Università degli Studi di Padova
Dipartimento di Matematica "Tullio Levi-Civita"
Corso di Laurea Magistrale in Matematica

The P- Partition Generating Function and The Quasisymmetric Basis ϕ

Supervisor : Prof. Marco Andrea Garuti
Djenabou Karimatou : 1188578

11 dicembre 2020

Abstract

An acyclic directed graph can be viewed as a (labelled) poset (P, ω). To the latter, one can associate a (P, ω)-partition generating function which is a quasisymmetric function. We propose two expansions of this function in the recently introduced type-2 quasisymmetric power sums basis ϕ and derive the leading coefficient of some types of posets.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Karimatou Djenabou, 7 June 2020.

Contents

1. Introduction

1.1 Research context

In discrete mathematics, graphs are one of the structures that have been intensively studied. Those studies have in particular helped to characterize, define certain properties of graphs and compute some graphical invariants, thus facilitating their manipulation. In that direction, (quasi)symmetric functions and graph polynomials such as Stanley's chromatic symmetric function play an essential role.

The chromatic polynomial is a polynomial function in the number of colors that gives the number of graph colourings. Stanley's chromatic symmetric function is a generalization of the classical chromatic polynomial. Given a simple graph (a graph with no loops and no multiple edges) $G = (V, E)$, the *Stanley's chromatic* symmetric function of G is defined as follows [10]: Let $\mathbb{P} = \{1, 2, ...\}$ be the set of positive integers (here, we can view elements of $\mathbb{P}$ as distinct colours). A function $\kappa : V \to \mathbb{P}$ is called a *proper* $\mathbb{P}$-colouring of G if $\kappa(i) \neq \kappa(j)$ whenever $(i, j) \in E$, i.e., adjacent vertices have different colours. Then the Stanley's chromatic symmetric function is

$$\mathbf{X}_G(X) := \sum_{\kappa} X_\kappa; \tag{1.1.1}$$

where the sum is over all proper $\mathbb{P}$-colouring κ of G, $X = (x_1, x_2, x_3, ...)$ are commuting indeterminates, $X_\kappa = x_{\kappa(v_1)} x_{\kappa(v_2)} x_{\kappa(v_3)} \dots x_{\kappa(v_d)}$ and $d = |V|$ is the number of vertices of G. It can be seen that $\mathbf{X}_G(x)$ is a symmetric and homogeneous function of degree d. For instance, considering the single edged graph $G : \circ\!\!-\!\!\!-\!\!\!-\!\!\circ$, we have $\mathbf{X}_G(X) = \sum_{i \neq j} x_i x_j$.

Having a graph invariant, one of the natural questions to consider is whether this invariant determines the graph. Considering the chromatic symmetric polynomial, the answer to this question is "no" and thus it is possible for two non-isomorphic graphs to have the same chromatic symmetric function. An example using graphs with five vertices is given in [10]. However, although $\mathbf{X}_G$ is not a complete isomorphism invariant, it is a stronger isomorphism invariant than the chromatic polynomial χ_G. For instance, $\chi_G(x) = x(x-1)^{n-1}$ for all trees on n vertices, while some families of trees can be determined by their chromatic symmetric function $\mathbf{X}_G$ as proved in [6]. More generally, Stanley [10] conjectured that the chromatic symmetric function distinguishes trees. This conjecture continues to inspire research and is an important motivation to us.

Results on graphs derived from their chromatic functions mostly consider the expansion of $\mathbf{X}_G$ in the various bases of the ring *Sym* of symmetric functions. We will consider the refinement of $\mathbf{X}_G$ in the ring *QSym* of quasisymmetric functions (with coefficients in $\mathbb{Q}[X]$) and extend from the class of labelled graphs to directed graphs. We work with quasisymmetric functions because complex combinatorial objects tend to have simpler formulas when considered into symmetric functions, and expanding from symmetric to quasisymmetric functions can provide new insights about those objects and reduce the complexity of proofs. Thus it is natural to combine the study of graphs polynomials with the theory of quasisymmetric functions (which we will overview in the next chapter).

In that sense, Shareshian and Wachs [8] defined in 2016, the refinement of Stanley's chromatic symmetric function called the *chromatic quasisymmetric function*. The definition of the chromatic quasisymmetric function only involves an additional parameter t in the definition of the chromatic symmetric function and considers labelled graphs. It is given by

$$\mathbf{X}_G(X,t) := \sum_{\kappa} t^{asc(\kappa)} X_\kappa; \tag{1.1.2}$$

where $asc(\kappa) := |(u,v) \in E : u < v$ and $\kappa(u) < \kappa(v)|$ is the number of ascents and the sum runs over all proper colourings κ of G. It can be seen that when $t = 1$ we recover the chromatic symmetric function. Also, for some type of graphs, the chromatic quasisymmetric function is symmetric. Various proprieties of $\mathbf{X}_G(X,t)$ and resulting results for graphs are presented in [8]. Moreover, Ellzey [3] considered an extended version of the chromatic quasisymmetric function to directed graphs. Recall that a *directed graph* (*digraph* for short) is a graph in which the edges have directions. If we denote by $\overrightarrow{G} = (V,E)$ a directed graph, then its quasisymmetric function $\mathbf{X}_{\overrightarrow{G}}(X,t)$ has the same expression as for labelled graphs in equation 1.1.2 except that

$$asc(\kappa) = |(u,v) \in E : \kappa(u) < \kappa(v)|$$

as there is no labelling considered. Therein, expansions of $\mathbf{X}_{\overrightarrow{G}}(X,t)$ into the fundamental quasisymmetric basis, the power sum and elementary symmetric bases were considered.

We can now transition from digraphs to partially ordered sets (posets). We make this change due to the fact that the set of vertices of an acyclic directed graph (directed tree) with the reachability order forms a poset and that for a directed tree $\overrightarrow{T} = (V,E)$, each proper colouring that contributes to the highest power of t (i.e, the coefficient of the highest power of t) in its chromatic quasisymmetric function can be considered an order preserving P-partition of the poset P of $\overrightarrow{T}$. Thus, from this observation, distinguishing directed trees through their chromatic quasisymmetric functions would be implied by distinguishing posets P through their P-partition generating functions. We therefore wish to study the expansion of P-partition generating function in various bases. The expansion in the monomial and fundamental quasisymmetric bases is well known.

More recently, Ballantine et al. [2] studied two types (type 1: Ψ and type 2: Φ) of quasisymmetric power sum bases. Liu and Weselcouch [4] studied the expansion of the (P,w)-partition generating function $K_{(P,w)}$ of a labelled poset (P,w) (here, we deliberately opted to not define labelled poset (P,w) and its (P,w)-partition generating function as we will have an entire section dedicated to it in the next chapter) into the Ψ power sum basis and proved its irreducibility when the poset is naturally labelled. In contract, expansions of $K_{(P,w)}$ in the quasisymmetric power sum ϕ basis have not yet been explored and this constitutes the essence of this project.

1.2 Motivation and goals

The main motivation behind the study of (P, w)-partition generating functions in the quasisymmetric power sum ϕ basis is to examine the analogue to Stanley's conjecture for the case of directed trees. Note that this conjecture is still open and attempted proofs have been mostly computational; it has been verified for trees up to 23 vertices (see [9]). Thus, in this quest, our principal objective will be the expansion of the (P, w)-partition generating function $K_{(P,w)}$ of a labelled poset (P, w) into the quasisymmetric power sum basis ϕ. Secondary goals would be to derive some general results on some particular types of posets and to examine to what extend one can distinguish posets of directed trees in general by this extension.

1.3 Thesis Outline

The remaining part of this thesis is structured as follows: In Chapter 2, we give the necessary mathematical background required to follow this project. Thus, we will present some concepts, notions and results related to labelled partially ordered sets, P-partitions and their generating functions, and quasisymmetric functions. Chapter 3 constitutes the main part of this project. There, we provide results on the P-partitions generating function for some types of posets by considering the quasisymmetric power sum basis ϕ. Most importantly will be the expansion of the (P, w)-partitions generating function of labelled posets (P, w) in this ϕ basis. In Chapter 4, we draw the conclusions of our work, highlighting the main elements we introduced and proposing further work on the present project.

2. Background

Here we introduce the necessary mathematical tools to present our results. More precisely, we present concepts related to partially ordered sets, quasisymmetric functions and P-partition generating functions.

2.1 Partially ordered sets (Posets)

The notion of poset is the formalization and generalization of the concept of ordering the elements of a set. Posets play an important role in combinatorics as the study of many combinatorial structures can be reduced to that of posets, which have a well-developed theory.

Definition 2.1.1. A *poset* is a set P equipped with a binary relation $\leq_P$ ('the partial order') which is reflexive, transitive and antisymmetric. That is every $x, y, z \in P$ satisfy the following:

- $x \leq_P x$,
- if $x \leq_P y$ and $y \leq_P z$ then $x \leq_P z$,
- if $x \leq_P y$ and $y \leq_P x$ then $x = y$.

From this definition, we can note that there can exist non-comparable elements inside a poset. When all the elements of a poset are comparable, the set is said to be a '*totally ordered set*' or a '*chain*', while if no two distinct elements of the poset are comparable, the poset is said to be an '*antichain*'.

2.1.2 Examples.

(a). An edgeless graph. In fact this is an example of an antichain poset.

(b). $\mathbb{P} = \{1, 2, 3, \dots\}$ with its usual order $\leq$. In fact this is an example of a totally ordered set.

(c). Let $\overrightarrow{T} = (V = (v_1, v_2, v_3, v_4, v_5),\ E = \{(v_1, v_3), (v_1, v_5), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_5)\})$ be a directed tree. Then its set of vertices together with the reachability order forms a poset.

(d). Let $A = \{x, y, z\}$ and consider its power set $\mathcal{P}(A)$. Then $\mathcal{P}(A)$ together with the containment "$\subseteq$" order forms a poset.

Posets can be graphically visualized by their Hasse diagram. A *Hasse diagram* of a poset P consists of all the elements of P linked according to the *cover* relation and considering an upward orientation. That is, if $x \leq_P y$ in P, then x appears in a lower position to y in the diagram and a line segment is drawn between these two elements if and only if y *covers* x i.e., when there is no $z \in P$ such that $x \leq_P z \leq_P y$. The Hasse diagrams of the posets in Examples 2.1.2 can be seen in Figure 2.1.

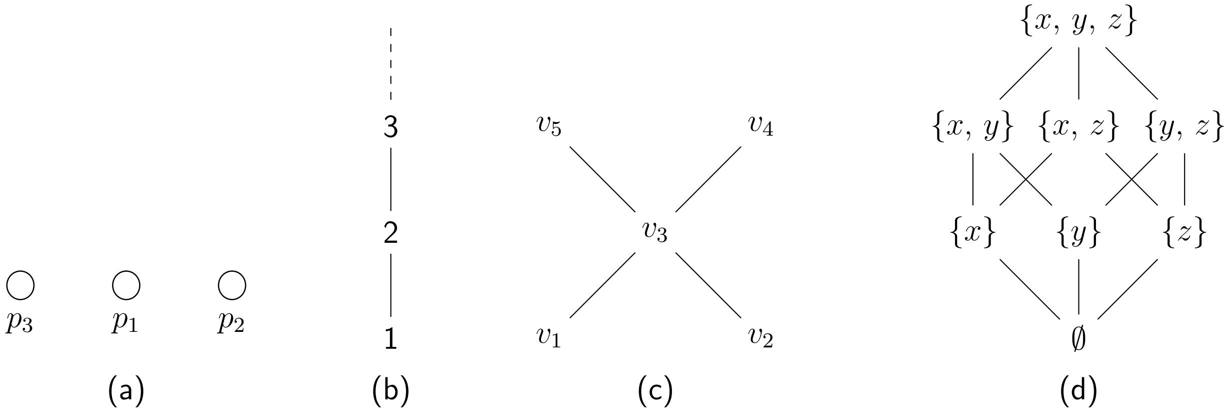


Figure 2.1: Hasse diagrams of posets of Examples 2.1.2

Moreover, given a (finite) poset P one can associate a labelling. A *labelling* of P is a bijection $\omega : P \to \{1, 2, ..., n\}$; where $n = |P|$. A poset P with an associated labelling ω is called a *labelled poset* and denoted by (P, ω). Given a labelled poset (P, ω) and $x, z \in P$, if y covers x and $\omega(x) < \omega(y)$, the relation $x \leq_P y$ is said to be *weak* while when $\omega(x) > \omega(y)$ the relation is said *strict* or *strong*. The weak and strict relations are usually represented in the Hasse diagram respectively by a single and a double line. A labelled poset consisting only of weak relations is said to be *naturally labelled* and will be denoted by just P. Graphically, we represent labelled posets by their Hasse diagram in which each node is annotated by its labelling. Figure 2.2 below is an example of a labelled poset with two weak relations and one strict relation.

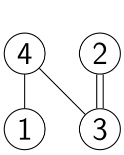


Figure 2.2: An example of a labelled poset (P, ω).

Moreover, operations such as the disjoint union and some natural involutions can be performed on posets [7]. If (P_1, ω_1) and (P_2, ω_2) are labelled posets, then their *disjoint union* (P, ω) is defined by $P = P_1 \sqcup P_2$ and for any $x \in P$, $\omega|_{P_1}(x) = \omega_1(x)$ and $\omega|_{P_2}(x) = \omega_2(x) + |P_1|$. Thus, P is the poset on the union of P_1 and P_2 such that $x \leq_P y$ if either $x, y \in P_1$ and $x \leq_{P_1} y$, or

$x, y \in P_2$ and $x \leq_{P_2} y$. A poset is called *connected* if is not a disjoint union of two non empty posets. The natural involutions on labelled posets (P, ω) that we will consider here are rotation of 180° denoted by $(P, \omega)^* = (P^*, \omega')$ and the application that switches weak and strict edges denoted by $\overline{(P, \omega)} = (P, \omega')$; where $\omega'(x) = |P| + 1 - \omega(x)$ and P^* is the poset P with the order between its elements reversed; i.e., if $x \leq_P y$ then $x \geq_{P^*} y$.

2.2 P-Partition Generating Functions

Definition 2.2.1. Let (P, w) be a labelled poset. A (P, w)-partition is a map $f : P \to \mathbb{P}$ such that for any $x, y \in P$:

- if $x \leq_P y$, then $f(x) \leq f(y)$; that is, f is order-preserving,
- if $x \leq_P y$ and $\omega(x) > \omega(y)$, then $f(x) < f(y)$.

One can note that the definition of (P, ω)-partitions relies on the assignment of strict and weak relations given by the labelling ω. Graphically, we represent (P, ω)-partitions of a poset (P, ω) by adding $f(x)$ next to the node representing x to the Hasse diagram of (P, ω). In Figure 2.3 below we give some (not all) of the (P, ω)-partitions of the labelled poset of Figure 2.2 in the codomain $\{1, 2, 3, 4\}$.

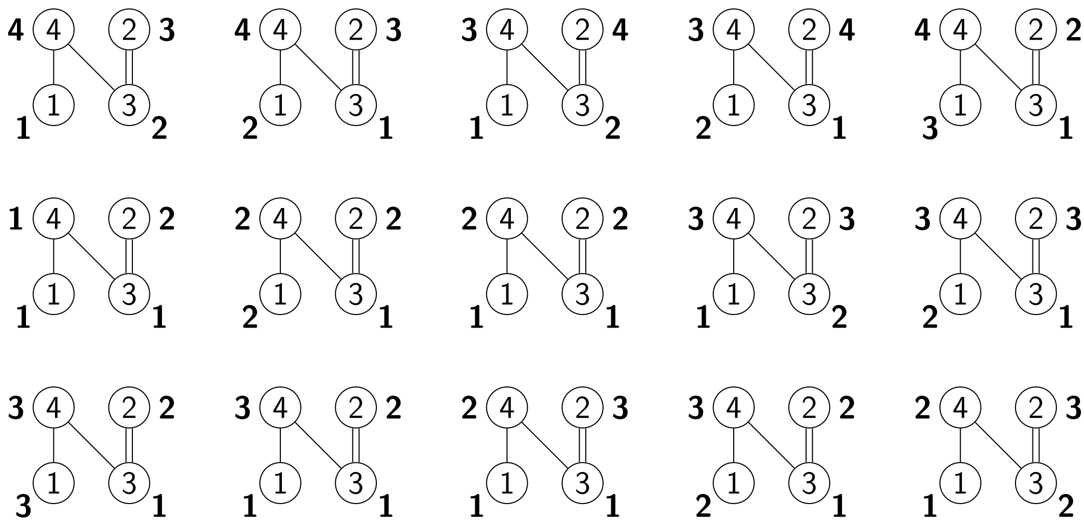


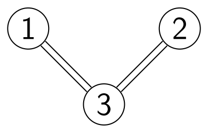
Figure 2.3: (P, ω)-partitions of the labelled poset (P, ω) of Figure 2.2.

(P, ω)-partitions are central to this project. Further informations on posets (labelled) and their (P, ω)-partitions can be found in [11]. Knowing what (P, ω)-partitions are, we can now define the main tool of interest of this work: the (P, ω)-partition generating function.

Definition 2.2.2. Let (P, ω) be a labelled poset. The (P, ω)-partition generating function $K_{(P,\omega)}$ for (P, ω) is the formal power series defined by:

$$K_{(P,\omega)} = K_{(P,\omega)}(x_1, x_2, \cdots) = \sum_{(P,\omega)\text{-partition } f} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots \tag{2.2.1}$$

Example 2.2.3. Considering the labelled poset (P, ω) below,



we have $K_{(P,\omega)}(x_1, x_2, x_3) = 2x_1x_2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2$.

When (P, ω) is a naturally labelled poset, we will denote by K_P its (P, ω)-partition generating function as all natural labellings of P give the same generating function.

Considering the operations on posets we mentioned earlier, we have the following proposition from[7].

Proposition 2.2.4. Let (P_1, ω_1) and (P_2, ω_2) be labelled posets and consider their disjoint union (P, ω). Then, $K_{(P,\omega)} = K_{(P_1,\omega_1)} K_{(P_2,\omega_2)}$,

Recall that our main goal consists of expanding $K_{(P,\omega)}$ into the power sum ϕ basis of *QSym*. Thus, quasisymmetric functions will be the object of the following section.

2.3 Symmetric and quasisymmetric Functions

Before giving an overview of symmetric and quasisymmetric functions, let us recall some definitions about integer compositions and partitions. In fact, to study a polynomial function, it seems natural to consider the different bases of the associated polynomial ring. The algebra of symmetric and quasisymmetric functions have several bases including those indexed respectively by integer *partitions* and integer *compositions* which are of particular interest in algebraic combinatorics. We will mainly focus on the bases of *Qsym*.

Let n be a positive integer.

1. A composition of n (denoted by $\alpha \vDash n$) is a sequence $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ of strictly positive integers which sum to n. That is, $\alpha_i > 0$ for all i and $\sum_{i=1}^{k} \alpha_i = n$. The elements

α_i are called parts. When the parts of α are in a weakly decreasing order, we say that α is a *partition* of n and denote it by $\alpha \vdash n$. For $\alpha \vDash n$, we will denote by $\ell(\alpha)$ the number of parts of α and by $\tilde{\alpha}$ the partition obtained from reordering the parts of α.

2. Let α and β be two compositions of n. Then, we say that β is a *refinement* of α or α is a *coarsening* of β (denoted by $\alpha \succeq \beta$ or $\beta \preceq \alpha$) if

$$\alpha = \left(\beta_1 + \cdots + \beta_{i_1}, \beta_{i_1+1} + \cdots + \beta_{i_1+i_2}, \cdots, \beta_{i_1+\cdots+i_{k-1}+1} + \cdots + \beta_{i_1+\cdots+i_k}\right).$$

In other words, α is obtained by adding consecutive parts of β together. In this case, we denote by $\beta^{(i)}$ the composition consisting of the parts of β that combine to give α_i.

As examples of compositions of 5 we can have $(1, 2, 2) \preceq (3, 2)$; $(2, 2, 1) \succeq (2, 1, 1, 1)$.

In addition, we should note that there is a natural bijection between compositions of n and subsets of $[n-1] = \{1, 2, \cdots n-1, \}$ given by:

- to any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \vDash n$, we associate the subset $\{\alpha_1, \alpha_1 + \alpha_2, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$ that we will denote by $Set(\alpha)$;
- and to any subset $A = \{a_1, a_2, \cdots, a_i\}$ of $[n-1]$ with $a_1 < a_2 < \cdots < a_i$, we associate the composition $\alpha = (a_1, a_2 - a_1, \cdots, a_i - a_{i-1}, n - a_i)$. We will denote this composition by $comp(A)$. For instance, for $\alpha = (3, 1, 2, 4)$, $Set(\alpha) = \{3, 4, 6\}$ and for $A = \{2, 5, 1\} \subset [6]$, $comp(A) = (1, 1, 3, 2)$.

2.3.1 Symmetric Functions. Let R be a commutative ring and $x_1, x_2, x_3, \cdots$ be commuting indeterminates.

A symmetric function f in the variables $x_1, x_2, ...$ with coefficients in R is a formal power series $f(x) \in R[[x_1, x_2, ...]]$ of bounded degree such that for any set of positive integers $\{a_1, a_2, ..., a_k\}$, the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} ... x_{i_k}^{a_k}$ in f is equal to the coefficient in f of $x_{j_1}^{a_1} x_{j_2}^{a_2} ... x_{j_k}^{a_k}$ for any sets $\{i_1, i_2, ..., i_k\}$ and $\{j_1, j_2, ..., j_k\}$ where the i_t's (respectively the j_t's) are distinct. For instance, $\sum_{i \neq j} x_i^2 x_j$ is symmetric while $\sum_{i<j} x_i^2 x_j$ is not.

The set of symmetric functions in $R[[x_1, x_2, ...]]$ forms a graded ring that we will denote *Sym*. In fact *Sym* $= \bigoplus_{n \in \mathbb{N}}$ *Sym*$_n$, where *Sym*$_n$ is a subgroup of *Sym* which consists of homogeneous functions of degree n in *Sym*. And f is said to be homogeneous of degree d if for any $a \in R, f(aX) = a^d f(X)$.

Sym has some natural bases indexed by integer partitions including the monomial basis, the complete homogeneous basis and the power sum basis. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. *The monomial basis* basis of *Sym* is denoted by $\{m_\lambda\}$ where

$$m_\lambda = \sum_{\substack{\alpha : \tilde{\alpha} = \lambda \\ 1 \leq i_1 < i_2 < \ldots < i_k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_k}^{\alpha_k}.$$

The *complete homogeneous basis* $\{h_\lambda\}$ is given by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k} \text{ with } h_{\lambda_i} = \sum_{\gamma \vdash \lambda_i} m_\gamma.$$

The *power sums basis* is $\{p_\lambda\}$ given by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k} \text{ with } p_{\lambda_i} = \sum_j x_j^{\lambda_i}.$$

Further information on these bases can be found in [12].

2.3.2 Quasisymmetric Functions. Quasisymmetric functions are generalizations of symmetric functions and are an important interest in algebraic combinatorics.

Definition 2.3.3. A quasisymmetric function $g(x) \in R[[x_1, x_2, x_3, \cdots]]$ is a formal power series of bounded degree such that for any set of positive integers $\{a_1, a_2, \cdots, a_k\}$, the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_k}^{a_k}$ in g is equal to the coefficient in g of $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 \cdots < j_k$.

The set of quasisymmetric functions *QSym* forms a graded ring. In fact $\mathit{QSym} = \bigoplus_{n \in \mathbb{N}} \mathit{QSym}_n$, where QSym_n is the set of $g \in \mathit{QSym}$ which are homogeneous function of degree n.

As example, the formal power series given by

$$\sum_{1 \leq i < j < k} \left(x_i^2 x_j x_k + x_i x_j^2 x_k + x_i x_j x_k^2 \right)$$

is both symmetric and quasisymmetric while

$$\sum_{1 \leq i < j < k} x_i^2 x_j x_k$$

is quasisymmetric but not symmetric. Most importantly, the (P, ω)-partition generating function of a labelled poset (P, ω) is a quasisymmetric function. In fact $K_{(P,\omega)} \in \mathbb{Q}[[x_1, x_2, x_3, \cdots]]$.

Qsym has some natural bases including the fundamental basis and the analogues to the ones of *Sym* (the monomial basis and the power sum basis).

i) The monomial and fundamental bases of *QSym*.

Let n be a positive integer and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \vDash n$. The quasisymmetric monomial function indexed by α is defined by:

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}. \tag{2.3.1}$$

For instance, we have $M_{(1,2,1)} = \sum_{i<j<k} x_i x_j^2 x_k$. The monomial basis of *QSym* is then $\{M_\alpha\}$ indexed by compositions α.

The quasisymmetric fundamental basis $\{F_\alpha\}$ indexed by compositions $\alpha \vDash n$ is given by:

$$F_\alpha = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in Set(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{2.3.2}$$

In term of the monomial basis, $F_\alpha = \sum_{\beta \preceq \alpha} M_\beta$. This implies that $M_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} F_\beta$ by the Möbius inversion formula.Thus, as an example we can have:

$$\begin{aligned} F_{(2,1)} &= \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \\ &= \sum_{i_1 < i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 = i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \\ &= \sum_{i<j<k} x_i x_j x_k + \sum_{i<j} x_i^2 x_j \\ &= M_{(1,1,1)} + M_{(2,1)} \end{aligned}$$

As $K_{(P,\omega)}$ is a quasisymmetric function, it should be possible to express it in the two bases above. It is clear that in terms of the monomial basis, the coefficient of M_α for any composition $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is the number of (P,ω)-partitions f such that $|f^{-1}(1)| = \alpha_1$, $|f^{-1}(2)| = \alpha_2$, $\cdots$, $|f^{-1}(k)| = \alpha_k$. For the expansion of $K_{(P,\omega)}$ in the fundamental basis, we need the notions of linear extensions of a poset, descent set and descent composition of a permutation.

A *descent set* of a permutation $\sigma \in S_n$ is the set given by $\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$ and it is denoted by $Des(\sigma)$. The *descent composition* of σ (denoted by $co(\sigma)$ is the composition of n given by $(d_1, d_2 - d_1, \cdots, d_k - d_{k-1}, n - d_k)$ with the d_i's are elements of $Des(\sigma)$ such that $d_1 < d_2 < \cdots < d_k$. That is $co(\sigma)$ is $comp(Des(\sigma))$. A *linear extension* of a labelled poset (P,ω) is a permutation of $\omega(P)$ that preserves the order in P. The set of all linear extensions of (P,ω) is denoted by $\mathcal{L}(P,\omega)$. Considering the poset of Figure 2.2, we have:

$$\mathcal{L}(P,\omega) = \{(1,3,2,4), (1,3,4,2), (3,1,2,4), (3,1,4,2), (3,2,1,4)\}.$$

Theorem 2.3.4. *([12]) Let (P,ω) be a labelled poset. Then,*

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{co(\sigma)}.$$

That is, the descent compositions of the elements of $\mathcal{L}(P,\omega)$ determine $K_{(P,\omega)}$. Considering the case of the poset of Figure 2.2, we thus have $K_{(P,\omega)} = F_{(2,2)} + F_{(3,1)} + F_{(1,3)} + F_{(1,2,1)} + F_{(1,1,2)}$. This theorem is very handy in determining the (P,ω)-partition generating function of a poset (P,ω) since in general the number of linear extensions of (P,ω) is far less that its number of (P,ω)-partitions as we can see from the poset of Figure 2.2 which has more than fifteen (P,ω)-partitions and five linear extensions.

Lemma 2.3.5. ([7]) Let (P, ω) be a labelled poset. Then,

(a) the descent sets of the linear extensions of $\overline{(P, \omega)}$ are the complements of the descent sets of the linear extensions of (P, ω);

(b) the descent compositions of the linear extensions of $(P, \omega)^*$ are the reverses of the descent compositions of the linear extensions of (P, ω).

ii) **The power sums bases of QSym**

Ballantine et al. [2] studied two types of power sums bases indexed by integer compositions; the Type 1 basis denoted by $\Psi = \{\Psi_\alpha\}$ and the Type 2 basis denoted by $\Phi = \{\Phi_\alpha\}$.

Type 1 power sums basis refines the power sums basis $\{p_\lambda\}$ of the ring of symmetric functions defined in [12], chapter 7 page 297. Alexandersson and Sulzgruber [1] studied in that basis the expansion of the generating function $K_{(P,\omega)}$ for naturally labelled poset (P, ω). Liu and Weselcouch [4] studied the expansion of $K_{(P,\omega)}$ in that basis and the irreducibility of $K_{(P,\omega)}$ for naturally labelled posets.

The Type 2 power sums basis Φ is also a refinement to the symmetric power sums basis. Its is given by the following expression:

$$\Phi_\alpha = z_\alpha \phi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{1}{sp(\alpha, \beta)} M_\beta. \tag{2.3.3}$$

Where

$$sp(\alpha, \beta) = \prod_i sp(\alpha^{(i)}) \text{ and } sp(\gamma) = \ell(\gamma)! \prod_j \gamma_j;$$

for α, β and γ compositions with α a refinement of β and

$$z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots k^{m_k} m_k!;$$

with m_i the number of occurrences of i in α. For instance, for $\alpha = (3, 2, 2)$ we have:

$$z_\alpha = 2^2 2! 3^1 1! = 24,$$

$$\Phi_{(3,2,2)} = 2M_{(3,2,2)} + M_{(3,4)} + M_{(5,2)} + \frac{1}{3} M_{(7)}.$$

In this project, we are interested in the unnormalized power sums basis $\{\phi_\alpha\}$. From 2.3.3, it follows the monomial quasisymmetric basis can be expanded into the ϕ basis as follows:

$$M_\beta = \sum_{\alpha \succeq \beta} (-1)^{\ell(\beta) - \ell(\alpha)} \frac{\prod_i \alpha_i}{\ell(\beta, \alpha)} \phi_\alpha; \tag{2.3.4}$$

where $\ell(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \ell(\beta^{(i)})$ for β a refinement of α. As mentioned, Φ is a refinement of the symmetric power sum basis.

Theorem 2.3.6. *[2, Theorem 3.17]*

$$p_\lambda = \sum_{\tilde{\alpha}=\lambda} \Phi_\alpha;$$

where p_λ is the power sum basis of Sym.

This overview on posets, their generating functions and quasisymmetric functions is all we need for this project. For further informations on these subjects we refer the reader to [12, 7, 2].

3. $K_{(P,\omega)}$ expansion in the ϕ-basis

In this chapter, beyond constructing the ϕ's expansion of $K_{(P,\omega)}$, we will first give some general results.

3.1 Definitions and Notations

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition of n and $\beta = (\beta_1, \beta_2, \ldots, \beta_q)$ a composition of m. We denote by:

- $\tilde{\alpha}$ the *partition* obtained by rearranging in a weakly decreasing order the parts of α;
- α^r the *reverse* of α, i.e., $\alpha^r = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$;
- $\alpha^c = comp(Set(\alpha)^c)$ the *complementary* of α;
- $\alpha^t = (\alpha^c)^r$ the *transpose* of α.
- $\alpha \sqcup\!\sqcup \beta$ be the set of compositions of $n+m$ that can be obtained by shuffling the parts of α and β in such a way that for any i, α_i appears before α_{i+1} and β_i appears before β_{i+1}.
- $\beta \vee \alpha$ (for $n = m$) the composition γ such that $\mathrm{Set}(\gamma) = \mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta)$. It is in fact the coarsest composition such that $\gamma \preceq \alpha$ and $\gamma \preceq \beta$; that is the coarsest refinement of both α and β. We will denote by $\beta \vee \alpha_i$ the resulting composition into which α_i is refined. If $\beta \succeq \alpha$ then $\beta \vee \alpha_i = \alpha_i$.

For instance, if $\alpha = (2,3,2)$ and $\beta = (3,4)$, then

$$\alpha^c = comp(\{2,5\}^c) = comp(\{1,3,4,6\}) = (1,2,1,2,1);\ Set(\alpha) \cup Set(\beta) = \{2,5\} \cup \{3\}$$

so that

$$\beta \vee \alpha = (2,1,2,2);\ \beta \vee \alpha_1 = \beta \vee \alpha_3 = (2) \text{ and } \beta \vee \alpha_2 = (1,2).$$

From now on, we denote by $L(P,\omega)$ the coefficient of ϕ_n when $K_{(P,\omega)}$ is expanded in the ϕ basis and call it the *leading coefficient* of (P,ω). For naturally labelled posets, we will omit the labelling ω and thus denote its partition generating function by K_P and its leading coefficient by $L(P)$.

3.2 Preliminary results

Recall that the unnormalized power sums quasisymmetric basis of Type 2 is given by

$$\phi_\alpha = \sum_{\beta \succeq \alpha} \frac{1}{sp(\alpha, \beta)} M_\beta; \tag{3.2.1}$$

with $sp(\alpha, \beta) = \prod_i sp(\alpha^{(i)})$ and $sp(\gamma) = \ell(\gamma)! \prod_j \gamma_j$.

Proposition 3.2.1. Let C_n be the naturally labelled chain poset. Then

$$K_{C_n} = \sum_{\alpha \models n} \phi_\alpha. \tag{3.2.2}$$

Proof. As we have a naturally labelled chain, the set of C_n-partitions are in bijection with the set of compositions of n. That is for each composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n, there a unique partition σ of C_n such that $|\sigma^{-1}(1)| = \alpha_1$, $|\sigma^{-1}(2)| = \alpha_2$, $\dots$, $|\sigma^{-1}(k)| = \alpha_k$. Thus,We have:

$$\begin{aligned} K_{C_n} = \sum_{\alpha \models n} M_\alpha = \sum_{\lambda \vdash n} m_\lambda = h_n & \\ &= \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \text{ by [12, Proposition 7.7.6]} \\ &= \sum_{\lambda \vdash n} \sum_{\tilde{\alpha} = \lambda} \phi_\alpha \text{ by Theorem 2.3.6} \\ &= \sum_{\alpha \models n} \phi_\alpha \end{aligned}$$

□

Proposition 3.2.2. If

$$K_P = \sum_{\alpha \models n} c_\alpha \phi_\alpha$$

then one has:

$$K_{P^*} = \sum_{\alpha \models n} c_\alpha \phi_{\alpha^r} \tag{3.2.3}$$

and

$$K_{\bar{P}} = \sum_{\alpha \models n} (-1)^{n - \ell(\alpha)} c_\alpha \phi_\alpha. \tag{3.2.4}$$

Proof. Let us first prove (3.2.3). Consider the involution R on quasi-symmetric functions defined on the monomial basis by: $R(M_\alpha) = M_{\alpha^r}$. Then from the definition of P^*, we deduce that $K_{P^*} = R(K_P)$. Moreover, using (3.2.1) and observing that $\mathrm{sp}(\alpha^r, \beta^r) = \mathrm{sp}(\alpha, \beta)$, we get that $R(\phi_\alpha) = \phi_{\alpha^r}$. Indeed,

$$\phi_{\alpha^r} = \sum_{\beta \succcurlyeq \alpha^r} \frac{1}{\mathrm{sp}(\alpha^r, \beta)} M_\beta = \sum_{\beta^r \succcurlyeq \alpha^r} \frac{1}{\mathrm{sp}(\alpha^r, \beta^r)} M_{\beta^r} = \sum_{\beta \succcurlyeq \alpha} \frac{1}{\mathrm{sp}(\alpha, \beta)} M_{\beta^r} = R(\phi),$$

implying (3.2.3).

For (3.2.4), let us consider [7, Lemma 3.6] which states that if $K_P = \sum_{\alpha \models n} d_\alpha F_\alpha$ then $K_{P^*} = \sum_{\alpha \models n} d_\alpha F_{\alpha^r}$ and $K_{\bar{P}} = \sum_{\alpha \models n} d_\alpha F_{\alpha^c}$. Recall the well-known involution ω on quasisymmetric functions defined by:

$$\begin{aligned} \omega : & Qsym \to Qsym \\ & F_\alpha \longmapsto F_{\alpha^t}. \end{aligned}$$

Then, we have

$$K_{\bar{P}^*} = \sum_{\alpha \models n} d_\alpha F_{(\alpha^c)^r} = \sum_{\alpha \models n} d_\alpha F_{\alpha^t} = \omega(K_P). \tag{3.2.5}$$

Besides, from [2, Section 4], we have

$$\omega(\phi_\alpha) = (-1)^{n-\ell(\alpha)} \phi_{\alpha^r}.$$

Thus, for $K_P = \sum_{\alpha \models n} c_\alpha \phi_\alpha$, we have by 3.2.5 that $K_{\bar{P}^*} = \sum_{\alpha \models n} c_\alpha \omega(\phi_\alpha) = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} c_\alpha \phi_{\alpha^r}$ and by applying (3.2.3) we get (3.2.4). □

Proposition 3.2.3. If (P, ω) is a disconnected poset, then $L((P, \omega)) = 0$.

Proof. Suppose that the poset (P, ω) is of size n and is the disjoint union of two posets (P_1, ω_1) with $|P_1| = n_1$ and (P_2, ω_2) with $|P_2| = n_2$: $(P, \omega) = (P_1, \omega_1) \sqcup (P_2, \omega_2)$. By [5, Proposition 4.6], we have that $K_{(P,\omega)} = K_{(P_1,\omega_1)} K_{(P_2,\omega_2)}$ and [2, equation (23)] tells us that the compositions appearing in the product of 2 ϕ functions $\phi_\alpha\, \phi_\beta$ are in $\alpha \sqcup\!\sqcup \beta$, the shuffle of α and β. But (n) does not appear in any $\alpha \sqcup\!\sqcup \beta$ with α composition of n_1 and β composition of n_2 since $n_1 < n$ and $n_2 < n$. So $\phi_{(n)}$ has coefficient 0 in $K_{(P_1,\omega_1)} K_{(P_2,\omega_2)}$. □

3.3 Expanding elements of the F-basis in the ϕ-basis

In this section, we will give the expression of F_β into the ϕ-basis for any composition β. First observe, any F_β is the partition generating function of a labelled chain. In fact, we know that any labelled chain has only one linear extension which is the labelling itself and by applying Theorem 2.3.4 it follows that the $K_{(C_n,\omega)}$ expansion for some chain C_n with associated labelling ω in the F-basis is just F_β for some composition β such that $\beta = co(\omega)$. The following theorem gives the ϕ expansion of any F_β and thus, for any labelled chain.

Theorem 3.3.1. *The coefficient of ϕ_α in the ϕ-expansion of $F_\beta = K_{(C_n,\omega)}$ is*

$$\prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\beta \vee \alpha_i)-1}}{\binom{\alpha_i - 1}{\ell(\beta \vee \alpha_i) - 1}}.$$

Equivalently,

$$F_\beta = \sum_{\alpha \vDash n} \phi_\alpha \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\beta\vee\alpha_i)-1}}{\binom{\alpha_i - 1}{\ell(\beta\vee\alpha_i)-1}} \tag{3.3.1}$$

or

$$F_\beta = \sum_{\alpha \vDash n} \phi_\alpha (-1)^{|\mathrm{Set}(\beta)\setminus\mathrm{Set}(\alpha)|} \prod_{i=1}^{\ell(\alpha)} \frac{1}{\binom{\alpha_i - 1}{\ell(\beta\vee\alpha_i)-1}}. \tag{3.3.2}$$

With this theorem, we fulfilled our goal of expanding $K_{(P,\omega)}$ into the ϕ-basis. In fact, knowing the expansion of each F_α into the ϕ-basis is sufficient to determine the full ϕ-expansion of any $K_{(P,\omega)}$ in the following way:

1. use Theorem 2.3.4 to expand $K_{(P,\omega)}$ in the F-basis;
2. use Theorem 3.3.1 to determine the ϕ-expansion of each F_α appearing the in F-expansion of $K_{(P,\omega)}$.

Note also that without Theorem 3.3.1, we can still expand the $K_{(P,\omega)}$ into the ϕ-basis by first using the M expansion of the F-basis and then the ϕ expansion of the M-basis from [2]. But this approach is cumbersome as it will include double summation (see equation 3.3.5) and hence the importance of Theorem 3.3.1. Let's illustrate this by the following example.

Example 3.3.2. Let $n \geq 3$ and Y_n denote the naturally labelled Y-shaped poset consisting of a chain of $n-2$ elements with two maximal elements adjoined to the top of the chain (see Figure 3.1).

Thus, the set of linear extensions of Y_n is just the possible labellings of Y_n; that is $\mathcal{L}(Y_n) = \{1, 2, \ldots, n-2, n-1, n\}, \{1, 2, \ldots, n, n-1\}$. Hence we have $K_{Y_n} = F_{co(\{1,2,\ldots,n-2,n-1,n\})} + F_{co(\{1,2,\ldots,n-2,n,n-1\})} = F_{(n)} + F_{(n-1,1)}$ by Theorem 2.3.4. Without using Theorem 3.3.1, we have

$$F_{(n)} = \sum_{\beta \preceq (n)} M_\beta = \sum_{\beta \vDash n} M_\beta = \sum_{\beta \vDash n} \sum_{\alpha \succeq \beta} (-1)^{\ell(\beta)-\ell(\alpha)} \frac{\prod_i \alpha_i}{\ell(\beta,\alpha)} \phi_\alpha$$

and a priori, we cannot directly deduce the coefficient of the ϕ_α's. But, by Theorem 3.3.1, since $(n) \succeq \alpha$ for all $\alpha \vDash n$, we have $(n) \vee \alpha_i = \alpha_i$. Thus, the coefficient of ϕ_α in the ϕ-expansion of F_n will be 1 for all $\alpha \vDash n$. Now to find the expansion of $F_{(n-1,1)}$, we can can consider two cases; the case for which $\alpha \in \{\gamma \vDash n : (n-1,1) \succeq \gamma\} = A$ and the case for which $\alpha \in \{\gamma \vDash n\} \setminus A$. In the former case, the coefficient of ϕ_α is 1 for all $\alpha \in A$ and the elements in A are just compositions of $n-1$ to which we append 1 at the end. For simplicity, let us restrict to the case $n = 4$.

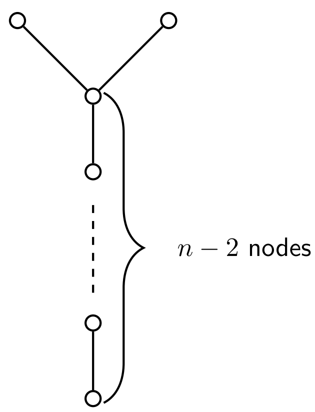


Figure 3.1: Y_n poset

Thus, for $\alpha \in \{(3,1), (2,1,1), (1,2,1), (1,1,1,1)\} = A$, the coefficient of ϕ_α in $F_{(n-1,1)}$ is 1.

So we are just left with the case for which $\alpha \in \{(4),(1,3),(2,2),(1,1,2)\}$. Thus, we get

$$\begin{aligned}
F_{31} &= \frac{(-1)^{|\mathrm{Set}(31)\setminus\mathrm{Set}(4)|}}{\binom{4-1}{\ell(31)-1}}\phi_4 + \phi_{31} + \frac{(-1)^{|\mathrm{Set}(31)\setminus\mathrm{Set}(13)|}}{\binom{3-1}{\ell(21)-1}}\phi_{13} + \frac{(-1)^{|\mathrm{Set}(31)\setminus\mathrm{Set}(22)|}}{\binom{2-1}{\ell(11)-1}}\phi_{22} \\
&\quad + \phi_{211} + \phi_{121} + \frac{(-1)^{|\mathrm{Set}(31)\setminus\mathrm{Set}(112)|}}{\binom{2-1}{\ell(11)-1}}\phi_{112} + \phi_{1111} \\
&= -\frac{1}{3}\phi_4 + \phi_{31} - \frac{1}{2}\phi_{13} - \phi_{22} + \phi_{211} + \phi_{121} - \phi_{112} + \phi_{1111}.
\end{aligned}$$

Thus

$$K_{Y_4} = \frac{2}{3}\phi_4 + 2\phi_{31} + \frac{1}{2}\phi_{13} + 2\phi_{211} + 2\phi_{121} + 2\phi_{1111}.$$

Proof of Theorem 3.3.1. Using the expansion of F_β in the M-basis and then the expansion of M_γ in the ϕ basis from [2] we get

$$F_\beta = \sum_{\gamma \preccurlyeq \beta} M_\gamma \tag{3.3.3}$$

$$= \sum_{\gamma \preccurlyeq \beta} \sum_{\alpha \succcurlyeq \gamma} (-1)^{\ell(\gamma)-\ell(\alpha)} \frac{\prod_i^{\ell(\alpha)} \alpha_i}{\ell(\gamma,\alpha)}\phi_\alpha, \tag{3.3.4}$$

where $\ell(\gamma,\alpha)$ denotes $\prod_{i=1}^{\ell(\alpha)} \ell(\gamma^{(i)})$. Reversing the double sum gives

$$F_\beta = \sum_{\alpha \vDash n} \left(\phi_\alpha (-1)^{\ell(\alpha)} \left(\prod_i^{\ell(\alpha)} \alpha_i \right) \sum_{\gamma \preccurlyeq \beta \vee \alpha} \frac{(-1)^{\ell(\gamma)}}{\ell(\gamma,\alpha)} \right). \tag{3.3.5}$$

Note that, if $\beta \vee \alpha = \beta' \vee \alpha$ for some β', then the coefficient of ϕ_α in F_β will be equal to that in $F_{\beta'}$. For example, taking $\beta' = (n)$ and any $\beta \vDash n$ such that $\beta \succeq \alpha$, the coefficient of ϕ_α in F_β and in $F_{\beta'}$ will be 1. Denote $\beta \vee \alpha$ by δ. Then comparing equation (3.3.1) and equation (3.3.5), it only remains to prove that for all α and δ with $\delta \preccurlyeq \alpha$, we have

$$(-1)^{\ell(\alpha)} \left(\prod_i^{\ell(\alpha)} \alpha_i \right) \sum_{\gamma \preccurlyeq \delta} \frac{(-1)^{\ell(\gamma)}}{\ell(\gamma,\alpha)} = \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\delta^{(i)})-1}}{\binom{\alpha_i - 1}{\ell(\delta^{(i)})-1}},$$

where $\delta^{(i)}$ refers to δ considered as a refinement of α or, in other words, $\delta^{(i)} = \beta \vee \alpha_i$. But

$$\prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\delta^{(i)})-1}}{\binom{\alpha_i - 1}{\ell(\delta^{(i)})-1}} = (-1)^{\ell(\alpha)} \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\delta^{(i)})}}{\binom{\alpha_i - 1}{\ell(\delta^{(i)})-1}}$$

So, we just have to show,

$$\left(\prod_i^{\ell(\alpha)} \alpha_i \right) \sum_{\gamma \preccurlyeq \delta} \frac{(-1)^{\ell(\gamma)}}{\ell(\gamma,\alpha)} = \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\delta^{(i)})}}{\binom{\alpha_i - 1}{\ell(\delta^{(i)})}} \tag{3.3.6}$$

Since, by definition, $\ell(\gamma,\alpha)$ is a product over the parts of α, we can rewrite the left-hand side of equation (3.3.6) to get

$$\prod_{i}^{\ell(\alpha)}\left(\alpha_i \sum_{\gamma^{(i)}\preccurlyeq\delta^{(i)}} \frac{(-1)^{\ell(\gamma^{(i)})}}{\ell(\gamma^{(i)},\alpha_i)}\right) = \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\delta^{(i)})}}{\binom{\alpha_i-1}{\ell(\delta^{(i)})-1}}.$$

Since α_i is just a composition of length 1, we have $\ell(\gamma^{(i)},\alpha_i)=\ell(\gamma^{(i)})$. So, using m in place of α_i, it suffices to show that for all $\delta \vDash m$, we have

$$\sum_{\gamma\preccurlyeq\delta}\frac{(-1)^{\ell(\gamma)}}{\ell(\gamma)} = \frac{(-1)^{\ell(\delta)}}{\binom{m}{\ell(\delta)}\ell(\delta)}. \tag{3.3.7}$$

Now, noting that the right-hand side only depends on $\ell(\delta)=l$ and the summand on the left-hand side only depends on $\ell(\gamma)=k$ and that there are $\binom{m-l}{k-l}$ compositions of length k that refine δ, the proof can be reduced to showing that

$$\sum_{k=l}^{m}\frac{(-1)^k}{k}\binom{m-l}{k-l} = \frac{(-1)^l}{\binom{m}{l}l},$$

and this constitutes our Lemma 3.3.3 below.

□

Lemma 3.3.3. For positive integers n and l with $l\le n$, we have

$$\sum_{k=l}^{n}\frac{(-1)^k}{k}\binom{n-l}{k-l} = \frac{(-1)^l}{\binom{n}{l}l}.$$

Proof. Considering the left-hand side and the change of variable $q=k-l$, we have

$$\sum_{k=l}^{n}\frac{(-1)^k}{k}\binom{n-l}{k-l} = \sum_{q=0}^{n-l}\frac{(-1)^{q+l}}{q+l}\binom{n-l}{q}. \tag{3.3.8}$$

We also know that for x,y positive integers, the Beta function

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{(x-1)!(y-1)!}{(x+y-1)!}. \tag{3.3.9}$$

Rewriting 3.3.9 using the binomial coefficient and setting $x = l$, $y = n - l + 1$, we get

$$\begin{aligned} B(l, n-l+1) = \frac{(l-1)!(n-l)!}{n!} &= \int_0^1 t^{l-1}(1-t)^{n-l}dt \\ &= \int_0^1 \sum_{q=0}^{n-l} \binom{n-l}{q} (-1)^q t^{q+l-1} dt \\ &= \sum_{q=0}^{n-l} \binom{n-l}{q} (-1)^q \left[\frac{t^{q+l}}{q+l}\right]_0^1 \\ &= \sum_{q=0}^{n-l} \binom{n-l}{q} \frac{(-1)^q}{q+l} \end{aligned} \tag{3.3.10}$$

Multiplying 3.3.10 by $(-1)^l$, we have our proof:

$$\sum_{q=0}^{n-l} \binom{n-l}{q} \frac{(-1)^{q+l}}{q+l} = \frac{(-1)^l (l-1)!(n-l)!}{n!} = \frac{(-1)^l}{l\binom{n}{l}}.$$

□

Corollary 3.3.4. For any composition β on n, the coefficient of ϕ_n in the expansion of F_β (call it leading coefficient of F_β) is:

$$L(F_\beta) = \frac{(-1)^{\ell(\beta)-1}}{\binom{n-1}{\ell(\beta)-1}}.$$

Proposition 3.3.5. Let $n \geq 3$. Then, the Leading coefficient of the Y_n poset (i.e., the coefficient of ϕ_n in the K_{Y_n}) is:

$$L(Y_n) = \frac{n-2}{n-1}.$$

Proof. By our example 3.3.2, we have $K_{Y_n} = F_n + F_{(n-1,1)}$. Then the coefficient of ϕ_n will be $L(F_n) + L(F_{(n-1,1)})$ and by Corollary 3.3.4 we have $L(F_n) = 1$ and $L(F_{(n-1,1)}) = \frac{-1}{n-1}$. Therefore $L(Y_n) = \frac{n-2}{n-1}$. □

With corollary 3.3.4, we have a way of determining the coefficient of ϕ_n in any $K_{(P,\omega)}$ by first considering Theorem 2.3.4 which states that

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{co(\pi)}$$

and then applying corollary 3.3.4. Having that in mind, we have determined the following way of getting the coefficient of any ϕ_α in the expansion of $K_{(P,\omega)}$ that includes the (P,ω)-partitions and the leadings coefficient of the subposets arising from them.

Theorem 3.3.6. *Let (P,ω) be a labelled poset. Suppose α is a composition of $n = |P|$ with $\ell(\alpha) \geq 2$. The coefficient of ϕ_α in the ϕ-expansion of $K_{(P,\omega)}$ is*

$$\sum_f \prod_{i=1}^{\ell(\alpha)} L(f^{-1}(i)) \tag{3.3.11}$$

where the sum is over all P-partitions of content α; $f^{-1}(i)$ denotes the labelled subposet of (P,ω) consisting of all elements mapped by f to i.

In Theorem 3.3.6, f is any P-partition; i.e., not only the (P,ω)-partitions. Thus, the f's do not have to satisfy the extra strictness conditions imposed by ω. However the labelled subposets $f^{-1}(i)$'s inherit strict and weak edges from (P,ω). This theorem is particularly helpful when one is looking for the coefficient of ϕ_α for some α for which we have almost all the subposets induced by the partitions of content α are disconnected or chains. For posets with few linear extensions, Theorem 3.3.1 might be more advantageous if we want all the coefficients. For instance, reconsidering the poset Y_4 in example 3.1, we instantly see that the coefficient of $\phi_{(2,2)}$ in K_{Y_4} is 0 since subposets induced by the partitions of contain $(2,2)$ are not all connected. The same, for the coefficient for $\phi_{(1,1,2)}$.

Proof of Theorem 3.3.6. From Theorem 2.3.4, we have:

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{co(\pi)}$$

where $co(\pi)$ denotes the composition determined by the descents of π. Applying Theorem 3.3.1, we get

$$\begin{aligned} K_{(P,\omega)} &= \sum_{\pi \in \mathcal{L}(P,\omega)} \sum_{\alpha \vDash n} \phi_\alpha \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(co(\pi) \vee \alpha_i)-1}}{\binom{\alpha_i - 1}{\ell(co(\pi)\vee\alpha_i)-1}} \\ &= \sum_{\alpha \vDash n} \phi_\alpha \sum_{\pi \in \mathcal{L}(P,\omega)} \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(co(\pi) \vee \alpha_i)-1}}{\binom{\alpha_i - 1}{\ell(co(\pi)\vee\alpha_i)-1}}. \end{aligned}$$

Thus the coefficient of ϕ_α in $K_{(P,\omega)}$ is

$$\sum_{\pi \in \mathcal{L}(P,\omega)} \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(co(\pi) \vee \alpha_i)-1}}{\binom{\alpha_i - 1}{\ell(co(\pi)\vee\alpha_i)-1}}.$$

And by Corollary 3.3.4, it becomes

$$\sum_{\pi \in \mathcal{L}(P,\omega)} \prod_{i=1}^{\ell(\alpha)} L(F_{co(\pi)\vee\alpha_i}). \tag{3.3.12}$$

Now let us directly apply Theorem 2.3.4 to the labelled subposet $f^{-1}(i)$ in the expression 3.3.11. Denote by ω_i the labelling ω restricted to $f^{-1}(i)$. A linear extension $\pi^{(i)}$ of the labelled poset $(f^{-1}(i), \omega_i)$ will be then a permutation of the (P, ω)-labels of the elements of $f^{-1}(i)$, and we can still define $co(f^{-1}(i), \omega_i)$ according to the descents of this permutation. That is

$$K_{(f^{-1}(i),\omega_i)} = \sum_{\pi^{(i)} \in \mathcal{L}(f^{-1}(i),\omega_i)} F_{co(\pi^{(i)})};$$

so that

$$L(f^{-1}(i)) = \sum_{\pi^{(i)} \in \mathcal{L}(f^{-1}(i),\omega_i)} L(F_{co(\pi^{(i)})}).$$

Thus, we get

$$\sum_f \prod_{i=1}^{\ell(\alpha)} L(f^{-1}(i)) = \sum_f \prod_{i=1}^{\ell(\alpha)} \sum_{\pi^{(i)} \in \mathcal{L}(f^{-1}(i),\omega_i)} L(F_{co(\pi^{(i)})}).$$

The $\pi^{(i)}$'s that appear in the second sum on the right-hand side can be concatenated to form a single $\pi \in \mathcal{L}(P, \omega)$. Conversely, any $\pi \in \mathcal{L}(P, \omega)$ can be partitioned into $\ell(\alpha)$ subpermutations $\pi^{(i)}$ according to the parts of α. For example, if $\alpha = (4, 2, 3)$ and $\pi = (815423697)$, then $\pi^{(1)} = (8154)$, $\pi^{(2)} = (23)$, and $\pi^{(3)} = (697)$. Moreover, f can be determined from π and α. In this setup, we get that $co(\pi^{(i)})$ is nothing more than $co(\pi) \vee \alpha_i$. Thus we get

$$\sum_f \prod_{i=1}^{\ell(\alpha)} L(f^{-1}(i)) = \sum_{\pi \in \mathcal{L}(P,\omega)} \prod_{i=1}^{\ell(\alpha)} L(F_{co(\pi) \vee \alpha_i}),$$

as required. □

Let us give one direct consequence of the theorem above. Let us define "tuft" to be the poset T_n with $n \geq 3$ elements which has a single minimal element covered by $n-1$ maximal elements (see Figure 3.3.7).

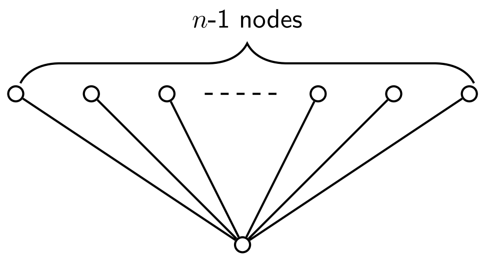


Figure 3.2: T_n poset

Then, we have the following relation.

Proposition 3.3.7. The leading coefficient in the ϕ expansion of K_{T_n} is given by

$$L(T_n) = nB^+(n-1); \tag{3.3.13}$$

where $B^+(n)$ denotes the Bernoulli numbers of the second kind. These are the Bernoulli numbers that follow the convention that $B^+(1) = +\frac{1}{2}$ (as opposed to $-\frac{1}{2}$ in the modern definition of the Bernoulli numbers, denoted $B^-(n)$).

Proof. Recall that $L(Q) = 0$ if Q is disconnected (Proposition 3.2.3), so the expression (3.3.11) can equivalently be restricted to a sum of those f for which $f^{-1}(i)$ is connected for all i. In particular, in the case of T_n, we can restrict to those T_n-partitions of content $(k, 1^{n-k})$ with $1 \leq k \leq n$. Here 1^{n-k} is the standard notation for $n-k$ copies of 1. Moreover, the minimal element is always mapped to 1 and thus for any $1 \leq k \leq n$ we have $\binom{n-1}{k-1}$ choices for elements mapped to 1 and $(n-k)!$ ways for mapping the rest of the elements. This implies that we have $\binom{n-1}{k-1}(n-k)!$ partitions of content $(k, 1^{n-k})$ for any $1 \leq k \leq n$. Thus, Theorem 3.3.6 gives us

$$K_{T_n} = L(T_n)\phi_n + \sum_{k=1}^{n-1} \binom{n-1}{k-1}(n-k)!L(T_k)\phi_{k,1^{n-k}}$$

which simplifies to

$$K_{T_n} = \sum_{k=1}^{n} \binom{n-1}{k-1}(n-k)!L(T_k)\phi_{k,1^{n-k}}. \tag{3.3.14}$$

Now consider the expansion of both sides of (3.3.14) in the monomial basis and, in particular, let us extract the coefficient of M_n. We know that this coefficient in K_{T_n} is 1.

From equation 2.3.3, we get that the coefficient of M_n in ϕ_α is

$$\frac{1}{\ell(\alpha)! \prod_i \alpha_i}.$$

Thus the coefficient of M_n in $\phi_{k,1^{n-k}}$ for $1 \leq k \leq n$ is

$$\frac{1}{(n-k+1)!\, k}.$$

So taking the coefficient of M_n on both sides of (3.3.14) yields

$$
\begin{aligned}
1 &= \sum_{k=1}^{n} \binom{n-1}{k-1} (n-k)! \frac{1}{(n-k+1)!\,k} L(T_k) \\
&= \frac{1}{n} L(T_n) + \sum_{k=1}^{n-1} \binom{n-1}{k-1} (n-k)! \frac{1}{(n-k+1)!\,k} L(T_k) \\
\text{Thus, } L(T_n) &= n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} (n-k)! \frac{n}{(n-k+1)!\,k} L(T_k) \\
&= n - \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{(n-k+1)} L(T_k) \\
&= n - \frac{1}{n+1} \sum_{k=1}^{n-1} \binom{n+1}{k} L(T_k) \qquad (3.3.15)
\end{aligned}
$$

It remains to show (3.3.13). Manipulating equation (3.3.15), we get

$$
\frac{L(T_n)}{n} = 1 - \frac{1}{n} \sum_{k=1}^{n-1} \binom{n}{k-1} \frac{L(T_k)}{k}. \qquad (3.3.16)
$$

On the other hand, the sequence $B^+(m)$ is characterized by the following recursion:

$$
B^+(m) = 1 - \sum_{j=0}^{m-1} \binom{m}{j} \frac{B^+(j)}{m-j+1};
$$

and by substituting $m = n-1$ and $j = k-1$, we get

$$
\begin{aligned}
B^+(n-1) &= 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \frac{B^+(k-1)}{n-k+1} \\
&= 1 - \frac{1}{n} \sum_{k=1}^{n-1} \binom{n}{k-1} B^+(k-1). \qquad (3.3.17)
\end{aligned}
$$

Comparing (3.3.16) and (3.3.17), together with the fact that $L(T_1) = B^+(0) = 1$, yields that $\frac{L(T_n)}{n} = B^+(n-1)$ and hence (3.3.13). $\square$

In summary, we have defined two expansions of $K_{(P,\omega)}$ in the ϕ-basis for labelled posets (P,ω) which follow from Theorem 3.3.1 and Theorem 3.3.6. Indeed, from Theorem 3.3.1, we have

$$
\boxed{K_{(P,\omega)} = \sum_{\beta \in co(\mathcal{L}(P,\omega))} \sum_{\alpha \vDash n} \phi_\alpha \prod_{i=1}^{\ell(\alpha)} \frac{(-1)^{\ell(\beta\vee\alpha_i)-1}}{\binom{\alpha_i-1}{\ell(\beta\vee\alpha_i)-1}}}. \qquad (3.3.18)
$$

From Theorem 3.3.6, we deduce that for any labelled poset (P, ω), the (P, ω)-partition generating function in the ϕ-basis is given by:

$$\boxed{K_{(P,\omega)} = L(P,\omega)\phi_n + \sum_{\substack{\alpha \vDash n \\ \alpha \neq (n)}} \phi_\alpha \sum_f \prod_{i=1}^{\ell(\alpha)} L(f^{-1}(i)),} \tag{3.3.19}$$

where for each α, f runs over all P-partitions of content α.

4. Conclusion and Perpectives

The aim of this work, was to define an expansion of the (P, ω)-partition generating function $K_{(P,\omega)}$ for labelled posets (P, ω) in the power sum basis ϕ of quasisymmetric functions. To that end, we have first defined the context in which this research project fits and given the motivation behind it. Then, we presented the mathematical tools necessary for the construction of our results. More precisely, we have talked about posets, their (P, ω)-partition generating function and (quasi)symmetric functions.

In Chapter 3, which is the main part of this project, we presented our results. Therein, we considered the recently studied power sum basis ϕ of the quasisymmetric functions and derived the leading coefficient of some posets in the ϕ-basis. Most importantly was the construction of two expansions of $K_{(P,\omega)}$ in the ϕ-basis. Lastly, we proved that the leading coefficient for "tuft" posets follows the Bernoulli numbers. Although we have some beautiful expressions of the leading coefficient of some posets, we do not yet have a combinatorial interpretation for the coefficients of ϕ_α in the expansion of $K_{(P,\omega)}$. However, knowing that the P-partition generating function does not distinguish posets in general and using the different expansions we defined here, can we determine some new classes of posets P that can be distinguished from their P-partition generating function?

Acknowledgements

I warmly thank my supervisors Prof. Jean-Christophe Aval and Prof. Peter McNamara for their guidance, time and commitment towards the completion of this project. Thank you also for always checking on me, I appreciated that. My great appreciation to Prof. Marco Garuti for giving me the opportunity to do the ALGANT Masters and to all the ALGANT staff for facilitating my stay over the past two years. A special thank to my parents, siblings and friends for their love and continuous support.

References

[1] Alexandersson, P. and Sulzgruber, R. (2018). p-partitions and p-positivity. *arXiv preprint arXiv:1807.02460*.

[2] Ballantine, C., Daugherty, Z., Hicks, A., Mason, S., and Niese, E. (2017). Quasisymmetric power sums. *arXiv preprint arXiv:1710.11613*.

[3] Ellzey, B. (2017). A directed graph generalization of chromatic quasisymmetric functions. *arXiv preprint arXiv:1709.00454*.

[4] Liu, R. I. and Weselcouch, M. (2019). p-partitions and quasisymmetric power sums. *arXiv preprint arXiv:1903.00551*.

[5] Malvenuto, C. (1993). *Produits et coproduits des fonctions quasi-symétriques et de l'algébre des descentes*. Départements de mathématiques et d'informatique, Université du Québec à Montréal, Ph.D. thesis.

[6] Martin, J. L., Morin, M., and Wagner, J. D. (2008). On distinguishing trees by their chromatic symmetric functions. *Journal of Combinatorial Theory, Series A*, 115(2):237–253.

[7] McNamara, P. R. and Ward, R. E. (2014). Equality of p-partition generating functions. *Annals of Combinatorics*, 18(3):489–514.

[8] Shareshian, J. and Wachs, M. L. (2016). Chromatic quasisymmetric functions. *Advances in Mathematics*, 295:497–551.

[9] Smith, I., Smith, Z., and Tian, P. (2015). Symmetric chromatic polynomial of trees. *arXiv preprint arXiv:1505.01889*.

[10] Stanley, R. P. (1995). A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111(1):166–194.

[11] Stanley, R. P. (2011). Enumerative combinatorics volume 1 second edition. *Cambridge studies in advanced mathematics*.

[12] Stanley, R. P. and Fomin, S. (1999). Enumerative combinatorics. vol. 2, volume 62 of. *Cambridge Studies in Advanced Mathematics*.