UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Laurea Magistrale in Matematica

Tesi di Laurea

RENORMALIZATION OF GIBBS STATES

Supervisors:
Prof. EVGENY VERBITSKIY
University of Leiden

Prof. PAOLO DAI PRA
University of Padova

Author:
ENRICO DI GASPERO
Student Number: 1130798

15 DECEMBER 2017

Contents

Chapter 1

Introduction

Equilibrium statistical mechanics is a branch of probability theory that has born between the end of 19th century and the beginning of the 20th century due to the work of Boltzmann and Gibbs. The latter, in particular, has been the fist one to introduced a statistical approach to thermodynamics to deduce collective macroscopic behaviors from individual microscopic information. Gibbs measures are the central object of this field, and the study of their existence and uniqueness is of great importance to understand the behaviour of a large number of physical infinite-volume models. In particular, the existence of more than one Gibbs measures is associated with statistical phenomena such as symmetry breaking and phase coexistence.

In order to further investigate the behaviour of these systems, theoretical physicists developed a powerful tool: renormalization transformations. Simply speaking, they allows systematic investigation of the changes of a physical system as viewed at different distance scales. However, during the second half of 20th century, it has been noticed that these kind of transformations should be applied carefully: indeed, they may be ill-defined and present some pathologies.

The aim of this work is double: first of all we want to present in details Gibbs formalism and the renormalization transformation. Second of all, we want to analyze one of the most famous examples in literature: the two-dimension Ising model.

In Chapter 2 we will introduce the reader to the Gibbs formalism, giving all the necessary definitions, such as those of Gibbs specification and Gibbs measure, and the main results. At the end of the Chapter, we will also give a brief description of the Ising Model.

Renormalization transformations and their pathologies will be described at the beginning of Chapter 3. Moreover, in this chapter, we will present two known approaches to study the decimation of the two-dimensional ferromagnetic Ising Model: the first one, given by Van Enter-Fernandez-Sokal, shows us that the renormalized measure is no longer Gibbs for low temperatures; the second one, due to Haller & Kennedy, proves that the renormalized measure is Gibbs even for temperature below the critical point. We will conclude this chapter presenting a new approach to study renormalized Gibbs measures introduced by Berghout and Verbitskiy

Chapter 4 is the central part of this master thesis and contains its originality. In this chapter, indeed, the author has applied Berghout & Verbitskiy method to the case of $b = 2$ decimation of the two dimensional ferromagnetic Ising model with zero external magnetic field. The result that has been found is the same proved by Haller & Kennedy, with the advantage of a simpler and more rigorous proof.

Chapter 2

Gibbs formalism

2.1 Interactions and Hamiltonians

Consider a lattice L and let $\mathcal{S}$ be the set of all finite subsets $\Lambda \Subset L$ ($\Subset$ indicates that the subset is finite). When the lattice is $\mathbb{Z}^d$, as in most of our cases, for every $n \in \mathbb{N}$ we will denote the n-th cube by $\Lambda_n = [-n, n]^d$.
To each site i of the lattice we attach the same finite measurable space $(E, \mathcal{E}, \mu_0)$, where the measure μ_0, called *a priori measure*, is the normalized uniform counting measure on E.

Example 2.1.1. In the Ising model of ferromagnetism the lattice L is $\mathbb{Z}^d$, while the state-space E is identified with the set $\{\pm 1\}$.

We will call *configuration space* the product space $(\Omega, \mathcal{F}, \mu) := (E^L, \mathcal{E}^{\otimes L}, \mu_0^{\otimes L})$; a *configuration* $\omega \in \Omega$ is then a collection of random variables $\{\omega_i\}_{i \in L}$, where each ω_i takes values in E. For each $\Delta \in \mathcal{S}$ we will denote by Ω_Δ the finite product space E^Δ and, for every configuration $\omega \in \Omega$, ω_Δ will be the finite configuration $\{\omega_i\}_{i \in \Delta}$, which is a finite collection of random variables. Finally, we define concatenated configurations by prescribing values on partitions of L, writing, for example, $\sigma_\Lambda \omega_{\Lambda^c}$ for the configuration which agrees with a configuration σ on Λ and with another configuration ω on the complement $\Lambda^c = L \setminus \Lambda$ of the set Λ.
The product σ-algebra $\mathcal{F} = \mathcal{E}^{\otimes L}$ is the smallest σ-algebra generated by the cylinders sets

$$C_{\sigma_\Lambda} = \{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\},$$

where $\Lambda \in \mathcal{S}$ is a finite subset of L and $\sigma_\Lambda \in \Omega_\Lambda$.
Since we are considering only finite state spaces E, a neighborhood of ω in the product topology is a set of configurations that agree with ω on some finite set of sites Λ, but are arbitrary outside Λ. We can now introduce the concept of local event and local misurability:

Definition 2.1.2. *For any $\Lambda \in \mathcal{S}$ we define the sub-σ-algebra $\mathcal{F}_\Lambda$ as the σ-algebra generated by:*

$$C_\Lambda = \{C_{\sigma_\Delta} : \sigma_\Delta \in \Omega_\Delta, \Delta \Subset \Lambda\}\,;$$

If in addition Λ is finite, any element of a sub-σ-algebra $\mathcal{F}_\Lambda$ will be called a ***local event****. Furthermore, we will say that a function f is* ***$\mathcal{F}_\Lambda$-measurable*** *if and only if it depends only on the values in Λ:*

$$f \in \mathcal{F}_\Lambda \iff (\omega_\Lambda = \sigma_\Lambda \Longrightarrow f(\omega) = f(\sigma))$$

Another important class of events is the class of those events which don't belong to any $\mathcal{F}_\Lambda$ for Λ finite:

Definition 2.1.3. *An event is said to be a* ***non-local event*** *if it is an element of the σ-algebra at infinity:*

$$\mathcal{F}_\infty := \bigcap_{\Lambda \in \mathcal{S}} \mathcal{F}_{\Lambda^c}. \tag{2.1.1}$$

We also introduce the following space of functions:

- $B(\Omega) = \mathcal{B}(\Omega, \mathcal{F})$ is the set of bounded measureable functions;
- $B_{loc}(\Omega) = \cup_{\Lambda \in \mathcal{S}} B(\Omega, \mathcal{F}_\Lambda)$ is the space of bounded local functions;
- $B_{ql}(\Omega) = \overline{B_{loc}(\Omega)}$ is the space of bounded quasilocal functions, where we will call a function *quasilocal* if it is the uniformly convergent limit of some sequence of local functions.

Remark 2.1.4. An equivalent characterization of a quasilocal function is the following:

$$\lim_{\Lambda \uparrow L} \sup_{\omega, \omega' \in \Omega, \omega_\Lambda = \omega'_\Lambda} |f(\omega) - f(\omega')| = 0\,: \tag{2.1.2}$$

Here the notion of convergence should be interpreted in the sense of "convergence along the net of finite subsets of L, directed by inclusion", that is:

$$\lim_{\Lambda \uparrow L} F(\Lambda) = \alpha$$

if for each $\varepsilon > 0$ there exists a finite subset $V_\varepsilon \subset L$ such that $|f(\Lambda) - \alpha| < \varepsilon$ whenever $\Lambda \supset V_\varepsilon$.

Remark 2.1.5. From the fact that E is finite follows that quasilocality is equivalent to continuity.

We want now to introduce the concept of Hamiltonian. From a physical point of view, a Hamiltonian is a function that assigns to a configuration ω its total energy $H(\omega)$, which is "computed" as the sum of the contributions of each subsystem; these contributions are described by the interaction: a family of function $\Phi = (\Phi_\Lambda)_{\Lambda \in \mathcal{S}}$ which assigns to every $\Lambda \in \mathcal{S}$ the energy of the configuration ω relative to Λ. Formally:

Definition 2.1.6. *An* ***interaction*** *(or potential) is a family $\Phi = (\Phi_\Lambda)_{\Lambda \in \mathcal{S}}$ of functions $\Phi_\Lambda : \Omega \to \mathbb{R}$, such that the function Φ_Λ is $\mathcal{F}_\Lambda$-measurable for each $\Lambda \in \mathcal{S}$. This means that $\Phi_\Lambda(\omega)$ depends only on the values of ω inside Λ:*

$$\Phi_\Lambda(\omega) = \Phi_\Lambda(\omega|_\Lambda).$$

Now we can define the so called Hamiltonian with free boundary conditions where the spins in the finite set Λ do not interact with the exterior:

Definition 2.1.7. *For every $\Lambda \in \mathcal{S}$ we define the* ***Hamiltonian*** $H^\Phi_{\Lambda,free}$ ***with free boundary conditions*** *as the function:*

$$H^\Phi_{\Lambda,free}(\omega) = \sum_{V \in \mathcal{S}, V \subset \Lambda} \Phi_V(\omega_V), \tag{2.1.3}$$

where Φ is the interaction considered.

Note that $H^\Phi_{\Lambda,free}$ is always well-defined since the sum involves only finitely terms. The free boundary condition does not fulfill our needs because we will need to let the finite volume Λ interact with the exterior volume; we should then add the contributions of the subsets which intersect Λ without being its subsets.

Definition 2.1.8. *Let Φ be a potential. For each $\Lambda \in \mathcal{S}$ we define the* ***Hamiltonian*** H^Φ_Λ ***with general external boundary conditions*** *as the function:*

$$H^\Phi_\Lambda(\omega) = \sum_{V \in \mathcal{S}, V \cap \Lambda \neq \emptyset} \Phi_V(\omega), \tag{2.1.4}$$

provided that the sum converges to a finite limit for all $\omega \in \Omega$. Furthermore we will define the ***Hamiltonian*** $H^\Phi_{\Lambda,\tau}$ ***with boundary condition*** *τ as:*

$$H^\Phi_{\Lambda,\tau}(\omega) = H^\Phi_\Lambda(\omega_\Lambda \tau_{\Lambda^c}). \tag{2.1.5}$$

For our purposes we will consider only a special class of potentials for which the convergence in 2.1.4 is assured:

Definition 2.1.9. *An interaction $\Phi = \{\Phi_V\}_V \in \mathcal{S}$ is called* ***uniformly absolutely convergent*** *(UAC) if:*

$$\|\Phi\| := \sup_{i\in L} \sum_{V\in\mathcal{S}, V\ni i} \sup_{\omega\in\Omega} |\Phi_V(\omega)| = \sup_{i\in L} \sum_{V\in\mathcal{S}, V\ni i} \|\Phi_V\|_\infty < \infty.$$

This particular class of interactions contains, for example, all the interactions which are simultaneously finite-range[1]and bounded[2]. This will be the case of the Ising Potential which will be described in Section 2.6.

2.2 Specifications

In this section we will introduce the notion of specification, which plays a central role in the construction of infinite-volume Gibbs Measures. We will follow the idea of Dobrushin [8] and Lanford and Ruelle [9] which consist in defining an infinite-volume Gibbs Measure as the measure whose conditional probabilities for finite subsystems Λ, conditioned on the configuration outside Λ, are given by the Boltzmann-Gibbs formula for the Hamiltonian H_Λ^Φ.

Remark 2.2.1. For nearest-neighbor potentials, as the Ising potential, is sufficient to condition on the spins of Λ^c which are adjacent to Λ.

To formalize mathematically this approach we need to define the notion of specification. Before doing it we recall what a probability kernel is:

Definition 2.2.2. *A* ***probability kernel*** *from a space $(\Omega, \mathcal{F})$ to another space $(\Omega', \mathcal{F}')$ is a map $\gamma : \Omega \times \mathcal{F}' \to [0, 1]$ satisfying:*

1. *for each fixed $\omega \in \Omega$, $\gamma(\omega, \cdot)$ is a probability measure on $(\Omega', \mathcal{F}')$;*
2. *for each fixed $A \in F'$, $\gamma(\cdot, A)$ is a $\mathcal{F}$-measurable function on Ω.*

One can naturally define the product of two probability kernels γ_1, γ_2, which is itself a probability kernel, by:

$$(\gamma_1\gamma_2)(\omega, A) := \int \gamma_2(\omega', A)\gamma_1(\omega, d\,\omega'). \tag{2.2.1}$$

Furthermore, if γ is a probability kernel from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$, γ maps each measure μ on $(\Omega, \mathcal{F})$ to $\mu\gamma$ on $(\Omega', \mathcal{F}')$ which is defined by:

$$(\mu\gamma)(A) := \int \gamma(\omega, A)\mu(d\,\omega). \tag{2.2.2}$$

[1]An interaction Φ has a finite-range if the supremum of the diameters of the sets A with $\Phi_A \neq 0$ is finite

[2]An interaction Φ is said to be bounded if every Φ_Λ is bounded.

In our case we need to specify a probability kernel γ_Λ from $(\Omega_{\Lambda^c}, \mathcal{F}_{\Lambda^c})$ to $(\Omega_\Lambda, \mathcal{F}_\Lambda)$. However, for technical reasons, it is more convenient to define the probability kernel from the full space $(\Omega, \mathcal{F})$ to itself. Because of this, in the following definition, we have to impose conditions (1) and (2):

Definition 2.2.3. *A **specification** is a family $\Gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ of probability kernels from $(\Omega, \mathcal{F})$ to itself, satisfying the following conditions:*

1. *for each $A \in \mathcal{F}$ the function $\gamma_\Lambda(\cdot, A)$ is $\mathcal{F}_{\Lambda^c}$-measurable;*
2. *γ_Λ is $\mathcal{F}_{\Lambda^c}$-proper, i.e. $\gamma_\Lambda(\omega, B) = \chi_B(\omega)$ for each $B \in \mathcal{F}_{\Lambda^c}$;*
3. *If $\Lambda \subset \Lambda'$ then $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$.*

We have introduced the notion of specification in order to prescribe the conditional probabilities for finite volumes Λ when conditioning on the infinite volume Λ^c; we will be interested in those measures whose finite-volume conditional probabilities coincides with the specification given. This are the so called measure consistent with the specification. Formally:

Definition 2.2.4. *A probability measure μ on Ω is said to be **consistent with the specification** $\Gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ if its conditional probabilities for finite subsystems are given by the $(\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ that is:*

$$\text{For each } \Lambda \in \mathcal{S} \text{ and } A \in \mathcal{F},\ \mathbb{E}_\mu[\chi_A(\cdot)|\mathcal{F}_{\Lambda^c}] = \gamma_\Lambda(\cdot, A)\ \mu\text{-a.e.}.$$

We will denote by $\mathcal{G}(\Gamma)$ the set of all measures consistent with Γ.

As we will see in the next section, we will focus our attention on a particular type of specification, those for which the spins inside a finite set Λ depends weakly on the spins that are far away from it. This requirement can be formally described as follows:

Definition 2.2.5. *A specification $\Gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ is said to be **quasilocal** if:*

$$f \in B_{ql}(\Omega) \Longrightarrow \gamma_\Lambda f \in B_{ql}(\Omega)$$

for each $\Lambda \in \mathcal{S}$.

Remark 2.2.6. On $\mathbb{Z}^d$, an equivalent condition for a specification $\Gamma = \{\gamma_\Lambda\}_{V \in \mathcal{S}}$ to be quasilocal is the following:

$$\sup_{\sigma_\Lambda \in \Omega_\Lambda} \sup_{\omega \in \Omega} \sup_{\eta, \xi \in \Omega} |\gamma_\Lambda(\omega_{\Lambda_n \setminus \Lambda} \eta_{\Lambda_n^c \setminus \Lambda}, \sigma_\Lambda) - \gamma_\Lambda(\omega_{\Lambda_n \setminus \Lambda} \xi_{\Lambda_n^c \setminus \Lambda}, \sigma_\Lambda)| \to 0, \tag{2.2.3}$$

for every $\Lambda \in \Omega$.

Finally we introduce two further desirable properties of specifications:

Definition 2.2.7. *A specification* $\Gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ *is said to be:*

- ***non-null*** *(with respect to* μ^0*) if, for each* $\Lambda \in \mathcal{S}$ *and each* $A \in \mathcal{F}_\Lambda$*,*

$$\mu(A) > 0 \Longrightarrow \gamma_\Lambda(\omega, A) > 0 \text{ for all } \omega \in \Omega;$$

- ***uniformly non-null*** *(with respect to* μ^0*) if, for each* $\Lambda \in \mathcal{S}$ *there exist constants* $0 < \alpha_\Lambda \leq \beta_\Lambda < \infty$ *such that:*

$$\alpha_\Lambda \mu(A) \leq \gamma_\Lambda(\omega, A) \leq \beta_\Lambda \mu(A)$$

for all $\omega \in \Omega$ *and all* $A \in \mathcal{F}_\Lambda$.

2.3 Gibbsian Specifications and Gibbs Measures

Consider a classical finite-volume statistical-mechanical system with configuration space Ω, Hamiltonian H and a priori measure μ_0. To this system one can associate the so-called Boltzmann-Gibbs distribution μ_{BG} at inverse temperature β which can be characterized in the following way:

$$d\mu_{BG}(\omega) = \frac{e^{-\beta H(\omega)}}{Z^{-1}} d\mu_0(\omega), \tag{2.3.1}$$

where

$$Z = \int e^{-\beta H(\omega)}\, d\mu_0(\omega). \tag{2.3.2}$$

The notion of Boltzmann-Gibbs measure can not immediately be extended to infinite-volume systems because the Hamiltonian H is, a priori, not well-defined. However, as we already said, we can adapt it to this case requiring that all the finite-volume conditional probabilities, conditioned on the exterior of the volume, coincide with the Boltzmann-Gibbs distribution for the finite volume considered. This led to the theory of Gibbs measures.

We start by defining formally what a Gibbs distribution and a Gibbs measure are. Let Φ be a UAC potential, we then define the partition function as:

$$Z_\Lambda^\Phi(\omega_{\Lambda^c}) = \sum_{\sigma_\Lambda \in \Omega_\Lambda} e^{-H_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda^c})}.$$

Definition 2.3.1. *The probability measure $\gamma_\Lambda^\Phi(\omega, \cdot)$ on $\mathcal{F}$ defined by*

$$\gamma_\Lambda^\Phi(\omega, A) = \frac{1}{Z_\Lambda^\Phi(\omega_{\Lambda^c})} \sum_{\sigma_\Lambda \in \Omega_\Lambda \cap A} e^{-H_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda^c})} \tag{2.3.3}$$

is called the ***Gibbs distribution*** *in volume Λ with boundary condition ω_{Λ^c} corresponding to the interaction Φ.*

It is easy to verify that the family $\Gamma^\Phi = \left\{\gamma_\Lambda^\Phi\right\}_\Lambda$ is indeed a specification (see for example [1, Chapter 2.1] or [5, Theorem 3.23]); it is called the *Gibbsian specification for the interaction* Φ.

Definition 2.3.2. *A measure μ on L consistent with Γ^Φ is called a* ***Gibbs measure*** *for the interaction Φ. We will denote by $\mathcal{G}(\Gamma^\Phi)$ (or $\mathcal{G}(\Phi)$) the set of all the Gibbs measures for the interaction Φ.*

We have the following existence result:

Theorem 2.3.3. *For a UAC potential Φ there exists at least one Gibbs measure, i.e., $\mathcal{G}(\Gamma^\Phi) \neq \emptyset$*

As a consequence of the choice of considering only UAC potentials, it follows that a Gibbsian specification is quasilocal (the details can be found in [1, Prop 2.24, Example 2.25]). Furthermore it is easy to prove that any Gibbsian specification is uniformly non-null.
A fundamental result of Kozlov [6] states that also the opposite is true: a specification $\Gamma = \{\gamma_\Lambda\}_{\Lambda \in \mathcal{S}}$ that is uniformly non-null and quasilocal is a Gibbsian specification for some UAC potential Φ, i.e. $\gamma = \gamma^\Phi$. Furthermore, in the case of state spaces E that are finite, which will be the case we will focus on, Kozlov [6] and Sullivan [7] have observed that a specification is non-null if and only if it is uniformly non-null.
We can summarize what we have just said in the following theorem, which is better known as the Gibbs-representation theorem:

Theorem 2.3.4. *Let Γ be a specification, and μ_0 a product measure. Then the following statements are equivalent:*

(i) there exists a UAC potential Φ such that Γ is the Gibbsian specification for Φ and μ_0;

(ii) Γ is quasilocal and uniformly non-null with respect to μ_0.

Moreover, if the single-spin space E is finite, then these are also equivalent to:

(iii) Γ is quasilocal and is non-null with respect to μ_0.

2.4 Uniqueness of Gibbs Measure

In this section we want to illustrate a result, proved by Dobrushin in 1968 [18], which establishes a sufficient condition for the uniqueness of the Gibbs measure on a system Ω. To do so we will follow the framework used by Georgii in his book [1, Chapter 8].
The idea is to analyze a given specification Γ by looking at the $L \times L$-matrix $C(\Gamma) = \{C_{i,j}(\Gamma)\}_{i,j\in L}$ that describes how much the conditional distribution $\gamma_i(\cdot|\omega)$ of σ_i depends on the value ω_j of the spin at j.
More specifically, let $\Gamma = \{\gamma_\Lambda\}_{\Lambda\in\mathcal{S}}$ be a specification with state space $(E, \mathcal{E})$ on an arbitrary lattice L; we define the **Dobrushin's interdependence matrix** $C(\Gamma) : L \times L \to \mathbb{R}$ by

$$C_{i,j}(\Gamma) := \sup_{\omega,\sigma\in\Omega:\omega_{L\setminus\{j\}}=\sigma_{L\setminus\{j\}}} \left\|\gamma_i^0(\cdot|\omega) - \gamma_i^0(\cdot|\sigma)\right\|, \tag{2.4.1}$$

where γ_i^0 are the projections of the specification Γ on the singletons in L; more precisely:

$$\gamma_i^0(A|\omega) = \gamma_{\{i\}}(\{\sigma_i \in A\}|\omega) \qquad (A \in \mathcal{E}, \omega \in \Omega). \tag{2.4.2}$$

We have to specify how the norm used in equation 2.4.1 is defined; there are several equivalent definition, for our purposes we can simply use the following:

$$\left\|\gamma_i^0(\cdot|\omega) - \gamma_i^0(\cdot|\sigma)\right\| := \max_{A\in\mathcal{E}} |\gamma_i^0(A|\omega) - \gamma_i^0(A|\sigma)|, \tag{2.4.3}$$

which, in the case of countable single-spin space E, can be rewritten in the following way:

$$\left\|\gamma_i^0(\cdot|\omega) - \gamma_i^0(\cdot|\sigma)\right\| = \frac{1}{2}\sum_{x\in E} |\gamma_i^0(\{x\}|\omega) - \gamma_i^0(\{x\}|\sigma)|. \tag{2.4.4}$$

We can now illustrate what it means for a specification to satisfy the Dobrushin's condition:

Definition 2.4.1. *Let Γ be a specification; Γ is said to satisfy Dobrushin's condition if it is quasilocal and*

$$c(\Gamma) := \sup_{i\in L} \sum_{j\in S} C_{i,j}(\Gamma) < 1. \tag{2.4.5}$$

This condition, as proved by Dobrushin, is sufficient to have uniqueness of the Gibbs measure:

Theorem 2.4.2. *If a specification Γ satisfies the Dobrushin's condition then $|\mathcal{G}(\Gamma)| \leq 1$.*

Remark 2.4.3. The condition anyway is not necessary. In the two dimensional ferromagnetic Ising model, for instance, for $\beta < \beta_c$[3] the Dobrushin's condition is not always satisfied, but we have uniqueness of the Gibbs measure.

We want to highlight that the proof presented by Georgii (see[1, Theorem 8.7]) requires the state space E to be the same for every node of the lattice L. However, in Chapter 4.2, we will need a more general version of this results: indeed, we will want to verify the Dobrushin condition for lattice were every single-spin space E to depend on the node to which it is attached:

$$\Omega = \prod_{j \in \mathbb{Z}^d} E_j \tag{2.4.6}$$

Anyway, Georgii's proof can be simply adapt to this case, as we will now show.

Before starting the proof we need to introduce a notion of comparison between two probabilities measure μ and $\tilde{\mu}$ on $(\Omega, \mathcal{F})$. We start by defining the oscillations at single sites for a bounded and quasilocal function. Let $f \in B_{\mathrm{ql}}(\Omega)$ and $j \in L$. The oscillation of f at j is defined by

$$\delta_j(f) := \sup_{\zeta,\eta \in \Omega, \zeta_{L\setminus\{j\}} = \eta_{L\setminus\{j\}}} |f(\zeta) - f(\eta)|.| \tag{2.4.7}$$

We have:

$$\delta_j(f) = \sup_{\omega \in \Omega} \delta(f_{j,\omega}), \tag{2.4.8}$$

where $f_{j,\omega} : E_j \to \mathbb{R}$ is defined by $f_{j,\omega}(x) = f(x\omega_{L\setminus\{j\}})$. Notice that each function $f_{j,\omega}$ is defined on E_j. In the original proof they were all defined on the same space E. This is the only adaptation we have to do in order to adapt Georgii's proof to our different framework.

Furthermore the following inequality holds:

$$\delta(f) := \sup_{\zeta,\eta} |f(\zeta) - f(\eta)| \leq \sum_{j \in L} \delta_j(f) \tag{2.4.9}$$

for all $f \in B_{\mathrm{ql}}(\Omega)$. We can now introduce the notion of local comparison:

Definition 2.4.4. *Let $\mu, \tilde{\mu} \in \mathcal{P}(\Omega, \mathcal{F})$ be given. We will say that a vector $a = (a_j)_{j \in L} \in [0, \infty[^L$ is an **estimate** for μ and $\tilde{\mu}$ if*

$$|\mu(f) - \tilde{\mu}(f)| \leq \sum_{j \in L} a_j \delta_j(f) \tag{2.4.10}$$

for all $f \in B_{ql}(\Omega)$.

[3] The reader who is not familiar with the Ising Model will find a description of it in Section 2.6.

Georgii, at this point, proves the following simple facts:

Remark 2.4.5.

1. The constant vector $a \equiv 1$ is always an estimate.
2. If (2.4.10) holds for all $f \in B_{\mathrm{loc}}(\Omega)$ only, then a is an estimate as well.
3. A coordinatewise limit of a sequence of estimates for μ and $\tilde{\mu}$ is also an estimate for μ and $\tilde{\mu}$.

Fix now two quasilocal specifications γ and $\tilde{\gamma}$, and let $\mu, \tilde{\mu}$ be Gibbs measures for $\gamma, \tilde{\gamma}$ respectively. For each $i \in L$ we let $b_i : \Omega \to [0, \infty[$ be a measurable function such that

$$\|\gamma_i^0(\cdot|\omega) - \tilde{\gamma}_i^0(\cdot|\omega)\| \leq b_i(\omega)$$

for all $\omega \in \Omega$. We have then the following Lemma whose proof we will omit:

Lemma 2.4.6. *Consider the situation described above, and suppose a is an estimate for μ and $\tilde{\mu}$. Define a vector $\overline{a}$ by*

$$\overline{a} := C(\Gamma)a + \tilde{\mu}(b).$$

That is, the i'th coordinate of $\overline{a}$ is:

$$\overline{a}_i = \sum_{j \in L} C_{i,j}(\Gamma) a_j + \tilde{\mu}(b_i).$$

Then $\overline{a}$ is also an estimate for μ and $\tilde{\mu}$.

We need now to introduce some of further notation. Consider a specification Γ; for every $n \geq 0$ we let

$$C^n(\Gamma) = \left\{C^n_{i,j}(\Gamma)\right\}_{i,j \in L}$$

denote the n'th power of the interaction matrix $C(\Gamma)$. We put:

$$D(\Gamma) = \left\{D_{i,j}(\Gamma)\right\}_{i,j \in L} = \sum_{n \geq 0} C^n(\Gamma).$$

We then have the following result:

Theorem 2.4.7. *Let Γ and $\tilde{\Gamma}$ be two specifications, and suppose Γ satisfies Dobruschin's condition. For each $i \in L$ we let b_i be a measurable function on Ω such that*

$$\|\gamma_i^0(\cdot|\omega) - \tilde{\gamma}_i^0(\cdot|\omega)\| \leq b_i(\omega)$$

for all $\omega \in \Omega$. If $\mu \in \mathcal{G}(\Gamma)$ and $\tilde{\mu} \in \mathcal{G}(\tilde{\Gamma})$ then

$$|\mu(f) - \tilde{\mu}(f)| \leq \sum_{i,j \in L} \delta_i(f) D_{i,j}(\Gamma) \tilde{\mu}(b_j)$$

for all $f \in B_{ql}(\Omega)$.

Proof. Let $C = C(\Gamma)$, $D = D(\Gamma)$ and $\tilde{b} = \left\{\mu(\tilde{b}_i)\right\}_{i\in L}$. We may assume that $\tilde{b}_i \leq 1$ for all $i \in L$, if not we replace b_i with $1 \wedge b_i$. We want to show that the vector $D\tilde{b}$ is an estimate for μ and $\tilde{\mu}$.
By Remark 2.4.5 (1) the constant vector $a \equiv 1$ is an estimate. We can then apply Lemma 2.4.6 to find that for each $n \geq 1$ the vector

$$a^{(n)} = C^n a + \sum_{k=0}^{n-1} C^k \tilde{b}$$

is an estimate. If we prove that $a^{(n)}$ tends to $D\tilde{b}$ coordinatewise when $n \to \infty$, then we conclude using 2.4.5 (3). Dobrushin's condition implies:

$$\sum_{j\in L} C^n_{i,j} \leq c(\Gamma)^n$$

for all $n \geq 0$, $i \in L$. As a consequence, the row sums of D are at most $1/(1-c(\Gamma))$; in particular, D has finite entries and $D\tilde{b}$ exists. Finally,

$$C^n a = \sum_{j\in L} C^n_{\cdot,j} \to 0$$

coordinatewise as $n \to \infty$. Hence $a^{(n)} \to D\tilde{b}$, and the proof is concluded. □

To prove Theorem 2.4.2 it is sufficient to apply the previous result to $\tilde{\Gamma} = \Gamma$: choosing $b_i \equiv 0$ for all $i \in L$, we see that $\mu(f) = \tilde{\mu}(f)$ for all $f \in B_{\text{loc}}(\Omega)$. Therefore $\mu = \tilde{\mu}$ whenever $\mu, \tilde{\mu} \in \mathcal{G}(\Gamma)$.

2.5 Markov processes

In this section we follow closely Chapter 3.1 of Georgii's book [1]. We want to discuss an easy example: the Markov specifications on the integers.
Let us choose $L = \mathbb{Z}$ and E a finite non-empty state space. We start with a definition:

Definition 2.5.1. *Consider a specification Γ with lattice $\mathbb{Z}$ and state space E. We will say that Γ is a **positive homogeneous Markov specification** if there is a function $g(\cdot,\cdot,\cdot) > 0$ defined on E^3 such that:*

$$\gamma_{\{i\}}(\sigma_i = x|\omega) = g(\omega_{i-1}, x, \omega_{i+1}),$$

for all $i \in \mathbb{Z}$, $y \in E$ and $\omega \in \Omega$.

The main result about Markov specifications says that each positive homogeneous Markov specification admits a unique Gibbs measure μ. In particular μ is a Markov chain with a positive transition matrix P which can be explicitly computed from Γ.
It is known that to every stochastic matrix $P = \{P(x,y)\}_{x,y\in E}$ with non-zero entries it is associated the unique distribution $\mu_P \in \mathcal{P}(\Omega,\mathcal{F})$ of the stationary Markov chain with transition matrix P. Furthermore μ_P is characterized by the condition:

$$\mu_P(\sigma_i = x_0, \sigma_{i+1} = x_1, \ldots, \sigma_{i+n} = x_n) = \alpha_P(x_0)P(x_0,x_1)\ldots P(x_{n-1},x_n), \quad (2.5.1)$$

where $i \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, $x_0, \ldots, x_n \in E$ and $\alpha_P \in]0,1[^E$ is the unique probability (row) vector satisfying the relation $\alpha P = \alpha_P$.
The following result is proven in [1]:

Theorem 2.5.2. *The relation*

$$\mathcal{G}(\Gamma) = \{\mu_P\}$$

establishes a one-to-one correspondence $\Gamma \leftrightarrow P$ *between the set of all positive homogeneous Markov specifications and the set of all stochastic matrices on* E *with non-vanishing entries. For given* P *the corresponding* Γ *is determined by the equation:*

$$\gamma_\Lambda(\sigma_\Lambda = \zeta|\omega) = \mu_P(\sigma_\Lambda = \zeta|\sigma_{\partial\Lambda} = \omega_{\partial\Lambda}), \quad (2.5.2)$$

with $\Lambda \in \mathcal{S}$, $\omega \in \Omega$, $\zeta \in E^\Lambda$ *and* $\partial\Lambda = \{i \in \mathbb{Z} \setminus \Lambda : |i-j| = 1 \text{ for some } j \in \Lambda\}$. *Conversely,* P *can be expressed in terms of the determining function* g *of* Γ *as:*

$$P(x,y) = Q(x,y)r(y)/qr(x), \quad (2.5.3)$$

with $x, y \in E$, $Q(x,y) = g(a,x,y)/g(a,a,y)$ *for some arbitrarily fixed* $a \in E$, q *is the largest positive eigenvalue of* $Q = \{Q(x,y)\}_{x,y\in E}$ *and* $r \in]0,\infty[^E$ *a corresponding right eigenvector.*

The right term of 2.5.3 can be also expressed explicitly for every $\Lambda \in \mathcal{S}$. Indeed, Λ is of the form

$$\Lambda = \cup_{k=1}^n \{i_k + 1, \ldots, i_k + n_K\},$$

for some $n \geq 1$, $i_k \in \mathbb{Z}$ and $n_k \in \mathbb{N}$ where the union is disjoint. Therefore, 2.5.1 gives:

$$\mu_P(\sigma_\Lambda = \zeta|\sigma_{\partial\Lambda} = \omega_{\partial\Lambda}) = \prod_{k=1}^n \frac{P(\omega_{i_k}, \zeta_{i_k+1})P(\omega_{i_k+1}, \zeta_{i_k+2})\ldots P(\omega_{i_k+n_k}, \zeta_{i_k+n_k+1})}{P^{n_k+1}(\omega_{i_k}, \omega_{i_k+n_k+1})},$$

where P^m is the m'th matrix power of P.
We want now to introduce the so called homogeneous nearet-neighbour potential:

Definition 2.5.3. *We will say that a nearest-neighbour potential* Φ *is* ***homogeneous*** *if there are two functions* $\phi_1 : E \to \mathbb{R}$ *and* $\phi_2 : E \times E \to \mathbb{R}$ *such that:*

$$\Phi_V(\sigma) = \begin{cases} \phi_1(\sigma_i) & \text{if } V = \{i\} \\ \phi_2(\sigma_i, \sigma_{i+1}) & \text{if } V = \{i, i+1\} \end{cases}$$

We then have the following characterization of the positive homogeneous Markov specifications:

Corollary 2.5.4. *A specification* Γ *is a positive homogeneous Markov specification if and only if is Gibbsian for some homogeneous nearest-neighbour potential* Φ*.*

2.6 The Ising Model

The Ising Model in one dimension, also called Ising chain, is a simplified model used to describe a ferromagnetic or anti-ferromagnetic substance. It was first introduce by W. Lenz in 1920 [10] and analyzed by E. Ising in 1924 [11]. The model is based on the following assumptions:

1. The substance modeled consist of spins which can assume two possible values: $+1$ and -1 ("up" and "down" orientation of the spin). So we set the single-spin space $E = \{\pm 1\}$;

2. The spins are disposed to form an infinite one dimension linear chain. In other words they are located at the sites of the lattice $L = \mathbb{Z}$;

3. The interaction among the spins is defined by the potential $\{\Phi_V\}_{V \in \mathcal{S}}$ with each $\Phi_V : E^L \to \mathbb{R}$ constructed in the following way:

$$\Phi_V^{J,h}(\omega) = \begin{cases} -h_i \omega_i & \text{if } V = \{i\} \\ -J_{i,j}\omega_i\omega_j & \text{if } V = \{i, j\} \\ 0 & \text{otherwise} \end{cases}$$

   where $J : L \times L \to \mathbb{R}$ is called the *coupling function* and $h : L \to \mathbb{R}$ the *external magnetic field*.

In the standard Ising model $J_{i,j} = 0$ unless i, j are nearest neighbour, if furthermore both J and h are constant we call it *homogeneous Ising model*. Finally it is said to be *ferromagnetic* when $J \geq 0$, *anti-ferromagnetic* when $J < 0$.

Remark 2.6.1. From a physical point of view, J plays the role of an inverse temperature, i.e., $1/J$ is proportional to the absolute temperature of the system.

We note that the potential of the standard Ising model is UAC, indeed one has:

$$\|\Phi\| = \sup_{i \in L} \{h_i + J_{i,i+1} + J_{i-1,i}\} < \infty.$$

For our purposes we will focus on the ferromagnetic, standard, homogeneous Ising Model. As showed in detail in [1, Chapter 3.2], in this case, for every $J, h \in \mathbb{R}$ there exists a unique Gibbs measure $\mu_{J,h} \in \mathcal{G}(\Phi^{J,h})$ which is characterized by:

$$\mu_{J,h}(\sigma_i) = [e^{-4J} + \sinh^2(h)]^{-\frac{1}{2}} \sinh(h), \tag{2.6.1}$$

for all $i \in \mathbb{Z}$.
It is interesting then to investigate the behaviour of the system in the low temperature limit. To do so, we multiply the potential $\Phi^{J,h}$ by a factor β, the inverse absolute temperature, and we consider $\mathcal{G}(\beta\Phi^{J,h}) = \mathcal{G}(\Phi^{\beta J,\beta h}) = \{\mu_{\beta J,\beta h}\}$ when β goes to infinity. What we find is that for $h > 0$ the measure $\mu_{\beta J,\beta h}$ converges weakly to δ_+, the Dirac measure concentrated on ω^+ (defined by $\omega_i^+ = 1$ $\forall i \in \mathbb{Z}$), and for $h < 0$ it converges to δ_-, the Dirac measure concentrated on $\omega^- = -\omega^+$. The behaviour is slightly different in the case of zero magnetic field: in this case, indeed, the limiting measure is the equidistribution on the set $\{\omega^+, \omega^-\}$. As a consequence of this fact we assist to a asymptotic loss of tail triviality: while each measure μ_J is either equal to 0 or 1 on $\mathcal{F}_\infty$, the limiting measure is not.
The situation gets worse in higher dimensions where we assist to a loss of tail triviality even for finite β's and this will lead to the existence of multiple Gibbs measures.
Consider for example the Ising Model in two dimensions with zero external magnetic field and coupling interaction fixed to 1; as showed in [1, Chapter 6.2], for β sufficiently large there exist two extremal measures $\mu_-^\beta, \mu_+^\beta \in \mathcal{G}(\beta\Phi)$. Furthermore we have:

$$\mu_-^\beta(\sigma_0) < 0 < \mu_+^\beta(\sigma_0),$$

with

$$\mu_+^\beta(\sigma_0) = [1 - (\sinh 2\beta)^{-4}]^{\frac{1}{8}}, \tag{2.6.2}$$

where the explicit formula for $\mu_+^\beta(\sigma_0)$ has been found by Yang in 1952 [12]. This last fact, from a physical point of view, means that at low temperatures the two-dimensional ferromagnetic Ising model admits an equilibrium with positive magnetization, even though there is no external magnetic field.
A stronger statement, based on the results from Ruelle [13] and Lebowitz and Martin-Löf [15] in 1972, says that $|\mathcal{G}(\beta\Phi)| > 1$ if and only if $\mu_+^\beta(\sigma_0) > 0$. This, with the fact that $\mu_+^\beta(\sigma_0)$ is a non-negative non-decreasing function of β, implies

that there exists a critical inverse temperature β_c such that $|\mathcal{G}(\beta\Phi)| = 1$ if $\beta < \beta_c$ and $|\mathcal{G}(\beta\Phi)| > 1$ if $\beta > \beta_c$. This critical value has been proved to have the following explicit expression:

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}). \tag{2.6.3}$$

Finally, Aizenman [16] in 1980 and Higuchi [17] in 1981 proved that μ_-^β, μ_+^β are the only extremal Gibbs measures for $\beta\Phi$, i.e., the set of all the Gibbs measures for the potential $\beta\Phi$ coincides with the interval $[\mu_-^\beta, \mu_+^\beta]$ for all $\beta > \beta_c$.

Chapter 3

Renormalization of Gibbs states

In this Chapter we want to present what a renormalization transformations is, together with the possible pathologies that can arise. Furthermore we will present two approaches to study these pathologies on the case of the two-dimensional ferromagnetic Ising model: the first one, described by Van Enter, Fernandez and Sokal in [2, Chapter 4], is a proof that the renormalized system does not admits Gibbs measures at low temperatures; the second one, introduced by Hallery & Kenendy [3], shows that the renormalized measure is Gibbs at high temperatures. We will conclude this Chapter presenting a new approach to the study of renormalized Gibbs states, which has been proposed by Berghout & Verbitskiy [4], .

3.1 Renormalization Transformations

In this section we want to describe the general framework for studying renormalization transformations (RTs).
More specifically, a renormalization transformation is a rule (which can be either deterministic or stochastic) that generate a configuration ω' of "block spins" given a configuration ω of "original spins". From a mathematical point of view this is given by a probability kernel $T(\omega \to \omega')$. This function is able to take any probability distribution $\mu(\omega)$ on the original spins and map it to a probability distribution $\mu'(\omega')$ of block spins in the following way:

$$\mu'(\omega') = (\mu T)(\omega') = \sum_{\omega} \mu(\omega) T(\omega \to \omega'). \tag{3.1.1}$$

We will now give a formal definition of this concept:

Definition 3.1.1. *A map T from an original system $(\Omega = E^{\mathbb{Z}^d}, \mathcal{F}, \mu)$ to an image (or renormalized) system $(\Omega' = (E')^{\mathbb{Z}^{d'}}, \mathcal{F}', \mu')$ is called a renormalization transformation if it satisfies the following assumptions:*

1. *T is a probability kernel from* $(\Omega, \mathcal{F})$ *to* $(\Omega', \mathcal{F}')$*;*

2. *T carries translation-invariant measures on* Ω *into translation-invariant measures on* Ω'[1]*.*

3. *T is strictly local in position space, with asymptotic volume compression factor* $K < \infty$*. This mean that there exists Van Hove sequences*[2] $\{\Lambda_n\}_n \subset \mathbb{Z}^d$ *and* $\{\Lambda'_n\}_n \subset \mathbb{Z}^{d'}$ *such that*

   (a) *The behavior of the image spins in* Λ'_n *depends only on the original spins in* Λ_n*, i.e. for each* $A \in \mathcal{F}'_{\Lambda'_n}$ *the function* $T(\cdot, A)$ *is* $\mathcal{F}_{\Lambda_n}$*-measureable;*

   (b) $\limsup_{n\to\infty} \frac{|\Lambda_n|}{|\Lambda'_n|} \leq K$*.*

Remark 3.1.2. Properties (1) and (2) make rigorous the equation (3.1.1): they guarantee that the map $\mu \mapsto \mu T$ is a well-defined map from the translational invariant measures on Ω into the translational invariant measures on Ω'.

We have given a very general definition even though in the cases we will analyze things will be less complicated. For example the first property allows the transformation to be stochastic while we will deal only with deterministic ones, where the image configuration ω' is a function $t(\omega)$ of the original one. Furthermore, in our cases, the spacial dimension and the configuration spaces will be the same, i.e., $d = d'$ and $\Omega = \Omega'$. To be precise we will limit ourselves to the study of the so-called decimation transformation.

3.1.1 Decimation Transformation

Let $\Omega' = \Omega$ and $d' = d$, and let b be an integer greater or equal then 2. We define the decimation transformation of parameter b as that deterministic map that "considers" only those spins belonging to the sub-lattice formed by the spins at distance b, i.e.,

$$\omega'_x = \omega_{bx}, \qquad \forall x \in \mathbb{Z}^d. \tag{3.1.2}$$

This is the classical definition of decimation transformation. However, for our purposes, we need to give also a different definition which is nothing but a different way to describe the same transformation. The idea is that one of dividing our spin's lattice into boxes of size b^d and define, on each of these boxes, a function $\pi^\square$ which,

[1] A measure μ is said to be translation invariant if $T_a\mu := \mu \circ \mathbf{T}_a^{-1} = \mu$ for any translation T_a of vector $a \in \mathbb{Z}^d$.

[2] A Van Hove sequence is a sequence of volumes Λ_n which grow in such a way that the surface-to-volume ratio tends to zero; for example we can require that $\lim_{n\to 0} |\partial^-\Lambda_n|/|\Lambda_n| = 0$, where $\partial^-\Lambda_n = \{x \in \Lambda : dist(x, \Lambda^c) \leq 1\}$.

from the original configuration, selects only one spin (the one that "survives" after the traditional decimation). In this way we have different single-spin states in the original and image system, but we have the advantage that the value of the image configuration at site i depends only on the original spins at the box i (while in the traditional way it depends on the spin at site bi).

Example 3.1.3. We would like to use the $b = 2$ decimation in the case of the two-dimensional Ising model to show the differences between the two way to see the decimation:

- **traditional decimation**: in this case both configuration spaces are the same:
$$\Omega = \Omega' = \{\pm 1\}^{\mathbb{Z}^2} .$$
For every $i \in \Omega$ we define $\pi : \Omega \to \Omega'$ in the following way:
$$\omega'_i = \pi(\omega)_i := \omega_{bi}$$

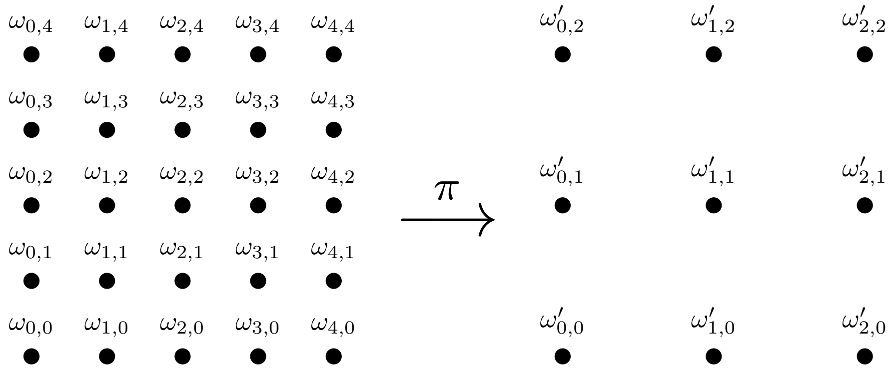

Figure 3.1: The b=2 traditional decimation

- **box decimation**: in this case we define $\Omega = \left\{\{\pm 1\}^4\right\}^{\mathbb{Z}^2}$ and $\Omega = \{\pm 1\}^{\mathbb{Z}^2}$. In this case we define the decimation function componentwise, and we define each component $\pi_i^{\square} : \{\pm 1\}^4 \to \{\pm 1\}$ in the following way:
$$\pi_i(\omega_i) = \omega_i^{0,0} (= \omega'_i),$$
where ω_i is formed by the four spins $\left\{\omega_i^{0,0}, \omega_i^{1,0}, \omega_i^{0,1}, \omega_i^{1,1}\right\}$.
In this way the lattice is formed by box and not by single spins:

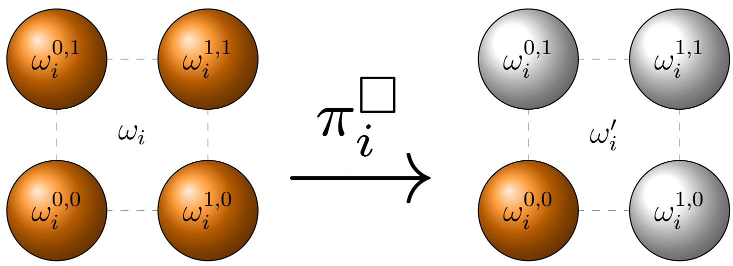

Figure 3.2: The π_i component of the b=2 box decimation

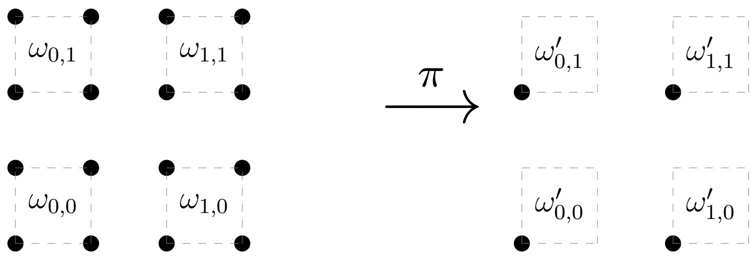

Figure 3.3: The b=2 box decimation

3.2 Possible pathologies

As we have seen in the previous section, it is easy to define the renormalization map on measures. However, for applications, it is more interesting to think of the renormalization transformation acting on Hamiltonians. A natural way to proceed is as follows: if μ is a Gibbs measure for the original system with Hamiltonian H, we may suppose that the renormalized measure μ' is the Gibbs measure for some Hamiltonian H'. This way of defining the renormalization for Hamiltonians is sketched in the following diagram:

$$\begin{array}{ccc} \mu & \xrightarrow{T} & \mu' := \mu T \\ \uparrow & & \downarrow \\ H & \xrightarrow{\mathcal{R}} & H' \end{array}$$

Since the relation between a Hamiltonian and its Gibbs measure is formally given by $\mu = const \times e^{-H}$, we can formally define the renormalization map on the space of Hamiltonians by:

$$H'(\omega') = (\mathcal{R}H)(\omega') = -\log(\sum_{\omega} e^{-H(\omega)} T(\omega \to \omega')) + const. \tag{3.2.1}$$

The problem is that H' given by (3.2.1) is defined only for finite-volume system, while we want to extend it to infinite-volume ones where it is ill-defined (in fact

its value is almost surely $\pm\infty$).
The first problem we have to deal with, is the fact that for the infinite-volume limit the Gibbs measure may be not unique. Consider, for instance, an Hamiltonian H for which there exist at least two distinct extremal translational-invariant Gibbs measure μ_1 and μ_2. When we apply the renormalization T as in (3.1.1), we find the renormalized measures $\mu'_1 = \mu_1 T$ and $\mu'_2 = \mu_2 T$ which, in principle, can be Gibbsian for two different renormalized Hamiltonians $H'_1 \neq H'_2$. This would mean that the renormalization map (R) on the space of the Hamiltonians may be multi-valued. However, as proved in [2, Theorem 3.4], this cannot happen (modulo physical equivalence in the DLR sense): if two initial Gibbs measures correspond to the same interaction Φ, then the renormalized measures are either both Gibbsian for the same renormalized interaction Φ', or else they are both non-Gibbsian.
More serious problems can arise from the downward vertical arrow of the diagram: indeed, although to a given μ' can correspond at most one Hamiltonian H', it can happen that to the given μ' there is no corresponding Hamiltonian H'. This means that it might happen that the images measure μ' is not a Gibbs measure for any Hamiltonian. This non-Gibbsianity of the renormalized measure is the only pathology that can arise for the renormalization map. In particular, it has been proved [19,20] that the the renormalization map is well-defined at high temperature while, in some cases, as we will see in the next section, it is ill-defined for low temperature.

3.3 Decimation of the 2D-Ising Model

In this section we want to present some results concerning the decimation of parameter 2 for the two dimensional ferromagnetic Ising Model with zero magnetic field. The first result, given by Van Enter, Fernandez and Sokal [2, Chapter 4.1.2], is about the non-gibbsianity of the renormalized measure at low temperature. The second result, given by Haller and Kennedy [3], is a proof of the fact that the renormalized measure can be Gibbs even for temperatures below the critical point ($\beta > \beta_c$).
In the following sections μ will be a Gibbs measure for the original system, T the traditional $b = 2$ decimation and $\nu = \mu T$ the renormalized measure. Under this transformation we introduce the following notation:

- *image spins* $(\mathbb{Z}^2)^{img}$: those spins (points of $\mathbb{Z}^2$) with both coordinates even;
- *internal spins* $(\mathbb{Z}^2)^{int}$: the remaining spins.

3.3.1 Non Gibbsianity: Van Enter-Fernandez-Sokal scenario

Van Enter, Fernandez and Sokal proved that for $\beta > \frac{1}{2}\cosh^{-1}(e^{2\beta_c}) \approx 1.73\beta_c$, the renormalized measure is not Gibbs for any potential. We want here to give a briefly description of their argument; further details can be found on their notes. The strategy of the proof is to show that, in the image system, the conditional expectations of $\sigma'_{0,0}$ (here the apex $'$ is used to refer to the image spins) are essentially discontinuous[3] as a function of the boundary conditions. This, indeed, for systems with a finite single-spin space, like the Ising model, is equivalent to non-quasilocality of the renormalized specification. Hence the renormalized measure ν is non-Gibbs. The proof of the essential discontinuity goes in four steps:

- Step 0: *Construction of the specification for the image system.*
  Here we want to compute the conditional probabilities $\nu(\cdot|\left\{\sigma'_j\right\}_{j\in(\Lambda')^c})$ for the image system; these probabilities can be seen as conditional probabilities on the original system $\mu(\cdot|\left\{\sigma'_j\right\}_{j\in\Lambda^c})$, where we are conditioning on a set Λ^c which is not cofinite, and therefore we can not use directly the DLR equations.
  However, we can proceed in the following way: we define the system restricted to the volume $\Delta := \mathbb{Z}^2 \setminus \Lambda^c$ with configuration space $\Omega_\Delta := \{\pm 1\}^\Delta$. The specification for the volume Δ with external spins set to σ_{Λ^c} is the family $\Pi^\sigma = \{\pi_V^\sigma\}_{V\in\mathcal{S},V\subset\Delta}$ defined by:

  $$\pi_V^\sigma(\sigma', A) = \pi_V(\sigma_{\Lambda^c}\sigma', A),$$

  where $\sigma' \in \Omega_\Delta$, $A \in \mathcal{F}_\Delta$ and $\{\pi_V^\sigma\}_V$ is the specification for the original system. At this point, if we consider μ^σ a regular conditional probability for μ given $\mathcal{F}_{\Lambda^c}$, it turns out that the measure $\mu^\sigma|_{\mathcal{F}_\Delta}$ is consistent with Π^σ for μ-a.e. σ [2, Proposition 2.25].

  Remark 3.3.1. Note that the we only know that μ^σ is some Gibbs measure for the restricted specification Π^σ: if there exist more than one Gibbs measures for Π^σ, then we can't know which one is μ^σ. Hence, all the bounds which will be proved in the next steps will have to be valid uniformly for all Gibbs measures for Π^σ.

  Remark 3.3.2. Note also that this computation of the conditional probabilities is valid only for μ-a.e. σ_{Λ^c}. This is why discontinuity of the conditional expectations of $\sigma'_{0,0}$ is not sufficient, and we need to show essential discontinuity.

[3] this means that no modification on a set of (μT)-measure zero can make them continuous.

- Step 1: *Selection of an image-spin configuration* $\omega'_{special}$.
  Since we want to prove that the conditional probabilities $\mu(\cdot|\sigma_{\Lambda^c})$ are essentially discontinuous functions of σ_{Λ^c}, we want to find a point ω' of essential discontinuity. The right configuration ω'_{special}, found by Griffiths and Pearce [21, 22], is the fully alternating configuration ω'_{alt} defined by:

$$\sigma'_{i_1,i_2} := (-1)^{i_1+i_2}.$$

From figure 3.4 we can see that each internal spin is adjacent either to two

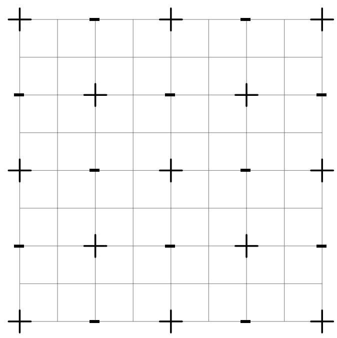

Figure 3.4: The alternating configuration

images spins of opposite sign (in which case the effective magnetic fields cancel) or else to no image spin at all. Therefore the modified object system is simply a ferromagnetic Ising model in a zero field decorated lattice. Integrating out the spins which have exactly two neighbors we find that for $\beta > \frac{1}{2}\cosh^{-1}(e^{2\beta_c}) \approx 1.73\beta_c$ the modify system with image-spin configuration ω'_{alt} has two distinct Gibbs extreme measures (called "+" and "-" phases).

- Step 2: *Study of a neighborhood of* $\omega'_{special} = \omega'_{alt}$.
  In this step we study the internal-spins system for a fixed image configuration ω' in a neighborhood of ω'_{alt}, and we show that the local magnetization $\langle\sigma_i\rangle_{\omega'}$ is a discontinuous function of ω'.
  To do so it is sufficient to show that there exists $\delta > 0$ such that in each neighborhood of ω'_{alt} the essential oscillation of $\langle\sigma_i\rangle_{\omega'}$ is at least δ. In particular, Van Enter, Fernandez and Sokal have proved that there exist $\delta > 0$ such that in each neighborhood $\mathcal{N}$ of ω'_{alt} there exist nonempty open sets

$\mathcal{N}_+, \mathcal{N}_- \subset \mathcal{N}$ and constants $c_+ > c_-$ with $c_+ - c_- \geq \delta$ such that:

$$\langle \sigma_i \rangle_{\omega'} \geq c_+ \text{whenever } \omega' \in \mathcal{N}_+$$
$$\langle \sigma_i \rangle_{\omega'} \leq c_- \text{whenever } \omega' \in \mathcal{N}_-$$

- Step 3: *Unfixing of the spin at the origin.*
  The final step is a slightly modification of the previous one. Indeed in Step 2 we have studied the system of the internal spins with all the image spins fixed to ω'_{alt}, but we need to study the system consisting of the internal spins plus the image spin inside Λ'. However, It is not necessary to do it for all the finite Λ' but it is sufficient to consider only the case of $\Lambda' = \{0\}$.
  The results is the same as before: after some computations we can show that the local magnetization at the origin is essentially greater than δ. More precisely, for every neighborhood $\mathcal{N}$ of ω'_{alt} there exist open sets A_+, A_- such that for all $\omega'_1 \in A_+$ and $\omega'_2 \in A_-$, we have

  $$\mathbb{E}_\nu \left[\sigma'_{0,0} | \left\{ \sigma'_{i,j} \right\}_{(i,j)\neq(0,0)} \right] (\omega'_2) - \mathbb{E}_\nu \left[\sigma'_{0,0} | \left\{ \sigma'_{i,j} \right\}_{(i,j)\neq(0,0)} \right] (\omega'_2) \geq \delta,$$

  with $\delta > 0$.
  This means that the conditional expectations of σ'_0 are essentially discontinuous as a function pf the boundary conditions. Which implies non Gibbsianity of the renormalized measure ν.

3.3.2 Gibbsianity: Haller & Kennedy proof

Haller and Kennedy proved that the renormalized measure ν is Gibbs for $\beta < 1.3645\beta_c$. We want now to present their method without reporting all the computations; the reader can find them on their paper.

The proof is based on the following result which has been proved by Haller and Kennedy:

Theorem 3.3.3. *If μ is a Gibbs measure such that there exist $C < \infty$ and $\lambda > 0$ such that for every finite $V \subset \mathbb{Z}^d$, all $n, m \in V$, every boundary condition τ_{V^c}, and every image spin configuration ω' one has:*

$$|\mu_V^{\omega'}(\sigma_n \sigma_m | \tau_{V^c}) - \mu_V^{\omega'}(\sigma_n | \tau_{V^c}) \mu_V^{\omega'}(\sigma_m | \bar{\tau}_{V^c})| \leq C e^{-\lambda \|n-m\|}, \tag{3.3.1}$$

then $\nu = \mu \circ \pi^{-1}$ is Gibbs.

Remark 3.3.4. The measures $\mu^{\omega'}$'s in the Theorem are the same introduced in the previous Section.

Furthermore, they show that if, on the system of the internal spins, the Dobrushin condition is satisfied uniformly in the block spins, then the hypothesis of their Theorem is satisfied:

Proposition 3.3.5. *If the Dobrushin condition is satisfied uniformly for every block spin configuration ω', i.e., $\sup_{\sigma'} C_{\sigma'} < 1$, then the hypothesis (3.3.1) of Theorem 3.3.3 is satisfied. Hence the renormalized measure ν is Gibbs.*

To prove this, they first tried to check immediately the Dobrushin condition on the internal-spins system. Unfortunately, the condition is not satisfied for any $\beta > \beta_c$. Indeed, after the $b = 2$ decimation, the system of the internal spins contains some spins which have 4 neighbours which are internal spins themselves. Since the interaction is a nearest-neighbour potential, it is clear that for these spins the Dobrushin condition is satisfied if and only if it was satisfied on the original lattice $\mathbb{Z}^2$. Hence, since in the original lattice the condition is not satisfied below the critical temperature, it follows that this direct approach will fail even on the modified lattice.
However, they overcame this obstacle by performing an intermediate step: instead of doing immediately the $b = 2$ renormalization they do two $b = \sqrt{2}$ renormalizations. The final decimated system is the same that we found with the one step renormalization; however, with this expedient, after the first renormalization we find a system where we have removed those spins that made the Dobrushin condition's test fail. Furthermore, for this intermediate system, it is easy to compute the renormalized Hamiltonian, and hence we can test the Dobrushin condition. With this different approach we find exactly what we were looking for: the Dobrushin condition is satisfied for $\beta < 1.3004\beta_c$.
This result can be improved: the central idea is to divide the renormalization in two passages; the factor $\sqrt{2}$ is not important. Indeed, performing a different intermediate decimation we can obtain better results. For instance, they tried the decimation presented in Figure 3.5. In the figure the blocked spins are indicated with the letter B, while the original spins that get integrated after the first decimation are denoted with a circle. They form clusters of five sites with no nearest neighbor interactions between such cluster, hence the sum over these spins can be done explicitly. Finally, the original spins that survived after the first step decimation are indicated with "X"'s.
With this different intermediate decimation they find that the Dobrushin condition on each modified system is satisfied for $\beta < 1.3645\beta_c$. .

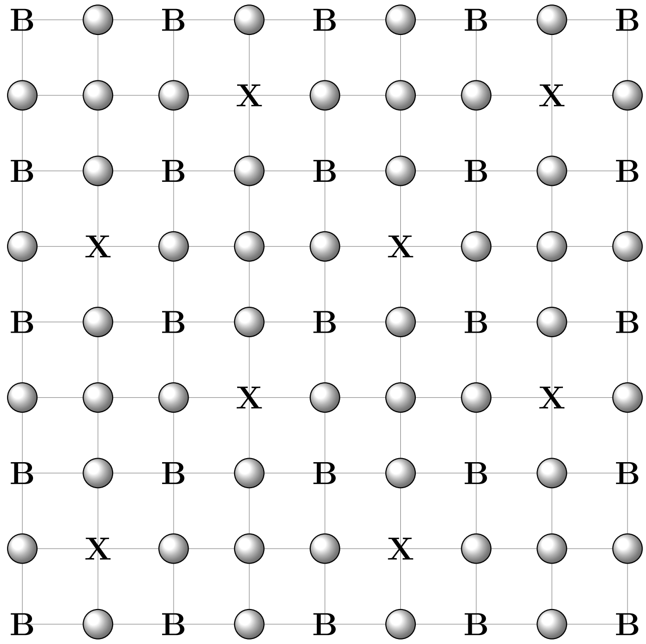

Figure 3.5: A different two step renormalization

3.4 Continuous measure disintegrations

In this section we will present a new approach to the study of renormalized pathologies for renormalized Gibbs measure. These approach has been studied by E. Verbitskiy and S. Berghout and has not been published yet. In Chapter 4 we will apply these results to the two-dimensional ferromagnetic Ising model.

Let $d \geq 1$ and $\mathscr{X} = \mathscr{A}^{\mathbb{Z}^d}$, $\mathscr{Y} = \mathcal{B}^{\mathbb{Z}^d}$, where $\mathscr{A}$, $\mathcal{B}$ are finite alphabets, and $|\mathscr{A}| > |\mathcal{B}|$. Suppose $\pi : \mathscr{A} \to \mathcal{B}$ is onto. We refer to π as a fuzzy coding map (factor). We use the same letter π to denote the (componentwise) extension of π to a mapping from $\mathscr{A}^V$ onto $\mathcal{B}^V$ for any subset $V \subseteq \mathbb{Z}^d$.

3.4.1 Conditional measures on fibres

For every image-spin configuration $y \in \mathscr{Y}$ we denote by $\mathscr{X}_y$ the fibre over y, i.e., the set of all the original configurations which are mapped by π into the fixed image configuration y:

$$\mathscr{X}_y = \pi^{-1}(y) \subset \mathscr{X}.$$

For every y, $\mathscr{X}_y$ is a closed, but not necessarily translation invariant subset of $\mathscr{X}$.

Remark 3.4.1. The fiber $\mathscr{X}_y$ can be seen as a lattice system itself. Indeed, in terms of [14], we have

$$\mathscr{X}_y = \prod_{n\in\mathbb{Z}^d} \mathscr{A}_{y_n},$$

where for every $b \in \mathcal{B}$, $\mathscr{A}_b = \{a \in \mathscr{A} : \pi(a) = b\}$.

We introduce now the central notion of *measure disintegration*, which is nothing but a collection of probabilities measures on the fibers $\mathscr{X}_y$ which respects the law of total expectation:

Definition 3.4.2. *A family of measures $\boldsymbol{\mu}_{\mathscr{Y}} = \{\mu_y\}_{y\in\mathscr{Y}}$ is called a family of conditional measures for μ on fibres $\mathscr{X}_y$ (or disintegration of μ) if*

(a) $\mu_y(\mathscr{X}_y) = 1$;

(b) for all $f \in L^1(\mathscr{X}, \mu)$, the map

$$y \to \int_{\mathscr{X}_y} f(x)\mu_y(dx)$$

is measurable and

$$\int_{\mathscr{X}} f(x)\mu(dx) = \int_{\mathscr{Y}}\int_{\mathscr{X}_y} f(x)\mu_y(dx)\nu(dy).$$

Remark 3.4.3. Note that

$$\int_{\mathscr{X}_y} f(x)\mu_y(dx) = \mathbb{E}\Big(f \,\Big|\, \pi^{-1}\mathfrak{B}(\mathscr{Y})\Big),$$

where $\pi^{-1}\mathfrak{B}(\mathscr{Y})$ is the σ-algebra of sets

$$\Big\{\pi^{-1}(C) : \ C \in \mathfrak{B}(\mathscr{Y})\Big\}$$

A first result, as one can easily expect, is that every measure admits at least one family of conditional measures on fibres, see [26]. Furthermore, as the following proposition says, we also have uniqueness of the measure disintegration (modulo a set of ν-measure zero):

Proposition 3.4.4. *If $\boldsymbol{\mu}_{\mathscr{Y}} = \{\mu_y\}$ and $\tilde{\boldsymbol{\mu}}_{\mathscr{Y}} = \{\tilde{\mu}_y\}$ are two families of conditional measures of μ on fibres $\{\mathscr{X}_y\}$. Then*

$$\nu\Big(\Big\{y : \mu_y \neq \tilde{\mu}_y\Big\}\Big) = 0.$$

Anyway, in order to study the renormalization of Gibbs measures, measure disintegrations are not enough; we need something more, a particular type of family of conditional measures. We need to introduce the so called *continuous measure disintegration*:

Definition 3.4.5. *A family of conditional measures* $\{\mu_y\}_{y\in\mathscr{Y}}$ *for* μ *on fibres* $\mathscr{X}_y$ *is called* ***continuous*** *if for every continuous* $f : \mathscr{X} \to \mathbb{R}$*, the map*

$$y \mapsto \int_{\mathscr{X}_y} f(x)\mu_y(dx)$$

is continuous on $\mathscr{Y}$*.*

While the existence of a measure disintegration is always granted, things are more complicated for continuous measure disintegrations. However, if for a given measure μ and 1-block transformation π there exists a continuous family of conditional measures, we have the following important result:

Theorem 3.4.6. *Suppose* $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$ *with* $\Phi \in \mathcal{B}_1(\mathscr{X})$*,* $\pi : \mathscr{X} \to \mathscr{Y}$ *is a 1-block factor. Suppose* μ *admits a continuous family* $\{\mu_y\}$ *of conditional measures on fibres* $\{\mathscr{X}_y\}$*. Then* $\nu = \mu \circ \pi^{-1}$ *is a Gibbs state on* $\mathscr{Y}$*.*

In other words the existence of a continuous measure disintegration is a sufficient condition for the renormalized measure ν to be Gibbs.

3.4.2 Tjur points

T. Tjur [24, 25] introduced the following procedure for construction of conditional measures on fibres, which we will now recall.
Suppose $y_0 \in \mathscr{Y}$, denote by D_{y_0} the set of pairs (V, B), where V is an open neighbourhood of y_0 and B is a measurable subset of V such that $\nu(B) > 0$. A pair (V_1, B_1) is said to be **closer** to y_0 than (V_2, B_2), denoted by $(V_1, B_1) \succcurlyeq (V_2, B_2)$, if $V_1 \subseteq V_2$. This relation gives a partial order on D_{y_0}. Moreover, $(D_{y_0}, \succcurlyeq)$ is upwards directed: for any two elements in D_{y_0} there exists a third element, which is closer to y_0 than both of them. For $(V, B) \in D_{y_0}$ define a measure μ^B on X as the conditional measure on $\pi^{-1}B$:

$$\mu^B(\cdot) = \mu(\cdot \,|\, \pi^{-1}B).$$

The set $\left\{\mu^B(\cdot) \,|\, (V, B) \in D_{y_0}\right\}$ is a **net**, or a **generalized sequence**, in the space of probability measures on $\mathscr{X}$. We want to study the limiting points of this set, which have been named **Tjur points**:

Definition 3.4.7. *If the limit (in the sense of net convergence)*

$$\mu^{y_0} = \lim_{D_{y_0} \ni (V,B) \uparrow \infty} \mu^B$$

exists and belongs to a set of probability measures on $\mathscr{X}$, *then* μ^{y_0} *is called the conditional distribution of* x, *given* $\pi(x) = y_0$; *and we say that* y_0 *is a Tjur point.*

Remark 3.4.8. The limit of a generalized sequence is understood in the net convergence sense, in the weak sense of measure convergence; more specifically, for any $\varepsilon > 0$ and $f \in C(\mathscr{X})$, there exists an open neighbourhood V of y_0 such that for any $B \subseteq V$ with $\nu(B) > 0$ one has

$$\Big| \int f(x)\mu^B(dx) - \int f(x)\mu^{y_0}(dx) \Big| < \varepsilon.$$

We want the reader to notice that Definition 3.4.7 requires that the limit, if it exists, is a probability measure. This is automatic when $\mathscr{X}$ is a compact space; however, for more general spaces, this is a non-trivial condition.
To sum up, we have constructed the limiting distributions $\{\mu^y\}$, provided that they exist, with the explicit hope to be the fibre measures in a conditional disintegration of μ. As we have discussed above, any disintegration $\{\mu_y\}$ of μ is defined ν-almost everywhere. Therefore, if the limiting distributions $\{\mu^y\}$ exist for ν-a.e. image-spin distribution y, i.e., if the set of Tjur points has full ν-measure, one should hope that $\{\mu^y\}$ could constitute a valid disintegration of the measure μ. Indeed, this is true, as the following result shows.

Theorem 3.4.9. [25, Theorem 5.1] *Suppose measures* μ^y *are defined for almost all* $y \in \mathscr{Y}$. *Then for any integrable* $f \in L^1(\mathscr{X}, \mu)$, f *is* μ^y*-integrable for almost all* y, *and the function* $y \mapsto \int f d\mu^y$ *is* ν*-integrable; furthermore*

$$\int_{\mathscr{X}} f(x)\mu(dx) = \int_{\mathscr{Y}} \Big[\int f(x)\mu^y(dx) \Big] \nu(dy).$$

Anyway, this is not completely satisfying for us. Indeed, as mentioned above, we are interested in finding a measure disintegration which is continuous. The interesting fact is that, as the following Theorem states, the function which assigns to a Tjur point y the distribution μ^y is continuous:

Theorem 3.4.10. [25, Theorem 4.1] *Denote by* $\mathscr{Y}_0$ *the set of all Tjur points in* $\mathscr{Y}$. *Then the map*

$$y \mapsto \mu^y$$

is continuous on $\mathscr{Y}_0$.

It follows that if the set of Tjur points has full measure, then we are able to construct a continuous measure disintegration $\{\mu^y\}$:

Theorem 3.4.11. [25, Theorem 7.1] *Suppose $\pi : \mathscr{X} \to \mathscr{Y}$ is a surjective 1-block factor. The following conditions are equivalent:*

(i) the family of measures $\{\mu_y\}$ on fibres $\{\mathscr{X}_y\}$ constitute a continuous disintegration of μ (c.f., Definition 3.4.5);

(ii) conditional distributions μ^y are defined for all $y \in \mathscr{Y}$ and $\mu^y = \mu_y$.

Furthermore, on lattice systems, one can validate the fact that all points are Tjur by checking uniform convergence of conditional probabilities. In particular it is sufficient to check the uniform convergence when one conditions on cylindric events in $\mathscr{Y}$.

Theorem 3.4.12. *The following conditions are equivalent*

(1) Every point $y \in \mathscr{Y}$ is a Tjur point; i.e., the limiting conditional distribution μ^y exists for all y;

(2) for all $y \in \mathscr{Y}$, the sequence of measures $\mu^{C_n^y}$ converges as $n \to \infty$ (to the limit μ^y), and the convergence is uniform in y: for every $\varepsilon > 0$ and $f \in C(\mathscr{X})$ there exists $N \geq 1$ such that for all $n \geq N$,

$$\Big| \int f(x)\mu^{C_n^y}(dx) - \int f(x)\mu^y(dx) \Big| < \varepsilon$$

for all $y \in \mathscr{Y}$.

3.4.3 Limiting conditional distributions and hidden phase transitions

As in the previous section, fix $y \in \mathscr{Y}$ and consider the net of conditional measures

$$\mathcal{N}_y = \Big\{ \mu^B(\cdot) = \mu(\cdot|\pi^{-1}B) : \quad (V, B) \in D_y \Big\}.$$

Since we are interested in the limiting points of these sets, we want to introduce a definition of accumulation point for the net $\mathcal{N}_y$:

Definition 3.4.13. *A measure $\tilde{\mu}$ is an accumulation point of the net $\mathcal{N}_y$ if for all $f \in C(\mathscr{X})$, $\varepsilon > 0$, and for every open set V containing y, there exists a set $B \subseteq V$, $\nu(B) > 0$, such that*

$$\Big| \int f(x)\mu^B(dx) - \int f(x)\tilde{\mu}(dx) \Big| < \varepsilon.$$

We will denote by $\overline{\mathfrak{M}}_y$ the set of all possible accumulation points of $\mathcal{N}_y$.

Remark 3.4.14. Clearly, since $\mathscr{X}$ is compact, $\overline{\mathfrak{M}}_y$ is not empty.

We then have the following result:

Theorem 3.4.15. *For every $y \in \mathscr{Y}$ the following holds:*

(a) $\overline{\mathfrak{M}}_y \neq \varnothing$ and for every $\lambda_y \in \overline{\mathfrak{M}}_y$, $\lambda_y(\mathscr{X}_y) = 1$.

(b) Suppose μ is a Gibbs measure on $\mathscr{X}$ for potential Φ, then $\overline{\mathfrak{M}}_y \subseteq \mathcal{G}_{\mathscr{X}_y}(\Phi)$, where $\mathcal{G}_{\mathscr{X}_y}(\Phi)$ is the set of Gibbs states on $\mathscr{X}_y$ for potential Φ.

Combining the previous result with the properties of Tjur points, we are able to formulate two easy corollaries.

Corollary 3.4.16. *If $\nu = \mu \circ \pi^{-1}$ is a renormalized Gibbs state and*

$$|\overline{\mathfrak{M}}_y| = 1$$

for all $y \in \mathscr{Y}$, then ν is Gibbs.

Clearly, $|\overline{\mathfrak{M}}_y| = 1$ for all $y \in \mathscr{Y}$, implies that all points in $\mathscr{Y}$ are Tjur. Hence we have a continuous measure disintegration $\{\mu_y\}$, and thus ν is Gibbs by Theorem 3.4.6.
However, the sufficient condition $|\overline{\mathfrak{M}}_y| = 1$ for all y is not easy to validate. Since $\overline{\mathfrak{M}}_y \subset \mathcal{G}_{\mathscr{X}_y}(\Phi)$ for all y, we also have the following weaker result.

Corollary 3.4.17. *If $\nu = \mu \circ \pi^{-1}$ is a renormalized Gibbs state and*

$$|\mathcal{G}_{\mathscr{X}_y}(\Phi)| = 1$$

for all $y \in \mathscr{Y}$, then ν is Gibbs.

Corollaries 3.4.16 and 3.4.17 give us two conditions which assure us the Gibbsianity of the renormalized measure ν. In particular, Berghout & Verbitskiy claim that the condition $|\mathcal{G}_{\mathscr{X}_y}(\Phi)| = 1$ for all y can be validated relatively easy in a number of examples.
In Chapter 4 we will try to use these results to check the Gibbsianity of decimated Markov (dim=1) and Ising (dim=2) models.

3.4.4 Conclusions

We can summarize the results of Berghout & Verbitskiy as follows:

1. The transformed measure $\nu = \mu \circ \pi^{-1}$ is Gibbs if there exists a continuous measure disintegration $\{\mu_y\}$ of μ;

2. Continuous measure disintegration $\{\mu_y\}$ of μ exists if and only if all points $y \in \mathscr{Y}$ are Tjur;

3. Requirement that all points $y \in \mathscr{Y}$ are Tjur is equivalent to uniform convergence of the sequence of measures $\mu^{C_n^y}$ as $n \to \infty$;

4. Condition of existence of a unique Gibbs measure for potential Φ on each fibre $\mathscr{X}_y$ implies that every point $y \in \mathscr{Y}$ is Tjur.

Hence if we want to show that $\nu = \mu \circ \pi^{-1}$ is Gibbs, it is **sufficient** to check any of the following conditions:

- uniform in $y \in \mathscr{Y}$ convergence of $\mu^{C_n^y}$ as $n \to \infty$;
- uniqueness of Gibbs measures for potential Φ on each fibre $\mathscr{X}_y$ for all y.

Chapter 4

Examples

In this Chapter we want to apply the approach presented in Chapter 3.4 to the two dimensional ferromagnetic Ising model with zero magnetic field. Before studying this case, however, we would like to present a simpler case: the decimation of Markov processes in one dimension.

4.1 Decimation of Markov Processes

Let $\Omega = E^{\mathbb{Z}}$ with $|E| < \infty$ and let π be the decimation transformation of scale parameter $b \geq 2$:

$$\begin{aligned} \pi\colon \Omega &\to \Omega \\ \{x_n\}_n &\mapsto \{y_n\}_n = \{x_{bn}\}_n \end{aligned}$$

As usual, we will denote by $\mathcal{S}$ the collection of all the finite subsets of Ω. Let $\{X_n\}_n$ be the (stationary) Markov chain with transition matrix $P > 0$ on the state space E. As stated in Chapter 2.5, there exists a unique positive vector α such that $\alpha P = \alpha$, and the stationary (shift-invariant) distribution of $\{X_n\}_n$ is given by:

$$\mathbb{P}(X_k = x_k, X_{k+1} = x_{k+1}, \ldots, X_{n-1} = x_{n-1}, X_n = x_n) = \alpha_{x_k} P_{x_k, x_{k+1}} \cdots P_{x_{n-1} x_n}.$$

Consider the resulting process $\{Y_n\}_n$ given by $Y_n = X_{bn}$ for every $n \in \mathbb{Z}, b \geq 2$.

Remark 4.1.1. Since $\{X_n\}_n$ is a Markov Process we have:

$$\mathbb{P}(X_n = j | X_0 = i) = P^n_{i,j}$$

We will indicate with $\mathbb{P}$ the law of $\{X_n\}_n$ and with $\mathbb{Q} = \mathbb{P} \circ \pi^{-1}$ the law of $\{Y_n\}_n$. We now want to illustrate three different approaches to prove that $\mathbb{Q}$, the law of the renormalized process $\{Y_n\}_n$, is a Gibbs measure.

4.1.1 First approach: direct computation

Since in this case the situation is pretty simple, we can explicitly write the re-normalized conditional probabilities and verify that the re-normalized law is Gibbs. Let us write down the conditional probabilities of the process $\{Y_n\}_n$:

$$\begin{aligned}\mathbb{Q}(Y_0 = y_0 | Y_1^m = y_1^m) =& \frac{\mathbb{Q}(Y_0 = y_0, Y_1 = y_1, \dots, Y_m = y_m)}{\mathbb{Q}(Y_1 = y_1, \dots Y_m = y_m)},\\ =& \frac{\mathbb{P}(X_0 = y_0, X_b = y_1, \dots, X_{bm} = y_m)}{\mathbb{P}(X_b = y_1, \dots X_{bm} = y_m)},\\ =& \frac{\alpha_{y_0} P^b_{y_0,y_1} \cdots P^b_{y_{m-1},y_m}}{\alpha_{y_1} P^b_{y_1,y_2} \cdots P^b_{y_{m-1},y_m}} = \frac{\alpha_{y_0} P^b_{y_0,y_1}}{\alpha y_1}\end{aligned}$$

$$\begin{aligned}\mathbb{Q}(Y_0 = y_0 | Y_1^m = y_1^m, Y_{-k}^{-1} = y_{-k}^{-1}) =& \frac{\mathbb{Q}(Y_{-k}^{-1} = y_{-k}^{-1}, Y_0 = y_0, Y_1^m = y_1^m)}{\sum_{\bar{y}_0} \mathbb{Q}(Y_{-k}^{-1} = y_{-k}^{-1}, Y_0 = \bar{y}_0, Y_1^m = y_1^m}\\ =& \frac{\alpha_{y_{-k}} P^b_{y_{-k},y_{-k+1}} \cdots P^b_{y_{-1},y_0} P^b_{y_0,y_1} \cdots P^b_{y_{m-1},y_m}}{\sum_{\bar{y}_0} \alpha_{y_{-k}} P^b_{y_{-k},y_{-k+1}} \cdots P^b_{y_{-1},\bar{y}_0} P^b_{\bar{y}_0,y_1} \cdots P^b_{y_{m-1},y_m}}\\ =& \frac{P^b_{y_{-1},y_0} P^b_{y_0,y_1}}{\sum_{\bar{y}_0} P^b_{y_{-1},\bar{y}_0} P^b_{\bar{y}_0,y_1}}.\end{aligned}$$

Therefore, since the conditional probabilities $\mathbb{Q}(Y_0 = y_0 | Y_1^m = y_1^m, Y_{-k}^{-1} = y_{-k}^{-1})$ are constant for $k, m \geq 1$, they converge uniformly for $k, m \to +\infty$.
Thus we have:

$$\begin{aligned}\mathbb{Q}(Y_0 = y_0 | Y_{-\infty}^{-1} = y_{-\infty}^{-1}, Y_1^{+\infty} = y_1^{+\infty}) =& \frac{P^b_{y_{-1},y_0} P^b_{y_0,y_1}}{\sum_{\bar{y}_0} P^b_{y_{-1},\bar{y}_0} P^b_{\bar{y}_0,y_1}}\\ =&: \gamma_{\{0\}}(y_{\{0\}} | y_{\{0\}^c}).\end{aligned} \tag{4.1.1}$$

Remark 4.1.2. For our purposes the singleton part of $\{\gamma_\Lambda\}_{\Lambda\in\mathcal{S}}$,i.e., the sub-family given by all the Λ's which are singletons, is in principle enough. Indeed, as a consequence of [1, Theorem 1.33], our specification is determined by its singleton part.

However, (4.1.1) can be easily generalized:

$$\begin{aligned}\mathbb{Q}(Y_k^m = y_k^m | Y_{-\infty}^{k-1} = y_{-\infty}^{k-1}, Y_{m+1}^{+\infty} = y_{m+1}^{+\infty}) =& \frac{P^b_{y_{k-1},y_k} \cdots P^b_{y_m,y_{m+1}}}{\sum_{\bar{y}_k^m} P^b_{y_{k-1},\bar{y}_k} \cdots P^b_{\bar{y}_m,y_{m+1}}}\\ =&: \gamma_{[k,m]}(y_{[k,m]}, y_{[k,m]^c}).\end{aligned} \tag{4.1.2}$$

which can be further extended to the case of a generic finite-subset of indexes:

$$\mathbb{Q}(Y_{k_1}^{m_1} = y_{k_1}^{m_1}, \ldots, Y_{k_n}^{m_n} = y_{k_n}^{m_n} | Y_{-\infty}^{k_1-1} = y_{-\infty}^{k_1-1}, Y_{m_1+1}^{k_2-1} = y_{m_1+1}^{k_2-1}, \ldots, Y_{m_{n-1}+1}^{k_n-1} = y_{m_{n-1}+1}^{k_n-1} Y_{m_n+1}^{+\infty} = y_{m_n+1}^{+\infty})$$
$$= \frac{P^b_{y_{k_1-1},y_{k_1}} \cdots P^b_{y_{m_1},y_{m_1+1}} P^b_{y_{k_2-1}y_{k_2}} \cdots P^b_{y_{m_2}y_{m_2+1}} \cdots P^b_{y_{k_n-1}y_{k_n}} \cdots P^b_{y_{m_n}y_{m_n+1}}}{\sum_{\bar{y}_{k_i}^{m_i}} P^b_{y_{k_1-1},\bar{y}_{k_1}} \cdots P^b_{\bar{y}_{m_1},y_{m_1+1}} P^b_{y_{k_2-1}\bar{y}_{k_2}} \cdots P^b_{\bar{y}_{m_2}y_{m_2+1}} \cdots P^b_{y_{k_n-1}\bar{y}_{k_n}} \cdots P^b_{\bar{y}_{m_n}y_{m_n+1}}} =: \gamma_\Lambda(y_\Lambda, y_{\Lambda^c}),$$

with $\Lambda = \bigcup_{i=i}^n [k_i, m_i]$ where the union is disjoint.
Clearly the family $\Gamma = \{\gamma_\Lambda\}_\Lambda \in \mathcal{S}$ we have constructed is a family of probability kernels from $(\Omega, \mathcal{F})$ to itself; let's prove that it is a specification:

- For each $A \in \mathcal{F}$, the function $\gamma_\Lambda(A, \cdot)$ is $\mathcal{F}_{\Lambda^c}$ measurable: this is trivial since $\gamma_\Lambda(A, y_{\Lambda^c})$, for A fixed, depends only on finitely many y_i's with $i \in \Lambda$; indeed it depends only on those i belonging to the set $\partial^+\Lambda := \{i : dist(i, \Lambda) = 1\}$.

- For each $B \in \mathcal{F}_{\Lambda^c}$, $\gamma_\Lambda(B, \tau) = \mathbb{1}_B(\tau)$: this is trivial from the way we have constructed $\{\gamma_\Lambda\}_\Lambda$. Indeed, by looking at (4.1.2) we notice immediately that for $y \in \mathcal{F}_{\Lambda^c}$ we have $\gamma_\Lambda(y|\tau_{\Lambda^c}) = 1$ (resp. $\gamma_\Lambda(y|\tau_{\Lambda^c}) = 0$) if $y_{\Lambda^c} = \tau_{\Lambda^c}$ (resp. $y_{\Lambda^c} \neq \tau_{\Lambda^c}$).

- If $\Lambda \subset \Lambda'$, then $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$: we prove the property for $\Lambda = \{0\}$ and $\Lambda' = \{-1, 0, 1\}$, the general case follows.

$$\begin{aligned}
\gamma_{\Lambda'}\gamma_\Lambda(\tau, \omega) &= \sum_{\sigma_{\Lambda'}} \gamma_\Lambda(\sigma, \omega)\gamma_{\Lambda'}(\tau, \sigma) \\
&= \sum_{\sigma_{-1}^1} \mathbb{1}_{\omega_{-1}\omega_1}(\sigma) \frac{P^b_{\sigma_{-1},\omega_0} P^b_{\omega_0,\sigma_1}}{\sum_{\overline{\omega}_0} P^b_{\sigma_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\sigma_1}} \cdot \frac{P^b_{\tau_{-2},\sigma_{-1}} P^b_{\sigma_{-1},\sigma_0} P^b_{\sigma_0,\sigma_1} P^b_{\sigma_1,\tau_2}}{\sum_{\overline{\omega}_{\Lambda'}} P^b_{\tau_{-2},\overline{\omega}_{-1}} P^b_{\overline{\omega}_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\overline{\omega}_1} P^b_{\overline{\omega}_1,\tau_1}} \\
&= \sum_{\sigma_0} \frac{P^b_{\omega_{-1},\omega_0} P^b_{\omega_0,\omega_1}}{\sum_{\overline{\omega}_0} P^b_{\omega_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\omega_1}} \cdot \frac{P^b_{\tau_{-2},\omega_{-1}} P^b_{\omega_{-1},\sigma_0} P^b_{\sigma_0,\omega_1} P^b_{\omega_1,\tau_2}}{\sum_{\overline{\omega}_{\Lambda'}} P^b_{\tau_{-2},\overline{\omega}_{-1}} P^b_{\overline{\omega}_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\overline{\omega}_1} P^b_{\overline{\omega}_1,\tau_1}} \\
&= \frac{\sum_{\sigma_0} P^b_{\omega_{-1},\sigma_0} P^b_{\sigma_0,\omega_1}}{\sum_{\overline{\omega}_0} P^b_{\omega_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\omega_1}} \cdot \frac{P^b_{\tau_{-2},\omega_{-1}} P^b_{\omega_{-1},\omega_0} P^b_{\omega_0,\omega_1} P^b_{\omega_1,\tau_2}}{\sum_{\overline{\omega}_{\Lambda'}} P^b_{\tau_{-2},\overline{\omega}_{-1}} P^b_{\overline{\omega}_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\overline{\omega}_1} P^b_{\overline{\omega}_1,\tau_1}} \\
&= \frac{P^b_{\tau_{-2},\omega_{-1}} P^b_{\omega_{-1},\omega_0} P^b_{\omega_0,\omega_1} P^b_{\omega_1,\tau_2}}{\sum_{\overline{\omega}_{\Lambda'}} P^b_{\tau_{-2},\overline{\omega}_{-1}} P^b_{\overline{\omega}_{-1},\overline{\omega}_0} P^b_{\overline{\omega}_0,\overline{\omega}_1} P^b_{\overline{\omega}_1,\tau_1}} = \gamma_{\Lambda'}(\tau, \omega).
\end{aligned}$$

So Γ is a specification; furthermore, it is wasy to see that Γ is local and non-null; hence $\mathbb{Q}$, which is consistent with this specification, is a Gibbs measure.

4.1.2 Second approach: uniform convergence of the measures on the fibres

This approach is based on one of the two conditions, presented in Chapter 3.4, which implies the Gibbsianity of the renormalized distribution: the uniform convergence of the conditional probability law $\mu_n^y(A) := \mu(A|y_{[-n,n]})$ on the fibres.

First of all we should slightly modify the model to adapt it to the set-up we have seen in Chapter 3. Indeed, while the co-domain of the decimation π is a sub-lattice of the domain, we want to deal with a transformation for which the domain and the co-domain are the same lattice. To do so we divide the lattice Ω into boxes containing b spins each as shown in Figure 4.1.

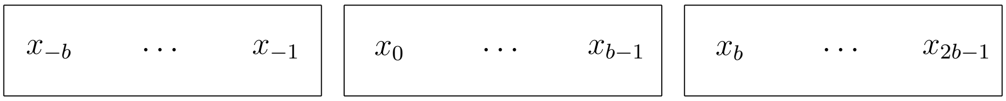

Figure 4.1: The box-lattice $\Omega_\Box$

We will denote the new lattice with $\Omega_\Box := \left\{A^b\right\}^{\mathbb{Z}}$.In this way the process $\{Y_n\}_n$ can be written in the form $Y_n = \pi_\Box(\hat{X}_n)$ where:

$$\hat{X}_n = (X_{bn}, \dots, X_{b(n+1)-1}),$$

and the function $\phi_\Box$ returns the first component of the vector $\hat{X}_n$.

Remark 4.1.3. Note that $\left\{\hat{X}_n\right\}_n$ is still a Markov chain with transition matrix $P^\Box > 0$ as well: if $x_0^{b-1}, \tilde{x}_0^{b-1}$ are two consecutive strings of length b then:

$$\begin{aligned} P^\Box_{x_0^{b-1},\tilde{x}_0^{b-1}} =& \mathbb{P}(\tilde{x}_0^{b-1}|x_0^{b-1}) = \mathbb{P}(\tilde{x}_0^{b-1}|x_{b-1}) = \frac{\mathbb{P}(x_{b-1}\tilde{x}_0^{b-1})}{\mathbb{P}(x_{b-1})} \\ =& \frac{\alpha_{x_{b-1}} P_{x_{b-1},\tilde{x}_0} P_{\tilde{x}_0,\tilde{x}_1} \cdots P_{\tilde{x}_{b-2},\tilde{x}_{b-1}}}{\alpha_{x_{b-1}}} \\ =& P_{x_{b-1},\tilde{x}_0} P_{\tilde{x}_0,\tilde{x}_1} \cdots P_{\tilde{x}_{b-2},\tilde{x}_{b-1}} > 0. \end{aligned}$$

We want to show that for every $A \in \mathcal{F}$ the sequences $\mu_n^y(A) := \mu(A|y_{[-n,n]})$ converges uniformly in y for n that goes to infinity. It is sufficient to verify this for a cylinder so we will consider an interval $[bk, bm-1]$ and we will compute $\mu_n^y(x_{[bk,bm-1]})$ for every x such that $x_i = y_{i/b}$ for every $i \in [bk, bm-1]$ multiple of b. Since we are considering the limit for n that goes to infinity we can suppose $[bk, bm-1] \subset [-nb, nb]$.

We can now compute the conditional probabilities $\mu_n^y(x_{bk}^{bm-1})$:

$$\begin{aligned}
\mu(x_{bk}^{bm-1}|y_{[-n,n]}) =& \mathbb{P}(X_{bk}^{bm-1}=x_{bk}^{bm-1}|X_{-nb}=y_n, X_{(-n+1)b}=y_{-n+1},\ldots,X_{nb}=y_n)\\
=&\frac{\mathbb{P}(X_{bk}^{bm-1}=x_{bk}^{bm-1}, X_{-nb}=y_n, X_{(-n+1)b}=y_{-n+1},\ldots,X_{nb}=y_n)}{\mathbb{P}(X_{-nb}=y_n, X_{(-n+1)b}=y_{-n+1},\ldots,X_{nb}=y_n)}\\
=&\frac{\alpha_{y_{-n}}P^b_{y_{-n}y_{-n+1}}\cdots P^b_{y_{k-1}x_{bk}}P_{x_{bk}x_{bk+1}}\cdots P_{x_{bm-2}x_{bm-1}}P_{x_{bm-1}y_m}\cdots P^b_{y_{n-1}y_n}}{\alpha_{y_{-n}}P^b_{y_{-n}y_{-n+1}}\cdots P^b_{y_{k-1}y_k}\cdots P^b_{y_{m-1}y_m}\cdots P^b_{y_{n-1}y_n}}\\
=&\frac{P_{x_{bk}x_{bk+1}}\cdots P_{x_{bm-2}x_{bm-1}}P_{x_{bm-1}y_m}}{P^b_{y_ky_{k+1}}\cdots P^b_{y_{m-1}y_m}}
\end{aligned}$$

which clearly does not depend on n, and hence the limit exists.
Note also that the limit is constant for n larger than a certain $N(A)$ and so the convergence is uniform in y hence, according to Theorem 3.4.12, the law of the re-normalized distribution is Gibbs.

4.2 Decimation of the 2D Ising model

We now turn our attention to the two-dimensional ferromagnetic Ising Model with inverse temperature β and zero magnetic field. We consider a Gibbs measure μ on $\mathcal{X}=\{\pm 1\}^{\mathbb{Z}^2}$ and π, the 2-decimation transformation. We want to find sufficient conditions for the renormalized measure $\nu=\mu\circ\pi^{-1}$ to be Gibbs.

Following the scheme of Chapter 3.4, we define $\mathcal{X}_\square=\left\{\{\pm 1\}^4\right\}^{\mathbb{Z}^2}$, $\mathcal{Z}_\square=\{\pm 1\}^{\mathbb{Z}^2}$. By looking at Figure 4.2 it is clear that the Gibbs measure μ on $\mathcal{X}$ gives, in a natural way, a Gibbs measure $\mu_\square$ on $\mathcal{X}_\square$ which interaction is given by:

$$\begin{aligned}
\Phi^\square_{\{(i,j)\}}(\sigma) &= \beta(\sigma^{0,0}_{i,j}\sigma^{0,1}_{i,j}+\sigma^{0,1}_{i,j}\sigma^{1,1}_{i,j}+\sigma^{1,1}_{i,j}\sigma^{1,0}_{i,j}+\sigma^{1,0}_{i,j}\sigma^{0,0}_{i,j}),\\
\Phi^\square_{\{(i,j),(i+1,j)\}}(\sigma) &= \beta(\sigma^{1,1}_{i,j}\sigma^{0,1}_{i+1,j}+\sigma^{1,0}_{i,j}\sigma^{0,0}_{i+1,j}),\\
\Phi^\square_{\{(i,j),(i,j+1)\}}(\sigma) &= \beta(\sigma^{0,1}_{i,j}\sigma^{0,0}_{i,j+1}+\sigma^{1,1}_{i,j}\sigma^{1,0}_{i,j+1})
\end{aligned}$$

and $\Phi^\square_\Lambda(\sigma)=0$ in all the other cases, i.e., Λ is neither a one-point set nor a nearest neighbour couple.
Consider now the tranformation $\pi^0_\square$ which, as showed in figure 4.3, selects the botton-left spin of each box-configuration:

$$\begin{array}{cccc}
\pi^0_\square: & \{\pm 1\}^4 & \longrightarrow & \{\pm 1\}\\
 & (\sigma^{0,0},\sigma^{0,1},\sigma^{1,0},\sigma^{1,1}) & \longmapsto & (\sigma^{0,0})
\end{array}$$

We will indicate by $\pi_\square$ the map from $\mathcal{X}_\square$ to $\mathcal{Z}_\square$ which is the coordinatewise extension of $\pi^0_\square$.

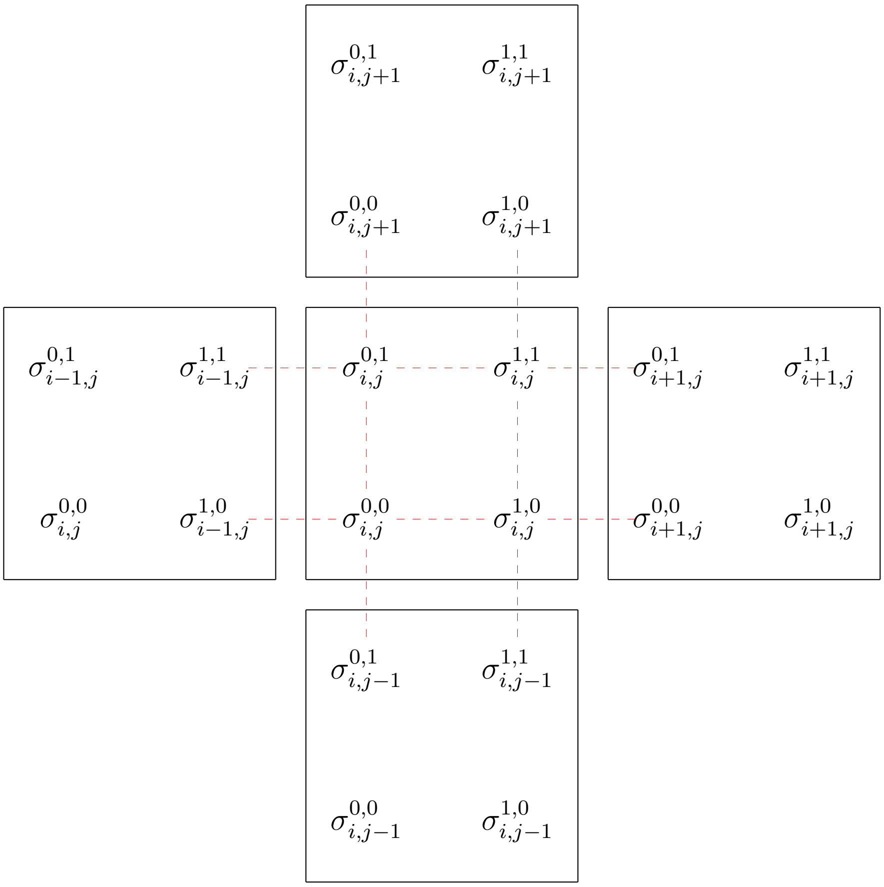

Figure 4.2: Interactions in the boxes' lattice $\mathcal{X}_\square$

For every $z \in \mathcal{Z}_\square$, $\mathcal{X}_\square^z$ is the fibre over z:

$$\mathcal{X}_\square^z = \pi^{-1}(z) \subset \mathcal{X}_\square.$$

According to Corollary 3.4.17, if we prove that for every $z \in \mathcal{Z}_\square$ we have uniqueness of the Gibbs measure on $X_\square^z$, then we have proved the Gibbsianity of the renormalized measure ν.
As we have seen in Chapter 2.4, a sufficient condition for an interaction to have a unique Gibbs measure is the Dobrushin's condition. We are then tempted to test the Dobrushin's condition on our fibre $\mathcal{X}_\square^z$ hoping to find $C(\Gamma_\square^z) \leq 1$ for $\beta < k\beta_c$ with $k > 1$. In this case, in fact, we would have proved that the non-Gibbsianity of the measure on the original Ising system doesn't imply the non-Gibbsianity of the renormalized measure. Indeed, for $\beta \in]\beta_c, k\beta_c[$ we would have non-Gibbsianity of the original measure μ but, at the same time, the renormalized measure ν would be Gibbs. Unfortunately, as already said in Chapter 3.3.2, this is not the case since

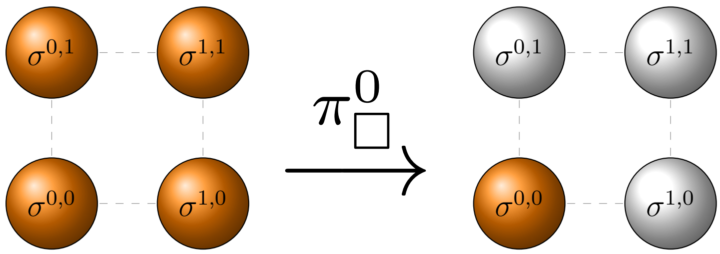

Figure 4.3: The decimation $\pi_{\Box}^{0}$

the test fails even for β_c. This is a consequence of the fact that in the original system $\mathcal{X}$, on which we build the "blocks' space" $\mathcal{X}_{\Box}$, after the decimation there are two classes of spins:

- original spins which have two nearest neighbors that are block spins and two that are original spins;

- original spins for which all the nearest neighbors are original spins.

For the spins belonging to the latter case, has it has also observed by Haller and Kennedy in [3], the quantity $\sum_{j\in\mathcal{X}} C_{i,j}(\Gamma)$ is greater than 1 even at β_c. As it is natural, this behaviour is transmitted even to the box-fibre $X_{\Box}^{z}$ as our tests with Matlab have confirmed.

As we have seen, Harry and Kennedy [3] proposed an expedient to overcome this problem: one has to sum out some of the original spins doing an intermediate decimation before testing the condition on the fibres.
We want to proceed like them by splitting the original decimation of parameter 2 into two decimations of parameter $\sqrt{2}$. However, we want to emphasize that the the choice of the factor $\sqrt{2}$ is not essential: the important thing is to divide the decimation into 2 steps!

We would like to describe rigorously this approach, adapting it to our different framework; indeed, while they applied it directly on the original configuration's set $\mathcal{X}$, we are working with the boxes' space $\mathcal{X}_{\Box}$.
As them, instead of doing directly the decimation of factor 2, we apply twice the decimation of parameter $\sqrt{2}$ as showed in figure 4.4.

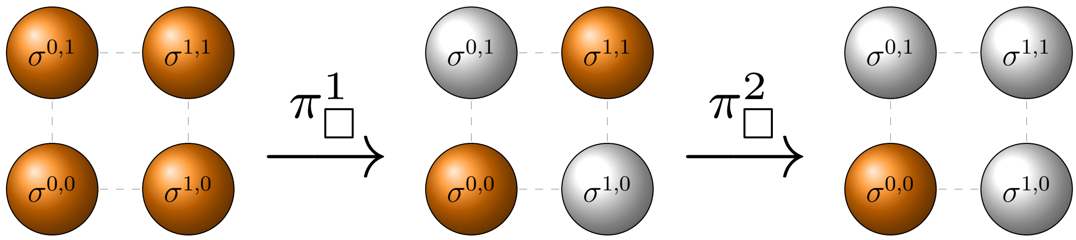

Figure 4.4: The two-steps decimation

Formally one has:

$$\pi^1_\square : \begin{array}{ccc} \{\pm 1\}^4 & \longrightarrow & \{\pm 1\}^2 \\ (\sigma^{0,0}, \sigma^{0,1}, \sigma^{1,0}, \sigma^{1,1}) & \longmapsto & (\sigma^{0,0}, \sigma^{1,1}) \end{array}$$

$$\pi^2_\square : \begin{array}{ccc} \{\pm 1\}^2 & \longrightarrow & \{\pm 1\} \\ (\sigma^{0,0}, \sigma^{1,1}) & \longmapsto & \sigma^{0,0} \end{array}$$

With an abuse of notation we will denote by $\pi^1_\square, \pi^2_\square$ the two componentwise extension of the two $\sqrt{2}$-decimations. Letting $\nu^1_\square = \mu_\square \circ (\pi^1_\square)^{-1}$ be the renormalized measure on $\mathcal{Y} = \left\{\{\pm 1\}^2\right\}^{\mathbb{Z}^d}$, we have $\nu_\square = \nu^1_\square \circ (\pi^2_\square)^{-1} = \mu_\square \circ \pi^{-1}_\square$. The same symbols, without the box $\square$, will refer to the decimations and measures for the "classical lattice" without blocks.

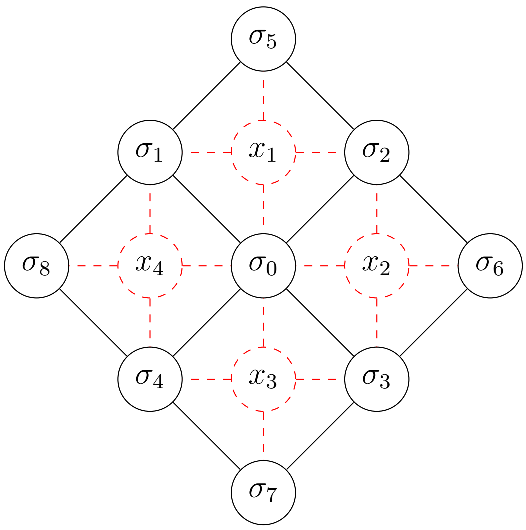

Figure 4.5:

Referring to Figure 4.5, where the red dotted lines indicate the spins and interactions that we want to integrate out, we have:

$$\begin{aligned}\nu_1(\sigma_0|\sigma_1,\ldots\sigma_8) &= \sum_{x_1^4}\mu(\sigma_0,x_1,\ldots,x_4|\sigma_1,\ldots,\sigma_8)\\ &= \sum_{x_1^4}\frac{1}{Z_\Lambda^\Phi(\sigma_1,\ldots,\sigma_8)}e^{H_\Lambda^\Phi(\sigma_0,\sigma_1,\ldots\sigma_8,x_1,\ldots x_4)},\end{aligned}$$

where Λ is the set of the nodes $\{\sigma_0,x_1,x_2,x_3,x_4\}$.

Remark 4.2.1. From the Gibbsianity of the measure μ it follows that I can condition on arbitrary configurations outside the volume $\{\sigma_0,x_1,\ldots,x_4,\sigma_1,\ldots,\sigma_8\}$ without any change in the result:

$$\begin{aligned}\nu_1(\sigma_0|\sigma_1,\ldots,\sigma_8,\text{ other }\sigma\text{'s}) &= \sum_{x_1^4}\mu(\sigma_0,x_1,\ldots,x_4|\sigma_1,\ldots,\sigma_8,\text{ other }\sigma\text{'s})\\ &= \sum_{x_1^4}\mu(\sigma_0,x_1,\ldots,x_4|\sigma_1,\ldots,\sigma_8)\\ &= \nu_1(\sigma_0|\sigma_1,\ldots\sigma_8).\end{aligned}$$

We can now proceed with the computations:

$$\begin{aligned}&\sum_{x_1^4}e^{H_\Lambda^\Phi(\sigma_0,\sigma_1,\ldots\sigma_8,x_1,\ldots x_4)} = \sum_{x_1^4}e^{\beta\sum\limits_{i\sim j}\sigma_i x_j} =\\ &=\sum_{x_1^4}\exp[\beta x_1(\sigma_0+\sigma_1+\sigma_2+\sigma_5)+\beta x_2(\sigma_0+\sigma_2+\sigma_3+\sigma_6)\\ &\quad+\beta x_3(\sigma_0+\sigma_3+\sigma_4+\sigma_7)+\beta x_4(\sigma_0+\sigma_1+\sigma_4+\sigma_8)]\\ &=\sum_{x_1^4}\{\exp[\beta x_1(\sigma_0+\sigma_1+\sigma_2+\sigma_5)]\cdot\exp[\beta x_2(\sigma_0+\sigma_2+\sigma_3+\sigma_6)]\\ &\quad\exp[\beta x_3(\sigma_0+\sigma_3+\sigma_4+\sigma_7)]\cdot\exp[\beta x_4(\sigma_0+\sigma_1+\sigma_4+\sigma_8)]\}\\ &=\sum_{x_1}\exp[\beta x_1(\sigma_0+\sigma_1+\sigma_2+\sigma_5)]\cdot\sum_{x_2}\exp[\beta x_2(\sigma_0+\sigma_2+\sigma_3+\sigma_6)]\\ &\quad\sum_{x_3}\exp[\beta x_3(\sigma_0+\sigma_3+\sigma_4+\sigma_7)]\cdot\sum_{x_4}\exp[\beta x_4(\sigma_0+\sigma_1+\sigma_4+\sigma_8)].\end{aligned}$$

Now we would like to be able to rewrite the summations of the exponential terms like exponential themselves. These is exactly what happens, as the following lemma confirms us:

Lemma 4.2.2. *For every $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{\pm 1\}$ we have:*

$$\sum_{\sigma_0 \in \{\pm 1\}} \exp[\beta \sigma_0(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] =$$
$$= \exp[a(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_4) + b\sigma_1\sigma_2\sigma_3\sigma_4 + c],$$

with:

$$\begin{aligned} a &= \frac{1}{8}\ln(\cosh(4\beta)), \\ b &= \frac{1}{8}\ln(\cosh(4\beta)) - \frac{1}{2}\ln(\cosh(2\beta)), \\ c &= \frac{1}{8}\ln(\cosh(4\beta)) + \frac{1}{2}\ln(\cosh(2\beta)) + \ln(2). \end{aligned}$$

Proof. Considering all the possible choices of $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{\pm 1\}$ we find three different constraints that must be satisfied:

$$\begin{cases} e^{4\beta} + e^{-4\beta} & = & e^{6a+b+c} \\ e^{2\beta} + e^{-2\beta} & = & e^{-b+c} \\ 2 & = & e^{-2a+b+c} \end{cases}$$

From the third equation we write a in function of b, c and we replaced it in the first equation:

$$\begin{cases} e^{4\beta} + e^{-4\beta} = \frac{1}{8}e^{4b+4c} \\ e^{2\beta} + e^{-2\beta} = e^{-b+c} \\ a = \frac{b+c-\ln(2)}{2} \end{cases}$$

Remembering that $\cosh(x) = \frac{e^x + e^{-x}}{2}$, we find that:

$$\begin{cases} \cosh(4\beta) = \frac{1}{16}e^{4b+4c} \\ \cosh(2\beta) = \frac{1}{2}e^{-b+c} \\ a = \frac{b+c-\ln(2)}{2} \end{cases}$$

We can now solve the system finding:

$$\begin{cases} a = \frac{1}{8}\ln(\cosh(4\beta)), \\ b = \frac{1}{8}\ln(\cosh(4\beta)) - \frac{1}{2}\ln(\cosh(2\beta)), \\ c = \frac{1}{8}\ln(\cosh(4\beta)) + \frac{1}{2}\ln(\cosh(2\beta)) + \ln(2). \end{cases}$$

□

Remark 4.2.3. This result is typical for solvable systems in general.

We can now apply the Lemma and conclude that:

$$\sum_{x_1^4} e^{H_\Lambda^\Phi(\sigma_0,\sigma_1,\dots\sigma_8,x_1,\dots x_4)} = \dots =$$
$$\begin{aligned}
=&\exp[a(\sigma_0\sigma_1+\sigma_0\sigma_2+\sigma_0\sigma_5+\sigma_1\sigma_2+\sigma_1\sigma_5+\sigma_2\sigma_5)+b\sigma_0\sigma_1\sigma_2\sigma_5+c]\\
&\cdot\exp[a(\sigma_0\sigma_2+\sigma_0\sigma_3+\sigma_0\sigma_6+\sigma_2\sigma_3+\sigma_2\sigma_6+\sigma_3\sigma_6)+b\sigma_0\sigma_2\sigma_3\sigma_6+c]\\
&\cdot\exp[a(\sigma_0\sigma_3+\sigma_0\sigma_4+\sigma_0\sigma_7+\sigma_3\sigma_4+\sigma_3\sigma_7+\sigma_4\sigma_7)+b\sigma_0\sigma_3\sigma_4\sigma_7+c]\\
&\cdot\exp[a(\sigma_0\sigma_1+\sigma_0\sigma_4+\sigma_0\sigma_8+\sigma_1\sigma_4+\sigma_1\sigma_8+\sigma_4\sigma_8)+b\sigma_0\sigma_1\sigma_4\sigma_8+c],
\end{aligned}$$

with a, b, c as in the Lemma.
Therefore we have found that:

$$\nu_1(\sigma_0|\sigma_1,\dots\sigma_8) = \frac{1}{Z_{\Lambda'}^{\Phi'}(\sigma_1,\dots,\sigma_8)} e^{\bar{H}_{\Lambda'}^{\Phi'}(\sigma)},$$

which means that the measure ν_1 is Gibbs for the Hamiltonian $\bar{H}$. Summarizing, starting from a Gibbs measure μ for the Ising Potential on the original system, we have found a Gibbs measure ν_1 for the first-step decimated system for which we are able to compute explicitly the Hamiltonian. In accordance with what found by Haller and Kennedy, using z_i's to refer to the block spins (the ones that will survive after the second decimation $\pi_\square^2$) and y_i's for the spins the survived after the decimation $\pi_\square^1$ but will be eliminated by $\pi_\square^2$, we see that the terms of the Hamiltonian $\bar{H}$ that involve the spin at the site 0 are:

$$\begin{aligned}
&2a(z_1+z_2+z_3+z_4)y_0\\
&+(a+bz_1z_2)y_0y_5+(a+bz_2z_3)y_0y_6\\
&+(a+bz_3z_4)y_0y_7+(a+bz_1z_4)y_0y_8.
\end{aligned}$$

From this Hamiltonian we can construct the corresponding Hamiltonian $\bar{H}_\square$ for the boxes' lattice; on the latter, indeed, the situation is the one described in Figure 4.6.
Therefore, the terms of the Hamiltonian that involve the spins at the block 0 are:

$$\begin{aligned}
&2a(z_1+z_2+z_3)y_0+2a(y_5+y_4+y_3)z_0+2ay_0z_0\\
&+(a+bz_1z_2)y_0y_1+(a+bz_2z_3)y_0y_2\\
&+(a+bz_3z_0)y_0y_3+(a+bz_0z_1)y_0y_5\\
&+(a+by_3y_4)z_0z_4+(a+by_4y_5)z_0z_5\\
&+az_0z_1+az_0z_3.
\end{aligned}$$

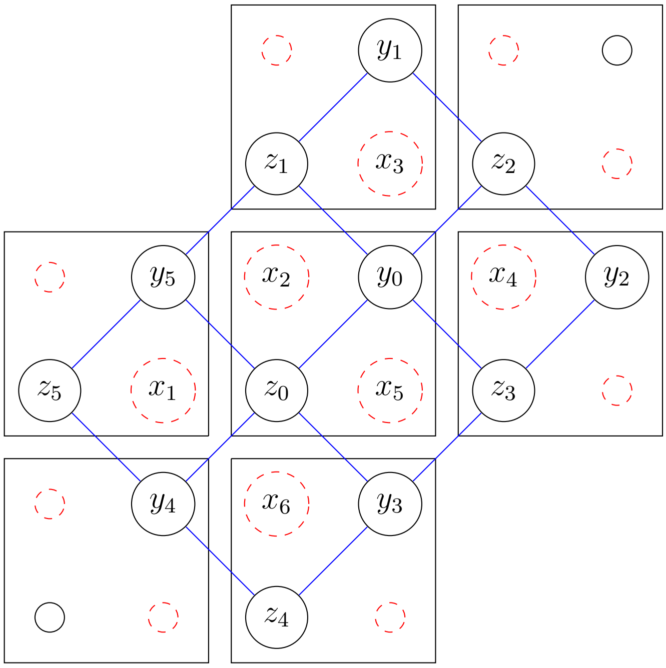

Figure 4.6:

Remark 4.2.4. Notice that when summing the contributions related to z_0 and the ones related to y_0, we have to pay attention not to add twice the contributions that involves both z_0 and y_0.

Now we have a Gibbs measure $\nu^1_\square$ for the Hamiltonian $\bar{H}_\square$ on $\mathcal{Y}_\square$ and a transformation $\pi^2_\square$ from $\mathcal{Y}_\square$ to $\mathcal{Z}_\square$. We can then construct the fibres $\mathcal{Y}^z_\square$ for every $z \in \mathcal{Z}_\square$ and test the Dobrushin condition on these fibres. We have done this with the help of a computer, the code can be found at appendix A, finding that the condition is satisfied for $\beta < 1.3005\beta_c$. For such β's we have then uniqueness of the Gibbs measure on the fibre $\mathcal{Y}^z_\square$ for every $z \in \mathcal{Z}_\square$, and therefore the renormalized measure $\nu = \nu_\square$ is Gibbs.

In conclusion, we have obtained a much simpler proof of the Gibbsianity of ν than the one given by Haller and Kennedy in [3]. Indeed, while for them the fact that the Dobrushin condition on the fibres is satisfied for $\beta < 1.3005\beta_c$ is a preliminary result, needed to prove the hypothesis of Theorem 3.3.3, for us it is enough to conclude.

Chapter 5

Conclusion

We have seen that the approach presented by Berghout & Verbitskiy can be applied effectively to the case of the $b = 2$ decimation for the two-dimensional ferromagnetic Ising model with zero external magnetic field. What we have showed is that, in this case, we can successfully validate Dobrushin's condition for each fibre. This implies the uniqueness of the Gibbs measures on the fibres and hence, by Corollary 3.4.17, we obtain the Gibbsianity of the renormalized Gibbs measure. However, the result obtained is not optimal: we have showed Gibbsianity for $\beta \leq 1.3005\beta_c$ while it is known that the renormalized measure is Gibbs even for higher values of the inverse-temperature β.
In accordance to what found by Haller & Kennedy, we believe that this result can be further improved considering a different intermediate decimation; in particular, using the scheme presented in Figure 3.5, we should be able to show Gibbsianity for $\beta \leq 1.3645\beta_c$. Anyway, since the Dobrushin's condition is only a sufficient but not necessary condition for the uniqueness of a Gibbs measure, we doubt that in this way we can obtain the optimal value for β. Therefore, even if our approach has produced significant results, we may want to find a different way to prove the uniqueness of the Gibbs measures on the fibres.
Further studies can be conduct to see if the sufficient condition given by Corollary 3.4.16 is also a necessary condition, i.e., if the Gibbsianity of the renormalized measure implies the uniqueness of the limiting measure on the fibres. For the moment, there are no evidences neither for nor against this conjecture. We believe that, in the case studied in this work, for $\beta > 1.73\beta_c$ one should be able to find two different limiting measure. The idea one should follow is the one of considering the limit given by two different sequences of boxes, each of them containing a different ratio of positive/negative spins. Indeed, the majority of one of the two spins should push the system to the corresponding phase.
Finally, it would be interesting to use this approach to analyze different models, like for example the Fuzzy-Potts model, for which it is still still not known the

precise behaviour of the renormalized measure.

Acknowledgements

I would like to express my gratitude to my supervisors, Professor Evgeny Verbitskiy and Professor Paolo Dai Pra, for their guidance, academic support and patience. Furthermore, I would like to thank my family, my girlfriend and my friends for their constant support throughout this entire journey.

Appendix A

Matlab Code

Here are reported the code used for the Matlab computations described in Chapter 4.2.

```
function P = CondProb(y1,y2,y3,y4,y5,y,p,q,r,s,t,J);
  a = log(cosh(4*J))/8;
  b = log(cosh(4*J))/8-log(cosh(2*J))/2;

  for x = -1:2:1
    i = (x+1)*(1/2);
    l = (y+1)*(1/2);
    Index = i+2*l+1;
    H = 2*a*(y1+y2+y3)*x+(a+b*y1*y2)*x*p
        +(a+b*y2*y3)*x*q+(a+b*y3*y)*x*r+(a+b*y*y1)*x*t
        +2*a*(t+s+r)*y+a*y*y1+a*y*y3+(a+b*r*s)*y*y4
        +(a+b*s*t)*y*y5+2*a*x*y;
    P(Index) = exp(H);
  end

  Sum = 0;
  for k = 1:length(P)
    Sum = Sum + P(k);
  end;
  for k = 1:length(P)
    P(k) = P(k)/Sum;
  end;
end;
```

Main program:

```
Jc = log(1+sqrt(2))/2;
```

```
J = 1.3005*Jc;

MAX = 0;

for y1 = -1:2:1
  for y2 = -1:2:1
    for y3 = -1:2:1
      for y4 = -1:2:1
        for y5 = -1:2:1
          for y6 = -1:2:1

            MAX1 = 0;
            MAX2 = 0;
            MAX3 = 0;
            MAX4 = 0;
            MAX5 = 0;

            for a = -1:2:1
              for b = -1:2:1
                for c = -1:2:1
                  for d = -1:2:1

                    P = CondProb(y1,y2,y3,y4,y5,y6,+1,a,b,c,d,J);
                    Q = CondProb(y1,y2,y3,y4,y5,y6,-1,a,b,c,d,J);
                    DIF = 1/2*sum(abs(P-Q));
                    if DIF > MAX1
                      MAX1 = DIF;
                    end

                    P = CondProb(y1,y2,y3,y4,y5,y6,a,+1,b,c,d,J);
                    Q = CondProb(y1,y2,y3,y4,y5,y6,a,-1,b,c,d,J);
                    DIF = 1/2*sum(abs(P-Q));
                    if DIF > MAX2
                      MAX2 = DIF;
                    end

                    P = CondProb(y1,y2,y3,y4,y5,y6,a,b,+1,c,d,J);
                    Q = CondProb(y1,y2,y3,y4,y5,y6,a,b,-1,c,d,J);
                    DIF = 1/2*sum(abs(P-Q));
                    if DIF > MAX3
                      MAX3 = DIF;
                    end
```

```
                  P = CondProb(y1,y2,y3,y4,y5,y6,a,b,c,+1,d,J);
                  Q = CondProb(y1,y2,y3,y4,y5,y6,a,b,c,-1,d,J);
                  DIF = 1/2*sum(abs(P-Q));
                  if DIF > MAX4
                    MAX4 = DIF;
                  end

                  P = CondProb(y1,y2,y3,y4,y5,y6,a,b,c,d,+1,J);
                  Q = CondProb(y1,y2,y3,y4,y5,y6,a,b,c,d,-1,J);
                  DIF = 1/2*sum(abs(P-Q));
                  if DIF > MAX5
                    MAX5 = DIF;
                  end

                end
              end
            end
          end

          SUM = MAX1 + MAX2 + MAX3 + MAX4 + MAX5;
          if SUM > MAX
            MAX = SUM;
          end
        end
      end
    end
  end
end
end
```

Bibliography

[1] Hans-Otto Georgii. *Gibbs Measures and Phase Transitions.* De Gruyter Studies in Mathematics 9, (2011).

[2] Aernout C.D. van Enter, Roberto Fernandez, Alan D. Sokal. *Regularity Properties and Pathologies of Position-Space Renormalization-Group*, 1992.

[3] Karl Haller, Tom Kennedy. *Absence of Renomarlization Group Pathologies Near the Critical Temperature. Two Examples*, 1995.

[4] S. Berghout, E. Verbitskiy, *Criteria for the preservation of Gibbsianity under renormalization*, Preprint, 2017.

[5] Arnaud Le Ny. *Introduction to (generalized) Gibbs measures*, Ensaios matematicos, 15:1-126, 2008.

[6] O. k. Kozlov. GIbbs description of a system of random variables. *Probl. Inform. Transmission*, 10:258-265, 1974.

[7] W. G. Sullivan. Potentials for almost Markovian random fields. *Commun. Math. Phys.*, 33:61-74, 1973.

[8] R. L. Dobrushin. *The Description of a random field by means of conditional probabilities and conditions of its regularity*, Theor. Prob. Appl., 13:197-224, 1968.

[9] O. E. Lanford III and D. Ruelle. *Observables at infinity and states with short range correlations in statistical mechanics*, Commun. Math. Phys., 13:194-215, 1969.

[10] W. Lenz *Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern*, Physik. Zeitschr. 21:613-615, 1920.

[11] E. Ising *Beitrag zur Theorie des Ferro- und Paramagnetismus*, Dissertation, Mathematisch-Naturwissenschaftliche Fakultät der Universität Hamburg, 1924.

[12] C. N. Yang, *The spontaneous magnetization of a two-dimensional Ising model*, Phys. Rev. 85:809-816, 1952.

[13] D. Ruelle, *On the use of "small external fields" in the problem of symmetry breakdown in statistical mechanics"*, Ann. Phys. 69:364-374, 1972.

[14] D. Ruelle, *Thermodynamic Formalism: The Mathematical Structure of Equilibrium Statistical Mechanics*, Cambridge Mathematical Library, Cambridge University Press., 2004, doi:10.1017/CBO9780511617546

[15] J. L. Lebowitz, A. Martin-Löf, *On the uniqueness of the equilibrium state for Ising spin systems*, CMP 25:276-282, 1972.

[16] M. Aizenman, *Translation invariance and instability of phase coexistence in the two-dimensional Ising system*, CMP 73:83-94, 1980.

[17] Y. Higuchi, *On the absence of non-translation invariant GIbbs states for the two-dimensional Ising model.* In: J. Fritz, J.L. Lebowitz and D.Szás (eds.), Random fields, Esztergom (Hungary) 1979. Amsterdam: north Holland, Vol. I, pp. 517-534, 1981.

[18] R. L. Dobrushin, *The description of a random field by means of conditional probabilities and condition of tis regularity*, Theor. Prob. Appl. 13:197-224, 1968.

[19] R. B. Israel. *Banach algebras and Kadanoff transformations*, in J. Fritz, J. L. Lebowitz, and D. Szász, editors, Random Fields (Esztergom, 1979), Vol. II, pages 593-608, North-Holland, Amsterdam, 1981.

[20] I. A. Kashapov, *Justification of the renormalization-group method*, Theor. Math. Phys., 42:184-186, 1980.

[21] R. B. Griffiths and P. A. Pearce, *Mathematical properties of position-space renormalization-group transformations*, J. Stat. Phys., 20:499-545, 1979.

[22] R. B. Griffiths, *Mathematical properties of renormalization-group transformations*, Physica, 106A:59-69, 1981.

[23] T. Tjur, *On the mathematical foundations of probability*, Institute of Mathematical Statistics, University of Copenhagen, Copenhagen, 1972. Lecture Notes, No. 1. MR0443014.

[24] T. Tjur, *Conditional probability distributions*, Institute of Mathematical Statistics, University of Copenhagen, Copenhagen, 1974. Lecture Notes, No. 2. MR0345151.

[25] T. Tjur, *A Constructive Definition of Conditional Distributions*, Preprint, Institute of Mathematical Statistics, University of Copenhagen, 1975.

[26] T. Bogenschutz and V. M. Gundlach, *Ruelle's transfer operator for random subshifts of finite type*, Ergodic Theory Dynam. Systems, 15:413-447, 1995, MR1336700.