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Study of general Monge-Ampère constraint conditions in variational models for prestrained plates

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Introduction and context

This Master Degree Thesis, conducted under the supervision of Professor M. R. Pakzad from University of Toulon, will focus specifically on variational studies in the context of what has been done in [4]. We will expand on that by studying specifically the existence of functions satisfying Monge-Ampère constraints $\det \nabla^2 u = f$ in the degenerate case, in which f is constantly 0 inside of a disk and greater than a strictly positive constant c outside of it.

We will use the main results of M. R. Pakzad, M. Lewicka and L. Mahadevan in [1] and [2], on the geometrical characterization of Monge-Ampère solutions, and T. Iwaniec and V. Šverák studies on integrable dilatation. As we tackle the problem of the existence of functions satisfying specific degenerate cases of Monge-Ampère constraints, we will also show how this can be connected to elasticity minimizers.

In chapter 1 we will see the context for this work, which collocates itself among the analytical studies on models for "incompatible elasticity problems" by Marta Lewicka, L. Mahadevan, Pablo Ochoa, Mohammad Reza Pakzad, among many others, on three-dimensional plate models generated from nonlinear elasticity (we cite in particular [4] and the pioneering work of Friesecke, James and Müller in [8]). The aim is conducting a study on the deformations in a thin (basically 2-dimensional) plate caused by residual strains in absence of an exterior force. Such a scenario may arise from inhomogeneous growth, plastic deformation, swelling or shrinkage driven by solvent absorption and can be used in the modeling of plastic films, polymer gels and the study of the shape of growing leaves to name a few.

In the second chapter we will state the conjecture we would like to prove and explain its relationship with the model illustrated in the previous section. We want to prove that there are no solutions to the Monge-Ampère problem with some specific degenerate constraints and will consider some ideas on how to expand to more general cases. Then, in the third chapter, we will see some useful past results on convexity, developability and integrable dilatation which will be necessary to prove our statement.

In chapter 4 and 5 we we will look at examples and prove some intermediate results. While they do not necessarily apply in a general case, these should clarify the structural issues one finds when claiming that the set of solutions to degenerate Monge-Ampère constraints is non-empty.

In chapter 6 we will provide some ideas which might fit the proof to the more general statement, which will hopefully be more complete in the near future.

1. The general context

Even though the problem considered in this paper is purely analytical, it makes sense to first clarify the setting of our study.

Let us illustrate the model in question first: we are considering 3d plates $\Omega^h = \Omega \times (-h/2, h/2)$ of varying thickness $0 < h << 1$ (with $\Omega \in \mathbb{R}^2$ open bounded) and we study the elastic energy $I_W^h(u^h)$ related to the deformation $u^h : \Omega^h \to \mathbb{R}^3$ which affects the plates after they have been prestrained.

For such a purpose, we first consider the activation process for each point given by such prestrain, which represents the spontaneous effects of instantaneous growth or deformation on the plate.

We represent such a process with a smooth invertible tensor $A^h : \overline{\Omega^h} \to \mathbb{R}^{3\times 3}$, which associates each point with their own instantaneous growth $A^h(x)$. We can write each deformation ∇u^h as the composition of the activation process and a consequent "rearrangement" of the material in a new shape in response to A^h. We describe this last process through an elastic tensor $F(x) = \nabla u^h(x)(A^h(x))^{-1}$ ([7] defines the model behind this tensor).

Now we can finally define the elastic energy functional related to the deformation, which only depends on F and has the form:

$$I_W^h(u^h) = \frac{1}{h}\int_{\Omega^h} W(F)dx = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h (A^h)^{-1})dx \tag{1}$$

for any given $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$. If we assume the elastic density function W to be regular enough (as we see in [4], normalization, frame independence properties and second order non-degeneracy of W must hold).

We further assume W regular enough by postulating the existence of a quadratic function $\mathcal{Q}_3$ such that $|W(\mathrm{Id} + F) - \mathcal{Q}_3(F)| \le \omega(|F|)|F|^2$ and we restrict ourselves to the study of a specific class of growth tensors A^h depending only on some smooth "stretching" tensor S_g and "bending" tensor B_g acting on Ω:

$$A^h(x', x_3) = \mathrm{Id}_3 + h^\gamma S_g(x') + h^{\gamma/2} x_3 B_g(x'),$$

for some scaling exponent $0 < \gamma < 2$. An activation tensor of this form allows us to deduce the induced metric on the plate:

$$G^h(x', x_3) = (\nabla u^h)^T \nabla u^h = (A^h)^T(A^h) = \mathrm{Id}_3 + 2h^\gamma \mathrm{sym}\ S_g(x') + 2h^{\gamma/2} x_3 \mathrm{sym}\ B_g(x'),$$

where terms with higher order (jointly in the variables x_3 and $h^{\gamma/2}$) have been excluded. We then get an energy functional of the form

$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W((\nabla u^h)(G^h)^{-1/2})dx \quad \forall u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3).$$

Then, by Γ-convergence of $\frac{1}{h^{\gamma+2}} E^h$ we can find a limiting functional

$$\mathcal{I}_f(v) = \frac{1}{12}\int_\Omega \mathcal{Q}_2(\nabla^2 v + (\mathrm{sym}\ B)_{2\times 2})$$

which bounds (1) from below as $h \to 0$. Such a functional is well defined over

$$\mathcal{A}_f = \{v \in W^{2,2}(\Omega); \det \nabla^2 v = f\},$$

which is the space of $W^{2,2}$ functions satisfying Monge-Ampère conditions with geometrical constraint $f = -\mathrm{curl}^T \mathrm{curl}\ S_{2\times 2}$ uniquely depending on the prestrain A^h. In particular, if $\mathcal{A}_f \neq \emptyset$, any sequence $u^h \in W^{1,2}$ that minimizes I_W^h as h goes to 0 converges to a minimizer v of the limiting funtional $\mathcal{I}_f(v)$.

From the given definition, studying the emptiness of $\mathcal{A}_f$ corresponds to the study of the existence of a function satisfying specific Monge-Ampère conditions. We will further look into this from the next section onwards. It is however also interesting to see that the above mentioned Γ-convergence only applies when we have a proper scaling of $\inf E^h \sim h^{\gamma+2}$. For that to apply, as seen in [11] and [12], we need the two following conditions to be simultaneously satisfied: (a) $\mathcal{A}_f$ is nonempty, (b) $\mathrm{curl}(\mathrm{sym}\ B)_{2\times 2} \not\equiv 0$ or $\mathrm{curl}^T \mathrm{curl}\ S_{2\times 2} + \det(\mathrm{sym}\ B)_{2\times 2} \not\equiv 0$. As such, the emptiness of $\mathcal{A}_f$ is also fundamental to study to determine whether our limit model is well defined or not.

In [4] are also provided some results for existence and uniqueness of minimizers on the limiting problem in $\Omega = B(0,1)$. With specific assumptions deriving from the isotropic elastic energy model in [8] we get a limiting functional of the form:

$$\mathcal{I}(v) = \int_{B(0,1)} |\nabla^2 v|^2 dx' \text{ subject to constraint: } \mathcal{A}_f = \{v \in W^{2,2}(B(0,1)), \det \nabla^2 v = f\}. \quad (2)$$

Given a function f regular enough, we want to study this minimization problem. We then call $\mathcal{I}_f(v)$ the restriction of the functional $\mathcal{I}(v)$ to the set $\mathcal{A}_f$.

It will be helpful in the following arguments to also consider the relaxed problem:

$$\mathcal{I}(v) = \int_{B(0,1)} |\nabla^2 v|^2 dx' \text{ subject to constraint: } \mathcal{A}_f^* = \{v \in W^{2,2}(B(0,1)), \det \nabla^2 v \geq f\}. \quad (3)$$

As above, we will call $\mathcal{I}_f^*(v)$ the restriction of the functional $\mathcal{I}(v)$ to the set $\mathcal{A}_f^*$

By studying minimizers for the relaxed case we can infer, under certain additional conditions, some properties of the minimizers of (2). In particular we can prove the existence of minimizers for both problems in the classical way by taking a minimizing sequence and using the lower semicontinuity of the functional $\mathcal{I}(v)$ as long as $\mathcal{A}_f$ and $\mathcal{A}_f^*$ are non-empty.

We can also prove (as done in [4], section 5, using a result from [9]) that for $f \geq c > 0$ we have uniqueness (up to an affine map) for the relaxed problem (3). On the other hand, uniqueness for the main problem is far more delicate.

The main result on uniqueness presented in [4] considers only radially symmetric functions. It is also important to require non-increasingness of our constraint funtion f:

Theorem 1.1 (Uniqueness, [4]). *Assume that $f \in L^2(B(0,1))$ is radially symmetric i.e.: $f = f(r)$ and $\int_0^1 rf^2(r)dr < \infty$. Assume further that $f \geq c > 0$, and that f is a.e. non-increasing, i.e. for a.e. $r \in [0,1]$ and a.e. $x \in [0,r]$ we have $f(r) \leq f(x)$.*

Then the problem (2) has a unique (up to an affine map) minimizer, which is radially symmetric and given by:

$$v_f(r) = \int_0^r \left(\int_0^s 2tf(t)dt \right)^{1/2} ds \quad (4)$$

It is important to notice that such a problem does not admit a uniqueness result in general. In particular for $f \equiv -1$ it can be checked that there exists a family of minimizers $v_\theta(x_1, x_2) = (cos\theta)\frac{x_1^2 - x_2^2}{2} + (sin\theta(x_1 x_2))$. Indeed this holds for any $f \leq c_0 < 0$ with $\Delta(\log|f|) = 0$ in Ω.

In this scenario, if the function space A_f is non-empty, the existence of the minimizers can be obtained by standard methods. However, once we lose the strict positivity condition on f on a part of the domain, even the existence of functions satisfying the constraint becomes non-trivial.

2. Description of the problem

The result we would like to prove is the following:

Conjecture 2.1. *Let $\Omega = B(0,R) \subset \mathbb{R}^2$ be a 2-dimensional disk and $f \in L^1(\Omega)$ be such that $f \geq c > 0$ on $B(0,R) \setminus B(0,r)$ for some $0 < r < R$ and $f \equiv 0$ on $B(0,r)$. Let us also assume $f \leq C$ a.e. for some $C \geq c > 0$. Then $\mathcal{A}_f = \emptyset$.*

It is interesting to see that the theory in [4] allows us to state that there cannot be any radially symmetric solution to such constraints, otherwise they would be of the same form as $v_f(r)$ defined in the previous chapter, which however does not belong to $W^{2,2}$ in this scenario.

The reason for which we believe this conjecture to hold is that, by analyzing the behavior of $\det \nabla^2 u$ on the outer annulus, but close to the boundary, because of our regularity assumptions there must exist at least a point of $\partial B(0,1)$ for which $\det \nabla^2 u(x) \to 0$ when x approaches that point from the outer annulus, contradicting the condition of strict positivity.

In general, the problem of verifying emptiness for $\mathcal{A}_f$ is fairly important in our general setting. As mentioned before, the existence of a function satisfying the Monge-Ampère constraint would ensure the existence of a minimizer to $\mathcal{I}_f$, also the nonemptiness of $\mathcal{A}_f$ is strictly necessary as a condition for the convergence of our model.

Furthermore, even if the previous consideration does not seem necessarily relevant in our model (since from theorem 1.5 of [4] we get $f \geq c > 0$ as long as the assumptions from our model are satisfied), studying situations with degenerate constraints can help us better understand the behavior of minimizers to the standard problem as f varies and approaches 0.

Let us look at an example, which for simplicity we will define on $\Omega = B(0,1)$. Consider the family of constraints:

$$f_\varepsilon = \varepsilon \chi_{(0,1/2]} + \chi_{(1/2,1]}$$

depending on the parameter $\varepsilon > 0$. They are radially symmetrical, but not non-increasing, so the theory we have discussed in the previous section will not work here. In fact we get $v_{f_\varepsilon} \in W^{2,2}(\Omega)$ and $\mathcal{I}(v_\varepsilon) \to \infty$ as $\varepsilon \to 0$ while $v_\psi = \frac{1}{2}r^2$, given by $\psi \equiv 1$, satisfies $\mathcal{I}(v_\psi) = 2\pi$ and is admissible for the relaxed problem $\mathcal{I}^*_{f_\varepsilon}$ for all $\varepsilon \leq 1$. So, for ε small enough, v_{f_ε} cannot be a minimizer of the relaxed problem (3) and thus the previously mentioned method does not give us any result.

On the other hand, let us consider a family of minimizers u_ε for $\mathcal{I}_{f_\varepsilon}$ with $0 < \varepsilon \leq 1$; we can deduce (by lower-semicontinuity of the energy functional) that if $\mathcal{I}(u_\varepsilon) \to M < +\infty$ for $\varepsilon \to 0$ (eventually up to a subsequence), then there would exist a subsequence u_{ε_j} converging to a function $u_0 \in W^{2,2}$ such that $\mathcal{I}(u_0) < +\infty$. Hence $\mathcal{A}_{f_0} \neq \emptyset$ and a minimizer to the degenerate problem would exist (here we call $f_0 = \chi_{(1/2,1]}$). A question which remains open (and will not be discussed in this report) is whether from such a minimizer, under certain assumptions, it would be possible to reconstruct a minimizing sequence for the non-degenerate problem with small values of ε.

The previous assertion implies that, if $\mathcal{A}_{f_0} = \emptyset$, there cannot be a subsequence of minimizers such that $\mathcal{I}(u_\varepsilon) \to M < +\infty$, or, in other words, that any sequence of minimizers u_ε for $\mathcal{I}_{f_\varepsilon}$ has to be such that $\mathcal{I}(u_\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

In the following pages, for simplicity, we will assume $f = \chi_{B(0,2) \setminus B(0,1)}$, however the ideas used in the following pages can be modified for different domains and constraints to fit the hypothesis of our conjecture, and it should be possible to adapt it to even more general cases as long as the domain over which f is 0 is convex, bounded and regular enough.

3. Known results

From here onwards let us assume by contradiction that there exists a function $u \in \mathcal{A}_f$, with $f = \chi_{B(0,2)\setminus B(0,1)}$ and prove that it cannot be in $W^{2,2}$.

We will now provide the already obtained results which we will be using in the next few sections. To prove that $u \in \mathcal{A}_f$ cannot exist we need to prove that the solutions for $\det \nabla^2 u = 0$ on the inner disk are not "compatible" with the classic Monge-Ampère solutions over the annulus (or in other words their sum is not $W^{2,2}$).

Let us first study the solutions to the degenerate problem on the disk. In [1] M. R. Pakzad gives several results concerning the developability of flat surfaces in $\mathbb{R}^3$ with $W^{2,2}$ regularity, providing us with a geometric characterization of the solutions to the Monge-Ampère conditions over convex domains. The following result corresponds to Proposition 1.1 in [1].

Theorem 3.1 ([1]). *Let Ω be a bounded regular convex domain in $\mathbb{R}^2$ with Lipschitz boundary and let $v \in W^{1,2}(\Omega, \mathbb{R}^2)$ be a map with almost everywhere symmetric singular (i.e., of zero determinant) gradient. Then for every point $x \in \Omega$, there exists either a neighborhood U of x, or a segment passing through it and joining $\partial\Omega$ at its both ends, on which v is constant.*

By taking $v = \nabla u$, we get geometrical conditions for the gradient of u on the inner disk; in particular u has to be linear along a family of segments with decreasing length as their distance from the center of the disk increases. In particular they converge to a degenerate segment of zero length, a limit point $z_0 \in \partial B(0,1)$. We are interested in the behavior of u close to z_0. As such we can without any loss of generality, place $z_0 = (1,0)$ via a rotation of the plane. We will also make the extra assumption of being in the case in which the segments close to z_0 parallel to each other and orthogonal to the horizontal axis (of the form $x_1 = k$ for some k close to 1) to simplify our first computations. The general case in which the segments are not parallel and this assumption does not hold, will not be treated in this report; however we will briefly discuss in the last section some ideas to modify our strategies for that case.

On the other hand, to study Monge-Ampère solutions over the annulus we will need to apply some results from the work of T. Iwaniec and V. Šverák. We will first begin by introducing some definitions.

Definition 3.2. Let $v \in W^{1,2}(\Omega, \mathbb{R}^2)$ and let $\det \nabla v \geq 0$ a.e. in Ω. We say that v has *integrable dilatation* iff, for a.e. $x \in \Omega$:

$$|\nabla v|^2(x) \leq K(x) \det \nabla v(x)$$

with some function $K \in L^1(\Omega)$.

In particular if $v \in W^{1,2}(\Omega, \mathbb{R}^2)$ and $\det \nabla v(x) \geq c > 0$, then v has integrable dilatation since $K(x) = \frac{1}{c}|\nabla v|^2(x) \in L^1(\Omega)$ and satisfies the above assumptions. An interesting result connects this definition with the following:

Definition 3.3. We say that a mapping $v \in C^0(\Omega, \mathbb{R}^2)$ is *connectedly locally one-to-one* iff it is locally one-to-one outside of a closed set $S \subset \Omega$ of measure zero, for which $\Omega \setminus S$ is connected.

Theorem 3.4 ([3]). *Let $v \in W^{1,2}(\Omega, \mathbb{R}^2)$ have integrable dilatation. Then there exists a homeomorphism $h \in W^{1,2}(\Omega', \Omega)$ and a holomorphic function $\varphi \in W^{1,2}(\Omega', \mathbb{R}^2 = \mathbb{C})$ such that:*

$$v = \varphi \circ h^{-1}.$$

In particular, v is either constant or connectedly locally one-to-one, and in the latter case the singular set is $S = h((\nabla\varphi)^{-1}\{0\})$.

The proof of this theorem can be found in [3]. In particular in our case we have that ∇u has integrable dilatation on the annulus and, as such, it is holomorphic up to a homeomorphism. This is why in the next chapter we will begin by assuming ∇u to be holomorphic to better understand the issues which do not allow such a function to exist. In the latter chapters we will then extend the proof to a more general case.

We will conclude this section with an important consequence of the above theorem, by using the following result of Šverák from [5].

Theorem 3.5 ([5]). *If $u \in W^{2,2}(\Omega)$ satisfies:*

$$\det \nabla^2 u(x) > 0$$

for a.e. $x \in \Omega$, then $u \in C^1(\Omega)$. If additionally $v = \nabla u$ is connectedly locally one-to-one, then modulo a global sign change, u is locally convex in Ω. In particular, when Ω is convex then u is either convex or concave in the whole Ω.

This result, combined with Theorem 3.4 has allowed M. Lewicka, L. Mahadevan and M. Reza Pakzad in [2] (theorem 2.3 in the article) to prove that any solution to the non-degenerate Monge-Ampère problem is locally convex. We remind that a function is *locally convex* if, for any point x in the domain, there exists a convex neighborhood V_x for which the function is (strictly) convex on V_x.

Theorem 3.6 (Convexity, [2]). *Let $u \in W^{2,2}(\Omega)$ be such that $\det \nabla^2 u = f$ in Ω, where $f : \Omega \to \mathbb{R}$, $f(x) \geq c_0 > 0$ for a.e. $x \in \Omega$. Then $u \in C^1(\Omega)$ and, modulo a global sign change, u is locally convex in Ω.*

and in particular, if Ω is convex, then u is globally strictly convex by Šverák's theorem.

In view of the previous results, any solution to $\det \nabla^2 u = f$, for f as in 2.1, has to both be developable on the inner disk $B(0,1)$ and locally convex on the annulus $B(0,2) \setminus \overline{B(0,1)}$. We will consider functions that satisfy these constraints in the following pages and we will see with some examples that the idea behind the conjecture is based on the inconsistencies they cause near the border $\partial B(0,1)$.

4. A basic case with holomorphic functions

In the previous section we saw that ∇u is holomorphic up to a homeomorphism on the annulus $B(0,2) \setminus \overline{B(0,1)}$ and that it is constant along segments touching the boundary of the unit disk. For now, instead of studying ∇u, we will consider a function $g : \mathbb{C} \to \mathbb{C} \cong \mathbb{R}^2$, holomorphic on $B(0,2) \setminus \overline{B(0,1)} \subset \mathbb{R}^2 = \mathbb{C}$, with similar properties. We ask for $g \in W^{1,2}(U)$ and $\det \nabla g = 0$ on the unit disk.

As such, we are asking g to be constant on segments in the unit disk just like ∇u, but to simplify, we will assume the segments to be parallel to each other and choose our coordinate system in such a way that they are orthogonal to the horizontal axis. We will call the extremes of the segments on the upper circle z_ε, where z_ε is the extreme of the segment of points with real part $1-\varepsilon$. The other extremes would then necessarily be $\overline{z}_\varepsilon = \frac{1}{z_\varepsilon}$.

We can compute that the length of the segment $[z_\varepsilon, \overline{z}_\varepsilon]$ is $2\varepsilon - \varepsilon^2$. As such, when $\varepsilon \to 0$, the lengths of the segments converge to zero and $z_\varepsilon \to z_0 = 1$.

Now, if we know that g is holomorphic on the open annulus $B(0,2) \setminus \overline{B(0,1)}$, we can express it in Laurent series over the annulus, with center 0, and get:

$$g(z) = \sum_{k=-\infty}^{+\infty} a_k z^k \tag{5}$$

for any $z \in B(0,2)\setminus\overline{B(0,1)}$. Now we would like to extend the holomorphic function $g|_{B(0,2)\setminus\overline{B(0,1)}}$ inside the unit disk. We will start by proving that its series expansion can be extended to the boundary.

Lemma 4.1. *Let $g(z) = \sum a_k z^k$ be holomorphic over $U = B(0,2) \setminus \overline{B(0,1)}$ and assume $g \in W^{1,2}(B(0,2))$. Let $\varphi \coloneqq g|_U$, then the trace of φ on the inner circle is in*

$$W^{\frac{1}{2},2}(\partial B(0,1)) \coloneqq \left\{ \psi \in L^2(\partial B(0,1)) \colon \psi(e^{i\theta}) = \sum_k b_k e^{i\theta k} \implies \sum_k |b_k|^2 |k| < +\infty \right\}$$

and can be written via Fourier series as

$$\varphi(re^{i\theta})|_{\partial B(0,1)} = S(\theta) = \sum_k a_k e^{i\theta k}, \text{ for a.e. } \theta \in [-\pi, \pi].$$

Proof. For any $1 < r < 2$ we can write the Laurent series coefficients for $g = \varphi$ as:

$$a_n = \frac{1}{2\pi i} \oint_{\partial B_r(0)} \frac{\varphi(y)}{y^{n+1}} dy$$

(which does not depend on the choice of r) and

$$\varphi(z) = \sum_n a_n z^n, \quad \varphi'(z) = \sum_n n a_n z^{n-1}$$

On the other hand for any $z \in \partial B_r(0)$ we have $z = re^{i\theta}$ for some $\theta \in [-\pi, \pi)$ and we can write φ (which is continuous on the circle) using the Fourier series as:

$$\varphi(re^{i\theta}) = \varphi_r(\theta) = \sum_k \int_{-\pi}^{\pi} \frac{\varphi_r(t) e^{i(\theta - t)k}}{2\pi} dt$$

and on the other hand using the same notation the Laurent series becomes:

$$\begin{aligned}
\varphi(z) = \sum_n a_n z^n &= \sum_n \frac{r^n e^{i\theta n}}{2\pi i} \oint_{\partial B_r(0)} \frac{\varphi(y)}{y^{n+1}} dy \\
&= \sum_n \frac{r^n e^{i\theta n}}{2\pi i} \int_{-\pi}^{\pi} \frac{\varphi_r(t)}{r^{n+1} e^{it(n+1)}} ire^{it} dt \\
&= \sum_n \int_{-\pi}^{\pi} \frac{\varphi_r(t) e^{i(\theta - t)n}}{2\pi} dt.
\end{aligned}$$

Hence the Laurent series terms are orthogonal (and the coefficients coincide up to a factor r^k with those of the Fourier series over any circle of radius $1 < r < 2$) so the following expressions, derived from the usual Parseval identity, hold:

$$\int_{-\pi}^{\pi} |\varphi_r(\theta)|^2 d\theta = \sum_k r^{2k} |a_k|^2,$$

$$\int_{-\pi}^{\pi} |\partial_\theta \varphi_r(\theta)|^2 d\theta = \int_{-\pi}^{\pi} \left| \sum_k ik \int_{-\pi}^{\pi} \frac{\varphi_r(t) e^{i(\theta - t)k}}{2\pi} dt \right|^2 d\theta = \sum_k k^2 r^{2k} |a_k|^2.$$

Now since we have $\varphi \in W^{1,2}(U) \subset L^2(U)$ we know that:

$$\sum_k |a_k|^2 \frac{2^{2k+1} - 1}{2k+1} = \sum_k \left(|a_k|^2 \int_1^2 r^{2k} dr \right) \leq \int_1^2 \int_{-\pi}^{\pi} |\varphi_r(\theta)|^2 r d\theta dr = \int_U |\varphi(z)|^2 dz < +\infty.$$

so the first sum is absolutely convergent and it holds:

$$\sum_{k \geq 0} |a_k|^2 \leq \sum_{k \in \mathbb{Z}} |a_k|^2 \frac{2^{2k+1} - 1}{2k+1} < +\infty \tag{6}$$

since $\frac{2^{2k+1}-1}{2k+1} \geq 1$ for all $k \geq 0$.

Similarly:

$$\begin{aligned} +\infty > \int_U |\varphi'(z)|^2 dz &= \\ &= \int_1^2 \int_{-\pi}^{\pi} \left| e^{i\theta} \partial_r \varphi + i \frac{e^{i\theta}}{r} \partial_\theta f \right|^2 r d\theta dr \\ &= \int_1^2 \int_{-\pi}^{\pi} \frac{1}{r^2} |\partial_\theta \varphi_r(\theta)|^2 r d\theta dr \\ &= \sum_k k^2 |a_k|^2 \int_1^2 r^{2k-1} dr \\ &= \sum_k k \frac{2^{2k} - 1}{2} |a_k|^2. \end{aligned}$$

hence:

$$\sum_{k<0} |a_k|^2 \leq \frac{8}{3} \sum_{k \in \mathbb{Z}} k \frac{2^{2k} - 1}{2} |a_k|^2 < +\infty \tag{7}$$

since $k \frac{2^{2k}-1}{2} \geq \frac{3}{8}$ for all $k < 0$.

So by combining (6) and (7) we get that

$$\sum_k |a_k|^2 < +\infty \tag{8}$$

By (8) we know that $S(\theta) := \sum_n a_n e^{i\theta n} \in L^2([\pi, \pi])$ through Parseval's identity and hence it is well defined almost everywhere.

We would like to now prove that S is the trace of φ on the unit circle, or in other words that $S(\arg(\cdot)) = \varphi|_{\partial B(0,1)}$. To see that, we can show that, for $r \to 1$, we have the convergence

$\varphi_r(\theta) = \varphi(re^{i\theta}) = \sum_k a_k e^{i\theta k} r^k \to \sum_k a_k e^{i\theta k} = S(\theta)$ in L^2; in fact we can see that for each $r = 1 + \varepsilon$, with $\varepsilon < 1$, we get:

$$\|\varphi_r - S\|_{L^2([-\pi,\pi])} = \sum_{k=-\infty}^{+\infty} |a_k|^2 \left|r^k - 1\right|^2 \leq \sum_{k=-\infty}^{-1} |a_k|^2 \left|1 - \frac{1}{1+\varepsilon}\right|^2 + \sum_{k=0}^{+\infty} |a_k|^2 \left|(1+\varepsilon)^k - 1\right|^2.$$

The first sum can be rewritten as:

$$\sum_{k=-\infty}^{-1} |a_k|^2 \left|1 - \frac{1}{1+\varepsilon}\right|^2 = \varepsilon^2 \sum_{k=-\infty}^{-1} |a_k|^2 \frac{1}{(1+\varepsilon)^2} \leq \frac{\varepsilon^2}{4} \|S\|_{L^2} \to 0$$

for $\varepsilon \to 0$ by using (8).

The second series can be instead rewritten (by expanding and using some straightforward inequalities) as:

$$\sum_{k=0}^{+\infty} |a_k|^2 \left|(1+\varepsilon)^k - 1\right|^2 \leq \sum_{k=0}^{+\infty} |a_k|^2 ((1+\varepsilon)^{2k} - 1) \leq \sum_{k=0}^{+\infty} |a_k|^2 k (2^{2k} - 1) < +\infty$$

by using (7) for all $0 < \varepsilon < 1$, hence by dominated convergence in L^2 we have that, since $\left|(1+\varepsilon)^k - 1\right|^2 \to 0$ for $\varepsilon \to 0$, then:

$$\sum_{k=0}^{+\infty} |a_k|^2 \left|(1+\varepsilon)^k - 1\right|^2 \xrightarrow[\varepsilon \to 0]{} 0$$

Hence $\varphi_r \to S$ in L^2 for $r \to 1$ and as such it converges a.e. for some sequence $r_n \to 1$.

On the other hand, let us fix a radial segment $I_{\tilde{z}}$ from 0 such that $\tilde{z} \in I_{\tilde{z}}$ for some point $\tilde{z} \in \partial B(0,1)$. On such a segment the function $r \to g(r\tilde{z})$ is in $W^{1,2}([0,2])$ by Fubini for almost every choice of $\tilde{z}$, otherwise by Fubini the $W^{1,2}$ norm of g would not be finite.

In particular, since we are now restricted to a 1-dimensional domain, we have that $W^{1,2}([0,2])$ embeds into $C^0((0,2))$ and $r \to g(r\tilde{z})$ has a continuous representation for a.e. $\tilde{z}$ on $(0,2)$ and as such it is bounded in a neighborhood of 1. So for $r \to 1^+$ we have that $\varphi|_{\partial B(0,1)} = \varphi_1 \coloneqq \lim_{r \to 1} \varphi_r$ and our trace is well defined by continuity in almost every point of $B(0,1)$. So necessarily for $r_n \to 1$ we get $\varphi_{r_n} \to \varphi_1$ a.e. on $B(0,1)$ and $\varphi_{r_n} \to S$ a.e. on $B(0,1)$, and as such $S = \varphi_1$ a.e..

Now we would like to show that $\left\|\varphi_1^{\frac{1}{2}}(\theta)\right\|_{L^2([\pi,\pi])} = \sum_k |k||a_k|^2 < +\infty$.

To do so we see that (7) already proves that $\sum_{k<0} |k||a_k|^2 < +\infty$ since

$$k\frac{2^{2k} - 1}{2} = |k|\left|\frac{2^{2k} - 1}{2}\right| \geq \frac{3}{8}|k|$$

for all $k < 0$. On the other hand we can see that $\frac{2^{2k+1}-1}{2k+1} \geq k$ eventually for $k \geq 0$, hence:

$$\sum_{k \geq 0} |k||a_k|^2 \leq C + \sum_{k \in \mathbb{Z}} |a_k|^2 \frac{2^{2k+1} - 1}{2k + 1} < +\infty$$

where C is a constant to balance out the (finite number of) terms for which the inequality $\frac{2^{2k+1}-1}{2k+1} \geq k$ does not hold (in reality we can actually see that it holds for all $k \geq 0$ and no additional constant C is needed). So we have proven the above mentioned identity.

$\square$

The previous result implies that the series $\sum a_k z^k$ can be expanded on the boundary (technically $\mathcal{H}^1$-a.e., but this will not be relevant).

Since we assumed g to be constant on vertical segments in $B(0,1)$, then by continuity we get $g(z) = g(\overline{z})$ for all $z \in \partial B(0,1)$. Hence for all such z we have $\sum a_k z^k = g(z) = g(\overline{z}) = \sum a_k \overline{z}^k$, or in other words for all $\theta \in [-\pi, \pi]$:

$$\sum_{k=-\infty}^{+\infty} a_k e^{i\theta k} = \sum_{k=-\infty}^{+\infty} a_k e^{-i\theta k} = \sum_{k=-\infty}^{+\infty} a_{-k} e^{i\theta k},$$

hence they are the same Fourier series and

$$a_k = a_{-k} \quad \text{for all} \quad k \in \mathbb{Z}. \tag{9}$$

So we can write:

$$g(z) = \sum_{k=0}^{+\infty} a_k \left(z^k + z^{-k}\right) \tag{10}$$

and, in particular, we can see that

$$g(z) = g\left(\frac{1}{z}\right) \quad \text{for all} \quad z \in U. \tag{11}$$

We already know that the Laurent series (10) is absolutely convergent for all $z \in U$, so, by (11) we get that it is absolutely convergent also for all $z \in \tilde{U} := B(0,1) \setminus \overline{B(0, \frac{1}{2})}$.

Let us now define

$$\tilde{g}(z) := \sum_{k=0}^{+\infty} a_k \left(z^k + z^{-k}\right)$$

for all $z \in B(0,2) \setminus \overline{B(0, \frac{1}{2})}$. It is absolutely convergent and hence holomorphic on U and $\tilde{U}$ and as such it must be absolutely convergent on the $\partial B(0,1)$ too. In fact, because of (9) we have that:

$$\sum_{k=0}^{+\infty} |a_k| \leq \sum_{k=0}^{+\infty} |a_k| \left|z^k\right| < +\infty$$

for $z \in U$, and the series is absolutely convergent on the unit circle and $\tilde{g}$ is holomorphic (and in particular continuous) over its whole domain. Furthermore g is continuous over $\partial B(0,1)$ and equal to $\tilde{g}$ almost everywhere on the unit circle, hence it must be that $g|_{\partial B(0,1)} = \tilde{g}|_{\partial B(0,1)}$.

Lemma 4.2. *Let $\tilde{g}$ be as above. Then necessarily*

$$\lim_{z \to 1} \det \nabla \tilde{g}(z) = 0.$$

Proof. Let us assume by contradiction $|\det \nabla \tilde{g}(1)| > 0$. Then there exists a neighborhood U of 1 over which $\tilde{g}$ is invertible by the Local Inversion Theorem. But then let $z \in \partial B(0,1) \cap U$, $z \neq 1$.

We have that $|z-1| = |\mathfrak{Re}(z) - 1|^2 + \mathfrak{Im}(z)^2 = |\overline{z} - 1|$, so $\overline{z} \in U$. Since $\overline{z} = \frac{1}{z}$ we have $\tilde{g}(\overline{z}) = \tilde{g}(z)$ and as such $\tilde{g}$ is not one-one. So $|\det \nabla \tilde{g}(1)| = 0$ and the statement follows by continuity.

□

Then we can now conclude: in fact we know that for all $z \in U$

$$\det \nabla \tilde{g} = \det \nabla^2 u \geq c > 0,$$

which by the previous lemma leads to a contradiction. Hence there cannot be such a function u and we have proven our case. Of course, this only holds in these very specific assumptions, nonetheless it helps to understand that we have to focus our attention on the behavior of the function and its gradient around z_0.

5. Non-holomorphic case

Let us now consider a situation in which the segments over which ∇u is constant are parallel to each other (at least close to z_0, same construction as in the last section), but the function u is not necessarily holomorphic on the external annulus ($h \neq \mathrm{id}$). As usual, we would like to change our coordinate system so that such segments are orthogonal to the horizontal axis in $\mathbb{R}^2$ and $z_0 = (1, 0)$.

To understand what changes in this scenario we will first consider a different situation in which our domain is $\tilde{\Omega} =]0,2[\times]-L,L[$ and with constraint $0 < c \le \tilde{f} \le C$ on $]1,2[\times]-L,L[$ and $\tilde{f} \equiv 0$ everywhere else. Ideally, close enough to z_0, $\partial B(0,1)$ becomes similar to its tangent in z_0 which is a vertical line, and should resemble the case we are considering right now. We will now study the behavior of u on the rectangle $R = [1, 1+\varepsilon] \times [-L, L]$ for some $\varepsilon > 0$ small.

Let $\tilde{u} \in W^{1,2}(\Omega)$ satisfy

$$\det \nabla^2 \tilde{u} = \tilde{f}.$$

In particular let us observe that now $\nabla\tilde{u}$ is continuous and constant along all segments with extremes $(a, -L)$ and (a, L) for $a \in]1-\varepsilon, 1[$. In particular it is constant a.e. on the segment $\sigma = [(1, -L), (1, L)]$ by continuity (we know by Fubini that it is $W^{1,2}$ on a.e. horizontal segment $x_1 = b$, and as such continuous on a.e. horizontal segment. Let us call $A_1 \subset \sigma$ the set of $\mathcal{H}^1$-measure 0 of the intersections of σ with the segments where this does not hold, on the other points we get the desired result by continuity). Let us now assume to simplify our calculations that $\nabla\tilde{u} = 0$ over σ (it will only change the center of the ball containing the image of $\nabla\tilde{u}$ we will be considering in the next paragraphs, but the proof will stay unchanged).

On the one hand by Theorem 3.5 and 3.6 we have that u is strictly convex and C^1 over R, hence $\nabla\tilde{u}$ is globally one-one on R and, as such, invertible over its image. So we have, by using the area and coarea formula together with the lower bound on $\tilde{f}$:

$$|\nabla\tilde{u}(R)| = \int_{\mathbb{R}^2} \chi_{\nabla\tilde{u}(R)}(y)dy = \int_{\mathbb{R}^2} \left(\int_{(\nabla\tilde{u})^{-1}(y)} d\mathcal{H}^0 \right) dy = \int_R \left|\det \nabla^2\tilde{u}(x)\right| dx \ge c|R| = 2cL\varepsilon. \tag{12}$$

We will now look an upper bound on $|\nabla\tilde{u}(R)|$: for any point $x = (x_1, x_2) \in R$ let $\sigma_{x_2} = (1, x_2)$ be its orthogonal projection on σ. Now let $\Lambda_\ell^\varepsilon \subset R$ be the horizontal segment constituted by all points $x = (x_1, x_2) \in R$ such that $x_2 = \ell$ and $|x - \sigma_\ell| < \varepsilon$. We have that $R = \bigcup_{\sigma_\ell \in \sigma} \Lambda_\ell^\varepsilon$. Now we know that, by Fubini:

$$\|\nabla\tilde{u}\|_{W^{1,2}(R)} = \int_{-L}^{L} \|\nabla\tilde{u}\|^2_{W^{1,2}(\Lambda_\ell^\varepsilon)} d\ell < +\infty,$$

and $\|\nabla\tilde{u}\|^2_{W^{1,2}(\Lambda^\varepsilon_\ell)} < +\infty$ for all ℓ such that $\sigma_\ell \in \sigma \setminus A_1$, $\nabla\tilde{u}$ is $W^{1,2}(\Lambda^\varepsilon_\ell)$. Furthermore, by absolute continuity of the norm over the strip R:

$$a(\varepsilon) := \int_{-L}^{L} \|\nabla\tilde{u}\|^2_{W^{1,2}(\Lambda^\varepsilon_\ell)} d\ell \xrightarrow{\varepsilon\to 0} 0. \tag{13}$$

We will now use the canonical embedding of $W^{1,2}$ into $C^{0,1/2}$ in dimension 1, where $C^{0,1/2}$ is the classic $\frac{1}{2}$-Hölder space. In particular, for all ℓ such that $\sigma_\ell \in \sigma\setminus A_1$, we have that $\nabla\tilde{u} \in C^{0,1/2}(\Lambda^\varepsilon_\ell)$ and consequently we get:

$$\sup_{x,y\in\Lambda^\varepsilon_\ell} \frac{|\nabla\tilde{u}(x) - \nabla\tilde{u}(y)|}{|x-y|^{\frac{1}{2}}} \le k\|\nabla\tilde{u}\|_{W^{1,2}(\Lambda^\varepsilon_\ell)}, \tag{14}$$

for all ℓ such that $\sigma_\ell \in \sigma \setminus A_1$.

Then we have, by using (14) and the fact that $\nabla\tilde{u}(\sigma_{x_2}) = 0$ for all $(x_1, x_2) \in R$:

$$m_\ell := \sup_{x\in\Lambda^\varepsilon_\ell} |\nabla\tilde{u}(x)| = \sup_{\substack{0<|x-\sigma_\ell|<\varepsilon \\ x_2=\ell}} |\nabla\tilde{u}(x) - \nabla\tilde{u}(\sigma_\ell)| \le k\varepsilon^{\frac{1}{2}}\|\nabla\tilde{u}\|_{W^{1,2}(\Lambda^\varepsilon_\ell)}. \tag{15}$$

By Chebyshev's inequality, using the definition of $a(\varepsilon)$ in (13), for all $\delta > 0$:

$$\left|\left\{\ell : \|\nabla\tilde{u}\|_{W^{1,2}(\Lambda^\varepsilon_\ell)} > \delta\right\}\right| \le \frac{a(\varepsilon)}{\delta^2}$$

Now let us define the following sets:

$$R_1 = [1, 1+\varepsilon] \times A_1,$$
$$R_2 = [1, 1+\varepsilon] \times \left([-L, L] \cap \left\{\ell : \|\nabla\tilde{u}\|_{W^{1,2}(\Lambda^\varepsilon_\ell)} > \delta\right\}\right),$$
$$R_3 = R \setminus (R_1 \cup R_2) = [1, 1+\varepsilon] \times \left([-L, L] \cap \left\{\ell : \|\nabla\tilde{u}\|_{W^{1,2}(\Lambda^\varepsilon_\ell)} \le \delta\right\}\right).$$

We have $R = R_1 \cup R_2 \cup R_3$ and $|\nabla\tilde{u}(R)| = |\nabla\tilde{u}(R_1)| + |\nabla\tilde{u}(R_2)| + |\nabla\tilde{u}(R_3)|$.

By the area and coarea formula, as done for the lower bound in the first part of the proof, we have that:

$$|\nabla\tilde{u}(R_1)| = \int_{R_1} \left|\det \nabla^2\tilde{u}(x)\right| dx = 0, \tag{16}$$

since $|R_1| = 0$; also:

$$|\nabla\tilde{u}(R_2)| = \int_{R_2} \left|\det \nabla^2\tilde{u}(x)\right| dx \le C|R_2| \le C\frac{a(\varepsilon)}{\delta^2}\varepsilon, \tag{17}$$

by using the bound $\tilde{f} \le C$ a.e..

Lastly, by (15), for all $x = (x_1, x_2) \in R_3$ we have the bound:

$$|\nabla\tilde{u}(x)| \le m_{x_2} \le \|\nabla\tilde{u}\|_{W^{1,2}(\Lambda^\varepsilon_\ell)}\varepsilon^{\frac{1}{2}} \le \delta\varepsilon^{\frac{1}{2}}.$$

As such $\nabla\tilde{u}(R_3) \subseteq B(0, \delta\varepsilon^{\frac{1}{2}})$ and

$$|\nabla\tilde{u}(R_3)| \le \left|B(0, \delta\varepsilon^{\frac{1}{2}})\right| = \pi\delta^2\varepsilon \tag{18}$$

So we get, by (16), (17) and (18):

$$|\nabla\tilde{u}(R)| \le \pi\delta^2\varepsilon + C\frac{a(\varepsilon)}{\delta^2}\varepsilon. \tag{19}$$

By combining (12) and (19), we have, for $\varepsilon \to 0$:

$$2cL\varepsilon \leq |\nabla\tilde{u}(R)| \leq \pi\delta^2\varepsilon + C\frac{a(\varepsilon)}{\delta^2}\varepsilon.$$

We can now fix δ small enough so that $2cL - \pi\delta^2 > 0$ (we observe that our choice of δ does not depend on ε) and get:

$$0 < (2cL - \pi\delta^2)\varepsilon \leq C\frac{a(\varepsilon)}{\delta^2}\varepsilon.$$

and since $\varepsilon > 0$:

$$0 < 2cL - \pi\delta^2 \leq C\frac{a(\varepsilon)}{\delta^2}.$$

which gives us a contradiction since $a(\varepsilon) \to 0$ for $\varepsilon \to 0$. □

So we understand that the idea is to study how the gradient behaves around z_0 and in particular to combine our assumptions of invertibility on the annulus and the behavior of the gradient on the border in order to get incompatible bounds. We will not be able to adapt the full proof to the circular case in this instance, but we will discuss the framework and main ideas behind it.

We have to first notice that in this new scenario our results on convexity do not globally hold on the annulus, and we only have local convexity since $U = B(0,2)\setminus\overline{B(0,1)}$ is not a convex domain. As such our proof changes quite a bit. We will make one further assumption for this example and ask that for all $x = (x_1, x_2) \in \overline{B(0,1)}$ it holds that $\nabla u(x_1, x_2) = (x_1, 0)$. This respects the assumptions of developability given by Theorem 3.1, since the gradient ∇u is constant along vertical segments, but also gives us some more regularity to work with.

Just as in the previous case, we will call $z_0 = (0,1)$. Now let $\sigma \subset B(0,1)$ be the arc of circle of length 2θ and with z_0 as its middle point. Let us take $\varepsilon > 0$ and consider $S_\sigma \subset B(0,1+\varepsilon)$, defined as the circular sector of $B(0,1+\varepsilon)$ of central angle 2θ containing σ and let $R = \left(B(0,1+\varepsilon)\setminus\overline{B(0,1)}\right)\cap S_\sigma$ will be the set we will be working with in this case.

By following the previous construction we first call each point in σ as σ_t with $t \in [-\theta,\theta]$ in such a way that $\sigma_{-\theta}$ and σ_θ are the extremes of σ and for each $t \in [-\theta,\theta]$ the arc contained in σ and delimited by $\sigma_{-\theta}$ and σ_t has length $t+\theta$ (and necessarily we get $\sigma_0 = z_0$). Now we call Λ_t^ε the segments obtained by taking the radius of $B(0,2)$ passing through σ_t and intersecting it with R.

To prove our lower bound we will use Brouwer's degree theory. We recall that a *regular value* of a function v on Ω is a point $y \in \Omega$ such that for any $x \in v^{-1}(y)$, x is a *regular point*, that is a point such that the differential of v computed at x, Dv_x, is surjective.

Definition 5.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded region, let $v\colon \overline{\Omega} \to \mathbb{R}^n$ be a smooth function and $y \notin v(\partial\Omega)$ a regular value of v. Then we can define the *Brouwer degree* of v at y as:

$$\deg(v,\Omega,y) := \sum_{x\in v^{-1}(y)} \operatorname{sgn}\det Dv(x)$$

By Sard's theorem we can define the Brower's degree of ∇u at almost every point in R. In particular we can use it to prove the following:

Theorem 5.2. *Let $R \subset \mathbb{R}^2$, $u : R \to \mathbb{R}$ be defined as above. Then there exists m such that $|\deg(\nabla u, R, y)| \leq m$ for a.e. $y \in \nabla u(R)$ and we have:*

$$\int_R \left|\det \nabla^2 u(x)\right| dx \leq m|\nabla u(R)|$$

Proof. The proof of the second part descends directly once again from the area and coarea formula. In fact we have:

$$\int_R \left|\det \nabla^2 u(x)\right| dx = \int_{\mathbb{R}^2} \left(\int_{(\nabla u)^{-1}(y)} d\mathcal{H}^0 \right) dy = \int_{\mathbb{R}^2} |\deg(\nabla u, R, y)| dy \le m |\nabla u(R)|$$

where in the last inequality we have used that $\deg(\nabla u, R, y) = 0$ for all $y \notin \nabla u(R)$ (see [6]).

Let us now prove that there is an m as in the hypothesis. We can use the same argument as in the holomorphic case to reflect our function ∇u on the inner disk. To do that we define a function g in a neighborhood V of z_0 such that $g(x) = \nabla u(x)$ for all $x \in U \cap V$ and for any $x = (x_1, x_2) \in B(0,1) \cap V$ it is defined in the following way:

$$g(x_1, x_2) = \nabla u \left(\frac{x_1}{(x_1^2 + x_2^2)}, \frac{-x_2}{(x_1^2 + x_2^2)} \right).$$

We see that the traces of the two definitions of g coincide (a.e.) on $\partial B(0,1)$. In fact we remind that by Fubini ∇u is continuous on almost every Λ_t^ε and as such it converges radially almost everywhere to its trace as $|x| \to 1$. On the other hand we notice that the transformation considered to define g on $B(0,1)$ preserves segments, in and particular it sends a radial segment into a radial segment. As such g is also continuous on almost every radial segment (radial w.r.t. to the origin 0) in $B(0,1) \cap V$, and radially converges on a.e. of these segments to its trace on $\partial B(0,1)$ as $|x| \to 1$. Then we can check that, for a.e. $(x_1, x_2) \in \partial B(0,1)$, and $r < 1$ such that $rx \in V$:

$$g(rx) = \nabla u \left(\frac{rx_1}{r^2(x_1^2 + x_2^2)}, \frac{-rx_2}{r^2(x_1^2 + x_2^2)} \right) = \nabla u \left(\frac{x_1}{r}, -\frac{x_2}{r} \right) \xrightarrow{r \to 1^-} \nabla u|_{\partial B(0,1)}(x_1, -x_2),$$

And for $r > 0$:

$$g(rx) = \nabla u(rx) \xrightarrow{r \to 1^+} \nabla u|_{\partial B(0,1)}(x_1, x_2) = \nabla u|_{\partial B(0,1)}(x_1, -x_2)$$

by the boundary conditions on ∇u given by the hypothesis of developability (which, in our case, translates in ∇u being constant on all vertical segments inside of $B(0,1)$ close to z_0). So the traces coincide and our newly defined function g is in $W^{1,2}(V)$.

We notice that $\det \nabla g(x) = f(x)$ for all $x \in U \cap V$ and

$$\det \nabla g(x_1, x_2) = \frac{(x_1^2 - x_2^2)^2}{(x_1^2 + x_2^2)^4} f \left(\frac{x_1}{(x_1^2 + x_2^2)}, \frac{-x_2}{(x_1^2 + x_2^2)} \right) \ge \tilde{c} > 0$$

as long as V is small enough around to have a bound on the coefficient. In particular that holds for a ball of radius small enough to not intersect the lines $x_1 = \pm x_2$.

Then by Theorem 3.4 we get that g is holomorphic up to a change of variables on V. We have assumed $g(z_0) = 0$ (otherwise we would just take $g(z) - g(z_0)$ instead). If we have that z_0 is a root of order m, then we get that there exist $\rho, \delta > 0$ such that $g(z) - w$ has exactly m roots in $B(z_0, \rho)$ for all $w \in B(0, \delta)$. In particular by continuity around z_0, by further restricting the radius of V, we may assume that $g(V) \subseteq B(0, \delta)$, and as such $g(z) - w$ has at most m zeroes in V for all $w \in g(V)$.

If we choose θ and ε small enough so that $R \subset V$ we get:

$$|\deg(\nabla u, R, y)| \le |\deg(g, V, y)| \le \left| \sum_{x \in g^{-1}(y)} \operatorname{sgn} \det Dg(x) \right| \le m.$$

□

So, from Theorem 5.2, we get that:

$$|\nabla u(R)| \geq \frac{1}{m}\int_R \det \nabla^2 u(x)dx \geq \frac{c}{m}|R| = \frac{c}{m}\pi\theta\varepsilon(2+\varepsilon) \geq 2\frac{c}{m}\pi\theta\varepsilon. \tag{20}$$

The other bound is trickier. We will not be able to provide a true bound in this project, however we will try to explain the idea behind it: as before for $\mathcal{H}^1$-a.e. $t \in [-\theta, \theta]$ (specifically for all $t \in [-\theta, \theta] \setminus A_1$, with $\mathcal{H}^1(A_1) = 0$) we have by Fubini and Morrey that:

$$m_t := \sup_{x \in \Lambda_t^\varepsilon} |\nabla u(x) - \nabla u(\sigma_t)| \leq \|\nabla u\|_{W^{1,2}(\Lambda_t^\varepsilon)} \varepsilon^{\frac{1}{2}} \tag{21}$$

For our assumptions on ∇u on $\overline{B(0,1)}$ we have that $\nabla u(\sigma_t) = (\cos t, 0)$.

For $t \in [-\theta, \theta]$ (and θ small) we have that $1 - \frac{\theta^2}{2} \leq \cos t \leq 1$, so:

$$\nabla u(\sigma_t) = (\cos t, 0) \subset \left\{(\xi, 0) : 1 - \frac{\theta^2}{2} \leq \xi \leq 1\right\} =: S.$$

Just as before we can also prove that:

$$a(\varepsilon) := \int_{-L}^{L} \|\nabla \tilde{u}\|^2_{W^{1,2}(\Lambda_t^\varepsilon)} dt \xrightarrow{\varepsilon \to 0} 0$$

Now the idea would be to choose $\delta > 0$ small (to fix later) and get that:

$$\left|\left\{t : \|\nabla u\|_{W^{1,2}(\Lambda_t^\varepsilon)} > \delta\right\}\right| \leq \frac{a(\varepsilon)}{\delta^2}$$

for some $\delta > 0$ and we define R_1, R_2 and R_3 similarly to before. However the issue is that, while the same bounds hold for R_1 and R_2, when trying to provide a bound for $|\nabla u(R_3)|$ we need to be a bit more accurate and use stricter inequalities.

In fact, since $\nabla u(\sigma_t) \in S$, we now see that on R_3 we have that $\nabla u(R_3) \subseteq S_{\delta\varepsilon^{1/2}}$, where $S_{\delta\varepsilon^{1/2}}$ is the set of points $x \in \mathbb{R}^2$ such that $d(x, S) \leq \delta\varepsilon^{\frac{1}{2}}$. If we were to roughly compute the area of this region, which is basically constituted by the union of two halves of a disk of radius $\delta\varepsilon^{\frac{1}{2}}$ and a rectangle of sides $2\delta\varepsilon^{\frac{1}{2}}$ and $|S| = \frac{\theta^2}{2}$, we would not get a direct bound. Some further considerations on the properties of u might be necessary to proceed further towards our goal.

The idea behind the computations would be to show that in the end each Λ_t^ε is sent by ∇u to some curve, whose length integrated on t should roughly correspond to the area of the rectangle considered above. These ideas will however only be verified in a future project.

6. Conclusions

We have proven for some specific cases that there is indeed no solution to the degenerate Monge-Ampère constraints we have provided and shown the ideas behind an extension of the result to a more general case.

The idea would be now to further expand the result. The examples above should clarify well enough the procedure one must adopt in general, except this time we will have that the segments on which ∇u is constant on the unit disk are not parallel.

We should be able to prove that the lower bound (20) on $|\nabla u(R)|$ is true with similar means as in the last case, as long as the region R is adjusted to the new geometrical configuration given by the developability assumptions. The bigger difference would be in the upper bound, since we no longer have the assumptions on the boundary values of ∇u on some arc σ containing z_0.

As long as the segments are parallel for a general ∇u, we could proceed by taking the segments in $B(0,1)$ that are parallel to the horizontal axis with extreme points on the unit circle. As before by Fubini on a.e. of those segments we get that ∇u is $W^{1,2}$ and as such $C^{0,1/2}$. The idea is that the values on those segments correspond to the values of ∇u on σ, and through the bounds given by the $C^{0,1/2}$ norm we can find some bounds on the region occupied by $\nabla u(\sigma)$ and proceed in a similar way as in the previous section.

This still takes into account that the segments on which ∇u is constant are parallel, while a problem may arise when they are not. However, to solve this, we could define a curve γ starting from z_0 and intersecting each segment Λ in a point x such that $\gamma(s) = x$ and $\gamma'(s) \perp \Lambda$. If we can prove γ to be regular enough then a possibility would be to proceed as mentioned above to find a bound on the values of ∇u on some arc σ and consequently find similar bounds on $|\nabla u(R)|$.

The above idea will be developed in future works. As of now I will conclude this project by thanking my supervisor, Professor Mohammad Reza Pakzad, for helping me throughout all this work, and my family and colleagues from Padova and Paris for their support.

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