University of Padua

DEPARTMENT OF MATHEMATICS "TULLIO LEVI-CIVITA"

Bachelor's Degree in Mathematics

Solving Solvable Quartic, Quintic and Sextic Equations

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"One who, treating such new subjects, taking a chance on such a strange road, pretty often difficulties presented themselves that I was unable to overcome. Even in these two memoirs, and especially in the second which is the more recent, the formula "I do not know" will often be found. The class of readers of whom I have spoken at the beginning will not fail to find something laughable there. Unhappily one cannot doubt that the most precious book of the greatest scientist will be that in which he tells us everything that he does not know; one cannot doubt that an author never betrays his readers so much as when he hides a difficulty."
— Évariste Galois (1811-1832) in the preface of *Deux mémoires d'Analyse pure*, October 8, 1831.

"Si deve prevedere che, trattandosi di soggetti talmente nuovi, azzardati in una veste così insolita che molto spesso si sono presentate delle difficoltà che non sono stato in grado di sormontare. Inoltre, in queste due memorie, e specialmente nella seconda, che è la più recente, troveremo spesso la formula "Non so". La classe dei lettori che ho menzionato all'inizio non mancherà di ridere di questo. Ciò accade perché, sfortunatamente, non pensiamo che il libro più prezioso del più sapiente sarebbe quello in cui egli dicesse tutto ciò che non sa; non comprendiamo che un autore non nuoce mai così tanto al suo lettore come quando dissimula una difficoltà."
— Évariste Galois (1811-1832) nella prefazione di *Deux mémoires d'Analyse pure*, October 8, 1831.

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Abstract

This thesis contains the result of K. Conrad, D. S. Dummit and T. R. Hagedorn about solving solvable polynomials of degree 4, 5 and 6 using Galois theory. First of all we will describe a procedure for figuring out the Galois groups of separable irreducible quartics (we are not going to derive the classical quartic formula by Ferrari). Then we will give general formulas for finding the roots of all irreducible quintic (sextic respectively) polynomials $f(x) \in \mathbb{Q}[x]$ with Gal(f)$= G_f$, where G_f is a transitive, solvable subgroup of S_5 (S_6 resp.).

Contents

Chapter 1

Introduction

1.1 Introduction

Given an irreducible polynomial $f(x) \in \mathbb{Q}[x]$, does there exist a formula for finding its roots using only the basic arithmetic operations and the taking of n-th roots? The answer to this classical question has been one of the main open problem in mathematics until the introduction and development of Galois theory in the beginning of 19th century. When such a formula exists, we say that the equation $f(x) = 0$ is solvable by radicals. If the same formula can be used for all polynomials $f(x)$ with degree n, we say that the general equation of degree n is solvable by radicals.

The quadratic formula, the Cardano's formula and the Ferrari's method show that the general equations of degree 2, 3 and 4 are solvable by radicals. Abel and Ruffini showed that the general equation of degree $n \geq 5$ is not solvable by radical. Even stronger, Galois theory established that for each $n \geq 5$, there are irreducible polynomials $f(x) \in \mathbb{Q}[x]$ of degree n which are not solvable by radicals. In fact, for $n \geq 5$, most irreducible polynomials $f(x)$ of degree n are insolvable by radicals.

We can ask whether such formulas exist when we restrict our attention to the class of polynomials which are solvable by radicals. We recall a fundamental theorem:

Theorem 1.1.1 (Galois, 1832)**.** *Let F be a field of characteristic zero, and let $f(x) \in F[x]$. The equation $f(x) = 0$ is solvable by radicals if and only if the Galois group $\mathrm{Gal}(f)$ of $f(x)$ is solvable.*

(see [7]) that let us switch our attention from the equation $f(x) = 0$ to the solvable subgroups of S_n ($n = \deg(f)$). We also know that when $f(x)$ is irreducible, $\mathrm{Gal}(f)$ is a transitive subgroup of S_n. We can now focus on this class of subgroups of S_n.

In this thesis, after a brief recall of some fundamental results and definitions, we are going to describe in Chapter 2 two methods (a 'classical' one and an 'alternative' one by Kappe and Warren (see as reference [6]) for figuring out the Galois groups of separable irreducible quartics. Then in Chapter 3 an explicit resolvent sextic is constructed which has a rational root if and only if the irreducible quintic $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbb{Q}[x]$ is solvable by radicals. When $f(x)$ is solvable by radicals, formulas for the roots are given in terms of p, q, r, s which produce the roots in a cyclic order. Finally, in Chapter 4 we show that there is a common formula for finding the roots of all irreducible sextic polynomials $f(x) \in \mathbb{Q}[x]$ with $\mathrm{Gal}(f) = G$ (transitive, solvable subgroup of S_6).

1.2 Theoretical background

In this Section we want to give some fundamental definitions and Theorems as a useful background for the results shown and proved in the following Chapters.

Theorem 1.2.1 ([6])**.** *Let $f(x) \in K[x]$ be a separable polynomial of degree n.*

1. *If $f(x)$ is irreducible in $K[x]$ then its Galois group over K has order divisible by n.*

2. *The polynomial $f(x)$ is irreducible in $K[x]$ if and only if its Galois group over K is a transitive subgroup of S_n.*

Definition 1 (Discriminant, [6],[7]). *If $f(x) \in K[x]$ factors in a splitting field as*

$$f(x) = c(x - r_1) \dots (x - r_n),$$

the discriminant of $f(x)$ is defined to be

$$\operatorname{disc}(f) = \prod_{i<j} (r_j - r_i)^2.$$

Theorem 1.2.2 ([6]). *Let $f(x) \in K[x]$ be a separable polynomial of degree n. If K does not have characteristic 2, the Galois group of $f(x)$ over K is a subgroup of A_n if and only if* $\operatorname{disc}(f)$ *is a square in K.*

Theorem 1.2.2 is why we will assume *our fields do not have characteristic 2.*

We now introduce the fundamental Theorem of Galois theory:

Theorem 1.2.3 (Fundamental Theorem of Galois theory, [7]). *Let K be a field and Ω/K be a Galois extension of K with Galois group $G = \operatorname{Gal}(\Omega/K)$. Than the subextensions of Ω/K are in one-to-one correspondence with the subgroups of G, i.e. the map $H \mapsto \Omega_H := \{\alpha \in \Omega \mid \sigma(\alpha) = \alpha,\ \forall \sigma \in H\}$ is a bijection from the set of subgroups of G to the set of subextensions of Ω/K,*

$$\{subgroups\ H \leq G\} \overset{1:1}{\longleftrightarrow} \{subextensions\ K \leq L \leq \Omega\}$$

with inverse $L \mapsto G_L = \operatorname{Gal}(\Omega/L) = \{\sigma \in G \mid \sigma|_L = id_L\}$. Moreover,

(a) the correspondence is inclusion-reversing:

$$H_1 \leq H_2 \Leftrightarrow \Omega_{H_1} \geq \Omega_{H_2} \text{ and } L_1 \leq L_2 \Leftrightarrow G_{L_1} \geq G_{L_2};$$

(b) indexes equal degrees:

$$\forall H_1 \leq H_2,\ (H_1 : H_2) = [\Omega_{H_2} : \Omega_{H_1}] \text{ and } \forall L_1 \leq L_2,\ (L_2 : L_1) = [G_{L_1} : G_{L_2}];$$

(c) $\forall \sigma \in G$, let $H^\sigma := \sigma H \sigma^{-1}$. Then

$$\Omega_{H^\sigma} = \sigma(\Omega_H) \text{ and } (G_L)^\sigma = G_{\sigma(L)};$$

(d) H is normal in $G \Leftrightarrow \Omega_H/K$ is normal (hence Galois) over K, in which case

$$\operatorname{Gal}(\Omega_H/K) = G/H.$$

Another fundamental Theorem of Galois theory is Theorem 1.1.1 in the previous Section.
Finally a Theorem about the splitting field of separable cubics:

Theorem 1.2.4 ([6]). *Let K not have characteristic 2 and $f(x) \in K[x]$ be a separable cubic with discriminant Δ. If r is one root of $f(x)$ then a splitting field of $f(x)$ over K is $K(r, \sqrt{\Delta})$. In particular, if $f(x)$ is a reducible cubic then its splitting field over K is $K(\sqrt{\Delta})$.*

Chapter 2

Quartics

2.1 Transitive subgroups of S_4

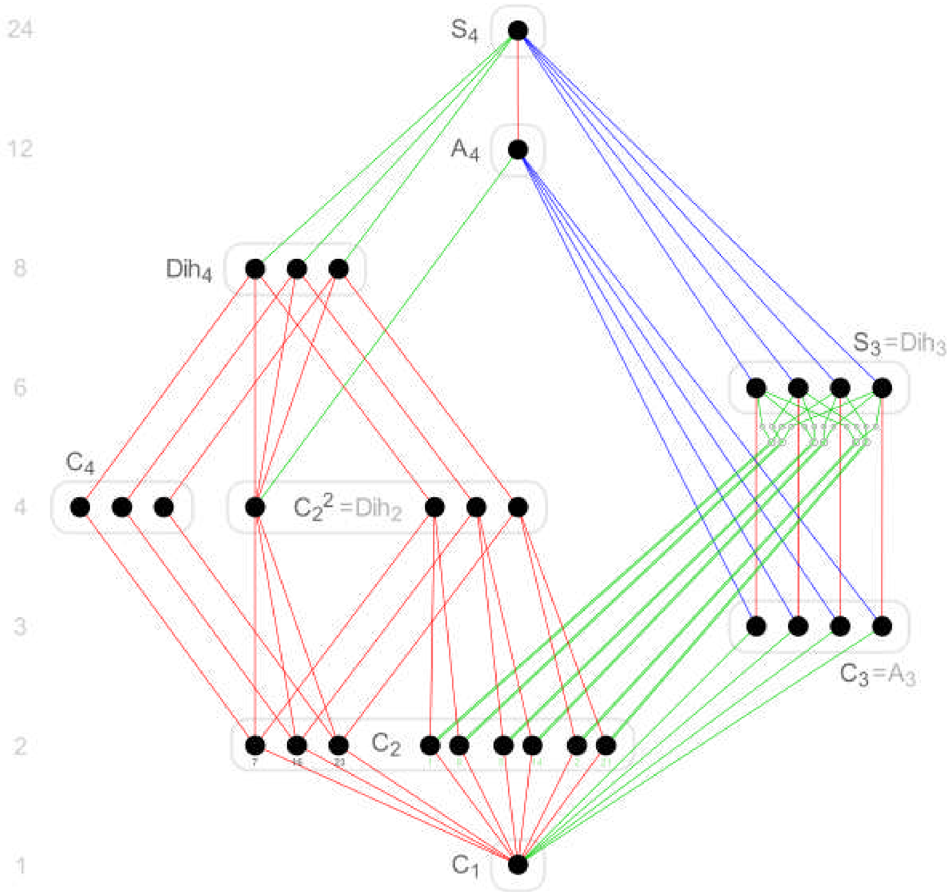


Figure 2.1: Subgroup diagram of S_4

To compute the Galois group G_f of a separable irreducible quartic $f(x) \in K[x]$, we first list all subgroups of S_4 in Figure 2.1. Among them, the candidates to be the Galois group are the transitive subgroups of S_4 such that $4 = \deg(f) \mid |G_f|$, by Theorem 1.2.1. These are (up to isomorphism):

G_f	S_4	A_4	D_4	C_4	$V \simeq C_2 \times C_2$
$\lvert G_f \rvert$	24	12	8	4	4

Table 2.1: Transitive subgroups of S_4

With the information given by Table 2.1 and Figure 2.1, we can now make some useful observations:

- D_4: Inside S_4 there are 3 transitive subgroups isomorphic to D_4, all conjugate to each other (from Sylow theorems, they are 2-Sylow subgroups):

$$\langle (1234), (13) \rangle, \ \langle (1324), (12) \rangle, \ \langle (1243), (14) \rangle.$$

- C_4: There are 3 transitive subgroups of S_4 isomorphic to C_4. These are the the only cyclic subgroups of order 4 in S_4 and they are conjugate to each other:

$$\langle (1234) \rangle, \ \langle (1324) \rangle, \ \langle (1243) \rangle.$$

- V: We write V for Klein's four-group $C_2 \times C_2$. There is only one transitive subgroup of S_4 isomorphic to V, that is:

$$\{(1), (12)(34), (13)(24), (14)(23)\}.$$

  V is the intersection of the 3 2-Sylow subgroups quote in the first point. There are other subgroups of S_4 that are isomorphic to V, but they are not transitive.

- The only transitive subgroups of S_4 inside A_4 are A_4 and V.
- The only transitive subgroups of S_4 with size divisible by 3 are S_4 and A_4.
- The only transitive subgroups of S_4 containing a transposition (a cycle of length 2) are S_4 and D_4.

2.2 Cubic resolvent

Let $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$ be monic, separable, irreducible, so $\mathrm{disc}(f) \neq 0$. Let r_1, r_2, r_3, r_4 be the roots of $f(x)$, so

$$f(x) = x^4 + ax^3 + bx^2 + cx + d = (x - r_1)(x - r_2)(x - r_3)(x - r_4) \in \Omega_f$$

where Ω_f is the splitting field of $f(x)$ over K.

It is known that the Galois group of a separable irreducible cubic polynomial $h(x) \in K[x]$ is determined by whether or not its discriminant $d = \mathrm{disc}(h)$ is a square in K, which can be thought of in terms of the associated quadratic polynomial $x^2 - d$ having a root in K. From this idea, we will see that the Galois group of a quartic polynomial depends on the behavior of an associated cubic polynomial.

We want to create a cubic polynomial with roots in Ω_f by finding an expression in the roots of $f(x)$ which only has 3 possible images under the Galois group. One such expression is: $x_1x_2 + x_3x_4$. In fact, if we define

$$\alpha = r_1r_2 + r_3r_4, \quad \beta = r_1r_3 + r_2r_4, \quad \gamma = r_1r_4 + r_2r_3$$

we can see that the group $S_4 = \mathrm{Sym}(\{r_1, r_2, r_3, r_4\})$ permutes $\{\alpha, \beta, \gamma\}$ transitively: $\alpha^{S_4} = \beta^{S_4} = \gamma^{S_4} = \{\alpha, \beta, \gamma\}$. The stabilizer of each of α, β, γ is a subgroup of S_4 of index $3 = |\{\alpha, \beta, \gamma\}|$, hence has order 8. So they must be the 3 2-Sylow subgroups. It follows that $\{\alpha, \beta, \gamma\}$ is fixed by their intersection, that is V from the previous observations. Therefore, if we consider the intermediate extension $K \leq K(\alpha, \beta, \gamma) \leq \Omega_f$, the subgroup $G_f \cap V$ of G_f fix $K(\alpha, \beta, \gamma)$.

In particular we have the following lemma:

$$\begin{array}{rl} & \Omega_f \\ G_f \cap V & \Big| \\ & K(\alpha,\beta,\gamma) \\ G_f/G_f \cap V & \Big| \\ & K \end{array}$$

Lemma 2.2.1 ([7])**.** *The fixed field of $G_f \cap V$ is $K(\alpha, \beta, \gamma)$. Hence $K(\alpha, \beta, \gamma)$ is Galois over K with Galois group $G_f/G_f \cap V$.*

Proof. The above discussion shows that the subgroup of G_f of elements fixing $K(\alpha, \beta, \gamma)$ is $G_f \cap V$, and so $\Omega_f^{G_f \cap V} = K(\alpha, \beta, \gamma)$ by the fundamental theorem of Galois theory. The remaining statements follow from the fundamental theorem using that V is normal. □

Definition 2 (Cubic resolvent, [7])**.** *Let $f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4) \in \Omega_f$,*

$$g(x) = (x - \alpha)(x - \beta)(x - \gamma) \in K(\alpha, \beta, \gamma)$$

is called cubic resolvent of f (α, β, γ defined as above).

We see that $\Omega_g = K(\alpha, \beta, \gamma)$ is the splitting field of $g(x)$ over K. Every permutation σ of the r_i (*a fortiori* $\forall \sigma \in G_f$) permutates α, β, γ, and so fixes $g(x)$: $g^\sigma = g$. We just prove that $g \in K[x]$ and $G_g = G_f/G_f \cap V$. More explicitly, we can express the coefficients of the cubic resolvent $g(x)$ in terms of the coefficients of the starting quartic $f(x)$:

Lemma 2.2.2 ([6])**.** *Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$, then*

$$g(x) = x^3 - bx^2 + (ac - 4d)x - (a^2d + c^2 - 4bd)$$

Moreover, $\text{disc}(f) = \text{disc}(g)$ *and $g(x)$ is separable since $f(x)$ is separable.*

Proof. (Sketch of proof) Expand $f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$ to express a, b, c, d in terms of r_1, r_2, r_3, r_4. Expand $g(x) = (x-\alpha)(x-\beta)(x-\gamma)$ to express the coefficients of g in terms of r_1, r_2, r_3, r_4, and substitute to express them in terms of a, b, c, d. To prove that the quartic and its cubic resolvent have the same discriminant, we just write the difference between the roots of $g(x)$. For example: $\alpha - \beta = (r_1r_2 + r_3r_4) - (r_1r_3 + r_2r_4) = (r_1 - r_4)(r_2 - r_3)$. Forming the other two differences, multiplying, and squaring, we obtain $\text{disc}(g) = \text{disc}(f)$. □

Remark: There is a second polynomial that can be found in the literature under the name of cubic resolvent for $f(X)$. In terms of the coefficients of $f(x)$, the cubic is: $x^3 - 2bx^2 + (b^2 + ac - 4d)x + (a^2d + c^2 - abc)$, whose roots are $(r_1 + r_2)(r_3 + r_4)$, $(r_1 + r_3)(r_2 + r_4)$, and $(r_1 + r_4)(r_2 + r_3)$. This amounts to exchanging additions and multiplications in the formation of the resolvent's roots.

Now let f be an irreducible quartic. Then G_f is one of the group in Table 2.1. These are the following possibilities for G_f:

$G_f \simeq$	$\lvert G_f \rvert$	$\lvert G_f \cap V \rvert$	$\lvert G_g \rvert$	$G_g \simeq G_f/G_f \cap V$
S_4	24	4	6	S_3
A_4	12	4	3	A_3
V	4	4	1	$\{1\}$
D_4	8	4	2	C_2
C_4	4	2	2	C_2

$$\begin{array}{rl} & \Omega_f \\ G_f \cap V & \Big| \\ & \Omega_g = K(\alpha,\beta,\gamma) \\ G_f/G_f \cap V \simeq G_g & \Big| \\ & K \end{array}$$

We now have 2 ways to decide between D_4 and C_4 as Galois group: the "classical" procedure and the Theorem proved by Kappe and Warren ([6]).

From the fundamental Theorem of Galois theory, we remind the following equalities:

- $|G_f \cap V| = (G_f \cap V : 1) = [\Omega_f : \Omega_g]$;
- $|G_g| = (G_f : G_f \cap V) = [\Omega_g : K]$.

2.3 "Classical" method

We can compute $|G_g|$ from the resolvent cubic g, because $G_g = \mathrm{Gal}(\Omega_g/K)$ and Ω_g is the splitting field of g. Once we know $|G_g|$ we can deduce G_f except in the case that is 2.

If $[\Omega_g : K] = 2$, then $G_f \cap V = V$ or C_2. We know that a separable polynomial $f(x) \in K[x]$ is irreducible if and only if G_f permutes the roots of f transitively. Only V acts transitively on the roots of f, and so $G = D_4$ or C_4 according as f is irreducible or not in $\Omega_g[x]$.

We can rewrite this "classical" procedure in the following Theorem:

Theorem 2.3.1 ([6])**.** *Let $f(x) \in K[x]$ be an irreducible quartic, where K does not have characteristic 2, and set $\Delta = \mathrm{disc}(f)$. Suppose Δ is not a square in K and $g(x)$ is reducible in $K[x]$, so G_f is D_4 or C_4.*

- *If $f(x)$ is irreducible over $K(\sqrt{\Delta})$ then $G_f = D_4$.*
- *If $f(x)$ is reducible over $K(\sqrt{\Delta})$ then $G_f = C_4$.*

Proof. We will make reference to the field diagrams in the proof of Theorem 2.4.4. When $G_f = D_4$, the field diagram in this case shows the splitting field of $f(x)$ over K is $K(r_1, \sqrt{\Delta})$. Since $[K(r_1, \sqrt{\Delta}) : K] = 8$, $[K(r_1, \sqrt{\Delta}) : K(\sqrt{\Delta})] = 4$, so $f(x)$ must be irreducible over $K(\sqrt{\Delta})$. When $G_f = C_4$, the splitting field of $f(x)$ over $K(\sqrt{\Delta})$ has degree 2, so $f(x)$ is reducible over $K(\sqrt{\Delta})$.

Because the different Galois groups imply different behaviour of $f(x)$ over $K(\sqrt{\Delta})$, these properties of $f(x)$ over $K(\sqrt{\Delta})$ tell us the Galois group. $\square$

The two versions of this classical method are equivalent. In fact we can prove that $\Omega_g = K(\alpha, \beta, \gamma)$ is equal to $K(\sqrt{\Delta})$, which implies that the behaviour of $f(x)$ over $\Omega_g[x]$ is the same as the behaviour of $f(x)$ over $K(\sqrt{\Delta})[x]$.

From Lemma 2.2.2 we know that $\mathrm{disc}(f) = \mathrm{disc}(g)$ and according to this information we can write: $\sqrt{\Delta} = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$ and in particular $\sqrt{\Delta} \in K(\alpha, \beta, \gamma)$. So $K(\sqrt{\Delta}) \subseteq K(\alpha, \beta, \gamma)$. If $G_f = D_4$ or $G_f = C_4$, we have in both cases $G_f \not\subseteq A_4$, so $\Delta \neq \square$ in K and $[K(\sqrt{\Delta}) : K] = 2$. We also know from the previous diagram that $|G_g| = [K(\alpha, \beta, \gamma) : K] = 2$, which means $\Omega_g = K(\alpha, \beta, \gamma) = K(\sqrt{\Delta})$.

2.4 "New" method

With the notation above:

Theorem 2.4.1 ([6])**.** *Let $f(x) \in K[x]$ be a quartic, G_f can be described in terms of whether or not $\mathrm{disc}(f)$ is a square in K and whether or not $g(x)$ factors in $K[x]$, according to the following table:*

$\mathrm{disc}(f)$ *in* K	$g(x)$ *in* $K[x]$	G_f
$\neq \square$	*irreducible*	S_4
$= \square$	*irreducible*	A_4
$\neq \square$	*reducible*	D_4 *or* C_4
$= \square$	*reducible*	V

Table 2.2

Proof. We check each row of the table in order.

- disc(f) is not a square and $g(x)$ is irreducible over K: Since $\mathrm{disc}(f) \neq \square$, $G_f \not\subseteq A_4$. Since $g(x)$ is irreducible over K and its roots are in the splitting field of $f(x)$ over K, adjoining a root of $g(x)$ to K gives us a cubic extension of K inside the splitting field of $f(x)$, so $|G_f|$ is divisible by 3. It's also divisible by 4, so $G_f = S_4$ or A_4, which implies $G_4 = S_4$.
- disc(f) is a square and $g(x)$ is irreducible over K: We have $G_f \subseteq A_4$ and $|G_f|$ divisible by 3 and 4, so $G_f = A_4$.

- disc(f) is not a square and $g(x)$ is reducible over K: Since disc$(f) \neq \square$, $G_f \not\subseteq A_4$, so G_f is S_4, D_4 or C_4. We will show $G_f \neq S_4$.

  What distinguishes S_4 from the other two choices for G_f is that S_4 contains 3-cycles. If $G_f = S_4$ then $(123) \in G_f$. Applying the hypothetical automorphism in the Galois group to the roots of $g(x)$ carries them through the single orbit:

$$r_1r_2 + r_3r_4 \mapsto r_2r_3 + r_1r_4 \mapsto r_3r_1 + r_2r_4 \mapsto r_1r_2 + r_3r_4.$$

  These numbers are distinct since $g(x)$ is separable. At least one root of $g(x)$ lies in K, so the G_f-orbit of that root is just itself, not three numbers. We have a contradiction.

- disc(f) is a square and $g(x)$ is reducible over K: The group G_f lies in A_4, so $G_f = V$ or $G_f = A_4$. We want to eliminate the second choice. As in the previous case, we can distinguish V from A_4 using 3-cycles: there are 3-cycles in A_4 but not in V. If there were a 3-cycle on the roots of $f(x)$ in G_f then applying it to a root of $g(x)$ shows all the roots of $g(x)$ are in single G_f-orbit, which is a contradiction since $g(x)$ is (separable and) reducible over K. Thus G_f contains no 3-cycles.

□

To make it more clear, Table 2.3 gives some examples of Galois group computations over $\mathbb{Q}$ using Theorem 2.4.1:

$f(x)$	disc(f)	$g(x)$	G_f
$x^4 - x - 1$	-283	$x^3 + 4x - 1$	S_4
$x^4 + 2x + 2$	$101 \cdot 4^2$	$x^3 - 8x - 4$	S_4
$x^4 + 8x + 12$	576^2	$x^3 - 48x - 64$	A_4
$x^4 + 3x + 3$	$21 \cdot 15^2$	$(x + 3)(x^2 - 3x - 3)$	D_4 or C_4
$x^4 + 5x + 5$	$5 \cdot 55^2$	$(x - 5)(x^2 + 5x + 5)$	D_4 or C_4
$x^4 + 36x + 63$	4320^2	$(x - 18)(x + 6)(x + 12)$	V

Table 2.3: Some examples

By Theorem 2.4.1, $g(x)$ is reducible over K only when G_f is D_4, C_4 or V. Looking at the examples in Table 2.3 of such Galois groups, we can make the following observation: $g(x)$ has one root in $\mathbb{Q}$ when G_f is D_4 or C_4 and all three roots are in $\mathbb{Q}$ when G_f is V. It is no coincidence:

Corollary 2.4.2 ([6]). *With the notation above, $G_f = V$ if and only if $g(x)$ splits completely over K and $G_f = D_4$ or C_4 if and only if $g(x)$ has a unique root in K.*

Proof. The condition for G_f to be V is: disc$(f) = \square$ and $g(x)$ is reducible over K. Since disc$(g) =$ disc(f), $G_f = V$ if and only if disc(g) is a square in K and $g(x)$ is reducible over K. By Theorem 1.2.4, a splitting field of $g(x)$ over K is $K(r, \sqrt{\text{disc}(g)})$, where r is any root of $g(x)$. Therefore $G_f = V$ if and only if $g(x)$ splits completely over K.

The condition for G_f to be D_4 or C_4 is: disc$(f) \neq \square$ and $g(x)$ is reducible over K. These conditions, by Theorem 1.2.4 for the cubic $g(x)$, are equivalent to $g(x)$ having a root in K but not splitting completely over K, which is the same as saying $g(x)$ has a unique root in K. □

As we said, Theorem 2.4.1 does not decide between Galois groups D_4 and C_4. The following theorem provides a partial way to do this over $\mathbb{Q}$, by checking the sign of the discriminant.

Theorem 2.4.3 ([6]). *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quartic. If $G_f = C_4$ then* disc$(f) > 0$. *Therefore if G_f is D_4 or C_4 and* disc$(f) < 0$, *$G_f = D_4$.*

Proof. If $G_f = C_4$, the splitting field of $f(x)$ over $\mathbb{Q}$ has degree 4. Any root of $f(x)$ already generates an extension of $\mathbb{Q}$ with degree 4, so the field generated over K by one root of $f(x)$ contains all the other roots. Therefore if $f(x)$ has one real root it has 4 rela roots: the number of real roots of $f(x)$ is either 0 or 4.

If $f(x)$ has 0 real roots then they fall into complex conjugate pairs, say z and $\overline{z}$ and w and $\overline{w}$. Then disc(f) is the square of

$$(z - \overline{z})(z - w)(z - \overline{w})(\overline{z} - w)(\overline{z} - \overline{w})(w - \overline{w}) = |z - w|^2|z - \overline{w}|^2(z - \overline{z})(w - \overline{w}) \qquad (2.1)$$

The differences $z - \overline{z}$ and $w - \overline{w}$ are purely imaginary (and non-zero, since z and w are not real), so their product is real and non-zero. Thus when we square (2.4), we find $\text{disc}(f) > 0$.

If $f(x)$ has 4 real roots then the product of the differences of its roots is real and non-zero, so $\text{disc}(f) > 0$. □

Let's give an example: the polynomial $x^4 + 4x^2 - 2$, which is irreducible by the Eisenstein criterion, has discriminant -18432 and cubic resolvent $x^3 - 4x^2 + 8x - 32 = (x-4)(x^2-8)$. Theorem 2.4.1 says its Galois group is D_4 or C_4. Since the discriminant is negative, Theorem 2.4.3 says the Galois group must be D_4.

Theorem 2.4.3 provides only a partial way to decide between D_4 and C_4. It does not distinguish the two possibilities when $\text{disc}(f) > 0$, since some polynomials with Galois group D_4 have positive discriminant. For example, we can't decide yet in Table 2.3 if $x^4 + 5x + 5$ has Galois group D_4 or C_4 over $\mathbb{Q}$.

We can finally prove the following

Theorem 2.4.4 (Kappe, Warren, 1989, [6]). *Let K be a field not of characteristic 2, $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$, and $\Delta = \text{disc}(f)$. Suppose $\Delta \neq \square$ in K and $g(x)$ is reducible in $K[x]$ with unique root $r' \in K$. Then $G_f = C_4$ if the polynomials $x^2 + ax + (b - r')$ and $x^2 - r'x + d$ split over $K(\sqrt{\Delta})$, while $G_f = D_4$ otherwise.*

Proof. Index the roots r_1, r_2, r_3, r_4 of $f(x)$ so that $r' = \alpha = r_1r_2 + r_3r_4$. Both D_4 and C_4, as subgroups of S_4, contain a 4-cycle. (The elements of order 4 in S_4 are 4-cycles). In Table 2.4 we describe the effect of each 4-cycle in S_4 on r' if the 4-cycle were in the Galois group. The (distinct) roots of $g(x)$ are in the second row, each appearing twice.

σ	(1234)	(1432)	(1243)	(1342)	(1324)	(1423)
$\sigma(r_1r_2 + r_3r_4)$	$r_2r_3 + r_4r_1$	$r_4r_1 + r_2r_3$	$r_2r_4 + r_1r_3$	$r_3r_1 + r_4r_2$	$r_3r_4 + r_2r_1$	$r_4r_3 + r_1r_2$

Table 2.4: Effect of a 4-cycle on r'

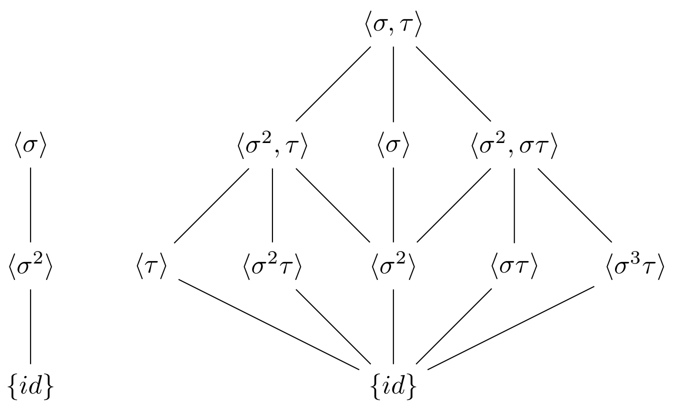
Since $r_1r_2 + r_3r_4$ is fixed by G_f, the only possible 4-cycles in G_f are (1324) and (1432). Both are in G_f since at least one is and they are inverses. Let $\sigma = (1324)$.

If $G_f = C_4$ then $G_f = \langle\sigma\rangle$. If $G_f = D_4$ then the observations in section 2.1 tell us $G_f = \langle(1324), (12)\rangle = \{(1), (1324), (12)(34), (1423), (12), (34), (13)(24), (14)(23)\}$ and the elements of G_f fixing r_1 are (1) and (34). Set $\tau = (34)$. Products of σ and τ as disjoints cycles are in Table 2.5.

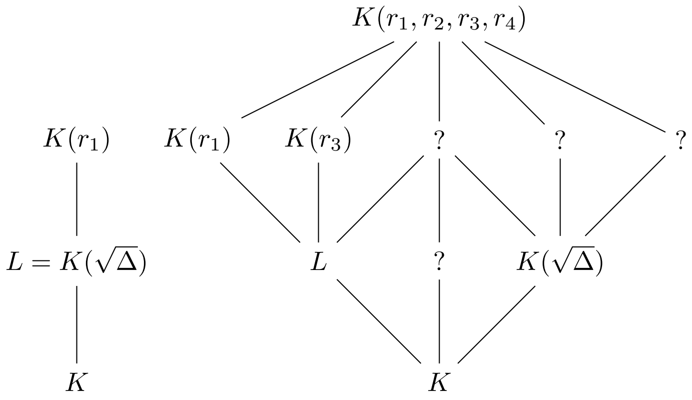
1	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
(1)	(1324)	(12)(34)	(1423)	(34)	(13)(24)	(12)	(14)(23)

Table 2.5: Products of σ and τ

Hence, if $G_f = D_4$ then $G_f = \langle(1324), (12)\rangle = \langle\sigma, \tau\rangle$. The subgroups of $\langle\sigma\rangle$ and $\langle\sigma, \tau\rangle$ look very different.



Corresponding to the above subgroup lattices we have the following subfield lattices of the splitting field, where L in both cases denotes the unique quadratic extension of K inside $K(r_1)$: if $G_f = C_4$ then L corresponds to $\langle\sigma^2\rangle$, while if $G_f = D_4$ then L corresponds to $\langle\sigma^2,\tau\rangle$. Since $\Delta \neq \square$ in K, $[K(\sqrt{\Delta}) : K] = 2$.



If $G_f = C_4$, then $L = K(\sqrt{\Delta})$ since there is only one quadratic extension of K in the splitting field.

If $G_f = D_4$, in the subgroup and subfield lattice diagrams above, we know $K(r_1)$ corresponds to $\langle\tau\rangle$, $K(r_3)$ corresponds to $\langle\sigma^2\tau\rangle$ and $K(\sqrt{\Delta})$ corresponds to $\langle\sigma^2,\sigma\tau\rangle$. Let's explain why: the degree $[K(r_1) : K] = 4$, so its corresponding subgroup in $D_4 = \langle\sigma,\tau\rangle$ has order 8/4=2 and $\tau = (34)$ fixes r_1 and has order 2. Similarly, $[K(r_3) : K] = 4$ and $\sigma^2\tau = (12)$ fixes r_3. The subgroup corresponding to $K(\sqrt{\Delta})$ is the even permutations in the Galois group, and that is $\{(1),(12)(34),(13)(24),(14)(23)\} = \langle\sigma^2,\sigma\tau\rangle$.

Although the two cases of G_f are different, we are going to develop some common ideas for both of them concerning the quadratic extensions $K(r_1)/L$ and L/K before we distinguish the two cases from each other. If $G_f = C_4$, $\mathrm{Gal}(K(r_1)/L) = \{1,\sigma^2\}$. If $G_f = D_4$, $\mathrm{Gal}(K(r_1)/L) = \langle\sigma^2,\tau\rangle/\langle\tau\rangle = \{1,\sigma^2\}$. So in both cases, the L-conjugate of r_1 is $\sigma^2(r_1) = r_2$ and the minimal polynomial of r_1 over L must be

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1r_2 \in L[x]$$

Therefore $r_1 + r_2$ and r_1r_2 are in L. Since $[K(r_1) : K] = 4$, this polynomial is not in $K[x]$:

$$r_1 + r_2 \notin K \text{ or } r_1r_2 \notin K. \tag{2.2}$$

If $G_f = C_4$ then $\mathrm{Gal}(L/K) = \langle\sigma\rangle/\langle\sigma^2\rangle = \{1,\overline{\sigma}\}$, and if $G_f = D_4$ then $\mathrm{Gal}(L/K) = \langle\sigma,\tau\rangle/\langle\sigma^2,\tau\rangle = \{1,\overline{\sigma}\}$. The coset of σ in $\mathrm{Gal}(L/K)$ represents the nontrivial coset both times, so $L^\sigma = K$. That is, an element of L fixed by σ is in K. Since $\sigma(r_1 + r_2) = r_3 + r_4$ and $\sigma(r_1r_2) = r_3r_4$, the polynomials

$$(x - (r_1 + r_2))(x - (r_3 + r_4)) = x^2 - (r_1 + r_2 + r_3 + r_4)x + (r_1 + r_2)(r_3 + r_4), \tag{2.3}$$

and

$$(x - r_1r_2)(x - r_3r_4) = x^2 - (r_1r_2 + r_3r_4)x + r_1r_2r_3r_4 \tag{2.4}$$

have coefficients in $L^\sigma = K$. The linear coefficient in (2.3) is a and the constant term is

$$(r_1 + r_2)(r_3 + r_4) = r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 = b - (r_1r_2 + r_3r_4) = b - r'$$

so (2.3) equals $x^2 + ax + (b - r')$. The quadratic polynomial (2.4) is $x^2 - r'x + d$. When $r_1 + r_2 \notin K$, (2.3) is irreducible in $K[x]$, so its discriminant is a non-square in K, and if $r_1 + r_2 \in K$ then (2.3) has a double root and its discriminant is 0. Similarly, (2.4) has a discriminant that is a non-square in K or is 0. Therefore the splitting field of (2.3) or (2.4) over K is either L or K and (2.2) tells us at least one of (2.3) and (2.4) has a non-square discriminant in K (so has splitting field L).

Since $r_1 + r_2$ and r_1r_2 are in L and $[L : K] = 2$, each one generates L over K if it is not in K. This happens for at least one of the two numbers, by (2.2).

First suppose $G_f = C_4$. Then $L = K(\sqrt{\Delta})$, so $x^2 + ax + (b - r')$ and $x^2 - r'x + d$ both split completely over $K(\sqrt{\Delta})$, since their roots are in L.

Next suppose $G_f = D_4$. Then $L \neq K(\sqrt{\Delta})$. By (2.2) at least one of (2.3) or (2.4) is irreducible over K, so its roots generate L over K and therefore are not in $K(\sqrt{\Delta})$. Thus the polynomial in (2.3) or (2.4) will be irreducible over $K(\sqrt{\Delta})$ if it's irreducible over K.

Since the conclusions about the two quadratic polynomials over $K(\sqrt{\Delta})$ are different depending on whether G_f is C_4 or D_4, these conclusions tell us the Galois group.

□

Corollary 2.4.5 ([6]). *When K does not have characteristic 2 and*

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

is an irreducible quaric in $K[x]$, define

$$\Delta = \mathrm{disc}(f) \text{ and } g(x) = x^3 - bx^2 + (ac - 4d)x - (a^2d + c^2 - 4bd).$$

The Galois group of $f(x)$ over K is described by Table 2.6.

Δ *in* K	$g(x)$ *in* $K[x]$	$(a^2 - 4(b - r'))\Delta$ *and* $(r'^2 - 4d)\Delta$	G_f
$\neq \Box$	*irreducible*		S_4
$= \Box$	*irreducible*		A_4
$\neq \Box$	*root* $r' \in K$	*at least one* $\neq \Box$ *in* K	D_4
$\neq \Box$	*root* $r' \in K$	*both* $= \Box$ *in* K	C_4
$= \Box$	*reducible*		V

Table 2.6: Galois groups distinction

Proof. The polynomials $x^2 + ax + (b - r')$ and $x^2 - r' + d$ split completely over $K(\sqrt{\Delta})$ if and only if their discriminants $a^2 - 4(b - r')$ and $r'^2 - 4d$ are squares in $K(\sqrt{\Delta})$. We saw in the proof of Theorem 2.4.4 that these discriminants are either 0 or nonsquares in K. A nonsquare in K is a square in $K(\sqrt{\Delta})$ if and only if its product with Δ is a square, and this is vacuously true for 0 also. □

In Table 2.7 we now give some examples of Galois group computations over $\mathbb{Q}$ using Corollary 2.4.5. In particular we list some quartic trinomials x^4+cx+d, all irreducible by Eisenstein criterion. If you pick a quartic in $\mathbb{Q}[x]$ at random it probably will be irreducible and have Galois group S_4, or perhaps A_4 if by chance the discriminant is a square, so we only list examples in Table 2.7 where the Galois group is smaller, which means the cubic resolvent is reducible. Since we choose a particular case where $a = b = 0$, $a^2 - 4(b - r')$ become $4r'$, so we need to decide when the rational numbers $4r'\Delta$ and $(r'^2 - 4d)\Delta$ are both squares in $\mathbb{Q}$:

$x^4 + cx + d$	Δ	$x^3 - 4dx - c^2$	$4r'\Delta$ and $(r'^2 - 4d)\Delta$	G_f
$x^4 + 3x + 3$	$21 \cdot 15^2$	$(x + 3)(x^2 - 3x - 3)$	$-56700,\ -14175$	D_4
$x^4 + 5x + 5$	$5 \cdot 55^2$	$(x - 5)(x^2 + 5x + 5)$	$550^2,\ 275^2$	C_4
$x^4 + 8x + 14$	$2 \cdot 544^2$	$(x - 8)(x^2 + 8x + 8)$	$4608^2,\ 2176^2$	C_4
$x^4 + 3x + 3$	$13 \cdot 1053^2$	$(x - 13)(x^2 + 13x + 13)$	$27378^2,\ 13689^2$	C_4

Table 2.7: Some examples of Galois group computations

Remark: A fundamental assumption before applying Corollary 2.4.5 is that the quartic must be irreducible. For example, $f(x) = x^4+4$ has discriminant $\Delta = \mathrm{disc}(f) = 128^2$ and cubic resolvent $g(x) = x^3 - 16x = x(x + 4)(x - 4)$. Such data (square discriminant, reducible resolvent) suggest the Galois group of $f(x)$ over $\mathbb{Q}$ is V, but $f(x)$ is reducible: it factors as $(x^2+2x+2)(x^2-2x+2)$. Both factors have discriminant 4, so the splitting field of $f(x)$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{-4}) = \mathbb{Q}(i)$ and the Galois group of $f(x)$ over $\mathbb{Q}$ is cyclic of order 2.

Chapter 3

Quintics

3.1 Transitive subgroups of S_5

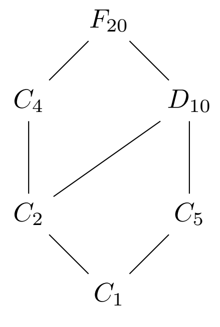
First of all, we want to identify the suitable subgroups of S_5 for being Galois group of an irreducible quintic $f(x) \in \mathbb{Q}[x]$. The candidates are the transitive subgroups of S_5, such that $5 = \deg(f) \mid |G_f|$, by Theorem 1.2.1. Up to isomorphism, these are:

G_f	S_5	A_5	F_{20}	D_5	C_5
$\lvert G_f \rvert$	120	60	20	10	5

Table 3.1: Transitive subgroups of S_5

We now want to study whether or not the equation $f(x) = 0$ is solvable by radicals. By Theorem 1.1.1, we have to choose the solvable groups from the ones in Table 3.1. Since S_5 and A_5 are not solvable, an irreducible quintic $f(x) \in \mathbb{Q}[x]$ is solvable by radicals if and only if the Galois group is contained in the Frobenius group F_{20}, i.e., if and only if the Galois group is isomorphic to F_{20}, to the dihedral group D_{10} of order 10, or the cyclic group C_5.

There are different notation in literature for some of this groups:



- $F_{20} = F_5 = C_5 \rtimes C_4 = \langle (12345), (2354) \rangle = \langle (12345), (1243) \rangle$;
- $D_{10} = D_5 = C_5 \rtimes C_2 = \langle (12345), (25)(34) \rangle$.

where the last equality gives just an example of possible generators of the group. More generally, for any prime p, a solvable subgroup of the symmetric group S_p whose order is divisible by p is contained in the normalizer of a Sylow p-subgroup of S_p. (In our case a Sylow p-subgroups is isomorphic to a copy of C_5 and its normalizer to a copy of F_{20}).

The purpose here is to give a criterion for the solvability of such a general quintic in terms of the existence of a rational root of an explicit associated resolvent sextic polynomial. When this is the case, we are going to give formulas for the roots analogous to Cardano's formulas for the general cubic and quartic polynomials and to determine the precise Galois group. In particular, the roots are produced in an order which is a cyclic permutation of the roots, which can be useful in other computations.

We work over the rationals $\mathbb{Q}$, but the results are valid over any field K of characteristic different from 2 and 5.

3.2 Criterion for the solvability

Let $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Q}[x]$ be a general quintic polynomial with roots r_1, r_2, r_3, r_4, r_5. Then

$$f(x) = x^5 - s_1x^4 + s_2x^3 - s_3x^2 + s_4x - s_5,$$

where the s_i are the elementary symmetric function in the roots r_i. This can be easily shown expanding $f(x) = \prod_{i=1}^{5}(x - r_i)$ and remembering the definition of the elementary symmetric function $s_i = \sum_{1\leq j_1<\cdots<j_i\leq 5} r_{j_1} \ldots r_{j_i}$. In our case: $s_1 = r_1 + r_2 + r_3 + r_4 + r_5$, $s_2 = r_1r_2 + r_1r_3 + r_1r_4 + r_1r_5 + r_2r_3 + r_2r_4 + r_2r_5 + r_3r_4 + r_3r_5 + r_4r_5$, ..., $s_5 = r_1r_2r_3r_4r_5$.

Let $F_{20} < S_5$ be the Frobenius group of order 20 with generators (12345) and (2354). Then the stabilizer of the element

$$\begin{aligned}\theta = \theta_1 =& r_1^2r_2r_5 + r_1^2r_3r_4 + r_2^2r_1r_3 + r_2^2r_4r_5 + r_3^2r_1r_5 \\ &+ r_3^2r_2r_4 + r_4^2r_1r_2 + r_4^2r_3r_5 + r_5^2r_1r_4 + r_5^2r_2r_3\end{aligned}$$

is precisely F_{20}. It follows that θ_1 satisfies a polynomial equation of degree 6 over $\mathbb{Q}(s_1, s_2, s_3, s_4, s_5)$ with conjugates $\theta^{S_5} = \{\theta, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$, where:

$$\begin{aligned}\theta_2 =&(123)\theta_1 \\ =& r_1^2r_2r_5 + r_1^2r_3r_4 + r_2^2r_1r_4 + r_2^2r_3r_5 + r_3^2r_1r_2 \\ &+ r_3^2r_4r_5 + r_4^2r_1r_5 + r_4^2r_2r_3 + r_5^2r_1r_3 + r_5^2r_2r_4; \\ \theta_3 =&(132)\theta_1 \\ =& r_1^2r_2r_3 + r_1^2r_4r_5 + r_2^2r_1r_4 + r_2^2r_3r_5 + r_3^2r_1r_5 \\ &+ r_3^2r_2r_4 + r_4^2r_1r_3 + r_4^2r_2r_5 + r_5^2r_1r_2 + r_5^2r_3r_4; \\ \theta_4 =&(12)\theta_1 \\ =& r_1^2r_2r_3 + r_1^2r_4r_5 + r_2^2r_1r_5 + r_2^2r_3r_4 + r_3^2r_1r_4 \\ &+ r_3^2r_2r_5 + r_4^2r_1r_2 + r_4^2r_3r_5 + r_5^2r_1r_3 + r_5^2r_2r_4; \\ \theta_5 =&(23)\theta_1 \\ =& r_1^2r_2r_4 + r_1^2r_3r_5 + r_2^2r_1r_5 + r_2^2r_3r_4 + r_3^2r_1r_2 \\ &+ r_3^2r_4r_5 + r_4^2r_1r_3 + r_4^2r_2r_5 + r_5^2r_1r_4 + r_5^2r_2r_3; \\ \theta_6 =&(13)\theta_1 \\ =& r_1^2r_2r_4 + r_1^2r_3r_5 + r_2^2r_1r_3 + r_2^2r_4r_5 + r_3^2r_1r_4 \\ &+ r_3^2r_2r_5 + r_4^2r_1r_5 + r_4^2r_2r_3 + r_5^2r_1r_2 + r_5^2r_3r_4.\end{aligned}$$

We are now ready to define the resolvent sextic (we will call it f_{20}) as the sextic polynomial with θ_i as a root. By computing the elementary symmetric functions of the θ_i, which are symmetric polynomials in r_1, r_2, r_3, r_4, r_5, it is a relatively straightforward matter to express these elements in terms of s_1, s_2, s_3, s_4, s_5 to determine the resolvent sextic f_{20}. By making a translation, we may assume $s_1 = 0$, i.e., that our quintic is

$$f(x) = x^5 + px^3 + qx^2 + rx + s,$$

in which case f_{20} is

$$\begin{aligned}
f_{20}(x) = x^6 &+ 8rx^5 + (2pq^2 - 6p^2r + 40r^2 - 50qs)x^4 \\
&+ (-2q^4 + 21pq^2r - 40p^2r^2 + 160r^3 - 15p^2qs - 400qrs + 125ps^2)x^3 \\
&+ (p^2q^4 - 6p^3q^2r - 8q^4r + 9p^4r^2 + 76pq^2r^2 - 136p^2r^3 + 400r^4 \\
&\qquad - 50pq^3s + 90p^2qrs - 1400qr^2s + 625q^2s^2 + 500prs^2)x^2 \\
&+ (-2pq^6 + 19p^2q^4r - 51p^3q^2r^2 + 3q^4r^2 + 32p^4r^3 + 76pq^2r^3 \\
&\quad - 256p^2r^4 + 512r^5 - 31p^3q^3s - 58q^5s + 117p^4qrs + 105pq^3rs \\
&\quad + 260p^2qr^2s - 2400qr^3s - 108p^5s^2 - 325p^2q^2s^2 + 525p^3es^2 \\
&\qquad + 2750q^2rs^2 - 500pr^2s^2 + 625pqs^3 - 3125s^4)x \\
&+ (q^8 - 13pq^6r + p^5q^2r^2 + 65p^2q^4r^2 - 4p^6r^3 - 128p^3q^2r^3 + 17q^4r^3 \\
&\quad + 48p^4r^4 - 16pq^2r^4 - 192p^2r^5 + 256r^6 - 4p^5q^3s - 12p^2q^5s \\
&\quad + 18p^6qrs + 12p^3q^3rs - 124q^5rs + 196p^4qr^2s + 590pq^3r^2s \\
&\quad - 160p^2qr^3s - 1600qr^4s - 27p^7s^2 - 150p^4q^2s^2 - 125pq^4s^2 \\
&\quad - 99p^5rs^2 - 725p^2q^2rs^2 + 1200p^3r^2s^2 + 3250q^2r^2s^2 \\
&\qquad - 2000pr^3s^2 - 1250pqrs^3 + 3125p^2s^4 - 9375rs^4).
\end{aligned} \tag{3.1}$$

For the particular case when $f(x) = x^5 + ax + b$, this polynomial is simply

$$\begin{aligned}
f_{20}(x) = x^6 &+ 8ax^5 + 40a^2x^4 + 160a^3x^3 + 400a^4x^2 \\
&+ (512a^5 - 3125b^4)x + (256a^6 - 9375ab^4).
\end{aligned}$$

We are now ready to give the criterion for the solvability of a general quintic polynomial.

Theorem 3.2.1 ([3]). *The irreducible quintic $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbb{Q}[x]$ is solvable by radicals if and only if the polynomial $f_{20}(x)$ in (3.1) has a rational root. If this is the case, the sextic $f_{20}(x)$ factors into the product of a linear polynomial and an irreducible quintic.*

Proof. The polynomial $f(x)$ is solvable if and only if the Galois group of $f(x)$, considered as a permutation group on the roots, is contained in the normalizer of some Sylow 5-subgroup in S_5. The normalizers of the six Sylow 5-subgroups in S_5 are precisely the conjugates of F_{20} above, hence are the stabilizers of the elements $\theta_1, ..., \theta_6$. It follows that $f(x)$ is solvable by radicals if and only if one of the θ_1 is rational. By renumbering the roots as $r_1, ..., r_5$, we may assume $\theta = \theta_1$ is rational, so that the Galois group of $f(x)$ is contained in the specific group F_{20} above, $F_{20} = \langle(12345), (1243)\rangle$. Since $f(x)$ is irreducible, the order of its Galois group is divisible by 5. It follows that the 5-cycle (12345) survives any specialization (this element generates the unique subgroup of order 5 in this F_{20}). Because this element is transitive on $\theta_2, ..., \theta_6$ (in fact cycling them as $\theta_2, \theta_6, \theta_3, \theta_4, \theta_5$), the remaining roots θ_i are roots of an irreducible quintic over $\mathbb{Q}(\theta) = \mathbb{Q}$. □

We now consider the question of solving for the roots of $f(x)$ when $f(x)$ is solvable, i.e., solving for the roots $r_1, \ldots, r_5$ in terms of radicals over the field $\mathbb{Q}(s_1, \ldots, s_5, \theta)$. We suppose the rational root of f_{20} is the root θ above. This determines an ordering of the roots r_i up to a permutation in F_{20}.

3.3 Lagrange resolvent

Let's introduce the Lagrange resolvent.

Definition 3 (Cyclic extension, [4]). *The extension K/F is said to be cyclic it is Galois with a cyclic Galois group.*

Lemma 3.3.1 ([4]). *Let F be a field with characteristic not dividing n which contains the n^{th} roots of unity. Then the extension $F(\sqrt[n]{a})$ for $a \in F$ is cyclic over F of order dividing n.*

Let now K be any cyclic extension of degree n over a field F of characteristic not dividing n which contains the n^{th} roots of unity. Let σ be a generator for the cyclic group $\text{Gal}(K/F)$.

Definition 4 (Lagrange resolvent, [4]). *For $\alpha \in K$ and any n^{th} root of unity ζ, define the Lagrange resolvent $(\alpha, \zeta) \in K$ by*

$$(\alpha, \zeta) = \alpha + \zeta\sigma(\alpha) + \zeta^2\sigma^2(\alpha) + \cdots + \zeta^{n-1}\sigma^{n-1}(\alpha).$$

If we apply the automorphism σ to (α, ζ) we obtain

$$\sigma(\alpha, \zeta) = \sigma\alpha + \zeta\sigma^2(\alpha) + \zeta^2\sigma^3(\alpha) + \cdots + \zeta^{n-1}\sigma^n(\alpha)$$

since ζ is an element of the base field F so is fixed by σ. We have $\zeta^n = 1$ in $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ (as group of the n^{th} roots of unity over $\mathbb{Q}$, under multiplication on the right, addition on the left) and $\sigma^n = 1$ in $\text{Gal}(K/F)$ so this can be written

$$\begin{aligned}
\sigma(\alpha, \zeta) &= \sigma\alpha + \zeta\sigma^2(\alpha) + \zeta^2\sigma^3(\alpha) + \cdots + \zeta^{n-1}(\alpha) \\
&= \zeta^{-1}(\alpha + \zeta\sigma\alpha + \zeta^2\sigma^2(\alpha) + \cdots + \zeta^{n-1}\sigma^{n-1}(\alpha)) \\
&= \zeta^{-1}(\alpha, \zeta). \qquad (3.2)
\end{aligned}$$

It follows that

$$\sigma(\alpha, \zeta)^n = (\zeta^{-1})^n(\alpha, \zeta)^n = (\alpha, \zeta)^n$$

so that $(\alpha, \zeta)^n$ is fixed by $\text{Gal}(K/F)$, hence is an element of F for any $\alpha \in K$.

Let ζ be a n^{th} root of unity. By the linear independence of the automorphisms $1, \sigma, \ldots, \sigma^{n-1}$, there is an element $\alpha \in K$ with $(\alpha, \zeta) \neq 0$. Iterating (3.2) we have

$$\sigma^i(\alpha, \zeta) = \zeta^{-i}(\alpha, \zeta), \quad i = 0, 1, \ldots,$$

and it follows that σ^i does not fix (α, ζ) for any $i < n$. Hence this element cannot lie in any proper subfield of K, so $K = F((\alpha, \zeta))$. Since we proved $(\alpha, \zeta)^n = a \in F$ above, we have $F(\sqrt[n]{a}) = F((\alpha, \zeta)) = K$. This proves the following converse of Lemma (3.3.2)

Lemma 3.3.2 ([4]). *Any cyclic extension of degree n over a field F of characteristic not dividing n which contains the n^{th} roots of unity is of the form $F(\sqrt[n]{a})$ for some $a \in F$.*

In our case, let ζ be a fixed primitive 5th root of unity and define the function fields $k = \mathbb{Q}(s_1, \ldots, s_5)$, $K = k(\theta)$ and $F = \mathbb{Q}(r_1, \ldots, r_5)$, so that $F(\zeta)/K$ is a Galois extension with $F_{20} \times (\mathbb{Z}/n\mathbb{Z})^\times$ as Galois group. Define the automorphism σ, τ and ω of F to be $\sigma = (12345)$ (trivial on constants), $\tau = (2354)$ (trivial on constants) and $\omega : \zeta \mapsto \zeta^3$ (trivial on $r_1, \ldots, r_5$).

Let $\Delta = \text{disc}(f)$ be the discriminant of the quintic $f(x)$ and $\sqrt{\Delta} = \prod_{i<j}(r_i - r_j)$ the fixed square root of Δ. Note that for a solvable quintic, the discriminant Δ is always positive: if the Galois group is dihedral or cyclic, then the Galois group is contained in A_5, so that Δ is actually a square; if the Galois group is the Frobenius group, then $\sqrt{\Delta}$ generates a quadratic extension which is a subfield of a cyclic quartic extension, so again $\Delta > 0$ (in fact, Δ is then the sum of two squares).

Define the usual Lagrange resolvents of the root r_1:

$$\begin{aligned}
(r_1, 1) &= r_1 + 1 \cdot \sigma(r_1) + 1^2 \cdot \sigma^2(r_1) + 1^3 \cdot \sigma^3(r_1) + 1^4 \cdot \sigma^4(r_1) \\
&= r_1 + r_2 + r_3 + r_4 + r_5 = s_1 = 0, \\
z_1 = (r_1, \zeta) &= r_1 + r_2\zeta + r_3\zeta^2 + r_4\zeta^3 + r_5\zeta^4, \\
z_2 = (r_1, \zeta^2) &= r_1 + r_2\zeta^2 + r_3\zeta^4 + r_4\zeta^+ r_5\zeta^3, \\
z_3 = (r_1, \zeta^3) &= r_1 + r_2\zeta^3 + r_3\zeta + r_4\zeta^4 + r_5\zeta^2, \\
z_4 = (r_1, \zeta^4) &= r_1 + r_2\zeta^4 + r_3\zeta^3 + r_4\zeta^2 + r_5\zeta,
\end{aligned}$$

so that

$$\begin{aligned}
r_1 &= (z_1 + z_2 + z_3 + z_4)/5, \\
r_2 &= (\zeta^4 z_1 + \zeta^3 z_2 + \zeta^2 z_3 + \zeta z_4)/5, \\
r_3 &= (\zeta^3 z_1 + \zeta z_2 + \zeta^4 z_3 + \zeta^2 z_4)/5, \qquad (3.3) \\
r_4 &= (\zeta^2 z_1 + \zeta^4 z_2 + \zeta z_3 + \zeta^3 z_4)/5, \\
r_5 &= (\zeta z_1 + \zeta^2 z_2 + \zeta^3 z_3 + \zeta^4 z_4)/5.
\end{aligned}$$

Write

$$(r_1, t) = r_1 + r_2t + r_3t^2 + r_4t^3 + r_5t^4$$

with an indeterminate t (so $t = \zeta$ gives the Lagrange resolvent z_1). Expanding $(r_1, t)^5$ gives

$$Z_1 = z_1^5 = (r_1, \zeta)^5 = l_0 + l_1\zeta + l_2\zeta^2 + l_3\zeta^3 + l_4\zeta^4 \tag{3.4}$$

where l_0 by definition is the sum of the terms in $(r_1, t)^5$ involving powers t^i of t with i divisible by 5, l_1 is the sum of the terms with $i \equiv 1 \bmod 5$, and so forth. Explicitly,

$$\begin{aligned}
l_0 = {} & 30r_2r_4r_5^2 + 20r_1r_4r_5^3 + 20r_1^3r_2r_5 + 20r_2r_3r_5^3 + r_2^5 + r_5^5 \\
& + r_1^5 + r_3^5 + r_4^5 + 20r_1^3r_3r_4 + 30r_1^2r_2^2r_4 + 30r_1^2r_2r_3^2 + 20r_1r_2^3r_3 \\
& + 30r_1^2r_3r_5^2 + 30r_1^2r_4^2r_5 + 30r_2^2r_3^2r_5 + 30r_2^2r_3r_4^2 \\
& + 20r_2^3r_4r_5 + 20r_2r_3^3r_4 + 20r_1r_2r_4^3 + 30r_1r_2^2r_5^2 + 30r_1r_3^2r_4^2 \\
& + 20r_1r_3^3r_5 + 120r_1r_2r_3r_4r_5 + 30r_3^2r_4r_5^2 + 20r_3r_4^3r_5, \\
l_1 = {} & 20r_1r_3r_4^3 + 30r_1^2r_4r_5^2 + 5r_1^4r_2 + 10r_1^3r_4^2 + 10r_1^2r_3^3 \\
& + 5r_2^4r_3 + 10r_2^2r_4^3 + 5r_3^4r_4 + 10r_2^3r_5^2 + 10r_3^2r_5^3 + 5r_4^4r_5 \\
& + 5r_1r_5^4 + 20r_1^3r_3r_5 + 30r_1^2r_2^2r_5 + 30r_1r_2^2r_3^2 + 20r_1r_2^3r_4 \\
& + 30r_2r_3^2r_4^2 + 20r_2r_3^3r_5 + 20r_2r_4r_5^3 + 30r_3r_4^2r_5^2 + 60r_1^2r_2r_3r_4 \\
& + 60r_2^2r_3r_4r_5 + 60r_1r_2r_4^2r_5 + 60r_1r_2r_3r_5^2 + 60r_1r_3^2r_4r_5, \\
l_2 = {} & 20r_1^3r_4r_5 + 10r_1^3r_2^2 + 5r_1^4r_3 + 10r_2^3r_3^2 + 5r_2^4r_4 + 10r_1^2r_5^3 \\
& + 10r_3^3r_4^2 + 5r_1r_4^4 + 5r_3^4r_5 + 5r_2r_5^4 + 10r_4^3r_5^2 + 30r_1^2r_2r_4^2 \\
& + 30r_1^2r_3^2r_4 + 20r_1r_2r_3^3 + 20r_1r_2^3r_5 + 30r_2^2r_3r_5^2 \\
& + 20r_2r_3r_4^3 + 30r_2^2r_4^2r_5 + 30r_2r_3^2r_5^2 + 60r_1^2r_2r_3r_5 + 60r_1r_2^2r_3r_4 \\
& + 60r_1r_2r_4r_5^2 + 60r_2r_3^2r_4r_5 + 60r_1r_3r_4^2r_5 + 20r_3r_4r_5^3, \\
l_3 = {} & 20r_2^3r_3r_4 + 20r_3^3r_4r_5 + 5r_1^4r_4 + 10r_1^2r_2^3 + 10r_1^3r_5^2 + 10r_2^2r_3^3 \\
& + 5r_2^4r_5 + 5r_1r_3^4 + 5r_2r_4^4 + 10r_3^2r_4^3 + 5r_3r_5^4 + 10r_4^2r_5^3 \\
& + 20r_1^3r_2r_3 + 30r_1^2r_3r_4^2 + 30r_1^2r_3^2r_5 + 30r_1r_2^2r_4^2 + 30r_2r_3^2r_5^2 \\
& + 30r_2^2r_4r_5^2 + 20r_1r_2r_5^3 + 20r_1r_4^3r_5 + 60r_1^2r_2r_4r_5 \\
& + 60r_1r_2r_3^2r_4 + 60r_1r_2^2r_3r_5 + 60r_2r_3r_4^2r_5 + 60r_1r_3r_4r_5^2, \\
l_4 = {} & 30r_1^2r_2r_5^2 + 5r_1^4r_5 + 10r_1^3r_3^2 + 5r_1r_2^4 + 5r_2r_3^4 + 10r_1^2r_4^3 \\
& + 10r_2^3r_4^2 + 10r_2^2r_5^3 + 5r_3r_4^4 + 10r_3^3r_5^2 + 5r_4r_5^4 + 20r_1^3r_2r_4 \\
& + 30r_1^2r_2^2r_3 + 30r_2^2r_3^2r_4 + 20r_2^3r_3r_5 + 20r_1r_3^3r_4 + 20r_2r_4^3r_5 \\
& + 30r_3^2r_4^2r_5 + 20r_1r_3r_5^3 + 30r_1r_4^2r_5^2 + 60r_1^2r_3r_4r_5 \\
& + 60r_1r_2^2r_4r_5 + 60r_1r_2r_3r_4^2 + 60r_1r_2r_3^2r_5 + 60r_2r_3r_4r_5^2.
\end{aligned} \tag{3.5}$$

(Note also that setting $t = 1$ shows that

$$l_0 + l_1 + l_2 + l_3 + l_4 = (r_1 + r_2 + r_3 + r_4 + r_5)^5.$$

In particular, if $s_1 = 0$, we have $l_0 = -l_1 - l_2 - l_3 - l_4$.)

Similarly we have

$$\begin{aligned}
Z_2 &= z_2^5 = l_0 + l_3\zeta + l_1\zeta^2 + l_4\zeta^3 + l_2\zeta^4, \\
Z_3 &= z_3^5 = l_0 + l_2\zeta + l_4\zeta^2 + l_1\zeta^3 + l_3\zeta^4, \\
Z_4 &= z_4^5 = l_0 + l_4\zeta + l_3\zeta^2 + l_2\zeta^3 + l_1\zeta^4.
\end{aligned}$$

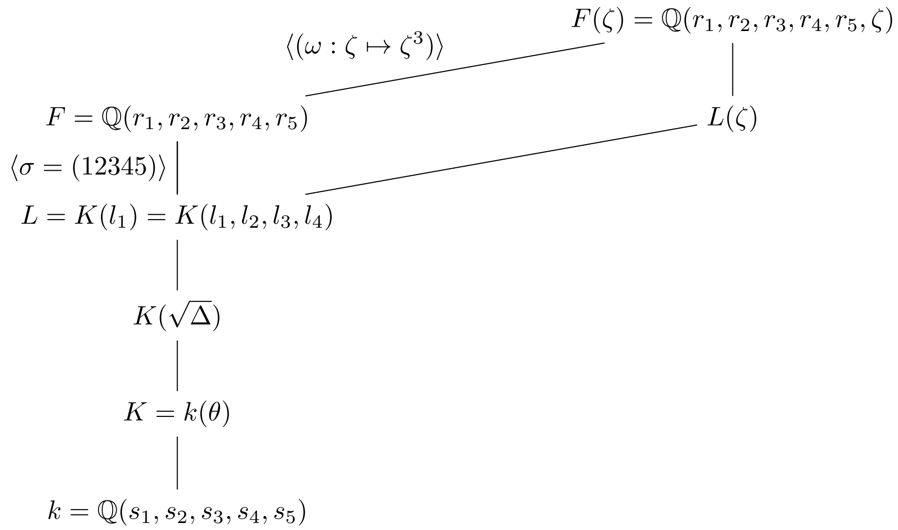
The Galois action over K on these elements is the following: The elements l_0, l_1, l_2, l_3, l_4 are contained in the field F and are fixed by σ;

$$\tau l_0 = l_0, \quad \tau l_1 = l_2, \quad \tau l_2 = l_4, \quad \tau l_3 = l_1, \quad \tau l_4 = l_3,$$

and the action on the Lagrange resolvents is given by

$$\begin{aligned} \sigma z_1 &= \zeta^4 z_1, \quad \tau z_1 = \omega z_1 = z_3, \\ \sigma z_2 &= \zeta^3 z_2, \quad \tau z_2 = \omega z_2 = z_1, \\ \sigma z_3 &= \zeta^2 z_3, \quad \tau z_3 = \omega z_3 = z_4, \\ \sigma z_4 &= \zeta^1 z_4, \quad \tau z_4 = \omega z_4 = z_2. \end{aligned} \tag{3.6}$$

It follows that $l_0 \in K$ and that l_1, l_2, l_3, l_4 are the roots of a quartic polynomial over K, and the field $L = K(l_1) = K(l_1, l_2, l_3, l_4)$ is a cyclic extension of K of degree 4 (with Galois group generated by the restriction of $\tau = (2354)$. The unique quadratic subfield of L over K is the field $K(\sqrt{\Delta})$. The field diagram is the following:



Since the Galois group of L/K is cyclic of degree 4, it follows that l_1, l_2, l_3, l_4 are the roots of a quartic over K which factors over $K(\sqrt{\Delta})$ into the product of two conjugate quadratics:

$$[x^2 + (T_1 + T_2\sqrt{\Delta})x + (T_3 + T_4\sqrt{\Delta})][x^2 + (T_1 + T_2\sqrt{\Delta})x + (T_3 - T_4\sqrt{\Delta})] \tag{3.7}$$

with $T_1, T_2, T_3, T_4 \in K$. The roots of one of these two quadratic factors are $\{l_1, l_4(= \tau^2 l_1)\}$, and the roots of the other are the conjugates $\{l_2(= \tau l_1), l_3(= \tau^3 l_1)\}$ for the specific l_i defined in equations (3.5). We may fix the order of the factors and determine the coefficients T_i explicitly by assuming that the roots of the first factor in (3.7) are $\{l_1, l_4\}$. Then

$$\begin{gathered} l_1 + l_4 = -T_1 - T_2\sqrt{\Delta}, \quad l_2 + l_3 = -T_1 + T_2\sqrt{\Delta}, \\ l_1 l_4 = T_3 + T_4\sqrt{\Delta}, \quad l_2 l_3 = T_3 - T_4\sqrt{\Delta}, \end{gathered}$$

which defines the T_i as explicit rational functions in $r_1, \ldots, r_5$. Writing these elements as linear combinations of $1, \theta, \theta^2, \ldots, \theta^5$ with symmetric functions as coefficients would be relatively more straightforward if $\mathbb{Z}[s_1, \ldots, s_5][\theta]$ were integrally closed in K, but unfortunately this is not the case. We proceed as follows. In a relation of the form

$$P = \alpha_0 + \alpha_1\theta + \alpha_2\theta^2 + \alpha_3\theta^3 + \alpha_4\theta^4 + \alpha_5\theta^5,$$

where the α_i are rational symmetric functions, if we apply the automorphisms (123) and (12) (which generate a complement to F_{20} in S_5 and so give the automorphisms of $K = k(\theta)$), we obtain the system of equations

$$\begin{aligned} P &= \alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_1^2 + \alpha_3\theta_1^3 + \alpha_4\theta_1^4 + \alpha_5\theta_1^5, \\ (123)P &= \alpha_0 + \alpha_1\theta_2 + \alpha_2\theta_2^2 + \alpha_3\theta_2^3 + \alpha_4\theta_2^4 + \alpha_5\theta_2^5, \\ (132)P &= \alpha_0 + \alpha_1\theta_3 + \alpha_2\theta_3^2 + \alpha_3\theta_3^3 + \alpha_4\theta_3^4 + \alpha_5\theta_3^5, \\ (12)P &= \alpha_0 + \alpha_1\theta_4 + \alpha_2\theta_4^2 + \alpha_3\theta_4^3 + \alpha_4\theta_4^4 + \alpha_5\theta_4^5, \\ (23)P &= \alpha_0 + \alpha_1\theta_5 + \alpha_2\theta_5^2 + \alpha_3\theta_5^3 + \alpha_4\theta_5^4 + \alpha_5\theta_5^5, \\ (13)P &= \alpha_0 + \alpha_1\theta_6 + \alpha_2\theta_6^2 + \alpha_3\theta_6^3 + \alpha_4\theta_6^4 + \alpha_5\theta_6^5, \end{aligned}$$

from which we may solve for the α_i using Cramer's rule. The denominator appearing in Cramer's rule is the Vandermonde determinant $-\prod_{i<j}(\theta_i - \theta_j)$, and it is not difficult to see that this is $(\sqrt{\Delta})^3 F$, where F is a symmetric polynomial. In particular, if P is a polynomial, this gives a bound for the denominator necessary for the rational symmetric functions α_i (since then the numerator in Cramer's rule is a polynomial).

3.4 Ordering the resolvents

Once we have defined all this variables, our goal is to find l_1, l_2, l_3, l_4 as roots of (3.7) and from the irreducibility of this polynomial to determine G_f. Then we want to determine the roots r_i of $f(x)$ using (3.3) and to do so we have to find a way to determine z_i. This is just a brief idea of what we have to do and what we will find in the final Theorem 3.4.2.

If we write

$$l_0 = (a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5)/F \tag{3.8}$$

and

$$T_1 = (b_{10} + b_{11}\theta + b_{12}\theta^2 + b_{13}\theta^3 + b_{14}\theta^4 + b_{15}\theta^5)/(2F), \tag{3.9}$$
$$T_2 = (b_{20} + b_{21}\theta + b_{22}\theta^2 + b_{23}\theta^3 + b_{24}\theta^4 + b_{25}\theta^5)/(2\Delta F), \tag{3.10}$$
$$T_3 = (b_{30} + b_{31}\theta + b_{32}\theta^2 + b_{33}\theta^3 + b_{34}\theta^4 + b_{35}\theta^5)/(2F), \tag{3.11}$$
$$T_4 = (b_{40} + b_{41}\theta + b_{42}\theta^2 + b_{43}\theta^3 + b_{44}\theta^4 + b_{45}\theta^5)/(2\Delta F), \tag{3.12}$$

the values can be found explicitly for the general polynomial $f(x) = x^5 + px^3 + qx^2 + rx + s$ in terms of p, q, r, s. We will give them only for the particular case when $f(x) = x^5 + ax + b$. These values are

$$T_1 = \frac{(215a^5 - 15625b^4 + 768a^4\theta + 416a^3\theta^2 + 112a^2\theta^3 + 24a\theta^4 + 4\theta^5)}{(50b^3)}, \tag{3.13}$$

$$T_2 = \frac{(3840a^5 - 78125b^4 + 4480a^4\theta + 2480a^3\theta^2 + 760a^2\theta^3 + 140a\theta^4 + 30\theta^5)}{(512a^5b + 6250b^5)}, \tag{3.14}$$

$$T_3 = \frac{(-18880a^5 + 781250b^4 - 34240a^4\theta - 21260a^3\theta^2 - 5980a^2\theta^3 - 1255a\theta^4 - 240\theta^5)}{(2b^2)}, \tag{3.15}$$

$$T_4 = \frac{(68800a^5 + 25000a^4\theta + 11500a^3\theta^2 + 3250a^2\theta^3 + 375a\theta^4 + 100\theta^5)}{(512a^5 + 6250b^4)}. \tag{3.16}$$

If we compute these expressions in terms of our given rational θ, and choose a specific δ as our square root of $\Delta = \mathrm{disc}(f)$, then the roots of the quadratics in (3.7) give us $\{l_1, l_4\}$ and $\{l_2, l_3\}$, up to a permutation of the two pairs. This is not sufficient to solve for the resolvents Z_1, Z_2, Z_3, Z_4, however, since for example if our choice of the roots in fact corresponds to $\{l_1, l_3, l_2, l_4\}$, then we do not simply obtain a permutation of the Z_i (this permutation is not obtained by an element of F_{20}). This difficulty is overcome by introducing an ordering condition. For this, observe that $(l_1 - l_4)(l_2 - l_3) = \eta\delta$ for some element $\eta \in K$. Computing this element as before, we write

$$\eta = (o_0 + o_1\theta + o_2\theta^2 + o_3\theta^3 + o_4\theta^4 + o_5\theta^5)/(\Delta F), \tag{3.17}$$

where again the values $o_1, \ldots, o_5$ can be found for general $f(x)$. As above, we will give them for the special case of $f(x) = x^5 + ax + b$. We have

$$\eta = \frac{(-1036800a^5 + 488828125b^4 - 2280000a^4\theta - 1291500a^3\theta^2 - 399500a^2\theta^3 - 76625a\theta^4 - 16100\theta^5)}{(256a^5 + 3125b^4)}$$

For any specific quintic $f(x)$, choose a square root δ' of the discriminant Δ, then define the roots of the first quadratic in (3.7) to be l_1' and l_4', and the roots of the second quadratic to be l_2' and l_3', ordered so that $(l_1' - l_4')(l_2' - l_3') = \eta\delta'$. If our choice of square root δ' is the same as that corresponding to δ determined by the ordering of the roots above, then our choice of l_1', l_2', l_3', l_4' is either l_1, l_2, l_3, l_4 or l_4, l_3, l_2, l_1. If our choice of square root δ' corresponds to $-\delta$, then our choice of l_1', l_2', l_3', l_4' is either l_2, l_4, l_1, l_3 or l_3, l_1, l_4, l_2. The corresponding resolvents computed in (3.4) are then simply permuted (namely, (Z_1, Z_2, Z_3, Z_4), (Z_4, Z_3, Z_2, Z_1), (Z_3, Z_1, Z_4, Z_2), (Z_2, Z_4, Z_1, Z_3), respectively), which will simply permute the order of the roots r_i in (3.3), as we shall see.

It remains to consider the choice of the fifth roots of the Z_i to obtain the resolvents z_i. We now show that, given $Z_1 = z_1^5$, each of the five possible choices for z_1 uniquely defines the choices for z_2, z_3, z_4, hence uniquely defines the five roots of the quintic.

Consider the expressions z_1z_4 and z_2z_3, which by the explicit Galois actions above are fixed by $\sigma, \tau\omega^{-1}$ and τ^2, hence are elements of the corresponding fixed field $K(\delta\sqrt{5})$.

As mentioned above, the discriminant Δ for any solvable quintic is a positive rational number. It follows that under any specialization, the elements z_1z_4 and z_2z_3 are elements of the field $\mathbb{Q}(\sqrt{5\Delta})$. Since the z_i are uniquely defined up to multiplication by a fifth root of unity, this uniquely determines z_4 given z_1, and z_3 given z_2. It remains to see how z_2 is determined by z_1. Consider now the elements $z_1z_2^2, z_3z_1^2, z_4z_3^2, z_2z_4^2$, which are invariant under σ and cyclically permuted by both τ and ω. It follows that these are the roots of a cyclic quartic over K, and that in particular

$$z_1z_2^2 + z_4z_3^2 = u + v\delta\sqrt{5}, \qquad z_3z_1^2 + z_2z_4^2 = u - v\delta\sqrt{5} \tag{3.18}$$

for some $u, c \in K$, where $\sqrt{5}$ is defined by the choice of ζ: $\zeta + \zeta^{-1} = (-1 + \sqrt{5})/2$.

Lemma 3.4.1 ([3]). *Given z_1, there is a unique choice of z_2, z_3, z_4 such that $z_1z_4, z_2z_3 \in K(\delta\sqrt{5})$ and such that the two equations in (3.18) are satisfied.*

Proof. We have already seen that z_1 uniquely determines z_4 and that z_2 uniquely determines z_3 by the conditions $z_1z_4, z_2z_3 \in K(\delta\sqrt{5})$. It remains to show that z_1 uniquely defines z_2 subject to the equations in (3.18).

If z_2 were replaced be εz_2 for some nontrivial fifth root of unity ε, then z_3 would be replaced by $\bar{\varepsilon}z_3$ (where $\varepsilon\bar{\varepsilon} = 1$), since their product must lie in $K(\delta\sqrt{5})$. If this new choice for z_2 and z_3 (together with the fixed z_1 and z_4) also satisfied the equations in (3.18), we would have

$$z_1z_2^2 + z_4z_3^2 = u + v\delta\sqrt{5}, \quad \text{and} \quad z_1(\varepsilon z_2)^2 + z_4(\bar{\varepsilon}z_3)^2 = u + v\delta\sqrt{5},$$

$$z_3z_1^2 + z_2z_4^2 = u - v\delta\sqrt{5}, \quad \text{and} \quad (\bar{\varepsilon}z_3)z_1^2 + (\varepsilon z_2)z_4^2 = u - v\delta\sqrt{5}.$$

Equating the expressions for $u + v\delta\sqrt{5}$ gives

$$\frac{z_1z_2^2}{z_4z_3^2} = -\frac{1-\bar{\varepsilon}^2}{1-\varepsilon^2} = \frac{1}{\varepsilon^2},$$

and equating the expressions for $u - v\delta\sqrt{5}$ gives

$$\frac{z_1^2z_3}{z_4^2z_2} = -\frac{1-\varepsilon}{1-\bar{\varepsilon}} = \varepsilon.$$

These two equations give $(z_1/z_4)^5 = 1$, which implies that z_1/z_4 is a fifth root of unity. This is a contradiction, since this element generates a quintic extension of $L(\zeta)$ which survives any specialization (the order of the Galois group of the irreducible $f(x)$ is divisible by 5), and completes the proof. □

The elements u and v are computed as before:

$$u = -25q/2,$$
$$v = (c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + c_5\theta^5)/(2\Delta F),$$

where the coefficient c_i for the general $f(x)$ can be found. We will give them for the special case of $f(x) = x^5 + ax + b$:

$$\begin{aligned} u =& 0, \\ v =& (-2048a^7 + 25000a^2b^4 - 3072a^6\theta - 6250ab^4\theta - 1664a^5\theta^2 - \\ & - 3125b^4\theta^2 - 448a^4\theta^3 - 96a^3\theta^4 - 16a^2\theta^5)/(32000a^5b^3 + 390625b^7), \end{aligned}$$

We are now ready for the final theorem:

Theorem 3.4.2 ([3]). *Suppose the irreducible polynomial $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbb{Q}[x]$ is solvable by radicals, and let θ be the unique rational root of the associated resolvent sextic f_{20} as in Theorem 3.2.1. Fix any square root δ of the discriminant Δ of $f(x)$ and fix any primitive fifth root of unity ζ. Define l_0 as in equation (3.8), and define l_1, l_4 and l_2, l_3 to be the roots of the quadratic factors in (3.7), subject to the condition $(l_1 - l_4)(l_2 - l_3) = \eta\delta$ in (3.17). Then the Galois group of $f(x)$ is:*

(a) the Frobenius group of order 20 if and only if the discriminant Δ of $f(x)$ is not a square, which occurs if and only if the quadratic factors in (3.7) are irreducible over $\mathbb{Q}(\sqrt{\Delta})$,

(b) the dihedral group of order 10 if and only if Δ is a square and the rational quadratics in (3.7) are irreducible over $\mathbb{Q}$,

(c) the cyclic group of order 5 if and only if Δ is a square and the rational quadratics in (3.7) are reducible over $\mathbb{Q}$.

Let z_1 be any fifth root of Z_1 in (3.4), and let z_2, z_3, z_4 be the corresponding fifth roots of Z_2, Z_3, Z_4 as in the lemma above. Then the formulas (3.3) give the roots of $f(x)$ in terms of the radicals and r_1, r_2, r_3, r_4, r_5 are permuted cyclically by some 5-cycle in the Galois group

Proof. The conditions in *(a)* to *(c)* are simply restatements of the structure of the field $L = K(l_1) = K(l_1, l_2, l_3, l_4)$ under specialization. We have already seen that the choice of δ and the roots l_i, of the quadratics determines the Z_i up to an ordering: (Z_1, Z_2, Z_3, Z_4) or (Z_4, Z_3, Z_2, Z_1) if the choice of δ is the same as that in the computations above, and (Z_3, Z_1, Z_4, Z_2) or (Z_2, Z_4, Z_1, Z_3) if the choice of δ is the negative of that used in the computations above. It is easy to check that the corresponding resolvents z_i are then simply $(z_1, z_2, z_3, z_4), (z_4, z_3, z_2, z_1), (z_3, z_1, z_4, z_2)$, and (z_2, z_4, z_1, z_3), respectively (this is the action of the automorphism $\tau = (2354)$ above). The formulas (3.3) then give the roots r_i in the orders $(r_1, r_2, r_3, r_4, r_5), (r_1, r_5, r_4, r_3, r_2), (r_1, r_3, r_5, r_2, r_4)$, and $(r_1, r_4, r_2, r_5, r_3)$, respectively. In terms of the 5-cycle $\sigma = (12345)$ above, these correspond to cyclic permutations by $\sigma, \sigma^{-1}, \sigma^2$ and σ^3, respectively. Finally, any choice of primitive fifth root of unity ζ produces precisely the same permutations of the roots r_i, so the roots of $f(x)$ are produced in a cyclic ordering independent of all choices. □

We now give some examples of Galois group and roots computations

1. Let $f(x) = x^5 + 15x + 12$, whose discriminant is $\Delta = 2^{10}3^4 5^5$. The corresponding resolvent sextic $f_{20}(x)$ is the polynomial

$$x^6 + 120x^5 + 9000x^4 + 540000x^3 + 20250000x^2 + 324000000x,$$

which clearly has $\theta = 0$ as a root. It follows that the Galois group of $f(x)$ is the Frobenius group F_{20} and that $f(x)$ is solvable by radicals. With $\delta = 7200\sqrt{5}$, where $\zeta + \zeta^{-1} = (-1+\sqrt{5})/2$, the roots l_1, l_2, l_3, l_4 of the quadratics in (3.7) (subject to the ordering condition in (3.17)) are

$$l_1 = -375 - 750\sqrt{5} + 75i\sqrt{625 + 29\sqrt{5}},$$
$$l_4 = -375 - 750\sqrt{5} - 75i\sqrt{625 + 29\sqrt{5}},$$
$$l_1 = -375 + 750\sqrt{5} - 75i\sqrt{625 - 29\sqrt{5}},$$
$$l_1 = -375 + 750\sqrt{5} + 75i\sqrt{625 - 29\sqrt{5}}.$$

Then

$$Z_1 = -1875 - 75\sqrt{1635 + 385\sqrt{5}} + 75\sqrt{1635 - 385\sqrt{5}},$$
$$Z_4 = -1875 + 75\sqrt{1635 + 385\sqrt{5}} - 75\sqrt{1635 - 385\sqrt{5}},$$
$$Z_2 = 5625 - 75\sqrt{1490 + 240\sqrt{5}} - 75\sqrt{1490 - 240\sqrt{5}},$$
$$Z_3 = 5625 + 75\sqrt{1490 + 240\sqrt{5}} + 75\sqrt{1490 - 240\sqrt{5}}.$$

Viewing these as real numbers, and letting z_1 be the real fifth root of Z_1, we conclude that the corresponding z_2, z_3 and z_4 are the real fifth roots of Z_2, Z_3 and Z_4, respectively, and then (3.3) gives the roots of $f(x)$. For example, the sum of the real fifth roots of Z_1, Z_2, Z_3, Z_4 above gives five times the (unique) real root of $f(x)$.

2. Let $f(x) = x^5 - 5x + 12$, whose discriminant is $\Delta = 2^{12}5^6$. The corresponding resolvent sextic $f_{20}(x)$ is the polynomial

$$x^6 - 40x^5 + 1000x^4 + 20000x^3 + 250000x^2 - 66400000x + 976000000,$$

which has $\theta = 40$ as a root, so that $f(x)$ has a solvable Galois group. Since in this case the quadratic factors in (3.7) are $x^2 + 1250x + 6015625$ and $x^2 - 3750x + 4921875$, which are irreducible over $\mathbb{Q}$, it follows that the Galois group of $f(x)$ is the dihedral group of order 10. If $\delta = 8000$, the roots l_1, l_2, l_3, l_4 of the quadratics in (3.7) (subject to the ordering condition in (3.17)) are

$$\begin{aligned} l_1 &= -625 + 750\sqrt{-10}, \\ l_4 &= -625 - 750\sqrt{-10}, \\ l_2 &= 1875 + 375\sqrt{-10}, \\ l_2 &= 1875 - 375\sqrt{-10}. \end{aligned}$$

Then

$$\begin{aligned} Z_1 &= -3125 - 1250\sqrt{5} - \frac{750}{2}\sqrt{100 + 20\sqrt{5}} - \frac{375}{2}\sqrt{100 - 20\sqrt{5}}, \\ Z_4 &= -3125 - 1250\sqrt{5} + \frac{750}{2}\sqrt{100 + 20\sqrt{5}} + \frac{375}{2}\sqrt{100 - 20\sqrt{5}}, \\ Z_2 &= -3125 + 1250\sqrt{5} + \frac{750}{2}\sqrt{100 + 20\sqrt{5}} - \frac{375}{2}\sqrt{100 - 20\sqrt{5}}, \\ Z_3 &= -3125 + 1250\sqrt{5} - \frac{750}{2}\sqrt{100 + 20\sqrt{5}} + \frac{375}{2}\sqrt{100 - 20\sqrt{5}}. \end{aligned}$$

Again viewing these as real numbers, and letting z_1 be the real fifth root of Z_1, we conclude that the corresponding z_2, z_3 and z_4 are the real fifth rots of Z_2, Z_3 and Z_4, respectively, and then (3.3) gives the roots of $f(x)$. For example, the sum of the real fifth roots of Z_1, Z_2, Z_3, Z_4 above again gives five times the (unique) real root in this example.

3. Let $f(x) = x^5 - 110x^3 - 55x^2 + 2310x + 979$, whose discriminant is $\Delta = 5^{20}11^4$. The corresponding resolvent sextic $f_{20}(x)$ is the polynomial

$$\begin{aligned} x^6 + 18480x^5 + 47764750x^4 - 580262760000x^3 - 1796651418959375x^2 \\ + 2980357148316659375x - 3602606856446967187 5, \end{aligned}$$

which has $\theta = -9955$ as a root, so that $f(x)$ ha a solvable Galois group. Since in this case the quadratic factors in (3.7) are $(x - 797500)(x + 61875)$ and $(x - 281875)(x + 405625)$, it follows that the Galois group of $f(x)$ is the cyclic group of order 5. If $\delta = 5^{10}11^2$, the roots l_1, l_2, l_3, l_4 of the quadratics in (3.7) (subject to the ordering condition in (3.17)) are

$$\begin{aligned} l_1 &= 797500, \\ l_2 &= -61875, \\ l_3 &= 281875, \\ l_4 &= -405625. \end{aligned}$$

Then

$$\begin{aligned} Z_1 &= 5^5 11(41\zeta + 26\zeta^2 + 6\zeta^3 + 16\zeta^4), \\ Z_2 &= 5^5 11(6\zeta + 41\zeta^2 + 16\zeta^3 + 26\zeta^4), \\ Z_3 &= 5^5 11(26\zeta + 16\zeta^2 + 41\zeta^3 + 6\zeta^4), \\ Z_4 &= 5^5 11(16\zeta + 6\zeta^2 + 26\zeta^3 + 41\zeta^4), \end{aligned}$$

Here,

$$u + v\delta = \frac{1375 + 6875\sqrt{5}}{2}, \quad u - v\delta = \frac{1375 - 6875\sqrt{5}}{2},$$

so with z_1 any fifth root of Z_1, z_4 is the fifth root of Z_4 such that $z_1 z_4$ is real, and z_2, z_3 are the fifth roots of Z_2, Z_3 whose product is real and which satisfy $z_3 z_1^2 + z_2 z_4^2 = (1375 - 6875\sqrt{5})/2$.

Chapter 4

Sextics

4.1 Introduction and notation

Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 6 and $\mathrm{Gal}(f) = G_f$ its Galois group. If we number the roots r_i of $f(x)$, then we can embed G_f as a transitive subgroup of S_6 through its action upon the r_i. Since changing the numbering of the roots conjugates the embedding of G_f in S_6, G_f is not a well-defined function of f.

Let $\overline{\mathrm{Gal}}(f) = \overline{G}_f$ be the S_6-conjugacy class of G_f and let Σ_6 be the S_6-conjugacy classes of transitive subgroups of S_6. Then for each irreducible polynomial f, $\overline{G}_f$ gives a well-defined element of Σ_6. Given $G \in \Sigma_6$, we let Γ_G be the set of all irreducible polynomials $f(x) \in \mathbb{Q}[x]$ with degree 6 such that $\overline{G}_f = G$.

Given $G \in \Sigma_n$, we say that the general equation of type (n, G) is *explicitly solvable* by radicals if (we will recall this definition in Theorem 4.4.5 of Section 4.4):

(i) There are formulas $z_1(t_i), z_2(t_i), \ldots, z_n(t_i)$ using only the basic arithmetic operations and radicals in variables $t_1, t_2, \ldots, t_m$;

(ii) A number field K, $[K : \mathbb{Q}] < \infty$, and bounded algorithm which associates to each $f \in \Gamma_G$, numbers $\hat{t}_1(f), \hat{t}_2(f), \ldots, \hat{t}_n(f) \in K$ such that $z_1(\hat{t}_i(f)), \ldots, z_n(\hat{t}_i(f))$ are the roots of f.

4.2 Transitive subgroups of S_6

We know that given $f(x)$ an irreducible sextic polynomial, its Galois group $\mathrm{Gal}(f)$ is a transitive subgroup of S_6. The candidates are the subgroup of S_6 such that $6 = \deg(f) \mid |G_f|$. Up to isomorphism, these are:

G_f	S_6	A_6	H_{120}	G_{72}	Γ_{60}	G_{48}	Γ_{36}	G_{36}
$\lvert G_f \rvert$	720	360	120	72	60	48	36	36
G_f	Γ_{24}	G_{24}	H_{24}	G_{18}	Γ_{12}	G_{12}	C_6	H_6
$\lvert G_f \rvert$	24	24	24	18	12	12	6	6

Table 4.1: Transitive subgroups of S_6

We can say "up to isomorphism" because Cayley and Cole (see [1],[2]) proved that each transitive subgroup of S_6 is conjugate in S_6 to one of sixteen non-isomorphic groups in Table (4.1). With the only exception of S_6 and A_6, in the notation above the subscript will denote the number of elements in the group. The groups $H_{m!}$ are isomorphic to S_m and the use of the notation Γ_m indicates that $\Gamma_m = G_{2m} \cap A_6$, with the exception that $\Gamma_{60} = H_{120} \cap A_6$. One can also show that $\Gamma_{12} \cong A_4$ and $\Gamma_{60} \cong A_5$.

The four maximal transitive subgroups of S_6 are S_6, H_{120}, G_{72} and G_{48}. We now explicitly describe their generators and subgroups.

- H_{120} is generated by the elements $(1452), (16524)$ and (143562) and is isomorphic to S_5. $\Gamma_{60} = H_{120} \cap A_6$ is a subgroup of H_{120} of index 2 and is isomorphic to A_5.

- We now consider G_{72} and its subgroups Γ_{36}, G_{36} and G_{18}. Let $X = \{1, 3, 5\}, Y = \{2, 4, 6\}$ and let Sym_Z denote the symmetric group of a set Z. We regard Sym_X and Sym_Y as subgroups of S_6. Since $\sigma = (12)(34)(56) \in S_6$ acts on $\mathrm{Sym}_X \times \mathrm{Sym}_Y \subset S_6$ by conjugation, we can define the semi-direct product

  $$G_{72} = (\mathrm{Sym}_X \times \mathrm{Sym}_Y) \rtimes \langle \sigma \rangle \subset S_6.$$

  It is the stabilizer in S_6 of the set $S = \{X, Y\}$ and is generated by (13), (15) and σ.

  Now $\Gamma_{36} = G_{72} \cap A_6$ is the subgroup of A_6 stabilizing S.

  The subgroup G_{36} is defined by $G_{36} = [A_6 \cap (\mathrm{Sym}_X \times \mathrm{Sym}_Y)] \rtimes \langle \sigma \rangle$. It is generated by (13)(24), (135) and σ.

  Finally, let G_{18} be the subgroup defined by $G_{18} = (A_X \times A_Y) \rtimes \langle \sigma \rangle$, where A_Z is the alternating subgroup of Sym_Z, for a set Z. The group G_{18} is generated by (135) and σ.

- We now describe G_{48} and its transitive subgroups. Let $X = \{1, 2\}, Y = \{3, 4\}, Z = \{5, 6\}$ and $T = \{X, Y, Z\}$. We define G_{48} to be the stabilizer of T in S_6. It is generated by the elements (12), (13)(24) and (135)(246). The subgroup $H = \mathrm{Sym}_X \times \mathrm{Sym}_Y \times \mathrm{Sym}_Z \cong (\mathbb{Z}/3\mathbb{Z})^3$ is generated by the cycles (12), (34) and (56). It is a normal subgroup of G_{48} of order 8 and $G_{48}/H \cong \mathrm{Sym}_T$. We have $G_{48} \cong \mathrm{Sym}_T \ltimes H$.

  To define the subgroups of G_{48} we introduce two characters on G_{48}. Let $\alpha : G_{48} \to \{\pm 1\}$ be the restriction from S_6 to G_{48} of the sign homomorphism $S_6 \to \{\pm 1\}$. We let $\alpha_1 : G_{48} \to \{\pm 1\}$ be the composition of $G_{48} \to G_{48}/H \cong \mathrm{Sym}_T$ and the sign homomorphism $\mathrm{Sym}_T \to \{\pm 1\}$.

  We now define the three subgroups Γ_{24}, G_{24} and H_{24} to be the kernels in G_{48} of the respective homomorphism α, α_1 and $\alpha\alpha_1$.

  We define $\Gamma_{12} = \Gamma_{24} \cap G_{24} (= \Gamma_{24} \cap H_{24} = G_{24} \cap H_{24})$. In terms of generators, these groups are easily described. For example, G_{24} is generated by (12) and (135)(246), and Γ_{12} is generated by (135)(246) and (12)(34). We have $G_{24} \cong H_{24} \cong S_4$, but G_{24} and H_{24} are not conjugate groups in S_4.

- Finally, we describe G_{12} and its two transitive subgroups C_6 and H_6 of order 6. We have $G_{12} = G_{72} \cap G_{48}$. It is generated by the cycles (135)(246), (13)(24) and (12)(34)(56).

  We denote by C_6 the cyclic subgroup generated by (145236) and by H_6 the group generated by (135)(246) and (12)(36)(45). We have $H_6 \cong S_3$.

The set Σ_6 has 16 elements and the representative of each conjugacy class are given in Table (4.1). Twelve of these groups are solvable (all 16 except for S_6, A_6, $H_{120} \cong S_5$ and $\Gamma_{60} \cong A_5$), and there are two maximal solvable groups, G_{72} and G_{48}. When G_f is solvable, we have

$$G_f \subseteq G_{72} \quad \text{or} \quad G_f \subseteq G_{48}.$$

We will give some subgroup relations between the transitive subgroups of S_6 in the following figure:

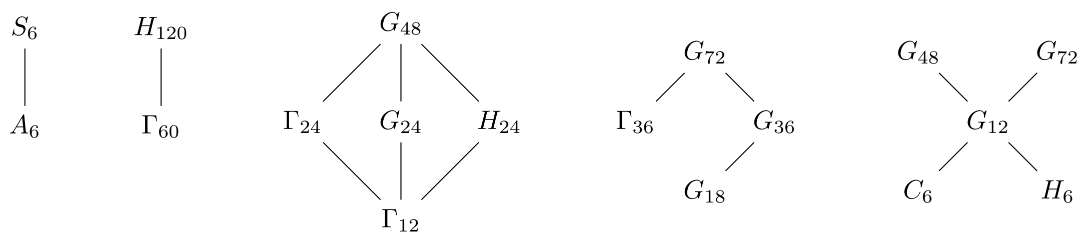


Figure 4.1

4.3 Galois resolvents

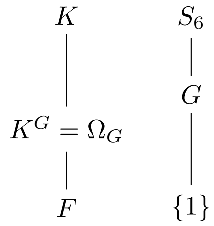
First of all, let's give some notation. Let $x_1, \dots, x_6$ be indeterminates over $\mathbb{Q}$, $R = \mathbb{Q}[x_1, \dots, x_6]$ and $K = \mathbb{Q}(x_1, \dots, x_6)$ the quotient field of R. We let $\sigma \in S_6$ act on K via $\sigma(x_i) = x_{\sigma(i)}$. Let $F = K^{S_6} = \Omega_{S_6}$ be the field of the elements of K fixed by S_6. Then $F = \mathbb{Q}(s_1, \dots, s_6)$, where

$$s_1 = x_1 + \cdots + x_6, \quad s_2 = \sum_{i<j} x_i x_j, \quad \dots \quad s_6 = x_1 x_2 \dots x_6,$$

are the symmetric polynomials in the x_i. K/F is a Galois extension with Galois group S_6. Given $\theta \in K$, we let $\mathrm{Stab}(\theta) = \{\sigma \in S_6 \mid \sigma(\theta) = \theta\}$. If $\theta \in K$ is a polynomial and $\mathrm{Stab}(f) = G$, we call θ a G-polynomial. If $G \subset S_6$, then K/K^G is a Galois extension with group G, and $K^G = F(\theta)$ for some $\theta \in K$ with $\mathrm{Stab}(\theta) = G$.

Now θ will have $m = [S_6 : G]$ conjugates $\theta = \theta_1, \dots, \theta_m$ in K. The Galois resolvent of θ is defined as

$$F_\theta(x) = \prod_{i=1}^{m} (x - \theta_i) \in F[x].$$



$F_\theta(x)$ has degree m and is the product of distinct irreducible factors in $K^H[x]$ for each $H \subset S_6$. Let X be the set of left G-cosets in S_n. The group H acts on X by left multiplication. Elementary group theory shows that the degrees of the irreducible factors of $F_\theta(x)$ in $K^H[x]$ are given by the lengths of the H-orbits in X. Hence, the set of degrees of the irreducible factors of $F_\theta(x)$ is independent of the choice of θ and depends only on G.

We now study three particular Galois resolvents. We denote by $F_2(x), F_{10}(x)$ and $F_{15}(x)$ the Galois resolvents corresponding to the pairs (G, θ_G), for $G = A_6$, $\theta_{A_6} = \prod_{i<j}(x_i - x_j)$; $G = G_{72}$, $\theta_{72} = (x_1 + x_3 + x_5)(x_2 + x_4 + x_6)$; and $G = G_{48}$, $\theta_{48} = x_1x_2 + x_3x_4 + x_5x_6$. Whenever there is no ambiguity, we will often write F_d instead of F_G, where d will be the index $[S_6 : G]$ (for example, $F_2 = F_{A_6}$ and $[S_6 : A_6] = 2$). The degree of $F_d(x)$ is d. Table (4.2) indicates the degrees of the irreducible factors of these resolvents in $K^H[x] = \mathbb{Q}(x_1 \dots, x_6)^H$, for all transitive subgroups $H \subset S_6$.

Group G	$F_2(x)$	$F_{10}(x)$	$F_{15}(x)$
S_6	2	10	15
A_6	1,1	10	15
H_{120}	2	10	10,5
Γ_{60}	1,1	10	10,5
G_{72}	2	9,1	9,6
Γ_{36}	1,1	9,1	9,6
G_{36}	2	9,1	9,3,3
G_{18}	2	9,1	9,3,3
G_{48}	2	6,4	8,6,1
Γ_{24}	1,1	6,4	8,6,1
G_{24}	2	6,4	8,6,1
H_{24}	2	6,4	6,4,4,1
Γ_{12}	1,1	6,4	6,4,4,1
G_{12}	2	6,3,1	6,3,3,2,1
C_6	2	6,3,1	6,3,3,2,1
H_6	2	3,3,3,1	3,3,3,3,1,1,1

Table 4.2

We now introduce the notion of specialization. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial. Choose a numbering $r_1, \dots, r_6$ of the roots of f so that the corresponding embedding $G_f \hookrightarrow S_6$ is one of the groups listed in Table (4.1). We will need to distinguish between the action of S_6 on x_i and the action of G_f on the roots r_i. Let L be the splitting field of $f(x)$ over $\mathbb{Q}$. Then let $\hat{\alpha} : R = \mathbb{Q}[x_1 \dots, x_6] \to L$ be the homomorphism defined by $\hat{\alpha}(x_i) = r_i$. Given $\theta \in R$, we let $\hat{\theta}$ denote the image $\hat{\alpha}(\theta) \in L$. If $g(x) = \sum_i a_i x^i \in R[x]$, we let $\hat{g}(x) = \sum_i \hat{a}_i x^i$. We will often use the following simple observation: If $\theta \in R$ is invariant under the action of $G(f) \subset S_6$, then $\hat{\theta} \in \mathbb{Q}$. Similarly, if the coefficients of $g(x)$ are G_f-invariant, then $\hat{g}(x) \in \mathbb{Q}[x]$. In particular, for each $G \subset S_6$ and G-polynomial $\theta \in R$, we have $\hat{F}_\theta(x) \in \mathbb{Q}[x]$.

It is important to remember that all specializations are with respect to $f(x)$ and a given numbering of the r_i. However, the specialization of a Galois resolvent can be computed without knowing r_i or their numbering. Let θ be a G-polynomial for some $G \subset S$ and $F_\theta(x)$ be its Galois resolvent. Then the coefficients of $\hat{F}_\theta(x)$ are polynomials in the coefficients a_i of $f(x)$. Hence $\hat{F}_\theta(x)$ can be computed by knowing only $f(x)$.

We now study several specific specializations of Galois resolvents. Again, for ease of notation, we will often write $f_G(x)$ or $f_d(x)$ (if $d = [S_6 : G]$) instead of $\hat{F}_G(x)$. For example, $f_2(x) = \hat{F}_2(x) = x^2 - \Delta$, where Δ is the discriminant of $f(x)$. Hence determining whether $f_2(x)$ has rational roots is equivalent to determining whether $G_f \subset A_6$. We will use $f_{10}(x) = \hat{F}_{10}(x)$ and $f_{15}(x) = \hat{F}_{15}(x)$ to draw similar conclusion about G_f. The coefficients of $f_2(x), f_{10}(x)$ and $f_{15}(x)$ are symmetric polynomials in the r_i and can be expressed as polynomials in the coefficients $a_1, \dots, a_6$ of $f(x)$. Let now write $f_{10}, f_{15} \in \mathbb{Q}[x]$ as

$$f_{10} = x^{10} + \sum_{i=1}^{10} (-1)^i b_i x^{10-i}, \quad f_{15} = x^{15} + \sum_{i=1}^{15} (-1)^i c_i x^{15-i}$$

where the coefficients $b_i, c_i \in \mathbb{Q}$ are defined in the Appendix in function of the coefficients $a_1, \dots, a_6$ of $f(x)$. We also give an explicit formula for the discriminant Δ.

Let us now review how Galois resolvents can be used to calculate G_f. Let $G \subset S_6$ and choose a G-polynomial $\theta_G \in \mathbb{Q}[x_1, \dots, x_6]$. Let $F_{\theta_G}(x)$ be the Galois resolvent. We will write $G_f \subset_c G$ if G_f is conjugate in S_6 to a subgroup of G. It can be easily shown:

Proposition 1 ([5]). *If $G_f \subset_c G$, then $\hat{F}_{\theta_G}(x) \in \mathbb{Q}[x]$ has a rational root. Conversly, if $\hat{F}_{\theta_G}(x)$ has a rational root with multiplicity one, then $G_f \subset_c G$.*

By assuming that G_f is one of the 16 groups in Table (4.1), we can replace $\subset_c$ in Proposition 1 by $\subset$ when $G = G_{72}, G_{48}$ or A_6. If for each transitive subgroup $G \subset S_6$, there exists $\theta_G \in \mathbb{Q}[x_1, \dots, x_6]$ such that the specialization $\hat{F}_{\theta_G}(x)$ always has distinct roots, then Proposition 1 would solve the problem of determining Galois groups.

The key to our approach is that we can choose θ_G for $G = G_{72}$, G_{48}, so that $\hat{F}_{\theta_G}$ has a rational root with multiplicity one most of the time. And in the remaining cases, we can use the factorization of $\hat{F}_{\theta_G}(x)$ to determine whether $G_f \subset G$. Then, once we know whether $G_f \subset G_{72}$ and $G_f \subset G_{48}$, we can use other criteria to determine G_f precisely. The three Galois resolvents we will use are $f_2(x) = x^2 - \Delta$, $f_{10}(x)$ and $f_{15}(x)$.

We now consider the factorization of $f_2(x)$, $f_{10}(x)$, $f_{15}(x)$ in $\mathbb{Q}[x]$ when $G_f = G$, for each transitive subgroup G of S_6. The factorization of $f_2(x)$ is easy to determine. $f_2(x) = x^2 - \Delta$ has a rational root if and only if $G_f \subset A_6$. And since $\Delta \neq 0$, these statements are equivalent to Δ being a square in $\mathbb{Q}$. For the other cases, we will make heavy use of the well-known lemma:

Lemma 4.3.1 ([5]). *Let $F_{\theta_G}(x)$ be the Galois resolvent associated to $G \subset S_6$. Assume that $f(x)$ is an irreducible polynomial with G_f. If $F(x)$ is an irreducible factor of the Galois resolvent $F_{\theta_G}(x)$ in $K^{G_f}[x]$, the $\hat{F}(x)$ is either an irreducible polynomial or the power of a linear polynomial in $\mathbb{Q}[x]$.*

We now consider the factorization of f_{10} in $\mathbb{Q}[x]$. We can recall that $f_{10} = \hat{F}_{10}(x)$. Let $\theta_{(abc)(def)} = (x_a + x_b + x_c)(x_d + x_e + x_f)$. Then the ten roots of $F_{10}(x)$ are the $\theta_{(abc)(def)}$, where $\{(abc)(def)\}$ ranges over all ten partition of $\{1, \dots, 6\}$ into two sets of three element each. Since $\theta_{(135)(246)} = \theta_{72}$ is a G_{72}-polynomial, if $G_f \subset G_{72}$, then $\theta = \hat{\theta}_{(135)(246)}$ is a rational root of $f_{10}(x)$.

Proposition 2 ([5]). *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial and assume that G_f is one of the groups in Table (4.1).*

(a) $G_f \subset G_{72} \iff f_{10}(x)$ has a rational root. When this holds, $f_{10}(x)$ has a rational root with multiplicity one.

(b) If $F(x)$ is an irreducible factor of $F_{10}(x)$ in $K^{G_f}[x]$ of degree ≥ 4, then $\hat{F}(x)$ is an irreducible factor of f_{10} in $\mathbb{Q}[x]$.

Proof. We first prove *(b)*. By Table (4.2), we can assume that $G_f \neq H_6$. Inspection then shows that whenever $F_{10}(x)$ has an irreducible factor $F(x)$ of degree 6, 9 or 10, then $F(x)$ has $\theta_{(123)(456)}$, $\theta_{(124)(356)}$ and $\theta_{(125)(346)}$ as roots. By Lemma 4.3.1, $\hat{F}(x)$ is either irreducible in $\mathbb{Q}[x]$ or

$$\hat{\theta}_{(123)(456)} = \hat{\theta}_{(124)(356)} = \hat{\theta}_{(125)(346)}.$$

Assume that the latter holds. Then the first equality shows that $r_1 + r_2 = r_5 + r_6$, the second gives $r_1 + r_2 = r_3 + r_6$ and we obtain the contradiction $r_3 = r_5$. Hence $\hat{F}(x) \in \mathbb{Q}[x]$ is irreducible. By process of elimination, we can now assume that $F(x)$ is the irreducible factor of degree four occurring when $G_f \subset G_{48}$, $G_f \not\subset G_{72}$. The roots of $F(x)$ are then $\theta_{(135)(246)}$, $\theta_{(136)(245)}$, $\theta_{(145)(236)}$ and $\theta_{(146)(235)}$. Again, by Lemma 4.3.1, if $\hat{F}(x)$ is not irreducible, then $\hat{\theta}_{(135)(246)} = \hat{\theta}_{(136)(245)}$ and $\hat{\theta}_{(145)(236)} = \hat{\theta}_{(146)(235)}$. Hence, $r_1 + r_3 = r_2 + r_4$ and $r_1 + r_4 = r_2 + r_3$ and we obtain the contradiction $r_1 = r_2$. Thus $\hat{F}(x)$ is irreducible and *(b)* is proved.

We now prove *(a)*. The direction ($\Longrightarrow$) follows from Proposition 1. Now ($\Longleftarrow$) follows from *(b)* since if $G_f \not\subset G_{72}$, then $f_{10}(x)$ does not have a rational root. Hence the equivalence in *(a)* is proved. We now show that when it holds, $f_{10}(x)$ has a rational root with multiplicity one. When $G_f \subset G_{72}$, $G_f \not\subset G_{48}$, then by *(b)*, f_{10} has a rational root with multiplicity one. Hence we can assume that $G_f = G_{12}$, C_6 or H_6. We will show that in each case, f_{10} has a rational root with multiplicity one. We first consider the case when $G_f = H_6$. It suffices to show that if the specialization $\hat{F}(x)$ of an irreducible cubic factor $F(x)$ of $F_{10}(x)$ equals $(x - a)^3$, then $a \neq \theta$ $(= \hat{\theta}_{(135)(246)})$. Inspection shows that the roots of the three irreducible cubic factors are given by the sets

$$\{\hat{\theta}_{(136)(245)},\ \hat{\theta}_{(145)(236)},\ \hat{\theta}_{(146)(235)}\},\ \{\hat{\theta}_{(132)(456)},\ \hat{\theta}_{(126)(354)},\ \hat{\theta}_{(156)(234)}\},$$

$$\{\hat{\theta}_{(134)(256)},\ \hat{\theta}_{(146)(235)},\ \hat{\theta}_{(145)(236)}\}.$$

If $a = \theta$, then either

$$\hat{\theta}_{(135)(246)} = \hat{\theta}_{(136)(245)} = \hat{\theta}_{(145)(236)} = \hat{\theta}_{(146)(235)},$$

or

$$\hat{\theta}_{(135)(246)} = \hat{\theta}_{(132)(456)} = \hat{\theta}_{(126)(354)} = \hat{\theta}_{(156)(234)},$$

or

$$\hat{\theta}_{(135)(246)} = \hat{\theta}_{(134)(256)} = \hat{\theta}_{(146)(235)} = \hat{\theta}_{(145)(236)}.$$

Proceeding as in the second part of the proof for *(b)*, we obtain a contradiction in all three cases. The cases when $G_f = G_{12}$, C_6 are similar. Hence *(a)* is proved. □

More generally, for any Galois resolvents coming from a G_{72}-polynomial, we can show

Proposition 3 ([5]). *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial and assume that G_f is one of the groups in Table (4.1).*

(a) Let $\theta \in \mathbb{Q}[x_1, \ldots, x_6]$ be a G_{72}-polynomial. If $G_f \subset G_{72}$, then $\hat{\theta} \in \mathbb{Q}$ and is the unique root of $\hat{F}_\theta(x) \in \mathbb{Q}[x]$ occurring with multiplicity 1, 4, 7 or 10.

(b) Let $\theta \in \mathbb{Q}[x_1, \ldots, x_6]$ be a G_{48}-polynomial. If $G_f \subset G_{48}$, $G_f \not\subset G_{72}$, then $\hat{\theta} \in \mathbb{Q}$ and is the unique root of $\hat{F}_\theta(x) \in \mathbb{Q}[x]$ with multiplicity 1, 5, 7, 9, 11 or 15.

Proof. We prove *(a)*. Using Table (4.2) and Lemma 4.3.1, for each possible Galois group $G_f = G \subset G_{72}$, one can determine the possible decomposition of $\hat{F}_\theta(x)$ in $\mathbb{Q}[x]$. For each possible decomposition, inspection shows that there exists a positive integer n such that $\hat{\theta}$ is the unique root r of $\hat{F}_\theta(x)$ with multiplicity n. The list of such n is the list in *(a)*. Since no other root can have this multiplicity, *(a)* is proved. The proof of *(b)* is the same. The restriction $G_f \subset G_{48}$, $G_f \not\subset G_{72}$ occurs because when $G_f \subset G_{12}$, it can be the case that $\hat{\theta}$ cannot be determined because there are multiple roots with the same multiplicity. □

We now consider the factorization of $f_{15}(x) =$ in $\mathbb{Q}[x]$. Let

$$\theta_{(ab)(cd)(ef)} = x_a x_b + x_c x_d + x_e x_f.$$

The roots of $F_{15}(x) = F_{\theta_{48}}(x)$ are the fifteen conjugates of θ_{48} listed in Table (4.3).

$\theta_{(12)(34)(56)}$	$\theta_{(12)(35)(46)}$	$\theta_{(12)(36)(45)}$
$\theta_{(13)(24)(56)}$	$\theta_{(13)(25)(46)}$	$\theta_{(13)(26)(45)}$
$\theta_{(14)(23)(56)}$	$\theta_{(14)(25)(36)}$	$\theta_{(14)(26)(35)}$
$\theta_{(15)(23)(46)}$	$\theta_{(15)(24)(36)}$	$\theta_{(15)(26)(34)}$
$\theta_{(16)(23)(45)}$	$\theta_{(16)(24)(35)}$	$\theta_{(16)(25)(34)}$

Table 4.3: Roots of $F_{15}(x)$

We have:

Proposition 4 ([5])**.** *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial. If $F(x)$ is an irreducible factor of $F_{15}(x)$ in $K^{G_f}[x]$ with degree $d \geq 6$, then $\hat{F}(x) \in \mathbb{Q}[x]$ is irreducible.*

Proof. Assume $\hat{F}(x)$ is reducible. Then $\hat{F}(x) = (x - a)^d$ for some $a \in \mathbb{Q}$, by Lemma 4.3.1. Since $\deg F(x) \geq 6$, two of the roots of $F(x)$ must be $\theta_{(1b)(cd)(ef)}$, $\theta_{(1b)(ce)(df)}$, for some permutation $b,\ c,\ d,\ e,\ f$ of the numbers $2, \ldots, 6$. Since $\hat{\theta}_{(1b)(cd)(ef)} = \hat{\theta}_{(1b)(ce)(df)}$, we have $(r_c - r_f)(r_d - r_e) = 0$ and thus, either the contradiction $r_c = r_f$ or the contradiction $r_d = r_e$. Hence $\hat{F}(x)$ is irreducible. □

We are now ready to give our criterion to determine whether G_f is solvable, as a corollary of the following Theorem.

Theorem 4.3.2 ([5])**.** *If $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial, then*

(a) $G_f \subset G_{72} \iff f_{10}(x)$ *has a rational root.*

(b) $G_f \subset G_{48} \iff$ *one of the following statements holds:*

- *(i)* $f_{15}(x)$ *has a rational root with multiplicity* $\neq 3,\ 5$.
- *(ii)* $f_{15}(x)$ *has a rational root with multiplicity three and* $f_{10}(x)$ *has either an irreducible cubic factor or at least two distinct linear factors.*
- *(iii)* $f_{15}(x)$ *has a rational root with multiplicity five and* $f_{10}(x)$ *is reducible.*

Proof. Part *(a)* is proved in Proposition 2*(a)*. We now prove *(b)*($\Longrightarrow$). Assume $G_f \subset G_{48}$. By Proposition 1, $f_{15}(x)$ has a rational root. If $f_{15}(x)$ has a rational root with multiplicity $\neq 3, 5$, then condition *(i)* holds. We now assume that *(i)* does not hold. If $f_{15}(x)$ has a rational root with multiplicity 3, then by Proposition 4 and Table 4.2, either $G_f = G_{12},\ C_6$ or H_6. Inspection, along the lines of the proof of Proposition 2*(b)*, then shows that the criterion in condition *(ii)* holds. If $G_f \subset G_{48}$ and $f_{15}(x)$ does not have any rational roots except with multiplicity 5, then by Proposition 4 we have $G_f = H_{24}$ or Γ_{12}. Then since $F_{10}(x)$ is reducible in $K^G[x]$ when $G = H_{24}$ or Γ_{12}, $f_{10}(x)$ is reducible. Hence *(iii)* holds.

We now prove *(b)*($\Longleftarrow$). Assume that condition *(i)* holds. Then by Proposition 4, we must have $G_f \subset G_{48}$. Assume now that condition *(ii)* holds. Since there is a root with multiplicity 3, then by Table 4.2 and Proposition 4, we must have $G_f = G_{36},\ G_{18}$ or $G_f \subset G_{12}$. But Proposition 2 shows that $f_{10}(x)$ contains an irreducible factor of degree 9 when $G_f = G_{36},\ G_{18}$. Hence $G_f \subset G_{12} \subset G_{48}$. Finally, assume that condition *(iii)* holds. If $f_{15}(x)$ has a root with multiplicity 5 then $G_f = H_{120},\ \Gamma_{60}$ or $G_f \subset G_{48}$ by Lemma 4.3.1. Since $f_{10}(x)$ is reducible, by Proposition 2 we have $G_f \neq H_{120},\ \Gamma_{60}$. Hence $G_f \subset G_{48}$ and the theorem is proven. □

Corollary 4.3.3 ([5])**.** *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial. Then G_f is solvable $\iff$ one of the following statements holds:*

(a) $f_{10}(x)$ *has a rational root.*

(b) $f_{15}(x)$ *has a rational root with multiplicity* $\neq 5$.

(c) $f_{15}(x)$ *has a rational root with multiplicity five and* $f_{10}(x)$ *is the product of irreducible quartic and sextic polynomials.*

Proof. G_f is solvable if and only if $G_f \subset G_{72}$ or $G_f \subset G_{48}$. Hence Corollary 4.3.3 follows from Theorem 4.3.2, Table 4.2 and the observation that $f_{15}(x)$ can only have a rational root with multiplicity three when $G_f \subset G_{72}$ or $G_f \subset G_{48}$. □

Once it is known by Theorem 4.3.2 that G_f is not solvable, it is easy to determine G_f. We have

Proposition 5 ([5]). *Let $f(x) \in \mathbb{Q}[x]$ be a non-solvable irreducible sextic polynomial. Then*

(a) $G_f \cong S_6 \iff f_{15}(x)$ is irreducible in $\mathbb{Q}[x]$ and Δ is not a square in $\mathbb{Q}$.

(b) $G_f \cong A_6 \iff f_{15}(x)$ is irreducible in $\mathbb{Q}[x]$ and Δ is a square in $\mathbb{Q}$.

(c) $G_f \cong H_{120} \iff f_{15}(x)$ is reducible in $\mathbb{Q}[x]$ and Δ is not a square in $\mathbb{Q}$.

(d) $G_f \cong \Gamma_{60} \iff f_{15}(x)$ is reducible in $\mathbb{Q}[x]$ and Δ is a square in $\mathbb{Q}$.

Proof. We have $G_f \cong S_6,\ A_6,\ H_{120}$ or Γ_{60}. By Proposition 4, $f_{15}(x)$ is irreducible $\iff G_f \cong S_6$ or A_6. The discriminant Δ distinguishes the remaining cases. □

4.4 Solving the sextic: $G_f \subseteq G_{48}, G_f \not\subseteq G_{72}$

Once it is proved Theorem 4.3.2, we now assume that $G_f \subseteq G_{48}$ and $G_f \not\subseteq G_{72}$. We first explain how to determine G_f, then how to determine the roots of the sextic $f(x)$ and finally how to explicitly determine the action of G_f on the roots.

We recall that by Proposition 3, we know the value of the rational root $\theta_1 = \hat{\theta}_{48} = r_1r_2 + r_3r_4 + r_5r_6$ of $f_{15}(x)$. We introduce the variables:

$$\begin{aligned} d_{12} = x_1 + x_2, \quad d_{34} = x_3 + x_4, \quad d_{56} = x_5 + x_6, \\ e_{12} = x_1x_2, \qquad e_{34} = x_3x_4, \qquad e_{56} = x_5x_6, \\ \chi_1 = (d_{12} - d_{34})(d_{34} - d_{56})(d_{56} - d_{12}), \\ \chi_2 = (e_{12} - e_{34})(e_{34} - e_{56})(e_{56} - e_{12}). \end{aligned} \tag{4.1}$$

Now $\chi_1^2,\ \chi_2^2$ are G_{48}-polynomials and by Proposition 3, the values of $\hat{\chi}_1^2,\ \hat{\chi}_2^2$ can be determined as roots of the Galois resolvents $f_{\chi_i^2}$. We now state some elementary properties of $\hat{\chi}_1,\ \hat{\chi}_2$.

Lemma 4.4.1 ([5]). *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic with $G_f = G_{48},\ \Gamma_{24},\ G_{24},\ H_{24}$ or Γ_{12}. Then*

(a) $\hat{\chi}_1 = 0 \iff \hat{d}_{12} = \hat{d}_{34} = \hat{d}_{56} = a_1/3$.

(b) $\hat{\chi}_2 = 0 \iff \hat{e}_{12} = \hat{e}_{34} = \hat{e}_{56} = \theta_1/3$.

(c) At least one of the $\hat{\chi}_i$ is non-zero. If both $\hat{\chi}_1,\ \hat{\chi}_2$ are non-zero, then $\hat{\chi}_1^2$ is a square in $\mathbb{Q} \iff \hat{\chi}_2^2$ is a square in $\mathbb{Q}$.

Proof. We first prove *(a)*. Suppose $\hat{\chi}_1 = 0$. Then one of the three factors of $\hat{\chi}_1$ must vanish. Assume that $\hat{d}_{12} = \hat{d}_{34}$. Then applying the automorphism $\sigma = (135)(246) \in \Gamma_{12} \subset G$, we obtain $\hat{d}_{34} = \hat{d}_{56}$, and $a_1 = 3\hat{d}_{12}$. Thus ($\Rightarrow$) is proved. The converse ($\Leftarrow$) is clear. Statement *(b)* is proved in the same way. We now prove *(c)*. Assume that $\hat{\chi}_1 = \hat{\chi}_2 = 0$. Then by *(a)*, *(b)*, the symmetric functions in r_1 and r_2 are rational numbers. But then

$$x^2 - \hat{d}_{12}x + \hat{e}_{12} = (x - r_1)(x - r_2)$$

is a rational quadratic factor of $f(x)$, contradicting the irreducibility of $f(x)$. Hence, at least one of the $\hat{\chi}_i$ is non-zero. Now suppose that both are non-zero. Since χ_1/χ_2 is fixed by G_{48}, $\hat{\chi}_1/\hat{\chi}_2 \in \mathbb{Q}^*$ and *(c)* is proved. □

By Lemma 4.4.1, we can define a non-zero number $\chi \in \mathbb{Q}$ by $\chi = \hat{\chi}_1^2$ if $\hat{\chi}_1 \neq 0$ and $\chi = \hat{\chi}_2^2$ otherwise. We recall that Δ is the discriminant of $f(x)$. We are now ready to prove the following Theorem to determine G_f.

Theorem 4.4.2 ([5]). *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic with $G_f = G_{48},\ \Gamma_{24},\ G_{24},\ H_{24}$ or Γ_{12}. Then*

(a) $G_f = G_{48} \iff$ none of the numbers $\chi,\ \Delta,\ \chi\Delta$ are squares in $\mathbb{Q}$.

(b) $G_f = \Gamma_{24} \iff \Delta$ is a square in $\mathbb{Q}$, but $\chi,\ \chi\Delta$ are not squares in $\mathbb{Q}$.

(c) $G_f = G_{24} \iff \chi$ is a square in $\mathbb{Q}$, but $\Delta,\ \chi\Delta$ are not squares in $\mathbb{Q}$.

(d) $G_f = H_{24} \iff \chi\Delta$ is a square in $\mathbb{Q}$, but $\chi,\ \Delta$ are not squares in $\mathbb{Q}$.

(e) $G_f = \Gamma_{12} \iff \chi,\ \Delta$ and $\chi\Delta$ are squares in $\mathbb{Q}$.

Proof. We first prove *(b)*. Let σ denote the element $(12)(34)(56) \in G_{48}$. Then $\sigma(\Delta) = -\Delta$ and the stabilizer of Δ in G_{48} is Γ_{24}. Since Δ is non-zero, we have that Δ is a rational square if and only if $G_f \subset \Gamma_{24}$. Since $\chi \neq 0$, $(13)(24)\chi = -\chi$ and $(13)(24)\chi\Delta = -\chi\Delta$, and G_{24} and H_{24} are the respective stabilizers of χ and $\chi\Delta$ in G_{48}, we obtain the corresponding statements *(c)*, *(d)* for G_{24} and H_{24}. Cases *(a)* and *(e)* then follow from cases *(b)*, *(c)* and *(d)* and the fact that $\Gamma_{12} = \Gamma_{24} \cap G_{24} \cap H_{24}$. □

We now show that there are general formulas for finding the roots of $f(x)$. Implicit in our approach will be the assumption that we can simplify algebraic numbers to determine whether they are rational numbers. First, we introduce some useful symmetric functions of the d_{ij} and e_{ij}. Some of these symmetric functions can be easily expressed in terms of a_i, θ_1. We have

$$\begin{aligned} a_1 &= \hat{d}_{12} + \hat{d}_{34} + \hat{d}_{56}, \\ a_2 - \theta_1 &= \hat{d}_{12}\hat{d}_{34} + \hat{d}_{34}\hat{d}_{56} + \hat{d}_{12}\hat{d}_{56}, \\ \theta_1 &= \hat{e}_{12} + \hat{e}_{34} + \hat{e}_{56}, \\ a_6 &= \hat{e}_{12}\hat{e}_{34}\hat{e}_{56} \end{aligned}$$

The other two symmetric functions, which are specializations of the two G_{48}-polynomials

$$\begin{aligned} D &= d_{12}d_{34}d_{56}, \\ E &= e_{12}e_{34} + e_{34}e_{56} + e_{12}e_{56}, \end{aligned} \tag{4.2}$$

are not easily expressed. By Proposition 3, $\hat{D},\ \hat{E}$ can be determined as rational roots of their Galois resolvents $\hat{F}_D(x),\ \hat{F}_E(x) \in \mathbb{Q}[x]$. Let

$$g_2(x) = x^3 - a_1x^2 + (a_2 - \theta_1)x - \hat{D} \in \mathbb{Q}[x], \tag{4.3}$$

$$g_3(x) = x^3 - \theta_1x^2 + \hat{E}x - a_6 \in \mathbb{Q}[x]. \tag{4.4}$$

Let ω be a primitive cubic root of unity. Formulas $y_g(\omega^i)$ for finding the roots of a cubic polynomial g are given in the Appendix (Lemma A.0.1). Define $l_i = y_{g_2}(\omega^i),\ m_i = y_{g_3}(\omega^i)$ for $i = 1, 2, 3$. The l_i (resp. m_i) are the roots of $g_2(x)$ (resp. $g_3(x)$). We than have

$$\{l_1, l_2, l_3\} = \{\hat{d}_{12}, \hat{d}_{34}, \hat{d}_{56}\}, \quad \{m_1, m_2, m_3\} = \{\hat{e}_{12}, \hat{e}_{34}, \hat{e}_{56}\}$$

We note that we do not yet know haw to identify the l_i (resp. m_i) with the $\hat{d}_{ij}$ (resp. $\hat{e}_{ij}$). Finally, let us define for $k = 1, 2$ the two G_{48}-polynomials:

$$h_{1k} = d_{12}^k e_{12} + d_{34}^k e_{34} + d_{56}^k e_{56}. \tag{4.5}$$

Since h_{11}, h_{12} are G_{48}-polynomials, $\hat{h}_{11}, \hat{h}_{12} \in \mathbb{Q}$ when $G_f \subseteq G_{48}$.

Proposition 6 ([5]). *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial with $G_f \subseteq G_{48}$. Assume that the values of $\theta_1, l_i, m_i, \hat{h}_{11}$ and $\hat{h}_{12}$ are known. Then there is an effective algorithm for determining $\sigma \in S_3$ such that for each i, l_i and $m_{\sigma(i)}$ correspond to the same pair of roots. In other words, we can find σ such that*

$$\Big\{(l_i, m_{\sigma(i)})\Big\}_{i=1,2,3} = \Big\{(\hat{d}_{12}, \hat{e}_{12}), (\hat{d}_{34}, \hat{e}_{34}), (\hat{d}_{56}, \hat{e}_{56})\Big\} \tag{4.6}$$

Before reading through the proof of Proposition 6, we now introduce some additional notation and prove a lemma needed to prove Proposition 6. Let $k = 1$ or 2. Define

$$\begin{aligned} & & j_{1k} &= d_{12}^k e_{12} + d_{34}^k e_{56} + d_{56}^k e_{34} \\ h_{2k} &= d_{12}^k e_{34} + d_{34}^k e_{56} + d_{56}^k e_{12}, & j_{2k} &= d_{12}^k e_{56} + d_{34}^k e_{34} + d_{56}^k e_{12} \\ h_{3k} &= d_{12}^k e_{56} + d_{34}^k e_{12} + d_{56}^k e_{34}, & j_{3k} &= d_{12}^k e_{34} + d_{34}^k e_{12} + d_{56}^k e_{56} \end{aligned} \tag{4.7}$$

Fix $k = 1, 2$. Then for all groups G in Figure 4.1 with $G \subseteq G_{48}$, the set $\{h_{2k}, h_{3k}\}$ is G-stable and the set $\{j_{1k}, j_{2k}, h_{3k}\}$ is a G-orbit in $\mathbb{Q}[x_1, \ldots, x_6]$.

Since G_f satisfies this condition, we have

$$\hat{h}_{2k} \in \mathbb{Q} \Longleftrightarrow \hat{h}_{3k} \in \mathbb{Q}, \tag{4.8}$$

and

$$\hat{j}_{ik} \in \mathbb{Q} \text{ for some } i \Longleftrightarrow \hat{j}_{ik} \in \mathbb{Q} \text{ for all } i. \tag{4.9}$$

When the last case occurs, $\hat{j}_{1k} = \hat{j}_{2k} = \hat{j}_{3k}$.

Lemma 4.4.3 ([5])**.** *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial with $G_f \subseteq G_{48}$. Let l_i, m_i, h_{ik}, j_{ik} be defined as above. Fix $k = 1, 2$. If the l_i are distinct and the m_i are distinct, then*

$$\hat{h}_{ik} \notin \left\{\hat{j}_{1k}, \hat{j}_{2k}, \hat{j}_{3k}\right\}.$$

Proof. We first assume that $k = 1$. If $\hat{h}_{11} = \hat{j}_{11}$, then $(\hat{d}_{34} - \hat{d}_{56})(\hat{e}_{34} - \hat{e}_{56}) = 0$ and $\hat{d}_{34} = \hat{d}_{56}$ or $\hat{e}_{34} = \hat{e}_{56}$. But this contradicts either the distinctness of the l_i or that of the m_i. Hence $\hat{h}_{11} \neq \hat{j}_{11}$. The other cases are proved similarly. Now suppose that $k = 2$ and that $\hat{h}_{12}$ equals one of the $\hat{j}_{i2}$. Then $\hat{j}_{i2} \in \mathbb{Q}$ for $i = 1, 2, 3$. Hence $\hat{j}_{12} = \hat{j}_{22} = \hat{j}_{32}$. Now the same argument as when $k = 1$ shows that $\hat{h}_{12} = \hat{j}_{12}$ implies that $\hat{d}_{34} = -\hat{d}_{56}$. Similarly $\hat{h}_{12} = \hat{j}_{22}$ implies that $\hat{d}_{12} = -\hat{d}_{56}$. Hence $\hat{d}_{12} = \hat{d}_{34}$ and the l_i are not distinct, contradicting the hypothesis. Hence the lemma is proved. ☐

Finally, for $k = 1, 2,\ \sigma \in S_3$, define

$$p_{k\sigma} = l_1^k m_{\sigma(1)} + l_2^k m_{\sigma(2)} + l_3^k m_{\sigma(3)}. \tag{4.10}$$

The $p_{k\sigma}$ have the property that $\{p_{k\sigma} \mid \sigma \in S_3\} = \{\hat{h}_{ik}, \hat{j}_{ik} \mid i = 1, 2, 3\}$. The following lemma is essential

Lemma 4.4.4 ([5])**.** *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible sextic polynomial with $G_f \subseteq G_{48}$. Let $l_i, m_i, p_{k\sigma}$ be defined as above. Assume that the l_i are distinct and the m_i are distinct. Then there exists a unique $\sigma \in S_3$ such that $p_{1\sigma} = \hat{h}_{11}$ and $p_{2\sigma} = \hat{h}_{12}$.*

Proof. By definition of $l_i,\ m_i$, the equations $p_{j\sigma} = \hat{h}_{1j}$, for $j = 1, 2$, have at least one solution σ. We now establish uniqueness. Assume that we have $\sigma_1, \sigma_2 \in S_3$ with $p_{k\sigma_1} = p_{k\sigma_2} = \hat{h}_{1k}$ for $k = 1, 2$. Then the three equations

$$\begin{aligned} x + y + z &= 0 \\ l_1 x + l_2 y + l_3 z &= 0 \\ l_1^2 x + l_2^2 y + l_3^2 z &= 0 \end{aligned} \tag{4.11}$$

have the non-zero solution

$$(x, y, z) = (m_{\sigma_1(1)} - m_{\sigma_2(1)}, m_{\sigma_1(2)} - m_{\sigma_2(2)}, m_{\sigma_1(3)} - m_{\sigma_2(3)}).$$

But since the determinant

$$\Delta' = \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ l_1^2 & l_2^2 & l_3^2 \end{vmatrix} = -\prod_{i<j} (l_i - l_j)$$

is non-zero as the l_i are distinct, the only solution to (4.11) is the trivial solution. Since the m_i are distinct, we have $\sigma_1 = \sigma_2$ and the lemma is proved. ☐

We can now prove Proposition 6.

Proof. We first note that (4.6) is satisfied for at least one $\sigma \in S_3$, and to prove the proposition, we need only show how to determine σ. We first consider the case when two of the l_i coincide. Then the action of $(135)(246) \in G_f$ shows that $l_1 = l_2 = l_3$. Similarly, if two of the m_i coincide, then they are all equal. In either case, the proposition holds trivially by letting $\sigma = (1)$. We now consider the case when the l_i are m_i are distinct. Now (4.6) has at least one solution $\sigma \in S_3$ and any solution is a solution to the equations $p_{1\sigma} = \hat{h}_{11}$ and $p_{2\sigma} = \hat{h}_{12}$. By Lemma 4.4.4, these equations have a unique solution. Hence by comparing the value of $p_{j\sigma}$ to those of the known constants $\hat{h}_{11}, \hat{h}_{12}$, we can determine $\sigma \in S_3$ satisfying (4.6) and the proposition is proved. □

Proposition 6 is the key to solving the sextic when $G_f \subseteq G_{48}$, $G_f \not\subseteq G_{72}$.

Theorem 4.4.5 ([5]). *Let G be ine of the transitive, solvable subgroups $G_{48}, \Gamma_{24}, G_{24}, H_{24}, \Gamma_{12}$ of S_6. Let $\overline{G}$ be its conjugacy class in Σ_6.*

(a) The general equation of type $(6, \overline{G})$ is explicitly solvable by radicals.

(b) The formulas $z_i(t_j)$ in (a) can be numbered so that for each $f \in \Gamma_{\overline{G}}$, the Galois action of $\tau \in G_f$ on the roots $z_i = z_i(\hat{t}_j(f))$ is given by $\tau(z_i) = z_{\tau(i)}$.

Proof. (a) Given an irreducible sextic polynomial $f \in \Gamma_{\overline{G}}$ with $G_f = G$, define the polynomials g_2, g_3 as in (4.3), (4.4). Let $y_g(a)$ be defined as in Lemma A.0.1 in the Appendix, ω a primitive cubic root of unity and define $z(f, a, b, \varepsilon) = \frac{1}{2}(y_{g_2}(a) + \varepsilon\sqrt{y_{g_2}(a)^2 - 4y_{g_3}(b)})$. Let l_i, m_i be the roots of g_2, g_3 defined following (4.3), (4.4). By Proposition 3, we can determine the values of $\theta_1 = \hat{\theta}_{48}$, $\hat{h}_{11}$, $\hat{h}_{12}$. By Proposition 6, one can calculate $\sigma \in S_3$ such that (4.6) holds. Define

$$z_{2i-1} = z(f, \omega^i, \omega^{\sigma(i)}, 1), \quad z_{2i} = z(f, \omega^i, \omega^{\sigma(i)}, -1), \quad \text{for } i = 1, 2, 3. \tag{4.12}$$

Then

$$\big\{\{r_1, r_2\}, \{r_3, r_4\}, \{r_5, r_6\}\big\} = \big\{\{z_1, z_2\}, \{z_3, z_4\}, \{z_5, z_6\}\big\}. \tag{4.13}$$

Hence $z(f, a, b, \varepsilon)$ provides formulas for the roots $\{z_i\}$ of f in terms of the variables a, b, ε, $\hat{D}$, $\hat{E}$ and θ_1. Since there is a finite algorithm for calculating their values given f and all values in $K = \mathbb{Q}[\omega]$, *(a)* is proved.

To prove *(b)*, it suffices to show that for any z_i arising from the formulas in *(a)*, there is an automorphism $\alpha : G_f \to G_f$ satisfying $\alpha(\sigma)(z_i) = z_{\sigma(i)}$ for $\sigma \in G_f$. Then, by twisting the Galois action by α, *(b)* holds. Now the roots r_i were initially chosen so that G_f was one of the five subgroups $G_{48}, \Gamma_{24}, G_{24}, H_{24}, \Gamma_{12}$ and have the property that $\sigma(r_i) = r_{\sigma(i)}$ for $\sigma \in G_f$. Now the proof of *(a)* shows (4.13) holds. Hence, there exists $\tau \in G_{48}$ such that $z_i = r_{\tau(i)}$ and consequently, $\sigma(r_i) = z_{\tau^{-1}\sigma\tau(i)}$ for $\sigma \in G_f$.. Since τ normalizes each of the groups $G_{48}, \Gamma_{24}, G_{24}, H_{24}$ and Γ_{12}, we have $\tau\sigma\tau^{-1} \in G_f$ and $(\tau\sigma\tau^{-1})(z_i) = z_{\sigma(i)}$. Hence, letting $\alpha(\sigma) = \tau\sigma\tau^{-1}$ gives the desired map. □

The following lemma follows trivially from the proof of Theorem 4.4.5

Lemma 4.4.6 ([5]). *Let $G \subseteq G_{48}$ be one of the transitive groups in Figure 4.1 and let $\overline{G}$ be its conjugacy class in Σ_6. Suppose that $f \in \Gamma_{\overline{G}}$ is an irreducible sextic with $G_f = G$, let r_i be the corresponding numbering of the roots, and suppose that the values of θ_1, $\hat{D}$, $\hat{E}$, l_i, m_i, $\hat{h}_{11}$, $\hat{h}_{12}$ are known. Let $\sigma \in S_3$ be the element determined by Proposition 6 and let z_i be defined as in (4.12). Then*

$$\big\{\{r_1, r_2\}, \{r_3, r_4\}, \{r_5, r_6\}\big\} = \big\{\{z_1, z_2\}, \{z_3, z_4\}, \{z_5, z_6\}\big\}.$$

In conclusion, we now summarize this section. Formulas for finding the roots of an irreducible sextic $f(x) \in \mathbb{Q}[x]$ whan $G_f \subseteq G_{48}$, $G_f \not\subseteq G_{72}$:

1. Let $f(x) = x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6$.
2. Use Theorem 4.3.2 to determine $G_f \subseteq G_{48}$ and $G_f \not\subseteq G_{72}$. Let θ_1 be the unique rational root of $f_{15}(x)$ with multiplicity 1, 5, 7 or 9.
3. For $z = D$, E, h_{11}, h_{12} defined as in (4.1), (4.2), (4.5), calculate the G_{48}-resolvent $F_z(x)$. For each z, let $\hat{z} \in \mathbb{Z}$ be the unique rational root of $\hat{F}_z(x)$ with multiplicity 1, 5, 7 or 9.

4. Let ω be a primitive cubic root of unity and let $y_g(\omega^i)$ be the formulas in Lemma A.0.1 in the Appendix.

5. Let $l_i = y_{g_2}(\omega^i)$ be the three roots of the cubic polynomial $g_2 = x^3 - a_1x^2 + (a_2 - \theta_1)x - \hat{D}$ and let $m_i = y_{g_3}(\omega^i)$ be the three roots of the cubic polynomial $g_3 = x^3 - \theta_1 x^2 + \hat{E}x - a_6$.

6. For $k = 1, 2$, $\sigma \in S_3$, define $p_{k\sigma} = \sum_{i=1}^{3} l_i^k m_{\sigma(i)}$.

7. (a) If $l_i = l_j$ or $m_i = m_j$ for some $i \neq j$, let $\sigma = 1$.

   (b) Otherwise, let σ be the unique element of S_3 with $p_{1\sigma} = \hat{h}_{11}$, $p_{2\sigma} = \hat{h}_{12}$.

8. Define $z(f, a, b, \varepsilon) = \frac{1}{2}(y_{g_2}(a) + \varepsilon\sqrt{y_{g_2}(a)^2 - 4y_{g_3}(b)})$. Let $z_{2i-1} = z(f, \omega^i, \omega^{\sigma(i)}, 1)$ for $i = 1, 2, 3$ and $z_{2i} = z(f, \omega^i, \omega^{\sigma(i)}, -1)$ for $i = 1, 2, 3$.

9. The z_i are formulas for the roots of $f(x)$ in the variables a_i, θ_1, $\hat{D}$, $\hat{E}$. The formulas use only the basic arithmetic operations and radicals. The Galois action of $\tau \in G_f$ on the z_i is given by $\tau(z_i) = z_{\tau(i)}$.

We now give example on an irreducible sextic polynomial $f(x) \in \mathbb{Q}[x]$ where we calculate $\mathrm{Gal}_f = G_f$, the roots z_i of f and the Galois action on the z_i. Let

$$f(x) = x^6 + x^4 - x^3 - 2x^2 + 3x - 1 \in \mathbb{Q}[x].$$

One can calculate that $f_{10}(x)$ factors into irreducible polynomials as

$$f_{10}(x) = (x^4 - 2x^3 - x^2 + 71x + 1)(x^6 - 4x^5 + 20x^4 - 30x^3 + 60x^2 - 15x + 1).$$

The resolvent $f_{15}(x)$ has the $\theta_1 = \hat{\theta}_{48} = 0$ as its unique rational root. Its factorization into irreducible polynomials is given by

$$f_{15}(x) = x(x^6 - x^5 + 4x^4 + 19x^3 - 46x^2 - 82x - 31)\cdot \\ \cdot (x^8 - 2x^7 + 9x^6 - 4x^5 - 25x^4 + 53x^3 - 144x^2 - 74x + 877).$$

By Theorem 4.3.2, $G_f \subseteq G_{48}$ and $G_f \not\subseteq G_{72}$. Looking at the possible factorizations of $F_{15}(x)$ (and thus f_{15}) in Table 4.2, we canfurther conclude that $G_f = G_{48}$, Γ_{24} or G_{24}. Using the formula for Δ in the Appendix, one calculates that $\Delta = 66309 = 69(31)^2$. Similarly, one can calculate the values $\hat{\chi}_1^2 = \hat{\chi}_2^2 = -31$, where χ_i is defined following 4.1. We thus let $\chi = -31$ and by Theorem 4.4.2, we conclude that $\mathrm{Gal}(f) = G_f = G_{48}$.

To find formulas for the roots of f, by using the algorithm listed above, we must calculate $\hat{D}$, $\hat{E}$ as roots of their G_{48}-resolvents. We find $\hat{D} = -1$ and $\hat{E} = 1$. Following the definitions in (4.3), (4.4), we have

$$g_2(x) = g_3(x) = x^3 + x + 1.$$

Letting $y_g(\omega^i)$ be defined as above, we let $l_i = m_i = y_{g_2}(\omega^i)$. We have

$$l_1 = m_1 = d_1\omega + d_2\omega^2,$$
$$l_2 = m_2 = d_1\omega^2 + d_2\omega,$$
$$l_3 = m_2 = d_1 + d_2,$$

where

$$d_1 = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{31}{108}}}, \quad d_2 = -\frac{1}{3d_1}$$

Define $p_{k\sigma}$, $k = 1, 2$, $\sigma \in S_3$ as in Step 6 of the algorithm above. Calculation shows $p_{1(1)} = -2$, $p_{1(123)} = p_{1(132)} = 1$, and $p_{2(1)} = -3$ is the only integral value amongst the $p_{2\sigma}$. Hence, without needing to calculate $\hat{h}_{11}$, $\hat{h}_{12}$ (which equal -2, -3 respectively), by Step 7 of the algorithm, we can determine that $\sigma = 1$. Then for each $i = 1, 2, 3$ let z_{2i-1}, z_{2i} be the two roots of the polynomial $x^2 - l_i x + m_i$. The z_i are the roots of f and the Galois action of $\tau \in G_{48}$ on the z_i is given by $\tau(z_i) = z_{\tau(i)}$.

4.5 Solving the sextic: $G_f \subseteq G_{72}, G_f \not\subseteq G_{48}$

For this and the next Section we are just going to give the main idea behind how to determine G_f and how to find the roots of $f(x)$.

Once it is proved Theorem 4.3.2, we now assume that $G_f \subseteq G_{72}$ and $G_f \not\subseteq G_{48}$. We first explain how to determine G_f. Let

$$\beta_1 = (x_1 - x_3)(x_3 - x_5)(x_5 - x_1),$$
$$\beta_2 = (x_2 - x_4)(x_4 - x_6)(x_6 - x_2),$$
$$\delta = \beta_1 + \beta_2, \quad \mu = \beta_1 - \beta_2,$$
$$M = \delta^2 + \mu^2, \quad N = \delta^2\mu^2$$

Now M and N are G_{72}-polynomials. Let $f_M(x)$, $f_N(x)$ be the specializations of their Galois resolvents. $\hat{M}$, $\hat{N}$ are rational roots of $f_M(x)$, $f_N(x)$, respectively, which can be determined by Proposition 3. Let

$$g(x) = x^2 - \hat{M}x + \hat{N} \in \mathbb{Q}[x],$$

and recall Δ is the discriminant of $f(x)$.

Theorem 4.5.1. *Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible sextic with $G_f = G_{72}$, G_{36}, Γ_{36} or G_{18}. Then*

(a) $G_f = \Gamma_{36} \iff \Delta$ is a square in $\mathbb{Q}$.

(b) $G_f = G_{72} \iff \Delta$ is not a square in $\mathbb{Q}$ and $g(x) \in \mathbb{Q}[x]$ is irreducible.

(c) $G_f \subseteq G_{36} \iff \Delta$ is not a square in $\mathbb{Q}$ and $g(x) \in \mathbb{Q}[x]$ is reducible.

Proof. We first prove *(a)*. Out of the four groups, only Γ_{36} is a subset of A_6. Hence Δ is a rational square $\iff G_f \subset A_6 \cap G_{72} \iff G_f = \Gamma_{36}$ and *(a)* is proved.

We now prove *(b)*. By *(a)*, we can assume that Δ is not a rational square and $G_f = G_{72}$, G_{36} or G_{18}. Since β_1, $\beta_2 \neq 0$, we have $\delta^2 \neq \mu^2$. Since δ^2, μ^2 are G_{36}-polynomials which are permuted by the action of (24)$\in G_{72}$, we have $g(x)$ is irreducible if and only if (24)$\in G_f$. Hence *(b)* is proved and *(c)* follows immediately. ☐

The next step is determine a criteria to distinguish between the cases $G_f = G_{36}$ and $G_f = G_{18}$. After some work and the introduction of new variables this criteria can be found.

Finally, we turn our attention to finding the roots of $f(x)$. This formulas do not depend upon precisely knowing G_f.

Proposition 7 ([5])**.** *Let $G \subseteq G_{72}$ be one of the groups in Figure 4.1. Let $\overline{G} \in \Sigma_6$ be the conjugacy class containing G.*

1. *The general equation of type $(6, \overline{G})$ is explicitly solvable by radicals.*
2. *If $G_{18} \subseteq G \subseteq G_{72}$, then the formulas $z_i(t_j)$ and the algorithm can be chosen so that for each $f \in \Gamma_{\overline{G}}$, the Galois action of $\tau \in G_f$ on the roots $z_i = z_i(\hat{t}_j(f))$ is given by $\tau(z_i) = z_{\tau(i)}$.*

To see the proof of Proposition 7 and the other details we remind to check Thomas R. Hagedorn article ([5]).

4.6 Solving the sextic: $G_f \subseteq G_{12}$

As said previously, we now give the main idea of the case $G_f \subseteq G_{12}$ describe in the article ([5]). This case is strictly connected with the previous one.

When $G_f \subseteq G_{12}$, we can easily find a criteria for determining G_f among G_{12}, C_6 and H_6 using the same notation introduced in Section 4.5. Problems come from the fact that the crucial variables determining G_f can be found as a set of roots but can not be distinguished with the same facility. Some work is done to say that this variables can effectively be computed.

As $G_f \subset G_{72}$, Proposition 7 shows that there are formulas for finding the roots z_i of $f(x)$. They can be calculated using the same algorithm described for the case $G_f \subseteq G_{72}, G_f \not\subseteq G_{48}$.

Finally, we show how to explicitly exhibit the Galois action of G_f on the roots z_i of $f(x)$. Unlike in Sections 4.4 and 4.5, the algorithm will depend upon G_f.

Appendix A

Notation for sextics

We first give formulas for the roots of a cubic polynomial:

Lemma A.0.1. *Let* $f(x) = x^3 + a_1x^2 + a_2x + a_3 \in \mathbb{Q}[x]$ *be a cubic polynomial and define*

$$c_1 = \frac{1}{6}(a_1a_2 - 3a_3) - \frac{1}{27}a_1^3, \qquad c_2 = \frac{1}{3}a_2 - \frac{1}{9}a_1^2,$$

$$d_1 = \sqrt[3]{c_1 + \sqrt{c_2^3 + c_1^2}}, \qquad d_2 = -\frac{c_2}{d_1}.$$

Letting $\overline{\alpha}$ *denote the complex conjugate of* α*, define*

$$y_f(\alpha) = -\frac{a_1}{3} + d_1\alpha + d_2\overline{\alpha}.$$

The roots of f *are then* $y_f(1)$, $y_f(\omega)$ *and* $y_f(\omega^2)$*, where* ω *is a primitive cubic root of unity.*

Now let $f(x) = x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6 \in \mathbb{Q}[x]$ be an irreducible sextic and $f_{10}(x),\ f_{15}(x) \in \mathbb{Q}[x]$ be the rational polynomials defined by

$$f_{10}(x) = x^{10} + \sum_{i=1}^{10}(-1)^i b_i x^{10-i}, \quad f_{15}(x) = x^{15} + \sum_{i=1}^{15}(-1)^i c_i x^{15-i}.$$

Now we give explicit formulas for the rational numbers b_i, c_i and for the discriminant Δ in terms of the coefficients a_i of $f(x)$.

(For your convenience, we report them in the following pages directly from the Appendix of the original article and we remind to consult it for more details (Thomas R. Hagedorn, [5])).

$$
\begin{aligned}
\Delta = {} & 108a_4^3a_3^4a_6 - 27a_4^2a_3^4a_5^2 - 3750a_5^5a_2a_3 - 1350a_6a_3^3a_5^3 - 22500a_6a_5^4a_4 \\
& + 320a_6a_1^4a_5^4 + 1500a_6a_5^4a_2^2 - 8748a_3^4a_6^3 + 34992a_3^2a_6^4 - 13824a_4^3a_6^3 \\
& - 13824a_2^3a_6^4 + 256a_1^5a_5^5 - 4860a_4a_2a_3^4a_6^2 - 630a_4a_2a_3^3a_5^3 \\
& + 3888a_4a_2a_3^2a_6^3 - 192a_4a_2a_1^4a_5^4 + 16a_4^4a_2^3a_1^2a_6 \\
& + 8208a_4^2a_2^2a_3^2a_6^2 - 6a_4^2a_2^2a_1^3a_5^3 + 560a_4^2a_2^2a_5^3a_3 + 4816a_4^3a_2^2a_1^2a_6^2 \\
& + 24a_4^2a_2^3a_1a_5^3 + 4816a_4^2a_2^3a_6a_5^2 - 4a_4^3a_2^3a_1^2a_5^2 - 6480a_4^2a_2a_1^2a_6^3 \\
& - 6480a_4a_2^2a_6^2a_5^2 + 1020a_4a_2^2a_1^2a_5^4 - 64a_4^4a_2^4a_6 - 4352a_4^3a_2^3a_6^2 \\
& + 16a_4^3a_2^4a_5^2 - 17280a_4^2a_2^2a_6^3 + 62208a_4a_2a_6^4 + 512a_4^5a_2^2a_6 \\
& - 128a_4^4a_2^2a_5^2 + 512a_4^2a_2^5a_6^2 - 900a_4a_2^3a_5^4 + 2000a_4^2a_2a_5^4 \\
& + 9216a_4^4a_2a_6^2 + 9216a_4a_2^4a_6^3 + 1500a_4^2a_1^4a_6^3 - 32400a_4a_1^2a_6^4 \\
& - 36a_4^3a_1^3a_5^3 + 108a_4^5a_1^4a_6 - 27a_4^4a_1^4a_5^2 - 50a_4^2a_1^2a_5^4 - 192a_4^4a_1^2a_6^2 \\
& + 27000a_6^2a_5^3a_3 - 1350a_3^3a_1^3a_6^3 + 38880a_6^4a_1a_5 + 540a_6^3a_1^2a_5^2 \\
& - 32400a_6^3a_5^2a_2 + 27000a_3a_1^3a_6^4 + 410a_6^2a_1^3a_5^3 - 8640a_3^2a_2^3a_6^3 \\
& + 43200a_1^2a_2^2a_6^4 + 43200a_5^2a_4^2a_6^2 - 8640a_4^3a_3^2a_6^2 - 192a_2^4a_5^2a_6^2 \\
& - 22500a_2a_1^4a_6^4 - 900a_1a_5^4a_3^3 - 128a_1^4a_5^4a_3^2 + 2000a_1^2a_5^5a_3 \\
& - 1600a_1^3a_5^5a_2 + 2250a_1a_5^5a_2^2 - 2500a_1a_5^5a_4 + 2250a_4a_3^2a_5^4 + 825a_3^2a_2^2a_5^4 \\
& - 1600a_4^3a_5^3a_3 + 19800a_4a_2a_6a_5^3a_3 - 46656a_6^5 + 2808a_4a_2^2a_3^3a_1a_6^2 \\
& + 2808a_4^2a_2a_3^3a_5a_6 - 4536a_4^2a_2a_3^2a_1^2a_6^2 - 22896a_4a_2a_3^2a_6^2a_1a_5 \\
& + 356a_4a_2^2a_3^2a_1a_5^3 - 4536a_4a_2^2a_3^2a_6a_5^2 + 144a_4a_2^3a_3^2a_1^2a_6^2 \\
& + 18a_4a_2^3a_3a_1^2a_5^3 - 3456a_4a_2^2a_3a_1a_6^3 - 13040a_4^2a_2a_1^3a_5a_6^2
\end{aligned}
$$

$$-5760a_4^3a_2a_1a_3a_6^2 - 5760a_4a_2^3a_3a_5a_6^2 - 3456a_4^2a_2a_5a_3a_6^2$$
$$+1020a_4^2a_2a_1^4a_5^2a_6 - 746a_4^2a_2a_1^2a_5^3a_3 - 2050a_4a_2a_1^4a_5a_3a_6^2$$
$$-80a_4a_2a_3^2a_1^3a_5^3 - 630a_4a_2a_3^3a_1^3a_6^2 + 31968a_4a_2a_6^3a_1a_5$$
$$+8748a_4a_2a_6^2a_1^2a_5^2 + 19800a_4a_2a_3a_1^3a_6^3 - 2050a_4a_2a_1a_5^4a_3$$
$$-1584a_4^2a_2^2a_3^2a_6a_1a_5 - 2496a_4^2a_2^3a_3a_1a_6^2 + 24a_4^3a_2^2a_1^3a_5a_6$$
$$+320a_4^4a_2^2a_1a_3a_6 - 80a_4^3a_2^2a_1a_5^2a_3 + 320a_4^2a_2^4a_3a_5a_6 - 2496a_4^3a_2^2a_5a_3a_6$$
$$+15264a_4^2a_2^2a_6^2a_1a_5 - 5428a_4^2a_2^2a_6a_1^2a_5^2 + 560a_4^2a_2^2a_3a_1^3a_6^2$$
$$-96a_4^3a_2^3a_6a_1a_5 - 80a_4^2a_2^3a_3a_1^2a_6a_5 + 356a_4^2a_2a_3^2a_1^3a_5a_6$$
$$+10152a_4a_2^2a_3a_1^2a_6^2a_5 - 746a_4a_2^2a_3a_1^3a_6a_5^2 + 3272a_4a_2^3a_3a_1a_6a_5^2$$
$$+3272a_4^3a_2a_1^2a_5a_3a_6 + 9768a_4a_2a_6a_1^3a_5^3 - 72a_4a_2^4a_3a_5^3$$
$$-576a_4a_2^4a_3^2a_6^2 - 10560a_4a_2^3a_1^2a_6^3 + 160a_4^3a_2a_1a_5^3 - 10560a_4^3a_2a_5^2a_6$$
$$-900a_4^3a_2a_1^4a_6^2 - 576a_4^5a_2a_1^2a_6 + 144a_4^4a_2a_1^2a_5^2 - 576a_4^4a_2a_3^2a_6$$
$$+144a_4^3a_2a_3^2a_5^2 - 576a_4a_2^5a_5^2a_6 + 2000a_4a_2^2a_1^4a_6^3 - 128a_4^2a_2^4a_1^2a_6^2$$
$$+162a_4a_1^2a_3^4a_6^2 + 24a_4a_1^2a_3^3a_5^3 - 27540a_4a_1^2a_3^2a_6^3 + 825a_4^2a_1^4a_3^2a_6^2$$
$$+2250a_4^2a_1^5a_5a_6^2 - 120a_4^3a_1^3a_3a_6^2 + 144a_4^2a_1^4a_5^3a_3 - 1800a_4a_1^3a_6^3a_5$$
$$-1700a_4a_1^4a_6^2a_5^2 - 3750a_4a_1^5a_3a_6^3 + 160a_4a_1^3a_5^4a_3 - 1600a_4a_1^5a_6a_5^3$$
$$+248a_4^3a_1^2a_5^2a_6 + 24a_4^4a_1^2a_3^2a_6 - 6a_4^3a_1^2a_3^2a_5^2 + 144a_4^4a_1^3a_5a_6$$
$$+21384a_3^3a_1a_2a_6^3 + 21384a_3^3a_5a_4a_6^2 + 15552a_3^2a_6^3a_1a_5$$
$$-27540a_3^2a_6^2a_5^2a_2 - 9720a_3^2a_1^2a_2^2a_6^3 - 77760a_3a_1a_2a_6^4$$
$$+46656a_1a_4^2a_3a_6^3 + 46656a_3a_2^2a_5a_6^3 - 77760a_5a_4a_3a_6^3$$
$$+2250a_1^4a_5a_3a_6^3 - 1800a_6^2a_1a_5^3a_2 + 248a_1^2a_2^3a_5^2a_6^2 - 21888a_1a_2^3a_5a_6^3$$
$$+15600a_1^3a_2^2a_6^3a_5 - 21888a_5a_1a_4^3a_6^2 - 6318a_1a_5a_3^4a_6^2$$
$$+15417a_1^2a_5^2a_3^2a_6^2 + 560a_1^2a_5^4a_3^2a_2 + 144a_1^3a_5^4a_3a_2^2 + 2000a_1^5a_5^2a_3a_6^2$$
$$-900a_1^4a_5a_3^3a_6^2 - 630a_1a_5^4a_3a_2^3 + 1020a_1a_5^3a_4^2a_3^2 + 144a_1a_5^3a_2^4a_6$$
$$+2250a_6a_1a_5^4a_3 - 1700a_6a_1^2a_5^4a_2 - 120a_6a_3a_2^3a_5^3 + 15600a_6a_1a_4^2a_5^3$$
$$-9720a_6a_4^2a_3^2a_5^2 + 10152a_4^2a_2a_6a_1a_5^2a_3 - 13040a_4a_2^2a_6a_1a_5^3$$
$$+144a_4a_2^4a_1^2a_5^3a_6 - 640a_4a_2^4a_1a_5a_6^2 + 160a_4a_2^3a_1^3a_6^2a_5$$

$$-72a_4^4a_2a_1^3a_3a_6 + 18a_4^3a_2a_1^3a_3a_5^2 - 640a_4^4a_2a_5a_1a_6$$
$$-12330a_4a_1^2a_6a_5^3a_3 - 108a_4^2a_1^2a_3^3a_5a_6 + 1980a_4a_1^3a_3^2a_6^2a_5$$
$$-2412a_4a_1^2a_3^2a_6a_5^2a_2 + 16632a_4^2a_1^2a_5a_3a_6^2 - 630a_4^3a_1^4a_5a_3a_6$$
$$-682a_4^2a_1^3a_6a_5^2a_3 - 31320a_3a_1^2a_2a_6^3a_5 - 12330a_3a_1^3a_2a_6^2a_5^2$$
$$+16632a_3a_1a_2^2a_6^2a_5^2 - 31320a_6^2a_1a_4a_5^2a_3 + 3942a_1^2a_5a_3^3a_2a_6^2$$
$$+3942a_1a_5^2a_3^3a_4a_6 + 1020a_1^3a_5a_3^2a_2^2a_6^2 + 560a_1^4a_5^2a_3^2a_4a_6$$
$$+160a_1^4a_5^3a_3a_2a_6 - 4464a_1a_5a_3^2a_2^3a_6^3 - 4464a_1a_5a_4^3a_3^2a_6$$
$$+1980a_6a_3^2a_1a_5^3a_2 - 682a_6a_3a_1^2a_2^2a_5^3 + 3125a_5^6 + 16a_3^3a_2^3a_5^3$$
$$+108a_3^4a_2^3a_6^2 + 16a_3^4a_1^3a_5^3 + 108a_3^5a_1^3a_6^2 - 27a_2^4a_5^4a_1^2 + 256a_2^5a_1^2a_6^3$$
$$+5832a_1a_4^2a_3^3a_6^2 + 768a_1a_4^5a_3a_6 - 192a_1a_4^4a_5^2a_3 + 162a_3^4a_2a_5^2a_6$$
$$+24a_3^2a_2^4a_5^2a_6 + 16a_3^2a_2^3a_4^3a_6 + 2250a_3^2a_2a_1^4a_6^3 + 6912a_3a_2^4a_1a_6^3$$
$$-72a_3^4a_2a_1a_5^3 - 486a_3^5a_2a_1a_6^2 + 768a_3a_2^5a_5a_6^2 - 1600a_3a_2^3a_1^3a_6^3$$
$$-4a_3^2a_2^3a_4^2a_5^2 - 27a_3^4a_2^2a_1^2a_6^2 - 4a_3^3a_2^2a_1^2a_5^3 + 16a_3^3a_1^3a_4^3a_6$$
$$-4a_3^3a_1^3a_4^2a_5^2 + 5832a_5a_3^3a_2^2a_6^2 + 6912a_5a_4^4a_3a_6 - 1024a_4^6a_6$$
$$+256a_4^5a_5^2 + 108a_2^5a_5^4 - 1024a_2^6a_6^3 + 108a_3^5a_5^3 + 729a_3^6a_6^2$$
$$-72a_3^3a_2a_4^3a_1a_6 + 18a_3^3a_2a_4^2a_1a_5^2 - 108a_3^3a_2^2a_5^2a_6a_1 - 6a_3^2a_2^3a_5^2a_1^2a_6$$
$$-4a_3^2a_2^2a_4^3a_1^2a_6 + 324a_3^4a_2a_4a_6a_1a_5 - 72a_3^3a_2^3a_4a_5a_6$$
$$-192a_3a_2^4a_1^2a_5a_6^2 + 24a_3^3a_2a_1^3a_6a_5^2 + a_3^2a_2^2a_4^2a_1^2a_5^2 + 18a_3^3a_2^2a_4a_1^2a_6a_5$$
$$-72a_3^4a_1^3a_4a_6a_5 - 486a_4a_3^5a_5a_6 + 3125a_1^6a_6^4$$

$$b_1 = 6a_2$$
$$b_2 = 15a_2^2 + 3a_1a_3 - 6a_4$$
$$b_3 = 20a_2^3 + 15a_1a_2a_3 - 3a_3^2 + \left(-a_1^2 - 22a_2\right)a_4 - 11a_1a_5 + 66a_6$$
$$b_4 = 15a_2^4 + 30a_1a_2^2a_3 + \left(3a_1^2 - 12a_2\right)a_3^2 + \left(-3a_1^2a_2 - 28a_2^2 - 13a_1a_2\right)a_4$$
$$+ a_4^2 + \left(-3a_1^3 - 47a_1a_2 + 36a_3\right)a_5 + \left(58a_1^2 + 138a_2\right)a_6$$
$$b_5 = 6a_2^5 + 30a_1a_2^3a_3 + 12a_1^2a_2a_3^2 - 18a_2^2a_3^2 - 6a_1a_3^3 - 2a_1^2a_2^2a_4$$
$$- 12a_2^3a_4 - 2a_1^3a_3a_4 - 36a_1a_2a_3a_4 + 12a_3^2a_4 - 4a_1^2a_4^2 - 12a_2a_4^2$$
$$- 13a_1^3a_2a_5 - 78a_1a_2^2a_5 + 3a_1^2a_3a_5 + 96a_2a_3a_5 + 63a_1a_4a_5$$

$$- 123a_5^2 + 11a_1^4a_6 + 156a_1^2a_2a_6 + 84a_2^2a_6 - 57a_1a_3a_6 + 114a_4a_6$$

$$\begin{aligned}
b_6 = {} & a_2^6 + 15a_1a_2^4a_3 + 18a_1^2a_2^2a_3^2 - 12a_2^3a_3^2 + a_1^3a_3^3 - 18a_1a_2a_3^3 + 3a_3^4 \\
& + 2a_1^2a_2^3a_4 + 2a_2^4a_4 - 4a_1^3a_2a_3a_4 - 30a_1a_2^2a_3a_4 - 6a_1^2a_3^2a_4 \\
& + 20a_2a_3^2a_4 - a_1^4a_4^2 - 20a_1^2a_2a_4^2 - 26a_2^2a_4^2 + 10a_1a_3a_4^2 + 24a_4^3 \\
& - 22a_1^3a_2^2a_5 - 62a_1a_2^3a_5 - 2a_1^4a_3a_5 + 88a_2^2a_3a_5 + 46a_1a_3^2a_5 \\
& + 32a_1^3a_4a_5 + 140a_1a_2a_4a_5 - 138a_3a_4a_5 - 111a_1^2a_5^2 - 94a_2a_5^2 \\
& + 33a_1^4a_2a_6 + 156a_1^2a_2^2a_6 + 20a_2^3a_6 - 3a_1^3a_3a_6 - 228a_1a_2a_3a_6 \\
& + 138a_3^2a_6 + 113a_1^2a_4a_6 + 88a_2a_4a_6 - 43a_1a_5a_6 + 129a_6^2
\end{aligned}$$

$$\begin{aligned}
b_7 = {} & 3a_1a_2^5a_3 + 12a_1^2a_2^3a_3^2 - 3a_2^4a_3^2 + 3a_1^3a_2a_3^3 - 18a_1a_2^2a_3^3 - 3a_1^2a_3^4 \\
& + 6a_2a_3^4 + 3a_1^2a_2^4a_4 + 2a_2^5a_4 - 4a_1a_2^3a_3a_4 - a_1^4a_3^2a_4 \\
& - 14a_1^2a_2a_3^2a_4 + 4a_2^2a_3^2a_4 + 14a_1a_3^3a_4 - 4a_1^4a_2a_4^2 - 28a_1^2a_2^2a_4^2 \\
& - 12a_2^3a_4^2 - 2a_1^3a_3a_4^2 + 6a_1a_2a_3a_4^2 - 2a_3^2a_4^2 + 30a_1^2a_4^3 + 16a_2a_4^3 \\
& - 18a_1^3a_2^3a_5 - 23a_1a_2^4a_5 - 8a_1^4a_2a_3a_5 - 10a_1^2a_2^2a_3a_5 + 32a_2^3a_3a_5 \\
& + 17a_1^3a_3^2a_5 + 82a_1a_2a_3^2a_5 - 36a_3^3a_5 + 4a_1^5a_4a_5 + 72a_1^3a_2a_4a_5 \\
& + 90a_1a_2^2a_4a_5 - 72a_1^2a_3a_4a_5 - 82a_2a_3a_4a_5 - 76a_1a_4^2a_5 \\
& - 36a_1^4a_5^2 - 76a_1^2a_2a_5^2 - 44a_2^2a_5^2 + 10a_1a_3a_5^2 + 94a_4a_5^2 \\
& + 38a_1^4a_2^2a_6 + 80a_1^2a_2^3a_6 + 10a_2^4a_6 + 4a_1^5a_3a_6 - 60a_1^3a_2a_3a_6 \\
& - 230a_1a_2^2a_3a_6 + 52a_1^2a_3^2a_6 + 186a_2a_3^2a_6 + 48a_1^4a_4a_6 \\
& + 76a_1^2a_2a_4a_6 - 36a_2^2a_4a_6 + 34a_1a_3a_4a_6 + 80a_4^2a_6 - 88a_1^3a_5a_6 \\
& + 184a_1a_2a_5a_6 - 342a_3a_5a_6 + 74a_1^2a_6^2 + 132a_2a_6^2
\end{aligned}$$

$$\begin{aligned}
b_8 = {} & 3a_1^2a_2^4a_3^2 + 3a_1^3a_2^2a_3^3 - 6a_1a_2^3a_3^3 - 6a_1^2a_2a_3^4 + 3a_2^2a_3^4 + 3a_1a_3^5 \\
& + a_1^2a_2^5a_4 + 4a_1^3a_2^3a_3a_4 + 3a_1a_2^4a_3a_4 - a_1^4a_2a_3^2a_4 - 10a_1^2a_2^2a_3^2a_4 \\
& - 4a_2^3a_3^2a_4 + a_1^3a_3^3a_4 + 12a_1a_2a_3^3a_4 - 6a_3^4a_4 - 5a_1^4a_2^2a_4^2 - 12a_1^2a_2^3a_4^2 \\
& + a_2^4a_4^2 - a_1^5a_3a_4^2 - 12a_1^3a_2a_3a_4^2 - 2a_1a_2^2a_3a_4^2 + 13a_1^2a_3^2a_4^2 \\
& + 12a_2a_3^2a_4^2 + 10a_1^4a_4^3 + 28a_1^2a_2a_4^3 - 8a_2^2a_4^3 - 24a_1a_3a_4^3 + 16a_4^4 \\
& - 7a_1^3a_2^4a_5 - 3a_1a_2^5a_5 - 11a_1^4a_2^2a_3a_5 - 8a_1^2a_2^3a_3a_5 + 4a_2^4a_3a_5 \\
& + a_1^5a_3^2a_5 + 29a_1^3a_2a_3^2a_5 + 42a_1a_2^2a_3^2a_5 - 15a_1^2a_3^3a_5 - 24a_2a_3^3a_5
\end{aligned}$$

$$
\begin{aligned}
&+ 9a_1^5a_2a_4a_5 + 48a_1^3a_2^2a_4a_5 + 12a_1a_2^3a_4a_5 - 9a_1^4a_3a_4a_5 \\
&- 33a_1^2a_2a_3a_4a_5 - 6a_2^2a_3a_4a_5 - 9a_1a_3^2a_4a_5 \\
b_9 = {}& a_1^3a_2^3a_3^3 - 3a_1^2a_2^2a_3^4 + 3a_1a_2a_3^5 - a_3^6 + 2a_1^3a_2^4a_3a_4 + a_1^4a_2^2a_3^2a_4 \\
&-2a_1^2a_2^3a_3^2a_4 - 2a_1^3a_2a_3^3a_4 - 2a_1a_2^2a_3^3a_4 + a_1^2a_3^4a_4 + 2a_2a_3^4a_4 \\
&-2a_1^4a_2^3a_4^2 - 3a_1^5a_2a_3a_4^2 - 10a_1^3a_2^2a_3a_4^2 + 2a_1a_2^3a_3a_4^2 + 3a_1^4a_3^2a_4^2 \\
&+22a_1^2a_2a_3^2a_4^2 - 2a_2^2a_3^2a_4^2 - 10a_1a_3^3a_4^2 + a_1^6a_4^3 + 10a_1^4a_2a_4^3 \\
&-2a_1^2a_2^2a_4^3 - 10a_1^3a_3a_4^3 - 8a_1a_2a_3a_4^3 + 8a_3^2a_4^3 + 8a_1^2a_4^4 - a_1^3a_2^5a_5 \\
&-6a_1^4a_2^3a_3a_5 - a_1^2a_2^4a_3a_5 + a_1^5a_2a_3^2a_5 + 15a_1^3a_2^2a_3^2a_5 + 6a_1a_2^3a_3^2a_5 \\
&-a_1^4a_3^3a_5 - 10a_1^2a_2a_3^3a_5 - 4a_2^2a_3^3a_5 + a_1a_3^4a_5 + 6a_1^5a_2^2a_4a_5 \\
&+8a_1^3a_2^3a_4a_5 - a_1a_2^4a_4a_5 + 2a_1^4a_2a_3a_4a_5 - 10a_1^2a_2^2a_3a_4a_5 \\
&+2a_2^3a_3a_4a_5 - 7a_1^3a_3^2a_4a_5 + 10a_1a_2a_3^2a_4a_5 - 6a_3^3a_4a_5 - 13a_1^5a_4^2a_5 \\
&-6a_1^3a_2a_4^2a_5 + 4a_1a_2^2a_4^2a_5 - 10a_1^2a_3a_4^2a_5 - 8a_2a_3a_4^2a_5 \\
&-4a_1^6a_2a_5^2 - 4a_1^4a_2^2a_5^2 + 2a_1^2a_2^3a_5^2 - a_2^4a_5^2 + 4a_1^5a_3a_5^2 \\
&-11a_1^3a_2a_3a_5^2 - 20a_1a_2^2a_3a_5^2 + 14a_1^2a_3^2a_5^2 + 16a_2a_3^2a_5^2 + 19a_1^4a_4a_5^2 \\
&+6a_1^2a_2a_4a_5^2 + 6a_2^2a_4a_5^2 + 18a_1a_3a_4a_5^2 - 8a_4^2a_5^2 - 7a_1^3a_5^3 \\
&+6a_1a_2a_5^3 - 14a_3a_5^3 + 7a_1^4a_2^4a_6 + 4a_1^2a_2^5a_6 - 2a_1^5a_2^2a_3a_6 \\
&-28a_1^3a_2^3a_3a_6 - 9a_1a_2^4a_3a_6 + a_1^6a_3^2a_6 + 42a_1^2a_2^2a_3^2a_6 + 6a_2^3a_3^2a_6 \\
&+5a_1^3a_3^3a_6 - 30a_1a_2a_3^3a_6 + 12a_3^4a_6 + 10a_1^6a_2a_4a_6 - 4a_1^4a_2^2a_4a_6 \\
&-20a_1^2a_2^3a_4a_6 + 2a_2^4a_4a_6 - 2a_1^5a_3a_4a_6 - 10a_1^3a_2a_3a_4a_6 \\
&+30a_1a_2^2a_3a_4a_6 + 26a_1^2a_3^2a_4a_6 - 4a_2a_3^2a_4a_6 + 14a_1^4a_4^2a_6 \\
&+36a_1^2a_2a_4^2a_6 - 16a_2^2a_4^2a_6 - 56a_1a_3a_4^2a_6 + 32a_4^3a_6 - 4a_1^7a_5a_6 \\
&-6a_1^5a_2a_5a_6 + 44a_1^3a_2^2a_5a_6 + 16a_1a_2^3a_5a_6 - 14a_1^4a_3a_5a_6 \\
&-74a_1^2a_2a_3a_5a_6 - 26a_2^2a_3a_5a_6 + 4a_1a_3^2a_5a_6 - 36a_1^3a_4a_5a_6 \\
&-24a_1a_2a_4a_5a_6 + 24a_3a_4a_5a_6 + 30a_1^2a_5^2a_6 + 20a_2a_5^2a_6 - 6a_1^6a_6^2 \\
&+48a_1^4a_2a_6^2 - 26a_1^2a_2^2a_6^2 + 4a_2^3a_6^2 - 52a_1^3a_3a_6^2 + 72a_1a_2a_3a_6^2 \\
&+48a_3^2a_6^2 + 72a_1^2a_4a_6^2 - 16a_2a_4a_6^2 - 32a_1a_5a_6^2 + 64a_6^3 \\
b_{10} = {}& a_1^4a_2^3a_3^2a_4 - 3a_1^3a_2^2a_3^3a_4 + 3a_1^2a_2a_3^4a_4 - a_1a_3^5a_4 - 2a_1^5a_2^2a_3a_4^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4a_1^4a_2a_3^2a_4^2 + a_1^2a_2^2a_3^2a_4^2 - 2a_1^3a_3^3a_4^2 - 2a_1a_2a_3^3a_4^2 + a_3^4a_4^2 + a_1^6a_2a_4^3\\
&- a_1^5a_3a_4^3 - 2a_1^3a_2a_3a_4^3 + 2a_1^2a_3^2a_4^3 + a_1^4a_4^4 - a_1^4a_2^4a_3a_5 + 3a_1^3a_2^3a_3^2a_5\\
&- 3a_1^2a_2^2a_3^3a_5 + a_1a_2a_3^4a_5 + a_1^5a_2^3a_4a_5 + a_1^6a_2a_3a_4a_5 - a_1^4a_2^2a_3a_4a_5\\
&- a_1^2a_2^3a_3a_4a_5 - a_1^5a_3^2a_4a_5 + a_1^3a_2a_3^2a_4a_5 + 3a_1a_2^2a_3^2a_4a_5 - a_1^2a_3^3a_4a_5\\
&- 2a_2a_3^3a_4a_5 - a_1^7a_4^2a_5 - a_1^5a_2a_4^2a_5 + a_1^3a_2^2a_4^2a_5 - a_1^4a_3a_4^2a_5\\
&- 2a_1^2a_2a_3a_4^2a_5 - a_1^6a_2^2a_5^2 + a_1^4a_2^3a_5^2 + a_1^5a_2a_3a_5^2 - 5a_1^3a_2^2a_3a_5^2\\
&- a_1a_2^3a_3a_5^2 + 5a_1^2a_2a_3^2a_5^2 + a_2^2a_3^2a_5^2 - a_1a_3^3a_5^2 + 2a_1^6a_4a_5^2\\
&- a_1^4a_2a_4a_5^2 + 2a_1^2a_2^2a_4a_5^2 + 5a_1^3a_3a_4a_5^2 - 2a_1a_2a_3a_4a_5^2 + 2a_3^2a_4a_5^2\\
&- 2a_1^2a_4^2a_5^2 - a_1^5a_5^3 + a_1^3a_2a_5^3 + a_1a_2^2a_5^3 - 3a_1^2a_3a_5^3 - 2a_2a_3a_5^3\\
&+ a_5^4 + a_1^4a_2^5a_6 - a_1^5a_2^3a_3a_6 - 3a_1^3a_2^4a_3a_6 + 3a_1^4a_2^2a_3^2a_6\\
&+ 3a_1^2a_2^3a_3^2a_6 - 3a_1^3a_2a_5^3a_6 - a_1a_2^2a_3^3a_6 + a_1^2a_3^4a_6 + 3a_1^6a_2^2a_4a_6\\
&- 6a_1^4a_2^3a_4a_6 + a_1^2a_2^4a_4a_6 + a_1^7a_3a_4a_6 - 10a_1^5a_2a_3a_4a_6\\
&+ 15a_1^3a_2^2a_3a_4a_6 - 2a_1a_2^3a_3a_4a_6 + 10a_1^4a_3^2a_4a_6 - 14a_1^2a_2a_3^2a_4a_6\\
&+ 2a_2^2a_3^2a_4a_6 + 8a_1a_3^3a_4a_6 + 10a_1^4a_2a_4^2a_6 - 6a_1^2a_2^2a_4^2a_6\\
&- 16a_1^3a_3a_4^2a_6 + 8a_1a_2a_3a_4^2a_6
\end{aligned}
$$

$$c_1 = 3a_2$$

$$c_2 = 3a_2^2 + 3a_1a_3 - 6a_4$$

$$c_3 = a_2^3 + 6a_1a_2a_3 + 3a_3^2 + 3a_1^2a_4 - 20a_2a_4 - 7a_1a_5 + 42a_6$$

$$
\begin{aligned}
c_4 = {}& 3a_1a_2^2a_3 + \left(3a_1^2 + 6a_2\right)a_3^2 + \left(6a_1^2 - 22a_2\right)a_2a_4 - 10a_1a_3a_4 + 7a_4^2\\
&+ \left(3a_1^3 - 26a_1a_2 + 9a_3\right)a_5 + \left(-8a_1^2 + 120a_2\right)a_6
\end{aligned}
$$

$$
\begin{aligned}
c_5 = {}& \left(3a_1^2 + 3a_2\right)a_2a_3^2 + 6a_1a_3^3 + \left(3a_1^2 - 8a_2\right)a_2^2a_4\\
&+ \left(6a_1^3 - 24a_1a_2 - 12a_3\right)a_3a_4 + \left(-12a_1^2 + 35a_2\right)a_4^2 + 6a_1^3a_2a_5\\
&- 31a_1a_2^2a_5 + \left(-13a_1^2 + 15a_2\right)a_3a_5 + 23a_1a_4a_5 + 21a_5^2\\
&+ \left(3a_1^4 + -32a_1^2a_2 + 126a_2^2 + 111a_1a_3 - 222a_4\right)a_6
\end{aligned}
$$

$$
\begin{aligned}
c_6 = {}& a_1^3a_3^3 + 6a_1a_2a_3^3 + 3a_3^4 + 6a_1^3a_2a_3a_4 - 14a_1a_2^2a_3a_4 + 4a_1^2a_5^2a_4\\
&- 28a_2a_3^2a_4 + 3a_1^4a_4^2 - 28a_1^2a_2a_4^2 + 50a_2^2a_4^2 - 5a_1a_3a_4^2 + 8a_4^3\\
&+ 3a_1^3a_2^2a_5 - 12a_1a_2^3a_5 + 6a_1^4a_3a_5 - 36a_1^2a_2a_3a_5 + 4a_2^2a_3a_5
\end{aligned}
$$

$$+ a_1a_3^2a_5 - 26a_1^3a_4a_5 + 103a_1a_2a_4a_5 - 21a_3a_4a_5 + 23a_1^2a_5^2 + 17a_2a_5^2$$
$$+ 6a_1^4a_2a_6 - 40a_1^2a_2^2a_6 + 56a_2^3a_6 - 16a_1^3a_3a_6 + 225a_1a_2a_3a_6$$
$$+ 57a_3^2a_6 + 91a_1^2a_4a_6 - 602a_2a_4a_6 - 151a_1a_5a_6 + 453a_6^2$$
$$c_7 = 3a_1^2a_3^4 + 3a_2a_3^4 + 3a_1^4a_3^2a_4 + 2a_1^2a_2a_3^2a_4 - 16a_2^2a_3^2a_4 - 8a_1a_3^3a_4$$
$$+ 3a_1^4a_2a_4^2 - 16a_1^2a_2^2a_4^2 + 22a_2^3a_4^2 - 8a_1^3a_3a_4^2 + 6a_1a_2a_3a_4^2$$
$$+ 5a_3^2a_4^2 + 5a_1^2a_4^3 - 4a_2a_4^3 + 6a_1^4a_2a_3a_5 - 23a_1^2a_2^2a_3a_5 - 2a_2^3a_3a_5$$
$$+ a_1^3a_3^2a_5 - 28a_1a_2a_3^2a_5 + 9a_3^3a_5 + 6a_1^5a_4a_5 - 66a_1^3a_2a_4a_5$$
$$+ 144a_1a_2^2a_4a_5 + 12a_1^2a_3a_4a_5 - 20a_2a_3a_4a_5 + 16a_1a_4^2a_5$$
$$- 14a_1^4a_5^2 + 91a_1^2a_2a_5^2 - 49a_2^2a_5^2 + 20a_1a_3a_5^2 - 109a_4a_5^2$$
$$+ 3a_1^4a_2^2a_6 - 16a_1^2a_2^3a_6 + 8a_2^4a_6 + 6a_1^5a_3a_6 - 48a_1^3a_2a_3a_6$$
$$+ 134a_1a_2^2a_3a_6 + 95a_1^2a_3^2a_6 + 120a_2a_3^2a_6 - 28a_1^4a_4a_6$$
$$+ 263a_1^2a_2a_4a_6 - 588a_2^2a_4a_6 - 409a_1a_3a_4a_6 + 340a_4^2a_6$$
$$+ 88a_1^3a_5a_6 - 529a_1a_2a_5a_6 + 207a_3a_5a_6 - 149a_1^2a_6^2 + 1173a_2a_6^2$$
$$c_8 = 3a_1a_3^5 + 8a_1^3a_3^3a_4 - 12a_1a_2a_3^3a_4 - 6a_3^4a_4 + 3a_1^5a_3a_4^2 - 12a_1^3a_2a_3a_4^2$$
$$+ 12a_1a_2^2a_3a_4^2 - 23a_1^2a_3^2a_4^2 + 28a_2a_3^2a_4^2 - 6a_1^4a_4^3 + 28a_1^2a_2a_4^3$$
$$- 40a_2^2a_4^3 + 20a_1a_3a_4^3 - 17a_4^4 + 3a_1^5a_3^2a_5 - 4a_1^3a_2a_3^2a_5 - 28a_1a_2^2a_3^2a_5$$
$$- 5a_1^2a_3^3a_5 + 6a_1^5a_2a_4a_5 - 40a_1^3a_2^2a_4a_5 + 64a_1a_2^3a_4a_5 - 20a_1^4a_3a_4a_5$$
$$+ 38a_1^2a_2a_3a_4a_5 + 32a_2^2a_3a_4a_5 - 2a_1a_3^2a_4a_5 + 22a_1^3a_4^2a_5$$
$$- 24a_1a_2a_4^2a_5 + 42a_3a_4^2a_5 + 3a_1^6a_5^2 - 38a_1^4a_2a_5^2 + 122a_1^2a_2^2a_5^2$$
$$- 72a_2^3a_5^2 + 27a_1^3a_3a_5^2 - 2a_1a_2a_3a_5^2 + 9a_3^2a_5^2 + 18a_1^2a_4a_5^2$$
$$- 236a_2a_4a_5^2 - 48a_1a_5^3 + 6a_1^5a_2a_3a_6 - 32a_1^3a_2^2a_3a_6 + 16a_1a_2^3a_3a_6$$
$$- 2a_1^4a_3^2a_6 + 74a_1^2a_2a_3^2a_6 + 72a_2^2a_3^2a_6 + 138a_1a_3^3a_6 + 6a_1^6a_4a_6$$
$$- 76a_1^4a_2a_4a_6 + 272a_1^2a_2^2a_4a_6 - 224a_2^3a_4a_6 + 146a_1^3a_3a_4a_6$$
$$- 768a_1a_2a_3a_4a_6 - 276a_3^2a_4a_6 - 220a_1^2a_4^2a_6 + 920a_2a_4^2a_6$$
$$- 30a_1^5a_5a_6 + 312a_1^3a_2a_5a_6 - 768a_1a_2^2a_5a_6 - 370a_1^2a_3a_5a_6$$
$$+ 648a_2a_3a_5a_6 + 380a_1a_4a_5a_6 + 288a_5^2a_6 + 22a_1^4a_6^2 - 344a_1^2a_2a_6^2$$
$$c_9 = a_3^6 + 7a_1^2a_3^4a_4 - 8a_2a_3^4a_4 + 7a_1^4a_3^2a_4^2 - 25a_1^2a_2a_3^2a_4^2 + 24a_2^2a_3^2a_4^2$$

$$-14a_1a_3^3a_4^2 + a_1^6a_4^3 - 8a_1^4a_2a_4^3 + 24a_1^2a_2^2a_4^3 - 28a_2^3a_4^3 - 14a_1^3a_3a_4^3$$
$$+30a_1a_2a_3a_4^3 + 14a_3^2a_4^3 + 14a_1^2a_4^4 - 39a_2a_4^4 + 7a_1^4a_3^3a_5 - 23a_1^2a_2a_3^3a_5$$
$$+8a_2^2a_3^3a_5 - a_1a_3^4a_5 + 6a_1^6a_3a_4a_5 - 34a_1^4a_2a_3a_4a_5 + 34a_1^2a_2^2a_3a_4a_5$$
$$+30a_2^3a_3a_4a_5 - 39a_1^3a_3^2a_4a_5 + 77a_1a_2a_3^2a_4a_5 - 21a_3^3a_4a_5 - 19a_1^5a_4^2a_5$$
$$+102a_1^3a_2a_4^2a_5 - 134a_1a_2^2a_4^2a_5 + 37a_1^2a_3a_4^2a_5 + 88a_2a_3a_4^2a_5$$
$$-46a_1a_4^3a_5 + 3a_1^6a_2a_5^2 - 24a_1^4a_2^2a_5^2 + 54a_1^2a_2^3a_5^2 - 26a_2^4a_5^2$$
$$-12a_1^5a_3a_5^2 + 58a_1^3a_2a_3a_5^2 - 27a_1a_2^2a_3a_5^2 + 42a_1^2a_3^2a_5^2 - 22a_2a_3^2a_5^2$$
$$+37a_1^4a_4a_5^2 - 68a_1^2a_2a_4a_5^2 - 128a_2^2a_4a_5^2 - 136a_1a_3a_4a_5^2 + 144a_4^2a_5^2$$
$$-35a_1^3a_5^3 + 14a_1a_2a_5^3 - 49a_3a_5^3 + 3a_1^6a_3^2a_6 - 10a_1^4a_2a_3^2a_6$$
$$-22a_1^2a_2^2a_3^2a_6 + 6a_2^3a_3^2a_6 + 29a_1^3a_3^3a_6 + 141a_1a_2a_3^3a_6 + 69a_3^4a_6$$
$$+6a_1^6a_2a_4a_6 - 48a_1^4a_2^2a_4a_6 + 96a_1^2a_2^3a_4a_6 - 16a_2^4a_4a_6$$
$$-24a_1^5a_3a_4a_6 + 210a_1^3a_2a_3a_4a_6 - 338a_1a_2^2a_3a_4a_6 - 87a_1^2a_3^2a_4a_6$$
$$-650a_2a_3^2a_4a_6 + 66a_1^4a_4^2a_6 - 540a_1^2a_2a_4^2a_6 + 964a_2^2a_4^2a_6$$
$$+368a_1a_3a_4^2a_6 - 300a_4^3a_6 + 6a_1^7a_5a_6 - 86a_1^5a_2a_5a_6 + 388a_1^3a_2^2a_5a_6$$
$$-524a_1a_2^3a_5a_6 + 148a_1^4a_3a_5a_6 - 920a_1^2a_2a_3a_5a_6 + 722a_2^2a_3a_5a_6$$
$$+77a_1a_3^2a_5a_6 - 212a_1^3a_4a_5a_6 + 904a_1a_2a_4a_5a_6 + 246a_3a_4a_5a_6$$
$$+224a_1^2a_5^2a_6 + 112a_2a_5^2a_6 - 16a_1^6a_6^2 + 146a_1^4a_2a_6^2 - 410a_1^2a_2^2a_6^2$$
$$+128a_2^3a_6^2 - 453a_1^3a_3a_6^2 + 2514a_1a_2a_3a_6^2 - 600a_3^2a_6^2 + 530a_1^2a_4a_6^2$$
$$-3428a_2a_4a_6^2 - 616a_1a_5a_6^2 + 1232a_6^3$$

$$c_{10} = 2a_1a_3^5a_4 + 5a_1^3a_3^3a_4^2 - 8a_1a_2a_3^3a_4^2 - 2a_3^4a_4^2 + 2a_1^5a_3a_4^3 - 8a_1^3a_2a_3a_4^3$$
$$+6a_1a_2^2a_3a_4^3 - 8a_1^2a_3^2a_4^3 + 6a_2a_3^2a_4^3 - 2a_1^4a_4^4 + 6a_1^2a_2a_4^4$$
$$-5a_2^2a_4^4 + a_1a_3a_4^4 + 2a_4^5 + 6a_1^3a_3^4a_5 - 20a_1a_2a_3^4a_5 + 12a_1^5a_3^2a_4a_5$$
$$-69a_1^3a_2a_3^2a_4a_5 + 93a_1a_2^2a_3^2a_4a_5 - 31a_1^2a_3^3a_4a_5 + 13a_2a_3^3a_4a_5$$
$$+3a_1^7a_4^2a_5 - 28a_1^5a_2a_4^2a_5 + 88a_1^3a_2^2a_4^2a_5 - 96a_1a_2^3a_4^2a_5$$
$$-40a_1^4a_3a_4^2a_5 + 128a_1^2a_2a_3a_4^2a_5 - 12a_2^2a_3a_4^2a_5 + 69a_1a_3^2a_4^2a_5$$
$$+51a_1^3a_4^3a_5 - 130a_1a_2a_4^3a_5 - 18a_3a_4^3a_5 + 3a_1^7a_3a_5^2 - 22a_1^5a_2a_3a_5^2$$
$$+41a_1^3a_2^2a_3a_5^2 - 7a_1a_2^3a_3a_5^2 - 13a_1^4a_3^2a_5^2 + 83a_1^2a_2a_3^2a_5^2 - 56a_2^2a_3^2a_5^2$$

$$
\begin{aligned}
&+20a_1a_3^3a_5^2 - 20a_1^6a_4a_5^2 + 140a_1^4a_2a_4a_5^2 - 251a_1^2a_2^2a_4a_5^2 \\
&+54a_2^3a_4a_5^2 + 86a_1^3a_3a_4a_5^2 - 224a_1a_2a_3a_4a_5^2 - 121a_3^2a_4a_5^2 \\
&-201a_1^2a_4^2a_5^2 + 372a_2a_4^2a_5^2 + 19a_1^5a_5^3 - 120a_1^3a_2a_5^3 \\
&+169a_1a_2^2a_5^3 - 126a_1^2a_3a_5^3 + 4a_2a_3a_5^3 + 381a_1a_4a_5^3 - 353a_5^4 \\
&+6a_1^5a_3^3a_6 - 27a_1^3a_2a_3^3a_6 - 7a_1a_2^2a_3^3a_6 + 79a_1^2a_3^4a_6 + 81a_2a_3^4a_6 \\
&+6a_1^7a_3a_4a_6 - 44a_1^5a_2a_3a_4a_6 + 64a_1^3a_2^2a_3a_4a_6 + 40a_1a_2^3a_3a_4a_6 \\
&+43a_1^4a_3^2a_4a_6 + 46a_1^2a_2a_3^2a_4a_6 - 418a_2^2a_3^2a_4a_6 - 313a_1a_3^3a_4a_6 \\
&-20a_1^6a_4^2a_6 + 172a_1^4a_2a_4^2a_6 - 456a_1^2a_2^2a_4^2a_6 + 408a_2^3a_4^2a_6 \\
&-261a_1^3a_3a_4^2a_6 + 626a_1a_2a_3a_4^2a_6 + 232a_3^2a_4^2a_6 + 260a_1^2a_4^3a_6 \\
&-636a_2a_4^3a_6 + 6a_1^7a_2a_5a_6 - 56a_1^5a_2^2a_5a_6 + 160a_1^3a_2^3a_5a_6 \\
&-128a_1a_2^4a_5a_6 - 28a_1^6a_3a_5a_6 + 282a_1^4a_2a_3a_5a_6 - 706a_1^2a_2^2a_3a_5a_6 \\
&+328a_2^3a_3a_5a_6 - 109a_1^3a_3^2a_5a_6 - 136a_1a_2a_3^2a_5a_6 + 243a_3^3a_5a_6 \\
&+128a_1^5a_4a_5a_6 - 882a_1^3a_2a_4a_5a_6 + 1316a_1a_2^2a_4a_5a_6 \\
&+718a_1^2a_3a_4a_5a_6 + 56a_2a_3a_4a_5a_6 - 870a_1a_4^2a_5a_6 - 180a_1^4a_5^2a_6 \\
&+1058a_1^2a_2a_5^2a_6 - 1030a_2^2a_5^2a_6 + 55a_1a_3a_5^2a_6
\end{aligned}
$$

$$
\begin{aligned}
c_{11} = {}& a_1^2a_3^4a_4^2 + 2a_2a_3^4a_4^2 + a_1^4a_3^2a_4^3 + 2a_1^2a_2a_3^2a_4^3 - 12a_2^2a_3^2a_4^3 + 2a_1a_3^3a_4^3 \\
&+2a_1^4a_2a_4^4 - 12a_1^2a_2^2a_4^4 + 17a_2^3a_4^4 + 2a_1^3a_3a_4^4 - 12a_1a_2a_3a_4^4 - 9a_3^2a_4^4 \\
&-9a_1^2a_4^5 + 28a_2a_4^5 + 2a_1^2a_3^5a_5 - 6a_2a_3^5a_5 + 8a_1^4a_3^3a_4a_5 \\
&-37a_1^2a_2a_3^3a_4a_5 + 38a_2^2a_3^3a_4a_5 - 25a_1a_3^4a_4a_5 + 5a_1^6a_3a_4^2a_5 \\
&-36a_1^4a_2a_3a_4^2a_5 + 79a_1^2a_2^2a_3a_4^2a_5 - 54a_2^3a_3a_4^2a_5 - 49a_1^3a_3^2a_4^2a_5 \\
&+123a_1a_2a_3^2a_4^2a_5 + 48a_3^3a_4^2a_5 - 10a_1^5a_4^3a_5 + 47a_1^3a_2a_4^3a_5 \\
&-44a_1a_2^2a_4^3a_5 + 82a_1^2a_3a_4^3a_5 - 138a_2a_3a_4^3a_5 - 11a_1a_4^4a_5 + 5a_1^6a_3^2a_5^2 \\
&-37a_1^4a_2a_3^2a_5^2 + 73a_1^2a_2^2a_3^2a_5^2 - 25a_2^3a_3^2a_5^2 + 25a_1a_2a_3^3a_5^2 + 6a_3^4a_5^2 \\
&+3a_1^8a_4a_5^2 - 32a_1^6a_2a_4a_5^2 + 119a_1^4a_2^2a_4a_5^2 - 169a_1^2a_2^3a_4a_5^2 \\
&+48a_2^4a_4a_5^2 - 34a_1^5a_3a_4a_5^2 + 206a_1^3a_2a_3a_4a_5^2 - 242a_1a_2^2a_3a_4a_5^2 \\
&+53a_1^2a_3^2a_4a_5^2 - 179a_2a_3^2a_4a_5^2 + 102a_1^4a_4^2a_5^2 - 477a_1^2a_2a_4^2a_5^2 \\
&+372a_2^2a_4^2a_5^2 - 58a_1a_3a_4^2a_5^2 - 48a_4^3a_5^2 - 7a_1^7a_5^3 + 64a_1^5a_2a_5^3
\end{aligned}
$$

$$
\begin{aligned}
&-191a_1^3a_2^2a_5^3 + 195a_1a_2^3a_5^3 + 17a_1^4a_3a_5^3 - 148a_1^2a_2a_3a_5^3 + 125a_2^2a_3a_5^3\\
&-181a_1a_3^2a_5^3 - 244a_1^3a_4a_5^3 + 905a_1a_2a_4a_5^3 + 378a_3a_4a_5^3 + 336a_1^2a_5^4\\
&-1153a_2a_5^4 + 6a_1^4a_3^4a_6 - 29a_1^2a_2a_3^4a_6 + 6a_2^2a_3^4a_6 + 81a_1a_3^5a_6\\
&+10a_1^6a_3^2a_4a_6 - 77a_1^4a_2a_3^2a_4a_6 + 143a_1^2a_2^2a_3^2a_4a_6 - 32a_2^3a_3^2a_4a_6\\
&+132a_1^3a_3^3a_4a_6 - 309a_1a_2a_3^3a_4a_6 - 162a_3^4a_4a_6 + 3a_1^8a_4^2a_6\\
&-32a_1^6a_2a_4^2a_6 + 108a_1^4a_2^2a_4^2a_6 - 120a_1^2a_2^3a_4^2a_6 + 24a_2^4a_4^2a_6\\
&+26a_1^5a_3a_4^2a_6 - 141a_1^3a_2a_3a_4^2a_6 + 182a_1a_2^2a_3a_4^2a_6 - 338a_1^2a_3^2a_4^2a_6\\
&+644a_2a_3^2a_4^2a_6 - 88a_1^4a_4^3a_6 + 524a_1^2a_2a_4^3a_6 - 672a_2^2a_4^3a_6\\
&-20a_1a_3a_4^3a_6 + 258a_4^4a_6 + 6a_1^8a_3a_5a_6 - 54a_1^6a_2a_3a_5a_6\\
&+144a_1^4a_2^2a_3a_5a_6 - 112a_1^2a_2^3a_3a_5a_6 + 56a_2^4a_3a_5a_6 + 42a_1^5a_3^2a_5a_6\\
&-45a_1^3a_2a_3^2a_5a_6 - 355a_1a_2^2a_3^2a_5a_6 - 184a_1^2a_3^3a_5a_6 + 291a_2a_3^3a_5a_6\\
&-42a_1^7a_4a_5a_6 + 414a_1^5a_2a_4a_5a_6 - 1254a_1^3a_2^2a_4a_5a_6\\
&+1136a_1a_2^3a_4a_5a_6 - 294a_1^4a_3a_4a_5a_6 + 918a_1^2a_2a_3a_4a_5a_6\\
&-98a_2^2a_3a_4a_5a_6 + 1088a_1a_3^2a_4a_5a_6 + 396a_1^3a_4^2a_5a_6\\
&-1334a_1a_2a_4^2a_5a_6 - 1488a_3a_4^2a_5a_6 + 69a_1^6a_5^2a_6 - 678a_1^4a_2a_5^2a_6\\
&+1950a_1^2a_2^2a_5^2a_6 - 1670a_2^3a_5^2a_6 + 434a_1^3a_3a_5^2a_6 - 721a_1a_2a_3a_5^2a_6\\
&-48a_3^2a_5^2a_6 - 916a_1^2a_4a_5^2a_6 + 2282a_2a_4a_5^2a_6 - 32a_1a_5^3a_6 + 3a_1^8a_2a_6^2\\
&-32a_1^6a_2^2a_6^2 + 112a_1^4a_2^3a_6^2 - 96a_1^2a_2^4a_6^2 - 112a_2^5a_6^2 - 16a_1^7a_3a_6^2\\
&+168a_1^5a_2a_3a_6^2 - 592a_1^3a_2^2a_3a_6^2 + 648a_1a_2^3a_3a_6^2 - 349a_1^4a_3^2a_6^2\\
&+1567a_1^2a_2a_3^2a_6^2 - 534a_2^2a_3^2a_6^2 - 648a_1a_3^3a_6^2 + 29a_1^6a_4a_6^2\\
&-201a_1^4a_2a_4a_6^2 + 320a_1^2a_2^2a_4a_6^2 + 128a_2^3a_4a_6^2 + 972a_1^3a_3a_4a_6^2\\
&-3786a_1a_2a_3a_4a_6^2 + 1296a_3^2a_4a_6^2 - 572a_1^2a_4^2a_6^2 + 2414a_2a_4^2a_6^2\\
&+3a_1^5a_5a_6^2 + 163a_1^3a_2a_5a_6^2 - 308a_1a_2^2a_5a_6^2 - 592a_1^2a_3a_5a_6^2\\
&-1068a_2a_3a_5a_6^2 + 1232a_1a_4a_5a_6^2 + 96a_5^2a_6^2 + 10a_1^4a_6^3 - 782a_1^2a_2a_6^3\\
&+2040a_2^2a_6^3 + 1296a_1a_3a_6^3 - 2592a_4a_6^3
\end{aligned}
$$

$$
\begin{aligned}
c_{12} = {}& 2a_1a_2a_3^3a_4^3 + 2a_3^4a_4^3 + 2a_1^3a_2a_3a_4^4 - 7a_1a_2^2a_3a_4^4 + 6a_1^2a_3^2a_4^4\\
&-14a_2a_3^2a_4^4 + 2a_1^4a_4^5 - 14a_1^2a_2a_4^5 + 22a_2^2a_4^5 - 10a_1a_3a_4^5\\
&+9a_4^6 + 2a_1^3a_3^4a_4a_5 - 5a_1a_2a_3^4a_4a_5 - 9a_3^5a_4a_5 + 2a_1^5a_3^2a_4^2a_5\\
&-8a_1^3a_2a_3^2a_4^2a_5 + 6a_1a_2^2a_3^2a_4^2a_5 - 37a_1^2a_3^3a_4^2a_5 + 62a_2a_3^3a_4^2a_5
\end{aligned}
$$

$$+4a_1^5a_2a_4^3a_5 - 28a_1^3a_2^2a_4^3a_5 + 48a_1a_2^3a_4^3a_5 - 16a_1^4a_3a_4^3a_5$$
$$+74a_1^2a_2a_3a_4^3a_5 - 88a_2^2a_3a_4^3a_5 + 56a_1a_3^2a_4^3a_5 - 7a_1^3a_4^4a_5$$
$$+50a_1a_2a_4^4a_5 - 51a_3a_4^4a_5 + 3a_1^5a_3^3a_5^2 - 20a_1^3a_2a_3^3a_5^2 + 32a_1a_2^2a_3^3a_5^2$$
$$+9a_2a_3^4a_5^2 + 4a_1^7a_3a_4a_5^2 - 38a_1^5a_2a_3a_4a_5^2 + 122a_1^3a_2^2a_3a_4a_5^2$$
$$-140a_1a_2^3a_3a_4a_5^2 - 32a_1^4a_3^2a_4a_5^2 + 172a_1^2a_2a_3^2a_4a_5^2 - 112a_2^2a_3^2a_4a_5^2$$
$$-4a_1a_3^3a_4a_5^2 - 17a_1^6a_4^2a_5^2 + 126a_1^4a_2a_4^2a_5^2 - 256a_1^2a_2^2a_4^2a_5^2$$
$$+108a_2^3a_4^2a_5^2 + 111a_1^3a_3a_4^2a_5^2 - 412a_1a_2a_3a_4^2a_5^2 + 55a_3^2a_4^2a_5^2$$
$$-24a_1^2a_4^3a_5^2 - 104a_2a_4^3a_5^2 + a_1^9a_5^3 - 12a_1^7a_2a_5^3 + 54a_1^5a_2^2a_5^3$$
$$-106a_1^3a_2^3a_5^3 + 72a_1a_2^4a_5^3 - 9a_1^6a_3a_5^3 + 66a_1^4a_2a_3a_5^3 - 136a_1^2a_2^2a_3a_5^3$$
$$+144a_2^3a_3a_5^3 - 19a_1^3a_3^2a_5^3 - 154a_1a_2a_3^2a_5^3 - 90a_3^3a_5^3 + 80a_1^5a_4a_5^3$$
$$-554a_1^3a_2a_4a_5^3 + 878a_1a_2^2a_4a_5^3 - 68a_1^2a_3a_4a_5^3 + 872a_2a_3a_4a_5^3$$
$$+78a_1a_4^2a_5^3 - 116a_1^4a_5^4 + 856a_1^2a_2a_5^4 - 1440a_2^2a_5^4 - 157a_1a_3a_5^4$$
$$-94a_4a_5^4 + 2a_1^3a_3^5a_6 - 9a_1a_2a_3^5a_6 + 27a_3^6a_6 + 8a_1^5a_3^3a_4a_6$$
$$-52a_1^3a_2a_3^3a_4a_6 + 68a_1a_2^2a_3^3a_4a_6 + 111a_1^2a_3^4a_4a_6 - 198a_2a_3^4a_4a_6$$
$$+4a_1^7a_3a_4^2a_6 - 44a_1^5a_2a_3a_4^2a_6 + 144a_1^3a_2^2a_3a_4^2a_6 - 136a_1a_2^3a_3a_4^2a_6$$
$$+56a_1^4a_3^2a_4^2a_6 - 334a_1^2a_2a_3^2a_4^2a_6 + 460a_2^2a_3^2a_4^2a_6 - 162a_1a_3^3a_4^2a_6$$
$$+8a_1^6a_4^3a_6 - 92a_1^4a_2a_4^3a_6 + 336a_1^2a_2^2a_4^3a_6 - 368a_2^3a_4^3a_6 - 18a_1^3a_3a_4^3a_6$$
$$+40a_1a_2a_3a_4^3a_6 + 56a_3^2a_4^3a_6 - 72a_1^2a_4^4a_6 + 320a_2a_4^4a_6 + 8a_1^7a_3^2a_5a_6$$
$$-70a_1^5a_2a_3^2a_5a_6 + 166a_1^3a_2^2a_3^2a_5a_6 - 64a_1a_2^3a_3^2a_5a_6 + 106a_1^4a_3^3a_5a_6$$
$$-376a_1^2a_2a_3^3a_5a_6 - 9a_1a_3^4a_5a_6 + 6a_1^9a_4a_5a_6 - 72a_1^7a_2a_4a_5a_6$$
$$+304a_1^5a_2^2a_4a_5a_6 - 528a_1^3a_2^3a_4a_5a_6 + 320a_1a_2^4a_4a_5a_6$$
$$+42a_1^6a_3a_4a_5a_6 - 152a_1^4a_2a_3a_4a_5a_6 - 88a_1^2a_2^2a_3a_4a_5a_6$$
$$+224a_2^3a_3a_4a_5a_6 - 432a_1^3a_3^2a_4a_5a_6 + 1784a_1a_2a_3^2a_4a_5a_6$$
$$+234a_3^3a_4a_5a_6 - 90a_1^5a_4^2a_5a_6 + 508a_1^3a_2a_4^2a_5a_6 - 680a_1a_2^2a_4^2a_5a_6$$
$$+454a_1^2a_3a_4^2a_5a_6 - 2264a_2a_3a_4^2a_5a_6 + 148a_1a_4^3a_5a_6 - 22a_1^8a_5^2a_6$$
$$+256a_1^6a_2a_5^2a_6 - 1058a_1^4a_2^2a_5^2a_6 + 1792a_1^2a_2^3a_5^2a_6 - 1008a_2^4a_5^2a_6$$
$$-178a_1^5a_3a_5^2a_6 + 998a_1^3a_2a_3a_5^2a_6 - 1430a_1a_2^2a_3a_5^2a_6 + 592a_1^2a_3^2a_5^2a_6$$
$$-612a_2a_3^2a_5^2a_6 + 322a_1^4a_4a_5^2a_6 - 1764a_1^2a_2a_4a_5^2a_6 + 2764a_2^2a_4a_5^2a_6$$

$$
\begin{aligned}
& -1244a_1a_3a_4a_5^2a_6 + 284a_4^2a_5^2a_6 - 152a_1^3a_5^3a_6 + 144a_1a_2a_5^3a_6 \\
& +1224a_3a_5^3a_6 + 3a_1^9a_3a_6^2 - 32a_1^7a_2a_3a_6^2 + 112a_1^5a_2^2a_3a_6^2 \\
& -96a_1^3a_2^3a_3a_6^2 - 112a_1a_2^4a_3a_6^2 + 28a_1^6a_3^2a_6^2 - 278a_1^4a_2a_3^2a_6^2 \\
& +664a_1^2a_2^2a_3^2a_6^2 - 144a_2^3a_3^2a_6^2 - 52a_1^3a_3^3a_6^2 + 234a_1a_2a_3^3a_6^2 - 324a_3^4a_6^2 \\
& -22a_1^8a_4a_6^2 + 224a_1^6a_2a_4a_6^2 - 688a_1^4a_2^2a_4a_6^2 + 416a_1^2a_2^3a_4a_6^2 \\
& +608a_2^4a_4a_6^2 - 338a_1^5a_3a_4a_6^2 + 2308a_1^3a_2a_3a_4a_6^2 - 3584a_1a_2^2a_3a_4a_6^2 \\
& -942a_1^2a_3^2a_4a_6^2 + 1260a_2a_3^2a_4a_6^2 + 228a_1^4a_4^2a_6^2 - 1220a_1^2a_2a_4^2a_6^2 \\
& +1040a_2^2a_4^2a_6^2 + 1782a_1a_3a_4^2a_6^2 - 1012a_4^3a_6^2 + 26a_1^7a_5a_6^2 \\
& -318a_1^5a_2a_5a_6^2 + 1160a_1^3a_2^2a_5a_6^2 - 1152a_1a_2^3a_5a_6^2 + 560a_1^4a_3a_5a_6^2 \\
& -2996a_1^2a_2a_3a_5a_6^2 + 2592a_2^2a_3a_5a_6^2 - 252a_1a_3^2a_5a_6^2 - 830a_1^3a_4a_5a_6^2 \\
& +3384a_1a_2a_4a_5a_6^2 - 792a_3a_4a_5a_6^2 + 944a_1^2a_5^2a_6^2 - 2880a_2a_5^2a_6^2 \\
& +170a_1^6a_6^3 - 1216a_1^4a_2a_6^3 + 2384a_1^2a_2^2a_6^3
\end{aligned}
$$

$$
\begin{aligned}
c_{13} = {} & a_2^2a_3^2a_4^4 + 2a_1a_3^3a_4^4 + a_1^2a_2^2a_4^5 - 4a_2^3a_4^5 + 2a_1^3a_3a_4^5 - 6a_1a_2a_3a_4^5 \\
& -2a_3^2a_4^5 - 2a_1^2a_4^6 + a_2a_4^6 + 3a_1^2a_2a_3^3a_4^2a_5 - 6a_2^2a_3^3a_4^2a_5 \\
& -9a_1a_3^4a_4^2a_5 + 3a_1^4a_2a_3a_4^3a_5 - 18a_1^2a_2^2a_3a_4^3a_5 + 26a_2^3a_3a_4^3a_5 \\
& -6a_1^3a_3^2a_4^3a_5 + 7a_1a_2a_3^2a_4^3a_5 + 9a_3^3a_4^3a_5 + 3a_1^5a_4^4a_5 - 30a_1^3a_2a_4^4a_5 \\
& +69a_1a_2^2a_4^4a_5 + 17a_2a_3a_4^4a_5 + 23a_1a_4^5a_5 + a_1^4a_3^4a_5^2 - 6a_1^2a_2a_3^4a_5^2 \\
& +9a_2^2a_3^4a_5^2 + a_1^6a_3^2a_4a_5^2 - 9a_1^4a_2a_3^2a_4a_5^2 + 30a_1^2a_2^2a_3^2a_4a_5^2 \\
& -42a_2^3a_3^2a_4a_5^2 - 25a_1^3a_3^3a_4a_5^2 + 93a_1a_2a_3^3a_4a_5^2 + 3a_1^6a_2a_4^2a_5^2 \\
& -23a_1^4a_2^2a_4^2a_5^2 + 48a_1^2a_2^3a_4^2a_5^2 - 18a_2^4a_4^2a_5^2 - 23a_1^5a_3a_4^2a_5^2 \\
& +179a_1^3a_2a_3a_4^2a_5^2 - 333a_1a_2^2a_3a_4^2a_5^2 + 75a_1^2a_3^2a_4^2a_5^2 - 84a_2a_3^2a_4^2a_5^2 \\
& +3a_1^4a_4^3a_5^2 + 34a_1^2a_2a_4^3a_5^2 - 164a_2^2a_4^3a_5^2 - 182a_1a_3a_4^3a_5^2 - 5a_4^4a_5^2 \\
& +a_1^8a_3a_5^3 - 11a_1^6a_2a_3a_5^3 + 47a_1^4a_2^2a_3a_5^3 - 90a_1^2a_2^3a_3a_5^3 + 54a_2^4a_3a_5^3 \\
& -11a_1^5a_3^2a_5^3 + 61a_1^3a_2a_3^2a_5^3 - 21a_1a_2^2a_3^2a_5^3 - 75a_1^2a_3^3a_5^3 - 99a_2a_3^3a_5^3 \\
& -13a_1^7a_4a_5^3 + 116a_1^5a_2a_4a_5^3 - 330a_1^3a_2^2a_4a_5^3 + 306a_1a_2^3a_4a_5^3 \\
& +40a_1^4a_3a_4a_5^3 - 483a_1^2a_2a_3a_4a_5^3 + 858a_2^2a_3a_4a_5^3 + 285a_1a_3^2a_4a_5^3 \\
& -37a_1^3a_4^2a_5^3 + 252a_1a_2a_4^2a_5^3 + 117a_3a_4^2a_5^3 + 29a_1^6a_5^4 - 278a_1^4a_2a_5^4
\end{aligned}
$$

$$
\begin{aligned}
&+855a_1^2a_2^2a_5^4 - 864a_2^3a_5^4 + 113a_1^3a_3a_5^4 - 219a_1a_2a_3a_5^4 - 234a_3^2a_5^4 \\
&-69a_1^2a_4a_5^4 - 222a_2a_4a_5^4 + 132a_1a_5^5 + 2a_1^4a_3^4a_4a_6 - 9a_1^2a_2a_3^4a_4a_6 \\
&+27a_1a_3^5a_4a_6 + 2a_1^6a_3^2a_4^2a_6 - 15a_1^4a_2a_3^2a_4^2a_6 + 24a_1^2a_2^2a_3^2a_4^2a_6 \\
&+6a_2^3a_3^2a_4^2a_6 + 28a_1^3a_3^3a_4^2a_6 - 81a_1a_2a_3^3a_4^2a_6 - 27a_3^4a_4^2a_6 \\
&-4a_1^4a_2^2a_4^3a_6 + 24a_1^2a_2^3a_4^3a_6 - 32a_2^4a_4^3a_6 - 7a_1^5a_3a_4^3a_6 \\
&+22a_1^3a_2a_3a_4^3a_6 + 10a_1a_2^2a_3a_4^3a_6 - 59a_1^2a_3^2a_4^3a_6 - 18a_2a_3^2a_4^3a_6 \\
&+27a_1^4a_4^4a_6 - 136a_1^2a_2a_4^4a_6 + 174a_2^2a_4^4a_6 + 145a_1a_3a_4^4a_6 - 118a_4^5a_6 \\
&+6a_1^6a_3^3a_5a_6 - 49a_1^4a_2a_3^3a_5a_6 + 102a_1^2a_2^2a_3^3a_5a_6 - 18a_2^3a_3^3a_5a_6 \\
&+81a_1^3a_3^4a_5a_6 - 270a_1a_2a_3^4a_5a_6 + 6a_1^8a_3a_4a_5a_6 - 74a_1^6a_2a_3a_4a_5a_6 \\
&+294a_1^4a_2^2a_3a_4a_5a_6 - 408a_1^2a_2^3a_3a_4a_5a_6 + 120a_2^4a_3a_4a_5a_6 \\
&+117a_1^5a_3^2a_4a_5a_6 - 708a_1^3a_2a_3^2a_4a_5a_6 + 942a_1a_2^2a_3^2a_4a_5a_6 \\
&-27a_1^2a_3^3a_4a_5a_6 + 540a_2a_3^3a_4a_5a_6 + 12a_1^7a_4^2a_5a_6 \\
&-104a_1^5a_2a_4^2a_5a_6 + 328a_1^3a_2^2a_4^2a_5a_6 - 396a_1a_2^3a_4^2a_5a_6 \\
&-139a_1^4a_3a_4^2a_5a_6 + 864a_1^2a_2a_3a_4^2a_5a_6 - 1002a_2^2a_3a_4^2a_5a_6 \\
&-387a_1a_3^2a_4^2a_5a_6 - 92a_1^3a_4^3a_5a_6 + 136a_1a_2a_4^3a_5a_6 + 450a_3a_4^3a_5a_6 \\
&+3a_1^{10}a_5^2a_6 - 40a_1^8a_2a_5^2a_6 + 206a_1^6a_2^2a_5^2a_6 - 506a_1^4a_2^3a_5^2a_6 \\
&+576a_1^2a_2^4a_5^2a_6 - 216a_2^5a_5^2a_6 + 19a_1^7a_3a_5^2a_6 - 136a_1^5a_2a_3a_5^2a_6 \\
&+316a_1^3a_2^2a_3a_5^2a_6 - 318a_1a_2^3a_3a_5^2a_6 - 152a_1^4a_3^2a_5^2a_6 + 999a_1^2a_2a_3^2a_5^2a_6 \\
&-1260a_2^2a_3^2a_5^2a_6 + 270a_1a_3^3a_5^2a_6 - 26a_1^6a_4a_5^2a_6 + 222a_1^4a_2a_4a_5^2a_6 \\
&-924a_1^2a_2^2a_4a_5^2a_6 + 1644a_2^3a_4a_5^2a_6 + 569a_1^3a_3a_4a_5^2a_6 \\
&-2208a_1a_2a_3a_4a_5^2a_6 - 540a_3^2a_4a_5^2a_6 + 192a_1^2a_4^2a_5^2a_6 \\
&-204a_2a_4^2a_5^2a_6 - 2a_1^5a_5^3a_6 + 238a_1^3a_2a_5^3a_6 - 744a_1a_2^2a_5^3a_6 \\
&-1203a_1^2a_3a_5^3a_6 + 3852a_2a_3a_5^3a_6 + 336a_1a_4a_5^3a_6 - 792a_5^4a_6 \\
&+3a_1^8a_3^2a_6^2 - 30a_1^6a_2a_3^2a_6^2 + 102a_1^4a_2^2a_3^2a_6^2 - 535a_1^3a_3a_4^2a_6^2 \\
&+1782a_1a_2a_3a_4^2a_6^2 + 216a_3^2a_4^2a_6^2 + 326a_1^2a_4^3a_6^2 - 900a_2a_4^3a_6^2 \\
&-23a_1^9a_5a_6^2 + 276a_1^7a_2a_5a_6^2 - 1228a_1^5a_2^2a_5a_6^2 + 2368a_1^3a_2^3a_5a_6^2 \\
&-1632a_1a_2^4a_5a_6^2 - 287a_1^6a_3a_5a_6^2 + 2232a_1^4a_2a_3a_5a_6^2 \\
&-5214a_1^2a_2^2a_3a_5a_6^2 + 3528a_2^3a_3a_5a_6^2 - 639a_1^3a_3^2a_5a_6^2
\end{aligned}
$$

$$
\begin{aligned}
&+1080a_1a_2a_3^2a_5a_6^2 + 402a_1^5a_4a_5a_6^2 - 2542a_1^3a_2a_4a_5a_6^2 \\
&+4008a_1a_2^2a_4a_5a_6^2 + 1566a_1^2a_3a_4a_5a_6^2 - 2160a_2a_3a_4a_5a_6^2 \\
&-1224a_1a_4^2a_5a_6^2 - 793a_1^4a_5^2a_6^2 + 4200a_1^2a_2a_5^2a_6^2 - 5472a_2^2a_5^2a_6^2 \\
&-1080a_1a_3a_5^2a_6^2 + 2160a_4a_5^2a_6^2 - 20a_1^8a_6^3 + 294a_1^6a_2a_6^3 \\
&-1376a_1^4a_2^2a_6^3 + 2472a_1^2a_2^3a_6^3 - 1440a_2^4a_6^3 - 118a_1^5a_3a_6^3 \\
&+774a_1^3a_2a_3a_6^3 - 1080a_1a_2^2a_3a_6^3 + 216a_1^2a_3^2a_6^3 + 444a_1^4a_4a_6^3 \\
&-2412a_1^2a_2a_4a_6^3 + 2160a_2^2a_4a_6^3 + 432a_1a_3a_4a_6^3 - 432a_4^2a_6^3 \\
&+288a_1^3a_5a_6^3 - 432a_1^2a_6^4
\end{aligned}
$$

$$
\begin{aligned}
c_{14} = {}& 2a_2a_3^2a_4^5 + 2a_1^2a_2a_4^6 - 8a_2^2a_4^6 + a_1a_3a_4^6 - 4a_4^7 + a_1a_2^2a_3^2a_4^3a_5 \\
&+2a_1^2a_3^3a_4^3a_5 - 15a_2a_3^3a_4^3a_5 + a_1^3a_2^2a_4^4a_5 - 4a_1a_2^3a_4^4a_5 + 2a_1^4a_3a_4^4a_5 \\
&-24a_1^2a_2a_3a_4^4a_5 + 64a_2^2a_3a_4^4a_5 - 16a_1a_3^2a_4^4a_5 - 7a_1^3a_4^5a_5 \\
&+27a_1a_2a_4^5a_5 + 39a_3a_4^5a_5 + a_1^3a_2a_5^3a_4a_5^2 - 3a_1a_2^2a_3^3a_4a_5^2 \\
&-9a_1^2a_3^4a_4a_5^2 + 27a_2a_3^4a_4a_5^2 + a_1^5a_2a_3a_4^2a_5^2 - 8a_1^3a_2^2a_3a_4^2a_5^2 \\
&+15a_1a_2^3a_3a_4^2a_5^2 - 7a_1^4a_3^2a_4^2a_5^2 + 51a_1^2a_2a_3^2a_4^2a_5^2 - 114a_2^2a_3^2a_4^2a_5^2 \\
&+72a_1a_3^3a_4^2a_5^2 + 3a_1^6a_4^3a_5^2 - 32a_1^4a_2a_4^3a_5^2 + 103a_1^2a_2^2a_4^3a_5^2 - 90a_2^3a_4^3a_5^2 \\
&+50a_1^3a_3a_4^3a_5^2 - 166a_1a_2a_3a_4^3a_5^2 - 129a_3^2a_4^3a_5^2 - 2a_1^2a_4^4a_5^2 \\
&-49a_2a_4^4a_5^2 + 2a_1^3a_2^2a_3^2a_5^3 - 9a_1a_2^3a_3^2a_5^3 - 12a_1^4a_3^3a_5^3 + 63a_1^2a_2a_3^3a_5^3 \\
&-27a_2^2a_3^3a_5^3 - 81a_1a_3^4a_5^3 + a_1^7a_2a_4a_5^3 - 7a_1^5a_2^2a_5^3 + 12a_1^3a_2^3a_4a_5^3 \\
&-17a_1^6a_3a_4a_5^3 + 158a_1^4a_2a_3a_4a_5^3 - 426a_1^2a_2^2a_3a_4a_5^3 + 324a_2^3a_3a_4a_5^3 \\
&+67a_1^3a_3^2a_4a_5^3 + 159a_1a_2a_3^2a_4a_5^3 + 135a_3^3a_4a_5^3 + 8a_1^5a_4^2a_5^3 \\
&-64a_1^3a_2a_4^2a_5^3 + 93a_1a_2^2a_4^2a_5^3 - 42a_1^2a_3a_4^2a_5^3 + 336a_2a_3a_4^2a_5^3 \\
&+57a_1a_4^3a_5^3 - 4a_1^8a_5^4 + 43a_1^6a_2a_5^4 - 174a_1^4a_2^2a_5^4 + 324a_1^2a_2^3a_5^4 \\
&-243a_2^4a_5^4 - 7a_1^5a_3a_5^4 - 13a_1^3a_2a_3a_5^4 + 99a_1a_2^2a_3a_5^4 + 153a_1^2a_3^2a_5^4 \\
&-432a_2a_3^2a_5^4 + 11a_1^4a_4a_5^4 + 48a_1^2a_2a_4a_5^4 - 162a_2^2a_4a_5^4 - 210a_1a_3a_4a_5^4 \\
&-60a_4^2a_5^4 - 34a_1^3a_5^5 + 72a_1a_2a_5^5 + 216a_3a_5^5 + 2a_1^3a_2a_3^3a_4^2a_6 \\
&-9a_1a_2^2a_3^3a_4^2a_6 + 27a_2a_3^4a_4^2a_6 + 2a_1^5a_2a_3a_4^3a_6 - 16a_1^3a_2^2a_3a_4^3a_6 \\
&+32a_1a_2^3a_3a_4^3a_6 - 8a_1^4a_3^2a_4^3a_6 + 64a_1^2a_2a_3^2a_4^3a_6 - 138a_2^2a_3^2a_4^3a_6
\end{aligned}
$$

$$
\begin{aligned}
&-27a_1a_3^3a_4^3a_6 - 6a_1^6a_4^4a_6 + 50a_1^4a_2a_4^4a_6 - 136a_1^2a_2^2a_4^4a_6 \\
&+128a_2^3a_4^4a_6 - 23a_1^3a_3a_4^4a_6 + 69a_1a_2a_3a_4^4a_6 + 27a_3^2a_4^4a_6 + 45a_1^2a_4^5a_6 \\
&-122a_2a_4^5a_6 + 2a_1^5a_3^4a_5a_6 - 15a_1^3a_2a_3^4a_5a_6 + 27a_1a_2^2a_3^4a_5a_6 \\
&+27a_1^2a_3^5a_5a_6 - 81a_2a_3^5a_5a_6 + 2a_1^7a_3^2a_4a_5a_6 - 24a_1^5a_2a_3^2a_4a_5a_6 \\
&+84a_1^3a_2^2a_3^2a_4a_5a_6 - 84a_1a_2^3a_3^2a_4a_5a_6 + 59a_1^4a_3^3a_4a_5a_6 \\
&-342a_1^2a_2a_3^3a_4a_5a_6 + 432a_2^2a_3^3a_4a_5a_6 + 27a_1a_3^4a_4a_5a_6 \\
&-8a_1^5a_2^2a_4^2a_5a_6 + 56a_1^3a_2^3a_4^2a_5a_6 - 96a_1a_2^4a_4^2a_5a_6 + 24a_1^6a_3a_4^2a_5a_6 \\
&-196a_1^4a_2a_3a_4^2a_5a_6 + 474a_1^2a_2^2a_3a_4^2a_5a_6 - 312a_2^3a_3a_4^2a_5a_6 \\
&+15a_1^3a_3^2a_4^2a_5a_6 + 36a_1a_2a_3^2a_4^2a_5a_6 - 81a_3^3a_4^2a_5a_6 + 10a_1^5a_4^3a_5a_6 \\
&-10a_1^3a_2a_4^3a_5a_6 - 108a_1a_2^2a_4^3a_5a_6 - 158a_1^2a_3a_4^3a_5a_6 \\
&+600a_2a_3a_4^3a_5a_6 - 83a_1a_4^4a_5a_6 + 2a_1^9a_3a_5^2a_6 - 25a_1^7a_2a_3a_5^2a_6 \\
&+115a_1^5a_2^2a_3a_5^2a_6 - 1134a_2a_3^2a_4a_5^2a_6 - 232a_1^3a_2^3a_3a_5^2a_6 \\
&+180a_1a_2^4a_3a_5^2a_6 + 29a_1^6a_3^2a_5^2a_6 - 228a_1^4a_2a_3^2a_5^2a_6 + 579a_1^2a_2^2a_3^2a_5^2a_6 \\
&-594a_2^3a_3^2a_5^2a_6 + 24a_1^3a_3^3a_5^2a_6 + 135a_1a_2a_3^3a_5^2a_6 + 81a_3^4a_5^2a_6 \\
&+a_1^8a_4a_5^2a_6 - 16a_1^6a_2a_4a_5^2a_6 + 130a_1^4a_2^2a_4a_5^2a_6 - 492a_1^2a_2^3a_4a_5^2a_6 \\
&+648a_2^4a_4a_5^2a_6 - 90a_1^5a_3a_4a_5^2a_6 + 526a_1^3a_2a_3a_4a_5^2a_6 \\
&-654a_1a_2^2a_3a_4a_5^2a_6 + 9a_1^2a_3^2a_4a_5^2a_6 - 82a_1^4a_4^2a_5^2a_6 + 510a_1^2a_2a_4^2a_5^2a_6 \\
&-606a_2^2a_4^2a_5^2a_6 + 225a_1a_3a_4^2a_5^2a_6 + 138a_4^3a_5^2a_6 + 35a_1^7a_5^3a_6 \\
&-336a_1^5a_2a_5^3a_6 + 1078a_1^3a_2^2a_5^3a_6 - 1152a_1a_2^3a_5^3a_6 + 468a_1^4a_3a_5^3a_6 \\
&-2562a_1^2a_2a_3a_5^3a_6 + 3348a_2^2a_3a_5^3a_6 + 486a_1a_3^2a_5^3a_6 - 302a_1^3a_4a_5^3a_6 \\
&+720a_1a_2a_4a_5^3a_6 - 540a_3a_4a_5^3a_6 + 516a_1^2a_5^4a_6 - 1296a_2a_5^4a_6 \\
&+3a_1^7a_3^3a_6^2 - 29a_1^5a_2a_3^3a_6^2 + 96a_1^3a_2^2a_3^3a_6^2 - 108a_1a_2^3a_3^3a_6^2 \\
&-27a_1^2a_2a_3^4a_6^2 + 81a_2^2a_3^4a_6^2 + 2a_1^9a_3a_4a_6^2 - 32a_1^7a_2a_3a_4a_6^2 \\
&+184a_1^5a_2^2a_3a_4a_6^2 - 448a_1^3a_2^3a_3a_4a_6^2 + 384a_1a_2^4a_3a_4a_6^2 + 5a_1^6a_3^2a_4a_6^2 \\
&-78a_1^4a_2a_3^2a_4a_6^2 + 288a_1^2a_2^2a_3^2a_4a_6^2 - 216a_2^3a_3^2a_4a_6^2 + 27a_1^3a_3^3a_4a_6^2 \\
&-162a_1a_2a_3^3a_4a_6^2 + 10a_1^8a_4^2a_6^2 - 80a_1^6a_2a_4^2a_6^2 + 160a_1^4a_2^2a_4^2a_6^2 \\
&+96a_1^2a_2^3a_4^2a_6^2 - 384a_2^4a_4^2a_6^2 + 54a_1^5a_3a_4^2a_6^2 - 221a_1^3a_2a_3a_4^2a_6^2 \\
&+144a_1a_2^2a_3a_4^2a_6^2 + 81a_1^2a_3^2a_4^2a_6^2 - 54a_2a_3^2a_4^2a_6^2 - 53a_1^4a_4^3a_6^2
\end{aligned}
$$

$$
\begin{aligned}
&+62a_1^2a_2a_4^3a_6^2 + 336a_2^2a_4^3a_6^2 - 54a_1a_3a_4^3a_6^2 - 27a_4^4a_6^2 + 3a_1^{11}a_5a_6^2 \\
&-44a_1^9a_2a_5a_6^2 + 256a_1^7a_2^2a_5a_6^2 - 736a_1^5a_2^3a_5a_6^2 + 1040a_1^3a_2^4a_5a_6^2 \\
&-576a_1a_2^5a_5a_6^2 + 26a_1^8a_3a_5a_6^2 - 278a_1^6a_2a_3a_5a_6^2 + 1048a_1^4a_2^2a_3a_5a_6^2 \\
&-1608a_1^2a_2^3a_3a_5a_6^2 + 864a_2^4a_3a_5a_6^2 + 31a_1^5a_3^2a_5a_6^2 - 93a_1^3a_2a_3^2a_5a_6^2 \\
&-108a_1a_2^2a_3^2a_5a_6^2 - 297a_1^2a_3^3a_5a_6^2 + 648a_2a_3^3a_5a_6^2 - 115a_1^7a_4a_5a_6^2 \\
&+1048a_1^5a_2a_4a_5a_6^2 - 3080a_1^3a_2^2a_4a_5a_6^2 + 2880a_1a_2^3a_4a_5a_6^2 \\
&-554a_1^4a_3a_4a_5a_6^2 + 2448a_1^2a_2a_3a_4a_5a_6^2 - 1728a_2^2a_3a_4a_5a_6^2 \\
&+108a_1a_3^2a_4a_5a_6^2 + 807a_1^3a_4^2a_5a_6^2 - 2844a_1a_2a_4^2a_5a_6^2 + 324a_3a_4^2a_5a_6^2 \\
&+103a_1^6a_5^2a_6^2 - 1048a_1^4a_2a_5^2a_6^2 + 3318a_1^2a_2^2a_5^2a_6^2 - 3240a_2^3a_5^2a_6^2 \\
&+39a_1^3a_3a_5^2a_6^2 - 540a_1a_2a_3a_5^2a_6^2 - 648a_3^2a_5^2a_6^2 - 810a_1^2a_4a_5^2a_6^2 \\
&+4104a_2a_4a_5^2a_6^2 - 648a_1a_5^3a_6^2 - 8a_1^{10}a_6^3 + 88a_1^8a_2a_6^3 - 352a_1^6a_2^2a_6^3 \\
&+608a_1^4a_2^3a_6^3 - 384a_1^2a_2^4a_6^3 - 29a_1^7a_3a_6^3 + 200a_1^5a_2a_3a_6^3 \\
&-492a_1^3a_2^2a_3a_6^3 + 432a_1a_2^3a_3a_6^3 - 27a_1^4a_3^2a_6^3 + 378a_1^2a_2a_3^2a_6^3 \\
&-648a_2^2a_3^2a_6^3 - 6a_1^6a_4a_6^3 + 228a_1^4a_2a_4a_6^3 - 936a_1^2a_2^2a_4a_6^3 \\
&+864a_2^3a_4a_6^3 - 270a_1^3a_3a_4a_6^3 + 648a_1a_2a_3a_4a_6^3 + 162a_1^2a_4^2a_6^3 \\
&-216a_2a_4^2a_6^3 - 18a_1^5a_5a_6^3 + 72a_1^3a_2a_5a_6^3 + 756a_1^2a_3a_5a_6^3 \\
&-1296a_2a_3a_5a_6^3 - 864a_1a_4a_5a_6^3 + 1296a_5^2a_6^3 + 189a_1^4a_6^4 \\
&-1080a_1^2a_2a_6^4 + 1296a_2^2a_6^4
\end{aligned}
$$

$$
\begin{aligned}
c_{15} = {} & a_3^2a_4^6 + a_1^2a_4^7 - 4a_2a_4^7 + a_1a_2a_3^2a_4^4a_5 - 9a_3^3a_4^4a_5 + a_1^3a_2a_4^5a_5 \\
&-4a_1a_2^2a_4^5a_5 - 10a_1^2a_3a_4^5a_5 + 38a_2a_3a_4^5a_5 + 2a_1a_4^6a_5 + a_2^3a_3^2a_4^2a_5^2 \\
&+a_1^3a_3^3a_4^2a_5^2 - 9a_1a_2a_3^3a_4^2a_5^2 - 27a_3^4a_4^2a_5^2 + a_1^2a_2^3a_4^3a_5^2 - 4a_2^4a_4^3a_5^2 \\
&+a_1^5a_3a_4^3a_5^2 - 14a_1^3a_2a_3a_4^3a_5^2 + 38a_1a_2^2a_3a_4^3a_5^2 + 33a_1^2a_3^2a_4^3a_5^2 \\
&-117a_2a_3^2a_4^3a_5^2 + 2a_1^4a_4^4a_5^2 + 18a_1^2a_2a_4^4a_5^2 - 31a_2^2a_4^4a_5^2 \\
&-16a_1a_3a_4^4a_5^2 - 3a_4^5a_5^2 + a_1^2a_2^2a_3^3a_5^3 - 4a_2^3a_3^3a_5^3 - 4a_1^3a_3^4a_5^3 \\
&+18a_1a_2a_3^4a_5^3 - 27a_3^5a_5^3 + a_1^4a_2^2a_3a_4a_5^3 - 9a_1^2a_2^3a_3a_4a_5^3 \\
&+18a_2^4a_3a_4a_5^3 - 4a_1^5a_3^2a_4a_5^3 + 38a_1^3a_2a_3^2a_4a_5^3 - 80a_1a_2^2a_3^2a_4a_5^3 \\
&-33a_1^2a_3^3a_4a_5^3 + 117a_2a_3^3a_4a_5^3 + a_1^7a_4^2a_5^3 - 10a_1^5a_2a_4^2a_5^3 \\
&+33a_1^3a_2^2a_4^2a_5^3 - 33a_1a_2^3a_4^2a_5^3 - 18a_1^4a_3a_4^2a_5^3 - 120a_1^2a_2a_3a_4^2a_5^3
\end{aligned}
$$

$$+157a_2^2a_3a_4^2a_5^3 + 33a_1a_3^2a_4^2a_5^3 + 12a_1^3a_4^3a_5^3 + 41a_1a_2a_4^3a_5^3$$
$$+22a_3a_4^3a_5^3 + a_1^6a_2^2a_5^4 - 9a_1^4a_2^3a_5^4 + 27a_1^2a_2^4a_5^4 - 27a_2^5a_5^4$$
$$-4a_1^7a_3a_5^4 + 38a_1^5a_2a_3a_5^4 - 117a_1^3a_2^2a_3a_5^4 - 117a_1a_2^3a_3a_5^4$$
$$-31a_1^4a_3^2a_5^4 + 157a_1^2a_2a_3^2a_5^4 - 186a_2^2a_3^2a_5^4 + 18a_1a_3^3a_5^4 + 2a_1^6a_4a_5^4$$
$$-16a_1^4a_2a_4a_5^4 + 33a_1^2a_2^2a_4a_5^4 - 18a_2^3a_4a_5^4 - 41a_1^3a_3a_4a_5^4$$
$$-86a_1a_2a_3a_4a_5^4 - 36a_3^2a_4a_5^4 + 6a_1^2a_4^2a_5^4 - 68a_2a_4^2a_5^4 - 3a_1^5a_5^5$$
$$+22a_1^3a_2a_5^5 - 36a_1a_2^2a_5^5 - 68a_1^2a_3a_5^5 + 168a_2a_3a_5^5 + 40a_1a_4a_5^5$$
$$+32a_5^6 + a_1^2a_2^2a_3^2a_4^3a_6 - 4a_2^3a_3^2a_4^3a_6 - 2a_1^3a_3^3a_4^3a_6 + 9a_1a_2a_3^3a_4^3a_6$$
$$+a_1^4a_2^2a_4^4a_6 - 8a_1^2a_2^3a_4^4a_6 + 16a_2^4a_4^4a_6 - 2a_1^5a_3a_4^4a_6$$
$$+17a_1^3a_2a_3a_4^4a_6 - 36a_1a_2^2a_3a_4^4a_6 - 3a_1^2a_3^2a_4^4a_6 - 4a_1^4a_4^5a_6$$
$$-17a_1^2a_2a_4^5a_6 - 4a_2^2a_4^5a_6 - 3a_1a_3a_4^5a_6 + a_1^4a_2a_3^3a_4a_5a_6$$
$$+9a_1^2a_2^2a_3^3a_4a_5a_6 + 18a_2^3a_3^3a_4a_5a_6 + 9a_1^3a_3^4a_4a_5a_6$$
$$-27a_1a_2a_3^4a_4a_5a_6 + a_1^6a_2a_3a_4^2a_5a_6 - 14a_1^4a_2^2a_3a_4^2a_5a_6$$
$$+60a_1^2a_2^3a_3a_4^2a_5a_6 - 80a_2^4a_3a_4^2a_5a_6 + 8a_1^5a_3^2a_4^2a_5a_6$$
$$-63a_1^3a_2a_3^2a_4^2a_5a_6 + 117a_1a_2^2a_3^2a_4^2a_5a_6 - 27a_2a_3^3a_4^2a_5a_6$$
$$-3a_1^7a_4^3a_5a_6 + 24a_1^5a_2a_4^3a_5a_6 - 54a_1^3a_2^2a_4^3a_5a_6 + 24a_1a_2^3a_4^3a_5a_6$$
$$+6a_1^4a_3a_4^3a_5a_6 - 54a_1^2a_2a_3a_4^3a_5a_6 + 138a_2^2a_3a_4^3a_5a_6$$
$$+54a_1a_3^2a_4^3a_5a_6 + 28a_1^3a_4^4a_5a_6 - 137a_1a_2a_4^4a_5a_6 + 9a_3a_4^4a_5a_6$$
$$-a_1^4a_2^2a_3^2a_5^2a_6 + 6a_1^2a_2^3a_3^2a_5^2a_6 - 6a_2^4a_3^2a_5^2a_6 + 3a_1^3a_2a_3^3a_5^2a_6$$
$$-27a_1a_2^2a_3^3a_5^2a_6 + 81a_2a_3^4a_5^2a_6 + a_1^8a_2a_4a_5^2a_6 - 15a_1^6a_2^2a_4a_5^2a_6$$
$$+80a_1^4a_2^3a_4a_5^2a_6 - 180a_1^2a_2^4a_4a_5^2a_6 + 144a_2^5a_4a_5^2a_6 + 9a_1^7a_3a_4a_5^2a_6$$
$$-64a_1^5a_2a_3a_4a_5^2a_6 + 119a_1^3a_2^2a_3a_4a_5^2a_6 - 26a_1a_2^3a_3a_4a_5^2a_6$$
$$+3a_1^4a_3^2a_4a_5^2a_6 + 135a_1^2a_2a_3^2a_4a_5^2a_6 - 378a_2^2a_3^2a_4a_5^2a_6$$
$$-108a_1a_3^3a_4a_5^2a_6 - 38a_1^4a_2a_4^2a_5^2a_6 + 210a_1^2a_2^2a_4^2a_5^2a_6$$
$$-286a_2^3a_4^2a_5^2a_6 - 94a_1^3a_3a_4^2a_5^2a_6 + 261a_1a_2a_3a_4^2a_5^2a_6 - 108a_3^2a_4^2a_5^2a_6$$
$$+20a_1^2a_4^3a_5^2a_6 + 210a_2a_4^3a_5^2a_6 - 4a_1^9a_5^3a_6 + 49a_1^7a_2a_5^3a_6$$
$$-227a_1^5a_2^2a_5^3a_6 + 468a_1^3a_2^3a_5^3a_6 - 360a_1a_2^4a_5^3a_6 - 35a_1^6a_3a_5^3a_6$$
$$+313a_1^4a_2a_3a_5^3a_6 - 905a_1^2a_2^2a_3a_5^3a_6 + 840a_2^3a_3a_5^3a_6 - 11a_1^3a_3^2a_5^3a_6$$

$$
\begin{aligned}
&+18a_1a_2a_3^2a_5^3a_6 + 216a_3^3a_5^3a_6 + 49a_1^5a_4a_5^3a_6 - 282a_1^3a_2a_4a_5^3a_6 \\
&+416a_1a_2^2a_4a_5^3a_6 + 126a_1^2a_3a_4a_5^3a_6 - 468a_2a_3a_4a_5^3a_6 \\
&-192a_1a_4^2a_5^3a_6 - 54a_1^4a_5^4a_6 + 308a_1^2a_2a_5^4a_6 - 456a_2^2a_5^4a_6 \\
&+72a_1a_3a_5^4a_6 + 144a_4a_5^4a_6 + a_1^6a_3^4a_6^2 - 9a_1^4a_2a_3^4a_6^2 + 27a_1^2a_2^2a_3^4a_6^2 \\
&-27a_2^3a_3^4a_6^2 + a_1^8a_3^2a_4a_6^2 - 15a_1^6a_2a_3^2a_4a_6^2 + 80a_1^4a_2^2a_3^2a_4a_6^2 \\
&-180a_1^2a_2^3a_3^2a_4a_6^2 + 144a_2^4a_3^2a_4a_6^2 + 3a_1^5a_3^3a_4a_6^2 - 27a_1^3a_2a_3^3a_4a_6^2 \\
&+54a_1a_2^2a_3^3a_4a_6^2 - 2a_1^8a_2a_4^2a_6^2 + 24a_1^6a_2^2a_4^2a_6^2 - 104a_1^4a_2^3a_4^2a_6^2 \\
&+192a_1^2a_2^4a_4^2a_6^2 - 128a_2^5a_4^2a_6^2 + 5a_1^7a_3a_4^2a_6^2 - 58a_1^5a_2a_3a_4^2a_6^2 \\
&+212a_1^3a_2^2a_3a_4^2a_6^2 - 240a_1a_2^3a_3a_4^2a_6^2 + 12a_1^4a_3^2a_4^2a_6^2 - 27a_1^2a_2a_3^2a_4^2a_6^2 \\
&-54a_2^2a_3^2a_4^2a_6^2 + 8a_1^6a_4^3a_6^2 - 31a_1^4a_2a_4^3a_6^2 - 60a_1^2a_2^2a_4^3a_6^2 \\
&+224a_2^3a_4^3a_6^2 + 4a_1^3a_3a_4^3a_6^2 + 18a_1a_2a_3a_4^3a_6^2 + 9a_1^2a_4^4a_6^2 \\
&-27a_2a_4^4a_6^2 + a_1^{10}a_3a_5a_6^2 - 14a_1^8a_2a_3a_5a_6^2 + 80a_1^6a_2^2a_3a_5a_6^2 \\
&-232a_1^4a_2^3a_3a_5a_6^2 + 336a_1^2a_2^4a_3a_5a_6^2 - 192a_2^5a_3a_5a_6^2 + a_1^7a_3^2a_5a_6^2 \\
&-13a_1^5a_2a_3^2a_5a_6^2 + 60a_1^3a_2^2a_3^2a_5a_6^2 - 72a_1a_2^3a_3^2a_5a_6^3 - 27a_1^2a_2a_3^3a_5a_6^2 \\
&+10a_1^9a_4a_5a_6^2 - 122a_1^7a_2a_4a_5a_6^2 + 544a_1^5a_2^2a_4a_5a_6^2 \\
&-1048a_1^3a_2^3a_4a_5a_6^2 + 736a_1a_2^4a_4a_5a_6^2 + 31a_1^6a_3a_4a_5a_6^2 \\
&-209a_1^4a_2a_3a_4a_5a_6^2 + 342a_1^2a_2^2a_3a_4a_5a_6^2 - 72a_2^3a_3a_4a_5a_6^2 \\
&-54a_1^3a_3^2a_4a_5a_6^2 + 324a_1a_2a_3^2a_4a_5a_6^2 - 93a_1^5a_4^2a_5a_6^2 \\
&+695a_1^3a_2a_4^2a_5a_6^2 - 1188a_1a_2^2a_4^2a_5a_6^2 - 162a_1^2a_3a_4^2a_5a_6^2 \\
&+108a_2a_3a_4^2a_5a_6^2 + 3a_1^8a_5^2a_6^2 + 8a_1^6a_2a_5^2a_6^2 - 192a_1^4a_2^2a_5^2a_6^2 \\
&+622a_1^2a_2^3a_5^2a_6^2 - 600a_2^4a_5^2a_6^2 + 8a_1^5a_3a_5^2a_6^2 - 45a_1^3a_2a_3a_5^2a_6^2 \\
&+108a_1a_2^2a_3a_5^2a_6^2 + 108a_1^2a_3^2a_5^2a_6^2 - 648a_2a_3^2a_5^2a_6^2 + 35a_1^4a_4a_5^2a_6^2 \\
&-666a_1^2a_2a_4a_5^2a_6^2 + 1512a_2^2a_4a_5^2a_6^2 + 432a_1a_3a_4a_5^2a_6^2 + 136a_1^3a_5^3a_6^2 \\
&-360a_1a_2a_5^3a_6^2 - 432a_3a_5^3a_6^2 + a_1^{12}a_6^3 - 16a_1^{10}a_2a_6^3 + 104a_1^8a_2^2a_6^3 \\
&-352a_1^6a_2^3a_6^3 + 656a_1^4a_2^4a_6^3 - 640a_1^2a_2^5a_6^3 + 256a_2^6a_6^3 + 2a_1^9a_3a_6^3.
\end{aligned}
$$

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