B. Frenkel, J. Lepowsky, and A. Meurman. A natural representation of the fischer-griess monster with the modular function j as character. *Proceedings of the National Academy of Sciences of the United States of America*, 81(10):3256–3260, 1984.
- [8] Igor Frenkel, James Lepowsky, and Arne Meurman. A moonshine module for the monster. In James Lepowsky, S. Mandelstam, and I. M. Singer, editors, *Math. Sci. Res. Inst. Publ.*, volume 3, pages 231–273. Springer, 1985.
- [9] Matthias R. Gaberdiel. An Introduction to conformal field theory. *Rept. Prog. Phys.*, 63:607–667, 2000.
- [10] Katrin Wendland. *Moduli spaces of unitary conformal field theories*. PhD thesis, Bonn U., 2000.
- [11] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.
- [12] R Blumenhagen and E Plauschinn. Introduction to conformal field theory: With applications to string theory, vol. 779 of lect. *Notes Phys. Springer*, 2009.
- [13] Paul H. Ginsparg. APPLIED CONFORMAL FIELD THEORY. In *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988*, pages 1–168, 1988.
- [14] K. Wendland. Superconformal orbifolds involving the fermion number operator. *Int. J. Mod. Phys.*, A19:3637–3667, 2004.
- [15] Adel Bilal. Introduction to supersymmetry. 2001.

- [16] Edward Witten. Elliptic Genera and Quantum Field Theory. *Commun. Math. Phys.*, 109:525, 1987.
- [17] W. Lerche, B. E. W. Nilsson, A. N. Schellekens, and N. P. Warner. Anomaly Cancelling Terms From the Elliptic Genus. *Nucl. Phys.*, B299:91–116, 1988.
- [18] Per Kraus. Lectures on black holes and the AdS(3) / CFT(2) correspondence. *Lect. Notes Phys.*, 755:193–247, 2008.
- [19] Tohru Eguchi, Hiroshi Ooguri, Anne Taormina, and Sung-Kil Yang. Superconformal algebras and string compactification on manifolds with $su(n)$ holonomy. *Nuclear Physics B*, 315(1):193 – 221, 1989.
- [20] Paul S. Aspinwall. K3 surfaces and string duality. In *Differential geometry inspired by string theory*, pages 421–540, 1996. [1(1996)].
- [21] Curtis G. Callan, Jr., Jeffrey A. Harvey, and Andrew Strominger. World sheet approach to heterotic instantons and solitons. *Nucl. Phys.*, B359:611–634, 1991.
- [22] Luis Alvarez-Gaume and Daniel Z. Freedman. Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model. *Commun. Math. Phys.*, 80:443, 1981.
- [23] Paul S. Aspinwall and David R. Morrison. String theory on K3 surfaces. pages 703–716, 1994. [AMS/IP Stud. Adv. Math.1,703(1996)].
- [24] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Symmetries of k3 sigma models. *Commun. number theory phys.*, 6(1), 2012.
- [25] Miranda C. N. Cheng, Sarah M. Harrison, Roberto Volpato, and Max Zimet. K3 String Theory, Lattices and Moonshine. 2016.
- [26] Gerald Hoehn and Geoffrey Mason. The 290 fixed-point sublattices of the Leech lattice. *J. Algebra*, 448:618–637, 2016.
- [27] Vassilis Anagiannis and Miranda C. N. Cheng. TASI Lectures on Moonshine. 2018.
- [28] R. E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Inventiones Mathematicae*, 109:405, 1992.
- [29] Miranda C. N. Cheng. K3 Surfaces, N=4 Dyons, and the Mathieu Group M24. *Commun. Num. Theor. Phys.*, 4:623–658, 2010.
- [30] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu twining characters for K3. *JHEP*, 09:058, 2010.
- [31] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu moonshine in the elliptic genus of k3. *Journal of High Energy Physics*, 2010(10):62, Oct 2010.
- [32] Terry Gannon. Much ado about mathieu. *Advances in Mathematics*, 301:322 – 358, 2016.
- [33] Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral Moonshine. *Commun. Num. Theor. Phys.*, 08:101–242, 2014.
- [34] Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral Moonshine and the Niemeier Lattices. 2013.
- [35] John F. R. Duncan, Michael J. Griffin, and Ken Ono. Proof of the Umbral Moonshine Conjecture. 2015.

- [36] Jeffrey A. Harvey and Sameer Murthy. Moonshine in Fivebrane Spacetimes. *JHEP*, 01:146, 2014.
- [37] Jeffrey A. Harvey, Sameer Murthy, and Caner Nazaroglu. ADE Double Scaled Little String Theories, Mock Modular Forms and Umbral Moonshine. *JHEP*, 05:126, 2015.
- [38] Shamit Kachru, Natalie M. Paquette, and Roberto Volpato. 3D String Theory and Umbral Moonshine. *J. Phys.*, A50(40):404003, 2017.
- [39] Max Zimet. Umbral Moonshine and String Duality. 2018.
- [40] Natalie M. Paquette, Roberto Volpato, and Max Zimet. No More Walls! A Tale of Modularity, Symmetry, and Wall Crossing for 1/4 BPS Dyons. *JHEP*, 05:047, 2017.
- [41] Werner Nahm and Katrin Wendland. A Hiker’s guide to K3: Aspects of N=(4,4) superconformal field theory with central charge $c = 6$. *Commun. Math. Phys.*, 216:85–138, 2001.
- [42] Vincent Bouchard. Lectures on complex geometry, Calabi-Yau manifolds and toric geometry. 2007.
- [43] J. H. Conway, N. J. A. Sloane, and E. Bannai. *Sphere-packings, Lattices, and Groups*. Springer-Verlag, Berlin, Heidelberg, 1987.