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Tesi di Laurea

An L^p-error study of Fourier-based quantization applied to option pricing

Studio sull'errore L^p della quantizzazione Fourier-based applicata al prezzaggio di opzioni

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Abstract

Financial markets have always been led by uncertainty, therefore Probability theory is an essential tool to model them and to tackle pricing and hedging of options. Specifically, instruments used also in Statistical Physics and Complex Systems, such as stochastic processes (Brownian motion in particular), are the core of models for the dynamics of underlying assets. As these models are based on stochastic differential equations, their solutions cannot be always found analytically. In such cases pricing, which in practice is equivalent to computing a (conditional) expectation, is performed, e.g., via numerical procedures, such as Monte-Carlo or quantization, to cite a few of them. Quantization is a discretization procedure: when applied to a (continuous) random variable, it produces a set of points which optimally (in a sense to be made precise) approximates the original distribution. The aim of this work is presenting and discussing quantization techniques. Moreover, the analysis is focused on a particular quantization method, based on the Fourier transform, and on its L^p error. As an application, in the famous Black-Scholes model a study of the L^p quantization error, for different values of $p \geq 1$ is performed, with a special focus on pricing.

Sommario

I mercati finanziari sono stati da sempre guidati dall'incertezza, perciò la teoria delle probabilità è uno strumento essenziale per modellizzarli e affrontare il prezzaggio e la copertura delle opzioni. Nello specifico, gli strumenti usati anche in Fisica Statistica e Sistemi Complessi, come i processi stocastici (il moto browniano in particolare), sono il nucleo dei modelli per la dinamica degli assets sottostanti. In questi casi il prezzaggio, che in pratica è equivalente a calcolare un'aspettazione (condizionale), è effettuato ad esempio tramite metodi numerici, come Monte-Carlo o quantizzazione, per citarne alcuni. La quantizzazione è una procedura di discretizzazione: quando applicata a una variabile aleatoria (continua), genera una serie di punti che ottimamente (in un senso che definiremo in seguito) approssimano la distribuzione originale. Lo scopo di questo lavoro è presentare e discutere queste tecniche di quantizzazione. Inoltre, l'analisi è focalizzata su un particolare metodo di quantizzazione, basato sulla trasformata di Fourier, e sul suo errore L^p. Come applicazione viene mostrato uno studio dell'errore di quantizzazione L^p nel famoso modello di Black-Scholes, per diversi valori di $p \geq 1$, con uno speciale focus sul prezzaggio.

Contents

Introduction

Financial markets are multi-agent systems, where plenty of buyers and sellers interact to exchange any type of goods, called assets. In order for investors to hedge themselves, a class of financial instruments called options was invented. Basically, these tools are contracts that enable the holder to buy (or sell) an asset, called underlying (it can be a commodity, a stock, or whatever), at a predetermined price on a specific date.
The option price should be fair, in the sense that the probability to profit is equal for both, the underwriter and the holder. To pursue this purpose, many models have been developed, so-called option pricing models, which consist of stochastic differential equations. These equations depend on some parameters and, most important, on the underlying asset price, for this reason these contracts are also called derivatives. Stochastic calculus is essential to describe the probability nature of the underlying asset price. A stochastic differential equation, which has not an analytical solution, can be solved via numerical methods, for instance Monte-Carlo or quantization.
The quantization technique is a fundamental concept in signal processing theory that involves the representation of continuous signals using a finite number of discrete values. It has its root from the 1950s decade in the Bell laboratories, where its usage consisted in optimally discretizing a continuous (stationary) signal in view of its transmission. Although Signal Processing was its original application field, several fields started to adopt it after its discovery, such as audio and image processing, information theory, pattern recognition, operations research, and our research topic, mathematical finance.
In financial engineering, quantization techniques are commonly used in option pricing models to approximate the continuous stochastic process of the underlying asset price with a finite set of discrete states. The idea behind this approach is to reduce the complexity of the underlying stochastic process while maintaining the accuracy of the pricing model.
This thesis will focus on a particular quantization method that involves the Fourier transform, developed by Callegaro et al. (2019), and we will test this method on the Black-Scholes model for option pricing, whose solutions are already known since they can be found analytically. Because of that, we are allowed to use them as a benchmark to test the outputs of the quantization method. Specifically, the aim of this thesis is to study the results of this method for different values of p in the L^p norm.
At first, we want to see if the guessings, made in Callegaro et al. (2019, Sect. 2.4), are true and how the quantization grids behave for different p-values. Furthermore, the main goal consists of understanding if the pricing of European Call options gets better by changing the p-norm for the values of the strike price lower or higher than the underlying asset price.
The thesis is organized as follows: Chapter 2 builds some basic concepts of stochastic processes and Brownian motion to better understand what is an option pricing model, such as Black-Scholes and gives some vocabulary about financial terms. Chapter 3 starts from the key bases of probability theory and explains how the quantization method works and the contribution of Fourier transform. Chapter 4 reports the calculation made and exposes the results achieved with some comments and considerations. Finally, in Chapter 5 we express our conclusion.

Brownian motion and option pricing models

2.1 Brownian motion: definitions and applications

In its origin, Brownian motion models the seemingly random and unpredictable movement of particles suspended in a fluid or gas, which was first observed by the botanist Robert Brown in 1827. Brownian motion has since become a fundamental tool in Physics for the description of random motion, i.e. random walk. Due to its ductility to describe phenomena that involve randomness and probability, it has found applications in a wide range of fields, including Chemistry, Biology, Finance, and Engineering.
The first pioneering work was done by Einstein (1905) in his famous paper "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen" (translation: "On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat") where it has helped to establish the existence of atoms and molecules. It is, noticeably, one of the first applications of this concept outside Physics by Louis Bachelier in his "The Theory of Speculation" (1900), who studied the potentiality of this tool in financial markets. The first one who developed a rigorous mathematical theory of Brownian motion was Norbert Wiener in 1923.

For the aims of this thesis, we are going to give some definitions of stochastic processes and Brownian motion, in such a manner we have the mathematical basis to understand an option pricing model. For the mathematical references, the main sources are Mikosch (1998) and Björk (2009).
First, let us define what a stochastic process is.

Definition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T \subset \mathbb{R}$. A **stochastic process** is a collection of random variables $(X_t,\, t \in T) = (X_t(\omega),\, t \in T,\, \omega \in \Omega)$ defined on a common Ω.*

Recall that Ω is the sample space and ω is a possible outcome, t is the time variable. For our purposes, we can take T as the interval $[0, +\infty)$. It is evident that $X_t(\omega)$ is a function of two variables:

- fixing t, i.e. considering each time step, $X(\omega)$ is a random variable
- fixing ω, X_t is a sequence of real numbers called realizations, which represents a trajectory or a sample path of the process X.

Brownian motion is then a particular type of stochastic process, which enjoys some properties listed below.

Definition 2. *A **Brownian motion** (or Wiener process) $B = (B_t,\, t \in [0, +\infty))$ is a stochastic process which satisfies:*

1. *For $0 \leq s < t$, the random variable $(B_t - B_s)$ has a normal distribution $\mathcal{N}(0, t-s)$* [1]*;*

2. *B has independent increments;* [2]

3. *$B_0 = 0$;*

4. *$P(B \in C[0, +\infty)) = 1$, it means that the sample paths of B are continuous with probability 1, i.e. there are no "jumps".*

Considering a standard Brownian motion B_t, the process

$$X_t = rt + \sigma B_t, \qquad t \in [0, T]$$

is called *Brownian motion with drift*, where $r \in \mathbb{R}$ is the drift and $\sigma > 0$ is the diffusion coefficient.

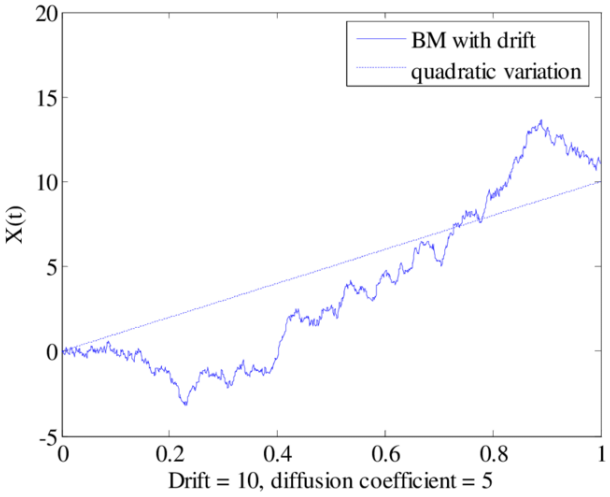


Figure 2.1: Example of Brownian motion with drift from Kasumo (2011).

This concept takes us closer to the definition of Geometric Brownian Motion, a stochastic process largely used in Finance, where it has a key role in the formulation of option pricing models. Let us report the definition as in Björk (2009, Sect. 5.2).

Definition 3. *A **Geometric Brownian motion** is a stochastic process X which satisfies the stochastic differential equation (SDE):*

$$dX_t = rX_t dt + \sigma X_t dB_t \qquad X_0 = x_0, \tag{2.1}$$

where $r \in \mathbb{R}$ is the drift, $\sigma > 0$ is the diffusion coefficient and B is a standard Brownian motion.

[1] For the Gaussian distribution is used the notation $\mathcal{N}(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance.

[2] X has independent increments when $\forall t_i \in T$ with $t_1 < ... < t_n$ and $n \geq 1$, $X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent random variables.

It is known that the corresponding deterministic linear equation has as a solution an exponential function of time. Moreover, in stochastic calculus, the Itô formula expresses the differential for a generic function $Z_t = f(t, X_t)$, where f is a sufficient differentiable function. We report it below as in Björk (2009, Th. 4.10),

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dB_t. \tag{2.2}$$

In such a manner, we can define $Z_t = \log X_t$ where it is assumed that X is strictly positive (see Prop. 1) and applying the Itô formula to it, the result is:

$$dZ_t = \left(r - \frac{1}{2}\sigma^2 \right) dt + \sigma dB_t \qquad Z_0 = \log x_0,$$

and this equation can be integrated directly since the integral doesn't require existence hypothesis because the volatility σ is constant. Thus, we obtain

$$Z_t = \log x_0 + \left(r - \frac{1}{2}\sigma^2 \right) t + \sigma B_t.$$

After these calculations, the next Proposition is pretty clear.

Proposition 1. *The solution of* (3) *is*

$$X_t = x_0 \exp\left\{ \left(r - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right\}. \tag{2.3}$$

Here we can see an example of a trajectory of a Geometric Brownian motion.

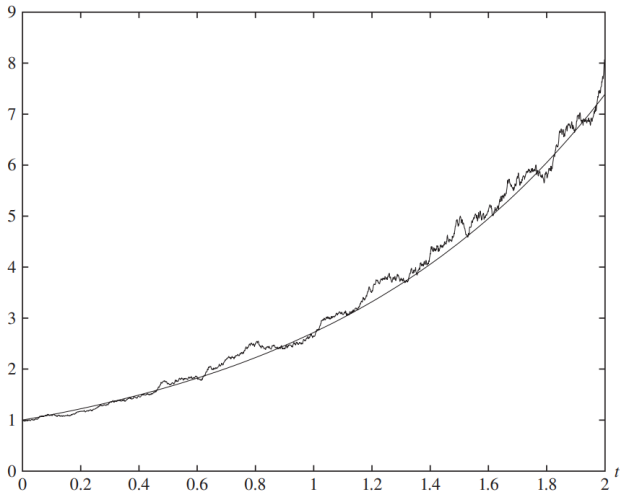


Figure 2.2: Example of Geometric Brownian motion from Björk (2009): $r = 1$, $\sigma = 0.2$ and $x_0 = 1$.

2.2 Financial assets and options

Let us now introduce some basic concepts about financial markets. Any regulated market consists of goods and people who want to trade them. In particular, sellers are interested to

sell at the highest possible price and, on the other hand, buyers are interested to buy at the lowest possible price. Nowadays, most trading assets are represented by commodities, stocks, and cryptocurrencies. A special instrument that helps investors to deal with the risk that an asset's price can increase or decrease too much is called an option. Options are financial contracts that give the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (strike price) and time (maturity). Options can be used to hedge against risk or to speculate on market movements, making them a valuable tool for investors. They can be divided into two categories: call options and put options.

- A **Call option** gives the holder the right to buy the underlying asset at a predetermined price.
- A **Put option** gives the holder the right to sell the underlying asset at a predetermined price.

Within these two categories, there are several different types of options, including European options, American options, and exotic options.
For the aims of this thesis, we will focus only on European call options. Let us give a definition as in Björk (2009):

Definition 4. *A **European Call option** on the underlying asset S, with strike price (or exercise price) K and maturity (or exercise date) T is a contract written at time $t = 0$ which respects the following properties:*

- *the holder of the contract has the right to buy the underlying asset S at the price $K > 0$ from the underwriter of the option;*
- *the holder of the option has no obligation to buy the underlying asset S;*
- *the right to buy the underlying asset S at the price K can only be exercised at the precise time $t = T$.*

A Put option, instead, gives to the holder the right to sell the underlying asset S at the strike price K.

Remark 1. *The difference between European and American options is that for the first ones, the right can only be exercised at exactly the date of expiration $t = T$; for the second ones, the right can be exercised at any time before the maturity, i.e. $0 < t \leq T$.*

The payoff of our European call option is given from the following formula:

$$X^{call} = (S_T - K)^+ = \max(S_T - K,\, 0) = \begin{cases} S_T - K & \text{if} \quad S_T > K \\ 0 & \text{if} \quad S_T \leq K, \end{cases} \tag{2.4}$$

where S_T is the underlying asset price at the time $t = T$ and K is the strike price. In this sense, an option is called a derivative asset since its value depends on the underlying asset. In financial terms, at a fixed time t a Call option is said to be

- "in the money" if $S_t > K$,
- "out of the money" if $S_t < K$,

- "at the money" if $S_t = K$.

Now the challenge consists in determining the right price [3] for the option. Since it is known only the underlying asset price at time $t = 0$, namely S_0, we should do a future stochastic claim to estimate the random variable S_T, the value of the underlying asset at the maturity T. In order to solve this problem, it is necessary to build some assumptions (more or less restrictive) and create a stochastic model to estimate the fair option price.

2.3 Option pricing models: Black-Scholes equation

One of the most widely used models for option pricing is the Black-Scholes model, which was developed by Fischer Black and Myron Scholes in 1973. Since the equation, for the evolution of S, is analytically solvable, it is easier to work with it and compare its solutions with other methods for solving stochastic differential equations (SDE). For these reasons, this model will be the benchmark for our quantization method resolution. But first, let us give a quick overview of options pricing models and then go deeper into the Black-Scholes solution formula. There are several types of option pricing models, each with its own assumptions, inputs, and outputs. They can be in discrete time, as the binomial model, or in continuous time, as Black-Scholes itself, or other more complex models. Just to cite a few of them, they are the Heston model and double-Heston, the Bates model, and so on. The application of the quantization method to these models can be found in Callegaro et al. (2019).
The hypothesis we need to assume for the Black-Scholes model, as reported in Hull (2003), are the following ones:

- The underlying asset price follows a Geometric Brownian motion.
- There are no transaction costs or taxes.
- The market is efficient [4] and there are no arbitrage opportunities.
- The risk-free interest rate and volatility are constant and known.
- The option is European-style.

[3] i.e. a price that excludes any arbitrage opportunities
[4] The price of an asset reflects all the available information

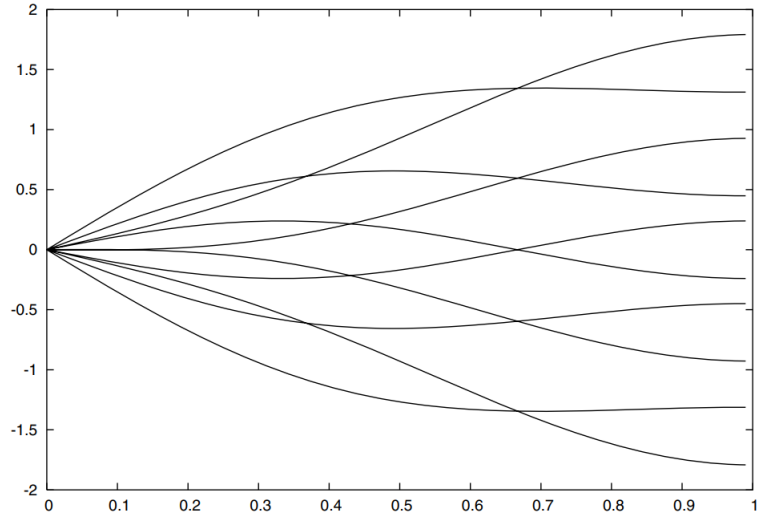


Figure 2.3: Example of the functional quantization for a standard Brownian motion with $N = 10$, from Pagès and Printems (2005).

Let us, from now on, uniform the notation and indicate the underlying asset price at time t as S_t, as in Section 3.3, where S_T is the price at the maturity and S_0 the initial price. The stochastic differential equation that Black and Scholes found is:

Proposition 2.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.5}$$

where $V(S,t)$ is the price of our option, which it can be called $C(S,t)$ in the case of a Call option, r is the risk-free interest rate, and σ the volatility of our underlying asset which follow a Geometric Brownian motion.

Proof See the derivation in Hull (2003, Sect. 15.6).
To find the price for a European call option, we use a risk-neutral valuation approach, as in Björk (2009, Sect. 7.4), which means that the option price can be written as:

$$C(t) = e^{-r(T-t)}\mathbb{E}[\max(S_T - K, 0)]. \tag{2.6}$$

The economic interpretation of the formula (2.6) is that the price of the derivative is calculated by taking the expectation of the final payment $\mathbb{E}[\max(S_T - K, 0)]$ and then discounting this expected value for the present value using the discount factor $e^{-r(T-t)}$.
Considering that, the solution can be expressed in the next proposition.

Proposition 3. *The price of a European call option with strike price K and time of maturity T is*

$$C = S_0 N(d_1) - e^{-r(T-t)} K N(d_2) \tag{2.7}$$

where $N(d_1)$ and $N(d_2)$ are the cumulative distribution functions of the standard normal distribution $\mathcal{N}(0,1)$ and

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Proof See the derivation in Björk (2009, Sect. 7.5).
In Chapter 4, we will compare the analytical solutions of the Black-Scholes formula (2.7) with the quantization method applied to this model, explained in Chapter 3.

Quantization method

3.1 Concepts from probability theory

At first, it is useful to introduce the probability background and some definitions.
Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^d$-valued random vector X with distribution function $\mu : \mathbb{R}^d \to [0, 1]$.
To fix the ideas, let us introduce a few key concepts (see Klenke (2013)).

Definition 5. *A **probability space** is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where:*

- *Ω is a set of all possible elementary outcomes ω;*
- *$\mathcal{F}$ is a σ-algebra [1];*
- *$\mathbb{P}$ is a probability measure on $\mathcal{F}$ [2].*

A practical example to better understand this definition is related to the random experiment of tossing a double-face coin. If we flip the coin, only two outcomes are possible, head (H) or tail (T). This means that $\Omega = \{H, T\}$ and $\mathcal{F}$ is the power set of Ω [3], i.e. $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$. The probability measures are $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}$ and $\mathbb{P}(\{H, T\}) = 1$.

On Section 3.3, we will use a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the filtration satisfies the usual hypotheses [4].
A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a standard probability space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, which is defined as following:

Definition 6. *A **filtration** on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-algebras of $\mathcal{F}$, i.e., $\mathcal{F}_t \subset \mathcal{F}$ is a σ-algebra $\forall t \in [0, T]$ and if $s \leq t$ with $s, t \in [0, T]$, $\mathcal{F}_s \subset \mathcal{F}_t$.*

In simpler terms, a filtered probability space is a way of adding more information to a basic probability space, by defining a series of progressively more detailed "levels" of information. For instance, in the stock market, we might define a filtered probability space that includes information about stock prices at various points in time, with each "level" of the set adding more information about the future price movements of the stock. This makes this concept useful in various fields of probability theory and stochastic calculus, where it allows us to model complex systems that evolve over time, such as financial markets or future weather conditions.

[1] $\mathcal{F}$ is a σ-algebra if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complements and countable unions.

[2] $\mathbb{P} : \mathcal{F} \to [0, 1]$ where $\mathbb{P}$ is normalized and σ-addictive.

[3] Only because the outcome set is discrete and finite.

[4] Filtration $(\mathcal{F}_t, \mathbb{P})$ satisfies the usual condition if $\mathcal{F}_0$ is complete in $\mathbb{P}$ and $\forall t \geq 0 \quad \mathcal{F}_t = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$ (right-continuous)

To conclude, the $\mathbb{R}^d$-valued random vector X is a d-dimensional vector $X = (X_1, ..., X_d)$ where each component X_i is a real-valued random variable [5]. A definition can be expressed as follow:

Definition 7. *A **distribution function** $\mu(x)$ for a random vector $X \in \mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is:*

$$\mu(x) := \mathbb{P}(X_1 \leq x_1, ..., X_d \leq x_d) = \mathbb{P}(\{\omega \in \Omega : X_1(\omega) \leq x_1, ..., X_d(\omega) \leq x_d\}),$$

where $x = (x_1, ...x_d) \in \mathbb{R}^d$.

In our case, we will use random variables with a Gaussian distribution function $\mathcal{N}(\mu, \sigma^2)$ with mean μ and variance σ^2, which implies that they have a probability density function

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

3.2 Quantization: some definitions

Now, we are ready to give some definitions for the quantization method. In this section, the main references for the theoretical background are Callegaro et al. (2019), Pagès (2015), and Pagès (2018).

Definition 8. *A **quantization grid** (also called N-quantizer) $\Gamma^N = \{x_1, ..., x_N\} \subset \mathbb{R}^d$ ($d \geq 1$), is a subset of size at most N, with $N \geq 1$, having pairwise distinct components.*

Definition 9. *A Γ-valued **quantization function** (also called quantizer) is a Borel function $q : \mathbb{R}^d \to \Gamma$. We can omit N in the notation of Γ if we keep it fixed.*

Thus, it is possible to approximate the random vector X with the quantization function $q(X)$, taking N values in $\Gamma \subset \mathbb{R}^d$ and the output will have an error of $\|X - q(X)\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. In order to get the best approximation of X, the choice of Γ and q should minimize this error.

Lemma 1. *Considering a random variable $\xi \in \mathbb{R}^d$, for every quantization function*

$$\|\xi - q(\xi)\| \geq dist(\xi, \Gamma) := \inf_{i \in \{1,...,N\}} |\xi - x_i|$$

Proposition 4. *The error is minimized, i.e., it is equal to the distance above if and only if q is the nearest neighbor projection on Γ, i.e., if*

$$q(\xi) = Proj_\Gamma(\xi) := \sum_{i=1}^{N} x_i \mathbb{1}_{C_i(\Gamma)}(\xi), \tag{3.1}$$

where $(C_i(\Gamma))_{1 \leq i \leq N}$ is a Borel partition of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, called Voronoï partition.

Definition 10. *A **Voronoï partition** is a Borel partition that satisfies*

$$C_i(\Gamma) \subset \left\{\xi \in \mathbb{R}^d : |\xi - x_i| \leq \min_{1 \leq j \leq N} |\xi - x_j|\right\} \quad \forall i = 1, ..., N. \tag{3.2}$$

[5] It is a real-valued function defined on the sample space Ω.

Observation 1. *When the norm is Euclidean, the Voronoï cells are convex sets and they satisfy*

$$\left\{\xi \in \mathbb{R}^d : |\xi - x_i| < \min_{1\le j\le N, j\neq i} |\xi - x_j|\right\} = \mathring{C}_i(\Gamma) \subset \bar{C}_i(\Gamma) = \left\{\xi \in \mathbb{R}^d : |\xi - x_i| = \min_{1\le j\le N} |\xi - x_j|\right\} \tag{3.3}$$

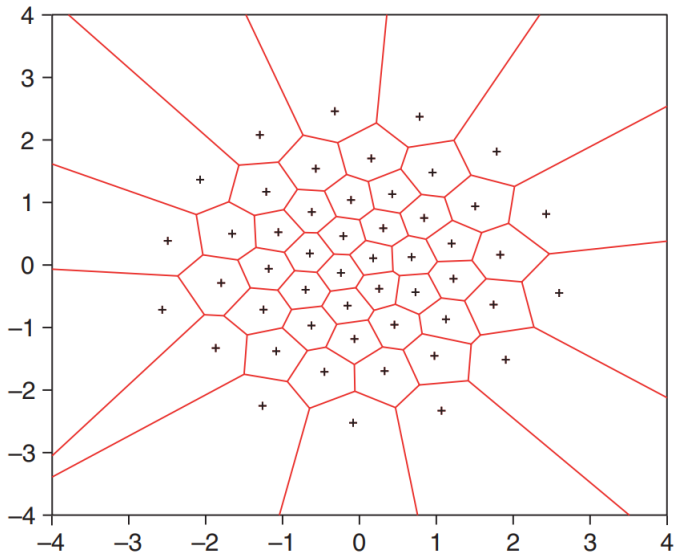


Figure 3.1: Example of optimal grid (points) and Voronoï cells (red edges) for a bivariate (standard) Gaussian distribution with $N = 50$ from Callegaro et al. (2015)

Thus, for a given random vector X having values in $\mathbb{R}^d$, it will be used the following notation for the Voronoï Γ-quantization of X:

$$\hat{X}^{\Gamma} = Proj_{\Gamma}(X).$$

To notice the difference between the real distribution and the quantization function, it is useful to define a quantization error

Definition 11. *The L^p-**mean quantization error** $e_{p,N}(X, \Gamma)$, for $p \in [1, +\infty)$, induced by a grid Γ is*

$$e_{p,N}(X, \Gamma) := \|X - \hat{X}^{\Gamma}\|_{L^p(\mathbb{P})} = \|\min_{1\le i\le N} |X - x_i|\|_{L^p(\mathbb{P})} = \left(\int_{\mathbb{R}^d} \min_{1\le i\le N} |\xi - x_i|^p \mu(d\xi)\right)^{1/p}. \tag{3.4}$$

Definition 12. *An $\mathbb{R}^d$-valued random vector X is said to be in $L^p_{\mathbb{R}^d}(\mathbb{P})$, where $p \in [1, +\infty)$ and $\mathbb{P}$ a probability measure, if its p-th moment is finite, i.e.,*

$$\|X\|_p := (\mathbb{E}[|X|]^p)^{1/p} < \infty$$

where $\mathbb{E}$ is the expected value operator and $|X|^p$ is the p-th power of the absolute value of X.

The idea to optimally choose Γ consists of searching a grid with size at most N, which optimally represents the distribution μ of X in a L^p-sense, when $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$. For this purpose, it is necessary to define:

Definition 13. *Let $p \in [1, +\infty)$ and $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$, the L^p-**distortion function** D_p is an $\mathbb{R}_+$-valued function defined on $(\mathbb{R}^d)^N$ by*

$$D_p : (x_1, x_2, ..., x_N) \longmapsto \mathbb{E}\left[\min_{1\leq i\leq N} |X - x_i|^p\right] = [e_{p,N}(X, \Gamma)]^p \tag{3.5}$$

In order to find a grid Γ which minimizes the L^p-mean quantization error $e_{p,N}(X, \Gamma)$, the problem consists in computing

$$e_{p,N}(X) := \inf_{\Gamma\subset\mathbb{R}^d, card(\Gamma)\leq N} e_{p,N}(X, \Gamma) = \inf_{(x_1,...,x_N)\in(\mathbb{R}^d)^N} D_p(x_1, x_2, ..., x_N)^{1/p}, \tag{3.6}$$

where $card(\Gamma)$ is the cardinality of the grid $\Gamma \subset \mathbb{R}^d$ and the last equality is true because a grid Γ with $card(\Gamma)$ less than N elements can always be represented by an N-tuple, in which some elements of the original grid can be duplicated.
Now, the question is about proving the existence of at least an optimal one-dimensional grid (that is the case we are interested in) and what an eventual procedure to find it should look like.
For the first issue, the answer is in the following proposition

Proposition 5. *Given $p \in [1, +\infty)$ and $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$*

(a) there exist at least one minimum Γ^* grid for the distortion function D_p;

(b) if $card(supp(\mu)) \geq N$, Γ^* has pairwise distinct components;

(c) $\lim_{N\to\infty} e_{p,N}(X) = 0$.

Proof See Pagès (2015, sect. 2.1).

Observation 2. *In the one-dimensional case, when μ is absolutely continuous with respect to a log-concave density, there exists a unique optimal quantization grid at level N, see Pagès (2015, sect. 2).*

Given that an optimal grid exists, the procedure goes through the differentiability of the distortion function and its p-stationary points.

Proposition 6. *Using the Euclidean norm $\|\cdot\|$ and $p \in [1, +\infty)$ [6] , the L^p-distortion function is differentiable at any N -tuple having pairwise distinct components $\Gamma = \{x_1, ..., x_N\}$ and the gradient formula is:*

$$\begin{aligned}\nabla D_p(x_1, ..., x_N) &= p\left(\int_{C_i(\Gamma)} \frac{x_i - \xi}{|x_i - \xi|}|x_i - \xi|^{p-1}\mu(d\xi)\right)_{1\leq i\leq N} \\ &= p\left(\mathbb{E}[\mathbb{1}_{X\in C_i(\Gamma)}\frac{x_i - X}{|x_i - X|}|x_i - X|^{p-1}]\right)_{1\leq i\leq N}\end{aligned}$$

[6] For $p = 1$, it exists an extension under appropriate continuity and integrability assumptions on the distribution μ, see Pagès (2015).

So, it follows the definition of p-stationary quantizers.

Definition 14. *Let $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$ a random vector with distribution μ. A quantization grid $\Gamma = \{x_1, x_2, ..., x_N\} \subset \mathbb{R}^d$ is p-stationary if the following two conditions are satisfied:*

(i) μ-negligibility of the boundary of the Voronoï diagram: $\mu\left(\bigcup_{1\le i\le N} \partial C_i(\Gamma)\right) = 0$;

(ii) Self-consistency of the centroids: $\forall i \in \{1, ..., N\}, \quad \mathbb{E}[\mathbb{1}_{X\in C_i(\Gamma)} \frac{x_i - X}{|x_i - X|} |x_i - X|^{p-1}] = 0.$

In case the differentiability of the gradient is true, the classical Newton-Raphson procedure is allowed. Indeed, it will be shown how, thanks to this algorithm, we will be able to numerically solve a nonlinear system of N equations to obtain these sub-optimal grids.
In the following section, the focus will be on p-stationary quantizers and how Fourier transform play a crucial role in this study.

3.3 Fourier transform in quantization

Let us now introduce some assumptions.
We denote the value of an asset by $S = (S_t)_{t\in[0,T]}$ where $S_T \ge 0$ at time $T > 0$ and T is the maturity of the option, and for plain vanilla option $X = S_T$ e.g. $d = 1$. Let us define the conditional characteristic function $\phi_{t,T}(u)$ of $\log(S_T)$, for $t \in [0, T]$:

$$\phi_{t,T}(u) := \mathbb{E}[e^{iu\log(S_T)}|\mathcal{F}_t], \quad u \in \mathbb{R}$$

which is known (if not, it can be computed efficiently). For $t = 0$, it is used the notation $\phi_T(u) := \phi_{0,T}(u)$.
As Shephard demonstrated in his paper (Shephard, 1991), whenever ϕ_T is Lebesgue integrable, it is possible to compute the density and the cumulative distribution function of S_T, using ϕ_T through Fourier inversion. We report the theorem from Klenke (2013)

Theorem 1. *(Fourier inversion) Let $X \in L^p_{\mathbb{R}}(\mathbb{P})$ a random vector with distribution μ and ϕ its characteristic function, defined as*

$$\phi(u) := \mathbb{E}[e^{iuX}] = \int_{\mathbb{R}} e^{iux}\mu(dx).$$

Then, if $\int_{\mathbb{R}} |\phi(u)|du < +\infty$, ϕ admits limited density g, such that $\mu(dx) = g(x)dx$ and

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iux}\phi(u)du.$$

Proof See Klenke (2013, pag. 300).
An application of this theorem is in the following equations:

$$\mathbb{P}(S_T \in dz) = \left(\frac{1}{\pi}\frac{1}{z}\int_0^\infty Re(e^{-iu\log(z)}\phi_T(u))du\right)dz =: f(z)dz \tag{3.7}$$

$$\mathbb{P}(S_T \le z) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\log(z)}\phi_T(u)}{iu}\right)du, \quad z \in (0, +\infty). \tag{3.8}$$

This is the contribution of Fourier transform analysis to quantization. The Fourier transform inversion formula above will allow us to write the density of the random variable S_T in terms of

the characteristic function, in order to make it explicit in the distortion function. This method makes the quantization of random variables efficient, in case they admit a characteristic function in closed form.

3.4 Master Equation

In this Chapter, we are going to obtain the *Master Equation*, whose solutions will lead us to the sub-optimal (stationary) quantizers.
As anticipated in Section 3.3, the procedure consists of writing the distortion function using the Fourier transform method to represent the price density, and differentiating it.

At first, in the one-dimensional case the grid $\Gamma = \{x_1, ...x_N\}$ is a set of points in the positive number line since we assume $S_T \geq 0$. Thus the Voronoï cells are intervals such as $C_j(\Gamma) = [x_j^-, x_j^+]$, where x_j^- and x_j^+ are

$$x_j^- = \frac{1}{2}(x_{j-1} + x_j) \quad \forall j = 2, ..., N, \qquad x_j^+ = \frac{1}{2}(x_j + x_{j+1}) \quad \forall j = 1, ..., N-1,$$

and

$$x_1^- = 0 \qquad x_N^+ = +\infty.$$

The distortion function can be written as

$$D_p(\Gamma) = D_p(x_1, ..., x_N) = \sum_{j=1}^{N} \int_{C_i(\Gamma)} |z - x_j|^p d\mathbb{P}_{S_T}(z). \tag{3.9}$$

where $\mathbb{P}_{S_T}$ is the distribution of S_T.
Since the density of S_T is assumed to be continuous and concentrated on $(0, +\infty)$, it is possible to differentiate the distortion function under the integral sign.
Thus, the procedure consists of solving the N-dimensional system given by the gradient of the distortion function $\nabla D(\Gamma) = 0$, whose equations are explicitly expressed by the next theorem:

Theorem 2. *(**The Master Equation**) Let $p \in [1, +\infty)$. Then the N-quantization grid $\Gamma = x_1, ..., x_N$ is p-stationary for the distortion function (3.9) if for all $j = 1, ..., N$*

$$\int_0^{+\infty} Re\left[\phi_T(u) e^{-iu\log(x_j)} \left(\bar{\beta}\left(\frac{x_j^-}{x_j}, -iu, p\right) - \bar{\beta}\left(\frac{x_j}{x_j^+}, 1 - p + iu, p\right)\right)\right] du = 0 \tag{3.10}$$

where for $a \in \mathbb{C}$, $Re(b) > 0$ and $x \in (0, 1)$, the function $\bar{\beta}$ is defined as [7]

$$\bar{\beta}(x, a, b) := \int_x^1 t^{a-1}(1-t)^{b-1} dt. \tag{3.11}$$

Proof See Callegaro et al. (2019, Appendix A).

[7] The function $\bar{\beta}(x, a, b)$ can be expressed as

$$\bar{\beta}(x, a, b) = \beta(a, b) - \beta(x, a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt - \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

3.5 The Newton-Raphson Algorithm

Now the last step is to solve the nonlinear system given in eq. (3.10), in order to find the p-stationary grid. Since this system is not directly solvable, we use the fact that the derivatives of every equation can be computed analytically and as a consequence, it is efficient to implement a Newton–Raphson algorithm to find the solution. In order to keep the notation simpler, we will use L_j to indicate the x_j-derivative of the distortion function in eq. (3.9), which is

$$L_j(\Gamma) := \frac{p}{\pi} x_j^{p-1} \int_0^{+\infty} Re\left[\phi_T(u) e^{-iu\log(x_j)} \left(\bar{\beta}\left(\frac{x_j^-}{x_j}, -iu, p\right) - \bar{\beta}\left(\frac{x_j}{x_j^+}, 1-p+iu, p\right)\right)\right] du. \tag{3.12}$$

As we specified in the section 3.4, the goal is to solve the N-dimensional system $L(\Gamma) = 0$ (called $\nabla D(\Gamma) = 0$ with the old notation), and to do that with the algorithm it is necessary to compute the $N \times N$ Jacobian matrix ∇L, which is basically the Hessian matrix of the distortion function. As it is evident, L_j depends only on x_{j-1}, x_j and x_{j+1} and these property makes ∇L a tridiagonal matrix [8].
To conclude, the following theorem shows explicitly how to make the algorithm work with the calculation's expression to find the final solution, i.e. the sub-optimal (stationary) quantization grid Γ.

Theorem 3. *(The Fourier Quantization Algorithm) Let $p \in [1, +\infty)$. Starting with an N-quantization grid $\Gamma^{(0)} = x_1, ..., x_N$, the recursive formula of the Newton-Raphson algorithm is*

$$\Gamma^{(n+1)} = \Gamma^{(n)} - \left(\nabla L(\Gamma^{(n)})\right)^{-1} \dot{L}(\Gamma^{(n)}), \quad n = 0, 1, ...$$

where the expressions for the non-zero elements of the tridiagonal matrix $\nabla L(\Gamma^{(n)})$ are:
UPPER DIAGONAL

$$\nabla L_{j,j+1}(\Gamma) = -\frac{p}{2\pi}\frac{1}{x_j^+}\left(\frac{x_{j+1}-x_j}{2}\right)^{p-1} \int_0^{+\infty} Re\left[\phi_T(u) e^{-iu\log(x_j^+)}\right] du,$$
$$j = 1, ..., N-1;$$

LOWER DIAGONAL

$$\nabla L_{j,j-1}(\Gamma) = -\frac{p}{2\pi}\frac{1}{x_j^-}\left(\frac{x_j-x_{j-1}}{2}\right)^{p-1} \int_0^{+\infty} Re\left[\phi_T(u) e^{-iu\log(x_j^-)}\right] du,$$
$$j = 2, ..., N;$$

MAIN DIAGONAL

$$\nabla L_{j,j}(\Gamma) = \frac{p(p-1)}{\pi} x_j^{p-2} \int_0^{+\infty} Re\left[\phi_T(u) e^{-iu\log(x_j)} \left(\bar{\beta}\left(\frac{x_j^-}{x_j}, -iu, p-1\right)\right.\right.$$
$$\left.\left. + \bar{\beta}\left(\frac{x_j}{x_j^+}, 2-p+iu, p-1\right)\right) du + \nabla L_{j,j+1}(\Gamma) + \nabla L_{j,j-1}(\Gamma),$$
$$j = 1, ..., N.$$

[8] It has nonzero elements only on the main diagonal, the diagonal upon the main diagonal, and the diagonal below the main diagonal

Proof See Callegaro et al. (2019, Appendix B).

Data analysis and simulation

4.1 Calculation method

As it is specified in the assumptions of the Black-Scholes model (2.3), we assume the underlying asset follows a Geometric Brownian motion; it means that the random variable S_T representing the underlying asset price at the maturity T is given by

$$S_T = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(B_T - B_t\right)\right\}. \tag{4.1}$$

We can use the trick as in Björk (2009, Sect. 7.5) and call the exponent as a unique random variable Z, in such a way it is possible to express $S_T = S_0 e^Z$. Thus, this Z will have a Gaussian distribution $\mathcal{N}[(r - \frac{1}{2}\sigma^2)(T-t),\, \sigma^2(T-t)]$. Considering an initial time of $t = 0$ and taking the logarithm of S_T, the underlying asset price will follow the above Gaussian distribution:

$$(\log S_T) \sim \mathcal{N}\left(\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)T,\, \sigma^2 T\right). \tag{4.2}$$

Once an N-optimal quantization grid $\Gamma = \{x_1, ..., x_N\}$ for the random variable S_T is given (by the quantization grid of Z), it is easy to find the formula for the pricing of our European call option. Assuming a Lipschitz continuous function $F : \mathbb{R}^d \to \mathbb{R}$, we can exploit the following cubature formula as in Pagès (2018, Chapter 5.2)

$$\mathbb{E}[F(S_T)] \cong \mathbb{E}[F(\hat{S}_T)] = \sum_{l=1}^{N} F(x_l)\mathbb{P}(\hat{S}_T = x_l), \tag{4.3}$$

making all the calculations explicitly

$$\mathbb{E}[\max(S_T - K, 0)] = \mathbb{E}[\max(S_0 e^Z - K, 0)] \cong \sum_{l=1}^{N} \max(S_0 e^{x_l} - K, 0)\mathbb{P}(\hat{S}_T = x_l), \tag{4.4}$$

where $\Gamma = \{x_1, ..., x_N\}$ is the quantization grid of the Gaussian distribution $\mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)T,\, \sigma^2 T\right)$. Let us call the mean and the standard deviation of this Gaussian, a and b, such as $\mathcal{N}(a,\, b^2)$, in such a manner it is easier to calculate the probability $\mathbb{P}(\hat{S}_T = x_l)$. Fixing a point of the quantization grid x_i and its Voronoï cell C_i, the work consists of calculating the probability that the random variable $Z = a + b\bar{Z}$ with distribution $\mathcal{N}(a,\, b^2)$ is in the cell C_i, this means

$$\mathbb{P}\left(Z \in C_i\right) = \mathbb{P}\left(a + b\bar{Z} \in C_i\right) = \mathbb{P}\left(x_i^- < a + b\bar{Z} < x_i^+\right) = \mathbb{P}\left(\frac{x_i^- - a}{b} < \bar{Z} < \frac{x_i^+ - a}{b}\right)$$

Since $\bar{Z}$ is a standard variable with distribution $\mathcal{N}(0,\, 1)$, it is trivial to calculate this probability using the Cumulative Distribution Function [1]:

[1] $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$

$$\mathbb{P}\left(Z \in C_i\right) = \Phi\left(\frac{x_i^+ - a}{b}\right) - \Phi\left(\frac{x_i^- - a}{b}\right). \tag{4.5}$$

4.2 Results and considerations

Now we are ready to calculate the price of our European call option, using as an input the quantization grid (choosing an appropriate N), the maturity T, the initial underlying asset price S_0, the risk-free interest rate r and the volatility σ.

We consider some ideal European Call options with maturity $T = 1$ year written on an underlying with initial price $S_0 = 100$, interest rate $r = 2\%$, volatility $\sigma = 20\%$ and strike K ranging between 20 and 150, with a step of 10. The grid size is fixed at N = 120 and we study the result for $1 \leq p \leq 15$. The quantization grids are given by MATLAB code, developed in Callegaro et al. (2019); meanwhile for the price computations and the graphs, we implement them using Python.

In the tables below, we give some examples of the prices calculated, comparing with the benchmark (the prices obtained from the analytical formula of Black-Scholes) with the relative error expressed in basis points [2] , computed as:

$$Rel\ error\ (bs) = \frac{|X_{bench}^{Call} - X_{quant}^{Call}|}{X_{bench}^{Call}} \cdot 10^4 \tag{4.6}$$

Only the results for the grid with $p = 2, 5, 10$, and 15 are reported, since it would have taken too much space writing them for every p.

K	Benchmark	2-Quantization	2-Rel error (bs)	5-Quantization	5-Rel error (bs)
20	80.3960	82.0255	202.6849	82.0306	203.3147
30	70.5940	72.0255	202.7781	72.0306	203.4954
40	60.7921	62.0255	202.9015	62.0306	203.7345
50	50.9907	52.0262	203.0761	52.0313	204.0696
60	41.2069	42.0452	203.4485	42.0506	204.7566
70	31.5766	32.2224	204.5318	32.2302	207.0197
80	22.5429	23.0104	207.4045	23.0013	203.3756
90	14.8065	15.1258	215.6410	15.1437	227.7151
100	8.9160	9.1204	229.2146	9.1432	254.7432
110	4.9439	5.0513	217.3717	5.0860	287.5367
120	2.5469	2.6162	272.1618	2.6095	245.5858
130	1.2320	1.2642	261.2176	1.2730	332.7150
140	0.5655	0.5806	266.6804	0.5850	344.1790
150	0.2487	0.2574	351.4118	0.2619	533.9466

Table 4.1: Call option prices for a 2-quantization and a 4-quantization grid.

[2] One basis point corresponds to 0.01%

K	Benchmark	10-Quantization	10-Rel error (bs)	15-Quantization	15-Rel error (bs)
20	80.3960	82.0360	203.9831	82.0408	204.5845
30	70.5940	72.0360	204.2566	72.0408	204.9415
40	60.7921	62.0360	204.6184	62.0408	205.4138
50	50.9907	52.0367	205.1315	52.0415	206.0819
60	41.2069	42.0559	206.0353	42.0612	207.3161
70	31.5766	32.2363	208.9444	32.2462	212.0723
80	22.5429	23.0175	210.5573	23.0463	223.3140
90	14.8065	15.1420	226.6034	15.0926	193.2440
100	8.9160	9.1673	281.8577	9.1891	306.2782
110	4.9439	5.0386	191.6585	5.0692	253.4590
120	2.5469	2.6511	408.8575	2.6061	232.3973
130	1.2320	1.2788	379.8014	1.2848	428.0761
140	0.5655	0.5682	47.9883	0.5978	571.7990
150	0.2487	0.2545	234.6524	0.2645	635.2484

Table 4.2: Call option prices for a 10-quantization and a 15-quantization grid.

To show all the grids used, they are plotted into a graph, which displays how they spread when the p increases.

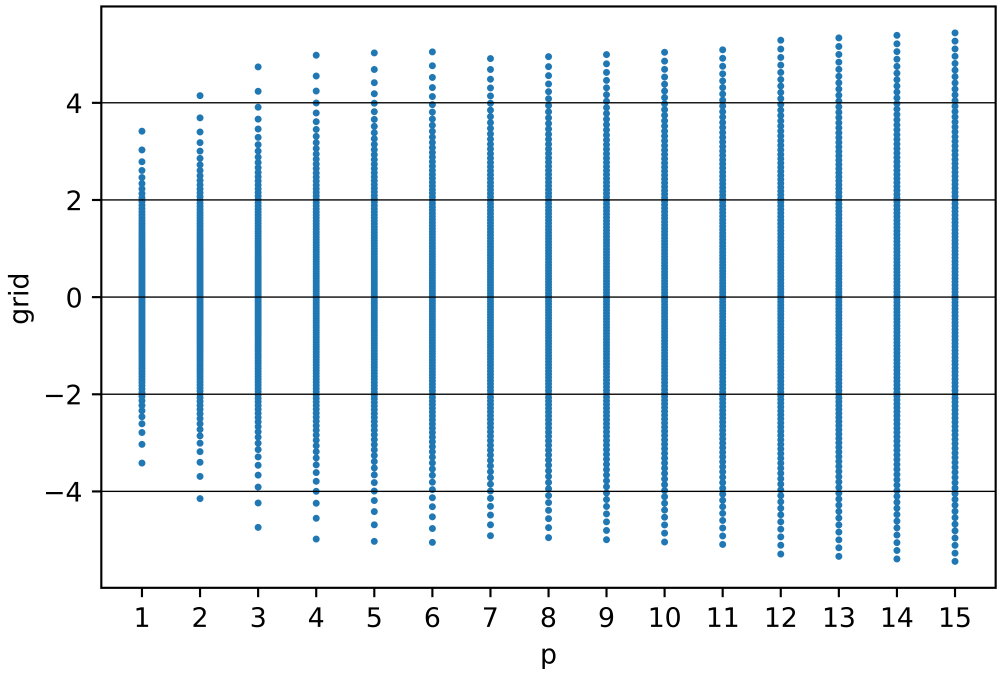
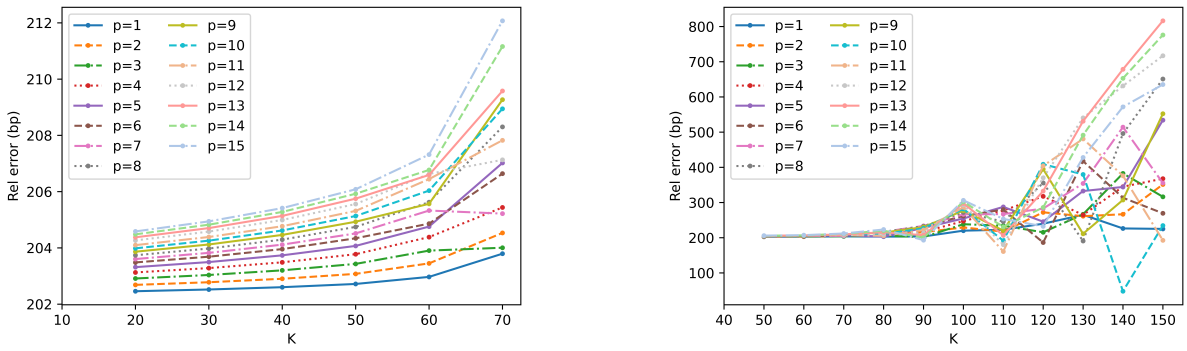


Figure 4.1: Comparison of the different grids used

The main work is shown in graphs 4.2(a) and 4.2(b), where the relative error in basis point is related to the strike price K for every $p \in [1, 15]$. The two versions make more clear what happens "in the money", i.e. for $K < S_0$, and "out of the money", i.e. for $K > S_0$.



(a) Errors for $20 \leq K \leq 70$

(b) Errors for $50 \leq K \leq 150$

Figure 4.2: Comparison of the relative errors, in basis points, "in the money" and "out of the money".

As it is evident in 4.1, the points in the grids are more sparse around the origin at the increasing of p, but not linearly. In contrast to what is written in Callegaro et al. (2019, Sect. 2.4), the grids start with a progressive spreading for the first p-values but they stop around $p = 6$. Here, they decrease a little to enlarge again in the end. However, we have to admit that when p rises, the calculation of the quantization grids by the MATLAB code starts to become inaccurate and the algorithm unstable. This happens because the algorithm requires to invert a tridiagonal matrix and for a certain p-value the matrix is closed to be singular or badly scaled. In this sense, it is more difficult for us to make consistent claims about the results for big p-values.
Our initial hypothesis was based on this conjecture, which means that with a larger p, the grids are wider and so we can have more information on the tails of our distribution. In this sense, we expected to price better the European Call options for K "deep in the money" or "deep out of the money". How it is visible from the error graphs 4.2, it reveals to be not true tough. Indeed, looking at the graph 4.2(a) with smaller strike prices, the better pricing shows up to be at small p-values. The same observations can be made for the graph 4.2(b) with bigger strike prices, but here instability of the algorithm and a payoff close to zero makes this data less precise.
All things considered, we may guess how the p-norm of an L^p space influences the princing of a European Call option with respect to the power of its payoff. According to this guessing, using the quantization method Fourier-based, a European Call option with a linear payoff

$$X^{Call} = \max(S_T - K, 0)$$

has its best pricing at $p = 1$; instead for a power payoff

$$X^{Call} = \max(S_T^n - K, 0)$$

the pricing that minimizes the relative error should be for an L^p space with $p = n$.

Conclusion

In this thesis, we introduce Black-Scholes model and its assumptions for European Call options. Furthermore, we explored how the stochastic equation of this model can be solved by the new quantization method based on the Fourier transform.
As we knew since the beginning from Callegaro et al. (2019), the changing of the p-norm in our L^p space, used in the quantization method, can modify the distribution of the quantization grids. In this sense, we confirm the guessing in Callegaro et al. (2019, Sect. 2.4) about the grid dynamics and its spreading at the rising of p, but for large p-values data are too inaccurate to make a hypothesis for a precise trend.
Regarding the pricing, we wondered if the accuracy could change with different p-values. Specifically, our hypothesis was about payoff with a strike price very far from the underlying asset price, i.e. in the tails. So we were particularly interested in such cases very "in" or "out of the money". Our expectations were about the fact that a more spread quantization grid would have given more information about tails events.
Nevertheless, our simulations show as the best pricing is made for lower p-value, in contrast with our previous assumptions. To conclude, we make a guess that the p-norm of an L^p space could be a key factor for the pricing of options with a power payoff.

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