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A worldsheet description of black hole microstates

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Introduction

From their conception in the first decades of the twentieth century, General Relativity and Quantum Mechanics have made possible the most significant achievements in the description of our Universe. The explanation of a considerable number of physical phenomena at the largest scales has been possible only within the framework of General Relativity. On the other hand, Quantum Mechanics provides an indispensable tool when attempting to understand the laws of nature at the atomic and subatomic scales. Both of these theories succeed when Classical Physics fails and have nowadays passed plentiful experimental tests. However, we still lack a unified theory of Quantum Gravity, that could encompass both Quantum Mechanics and General Relativity. Remarkably, without such a theory, the full understanding of some physical objects is not possible: for instance, in the study of black holes, we cannot disregard any of the two theories. For this reason, black holes are a sizeable part of the studies aiming to get an insight into Quantum Gravity.
Briefly, black holes are classical solutions of the Einstein's equations, characterized by a region of spacetime, the interior of black holes, into which particles can fall, but no more exit. This region is surrounded by a surface called event horizon. As pointed out for the first time by Bekenstein and Hawking, black holes exhibit thermodynamic properties: Quantum Mechanics comes into play when trying to understand them. For instance, an entropy can be defined for black holes. Due to the Boltzmann equation we expect that it increases with the growing of the number of microstates. Black holes turn out to have a very large entropy, but in General Relativity they are fully characterized by just a few quantities (mass, charge and angular momentum), which do not suffice to predict the huge number of expected microstates. Furthermore, black holes have some temperature and are then, supposed to radiate: the emission of the so-called Hawking radiation is not possible at the classical level and requires a semi-classical approach. On top of that, through Hawking emission, a black hole can fully evaporate, leaving just thermal radiation in its place. A pure state (black holes are completely determined by the aforementioned parameters of mass, charge and spin) has given rise to a mixed state as a remnant. Therefore, the process in between has caused the loss of part of the initial information and cannot be unitary: this conflicts with Quantum Mechanics, which would only allow unitary evolution. This inconsistency is known as the information paradox.
A sound unified theory of Quantum Gravity has to explain these issues and String Theory is the most promising candidate. As a matter of fact, Strominger and Vafa [1] performed the counting of microstates for a particular black hole configuration arising in string theory. As a result, they reproduced the Bekenstein-Hawking prediction through the Boltzmann equation. Mathur [2,3], instead, designed peculiar string configurations for microstates, which were dubbed 'fuzzballs'. Their most remarkable property consists in the replacement of the horizon with a completely smooth structure across which radiation is emitted unitarily.
All these progresses have been achieved within the low energy effective field theory for strings, i. e. supergravity. The purpose of this thesis is to discuss black hole microstates from another perspective. Indeed, we employ an exactly solvable model to construct the worldsheet string theory of a particular realization of the fuzzball microstates. This tool is known as the Wess-Zumino-Witten model [4–6], which allows to describe the dynamics of strings on curved backgrounds. The suitable

manifold for our study is the coset G/H, with

$$G = SL(2,\mathbb{R}) \times SU(2) \times \mathbb{R}^t \times S^1 \times \mathbb{T}^4, \qquad H = U(1)_L \times U(1)_R.$$

The Lie group G has twelve dimensions. The gauging of the H subgroup reduces them down to the critical ten dimensions. This way, we can not only reproduce the already known results of supergravity but most importantly, obtain new information about the spectrum of the theory. In particular, the BRST quantization procedure enables us to find some constraints, which once solved, provide the physical vertex operators.

Outline of the thesis

In Chapter 1 we review the most general aspects of the bosonic string theory, including the lightcone gauge quantization. We dwell on what is more useful in the following: conformal field theories in two dimensions with particular focus on the Virasoro algebra and the vertex operators, the Polyakov path integral and the BRST quantization. Then, after a small detour about supersymmetry for point-like particles and a brief description of the superspace, the superstring theory is treated. The Clifford algebra, the action of the theory in the superconformal gauge and the spectrum are outlined, together with a pause upon the conformal field theories for fermions and ghosts, with attention to the BRST quantization and the bosonization for the superstrings. Eventually, the non-linear sigma model and the different superstring theories are introduced, highlighting the dualities that link them.

Chapter 2 is devoted to the description of black holes in the framework of supergravity. Their thermodynamic properties are discussed, then we deal with two BPS configurations of strings and branes, which are stable due to supersymmetry: the three-charge black hole (D1-D5-P in IIB supergravity or M2-M2-M2 in 11 dimensional supergravity, according to the duality frame) and the four-charge black hole (D2-D2-D2-D6 in IIA-supergravity). Their entropy is computed through the Bekenstein-Hawking formula, after we have determined the area of the event horizon. Then, in the regime of very small string coupling, we count the number of microstates giving rise to the same macrostate in the three-charge black hole. In particular, we have to count the different ways in which the momentum units can be divided among open strings stretching between the D1 and the D5 branes. Then we apply the Boltzmann equation: the result coincides with the Bekenstein-Hawking prediction. Finally, the fuzzball proposal is introduced for the NS1-P system in IIB-type supergravity: the momentum charge can be regarded as the propagation of a transverse displacement profile along the string. The set of all the possible displacement profiles constitutes the statistical ensemble of microstates giving rise to the same macrostate. At the end, one particular displacement whose shape is a helix, is treated in full details.

The worldsheet theory for this particular microstate is constructed in Chapter 3. After illustrating the general features of the Wess-Zumino-Witten model, we deal with the configuration of NS5 branes in IIB theory, disposed on a circle. Its metric is found first with supergravity techniques and then with the gauged Wess-Zumino-Witten model on the manifold G defined above. The results exactly coincide. When these branes are given some momentum, we end up with the structure of a round supertube, whose metric can be equivalently computed in supergravity and within the Wess-Zumino-Witten model.

In Chapter 4, some aspects of the spectrum of the theory are analyzed. For this purpose, the fundamental properties of the current algebra and the Sugawara construction are illustrated both in the general theory and for each of the factors of the group manifold for our fuzzball microstate. Then the reparametrization and the gauging ghosts are introduced, with which we can construct the full BRST charge. Requiring the BRST invariance, we obtain the constraints that the physical vertex operators must satisfy in the Neveu-Schwarz sector and the solutions are then reported. This is the main result of this thesis.

Chapter 1

The toolkit: string theory

1.1 The bosonic string theory in Minkowski spacetime

The fundamental objects in string theory are not pointlike particles, but the one-dimensional *relativistic strings*, propagating in some *ambient* or *target space* as the time flows. In this chapter the ambient space is identified with the D-dimensional Minkowski spacetime, with metric $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, ..., +)$.

1.1.1 An action for strings

The motion of the relativistic string can be understood as the extension of the dynamics of the relativistic point-particle when one parameter is added. In some coordinate framework x^μ, the one-dimensional path travelled by a relativistic particle can be written as $X^\mu = X^\mu(\tau)$, where τ is the only (dimensionless) parameter needed and $\mu = 0, 1, ..., D-1$. In this writing time and space are on equal footing and the explicit Poincaré invariance of the equations can be accomplished. We also require the physical quantities not to depend on the choice of the parameter τ of the worldline. The simplest action for a relativistic point-particle would then read

$$\mathcal{S}_{pp} = -m \int d\tau (-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu)^{1/2}, \tag{1.1}$$

where a dot denotes the derivative with respect to τ. Up to the mass of the particle as a pre factor, this is nothing but the proper time along the particle's worldline. In order to deal with massless particles, we introduce the einbein $e(\tau) = (-g_{\tau\tau}(\tau))^{1/2}$, where $g_{\tau\tau}$ is the one-dimensional metric on the wordline, and the action becomes

$$\mathcal{S'}_{pp} = \frac{1}{2} \int d\tau (e^{-1} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - em^2). \tag{1.2}$$

For massive particles, this can be proved to be equivalent to (1.1) after integrating the einbein out. (1.2) is invariant under reparametrization and is Poincaré invariant as well. Compared to (1.1), it works for massless particle and its quantization within the pah-integral formalism leads to easier computations.

The string extends in one spatial dimension, then it sweeps out a two-dimensional *worldsheet* in the target space, $X^\mu(\tau, \sigma)$, parametrized by a time-like coordinate τ (completely analogous to the parameter for the particle's worldlines) and a further space-like coordinate σ identifying the points along the string. These two coordinates are dimensionless and as a whole, they are usually named $\sigma^a = (\tau, \sigma)$. The action for the string needs to be invariant under Poincaré transformations on the D-dimensional Minkowski spacetime (global transformation on the worldsheet) and diffeomorphisms on the worldsheets, i.e. $\sigma^a \to \tilde{\sigma}^a(\sigma)$ reparametrizations (gauge, i.e. local transformations on the worldsheet). Exactly as the action (1.1) is proportional to the length of the worldline, we expect the action for the string to be proportional to the area of the worldsheet. The latter depends on the induced metric, i.e. the pull-back of the flat metric on the Minkowski spacetime, on the worldsheet

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b}\eta_{\mu\nu}$$

and reads

$$\mathcal{S}_{NG} = -T\int_M d^2\sigma\sqrt{-\det\gamma}, \tag{1.3}$$

which is the *Nambu-Goto action* for the relativistic string. Here, M denotes the worldsheet and the pre-factor T goes under the name of *tension* of the string and has the dimensions of energy per length unit. By historical reasons, it can be alternatively expressed as

$$T = \frac{1}{2\pi\alpha'} \tag{1.4}$$

where α' is the *universal Regge slope* and is linked to the string length scale l_s through the identity

$$\alpha' = {l_s}^2.$$

The equation of motion for X^μ arising from the Nambu-Goto action is

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) = 0. \tag{1.5}$$

The path-integral formalism for the string quantization can be more easily performed if (1.3) is exchanged with the *Polyakov action*

$$\mathcal{S}_P = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu}. \tag{1.6}$$

Here g^{ab} is a dynamical metric on the worldsheet, exactly as $e(\tau)$ in (1.2), and has its own equations of motion (differently from γ_{ab} which is completely determined by the equations of motion for X^μ). These can be found varying the Polyakov action with respect to the metric:

$$g_{ab} = \frac{1}{g^{cd}\ \partial_c X\cdot\partial_d X}\ \partial_a X\cdot\partial_b X \equiv 2f(\sigma)\ \partial_a X\cdot\partial_b X = 2f(\sigma)\ \gamma_{ab}, \tag{1.7}$$

where the dot denotes the Minkowski scalar product. The function f is a conformal factor between γ_{ab} and g_{ab}; its argument σ for the function f is a compact notation for $\sigma^a = (\tau,\sigma)$: if not otherwise specified, this convention is going to be kept throughout this chapter.
The equation of motion for X^μ instead, reads

$$\partial_a(\sqrt{-g}g^{ab}\partial_b X^\mu) = 0 \tag{1.8}$$

and due to (1.7) is equivalent to (1.5). Both the actions (1.3) and (1.6) lead to the same equations of motion for the X^μ fields and hence (as (1.1) and (1.2)), they are classically equivalent. Nonetheless, the Polyakov action exhibits a further gauge symmetry on the worldsheet, compared to the Nambu-Goto one: as a matter of fact, it is invariant under

$$g_{ab}(\sigma) \to \Omega(\sigma)^2 g_{ab}(\sigma),$$

which is the so-called *Weyl symmetry.* Exploiting the latter and the reparametrization invariance, we completely fix the degrees of freedom of the worldsheet metric which in fact, can be set to be η_{ab}. In this gauge, the Polyakov action reduces to the action for D scalar fields

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\ \partial_a X\cdot\partial^a X. \tag{1.9}$$

The equation of motion for X^μ is the free wave equation

$$\partial_a \partial^a X^\mu = 0, \tag{1.10}$$

whereas in this gauge, (1.7) can be recast in terms of the stress-energy tensor T_{ab}:

$$T_{ab} \equiv -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\delta \mathcal{S}_P}{\delta g^{ab}} = \partial_a X \cdot \partial_b X - \frac{1}{2}\eta_{ab}\eta^{cd}\, \partial_c X \cdot \partial_d X = 0. \tag{1.11}$$

This equation can be regarded as a constraint (also known as Virasoro constraint) on the solutions of (1.8) and explicitly reads

$$T_{01} = \dot{X} \cdot X' = 0, \qquad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0, \tag{1.12}$$

where $\dot{X} = \partial_\tau X$ and $X' = \partial_\sigma X$.
Introducing the lightcone coordinates on the worldsheet $\sigma^\pm = \tau \pm \sigma$, (1.10) reads

$$\partial_+ \partial_- X^\mu = 0, \tag{1.13}$$

whose most general solution can be written in terms of left-moving and right-moving modes

$$X^\mu(\sigma, \tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-). \tag{1.14}$$

The constraint equations (1.12) are, instead, written as

$$\partial_+ X^\mu \partial_+ X_\mu = 0 = \partial_- X^\mu \partial_- X_\mu. \tag{1.15}$$

1.1.2 Closed and open strings

Strings can be closed or open. Regardless of this distinction, the dynamics of each point along the string only depends on local physics and is ruled by the Polyakov action and its equation of motion. The only difference lies in the behaviour at the boundaries.

When the periodicity condition

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) \tag{1.16}$$

holds, we are dealing with the *closed strings*. In this case, the two addends of the general solution (1.14) can be Fourier expanded as

$$\begin{aligned} X^\mu{}_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \\ X^\mu{}_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}. \end{aligned} \tag{1.17}$$

Let us notice that these are not individually periodic in σ, while the sum is, as it should. x^μ and p^μ can be identified as respectively the position and the momentum of the center of mass of the string. Since $X^\mu(\tau, \sigma)$ fields are real-valued,

$$\alpha^\mu_n = (\alpha^\mu_{-n})^*, \qquad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^*.$$

If the condition (1.16) is not fulfilled, *open strings*, whose endpoints are not bound together, come into play. Let us suppose the worldsheet parameters to be $\sigma \in [0, \pi]$ and $\tau \in [\tau_i, \tau_f]$. By varying the Polyakov action in conformal gauge (1.9), we get

$$\begin{aligned} \delta S &= -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma\ \partial_\alpha X^\mu \partial^\alpha \delta X_\mu = \\ &= \frac{1}{2\pi\alpha'}\Big[\int d^2\sigma\ (\partial^\alpha\partial_\alpha X^\mu)\delta X_\mu + \int_0^\pi d\sigma\ \partial_\tau X^\mu \delta X_\mu - \int_{\tau_i}^{\tau_f} d\tau\ \partial_\sigma X^\mu \delta X_\mu\Big]. \end{aligned}$$

We employ the principle of least action, i.e. we require $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$. Therefore, in order to come up with the equation of motion (1.10) for the open strings, we need to demand that either $\partial_\sigma X^\mu = 0$ for $\sigma = 0, \pi$ (Neumann boundary condition) or $\delta X^\mu = 0$. Whereas in the former case, the boundaries are free to move, in the latter, they are stuck at some constant position in the μ - th coordinate. In the most general case, both kinds of boundary conditions are imposed in different spacetime coordinates, that is,

$$\partial_\sigma X^A = 0 \quad \text{for } A = 0, \ldots, p; \qquad X^I = c^I \quad \text{for } I = p+1, \ldots, D-1. \tag{1.18}$$

since of course X^0 is a timelike coordinate and cannot be fixed (otherwise we get the so-called instantons). The endpoints of the open strings are then constrained to lie on a $(p+1)$-dimensional hypersurface, which is named Dp-brane (D stands for Dirichlet whilst p is the number of spacelike dimensions of the brane). A splitting of the original Lorentz group also occurs

$$SO(1, D-1) \to SO(1, p) \times SO(D-p-1).$$

As for closed strings, we can perform a mode expansion of the general solution (1.14):

$$\begin{aligned} X^\mu{}_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \\ X^\mu{}_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}. \end{aligned} \tag{1.19}$$

Imposing (1.18) we find out that only one set of oscillators is independent. Indeed, for the Neumann boundary conditions we have that

$$\alpha^A_n = \tilde{\alpha}^A_n, \tag{1.20}$$

whereas for Dirichlet boundary conditions,

$$x^I = c^I, \quad p^I = 0, \quad \alpha^I_n = -\tilde{\alpha}^I_n. \tag{1.21}$$

1.1.3 Lightcone gauge quantization of the relativistic string

The quantization of the relativistic string can be accomplished exploiting one of the usual equivalent methods that make gauge theories quantum: covariant quantization (the analogue of Gupta-Bleuler quantization for QED), light-cone gauge quantization and the path-integral quantization. In this section we focus on the second choice, in which we first solve the constraints and then quantize the physical degrees of freedom. The third procedure will be instead, followed in Section 1.3. The quantizations of open and closed strings are fairly similar: we first devote to the closed string and eventually, will highlight the slight differences between the two cases.

Let us first notice that the Weyl and diffeomorphism symmetries do not completely fix the metric g_{ab}. Indeed, transformations of the kind

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+), \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \tag{1.22}$$

bring some overall prefactor to the flat worldsheet metric in lightcone coordinates

$$ds^2 = -d\sigma^+ d\sigma^-. \tag{1.23}$$

A Weyl rescaling can bring the worldsheet metric again in the form (1.23). This residual gauge symmetry surviving the gauge fixing is a zero-measure subset of all possible gauge transformations. Nonetheless it has to be considered in the counting of the degrees of freedom. As a matter of fact, the general solution of (1.13) is made up of left and right-moving waves then in total $2D$ functions. These are however, constrained by (1.15) and then only $2D-2$ are actual degrees of freedom.

Eventually fixing the redundancy given in (1.22), reduces the number of independent functions to $2(D-2)$: these remaining degrees of freedom are the physical transverse fluctuations of the string.

We can define the analogue of the lightcone coordinates on the D-dimensional Minkowski spacetime:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}), \tag{1.24}$$

where we have selected one particular spacelike direction. At the classical level, this is harmless as far as Poincaré symmetry in the ambient spacetime is concernerd, but when lifting to a quantum theory this might really spoil the Poincaré invariance.
In the new coordinate frame, the Minkowski metric gets

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \tag{1.25}$$

The gauge redundancy (1.22) can be fixed identifying, up to some prefactors and a constant, τ with X^+:

$$X^+(\tau,\sigma) = x^+ + \alpha' p^+ \tau. \tag{1.26}$$

This gauge choice is exactly the *lightcone gauge*. x^+ cannot be thought as an actual physical coordinate since it can be reabsorbed through a shift in τ. Furthermore, imposing the Virasoro constraints in the form (1.15) we get to know that

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i, \qquad 2\partial_- X^- \partial_- X^+ = \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i. \tag{1.27}$$

Using (1.26) and the expressions (1.14) and (1.17), the former of these equations leads to

$$\tilde{\alpha}_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2}\tilde{\alpha}_{n-m}^i\tilde{\alpha}_m^i, \qquad \frac{\alpha'}{2}p^- = \frac{1}{2p^+}\sum_{i=1}^{D-2}\Big(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0}\tilde{\alpha}_n^i\tilde{\alpha}_{-n}^i\Big) \tag{1.28}$$

and analogously, from the second one, we learn that

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_m^i, \qquad \frac{\alpha'}{2}p^- = \frac{1}{2p^+}\sum_{i=1}^{D-2}\Big(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0}\alpha_n^i\alpha_{-n}^i\Big) \tag{1.29}$$

Let us notice that due to the constraints, except the integration constant x^-, all fields determining X^-, i.e. p^-, α_n^- and $\tilde{\alpha}_n^-$ are fully determined by the α_n^i's and the $\tilde{\alpha}_n^i$'s, which are the actual physical modes. As a support for this statement, we can compute the mass of a closed string using (1.28) or (1.29):

$$M_{cl}^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i, \tag{1.30}$$

which indeed only depends on the transverse $2(D-2)$ modes. The fact that the same quantity can be expressed both in terms of right-moving or left-moving oscillator modes is also known as *level matching*. It implies that the number of excitations in the right-moving sector has to be equal to those of the left-moving sector. In the end, the actual physical quantities are the transverse oscillator modes.

The light-cone gauge quantization for the closed string is carried out on just the physical degrees of freedom, i.e. the transverse oscillator modes, x^i, p^i, p^+ and x^-. The non-vanishing commutation relations are given by

$$[x^i, p^j] = i\delta^{ij}, \qquad [x^-, p^+] = -i, \qquad [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}. \tag{1.31}$$

The Hilbert space for the excitations of a single string is just the Fock space built upon a vacuum state $|0;p\rangle$, defined such that

$$\hat{p}^\mu |0;p\rangle = p^\mu |0;p\rangle , \qquad \alpha^i_n |0;p\rangle = \tilde{\alpha}^i_n |0;p\rangle = 0 \ \text{ for } n > 0.$$

All the other states originate from acting on this vacuum with the raising operators α^i_{-n} or $\tilde{\alpha}^i_{-n}$, $n > 0$, In the lightcone gauge quantization, the issue of negative-normed states has been implicitly healed while identifying the physical states, therefore there are no ghosts to deal with. Let us notice that $|0;p\rangle$ is not a zero-string state, but rather a state with one single string of momentum p^μ and no transverse oscillations: the raising and lowering operators do not create or destroy strings but excitations on the transverse directions. As an aside, the string is not supposed to live in the $X^0 - X^{D-1}$ plane, thus "transverse" oscillations might not be transverse to the string itself.

In the framework of a quantum theory, the identifications (1.28) and (1.29) do not lead to the mass-shell condition (1.30): due to the commutation relations (1.31), indeed,

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n &= \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n<0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n = \\ &=\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n<0}\big[\alpha^i_n\alpha^i_{-n} - n\big] + \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n = \\ &=\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \frac{D-2}{2}\sum_{n>0} n.\end{aligned}$$

The normal-ordering, then, has given rise to some further constant compared to (1.30). This is clearly divergent: its renormalization was first devised by Ramanujan and gives

$$\sum_{n>0} n = -\frac{1}{12}.$$

Therefore, the mass of the closed string in the quantum theory is

$$M_{cl}{}^2 = \frac{4}{\alpha'}\Big(N - \frac{D-2}{24}\Big) = \frac{4}{\alpha'}\Big(\tilde{N} - \frac{D-2}{24}\Big), \tag{1.32}$$

where we have defined

$$N = \sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n, \qquad \tilde{N} = \sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n. \tag{1.33}$$

These are akin to the number operators of two harmonic oscillators, with the difference that the α^i_n and $\tilde{\alpha}^i_n$ satisfy the commutation relations (1.31) instead of the standard $[a_n, a^\dagger_m] = \delta_{mn}$. As a consequence, N and $\tilde{N}$ count the number of excitations weighted with the corresponding n and not just the number of excitations.

The lowest-energy state in the spectrum corresponds to the level $N = 0 = \tilde{N}$, i.e. the string with no excitations. The squared-mass of this state results to be negative, hence we are dealing with a tachyon, which is a moot point for the bosonic string theory. However, when fermions are added to the worldsheet, tachyons do not appear: the superstring overcomes this problem.
Due to the level matching condition, states of level $N = \tilde{N} = 1$ need to have one excitation both in the right and in the left-moving sectors:

$$\xi_{ij}(X)\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}|0;p\rangle , \tag{1.34}$$

where the pre-factor $\xi_{ij}(X)$ is the polarization two-tensor. Since i, j run from 1 to $D-2$, (1.34) represents $(D-2)^2$ different excitations. This is also the number of degrees of freedom of ξ_{ij} which is indeed, a generic $(D-2)\times(D-2)$ tensor. Using (1.32), we learn that in this case

$$M^2_{cl.} = \frac{4}{\alpha'}\Big(1 - \frac{D-2}{24}\Big). \tag{1.35}$$

We would like the states (1.34) to sit in some representation of the Lorentz symmetry $SO(1, D-1)$, which the choice of lightcone coordinates (1.24) seems to have spoiled. Resorting to Wigner's classification of representations of the Poincaré group, we go to the rest-frame of a massive particle, where the momentum is $p^\mu = (p, 0, \ldots, 0)$: a state for a massive particle should then transform in some representation of $SO(D-1)$ little group of the full Lorentz group. Since we cannot package $(D-2)^2$ states in any representation of $SO(D-1)$ we shift to massless particles, which lack a rest frame and hence we choose $p^\mu = (p, \ldots, p)$. This time the little group is $SO(D-2)$, whose $(D-2)^2$-dimensional representation is the double vector representation. Therefore, level-1 state of string excitations must be massless. Imposing that the mass (1.35) vanishes, we can determine the dimension of the ambient space such that Lorentz invariance is preserved at the quantum level, known as *critical dimension*:

$$D = 26. \tag{1.36}$$

This upshot forbids quantum anomalies of the Poincaré group, as could be more rigorously proven using Lorentz generators. In Section 1.3, instaed, we are providing a further alternative proof by employing tools borrowed from conformal field theory. Higher level states are massive and fit some representation of $SO(D-1)$ (for instance level-2 states sit in its traceless symmetric representation) with no other requirements about the dimension.

The states (1.34) transform then in the $\vec{24} \otimes \vec{24}$ representation and can be decomposed in three irreducible representations: one symmetric traceless, one anti-symmetric and one singlet referring to the trace. Accordingly the polarization two-tensor can be decomposed in one symmetric traceless part $G_{\mu\nu}(X)$, an antisymmetric one $B_{\mu\nu}(X)$ and a one-dimensional trace part $\Phi(X)$. Both properties of the first field are common to classical gravitational waves and by a theorem of Weinberg's, it can be identified with the the metric of the spacetime or the graviton in the quantum theory. $B_{\mu\nu}$ is named *Kalb-Ramond field* and since antisymmetric, is a 2-form. It can be regarded as a gauge potential, whose field strength $H = dB$ is invariant under

$$B \to B + d\Lambda,$$

with Λ one-form. $\Phi(X)$ is instead, the *dilaton* and is a scalar field under Poincaré transformations.

The quantization of the open strings is quite the same, once the physical degrees of freedom have been identified and promoted to operators: since (1.21) fixes x^I and p^I and with (1.20) selects just one set of harmonic oscillators, the physical degrees of freedom are x^a, p^a and α_n^μ. The spacetime lightcone coordinates are defined to be

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^p)$$

and the mass for the states gets

$$M_{op}^2 = \frac{1}{\alpha'}\Big(\sum_{i=1}^{p-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{i=p+1}^{D-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - \frac{D-2}{24}\Big)$$

The $SO(1,p) \times SO(D-p-1)$ invariance is preserved for the quantum theory if $D = 26$, which is the critical dimension for the closed string as well. The ground state $|0,p\rangle_{op}$ of the spectrum of the open strings is again a tachyon since its square mass is negative. The first excited states are massless. $\alpha_{-1}^A |0,p\rangle_{op}$, with $A = 1, \ldots, p-1$, transform under $SO(1,p)$ on the brane can be regarded as excitations of a gauge field living on the brane. $\alpha_{-1}^I |0,p\rangle_{op}$ transform under $SO(D-p-1)$ and are instead scalars under $SO(1,p)$ and can be physically thought as transverse fluctuations of the D-brane. D-branes are, indeed, dynamical objects.

1.2 Conformal field theories in 2 dimensions

In the last decades, conformal field theories (CFT) have witnessed increasing interest and attention in different sectors of Theoretical Physics, such as Statistical Mechanics and High Energy Physics.

In particular, conformal field theories are defined on the worldsheets of strings and play a crucial role in the AdS-CFT correspondence. In this Section, after reviewing some general aspects, we are discussing the most remarkable classical and quantum features of 2-dimensional CFTs.

1.2.1 General properties

Given a d-dimensional space with a metric $g_{\mu\nu}$ in a certain frame of coordinates, a *conformal transformation* is a change of coordinates $x \to x'$ such that

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{1.37}$$

These transformations do not preserve the lengths, but the angles between vectors in the spacetime:

$$\frac{v \cdot w}{\sqrt{(v^2 w^2)}},$$

where $v \cdot w = g_{\mu\nu} v^\mu w^\nu$. For an infinitesimal coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu(x)$, the metric transforms according to

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu), \tag{1.38}$$

where we have stopped at first order in ε. In order that (1.37) is fulfilled by this transformation, we have to require $\delta g_{\mu\nu} \propto g_{\mu\nu}$, which in particular reads

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\partial_\alpha \varepsilon^\alpha) g_{\mu\nu} \equiv f(x) g_{\mu\nu}. \tag{1.39}$$

From here on out, we specialize to the flat metric with Euclidean signature, i.e. $g_{\mu\nu} = \delta_{\mu\nu}$; the treatment is analogous for a Minkowski metric, except the explicit form of $g_{\mu\nu}$. By applying an extra derivative ∂_ρ on both sides of (1.39), permuting the indices and combining the results linearly, we get

$$2\partial_\mu \partial_\nu \varepsilon_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\nu\rho} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f. \tag{1.40}$$

Contracting with $\delta^{\mu\nu}$ we arrive at

$$2 \,\Box \varepsilon_\mu = (2-d) \partial_\mu f.$$

We apply ∂_ν on both sides of this equation and $\Box$ on both sides of (1.39) to get

$$(2-d) \partial_\mu \partial_\nu f = \delta_{\mu\nu} \Box f,$$

which upon contraction with $\delta^{\mu\nu}$, eventually reads

$$(d-1) \Box f = 0.$$

Therefore, we can infer that in $d = 1$ any transformation is conformal: trivially, no notion of angles can exist in this case. In $d \geq 3$, instead, $f(x)$ will be allowed to be at most linear in the coordinates and thus, the most general coordinate variation will be

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu}, \tag{1.41}$$

with all the coefficients independent on x.
We can recognize the meaning of each addend in (1.41). The first term generates spacetime translations. The linear term gives rise to finite dilations and rigid rotations (in Minkowski spacetime, these would be rigid Lorentz transformations)

$$x'^\mu = \alpha x^\mu, \quad \alpha \in \mathbb{R} \qquad\qquad x'^\mu = M^\mu_\nu x^\nu, \quad M^\mu{}_\nu \in SO(d).$$

Eventually, (1.40) the quadratic term takes us to the *special conformal transformations* whose fnite expression is

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}.$$

From the equivalent writing

$$\frac{x'^\mu}{x'^2} = \frac{x'^\mu}{x'^2} - b^\mu,$$

we can infer that these transformations are but a particular inversion preceeded and followed by a translation.

1.2.2 2-dimensional CFT

We deal with 2-dimensional CFTs in more detail. This is why we shift from the general Greek indices μ, ν to the two Latin indices $a, b = 1, 2$ from here on along this section. Although on a worldsheet the metric has a Minkowskian signature, we keep the metric Euclidean: the link between the two choices is nothing but a Wick rotation.

In two dimensions, equation (1.39) reads

$$\partial_1 \varepsilon_1 = \partial_2 \varepsilon_2, \qquad \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1,$$

which are the Cauchy-Riemann equations identifying a holomorphic function $\varepsilon(z) = \varepsilon^1 + i\varepsilon^2$ and its antiholomorphic counterpart $\bar{\varepsilon}(\bar{z}) = \varepsilon^1 - i\varepsilon^2$, in the complex coordinates

$$z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2. \tag{1.42}$$

In these coordinates, the derivatives are usually denoted as

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

and the components of the vectors are analogously given by

$$v_z = \frac{1}{2}(v_1 - iv_2), \qquad v_{\bar{z}} = \frac{1}{2}(v_1 + iv_2). \tag{1.43}$$

The flat Eucidean metric is given by

$$ds^2 = dzd\bar{z}. \tag{1.44}$$

Its components will be denoted as $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$, $g_{zz} = g_{\bar{z}\bar{z}} = 0$. The integration measure will be $dzd\bar{z} = 2dx^1dx^2$ and the delta function is defined such that $\int d^2z\delta(z, \bar{z}) = 1$. The Levi-Civita anti-symmetric tensor is defined employing the Jacobian and hence $\varepsilon_{z\bar{z}} = \frac{1}{2}i$. This is a full-fledged tensor, then its indices are raised and lowered using the metric (1.44).
A general two-dimensional conformal transformation will be just the holomorphic change

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}) : \tag{1.45}$$

under this change of coordinates, indeed,

$$ds^2 = dzd\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 dzd\bar{z}.$$

The infinitesimal version of $z \to f(z)$ is

$$z \to z + \varepsilon(z), \qquad \varepsilon(z) = \sum_{n=-\infty}^{+\infty} c_n z^{n+1}, \tag{1.46}$$

where the second expression is the Laurent expansion around $z = 0$. Something completely analogous also holds for the anti-holomorphic transformation in (1.45). We can therefore identify the generators of a conformal transformation:

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{1.47}$$

which satisfy the commutation relations

$$[l_m, l_n] = (m-n)l_{m+n}, \qquad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \qquad [l_m, \bar{l}_n] = 0, \tag{1.48}$$

which is the so-called *Witt algebra*. Due to the last equation, $\{l_n\}$ and $\{\bar{l}_n\}$ are two independent algebras, hence we can regard z and $\bar{z}$ as two independent coordinates.

To define a global conformal field theory, we have to verify that the vector fields generating the holomorphic transformations

$$v(z) = \sum_{n=-\infty}^{+\infty} c_n z^{n+1}\partial_z$$

are well-defined over all the Riemann sphere $S^2 = \mathbb{C} \cup \infty$: to avoid issues with singularities at $z \to 0$, we have to impose $n \geq -1$. With the transformation $z = -\frac{1}{w}$ we get

$$v(z) = \sum_{n=-\infty}^{+\infty} c_n \Big(-\frac{1}{w}\Big)^{n+1} \Big(\frac{dz}{dw}\Big)^{-1} \partial_w = \sum_{n=-\infty}^{+\infty} c_n \Big(-\frac{1}{w}\Big)^{n-1} \partial_w,$$

which is well-defined at $z \to +\infty$, i.e. $w \to 0$, for $n \leq 1$. The global conformal algebra is then generated by $\{l_{-1}, l_0, l_1\}$, $\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_{-1}\}$: l_{-1} and $\bar{l}_{-1}$ generate the translations, $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ generate respectively the dilations and rotations, whereas l_1 and $\bar{l}_1$ generate the special conformal transformations. A general global conformal transformation will be given by

$$z \to \frac{az+b}{cz+d}, \tag{1.49}$$

with $a, b, c, d \in \mathbb{C}$. With no other constraints on the parameters, this map would be in $GL(2, \mathbb{C})$. However, (1.49) is left invariant by $(a, b, c, d) \to \lambda(a, b, c, d)$: to identify the not redundant degrees of freedom we can impose $ad - bc = 1$, i.e. we restrict to $SL(2, \mathbb{C})$. Nonetheless, $(a, b, c, d) \to (-a, -b, -c, -d)$ still leads to the same map: the final group of transformations like (1.49) is then, $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{I, -I\}$.

1.2.3 Classical Noether currents for conformal transformations

Global translations $x^a \to x^a + \varepsilon^a$ are conformal, so for CFTs we can define the corresponding conserved current, i.e. the *stress-energy tensor* T^{ab}. It can be derived by promoting the constant parameter ε^a to a local variable $\varepsilon^a(x)$. By definition of a conserved current under translations, if $\mathcal{S}$ is the action of our CFT,

$$\delta\mathcal{S} = \int d^2x T^{ab}\partial_a\varepsilon_b. \tag{1.50}$$

Indeed, since $\delta\mathcal{S} = 0$ for any ε, $\partial_a T^{ab} = 0$. The action is also invariant for local symmetries, hence we can infer that its vanishing variation under this kind of transformations is the sum of the variation under the coordinate change and the variation under the metric change (1.38). Therefore the variation with respect to the coordinates must be opposite to the variation with respect to the metric. In our case, even though promoting ε to a local variable, we are actually dealing with a global change of coordinates; anyway we expect the variation of the action to have the same expression as if we worked with local coordinate transformations. Then,

$$\delta\mathcal{S} = -\int d^2x \frac{\partial\mathcal{S}}{\partial g_{ab}}\delta g_{ab} = 2\int d^2x \frac{\partial\mathcal{S}}{\partial g_{ab}}\partial_\alpha\varepsilon_\beta \tag{1.51}$$

where (1.38) has been used. By comparing with (1.50) and fixing some normalization constant, we end up with

$$T_{ab} = -\frac{4\pi}{\sqrt{g}}\frac{\partial\mathcal{S}}{\partial g^{ab}}, \tag{1.52}$$

which coincides with the definition (1.11).
Moreover, invariance under scale transformations of the metric (no coordinates involved) constrains the stress-energy tensor to be traceless. Indeed, under a scale transformation, $\delta g^{ab} = \varepsilon g^{ab}$

$$\delta\mathcal{S} = \int d^2\sigma \frac{\partial\mathcal{S}}{\partial g_{ab}}\delta g_{ab} = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\varepsilon T^\alpha{}_\alpha = 0 \Rightarrow T^\alpha{}_\alpha = 0.$$

Notice that, consistently, (1.11) is traceless. This property implies that

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$

and by the conservation equation $\partial_a T^{ab} = 0$, we get

$$\bar{\partial} T_{zz} = 0, \qquad \partial T_{\bar{z}\bar{z}} = 0.$$

Thus the two non-vanishing components of the stress-energy tensor are respectively holomorphic and anti-holomorphic. In the folowing we are going to label them as $T_{zz}(z) \equiv T(z)$ and $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$.

The stress-energy tensor also allows to define the Noether current for any infinitesimal conformal change of coordinates

$$z \to z + \varepsilon(z), \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z}).$$

By employing again the compensating metric trick of (1.51) we arrive at

$$\delta \mathcal{S} = -\int d^2x \frac{\partial \mathcal{S}}{\partial g^{ab}} \delta g^{ab} = -\frac{1}{2\pi} \int d^2x T_{ab} \partial^\alpha \delta x^\beta = -\frac{1}{2\pi} \int d^2x (T \bar{\partial} \varepsilon + \bar{T} \partial \bar{\varepsilon}), \tag{1.53}$$

wihch of course vanishes when ε is holomorphic and $\bar{\varepsilon}$ is antiholomorphic (as expected since we are dealing with a symmetry). Since z and $\bar{z}$ variables are independent, let us focus on the symmetry $\delta z = \varepsilon(z)$, $\delta \bar{z} = 0$. Exactly as done in (1.51), we perform the promotion $\varepsilon(z) \to \varepsilon(z, \bar{z}) = \varepsilon(z) f(\bar{z})$. Consequently, from the vanishing of we can identify the holomorphic and the antiholomorphic currents

$$J(z) = T(z)\varepsilon(z), \qquad \bar{J}(\bar{z}) = \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}) \tag{1.54}$$

with $\bar{J}$ arising from the analogous procedure for the symmetry $\delta z = 0$, $\delta \bar{z} = \bar{\varepsilon}(\bar{z})$.

1.2.4 Ward identity and the primary operators

Turning to the quantum theory the conservation of a current gets expressed inside correlation functions in the so-called *Ward identity.* In the path-integral approach to quantum field theories, the invariant object under a symmetry is the partition function

$$Z = \int \mathcal{D}\phi \; e^{-\mathcal{S}[\phi]},$$

where $\phi(x)$ is a short notation for all fields. Let us work with the correlation function

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; e^{-\mathcal{S}(\phi)} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n), \tag{1.55}$$

where $J^a(x)$ is the classically conserved current and $\mathcal{O}_i(x_i)$ are local operators depending on the fields ϕ. Under a certain infinitesimal symmetry,

$$\phi' = \phi + \varepsilon \delta\phi, \qquad \mathcal{O}'_i = \mathcal{O}_i + \varepsilon \delta \mathcal{O}_i,$$

with ε independent on x. By the standard promotion of this parameter to a local variable, imposing ε to be supported only away from the insertions of $\mathcal{O}_i$ operators (so that they do not vary under this symmetry), (1.55) gets

$$\begin{aligned}\frac{1}{Z'} \int \mathcal{D}\phi' \; e^{-\mathcal{S}[\phi']} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) &= \int \mathcal{D}\phi \exp\Big(-\mathcal{S}[\phi] - \frac{1}{2\pi} \int d^2x \; J^a \partial_a \varepsilon\Big) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) = \\ &= \int \mathcal{D}\phi \; e^{-\mathcal{S}[\phi]} \Big(1 - \frac{1}{2\pi} \int d^2x \; J^a \partial_a \varepsilon\Big) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n),\end{aligned}$$

where we have exploited the definition of the variation of the action in terms of a Noether current, with $1/2\pi$ just a convention. Imposing the correlator to be invariant under this symmetry, we end up with the conservation equation

$$\langle \partial_a J^a(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = 0$$

To evaluate this expression when x approaches the insertion of one operator, w.l.o.g. $\mathcal{O}_1(x_1)$, we choose the support of ε to also include x_1, for instance ε constant in its support and vanishing in the outside. The same procedure, stopping at the first order in ε, leads to the *Ward identity*

$$-\frac{1}{2\pi}\int_{\varepsilon} d^2x\partial_a \langle J^a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \langle\delta\mathcal{O}_1(\sigma_1)\dots\rangle\,, \tag{1.56}$$

with the integral calculated on the support of ε.
This can be specialized to conformal symmetries. Denoting by $\hat{n}^a = (dx^2, -dx^1)$ the normal unit vector to the boundary of the support of ε, the Stokes' theorem allows to write

$$\int_{\varepsilon} d^2x\;\partial_a J^a(x) = \oint_{\partial\varepsilon} J_a\hat{n}^a = \oint_{\partial\varepsilon}\big(J_1 dx^2 - J_2 dx^1\big) = -i\oint_{\partial\varepsilon}\big(J_z dz - J_{\bar{z}}d\bar{z}\big).$$

where in the last step we have used the definitions (1.42) and (1.43). We insert this in (1.56) and use the Cauchy residue theorem. Splitting the holomorphic and the antiholomorphic components of the transformations

$$\begin{aligned}\delta\mathcal{O}_1(x_1) &= -\mathrm{Res}[J_z(z)\mathcal{O}_1(x_1)] = -\mathrm{Res}[\varepsilon(z)T(z)\mathcal{O}_1(x_1)]\\ \delta\mathcal{O}_1(x_1) &= -\mathrm{Res}[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_1(x_1)] = -\mathrm{Res}[\bar{\varepsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_1(x_1)]\end{aligned} \tag{1.57}$$

where the last expression holds using the conserved currents under the conformal transformations (1.54).
These residues can be easily read off from the operator product expansion (OPE) between the current and the operator or, analogously, from the OPE of the stress-energy tensor and the operator. OPEs between two local fields in CFT are defined as the expansion

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum_k C^k_{ij}(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}), \tag{1.58}$$

where the coefficients can only depend on the separation between the two points, due to translational invariance. In quantum field theory, operators are always meant to stay inside correlator, hence we are implictily assuming that in (1.58), operators are time-ordered. Therefore, we expect that $\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \mathcal{O}_j(w,\bar{w})\mathcal{O}_i(z,\bar{z})$, at most with a "-" in front of the right-hand side, if we are dealing with Grassmannian variables.
OPE can also come in handy for identifying the so-called *primary* operators, which by definition, are such that their OPEs with the stress-energy tensor have at most second-order poles:

$$\begin{aligned}T(z)\mathcal{O}(w,\bar{w}) &= h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots\\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \dots.\end{aligned} \tag{1.59}$$

where $\dots$ stand for non-singular terms which we are not interested in. Because of (1.57) (considering only $\delta z = \varepsilon(z)$, but for the antiholomorphic part, the reasonings are the same)

$$\delta\mathcal{O}(w,\bar{w}) = -\mathrm{Res}\Big[\varepsilon(z)\Big(h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2}+\frac{\partial\mathcal{O}(w,\bar{w})}{z-w}+\dots\Big)\Big] = -h\varepsilon'(w)\mathcal{O}(w,\bar{w})-\varepsilon(w)\partial\mathcal{O}(w,\bar{w}), \tag{1.60}$$

where in the last step we have Taylor-expanded $\varepsilon(z)$ around $z = w$ and computed the residue. By integrating, we get the transformation rule of a primary operator under a finite conformal transformation $(z,\bar{z}) \to (\tilde{z},\bar{\tilde{z}})$

$$\mathcal{O}(z,\bar{z}) \to \tilde{\mathcal{O}}(\tilde{z},\bar{\tilde{z}}) = \Big(\frac{\partial\tilde{z}}{\partial z}\Big)^{-h}\Big(\frac{\partial\bar{\tilde{z}}}{\partial\bar{z}}\Big)^{-\tilde{h}}\mathcal{O}(z,\bar{z}), \tag{1.61}$$

where we have also added the piece referring to the antiholomorphic transformation. h and $\tilde{h}$ are real numbers, named *conformal weights* of the operator. They can be identified as the eigenvalues with respect to the generators l_0 and $\bar{l}_0$ of the global conformal algebra, of some state corresponding to $\mathcal{O}$. Thereafter, the spin $s = h - \tilde{h}$ and the scaling dimension $\Delta = h + \tilde{h}$ (nothing but the naive dimension of the operator in classical field theory) are the eigenvalues of the same eigenstate, with respect to rotations and dilations. As we are going to prove in the next Section, unitarity constrains them to be $h, \tilde{h} \geq 0$.

Conformal weights do not charachterize only primary fields, but all fields whose OPE with the stress-energy tensor has the form (1.60) plus some higher order singularity. One example is the stress-energy tensor, which has conformal dimension 2 (its integral over a one-dimensional space is an energy) and spin 2 as well (it is a symmetric 2-tensor). Hence, the component T has weight $(2, 0)$ and $\bar{T}$ has weight $(0, 2)$ and the most general TT OPE will be

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \tag{1.62}$$

The first and second-order pole coefficients arise from the conformal weight of T. $\propto (z-w)^{-3}$ contribution would violate the commutation property of the OPE, thus it does not appear, whereas the coefficient of the fourth order pole is dimensionless (so that each addend has dimension 4) and is named *central charge*: as long as it does not vanish, the stress-energy tensor cannot be a primary field. Higher order singularities cannot be written since in unitary CFTs, $h, \tilde{h} \geq 0$ and then operators with negative dimension, necessary to provide the possible higher-order singularity terms with the correct dimension, cannot exist.

Let us recall that local operators in OPEs are implicitly assumed to appear in correlators and hence they are time-ordered. A time-ordered product of operators can be computed exploiting the Wick theorem

$$T(\mathcal{O}_1(x_1), \ldots, \mathcal{O}_n(x_n)) =: \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) : + \sum \text{contractions} \tag{1.63}$$

where the sum is over all contractions, i.e. over all possible ways to substitute one, two or more pairs of operators in the normal-ordered product, with the propagator of their two fields. Normal-ordered terms are regular and usually omitted in OPEs, since this operation erases singularities. Let us discuss whether X^μ fields are primary or not. The propagator is given by

$$\langle X^\mu(x_1) X^\nu(x_2) \rangle = -\delta^{\mu\nu} \frac{\alpha'}{2} \ln{(x_1 - x_2)^2}. \tag{1.64}$$

We can prove this by considering the action for these fields in conformal gauge (1.9) in the Euclidean

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int_M d^2x \; \partial_a X^\rho \partial^a X^\sigma \delta_{\rho\sigma}. \tag{1.65}$$

Since total derivatives in the path-integral vanish, we get

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X_\mu(x_1)} \big[e^{-\mathcal{S}} X^\nu(x_2)\big] = \int \mathcal{D}X \Big[\frac{1}{2\pi\alpha'} \partial^2 X^\mu(x_1) X^\nu(x_2) + \delta(x_1 - x_2)\Big],$$

which gives (1.64), remembering that

$$\partial^2 \ln(x_1 - x_2)^2 = 4\pi\delta(x_1 - x_2).$$

Due to Wick's theorem, the propagator is the only singular part in the OPE, hence

$$X^\mu(x_1) X^\nu(x_2) = -\delta^{\mu\nu} \frac{\alpha'}{2} \ln(x_1 - x_2)^2$$

Since X^μs along different directions do not give rise to singular terms, we just focus on one direction. Shifting to complex coordinates and only cosidering the left moving part of the solution $X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$ we get the OPEs

$$X(z)X(w) = -\frac{\alpha'}{2} \ln(z-w) + \ldots, \qquad \partial X(z) \partial X(w) = -\frac{\alpha'}{2} \frac{1}{(z-w)^2} + \ldots \tag{1.66}$$

The stress-energy tensor for just one direction (normal-ordered so that its v.e.v. vanishes)

$$T(z) = -\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \tag{1.67}$$

satisfies the OPE

$$T(z)\partial X(w) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) : \partial X(w) \sim -\frac{2}{\alpha'}\partial X(z)\Big(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\Big) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w},$$

which is computed using the Wick's theorem ("$\sim$" means up to non-singular terms, such as the normal-ordered product). The prefactor 2 refers to the number of possible contractions and in the last step we have Taylor expanded $\partial X(z)$ around $z = w$. We learn thus, that ∂X is a primary field with conformal weights $(1, 0)$. Moreover,

$$\begin{aligned} T(z)T(w) &= \frac{1}{\alpha'^2} : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \\ &\sim \frac{2}{\alpha'^2}\Big(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\Big)^2 - \frac{4}{\alpha'^2}\frac{\alpha'}{2}\frac{: \partial X(z)\partial X(z) :}{(z-w)^2} = \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \end{aligned}$$

and we find again what we already knew from general arguments: the holomorphic component of the stress-energy tensor is not a primary field and has conformal weight $(2, 0)$. The central charge for each $X^\mu(z)$ field is $c = 1$.

The only other conformal field in the free bosonic model is the normal ordered exponential of $X^\mu(z, \bar{z})$. The corresponding conformal weights are given by $h = \tilde{h} = \alpha' k^2/4$, as we can read off from the OPE with the stress-energy tensor (1.67) and its antiholomorphic counterpart. Indeed,

$$\begin{aligned} \partial X(z) : e^{ikX(w)} : &= \sum_{n=0}^{+\infty} \frac{(ik)^n}{n!} \partial X(z) : X(w)^n : \sim \\ &\sim \sum_{n=1}^{+\infty} \frac{(ik)^n}{(n-1)!} : X(w)^{n-1} : \Big(-\frac{\alpha'}{2}\frac{1}{z-w}\Big) = \\ &= -\frac{i\alpha' k}{2}\frac{: e^{ikX(w)} :}{z-w} \end{aligned} \tag{1.68}$$

and then

$$\begin{aligned} T(z) : e^{ikX(w)} : &= -\frac{1}{\alpha'} : \partial X(z)\partial X(z) :: e^{ikX(w)} := \\ &= \frac{\alpha' k^2}{4}\frac{: e^{ikX(w)} :}{(z-w)^2} + ik\frac{: \partial X(z) e^{ikX(w)} :}{z-w} = \\ &= \frac{\alpha' k^2}{4}\frac{: e^{ikX(w)} :}{(z-w)^2} + \frac{\partial_w : e^{ikX(w)} :}{z-w}, \end{aligned} \tag{1.69}$$

as expected for a conformal field of weight $h = \alpha' k^2/4$. Analogous OPEs also hold in the antiholomorphic sector.

1.2.5 Virasoro algebra

The quantization of a CFT can be accomplished through the so-called *radial quantization*. Let us name (τ, σ) the coordinates of the the Euclidean plane where the theory is defined. We can then get a cylinder by compactifying one direction (which after Wick rotating, can be identified with the spacelike component in the plane with Minkowski metric) $\sigma \sim \sigma + 2\pi$. After that, we can perform a map between this cylinder (parametrized by w) and the complex plane (parametrized by z):

$$w = \sigma + i\tau, \quad z = e^{-iw}, \qquad \sigma \in [0, 2\pi[.$$

Accordingly, $\tau \to -\infty$ on the cylinder corresponds to the point $z = 0$. On the cylinder the states live on slices of constant σ and evolve with $H = \partial_\tau$. On the complex plane, this operator coincides with $D = z\partial + \bar{z}\bar{\partial}$, the generator of the dilations: constant τ closed lines on the cylinder corresponds to constant radius circles on the complex plane.
Let us name $T(z)$ and $\bar{T}(\bar{z})$ the components of the stress-energy tensor on the complex plane: they are respectively holomorphic and antiholomorphic thereafter we can write their Laurent expansions

$$T(z) = \sum_{m=-\infty}^{+\infty} \frac{L_m}{z^{m+2}}, \qquad \bar{T}(\bar{z}) = \sum_{m=-\infty}^{+\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}. \tag{1.70}$$

If we choose a contour of constant radius and only containing z as operator insertion, the coefficients of these expansions can be written as

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z), \qquad \tilde{L}_n = \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}). \tag{1.71}$$

Recalling (1.54), we have but constant-time integrals (on the cylinder) of the conserved currents under the conformal transformation $\delta z = z^{n+1}$, hence these can be identified as conserved charges. We can meaningfully calculate their algebra on the complex plane:

$$\big[L_m, L_n\big] = L_m L_n - L_n L_m = \Big(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\Big)\, z^{m+1} w^{n+1} T(z) T(w).$$

As usual in quantum theories, products such $L_m L_n$ sit inside correlators and are time-ordered. On the comlex plane this amounts to radially order the operators: in the first addend we are implicitly assuming $|z| > |w|$ and viceversa in the second one. By deforming the contours we eventually get

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i}\, z^{m+1} w^{n+1} T(z) \\
&= \oint \frac{dw}{2\pi i}\, \mathrm{Res}\Big[z^{m+1} w^{n+1} \Big(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots\Big)\Big] = \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \Big[w^{m+1} \partial T(w) + 2(m+1) w^m T(w) + \frac{c}{12} m(m^2-1) w^{m-2}\Big] = \\
&= (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}
\end{aligned} \tag{1.72}$$

where in the last step we have integrated the first term by parts and recognized for the first two terms the expression for L_{m+n}. This is the *Virasoro algebra*, which can be thought as the quantum version of the Witt algebra (1.48). We recall that on the worldsheet the residue symmetry was the succession of a coordinates reparametrization and a Weyl rescaling: the extra term in (1.72) compared to (1.48) is exactly due to this Weyl rescaling. Hence a CFT is defined on the worldsheet, whose symmetries are the compositions of a coordinate change and a Weyl rescaling. They are gauge symmetries.
Let us notice that the subalgebra of L_{-1}, L_0 and L_1 is

$$[L_{\pm 1}, L_0] = \pm L_{\pm}, \qquad [L_1, L_{-1}] = 2L_0,$$

which means that the global conformal group $SL(2,\mathbb{C})$ generated by $L_{-1,0,1}$ is a symmetry group at the quantum level as well.
In order to study the representation of the Virasoro algebra let us start from $|\psi\rangle$, common eigenstate of L_0 and $\tilde{L}_0$ (which exists since these operators commute)

$$L_0 |\psi\rangle = h |\psi\rangle, \quad \tilde{L}_0 |\psi\rangle = \tilde{h} |\psi\rangle, \tag{1.73}$$

where h and $\tilde{h}$ are the energy eigenvalues. Using (1.72), we get that for any n,

$$L_0 L_n |\psi\rangle = (L_n L_0 - n L_n) |\psi\rangle = (h - n) L_n |\psi\rangle ,$$

so L_n operators raise or lower the energy of the state, respectively when $n < 0$ or $n > 0$. If the spectrum is bounded from below, we are able to find some states whose energy cannot be lowered, i.e. such that

$$L_n |\psi\rangle = \tilde{L}_n |\psi\rangle = 0 \qquad \text{for } n > 0. \tag{1.74}$$

These states are named *primary* or in the language of representation theory, *highest weight states*, since mathematicians usually invert the sign of L_0. All the other states of the representation of the Virasoro algebra arise acting on the primary states with the raising operators L_{-n}, for $n > 0$: these states are named *descendants* and they make up the so-called *Verma module* of the given primary state. This is an irreducible representation of the Virasoro algebra and the spectrum of the primary states suffices to know the spectrum of the thorough theory. We can identify the vacuum as the primary state of minimum energy. As we are going to show in this Subsection, $h, \tilde{h} \geq 0$, so the minimum energy is achieved when $h = \tilde{h} = 0$: the vacuum state is then such that

$$L_n |0\rangle = 0 = \bar{L}_n |0\rangle \qquad \text{for } n \geq 0.$$

Moreover, the regularity of

$$T(z) |0\rangle = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2} |0\rangle$$

(we are using (1.71)) at $z = 0$ requires that

$$L_n |0\rangle = 0 \qquad \text{for } n \geq -1,$$

which contains the property that the vacuum is invariant under the global conformal group $SL(2, \mathbb{C})$. We can exploit the definition of adjoint of the local operator $A(z, \bar{z})$

$$[A(z, \bar{z})]^\dagger = A\Big(\frac{1}{\bar{z}}, \frac{1}{z}\Big) \frac{1}{\bar{z}^{2h}} \frac{1}{z^{2\bar{h}}} \tag{1.75}$$

to infer that $L_n = L_{-n}^\dagger$.[1] As a consequence,

$$\langle 0| T(z) |0\rangle = 0 = \langle 0| \bar{T}(\bar{z}) |0\rangle .$$

The Virasoro algebra also allows to show that $h, \tilde{h} \geq 0$ and in any non-trivial theory $c \geq 0$. Indeed, due to unitarity of the Hilbert space of the quantum conformal field theory the descendant field $L_{-1} |\psi\rangle$, with $|\psi\rangle$ primary, has to satisfy the condition

$$\big| L_{-1} |\psi\rangle \big|^2 = \langle \psi| L_{+1} L_{-1} |\psi\rangle = \langle \psi| [L_{+1}, L_{-1}] |\psi\rangle = 2h \langle \psi|\psi\rangle \geq 0$$

and since $\langle \psi|\psi\rangle$, $h \geq 0$. By the same reason, $\tilde{h} \geq 0$. Again imposing the unitarity of the Hilbert space, for $n > 0$,

$$| L_{-n} |0\rangle |^2 = \langle 0| [L_n, L_{-n}] |0\rangle = \frac{c}{12} n(n^2 - 1) \geq 0$$

therefore $c > 0$ ($c = 0$ is the trivial theory with only the vacuum as a state).
In a CFT, an isomorphism between states and local operators holds, despite the former live on a spatial slice at fixed time and the latter at a fixed point in time and also in space. The identification

[1] Indeed, from (1.75) and (1.71), we see that

$$T^\dagger(z) = T\Big(\frac{1}{\bar{z}}\Big) \frac{1}{\bar{z}^4} \Rightarrow \sum_{m=-\infty}^{+\infty} \frac{L_m^\dagger}{\bar{z}^{m+2}} = \sum_{m=-\infty}^{+\infty} \frac{L_m}{\bar{z}^{-m-2}} \frac{1}{\bar{z}^4} = \sum_{m=-\infty}^{+\infty} \frac{L_{-m}}{\bar{z}^{m+2}}.$$

is possible because the far past on the cylinder can be mapped to the origin of the complex plane: states defined at the infinite past, then, can be matched with local fields at the origin. Therefore, the in-states of the theory can be defined as

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\,, \tag{1.76}$$

where $\phi(z,\bar{z})$ is a local operator. For the out-states $\tau \to +\infty$ and then $z \to +\infty$ or analogously $w = 1/z \to 0$. One can prove that

$$\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger\,,$$

where the Hermitean conjugation for the local field in (1.76), is defined as in (1.75). When $\phi(z,\bar{z})$ is a primary field of conformal weights $(h,\tilde{h})$, we get that the state

$$|h\rangle = \phi(0,0)\,|0\rangle \tag{1.77}$$

is primary. Let us show this for the holomorphic part. Making use of the OPEs (1.59), we get that

$$[L_n,\phi(w)] = \oint \frac{dz}{2\pi i} z^{n+1}T(z)\phi(w) = h(n+1)w^n\phi(w) + w^{n+1}\partial\phi(w)$$

and then, since $L_n\,|0\rangle = 0$,

$$L_0\,|h\rangle = h\,|h\rangle\,, \qquad L_n\,|h\rangle = 0 \quad \text{for } n>0,$$

which are the conditions (1.73) and (1.74) for primary states. Primary states are then tightly linked to primary operators and the labels h and $\tilde{h}$ in (1.59) and (1.73) coincide meaningfully. Moreover, we can organize the local operators of a conformal field theory in families which are the operator analogue of Verma modules: they indeed, contain one primary operator and a set of secondary fields which are its descendants.

1.2.6 Vertex operators

The primary operators appearing in (1.77) are also named *vertex operators*. However, in order to preserve the gauge invariance for diffeomorphisms (which from the active perspective, are displacements of the insertion points along the worldsheet), we expect them not to be defined on a specific point, but rather, to be integrated over the whole worldsheet:

$$V \propto \int d^2z\ \phi(z,\bar{z}). \tag{1.78}$$

Moreover, the conformal weights of vertex operators have to be $(1,1)$ so that the Weyl symmetry of V is preserved.
Vertex operators allow to recover the spectrum of conformal field theories. For instance, let us focus on the X^μ fields for the closed bosonic string. The vertex operator for the lowest-energy state is given by $:e^{ik\cdot X(z,\bar{z})}:$, then the tachyon state can be written as

$$|0;k\rangle =: e^{ik\cdot X(0,0)} : |0\rangle\,. \tag{1.79}$$

In order that the conformal weights $h = \tilde{h} = \alpha' k^2/4$ are equal to one, we need to impose the further condition that $M_{cl}^2 = -k^2 = -\frac{\alpha'}{4}$ which is the already known mass for a tachyon. Moreover, by using one expression of the momentum operator as a contour integral and the OPE (1.68), we can compute

$$p^\mu\,|0;k\rangle = \frac{2}{\alpha'}\oint \frac{dz}{2\pi i}\ i\partial X_\mu(z) : e^{ik\cdot X(0,0)} : |0\rangle = k^\mu\,|0;k\rangle\,.$$

The first excited state is given by

$$|k,\xi\rangle = -\frac{2}{\alpha'}\xi_{\mu\nu}(k) \lim_{z,\bar{z}\to 0} : \partial X^\mu(z)\bar{\partial}\bar{X}^\nu(\bar{z})e^{ik\cdot X(z,\bar{z})} : |0\rangle\,,$$

with $\xi_{\mu\nu}$ polarization tensor. From the OPE

$$T(z) : \xi_{\mu\nu}(k)\partial X^\mu(w)\bar{\partial}\bar{X}^\nu(\bar{w})e^{ik\cdot X(w,\bar{w})} : \sim -\frac{i\alpha'}{2}\frac{k^\mu\xi_{\mu\nu}}{(z-w)^3} : \bar{\partial}^\nu\bar{X}(\bar{w})e^{ik\cdot X(w,\bar{w})} : + \\ + \Big[\frac{\frac{\alpha'}{4}k^2+1}{(z-w)^2} + \frac{\partial_w}{z-w}\Big]\xi_{\mu\nu}(k) : \partial X^\mu(w)\bar{\partial}\bar{X}^\nu(\bar{w})e^{ik\cdot X(w,\bar{w})} :,$$

we infer the conditions so that the vertex operator is primary and with conformal weight one:

$$k^\mu\xi_{\mu\nu}(k) = 0, \qquad M_{cl}^2 = -k^2 = 0,$$

i.e. we recover the transversality of the polarization tensor and the mass-shell condition for the first excited states, which we already knew to be massless.

1.3 The path-integral quantization

The central charge is linked to the so-called *Weyl anomaly*, i.e. the fact that non-trivial quantum conformal field theory do not mantain Weyl invariance. This is quantified by

$$\langle T^\alpha{}_\alpha\rangle = -\frac{c}{12}R. \tag{1.80}$$

The Ricci scalar is not gauge invariant since a Weyl scaling changes the metric: this v.e.v. stops being a physical quantity as we would instead demand for conformal field theories. This issue can be healed by introducing the ghost fields.

1.3.1 Polyakov path integral

The dynamical fields for the Polyakov action in the Euclidean ((1.6) with $\eta_{\mu\nu} = \delta_{\mu\nu}$ and no "-" in front) are the coordinates X^μ and the metric g_{ab}. The partition function, then, reads

$$Z = \frac{1}{\text{Vol}}\int \mathcal{D}g\mathcal{D}X e^{-\mathcal{S}_P[X,g]}. \tag{1.81}$$

Gauge symmetries on the worldsheet are the diffeomorphisms and the Weyl invariance, whose general transformation can be written as

$$g_{ab}(\sigma) \to g^\zeta_{ab}(\sigma') = e^{2\omega(\sigma)}\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}g_{cd}.$$

The prefactor in (1.81) is the volume of the so-called gauge orbits, i.e. the curves in the configuration space collecting all gauge-equivalent metrics: we want to path-integrate just on the physically inequivalent degrees of freedom, i.e. one representative $\hat{g}$ on each orbit. Let us fix some metric g: there exists exactly one gauge transformation ζ such that $g = \hat{g}^\zeta$. We can thus define

$$\int \mathcal{D}\zeta\ \delta(g - \hat{g}^\zeta) = \Delta^{-1}_{FP}[g] \tag{1.82}$$

where $\Delta_{FP}[g]$ is named *Faddeev-Popov determinant* and can be thought as the Jacobian of the expression inside the δ functional. Let us notice that the measure $\mathcal{D}\zeta$ is gauge invariant as well as $\Delta_{FP}[g]$. We can calculate the partition function for the physical metric $\hat{g}$:

$$\begin{aligned} Z[\hat{g}] &= \frac{1}{\text{Vol}}\int \mathcal{D}\zeta\mathcal{D}X\mathcal{D}g\ \Delta_{FP}[g]\delta(g-\hat{g}^\zeta)e^{-\mathcal{S}_P[X,g]} \\ &= \frac{1}{\text{Vol}}\int \mathcal{D}\zeta\mathcal{D}X\ \Delta_{FP}[\hat{g}^\zeta]e^{-\mathcal{S}_P[X,\hat{g}^\zeta]} \\ &= \frac{1}{\text{Vol}}\int \mathcal{D}\zeta\mathcal{D}X\ \Delta_{FP}[\hat{g}]e^{-\mathcal{S}_P[X,\hat{g}]} \\ &= \int \mathcal{D}X\ \Delta_{FP}[\hat{g}]e^{-\mathcal{S}_P[X,\hat{g}]}. \end{aligned} \tag{1.83}$$

In the second step we have exploited the gauge invariance of the action, the metric and Δ_{FP}. In the last step we have simply noticed that nothing depends on ζ anymore and Vol $= \int \mathcal{D}\zeta$. The Faddeev-Popov determinant can be expressed in function of some Grassmannian-valued scalar fields, which are also known as *ghosts*: the traceless symmetric worldsheet tensor b_{ab} and c^{α}. In the Euclidean

$$\Delta_{FP}[g] = \int \mathcal{D}b\mathcal{D}c \; e^{-\mathcal{S}_{\text{ghost}}[b,c,\hat{g}]}, \tag{1.84}$$

thereafter

$$Z[\hat{g}] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c \; \Delta_{FP}[\hat{g}] e^{-\mathcal{S}_P[X,\hat{g}]-\mathcal{S}_{\text{ghost}}[b,c,\hat{g}]}.$$

The action for ghosts reads

$$\mathcal{S}_{\text{ghost}} = \frac{1}{2\pi}\int d^2\sigma \; \sqrt{g} b_{ab}\nabla^a c^b. \tag{1.85}$$

We choose to work in the conformal gauge $\hat{g}_{ab} = e^{2\omega}\delta_{ab}$ ad in complex coordinates. Since b_{ab} is traceless and symmetric, $b_{z\bar{z}} = 0$. We also notice that

$$\Gamma^{z}_{\bar{z}\alpha} = \frac{1}{2} g^{z\bar{z}}\big(\partial_{\bar{z}} g_{\alpha\bar{z}} + \partial_\alpha g_{\bar{z}\bar{z}} - \partial_{\bar{z}} g_{\bar{z}\alpha}\big) = 0 = \Gamma^{\bar{z}}_{z\alpha}$$

and then the ghost action in complex coordinates is

$$\mathcal{S}_{\text{ghost}} = \frac{1}{2\pi}\int d^2z\big(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\big) = \frac{1}{2\pi}\int d^2z\big(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\big) = \frac{1}{2\pi}\int d^2z \;\big(b\bar{\partial}c + \bar{b}\partial\bar{c}\big). \tag{1.86}$$

In the last step we have renamed the variables:

$$b = b_{zz}, \quad \bar{b} = b_{\bar{z}\bar{z}}, \quad c = c^z, \quad \bar{c} = c^{\bar{z}}.$$

b and c are holomorhic, $\bar{b}$ and $\bar{c}$ are anti-holomrphic. Indeed their equations of motion are

$$\bar{\partial}b = \partial\bar{b} = \partial\bar{c} = \bar{\partial}b = 0. \tag{1.87}$$

Along with the definition (1.52), the action (1.85) allows to compute the stress energy tensor in the conformal gauge. The tracelessness of b_{ab} and the equations of motion (1.87) have to be used and eventually we arrive at

$$T = 2 : \partial c b : + : c\partial b :, \qquad \bar{T} = 2 : \bar{\partial}\bar{c}\bar{b} : + : \bar{c}\bar{\partial}\bar{b} :, \tag{1.88}$$

where the normal ordering has been implemented as well.
In order to discover the central charge of the ghost system, we determine the OPEs.

$$0 = \int \mathcal{D}b\mathcal{D}c \; \frac{\delta}{\delta b(\sigma)}\big[e^{-\mathcal{S}_{\text{ghost}}} b(\sigma')\big] = \int \mathcal{D}b\mathcal{D}c\Big[-\frac{1}{2\pi}\bar{\partial}c(\sigma)b(\sigma') + \delta(\sigma-\sigma')\Big],$$

therefore, by using $\bar{\partial}(1/z) = 2\pi\delta(z,\bar{z})$, we end up with

$$b(z)c(w) = \frac{1}{z-w} + \dots, \qquad c(w)b(z) = \frac{1}{w-z} + \dots. \tag{1.89}$$

The second OPE arises from the Fermi statistics. Furthermore, again using Wick's theorem,

$$T(z)c(w) = 2 : \partial c(z)b(z) : c(w) + : c(z)\partial b(z) : c(w) = -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w},$$
$$T(z)b(w) = 2 : \partial c(z)b(z) : b(w) + : c(z)\partial b(z) : b(w) = \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}.$$

As a consequence, c has conformal weight -1, whereas b has conformal weight 2. These could also be deduced by the tensor structure of the ghosts under diffeomorphism: the ghosts are indeed,

neutral under Weyl transformations.
To infer the central charge of ghosts, we compute

$$\begin{aligned}T(z)T(w) &= 4 : \partial c(z)b(z) :: \partial c(w)b(w) : +2 : \partial c(z)b(z) :: c(w)\partial b(w) : \\ &+ 2 : c(z)\partial b(z) :: \partial c(w)b(w) : + : c(z)\partial b(z) :: c(w)\partial b(w) := \\ &= \frac{-4}{(z-w)^4} + \frac{4 : \partial c(z)b(w) :}{(z-w)^2} - \frac{4 : b(z)\partial c(w) :}{(z-w)^2} + \\ &- \frac{4}{(z-w)^4} + \frac{2 : \partial c(z)\partial b(w) :}{z-w} - \frac{4 : b(z)c(w) :}{(z-w)^3} + \\ &- \frac{4}{(z-w)^4} - \frac{4 : c(z)b(w) :}{(z-w)^3} + \frac{2 : \partial b(z)\partial c(w) :}{z-w} + \\ &- \frac{1}{(z-w)^4} - \frac{: c(z)\partial b(w) :}{(z-w)^2} + \frac{: \partial b(z)c(w) :}{(z-w)^2} = \\ &= \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots\end{aligned} \tag{1.90}$$

and we can read off that

$$c = -26.$$

Avoiding the Weyl anomaly is possible if we add any kind of matter degrees of freedom whose central charge amounts to $c_m = 26$. For instance, in the free theory, each coordinate X^μ is such that $c = 1$ and hence we should add $D = 26$ of these: we have gained an alternative determination of the critical dimension of the ambient space. On the other hand, this is just an example and any other choice cancelling the central charge of ghosts is allowed.

1.3.2 BRST quantization

Compared to the covariant and the lightcone gauge instances, the Polyakov quantization introduces new degrees of freedom, the ghosts, and involves all X^μ fields, without distinguishing between transverse and longitudinal polarizations. In this subsection, we focus on the tool properly conceived in order to identify the physical states within the path-integral quantization of theories with local gauge symmetries: *BRST quantization*. Generally speaking, the generators of the gauge symmetries make up a finite dimensional Lie algebra $\mathfrak{g}$

$$[K_i, K_j] = i f_{ij}{}^k K_k, \qquad i, j, k = 1, \ldots \dim\mathfrak{g}, \tag{1.91}$$

where $f_{ij}{}^k$ are the structure constants of the Lie algebra. We can introduce the anticommuting fields c^i and b_i, collectively named ghosts, such that

$$\{c^i, c^j\} = 0, \qquad \{b_i, b_j\} = 0, \qquad \{c^i, b_j\} = \delta^i{}_j. \tag{1.92}$$

They enter the BRST charge, a conserved operator which acts on all fields as a fermionic transformation and reads

$$Q = c^i\Big(K_i - \frac{1}{2} f_{ij}{}^k c^j b_k\Big) = c^i\Big(K_i + \frac{1}{2} K_i^{b,c}\Big). \tag{1.93}$$

The fields c^i replace the gauge parameters, whereas the fields b_i are needed so that the charge is nilpotent. Indeed, due to the anticommutation relations (1.92),

$$\begin{aligned}Q^2 &= c^i\Big(K_i - \frac{1}{2} f_{ij}{}^k c^j b_k\Big) c^l \Big(K_l - \frac{1}{2} f_{lm}{}^n c^m b_n\Big) = \\ &= \frac{1}{2} f_{il}^m c^i c^l K_m - \frac{1}{2} f_{ij}^k c^i c^j \{b^k, c^l\} K_l + \frac{1}{4} f_{ij}{}^k f_{lm}{}^n c^i c^j b^k c^l c^m b^n = \\ &= \frac{1}{4} f_{ij}{}^l f_{lmn} c^i c^j c^m b^n - \frac{1}{4} f_{ijk} f_{lmn} c^i c^j c^l b^k c^m b^n = \\ &= \frac{1}{2} f_{ij}{}^l f_{lmn} c^i c^j c^m b^n + \frac{1}{4} f_{ijk} f_{lmn} c^i c^j c^l c^m b^k b^n = \\ &= \frac{1}{2} f_{[ij|}{}^l f_{l|m]n} c^i c^j c^m b^n = 0,\end{aligned}$$

where the last step is due to the Jacobi identity of the structure constants. As long as ghosts and K_i generators are Hermitian then Q is Hermitian as well. Moreover, one can define the ghost number operator

$$N_g = -\sum_{i=1}^{\dim\mathfrak{g}} b_i c^i, \tag{1.94}$$

according to which c^i and Q have ghost number $+1$ whilst b_i has ghost number -1. BRST invariant states $|\phi\rangle$ are BRST-closed, i.e. such that

$$Q\,|\phi\rangle = 0, \tag{1.95}$$

are gauge invariant and then they could be physical states. Due to the nilpotency of Q, states of the kind $|\phi\rangle = Q\,|\chi\rangle$ satisfy the condition (1.95). but at the same time, they have vanishing norm as Q is Hermitian and nilpotent. As a consequence, the physical states have to be BRST-closed but not BRST-exact, that is, they have to be such that

$$Q\,|\phi\rangle = 0, \qquad |\phi\rangle \neq Q\,|\chi\rangle\,. \tag{1.96}$$

If two states $|\phi\rangle$ and $|\phi'\rangle$ are such that

$$|\phi\rangle = |\phi'\rangle + Q\,|\chi\rangle\,,$$

they are physically equivalent. The BRST equivalence classes are named *BRST cohomology classes* and S-matrix elements will be independent on the representative chosen for each class. For vertex operators $\Phi(z,\bar z)$, such that $|\phi\rangle = \lim_{z,\bar z\to 0}\Phi(z,\bar z)\,|0\rangle$, the condition (1.96) for states translates into $[Q,\Phi]=0$: operators satisfying this condition describe physical states.

In the case of the bosonic string theory, the gauge Lie algebra is the infinite dimensional Virasoro algebra. The ghosts can be expanded in modes as

$$c(z) = \sum_{n=-\infty}^{+\infty} c_n z^{-n+1}, \qquad b(z) = \sum_{n=-\infty}^{+\infty} b_n z^{-n-2} \tag{1.97}$$

with Hermicity conditions ${b_n}^\dagger = b_{-n}$ and ${c_n}^\dagger = c_{-n}$. Identifying $c^m = c_{-m}$ the BRST charge mimics (1.93) and can written as

$$Q = \sum_m :c_{-m}\left[L_m^X + \frac{1}{2}L_m^{b,c}\right]: \tag{1.98}$$

The normal ordering has been introduced to avoid any singular term. Moreover, the superscripts X and b,c refer respectively to the matter and to the ghosts sectors. By exploiting the expansions (1.70) and (1.97) we can verify that this definition of the BRST charge is equivalent to the contour integral

$$Q = \oint_{C_0}\frac{dz}{2\pi i} :c(z)\left[T^X(z) + \frac{1}{2}T^{b,c}(z)\right]: := \oint_{C_0} j_{BRST}(z). \tag{1.99}$$

The BRST current $j_B(z)$ is defined up to a harmless total derivative and its most general expression is

$$j_{BRST} = cT^X + \frac{1}{2} :cT^{b,c}: + k\partial^2 c = cT^X + \frac{1}{2} :bc\partial c: + k\partial^2 c. \tag{1.100}$$

The dimension and the ghost number of the last summand is indeed consistent with the first two. Moreover, by defining the total stress-energy tensor as $T = T^X + T^{b,c}$, we can calculate the OPE

$$T(z)j_B(w) \sim \frac{D/2-6k-4}{(z-w)^4} + \frac{(3-2k)\partial c(w)}{(z-w)^3} + \frac{j_B(w)}{(z-w)^2} + \frac{\partial j_B(w)}{(z-w)}$$

we can fix the constant k requiring the BRST current to be a primary field of conformal weight one: the vanishing of the third and fourth order poles leads to

$$k = 3/2, \qquad D = 26,$$

then we have found again the requirement of a critical dimension 26. In addition, this is necessary to ensure the nilpotency of the BRST charge. Indeed,

$$j_B(z)j_B(w) \sim -\frac{D-18}{2(z-w)^3}c(w)\partial c(w) - \frac{D-18}{4(z-w)^2}c(w)\partial^2 c(w) - \frac{D-26}{12(z-w)}c(w)\partial^3 c(w) \quad (1.101)$$

and then

$$Q^2 = \frac{1}{2}\{Q,Q\} = \frac{1}{2}\oint_{C_w}\frac{dz}{2\pi i}j_B(z)j_B(w) = 0$$

only if the residue of (1.101) vanishes, i.e. $D = 26$.
In order to identify physical states in the bosonic string theory we need to implement the condition (1.96), which in terms of local vertex operators[2] $|\phi\rangle = \phi(0)|0\rangle$ demands that the commutator

$$[Q, \phi(z)] = \oint_{C_z}\frac{dw}{2\pi i}j_B(w)\phi(z)$$

vanishes or is a total derivative: indeed, vertex operators are integrated over the insertion points. Therefore, since $\phi(z)$ is a conformal field of weight h,

$$\begin{aligned}
[Q,\phi(z)] &= \oint_{C_z}\frac{dw}{2\pi i}j_B(w)\phi(z) = \\
&= \oint_{C_z}\frac{dw}{2\pi i}c(w)T^{\phi}(w)\phi(z) = \\
&= \oint_{C_z}\frac{dw}{2\pi i}c(w)\Big[\frac{h\phi(z)}{(w-z)^2} + \frac{\partial\phi(z)}{w-z}\Big] = \\
&= h\partial c\phi(z) + c\partial\phi(z),
\end{aligned}$$

which is a total derivative if $h = 1$. We have recovered the known condition that local vertex operators must have conformal weight one. In the ghost sector, instead, we have two degenerate vacua, $|\uparrow\rangle$ and $|\downarrow\rangle$, such that

$$c_0|\uparrow\rangle = 0, \qquad b_0|\downarrow\rangle = 0, \qquad b_0|\uparrow\rangle = |\downarrow\rangle, \qquad c_0|\downarrow\rangle = |\uparrow\rangle.$$

Moreover, if $|0\rangle$ is the invariant vacuum under $SL(2,\mathbb{C})$,

$$c_1|0\rangle = |\downarrow\rangle, \qquad c_0c_1|0\rangle = |\uparrow\rangle.$$

States can be then built starting from one of these two ground states for ghosts. However, for each BRST cohomology class we can choose a representative of the form

$$|\psi\rangle = |\phi\rangle_X \otimes |\downarrow\rangle = |\phi\rangle_X \otimes \big(c_1|0\rangle\big) \quad (1.102)$$

with $|\phi\rangle_X$ a highest weight state of the Virasoro algebra for matter. Indeed, for this kind of states, we have BRST invariance, i.e.

$$Q|\psi\rangle = \Big(c_0(L_0^X - 1) + \sum_{n>0} c_{-n}L_n^X\Big)|\psi\rangle = 0$$

if the already known conditions for physical states hold:

$$(L_0^X - 1)|\phi\rangle_X = 0, \qquad L_n^X|\psi\rangle_X = 0 \quad \text{for } n > 0.$$

[2]Let us focus on the holomorphic sector, in the antiholomorphic case the procedure is completely analogous.

States built on the other ghost vacuum instead, would lose the first condition since $c_0 |\uparrow\rangle = 0$: physical states are then such that $b_0 |\psi\rangle = 0$. Lastly, we have to demand BRST closure but not exactness of states: this amounts to further require the mass condition (1.32), which had already been found in the framework of lightcone gauge quantization.
Let us notice that by (1.97), $c_1 = c(0)$. Vertex operators of physical states (1.102) are then

$$\psi(z) = c(z)\phi(z).$$

Since the vertex operator of the matter sector has conformal weight one, we find that

$$[Q, \psi(z)] = (h-1) : \partial c c \phi(z) := 0$$

This is not a total derivative and integration over the insertion points is not needed to have BRST invariance. Adding the antiholomorphic part, we end up then, with two kinds of vertex operators: the integrated ones, defined in (1.78) as $\int d^2 z \phi(z, \bar{z})$ and such that

$$[Q, \phi(z, \bar{z})] = \partial(c\phi)(z, \bar{z}), \qquad [\bar{Q}, \phi(z, \bar{z})] = \bar{\partial}(\bar{c}\phi)(z, \bar{z})$$

and the unintegrated ones, which are given by

$$\psi(z, \bar{z}) = c(z)\bar{c}(\bar{z})\phi(z, \bar{z}) \tag{1.103}$$

and are such that

$$[Q, \psi(z, \bar{z})] = 0, \qquad [\bar{Q}, \psi(z, \bar{z})] = 0.$$

1.4 The superstrings

Roughly speaking, *supersymmetry* (SUSY) is a symmetry exchanging bosons and fermions, which can be employed when aiming at adding fermionic excitations on the worldsheet. The theories obtained matching SUSY with the bosonic string theory already presented in the above chapter, are dubbed *superstring theories* and most of them are tachyon-free, i.e. not affected by instability of the vacuum. Superstring theories will be discussed in Subsection 1.4.3. The first two subsections are instead, thought as a preliminary study.

1.4.1 A worldline theory for fermions

A first insight into the worldline fermionic theory can be gained when trying to infer the dynamics of fermionic point-like relativistic particles from the bosonic one. The latter is described by the action (1.1), in which time and space are on the same footing in order that Lorentz invariance is ensured. The invariance under reparametrizations can be identified as the gauge symmetry necessary in order that despite the further degree of freedom linked to the time, only $D-1$ degrees of freedom are the actual physical ones. In the Hamiltonian formalism, the canonical momenta

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m\dot{X}_\mu}{\sqrt{-\dot{x}}^2} \tag{1.104}$$

satisfy the constraint

$$H \equiv p_\mu p^\mu + m^2 = 0. \tag{1.105}$$

Thereafter, we are working with a Hamiltonian constrained system, whose dynamics can solely take place on the hypersurface determined by (1.105). Introducing the Lagrange mutliplier $e(\tau)$ to implement the constraint, (1.1) gets

$$\mathcal{S}_{pp}[x^\mu(\tau), p_\mu(\tau), e(\tau)] = \int d\tau \Big(p_\mu \dot{X}^\mu - \frac{e}{2} \big(p_\mu p^\mu + m^2 \big) \Big) \tag{1.106}$$

and indeed the e.o.m. for $e(\tau)$ is nothing but (1.105). In this framework, $H = 0$ is a *first class constraint*: H is indeed, a generator of gauge symmetries i.e. its orbits lie on the constraint hypersurface and are sets of physically equivalent points that can be identified one with another. Furthermore, the canonical Hamiltonian of this system vanishes hence the constraint H is invariant under time evolution and the constraint hypersurface is time invariant.
The Hamiltonian constrained systems can be quantized in three ways:

- imposing some gauge-fixing conditions $F^\alpha = 0$, besides the constraint $C^\alpha = 0$, so that one representative for each orbit is chosen and performing the standard covariant quantization on the outcoming further reduced phase space;
- the *Dirac method*, in which the constraint functions C^α are promoted to operators selecting the physical states $|\psi_{\text{phys}}\rangle$ as such that
$$\hat{C}^\alpha |\psi_{\text{phys}}\rangle = 0 \qquad \text{for all } \alpha$$
or in the weaker form
$$\langle\psi_{\text{phys}}| \hat{C}^\alpha |\psi'_{\text{phys}}\rangle = 0;$$
- *BRST method* in the path-integral approach, with the introduction of ghosts and selecting the physical states as belonging to the cohomology of the real, anticommuting and nilpotent BRST charge.

Since the Hamiltonian vanishes, the Schrödinger equation gets

$$i\hbar\frac{\partial}{\partial\tau} |\phi\rangle = 0.$$

Let us choose the second method to quantize (1.106). The physical states of the quantum theory for a relativistic point-particle are demanded to fulfill (1.105) at the operator level

$$\left(\hat{p}^\mu\hat{p}_\mu + m^2\right) |\phi\rangle = 0$$

which working in terms of the wave function $\phi(x) = \langle x^\mu|\phi\rangle$ is the Klein-Gordon equation

$$\left(-\partial_\mu\partial^\mu + m^2\right)\phi(x) = 0.$$

The e.o.m. for the canonical momenta is

$$p_\mu = e^{-1}\dot{X}_\mu,$$

which once inserted in (1.106) gives (1.2). Let us notice that the Lagrange multiplier coincides with the einbein and can be consistently written as $e(\tau)$.

In order to deal with spin-1/2 relativistic particles starting from relativistic scalar point particles, we introduce the real anti-commuting superpartners ψ^μ of the bosonic fields X^μ. For massless particles, the first class constraint (1.105) extends to

$$H = \frac{1}{2}p^2, \qquad Q = p_\mu\psi^\mu, \tag{1.107}$$

generating the gauge transformations on the hypersurface. Using the Poisson brackets

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu, \qquad \{\psi^\mu, \psi_\nu\} = -i\delta^\mu{}_\nu, \tag{1.108}$$

we end up with $\mathcal{N} = 1$ susy algebra, given by

$$\{Q, Q\} = -2iH. \tag{1.109}$$

The symplectic structure (1.108) naturally arises from the action describing spin-1/2 realtivistic massless particles

$$\mathcal{S} = \int d\tau \Big(p_\mu \dot{X}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu - eH - i\chi Q \Big) \tag{1.110}$$

where the constraints are implemented introducing the Lagrange multipliers (e, χ), which are named respectively einbein (commuting) and gravitino (anticommuting). The variation of the action with respect to e and χ leads to the e.o.m.

$$H = 0, \qquad Q = 0.$$

The quantization takes place promoting the phase space variables (x, p, ψ) to operators; in the framework of the covariant quantization, from the Poisson brackets (1.108), we can get the (anti)commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu{}_\nu, \qquad \{\hat{\psi}^\mu, \hat{\psi}^\nu\} = \eta^{\mu\nu}, \tag{1.111}$$

with all the others vanishing. Let us notice that the second expression in (1.111) is nothing but the Clifford algebra once we identify

$$\hat{\psi}^\mu \to \frac{1}{\sqrt{2}}\gamma^\mu. \tag{1.112}$$

The Hilbert space of physical states $|\Psi\rangle$ can be determined via the Dirac method, imposing the constraints at the operator level

$$\hat{Q}\,|\Psi\rangle = \hat{p}_\mu \hat{\psi}^\mu \,|\Psi\rangle = 0, \tag{1.113}$$

which for $\langle x^\mu|\Psi\rangle = \Psi(x)$ reads

$$\gamma^\mu \partial_\mu \Psi(x) = 0,$$

i.e. the Dirac equation for a massless field. The constraint $\hat{H}\,|\Psi\rangle = 0$, giving

$$\partial_\mu \partial^\mu \Psi(x) = 0$$

is already guaranteed by (1.113) as $\hat{Q}^2 = \hat{H}$, due to (1.109).
For massive pointlike particles, instead, the action is given by

$$\mathcal{S} = \int d\tau \Big(p_\mu \dot{X}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu + \frac{i}{2}\psi^5 \dot{\psi}^5 - \frac{e}{2}\big(p_\mu p^\mu + m^2\big) - i\chi\big(p_\mu \psi^\mu + m\psi^5\big)\Big).$$

In $D = 4$, the identification $\hat{\psi}^5 = \gamma^5$ can be made and hence at the quantum level, the SUSY constraint gives

$$\big(-i\gamma^\mu \partial_\mu + m\gamma^5\big)\Psi(x) = 0. \tag{1.114}$$

Upon recognizing an equivalent set of gamma matrices $\tilde{\gamma}^\mu = -i\gamma^5\gamma^\mu$ which fulfil the Clifford algebra, we get the Dirac equation for massive fields

$$\big(\tilde{\gamma}^\mu \partial_\mu + m\big)\Psi(x) = 0.$$

In the final analysis, we have obtained the dynamics of fermionic point particles from that of bosonic particles.

1.4.2 Super Riemann surfaces

Supersymmetric strings can be studied in total analogy with the bosonic strings, except the fact that the gauge symmetry on the worldsheet shifts from the conformal invariance to the *superconformal invariance*. Accordingly, the sum over Riemann surfaces involved in the scattering amplitudes becomes a sum over *super Riemann surfaces*, also known as *superspaces* or *supermanifolds*. Let us focus on one-dimensional complex supermanifolds: these are locally described by the *superco-ordinates* $\boldsymbol{z} = (z, \theta)$, where θ is an anticommuting Grassmann coordinate such that $\theta^2 = 0$ and representing the fermionic degrees of freedom. Let us define the superderivatives

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\frac{\partial}{\partial\bar{z}}, \tag{1.115}$$

such that $D^2 = \frac{\partial}{\partial z}$ and $\bar{D}^2 = \frac{\partial}{\partial\bar{z}}$. Super holomorphic functions fulfill the condition $\bar{D}f = 0$ and can be expanded as $f(\boldsymbol{z}) = f_0(z) + \theta f_1(z)$. Let us restrict to this kind of fields: the same procedure can be carried out for super antiholomorphic fields, too.
Super analytic maps $\boldsymbol{z} \to \tilde{\boldsymbol{z}}(\boldsymbol{z}) = (\tilde{z}(z,\theta), \tilde{\theta}(z,\theta))$ transforming superderivatives homogeneously, i.e.

$$D = (D\tilde{\theta})\tilde{D},$$

are named superconformal transformations and can be identified as the transition functions between coordinate patches within the supermanifold. Primary superfields $\phi(\boldsymbol{z})$ are the supersymmetric counterpart of primary fields and under supeconformal transformations change according to

$$\phi(\boldsymbol{z}) = \tilde{\phi}(\tilde{\boldsymbol{z}})(D\tilde{\theta})^{2h}, \tag{1.116}$$

where h is the superconformal weight of the field. As well as super analytic functions, superconformal fields can be expanded as $\phi(\boldsymbol{z}) = \phi_0(z) + \theta\phi_1(z)$, where ϕ_0 and ϕ_1 are primary fields with weights h and $h + 1/2$, respectively (the consistency of these weights is granted by the fact that $[\theta] = -1/2$). At the quantum level, ϕ_0 shares the same statistics as ϕ, whilst ϕ_1 follows the opposite one. Under the infinitesimal superconformal transformation

$$\boldsymbol{z} \to \tilde{\boldsymbol{z}} = \boldsymbol{z} + \delta\boldsymbol{z}, \qquad \delta\boldsymbol{z} = v(\boldsymbol{z}) = \delta z + \theta\delta\theta, \tag{1.117}$$

superconformal fields transform according to the infinitesimal version of (1.116)

$$\delta_v\phi = \big(v\partial + \frac{1}{2}DvD + h\partial v\big)\phi. \tag{1.118}$$

The integration over the Grassmannian coordinate is given by

$$\int d\theta\ \theta = 1, \qquad \int d\theta\ 1 = 0 \tag{1.119}$$

and the integral

$$f(\boldsymbol{z}_1, \boldsymbol{z}_2) = \int_{\boldsymbol{z}_2}^{\boldsymbol{z}_1} d\boldsymbol{z}\ \omega(\boldsymbol{z})$$

is by definition such that

$$f(\boldsymbol{z}_2, \boldsymbol{z}_2) = 0, \qquad D_1 f(\boldsymbol{z}_1, \boldsymbol{z}_2) = \omega(\boldsymbol{z}_1).$$

Superconformal symmetries also include supertranslations: suitable coordinates for super translation invariant functions are

$$\theta_{12} = \theta_1 - \theta_2 = \int_{\boldsymbol{z}_2}^{\boldsymbol{z}_1} d\boldsymbol{z}, \qquad z_{12} = z_1 - z_2 - \theta_1\theta_2 = \int_{\boldsymbol{z}_2}^{\boldsymbol{z}_1} d\boldsymbol{z} \int_{\boldsymbol{z}_2}^{\boldsymbol{z}} d\boldsymbol{z}'.$$

The quantum superconformal transformations are generated by the super stress-energy tensor

$$T(\boldsymbol{z}) = T_F(z) + \theta T_B(z), \tag{1.120}$$

where T_B stands for the holomorphic component of the bosonic stress-energy tensor whilst T_F is its super partner of conformal dimension $(3/2, 0)$. Indeed, we get that under superconformal transformations, any field changes as follows:

$$\delta_v\phi(\boldsymbol{z}_2) = \frac{1}{2\pi i}\oint_C d\boldsymbol{z}_1 v(\boldsymbol{z}_1)T(\boldsymbol{z}_1)\phi(\boldsymbol{z}_1), \tag{1.121}$$

where the integral is performed over a closed contour turning once around $\boldsymbol{z}_2$. By comparison with (1.118) and exploiting the super Cauchy formulas

$$\frac{1}{2\pi i}\oint_C d\boldsymbol{z}_1\ f(\boldsymbol{z}_1)\theta_{12}z_{12}^{-n-1}=\frac{1}{n!}\partial_2^n f(\boldsymbol{z}_2),\qquad \frac{1}{2\pi i}\oint_C d\boldsymbol{z}_1\ f(\boldsymbol{z}_1)z_{12}^{-n-1}=\frac{1}{n!}\partial_2^n D_2 f(\boldsymbol{z}_2)$$

with C winding once around $\boldsymbol{z}_2$, we arrive at the OPE

$$T(\boldsymbol{z}_1)\phi(\boldsymbol{z}_2)\sim\frac{\theta_{12}}{z_{12}^2}h\phi(\boldsymbol{z}_2)+\frac{1/2}{z_{12}}D_2\phi+\frac{\theta_{12}}{z_{12}}\partial_2\phi. \tag{1.122}$$

Moreover, the super stress energy tensor is a non-primary superfield of weight 3/2. Its TT OPE are

$$T(\boldsymbol{z}_1)T(\boldsymbol{z}_2)\sim\frac{c}{4}\frac{1}{z_{12}^3}+\frac{3}{2}\frac{\theta_{12}}{z_{12}^2}T(\boldsymbol{z}_2)+\frac{1/2}{z_{12}}D_2T(\boldsymbol{z}_2)+\frac{\theta_{12}}{z_{12}}\partial_2\phi(\boldsymbol{z}_2) \tag{1.123}$$

where c is the central charge. The decomposition (1.120) allows to write the OPE for each component

$$\begin{aligned}
T_B(z_1)T_B(z_2)&\sim\frac{3c/4}{(z_1-z_2)^4}+\frac{2}{(z_1-z_2)^2}T_B(z_2)+\frac{1}{z_1-z_2}\partial_2T_B(z_2),\\
T_B(z_1)T_F(z_2)&\sim\frac{3/2}{(z_1-z_2)^2}T_F(z_2)+\frac{1}{z_1-z_2}\partial_2T_F(z_2),\\
T_F(z_1)T_F(z_2)&\sim\frac{c/4}{(z_1-z_2)^3}+\frac{1/2}{z_1-z_2}T_B(z_2),
\end{aligned}$$

among which the first one coincides with (1.62). As done in (1.70) we can expand both the components in the Laurent series

$$T_B(z)=\sum_{m=-\infty}^{+\infty}\frac{L_m}{z^{m+2}},\qquad T_F(z)=\sum_{m=-\infty}^{+\infty}\frac{G_m}{2z^{m+3/2}}, \tag{1.124}$$

where the coefficients satisfy the so-called *Ramond algebra*

$$\begin{aligned}
[L_m,L_n]&=(m-n)L_{m+n}+\frac{c}{8}(m^3-m)\delta_{m+n,0},\\
[L_m,G_n]&=\Big(\frac{1}{2}m-n\Big)G_{m+n}\\
\{G_m,G_n\}&=2L_{m+n}+\frac{1}{2}c\Big(m^2-\frac{1}{4}\Big)
\end{aligned} \tag{1.125}$$

(let us notice that the first relation is nothing but the Virasoro commutation relation (1.72)).

1.4.3 Superstring theories

Superstring theories arise when a fermionic sector is added to the bosonic Polyakov action (1.6). The characters of this new sector are the 2-dimensional worldsheet spinors, which are vectors of the representation space of any representation of the Clifford algebra

$$\{\gamma^a,\gamma^b\}_{AB}=2\eta^{ab}\mathbb{I}_{AB}, \tag{1.126}$$

with the worldsheet metric given by η^{ab}. Since they are worldsheet indices, $a,b=0,1$, whereas the range of values for the spinor indices A and B depends on the dimension of the representation of the Clifford algebra. Spinors transform under the spinorial representation of the Lorentz group, i.e.

$$\psi_A\to S_{AB}\psi_B,\qquad S_{AB}=\big[\exp\big(i\omega_{ab}\frac{i}{4}[\gamma^a,\gamma^b]\big)\big]_{AB}. \tag{1.127}$$

Let us consider the 2-dimensional representation of the Clifford algebra generated by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If both the components of the spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

are real, then we have a *Majorana spinor*. Furthermore, if $\gamma = \gamma^0\gamma^1$,

$$\gamma \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \quad \gamma \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}, \tag{1.128}$$

so the two components of ψ have definite chirality and are then *Weyl spinors*.
The action for the fermionic sector of superstrings arises building the supersymmetric correspondent of the Polyakov action (1.6). The supersymmetric partners of X^μ are the fields ψ^μ, Weyl-Majorana spinors whose components are Grassmann-valued spacetime vectors. Exactly as done in the point-particle case of Subsection 1.4.1, the zweibein and the gravitino Lagrange multipliers are added to the worldsheet in order to impose some constraints fixing the reparametrization invariance. We can fix this redundancy by imposing some gauge fixing condition, for instance the superconformal gauge. With this choice the full action gives

$$\mathcal{S} = \mathcal{S}_B + \mathcal{S}_F = -\frac{1}{4\pi}\int d^2\sigma \left[\frac{1}{\alpha'}\partial_a X^\mu \partial^a X_\mu + i\bar{\psi}^\mu_A \gamma^a_{AB}\partial_a \psi_{\mu,B}\right], \tag{1.129}$$

with the first addend given by the Polyakov action (1.9) in the gauge with flat metric. In the worldsheet lightcone coordinates $\sigma^\pm = \tau \pm \sigma$, (1.129) gets

$$\mathcal{S} = \frac{1}{2\pi}\int d^2\sigma \left[\frac{2}{\alpha'}\partial_+ X^\mu \partial_- X_\mu + i\big(\psi^\mu_+ \partial_- \psi_{+,\mu} + \psi^\mu_- \partial_+ \psi_{-,\mu}\big)\right]. \tag{1.130}$$

This is invariant under the residual symmetry

$$\sqrt{\frac{2}{\alpha'}}\delta X^\mu = i(-\varepsilon_- \psi^\mu_+ + \varepsilon_+ \psi^\mu_-), \qquad \delta\psi^\mu_\pm = \pm\sqrt{\frac{2}{\alpha'}}\varepsilon_\mp \partial_\pm X^\mu, \tag{1.131}$$

provided that the components of the infinitesimal Majorana spinor ε_A satisfy the conditions

$$\partial_+\varepsilon_+ = 0, \qquad \partial_-\varepsilon_- = 0, \tag{1.132}$$

i.e. they are chiral, in the sense that $\varepsilon_+ = \varepsilon_+(\sigma_-)$ and $\varepsilon_- = \varepsilon_-(\sigma_+)$. (1.131) is a supersymmetry because it relates bosonic and fermionic degrees of freedom.
The classical dynamics of the bosonic sector exactly coincides with that of the bosonic string. For the fermionic sector, instead, let us consider the variation of (1.130) under a small variation of the fermionic fields ψ,

$$\begin{aligned}
\delta\mathcal{S} = \delta\mathcal{S}_F = & \frac{i}{2\pi}\int d^2\sigma\ \big(\delta\psi_+\partial_-\psi_+ + \psi_+\partial_-\delta\psi_+ + \delta\psi_-\partial_+\psi_- + \psi_-\partial_+\delta\psi_-\big) = \\
= & \frac{i}{2\pi}\int d^2\sigma\ \big[\delta\psi_+\big(\partial_-\psi_+ + \partial_-\psi_+\big) + \partial_-\big(\psi_+\delta\psi_+\big) + \delta\psi_-\big(\partial_+\psi_- + \partial_+\psi_-\big) + \partial_+\big(\psi_-\delta\psi_-\big)\big] = \\
= & \frac{i}{2\pi}\int d^2\sigma\ \Big[2\delta\psi_+\partial_-\psi_+ + 2\delta\psi_-\partial_+\psi_- + \frac{1}{2}\partial_\sigma\big(\psi_-\delta\psi_- - \psi_+\delta\psi_+\big) + \frac{1}{2}\partial_\tau\big(\psi_-\delta\psi_- + \psi_+\delta\psi_+\big)\Big].
\end{aligned}$$

The first addend vanishes once the equations of motion are identified to be

$$\partial_-\psi_+ = 0, \qquad \partial_+\psi_- = 0. \tag{1.133}$$

The Weyl spinors are then chiral also in the sense that $\psi_\pm = \psi_\pm(\sigma^\pm)$. The third addend vanishes, since the variational principle imposes that at time boundaries the variations of the fields vanish. The second term, instead, reads

$$\frac{i}{4\pi}\int d\tau \left[\psi_-\delta\psi_- - \psi_+\delta\psi_+\right]\Big|_{\sigma=0}^{\sigma=l}. \tag{1.134}$$

In the closed string sector, this vanishes as long as

$$\psi_+\delta\psi_+ - \psi_-\delta\psi_-\big|_{\sigma=0} = \psi_+\delta\psi_+ - \psi_-\delta\psi_-|_{\sigma=l},$$

which is fulfilled if

$$\psi_\pm(\sigma+l) = e^{2\pi i\phi_\pm}\psi_\pm(\sigma), \qquad \phi_\pm = 0, 1/2.$$

In the case $\phi = 0$ we are dealing with the *Ramond sector* for closed strings, whereas $\phi = 1/2$ is the so-called *Neveu-Schwarz sector*. Thereafter, we end up with 4 different sectors, since each of the two Weyl components can assume one or the other value for ϕ.
In the open string sector, instead, for both the boundaries we need to impose

$$\psi_+^\mu(\sigma) = \pm\psi_-^\mu(\sigma), \qquad \sigma = 0, \sigma = l.$$

This requirement must be invariant under the supersymmetry (1.131). To ensure this, the boundary conditions for the fermionic fields must be properly chosen for each direction μ according to the Neumann or Dirichlet boundary conditions for the bosonic fields X^μ at the two endpoints. In any case, for each direction periodicity or antiperiodicity of the fermionic fields is allowed and the two possibilities give rise to the Ramond or the Neveu-Schwarz sectors respectively.

The flat metric on the worldsheet is just one gauge possibility: when the worldsheet metric is the general h_{ab}, we really end up with a theory of gravity on the worldsheet, coupled with supersymmetry, which is called *supergravity*. This theory is invariant under the *local supersymmetry*, i.e. supersymmetry along with invariance under local diffeomorphisms and Weyl transformations. Once the metric is fixed to be flat, super-conformal transformations represent the only remaining symmetry and are such that the supersymmetry is only chiral (as highlighted in (1.132)). The conserved currents under super-conformal transformations are the stress-energy tensor T_{ab} and the super-current J_a. On them, super-Virasoro constraints must be imposed:

$$T_{\pm\pm} = 0, \qquad J_\pm = 0.$$

These allow to perform the covariant quantization in lightcone gauge of the superstrings and determine their spectrum. Within this framework, the critical dimension of the superstring theories can be determined, with the upshot

$$D = 10. \tag{1.135}$$

The light-cone gauge quantization also provides the spectrum for both the open and the closed strings in the Neveu-Schwarz and the Ramond sectors. For the open strings in the NS sector, the lowest-mass state is a tachyonic spacetime scalar of $SO(9)$. The first excited level is massless and is an eight-dimensional vector along the eight transverse directions of the spacetime. States with higher energy are massive bosons in tensor representation of $SO(9)$. In the Ramond sector, instead, the ground state is already massless and tachyon-free. It is made up of two Weyl spinors with opposite chirality, which are Lorentz vectors in $SO(8)$. States with higher energy are massive spinors in some irreducible representation of $SO(9)$.
With respect to closed strings, instead, we have to recall that the left and the right-moving sectors are completely independent, up to the level matching condition

$$\frac{\alpha'}{4}M^2 = (N-a) = (\tilde{N}-a),$$

with the normal-order constant $a = 0, 1/2$ respectively for the Ramond and Neveu-Schwarz sectors and N and $\tilde{N}$ the number operators of left and right-moving modes. For both kinds of modes, we can choose Ramond or Neveu-Schwarz boundary conditions and two states (labelled with $\pm$) of the so-called G-parity, accounting for the number of fermionic excitations and the chirality of the Weyl spinors. We would expect, then, 16 possible combinations. Nevertheless, the level matching condition allows only the coupling (NS_-, NS_-) (standard notation with the first member referring to right-moving modes and the second one to left-moving ones) and all the possibile combinations reduce to 10. The ground state is in the (NS_-, NS_-) sector and is a tachyon of mass $M^2 = -\frac{2}{\alpha'}$. All possible couples of sectors, except the latter, contain a massless state. In principle, all sectors may appear or not in a superstring theory, hence leading to 2^{10} different theories. However, the *GSO projection* (named after Gliozzi, Scherk and Olive) selects only some of them. In fact, on grounds of the absence of tachyons (tachyons would lead to an instable vacuum, promptly decaying and being then irrelevant) and CFT consistency (absence of branch cuts in the Ramond sector and modular invariance for the loop amplitudes), we can identify four types of theories: IIA, IIA', IIB and IIB'. In Table 1.1, we list the sectors appearing in IIB and IIA superstring theories, also writing the massless fields of their spectrum.

Theory	Sector	Fields
IIB	(NS_+, NS_+)	$\Phi, B_{\mu\nu}, G_{\mu\nu}$
	(R_+, R_+)	C_0, C_2, C_4
	(NS_+, R_+)	λ_a, ψ_a^μ
	(R_+, NS_+)	λ_a, ψ_a^μ

Theory	Sector	Fields
IIA	(NS_+, NS_+)	$\Phi, B_{\mu\nu}, G_{\mu\nu}$
	(R_+, R_-)	C_1, C_3
	(NS_+, R_+)	$\tilde{\lambda}_a$, $\tilde{\psi}_a^\mu$
	(R_+, NS_+)	λ_a, ψ_a^μ

Table 1.1: Sectors and spectrum of IIB, IIA superstring theories

The massless fields of the (NS_+, NS_+) sector are nothing but the tensors of the first excited state of the closed strings: the dilaton, the Kalb-Ramond 2-form and the graviton. The (R, R) sector, instead, shows bosonic antisymmetric forms, whose dimension is written as a subscript (notice that since they live in the transverse 8 dimensional space, C_4 is self-dual). The (R, NS) sector, instead, counts two massless fermions: λ_a is the spin-1/2 dilatino, whilst ψ_a^i is the spin-3/2 gravitino, i.e. the super-partner of the graviton. The two possible Weyl chiralities are distinguished by the presence or lack of the tilde. Two independent gravitinos give rise to two independent SUSY algebras: this is why these theories are type II. IIB superstring model is chiral, since left and right movers in the (R, R) sector share the same chirality. This does not hold for IIA superstring theory, which is indeed not chiral. Two equivalent models for superstrings are IIB' and IIA', which arise respectively from IIB and IIA, exchanging R_+ with R_-. Other 10-dimensional superstring theories are Type I theory and two heterotic string theories. A further supersymmetric theory, the M-theory, is instead, 11-dimensional.

1.4.4 A conformal field theory for fermions and the ghosts

The result of (1.135) can be alternatively inferred making use of the conformal field theory for the free Majorana fermion $\Psi(\sigma)$ (spinorial indices are understood and not explicit in this case) defined on the worldsheet. We just need to discover the central charge, following the analogous procedure adopted in the bosonic instance. The dynamics is determined by the action

$$\mathcal{S}_F = \frac{1}{2} g \int d^2\sigma \; \Psi^\dagger \gamma^0 \gamma^a \partial_a \Psi, \qquad a = 0, 1, \tag{1.136}$$

where the g is a general pre-factor that may appear. This is the form of the action which was used in (1.129), for D fermions instead of one. We shift to the coordinates $(z, \bar{z})$. Due to the equations of motion the two real components of the Majorana spinor $\Psi = (\psi, \bar{\psi})^T$ are respectively holomorphic and antiholomorphic. Through the calculation of the propagators between each of these components we get the OPEs

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g}\frac{1}{z-w}, \qquad \psi(z)\bar{\psi}(\bar{w}) \sim 0, \qquad \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) \sim \frac{1}{2\pi g}\frac{1}{\bar{z}-\bar{w}}. \tag{1.137}$$

Moreover, the holomorphic component of the stress-energy tensor for this action reads

$$T(z) = -\pi g : \psi(z)\partial\psi(z) : \tag{1.138}$$

We first compute

$$T(z)\psi(w) = -\pi g : \psi(z)\partial\psi(z) : \psi(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{(z-w)},$$

which allows to deduce that the conformal weight of $T(z)$ is $1/2$. Hence, we can calculate

$$T(z)T(w) = \pi^2 g^2 : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) :\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}.$$

By comparison with (1.62), we learn that the central charge for free Majorana fermions is $1/2$.

To consider all fields appearing in the superstring theory, we must take care of the ghosts arising in the fermion sector. In the path-integral quantization, indeed, we have to fix the invariance under superconformal transformations: in the boson sector this issue is already healed by the ghost system (b, c) and in the spirit of supersymmetry, the same should be performed for the fermions. For later convenience (cf. Subsection 4.3) the following treatment for ghosts aims to be as general as possible: the specialization to the fermionic sector of superstrings will take place at the end. The general action for the ghost system $(\boldsymbol{b}, \boldsymbol{c})$, in the holomorphic sector, is given by

$$S = \frac{1}{\pi}\int d^2z\, \boldsymbol{b}\bar{\partial}\boldsymbol{c}. \tag{1.139}$$

The bold notation is adopted in order to represent a general ghost system, not to be necessarily identified with the (b, c) ghosts of 1.3. Moreover, let us notice that the action for the latter system is equivalent to (1.139), as shown in (1.86). The fields $(\boldsymbol{b}, \boldsymbol{c})$ have conformal weight respectively λ and $1-\lambda$ and can be either fermions or bosons. Their OPEs are

$$\boldsymbol{c}(z)\boldsymbol{b}(w) \sim \frac{1}{z-w}, \qquad \boldsymbol{b}(z)\boldsymbol{c}(w) \sim \frac{\varepsilon}{z-w} \tag{1.140}$$

with $\varepsilon = +1$ for fermions and $\varepsilon = -1$ for bosons. The stress-energy tensor reads

$$T^g = -\lambda\boldsymbol{b}\partial\boldsymbol{c} + (1-\lambda)\partial(\boldsymbol{b})\boldsymbol{c}, \tag{1.141}$$

and we can compute the OPE

$$T^g(z)T^g(w) \sim \frac{-\varepsilon(6\lambda^2 - 6\lambda + 1)}{(z-w)^4} + \frac{2T^g(w)}{(z-w)^2} + \frac{\partial T^g(w)}{z-w} \tag{1.142}$$

Then, defining $Q = \varepsilon(1-2\lambda)$, the central charge for any ghost system is given by

$$c_g = -2\varepsilon(6\lambda^2 - 6\lambda + 1) = \varepsilon(1 - 3Q^2). \tag{1.143}$$

The properties of the ghost system (b, c) are recovered once we fix $\lambda = 2, \varepsilon = +1$. The ghosts (β, γ) for the fermionic sector instead, are commuting fields, then they are such that $\varepsilon = -1$ and $\lambda = 3/2$. As a consequence, $Q_{\beta\gamma} = 2$ and $c_{\beta\gamma} = 11$. Summing this result to $c_{bc} = -26$ we get that $c_{\text{ghosts}} = -15$. As a consequence, the Weyl anomaly is removed whenever the matter system consists of degrees of freedom whose central charge sums to 15. In a fully supersymmetric system, we expect the number of bosonic fields X^μ ($c = 1$) to equate the number of fermionic fields ψ^μ

($c=1/2$). In order to cancel the Weyl anomaly, we must impose that $D\big(\frac{1}{2}+1\big)=15$, getting that the critical dimension of the superstrings is indeed, $D=10$.
As in the case of the bosonic string, the BRST procedure allows to identify the physical states or equivalently, the physical vertex operators of the superstring theory. The generalization of (1.99) to a supersymmetric framework is

$$Q=\oint_{C_0}\frac{dz}{2\pi i}\Big[c(z)\Big(T^{X,\psi}(z)+\frac{1}{2}T^{b,c,\gamma,\beta}(z)\Big)-\gamma(z)\Big(T_F^{X,\psi}(z)+\frac{1}{2}T_F^{b,c,\beta,\gamma}(z)\Big)\Big]. \tag{1.144}$$

The bosonic stress-energy tensor for the fields X^μ and ψ^μ is the sum of (1.67) and (1.138) (with $g=1/2\pi$), whereas the supercurrent is given by

$$T_F^{X,\psi}=i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu. \tag{1.145}$$

From (1.141) we have that

$$T^{b,c}=-2b\partial c-(\partial b)c, \qquad T^{\beta,\gamma}=-\frac{3}{2}\beta\partial\gamma-\frac{1}{2}(\partial\beta)\gamma.$$

For a general supersymmetric system (b,c,β,γ) of ghosts, the supercurrent is

$$T_F^g=\frac{1}{2}b\gamma+(1-\lambda)(\partial\beta)c-(\lambda-\frac{1}{2})\beta\partial c, \tag{1.146}$$

with the parameter λ corresponding to (b,c). Hence, since in our case $\lambda=2$,

$$T_F^{b,c,\beta,\gamma}=\frac{1}{2}b\gamma-(\partial\beta)c-\frac{3}{2}\beta\partial c. \tag{1.147}$$

As a consequence, the BRST charge (1.144) can be split into three parts:

$$Q=Q_0+Q_1+Q_2, \tag{1.148}$$

with

$$Q_0=\oint_{C_0}\frac{dz}{2\pi i}\Big(cT^{X,\psi,\beta,\gamma}+c(\partial c)b\Big), \qquad Q_1=-\oint_{C_0}\frac{dz}{2\pi i}\gamma T_F^{X,\psi}. \tag{1.149}$$

In this definition, Q_0 is exactly the BRST charge of the bosonic theory (the commuting ghosts β and γ are included as extra matter fields) and Q_1 is the generator of the superconformal transformations with parameter γ. Therefore,

$$\begin{aligned}
Q_3&=\oint_{C_0}\frac{dz}{2\pi i}\Big[-\frac{c}{2}T^{\beta\gamma}+\frac{c}{2}T^{b,c}-\frac{\gamma}{2}T_F^{b,c,\beta,\gamma}\Big]=\\
&=\oint_{C_0}\frac{dz}{2\pi i}\Big[-\frac{c}{2}\Big(-\frac{3}{2}\beta\partial\gamma-\frac{1}{2}\partial\beta\gamma\Big)+\frac{c}{2}\Big(-2b\partial c-\partial bc\Big)-\frac{\gamma}{2}\Big(-\partial\beta c-\frac{3}{2}\beta\partial c+\frac{1}{2}b\gamma\Big)\Big]=\\
&=\oint_{C_0}\frac{dz}{2\pi i}\Big[-\frac{1}{4}b\gamma^2+\frac{3}{4}\partial(cb\gamma)=\\
&=-\oint_{C_0}\frac{dz}{2\pi i}\frac{1}{4}b\gamma^2,
\end{aligned} \tag{1.150}$$

since the contour integral of an exact differential vanishes.

Let us discuss one final point. In Subsection 1.3.2 we could not use $|0\rangle$, the invariant vacuum under $SL(2,\mathbb{C})$, as a vacuum for ghosts. This is in general due to the following properties of the ghost modes

$$\begin{aligned}
\boldsymbol{b}_n\,|0\rangle&=0 \quad \text{for } n\geq 1-\lambda,\\
\boldsymbol{c}_n\,|0\rangle&=0 \quad \text{for } n\geq \lambda.
\end{aligned}$$

Then, according to λ, the energy of the vacuum $|0\rangle$ could be lowered by ghosts. This happens both for (b, c) and (β, γ). The energy spectrum is then unbounded both above and below and the choice of the vacuum is arbitrary. This is nothing new for fermionic field theories, where the vacuum gets fixed once the level of the Fermi - sea is specified. In this framework, we have further, to fix the level of the Bose - sea. The possible vacua of the theory are denoted by $|q\rangle$ and are such that

$$\begin{aligned} \boldsymbol{b}_n\,|q\rangle &= 0 \quad \text{for } n > q - \varepsilon\lambda, \\ \boldsymbol{c}_n\,|q\rangle &= 0 \quad \text{for } n \geq -\varepsilon q + \lambda. \end{aligned}$$

The vertex operator producing the q - vacua from $|0\rangle$ is given by $: e^{q\phi(z)} :$. Indeed,

$$|q\rangle = e^{q\phi(0)}\,|0\rangle\,,$$

with $\phi(z)$ a holomorphic field such that for fermionic ghosts,

$$\boldsymbol{c}(z) = e^{\phi(z)}, \qquad \boldsymbol{b}(z) = e^{-\phi(z)} \tag{1.151}$$

whereas for bosonic ghosts

$$\boldsymbol{c}(z) = e^{\phi(z)}\eta(z), \qquad \boldsymbol{b}(z) = e^{-\phi(z)}\partial\xi(z). \tag{1.152}$$

In the latter instance, we need to add other fermionic fields since $e^{\pm\phi}$ are always fermions. Moreover, the following OPEs hold:

$$\phi(z)\phi(w) \sim -\ln(z-w), \qquad \xi(z)\eta(w) \sim \frac{1}{z-w}.$$

Finally, the conformal weight of the vertex operator $: e^{q\phi(w)} :$ is $\frac{1}{2}\varepsilon q(q+Q)$, as we deduce from

$$T(z) : e^{q\phi(w)} :\sim \Big[\frac{\frac{1}{2}\varepsilon q(q+Q) : e^{q\phi(w)} :}{(z-w)^2} + \frac{\partial_w : e^{q\phi(w)} :}{z-w}\Big]. \tag{1.153}$$

1.5 Strings on a curved background and their effective actions

The Polyakov action (1.6) can be naturally generalized when strings propagate in a target space with the general metric $G_{\mu\nu}(X)$:

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X). \tag{1.154}$$

If we write it as $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$, we can infer that the metric fluctuation just arises from a coherent superposition of the gravitons appearing in the string spectrum. Furthermore, $G_{\mu\nu}$ can be used to determine the coupling constants of the fluctuations of the bosonic fields X^μ. As a matter of fact, let us impose $g_{ab} = \eta_{ab}$ and consider the quantum fluctuations around the classical solution $X^\mu = \bar{x}^\mu$:

$$X^\mu(\sigma) = \bar{x}^\mu + \sqrt{\alpha'}Y^\mu(\sigma),$$

with $Y << 1$. The Lagrangian density of (1.154) gets

$$G_{\mu\nu}(X)\partial X^\mu \partial X^\nu = \alpha'\Big[G_{\mu\nu}(\bar{x}) + \sqrt{\alpha'}G_{\mu\nu,\omega}(\bar{x})Y^\omega + \frac{\alpha'}{2}G_{\mu\nu,\omega\rho}(\bar{x})Y^\omega Y^\rho\Big]\partial Y^\mu \partial Y^\nu, \tag{1.155}$$

with $G_{\mu\nu,\omega} = \frac{\partial G_{\mu\nu}}{\partial X^\omega} \sim \frac{1}{r_c}$. r_c is a characteristic radius of curvature and the corrections start being important as soon as

$$\frac{\sqrt{\alpha'}}{r_c} \sim 1,$$

which is the loop-expansion parameter in (1.155).
In the spectrum of bosonic closed strings, also the Kalb-Ramond field and the dilaton appear and we expect analogous couplings of the strings to these background fields. This will be given by

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{g}\Big[G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu g^{ab} + B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \varepsilon^{ab} + \alpha'\Phi(X)R^{(2)}\Big], \tag{1.156}$$

with $R^{(2)}$ the worldsheet Ricci scalar. This kind of action is usually called *non-linear sigma model.* The coupling to the B field is exactly modelled on the interaction action of a charged object and a gauge field, e.g. an electrically charged point particle and the e.m. gauge field. In (1.156), the interaction term is invariant under the gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu$$

and hence all the physics is encoded in the gauge invariant field strength $H = dB$.
The coupling to the dilaton, instead, seems to spoil Weyl symmetry, since the Ricci scalar is not invariant under Weyl transformations. Anyway, the presence of α' in front of this term hints that loop contributions arising from the first two couplings may somehow compensate this breaking of Weyl invariance. In quantum field theories, Weyl invariance takes place provided that the β-functions of the couplings vanish. In (1.156) the couplings are expressed in terms of the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, hence what is expected to vanish are β-functionals like

$$\beta_{\mu\nu}(G) \sim \mu\frac{\partial G_{\mu\nu}(X;\mu)}{\partial\mu},$$

where μ is the energy scale at which the interactions take place. Moreover, in CFTs, the breakdown of Weyl invariance is quantified by the v.e.v. of the trace of the stress-energy tensor, which for the non-linear sigma model (1.156) reads

$$\langle T^\alpha{}_\alpha\rangle = -\frac{1}{2\alpha'}\beta_{\mu\nu}(G)g^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{1}{2\alpha'}\beta_{\mu\nu}(B)\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta(\Phi)R^{(2)}. \tag{1.157}$$

At one loop, the β-functionals are:

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha'\mathcal{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu{}^{\lambda\kappa},\\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu},\\
\beta(\Phi) &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda},
\end{aligned}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor for the background metric and ∇_μ is the covariant derivative for the Levi-Civita connection of the same metric. In order to preserve Weyl invariance, we expect

$$\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0.$$

These can be regarded as the equations of motion for the background where strings propagate and arise from the action

$$\mathcal{S}_1 = \frac{1}{2{\kappa_0}^2}\int d^{26}X\sqrt{-G}e^{-2\Phi}\Big(\mathcal{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\Big), \tag{1.158}$$

with ${\kappa_0}^2 \sim {l_s}^24$ This is the *low-energy effective action* for spacetime fields, since in its e.o.m., the β-functionals at one loop level (so at high curvature radius) appear. Furthermore, (1.158) holds in the so-called string frame. The Einstein frame, with the standard Einstein-Hilbert term for the background metric and a canonical normalization for the dilaton, is achieved performing the relabelling

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)}G_{\mu\nu}(X),$$

Here, $\tilde{\Phi} = \Phi - \Phi_0$, with Φ_0 the asymptotic value of the dilaton. Therefore,

$$\tilde{\mathcal{S}_1} = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{G}}\Big(\tilde{\mathcal{R}} - \frac{1}{12}e^{-\tilde{\Phi}/3}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{6}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}\Big),$$

with $\kappa_{26}^2 = {\kappa_0}^2 e^{2\Phi_0} \sim {l_s}^{24} g_s^2$. On grounds of consistency with Einstein's General Relativity, we demand

$$8\pi G_N = l_p^{24} = \kappa^2$$

hence a weak string coupling $g_s << 1$ means that $l_s >> l_p$, i.e. quantum gravity effects are not important from a string viewpoint.

In the framework of superstrings, the effective action describes the dynamics of massless fields only. Focussing on type IIB and IIA superstrings,

$$\mathcal{S}_{IIA/IIB} = \mathcal{S}_1 + \mathcal{S}_R + \mathcal{S}_{CS},$$

with $\mathcal{S}_1$ describing the background fields in 10 dimensions, $\mathcal{S}_R$ the forms of the Ramond-Ramond sector and $\mathcal{S}_{CS}$ a topological (i.e. independent on the metric) Chern-Simons term. In particular,

$$\mathcal{S}_1 = \frac{1}{2\kappa_0^2}\int d^{10}X\sqrt{-G}e^{-2\Phi}\Big(\mathcal{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\Big)$$

is the same for both theories. Switching to a more compact writing for forms, in type IIA theory we have

$$\mathcal{S}_R + \mathcal{S}_{CS} = -\frac{1}{4{\kappa_0}^2}\int d^{10}X\big[F_2\wedge *F_2 + \tilde{F}_4\wedge *\tilde{F}_4\big] - \frac{1}{4{\kappa_0}^2}\int d^{10}X B_2\wedge F_4\wedge F_4,$$

with

$$F_2 = dC_1, \quad F_4 = dC_3, \quad \tilde{F}_4 = F_4 - C_1\wedge F_3.$$

In type IIB theory, instead, we have

$$\mathcal{S}_R + \mathcal{S}_{CS} = -\frac{1}{4{\kappa_0}^2}\int d^{10}X\big[F_1\wedge *F_1 + \tilde{F}_3\wedge *\tilde{F}_3 + \tilde{F}_5\wedge *\tilde{F}_5\big] - \frac{1}{4{\kappa_0}^2}\int d^{10}X\ C_4\wedge H_3\wedge F_3,$$

with

$$F_1 = dC_0, \quad F_3 = dC_2, \quad F_5 = dC_4, \quad \tilde{F}_3 = F_3 - C_0\wedge H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3.$$

Another example of supergravity theory is that arising from the M-theory. SUSY constrains the possible objects, fields and couplings of the theory. In this case, the action looks far simpler:

$$\mathcal{S} = \int d^{11}X\sqrt{-G}\big(\mathcal{R} + F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\big),$$

where $F_{\mu\nu\rho\sigma}$ are the components of the field-strength of a 3-potential C_3. As a point particle describing a 1-dimensional worldline is electrically coupled to the e.m. 1-potential, an object sweeping a 3-dimensional surface couples electrically to C_3. It is the M2-brane, with a 3-dimensional worldvolume Σ, whose coupling with the potential is

$$\mathcal{S}_{M_2} = N_{M_2}\int_\Sigma C_{012}dx^0dx^1dx^2.$$

In this expression, without loss of generality, the membrane is thought to extend in the directions x^0, x^1, x^2. The correct interpretation for N_{M_2}, instead, is the number of M2-branes. Also something akin to magnetic monopoles live in M-supergravity. In total analogy to magnetic monopoles in the electromagnetic theory, their charge can be found integrating the field strength over a sphere of a proper dimension: in e.m. we integrate $F_{\mu\nu}$ over S^2 since it is a 2-form, whilst in the present instance,

$$g_M = \int_{S^4} F_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma, \tag{1.159}$$

since the field-strength is a 4-form. S^4 is a hypersurface of $\mathbb{R}^5$: the magnetic monopole takes up the remaining non-transverse spacelike directions, hence is a $(5+1)$-dimensional object called M5-brane.

The objects living in the IIA supergravity can be determined from the quantization of the IIA string: the fundamental strings F1, NS5-branes and D-branes with an even number of spacelike directions appear. IIB supergravity displays the same zoo, with the difference that in this case, D-branes must have uneven space dimensions. As an alternative, the objects populating IIA supergravity arise from those of the 11-dimensional supergravity, upon compactification of one spacelike direction, e.g. x^{11}. The whole spacetime, then, appears with one dimension less and objects extending in this direction, will have worldvolumes with one dimension lower. For instance, the fundamental strings of IIA theory can originate after compactifying M2-branes of 11-dimensional supergravity whose worldvolume also stretches in x^{11} direction. M5-branes in 11 dimensions which do not extend along x^{11} give rise to IIA NS5 branes upon compactification. Lastly, IIA particles emerge from momentum waves propagating along the compact direction in 11d-supergravity (the momentum is quantized since the direction gets a circle). The potentials of IIA supergravity are obtained from C_3 or the metric $G_{\mu\nu}$ as

$$B_{\mu\nu} = C_{\mu\nu 11}, \qquad C_\mu = G_{\mu 11}.$$

The other components of $C_{\mu\nu\rho}$ give rise to C_3 in 10 dimensions, as well as the other components of $G_{\mu\nu}$ coincide with the components of the metric in 10 dimensions. The fields and the related charged objects are collected in Table 1.2: a p-potential is electrically coupled to (p-1)-branes and magnetically coupled to (7-p)-branes. Indeed, the (p+1)-dimensional field strength of a p-potential has to be integrated on a (p+1)-dimensional sphere, which lies in a (p+2)-dimensional transverse space. The worldvolume of the magnetic charge will take up $9-p-2 = 7-p$ space dimensions.

	IIA			IIB			
Potential	B	C_1	C_3	B	C_0	C_2	C_4
Electric	F1	D0	D2	F1	D(-1)	D1	D3
Magnetic	NS5	D6	D4	NS5	D7	D5	D3

Table 1.2: Potentials and corresponding charged objects in IIA and IIB supergravity

Let us underline that D3-branes are both electrically and magnetically charged objects: they are therefore, dyons. This is related to the fact that the dual field strength of $F_5 = dC_4$, whose components read

$$\tilde{F}_{\mu_1 \cdots \mu_5} = \frac{1}{5!} \sqrt{-g} \varepsilon_{\mu_1 \ldots \mu_5 \mu_6 \ldots \mu_{10}} F^{\mu_6 \cdots \mu_{10}},$$

coincides with F_5. Moreover, D(-1)-branes are also known as instantons, i.e. objects localized both in time and space.

1.5.1 Dualities

Superstring theories were devised independently one on another. Only in the 90s, physicists noticed that they were all equivalent and could be thought as limits of one more general theory. The maps between the different theories are called S and T dualities. We are going to dwell on those of their realizations which turn out to be useful in the following.

For instance, S-duality matches the type-I superstring theory to the SO(32) heterotic string theory and the type IIB superstring theory to itself. Nonetheless, let us focus on the latter case: S-duality exchanges the B field and the C_2 field, leaving C_4 invaried. Accordingly, F1 branes are paired up with D1 branes and NS5 branes with D5 branes. Moreover, S-duality is such that the coupling of the IIB theory gets

$$g \to g' = \frac{1}{g}.$$

Consequently, all parameters of the theory transform according to their dimension, i.e.

$$A \to A' = \frac{A}{g^{-[A]/2}},$$

where $[A]$ is the energy dimension of the parameter A.
Besides linking the two heterotic superstring theories, the T-duality connects the IIB and the IIA-type string theories. Let us consider a IIA fundamental string wrapped on a circle of radius R. On it, winding modes are defined as quanta contributing to the total mass of the string. Given that the string twists around the circle n_1 times and its tension is $T_{F_1} = 1/2\pi l_s^2$, its mass will be

$$M = \frac{2\pi n_1 R}{2\pi l_s^2} = \frac{n_1 R}{l_s^2}$$

and this mass comes in units of the fundamental mass R/l_s^2. On top of that, the string can be provided with momentum excitations, that on the compact direction come in quanta of $1/R$. As far as the supergravity fields are concerned, T-duality along one direction interchanges the momentum and winding modes of IIA and IIB theories, i.e.

$$P \leftrightarrow F1,$$

as well as their spectra

$$R/l_s^2 \leftrightarrow 1/\tilde{R}, \qquad \tilde{R}/l_s^2 \leftrightarrow 1/R,$$

with $\tilde{R}$ the radius of the circle in IIB. T-duality hence, allows to choose the proper duality-frame in which to perform perturbative string theory calculations. As long as $R >> l_s$ in IIA, the stringy nature of the theory is not manifest and the dynamics can be easily determined in the Supergravity framework. In the opposite case, a perturbative approach is nomore allowed, unless we shift to IIB theory, where $\tilde{R} >> l_s^2$ and perturbation theory can be safely employed. Additionally, let us consider a Dp-brane in IIA (IIB) with one of the directions of the worldvolume compactified on a circle. A T-duality along this compact direction will cancel this direction and give rise to a D(p-1)-brane living in IIB (IIA). Conversely, a T-duality along one transverse direction will add a new dimension to the worldvolume of the Dp-brane, which now gets a D(p+1)-brane. T-duality turns a continuous distribution of D-branes on the compact direction into a localized distribution of D-branes one on top of the other (since the direction of the smearing is nomore transverse after the duality) and viceversa. Ultimately, a T-duality along the compact direction with the identification $y \sim y + 2\pi R$, the string coupling of the theory transform as

$$g_s' = \frac{\sqrt{\alpha'}}{R} g_s.$$

Chapter 2

Black holes in supergravity

Black holes are classical solutions to the Einstein's equations of General Relativity. They are spacetimes characterized by a region from which no causal signals can reach an asymptotic observer sitting at infinity. The boundary of this region is named *event horizon*. Furthermore, classical black holes are characterized by a curvature *singularity*, where General Relativity breaks down. This singularity is always cloaked by an event horizon: "naked singularities" are thought to be non-physical. Furthermore, black holes have thermodynamic properties. A thorough understanding of this aspect cannot be acquired in a classical framework, hence black holes have proved to be a playground for Quantum Gravity theories.

2.1 Black holes thermodynamics

2.1.1 Laws of thermodynamics

The thermodynamic behaviour of black holes was for the first time highlighted by Bekenstein. His insight arose from the firm belief that thermodynamics laws are to be preserved in every physical system. As a matter of fact, black holes are the outcome of the collapse of matter carrying some entropy. An asymptotic observer would state that the total entropy of the universe decreases as matter disappears behind the event horizon, unless black holes are thought to have their own entropy. The demand for some entropy for black holes can be understood considering that despite these solutions are completely determined by a few parameters, such as mass, angular momentum and electric charge ("black holes have no hair"), several microstates can produce the same macroscopic state.
A first hint of what the entropy of black holes depends on, arose from Hawking's *area theorem*, according to which, under some conditions, in any physical process the area A of the event horizon of a black hole cannot decrease:

$$\Delta A \geq 0. \tag{2.1}$$

This was compared by Bekenstein to the second law of thermodynamics, stating that any physical process is such that that the total entropy of the Universe cannot decrease:

$$\Delta S \geq 0. \tag{2.2}$$

Therefore, Bekenstein guessed that the entropy should be a monotonic function of A/l_P^2 (the Planck length in the denominator is inserted for dimensional reasons). A rough calculation can confirm this first intuition. Indeed, let us suppose that N quanta are thrown into a Schwarzschild black hole, giving rise to a certain macrostate. The number of the corresponding microstates will grow exponentially with N (e.g. if quanta are spin-1/2 particles, then we will have 2^N microstates). Additionally, to fit behind the horizon, the energy of each quantum should be at least $1/r_S$, where

$r_S = 2GM$ is the Schwarzschild radius. Thereafter, due to Boltzmann law,

$$dS \sim dN \sim r_S dM \sim \frac{r_S dr_S}{G} \sim \frac{dA}{G}.$$

(2.1) is also known as the second law of thermodynamics for black holes. Actually, all the other laws of thermodynamics have an analogue in the framework of black holes. For instance, the zeroth law of thermodynamics claims that the temperature is everywhere the same in a system in equilibrium. A quantity which remains constant in black holes is the surface gravity on a Killing horizon[1]. Therefore, we can identify the surface gravity and the temperature of the black hole. Furthermore, given a system of temperature T rotating with angular velocity Ω and in a potential Φ, its variation of the energy will arise from the first law of thermodynamics

$$dE = TdS + dW = TdS + \Omega dJ + \Phi dQ, \tag{2.3}$$

where dJ and dQ are the infinitesimal variations of the angular momentum and the charge. If, instead, we focus on a stationary black hole with mass M, angular momentum J and electric charge Q, under an infinitesimal transformation, the mass will vary according to

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ, \tag{2.4}$$

with Ω_H the angular velocity of the horizon and Φ_H the co-rotating electric surface potential. (2.4) is regarded as the black hole version of the first law of thermodynamics. By comparing the latter with (2.3), we can identify

$$T = \alpha \frac{\kappa}{8\pi}, \qquad S = \frac{A}{\alpha},$$

confirming the intuitions about the zeroth and the second thermodynamics laws. Here, α is just a constant but exact calculations performed by Hawking allow to fix $\alpha = 4$ and the correct physical constants, getting

$$S = k_B \frac{A}{4 l_P^2}, \qquad T = \frac{\hbar}{c k_B} \frac{\kappa}{2\pi} \tag{2.5}$$

with $l_P^2 = G\hbar/c^3$. Ultimately, the third thermodynamics law states that a thermal system cannot reach zero temperature in a finite number of physical processes. Hence, a finite number of physical processes cannot render a black hole *extremal*, i.e. such that $\kappa = 0$.

2.1.2 Hawking radiation and the information paradox

Classically, nothing can escape the event horizon, radiation included. Hence, in a classical framework, black holes should be regarded as zero temperature objects and their thermodynamical properties are nothing more than a formal suggestion. Nonetheless, the presence of the Planck constant in (2.5), both for the entropy and the temperature turns out to be a hint that these two quantities can be fully understood in a quantum description of gravity. As a matter of fact, the computation of black hole microstates in [1] showed that String Theory, the current most promising candidate among quantum gravity theories, can explain the Bekenstein-Hawking entropy from a microscopic viewpoint. Furthermore, in 1974 Hawking proved that black holes truly radiate and a distant observer sees the emitted particles as a thermal distribution whose temperature is exactly (2.5). If this did not happen, a violation of the second law of thermodynamics would occurr. Indeed, let us consider a black hole immersed in a thermal bath with temperature $T_{\text{bath}} < T_{BH}$. Due to the enregy conservation, if there is no change in the angular momentum and in the charge of the black hole,

$$T_{BH} dS_{BH} + T_{\text{bath}} dS_{\text{bath}} = 0 \Rightarrow T_{\text{bath}} = -\frac{T_{BH} dS_{BH}}{dS_{\text{bath}}}. \tag{2.6}$$

[1]The surface gravity κ of a Killing horizon $\mathcal{N}$ can be identified as the acceleration of a static particle near the horizon as measured at the spatial infinity. If ξ^μ is a Killing vector orthogonal to $\mathcal{N}$, then $\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu$.

If matter can only fall into the black hole and not exit, $dS_{BH} > 0$ and $dS_{\text{bath}} < 0$. Hence, plugging (2.6) into $T_{\text{bath}} < T_{BH}$, we end up with $T_{BH}(dS_{BH} + dS_{\text{bath}}) < 0$ and then the total entropy of the Universe decreases. If instead, the black hole can radiate, $dS_{BH} < 0$ and $dS_{\text{bath}} > 0$ and we end up with $T_{BH}(dS_{BH} + dS_{\text{bath}}) > 0$ and the second thermodynamics law is preserved. In the last analysis, the entropy and the temperature of a black hole are actually physical quantities, describing a radiation emitting black body. Their expressions (2.5) hold for any values of mass, angular momentum and electric charge and in any spacetime dimension.
The emission from a black hole is named *Hawking radiation* and is due to quantum effects. A quantitative description was proposed by Hawking, who resorted to a semiclassical approach to the QFT of particles in the curved background sourced by the black hole. For a Schwarzschild black hole, $\kappa \sim 1/M$ and then $T \sim 1/M$: the smaller the mass of a black hole is, the speedier the radiation emission runs. Small black holes emit more than they might absorb, for instance, from CMB. In the end, a small enough black hole fully evaporates leaving just thermalized radiation in its place. From the pure state of matter collapsing beyond the horizon and feeding the black hole, a mixed state of thermalized particles is left: the process of blak hole evaporation does not preserve information and is then not unitary. This issue is called *information paradox*.

2.2 Supersymmetric black holes

Some solutions of supergravity exhibit the typical features of black holes, namely a singularity and an event horizon covering it. They are sourced by stable configurations of branes, sitting in different points of the transverse space. Indeed, D-branes have some tension, hence give rise to a metric, and as already highlighted for instance in Table 1.2, they couple to massless R-R fields. A further field sourced by D-branes is the dilaton ϕ. As an example, a D2-brane stretching on the directions 0, 1 and 2 will produce the fields

$$g = Z^{-1/2}\big(-dx_0^2 + dx_1^2 + dx_2^2\big) + Z^{1/2}\big(dx_3^2 + \cdots + dx_9^2\big), \qquad C_{012} = Z^{-1}, \qquad e^{\phi} = Z^{1/4}.$$

The metric is Lorentz invariant along the D2-brane directions and rotational symmetric in the transverse ones. Z is a function of spacetime coordinates obeying the Poisson equation in the transverse space $\mathbb{R}^7$:

$$\Delta_7 Z = \rho_{D_2}, \tag{2.7}$$

where ρ_{D_2} represents the density of D-branes. The general solution for (2.7) in a n-dimensional transverse space only depends on $r = |\vec{x}|$, where $\vec{x}$ is the n-dimensional vector. More precisely, it reads

$$Z(r) = \frac{A}{r^{n-2}} + C.$$

In the case of a stack of D2-branes sitting on a point in the transverse space (w.l.o.g., chosen to be the origin of coordinates), $n = 7$ and $\rho_{D_2} = N_{D_2}\delta(\vec{r}_7)$, thereafter

$$Z = C + \frac{N_{D_2}}{r^5}. \tag{2.8}$$

The parameter C can be set to one with a change of coordinates. If other stacks of branes are added in the transverse spacetime, then the different solutions sum up due to the linearity of the Poisson equation. From (2.8), we can infer that the metric coefficients diverge for $r = 0$: this is in fact, an actual singularity since the energy associated to the C-field blows up.

Charged point particles, like electrons, would fly apart when placed in some points of the space, due to the electric repulsion. Configurations of branes, instead, are stable due to supersymmetry, if they satisfy the so-called BPS bound (after Bogomolny-Prasad-Sommerfeld), such that

$$M = Q, \tag{2.9}$$

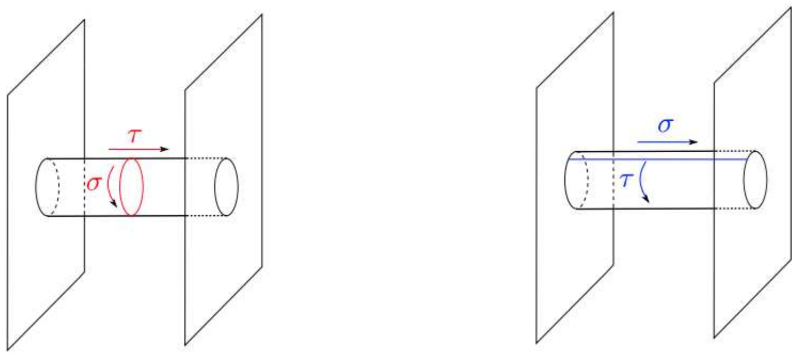

Figure 2.1: The same worldsheet between two parallel D-branes can be interpreted as an exchange of a closed string or a loop in open string theory. The image is taken from [19].

where M is the total mass of the brane and Q is its charge under the corresponding gauge R-R field and is defined as in (1.159) (manifestly, changing the dimension of the field strength according to the dimension of the D-branes involved). For any charged object, the relation $M \geq Q$ holds and (2.9) can be regarded as its limiting case. The latter is a consequence of the invariance of the system under certain supersymmetry transformations. Objects satisfying the BPS bound are said extremal: their temperature vanishes and hence they do not radiate. Multi-center solutions are then stable since they do not lose energy: their gravitational attraction exactly compensates their electrostatic repulsion. This is illustrated in Figure 2.1. The two diagrams represent two parallel branes (a case we are going to deal with in Chapter 3), whose interaction has a twofold interpretation: the exchange of a closed string, meaning gravitational interaction, or a loop of open strings, i.e. vacuum fluctuations which is the analogue of the Casimir energy for a photon field. As already stressed, these two kinds of interactions exactly cancel in order that the stability of multi-center solutions is granted. The dual intepretation of this diagram is at the root of the gauge/gravity duality.

We are going to deal with some examples of supersymmetric black holes. Their entropy is turning out to be independent on the coupling g_s^2 of closed strings with open strings or D-branes, hence we choose to work in the limit $g_s << 1$, such that the quantum effects due to the stringy nature of black holes can be safely neglected. Nonetheless, the actual coupling of N D-branes with the closed strings is $g_s N$: D-branes can therefore, modify the spacetime metric if $g_s N$ is finite, despite $g_s << 1$. Furthermore, the area of black hole horizons depends on $g_s N$: if $g_s N << 1$, it is small in string units and the horizons could not be studied in the supergravity framework. In order to study black holes close to the horizon, then, we need to impose $g_s N >> 1$. Anyway, we choose to work in the $g_s N << 1$ regime, so that a free classical theory for D-branes and open or closed strings can be employed to compute the entropy.

2.2.1 Three-charge black hole

Let us work in the IIA-type theory and a NS1 string wrapping around a compact S^1, whose direction is labelled as y:

$$y \sim y + 2\pi R.$$

The supergravity solution will be given by

$$g = H_1^{-1}\left(-dt^2 + dy^2\right) + \sum_{i=1}^{8} dx_i dx_i, \qquad e^{2\phi} = H_1^{-1}, \qquad H_1 = 1 + \frac{Q_1}{r^6},$$

where the exponent in the last expression is due to the 8 dimensions of the transverse space. As $r \to 0$, the dilaton field approaches $-\infty$ and hence the Einstein length of the compact circle vanishes. In a more physical view, this is a consequence of the string tension (or the tension of the corresponding M2-brane in the M-theory), which leads the circle on which it is wrapped, to collapse. Moreover, since the only hypersurface candidate to be a horizon is determined by $r = 0$, its Einstein area can only vanish. To stabilize the dilaton, we could add a NS5-brane wrapping around the y compact direction and the torus T^4. In this case, indeed,

$$H_1 = 1 + \frac{Q_1}{r^2}, \qquad H_5 = 1 + \frac{Q_5}{r^2}, \qquad e^{2\phi} = \frac{H_5}{H_1} \to \frac{Q_5}{Q_1} \ \text{ for } \ r \to 0.$$

Nevertheless, since both the string and the brane wind around y, this direction is again squeezed to zero and the black hole has no horizon area.
Finally, a stable black hole is built up once another charge is added to this system: a momentum P along the compact S^1. A three-charge black hole has then appeared. After a T-duality along one direction of the torus and a S-duality, we get:

$$NS1NS5P\ (IIA) \ \overset{T}{\to}\ NS1NS5P\ (IIB) \ \overset{S}{\to}\ D1D5P(IIB).$$

This is one of the equivalent writings of the three charges in different duality frames and also arises starting from the configurations with three orthogonal M2 branes in the 11d-supergravity. $x_1, ..., x_6$ directions are compact and when not extended along, the branes are smeared along these compact directions:

$\mathrm{M2}_1$	0	1	2	-	-	-	-
$\mathrm{M2}_2$	0	-	-	3	4	-	-
$\mathrm{M2}_3$	0	-	-	-	-	5	6

The transverse space is then 4-dimensional and the harmonic functions are

$$Z_1 = 1 + \frac{Q_1}{r^2}, \quad Z_2 = 1 + \frac{Q_2}{r^2}, \qquad Z_3 = 1 + \frac{Q_3}{r^2}.$$

These enter the supergravity solution one independently on the other since we deal with a supersymmetric configuration:

$$\begin{aligned} ds^2 = &- (Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}\big(dx_7^2 + \cdots + dx_{10}^2\big) + \\ &+ \frac{\big(Z_2Z_3\big)^{1/3}}{{Z_1}^{2/3}}\big(dx_1^2 + dx_2^2\big) + \frac{\big(Z_1Z_3\big)^{1/3}}{{Z_2}^{2/3}}\big(dx_3^2 + dx_4^2\big) + \frac{\big(Z_1Z_2\big)^{1/3}}{{Z_2}^{2/3}}\big(dx_5^2 + dx_6^2\big), \end{aligned}$$

which for large r is the product of a 5-dimensional Minkowski spacetime and a six-torus with constant radii, i.e. the compactification of the flat 11-dimensional spacetime to a Minkoswki 5-dimensional spacetime. $r \to 0$, instead, is an actual curvature singularity, which is to be identified with the horizon. Close to $r \to 0$, the metric gets

$$\begin{aligned} ds^2 = &- \frac{r^4}{\big(Q_1Q_2Q_3\big)^{2/3}}dt^2 + \big(Q_1Q_2Q_3\big)^{1/3}\frac{dr^2}{r^2} + \big(Q_1Q_2Q_3\big)^{1/3}d\Omega_3^2 + \\ &+ \frac{\big(Q_2Q_3\big)^{1/3}}{{Q_1}^{2/3}}\big(dx_1^2 + dx_2^2\big) + \frac{\big(Q_1Q_3\big)^{1/3}}{{Q_2}^{2/3}}\big(dx_3^2 + dx_4^2\big) + \frac{\big(Q_1Q_2\big)^{1/3}}{{Q_2}^{2/3}}\big(dx_5^2 + dx_6^2\big). \end{aligned} \tag{2.10}$$

By imposing $\rho = r^2$, we can realize that the near-horizon geometry is $AdS_2 \times S^3 \times T^6$, with T^6 a torus of constant radii. Furthermore, from the gauge 3-potentials

$$C_{012} = Z_1^{-1}, \qquad C_{034} = Z_2^{-1}, \qquad C_{056} = Z_3^{-1},$$

we can infer the parameters Q_i. Let us focus on the harmonic function for the first brane. We have that the only non-vanishing component of the field-strength (up to permutations of the indices) is

$$F_{012r} = \partial_r C_{012} = \partial_r Z_1^{-1} = \frac{2r}{Q_1}.$$

Labelling as θ_1, θ_2, θ_3 the three angles of S^3, the only non vanishing components (up to permutations) of the corresponding Hodge dual $\tilde{F}_7 = \star_{11} F_4$ are

$$\tilde{F}_{3456\theta_1\theta_2\theta_3} = \sqrt{-g}\varepsilon_{3456\theta_1\theta_2\theta_3 012r} g^{00} g^{11} g^{22} g^{rr} F_{012r} = 2Q_1,$$

where the metric (2.10) has been employed. $\tilde{F}_7$ can be integrated on the 7-dimensional transverse space of the M2-brane $\Sigma = T^4_{3456} \times S^3$, giving the number of M2-branes:

$$\left(2\pi l_P\right)^6 N_1 = \int_\Sigma dx_3 dx_4 dx_5 dx_6 d\Omega_3 \; \tilde{F}_{3456\theta_1\theta_2\theta_3} = (2\pi)^6 L_3 L_4 L_5 L_6 Q_1,$$

where L_i are the radii of the circles of the 6-dimensional torus and we have used

$$\int d\Omega_3 = 2\pi^2.$$

Finally,

$$Q_1 = \frac{N_1 (l_P)^6}{L_3 L_4 L_5 L_6}, \qquad Q_2 = \frac{N_2 (l_P)^6}{L_1 L_2 L_5 L_6} \qquad Q_3 = \frac{N_3 (l_P)^6}{L_1 L_2 L_3 L_4}. \tag{2.11}$$

The entropy of this black hole can be computed using the Bekenstein-Hawking formula (2.5), after imposing $\hbar, c, k_B = 1$:

$$S = \frac{A}{4G_N}.$$

This formula holds for any spacetime dimension D, provided that G_N has the proper expression

$$16\pi G_N = (2\pi)^{D-3} l_P^{D-2}.$$

The area of the horizon can be computed resorting to the near-horizon metric (2.10) for fixed time and radius $r = 0$:

$$\begin{aligned} A &= \int_{S^3 \times T^6} \sqrt{g} = \int_{S^3} \sqrt{g_{S^3}} \int_{T^6} \sqrt{g_{T^6}} = \\ &= \sqrt{Q_1 Q_2 Q_3} \int d\Omega_3 \int dx_1 \dots dx_6 = \\ &= 2\pi^2 \sqrt{Q_1 Q_2 Q_3} \prod_{i=1}^{6} L_i = \\ &= 2\pi^2 (2\pi)^6 (l_P)^9 \sqrt{N_1 N_2 N_3}. \end{aligned}$$

In the last step, we have exploited (2.11) formulae, which indeed, have been obtained using the near-horizon geometry. Ultimately, using the Newton constant for $D = 11$, the entropy reads

$$S = 2\pi\sqrt{N_1 N_2 N_3}. \tag{2.12}$$

As already alluded before this subsection, the entropy of the black hole does not depend on couplings or the parameters of the torus upon which the compactification takes place: different black holes with different parameters can have the same entropy. This feature is a consequence of extremality and ultimately, on supersymmetry. Accordingly, T-dualities changing the torus parameters would lead to different black holes with the same entropy. After reduction to the IIA-type theory along x_6, we get a D2-D2-F1 system. Then, performing three T-dualities along x_1, x_2 and x_5 we end up with the three-charge black hole in IIB-type theory

D_1	0	-	-	-	-	5
D_5	0	1	2	3	4	5
P	0	-	-	-	-	5

Thereby, (2.12) is the entropy of the three charge black hole D1-D5-P, after the proper relabelling:

$$S = 2\pi\sqrt{N_1 N_5 N_p} \tag{2.13}$$

. The metric sourced by this configuration in the framework of IIB-supergravity is

$$ds^2 = \big(Z_1 Z_5\big)^{-1/2} Z_P^{-1}\big(-dt^2 + dx_5^2\big) + Z_1^{1/2} Z_5^{-1/2}\big(dx_1^2 + ... + dx_4^2\big) + \big(Z_1 Z_5\big)^{1/2}\big(dx_7^2 + ... + dx_{10}^2\big),$$

with $Z_i = 1 + g_s N_i/r^2$. The two-charge version D1-D5 arises when $Z_P = 1$. We get again a black hole with a horizon at $r = 0$. The near-horizon geometry reads

$$ds^2 = r^2\big(-dt^2 + dx_5^2\big) + \frac{dr^2}{r^2} + d\Omega_3^2 + N_1^{1/2} N_5^{-1/2}\big(dx_1^2 + ... + dx_4^2\big).$$

As already argued, the horizon area vanishes:

$$A\big|_{r=0} = \int_{\mathbb{R}\times S^3\times T^4} \sqrt{g} = 2\pi^2(2\pi)^4\frac{N_1}{N_5}\int_{\mathbb{R}} r dx_5 = 0.$$

2.2.2 Four-charge black hole

Another noteworthy supersymmetric black hole is the four-charge black hole which can be constructed in IIA-supergravity and counts three orthogonal D2-branes and a D6-brane stretching along the worldvolumes of the former. Additionally, the D2-branes are smeared in their four transverse directions belonging to worldvolume of the D6-brane. More explicitly, the setup is the following:

$D2_1$	0	1	2	-	-	-	-
$D2_2$	0	-	-	3	4	-	-
$D2_3$	0	-	-	-	-	5	6
$D6_4$	0	1	2	3	4	5	6

and consequently the metric is

$$\begin{aligned} ds^2 = &-\big(Z_1 Z_2 Z_3 Z_4\big)^{-1/2} dt^2 + \big(Z_1 Z_2 Z_3 Z_4\big)^{1/2}\big(dx_7^2 + dx_8^2 + dx_9^2\big) + \\ &+ \frac{\big(Z_2 Z_3\big)^{1/2}}{\big(Z_1 Z_4\big)^{1/2}}\big(dx_1^2 + dx_2^2\big) + \frac{\big(Z_1 Z_3\big)^{1/2}}{\big(Z_2 Z_4\big)^{1/2}}\big(dx_3^2 + dx_4^2\big) + \frac{\big(Z_1 Z_2\big)^{1/2}}{\big(Z_3 Z_4\big)^{1/2}}\big(dx_5^2 + dx_6^2\big). \end{aligned} \tag{2.14}$$

As in the three-charge case, each of the branes contribute to the metric independently on one another on grounds of supersymmetry. Due to these smearings, the harmonic functions for the D2-branes do not depend on x_1, ..., x_6 directions, as well as the harmonic function for the D6-brane. In particular, the transverse space is 3-dimensional and then

$$\Delta_3 Z_i = 0, \qquad Z_i = 1 + \frac{Q_i}{r}.$$

Hence, we can compactify these directions in a six-torus T^6. A T-duality along each of x_1, ..., x_6 leads to another duality framework in which the four charges are D4 - D4 - D4 - D0.
The extremality of the solution (2.14) can be read in its asymptotic version, which is nothing but the compactification of a 10-dimensional supergravity theory to . For large r, indeed,

$$g_{tt} = 1 - \frac{1}{2}\frac{Q_1 + Q_2 + Q_3 + Q_4}{r}.$$

With the ADM (Arnowit, Deser, Misner) prescription in mind, we compare the last expression with the Schwarzschild's $g_{tt} = 1 - G_N M/2r$. Hence,

$$G_N M = Q_1 + Q_2 + Q_3 + Q_4,$$

which is exactly the BPS bound (2.9) defining the extremal black holes. Supersymmetry, then, ensures the stability of the configuration of branes producing the metric (2.14).
For small radii, the four-charge black hole metric reads

$$\begin{aligned} ds^2 = & -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}\left(dx_7^2 + dx_8^2 + dx_9^2\right) + \frac{\left(Q_2Q_3\right)^{1/2}}{\left(Q_1Q_4\right)^{1/2}}\left(dx_1^2 + dx_2^2\right) + \\ & + \frac{\left(Q_1Q_3\right)^{1/2}}{\left(Q_2Q_4\right)^{1/2}}\left(dx_3^2 + dx_4^2\right) + \frac{\left(Q_1Q_2\right)^{1/2}}{\left(Q_3Q_4\right)^{1/2}}\left(dx_5^2 + dx_6^2\right) = \\ = & -\frac{r^2}{R^2}dt^2 + R^2\frac{dr^2}{r^2} + R^2 d\Omega_2^2 + ds^2\left(T^6\right), \end{aligned}$$

where T^6 represents the six-torus of constant radii in the compact directions x_1, ... x_6. The four-charge geometry arising for $r \to 0$ is then $AdS_2 \times S^2 \times T^6$. Besides, the $r = 0$ hypersurface is such that $g_{tt} \to 0$ and can be thus identified as a black hole horizon: we are really dealing with an extremal black hole in four dimensions. Moreover, in the particular case in which the charges of all the branes are equal, i.e.

$$Q_i = Q, \qquad Z_i = Z,$$

the metric (2.14) becomes

$$ds^2 = -Z^{-2}dt^2 + Z^2dr^2 + Z^2r^2d\Omega_2^2 + ds^2(T^6), \tag{2.15}$$

which neglecting the non-physical dimensions wrapped on T^6, can be identified as a four-dimensional charged and not spinning black hole: the Reissner-Nordström solution of Einstein's equations. Indeed, the latter reads

$$ds_{RN}^2 = -\left(1 - \frac{2M}{Q} + \frac{Q}{\rho}\right)^2 dt^2 + \left(1 - \frac{2M}{\rho} + \frac{Q}{\rho}\right)^{-2} + \rho^2 d\Omega_2^2,$$

which in the exremality limit $M = Q$, boils down to

$$ds_{RN}^2 = -\left(1 - \frac{Q}{\rho}\right)^2 dt^2 + \left(1 - \frac{Q}{\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega_2^2.$$

This metric is equivalent as can be proved imposing

$$r = \rho - Q.$$

In the "isotopic coordinate" ρ, the horizon $r = 0$ lies at $\rho = Q$ and then the entropy of the black hole is

$$S \sim 4\pi\rho^2 = 4\pi Q^2 \sim 4\pi\sqrt{Q_1Q_2Q_3Q_4},$$

where the last expression is the generalization to the case in which all charges are different from one another. Let us finally notice that the coordinate $r > 0$ for the transverse space is nomore useful to investigate the black hole behind the horizon, where one has instead, to resort to the isotropic ρ to get from the horizon $\rho = Q$ down to the singularity $\rho = 0$.

2.3 First counting of black hole microstates

One of the most astonishing achievements of the string theory was the explanation of the microscopic origin of the Bekenstein-Hawking entropy (2.5). This was pointed out for the first time by Strominger and Vafa [1]. In this Section, we are showing this for the three-charge black hole in IIB-type theory D1-D5-P, with D1 and P on one of the directions of the worldvolume of the D5-brane, e. g. x_5, which is compactified on a circle S^1 of radius R. We will focus on a set of open strings stretching between D1 and D5 (the contribution of open strings with both endpoints either on D1 or D5 is subleading). In the regime $g_s \ll 1$ and $E \ll M_s$ we just have point particles moving along the circle. Furthermore, if $g_s N \ll 1$, open strings are free and their total momentum P is described by the wavefunction

$$\psi(x_5) = \sum_n e^{-nx_5/R}, \tag{2.16}$$

where x_5 is the coordinate along the circle and n is the number of momentum units. Besides, let us name $x^{(1)}$ and $x^{(5)}$ the x_5 coordinates of both the endpoints of one open strings, respectively lying on D1 and D5. If D1 wraps N_1 times and D5 wraps N_5 times around S^1, in order that the wavefunction of a single open string is single-valued, we must require that [2]

$$\psi(x^{(1)}, x^{(5)}) = \psi(x^{(1)} + 2\pi N_1 N_5 R,\ x^{(5)} + 2\pi N_1 N_5 R).$$

Hence the unit of the quantized momentum for each string is $1/RN_1N_5$ and

$$\psi(x_5) \sim \sum_n e^{-nx_5/(RN_1N_5)}.$$

Given a total momentum $p = N_p/R$ we would like to count the number of ways it can be partitioned among the open strings stretching between D1 and D5, that is to say, in how many ways we can choose the number of open strings n_m for each momentum m/RN_1N_5, m positive integer, such that

$$\sum_{m=1}^{\infty} \frac{n_m m}{N_1 N_5 R} = \frac{N_p}{R}.$$

The issue amounts to counting the partitions of the integer

$$M \equiv N_1 N_5 N_p = \sum_{m=1}^{\infty} n_m m$$

and can be solved using the partition function

$$Z = (1 + q + q^2 + ...)(1 + q^2 + q^4 + ...)(1 + q^3 + q^6 + ...)(...) = \prod_{m=1}^{\infty} \frac{1}{1 - q^m}, \tag{2.17}$$

where the second expression holds when $q < 1$. Indeed,

$$Z = 1 + q + 2q^2 + 3q^3 + ...$$

and one may realize that the coefficient in front of each monomial is the number of partitions of the corresponding exponent. Each bracket in (2.17) represents the sum over all the possible numbers of momentum excitations n_m for each level m, which are supposed to be any positive integer. Actually, when dealing with fermionic string excitations, only $n_m = 0, 1$ are allowed. The fermionic partition function then reads

$$Z = \prod_{m=1}^{\infty} (1 + q^m). \tag{2.18}$$

A further partitioning occurs among the different bosonic and fermionic modes, whose number c is equal for both statistics due to supersymmetry. The thorough partition function, then, is

$$Z = \Big[\prod_{m=1}^{\infty} \Big(\frac{1 + q^m}{1 - q^m} \Big) \Big]^c. \tag{2.19}$$

[2]This is really correct provided that N_1 and N_5 are coprime. If they are not, we can anyway choose $m << N_1, N_5$ such that $N_1 - m$ and N_5 are coprime and the leading contribution to the entropy will be due to N_1 and N_5.

In the canonical ensemble,

$$q = e^{-\beta}$$

and thus (2.17) and (2.18) are really partition functions since we are summing the Boltzmann factors for each energy eigenvalue, weighted with the corresponding multiplicity. In the canonical ensemble, the entropy reads

$$S = \log Z + \beta \langle m \rangle \,, \tag{2.20}$$

where $\langle m \rangle$ is the expectation value for the momentum level. Due to (2.19) and working in the high-temperature limit such that $\beta << 1$, we get

$$\begin{aligned} \log Z &= c \sum_{m=1}^{\infty} \big(\log(1+q^m) - \log(1-q^m)\big) = \\ &= c \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \Big(\frac{(-1)^{k-1}(q^m)^k}{k} + \frac{(q^m)^k}{k}\Big) = \\ &= 2c \sum_{k=1}^{\infty} \frac{1}{2k-1} \sum_{m=1}^{\infty} (q^{2k-1})^m = \\ &= 2c \sum_{k=1}^{\infty} \frac{1}{2k-1} \frac{q^{2k-1}}{1-q^{2k-1}}, \end{aligned}$$

where in the second step the Taylor expansion for $\log(1+x)$ has been exploited. In the high-temperature limit,

$$q \sim 1 - \beta, \qquad q^{2k-1} \sim 1 - (2k-1)\beta$$

and accordingly [3],

$$\log Z \sim 2c \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \frac{1 - 2k\beta + \beta}{\beta} \sim \frac{2c}{\beta} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{2c}{\beta}\frac{\pi^2}{8}.$$

In the canonical ensemble,

$$\langle m \rangle = -\frac{\partial}{\partial \beta} \log Z = \frac{c\pi^2}{4\beta^2},$$

which in the case $\langle m \rangle = M$, can be inverted giving

$$\beta = \sqrt{\frac{c\pi^2}{4M}}.$$

Additionally, the strings between D1 and D5 are such that $c = 4$, exactly as the number of dimensions of T^4 in which the strings can show excitations. Finally the entropy (2.20) reads

$$S = \frac{2\pi^2}{\beta} = 2\pi\sqrt{M} = 2\pi\sqrt{N_1 N_5 N_p}$$

and we have really reproduced the entropy (2.13) of the three-charge black hole.

[3] Let us recall the value of the Riemann ζ-function $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and then

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{(k)^2} = \frac{\pi^2}{8}$$

2.4 The fuzzball proposal

The counting of the microstates exactly predicting the Bekenstein-Hawking entropy is one sensational success of the string theory. Nonetheless, the resolution of other issues such as the information paradox, requires the knowledge of what these microstates really look like. The *fuzzball proposal* by Mathur [2] is one remarkable hypothesis about the nature of these microstates.

In order to grasp the main features of this microstate description, let us focus on two-charge supergravity solutions. A two-charge black hole may be regarded as a three-charge black hole with one vanishing charge, for instance $N_5 = 0$: consequently, one could expect that due to (2.13), the entropy of these black holes vanishes. However, a counting of the microstates can be also performed for two-charge black holes, with the nonzero upshot $S \sim \sqrt{N_1 N_p}$. To solve this issue, we define $u = t + y$, $v = t - y$ and write the *naive* metric of NS1-P

$$ds^2 = Z^{-1}\big[-dudv + Kdv^2\big] + \sum_{i=1}^{4} dx_i dx_i + \sum_{a=6}^{9} dz_a dz_a, \tag{2.21}$$

with z_a the four compact directions of NS5-worldvolume (besides the compact y), x_i the transverse directions and

$$Z = 1 + \frac{Q_1}{r^2}, \qquad K = \frac{Q_p}{r^2}.$$

This metric was found imposing $N_5 = 0$ in the NS1-NS5-P solution, but is not a trustable description for NS1-P, since it only holds provided that the sources are fixed at $r = 0$ in the transverse space. As a matter of fact, if a NS1 string wraps N_1 times along a circle and a momentum wave propagates across the same direction, the momentum P appears as vibration modes of the string, necessarily along the eight transverse directions and the sources will not sit on a fixed position as the wave goes by. We only take care of displacements parallel to the non-compact transverse directions x^i, which in general can be different for each strand (s) of the string (but all carrying momentum in the direction y): we label $\vec{F}^{(s)}(t-y)$ the displacement and $Q_1^{(s)} = Q_1/N_1$ the charge of each strand. If all strands are mutually BPS, we end up with the metric

$$\begin{gathered}
ds^2 = Z^{-1}\big[-dudv + Kdv^2 + 2A_i dx_i dv\big] + \sum_{i=1}^{4} dx_i dx_i + \sum_{a=6}^{9} dz_a dz_a, \\
Z = 1 + \sum_{s=1}^{N_1} \frac{Q_1}{|\vec{x} - \vec{F}(t-y)|^2}, \quad K = \sum_{s=1}^{N_1} \frac{Q_1 |\dot{\vec{F}}(t-y)|}{|\vec{x} - \vec{F}(t-y)|^2}, \quad A_i = \sum_{s=1}^{N_1} -\frac{Q_1 |\dot{\vec{F}}(t-y)|}{|\vec{x} - \vec{F}(t-y)|^2}.
\end{gathered} \tag{2.22}$$

In the limit in which neighbouring strands have very similar displacement profiles, they will give very close contributions to the harmonic functions and the sums in (2.22) can be substituted with an integral:

$$\sum_{s=1}^{N_1} \to \int_{s=0}^{N_1} ds = \int_{y=0}^{2\pi R N_1} \frac{ds}{dy} dy = \frac{1}{2\pi R} \int_{v=0}^{L_T} dv,$$

where in the last step we have exploited that $v = t - y$ hence the integral can be equivalently computed over v. Apart from that, we have used the knowledge that $y = 2\pi R s$ is the position of the final endpoint of each strand and $L_T = 2\pi R N_1$ is the total length. As a consequence, the harmonic functions in (2.22) become

$$Z = 1 + \frac{Q_1}{L_T} \int_0^{L_T} \frac{dv}{|\vec{x} - \vec{F}(v)|^2}, \quad K = \frac{Q_1}{L_T} \int_0^{L_T} \frac{dv (\dot{F}(v))^2}{|\vec{x} - \vec{F}(v)|^2}, \quad A_i = -\frac{Q_1}{L_T} \int_0^{L_T} \frac{dv \dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2}. \tag{2.23}$$

We can finally perform a chain of dualities, mapping this NS1-P (IIB) solution to D1-D5 (IIB) (at every step, we remain in IIB-type supergravity since we perform a S-duality or an even number of

T-dualities):

$$NS1(5)\ P(5) \overset{S}{\to} D1(5)\ P(5) \overset{T_{6789}}{\to} D5(56789)\ P(5) \overset{S}{\to}$$
$$\overset{S}{\to} NS5(56789)\ P(5) \overset{T_{56}}{\to} NS5(56789)NS1(5) \overset{S}{\to} D5(56789)D1(5),$$

where the numbers in brackets refer to the directions of the worldvolume of each object involved ($x^6, ...x^9$ are the directions of the torus T^4 on which D5 winds, $x^1, ..., x^4$ are the directions of the transverse space, whereas $y = x^5$ is the compact dimension of the starting NS1 and P). We are going to track the transformation under these dualities, of some parameters: the coupling g_s, the scale Q_1 appearing in the harmonic functions (2.23), the radii R and R_6 of the compact dimensions x^5 and x^6 and the volume $(2\pi)^4 V$ of the torus T^4.

$$\begin{pmatrix} g \\ Q_1 \\ R \\ R_6 \\ V \end{pmatrix} \overset{S}{\to} \begin{pmatrix} 1/g \\ Q_1/g \\ R/\sqrt{g} \\ R_6/\sqrt{g} \\ V/g^2 \end{pmatrix} \overset{T6789}{\to} \begin{pmatrix} g/V \\ Q_1/g \\ R/\sqrt{g} \\ \sqrt{g}/R_6 \\ g^2/V \end{pmatrix} \overset{S}{\to} \begin{pmatrix} V/g \\ Q_1 V/g^2 \\ R\sqrt{V}/g \\ \sqrt{V}/R_6 \\ V \end{pmatrix} \overset{T56}{\to} \begin{pmatrix} R_6/R \\ Q_1 V/g^2 \\ g/(R\sqrt{V}) \\ R_6/\sqrt{V} \\ R_6^2 \end{pmatrix} \overset{S}{\to} \begin{pmatrix} R/R_6 \\ Q_1 V R/(g^2 R_6) \\ g/(\sqrt{R R_6 V}) \\ \sqrt{R_6 R/V} \\ R^2 \end{pmatrix} \equiv \begin{pmatrix} g' \\ Q_5' \\ R' \\ R_6' \\ V' \end{pmatrix}.$$

After dualities, the charge Q_1 has been renamed Q_5' since it is now the charge of the D5-branes. The harmonic function for the NS1 string (at large r, since close to the NS1 string we should rely on (2.22) or (2.23))

$$Z \approx 1 + \frac{Q_1}{r^2}$$

after dualities becomes

$$Z \approx 1 + \frac{Q_5'}{r^2},$$

with

$$Q_5' = \frac{VR}{g_s^2 R_6} Q_1 \equiv \mu^2 Q_1. \tag{2.24}$$

$[Q] = -2$ and after this duality chain, all lengths gets scaled by a factor μ. Moreover,

$$Q_5' = \mu^2 Q_1 = \mu^2 \frac{g_s^2 N_1}{V} = g' N_1$$

as expected for brane sources. In the end, after all these dualities, the metric (2.22) with the harmonic functions (2.23), gives

$$ds^2 = \sqrt{\frac{H}{1+K}}\left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2\right] + \sqrt{\frac{1+K}{H}} \sum_{i=1}^{4} dx_i dx_i + \sqrt{H(1+K)} \sum_{a=6}^{9} dz_a dz_a,$$
$$Z = 1 + \frac{\mu^2 Q_1}{\mu L_T} \int_0^{\mu L_T} \frac{dv}{|\vec{x} - \mu \vec{F}(v)|^2}, \quad K = \frac{\mu^2 Q_1}{\mu L_T} \int_0^{\mu L_T} \frac{dv (\dot{\mu F}(v))^2}{|\vec{x} - \mu \vec{F}(v)|^2}, \quad A_i = -\frac{\mu^2 Q_1}{\mu L_T} \int_0^{\mu L_T} \frac{dv \mu \dot{F}_i(v)}{|\vec{x} - \mu \vec{F}(v)|^2}, \tag{2.25}$$

where

$$dB = - \star_4 dA. \tag{2.26}$$

On the other hand, the naive geometry for the D1-D5 system (again arising after the same duality chain on the naive metric (2.21)) reads

$$ds^2_{naive} = \frac{1}{\sqrt{(1+\frac{Q_1'}{r^2})(1+\frac{Q_5'}{r^2})}}[-dt^2 + dy^2] + \sqrt{(1+\frac{Q_1'}{r^2})(1+\frac{Q_5'}{r^2})} dx_i dx_i + \sqrt{\frac{1+\frac{Q_1'}{r^2}}{1+\frac{Q_1'}{r^2}}} dz_a dz_a. \tag{2.27}$$

A remarkable hypersurface in the tranverse space is given by $|\vec{x}| = \sqrt{\alpha'}$. In the outer part, the metric (2.25) for the D1-D5 system boils down to (2.27): for instance both are flat at infinity.

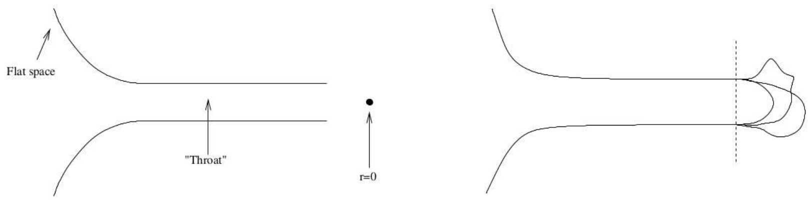

Figure 2.2: On the left, the naive geometry of the D1-D5 system with the singularity after the throat, on the right the actual geometry with a sketch of the hair over the smooth "cap". The dashed line represents the $|\vec{x}| = \sqrt{\alpha'}$ hypersurface. The image is taken from [2].

However, in the inner part, different features appear. Indeed, as depicted in Figure 2.2, the latter comes up with a singularity for $r = 0$ after a throat and a horizon, whereas the former is completely regular: the points on the curve $\vec{x} = \mu\vec{F}(v)$ in the transverse space represent just coordinate singularities. As a matter of fact, if we choose to parametrize the four-dimensional transverse space with the spherical coordinates (ρ, θ, ϕ) for the space orthogonal to the curve and, once we fix $\vec{x}_0 = \mu\vec{F}(v_0)$ the coordinate

$$z \approx \mu|\dot{F}(v_0)|(v - v_0)$$

labels the point along the curve, one can show that the metric ends in a smooth 'cap' for $r = 0$ and not in a singularity. All the different profile functions $\vec{F}(v)$ will give rise to different shapes of the caps which all together make up the statistical ensemble responsible for the entropy of the two-charge black-hole. We have then identified the hair that distinguishes the microstates giving rise to the same macrostate and this is why in this framework, black holes are fuzzballs. These lack a horizon in the traditional sense, but their entropy can be anyway calculated through the Bekenstein-Hawking formula, if the horizon is identified with the hypersurface $|\vec{x}| = \sqrt{\alpha'}$: the result agrees with the prediction $S \sim \sqrt{N_1 N_p}$. All the differences between microstates disappear, instead, when probing physics at higher length scales than $\sqrt{\alpha'}$: this coarse-grained view is tantamount to the traditional description of black holes with a horizon covering a singularity. In this sense, also the lack of the horizon for fuzzballs can be understood. As a matter of fact, due to the Bekenstein-Hawking formula, horizons are always thought to be linked to entropy. However, entropy is a statistical quantity and would be meaningless for a single profile of the cap, as we may define the entropy for a macrostate of a gas of particles, but not for a single microstate with particles with definite positions and momenta $\{\vec{x}_i, \vec{p}_i\}$. As a final benefit, the emission of the Hawking radiation from black holes is a unitary process, without information loss.

2.4.1 One example of displacement

Let us choose one particular case of vibration profile for the string in the NS1-P duality frame:

$$F_1 = \hat{a}\cos\omega v, \quad F_2 = \hat{a}\sin\omega v, \quad F_3 = F_4 = 0. \tag{2.28}$$

This represents a uniform helix in the $\big(x_1, x_2, y\big)$ space. With the angular velocity

$$\omega = \frac{1}{N_1 R},$$

each point on the string turns around the circle in (x_1, x_2) just once. Hence the wavelength is maximum and all the energy of the wave is stored in the minimum energy possible. Let us make use of the polar coordinates in the $\vec{x}$ space:

$$x_1 + ix_2 = ze^{i\tilde{\phi}}, \qquad x_3 + ix_4 = we^{i\tilde{\psi}}, \qquad\qquad \text{with } \; z = \tilde{r}\sin\tilde{\theta}, \qquad w = \tilde{r}\cos\tilde{\theta}$$

and compute the harmonic functions (2.23). For instance, if we define $\xi = \omega v$, we get

$$\begin{aligned} Z &= 1 + \frac{Q_1}{2\pi} \int_0^{2\pi} \frac{d\xi}{(x_1 - \hat{a}\cos\xi)^2 + (x_2 - \hat{a}\sin\xi)^2 + x_3^2 + x_4^2} = \\ &= 1 + \frac{Q_1}{2\pi} \int_0^{2\pi} \frac{d\xi}{z^2 + w^2 + \hat{a}^2 - 2z\hat{a}\cos(\xi - \tilde{\phi})} = \\ &= 1 + \frac{Q_1}{[(z^2 + w^2 + \hat{a}^2) - 4z^2\hat{a}^2]} = \\ &= 1 + \frac{Q_1}{[(\tilde{r}^2 + \hat{a}^2)^2 - 4\tilde{r}^2 \sin^2\tilde{\theta}\hat{a}^2]}, \end{aligned} \tag{2.29}$$

where the integration has been performed exploiting

$$\int_0^{2\pi} \frac{d\alpha \cos^n \alpha}{1 + a\cos\alpha} = \frac{2\pi}{\sqrt{1-a^2}} \Big(\frac{\sqrt{1-a^2} - 1}{a} \Big)^n .$$

After the redefinition

$$\tilde{r} = \sqrt{r^2 + \hat{a}^2 \sin^2\theta}, \qquad \cos\tilde{\theta} = \frac{r\cos\theta}{\sqrt{r^2 + \hat{a}^2\sin^2\theta}},$$

the harmonic function (2.29) yields

$$Z = 1 + \frac{Q_1}{r^2 + \hat{a}^2\cos^2\theta}.$$

With analogous steps we obtain

$$K = \frac{\hat{a}^2}{N_1 R^2} \frac{Q_1}{(r^2 + \hat{a}^2\cos^2\theta)} \equiv \frac{Q_p}{(r^2 + \hat{a}^2\cos^2\theta)}, \qquad A_{\tilde{\phi}} = \frac{\partial x_1}{\partial\tilde{\phi}} A_{x_1} + \frac{\partial x_2}{\partial\tilde{\phi}} A_{x_2} = -\frac{Q_1 \hat{a}^2}{R N_1} \frac{\sin^2\theta}{(r^2 + \hat{a}^2\cos^2\theta)}.$$

With the duality chain to the D1-D5 system, lengths scale up by the factor μ defined in (2.24). The displacement profile hence, becomes $\mu\vec{F}$ and the harmonic functions can be written as

$$Z' = 1 + \frac{Q_5'}{(r^2 + \hat{a}^2\cos^2\theta)}, \qquad K' = \mu^2 \frac{Q_p}{(r^2 + \hat{a}^2\cos^2\theta)} = \frac{Q_1'}{(r^2 + \hat{a}^2\cos^2\theta)},$$

where Q_5' coincides with what defined in (2.24), Q_p has changed subscript since through dualities the momentum gets a D1-brane, the last parameter B_i has been found using (2.26). In the end, using (2.25) and identifying

$$f = r^2 + a^2\cos^2\theta, \qquad a \equiv \mu\hat{a} = \frac{Q_1' Q_5'}{R'}. \qquad h = \Big[\Big(1 + \frac{Q_1'}{f}\Big)\Big(1 + \frac{Q_5'}{f}\Big) \Big]^{1/2},$$

we can write the metric for the D1-D5 system with this particular choice of string vibrations:

$$\begin{aligned} ds^2 = &-\frac{1}{h}(dt^2 - dy^2) + hf\Big(d\theta^2 + \frac{dr^2}{r^2 + a^2}\Big) - \frac{2a\sqrt{Q_1' Q_5'}}{hf} \big(\cos^2\theta dy d\psi + \sin^2\theta dt d\phi\big) \\ &+ h\Big[\Big(r^2 + \frac{a^2 Q_1' Q_5' \cos^2\theta}{h^2 f^2}\Big) \cos^2\theta d\psi^2 + \Big(r^2 + a^2 - \frac{a^2 Q_1' Q_5' \sin^2\theta}{h^2 f^2}\Big) \sin^2\theta d\phi^2\Big] + \sqrt{\frac{Q_1' + f}{Q_5' + f}} dz_a dz_a. \end{aligned} \tag{2.30}$$

As expected, this is asymptotically flat. Instead, for $r \ll (Q_1' Q_5')^{1/4}$, the metric is given by

$$\begin{aligned} ds^2 = &\sqrt{Q_1' Q_5'}\Big[-(r'^2 + 1)\frac{dt^2}{R^2} + r'^2\frac{dy^2}{R^2} + \frac{dr'^2}{r'^2 + 1}\Big] + \\ &+ \sqrt{Q_1' Q_5'}\big[d\theta^2 + \cos^2\theta d\psi'^2 + \sin^2\theta d\phi'^2\big] + \sqrt{\frac{Q_1'}{Q_5'}} dz_a dz_a, \end{aligned} \tag{2.31}$$

where the following identifications have been used:

$$r' = r - a, \qquad \psi' = \psi - \frac{a}{\sqrt{Q_1' Q_5'}} y, \qquad \phi' = \phi - \frac{a}{\sqrt{Q_1' Q_5'}} t.$$

The geometry of the metric (2.31) is locally that of $AdS_3 \times S^3 \times \mathbb{T}^4$. In the following Chapter, this peculiar instance of microstate will be described in the full worldsheet theory. However, instead of D1-D5 we will focus on the duality frame in which the charges are NS5-P.

Chapter 3

A worldsheet theory for microstates

The fuzzball description of black holes was devised in the framework of supergravity, which only involves the massless excitations of strings. However, we expect that a UV complete theory with the full spectrum of string modes should exist for the different displacement profiles. In this Chapter, the gauged Wess-Zumino-Witten model of strings propagating on group manifolds are proved to furnish this worldsheet theory for a particular microstate: the focus is kept on the special example introduced in Subsection 2.4.1.

3.1 Wess-Zumino-Witten model

The target space in which string propagate can be a flat Minkowski spacetime or a more general manifold with any metric and curvature. In the latter case, the action describing the dynamics of strings is the non-linear sigma model (1.156). The realization of this for group manifolds or Lie groups, is the so-called *Wess-Zumino-Witten model*. The treatment of its main aspects both in this Section and in Section 4.1, follows [26].

Let us consider a quantum field theory defined on the Riemann sphere S^2 provided with the Euclidean metric. A generic field $g(z,\bar{z})$ is a smooth map from points of the sphere to a Lie group G:

$$g : S^2 \to G. \tag{3.1}$$

We choose some representation of the Lie group, so $g(z,\bar{z})$ can be thought as a matrix-valued field. When G is semi-simple, that is to say, it can be written as the product of simple Lie groups, a non-degenerate and invariant trace can be defined on the corresponding Lie algebra $\mathfrak{g}$ and the scalar product for any $X, Y \in \mathfrak{g}$ is defined as

$$\langle X, Y\rangle = \mathrm{Tr}(X, Y).$$

Given a real parameter λ, we can define the action of the *principal chiral model*

$$\mathcal{S}_0 = \frac{1}{4\lambda^2}\int_{S^2} d^2z\ \mathrm{Tr}\big(g^{-1}\partial_\mu g g^{-1}\partial^\mu g\big), \tag{3.2}$$

where $g^{-1}\partial_\mu g$ belongs to the Lie algebra and the trace is then well defined. This action is the natural generalization for group manifolds, of the component (1.154) of the non-linear sigma model. Since the coupling λ is dimensionless, (3.2) defines a conformal field theory at the classical level. Moreover, this theory is invariant under the global symmetry

$$G \times G : g(z,\bar{z}) \to g_L g(z,\bar{z}) g_R^{-1}. \tag{3.3}$$

We calculate the equation of motion through the variational principle. Making use of

$$\delta\big(gg^{-1}\big) = 0 \Rightarrow \delta g^{-1} = -g^{-1}\delta g g^{-1}$$

and of the ciclicity of the trace, we get

$$\begin{aligned}\delta\mathcal{S}_0 &= \frac{1}{2\lambda^2}\int_{S^2} d^2z\,\mathrm{Tr}\big((-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\delta\partial_\mu g)g^{-1}\partial^\mu g\big) = \\ &= \frac{1}{2\lambda^2}\int_{S^2} d^2z\,\mathrm{Tr}\big((\partial_\mu g^{-1}\delta g + g^{-1}\partial_\mu\delta g)g^{-1}\partial^\mu g\big) = \\ &= \frac{1}{2\lambda^2}\int_{S^2} d^2z\,\mathrm{Tr}\big(\partial_\mu(g^{-1}\delta g)g^{-1}\partial^\mu g\big) = \\ &= -\frac{1}{2\lambda^2}\int_{S^2} d^2z\,\mathrm{Tr}\big(g^{-1}\delta g\partial_\mu(g^{-1}\partial^\mu g)\big),\end{aligned} \tag{3.4}$$

where in the last step we have integrated by parts. The trace is non-degenerate, therefore the vanishing of $\delta\mathcal{S}_0$ for each variation of the field implies the classical equation of motion

$$\partial_\mu\big(g^{-1}\partial^\mu g\big) = 0, \tag{3.5}$$

which is the statement of the conservation of the current $J^\mu = g^{-1}\partial^\mu g$. Moreover, (3.5) implies that

$$0 = g\partial_\mu\big(g^{-1}\partial^\mu g\big)g^{-1} = \partial_\mu\big(\partial^\mu g g^{-1}\big),$$

therefore $\tilde{J}^\mu = \partial^\mu g g^{-1}$ is conserved as well. J^μ and $\tilde{J}^\mu$ are the conserved currents respectively corresponding to the right and to the left multiplication symmetry in (3.3). We can write the equation of motion (3.5) in complex coordinates (using the flat Euclidean metric (1.44) to lower the indices):

$$\partial J^z + \bar{\partial}J^{\bar{z}} = \partial J_{\bar{z}} + \bar{\partial}J_z = 0 \tag{3.6}$$

If the two addends vanished separately, then J_z and $J_{\bar{z}}$ would be respectively holomorphic and antiholomorphic and the theory would be conformal at the quantum level as well. However, this does not occur, since the vanishing of both terms in (3.6) is equivalent to require that

$$\partial J_{\bar{z}} - \bar{\partial}J_z = \partial_\mu\big(\varepsilon^{\mu\nu}J_\nu\big) = 0.$$

On the other hand, let us realize that J^μ can be regarded as the gauge potential for the gauge group G, in case $A_\mu = 0$: it is a pure gauge potential, then its field-strength vanishes

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0.$$

As a consequence,

$$\partial_\mu\big(\varepsilon^{\mu\nu}J_\nu\big) = \frac{1}{2}\varepsilon^{\mu\nu}\big(\partial_\mu J_\nu - \partial_\nu J_\mu\big) = -\frac{1}{2}\varepsilon^{\mu\nu}\big[J_\mu, J_\nu\big],$$

which is zero only for Abelian Lie algebras. In the most general case, the conformal invariance of the principal chiral model is spoiled at the quantum level.

A new term was added to (3.2) in order to recover conformal invariance at the quantum level [4–6]. It is known as *Wess-Zumino-Witten term* and reads

$$\Gamma = -\frac{i}{12\pi}\int_B d^3y\;\varepsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\big(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\big). \tag{3.7}$$

In this expression, B is a three-dimensional manifold such that $\partial B = S^2$. Accordingly, g has to be regarded as an extension of the smooth maps appearing in (3.2) defined on the boundary S^2, which only in this paragraph we choose to call $\hat{g}$. G is a semi-simple Lie group and then each transformation $\hat{g}$ is in the null second fundamental group: in other words it is homotopic to the constant map. Constant maps on S^2 can be always extended in the interior B, but this extension is not unique: homotopies can provide small perturbations in g without changing the value at the boundary. However, under this local variation

$$\begin{aligned}\delta\Gamma &= -\frac{i}{4\pi}\int_B d^3y\ \varepsilon_{\alpha\beta\gamma}\operatorname{Tr}\big((-g^{-1}\delta g g^{-1}\partial^\alpha g + g^{-1}\partial^\alpha\delta g)g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\big) = \\ &= -\frac{i}{4\pi}\int_B d^3y\ \partial^\alpha\operatorname{Tr}\big(g^{-1}\delta g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\big) = \\ &= -\frac{i}{4\pi}\int_{S^2} d^2x\ \varepsilon_{\alpha\beta}\operatorname{Tr}\big(g^{-1}\delta g g^{-1}\partial^\alpha g g^{-1}\partial^\beta g\big) = \\ &= -\frac{i}{4\pi}\int_{S^2} d^2x\ \varepsilon_{\alpha\beta}\operatorname{Tr}\big(g^{-1}\delta g\partial^\alpha(g^{-1}\partial^\beta g)\big),\end{aligned} \tag{3.8}$$

where the second step is due to Stokes' theorem. In the end, since $\delta g = 0$ on S^2, then also $\delta\Gamma = 0$. The three-dimensional interior where to define the extension can globally change as well: a compact two-dimensional space delimits two distinct three-manifolds which in our instance we name B and $\tilde{B}$. Given the two extensions (g, B) and $(\tilde{g}, \tilde{B})$, we can glue them along the common boundary and define

$$(g, \tilde{g}) : (B \cup \tilde{B})/\partial B \approx S^3 \to G$$

Maps of this kind are classified up to homotopy by the third homotopy group $\pi_3(G)$. A theorem by Bott states that any map $S^3 \to G$, with G simple and compact, is homotopic to $S^3 \to SU(2) \cong S^3$, where $SU(2)$ is a subgroup of G. In practice we are thus working with maps $S^3 \to S^3$, which are classified up to homotopy by $\pi_3(S^3) = \mathbb{Z}$, i. e. according to the number of times S^3 wraps around itself. The variation of (3.7) under the two different choices of the manifold B is the integral

$$\Delta\Gamma = \Gamma[g] - \Gamma[\tilde{g}] = \int_B(\ldots) - \Big(-\int_{\tilde{B}}(\ldots)\Big) = \int_B(\ldots) + \int_{\tilde{B}}(\ldots) = \int_{B\cup\tilde{B}=S^3}(\ldots),$$

with the dots standing for the integrand of (3.7). The crucial '-' sign in front of the integral in $\Gamma[\tilde{g}]$ is due to the fact that S^2 bounds $\tilde{B}$ with an orientation opposite to B. We can focus on the identity map

$$g(y) = y^0 - iy^k\sigma_k,$$

with $y \in S^3 \subset \mathbb{R}^4$ and σ^k the Pauli matrices. $g(y)$ wraps once around S^3. Then, $g^{-1}\partial^k g = -i\sigma^k$ and since the integrand gets independent on the coordinates y, we can substitute the the integral with the volume of S^3:

$$\begin{aligned}\Delta\Gamma &= -\frac{i}{12\pi}\int_{S^3} d^3y\ \varepsilon_{\alpha\beta\gamma}\operatorname{Tr}\big(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\big) = \\ &= -\frac{i}{12\pi}(-i)^3 2\pi^2\sum_{i,j,k}\varepsilon_{ijk}\operatorname{Tr}\big(\sigma^i\sigma^j\sigma^k\big) = \\ &= \frac{\pi}{12}\sum_{i,j,k}\varepsilon_{ijk}\operatorname{Tr}([\sigma^i, \sigma^j]\sigma^k) \\ &= \frac{\pi}{12}2i\sum_{i,j,k}\varepsilon_{ijk}\operatorname{Tr}(\varepsilon^{ijl}\sigma_l\sigma_k) = \\ &= \frac{i\pi}{6}\sum_k 2\operatorname{Tr}(\sigma^k\sigma_k) = 2\pi i.\end{aligned}$$

Let us define the full action of the model

$$\mathcal{S}[g] = \mathcal{S}_0[g] + k\Gamma[g], \tag{3.9}$$

with the real coupling k. This action is then not invariant under global topological transformations, because Γ gets shifted. However, in a quantum theory, the Boltzmann weight $e^{-\mathcal{S}[g]}$ is what really matters: since $\Delta\Gamma = 2\pi i$, it will be single-valued as long as $k \in \mathbb{Z}$. The coupling k is called the

level of the model and is then quantized for compact Lie groups (for non-compact groups, this reasoning does not hold and there might not be any quantization of the level).
The equation of motion of (3.9) is found varying both terms in the action. Summing up the results in (3.4) and (3.8), we get that

$$\begin{aligned}0 =& -\frac{1}{2\lambda^2}\partial_\alpha\big(g^{-1}\partial^\alpha g\big) - \frac{i}{4\pi}\varepsilon_{\alpha\beta}\partial^\alpha(g^{-1}\partial^\beta g) = \\ =&\Big(1+\frac{k\lambda^2}{2\pi}\Big)\partial(g^{-1}\bar\partial g) + \Big(1-\frac{k\lambda^2}{2\pi}\Big)\bar\partial(g^{-1}\partial g),\end{aligned} \tag{3.10}$$

since $g^{z\bar z} = g^{\bar z z} = 2$ and $\varepsilon_{z\bar z} = -\varepsilon_{\bar z z} = i/2$. The Wess-Zumino-Witten model really arises when imposing

$$\lambda^2 = \frac{2\pi}{k}. \tag{3.11}$$

This choice implies that $k \in \mathbb{Z}_0$: $k < 0$ would cause some issues with the convergence of the path-integral. Apart from that, the calculation of the β-function at one-loop level shows that under (3.11), the action (3.9) describes a conformal field theory. When (3.11) does not hold, the coupling λ, despite dimensionless, would be scale-dependent at the quantum level (not k, which is a fixed integer) and the theory could not be conformal (it is asymptotically free). This way, we infer the existence of one antiholomorphic current and one holomorphic current:

$$\bar J = kg^{-1}\bar\partial g, \qquad J = -k\partial g g^{-1}. \tag{3.12}$$

The former is promptly read off from (3.10), whereas the latter is a consequence:

$$\bar\partial\big(\partial g g^{-1}\big) = g\partial(g^{-1}\bar\partial g)g^{-1} = 0.$$

These holomorphic and antiholomoprhic currents correspond to the new local symmetry

$$G(z)\times G(\bar z) : g(z,\bar z) \to g_L(z)g(z,\bar z)g_R^{-1}(\bar z). \tag{3.13}$$

3.2 Branes on a circle

The purpose of this Section is the study of one single kind of charge, showing for the first time how the Wess-Zumino-Witten model just introduced, can be a powerful tool to recover in a stringy fashion the supergravity results.

3.2.1 The supergravity calculation

Let us work in the framework of the 10-dimensional IIB-type supergravity theory, with n_5 parallel NS5-branes. The transverse space is the Euclidean 4-dimensional space E^4, in which the positions of the branes are named $\vec x = \vec x_m$, $m = 0, ..., n_5 - 1$. The tranverse directions to the branes are labelled as $i = 1, ..., 4$, whereas the parallel directions are x^μ, with $\mu = 0, 5, ..., 9$. The mth brane sources a deformation in the spacetime metric, given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + Z_m(\vec x)dx^i dx_i, \tag{3.14}$$

where $\eta_{\mu\nu}dx^\mu dx^\nu$ stands for the Minkowski metric in the parallel directions, $dx^i dx_i$ for the Euclidean one in the transverse directions. $Z_m(x^i)$ is a harmonic function with respect to the transverse E^4:

$$Z_m(\vec x) = 1 + \frac{\alpha'}{|\vec x - \vec x_m|^2}, \tag{3.15}$$

where α' has been written for completeness, but from here on it is set to $\alpha' = 1$. A generic choice of the positions $\vec x_m$ breaks the $SO(4)$ symmetry of the transverse space, but we require that the branes sit on a circle of radius a lying on the plane (x_1, x_2) in the positions $\vec x_m =$

$(a\cos\phi_m, a\sin\phi_m, 0, ..., 0)$, with $\phi_m = 2\pi m/n_5$. $SO(4)$ symmetry is then broken but a $SO(2)\times Z_{n_5}$ symmetry survives. We factorize the transverse space as $E^4 = \mathbb{R}^2\times\mathbb{R}^2 \simeq \mathbb{C}\times\mathbb{C}$ and due to linearity, the full harmonic function of the system can be written as

$$Z_5(\vec{x}) = 1 + \sum_{m=0}^{n_5-1} \frac{1}{|x_1 + ix_2 - ae^{i\phi_m}|^2 + |x_3 + ix_4|^2}. \tag{3.16}$$

For the two complex planes we choose the parametrization

$$x_1 + ix_2 = re^{i\phi}, \qquad x_3 + ix_4 = Re^{i\psi}. \tag{3.17}$$

Therefore

$$\begin{aligned} Z_5(\vec{x}) &= 1 + \sum_{m=0}^{n_5-1} \frac{1}{R^2 + r^2 + a^2 - 2ar\cos(2\pi m/n_5 - \phi)} = \\ &= 1 + \frac{1}{2ar}\sum_{m=0}^{n_5-1} \frac{1}{\cosh\chi - \cos(2\pi m/n_5 - \phi)}, \end{aligned}$$

where we have defined

$$\cosh\chi = \frac{R^2 + r^2 + a^2}{2ar}. \tag{3.18}$$

This is a consistent definition since $R^2+r^2+a^2-2ar = R^2+(r-a)^2 \geq 0$, then $R^2+r^2+a^2 \geq 2ar$ and $\frac{R^2+r^2+a^2}{2ar} \geq 1$. Remembering that $\cos(z) = \cosh(iz)$ and exploiting the generalization of prosthaphaeresis formulas to hyperbolic functions we obtain that

$$Z_5(\vec{x}) = 1 + \frac{1}{2ar}\sum_{m=0}^{n_5-1} \frac{1}{2\sinh(\frac{\chi + i(2\pi m/n_5 - \phi)}{2})\sinh(\frac{\chi - i(2\pi m/n_5 - \phi)}{2})},$$

which by the generalized Dirichlet series

$$\mathrm{csch}(z) = \frac{1}{\sinh z} = 2e^{-z}\sum_{n=0}^{\infty} e^{-2nz}, \qquad z \in \mathbb{C}, \mathrm{Re}\, z > 0, \tag{3.19}$$

gets

$$\begin{aligned} Z_5(\vec{x}) &= 1 + \frac{e^{-\chi}}{ar}\sum_{n,l=0}^{\infty}\sum_{m=0}^{n_5-1} e^{-(n+l)\chi} e^{i(n-l)(2\pi m/n_5 - \phi)} = \\ &= 1 + \frac{e^{-\chi}}{ar}\sum_{n,l=0}^{\infty} e^{-(n+l)\chi} e^{-i(n-l)\phi} \sum_{m=0}^{n_5-1} e^{i(n-l)(2\pi m/n_5)}. \end{aligned} \tag{3.20}$$

The finite sum over m vanishes, except for $n - l = sn_5$, with $s \in \mathbb{Z}$. In this case,

$$\sum_{m=0}^{n_5-1} e^{i(n-l)(2\pi m/k)} = \sum_{m=0}^{n_5-1} e^{i2\pi sm} = n_5.$$

The set of all $(n = l + sn_5, l)$ pairs with n, l positive integers and $s \in \mathbb{Z}$ can be considered as the union of the sets of pairs with $s = 0$, $s > 0$ and $s < 0$. The latter can be obtained from the second one with the transformation

$$(n = \alpha + sn_5, l = \alpha) \rightarrow (n = \beta - sn_5 = \alpha, l = \beta), \qquad \alpha, \beta, s > 0.$$

This map exchanges the two elements of each pair: $n+l = 2\alpha + sn_5$ in both cases, whereas the difference $n-l = sn_5 \to -sn_5$. Therefore,

$$\begin{aligned} Z_5(\vec{x}) &= 1 + \frac{n_5 e^{-\chi}}{ar} \sum_{n,l=0}^{\infty} e^{-(n+l)\chi} e^{-i(n-l)\phi} \\ &= 1 + \frac{n_5 e^{-\chi}}{ar} \Big(\sum_{\alpha=0}^{\infty} e^{-2\alpha\chi} + \sum_{\alpha=0}^{\infty} e^{-2\alpha\chi} \sum_{+,-}\sum_{s=1}^{\infty} e^{-sn_5\chi \pm isn_5\phi}\Big) \\ &= 1 + \frac{n_5}{2ar\sinh\chi} \Lambda_{n_5}(\chi,\phi), \end{aligned} \tag{3.21}$$

where we have used again (3.19). We have also defined

$$\begin{aligned} \Lambda_{n_5}(\chi,\phi) &\equiv 1 + \sum_{+,-}\sum_{s=1}^{\infty} e^{-sn_5\chi \pm isn_5\phi} \\ &= \frac{1}{2}\Big(1 + 2\sum_{s=1}^{\infty} e^{-sn_5\chi - isn_5\phi} + 1 + 2\sum_{s=1}^{\infty} e^{-sn_5\chi + isn_5\phi}\Big) \\ &= \frac{1}{2}\Big(\coth\big(\frac{n_5}{2}(\chi + i\phi)\big) + \coth\big(\frac{n_5}{2}(\chi - i\phi)\big)\Big), \end{aligned} \tag{3.22}$$

since the Dirichlet generalized series for the hyperbolic cotagent is

$$\coth(z) = \frac{1}{\tanh z} = 1 + 2\sum_{n=1}^{\infty} e^{-2nz}, \qquad z \in \mathbb{C}, \mathrm{Re}\, z > 0. \tag{3.23}$$

We can further simplify (3.22):

$$\begin{aligned} \Lambda_k(\chi,\phi) &= \frac{\cosh\big(\frac{n_5}{2}(\chi+i\phi)\big)\sinh\big(\frac{n_5}{2}(\chi-i\phi)\big) + \cosh\big(\frac{n_5}{2}(\chi-i\phi)\big)\sinh\big(\frac{n_5}{2}(\chi+i\phi)\big)}{2\sinh\big(\frac{n_5}{2}(\chi+i\phi)\big)\sinh\big(\frac{n_5}{2}(\chi-i\phi)\big)} \\ &= \frac{\sinh(n_5\chi)}{\cosh(n_5\chi) - \cos(n_5\phi)}, \end{aligned} \tag{3.24}$$

where in the last step we have employed the addition formula and the generalization of the Werner formulas for hyperbolic functions.
We can eventually parametrize the radii (r, R) introduced in (3.17) as

$$r = a\cosh\rho\sin\theta, \quad R = a\sinh\rho\cos\theta, \tag{3.25}$$

with $r \geq 0$, $0 \leq \theta \leq \pi/2$. This leads to

$$\begin{aligned} (2ar\sinh\chi)^2 &= (2ar\cosh\chi)^2 - (2ar)^2 \\ &= (R^2 + r^2 + a^2)^2 - (2ar)^2 \\ &= a^4\big[(\sinh^2\rho\cos^2\theta + \cosh^2\rho\sin^2\theta + 1)^2 - 4\cosh^2\rho\sin^2\theta\big] \\ &= a^4(\cosh^2\rho - \sin^2\theta)^2. \end{aligned}$$

where in the second step the definition (3.18) has been exploited. Then, working in the NS5-brane decoupling limit, we drop the one in the harmonic function $Z_5(\vec{x})$ and we end up with:

$$Z_5(\vec{x}) = \frac{n_5}{a^2(\cosh^2\rho - \sin^2\theta)}\Lambda_{n_5}(\chi,\phi), \qquad \Lambda_{n_5}(\chi,\phi) = \frac{\sinh(n_5\chi)}{\cosh(n_5\chi) - \cos(n_5\phi)}, \tag{3.26}$$

$$\begin{aligned} {ds_\perp}^2 &= Z_5(\vec{x}) dx^i dx_i \\ &= Z_5(\vec{x})(dr^2 + dR^2 + r^2 d\phi^2 + R^2 d\psi^2) \\ &= Z_5(\vec{x})\ a^2(\cosh^2\rho - \sin^2\theta)\big[d\rho^2 + d\theta^2 + \frac{\tanh^2\rho d\phi^2 + \tan^2\theta d\psi^2}{1 + \tan^2\theta\tanh^2\rho}\big] \quad . \\ &= n_5\Lambda_{n_5}(d\rho^2 + d\theta^2) + a^2 Z_5\Big(\sin^2\theta\cosh^2\rho\ d\phi^2 + \cos^2\theta\sinh^2\rho\ d\psi^2\Big). \end{aligned} \tag{3.27}$$

In the case in which the NS5-branes are smeared on the circle, i.e. $n_5 \to +\infty$,

$$\Lambda_{n_5}(\chi,\phi) = \frac{\sinh(n_5\chi)}{\cosh(n_5\chi) - \cos(n_5\phi)} \approx \frac{e^{n_5\chi}}{e^{n_5\chi}} = 1$$

and the transverse metric gets

$$ds_\perp{}^2 = n_5(d\rho^2 + d\theta^2) + \frac{n_5}{(\cosh^2\rho - \sin^2\theta)}\Big(\sin^2\theta\cosh^2\rho\; d\phi^2 + \cos^2\theta\sinh^2\rho\; d\psi^2\Big). \qquad (3.28)$$

3.2.2 The gauged WZW model

The metric for a continuous distribution of branes on a circle can be alternatively calculated making use of the Wess-Zumino-Witten model. The target space of the model is the 10+2-dimensional group manifold

$$\big(SL(2,\mathbb{R})\times SU(2)\big)\times\big(\mathbb{R}_t\times S^1_{\tilde y}\times\mathbb{T}^4\big). \qquad (3.29)$$

The last three factors refer to the worldvolume of the NS5-branes: the five space dimensions are wrapped on circles. For later purposes, one of them, $\tilde y$, is separately highlighted.
Let us focus on the first two factors

$$G = SL(2,\mathbb{R})\times SU(2) \cong SU(1,1)\times SU(2).$$

These describe the transverse space. However, they form a 6-dimensional manifold: in order to recover the 4-dimensional transverse space and its metric (3.28), we are required to identify the points on a 2-dimensional manifold subset of G, employing an extension of the WZW theory: the gauged Wess-Zumino-Witten model.

Let us package the elements of G as $\mathcal{G} = \mathrm{diag}(g', g) \in G$. Using (3.2), (3.7) and (3.11) then the action of the WZW model (3.9) for the target space G is given by

$$\mathcal{S}_{WZW} = \frac{k}{8\pi}\int_{S^2} d^2z\ \mathrm{Tr}[\mathcal{G}^{-1}\partial^\mu\mathcal{G}\mathcal{G}^{-1}\partial_\mu\mathcal{G}] - \frac{ik}{12\pi}\int_B d^3y\ \varepsilon_{\alpha\beta\gamma}\mathrm{Tr}[\mathcal{G}^{-1}\partial^\alpha\mathcal{G}\mathcal{G}^{-1}\partial^\beta\mathcal{G}\mathcal{G}^{-1}\partial^\gamma\mathcal{G}], \quad (3.30)$$

where k is the quantized level of the model. The elements of both factors of the target space can be expressed via the Euler angle parametrization:

$$g' = e^{i\frac{\sigma_3}{2}\tilde\Phi_L}e^{\frac{\sigma_1}{2}S}e^{i\frac{\sigma_3}{2}\tilde\Phi_R}, \qquad g = e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R}. \qquad (3.31)$$

Moreover, $\mathcal{G}$ is diagonal, thus we can treat the two group factors separately. Let us focus on $SU(2)$ and calculate the expression of (3.30) in the Euler angle parametrization. We have to multiplicate two or three factors of the form

$$g^{-1}\partial_\mu g = e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}\Big[i\frac{\sigma_3}{2}\partial_\mu\Phi_L e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R} + e^{i\frac{\sigma_3}{2}\Phi_L}i\frac{\sigma_1}{2}\partial_\mu\Omega e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R} + \\ + e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}i\frac{\sigma_3}{2}\partial_\mu\Phi_R e^{i\frac{\sigma_3}{2}\Phi_R}\Big]$$

and compute the traces. Most of these vanish, except those of the kind:

$$\mathrm{Tr}\Big[e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}i\frac{\sigma_3}{2}\partial_\mu\Phi_L e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_3}{2}\Phi_L}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_R}i\frac{\sigma_3}{2}\partial^\mu\Phi_L e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R}\Big] = \\ = -\frac{1}{4}\ \mathrm{Tr}[\mathbb{I}_2]\ \partial^\mu\Phi_L\partial_\mu\Phi_L = -\frac{1}{2}\partial^\mu\Phi_L\partial_\mu\Phi_L;$$

$$\mathrm{Tr}\Big[e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}i\frac{\sigma_3}{2}\partial_\mu\Phi_L e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}i\frac{\sigma_3}{2}\partial^\mu\Phi_R e^{i\frac{\sigma_3}{2}\Phi_R}\Big] = \\ = -\frac{1}{4}\ \mathrm{Tr}[e^{-i\frac{\sigma_1}{2}\Omega}\sigma_3 e^{i\frac{\sigma_1}{2}\Omega}\sigma_3]\ \partial^\mu\Phi_L\partial_\mu\Phi_R = \\ = -\frac{1}{4}\ \mathrm{Tr}\Big[\Big(\cos(\frac{\Omega}{2})\mathbb{I}_2 - i\sin(\frac{\Omega}{2})\sigma_1\Big)\sigma_3\Big(\cos(\frac{\Omega}{2})\mathbb{I}_2 + i\sin(\frac{\Omega}{2})\sigma_1\Big)\sigma_3\Big]\partial^\mu\Phi_L\partial_\mu\Phi_R = \\ = -\frac{1}{4}\mathrm{Tr}\Big[\cos^2(\frac{\Omega}{2})\mathbb{I}_2 - \sin^2(\frac{\Omega}{2})\mathbb{I}_2\Big]\partial^\mu\Phi_L\partial_\mu\Phi_R = -\frac{1}{2}\cos\Omega\ \partial^\mu\Phi_L\partial_\mu\Phi_R;$$

$$\mathrm{Tr}\Big[e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}i\frac{\sigma_3}{2}\partial^\alpha\Phi_L e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_3}{2}\Phi_L}i\frac{\sigma_1}{2}\partial^\beta\Omega_L e^{i\frac{\sigma_1}{2}\Omega}e^{i\frac{\sigma_3}{2}\Phi_R}\times$$
$$\times\, e^{-i\frac{\sigma_3}{2}\Phi_R}e^{-i\frac{\sigma_1}{2}\Omega}e^{-i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_3}{2}\Phi_L}e^{i\frac{\sigma_1}{2}\Omega}i\frac{\sigma_3}{2}\partial^\gamma\Phi_R e^{i\frac{\sigma_3}{2}\Phi_R}\Big] =$$
$$= -\frac{1}{8}\ \mathrm{Tr}[e^{-i\frac{\sigma_1}{2}\Omega}\sigma_2 e^{i\frac{\sigma_1}{2}\Omega}\sigma_3]\ \partial^\alpha\Phi_L\partial^\beta\Omega\partial^\gamma\Phi_R =$$
$$= -\frac{1}{8}\ \mathrm{Tr}\Big[\Big(\cos(\frac{\Omega}{2})\mathbb{I} - i\sin(\frac{\Omega}{2})\sigma_1\Big)\sigma_3\Big(\cos(\frac{\Omega}{2})\mathbb{I} + i\sin(\frac{\Omega}{2})\sigma_1\Big)\sigma_3\Big]\partial^\alpha\Phi_L\partial^\beta\Omega\partial^\gamma\Phi_R =$$
$$= -\frac{1}{8}\mathrm{Tr}\Big[2\cos(\frac{\Omega}{2})\sin(\frac{\Omega}{2})\mathbb{I}\Big]\partial^\alpha\Phi_L\partial^\beta\Omega\partial^\gamma\Phi_R = -\frac{1}{4}\sin\Omega\ \partial^\alpha\Phi_L\partial^\beta\Omega\partial^\gamma\Phi_R.$$

As a consequence, for the $SU(2)$ factor we get that

$$\frac{k}{8\pi}\int_{S^2} d^2z\ \mathrm{Tr}[g^{-1}\partial^\mu g g^{-1}\partial_\mu g] = \frac{k}{8\pi}\int_{S^2} d^2z\ \big[-\frac{1}{2}\partial^\mu\Phi_L\partial_\mu\Phi_L - \frac{1}{2}\partial^\mu\Omega\partial_\mu\Omega - \frac{1}{2}\partial^\mu\Phi_R\partial_\mu\Phi_R - \cos\Omega\ \partial^\mu\Phi_L\partial_\mu\Phi_R\big]$$

and

$$\begin{aligned}-\frac{ik}{12\pi}\int_B d^3y\ \varepsilon_{\alpha\beta\gamma}\mathrm{Tr}[g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g] &= \frac{6ik}{12\pi}\frac{1}{4}\int_B d^3y\ \varepsilon_{\alpha\beta\gamma}\sin\Omega\ \partial^\alpha\Phi_L\partial^\beta\Omega\partial^\gamma\Phi_R =\\ &= \frac{ik}{8\pi}\int_B d^3y\ \varepsilon_{\alpha\beta\gamma}\partial^\gamma(\cos\Omega\ \partial^\alpha\Phi_L\partial^\beta\Phi_R) =\\ &= \frac{ik}{8\pi}\int_{S^2} d^2z\ \varepsilon_{\alpha\beta}\partial^\alpha\Phi_L\partial^\beta\Phi_R\cos\Omega.\end{aligned}$$

where in the last line we have exploited the Stokes' theorem. Summing the two contributions up and expressing the full result in the coordinates $(z,\bar{z})$ on the Riemann sphere (remembering $g^{z\bar{z}} = 2 = g^{\bar{z}z}$, $g^{zz} = 0 = g^{\bar{z}\bar{z}}$ and $\varepsilon_{z\bar{z}} = \frac{i}{2}$), we get

$$\begin{aligned}\mathcal{S}_{WZW}[g] &= \frac{k}{8\pi}\int_{S^2} d^2z\big[-\frac{1}{2}\partial^\mu\Phi_L\partial_\mu\Phi_L - \frac{1}{2}\partial^\mu\Omega\partial_\mu\Omega - \frac{1}{2}\partial^\mu\Phi_R\partial_\mu\Phi_R - \cos\Omega\ \big(\partial^\mu\Phi_L\partial_\mu\Phi_R - i\varepsilon_{\alpha\beta}\partial^\alpha\Phi_L\partial^\beta\Phi_R\big)\big]\\ &= -\frac{k}{4\pi}\int_{S^2} d^2z\big[\partial\Phi_L\bar{\partial}\Phi_L + \partial\Omega\bar{\partial}\Omega + \partial\Phi_R\bar{\partial}\Phi_R + 2\cos\Omega\bar{\partial}\Phi_L\partial\Phi_R\big].\end{aligned} \tag{3.32}$$

We choose to change the parametrization in (3.31) in the following way:

$$g' = e^{i(\tau+\sigma)\frac{\sigma_3}{2}}e^{\rho\sigma_1}e^{i(\tau-\sigma)\frac{\sigma_3}{2}}, \qquad g = e^{i(\psi+\phi)\frac{\sigma_3}{2}}e^{i\theta\sigma_1}e^{i(\psi-\phi)\frac{\sigma_3}{2}}. \tag{3.33}$$

Accordingly,

$$\Phi_L = \psi+\phi, \qquad \Omega = 2\theta, \qquad \Phi_R = \psi-\phi$$

and then

$$\begin{aligned}\mathcal{S}_{WZW}[g] &= -\frac{k}{4\pi}\int_{S^2} d^2z\big[(\partial\psi+\partial\phi)(\bar{\partial}\psi+\bar{\partial}\phi) + 4\partial\theta\bar{\partial}\theta + (\partial\psi-\partial\phi)(\bar{\partial}\psi-\bar{\partial}\phi)+\\ &\qquad + 2\cos(2\theta)(\bar{\partial}\psi+\bar{\partial}\phi)(\partial\psi-\partial\phi) =\\ &= -\frac{k}{4\pi}\int_{S^2} d^2z\big[4\partial\theta\bar{\partial}\theta + 2(1-\cos(2\theta))\partial\phi\bar{\partial}\phi + 2(1+\cos(2\theta))\partial\psi\bar{\partial}\psi\\ &\qquad - 2\cos(2\theta)(\partial\phi\bar{\partial}\psi - \partial\psi\bar{\partial}\phi)\big] =\\ &= -\frac{k}{\pi}\int_{S^2} d^2z\big[(\partial\theta\bar{\partial}\theta + \sin^2\theta\partial\phi\bar{\partial}\phi + \cos^2\theta\partial\psi\bar{\partial}\psi) - \cos^2\theta(\partial\phi\bar{\partial}\psi - \partial\psi\bar{\partial}\phi)\big],\end{aligned} \tag{3.34}$$

where we have neglected a constant contribution in the last term which can be identified as the component of a 2-form potential.
The same procedure for the $SU(1,1)$ factor leads to the full WZW action

$$\begin{aligned}\mathcal{S}_{WZW} &= \mathcal{S}_{WZW}[g'] - \mathcal{S}_{WZW}[g] =\\ &= \frac{k}{\pi}\int_{S^2} d^2z\big[(\partial\rho\bar{\partial}\rho + \sinh^2\rho\ \partial\sigma\bar{\partial}\sigma - \cosh^2\rho\ \partial\tau\bar{\partial}\tau) - \cosh^2\rho(\partial\tau\bar{\partial}\sigma - \partial\sigma\bar{\partial}\tau)+\\ &\qquad + (\partial\theta\bar{\partial}\theta + \sin^2\theta\ \partial\phi\bar{\partial}\phi + \cos^2\theta\ \partial\psi\bar{\partial}\psi) - \cos^2\theta(\partial\phi\bar{\partial}\psi - \partial\psi\bar{\partial}\phi)\big]\big],\end{aligned} \tag{3.35}$$

where the sign between the contributions of the two group factors relies on the requirement of one timelike (τ) and five spacelike directions.

Let us notice that the action (3.35) is invariant under the symmetries associated to four Killing vectors: ∂_τ, ∂_σ, ∂_ψ and ∂_ϕ. We can consider the combinations

$$\xi_1 = \big(\partial_\tau - \partial_\phi\big) + \big(\partial_\sigma - \partial_\psi\big), \qquad \xi_2 = \big(\partial_\tau - \partial_\phi\big) - \big(\partial_\sigma - \partial_\psi\big),$$

which are Killing vectors as well. These generate the subgroup $U(1)_L \times U(1)_R$ of the target space and when identifying the points sitting along these directions, we really obtain a four-dimensional transverse space. In practice, we identify all points on G linked by the transformation

$$(g, g') \to \big(e^{\frac{i}{2}\lambda\sigma_3} g e^{\frac{i}{2}\xi\sigma_3}, e^{-\frac{i}{2}\lambda\sigma_3} g' e^{\frac{i}{2}\xi\sigma_3}\big). \tag{3.36}$$

Due to the gauging of the Lie group, we can define two $U(1)$-gauge fields, denoted as $\mathcal{A}$ and $\bar{\mathcal{A}}$, whose gauge transformation is

$$\mathcal{A} \to \mathcal{A} + \partial\xi \qquad \bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\lambda. \tag{3.37}$$

These couple to the conserved currents

$$\begin{aligned} J^{sl} &= -ik\,\mathrm{Tr}[T_3 \partial g g^{-1}] = k\big(\cosh^2\rho\;\partial\tau - \sinh^2\rho\;\partial\sigma\big),\\ \bar{J}^{sl} &= -ik\,\mathrm{Tr}[T_3 g^{-1}\bar{\partial} g] = k\big(\cosh^2\rho\;\bar{\partial}\tau + \sinh^2\rho\;\partial\sigma\big),\\ J^{su} &= -ik\,\mathrm{Tr}[T_3 \partial g' g'^{-1}] = k(\cos^2\theta\;\partial\psi + \sin^2\theta\;\partial\phi),\\ \bar{J}^{su} &= -ik\,\mathrm{Tr}[T_3 g'^{-1}\bar{\partial} g'] = k\big(\cos^2\theta\;\bar{\partial}\psi - \sin^2\theta\;\bar{\partial}\phi\big), \end{aligned} \tag{3.38}$$

defined according to (3.12). In all definitions, $T_3 = \sigma_3/2$, which is the generator of both subgroups $U(1)_L$ and $U(1)_R$ in (3.36). The currents (3.38) are calculated exploiting the very analogous reasoning that led to (3.32). Finally, the gauge action reads

$$\mathcal{S}_{\text{gauge}} = \frac{1}{\pi}\int_{S^2} d^2z\; J_\mu \mathcal{A}^\mu = \frac{1}{\pi}\int_{S^2} d^2z\;\big[\mathcal{A}(\bar{J}^{sl} - \bar{J}^{su}) + \bar{\mathcal{A}}(J^{sl} + J^{su}) + B(\mathcal{A}, \bar{\mathcal{A}})\big], \tag{3.39}$$

where the "-" sign for $\bar{J}^{su}$ is due to the different sign between the two groups, in the exponent for the $U(1)_L$ transformation in (3.36). The term $B(\mathcal{A}, \bar{\mathcal{A}})$ is added to ensure the actual gauge invariance.
The total action of the gauged WZW model is then

$$\mathcal{S}_{\text{tot}} = \mathcal{S}_{WZW} + \mathcal{S}_{\text{gauge}}, \tag{3.40}$$

where

$$\begin{aligned} \mathcal{S}_{\text{gauge}} &= \mathcal{S}_{\text{gauge}}[g'] + \mathcal{S}_{\text{gauge}}[g] = \\ &= \frac{k}{\pi}\int_{S^2} d^2z\big[\mathcal{A}\big(\cosh^2\rho\;\bar{\partial}\tau + \sinh^2\rho\;\bar{\partial}\sigma\big) + \big(\cosh^2\rho\;\partial\tau + \sinh^2\rho\;\partial\sigma\big)\bar{\mathcal{A}} - \frac{\cosh 2\rho}{2}\mathcal{A}\bar{\mathcal{A}} + \\ &\quad - \mathcal{A}\big(\cos^2\theta\;\bar{\partial}\psi - \sin^2\theta\;\bar{\partial}\phi\big) + \big(\cos^2\theta\;\partial\psi + \sin^2\theta\;\partial\phi\big)\bar{\mathcal{A}} - \frac{\cos 2\theta}{2}\mathcal{A}\bar{\mathcal{A}}\big]. \end{aligned} \tag{3.41}$$

In order to show this action is honestly gauge invariant, we choose to impose $\lambda = \alpha + \beta$, $\xi = \alpha - \beta$, with $\alpha = \alpha(z, \bar{z})$ and $\beta = \beta(z, \bar{z})$. Accordingly, the gauge transformations of the Euler angles of the parametrization (3.33) and of the gauge fields become

$$\begin{gathered} \psi \to \psi - \beta, \qquad \phi \to \phi - \alpha, \qquad \tau \to \tau + \alpha, \qquad \sigma \to \sigma + \beta, \\ \mathcal{A} \to \mathcal{A} + \partial(\alpha - \beta) \quad \bar{\mathcal{A}} \to \bar{\mathcal{A}} + \partial(\alpha + \beta) \end{gathered} \tag{3.42}$$

and we can calculate the variation of the total action under this transformation, for each subgroup. For this proof, we employ the second expression of the WZW action given in (3.34) and not the last one which was used to write (3.35). Thus

$$
\begin{aligned}
\delta\mathcal{S}_{tot}[g] &= -\delta\mathcal{S}_{WZW}[g] + \delta\mathcal{S}_{gauge}[g] = \\
&= \frac{k}{\pi}\int_{S^2} d^2z\big[\sin^2\theta\,(-\partial\alpha\bar\partial\phi - \bar\partial\alpha\partial\phi + \partial\alpha\bar\partial\alpha) + \cos^2\theta\,(-\partial\beta\bar\partial\psi - \partial\psi\bar\partial\beta + \partial\beta\bar\partial\beta) + \\
&- \frac{1}{2}(\cos^2\theta - \sin^2\theta)(-\partial\phi\bar\partial\beta - \partial\alpha\bar\partial\psi + \partial\alpha\bar\partial\beta + \partial\beta\bar\partial\phi + \partial\psi\bar\partial\alpha - \partial\beta\bar\partial\alpha) + \\
&- \mathcal{A}(-\cos^2\theta\bar\partial\beta + \sin^2\theta\bar\partial\alpha) - (\cos^2\theta(\bar\partial\psi - \bar\partial\beta) - \sin^2\theta(\bar\partial\phi - \bar\partial\alpha))(\partial\alpha - \partial\beta) + \\
&- \cos^2\theta\;\partial\beta\bar{\mathcal{A}} - \partial\alpha\sin^2\theta\bar{\mathcal{A}} + (\bar\partial\alpha + \bar\partial\beta)(\cos^2\theta(\partial\psi - \partial\beta) + \sin^2\theta(\partial\phi - \partial\alpha)) + \\
&- \frac{1}{2}(\cos^2\theta - \sin^2\theta)(\partial\alpha - \partial\beta)\bar{\mathcal{A}} - \frac{1}{2}(\cos^2\theta - \sin^2\theta)\mathcal{A}(\bar\partial\alpha + \bar\partial\beta) + \\
&- \frac{1}{2}(\cos^2\theta - \sin^2\theta)(\partial\alpha - \partial\beta)(\bar\partial\alpha + \bar\partial\beta)\big] = \\
&= \frac{k}{\pi}\int_{S^2} d^2z\Big[\frac{\sin^2\theta}{2}\big(-\partial\alpha\bar\partial\alpha - \partial\phi\bar\partial\beta - \partial\beta\bar\partial\phi + \partial\phi\bar\partial\beta - \partial\alpha\bar\partial\psi + \partial\psi\bar\partial\alpha - \partial\beta\bar\partial\beta\big) + \\
&+ \frac{\cos^2\theta}{2}\big(-\partial\alpha\bar\partial\alpha - \partial\beta\bar\partial\beta - \partial\alpha\bar\partial\psi + \partial\psi\bar\partial\alpha - \partial\beta\bar\partial\phi + \partial\phi\bar\partial\beta\big) + \frac{\mathcal{A}}{2}\big(\bar\partial\beta - \bar\partial\alpha\big) + \frac{\bar{\mathcal{A}}}{2}\big(-\partial\beta - \partial\alpha\big)\Big] = \\
&= \frac{k}{2\pi}\int_{S^2} d^2z\Big[\big(-\partial\alpha\bar\partial\alpha - \partial\beta\bar\partial\beta - \partial\alpha\bar\partial\psi + \partial\psi\bar\partial\alpha - \partial\beta\bar\partial\phi + \partial\phi\bar\partial\beta\big) + \mathcal{A}\big(\bar\partial\beta - \bar\partial\alpha\big) + \bar{\mathcal{A}}\big(-\partial\beta - \partial\alpha\big)\Big] = \\
&= \frac{k}{2\pi}\int_{S^2} d^2z\Big[-\partial\alpha\bar\partial\alpha - \partial\beta\bar\partial\beta + \mathcal{A}\big(\bar\partial\beta - \bar\partial\alpha\big) + \bar{\mathcal{A}}\big(-\partial\beta - \partial\alpha\big)\Big],
\end{aligned}
$$

where in the last step four terms have netted out to zero since they can be pairwise identified after integrating by parts.
Following the same steps for the other subgroup we obtain

$$
\begin{aligned}
\delta\mathcal{S}_{tot}[g'] &= \delta\mathcal{S}_{WZW}[g'] + \delta\mathcal{S}_{gauge}[g'] = \\
&= \frac{k}{\pi}\int_{S^2} d^2z\big[\sinh^2\rho\,(\partial\beta\bar\partial\sigma + \partial\sigma\bar\partial\beta + \partial\beta\bar\partial\beta) - \cosh^2\rho\,(\partial\alpha\bar\partial\tau + \partial\tau\bar\partial\alpha + \partial\alpha\bar\partial\alpha) + \\
&- \frac{1}{2}(\cosh^2\rho + \sinh^2\rho)(\partial\tau\bar\partial\beta + \partial\alpha\bar\partial\beta + \partial\alpha\bar\partial\sigma - \partial\beta\bar\partial\tau - \partial\sigma\bar\partial\alpha - \partial\beta\bar\partial\alpha) + \\
&+ \mathcal{A}(\cosh^2\rho\;\bar\partial\alpha + \sinh^2\rho\;\bar\partial\beta) + (\cosh^2\rho\;(\bar\partial\tau + \bar\partial\alpha) + \sinh^2\rho(\bar\partial\sigma + \bar\partial\beta))(\partial\alpha - \partial\beta) + \\
&+ (\cosh^2\rho\;\partial\alpha - \sinh^2\rho\;\partial\beta)\bar{\mathcal{A}} + (\bar\partial\alpha + \bar\partial\beta)(\cosh^2\rho(\partial\tau + \partial\alpha) - \sinh^2\rho(\partial\sigma + \partial\beta)) + \\
&- \frac{1}{2}(\cosh^2\rho + \sinh^2\rho)(\partial\alpha - \partial\beta)\bar{\mathcal{A}} - \frac{1}{2}(\cosh^2\rho + \sinh^2\rho)\mathcal{A}(\bar\partial\alpha + \bar\partial\beta) + \\
&- \frac{1}{2}(\cosh^2\rho + \sinh^2\rho)(\partial\alpha - \partial\beta)(\bar\partial\alpha + \bar\partial\beta)\big] = \\
&= \frac{k}{\pi}\int_{S^2} d^2z\Big[\frac{\sinh^2\rho}{2}\big(\partial\beta\bar\partial\tau - \partial\tau\bar\partial\beta + \partial\alpha\bar\partial\sigma - \partial\sigma\bar\partial\alpha - \partial\alpha\bar\partial\alpha - \partial\beta\bar\partial\beta\big) + \\
&- \frac{\cosh^2\rho}{2}\big(\partial\beta\bar\partial\tau - \partial\tau\bar\partial\beta + \partial\alpha\bar\partial\sigma - \partial\sigma\bar\partial\alpha - \partial\alpha\bar\partial\alpha - \partial\beta\bar\partial\beta\big) + \frac{\mathcal{A}}{2}\big(\bar\partial\alpha - \bar\partial\beta\big) + \frac{\bar{\mathcal{A}}}{2}\big(\partial\beta + \partial\alpha\big)\Big] = \\
&= \frac{k}{2\pi}\int_{S^2} d^2z\Big[\partial\alpha\bar\partial\alpha + \partial\beta\bar\partial\beta + \mathcal{A}\big(\bar\partial\alpha - \bar\partial\beta\big) + \bar{\mathcal{A}}\big(\partial\beta + \partial\alpha\big)\Big],
\end{aligned}
$$

which exactly cancels with $\delta\mathcal{S}_{tot}[g]$: (3.40) is thus an actual gauge invariant action.

Finally, we can integrate the gauge fields out, performing the standard top-down procedure of effective field theories. The equations of motion of each component of the gauge field can be found varying (3.41) with respect to the other component:

$$
\begin{aligned}
\mathcal{A} &= \frac{2}{\cosh 2\rho + \cos 2\theta}\left[\cosh^2\rho\ \partial\tau - \sinh^2\rho\ \partial\sigma + \cos^2\theta\ \partial\psi + \sin^2\theta\ \partial\phi\right],\\
\bar{\mathcal{A}} &= \frac{2}{\cosh 2\rho + \cos 2\theta}\left[\cosh^2\rho\ \bar\partial\tau + \sinh^2\rho\ \bar\partial\sigma - \cos^2\theta\ \bar\partial\psi + \sin^2\theta\ \bar\partial\phi\right].
\end{aligned}
\tag{3.43}
$$

Substituting these expressions in (3.41) and working in the gauge such that $\tau = 0 = \sigma$ we get the effective action

$$
\begin{aligned}
\mathcal{S} = \frac{n_5}{\pi}\int_{S^2} d^2z\Big[&\partial\rho\bar\partial\rho + \partial\theta\bar\partial\theta + \sin^2\theta\ \partial\phi\bar\partial\phi + \cos^2\theta\ \partial\psi\bar\partial\psi - \cos^2\theta\Big(\partial\phi\bar\partial\psi - \partial\psi\bar\partial\phi\Big) + \\
&+ \frac{2}{\cosh 2\rho + \cos 2\theta}\Big(\cos^2\theta\ \partial\psi + \sin^2\theta\ \partial\phi)(\sin^2\theta\ \bar\partial\phi - \cos^2\theta\ \bar\partial\psi\Big)\Big],
\end{aligned}
$$

where we have fixed the level of the model $k = n_5$. Since

$$
\cosh 2\rho = 2\cosh^2\rho - 1, \qquad \cos 2\theta = 1 - 2\sin^2\theta,
$$

the action eventally turns out to be

$$
\begin{aligned}
\mathcal{S} =& \frac{n_5}{\pi}\int_{S^2} d^2z\Big[\partial\rho\bar\partial\rho + \partial\theta\bar\partial\theta + \sin^2\theta\Big(1 + \frac{\sin^2\theta}{\cosh^2\rho - \sin^2\theta}\Big)\partial\phi\bar\partial\phi + \cos^2\theta(1 - \frac{\cos^2\theta}{\cosh^2\rho - \sin^2\theta})\partial\psi\bar\partial\psi\\
&+ \cos^2\theta\Big(1 + \frac{\sin^2\theta}{\cosh^2\rho - \sin^2\theta}\Big)\Big(\partial\psi\bar\partial\phi - \partial\phi\bar\partial\psi\Big)\Big] =\\
=& \frac{n_5}{\pi}\int_{S^2} d^2z\Big[\partial\rho\bar\partial\rho + \partial\theta\bar\partial\theta + \frac{n_5}{\Sigma}\Big(\sin^2\theta\cosh^2\rho\ \partial\phi\bar\partial\phi + \cos^2\theta\sinh^2\rho\ \partial\psi\bar\partial\psi\Big)\\
&+ \frac{n_5\cos^2\theta\cosh^2\rho}{\Sigma}\Big(\partial\psi\bar\partial\phi - \partial\phi\bar\partial\psi\Big)\Big],
\end{aligned}
\tag{3.44}
$$

where we have defined

$$
\Sigma = n_5(\cosh^2\rho - \sin^2\theta). \tag{3.45}
$$

The metric appearing in (3.44) exactly coincides with (3.28), which was got in the supergravity framework.

3.3 The round supertubes

In order to get a two-charge black hole, we should add another charge to the system of NS5-branes on a circle. This is accomplished once a momentum is added along the compact direction $\tilde{y}$. This momentum arises together with an angular momentum, after a boost-like transformation. These systems of rotating branes are called *round supertubes*. This way, we are dealing with a NS5-P black hole, which under a T-duality along $\tilde{y}$ gets a NS5-NS1 black hole, exactly the perturbation profile giving the particular microstate subject of attention in 2.4.1.

3.3.1 Supergravity solutions

Let us call $R_{\tilde{y}}$ the radius of the compactification of $\tilde{y}$. We can implement twisted boundary conditions along this compact direction, for the transverse dimensions:

$$
\vec{x}_m(v + 2\pi R_{\tilde{y}}) = \vec{x}_{\sigma(m)}(v).
$$

Here σ represents a cyclic permutation of the centres where branes sit, labelled with m: it consists of a shift by k positions around the circle in (x_1, x_2) as $\tilde{y}$ completes a turn around $S^1_{\tilde{y}}$. Then, in short, the branes wrap around the $(\tilde{y}, \phi)$ torus. In order that a brane reaches its starting mth place after a certain number of full laps of $\tilde{y}$ (i.e. completes a path along both circles of the torus), a shift of $lcm(k, n_5)$ positions is needed. In the meanwhile, the branes have described $lcm(k, n_5)/k$ rotations around $S^1_{\tilde{y}}$. This is exactly the number of times that a single brane crosses a surface at fixed $\tilde{y}$. The number of independent strands making up the supertube is then $n_5 k/lcm(n_5, k) = gcd(n_5, k)$. Trivially, when k and n_5 are coprime, there is a single strand wrapping n_5 times around the circle. Figure 3.1 shows an example of supertube.

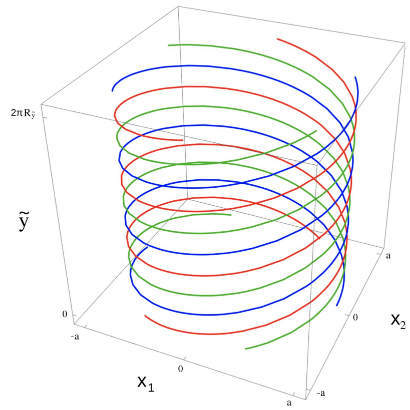

Figure 3.1: A supertube generated by $n_5 = 6$ NS5-branes in the case $k = 9$. The number of independent strands is $gcd(n_5, k) = 3$.

The profile of supertubes in the transverse space is described by

$$x^1_m + ix^2_m = ae^{i\phi_m}, \qquad \phi_m = \frac{k}{n_5}\frac{t+\tilde{y}}{R_{\tilde{y}}} + \frac{2\pi m}{n_5} \tag{3.46}$$

and of course $x^3_m = 0 = x^4_m$. The supergravity solution depends on the harmonic function Z_5 defined in (3.16), with the difference that the expression for the angle ϕ is that given in (3.46). The upshot is

$$Z_5 = 1 + \frac{n_5^2}{a^2\Sigma}\Lambda_{n_5}, \qquad Z_p = \frac{k^2}{n_5 R_{\tilde{y}}\Sigma}\Lambda_{n_5}.$$

Here Σ is given by (3.45) and the correct expression for Λ_{n_5} is (3.24) with the substitution $\phi \to \phi + k(t+\tilde{y})/(R_{\tilde{y}} n_5)$. We work in the smeared limit $n_5 \to +\infty$, such that anyway, $\Lambda_{n_5} \approx 1$. The supergravity solution turns out to be

$$\begin{aligned} ds^2 &= -dudv + n_5\Big(d\rho^2 + d\theta^2\Big) + \frac{n_5^2}{\Sigma}\Big[\sin^2\theta\cosh^2\rho d\phi^2 + \cos^2\theta\sinh^2\rho d\psi^2 + \\ &\quad + \frac{2k}{n_5 R_{\tilde{y}}}\sin^2\theta dv d\phi + \frac{k^2}{n_5^2 R_{\tilde{y}}}dv^2\Big] + dz_a dz^a, \\ e^{2\Phi} &= \frac{n_5^2 g_s^2}{a^2\Sigma}, \quad B_{\psi\phi} = \frac{n_5^2\cos^2\theta\cosh^2\rho}{\Sigma}, \quad B_{\psi v} = \frac{n_5 k\cos^2\theta}{R_{\tilde{y}}\Sigma}, \end{aligned} \tag{3.47}$$

which is nothing but the metric (2.30) in the duality frame where the system is NS5-P.

3.3.2 WZW description

The solution (3.47) can be obtained availing of the gauged WZW model. Let us work in the case such that $\tilde{y}$ is opened up and is not compact. It suffices to extend the gauge action (3.41) with a second term representing the spiralling of the fivebranes around the $\tilde{y} - \phi$ cylinder. The $(t, \tilde{y})$ part of the action then reads

$$\mathcal{S}_{t\tilde{y}} = \frac{1}{2\pi}\int d^2z\Big[-(\partial u\bar{\partial}v + \partial v\bar{\partial}u) + 2\alpha(\mathcal{A}\bar{\partial}v + \bar{\mathcal{A}}\partial v)\Big]. \tag{3.48}$$

Let us notice that in the coordinates defined in (3.17) and (3.25), the circle where branes sit is identified imposing $\rho = 0, \theta = \pi/2$. In this case the action (3.41) is nomore quadratic, but linear in the gauge fields, which can be now regarded as Lagrange multipliers defining a constraint. When also adding (3.48) to the gauge action, this constraint reads

$$n_5 d\phi + \alpha dv = 0, \tag{3.49}$$

which is exactly what we need to ensure a shift of branes in the ϕ coordinate.
As well as in the static configuration of branes, we can integrate out the gauge fields, to get

$$\begin{aligned}\mathcal{S} = &\frac{1}{2\pi}\int d^2z\Big[-\partial u\bar{\partial}v - \partial v\bar{\partial}u\Big] + \frac{n_5}{\pi}\int d^2z\,\Big[\partial\rho\bar{\partial}\rho + \partial\theta\bar{\partial}\theta + \\ &+ \frac{n_5}{\Sigma}\big(\sin^2\theta\cosh^2\rho\,\partial\phi\bar{\partial}\phi + \cos^2\theta\sinh^2\rho\,\partial\psi\bar{\partial}\psi\big) + \frac{n_5\cos^2\theta\cosh^2\rho}{\Sigma}\big(\partial\psi\bar{\partial}\phi - \partial\phi\bar{\partial}\psi\big) + \\ &+ \frac{\alpha}{\Sigma}\big(\sin^2\theta(\partial\phi\bar{\partial}v + \partial v\bar{\partial}\phi) + \cos^2\theta(\partial\psi\bar{\partial}v - \partial v\bar{\partial}\psi)\big) + \frac{\alpha^2}{n_5\Sigma}\partial v\bar{\partial}v\Big].\end{aligned} \tag{3.50}$$

The parameter Σ coincides with the coefficient of the quadratic term in $\mathcal{A}\bar{\mathcal{A}}$ and is again

$$\Sigma = n_5(\cosh^2\rho - \sin^2\theta).$$

When $\tilde{y}$ is again compact, we can choose

$$\alpha = \frac{k}{R_{\tilde{y}}} \tag{3.51}$$

and the dynamics (3.46) is obtained. The action (3.50), then, gives rise to a metric and a Kalb-Ramond field coinciding with (3.47).

The Wess-Zumino-Witten model also allows to recover the supergravity solution arising from (2.31) after a S-duality is performed and the NS5-NS1 system is then obtained. To carry out this task, we just need a T-duality along $\tilde{y}$, so that $R_{\tilde{y}} \to R_y = 1/R_{\tilde{y}}$ and the NS5-P supertube becomes a NS5-NS1 supertube. In this second instance, the (t, y) action reads

$$\mathcal{S}_{ty} = \frac{1}{2\pi}\int d^2z\Big[-\big(\partial u\bar{\partial}v + \partial v\bar{\partial}u\big) + 2\alpha\big(\mathcal{A}\bar{\partial}u + \bar{\mathcal{A}}\partial v\big) - 2\alpha^2\mathcal{A}\bar{\mathcal{A}}\Big]. \tag{3.52}$$

The coefficient of the quadratic piece in the gauge fields now changes and gets

$$\Sigma = \alpha^2 + n_5\big(\cosh^2\rho - \sin^2\theta\big).$$

We can again integrate out the gauge fields, finally obtaining the action

$$\begin{aligned}\mathcal{S} = &\frac{1}{2\pi}\int d^2z\Big[-\partial u\bar{\partial}v - \partial v\bar{\partial}u\Big] + \frac{n_5}{\pi}\int d^2z\,\Big[\partial\rho\bar{\partial}\rho + \partial\theta\bar{\partial}\theta + \\ &+ \frac{\cos^2\theta(\alpha^2 + n_5\cosh^2\rho)}{\Sigma}\big(\partial\psi\bar{\partial}\phi - \partial\phi\bar{\partial}\psi\big) + \frac{\sin^2\theta}{\Sigma}(\alpha^2 + n_5\cosh^2\rho)\partial\phi\bar{\partial}\phi + \\ &+ \frac{\cos^2\theta}{\Sigma}(\alpha^2 + n_5\sinh^2\rho)\partial\psi\bar{\partial}\psi + \frac{\alpha\cos^2\theta}{\Sigma}\big(\partial\psi\bar{\partial}u - \partial v\bar{\partial}\psi\big) + \frac{\alpha\sin^2\theta}{\Sigma}(\partial\phi\bar{\partial}u + \partial v\bar{\partial}\phi) + \frac{\alpha^2}{n_5\Sigma}\partial v\bar{\partial}u\Big],\end{aligned} \tag{3.53}$$

which can be identified with the NS5-branes decoupling limit of the NS5-NS1 supergravity solution.

The three instances of gauge actions (3.41), (3.41) + (3.48) and (3.41) + (3.52) can be regarded as three different specific realizations of the most general $U(1)_L \times U(1)_R$ gauge theory obtained making use of the currents

$$\begin{aligned}U(1)_L &: \mathcal{I} = l_1 J^{sl} + l_2 J^{su} + l_3\partial t + l_4\partial y,\\ U(1)_R &: \bar{\mathcal{I}} = r_1\bar{J}^{sl} + r_2\bar{J}^{su} + r_3\bar{\partial}t + r_4\bar{\partial}y.\end{aligned} \tag{3.54}$$

Moreover, we impose that these currents are null, i. e.

$$\frac{n_5}{2}\left(-l_1^2+l_2^2\right)-\frac{1}{2}l_3^2+\frac{1}{2}l_4^2=0, \qquad \frac{n_5}{2}\left(-r_1^2+r_2^2\right)-\frac{1}{2}r_3^2+\frac{1}{2}r_4^2=0. \tag{3.55}$$

as we explain in (4.52).

The gauge action to be added to the Wess-Zumino-Witten one is indeed

$$\mathcal{S}_{\text{gauge}}=\frac{1}{\pi}\int_{S^2} d^2z\ [\mathcal{A}\bar{\mathcal{I}}+\bar{\mathcal{A}}\mathcal{I}-\Sigma\mathcal{A}\bar{\mathcal{A}}], \tag{3.56}$$

with

$$\Sigma=\frac{1}{2}\left[n_5\left(l_1r_1\cosh 2\rho-l_2r_2\cos 2\theta\right)+l_3r_3-l_4r_4\right].$$

Different choices of the parameters r_i and l_i satisfying the null condition (3.55), lead to different configurations of supertubes, as illustrated in [30–32]. For example, the static NS5-branes are obtained for

$$l_1=l_2=1, \quad l_3=l_4=0, \quad r_1=-r_2=1, \quad r_3=-r_4=0, \tag{3.57}$$

the rounding branes in the NS5-P duality frame correspond to

$$l_1=l_2=1, \quad l_3=-l_4=-\frac{k}{R_{\tilde{y}}}, \quad r_1=-r_2=1, \quad r_3=-r_4=-\frac{k}{R_{\tilde{y}}}, \tag{3.58}$$

whereas the rounding branes in the NS5-NS1 duality frame, arise from

$$l_1=l_2=1, \quad l_3=l_4=-kR_y, \quad r_1=-r_2=1, \quad r_3=-r_4=-kR_y, \tag{3.59}$$

where $R_y=\frac{1}{R_{\tilde{y}}}$ is the radius of the compact circle after the T-duality ($\alpha'=1$). The most general choice consistent with (3.55), allows to recover the already known supergravity solutions for the three-charge black holes, whose horizon is macroscopic and not of the order of the string length.

Chapter 4

The spectrum of the worldsheet theory

Wess-Zumino-Witten models are conformal field theories also at the quantum level and we can determine their spectrum of states. The purpose of this Chapter is to determine the vertex operators giving rise to the states in the worldsheet theory so far developed for the particular studied microstate.

4.1 A quantum treatment of Wess-Zumino-Witten models

4.1.1 The current algebra

For the Wess-Zumino-Witten models, (3.12) defines the left and right-moving currents

$$J = J^a T^a = -k\partial g g^{-1} \qquad \bar{J} = k g^{-1}\bar{\partial} g. \tag{4.1}$$

where T^a, $a = 1, ..., \dim \mathfrak{g}$, are the generators of the Lie algebra satisfying the algebra

$$[T^a, T^b] = i f^{abc} T^c. \tag{4.2}$$

We see that, since k is dimensionless, J and $\bar{J}$ have conformal weight $(1,0)$ and $(0,1)$.
Let us study the algebra for the current components. The fields $g_L(z)$ and $g_R(\bar{z})$ appearing in the local gauge transformations (3.13)

$$G(z)\times G(\bar{z}) : g(z,\bar{z}) \to g_L(z) g(z,\bar{z}) g_R^{-1}(\bar{z}) \tag{4.3}$$

are valued in the Lie group G. Therefore, at the infinitesimal level,

$$g_L = 1 + w^a T^a, \qquad g_R(\bar{z}) = 1 + \bar{w}^a T^a,$$

with w^a and $\bar{w}^a$ respectively holomorphic and antiholomorphic and (4.3) is given by

$$g \to g + \delta_w g + \delta_{\bar{w}} g = g + wg - g\bar{w} = -k\partial w + [w, J]. \tag{4.4}$$

Let us focus on the left-moving part:

$$\delta_w J = -k\big[\partial(wg)g^{-1} - \partial g g^{-1}(wg)g^{-1}\big] = -k\partial w + [w, J],$$

which in components is equivalent to

$$\delta_w J^a = -k\partial w^a + i f^{abc} w_b J_c.$$

Moreover, the Ward identity (1.57) implies that

$$\delta_w J^a = -\oint_{C_0} \frac{dz}{2\pi i} w_b(z) J^b(z) J^a(0).$$

Comparing these last two expressions, we end up with the OPE

$$J^a(z)J^b(w) \sim \frac{k\eta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c}{(z-w)}, \tag{4.5}$$

which is the *current algebra* we were looking for. Here, η^{ab} is the Killing invariant tensor of the Lie algebra. The antiholomorphic counterpart is instead given by

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\eta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{i f^{ab}{}_c J^c}{(\bar{z}-\bar{w})}. \tag{4.6}$$

4.1.2 The Sugawara construction

For any quantum conformal field theory, the stress-energy tensor satisfies the OPE (1.62)

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{4.7}$$

with c central charge. We want to prove that a stress-energy tensor satisfying the OPE (4.7) can be also defined for Wess-Zumino-Witten models and then we would like to determine the corresponding central charge. The procedure we follow is known as the *Sugawara construction*. The classical stress-energy tensor for a WZW model is given by

$$T(z) = \frac{1}{2k}\eta_{ab} J^a(z) J^b(z),$$

where as usual, k is the level of the model. In the quantum theory, the normal ordering is required in order to avoid the short-distance singularities. Then the most general form of the stress energy tensor reads

$$T(z) = \gamma\eta_{ab} : J^a(z)J^b(z) := \frac{\gamma}{2\pi i}\oint_{C_z} \frac{dx}{x-z} J^a(x) J_a(z), \tag{4.8}$$

with the integration only selecting the constant part and removing singularities and γ s a generic prefactor we are going to determine. (4.8) is also known as the *Sugawara stress-energy tensor*. The first OPE to compute is

$$\begin{aligned}
J^a(z)T(w) &= \frac{\gamma}{2\pi i}\oint_{C_w} \frac{dx}{x-w} J^a(z) : J^b(x) J_b(w) := \\
&\sim \frac{\gamma}{2\pi i}\oint_{C_w} \frac{dx}{x-w}\Big[\Big(\frac{k\eta^{ab}}{(z-x)^2} + \frac{i f^{ab}{}_c J^c(x)}{z-x}\Big) J_b(w) + J^b(x)\Big(\frac{k\delta^a{}_b}{(z-w)^2} + \frac{i f^{ab}{}_c}{z-w}\Big)\Big] = \\
&= \frac{\gamma}{2\pi i}\oint_{C_w} \frac{dx}{x-w}\Big[\frac{k\eta^{ab} J_b(w)}{(z-x)^2} + \frac{k\eta^{ab} J_b}{(z-w)^2} + \\
&\qquad + \frac{i f^{ab}{}_c}{z-x}\Big(\frac{i f^{cb}{}_d J^d(w)}{x-w} + \big(J^c J^b\big)(w)\Big) + \frac{i f^{ab}{}_c}{z-w}\Big(\frac{i f^{bc}{}_d J^d(w)}{x-w} + \big(J^b J^c\big)(w)\Big)\Big] = \\
&= \gamma\Big(\frac{2k\eta^{ab} J_b(w)}{(w-z)^2} - \frac{f^{ab}{}_c f^{cb}{}_d J^d(w)}{(w-z)^2}\Big),
\end{aligned}$$

where in the last step, the antisymmetry of the structure constants has been exploited to cancel the fourth and the sixth terms. In group theory, the quadratic Casimir of the adjoint representation is the double of the so-called dual Coxeter number $h^\vee$ Then,

$$f^{ab}{}_c f^{bc}{}_d = 2h^\vee \eta^{ad}$$

and accordingly,

$$T(z)J^a(w) = J^a(w)T(z) \sim 2\gamma(k+h^\vee)\frac{J^a(z)}{(w-z)^2} = 2\gamma(k+h^\vee)\Big(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\Big). \tag{4.9}$$

As already stressed, the holomorphic current has conformal weight $(1,0)$ and this does not change at the quantum level. Then, we have to impose that

$$\gamma = \frac{1}{2(k+h^\vee)} \tag{4.10}$$

and we finally obtain

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}.$$

Finally, in order to verify that the theory is indeed conformal, we calculate

$$\begin{aligned} T(z)T(w) &= \frac{1}{4\pi i(k+h^\vee)}\oint_{C_w}\frac{dx}{x-w}\big(T(z):J^a(x)J_a(w):\big) \sim \\ &\sim \frac{1}{4\pi i(k+h^\vee)}\oint_{C_w}\frac{dx}{x-w}\Big[\Big(\frac{J^a(x)}{(z-x)^2}+\frac{\partial J^a(x)}{z-x}\Big)J_a(w) + J^a(x)\Big(\frac{J_a(w)}{(z-w)^2}+\frac{\partial J_a(w)}{z-w}\Big)\Big] \sim \\ &\sim \frac{(3-2+0+0)k\dim\mathfrak{g}}{2(k+h^\vee)} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \end{aligned}$$

which really coincides with (4.7) as long as the central charge is identified as

$$c = \frac{k\ \dim\mathfrak{g}}{k+h^\vee}. \tag{4.11}$$

This also prove that WZW models are conformal field theories at the quantum level as well.

4.1.3 The affine algebra and representations

Focussing on the left moving sector, the holomorphicity of currents allows their Laurent expansion around the origin:

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \tag{4.12}$$

The expansion modes satisfy the following algebraic relation:

$$\begin{aligned} [J^a_m, J^b_n] &= \frac{1}{(2\pi i)^2}\Big(\oint_{C_0}dz\oint_{C_0}dw - \oint_{C_0}dw\oint_{C_0}dz\Big)\ z^m w^n J^a(z)J^b(w) = \\ &= \frac{1}{(2\pi i)^2}\oint_{C_0}dw\oint_{C_w}dz\ z^m w^n\Big(\frac{k\eta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_cJ^c}{(z-w)}\Big) = \\ &= \frac{1}{2\pi i}\oint_{C_0}dw\big(km\eta^{ab}w^{m+n-1} + if^{ab}{}_cJ^c(w)w^{m+n}\big) = \\ &= km\eta^{ab}\delta_{m+n,0} + if^{ab}{}_cJ^c_{m+n}. \end{aligned} \tag{4.13}$$

In the first step, the radial ordering (i.e. time ordering on the cylinder) is understood and we have used the same trick of contour deformation introduced in (1.72). This relation defines the *affine Kac-Moody algebra* associated to the Lie algebra $\mathfrak{g}$ and usually denoted as $\mathfrak{g}_k$ or $\hat{\mathfrak{g}}_k$(k represents the level). The Kac-Moody algebra is not an actual symmetry of the theory, due to the additional central term compared to the standard symmetry algebra. Only the zero-modes are such that

$$[J^a_0, J^b_0] = if^{ab}{}_cJ^c_0.$$

The right-moving modes make up an analogous algebra to (4.13). Since $[J^a_n, \bar{J}^b_m] = 0$, the two algebras are independent. As a final remark, since the stress-energy tensor is holomorphic, we can expand it in a Laurent series, whose modes can be expressed as

$$L_m = \gamma \sum_{n\in\mathbb{Z}} \eta_{ab} : J^a_n J^b_{m-n} := \gamma\eta_{ab}\Big(\sum_{n\leq -1} J^a_n J^b_{m-n} + \sum_{n\geq 0} J^a_{m-n} J^b_n \Big). \tag{4.14}$$

These satisfy the Virasoro algebra (1.72)

$$[L_m, L_n] = \frac{c}{12} m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n}, \tag{4.15}$$

with the central charge exactly coinciding with (4.11). Moreover, the Virasoro generators are linked to the modes of the J^a currents by the algebraic relation

$$[L_m, J^a_n] = -nJ^a_{m+n}. \tag{4.16}$$

One sometimes refers to the affine Kac-Moody algebra as *spectrum generating algebra*, since it plays a central role in the representations of the states of the WZW model. Indeed, for compact Lie groups (e. g. $SU(2)$), we can define a primary or highest weight state $|j\rangle$, such that

$$J^a_n |j\rangle = 0 \quad \text{for } n > 0, \tag{4.17}$$

exactly as in the case of the representations of the Virasoro algebra. All the states of the Hilbert space of the theory are obtained once we act with the modes J^a_{-n}, $n > 0$ and so we get the Verma module

$$\{J^{a_1}_{-n_1} \dots J^{a_p}_{-n_p} |j\rangle\}, \tag{4.18}$$

corresponding to the highest weight state $|j\rangle$. In this expression, $p = \dim\mathfrak{g}$ is the finite dimension of the Lie algebra. Primary states transform under a representation of the J^a_0, which as already stated satisfy the standard Lie algebra and are then generators: primary states then transform under some representation of the Lie group. As a last remark, let us calculate the weight of the highest weight state, acting with L_0 on it:

$$\begin{aligned} L_0 |j\rangle &= \frac{1}{2(k+h^\vee)} \Big(\sum_{n\leq -1} J^a_n J^a_{-n} + \sum_{n\geq 0} J^a_{-n} J^a_n \Big) |j\rangle = \\ &= \frac{1}{2(k+h^\vee)} J^a_0 J^a_0 |j\rangle = \\ &= \frac{\mathcal{C}(j)}{2(k+h^\vee)} |j\rangle \end{aligned} \tag{4.19}$$

where $\mathcal{C}(j)$ denotes the quadratic Casimir of the zero-mode representation. Moreover, due to (4.16), each time we act with some J^a_n on some state, its weight increases by n units and then, a general state in the Verma module will have weight given by

$$L_0 \Big(J^{a_1}_{-n_1} \dots J^{a_p}_{-n_p} |j\rangle \Big) = \Big(\frac{\mathcal{C}(j)}{2(k+h^\vee)} + \sum_{m=1}^{p} n_m \Big) |j\rangle \,.$$

4.2 The group manifold for the fuzzball microstate

The Lie group where the dynamics of the microstate geometry of Chapter 3 takes place, is given by the coset G/H, with[1]

$$G = \big(SL(2,\mathbb{R}) \times SU(2)\big) \times \big(\mathbb{R}_t \times S^1_{\tilde{y}} \times \mathbb{T}^4\big), \qquad H = U(1)_L \times U(1)_R. \tag{4.20}$$

[1]Let us notice the change of notation compared to (3.29), where G represents just the first two group manifolds.

In this Section, we focus on G: the gauging procedure is explained and carried out in the following. The conserved currents J^a_{sl} and J^a_{su} (together with their right-moving counterparts), generate the affine algebras for the first two factors. These are exactly the generalization of (3.38) (except for some harmless pre-factor), for any component:[2]

$$J^a = n_5 \,\mathrm{Tr}\big[T^a D g g^{-1}\big], \qquad \bar{J}^a = n_5 \,\mathrm{Tr}\big[\big(T^a\big)^* D g g^{-1}\big], \tag{4.21}$$

where the generators can alternatively correspond to the algebra $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{R})$. Furthermore, the level has been fixed to be the level of the model, i. e. the number n_5 of the NS5-branes along the circle. We choose to work in a supersymmetric framework, hence the standard derivatives have been replaced by the superderivatives defined in (1.115). Then for each current J^a, one supersymmetric partner ψ^a is defined.
In this Section we collect the current algebras and the representations of each of the factors of the Lie group G in (4.20), in order that in the following, we are able to deal with the spectrum of our worldsheet model.

4.2.1 SU(2)

The structure constants of this group are given by $f^{abc} = \varepsilon^{abc}$. We choose to work in the basis $\{J^{\pm} = J^1 \pm iJ^2, J^3\}$ and then the current algebra (4.5) becomes

$$J^a_{su}(z) J^b_{su}(w) \sim \frac{\frac{n_5}{2}\delta^{ab}}{(z-w)^2} + \frac{i\varepsilon^{ab}{}_c J^c_{su}(w)}{z-w}. \tag{4.22}$$

The OPE of fermions mimics (1.137) (except one pre-factor we are going to discuss):

$$\psi^a_{su}(z)\psi^b_{su}(w) \sim \frac{\frac{n_5}{2}\delta^{ab}}{z-w}. \tag{4.23}$$

In these expressions, we notice the appearance of one factor of 1/2 in front of the level, as usual when choosing this new basis. Moreover, $i\varepsilon^{+-3} = 2$ and the Cartan-Killing form in this basis is such that $\delta^{+-} = 2$ and $\delta^{33} = 1$. The signature is $(+++)$ since we lack time-like directions along this manifold. In supersymmetric models, the components J^a are usually written in terms of two independent contributions:

$$J^a_{su} = j^a_{su} + \hat{\jmath}^a_{su}, \qquad \hat{\jmath}^a_{su} = -\frac{i}{n_5}\varepsilon^{ab}{}_c \psi^b_{su}\psi^c_{su}.$$

We can exploit the OPE (4.23) to calculate

$$\hat{\jmath}^a_{su}(z)\hat{\jmath}^b_{su}(w) \sim \frac{\delta^{ab}}{(z-w)^2} + \frac{i\varepsilon^{ab}{}_c \hat{\jmath}^c_{su}(w)}{z-w}, \qquad j^a_{su}(z) j^b_{su}(w) \sim \frac{\frac{n_5-2}{2}\delta^{ab}}{(z-w)^2} + \frac{i\varepsilon^{ab}{}_c j^c_{su}(w)}{z-w}.$$

The latter arises simply so that the total current J^a_{su} satisfies (4.22) (j_{su} and $\hat{\jmath}_{su}$ are indeed independent and have vanishing OPE). We see then, that the currents j^a_{su} generate a bosonic Wess-Zumino-Witten model of level $n_5 - 2$, whereas the model generated by the currents $\hat{\jmath}^a_{su}$ has level 2. In the supersymmetric framework, the stress-energy tensor can be written as in (1.120). For the Lie group $SU(2)$, the bosonic part of the super stress-energy tensor

$$T^{su} = \frac{1}{n_5}\Big[\delta_{ab} j^a_{su} j^b_{su} - \delta_{ab}\psi^a_{su}\partial\psi^b_{su}\Big] \tag{4.24}$$

arises from (4.8) and (1.138), whereas the supercurrent reads

$$G^{su} = \frac{2}{n_5}\Big[\delta_{ab}\psi^a_{su} j^b_{su} - \frac{i}{3n_5}\varepsilon_{abc}\psi^a\psi^b\psi^c\Big]. \tag{4.25}$$

[2] Indeed, let us notice that the definition (3.38) only involved the third component of the current.

Let us state that for $SU(2)$, $\dim\mathfrak{g} = 3$ and the dual Coxeter number is $h^\vee = 1$. Moreover, the three added fermions are free, each with central charge $+1/2$. Therefore, from (4.11), if $k = n_5 - 2$, the central charge associated with the matter fields in the SU(2) factor is

$$c_{su} = \frac{3(n_5 - 2)}{n_5} + \frac{3}{2}. \tag{4.26}$$

Eventually, let us write the Kac-Moody algebra for the SU(2) Wess-Zumino-Witten model of level $n_5 - 2$ in this basis:

$$\begin{aligned}
[j^3_{su,m}, j^3_{su,n}] &= \frac{n_5 - 2}{2} m\delta_{m+n,0}, \\
[j^3_{su,m}, j^\pm_{su,n}] &= \pm j^\pm_{su,n+m}, \\
[j^+_{su,m}, j^-_{su,n}] &= (n_5 - 2)m\delta_{m+n,0} + 2j^3_{su,m+n}.
\end{aligned} \tag{4.27}$$

The highest weight states for the bosonic theory $SU(2)_{n_5-2}$ have to fulfil (4.17) and to sit in a unitary irreducible representation of $j^a_{su,0}$. This kind of representations are labelled by the quantum numbers j', positive and half-integer, and $m' = -j', -j'+1, \ldots, j'$. In particular,

$$\begin{aligned}
j^a_{su,n}\left|j', m'\right\rangle &= 0 \qquad\qquad \text{for } n > 0, \\
j^3_{su,0}\left|j', m'\right\rangle &= m'\left|j', m'\right\rangle.
\end{aligned} \tag{4.28}$$

Due to the algebraic rules (4.27), the operators $j^\pm_{su,0}$ are such that they respectively raise or lower the quantum number m' by one unit. The particular chosen normalization is given by

$$j^\pm_{su,0}\left|j', m'\right\rangle = (\pm m' + j' + 1)\left|j', m' \pm 1\right\rangle. \tag{4.29}$$

Eventually, the states $|j', j'\rangle$ and $|j', -j'\rangle$ are such that

$$j^+_{su,0}\left|j', j'\right\rangle = 0, \qquad j^-_{su,0}\left|j', -j'\right\rangle = 0. \tag{4.30}$$

Let us notice that these two states do not fulfil the condition (4.29). Moreover, on grounds of (4.19), we can infer that the weight of all these highest weight states is $j(j+1)/n_5$. This is also the weight of the primary vertex operator giving rise to this state. Let us name $\Phi^{su}_{j',m',\bar{m}'}(w, \bar{w})$ the vertex operators giving rise to these states: in this writing, we are also adding the label $\bar{m}'$ just to recall that the same procedure can be performed for the antiholomorphic part. Then the requirements (4.28) and (4.29) can be rephrased in terms of the OPEs

$$j^3_{su}(z)\Phi^{su}_{j',m',\bar{m}'}(w, \bar{w}) \sim \frac{m'\Phi^{su}_{j',m',\bar{m}'}(w, \bar{w})}{z - w}, \tag{4.31}$$

$$j^\pm_{su}\Phi^{su}_{j',m',\bar{m}'}(w, \bar{w}) \sim \frac{(\pm m' + j' + 1)\Phi^{su}_{j',m'\pm 1,\bar{m}'}(w, \bar{w})}{z - w}, \tag{4.32}$$

$$T^{su}(z)\Phi^{su}_{j',m',\bar{m}'}(w, \bar{w}) \sim \frac{j'(j'+1)}{n_5}\frac{\Phi^{su}_{j',m',\bar{m}'}}{(z-w)^2} + \frac{\partial\Phi^{su}_{j',m',\bar{m}'}}{z - w}. \tag{4.33}$$

4.2.2 SL(2, ℝ)

The procedure that led to the collection of properties of the previous Subsection can be also followed to infer the analogous relations for the factor $SL(2, \mathbb{C})$ that we are going to illustrate. Nonetheless, some minor differences emerge and they are going to be stressed in the following.
The structure constants are $f^{abc} = \varepsilon^{abc}$. Working again in the basis $\{J^\pm = J^1 \pm iJ^2, J^3\}$, the current algebra (4.5) and the OPE (1.137) give

$$J^a_{sl}(z)J^b_{sl}(w) \sim \frac{\frac{n_5}{2}\eta^{ab}}{(z-w)^2} + \frac{i\varepsilon^{ab}{}_c J^c_{sl}(w)}{z - w}, \qquad \psi^a_{sl}(z)\psi^b_{sl}(w) \sim \frac{\frac{n_5}{2}\eta^{ab}}{z - w}. \tag{4.34}$$

In this basis, $i\varepsilon^{+-3} = 2$ and the Cartan-Killing form is such that $\eta^{+-} = 2$ and $\eta^{33} = -1$. The different signature compared to $SU(2)$ is due to the presence of one time-like direction along the manifold. The usual decomposition of supersymmetric frameworks is again performed:

$$J^a_{sl} = j^a_{sl} + \hat{j}^a_{sl}, \qquad \hat{j}^a_{sl} = -\frac{i}{n_5}\varepsilon^{ab}{}_c\psi^b_{sl}\psi^c_{sl}.$$

The fermionic OPE (4.34) allows to compute

$$\hat{j}^a_{sl}(z)\hat{j}^b_{sl}(w) \sim -\frac{\eta^{ab}}{(z-w)^2} + \frac{i\varepsilon^{ab}{}_c\hat{j}^c_{sl}(w)}{z-w}, \qquad j^a_{sl}(z)j^b_{sl}(w) \sim \frac{\frac{n_5+2}{2}\delta^{ab}}{(z-w)^2} + \frac{i\varepsilon^{ab}{}_c j^c_{sl}(w)}{z-w}.$$

Then, in this instance, the Wess-Zumino-Witten model generated by the currents $\hat{j}^a_{su}$ has level -2 and consequently the bosonic model generated by j^a_{sl} has level n_5+2. The bosonic stress-energy tensor and the supercurrent are given by

$$T^{sl} = \frac{1}{n_5}\Big[\eta_{ab}j^a_{sl}j^b_{sl} - \delta_{ab}\psi^a_{sl}\partial\psi^b_{sl}\Big], \qquad G^{sl} = \frac{2}{n_5}\Big[\eta_{ab}\psi^a_{sl}j^b_{sl} - \frac{i}{3n_5}\varepsilon_{abc}\psi^a_{sl}\psi^b_{sl}\psi^c_{sl}\Big]. \tag{4.35}$$

Due to the different level, the central charge is now

$$c_{sl} = \frac{3(n_5+2)}{n_5} + \frac{3}{2}. \tag{4.36}$$

In the chosen basis, the affine algebra for the bosonic $SU(2)$ Wess-Zumino-Witten model of level n_5+2

$$\begin{aligned}
[j^3_{sl,m}, j^3_{sl,n}] &= -\frac{n_5+2}{2}m\delta_{m+n,0},\\
[j^3_{sl,m}, j^\pm_{sl,n}] &= \pm j^\pm_{sl,n+m},\\
[j^+_{sl,m}, j^-_{sl,n}] &= (n_5+2)m\delta_{m+n,0} - 2j^3_{sl,m+n}.
\end{aligned} \tag{4.37}$$

In full generality, since the Cartan-Killing metric is not positive definite, states with negative energy might appear in the spectrum of $SL(2,\mathbb{R})$ as well as states with negative norms. However, in some representations, the eigenvalues of L_0 are all bounded from below: this is the case of the *positive energy representations*, which follow the description in the Subsection 4.1.3. Indeed, some states are annihilated by all $j^{3,\pm}_n$ $(n \geq 1)$ and are then identified as the highest weight states. All the other states of the Hilbert space arise acting on these highest weight states with the current modes $j^{3,\pm}_{-n}$. The primary states are in an irreducible unitary representation of the zero modes $j^a_{sl,0}$. We know two discrete and two continuous representations of this kind (besides the trivial one). We adopt the discrete ones:

- Principal discrete representations (lowest weight): the Hilbert space of this kind of representations is
$$\mathcal{D}^+_j = \{|j,m\rangle : m = j, j+1, j+2, \dots\},$$
such that
$$j^-_{sl,0}\,|j,j\rangle = 0, \qquad j^3_{sl,0}\,|j,m\rangle = m\,|j,m\rangle\,.$$
Unitarity requires that j is real and positive. For representations of $SL(2,\mathbb{R})$, j has to be a half integer and such that $0 \leq j \leq n_5/2$.

- Principal discrete representations (highest weight): the Hilbert space of this kind of representations is
$$\mathcal{D}^-_j = \{|j,m\rangle : m = -j, -j-1, -j-2, \dots\},$$
such that
$$j^+_{sl,0}\,|j,j\rangle = 0, \qquad j^3_{sl,0}\,|j,m\rangle = m\,|j,m\rangle\,.$$
Unitarity requires that j is real and positive. For representations of $SL(2,\mathbb{R})$, j has to be a half integer and again, $0 \leq j \leq n_5/2$.

The Casimir of all the unitary representations of $SL(2,\mathbb{R})$ is $-j(j-1)$. Then, the expression (4.19), suggests that the weight of all these primary states (and then of the corresponding vertex operators) is $-j(j-1)/n_5$. Let us name $\Phi^{sl}_{j,m,\bar{m}}(w,\bar{w})$ the vertex operators giving rise to these states: then the properties listed up to now can be expressed in terms of the OPEs

$$j^3_{sl}(z)\Phi^{sl}_{j,m,\bar{m}}(w,\bar{w}) \sim \frac{m\Phi^{sl}_{j,m,\bar{m}}(w,\bar{w})}{z-w}, \tag{4.38}$$

$$j^{\pm}_{sl}\Phi^{sl}_{j,m,\bar{m}}(w,\bar{w}) \sim \frac{(m \mp (j-1))\Phi^{sl}_{j,m\pm 1,\bar{m}}(w,\bar{w})}{z-w}, \tag{4.39}$$

$$T^{sl}(z)\Phi^{sl}_{j,m,\bar{m}}(w,\bar{w}) \sim -\frac{j(j-1)}{n_5}\frac{\Phi^{sl}_{j,m,\bar{m}}}{(z-w)^2} + \frac{\partial\Phi^{sl}_{j,m,\bar{m}}}{z-w}. \tag{4.40}$$

4.2.3 $\mathbb{R}_t \times S^1_{\tilde{y}} \times \mathbb{T}^4$

The coordinates for this last factor are $(t, \tilde{y}, z^i)$ with $i = 1, \dots, 4$ since z^i are the coordinates of the torus, can be grouped into the unique set Y^i, with $i = 0, \dots, 5$, such that

$$Y^0 = t, \qquad Y^1 = \tilde{y} \qquad Y^i = z^i \equiv \vec{Y}, \text{ for } i = 2, \dots, 5.$$

The corresponding fermionic coordinates are named λ^i and we can employ the corresponding OPEs of flat space for free fields, i. e. (1.66) and (1.137) (we are imposing $\alpha' = 1$ and the pre-factor of the action (1.136) to be $g = 1/\pi$):

$$\partial Y^i(z)\partial Y^j(w) \sim \frac{1}{2}\frac{\eta^{ij}}{(z-w)^2}, \qquad \lambda^i(z)\lambda^j(w) \sim \frac{1}{2}\frac{\eta^{ij}}{z-w}, \tag{4.41}$$

with η^{ij} the Minkowski metric in Cartesian coordinates. The bosonic stress energy tensor coincides with the sum of (1.67) and (1.138):

$$T^{\mathbb{R}} = \partial Y^i \partial Y_i - \lambda^i \partial \lambda_i, \tag{4.42}$$

whereas the supercurrent is

$$G^{\mathbb{R}} = i\lambda^i \partial Y_i. \tag{4.43}$$

Since wee are dealing with six bosonic and six fermionic free fields, the central charge will be

$$c_{\mathbb{R}} = 6 \times 1 + 6 \times \frac{1}{2} = 9. \tag{4.44}$$

Finally, the vertex operator for the lowest energy state is $:e^{ik_iY^i}:$, as we can read from (1.79). We can compute the OPE

$$i\partial Y^i(z) :e^{ik_jY^j(w)}: \sim \frac{k^i}{2(z-w)} :e^{ik_jY^j(w)}:, \tag{4.45}$$

from which we derive

$$T^{\mathbb{R}}(z)e^{ik_iY^i(w)} \sim \frac{k^2}{2}\frac{:e^{ik_jY^j(w)}:}{(z-w)^2} + ik_i\frac{:\partial Y^i e^{ik_jY^j(w)}:}{z-w}. \tag{4.46}$$

4.3 The need for the ghosts and the coset

A consistent quantization of strings demands the cancellation of the Weyl anomaly, which in turn, requires the vanishing of the total central charge of the studied system. The central charges of the matter sector were found in the previous Section (cf. equations (4.26), (4.36) and (4.44)). Their sum is

$$c_m = c_{sl} + c_{su} + c_{\mathbb{R}} = 18, \tag{4.47}$$

then we need something with negative central charge: the further fields to add are the ghosts. Nonetheless, the worldsheet reparametrization ghosts are not sufficient: their central charge is indeed, $c_g = -15$. The second set of ghosts arises from the gauging of some directions of the spacetime. As in the reparametrization case, we have two ghost systems: the anticommuting fields $(\hat{c}, \hat{b})$ and the commuting $(\hat{\gamma}, \hat{\beta})$. The parameters λ have to be respectively $\lambda = 1$ and $\lambda = 1/2$. The properties of each of the four ghost systems (anticommuting and commuting reparametrization and gauge ghosts) arise from the discussion in the Subsection 1.4.4. In particular the central charge of the two systems of gauging ghosts can be calculated through (1.143), giving

$$c_{\text{g,gauge}} = c_{\hat{c},\hat{b}} + c_{\hat{\gamma},\hat{\beta}} = -2 - 1 = -3.$$

The gauging of the target group of the Wess-Zumino-Witten model is then necessary in order to erase the Weyl anomaly: this is nothing but the requirement that the dimension of the ambient spacetime has to be critical. As a final remark, let us state the conformal weights of the vertex operator for the Bose systems. For the reparametrization bosonic ghosts (β, γ), we have $\lambda = 3/2$ and then, by (1.153) and the definition of the parameter Q within (1.143), the conformal weight is

$$h = \frac{1}{2}\varepsilon q(q + Q) = -\frac{1}{2}q(q + 2), \tag{4.48}$$

whereas for the gauging commuting ghosts $(\hat{\beta}, \hat{\gamma})$, since $\lambda = 1/2$, we obtain

$$\hat{h} = \frac{1}{2}\varepsilon q(q + Q) = -\frac{1}{2}q^2. \tag{4.49}$$

The target space for the worldsheet description of the fuzzball microstates is not the whole G given by (4.20), but the coset G/H, with H subgroup of G such that

$$H = U(1)_L \times U(1)_R. \tag{4.50}$$

In a more geometrical perspective, we are obtaining a ten-dimensional space, identifying all the points along two directions of the manifold G, which in the current description of the Lie group, are determined by the currents (both in the bosonic and in the fermionic sectors)

$$\begin{aligned}
U(1)_L : &\quad \mathcal{I} = l_1 J^3_{sl} + l_2 J^3_{su} + i l_3 \partial t + i l_4 \partial y, \\
U(1)_R : &\quad \bar{\mathcal{I}} = r_1 \bar{J}^3_{sl} + r_2 \bar{J}^3_{su} + i r_3 \bar{\partial} t + i r_4 \bar{\partial} y, \\
U(1)_L : &\quad \Psi = l_1 \psi^3_{sl} + l_2 \psi^3_{su} + l_3 \lambda^t + l_4 \lambda^y, \\
U(1)_R : &\quad \bar{\Psi} = r_1 \bar{\psi}^3_{sl} + r_2 \bar{\psi}^3_{su} + r_3 \bar{\lambda}^t + r_4 \bar{\lambda}^y.
\end{aligned} \tag{4.51}$$

Let us notice a change of notation compared to the notation of 4.2.3. Here and till the end of the Chapter, y is still compact. Moreover, a conventional i appears in front of ∂t, ∂y, $\bar{\partial} t$ and $\bar{\partial} y$, compared to (3.54). This also changes some signs in (4.52) compared to (3.55). The objects appearing in these currents, with all their properties, are defined in Section 4.2. These currents generate the gauge transformations $U(1)_L \times U(1)_R$. We impose that these currents have to be null with respect to the scalar product generated for each sector by the corresponding Cartan-Killing metric:

$$\begin{aligned}
\langle \mathcal{I}, \mathcal{I} \rangle &= l_1^2 \langle J^3_{sl}, J^3_{sl} \rangle + l_2^2 \langle J^3_{su}, J^3_{su} \rangle + l_3^2 \langle \partial t, \partial t \rangle + l_4^2 \langle \partial y, \partial y \rangle = \frac{n_5}{2}\left(-l_1^2 + l_2^2\right) + \frac{1}{2} l_3^2 - \frac{1}{2} l_4^2 = 0, \\
\langle \bar{\mathcal{I}}, \bar{\mathcal{I}} \rangle &= r_1^2 \langle \bar{J}^3_{sl}, \bar{J}^3_{sl} \rangle + r_2^2 \langle \bar{J}^3_{su}, \bar{J}^3_{su} \rangle + l_3^2 \langle \bar{\partial} t, \bar{\partial} t \rangle + l_4^2 \langle \bar{\partial} y, \bar{\partial} y \rangle = \frac{n_5}{2}\left(-r_1^2 + r_2^2\right) + \frac{1}{2} r_3^2 - \frac{1}{2} r_4^2 = 0.
\end{aligned} \tag{4.52}$$

This way the left and right gaugings are independent and the gauge anomalies disappear.

The action of the null gauged Wess-Zumino-Witten model with the coset G/H as a target space, is then given by (3.40):

$$\mathcal{S}[g,\mathcal{A},\bar{\mathcal{A}}] = \mathcal{S}_0[g] + \mathcal{S}_{\text{gauge}}[g,\mathcal{A},\bar{\mathcal{A}}], \tag{4.53}$$

such that the first summand coincides with the full Wess-Zumino-Witten action (3.9)[3] for the factors $SL(2,\mathbb{R})$ and $SU(2)$ of G (in the Euler parametrization), whereas we have the trivial superstring theory on $\mathbb{R}^t \times S^1 \times \mathbb{T}^4$. The second term in (4.53) is instead (3.56):

$$\mathcal{S}_{\text{gauge}} = \frac{1}{\pi}\int_{S^2} d^2\boldsymbol{z}\ [\mathcal{A}\bar{\mathcal{I}} + \bar{\mathcal{A}}\mathcal{I} - \Sigma\mathcal{A}\bar{\mathcal{A}}]. \tag{4.54}$$

In the coordinates of the Euler parametrization (3.33), we have that

$$\Sigma = \frac{1}{2}\big[n_5\big(l_1 r_1 \cosh 2\rho - l_2 r_2 \cos 2\theta\big) + l_3 r_3 - l_4 r_4\big]. \tag{4.55}$$

Since we deal with a supersymmetric system, the integration coordinate z has been promoted to the supercoordinate $\boldsymbol{z} = (z,\theta)$. The same should be understood in the Wess-Zumino-Witten term, together with the condition that all derivatives should become superderivatives. The symbols $\mathcal{A}$ and $\bar{\mathcal{A}}$ denote the gauge fields for $U(1)_L$ and $U(1)_R$ respectively.

4.4 Path integral analysis

The gauged Wess-Zumino-Witten model (4.53) can be quantized with the help of the path integral. The fields in the game are $g \in G$ and the two gauge fields $\mathcal{A}$ and $\bar{\mathcal{A}}$, therefore the partition function reads

$$Z = \int [dg][d\mathcal{A}][d\bar{\mathcal{A}}]\ \exp[-n_5\mathcal{S}(g,\mathcal{A},\mathcal{A})]. \tag{4.56}$$

The gauge fields can be parametrized as

$$\mathcal{A} = hDh^{-1}, \qquad \bar{\mathcal{A}} = \bar{h}\bar{D}\bar{h}^{-1}, \tag{4.57}$$

with $h \in U(1)_L$, and $\bar{h} \in U(1)_R$. At this point, one formula comes in handy:

$$\mathcal{S}(g,\mathcal{A},\bar{\mathcal{A}}) = \mathcal{S}_0(h^{-1}g\bar{h}) - \mathcal{S}_0(h^{-1}\bar{h}). \tag{4.58}$$

It is easier to prove the latter by working with the general gauged Wess-Zumino-Witten action

$$\begin{aligned} I(g,\mathcal{A}) &= I_0(g) + \frac{1}{\pi}\int d^2\boldsymbol{z}\,\mathrm{Tr}\Big[-\mathcal{A}\bar{D}gg^{-1} + \bar{\mathcal{A}}g^{-1}Dg + g^{-1}\mathcal{A}g\bar{\mathcal{A}} - \mathcal{A}\bar{\mathcal{A}}\Big], \\ &= I_0(g) + \frac{1}{\pi}\int d^2\boldsymbol{z}\,\mathrm{Tr}\Big[-hDh^{-1}\bar{D}gg^{-1} + \bar{h}\bar{D}\bar{h}^{-1}g^{-1}Dg + g^{-1}hDh^{-1}g\bar{h}\bar{D}\bar{h}^{-1} - hDh^{-1}\bar{h}\bar{D}\bar{h}^{-1}\Big], \end{aligned} \tag{4.59}$$

with $I_0(g)$ representing the WZW action (3.9) and where in the second step the parametrization (4.57) has been used. Making use of the *Polyakov-Wiegmann identity*

$$I_0(gh) = I_0(g) + I_0(h) - \frac{1}{\pi}\int d^2\boldsymbol{z}\,\mathrm{Tr}[g^{-1}Dg\bar{D}hh^{-1}], \tag{4.60}$$

we obtain that

$$\begin{aligned} I(h^{-1}\bar{h}) &= I(h^{-1}) + I(\bar{h}) + \frac{1}{\pi}\int d^2\boldsymbol{z}\,\mathrm{Tr}\Big[hDh^{-1}\bar{h}\bar{D}\bar{h}^{-1}\Big], \\ I(h^{-1}g\bar{h}) &= I(h^{-1}) + I(g) + I(\bar{h}) - \frac{1}{\pi}\int d^2\boldsymbol{z}\,\mathrm{Tr}\Big[g^{-1}Dg\bar{D}\bar{h}\bar{h}^{-1} + hDh^{-1}\bar{D}gg^{-1} + hDh^{-1}g\bar{D}\bar{h}\bar{h}^{-1}g^{-1}\Big] \end{aligned}$$

[3] Notice the change of notation: from here on out, $\mathcal{S}$, is the action of the ungauged WZW model and no more just the action of the principal chiral model (3.2)

Subtracting the latter from the former, we find (4.59), thus proving (4.58). We insert this identity into the partition function and express it in terms of the fields h and $\bar{h}$. This leads to

$$Z = \int [dg][dh][d\bar{h}]\ \det D \det \bar{D}\ \exp[-n_5 \mathcal{S}_0(h^{-1}g\bar{h})]\ \exp[n_5 \mathcal{S}_0(h^{-1}\bar{h})].$$

The two determinants are the Jacobian of the transformation and as usual, can be written in terms of a ghost action:

$$Z = \int [dg][dh][d\bar{h}][d\hat{B}][d\hat{C}] \exp[-n_5 \mathcal{S}_0(h^{-1}g\bar{h})]\ \exp[n_5 \mathcal{S}_0(h^{-1}\bar{h})]\ \exp[-\mathcal{S}_{gh}(\hat{B},\hat{C})].$$

The fields $\hat{B}$ and $\hat{C}$ are nothing but a compact notation for the gauging ghosts that were introduced in the previous Section:

$$\hat{C} = \hat{c} + \theta\hat{\gamma}, \qquad \hat{B} = \hat{\beta} + \theta\hat{b}. \tag{4.61}$$

We write the action for both the left-moving and the right-moving sectors, using the extension to the supersymmetric case of (1.139):

$$\hat{\mathcal{S}}_{gh}(\hat{B},\hat{C}) = \frac{1}{\pi}\int d^2\boldsymbol{z}\big[\hat{B}\bar{D}\hat{C} - \bar{\hat{B}}D\bar{\hat{C}}\big]. \tag{4.62}$$

We can now fix the gauge by choosing a transformation such that h is the identity. Moreover, since we are dealing with a null subgroup, $\mathcal{S}_0(\bar{h}) = 0$. We add the action for the reparametrization ghosts packaged in the manner of (4.61) and we end up with the partition function

$$Z = \int [dg][dB][dC][d\hat{B}][d\hat{C}]\ \exp[-n_5\mathcal{S}_0(g) - \mathcal{S}_{gh}(B,C) - \hat{\mathcal{S}}_{gh}(\hat{B},\hat{C})]. \tag{4.63}$$

4.5 The spectrum

4.5.1 BRST constraints on the vertex operators

The BRST prescription allows to identify the physical states and then the spectrum of the Wess-Zumino-Witten model. For each of the two kinds of redundancies to fix, we have to introduce one BRST charge. For the reparametrization invariance, this is given exactly by (1.148):

$$Q = Q_0 + Q_1 + Q_2, \tag{4.64}$$

with the three pieces given by (let us recall that G denotes the supercurrent according to the conventions of 4.2):

$$Q_0 = \oint_{C_0}\frac{dz}{2\pi i}\Big(cT^{X,\psi,\beta,\gamma} + c(\partial c)b\Big), \qquad Q_1 = -\oint_{C_0}\frac{dz}{2\pi i}\gamma G, \qquad Q_3 = -\oint_{C_0}\frac{dz}{2\pi i}\frac{1}{4}b\gamma^2. \tag{4.65}$$

For the gauging, instead, the BRST charge is

$$Q_{\mathcal{I}} = \oint_{C_0}\frac{dz}{2\pi i}[\hat{c}\mathcal{I} + \hat{\gamma}\Psi]. \tag{4.66}$$

This is a nilpotent charge. Indeed,

$$Q_{\mathcal{I}}^2 = \frac{1}{2}\{Q_{\mathcal{I}}, Q_{\mathcal{I}}\} = \frac{1}{2}\oint_{C_w}\frac{dz}{2\pi i}\big[\hat{c}(z)\hat{c}(w)\mathcal{I}(z)\mathcal{I}(w) + \hat{\gamma}(z)\hat{\gamma}(w)\Psi(z)\psi(w)\big].$$

Moreover, exploiting the OPEs collected in Section 4.2 we can compute

$$\mathcal{I}(z)\mathcal{I}(w) \sim \frac{\frac{n_5}{2}(-l_1^2 + l_2^2) + \frac{1}{2}l_3^2 - \frac{1}{2}l_4^2}{(z-w)^2},$$

which vanishes due to the condition of null gauging subgroup (4.52). The same also holds for the OPE $\Psi(z)\Psi(w)$.
To identify the physical states we work at the level of vertex operators $\mathcal{V}(z,\bar{z})$. In order that they give rise to physical states, they must belong to the cohomology class of both the BRST charges we have defined, i. e. they are BRST invariant

$$[Q, \mathcal{V}(z,\bar{z})] = 0, \qquad [Q_{\mathcal{I}}, \mathcal{V}(z,\bar{z})] = 0, \tag{4.67}$$

but they have not to be BRST exact. The general vertex operator of the theory in the Neveu-Schwarz sector can be written as

$$\mathcal{V}_q = e^{-q\phi} P(\psi, j, \dots)\bar{P}(\bar{\psi}, \bar{j}, \dots)\Phi^{sl}_{j,m,\bar{m}}\Phi^{su}_{j',m',\bar{m}'} e^{iEt} e^{iPy} e^{i\vec{k}\cdot\vec{Y}} \tag{4.68}$$

In particular, $P(\psi, j, \dots)$ denotes a generic polynomial depending on the fields of the theory, $\Phi^{sl}_{j,m,\bar{m}}$ and $\Phi^{su}_{j',m',\bar{m}'}$ are bosonic primary operators of the string theory on $SL(2,\mathbb{R})$ and $SU(2)$. The exponentials are the vertex operators of the bosonic string theory on $R \times S^1 \times \mathbb{T}^4$ and $e^{-q\phi}$ represents the vertex operator creating the q-vacuum state for reparametrization ghosts from the $SL(2,\mathbb{C})$ invariant vacuum. We choose to work in the $q = -1$ picture.
The simplest state to study is the tachyon, which does not include any excitation of fields, hence it is just the vertex operator

$$\mathcal{V}_{-1} = e^{-\phi}\Phi^{sl}_{j,m,\bar{m}}\Phi^{su}_{j',m',\bar{m}'} e^{iEt} e^{iPy} e^{i\vec{k}\cdot\vec{Y}}. \tag{4.69}$$

The conditions (4.67) have to be satisfied by the physical tachyon vertex operator. The first condition is equivalent to require that the conformal weight of $\mathcal{V}_{-1}$ is 1. From the expression (4.48), we deduce that the conformal weight of the vertex operator for reparametrization ghosts, when $q = -1$, is 1/2. We read the conformal weights of the other bosonic primaries from the OPEs (4.33), (4.40) and (4.46). Summing all contributions we end up with

$$-\frac{j(j-1)}{n_5} + \frac{j'(j'+1)}{n_5} - \frac{1}{2}E^2 + \frac{1}{2}P_y^2 + \frac{1}{2}|\vec{k}|^2 + \frac{1}{2} = 1. \tag{4.70}$$

This is the mass-shell condition for tachyons. Indeed, when there are no excitations on $SL(2,\mathbb{R})$ and $SU(2)$ we end up with a mass-square $-1/2$, which really coincides with the mass of the tachyon (in $\alpha' = 1$ units). The second condition in (4.67), instead, imposes that

$$\begin{aligned} l_1 m + l_2 m' + \frac{l_3}{2}E + \frac{l_4}{2}P_y &= 0, \\ r_1\bar{m} - r_2\bar{m}' + \frac{r_3}{2}E + \frac{r_4}{2}P_y &= 0, \end{aligned} \tag{4.71}$$

where we have used the OPEs (4.31), (4.38) and (4.45). Let us notice that in principle, we should have two different momentum eigenvalues on the left and the right-moving sectors. Indeed, since the direction y is compact, the momentum is quantized along that. In general, its eigenvalues would be

$$P_{y,L/R}y = \frac{n_y}{R} \pm w_y R_y,$$

where n_y is the number of momentum units, w_y the number of winding modes along this compact direction and R_y is the radius of S^1_y. However, we impose $w_y = 0$ and then the two momentum eigenvalues coincide.

In order to identify the constraints arising when fermions are present, let us write the vertex operator as

$$\mathcal{V}_{-1} = e^{-\phi}V, \qquad V = V_{sl} + V_{su} + V_{\mathbb{R}}. \tag{4.72}$$

The studied states involve the appearance of a twelve-dimensional polarization tensor ξ^μ in front of the vertex operators, together with the fermionic excitations. The components of the polarization tensor are split among the different factors of the spacetime in the following manner (we just focus on the holomorphic side):

$$\begin{aligned} V_{sl} &= \xi_b^{sl}\psi_{sl}^b\Phi_{j,m-b,\bar{m}}^{sl}\Phi_{j',m',\bar{m}'}^{su}e^{iEt}e^{iPy}e^{i\vec{k}\cdot\vec{Y}},\\ V_{su} &= \xi_b^{sl}\psi_{su}^b\Phi_{j,m,\bar{m}}^{sl}\Phi_{j',m'-b,\bar{m}'}^{su}e^{iEt}e^{iPy}e^{i\vec{k}\cdot\vec{Y}},\\ V_{\mathbb{R}} &= \xi_i\lambda^i\Phi_{j,m\bar{m}}^{sl}\Phi_{j,m',\bar{m}'}^{su}e^{iEt}e^{iPy}e^{i\vec{k}\cdot\vec{Y}} \end{aligned} \tag{4.73}$$

In particular, $b = +, -, 3$ represents the components of the fermions, whilst the index $i = 0, \dots 5$ refers to the directions of the six-dimensional $\mathbb{R}_t \times S_y^1 \times \mathbb{T}^4$. In the following we focus only on the holomorphic part and then the right moving indices will not be written.
For instance, we can impose that excitations only take place on AdS_3 and compute the physical constraints. Let us focus then, just on

$$V_{sl} = \left(\xi_+^{sl}\psi_{sl}^+\Phi_{j,m-1}^{sl} + \xi_-^{sl}\psi_{sl}^-\Phi_{j,m+1}^{sl} + \xi_3^{sl}\psi_{sl}^3\Phi_{j,m}^{sl}\right)\Phi_{j',m'}^{su}e^{iEt}e^{iPy}e^{i\vec{k}\cdot\vec{Y}} \tag{4.74}$$

and impose the conditions (4.67). Due to the invariance of the vertex operators under the BRST charge for reparametrizations we obtain two pieces, respectively arising from the summand Q_0 and Q_1. The former is again the mass-shell condition

$$-\frac{j(j-1)}{n_5} + \frac{j'(j'+1)}{n_5} - \frac{1}{2}E^2 + \frac{1}{2}P_y^2 + \frac{1}{2}|\vec{k}|^2 + \frac{1}{2} = \frac{1}{2}. \tag{4.75}$$

The further 1/2 (compared to (4.70)) in the left-hand side represents the weight of the fermion. We can notice that when the excitations on $SL(2,\mathbb{R})$ and $SU(2)$ disappear, we really have the vanishing mass for the first excited state. For the BRST invariance under Q_1, we must compute the OPEs of (4.74) with the supercurrent (4.35) of the $SL(2,\mathbb{R})$ factor

$$G^{sl} = \frac{2}{n_5}\Big[\eta_{ab}\psi_{sl}^a j_{sl}^b - \frac{i}{3n_5}\varepsilon_{abc}\psi_{sl}^a\psi_{sl}^b\psi_{sl}^c\Big]. \tag{4.76}$$

However, each term of the vertex operator already contains one fermion field. Therefore, when computing the OPEs of the vertex operator with the second addend in the supercurrent, we would get some first-order pole, not important in our computation. The relevant part of the supercurrent is then, just

$$\tilde{G}^{sl} = \frac{2}{n_5}\eta_{ab}\psi_{sl}^a j_{sl}^b = \frac{2}{n_5}\Big[\frac{1}{2}\Big(\psi_{sl}^+ j_{sl}^- + \psi_{sl}^- j_{sl}^+\Big) - \psi_{sl}^3 j_{sl}^3\Big].$$

We focus on the second-order pole of the OPE $\hat{G}^{sl}(z)V_{sl}(w)$ (for shortness we avoid writing the spectator operators $\Phi_{j',m'}^{su}e^{iEt}e^{iPy}e^{i\vec{k}\cdot\vec{Y}}$):

$$\begin{aligned} &\frac{2}{n_5}\Big[\frac{1}{2}\Big(\psi_{sl}^+ j_{sl}^- + \psi_{sl}^- j_{sl}^+\Big) - \psi_{sl}^3 j_{sl}^3\Big](z)\Big[\Big(\xi_+^{sl}\psi_{sl}^+\Phi_{j,m-1}^{sl} + \xi_-^{sl}\psi_{sl}^-\Phi_{j,m+1}^{sl} + \xi_3^{sl}\psi_{sl}^3\Phi_{j,m}^{sl}\Big)\Big](w) \sim \\ &\qquad \sim \frac{\xi_+^{sl}(m-j) + \xi_-^{sl}(m+j) + m\xi_3^{sl}}{(z-w)^2}, \end{aligned} \tag{4.77}$$

where we have used (4.34), (4.38) and (4.39). Imposing that its coefficient vanishes we end up with the constraint

$$\xi_+^{sl}(m-j) + \xi_-^{sl}(m+j) + m\xi_3^{sl} = 0. \tag{4.78}$$

We can now impose the invariance under the reparametrization BRST charge for the other two vertex operators in (4.73). This allows to get the constraints for the other components of the polarization tensor:

$$\xi_+^{su}(j'+m') + \xi_-^{su}(j'-m') + m'\xi_3^{su} = 0, \tag{4.79}$$

$$+\frac{1}{4}\big[-\xi_t E + \xi_y P + \vec{k}\xi\big] = 0, \tag{4.80}$$

which hold respectively when we have excitations only on $SU(2)$ or on $SL(2,\mathbb{R})$. The general vertex operator, however, contains excitations on any factor of the Lie group where the dynamics takes place: in this case, just the sum of the three contributions in (4.78), (4.79), (4.80) and not each single term, is required to vanish.
For the BRST invariance under the gauging, we have to look at the OPE of the BRST charge $Q_{\mathbb{I}}$ given in (4.66). It is made up of two terms, which yield two linearly independent OPEs (the two terms involve different gauging ghosts). We only need, then, look at the first-order pole of the OPEs of the currents $\mathcal{I}$ and Ψ with the full vertex operator V, sum of the three contributions in (4.73):

$$\begin{aligned}\mathcal{I}(z)(V_{sl}+V_{su}+V_{\mathbb{R}})(w) &\sim \frac{l_1 m + l_2 m' + l_3 E/2 + l_4 P_y/2}{z-w}(V_{sl}+V_{su}+V_{\mathbb{R}})(w)\\ \Psi(z)(V_{sl}+V_{su}+V_{\mathbb{R}})(w) &\sim \frac{-\frac{n_5}{2}l_1\xi_3^{sl}+\frac{n_5}{2}l_2\xi_3^{su}-\frac{1}{2}l_3\xi_t+\frac{1}{2}l_4\xi_y}{z-w}\Phi^{sl}_{j,m}\Phi^{su}_{j',m'}e^{-iEt}e^{iPy}e^{i\vec{k}\cdot\vec{Y}}.\end{aligned} \tag{4.81}$$

For the computation of the first OPE we have just used (4.34), (4.22) and (4.41). For the computation of the first OPE, besides (4.45), we have considered that V^{sl} and V^{su} are such that

$$J^3_{sl,0}V^{sl} = mV^{sl}, \qquad J^3_{su,0}V^{su} = mV^{su}. \tag{4.82}$$

This can be calculated explicitly by splitting $J^3 = j^3 + \hat{j}^3$. For instance, we find that:

$$j^3_{sl}(\xi^{sl}_+\psi^+_{sl}\Phi^{sl}_{j,m-1}) = (m-1)\xi^{sl}_+\psi^+_{sl}\Phi^{sl}_{j,m-1},$$

where we have again neglected the spectator operators and

$$\hat{j}^3_{sl}(\xi^{sl}_+\psi^+_{sl}\Phi^{sl}_{j,m-1}) = \mathrm{Res}\Big[-\frac{2i}{n_5}\varepsilon^3{}_{+-}(\psi^+_{sl}\psi^-_{sl})(z)(\xi^{sl}_+\psi^+_{sl}\Phi^{sl}_{j,m-1})(w)\Big] = (\xi^{sl}_+\psi^+_{sl}\Phi^{sl}_{j,m-1}),$$

where we have used the fact that $-i\varepsilon^{3-}{}_{-} = 1$ and (4.34). Accordingly, we impose the two residues in (4.81) to be zero:

$$\begin{aligned}&l_1 m + l_2 m' + l_3 E + l_4 P = 0,\\ &-\frac{n_5}{2}l_1\xi_3^{sl}+\frac{n_5}{2}l_2\xi_3^{su}-\frac{1}{2}l_3\xi_t+\frac{1}{2}l_4\xi_y = 0.\end{aligned} \tag{4.83}$$

4.5.2 The solutions for the constraints

The first two vertex operators in (4.73) are representations of the tensor product $1\otimes j$ of the three fermion vertex operators and the bosonic primaries $\Phi^{sl}_{j,m}$ or $\Phi^{su}_{j',m'}$ respectively. As suggested by (4.82), these vertex operators can be seen as representations of the current algebras of $SL(2,\mathbb{R})$ and $SU(2)$ with m as eigenvalue of the third component. For instance, for $SL(2,\mathbb{R})$, they satisfy

$$\begin{aligned}J^3_{sl,0}V^{sl}_{h,m} &= mV^{sl}_{h,m},\\ J^{\pm}_{sl,0}V^{sl}_{h,m} &= (m\mp(h-1))V^{sl}_{h,m\pm1},\end{aligned} \tag{4.84}$$

where besides m, the first quantum number h of the current algebra has been associated to the vertex operator. The allowed values for h are $j-1, j, j+1$ and the Clebsch-Gordan coefficients allow to express a representation $V^{sl}_{h,m}$ of this current algebra in terms of the elements of the tensor product with quantum number m. If, for example, $h = j-1$,

$$V^{sl}_{j-1,m} = a_m\psi^3_{sl}\Phi^{sl}_{j,m} + b_m\psi^+\Phi^{sl}_{h,m-1} + c_m\psi^-\Phi^{sl}_{h,m+1}. \tag{4.85}$$

Let us notice, comparing with (4.73), that the components of the polarization tensor are exactly the Clebsch-Gordan coefficients a_m, b_m, c_m. The procedure to determine them is the same for each

h and also for $SU(2)$. We illustrate it for the vertex operator $V^{sl}_{j-1,m}$. We apply $J^-_{sl} = j^-_{sl} + \hat{j}^-_{sl}$ to both sides of (4.85):

$$J^-_{sl} V^{sl}_{j-1,m} = (m+j-2)V^{sl}_{j-1,m-1} = (m+j-2)\big(a_{m-1}\psi^3_{sl}\Phi^{sl}_{j,m-1} + b_{m-1}\psi^+_{sl}\Phi^{sl}_{j,m-2} + c_{m-1}\psi^-_{sl}\Phi^{sl}_{j,m}\big),$$
$$J^-_{sl} V^{sl}_{j-1,m} = (a_m(m+j-1)+2b_m)\psi^3_{sl}\Phi^{sl}_{j,m-1} + b_m(m+j-2)\psi^+_{sl}\Phi^{sl}_{j,m-2} + (a_m + c_m(m+j))\psi^-_{sl}\Phi^{sl}_{j,m}.$$

Furthermore, let us consider (4.85) for $V_{j-1,m-1}$. If we act on both sides with $J^+_{sl} = j^+_{sl} + \hat{j}^+_{sl}$ we get that:

$$\begin{aligned} J^+_{sl} V^{sl}_{j-1,m-1} &= (m-j+1)V^{sl}_{j-1,m} = (m-j+1)\big(a_m\psi^3_{sl}\Phi^{sl}_{j,m} + b_m\psi^+_{sl}\Phi^{sl}_{j,m-1} + c_m\psi^-_{sl}\Phi^{sl}_{j,m+1}\big), \\ J^+_{sl} V^{sl}_{j-1,m-1} &= (a_{m-1}(m-j)-2c_m)\psi^3_{sl}\Phi^{sl}_{j,m} + \big(b_{m-1}(m-j-1) - a_{m-1}\big)\psi^+_{sl}\Phi^{sl}_{j,m-1} + \\ &\quad + c_{m-1}(m-j+1)\psi^-_{sl}\Phi^{sl}_{j,m+1}. \end{aligned}$$

We can equate the corresponding coefficients of both the expressions for $J^-_{sl}V^{sl}_{j-1,m}$ and perform the same identification for $J^+_{sl}V^{sl}_{j-1,m-1}$: we find a system of six equations for the six unknown quantities $a_m, b_m, c_m, a_{m-1}, b_{m-1}, c_{m-1}$. The found values can be inserted in (4.85), which up to an overall rescaling reads

$$V^{sl}_{j-1,m} = \psi^3_{sl}\Phi^{sl}_{j,m} - \frac{1}{2}\psi^+_{sl}\Phi^{sl}_{j,m-1} - \frac{1}{2}\psi^-_{sl}\Phi^{sl}_{j,m+1}. \tag{4.86}$$

Once we have determined all the Clebsch-Gordan coefficients we end up with:

$$\begin{aligned} V^{sl}_{j+1,m} &= (j+m)(j-m)\psi^3_{sl}\Phi^{sl}_{j,m} + \frac{1}{2}(j+m)(j+m-1)\psi^+_{sl}\Phi^{sl}_{j,m-1} + \frac{1}{2}(j-m-1)(j-m)\psi^-_{sl}\Phi^{sl}_{j,m+1}, \\ V^{sl}_{j,m} &= m\psi^3_{sl}\Phi^{sl}_{j,m} - \frac{1}{2}(j-1+m)\psi^+_{sl}\Phi^{sl}_{j,m-1} + \frac{1}{2}(j-1-m)\psi^-_{sl}\Phi^{sl}_{j,m+1}, \\ V^{sl}_{j-1,m} &= \psi^3_{sl}\Phi^{sl}_{j,m} - \frac{1}{2}\psi^+_{sl}\Phi^{sl}_{j,m-1} - \frac{1}{2}\psi^-_{sl}\Phi^{sl}_{j,m+1}, \\ V^{su}_{j'+1,m'} &= = \psi^3_{su}\Phi^{su}_{j',m'} - \frac{1}{2}\psi^+_{su}\Phi^{su}_{j',m'-1} + \frac{1}{2}\psi^-_{su}\Phi^{su}_{j',m'+1}, \\ V^{su}_{j',m'} &= m'\psi^3_{su}\Phi^{su}_{j',m'} + \frac{1}{2}(j'+1-m')\psi^+_{su}\Phi^{su}_{j',m'-1} + \frac{1}{2}(j'+1+m')\psi^-_{sl}\Phi^{sl}_{j',m'+1}, \\ V^{su}_{j'-1,m'} &= (j'-m')(j'+m')\psi^3_{su}\Phi^{su}_{j',m'} + \frac{1}{2}(j'-m')(j'-m'+1)\psi^+_{su}\Phi^{su}_{j',m'-1} + \\ &\quad - \frac{1}{2}(j'+m')(j'+m'+1)\psi^-_{sl}\Phi^{sl}_{j',m'+1}. \end{aligned} \tag{4.87}$$

Let us focus on the instance in which there are no excitations both on $\mathbb{R}_t \times S^1_y$ and on the torus. Starting from the expansions above, we would like to identify the physical vertex operators. The Clebsch-Gordan coefficients of $V^{sl}_{j+1,m}$ and $V^{sl}_{j-1,m}$ satisfy the constraint (4.78), as well as the coefficients of $V^{su}_{j'+1,m'}$ and $V^{su}_{j'-1,m'}$ fulfil (4.79). As a consequence, these are physical vertex operators when considered singularly, as long as the other constraints, i. e. (4.75) and (4.83), are also fulfilled. The states arising only from $V^{sl}_{j,m}$ or $V^{su}_{j',m'}$ are not physical since they are not consistent respectively with (4.78) and (4.79). Indeed, if we call ε^{sl}_b and ε^{sl}_b respectively the Clebsch-Gordan coefficients of $V^{sl}_{j,m}$ or $V^{su}_{j',m'}$, we obtain:

$$\begin{aligned} \varepsilon^{sl}_+(m-j) + \varepsilon^{sl}_-(m+j) + m\varepsilon^{sl}_3 &= j(j-1), \\ \varepsilon^{su}_+(j'+m') + \varepsilon^{su}_-(j'-m') + m'\varepsilon^{su}_3 &= j'(j'+1). \end{aligned} \tag{4.88}$$

However, let us notice that the vertex operator $\tilde{V} = V^{sl}_{j,m} - V^{su}_{j',m'}$ is a good candidate as a vertex operator. As a matter of fact, when the further requirement

$$j = j' + 1 \tag{4.89}$$

holds, the difference of the two constraints in (4.88) vanishes. This is a realization of the necessary and sufficient condition for the invariance of $aV^{sl} + bV^{su}$ under the BRST charge Q_1: the linear combination of (4.78) and (4.79) with the same coefficients a and b, has to vanish and not each of the constraints singularly. In addition to this, we notice that requiring the validity of (4.75) with excitations on $AdS_3 \times S_3$ only, is equivalent to require (4.89): any other combination of the vertex operators $V^{sl}_{j,m}$ and $V^{su}_{j,m}$ would have violated the mass shell condition. However, as explained in [35], this vertex operator is not physical: it is BRST exact and then it does not belong to the cohomology.
We can move, instead, to the case in which there are fermionic excitations on the $\mathbb{R}^t \times S^1_y$ sector. We know $P_y = n_y/R_y$ and then, due to the mass-shell condition for these massless excitations, $E = n_y/R_y$ as well. In the following we impose $R_y = 1$. The vertex operators, then, read

$$V^{\pm}_{ty} = \left(\lambda^t \pm \lambda^y\right) e^{-in_y(t-y)}. \tag{4.90}$$

We want to determine the corresponding physical vertex operators. The invariance under the BRST charge Q_1 amounts to calculate the OPE

$$G^{\mathbb{R}}(z) V^{\pm}_{ty}(w) \sim -\frac{i}{4(z-w)^2}(E \pm n_y) e^{-in_y(t-y)}, \tag{4.91}$$

where the supercurrent $G^{\mathbb{R}}$ is given by (4.43) and the OPEs (4.41) have been used. Since $E = n_y$, we have that V^{+}_{ty} cannot be BRST invariant. Nonetheless, the physical vertex operators will be constructed defining

$$\tilde{V}^{sl} = 2n_y V^{sl}_{j,m} - j(j-1)V^{+}_{ty}, \qquad \tilde{V}^{su} = 2n_y V^{su}_{j',m'} - j'(j'+1)V^{+}_{ty}. \tag{4.92}$$

This way, when considering the total supercurrent $G = G^{sl} + G^{su} + G^{\mathbb{R}}$, we get that

$$G(z)\tilde{V}^{sl}(w) \sim \frac{2n_y\left[\varepsilon^{sl}_{+}(m-j) + \varepsilon^{sl}_{-}(m+j) + m\varepsilon^{sl}_{3}\right] - 2n_y j(j-1)}{(z-w)^2} = 0,$$

where we have used (4.77), (4.88) and (4.91). Therefore, $\tilde{V}^{sl}$ and, for the analogous reason, $\tilde{V}^{su}$ are invariant under the reparametrization BRST operator. Let us shift to the constraints arising from the BRST gauging invariance. In particular, we would like to find combinations of the vertex operators so far defined, such that

$$\Psi(z) \cdot V(w) = 0. \tag{4.93}$$

In this expression, Ψ is given by (4.51) and the dot in between is a short notation for the residue of the OPE. Let us define

$$\begin{aligned}
c_1 &= \Psi \cdot V^{sl}_{j+1,m} = -l_1 \frac{n_5}{2}(j+m)(j-m), & c_2 &= \Psi \cdot V^{sl}_{j-1,m} = -l_1 \frac{n_5}{2}, \\
c_3 &= \Psi \cdot V^{su}_{j'-1,m'} = l_2 \frac{n_5}{2}(j'+m')(j'-m'), & c_4 &= \Psi \cdot V^{su}_{j'+1,m'} = l_2 \frac{n_5}{2}, \\
c_5 &= \Psi \cdot \tilde{V}^{sl} = -n_y n_5 m l_1 - \frac{1}{2} j(j-1)(l_3 - l_4), & c_6 &= \Psi \cdot \tilde{V}^{su} = n_y n_5 m' l_2 - \frac{1}{2} j'(j'+1)(l_3 - l_4).
\end{aligned} \tag{4.94}$$

To calculate these residues we have made use of the algebra for the fermions of the three factors and of the Clebsch-Gordan coefficients in (4.87). The physical vertex operators, BRST invariant but not exact, are then given by

$$\begin{aligned}
W^{sl}_1 &= V^{sl}_{j+1,m} - \frac{c_1}{c_5}\tilde{V}^{sl}, & W^{sl}_2 &= V^{sl}_{j-1,m} - \frac{c_2}{c_5}\tilde{V}^{sl}, \\
W^{su}_1 &= V^{su}_{j'+1,m'} - \frac{c_4}{c_6}\tilde{V}^{su}, & W^{su}_2 &= V^{su}_{j'-1,m'} - \frac{c_3}{c_6}\tilde{V}^{su}.
\end{aligned} \tag{4.95}$$

In particular, in the two-charge supertube case, the parameters l_i and r_i are given by (3.58) or (3.59) according to the duality frame. In this instance, then, the vertex operators are determined by (4.95), with the c_is calculated through (4.94) and imposing the condition (3.58) or (3.59).

Conclusions

The purpose of this thesis has been the study of a particular realization of the fuzzball microstates in a full worldsheet theory. In order to accomplish this goal, we have analyzed the most helpful aspects of String Theory: conformal field theories, Polyakov path integral, BRST quantization both for the bosonic and the supersymmetric strings. Furthermore, the non-linear sigma model has been introduced, together with the effective actions for the different types of superstring theories. The dualities have been illustrated in order to show the equivalence between all these theories.
After a discussion about the thermodynamic properties of black holes and some linked inconsistencies, the area of the horizon has been calculated in some black holes solutions in supergravity: the three-charge and four-charge BPS configurations. Through the Bekenstein-Hawking formula, then, we have determined the corresponding entropy. For the three-charge black hole the same outcome has been recovered with a suitable counting of the microstates producing the same macroscopic state and then applying the Boltzmann equation: this is one of the major successes of String Theory as a Quantum Gravity candidate. We have then, introduced the fuzzball microstates for the two-charge black holes, which in this framework emit the Hawking radiation unitarily.
One particular realization of fuzzballs has been described through the gauged Wess-Zumino-Witten model, an exactly solvable theory describing the dynamics of strings on a curved background. The latter used in the studied instance is the coset G/H of two properly chosen Lie groups. We have verified that the results coincide with the supergravity predictions, both in the case of one charge, i. e. NS5-branes on a circle, and of two charges, i. e. supertubes of NS5-branes. In this instance, however, the results have been obtained from the viewpoint of a worldsheet theory. The spectrum can be then constructed. We have outlined the fundamentals of the current algebra and the Sugawara stress-energy tensor, which have been adapted to the three factors of the Lie group G. We have performed the path-integral quantization of the worldsheet theory and then focussed on the Neveu-Schwarz sector of the spectrum. We have written the constraints for the physical vertex operators, arising from the reparametrization and gauge invariance of the theory and we have eventually found solutions for them. One possible extension of this topic would deal with the Ramond sector of the spectrum.

Compared to the supergravity approach, the worldsheet theory widens the knowledge of the fuzzball microstates and opens up the way for further progress. In particular:

- it allows to study the dynamical properties of the fuzzballs, for instance the absorption and the emission processes;
- it could be unavoidable in order to find typical microstates for three-charge black holes, which still lack a supergravity description;
- using the vertex operators determined for the two-charge black holes, correlators can be computed and compared with protected correlators in the dual D1-D5 conformal field theory.

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