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Covariant Loop Quantum Gravity,
An Introduction and some Mathematical Tools

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Abstract

The aim of this thesis is to give a presentation of Loop Quantum Gravity in its covariant form, also known as spinfoam approach, and to present the basic mathematical tools to access the theory.

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Chapter 1

Introduction

1.1 Historical Overview

Quantum gravity is the search for a theory that aims to merge two well established theories of the twenthieth century: General Relativity and Quantum Mechanics. The need for such a theory came clear early, in lue of the fact that the gravitational field, being a field, was expected to be quantized. Along the years, three main lines of research have been established:

- **The covariant line of research** is the attempt to build the theory as a quantum field theory of the fluctuations of the metric over a flat Minkowski space, this approach eventually led to string theory in the late eighties.

- **The canonical line of research** is the attempt to construct a quantum theory in which the Hilbert space carries a representation of the operators corresponding to the full metric, or some functions of the metric, without background metric to be fixed. The formal equations of the quantum theory were then written down by Wheeler and DeWitt in the middle sixties, but turned out to be too ill-defined. A well defined version of the same equations was successfully found only in the late eighties, with loop quantum gravity.

- **The sum over historiers line of research** is the attempt to use some version of Feynman's functional integral quantization to define the theory, leading eventually to the spin foam approach.

In this introducion we focus solely on the last two approaches and we try to sketch the evoultion of loop quantum gravity.
In the early thirties attempts are made in order to apply the quantization method of gauge theories to the linearized Einstein field equations. Later in the decade, Bronstein realizes that field quantization techniques must be generalized in such a way as to be applicable in the absence of a background geometry, in sharp contrast to the approach used in quantum electrodynamics [1].
At the beginning of the fifties starts the development of the "flat space quantization" of the gravitational field. The idea is to quantize the small fluctations around the Minkowski metric, that is, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. This idea represents the birth of the covariant approach.
On the other hand, Bergmann starts its program of phase space quantization of non linear field theories and problems raised by systems with constraints are studied too [2]. The canonical approach to quantum gravity is born.

Later in the decade, Charles Misner introduces the “Feynman quantization of general relativity” $Z = \int \exp(iS[g])dg$ [3], and so the three lines of research are established.
In 1959 Dirac has completely unraveled the canonical structure of GR [4], and two years later, Arnowit, Deser and Misner complete the so-called ADM formulation of GR [5], namely its hamiltonian version in appropriate variables, which greatly simplify the hamiltonian formulation and make its geometrical reading transparent. Following the ADM methods, in 1962 Peres writes the Hamilton-Jacobi formulation of GR [6]:

$$G^2(q_{ab}q_{cd} - \frac{1}{2}q_{ac}q_{bd})\frac{\delta S(q)}{\delta q_{ac}}\frac{\delta S(q)}{\delta q_{bd}} + \det(q)R[q] = 0 \ ,$$

which will lead to the Wheeler-DeWitt equation.
In 1967 Bryce DeWitt publishes the “Einstein-Schrödinger equation” [7]:

$$\left((\hbar G)^2(q_{ab}q_{cd} - \frac{1}{2}q_{ac}q_{bd})\frac{\delta}{\delta q_{ac}}\frac{\delta}{\delta q_{bd}} + \det(q)R[q]\right)\Psi(q) = 0 \ ,$$

which is known as the “Wheeler-DeWitt equation”. This equation comes with the so-called “problem of time” in quantum gravity, because the time variable disappears. To be fair, the time coordinate already disappears in the classical Hamilton-Jacobi form of GR, thus the fact that physical obsservables are coordinate independent is a genuine feature of any formulation of GR. But in the quantum context there is no single spacetime, as there is no trajectory for a quantum particle, and the very concepts of space and time become fuzzy.
In the seventies, Hawking announces the derivation of black hole radiation [8] and he states that a Schwarzschild black hole of mass M emits thermal radiation at the temperature

$$T = \frac{\hbar c^3}{8\pi kGM} \ ,$$

opening a new field of research in “black hole thermodynamics” and leading to the understanding of the statistical origin of the black hole entropy, which, for a Schwarzschild black hole, reads

$$S_{BH} = \frac{1}{4}\frac{c^3}{\hbar G}A \ ,$$

where A is the area of the black hole surface. Later in the decade, the Hawking radiation is rederived in a number of ways, strongly reinforcing its credibility.
In 1986 the connection formulation of GR is developed by Abhay Ashtekar [9](as opposed to the metric formulation), semplifying the canonical analysis in the sense that the constraints take a simpler form. Furthermore, the theory now takes the form of a $SU(2)$-theory, since the structure constants associated to the Poisson structure of the Ashtekar variables coincide with the structure constants of the su(2) algebra. In addition to this, there is a geometric interpretation of the “Ashtekar electric field”, namely, the field conjugate to the Ashtekar connection, in terms of area elements.
In 1988, Ted Jacobson and Lee Smolin find loop-like solutions to the Wheeler-DeWitt equation formulated in the connection formulation [10], that is, they present a large class of exact solutions to the hamiltonian constraint written in terms of Wilson loops. Based on these results, and on knot theory, the canonical approach gets new blood, and “loop quantum gravity” gets started. Let’s birefly summarize this important step and explain where the word “loop” in LQG comes from. The Jacobson-Smolin solutions are not physical states of quantum gravity, since they fail to satisfy the second equation of canonical quantum gravity (the first being the Wheeler-DeWitt equation), which demands states to be invariant under 3d diffeomorphisms. Then, soon afterwards, Smolin starts to think that since loops

up to diffeomorphisms mean knots, knots could play a role in quantum gravity. The solutions are written in terms of Wilson loops

$$\psi(\gamma) = \mathrm{Tr}\left[\mathcal{P}e^{\int_\gamma A}\right]\psi(A)dA\,, \tag{1.1}$$

moving from the connection representation to the loop representation means considering the "loop transform"

$$\psi_\gamma[A] = \mathrm{Tr}\left[\mathcal{P}e^{\int_\gamma A}\right]\,, \tag{1.2}$$

where dA is a diffeomorphism-invariant measure on the space of connections (constructed by Ashtekar and Lewandowski). This transform maps the $\psi(A)$ representation of quantum states in A space to the representation $\psi(\gamma)$ of quantum states in γ space, that is, in loop space. The advantages of moving to the loop basis are: $\psi(\gamma)$ depending only on the knot class of the loop γ solve the diffeomorphism constraint, that is, there is one independent solution for each knot; all such states where the loop does not self-intersect are exact solutions of all equations of quantum gravity (the partial result obtained by Smolin was that not self-intersecting loops gave rise to solution of the Hamiltonin constraint only). Later on, Jorge Pullin realizes that all solutions without nodes (intersections between loops) correspond to 3-geometries with zero volume, meaning therefore that nodes are essential to describe physical quantum geometry [11]. In 1995 the spin network orthonormal basis on the Hilbert space of loop quantum gravity is found [12], and a main main physical result is obtained: the computation of the eigenvalues of area and volume.
In 1996 the Bekenstein-Hawking black hole entropy is computed within loop quantum gravity [13], as well as within string theory.

1.1.1 The Problems Addressed

There are three major theoretical and conceptual problems that the theory addresses:

- **Quantum geometry**: What is a physical "quantum space"? That is, what is the mathematics that describes the quantum spacetime metric? LQG predicts that any measured physical area must turn out to be quantized and given by the spectrum (4.58).
- **Ultraviolet divergences of quantum field theory**: This is a major open problem in non-gravitational contexts. But it is a problem physically related to quantum gravity because the ultraviolet divergences appear in the calculations as effects of ultra-short trans-Planckian modes of the field. If physical space has a quantum discreteness at small scale, these divergences should disappear. In LQG the ultraviolet divergences are not present since there is a natural cut-off due to the discretized spectrum of the area, nevertheless infrared divergences could possibly arise by considering greater values of the spins, these are called "spikes". Interestingly, when one considers the theory with the presence of a cosmological constant, it can be shown that this provides an upper limit for the greatest value of a spin, thus resolving the problem of infrared divergences.
- **General covariant quantum field theory**: Loop gravity "takes seriously" general relativity, and explores the possibility that the symmetry on which general relativity is based (general

covariance) holds beyond the classical domain. Since standard quantum field theory is defined on a metric manifold, this means that the problem is to find a radical generalization of quantum field theory, consistent with full general covariance and with the physical absence of a background metric structure. In other words, loop gravity, before being a quantum theory of general relativity, is the attempt to define a general covariant quantum field theory.

1.1.2 Open Problems

- **Consistency**: With the cosmological constant, the transition amplitudes are finite at all orders and the classical limit of each converges to the truncation of classical limit of GR on a finite discretization of spacetime; in turn, these converge to classical GR when the discretization is refined. This gives a coherent approximation scheme. However the approximation scheme may go wrong if the quantum part of the corrections that one obtains refining the discretization is large. These can be called "radiative corrections", since they are somewhat similar to standard QFT radiative corrections: possibly large quantum effects effects that appear taking the next order in the approximation. It is not sufficient for these radiative corrections to be finite, for the approximation to be viable, they must also be small. Since the theory includes a large number, the ratio of the cosmological constant scale over the Planck scale (or over the observation scale), these radiative corrections a priori could be large.
- **Completeness**: The matter sector of the theory has not been sufficiently developed. In addition to this, the q-deformed version of the theory, that is, based on the quantum group $SU(2)_q$ (a one-parameter deformation of the representations of $SU(2)$), is very little developed. This version is utilized in order to introduce the cosmological constant but it's not clear if one can obtain the results of the $\Lambda = 0$ theory.

Chapter 2

Classical General Relativity

This chapter is devoted to a formulation of classical General Relativity more suitable for the discretization and the consequent quantization.

2.1 Tetrad-Connection formalism

As already carefully explained in the Appendix, which we refer to, it is possible to express the Einstein-Hilbert action in terms of the (co)tetrads and a Lorentz connection. Briefly we recall the main formulas. Tetrads are such that:

$$g_{\mu\nu}(x) = e^I_\mu(x) e^J_\nu(x) \eta_{IJ} \tag{2.1}$$

The metric is not affected if the tetrads undergo a local $SO(3,1)$ transformation; the Lorentz connection associated to this gauge invariance is a one-form with values in the Lie algebra $\mathrm{sl}(2,\mathbb{C})$, therefore it is antisymmetric:

$$\omega^{IJ}_\mu = -\omega^{JI}_\mu \tag{2.2}$$

The curvature of the connection is given by:

$$F^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J \ , \tag{2.3}$$

if the connection is torsionless then it can be shown to be unique, namely, the spin connection, or the Levi-Civita connection. In terms of these objects the Einstein-Hilbert action reads:

$$S[e] = \frac{1}{2} \int e^I \wedge e^J \wedge F^{KL} \epsilon_{IJKL}. \tag{2.4}$$

In order to rewrite the action in a more succint form we introduce the Hodge dual in the Minkowski space, that is, $F^\star_{IJ} := \star F_{IJ} := \frac{1}{2}\epsilon_{IJKL}F^{KL}$. Furthermore the 2-form $\Sigma^{IJ} := e^I \wedge e^J$ is called the Plebanski 2-form, and, suppressing contracted indices we get:

$$S[e] = \int e \wedge e \wedge F^\star \, . \tag{2.5}$$

We point out a difference between the Einstein-Hilbert action written in terms of the metric and the action written in terms of tetrads. In fact, if we write both in terms of tetrads we see that:

$$
\begin{aligned}
S_{EH}[e] &= \frac{1}{2}\int |\mathrm{det}e| R[e] d^4x\,, \\
S_T[e] &= \frac{1}{2}\int (\mathrm{det}e) R[e] d^4x\,.
\end{aligned}
\tag{2.6}
$$

The difference then amounts by a sign factor $s := \mathrm{sgn}(\mathrm{det}e)$. Therefore, when moving to the quantum case, where one takes the path-integral over tetrads, this sign translates to two different terms, namely:

$$
e^{-\frac{i}{\hbar}S_{EH}[g]} \qquad \text{and} \qquad e^{+\frac{i}{\hbar}S_{EH}[g]}\,. \tag{2.7}
$$

These two terms will reappear when dealing with the classical limit.
We can regard (2.5) also as a function of a tetrad and a Lorentz connection as independent fields, namely:

$$
S[e,\omega] = \int e\wedge e\wedge F[\omega]^{\star}\,, \tag{2.8}
$$

performing the variation with respect to the connection gives the torsionless condition and the variation with respect to the tetrad yelds Einstein equations. This polynomial action is referred to as the "tetrad-Palatini" action. It is possible to add another term respecting the given symmetries, this term has the form $\int e\wedge e\wedge F := \int e_I\wedge e_J\wedge F^{IJ}$. If we add this term with a coupling constant $1/\gamma$ (γ is known as the "Barbero-Immirzi constant") we get the following action:

$$
S[e,\omega] = \int e\wedge e\wedge F^{\star} + \frac{1}{\gamma}\int e\wedge e\wedge F\,. \tag{2.9}
$$

It can be shown that the second term has no effect on the equation of motion, because it vanishes when the connection is torsionless. Seeking a more compact form we observe that:

$$
\begin{aligned}
S[e,\omega] &= \int e\wedge e\wedge\left(F^{\star} + \frac{1}{\gamma}F\right) \\
&= \int e\wedge e\wedge\left(\star + \frac{1}{\gamma}\right)F \\
&= \int\left(\star\,(e\wedge e) + \frac{1}{\gamma}e\wedge e\right)\wedge F\,,
\end{aligned}
\tag{2.10}
$$

renaming the term in parentheses by $B := \left(\star\,(e\wedge e) + \frac{1}{\gamma}e\wedge e\right)$ we finally get:

$$
S[e,\omega] = \int B\wedge F\,. \tag{2.11}
$$

From this equation we can read out that, on a $t=0$ boundary, B is the derivative of the action with respect to $\partial\omega/\partial t$, therefore B is the momentum conjugate to the connection. More precisely, reintroducing the dimensionful constant $\frac{1}{8\pi G}$ in front of the action and going to a time gauge where the restriction of $\star\,(e\wedge e)$ on the boundary vanishes, the momentum is the 2-form on the boundary with values on $\mathrm{sl}(2,\mathbb{C})$, that is:

$$\Pi = \frac{1}{8\gamma\pi G} B\,. \tag{2.12}$$

2.2 Linear Simplicity Constraint

We consider now a spacelike boundary surface Σ, this is characterized by a vector which is normal to all the tangent vectors in Σ, we can write it as:

$$n_I = \epsilon_{IJKL} e^J_\mu e^K_\nu e^L_\rho \frac{\partial x^\mu}{\partial \sigma^1} \frac{\partial x^\nu}{\partial \sigma^2} \frac{\partial x^\rho}{\partial \sigma^3}\,, \tag{2.13}$$

where $\{\sigma^i\}$, $i = 1, 2, 3$ are the coordinates of the point $\sigma \in \Sigma$ and $x^\mu(\sigma)$ indicates the embedding of the boundary Σ into spacetime. By choosing a specific n_I we can focus on a fixed-time surface where $n_I = (1, 0, 0, 0)$. By doing so, the pull-back on Σ of the 2-form B can be decomposed into its electric $K^I = n_J B^{IJ}$ and magnetic $L^I = n_J (\star B)^{IJ}$ parts. Since B is antisymmetric, L^I and K^I do not have components normal to Σ, i.e. $n_I K^I = n_I L^I = 0$ and so they are three-dimensional vectors in Σ. In the gauge where $n_I = (1, 0, 0, 0)$ they are given by:

$$K^i = B^{i0}, \qquad L^i = \frac{1}{2}\epsilon^i_{jk} B^{jk}\,. \tag{2.14}$$

Now, from the definition of B we have that:

$$n_I B^{IJ} = n_I \left(\star (e \wedge e) + \frac{1}{\gamma} e \wedge e \right)^{IJ} = n_I \left(\epsilon^{IJ}_{KL} e^K \wedge e^L + \frac{1}{\gamma} e^I \wedge e^J \right), \tag{2.15}$$

on the boundary we have $n_I e^I|_\Sigma = 0$, therefore

$$n_I B^{IJ} = n_I (\star e \wedge e)^{IJ}\,. \tag{2.16}$$

Analogously:

$$n_I (B^\star)^{IJ} = n_I \left(\left(\frac{1}{\gamma} e \wedge e \right)^\star \right)^{IJ} = \frac{1}{\gamma} n_I (\star e \wedge e)^{IJ} = \frac{1}{\gamma} n_I B^{IJ}\,. \tag{2.17}$$

In conclusion, by definition of K^I and L^I we can notice that:

$$\vec{K} = \gamma \vec{L}\,. \tag{2.18}$$

This equation is called “linear symplicity constraint” and turns out to be a fundamental feature of covariant loop quantum gravity, indeed, it completely determines the dynamics of the theory.

2.3 Hamilton Function and Boundary Term

In writing the action on a compact region of spacetime we have to add a boundary term if we want to have a well-defined Hamilton function. In General Relativty, Gibbons and Hawkings have shown that the boundary term is given by:

$$S_{EH\ boundary} = \int_\Sigma k^{ab} q_{ab} \sqrt{q} d^3\sigma \, , \tag{2.19}$$

where k^{ab} is the extrinsic curvature of the boundary, q_{ab} is the three-metric induced by the embedding, q its determinant and σ are coordinates on the boundary. In the case of pure gravity without cosmological constant the Ricci scalar vanishes on the solution of the Einstein equations, therefore the bulk action vanishes and the Hamilton function is given by the boundary term:

$$S_{EH}[q] = \int_\Sigma k^{ab} q_{ab} \sqrt{q} d^3\sigma \, . \tag{2.20}$$

Notice that the Hamilton function is a functional of the boundary 3-metric, while the action is a functional of the 4-metric. Indeed, the Hamilton function represents a non-trivial functional to compute, because the extrinsic curvature $k^{ab}[q]$ is determined by the bulk solution singled out by the boundary intrinsic geometry, therefore it is going to be non-local. Knowing the general dependence of k^{ab} from q is equivalent to knowing the general solution of the Einstein equations.

2.4 ADM variables and Ashtekar variables

In order to approach a Hamiltonian formulation of General Relativity we introduce the so-called ADM variables and later on the Ashtekar variables.
The ADM variables are obtained by defining the following fields:

$$\begin{aligned} q_{ab} &= g_{ab} \, , \\ N_a &= g_{a0} \, , \\ N &= (-g_{00})^{-\frac{1}{2}} \, , \end{aligned} \tag{2.21}$$

where $a, b = 1, 2, 3$. N and N_a are called Lapse and Shift functions, q_{ab} is the three-metric. In these variables the line element reads

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dx^a dt + q_{ab} dx^a dx^b \, , \tag{2.22}$$

and the extrinsic curvature of a $t = constant$ surface is given by

$$k_{ab} = \frac{1}{2N}(\dot{q}_{ab} - D_{(a}N_{b)}) \tag{2.23}$$

where the dot indicates the derivative with respect to t and D_a is the covariant derivative of the three-metric. The action in terms of this variables takes the form

$$S[N, \overrightarrow{N}, q] = \int dt \int d^3x \sqrt{q} N \left(k_{ab}k^{ab} - k^2 + R[q] \right) \, , \tag{2.24}$$

where $k = k^a_a$ and $\sqrt{q} = \sqrt{\det q}$. From (2.24) one can read out the Lagrangian and the conjugate variables, thus obtaining an action written in hamiltonian form with the presence of constraints. Moving to the Ashtekar variables accounts for a simplification of these constraints and a better comprehension of the geometrical picture. Essentially, instead of dealing with tetrads on spacetime, we can introduce tetrads on each $t = constant$ surface. By doing so we have:

$$q_{ab}(x) = e^i_a(x) e^j_b(x) \delta_{ij} \,, \tag{2.25}$$

where q_{ab} is the 3-metric and $i, j = 1, 2, 3$ are flat indices. We can define also the triad version of the extrinsic curvature by:

$$k^a_i e^i_b := k_{ab} \,. \tag{2.26}$$

In this way, we can consider the following connection:

$$A^i_a = \Gamma^i_a[e] + \beta k^i_a \,, \tag{2.27}$$

where $\Gamma^i_a[e]$ is the torsionless spin connection of the triad and β is an arbitrary parameter, and the so-called "Ashtekar electric field":

$$E^a_i(x) = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c \,, \tag{2.28}$$

that is, the inverse of the triad multiplied by its determinant. What's remarkable about these two fields is that they satisfy the following Poisson brackets:

$$\left\{ A^i_a(x) \,,\, A^i_a(y) \right\} = 0 \tag{2.29}$$

and

$$\left\{ A^i_a(x) \,, E^b_j(y) \right\} = \beta \delta^b_a \delta^i_j \delta^3(x, y) \,. \tag{2.30}$$

Therefore A^i_a and E^a_i are canonically conjugate variables. This simplifies the canonical analysis, i.e. the expressions of the constraints are easier to read. From a geometrical point of view there is an important feature which has a counterpart also in the quantum theory. In fact, the field E^a_i has an interpretation in terms of the area element: choosing a two-surface S in a $t = constant$ hypersurface we have that:

$$A_S = \int_S d^2\sigma \sqrt{E^a_i n_a E^b_i n_b} \,. \tag{2.31}$$

Then, by introducing the 2-form

$$E^i = \frac{1}{2}\epsilon_{abc} E^{ai} dx^b dx^c \,, \tag{2.32}$$

we can write

$$A_S = \int_S |E| \,. \tag{2.33}$$

Now, in the limit where the surface is small, the quantity

$$E_S^i = \int_S E^i = \frac{1}{2}\epsilon^i_{jk} \int_S e^j \wedge e^k \,, \tag{2.34}$$

is a vector normal to the surface, whose length is the area of the surface. Therefore we can say that the momentum conjugate to the connection represents an area element, this fact still holds in the quantum case.

Chapter 3

Discretization

3.1 Lattice QCD and Regge Calculus

Let's consider a $SU(2)$ Yang-Mills theory in four dimensions. The Yang-Mills field is known to be the connection whose components are $A^i_\mu(x)$, where i is an index in the Lie algebra su(2). Explicitly we can write the connection as:

$$A(x) = A^i_\mu(x)\tau_i dx^\mu \,, \tag{3.1}$$

where τ_i provide a basis of su(2). In order to discretize such a theory Wilson suggests to fix a cubic lattice with N vertices connected by E edges, this of course breaks the Lorentz invariance of the theory, recovered only in a suitable limit. We call a the length of the lattice edges, this is determined by the flat metric. Then we associate to each oriented edge a group variable $U_\mathbf{e} \in SU(2)$ in the following way:

$$U_\mathbf{e} = \mathcal{P}e^{\int_\mathbf{e} A} \,, \tag{3.2}$$

where $\mathcal{P}e$ stands for the path-ordered exponential (see Appendix). The idea is then to use a discrete set of group variables in place of the continuous variable A. Starting from this group variables instead of the algebra variables it is possible to calculate physical quantities in the limit where $N \to \infty$ and $a \to 0$. Under a gauge transformation the group elements $U_\mathbf{e}$ transform "homogeneously", that is:

$$U_\mathbf{e} \to \lambda_{s_\mathbf{e}} U_\mathbf{e} \lambda^{-1}_{t_\mathbf{e}} \,, \tag{3.3}$$

where $s_\mathbf{e}$ and $t_\mathbf{e}$ are the initial and final vertices of the edge $\mathbf{e}$ (source and target), $\lambda_\mathbf{v}$ is an element of $SU(2)$. Therefore a gauge transformation can be thought as an element of $SU(2)^V$, where V is the number of vertices. Gauge transformations take place at each vertex.
From eq. (3.3) it is straightforward to see that if we take the ordered product of four group elements around a face $\mathbf{f}$

$$U_\mathbf{f} = U_{\mathbf{e}_1} U_{\mathbf{e}_2} U_{\mathbf{e}_3} U_{\mathbf{e}_4} \tag{3.4}$$

and we consider its trace, we get a gauge invariant quantity. In addition to this $U_\mathbf{f}$ is a discrete version of the connection, since it is the holonomy of the connection on the loop given by a square. Wilson has shown that the discrete action

$$S = \beta \sum_{\mathbf{f}} \mathrm{Tr} U_{\mathbf{f}} \; + \; c.c. \tag{3.5}$$

approximates the continuous action in the limit where a is small.

The Hamiltonian formulation of the discretized theory is particularly important, since it is going to have a counterpart in the quantum realm. We focus on a boundary, say spacelike, the hamiltonian coordinates are given by the group elements U_l on the boundary edges, called "links". The canonical configuration space is therefore $SU(2)^L$, where L is the number of links. The corresponding phase space is the cotangent space $T^*SU(2)^L$, the Poisson structure of this space is carefully explained in the Appendix. We denote the conjugate momentum of U_l by $L_l^i \in \mathrm{su}(2)$. The Poisson brackets are then given by:

$$\begin{aligned} &\{U_l \,,\, U_{l'}\} = 0 \,, \\ &\left\{U_l \,,\, L^i_{l'}\right\} = \delta_{ll'} U_l \tau^i \,, \\ &\left\{L^i_l \,,\, L^j_{l'}\right\} = \delta_{ll'} \epsilon^{ij}_k L^k_l \,, \end{aligned} \tag{3.6}$$

(no summation over l). The Hilbert space of the discrete theory can therefore be represented by states $\psi\left(U_l\right)$, i.e. functions on the configuration space. The space of these functions carries a natural scalar product which is invariant under the gauge tranformations on the boundary, this is given by the $SU(2)$ Haar measure:

$$(\phi \,,\, \psi) = \int_{SU(2)} dU_l \overline{\phi\left(U_l\right)} \psi\left(U_l\right) \,. \tag{3.7}$$

The boundary gauge transformations act at the nodes of the boundary and transform the states as follows

$$\psi\left(U_l\right) \to \psi\left(\lambda_{s_l} U_l \lambda^{-1}_{t_l}\right) \,, \quad \lambda_n \in SU(2). \tag{3.8}$$

We move on and introduce Regge calculus now. Tullio Regge introduced a discretization of General Relativity called "Regge calculus". We can summarize it as follows: a d-simplex is a generalization of a triangle or a tetrahedron to arbitrary dimensions, more precisely, it is the convex hull of its $d+1$ vertices. These vertices are connected by $d(d+1)/2$ line segments whose length L_s fully specify the shape of the simplex, i.e. its geometry.

A Regge space $(M \,, L_s)$ in d dimensions is a d-dimensional metric space obtained by gluing d-simplices along matching boundary $(d-1)$-simplices. For example, in two dimensions we can obtain a surface by gluing triangles, bounded by segments, which meet at points. In three dimensions we chop space into tetrahedra, bounded by triangles, in turn bounded by segments, which meet at points. In four dimensions we chop spacetime into 4-simplices, bounded by tetrahedra, in turn bounded by triangles, in turn bounded by segments, which meet at points. These structures are called triangulations. We can legitimately ask how curvature arises in a Regge space, since all these geometrical objects are flat. We consider the simplest case, that is $d = 2$ dimensions: it is easy to see that if we glue triangles around a common vertex, curvature arises in terms of a deficit angle, that is, the sum of all the angles insisting on a given vertex does not add up to 2π. In formulas:

$$\delta_P\left(L_s\right) = 2\pi - \sum_{\mathbf{t}} \theta_{\mathbf{t}}\left(L_s\right) \tag{3.9}$$

This fact admits a generalization to higher dimensions: gluing flat d-simplices can generate curvature on the $(d-2)$-simplices (sometimes called "hinges"). Now, a Riemannian manifold $(M\,, g)$ can be approximated arbitrarly well by a Regge manifold, in fact, for any $(M\,, g)$ and any ϵ we can find a $(M\,, L_s)$ with sufficiently many simplices such that, for any two points $x, y \in M$, the difference between the Riemannian distance and the Regge distance is smaller than ϵ. In order to dicretize General Relativity we need a discretized version of the action, the Regge action is defined as:

$$S_M\left(L_s\right) = \sum_h A_h\left(L_s\right)\delta\left(L_s\right) \,, \tag{3.10}$$

where the sum is over the hinges and A_h is the $(d-2)$-volume of the hinge h. Remarkably, this action converges to the Einstein-Hilbert action when the Regge manifold $(M\,, L_s)$ converges to the Riemann manifold $(M\,, g)$. The Regge action can be also rewritten as a sum over the d-simplices $\mathbf{v}$ of the triangulation: from (3.9) and (3.10) we have that

$$S_M\left(L_s\right) = 2\pi\sum_h A_h\left(L_s\right) - \sum_{\mathbf{v}} S_{\mathbf{v}}\left(L_s\right) \,, \tag{3.11}$$

where the action of a d-simplex is

$$S_{\mathbf{v}}\left(L_s\right) = \sum_h A_h\left(L_s\right)\theta_h\left(L_s\right) \,. \tag{3.12}$$

3.2 Discretization in 3D

The discretization used in LQG differs from the Regge one, because essentially lengths are constrained by inequalities (think about a triangle for instance), and it's difficult to implement a configuration space with such constraints. It is preferable then to consider also the "dual" of a triangulation, in three dimensions this is simply obtained by replacing each tetrahedra by a vertex sitting at its center, each face (a triangle) by an edge coming off the vertex and puncturing the triangle. Therefore, adjacent tetrahedra are replaced by vertices connected by edges. The dual of the triangulation Δ is denoted as Δ^* and the set of vertices, edges and faces is called a "2-complex" (denoted with $\mathcal{C}$). Thus, we are going to discretize classical GR on a 2-complex. One word about the boundary: if we consider a compact region of spacetime, the discretization Δ will induce a discretization of the boundary, formed by the boundary segments and the boundary triangles of Δ. Moving to Δ^* we realize that the boundary is formed now by the end points of the edges dual to the boundary triangles, which are called nodes, and the boundary of the faces dual to the boundary segments, together they form the graph Γ of the boundary. The boundary graph, by construction, is at the same time the boundary of the 2-complex and the dual of the boundary of the triangulation:

$$\Gamma = \partial\left(\Delta^*\right) = \left(\partial\Delta\right)^* \,. \tag{3.13}$$

Now, in 3 dimensions the gravitational field is described by a tetrad field $e^i = e^i_a dx^a$ and a $SO(3)$ connection $\omega^i_j = \omega^i_{aj} dx^a$, where $a, b, .. = 1, 2, 3$ are spacetime indices and $i, j = 1, 2, 3$ are internal

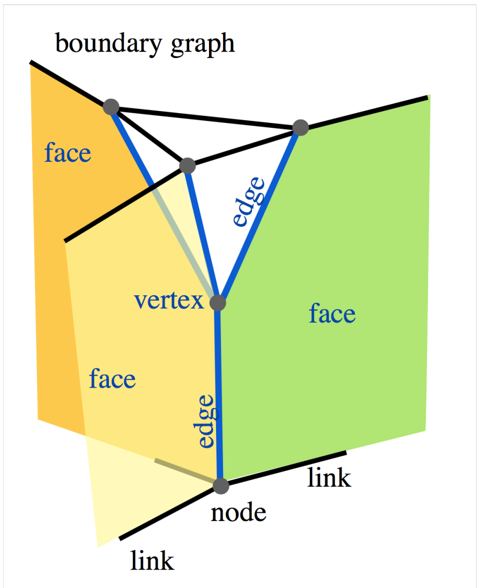

Figure 3.1: A 2-complex

indices. We discretize the connection as in Yang-Mills theory, that is, by assigning a $SU(2)$ element $U_{\mathbf{e}}$ to each edge $\mathbf{e}$ of the 2-complex. We discretize the triad by associating a vector L_s^i of $\mathbb{R}^3$ to each segment s of the original triangulation:

$$\begin{aligned} \omega &\longrightarrow U_{\mathbf{e}} = \mathcal{P}\exp \int_{\mathbf{e}} \omega \in SU(2) \\ e &\longrightarrow L_s^i = \int_s e^i \in \mathbb{R}^3 \,, \end{aligned} \tag{3.14}$$

in the LQG jargon $U_{\mathbf{e}}$ is called the "holonomy" (of the connection along the edge). The Einstein-Hilbert action can be approximated in terms of these objects. We have seen that under a gauge transformation the holonomy transforms "well", that is, as

$$U_{\mathbf{e}} \mapsto R_{s_e} U_{\mathbf{e}} R_{t_e}^{-1} \tag{3.15}$$

whereas the algebra values L^i apparently don't follow this rule. Nevertheless, it is possible to give a gauge equivalent definition of L^i in such a way that it transforms as the holonomy, as shown in [22]. The discretization approximates well the continuum theory when the curvature is small at the scale of the triangulation and the segments are straight lines. We notice that the norm of vector L_s^i associated with the segment s is the length of the segment, i.e.

$$L_s^2 = |\overrightarrow{L_s}|^2 \,. \tag{3.16}$$

Since each face $\mathbf{f}$ of the 2-complex corresponds to a segment $s = s_{\mathbf{f}}$ of the triangulation, we can view L_s^i as associated with the face: $L_{\mathbf{f}}^i = L_{s_{\mathbf{f}}}^i$. Furthermore, since $\mathbb{R}^3$ equipped with the usual cross product is isomorphic (as a Lie algebra) to su(2) we can express $L_{\mathbf{f}}^i$ as an element of su(2), that is:

$$L_{\mathbf{f}} = L_{\mathbf{f}}^i \tau_i \,. \tag{3.17}$$

Summarizing, the variables of the discretized theory are:

- An $SU(2)$ group element $U_{\mathbf{e}}$ for each edge $\mathbf{e}$ of the 2-complex;
- An su(2) algebra element $L_{\mathbf{f}}$ for each face $\mathbf{f}$ of the 2-complex.

In the four-dimensional case the approach will be the same, group and algebra variables associated to edges and faces.
Having discretized space, we need a discrete version of the action. The idea is to mimic the Regge action, knowing that in three dimensions curvature arises around segments and so the volume of the hinge is the length of the segment. The deficit angle is replaced by the curvature written in terms of the holonomy, as already remarked earlier we have curvature around a segment if the group element $U_{\mathbf{f}} = U_{\mathbf{e}_1} \cdots U_{\mathbf{e}_n}$ is different from the identity ($\mathbf{e}_1, \ldots, \mathbf{e}_n$ being the edges bounding the face $\mathbf{f}$). In this way, we can write the discretized action as follows:

$$\mathcal{S} = \frac{1}{8\pi G} \sum_{\mathbf{f}} \mathrm{Tr}\left(L_{\mathbf{f}} U_{\mathbf{f}}\right) , \tag{3.18}$$

Performing the variation of the action with respect to $L_{\mathbf{f}}$ and setting it to zero gives $U_{\mathbf{f}} = \mathbb{1}$, that is, flatness, which is equivalent to the continuous Einstein equations in three dimensions.

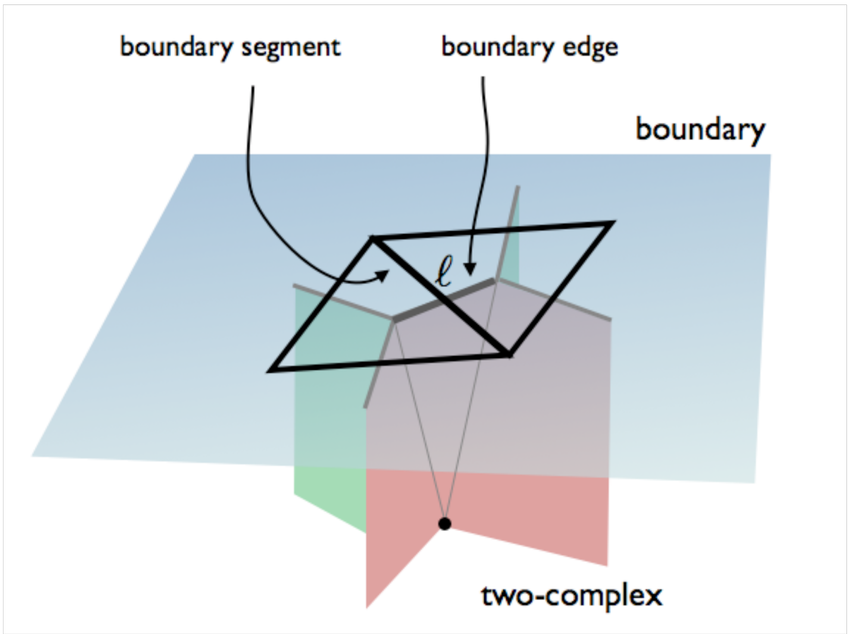

Figure 3.2: Boundary graph

We specify on the boundary now. On the boundary there are two kinds of variables: the group elements U_l of the boundary edges, namely, the links, and the algebra elements L_s of the boundary segment s. Notice that there is precisely one boundary segment s per each link l, and the two cross. We can therefore rename L_s as L_l whenever l is the link crossing the boundary segment s. In this way, the boundary variables are formed by a pair $(L_l\,, U_l) \in \mathrm{su}(2) \times SU(2)$ for each link l of the graph Γ. Therefore on the boundary we have a pair of conjugate variables at each link, the Poisson brackets are the ones already written in (3.6), only with a factor $8\pi G$ on the RHS coming from the action.

3.3 Discretization in 4D

Moving to the four-dimensional case we consider a triangulation Δ made of 4-simplices, the corresponding dual triangulation Δ^* has the following properties: a vertex is dual to a 4-simplex, an edge is dual to a tetrahedron, a face is dual to a triangle (for instance a face in the (x, y) plane is dual to a triangle in the (z, t) plane). Therefore, we have that a face of the 2-complex that touches the boundary is dual to a boundary triangle, this in turn is dual to a boundary link l. Geometrically, this link is the intersection of the face with the boundary. Thus, a boundary link l is obviously a boundary edge (by definition), but is also associated with a face $\mathbf{f}$ touching the boundary. From these considerations follows that we discretize the connection and the triad as

$$
\begin{aligned}
\omega &\longrightarrow U_{\mathbf{e}} = \mathcal{P}\mathrm{exp}\int_{\mathbf{e}} \omega \in SL(2,\mathbb{C}) \\
e &\longrightarrow B_{\mathbf{f}} = \int_{\mathbf{t_f}} B \quad \in \mathrm{sl}(2,\mathbb{C})\,,
\end{aligned}
\tag{3.19}
$$

where $B = \left((e\wedge e)^* + \frac{1}{\gamma}(e\wedge e)\right)$ is the 2-form defined in the action, and $\mathbf{t_f}$ is the triangle dual to the face $\mathbf{f}$.
The variables of the discretized theory are then:

• a group element $U_{\mathbf{e}}$ for each edge $\mathbf{e}$ of the 2-complex;
• an algebra element $B_{\mathbf{f}}$ for each face $\mathbf{f}$ of the 2-complex.

Pretty much the same as seen in the three-dimensional case. Therefore we call U_l the group elements associated with the boundary edges l, that is, the links of the boundary graph Γ, and B_l are the elements of a face bounded by the link l. There is a remarkable geometric interpretation of B_l: consider a triangle lying on the boundary, choose the tetrad field in the time gauge, that is, $e^0 = dt$ and $e^i = e^i_a dx^a$, the pull-back of $(e\wedge e)^*$ on the boundary vanishes and we are left with

$$
L^i_{\mathbf{f}} = \frac{1}{2\gamma}\epsilon^i_{\ jk}\int_{\mathbf{t_f}} e^j\wedge e^k\,. \tag{3.20}
$$

In the approximation in which the metric is constant on the triangle it follows then that the norm of $L^i_{\mathbf{f}}$ is proportional to the area of the triangle:

$$
|L_{\mathbf{f}}| = \frac{1}{\gamma}A_{\mathbf{t_f}}\,. \tag{3.21}
$$

Here we see an analogy between the vector $\overrightarrow{L_{\mathbf{f}}}$ and the vector $\overrightarrow{E_S}$ defined in terms of the Ashtekar variables.

Chapter 4

Quantization

4.1 3D Theory

In order to define the quantum theory two features are needed:

• a boundary Hilbert space that describes the quantum states of the boundary geometry;
• the transition amplitude for these boundary states; in the small $\hbar$ limit the transition amplitude must reproduce the exponential of the Hamilton function.

4.1.1 Hilbert Space

To construct the Hilbert space and the transition amplitude we proceed as follows: first, we discretize the classical theory; then, we study the quantum theory that corresponds to the discretized theory; finally we discuss the continuum limit.
We recall from the previous chapter that the discrete boundary geometry is described by a pair of variables for each link of the graph Γ: $(U_l\,, L_l) \in SU(2) \times \mathrm{su}(2)$. We are seeking the quantum version of these phase space variables, i.e. we are looking for operators U_l and L_l satisfying the quantum version of the Poisson brackets seen earlier:

$$\left[U_l\,,\, L^i_{l'}\right] = i\,(8\pi\hbar G)\,\delta_{ll'}U_l\tau^i \tag{4.1}$$

For this purpose we consider, as the Hilbert space, the space of square integrable functions on $SU(2)^L$:

$$\mathcal{H}_\Gamma = L_2\left[SU(2)^L\right]\;. \tag{4.2}$$

States are therefore wavefunctions $\psi(U_l)$ of L group elements U_l. The scalar product compatible with the $SU(2)$ structure is given by the group-invariant measure, that is, the Haar measure:

$$\langle\phi|\psi\rangle = \int_{SU(2)^L} dU_l\overline{\phi\left(U_l\right)}\psi\left(U_l\right)\;. \tag{4.3}$$

By doing so, U_l can be seen simply as a multiplicative operator acting as $U_{l'}\left(\psi(U_l)\right) = \psi\left(U_{l'}U_l\right)$.
Furthermore, as showed in the Appendix, on the Lie group $SU(2)$ it is defined a left-invariant vector

field, whose components are:

$$\left(J^i\psi\right) := -i\frac{d}{dt}\psi\left(Ue^{t\tau_i}\right)|_{t=0}\,. \tag{4.4}$$

Then, to get the correct operator satysfing (4.1) it is sufficient to scale the left-invariant vector field with the appropriate dimensionful factor:

$$L_l^i := (8\pi\hbar G)\,J_l^i\,. \tag{4.5}$$

One important consequence is that length is quantized. In fact, we recall that $L_s = |\overrightarrow{L_{\mathbf{f}}}|$, with $\mathbf{f}$ being the face dual to the segment s. This means that on the boundary we have $L_l = |\overrightarrow{L_l}|$ where l is the link crossing the boundary segment s. Therefore, since $\overrightarrow{J_l}$ is the generator of $SU(2)$, $|\overrightarrow{J_l}|^2$ is the $SU(2)$ Casimir, its eigenvalues are $j(j+1)$, j being an half-integer. Then we get the following spectrum for the operator L_l:

$$L_{j_l} = 8\pi\hbar G\sqrt{j_l(j_l+1)}\,, \tag{4.6}$$

for half-integers j_l.
We go on now with the definition of the boundary Hilbert space. Since the theory must be invariant under $SU(2)$ gauge transformations (taking place at nodes), we have to take that into account. Then, the gauge-invariant states must satisfy

$$\psi\left(U_l\right) = \psi\left(\Lambda_{s_l}U_l\Lambda_{t_l}^{-1}\right)\,, \quad \Lambda_{\mathrm{n}} \in SU(2)\,. \tag{4.7}$$

We can write equivalently

$$\overrightarrow{C_{\mathrm{n}}}\psi = 0 \tag{4.8}$$

for every node n of the boundary graph, where $\overrightarrow{C_{\mathrm{n}}}$ is the generator of $SU(2)$ transformations at the node n, i.e. :

$$\overrightarrow{C_{\mathrm{n}}} = \overrightarrow{L_{l_1}} + \overrightarrow{L_{l_2}} + \overrightarrow{L_{l_3}} = 0\,; \tag{4.9}$$

where l_1, l_2, l_3 are the three links emerging from the node n. This relation is called gauge constraint. From a geometrical standpoint the interpretation of this equation is straightforward: l_1, l_2, l_3 are three links that cross three segments which in turn bound a triangle, then, the condition (4.9) can be read as the closure condition satisfied by every triangle (since L_{l_i} represents the length of the segment s_i). It is worth noting that a similar result was obtained by Roger Penrose in 1971 [21], in his "spin-geometry theorem". Penrose observed that if we consider the operators $\overrightarrow{L}_l$, which are not gauge invariant, we can define a gauge invariant operator, called "Penrose metric operator", by

$$G_{ll'} = \overrightarrow{L}_l \cdot \overrightarrow{L}_{l'}\,, \tag{4.10}$$

where l and l' share the same source. The Casimir operators of $SU(2)$ are then given by

$$A_l^2 = \overrightarrow{L}_l \cdot \overrightarrow{L}_l\,. \tag{4.11}$$

The theorem states that the equations (4.10), (4.11) and (4.9) (which generalizes to a node with arbitrary valence), are sufficient to guarantee the existence of a flat polyhedron, such that the area of its faces is A_l and where $G_{ll'}$ is given by $G_{ll'} = A_l A_{l'} \cos\theta_{ll'}$, $\theta_{ll'}$ being the angle between the normals to the faces l and l'. More precisley, there exists a 3×3 metric tensor g_{ab}, $a, b = 1, 2, 3$ and normal to the faces $\overrightarrow{n}_l$, such that

$$G_{ll'} = g_{ab} n_l^a n_{l'}^b \tag{4.12}$$

and the length of these normals is equal to the area of the face. In conclusion, the algebraic structure of the momentum operators in $\mathcal{H}_\Gamma$ determine the existence of a metric at each node and therefore endows each quantum of space with a geometry. It is curious that these results reappered more than 20 years later in LQG.
Proceeding with the construction of the boundary Hilbert space, we consider the subspace of $\mathcal{H}_\Gamma$ where (4.7) is verified, which is a proper subspace, we call it $\mathcal{K}_\Gamma$ and write it as

$$\mathcal{K}_\Gamma = L_2 \left[SU(2)^L / SU(2)^N \right]_\Gamma . \tag{4.13}$$

Clearly L indicates the number of links, N is the number of nodes and the subscript Γ denotes the fact that the pattern of the $SU(2)^N$ transformations is dictated by the structure of the graph Γ. Let's study the structure of $\mathcal{K}_\Gamma$. On this Hilbert space the length operators L_l are gauge-invariant, furthermore, they form a complete commuting set. This means that a basis of $\mathcal{K}_\Gamma$ is given by the normalized eigenvectors of these operators, which we indicate as $|j_l\rangle$. An element of this basis is therefore determined by assigning a spin j_l to each link l of the graph. A graph with a spin assigned to each link is called a "spin network". The spin network states $|j_l\rangle$ form a basis of $\mathcal{K}_\Gamma$, this is called a *spin-network basis* and spans the quantum states of geometry.
More concretely, we can make use of the Peter-Weyl theorem to get a more intuitive picture of what's going on. In fact, we know that the Wigner matrices D^j_{mn} provide an orthogonal basis for the spin-j representation, that is:

$$\int dU \overline{D^{j'}_{m'n'}(U)} D^j_{mn}(U) = \frac{1}{d_j} \delta^{jj'} \delta_{mm'} \delta_{nn'} , \tag{4.14}$$

where $d_j = 2j + 1$ is the dimension of the j representation and dU is the $SU(2)$ Haar measure. In other words, the Hilbert space $L_2 [SU(2)]$ can be decomposed into a sum of finite dimensional subspaces of spin j, spanned by the basis states formed by the matrix elements of the Wigner matrices $D^j(U)$. This matrix is a map from the Hilbert space $\mathcal{H}_j$ to itself, therefore we can see $D^j(U)$ as an element of $\mathcal{H}_j \otimes \mathcal{H}_j^*$. Since we know that $\mathcal{H}_j \cong \mathcal{H}_j^*$, for notational convenience we omit the asterisk. All together this reads as

$$L_2 [SU(2)] = \bigoplus_j (\mathcal{H}_j \otimes \mathcal{H}_j) . \tag{4.15}$$

Having L links it is straightforward to consider the following:

$$L_2 \left[SU(2)^L \right] = \otimes_l \left[\oplus_j (\mathcal{H}_j \otimes \mathcal{H}_j) \right] = \oplus_{j_l} \otimes_l (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}) . \tag{4.16}$$

The two Hilbert spaces associated with a link can be seen as belonging to the two ends of the link, because each transforms according to the gauge transformation at one end. In order to see what's going on a node we can regroup the Hilbert spaces $\mathcal{H}_j$ in such a way that

$$L_2\left[SU(2)^L\right] = \oplus_{j_l} \otimes_n \left(\mathcal{H}_j \otimes \mathcal{H}_{j'} \otimes \mathcal{H}_{j''}\right) \,, \tag{4.17}$$

where j, j', j'' are the spins coming out from the node n. Next, we want the space of gauge-invariant states, thus we should restrict to the invariant part of the spaces transforming at the same node, that is:

$$L_2\left[SU(2)^L/SU(2)^N\right] = \oplus_{j_l} \otimes_n \mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right) \,. \tag{4.18}$$

From $SU(2)$ representation theory it is known that $\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right)$ does not exist unless the sum of three spins is an integer and the three spins satisfy the triangular inequality:

$$|j_1 - j_2| < j_3 < j_1 + j_2 \,. \tag{4.19}$$

If this condition holds then the invariant space is one-dimensional:

$$\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right) = \mathbb{C} \,. \tag{4.20}$$

Therefore

$$L_2\left[SU(2)^L/SU(2)^N\right] = \oplus_{j_l}\mathbb{C} \tag{4.21}$$

where the sum is restricted to the j_l that satisfy the triangular inequalities. Since spins are associated to the lengths of the sides of a triangle, and these are known to satisfy the triangular inequality, the resemblance with the geometrical picture holds nicely.
Then, a generic quantum state in loop quantum gravity is a superposition of spin-network states:

$$|\psi\rangle = \sum_{j_l} \mathcal{C}_{j_l} |j_l\rangle \,. \tag{4.22}$$

Summarizing, the spin network states $|j_l\rangle$:

- are an eigenbasis of all lengths operators;
- span the gauge-invariant Hilbert space;
- have a simple geometric interpretation: they just say how long the boundary links are.

Next, we would like to write the spin-network states $|j_l\rangle$ in the $\psi\left(U_l\right)$ representation, that is, compute the spin-network wavefunctions:

$$\psi_{j_l}\left(U_l\right) = \langle U_l | j_l\rangle \,. \tag{4.23}$$

This can be done explicitly by solving the eigenvalue equation for the length operators L_l

$$L_l \psi_{j_l}\left(U_l\right) = L_{j_l} \psi_{j_l}\left(U_l\right) \,. \tag{4.24}$$

It is possible to write a generic state $\psi(U) \in L_2[SU(2)]$ as a linear combination in the basis provided by the Wigner matrices, as follows:

$$\psi(U) = \sum_{jmn} \mathcal{C}_{jmn} D^j_{mn}(U) \,. \tag{4.25}$$

Therefore, in our case, a state $\psi\left(U_l\right) \in L_2[SU(2)^L]$ can be written as

$$\psi\left(U_l\right) = \sum_{j_i, m_i, n_i} \mathcal{C}_{j_1 \cdots j_L m_1 \cdots m_L n_1 \cdots n_L} D^{j_1}_{m_1 n_1}\left(U_{l_1}\right) \cdots D^{j_L}_{m_L n_L}\left(U_{l_L}\right) \,, \tag{4.26}$$

where $i = 1, \ldots, L$. A state invariant under a $SU(2)$ transformation must be invariant if we act with a transformation Λ_{n} taking place at the node n. This in turn acts on the three group elements of the three links that meet at the node. Since the Wigner matrices are representation matrices, the gauge transformation acts on the three corresponding indices, for this reason we have that, for the state to be invariant, $\mathcal{C}_{j_1 \cdots j_L m_1 \cdots m_L n_1 \cdots n_L}$ must be invariant when acted upon by a group transformation on the three indices corresponding to the same node. From representation theory it is known that, up to normalization, it exists only one invariant tensor with three indices in three $SU(2)$ representations, it is called the Wigner $3j$-symbol and is denoted as

$$\iota^{m_1 m_2 m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \,. \tag{4.27}$$

In this way, we can express any invariant state in the triple tensor product of representations of $SU(2)$ as

$$\iota^{m_1 m_2 m_3} = \sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle \,. \tag{4.28}$$

Going on, a gauge-invariant state must then have the form

$$\psi\left(U_l\right) = \sum_{j_1 \cdots j_L} \mathcal{C}_{j_1 \cdots j_L} \iota_1^{m_1 m_2 m_3} \cdots \iota_N^{m_{L-2} m_{L-1} m_L} D^{j_1}_{m_1 n_1}\left(U_{l_1}\right) \cdots D^{j_L}_{m_L n_L}\left(U_{l_L}\right) \tag{4.29}$$

where all the indices are contracted between the intertwiner ι and the Wigner matrices D. Don't let confuse yourself if you don't see any n-indices contracted, because the pattern of contraction is dictated by the structure of the graph (and so, broadly speaking, m's and n's are interchangeable, it is just the notation of Wigner matrices that keeps them different).
Seeking a more compact form we write a generic gauge-invariant state as

$$\psi\left(U_l\right) = \sum_{j_l} \mathcal{C}_{j_l} \psi_{j_l}\left(U_l\right) \,, \tag{4.30}$$

where

$$\psi_{j_l}\left(U_l\right) = \iota_1^{m_1 m_2 m_3} \cdots \iota_N^{m_{L-2} m_{L-1} m_L} D^{j_1}_{m_1 n_1}\left(U_{l_1}\right) \cdots D^{j_L}_{m_L n_L}\left(U_{l_L}\right) \tag{4.31}$$

are the orthogonal states labeled by a spin associated with each link. These are the spin-network wavefunctions. We can write them more compactly as

$$\langle U_l | j_l \rangle = \psi_{j_l}\left(U_l\right) = \bigotimes_{\mathrm{n}} \iota_{\mathrm{n}} \cdot \bigotimes_{l} D^{j_l}\left(U_l\right) \,. \tag{4.32}$$

4.1.2 Transition Amplitude

The next goal is to write down the transition amplitude of the three-dimensional theory. The transition amplitude is a function of the boundary states, therefore we assume that a triangulation Δ is fixed and we consider a boundary, which means considering the boundary graph $\Gamma = (\partial\Delta)^*$. We denote the transition amplitude expressed in terms of the "coordinates" as $W_\Delta(U_l) = \langle W_\Delta | U_l \rangle$ and the transition amplitude in terms of the "momenta" as $W_\Delta(j_l) = \langle W_\Delta | j_l \rangle$.
Notice that the transition matrix between the two basis is given precisely by the spin-network states. To compute the transition amplitude W_Δ of the theory discretized on the 2-complex, dual to Δ, we use the Feynman path integral. The amplitude is given by the integral over all classical configurations weighted by the exponential of the (discretized) classical action:

$$W_\Delta(U_l) = \mathcal{N} \int dU_{\mathbf{e}} \int dL_{\mathbf{f}} e^{\frac{i}{8\pi\hbar G} \sum_{\mathbf{f}} \mathrm{Tr}[U_{\mathbf{f}} L_{\mathbf{f}}]}, \tag{4.33}$$

where $\mathcal{N}$ is a normalization factor. Reabsorbing factors on the overall constant $\mathcal{N}$ and performing the integral over the momenta we obtain

$$W_\Delta(U_l) = \mathcal{N} \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \delta(U_{\mathbf{f}}) \,. \tag{4.34}$$

To compute this integral, we expand the delta function over the group in representations using

$$\delta(U) = \sum_j d_j \mathrm{Tr} D^{(j)}(U) \,, \tag{4.35}$$

where $d_j = 2j+1$ is the dimension of the spin-j representation. Therefore (4.34) turns out to be

$$\begin{aligned} W_\Delta(U_l) &= \mathcal{N} \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \left(\sum_j d_j \mathrm{Tr} D^j(U_{\mathbf{f}}) \right) \\ &= \mathcal{N} \sum_{\mathbf{f}} \left(\prod_{\mathbf{f}} d_{j_{\mathbf{f}}} \right) \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \mathrm{Tr} \left(D^{j_{\mathbf{f}}}(U_{1\mathbf{f}}) \cdots D^{j_{\mathbf{f}}}(U_{n\mathbf{f}}) \right) \,, \end{aligned} \tag{4.36}$$

where $U_{\mathbf{f}} = U_{1\mathbf{f}} \cdots U_{n\mathbf{f}}$. Now, if we focus our attention on one edge in particular, we notice that an edge bounds precisely three faces (because an edge is dual to a triangle, which is bounded by three segments, and segments are dual to faces). Therefore each $dU_{\mathbf{e}}$ integral is of the form

$$\int dU D^{j_{l_1}}_{m_1 n_1}(U) D^{j_{l_2}}_{m_2 n_2}(U) D^{j_{l_3}}_{m_3 n_3}(U) \,, \tag{4.37}$$

but, since the Haar measure is invariant on both sides, the result must be invariant in both set of indices. As we have seen before, there is only one such object, the Wigner $3j$-symbol, then

$$\begin{aligned}\int dU D^{j_{l_1}}_{m_1 n_1}(U) D^{j_{l_2}}_{m_2 n_2}(U) D^{j_{l_3}}_{m_3 n_3}(U) &= \iota^{m_1 m_2 m_3} \iota^{n_1 n_2 n_3} \\ &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} ,\end{aligned} \tag{4.38}$$

where, again, we emphasize that the m and n indices are dictated by the structure of the graph Γ. Thus, what we obtain in the end is nothing but $3j$-symbols contracted among themselves. More precisley, we observe that each edge produce two $3j$-symbols which we can view as located at the two ends of the edge, since their indices are contracted at the end (on a vertex). At each vertex there are four edges, therefore four $3j$-symbols contracted among themselves. The contraction must be $SU(2)$-invariant, so we are looking for an object which involves four $3j$-symbols and is invariant under a $SU(2)$ transformation, it turns out it exists and it is called the Wigner $6j$-symbol:

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} := \sum_{m_a, n_a} \prod_{a=1}^{6} g_{m_a n_a} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_5 \\ n_1 & m_4 & m_3 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_6 \\ n_2 & n_4 & m_6 \end{pmatrix} \begin{pmatrix} j_3 & j_5 & j_6 \\ n_3 & n_5 & n_6 \end{pmatrix} , \tag{4.39}$$

where

$$g_{mn} = \sqrt{2j+1} \begin{pmatrix} j & j & 0 \\ m & n & 0 \end{pmatrix} = \delta_{m,-n} (-1)^{j-m} . \tag{4.40}$$

After integrating over all internal edge-group variables, the group variables of the boundary are left. We can integrate these as well contracting with a boundary spin network state, obtaining [14]

$$W_\Delta(j_l) = \mathcal{N}_\Delta \sum_{j_\mathbf{f}} \prod_{\mathbf{f}} (-1)^{j_\mathbf{f}} d_{j_\mathbf{f}} \prod_{\mathbf{v}} (-1)^{J_\mathbf{v}} \{6j\} , \tag{4.41}$$

where the sum is over the association of a spin to each face, respecting the triangular inequalitites at all edges, $J_\mathbf{v} = \sum_{a=1}^{6} j_a$, and j_a are the spin of the faces adjacent to the vertex $\mathbf{v}$ (a vertex of the 2-complex is adjacent to six faces).

We can see the connection with general relativity in the classical limit (the continuum limit will be discussed in the four-dimensional case, which is more interesting). If we consider a single tetrahedron whose sides have length $L_a = j_a + 1/2$, it is possible to show [15] that, in the large j limit we have

$$\{6j\} \underset{j\to\infty}{\sim} \frac{1}{\sqrt{12\pi V}} \cos\left(S + \frac{\pi}{4}\right) . \tag{4.42}$$

Thus, by using the well known relation $e^{i\alpha} = \cos\alpha + i\sin\alpha$ we get the following

$$\{6j\} \underset{j\to\infty}{\sim} \frac{1}{2\sqrt{-12i\pi V}} e^{iS} + \frac{1}{2\sqrt{12i\pi V}} e^{-iS} . \tag{4.43}$$

We see therefore that two terms with opposite phase enter here, this is precisely the discussion we were addressing when dealing with the tetrad action.
If we consider only large spins we can disregard quantum discreteness and the sum over the spins is approximated by an integral over lengths in a Regge geometry. This is a discretization of a path

integral over geometries of the exponential of the Einstein-Hilbert action. Therefore (4.38) is a concrete implementation of the path-integral "sum over geometries" formal definition of quantum gravity:

$$Z \sim \int D[g] e^{\frac{i}{\hbar} \int \sqrt{-g} R} . \tag{4.44}$$

The next discussion addresses a topic that will be generalised to the four-dimensional case and it is particularly relevant. Consider a triangulation formed by a single tetrahedron τ, the boundary graph has again the shape of a tetrahedron, since we have four vertices obtained as the end points of the four edges puncturing the four faces of the original tetrahedron. The amplitude is then a function of the variables of the links of the graph. Let's label with $a, b = 1, 2, 3, 4$ the nodes of the graph and denote with $U_{ab} = U_{ba}^{-1}$ the boundary group elements. The transition amplitude is then a function $W(U_{ab})$. Notice that we have already constructed the 2-complex, which consists of the four edges puncturing the four faces and of the boundary links, therefore it is made of six faces (obtained by connecting the vertex sitting inside τ with the six boundary links). Using (4.34) and dropping the normalization we get:

$$W(U_{ab}) = \int dU_a \prod_{ab} \delta\left(U_a U_{ab} U_b^{-1}\right) . \tag{4.45}$$

Once this integrals are performed we obtain:

$$W(U_{ab}) = \delta(U_{12} U_{23} U_{31})\, \delta(U_{13} U_{34} U_{41})\, \delta(U_{23} U_{34} U_{42}) \tag{4.46}$$

Notice that each sequence of U_{ab} inside the deltas corresponds to an independent closed loop in the boundary graph. The interpretation of this amplitude is therefore immediate: the amplitude forces the connection to be flat on the boundary (by the very definition of the delta). More precisely, it is the three-dimensional connection which is flat, not the two-dimensional one living on the boundary. We can think of it as having a spacetime reference frame on each face that can be parallel transported along the boundary in such a way that any closed loop gives unity. In other words, $W(U_{ab})$ is just the gauge-invariant version of $\prod_{ab} \delta(U_{ab})$.
Notice in particularly that:

$$\langle W | \psi \rangle = \int dU_{ab} W(U_{ab})\, \psi(U_{ab}) = \int dU_a \psi\left(U_a U_b^{-1}\right) , \tag{4.47}$$

from which we read that W projects on the flat connections, averaged over the gauge orbits.
To achieve the important result we are aiming for we would like to see if everything is still consistent, that is, we know that the same amplitude in the spin representation is given by:

$$W(j_{ab}) = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} , \tag{4.48}$$

therefore, we expect to obtain the same result by considering

$$W(j_{ab}) = \int dU_{ab} \psi_{j_{ab}}(U_{ab})\, W(U_{ab}) . \tag{4.49}$$

Thus, by inserting the definitions we get

$$W\left(j_{ab}\right)=\int dU_{ab}\int dU_{a}\prod_{ab}\delta\left(U_{a}U_{ab}U_{b}^{-1}\right)\otimes_{a}\iota_{a}\cdot\prod_{ab}D^{j_{ab}}\left(U_{ab}\right)\,. \tag{4.50}$$

Performing the integral we obtain

$$W\left(j_{ab}\right)=\int dU_{a}\prod_{ab}\otimes_{a}\iota_{a}\cdot\prod_{ab}D^{j_{ab}}\left(U_{a}\right)D^{j_{ab}}\left(U_{b}^{-1}\right)\,. \tag{4.51}$$

It is possible to show that the overall result of this integral is:

$$W\left(j_{ab}\right)=\mathrm{Tr}\left[\otimes_{a}\iota_{a}\right]\,, \tag{4.52}$$

which coincides precisely with the $6j$-symbol (since it is the invariant contraction of four $3j$-symbols). This result tells us that the $6j$-symbol can be thought as the Fourier transform of the gauge-invariant delta functions on flat connections, in the Hilbert space associated with the tetrahedral graph. This can be written in the notation

$$W\left(j_{ab}\right)=\psi_{j_{ab}}\left(\mathbb{1}\right)\,, \tag{4.53}$$

or, by using the projector $P_{SU(2)}$ on the $SU(2)$ invariant part of a function, the vertex amplitude can be written as

$$\langle\psi_{\mathbf{v}}|W_{\mathbf{v}}\rangle=\left(P_{SU(2)}\psi_{\mathbf{v}}\right)\left(\mathbb{1}\right)\,, \tag{4.54}$$

where ψ_{v} is a state in the boundary of a vertex.

We can summarize the properties of the amplitude by pointing out the following features:

1. **Superposition principle**: this is the basic principle of quantum mechanics, the amplitude is given by the sum of elemetary amplitudes, that is, by a Feynman's sum over the possible paths σ:

$$\langle W|\psi\rangle=\sum_{\sigma}W(\sigma)\,. \tag{4.55}$$

2. **Locality**: the elementary amplitudes can be seen as products of amplitudes associated with spacetime points (in QFT the product is expressed as the exponential of an integral on spacetime):

$$W(\sigma)\sim\prod_{\mathbf{v}}W_{\mathbf{v}}\,. \tag{4.56}$$

3. **Local euclidean invariance**: the $6j$-symbol can be written as the projection on the $SU(2)$-invariant part of the state on the boundary graph of the vertex, i.e.

$$W_{\mathbf{v}}=\left(P_{SU(2)}\psi_{\mathbf{v}}\right)\left(\mathbb{1}\right)\,. \tag{4.57}$$

These properties will be found also in the 4-dimensional theory.

4.2 4D Theory

Following the same line of reasoning of the previous section and recalling the results obtained in the four-dimensional discretization we are ready to face the four-dimenisonal quantization.

4.2.1 Hilbert Space

The boundary Hilbert space we are interested in is obtained in the same manner of the three-dimensional one: the variables $B_l \in \mathrm{sl}(2,\mathbb{C})$ and $U_l \in SL(2,\mathbb{C})$ become operators in the quantum theory, the states are given by $\psi(U_l)$, functions on $SL(2,\mathbb{C})^L$ and the operator $B_l \in \mathrm{sl}(2,\mathbb{C})$ is realized as the generator of $SL(2,\mathbb{C})$ transformations. We recall that B on the boundary is split into its electric and magnetic parts, and these are constrained by $\overrightarrow{K} = \gamma \overrightarrow{L}$, therefore we expect this condition continues to hold, at least in the classical limit. Keeping this constraint in the quantum case has crucial consequences, it completley determines the dynamics of LQG.
Furthermore, we recall from eq. (3.21) that $|L_{\mathbf{f}}| = \frac{1}{\gamma} A_{\mathbf{t_f}}$ and this, together with eq (4.6), which now reads

$$L_{j_l} = 8\pi\hbar G\gamma\sqrt{j_l(j_l+1)}\,, \tag{4.58}$$

suggests that the scale of LQG is given by $L^2_{\text{loop}} = 8\pi\hbar G\gamma$. Then it can be stated, that, since the value of the Barbero-Immirzi constant γ is of order unity ($\gamma \sim 0.274067$ is the value fixed by the Bekenstein-Hawking entropy) the scale of LQG is of the same order of the Planck scale ($L^2_{\text{Planck}} = \hbar G$).
To begin with, we are interested in irreducible unitary representations of $SL(2,\mathbb{C})$, these are labeled by a positive real number p and a non-negative half-integer k. The Hilbert space $V^{(p,k)}$ of the (p,k) representation decomposes into irreducibles representations of $SU(2) \subset SL(2,\mathbb{C})$ as follows:

$$V^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j\,, \tag{4.59}$$

where $\mathcal{H}_j$ is the $2j+1$-dimensional space that carries the spin j irreducible representation of $SU(2)$. Therefore, we can choose a basis of states $|p,k;j,m\rangle$, with $j = k, k+1, \ldots$ and $m = -j, \ldots, j$. The quantum numbers (p,k) are related to the two Casimir operators of $SL(2,\mathbb{C})$ by

$$\begin{aligned} |\overrightarrow{K}|^2 - |\overrightarrow{L}|^2 &= p^2 - k^2 + 1\,, \\ \overrightarrow{K}\cdot\overrightarrow{L} &= pk\,, \end{aligned} \tag{4.60}$$

where j and m are the quantum numbers of $|\overrightarrow{L}|^2$ and L_z respectively. Now, taking into account the linear simplicity constraint for large quantum numbers means that

$$\begin{aligned} |\overrightarrow{K}|^2 - |\overrightarrow{L}|^2 &= \left(\gamma^2 - 1\right)|\overrightarrow{L}|^2\,, \\ \overrightarrow{K}\cdot\overrightarrow{L} &= \gamma|\overrightarrow{L}|^2\,, \end{aligned} \tag{4.61}$$

and so, by means of (4.59) we get

$$\begin{aligned} p^2 - k^2 + 1 &= \left(\gamma^2 - 1\right) j(j+1)\,, \\ pk &= \gamma j(j+1)\,. \end{aligned} \tag{4.62}$$

In the large quantum numbers limit we then obtain

$$\begin{aligned} p^2 - k^2 + 1 &= \left(\gamma^2 - 1\right) j^2 \,, \\ pk &= \gamma j^2 \,, \end{aligned} \tag{4.63}$$

which is solved by

$$\begin{aligned} p &= \gamma k \,, \\ k &= j \,. \end{aligned} \tag{4.64}$$

The first of these two equations is a restriction on the set of unitary representations, whereas the second one picks out a subspace within each representation (the lowest one).
Thus, the states that satisfy these relations have the form

$$|p.k; j, m\rangle = |\gamma j, j; j, m\rangle \,. \tag{4.65}$$

Clearly these states are in one-to-one correspondence with the states in the representations of $SU(2)$. It is legit then to introduce a map Y_γ defined by

$$\begin{aligned} Y_\gamma \ : \quad \mathcal{H}_j \ &\longrightarrow \ V^{(p=\gamma j,\, k=j)} \\ |j; m\rangle &\longmapsto |\gamma j, j; j, m\rangle \,, \end{aligned} \tag{4.66}$$

and all the vectors in the image of this map satisfy the linear simplicity constraint, in the sense that

$$\langle Y_\gamma \psi | \overrightarrow{K} - \gamma \overrightarrow{L} | Y_\gamma \phi \rangle = 0 \,, \tag{4.67}$$

holds in the large j limit. For this reason, we assume that the states of the four-dimensional theory are constructed from the states $|\gamma j, j; j, m\rangle$ alone.
The map Y_γ can be extended to a map from functions over $SU(2)$ to functions over $SL(2, \mathbb{C})$, namely

$$\begin{aligned} Y_\gamma \ : \qquad\qquad L_2[SU(2)] \ &\longrightarrow \ F[SL(2, \mathbb{C})] \\ \psi(h) = \sum_{jmn} c_{jmn} D^{(j)}_{mn}(h) &\longmapsto \psi(g) = \sum_{jmn} c_{jmn} D^{(\gamma j, j)}_{mn}(g) \,, \end{aligned} \tag{4.68}$$

This is the way to map $SU(2)$ spin-networks into $SL(2, \mathbb{C})$ spin-networks.
The physical states of quantum gravity are thus, essentially, $SU(2)$ spin-networks. This fact is consistent with the classical theory expressed in terms of the Ashtekar variables, which form the same kinematical phase space of a $SU(2)$ Yang-Mills theory.
Following the same line of reasoning of the three-dimensional case, we would like to find the gauge-invariant states. In order to do this we decompose the Hilbert space as

$$L_2 \left[SU(2)^L\right] = \otimes_l \left[\oplus_j \left(\mathcal{H}_j \otimes \mathcal{H}_j\right)\right] = \oplus_{j_l} \otimes_l \left(\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}\right) \tag{4.69}$$

and so

$$L_2 \left[SU(2)^L / SU(2)^N\right] = \oplus_{j_l} \otimes_n \mathrm{Inv}_{SU(2)} \left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}\right) \tag{4.70}$$

where clearly we have an additional factor due to the fact that now each edge is bounded by four faces, not three. The space $\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}\right)$ is not one-dimensional in general, it turns out that linearly independent invariant tensors in this space can be constructed as follows:

$$\iota_k^{m_1 m_2 m_3 m_4} = \begin{pmatrix} j_1 & j_2 & k \\ m_1 & m_2 & m \end{pmatrix} g_{mn} \begin{pmatrix} k & j_3 & j_4 \\ n & m_3 & m_4 \end{pmatrix}, \tag{4.71}$$

for any k that satisfies the triangular relations both with j_1, j_2 and j_3, j_4, more precisely:

$$\max[|j_1 - j_2|, |j_3 - j_4|] \leq k \leq \min[j_1 + j_2, j_3 + j_4]\,. \tag{4.72}$$

We denote these states with $|k\rangle$ and the invariant subspace as

$$\mathcal{K}_{j_1 \ldots j_4} := \mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}\right)\,, \tag{4.73}$$

whose dimension is therefore

$$\dim[\mathcal{K}_{j_1 \ldots j_4}] = \min[j_1 + j_2, j_3 + j_4] - \max[|j_1 - j_2|, |j_3 - j_4|] + 1\,. \tag{4.74}$$

It follows that a generic gauge-invariant state is a linear combination

$$\psi\left(U_l\right) = \sum_{j_l k_{\mathbf{n}}} \mathcal{C}_{j_l k_{\mathbf{n}}} \psi_{j_l k_{\mathbf{n}}}\left(U_l\right)\,, \tag{4.75}$$

of the orthogonal states

$$\psi_{j_l k_{\mathbf{n}}}\left(U_l\right) = \iota_{k_1}^{m_1 m_2 m_3 m_4} \cdots \iota_{k_N}^{m_{L-3} m_{L-2} m_{L-1} m_L} D^{j_1}_{m_1 n_1} \cdots D^{j_L}_{m_L n_L}\,. \tag{4.76}$$

The difference from the three-dimensional case is that the spin-networks of the four-dimensional case are labeled not only by spins, but also by an intertwine quantum number k associated to each node n. Using a more compact notation we denote the spin-network wave functions as

$$\psi_{j_l k_{\mathbf{n}}}\left(U_l\right) = \langle U_l | j_l, k_{\mathbf{n}}\rangle = \bigotimes_{\mathbf{n}} \iota_{k_{\mathbf{n}}} \cdot \bigotimes_{l} D^{j_l}\left(U_l\right)\,. \tag{4.77}$$

At a classical level, this residual geometric freedom at each node is described by the space of possible shapes of a tetrahedron with fixed areas, which is a two-dimensional space (coordinatized for instance by two opposite dihedral angles). This space can also be seen as the space of quadruplets of vectors satisfying the closure relation, with given areas, up to global rotations, the counting of the dimension gives: $4 \times 3 - 4 - 3 - 3 = 2$.
An observable on this space is given by the volume V of the tetrahedron, which is given by

$$V^2 = \frac{2}{9}\epsilon_{ijk} E^i E^j E^k\,. \tag{4.78}$$

where the operator $\overrightarrow{E}$ is associated with each link and it is given by

$$\overrightarrow{E}_l = 8\pi\gamma\hbar G \overrightarrow{L}_l\,. \tag{4.79}$$

The matrix elements of V can be computed in the $|k\rangle$ basis, then by diagonalization of this matrix it is possible to obtain the eigenvalues v and their correspondent eigentates $|v\rangle$ of the volume in each Hilbert space $\mathcal{K}_{j_1 \dots j_4}$. In this basis, the spin-network states can be written as

$$\psi_{j_l v_\mathbf{n}}(U_l) = \langle U_l | j_l, v_\mathbf{n} \rangle = \bigotimes_\mathbf{n} \iota_{v_\mathbf{n}} \cdot \bigotimes_l D^{j_l}(U_l) \ . \tag{4.80}$$

To summarize, the Hilbert space associated with the boundary graph Γ is given by

$$\mathcal{H}_\Gamma = L_2[SU(2)^L / SU(2)^N] \tag{4.81}$$

and spin-network states are denoted by $|\Gamma, j_l, v_\mathbf{n}\rangle$, where j_l is a spin associated with each link of the graph and $v_\mathbf{n}$ is a volume eigenvalue associated with each node of the graph.
This formalism is referred to as "spinfoam", where "foam" refers to a 2-complex and "spin" is obviuosly associated to the spin representation sitting on each edge.

4.2.2 Transition Amplitude

To complete the description of the full theory we need to write down the transition amplitude. First of all, we give an alternative form of the amplitude which will be more suitable for the four-dimensional case. We start from

$$Z = \int dU_\mathbf{e} \prod_\mathbf{f} \delta\left(U_{\mathbf{e}_1} \cdots U_{\mathbf{e}_n}\right) \ . \tag{4.82}$$

At this point, we introduce two group variables per each edge $\mathbf{e}$, that is, $U_\mathbf{e} = g_\mathbf{ve} g_{\mathbf{ev}'}$, where $g_\mathbf{ev} = g_\mathbf{ve}^{-1}$ is a variable associated with each couple vertex-edge. Thus, we can write

$$Z = \int dg_\mathbf{ve} \prod_\mathbf{f} \delta\left(g_\mathbf{ve} g_{\mathbf{ev}'} g_{\mathbf{v}'\mathbf{e}'} g_{\mathbf{e}'\mathbf{v}''} \cdots\right) \ . \tag{4.83}$$

Then we regroup the $g_\mathbf{ev}$ variables in a different way, namely we define $h_\mathbf{vf} = g_\mathbf{ev} g_{\mathbf{ve}'}$, where $\mathbf{e}$ and $\mathbf{e}'$ are the two edges coming out from the vertex $\mathbf{v}$ and bounding the face $\mathbf{f}$.

Clearly, the amplitude takes the form

$$Z = \int dh_\mathbf{vf} dg_\mathbf{ve} \prod_\mathbf{f} \delta\left(g_\mathbf{ve} g_{\mathbf{ev}'} g_{\mathbf{v}'\mathbf{e}'} g_{\mathbf{e}'\mathbf{v}''} \cdots\right) \prod_\mathbf{vf} \delta\left(g_{\mathbf{e}'\mathbf{v}} g_\mathbf{ve} h_\mathbf{vf}\right) \ . \tag{4.84}$$

This can be reorganised as a transition amplitude where a delta function glues the group element around each face:

$$Z = \int dh_\mathbf{vf} \prod_\mathbf{f} \delta\left(h_\mathbf{f}\right) \prod_\mathbf{v} A_\mathbf{v}\left(h_\mathbf{vf}\right) \ , \tag{4.85}$$

where $h_\mathbf{f} := \prod_{\mathbf{v} \in \partial \mathbf{f}} h_\mathbf{vf}$ is a group variable associated with a face.

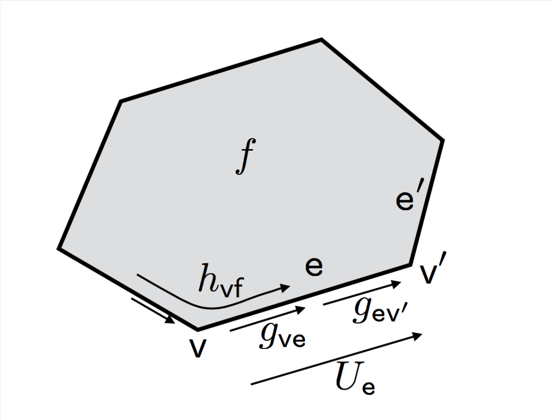

Figure 4.1: Splitting of the group elements

Furthermore, the vertex amplitude is defined by

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) := \int dg_{\mathbf{ve}} \prod_{\mathbf{f}} \delta\left(g_{\mathbf{e'v}} g_{\mathbf{ve}} h_{\mathbf{vf}}\right) . \tag{4.86}$$

The $SU(2)$ integrals in a vertex are $n = 4$, that is, one group element for each of the $n = 4$ edges coming out of the vertex. However, if one thinks about it, there is one redundant integral, because after integrating $n-1$ group variables the result is not affected by the last integration. We denote this fact by

$$\int_{SU(2)^n} dg'_{\mathbf{ve}} := \int_{SU(2)^{(n-1)}} dg_{\mathbf{ve_1}} \cdots dg_{\mathbf{ve_{n-1}}} . \tag{4.87}$$

In three dimensions this observation does not change anything: performing the last integral gives unity, since the volume of $SU(2)$ is just one, but in the four-dimensional case this turns out to be crucial, because $SL(2,\mathbb{C})$ is non-compact. If we expand the delta function in representations we get

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) = \sum_{j_{\mathbf{f}}} \int dg'_{\mathbf{ve}} \prod_{\mathbf{f}} \left(2j_{\mathbf{f}}+1\right) \mathrm{Tr}_{j_{\mathbf{f}}} \left[g_{\mathbf{e'v}} g_{\mathbf{ve}} h_{\mathbf{vf}}\right] , \tag{4.88}$$

where $\mathrm{Tr}_j(U) := \mathrm{Tr}\left[D^j(U)\right]$. Therefore the vertex amplitude is a function of one $SU(2)$ variable per face around the vertex. We can also picture this by drawing a sphere around the vertex, the intersection between this sphere and the 2-complex is a graph, $\Gamma_{\mathbf{v}}$. The vertex amplitude is then a function of the states in

$$\mathcal{H}_{\Gamma_{\mathbf{v}}} = L_2\left[SU(2)^6/SU(2)^4\right]_{\Gamma_{\mathbf{v}}} , \tag{4.89}$$

where $\Gamma_{\mathbf{v}}$ is the complete graph with four nodes and represents the boundary graph of the vertex. Therefore, we can express the transition amplitude in the following way:

$$W\left(h_l\right) = \int dh_{\mathbf{vf}} \prod_{\mathbf{f}} \delta\left(h_{\mathbf{f}}\right) \prod_{\mathbf{v}} A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) \,, \tag{4.90}$$

where the vertex amplitude is given by

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) = \mathcal{N} \sum_{j_{\mathbf{f}}} \int dg'_{\mathbf{ve}} \prod_{\mathbf{f}} \left(2j_{\mathbf{f}}+1\right) \mathrm{Tr}_{j_{\mathbf{f}}}\left[g_{\mathbf{e'v}} g_{\mathbf{ve}} h_{\mathbf{vf}}\right], . \tag{4.91}$$

We are ready now to treat the four-dimensional case.
We notice that the form of the transition amplitude is the same as in (4.84), since this only reflects the superposition principle, therefore the dynamics is contained in the vertex amplitude. The vertex amplitude in turn must be $SL(2,\mathbb{C})$-invariant in the four-dimensional case, but $A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right)$ can be regarded only as a function of $SU(2)$ group elements living on the graph of a node (which is on the boundary of a 4-simplex). To obtain the analogue of (4.87) in the four-dimensional case then we have to replace the $SU(2)$ integrals with $SL(2,\mathbb{C})$ ones and to map the $SU(2)$ group elements into the $SL(2,\mathbb{C})$ ones. In order to do that we make use of the Y_γ map, as follows:

$$A_{\mathbf{v}}(\psi) = \left(P_{SL(2,\mathbb{C})} Y_\gamma \psi\right)(\mathbb{1}) \tag{4.92}$$

which, more expilicitly, it reads

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) = \mathcal{N} \sum_{j_{\mathbf{f}}} \int dg'_{\mathbf{ve}} \prod_{\mathbf{f}} \left(2j_{\mathbf{f}}+1\right) \mathrm{Tr}_{j_{\mathbf{f}}}\left[Y_\gamma^\dagger g_{\mathbf{e'v}} g_{\mathbf{ve}} Y_\gamma h_{\mathbf{vf}}\right] \,, \tag{4.93}$$

where the trace is given by

$$\mathrm{Tr}_j\left[Y_\gamma^\dagger g Y_\gamma h\right] = \mathrm{Tr}_j\left[Y_\gamma^\dagger D^{(\gamma j,j)}(g) Y_\gamma D^{(j)}(h)\right] = \sum_{mn} D^{(\gamma j,j)}_{jm,jn}(g) D^{(j)}_{nm}(h)\,. \tag{4.94}$$

The vertex amplitude is then a function of one $SU(2)$ variable per face around the vertex. As seen before, we can picture a small sphere around a vertex, obtaining a graph $\Gamma_{\mathbf{v}}$, the vertex amplitude becomes thus a function of the states in

$$\mathcal{H}_{\Gamma_{\mathbf{v}}} = L_2\left[SU(2)^{10}/SU(2)^5\right]_{\Gamma_{\mathbf{v}}} \,. \tag{4.95}$$

The graph $\Gamma_{\mathbf{v}}$ is the complete graph with five nodes.

4.2.3 Continuum Limit

We have seen above the equations that describe the theory on a given graph Γ, obtained from a given 2-complex $\mathcal{C}$. This is a theory with a finite number of degrees of freedom, beacuse it corresponds to a truncation of classical general relativity, which is a theory with an infinte number of degrees of freedom. The full theory is approximated by choosing increasingly refined complexes $\mathcal{C}$ and $\Gamma = \partial\mathcal{C}$, where the refinement is chosen in relation to the desired precision, in analogy with a finite order in

perturbation theory in QED. More precisely: let Γ' be a subgraph of Γ, namely, a graph formed by a subset of nodes and links of Γ, then, there is a subspace $\mathcal{H}_{\Gamma'} \subset \mathcal{H}_{\Gamma}$ which is isomorphic to the loop-gravity Hilbert space of the graph Γ'. Indeed, this is formed by all the states $\psi(U_l) \in \mathcal{H}_{\Gamma}$ which are independent of the group elements U_l associated with the links l that are in Γ but not in Γ'. Equivalently, $\mathcal{H}_{\Gamma'}$ is the linear span of the spin-network states characterized by $j_l = 0$ for any l that is in Γ but not in Γ'.
Therefore, if we define the theory on Γ we have at our disposal a subset of states that captures the theory defined on the smaller graph Γ', in this way, the step from Γ' to Γ is a refinement of the theory. More precisely, the continuum limit can be defined by

$$Z(h_l) = \lim_{\mathcal{C}\to\infty} Z_{\mathcal{C}}(h_l)\,, \tag{4.96}$$

which is well defined in the sense of nets, because two-complexes form a partially ordered set with upper bound. Nevertheless, there is not a unique notion of limit at the present time, and it is often said that the approximation is good when the discretized theory approximates the continuum theory in the classical context, that is, when the degree of accuracy of the triangulation meets the desired expectations.
When dealing with the continuum limit it is natural then to ask what happens to the tranistion amplitude when refining the triangulation. The simplest case to analyze is considering a single tetrahedron τ and adding a point P inside it, then joining P to the four vertices of τ. In this way the original tetrahedron has been split into four smaller tetrahedra. If we call Δ_1 the original triangulation and Δ_4 the new one, it's clear that, when dealing with the respective 2-complexes, the refinement produces a "bubble", as shown in figure.

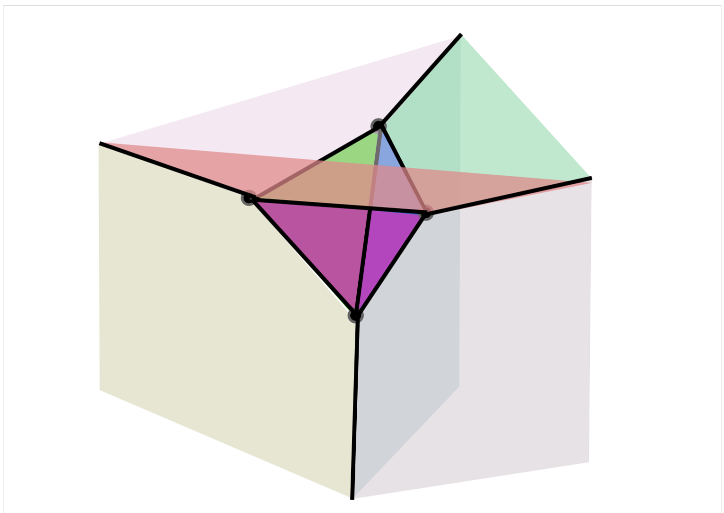

Figure 4.2: The graph Δ_4^*

Starting from (4.45) it is possible to compute the amplitude of this triangulation W_{Δ_4}. It can be shown [16] that the relation between W_{Δ_4} and the original W_{Δ_1} amounts to an infinite factor multipling the latter.
The appearance of the divergence is a manifestation of the standard quantum field theory divergences. It is strictly connected to the existence of the bubble. To see that this is the case, reconsider the same calculation in the spin representation. From eq. (4.41):

$$W_{\Delta_4}(j_{ab}) = \sum_{j_{ab}} \prod_{ab} d_{j_{ab}} \prod_{a} \{6j\} \,. \tag{4.97}$$

In general, in a sum like this the range of summation of the j_{ab} is restricted by the triangular identities. Since the boundary faces have finite spins, the only possibility for an internal face to have a large spin is to be adjacent, at each edge, to at least one other face with a large spin. In other words, a set of faces with arbitrary large spins cannot have boundaries. Therefore to have a sum which is not up to a maximum spin by the triangular identities the only possibility is to have a set of faces that form a surface without boundaries in the two complex. That is, a bubble. All this is very similar to the ultraviolet divergences in the Feynman expansion of a normal quantum field theory, where divergences are associated to loops, because the momentum is conserved at the vertices. Here, divergences are associated to bubbles, because angular momentum is conserved on the edges. A Feynman loop is a closed set of lines where arbitrary high momentum can circulate. A spinfoam divergence is a closed set of faces, that can have arbitrarily high spin. Notice however that in spite of the formal similarity there is an important difference in the physical interpretation of the two kinds of divergences. The Feynman divergences regards what happens at very small scale. On the contrary, the spinfoam divergences concern large spins, namely large geometries. Therefore they are not ultraviolet divergences, they are infrared. A way to get rid of these divergences is by considering the so-called "Turaev-Viro" amplitude, in which, instead of considering the group $SU(2)$, one chooses the group $SU(2)_q$ (q being a parameter), i.e. a one-parameter deformation of the algebra of the representations of $SU(2)$. The Turaev-Viro amplitude is given by:

$$W_q(j_l) = w_q^p \sum_{j_{\mathbf{f}}} \prod_{j_{\mathbf{f}}} (-1)^{j_{\mathbf{f}}} d_q(j_{\mathbf{f}}) \prod_{\mathbf{v}} (-1)^{J_{\mathbf{v}}} \{6j\}_q \,. \tag{4.98}$$

The remarkable fact, is that the dimension d_j^q has a maximum value [17], this finiteness makes the amplitude finite.
Furthermore, the parameter q can be put in relation with the cosmological constant $q = e^{i\sqrt{\Lambda}\hbar G}$ as shown in [18], thus relating the finiteness of the amplitude to the presence of the cosmological constant.

4.3 Classical Limit

The classical limit in covariant LQG is studied on the basis of the so-called coherent states: these are similar to wave packets in quantum mechanics, i.e. states in which both position and momentum are minimally spread. Geometrically, a tetrahedron is uniquely determined by giving six numbers, that is, the lengths of its sides, but we have seen before that a state associated to a node (and therefore to a tetrahedron) is characterized only by five numbers: four areas and the volume. In a sense, the geometry of the tetrahedron is fuzzy, in the same way angular momentum is, in quantum mechanics. Let's consider then a node n, we have a Hilbert space $\mathcal{H}_{\mathbf{n}}$, a basis of states is given by $|\iota_k\rangle$ defined in (4.70). It can be shown that these states are eigenstates of $\overrightarrow{L}_1 \cdot \overrightarrow{L}_2$, that is, they diagonalize the dihedral angle θ_{12} between the faces 1 and 2. We would like to find, given a classical tetrahedron, a quantum state whose dihedral angles are minimally spread around the classical variables. These states are called "intrinsic coherent states".

4.3.1 Intrinsic Coherent States

We recall that a tetrahedron is characterized by four vectors $\overrightarrow{E}_a$ (one per each face) whose length is the area of the correspondent face; in the quantum theory these are quantized and they are given by

$$\overrightarrow{E}_a = 8\pi G\hbar\gamma\overrightarrow{L}_a\,. \tag{4.99}$$

From the following commutation relations

$$\left[L_a^i, L_b^j\right] = i\delta_{ab}\epsilon^{ij}_k L_a^k \tag{4.100}$$

it is possible to show that the commutation relations between two dihedral angles are given by

$$\left[\overrightarrow{E}_1\cdot\overrightarrow{E}_2, \overrightarrow{E}_1\cdot\overrightarrow{E}_3\right] = i8\pi G\hbar\gamma\overrightarrow{E}_1\cdot(\overrightarrow{E}_2\times\overrightarrow{E}_3)\,. \tag{4.101}$$

Then, from this, it follows the Heisenberg relation

$$\Delta(\overrightarrow{E}_1\cdot\overrightarrow{E}_2)\cdot\Delta(\overrightarrow{E}_1\cdot\overrightarrow{E}_2) \geq \frac{1}{2}8\pi G\hbar\gamma|\langle\overrightarrow{E}_1\cdot(\overrightarrow{E}_2\times\overrightarrow{E}_3)\rangle|\,, \tag{4.102}$$

where $\langle A\rangle = \langle\iota|A|\iota\rangle$ and $\Delta A = \sqrt{\langle\iota|A^2|\iota\rangle - (\langle\iota|A|\iota\rangle)^2}$. Thus, we are aiming for states whose dispersion is small compared with their expectation value, that is

$$\frac{\Delta(\overrightarrow{E}_a\cdot\overrightarrow{E}_b)}{|\overrightarrow{E}_a||\overrightarrow{E}_b|} \ll 1 \qquad \forall a,b\,. \tag{4.103}$$

The first step is to consider $SU(2)$ coherent states. We start from a state of fixed total angular momentum j, $|j,m\rangle \in \mathcal{H}_j$ is then a basis of these states. Then, because $[L_x, L_y] = iL_z$, we have the Heisenber relation

$$\Delta L_x \Delta L_y \geq \frac{1}{2}|\langle L_z\rangle|\,, \tag{4.104}$$

which is satisfied by every state. A state that saturates this inequality can be shown to be given by $|j,j\rangle$.

Furthermore, there is an entire family of coherent states which can be obtained starting from the state $|j,j\rangle$, namely, by rotating the state by means of a matrix $R\in SO(3)$:

$$|j,\overrightarrow{n}\rangle = D_{\overrightarrow{n}}(R)|j,j\rangle\,, \tag{4.105}$$

where $\overrightarrow{n}$ is the direction obtain by starting from the z-axis and then applying the rotation. These coherent states can be expanded in terms of eigenstates of L_z as follows

$$|j, \vec{n}\rangle = \sum_m \phi_m(\vec{n})|j, m\rangle\,, \tag{4.106}$$

where $\phi_m(\vec{n}) = \langle j, m|D(R)|j, j\rangle = D^{(j)}(R)^j_m$.
One of the most important properties of the coherent states is that they provide a resolution of the identity, that is,

$$\mathbb{1}_j = \frac{2j+1}{4\pi}\int_{S^2} d^2\vec{n}|j, \vec{n}\rangle\langle j, \vec{n}|\,. \tag{4.107}$$

By means of these coherent states it is possible to describe a "coherent" tetrahedron, whose faces are described by coherent states. More precisely, let's consider the coherent state

$$|j_1, \vec{n}_1\rangle \otimes |j_2, \vec{n}_2\rangle \otimes |j_3, \vec{n}_3\rangle \otimes |j_4, \vec{n}_4\rangle \in \mathcal{H}_1 \otimes \cdots \mathcal{H}_4\,, \tag{4.108}$$

which is still a coherent state, since tensor products of coherent stantes are coherent, and project it down to its invariant part by means of

$$P\ :\ \mathcal{H}_1 \otimes \cdots \mathcal{H}_4 \to \mathrm{Inv}(\mathcal{H}_1 \otimes \cdots \mathcal{H}_4)\,. \tag{4.109}$$

Thus, we denote this coherent state as

$$||j_a, \vec{n}_a\rangle := P(|j_1, \vec{n}_1\rangle \otimes |j_2, \vec{n}_2\rangle \otimes |j_3, \vec{n}_3\rangle \otimes |j_4, \vec{n}_4\rangle)\,, \tag{4.110}$$

which is then the element of $\mathcal{H}_\Gamma$ that describes a semicalssical tetrahedron. More precisely, the projection can be explicitly implemented by the following

$$||j_a, \vec{n}_a\rangle = \int_{SO(3)} dR(|j_1, R\vec{n}_1\rangle \otimes |j_2, R\vec{n}_2\rangle \otimes |j_3, R\vec{n}_3\rangle \otimes |j_4, R\vec{n}_4\rangle)\,, \tag{4.111}$$

which can be translated in a $SU(2)$ integral as

$$||j_a, \vec{n}_a\rangle = \int_{SU(2)} dh(D^{j_1}(h)|j_1, \vec{n}_1\rangle \otimes D^{j_2}(h)|j_2, \vec{n}_2\rangle \otimes D^{j_3}(h)|j_3, \vec{n}_3\rangle \otimes D^{j_4}(h)|j_4, \vec{n}_4\rangle\,. \tag{4.112}$$

These states are also referred to as the "Livine-Speziale coherent intertwiners", since they are associated to a tetrahedron which is in turn associated to a node. It can be shown that these states can be expanded in any intertwiner basis:

$$||j_a, \vec{n}_a\rangle = \sum_k \Phi_k(\vec{n}_a)|\iota_k\rangle\,, \tag{4.113}$$

where the coefficients $\Phi_k(\vec{n}_a) = \iota^{m_1 m_2 m_3 m_4}\psi_{m_1}(\vec{n}_1)\cdots\psi_{m_4}(\vec{n}_4)$, for large j, have the form $\Phi_k(\vec{n}_a) \sim e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}}e^{ik\psi}$, i.e. they are concentrated around a single value k_0 which determines the value of the corresponding dihedral angle, and have a phase such that, when changing basis to a different intertwined basis, we still obtain a state concentrated around the same value.

For large j, these states satisfy the following properties

$$\langle \iota(n_l)|E_a \cdot E_b|\iota(n_l)\rangle \sim j_a j_b \vec{n}_a \cdot \vec{n}_b \tag{4.114}$$

and

$$\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|\vec{E}_a||\vec{E}_b|} \ll 1\,, \tag{4.115}$$

the last one proves that these are in fact coherent states.
Putting everything together, that is, combining coherent intertwiners at each node, we can define a coherent state in $\mathcal{H}_\Gamma$, which can be thought as a "wave packet" peaked on a classical triangulated geometry:

$$\psi_{j_l,\vec{n}_{s_l},\vec{n}_{t_l}}(U_l) = \otimes_l D^{(j_l)}(U_l) \cdot \otimes_{\mathbf{n}} \iota_{\mathbf{n}}(\vec{n}_l)\,. \tag{4.116}$$

4.3.2 Spinors

Coherent states provide a tool to perform the classical limit, but to reach that goal we need to exploit their relation with spinors. Spinors are the elements of the fundamental representation of $SU(2)$, namely, $\mathcal{H}_{\frac{1}{2}} = \mathbb{C}^2$, that coincides with the fundamental representation of $SL(2,\mathbb{C})$. We denote a spinor $\mathbf{z} \in \mathbb{C}^2$ by

$$\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} = z^A = |z\rangle\,. \tag{4.117}$$

The spinor $\mathbf{n} = (1,0)$ is the eigenvector of L_z with eigenvalue $\frac{1}{2}$ and unit norm, then, we can identify it with the state $|j = \frac{1}{2}, m = \frac{1}{2}\rangle$, which is a coherent state. Since all the coherent states in the $j = \frac{1}{2}$ representation are obtained by rotating $|\frac{1}{2}, \frac{1}{2}\rangle$ and since rotation preserves the norm of a spinor, it follows that all normalized spinors $\mathbf{n}$ describe coherent states in the fundamental representation, that is,

$$|\mathbf{n}\rangle = |\frac{1}{2}, \vec{n}\rangle\,. \tag{4.118}$$

Now, with each spinor $\mathbf{n} \in \mathbb{C}^2$ we can associate a three-dimensional real vector by

$$\vec{n} = \langle \mathbf{n}|\vec{\sigma}|\mathbf{n}\rangle\,, \tag{4.119}$$

therefore we have that

$$\langle \mathbf{n}|\vec{L}|\mathbf{n}\rangle = \langle \mathbf{n}|\frac{\vec{\sigma}}{2}|\mathbf{n}\rangle = \frac{1}{2}\vec{n} = j\vec{n}\,. \tag{4.120}$$

Normalized spinors are coherent states for the normalized three-vector they define. This result can be extended to any representation, because the tensor product of coherent states is a coherent state.

Thus, it makes sense to consider the followng state

$$|j,\mathbf{n}\rangle = \underbrace{\mathbf{n}\otimes\cdots\otimes\mathbf{n}}_{2j}\,, \tag{4.121}$$

which coincides with the spin-j representation, and is precisely the coherent state $|j,\overrightarrow{n}\rangle$ that satisfies

$$\langle j,\mathbf{n}|\overrightarrow{L}|j,\mathbf{n}\rangle = j\overrightarrow{n}\,. \tag{4.122}$$

Armed with spinors we can look for a different realization of the spin-j representation, namely, the finite-dimensional vector space $\mathcal{H}_j$ can be realized as the space of the totally symmetric polynomial functions $f(\mathbf{z})$ of degree $2j$. In order to see this, we recall that the spin-j representation space $\mathcal{H}_j$ can be realized by symmetric tensors $y^{A_1A_2\ldots A_{2j}}$ with $2j$ indices. Therefore, the corresponding polynomial function of $\mathbf{z}$ is simply

$$f(\mathbf{z}) = y^{A_1A_2\ldots A_{2j}}z_{A_1}\cdots z_{a_{2j}}\,. \tag{4.123}$$

This function satisfies the homogeneity condition

$$f(\lambda\mathbf{z}) = \lambda^{2j}f(\mathbf{z}) \tag{4.124}$$

and the $SU(2)$ action on these functions is given by

$$(Uf)(\mathbf{z}) = f(U^T\mathbf{z})\,. \tag{4.125}$$

We would like to see how coherent states look in this representation. For spin 1/2, a coherent state is represented by the linear function

$$f_{\mathbf{n}}(\mathbf{z}) \sim n^Az_A \sim \langle\mathbf{z}|\mathbf{n}\rangle\,, \tag{4.126}$$

up to normalization. If we take the symmetrized tensor product of this state with itself $2j$-times, we obtain the coherent state in the j representation in the following form (including normalization)

$$f^{(j)}_{\mathbf{n}}(\mathbf{z}) = \sqrt{\frac{2j+1}{\pi}}\langle\mathbf{z}|\mathbf{n}\rangle^{2j}\,. \tag{4.127}$$

This relaization of $SU(2)$ representation spaces turns out to be very useful to relate $SU(2)$ representations with $SL(2,\mathbb{C})$ unitary representations. We have seen previously that these representations were given by $V^{(p,k)}$, but again we would like to write these spaces in terms of functions of spinors $f(\mathbf{z})$, with $\mathbf{z}\in\mathbb{C}^2$. The representation (p,k) is defined on the space of the homogeneous functions of spinors that have the property

$$f(\lambda\mathbf{z}) = \lambda^{(-1+ip+k)}\overline{\lambda}^{(-1+ip-k)}f(\mathbf{z}) \tag{4.128}$$

and the $SL(2,\mathbb{C})$ action reads

$$gf(\mathbf{z}) = f(g^T\mathbf{z})\,. \tag{4.129}$$

The transition between the canonical basis and the spinor basis can be shown to be given by

$$f_m^j(\mathbf{z}) = \langle \mathbf{z}|p,k;j,m\rangle = \sqrt{\frac{2j+1}{\pi}}\langle \mathbf{z}|\mathbf{z}\rangle^{ip-1-j} D^j_{mk}(g(\mathbf{z})) \tag{4.130}$$

where

$$g(\mathbf{z}) = \left(\begin{array}{cc} z_0 & \overline{z}_1 \\ z_1 & \overline{z}_0 \end{array} \right) . \tag{4.131}$$

In these representations the scalar product between two functions is given by an integral in spinor space, that is, if f and g are functions of spinors, we have:

$$\langle f|g\rangle = \int \overline{f} g d\Omega \,, \tag{4.132}$$

where

$$d\Omega = \frac{i}{2}(z^0 dz^1 - z^1 dz^0) \wedge (\overline{z}^0 d\overline{z}^1 - \overline{z}^1 - d\overline{z}^0) \,. \tag{4.133}$$

These spinor representations are particurarly convenient because the Y_γ map takes a particurarly simple form in this language. Since the embedding of $\mathcal{H}_j$ in $V^{(p,k)}$ is given by

$$f(\mathbf{z}) \mapsto \langle \mathbf{z}|\mathbf{z}\rangle^{-1+ip-k} f(\mathbf{z}) \,, \tag{4.134}$$

we have then

$$Y_\gamma f(\mathbf{z}) = \langle \mathbf{z}|\mathbf{z}\rangle^{-1+(i\gamma-1)j} f(\mathbf{z}) \,. \tag{4.135}$$

This allows us to write the action of the Y_γ map on the coherent states:

$$\langle \mathbf{z}|Y_\gamma|j,\overrightarrow{n}\rangle = \sqrt{\frac{2j+1}{\pi}}\langle \mathbf{z}|\mathbf{z}\rangle^{-1+(i\gamma-1)j}\langle \mathbf{z}|\mathbf{n}\rangle^{2j} \,, \tag{4.136}$$

which we write also as

$$\langle \mathbf{z}|Y_\gamma|j,\overrightarrow{n}\rangle = \frac{\sqrt{2j+1}}{\sqrt{\pi}\langle \mathbf{z}|\mathbf{z}\rangle} \, e^{j[(i\gamma-1)\ln\langle \mathbf{z}|\mathbf{z}\rangle + 2\ln\langle \mathbf{z}|\mathbf{n}\rangle]} \,. \tag{4.137}$$

At this point, we would like to rewrite the amplitude in terms of spinors. In order to do so, we first recall that

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) = \sum_{j_{\mathbf{f}}} \int_{SL(2,\mathbb{C}} dg'_{\mathbf{ve}} \prod_{\mathbf{f}} \left(2j_{\mathbf{f}}+1\right) \mathrm{Tr}_{j_{\mathbf{f}}} \left[Y_\gamma^\dagger g_{\mathbf{e'v}} g_{\mathbf{ve}} Y_\gamma h_{\mathbf{vf}}\right] \,, \tag{4.138}$$

which can be written, dropping the subscript $\mathbf{v}$ and labeling the edges emerging from the vertex with

$a, b = 1, \ldots, 5$ and the faces adjacent to the vertices as ab,

$$A_{\mathbf{v}}\left(h_{ab}\right) = \sum_{j_{ab}} \int_{SL(2,\mathbb{C}} dg'_a \prod_{ab} \left(2j_{ab}+1\right) \mathrm{Tr}_{j_{ab}} \left[Y_\gamma^\dagger g_a^{-1} g_b Y_\gamma h_{ab}\right] . \tag{4.139}$$

The trace in the last equation can be written inserting two resolutions of the identity in terms of coherent states

$$\mathrm{Tr}_j \left[Y_\gamma^\dagger g g' Y_\gamma h\right] = \int_{S^2} d\vec{n}\, d\vec{m} \langle j, \vec{m}|Y_\gamma^\dagger g g' Y_\gamma|j, \vec{n}\rangle\langle j, \vec{n}|h|j, \vec{m}\rangle . \tag{4.140}$$

The first matrix element can be expressed in terms of spinors:

$$\langle j, \vec{m}|Y_\gamma^\dagger g g' Y_\gamma|j, \vec{n}\rangle = \int_{\mathbb{C}^2} d\Omega \langle Y_\gamma j, \vec{m}|g\mathbf{z}\rangle\langle g'^\dagger \mathbf{z}|Y_\gamma j, \vec{n}\rangle . \tag{4.141}$$

Using (4.133) and introducing the notation

$$\mathbf{Z} = g\mathbf{z}\,, \quad \mathbf{Z}' = g'^\dagger \mathbf{z}\,, \tag{4.142}$$

we obtain

$$\langle j, \vec{m}|Y_\gamma^\dagger g g' Y_\gamma|j, \vec{n}\rangle = \frac{2j+1}{\pi} \int_{\mathbb{C}^2} \frac{d\Omega}{\langle \mathbf{Z}|\mathbf{Z}\rangle\langle \mathbf{Z}'|\mathbf{Z}'\rangle} e^{j\, S(\mathbf{n},\mathbf{m},\mathbf{Z},\mathbf{Z}')} , \tag{4.143}$$

where

$$S(\mathbf{n}, \mathbf{m}, \mathbf{Z}, \mathbf{Z}') := \ln \frac{\langle \mathbf{Z}|\mathbf{m}\rangle^2 \langle \mathbf{Z}'|\mathbf{n}\rangle^2}{\langle \mathbf{Z}|\mathbf{Z}\rangle\langle \mathbf{Z}'|\mathbf{Z}'\rangle} + i\gamma \ln \frac{\langle \mathbf{Z}|\mathbf{Z}\rangle}{\langle \mathbf{Z}'|\mathbf{Z}'\rangle} . \tag{4.144}$$

We would like to insert this result in the expression of the amplitude. In order to do so, we choose a coherent state in $\mathcal{H}_{\Gamma_{\mathbf{v}}}$, that is, we pick a quadruplet of normalized vector $\vec{n}_{ab}$ for each node of $\Gamma_{\mathbf{v}}$, these define a state $|j_{ab}, \vec{n}_{ab}\rangle$. Therefore the amplitude takes the form:

$$A_{\mathbf{v}}\left(j_{ab}, \vec{n}_{ab}\right) \equiv \langle A_{\mathbf{v}}|j_{ab}, \vec{n}_{ab}\rangle = \int_{SL(2,\mathbb{C})} dg'_a \prod_{ab} \left(2j_{ab}+1\right) \langle j_{ab}, \mathbf{n}_{ab}|Y_\gamma^\dagger g_a^{-1} g_b Y_\gamma|j_{ba}, \mathbf{n}_{ba}\rangle . \tag{4.145}$$

Now, using the result in (4.134), we get

$$A_{\mathbf{v}}\left(j_{ab}, \vec{n}_{ab}\right) = \mu(j_{ab}) \int_{SL(2,\mathbb{C})} dg'_a \int_{\mathbb{C}^2} \frac{d\Omega_{ab}}{|\mathbf{Z}_{ab}||\mathbf{Z}_{ba}|} e^{\sum_{ab} j_{ab}\, S(\mathbf{n}_{ab},\mathbf{n}_{ba},\mathbf{Z}_{ab},\mathbf{Z}_{ba})} , \tag{4.146}$$

where $\mu(j_{ab}) = \prod_{ab} \frac{(2j_{ab}+1)^2}{\pi}$ and $\mathbf{Z}_{ab} = g_a \mathbf{z}_{ab}$ and $\mathbf{Z}_{ba} = g_b \mathbf{z}_{ab}$.
In order to perform the classical limit we have to take the limit of large quantum numbers, that is, when j_{ab} are large. In this limit, the integral in (4.143) can be computed using the saddle-point approximation, which, in d dimensions takes the form:

$$\int_{\mathbb{R}^d} dx^d g(x) e^{jf(x)} = \left(\frac{2\pi}{j}\right)^{\frac{d}{2}} (\det H_2 f)^{-\frac{1}{2}} g(x_0) e^{jf(x_0)} \left[1 + o\left(\frac{1}{j}\right)\right] , \tag{4.147}$$

where $H_2 f$ is the Hessian of f at the saddle point x_0, which is the point where the gradient of f

vanishes. Now, if f is real and negative, a large j gives a narrow gaussian around the maximum of f; if f is imaginary, when j is large, the exponential oscillates very rapidly and the integral is canceled out except for the points where the derivative of f vanishes.
For this reason, we start from the real part of the action, this is given by:

$$\mathrm{Re}[S] = \sum_{ab} \log \frac{|\langle \mathbf{Z}_{ab}|\mathbf{n}_{ab}\rangle|^2 \, |\langle \mathbf{Z}_{ba}|\mathbf{n}_{ba}\rangle|^2}{\langle \mathbf{Z}_{ab}|\mathbf{Z}_{ab}\rangle \langle \mathbf{Z}_{ba}|\mathbf{Z}_{ba}\rangle} \, . \tag{4.148}$$

The maximum is obtained when the logarithm vanishes, that is, when

$$\mathbf{n}_{ab} = e^{i\phi_{ab}} \frac{\mathbf{Z}_{ab}}{|\mathbf{Z}_{ab}|} \quad \mathbf{n}_{ba} = e^{i\phi_{ba}} \frac{\mathbf{Z}_{ba}}{|\mathbf{Z}_{ba}|} \, , \tag{4.149}$$

which, by definition of $\mathbf{Z}$, turns into

$$g_a^{-1} \mathbf{n}_{ab} = \frac{|\mathbf{Z}_{ba}|}{|\mathbf{Z}_{ab}|} e^{i\theta_{ab}} g_b^{-1} \mathbf{n}_{ba} \, . \tag{4.150}$$

At this point, we look at the extrema of the action under a variation of the spinor variables $\mathbf{z}_{ab}$. The explicit calculation gives

$$g_a \mathbf{n}_{ab} = \frac{|\mathbf{Z}_{ba}|}{|\mathbf{Z}_{ab}|} e^{i\theta_{ab}} g_b \mathbf{n}_{ba} \, . \tag{4.151}$$

Next, we consider a variation with respect to the group elements g_a and the spinor variables $\mathbf{z}_{ab}$. The former variation gives the action of the algebra elements, therefore the saddle-point equations for the group elements give the vanishing of the action of an infinitesimal $SL(2,\mathbb{C})$ transformation. This action can be decomposed into boosts and rotations, but in the relevant representations these are proportional, and so the needed invariance is only under rotations. In lue of (4.146), this can be moved from the variables $\mathbf{Z}$ (which contain the group elements) to the normals, thus obtaining [19]

$$\sum_b j_{ab} |\mathbf{n}_{ab}\rangle = 0 \, . \tag{4.152}$$

This equation shows exactly the closure conditions for the normal at each of the boundary nodes of the vertex graph. This is remarkable, because the initial set of normals is arbitrary; then, the dynamics suppreses all the possible sets of $\mathbf{n}_{ab}$ unless these satisfy the closure constraint at each node. Therefore, the normals define a proper tetrahedron τ_a at each node a of the vertex graph. We have then five tetrahedra in the vertex graph (which is the complete graph with five nodes), that is, one for each boundary node. These tetrahedra are three-dimensional objects, we can think of them as lying in a common three-dimensional surface Σ of Minkowski space, left invariant by the $SU(2)$ action. Now, a vector in Σ defines a surface in Σ to which it is normal, then, a Lorentz transformation can act on this surface and move it to an arbitrary (spacelike) surface. In terms of spinors, this action is given by the action an element of $SL(2,\mathbb{C})$ on the spinor associated to the surface. This reasoning allows to interpret (4.149) in the following way: there are five Lorentz transformations g_a that rotate the five tetrahedra τ_a in such a way that the b face of the tetrahedron τ_a is parallel to the a face of the tetrahedron τ_b. The value of the action at the saddle point can be shown to be given by [19]

$$S = i\gamma \sum_{ab} j_{ab} \Theta_{ab} \, , \tag{4.153}$$

where Θ_{ab} is the difference between the Lorentz transformations to the opposite sides of adjacent tetrahedra, that is, it is the dihedral angle between two tetrahedra. We recall that γj_{ab} is the area of the boundary faces of the 4-simplex, in units where $8\pi G\hbar = 1$, therefore, S on the critical point is the Regge action of the 4-simplex having the boundary geometry determined by the 10 areas j_{ab}.

4.3.3 Classical Limit versus Continuum Limit

The classical limit is obtained when considering a fixed triangulation and then taking the large-j limit of the transition amplitude, whereas the continuum limit is obtained by refining the 2-complex $\mathcal{C}$. The two procedures are obviously not equivalent, but the strategy to obtain the Hamilton function of General Relativity from the transition amplitude involves both. Indeed, one can perform the classical limit in the first place, thus obtaining the Regge Hamilton function, and then perform the continuum limit by considering more refined discretizations. The latter limit is known to converge to the General Relativity Hamilton function as mentioned earlier.
Now, the regimes where the classical limit is good in quantum gravity are those involving scales L that are much larger than the Planck scale:

$$L \gg L_{\text{Planck}} \,. \tag{4.154}$$

The regimes where the truncation is good are suggested by the Regge approximation, that is, the deficit angles have to be small. This happens when the scale of the discretization is small with respect to the curvature scale $L_{\text{curvature}}$:

$$L \ll L_{\text{curvature}} \,. \tag{4.155}$$

Therefore a triangulation with few cells, and, correspondingly, a two-complex with few vertices, provide an approximation in the regimes (determined by the boundary data) where the size of the cells considered is small with respect to the curvature scale (of the classical solution of the Einstein's equation determined by the given boundary data).
Refining the triangulation leads to including shorter length-scale degrees of freedom. But the physical scale of a spinfoam configuration is not given by the graph or the two complex. It is given by the size of its geometrical quantities, which is determined by the spins (and intertwiners). The same triangulation can represent both a small and a large size of spacetime. A large chunk of nearly flat spacetime can be well approximated by a coarse triangulation, while a small chunk of spacetime where the curvature is very high requires a finer triangulation. In other words, triangulations do not need to be uselessly fine, they need to be just as fine as to to capture the relevant curvature.

4.3.4 Extrinsic Coherent States

We would like to build, for practical applications, states which are coherent both in the intrinsic and extrinsic geometry, since we recall that the extrinsic curvature is the variable conjugate to the 3-metric in the ADM variables. In order to introduce extrinsic coherent states, we recall that a wave packet in quantum mechanics peaked on the phase space point (q, p) is of the form

$$\langle x|q,p\rangle \equiv \psi_{q,p}(x) = e^{-\frac{(x-q)^2}{2\sigma^2}+\frac{i}{\hbar}px} \,. \tag{4.156}$$

Its Fourier transform is proportional to

$$\langle k|q,p\rangle \sim e^{-\frac{(k-p/\hbar)^2}{2/\sigma^2}+iqk} \,. \tag{4.157}$$

We can rewrite this state also as

$$\psi_{q,p}(x) = e^{-\frac{(x-z)^2}{2\sigma^2}} \,, \tag{4.158}$$

where z is the complex variable given by

$$z = q - i\frac{\sigma^2}{\hbar}p \,. \tag{4.159}$$

We need to find the analogue of this state in $\mathcal{H}_\Gamma$. Starting from $L_2[SU(2)]$, we notice that a state peaked on group variables is given by a delta function:

$$\psi(U) = \delta(Uh^{-1}) \,, \tag{4.160}$$

$\psi(U)$ is a state sharp on the element $h \in SU(2)$. This state is, on the other hand, completely spread in the conjugate variable since

$$\delta(U) = \sum_j d_j \mathrm{Tr}_j[U] \,. \tag{4.161}$$

It is possible to obtain a state peaked on the value $j = 0$ by adding an exponential factor, more precisely,

$$\psi_{h,0}(U) = \sum_j d_j e^{-tj(j+1)} \mathrm{Tr}_j[Uh^{-1}] \tag{4.162}$$

is a state peaked on $U = h$ and $j = 0$. By complexifing the group variable it is possible to get a state peaked on a generic $j \neq 0$, in analogy with the wave packet seen before, where, in that case, the factor needed was $e^{ipx/\hbar}$. A complexification of $SU(2)$ is given by $SL(2,\mathbb{C})$, for this reason we consider the following state:

$$\psi_H(U) = \sum_j d_j e^{-tj(j+1)} \mathrm{Tr} D^{(j)}[UH^{-1}] \,, \tag{4.163}$$

where $H \in SL(2,\mathbb{C})$ is given by

$$H = e^{it\frac{E}{l_0^2}} h \,, \tag{4.164}$$

with $h \in SU(2)$ and $E \in \mathrm{su}(2)$. This state can be regarded as a wave packet peaked both on the group variable and its conjugate, since it is possible to show that:

$$\frac{\langle\psi_H|U|\psi_H\rangle}{\langle\psi_H|\psi_H\rangle} = h \quad , \quad \frac{\langle\psi_H|\vec{E}|\psi_H\rangle}{\langle\psi_H|\psi_H\rangle} = \vec{E} \,. \tag{4.165}$$

In order to generalize these states to spin-network states, it is necessary to make them invariant under $SU(2)$ at the nodes, therefore, an extrinsic coherent state on a graph Γ is labeled by a $SL(2,\mathbb{C})$ variable H_l associated with each link and is given by

$$\psi_{H_l}(U_l) = \int_{SU(2)} dh_{\mathbf{n}} \prod_l \sum_{j_l} d_{j_l} e^{-tj_l(j_l+1)} \mathrm{Tr} D^{(j_l)}[U_l h_{s_l} H_l^{-1} h_{t_l}^{-1}] \, . \tag{4.166}$$

Extrinsic coherent states represent the ideal tools when studying cosmology. More precisely, if we write the Hamilton function associated to a homogeneous and isotropic geometry, i.e. the one associated to the Friedmann-Lemaître metric, and then write down the expected form of the transition amplitude, it is possible to obtain the same behaviour starting from two extrinsic coherent states and then performing the classical limit [20] (large spins and saddle point).

Appendices

Appendix A

Lie Algebra

A.1 Left-Invariant Vector Fields

In this section we introduce the basic notions in order to address the standard formulation of a gauge theory from a mathemathical perspective. In addition to this we add some important issues concerning the quantization in LQG and a different formulation of General Relativity based on the so called tetrad fields.

Let's consider a Lie group G, a vector field $X \in \mathcal{T}(G)$ is called left-invariant if

$$l_{g*}X_{g\prime} = X_{gg\prime} \quad \forall g, g\prime \in G \tag{A.1}$$

where l_g is the left multiplication by g , i.e $l_g : G \to G$, $h \mapsto gh$.
We denote by $L(G)$ the vector space of left-invariant vector fields, one can easily show that it is a Lie subalgebra of $\mathcal{T}(G)$, this is due to the fact that X as in (1) is the field l_g-correlated to itself and so the Lie bracket of two left-invariant vector fields is still left-invariant.
In addition to this, one remarkable property is that $L(G)$ is isomorphic to T_eG, the latter being known as the Lie algebra of the Lie group G. The map that does the job is given by $i : T_eG \longrightarrow L(G)$, $A \mapsto L^A$, where $L^A \in L(G)$ is defined by ${L_g}^A := l_{g*}A$.
An important feature of left-invariant vector fields is that they are complete, in the sense that if $X \in L(G)$ then its integral curve is defined everywhere on $\mathbb{R}$, that is we have $\sigma^X : \mathbb{R} \longrightarrow G$ such that ${\sigma_*}^X(\frac{d}{dt}) = X$.
This fact allows us to define a map from T_eG to G called the *exponential map*, in the following way: first of all, the unique integral curve $t \mapsto \sigma^{L^A}(t)$ of $L^A \in L(G)$ such that $\sigma^{L^A}(0) = e$ and ${\sigma_*}^{L^A}(\frac{d}{dt})_0 = A$ is denoted by $t \mapsto \mathsf{exp}tA$, where $A \in T_eG$; then the *exponential map* is the map $\mathsf{exp} : T_eG \longrightarrow G$ defined by $\mathsf{exp} := \mathsf{exp}tA|_{t=1}$.
If we consider the case $G = GL(n, \mathbb{R})$ we can find a useful expression for a left-invariant vector field, introducing coordinates on $GL(n, \mathbb{R})$. First of all, we choose a coordinate system on $GL^+(n, \mathbb{R})$, which is the connected component containing matrices whose determinant is positive, then, in a neighbourhood of the identity we define:

$$x^{ij}(g) := g^{ij}, \quad g \in GL^+(n, \mathbb{R}), \quad i, j = 1, ..., n \ . \tag{A.2}$$

Now, let $A \in T_eG \cong M(n, \mathbb{R})$, we get:

$$L_g^A = \sum_{i,j=1}^{n} \left(L^A x^{ij}\right)_g \left(\frac{\partial}{\partial x^{ij}}\right)_g \tag{A.3}$$

and furthermore

$$(L^A x^{ij})_g = \frac{d}{dt}\left(x^{ij}(g\mathsf{exp}tA)\right)_{t=0}, \tag{A.4}$$

where we used the definition of integral curve:

$$X(f)_p = \frac{d}{dt}\left(f(\sigma^X(t))\right)_{t=0}, \tag{A.5}$$

where $\sigma^X(0) = p$.
Since $A \in M(n, \mathbb{R})$ is a matrix , it is possible to consider the curve $t \mapsto e^{tA}$ in $GL^+(n, \mathbb{R})$, where e^{tA} is the exponential of matrix defined by means of a series. Clearly, the tangent vector to this curve in $t = 0$ is the matrix A, furthermore, the curve defines a one-parameter subgroup of $GL^+(n, \mathbb{R})$, thus, because every one-parameter subgroup is necessariely of the form $\mathsf{exp}(tA)$ we have that:

$$e^{tA} = \mathsf{exp}tA, \quad \forall t \in \mathbb{R}, \quad \forall A \in T_e G. \tag{A.6}$$

By means of (A.6) we can rewrite (A.4) as:

$$\begin{aligned}(L^A x^{ij})_g = \frac{d}{dt}\left(x^{ij}(ge^{tA})\right)_{t=0} &= \sum_{k=1}^{n} \frac{d}{dt}\left(e^{tA}\right)^{kj}|_{t=0} \\ &= \sum_{k=1}^{n} g^{ik} A^{kj} = (gA)^{ij},\end{aligned} \tag{A.7}$$

thus we get the following expression for a left-invariant vector field on a matrix group:

$$L_g^A = \sum_{i,j=1}^{n} (gA)^{ij} \left(\frac{\partial}{\partial x^{ij}}\right)_g. \tag{A.8}$$

This expression will reappear later on, when we'll introduce the Poisson structure of T^*G, choosing $G = SU(2)$.

A.2 Left-Invariant One-Forms

We can move now to left-invariant one-forms. An n-form $\omega \in A^n(G)$ is left-invariant if

$$l_g^*\left(\omega_{g\prime}\right) = \omega_{g^{-1}g\prime}\,, \quad \forall g, g\prime \in G. \tag{A.9}$$

We have seen that $T_eG \cong L(G)$ via the map $i(A) = L^A$, we can therefore expect that $T_e^*G \cong L^*(G)$, that is to each $d \in T_e^*G$ is associated a left-invariant one-form λ^d defined by

$$\lambda_g^d := l_{g^{-1}}^*(d) \;\in T_e^*G \quad \forall g \in G. \tag{A.10}$$

It is possible to find an explicit relation between a left-invariant vector field and a left-invariant one-form, by contracting the latter with the previous one, as follows:

$$\begin{aligned} \left\langle \lambda^d, L^A \right\rangle_g &= l_{g^{-1}}^*(d)\left(L_g^A\right) = l_{g^{-1}}^*(d)\left(l_{*_g}(A)\right) \\ &= d\left(l_{*_g}^{-1} \circ l_{*_g}(A)\right) = \langle d, A\rangle \qquad \forall g \in G. \end{aligned} \tag{A.11}$$

We have seen how $L(G)$ is a Lie subalgebra of $\mathcal{T}(G)$, is there a similar result for the dual space $L^*(G)$, thought as the dual vector space? Let $\{E_1, ..., E_n\}$, $n = dimG$, a base of $L(G)$, then we have:

$$[E_\alpha, E_\beta] = \sum_{\gamma=1}^{n} C_{\alpha\beta}^{\gamma} E_\gamma, \tag{A.12}$$

where $C_{\alpha\beta}^{\gamma}$ are the srtucture constants of G with respect to the chosen basis. Therefore in $L^*(G)$ we have the corrispondent dual basis $\{\omega^1, ..., \omega^n\}$ of $L^*(G)$ which, by definiton, is such that: $\langle \omega^\alpha, E_\beta\rangle := \delta_\beta^\alpha$. We see at this point that, in general, given two vector fields, there is a natural way of obtaining a third one by combining the two via the commutator; on the other hand, there is not a natural way of doing the same thing using one-forms in place of vector fields. Nevertheless, we know that, given a one-form, we can obtain a two-form either by taking the differential or by making the external product of two one-forms. In the following we will use both these operations to find an equation that every left-invariant one-form satisfies. Taking the differential of a left-invariant one-form we get:

$$d\omega^\alpha(E_\beta, E_\gamma) = E_\beta\left(\langle \omega_\alpha, E_\gamma\rangle\right) - E_\gamma\left(\langle \omega^\alpha, E_\beta\rangle\right) - \langle \omega^\alpha, [E_\beta, E_\gamma]\rangle = -C_{\alpha\beta}^{\gamma}, \tag{A.13}$$

taking the external product of two left-invariant one-forms we get:

$$\omega^\delta \wedge \omega^\epsilon(E_\beta, E_\gamma) = \omega^\delta \otimes \omega^\epsilon(E_\beta, E_\gamma) - \omega^\epsilon \otimes \omega^\delta(E_\gamma, E_\beta) = \delta_\beta^\delta \delta_\gamma^\epsilon - \delta_\gamma^\delta \delta_\beta^\epsilon, \tag{A.14}$$

joining these two results we finally obtain:

$$d\omega^\alpha + \frac{1}{2}\sum_{\beta,\gamma=1}^{n} C_{\alpha\beta}^{\gamma} \omega^\beta \wedge \omega^\gamma = 0. \tag{A.15}$$

This equation is called the Cartan-Maurer equation and it is always satisfied by a left-invariant one-form. We will find again this equation when we will deal with T^*G.

A.3 Cartan-Maurer Form

Next, we are going to deal with the so called Cartan-Maurer form which we'll find in the context of gauge fields. The Cartan-Maurer form is the $L(G)$-valued one-form which assigns to each $v \in T_g G$ the left-invariant vector field on G whose element in g is precisely v. If we denote with $\langle \Xi, v \rangle$ the left-invariant vector field then we have:

$$\langle \Xi, v \rangle (g\prime) := l_{g\prime_*} \left(l_{g^{-1}*} v \right) \quad \forall v \in T_g G, \tag{A.16}$$

in particular, that means that:

$$\langle \Xi, L_g^A \rangle (g\prime) = L_{g\prime}^A, \tag{A.17}$$

furthermore, because $L(G) \cong T_e G$ we can associate to L_g^A the element $A \in T_e G$, to get $\langle \Xi, L_g^A \rangle = A$. The Cartan-Maurer form is clearly left-invariant.
In the special case in which $G = GL(n, \mathbb{R})$ we have seen that $L_g^A = (gA)^{ij} \left(\frac{\partial}{\partial x^{ij}} \right)$, thus:

$$\delta^{ij} = \left(\langle \Xi, L_g^{\mathbb{1}} \rangle \right)^{ij} = \Xi^{ik} \left(L_g^{\mathbb{1}} \right)^{kj} = \Xi^{ik} g^{kj}, \tag{A.18}$$

where $\mathbb{1} \in T_e G$ is the identity matrix. From (A.18) we can deduce that:

$$\Xi_g^{ij} = \sum_{k=1}^{n} \left(g^{-1} \right)^{ik} \left(dx^{kj} \right)_g . \tag{A.19}$$

The expressions for L_g^A and Ξ_g found in a coordinate system as in (A.2) are still valid for a general Lie matrix group.

A.3.1 Gauge Transofrmations and Cartan-Maurer Form

Let's end this section with an example, which turns out to be useful in the following.
Let $U \in \mathcal{M}$ be an open set, $\Omega : U \to G$, $\mathcal{M}$ is a m-dimensional differentiable manifold (in Yang-Mills theories it represents spacetime), G is the gauge group and so Ω is meant to be a gauge function which assigns to each point of $\mathcal{M}$ a gauge transformation. Obviously, on G is present the Cartan-Maurer form, then we can consider the pull-back of Ξ on $\mathcal{M}$ through Ω. If G is a matrix group, using coordinates as in (2.2) we can find an expression of $\Omega^* \Xi$ in the following way:

$$\begin{aligned}\left\langle \left(\Omega^*\Xi\right)_p^{ij}, \left(\frac{\partial}{\partial x^\mu}\right)_p \right\rangle &= \left\langle \Xi^{ij}, \Omega_*\left(\frac{\partial}{\partial x^\mu}\right)\right\rangle_{\Omega(p)} \\ &= \left\langle \sum_{k=1}^{n} \left(\Omega(p)^{-1}\right)^{ik} \left(dx^{kj}\right)_{\Omega(p)}, \Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_{\Omega(p)} \right\rangle \\ &= \sum_{k=1}^{n} \left(\Omega(p)^{-1}\right)^{ik} \Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p \left(x^{kj}\right) \\ &= \sum_{k=1}^{n} \left(\Omega(p)^{-1}\right)^{ik} \frac{\partial}{\partial x^\mu} x^{kj}\left(\Omega(p)\right),\end{aligned} \tag{A.20}$$

for each p belonging to a local chart whose domain is U.
Therefore we obtain the expression:

$$\left(\Omega^*\Xi\right)_p^{ij} = \sum_{\mu=1}^{m}\sum_{k=1}^{n} \left(\Omega^{-1}(p)\right)^{ik} \frac{\partial}{\partial x^\mu}\Omega^{kj}(p)\left(dx^\mu\right)_p, \tag{A.21}$$

which very often appears in the more succinct form:

$$\Omega^*\Xi = \Omega^{-1} d\Omega. \tag{A.22}$$

Appendix B

Principal Fibre Bundles

In this section we are going to explore the idea of a principal fibre bundle, the reason for that is because it's the main structure in guauge theories.

B.1 Principal Fibre Bundles

The idea of a principal fibre bundle is that of a fibre bundle in which fibres are diffeomorphic to a Lie group G and on which the same group G acts in such a way to "move points along the fibres".

The preliminary definition we have to give is that of a $G - bundle$: a bundle $(E, \pi, \mathcal{M})$ is a $G - bundle$ if E is a G-space (i.e. there is a G-action on E) and if $(E, \pi, \mathcal{M})$ is isomorphic to the bundle $(E, \rho, E/G)$ where E/G is the space of the orbits of the G-action on E and ρ is the projection on the orbit, to summarize we say that the following diagram has to commute:

$$\begin{array}{ccc} E & \xrightarrow{u} & E \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \rho} \\ \mathcal{M} & \xrightarrow{\simeq} & E/G \end{array} \tag{B.1}$$

The fact that $(E, \pi, \mathcal{M})$ and $(E, \rho, E/G)$ are isomorphic means that the fibres of E are the orbits of the G-action on E. If the action of G on E is free, that is, if $\forall p \in E$ we have $\{g \in G \mid pg = p\} = \{e\}$, then $(E, \pi, \mathcal{M})$ is said principal G-bundle and G is called the structure group of the bundle. The fact that the action of G is free implies that every orbit is homeomorphic to G, therefore it makes sense to say that $(E, \pi, \mathcal{M})$ is a fibre bundle with fibre G.

A principal fibre bundle which turns out to be very useful is the bundle of frames of a m-dimensional differentiable manifold $\mathcal{M}$. Let be $x \in \mathcal{M}$, $(b_1, ..., b_m)$ a basis of vectors in $T_x\mathcal{M}$, the total space $\mathcal{B}(\mathcal{M})$ of the bundle of frames is defined as the set of all frames at each point of $\mathcal{M}$, the projection $\pi : \mathcal{B}(\mathcal{M}) \to \mathcal{M}$ is defined by the function which sends each frame to the point it is attached. It is possible to introduce a free right action of $GL(m, \mathbb{R})$ on $\mathcal{B}(\mathcal{M})$ given by:

$$(b_1, ..., b_m)\, g := \left(\sum_{j_1=1}^{m} b_{j_1} g_{j_1 1}, ..., \sum_{j_m=1}^{m} b_{j_m} g_{j_m m} \right) \; \forall g \in GL(m, \mathbb{R}). \tag{B.2}$$

This action is clearly free, as one can verifies; the action corresponds to a change of basis in $T_x\mathcal{M}$, $x \in \mathcal{M}$. In addition to this, $\mathcal{B}(\mathcal{M})$ can be endowed with a differentiable structure, as follows: let $U \subset \mathcal{M}$ be a domain of a local chart on $\mathcal{M}$, whose coordinates we denote by $(x^1, ..., x^m)$, then each basis $b = (b_1, ..., b_m)$ of $T_x\mathcal{M}$, $x \in U$, can be written as

$$b_i = \sum_{j=1}^{m} b_i^j \left(\frac{\partial}{\partial x^j} \right)_x, \; i = 1, ..., m, \tag{B.3}$$

for a certain $b_i^j \in GL(m, \mathbb{R})$. We can therefore define the following map:

$$\begin{aligned} h \; : U \times GL(m, \mathbb{R}) &\longrightarrow \;\; \pi^{-1}(U) \\ (\, x \,,\, g \,) \qquad &\longmapsto \left(\sum_{j_1=1}^{m} g_1^{j_1} \left(\partial_{j_1}\right)_x, ..., \sum_{j_m=1}^{m} g_1^{j_m} \left(\partial_{j_m}\right)_x \right) \end{aligned} \tag{B.4}$$

and use $\left(x^1, ..., x^m; g_i^j\right)$ as coordinates in $\mathcal{B}(\mathcal{M})$. In this way, $\mathcal{B}(\mathcal{M})$ becomes a $m + m^2$ differentiable manifold.

Having said what is meant by a G-principal bundle, now we have to say what we mean by a principal map, i.e. a map between principal bundles. A bundle map (u, f) between a pair of G-principal bundles $(P, \pi, \mathcal{M})$ and $(P', \pi', \mathcal{M}')$ is said a principal map if $u \, : \, P \to P'$ is G-equivariant, that is, $u(pg) = pg \; \forall p \in P$, $\forall g \in G$; in other words the orbit O_p is sent to the orbit $O'_{u(p)}$, thus preserving the fibre structure of P and P'.
It is possible to generalize this definition to the case of a pair of principal bundles with different structure groups, say G and G', the requirement of G-equivariance is now implemented by adding a group homomrphism
$\Lambda \, : \, G \to G'$ and demanding that $u(pg) = u(p)\Lambda(g) \; \forall p \in P$, $\forall g \in G$.
This last property is important when we will deal with the so called spin connection: we will consider a principal bundle with structure group given by $G' = SO(3)$, the base space $\mathcal{M}$ will be a 3-dimensional Riemannian manifold and $G = \mathsf{Spin}(3, \mathbb{R})$, that is, the double cover of $SO(3)$, which coincides with SU(2), the universal cover of SO(3). Clearly, SO(3) and SU(2) are homomorphic, in addition to this we know also that they have isomorphic Lie algebras, i.e. $\mathsf{su}(2) \cong \mathsf{so}(3)$.
In the same fashion, if $\mathcal{M}$ is a Lorentzian manifold, $G' = SO(3,1)$ and $G = \mathsf{Spin}(3,1) \cong SL(2, \mathbb{C})$ and $SO(3,1)$ is homomorphic to $SL(2, \mathbb{C})$. When we'll introduce the connection on a principal bundle we'll see, as an application, the case of the principal bundle of orthonormal frames, which turns out to be a SO(3,1)-bundle, and then take the pull-back of the connection on the principal bundle of spinorial frames (hence the name, spin connection) which turns out to be a Spin(3,1)-bundle. We can safely do that because there is a principal map between these two principal bundles.

B.2 Tetrads

We go on now introducing an object that will find place in the action of the GR and also in the Poisson structure of LQG.

We shall consider the bundle of orthonormal frames, denoted as $\mathcal{O}(\mathcal{M})$, which can be seen as a subbundle of $\mathcal{B}(\mathcal{M})$, whose structure group $GL(m,\mathbb{R})$ has been reduced to $O(m.\mathbb{R})$ (this can be seen by introducing the concept of associated bundle). However, $\mathcal{O}(\mathcal{M})$ is a principal bundle in its own right. We choose a local orthonormal frame of $T\mathcal{M}$, or equivalently, a local section of $\mathcal{O}(\mathcal{M})$, denoted as $\{e_1,...,e_m\}$, this local frame is called $m-bein$ and each e_i is a *tetrad*. If we consider $T^*\mathcal{M}$ instead of $T\mathcal{M}$, each e^i is called a $co-tetrad$. The requirement of orthonormality reads:

$$\delta_{ij} = g_{\mu\nu}(x)e_i^\mu(x)e_j^\nu(x), \tag{B.5}$$

in the case of a Riemannian manifold, if $\mathcal{M}$ is a Lorentzian manifold we have instead

$$\eta_{ij} = g_{\mu\nu}(x)e_i^\mu(x)e_j^\nu(x), \tag{B.6}$$

where x belongs to the local domain of the m-bein.

B.3 Connection

In order to write the action of GR in another form we need to replace the metric with tetrads and connection (on a principal bundle). To introduce the idea of a connection we can follow this way of reasoning: we seek a vector field on a principal bundle P that lets us move from one fibre to another and not along the fibre. Now, in general, if G is a Lie group which acts on a differentiable manifold $\mathcal{M}$ by means of a right action $\delta : \mathcal{M}\times G \to \mathcal{M}$, $(p,g)\mapsto \delta(p,g) =: \delta_g(p)$, it is possible to define a vector field $X^A \in \mathcal{T}(\mathcal{M})$ induced by the action of the one-parameter subgroup $t\mapsto \mathsf{exp}(tA)$, $A\in T_eG$ (i.e. restricting the right action δ to those elements of G that can be written as the exponential of an element in T_eG). The vector field X^A is defined as follows:

$$X_p^A(f) := \frac{d}{dt} f\left(p\mathsf{exp}(tA)\right)|_{t=0}, \tag{B.7}$$

where $f\in C^\infty(\mathcal{M})$ and $pg := \delta_g(p)$. In other words, the curve through p given by $t\mapsto p\mathsf{exp}(tA)$ is the integral curve of X^A. The flux of X^A, denoted as ϕ_t^A is thus given by $\phi_t^A(p) = p\mathsf{exp}(tA) = \delta_{\mathsf{exp}(tA)}(p)$, that is $\phi_t^A = \delta_{\mathsf{exp}(tA)}$.

If, in lieu of $\mathcal{M}$, we consider a principal bundle P, where, as we know, is defined a right action of G, we can write down the vector field induced by this action. At this point, it is possible to show that the map $\iota : L(G)\to\mathcal{T}(P)$, $A\mapsto X^A$ is a Lie algebra homomorphism, that is $X^{[A,B]} = \left[X^A, X^B\right]$.

However, the vector fields X_p^A point along the fibre $\forall A\in T_eG$, that's because the right action moves a point along the fibre, by its very definition. In this context we say that X_p^A is a *vertical* vector, in the sense that it belongs to the vertical subspace V_pP of T_pP, which is defined by $V_pP := \{\tau \in T_pP | \pi_*\tau = 0\}$, from which it is clear that τ points along the fibre. Now, the map $A\mapsto X^A$ is an isomorphism of $L(G)$ onto V_pP because it is linear, injective (beacuse the action of G on P is free) and for dimensional reasons $dimV_pP = dimG = dimL(G)$.

Intuitively, then, it is justified the following definition of connection:

a connection on a prinicipal bundle $G\to P\to\mathcal{M}$ is an assignement to each point $p\in P$ of a subspace H_pP of T_pP such that

$$
\begin{aligned}
&(a)\;\; T_pP \cong V_pP \oplus H_pP \;\; \forall p \in P \\
&(b)\;\; \delta_{g_*}(H_pP) = H_{pg}P \;\; \forall g \in G,\; \forall p \in P;
\end{aligned}
\tag{B.8}
$$

this means that a connection is first of all a k-dimensional distribution on P, with $k = dimP - dimG$. We can therefore split a vector $\tau \in T_pP$ into two components, horizontal and vertical: $\tau = ver(\tau) + hor(\tau)$. The condition (b) guarantees that this operation is compatible with the right action on P, in the sense that: $\delta_{g_*}(\tau) = \delta_{g_*}hor(\tau) + \delta_{g_*}ver(\tau) = hor(\delta_{g_*}\tau) + ver(\delta_{g_*}\tau)$.
There is also an equivalent definition of a connection, less intuitive but which is very used to find explicit expressions in which the connection is involved. The alternative definition goes as follows: a connection can be associated to a $L(G)$-valued one-form on P in the following way, if $\tau \in T_pP$ we define

$$
\omega_p(\tau) := \iota^{-1}\left(ver(\tau)\right), \tag{B.9}
$$

where $\iota : L(G) \to V_pP$ is the isomorphism introduced before. From this definition it follows that:

$$
\begin{aligned}
&(i)\;\; \omega_p(X^A) = A,\; \forall p \in P,\; \forall A \in L(G), \\
&(ii)\;\; \delta_g^*\omega = \mathsf{Ad}_{g^{-1}}\omega,\; i.e.\; \left(\delta_g^*\omega\right)_p(\tau) = \mathsf{Ad}_{g^{-1}}\left(\omega_p(\tau)\right),\; \forall \tau \in T_pP,
\end{aligned}
\tag{B.10}
$$

where we recall that $C_g : G \to G$, $h \mapsto hgh^{-1}$ (C being the conjugate action), and $\mathsf{Ad}_g := dC_g$.
In particular, we notice that $\tau \in H_pP$ if and only if $\omega_p(\tau) = 0$. From this last equation it is clear that the connection is a sort of constraint on the space of vector fields on P, thought in this way, its counterimage $\omega_p^{-1}(0)\;\; \forall p \in P$ is exactly a distribution of horizontal vector fields.

B.4 Yang-Mills Fields and Gauge Transformations

At this point, we can establish the relation between a connection ω on a principal bundle, thought as a $L(G)$-valued one-form on P and the so called Yang-Mills fields, often introduced as functions on spacetime.
Usually, a Yang-Mills field is denoted as A^a_μ, where μ is a spacetime index and a is a Lie algebra index, therefore the following expression is meaningful:

$$
A(x) = \sum_{\mu=1}^{m} \sum_{a=1}^{dimG} A^a_\mu(x) E_a \left(dx^\mu\right)_x \tag{B.11}
$$

where $\{E_1, ..., E_{dimG}\}$ is a basis of $L(G)$, thus, locally, a Yang-Mills field corresponds to a $L(G)$-valued one-form. More precisely, let $\sigma : U \subset \mathcal{M} \to P$ be a local section of the principal bundle $G \to P \to \mathcal{M}$ on which is present also a connection one-form ω. We define the local σ-representative of ω as $\omega^U := \sigma^*\omega$, which is then a $L(G)$-valued one-form on U. Let then $h : U \times G \to \pi^{-1}(U) \subset P$ be the local trivialisation of P induced by σ, that is, $h(x, g) := \sigma(x)g$. As a consequence, it can be shown that if $(\alpha, \beta) \in T_{(x,g)}(U \times G) \cong T_xU \oplus T_gG$ we have that $h^*\omega$ can be written in terms of ω^U as follows:

$$
\left(h^*\omega\right)_{(x,g)}(\alpha, \beta) = \mathsf{Ad}_{g^{-1}}\left(\omega^U_x(\alpha)\right) + \Xi_g(\beta), \tag{B.12}
$$

where Ξ is the Cartan-Maurer form. Therefore we notice that, locally, a connection one-form ω is split into the sum of a $L(G)$-valued one-form on spacetime and a $L(G)$-valued one-form on the structure group G.

Next, we know that YM fields are subjected to local gauge transformations, where with gauge transformations we mean a principal automorphism of $G \to P \to \mathcal{M}$; if $\phi : P \to P$ is such a map then $\phi^*(\omega)$ is still a $L(G)$-valued one-form and $\phi^*(\omega)$ is called the gauge transform of ω. This one is the so called active version of gauge transformations. We can ask ourselves how ω^U changes if we choose a different section σ, that is we address the same issue adopting a passive view. Let's consider then two local sections of P, $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$, where $U_1, U_2 \subset \mathcal{M}$, $U_1 \cap U_2 \neq \emptyset$.
We call $A_\mu^{(1)}$ and $A_\mu^{(2)}$ the local representatives of ω with respect to σ_1 and σ_2. Then, if $\Omega : U_1 \cap U_2 \to G$ is the unique (because the action of G on P is free) local gauge function such that $\sigma_2(x) = \sigma_1(x)\Omega(x)$ we have from (B.12) that

$$A_\mu^{(2)}(x) = \mathsf{Ad}_{\Omega(x)^{-1}}\left(A_\mu^{(1)}(x)\right) + \left(\Omega^*\Xi\right)_\mu(x), \tag{B.13}$$

in the case where G is a matrix group we can write

$$A_\mu^{(2)}(x) = \Omega(x)^{-1}A_\mu^{(1)}(x)\Omega(x) + \Omega(x)^{-1}\partial_\mu\Omega(x). \tag{B.14}$$

How it reads a gauge transformations if instead we adopt an active view?
The answer is easy, if $\sigma : U \to P$ is a local section, $A := \sigma^*(\omega)$ and $\phi : P \to P$ is an automorphism of $G \to P \to \mathcal{M}$, we can consider the transformation $A \mapsto \sigma^*(\phi^*\omega) = (\phi \circ \sigma)^*\omega$. Comparing now A_μ with $A_\mu^{(1)}$ and $\sigma^*(\phi^*\omega)$ with $A^{(2)}$ it is clear that

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega(x)^{-1} + \Omega(x)\partial_\mu\Omega(x)^{-1}. \tag{B.15}$$

B.5 Analogies between $(\mathcal{M}, g)$ and $\mathcal{O}(\mathcal{M})$

B.5.1 Linear Connection and Connection one-form

At this point we can specialize to the case of the bundle of orthonormal frames $\mathcal{O}(\mathcal{M})$ on a differentiable manifold $\mathcal{M}$, where $\mathcal{M}$ is thought as spacetime. We have already established the relation between the metric g of a (pseudo-)Riemannian manifold and the tetrads e_i, we could ask then which is the relation between the Levi-Civita connection (defined on $(\mathcal{M}, g)$) and the corrispondent connection one-form on $\mathcal{O}(\mathcal{M})$.
In order to do that we begin by recalling that, if $\mathcal{M}$ is a differentiable manifold, a linear connection on $\mathcal{M}$ is a map:

$$\begin{aligned} \nabla : \ \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) &\longrightarrow \ \mathcal{T}(\mathcal{M}) \\ (X \ , \ Y) \quad &\longmapsto \ \nabla_X Y \end{aligned} \tag{B.16}$$

which is $\mathbb{R}$-linear in the second argument, C^∞-linear in the first one and that satisfies the Leibniz rule. Using local coordinates $\left(x^1, ..., x^m\right)$ on $U \subset \mathcal{M}$ we have that $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$ (summation of repeated indices implied), Γ^k_{ij} are called the Christoffel symbols of the linear connection ∇ and determine uniquely the connection. There is another equivalent definition of a linear connection, that states that a linear connection ω on $\mathcal{M}$ is a $\mathcal{T}(\mathcal{M})$-valued one-form, locally it is represented by a matrix of one-forms as follows: $(\omega)^k_j = \Gamma^k_{ij}dx^i$. In the context of (pseudo-)Riemmanian manifolds it is known that it exists a unique linear connection symmetric and compatible with the metric, called the Levi-Civita connection; in this case it is possible to show that the relation between Γ^k_{ij} and $(\omega)^k_j$ is given by:

$$g\left(\nabla_X e_i, e_j\right) = \omega^k_i(X)\eta_{kj} = \omega_{ij}(X), \tag{B.17}$$

where e_i are the tetrads. Let's see why it is so: from $\nabla_X e_i = \omega^k_i(X)e_k$ follows that $\nabla_\rho e_i = \omega^k_j(\partial_\rho)e_k$, thus

$$(\nabla_\rho e_i)^\mu = \partial_\rho e^\mu_i + \Gamma^\mu_{\rho\sigma}e^\sigma_i = \omega^k_j(\partial_\rho)e^\mu_k, \tag{B.18}$$

because $g_{\mu\nu}e^\mu_k e^\nu_j = e^\mu_k e_{j\mu} = \eta_{kj}$. Finally we get:

$$\omega_{ij}\left(\partial_\rho\right) = \left(\partial_\rho e^\mu_i\right)e_{j\mu} + \Gamma^\mu_{\rho\sigma}e^\sigma_i e_{j\mu}. \tag{B.19}$$

On the other hand we have that

$$\begin{aligned} g\left(\nabla_\rho e_i, e_j\right) = g_{\mu\nu}\left(\nabla_\rho e_i\right)^\mu e^\nu_j &= g_{\mu\nu}\left(\partial_\rho e^\mu_i + \Gamma^\mu_{\rho\sigma}e^\sigma_i\right)e^\nu_j = \\ &= \left(\partial_\rho e^\mu_i\right)e_{j\nu} + \Gamma^\mu_{\rho\sigma}e^\sigma_i e_{j\mu} \end{aligned} \tag{B.20}$$

from which it is clear that (B.17) holds.
Obviously, it is possible to write the Christoffel symbols in terms of the tetrads, it is sufficient to substitute in $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)$ the expression $g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}$, in doing so we find an expression of ω_{ij} solely in terms of e_i.
The conclusion is that we deal with the bundle of orthonormal frames on $\mathcal{M}$ in place of a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$. In physics, the metric represents the gravitational field and the independent components of $g_{\mu\nu}$ are 10, that equals the sum of the indepenent components of η_{ij} (6) and e_i (4).
The connection one-form ω on $\mathcal{O}(\mathcal{M})$ is often called *spin connection*, because one has in mind the G-principal bundle with $G = Spin(3,1) \cong SL(2,\mathbb{C})$, i.e. the universal cover of $SO(3,1)$. However, we know that homomorphic Lie groups have isomorphic Lie algebras, in this case $\mathsf{so}(3,1) \cong \mathsf{sl}(2,\mathbb{C})$, then the connection one-form doesn't change.

B.5.2 Torsion and Curvature

At this point we want to introduce another very important concept: the curvature of a connection. In particular, we shall focus our attention on the Levi-Civita connection, which, as remarked earlier, is the unique linear connection on a Riemannian manifold that is compatible with the metric and symmetric (i.e with null torsion).

Let's begin therefore by recalling how it is defined the torsion of a linear connection ∇:

$$\tau : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M}), \quad \tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \tag{B.21}$$

it is easily verified that τ is $C^\infty(\mathcal{M})$-linear in all the variables and so it can be regarded as a tensor field $\tau \in \mathcal{T}_1^2(\mathcal{M})$, furthermore τ is antisymmetric.
We have already established the relation between a linear connection on $\mathcal{M}$ and a connection one-form on the bundle of orthonormal frames, we would like to accomplish the same goal regarding the torsion. In order to do so we choose local coordinates $\{x^1, ..., x^m\}$, the torsion in coordinates reads:

$$\tau^\gamma_{\alpha\beta} = (\nabla_\alpha \partial_\beta)^\gamma - (\nabla_\beta \partial_\alpha)^\gamma - [\partial_\alpha, \partial_\beta]^\gamma = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}, \tag{B.22}$$

furthermore, in terms of the tetrads e_i, we have

$$\tau(\partial_\alpha, \partial_\beta) = \tau^\gamma_{\alpha\beta}\partial_\gamma = \tau^\gamma_{\alpha\beta} e^i_\gamma e_i = T^i_{\alpha\beta}, \tag{B.23}$$

where we have defined $T^i_{\alpha\beta} := \tau^\gamma_{\alpha\beta} e^i_\gamma = \left(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\right) e^i_\gamma$. It happens that $T^i_{\alpha\beta}$ are the components of the following 2-form:

$$T^i := \Gamma^\gamma_{\mu\nu} e^i_\gamma dx^\mu \wedge dx^\nu. \tag{B.24}$$

Thus, the information on the torsion τ is equally contained in the 2-forms T^i, $i = 1, ..., 4$. At this point we seek an expression of T^i purely in terms of the tetrads and the connection one-form ω on $\mathcal{O}(\mathcal{M})$.
Firstly, we observe that $dx^\mu = e^\mu_j e^j$, $dx^\nu = e^\nu_k e^k$, then:

$$T^i = \Gamma^\gamma_{\mu\nu} e^i_\gamma e^\nu_k \left(e^j \wedge e^k\right), \tag{B.25}$$

recalling eq. (B.19) we notice that:

$$\Gamma^\gamma_{\mu\nu} e^i_\gamma e^\nu_k = \omega^i_k(\partial_\mu) - (\partial_\mu e^\alpha_k) e^i_\alpha, \tag{B.26}$$

from which

$$\begin{aligned} T^i &= \omega^i_k(\partial_\mu) e^\mu_j \left(e^j \wedge e^k\right) - (\partial_\mu e^\alpha_k) e^i_\alpha e^\mu_j \left(e^j \wedge e^k\right) = \\ &= e^\mu_j \omega^i_k(\partial_\mu) \left(e^k \wedge e^j\right) + \left(\partial_\mu e^i_\alpha\right) e^\alpha_k e^\mu_j \left(e^j \wedge e^k\right), \end{aligned} \tag{B.27}$$

where we used the fact that $(\partial_\mu e^\alpha_k) e^i_\alpha = \partial_\mu \left(e^\alpha_k e^i_\alpha\right) - e^\alpha_k \partial_\mu e^i_\alpha = \partial_\mu \left(\eta^i_k\right) - \partial_\mu e^i_\alpha e^\alpha_k$.
Now the last steps:

$$e^\mu_j \omega^i_k(\partial_\mu) e^k = \omega^i_j, \tag{B.28}$$

in fact $e^\mu_j \omega^i_k(\partial_\mu) e^k_\nu = \omega^i_j(\partial_\nu)$ is true since $e^\mu_j e^k_\nu = \eta^k_j$, as one can easily verify.
Finally we observe that:

$$de^i = d\left(e^i_\alpha dx^\alpha\right) = \partial_\mu e^i_\alpha dx^\mu \wedge dx^\alpha = \partial_\mu e^i_\alpha e^\mu_j e^\alpha_k \left(e^j \wedge e^k\right), \tag{B.29}$$

thus we can rewrite T^i as follows:

$$T^i = de^i + \omega^i_j \wedge e^j. \tag{B.30}$$

Eq. (B.30) is the expression of the torsion of the connection one-form ω on $\mathcal{O}(\mathcal{M})$.
From the Riemannian geometry it is known that it exists a unique linear connection which is compatible with the metric and symmetric.
The fact that ∇ is compatible with the metric is already implicitly contained in the fact that ω_{ij} is antisymmetric, in fact from $g(\nabla_X e_i, e_j) = \omega_{ij}$ we deduce that $0 = X(g(e_i, e_j)) = \nabla_X(g(e_i, e_j)) = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j)$, thus ω_{ij} being antisymmetric (a necessary condition since ω is a $\mathsf{sl}(2, \mathbb{C})$-valued one-form) is equivalent to the fact that ∇ is compatible with the metric.
If, in addition to this, $\tau(X, Y) = 0 \quad \forall X, Y \in \mathcal{T}(\mathcal{M})$ then ∇ is the Levi-Civita connection, equivalently, if $T^i = 0 \quad \forall i = 1, ..., 4$ then ω is the Levi-Civita connection one-form on $\mathcal{O}(\mathcal{M})$.

It is known from GR that the curvature of the Levi-Civita connection is related to the gravitational force; in gauge theories, similarly, the curvature of a connection one-form is associated with the gauge interaction. In the tetrad-connection formalism GR resembles a gauge theory, for this reason we attempt to establish a relation between the Riemann tensor $R \in \mathcal{T}^3_1(\mathcal{M})$ defined by

$$R(X, Y, Z) := R_{XY}(Z, W) := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{B.31}$$

and the analogous of the curvature of a connection one-form.
Let's give then the definition of curvature in this context: if ω is a k-form on P, the exterior covariant derivative of ω is the horizontal $(k+1)$-form defined by:

$$D\omega := d\omega \circ hor, \tag{B.32}$$

that is, $D\omega(X_1, X_2, ..., X_{k+1}) = d\omega(horX_1, ..., horX_k)$, $\forall X_1, ..., X_{k+1}$ vector fields on P.
If ω is a connection one-form on P the curvature 2-form of ω is defined by

$$G := D\omega. \tag{B.33}$$

We mention a very important result: if $G = D\omega$ is the curvature 2-form of ω then, $\forall p \in P$, we have that:

$$G_p(X, Y) = d\omega_p(X, Y) + [\omega_p(X), \omega_p(Y)] \quad \forall X, Y \in \mathcal{T}(\mathcal{P}), \tag{B.34}$$

where [,] denotes the Lie brackets in $L(G)$.
Choosing a basis $\{E_1, ..., E_{dimG}\}$ of $L(G)$ we obtain that $\omega = \omega^a E_a$ and then:

$$G^a = d\omega^a + \frac{1}{2} C^a_{bc} \omega^b \wedge \omega^c, \tag{B.35}$$

where C^a_{bc} are the structure contants of $L(G)$ with respect ot the basis $\{E_1, ..., E_{dimG}\}$.

As already done previously, we can find what the curvature looks like when we consider its pullback by a local section. Let then $\sigma : U \to P$ be a local section, $A := \sigma^* \omega$ the local representative of ω, $F := \sigma^* G$ the local representative of G. From the properties of the pull back it follows that:

$$F^a = dA^a + \frac{1}{2} C^a_{bc} A^b \wedge A^c. \tag{B.36}$$

Introducing local coordinates on $U \subset \mathcal{M}$ we can write:

$$F^a_{\mu\nu} = \frac{1}{2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \right), \tag{B.37}$$

which is the familiar expression of the field strength found in gauge theories. It's not difficult, in this context, to prove the Bianchi identity $DG = 0$, in fact, from $DG_p (X, Y, Z) = dG_p (horX, horY, horZ)$ it's enough to apply the definition of external differential and notice that $G_p (horX, horY) = 0 \quad \forall X, Y \in \mathcal{T}(P)$. At this point, we choose as a basis of $\mathsf{sl}(2, \mathbb{C})$ the set $\{E_{IJ}\}$ of the antisymmetric matrices such that $\omega = \omega^{IJ} E_{IJ}$ and:

$$[E_{KL}, E_{MN}] = (\eta_{KM}\eta_{LN} - \eta_{LM}\eta_{KN})^{IJ} E_{IJ}. \tag{B.38}$$

Then we notice that, since $(\eta_{KM}\eta_{LN})^{IJ} = (\eta_{KM})^I_P (\eta_{LN})^{PJ} = \eta^I_K \eta_{MP} \eta^P_L \eta^J_N = \eta^I_K \eta_{ML} \eta^J_N$, we have $(\eta_{KM}\eta_{LN})^{IJ} \omega^{KM} \wedge \omega^{MN} = \omega^I_M \wedge \omega^{MJ}$, for this reason the curvature can be written as:

$$F^{IJ} = d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ}. \tag{B.39}$$

Now we want to find the relation between R and F. Starting from (B.39) it is possible to show that (after long and painful calculations)

$$R^\mu_{\nu\rho\sigma} = e^\mu_I e^J_\nu F^I_{J\rho\sigma}, \tag{B.40}$$

or, equivalently,

$$F^{IJ} = e^I_\mu e^J_\nu R^{\mu\nu}_{\rho\sigma} dx^\rho \wedge dx^\sigma. \tag{B.41}$$

We recall that the Ricci tensor is obtained by contracting the first and the third indeces of the Riemann tensor:

$$R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} = e^\mu_I e^J_\nu F^I_{J\mu\sigma}, \tag{B.42}$$

from which it follows that the Ricci scalar is given by:

$$R = g^{\nu\sigma} R_{\nu\sigma} = g^{\nu\sigma} e^\mu_I e^J_\nu F^I_{J\mu\sigma} = e^\mu_I e^\sigma_J F^{IJ}_{\mu\sigma} = \left(F^{IJ}\right)_{IJ}. \tag{B.43}$$

The last equality is due to the fact that:

$$F^{IJ} = \frac{1}{2} F^{IJ}_{\mu\sigma} dx^\mu \wedge dx^\sigma = \frac{1}{2} F^{IJ}_{\mu\sigma} e^\mu_K e^\sigma_L \left(e^K \wedge e^L\right) = \frac{1}{2} F^{IJ}_{KL} \left(e^K \wedge e^L\right) \tag{B.44}$$

where, again, the last equality holds by the very definition of differential form, F^{IJ}_{KL} are the compoponenents of the 2-form F^{IJ} in the basis of the cotetrads e^I. With the intention of writing the Einstein-Hilbert action we observe that:

$$\mathsf{det}\left(g_{\mu\nu}\right) = \mathsf{det}\left(e^T \eta e\right)_{\mu\nu} = \mathsf{det}^2(e)\mathsf{det}(\eta) = -\mathsf{det}^2(e), \tag{B.45}$$

from which it follows that $g := \mathsf{det}\left(g_{\mu\nu}\right) = -\mathsf{det}^2(e) =: -e^2$ and so $\sqrt{-g} = |e|$. Finally we notice that

$$\begin{aligned} \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} &= \frac{1}{2}\epsilon_{IJKL} F^{KL}_{MN} e^I \wedge e^J \wedge e^M \wedge e^N = \\ &= \frac{1}{2}\epsilon_{IJKL}\epsilon^{IJMN} F^{KL}_{MN} |e| d^4x = \\ &= -\left(\delta^M_K \delta^N_L - \delta^M_L \delta^N_K\right) F^{KL}_{MN} |e| d^4x = \\ &= -2|e| R d^4x, \end{aligned} \tag{B.46}$$

thanks to the fact that

$$\begin{aligned} e^I \wedge e^J \wedge e^M \wedge e^N &= e^I_\mu e^J_\nu e^M_\rho e^N_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \\ &= e^I_\mu e^J_\nu e^M_\rho e^N_\sigma \varepsilon^{\mu\nu\rho\sigma} d^4x = \\ &= \epsilon^{IJMN} |e| d^4x. \end{aligned} \tag{B.47}$$

If we define now $Tr\left(e \wedge e \wedge F\right) := \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}$ we can write the Einstein-Hilbert action S_{EH} as follows:

$$S_{EH} = \frac{1}{16\pi G}\int d^4x \sqrt{-g} R = -\frac{1}{32\pi G}\int Tr\left(e \wedge e \wedge F\right). \tag{B.48}$$

In this form S_{EH} is a functional of e and ω.

B.6 Holonomy

In order to approach the definition of holonomy it is necessary to explain what is meant by parallel transport on a principal bundle. The idea is to find a curve that lets us move from one fiber to the other, we have already seen that $\pi_* : H_pP \to T_{\pi(p)}\mathcal{M}$ is an isomorphism, then for each vector field $X \in \mathcal{T}\left(\mathcal{M}\right)$ it exists a unique vector field on P, denoted as $X^\uparrow$ such that, $\forall p \in P$, we have

$$\begin{aligned} &(a)\ \pi_*\left(X^\uparrow_p\right) = X_{\pi(p)} \\ &(b)\ ver\left(X^\uparrow_p\right) = 0\ , \end{aligned} \tag{B.49}$$

$X^\uparrow$ is called the horizontal lift of X. Intuitively the picture is quite clear: the integral curve on $\mathcal{M}$ of X is lifted to a curve on P which does precisely the job we are looking for. Indeed, we can define the horizontal lift of a curve $\alpha : [a,b] \to \mathcal{M}$ as the curve $\alpha^\uparrow : [a,b] \to P$ such that $\pi\left(\alpha^\uparrow(t)\right) = \alpha(t), \forall t \in [a,b]$ and that is horizontal, i.e. $ver\left[\alpha^\uparrow_*\left(\frac{d}{dt}\right)\right] = 0$. It is possible to show that $\forall p \in \pi^{-1}\{\alpha(a)\} \subset P$ it exists a unique horizontal lift of α such that $\alpha^{\uparrow}(a) = p$. As usual, it is useful to find an explicit expression for $\alpha^{\uparrow}$, which will naturally contain the connection one-form ω. In order to do so we can follow this line of reasoning: let's suppose that $\beta : [a, b] \to P$ is a lift of α (not necessarily horizontal), that is, $\pi(\beta(t)) = \alpha(t)\ \forall t \in [a, b]$ (thus the vector field $\beta_* \left(\frac{d}{dt}\right)$ will have nonzero vertical and horizontal components). Then, it exists a unique function $g : [a, b] \to G$ such that $\alpha^{\uparrow}(t) = \beta(t)g(t)$ (because, again, the action of G on P is free). It is worth considering the following factorisation:

$$\begin{array}{ll} [a,b] \xrightarrow{\beta\times g} & P \times G \xrightarrow{\delta} P \\ t & \mapsto (\beta(t), g(t)) \mapsto \beta(t)g(t) \end{array} \tag{B.50}$$

in this way (see [1], page 265, for further specifications), since $\omega\left(\alpha_*\left(\frac{d}{dt}\right)\right) = 0$, we find that

$$0 = \mathsf{Ad}_{g(t)^{-1}}\left(\omega_{\beta(t)}\left(\beta_*\left(\frac{d}{dt}\right)\right)\right) + \Xi_{g(t)}\left(g_*\left(\frac{d}{dt}\right)\right), \tag{B.51}$$

from which it follows that, for a matrix Lie group G,

$$0 = g(t)^{-1}\omega_{\beta(t)}\left(\beta_*\left(\frac{d}{dt}\right)\right)g(t) + g(t)^{-1}\frac{dg}{dt}. \tag{B.52}$$

This is the differential equation that determines the function $g(t)$, which turns the (general) lift β into a horizontal lift; $g(t)$ clearly depends on ω. Sometimes the function $g(t)$ is also called the parallel transport matrix.
Our next goal is to find $g(t)$, namely to resolve the differential equation (3.52). Before doing so, we have to make a choice on the function β, because up to now it is a generic lift of α. A natural choice is given by a local section of P, $\sigma : U \to P$, we recall that σ is needed also to have a local representative of ω, i.e. to deal with a Yang-Mills field. Let then be $\beta(t) := \sigma(\alpha(t))$, from which $\beta_*\left(\frac{d}{dt}\right) = \sigma_*\left(\alpha_*\left(\frac{d}{dt}\right)\right)$, then $\omega_{\beta(t)}\left(\beta_*\left(\frac{d}{dt}\right)\right) = (\sigma^*\omega)_{\alpha(t)}\left(\alpha_*\left(\frac{d}{dt}\right)\right)$ and $\sigma^* = \omega^U$ was named A (the Yang-Mills field).
With this notation eq. (3.52) becomes:

$$0 = \sum_{\mu=1}^{m} g(t)^{-1}A_\mu(\alpha(t))\,g(t)\frac{dx^\mu(\alpha(t))}{dt} + g(t)^{-1}\frac{dg(t)}{dt}, \tag{B.53}$$

where x^μ are local coordinates on $U \subset \mathcal{M}$. Choosing initial conditions on $t \mapsto g(t)$ as $g(a) = g_0 \in G$ we get:

$$g(t) = g_0 - \int_a^t ds A_\mu(\alpha(s))\,\dot{\alpha}^\mu(s)g(s), \tag{B.54}$$

which admits a solution in terms of the path-ordered integral:

$$\begin{aligned} g(t) = \left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu(\alpha(s))\,\dot{\alpha}^\mu(s)\right)g_0 := \Bigg(1 - \int_a^t ds A_\mu(\alpha(s))\,\dot{\alpha}^\mu(s) \\ + \int_a^t ds_1 \int_a^{s_1} ds_2 A_{\mu_1}(\alpha(s))\,A_{\mu_2}(\alpha(s))\,\dot{\alpha}^{\mu_1}(s)\dot{\alpha}^{\mu_2}(s) + \ldots\Bigg)g_0. \end{aligned} \tag{B.55}$$

Finally, we can conclude that the horizontal lift $\alpha^{\uparrow}$ is expressed, locally, in terms of σ by:

$$\alpha^{\uparrow}(t) = \sigma\left(\alpha(t)\right)\left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right) g_0. \tag{B.56}$$

A word about terminology, in LQG the function $g(t)$ is often referred to as the holonomy, though matematically speaking that's an abuse of language, as we shall see later.
Summarizing, we have seen that, in order to move from one fiber to another, we have to lift a curve on the base manifold in an horizontal fashion. The result of this operation depends on the connection one-form (even in the context of Riemannian manifolds the parallel transport depends on the Levi-Civita connection), since we are interested in a local expression we use a local section to lift the curve and to pullback the connection.

It is very important to know how the function $g(t)$ changes if A is subjected to an active gauge transformation, i.e. when $A_\mu(x) \mapsto \Omega(x) A_\mu(x)\Omega(x)^{-1} + \Omega\partial_\mu\Omega(x)^{-1}$. To see this, we shall consider Eq. (B.53) (adopting a compact notation):

$$0 = g^{-1}\Omega A_\mu \Omega^{-1} g\dot{\alpha}^\mu + g^{-1}\Omega\partial_\mu\Omega^{-1} g\dot{\alpha}^\mu + g^{-1}\Omega\Omega^{-1}\frac{dg}{dt}\ , \tag{B.57}$$

now we rewrite

$$\begin{aligned} g^{-1}\Omega\Omega^{-1}\frac{dg}{dt} =& g^{-1}\Omega\frac{d}{dt}\left(\Omega^{-1}g\right) - g^{-1}\Omega\left(\frac{d}{dt}\Omega^{-1}\right)g \\ =& g^{-1}\Omega\frac{d}{dt}\left(\Omega^{-1}g\right) - g^{-1}\Omega\partial_\mu\Omega^{-1}g\dot{\alpha}^\mu, \end{aligned} \tag{B.58}$$

inserting (B.58) back into (B.57) we get:

$$0 = g^{-1}\Omega A_\mu \Omega^{-1} g\dot{\alpha}^\mu + g^{-1}\Omega\frac{d}{dt}\left(\Omega^{-1}g\right), \tag{B.59}$$

denoting with $\tilde{g}(t) = \Omega^{-1}\left(\alpha(t)\right)g(t)$, finally we obtain:

$$0 = \tilde{g}^{-1}(t) A_\mu\left(\alpha(t)\right)\tilde{g}(t)^{-1} + \tilde{g}(t)^{-1}\frac{d\tilde{g}(t)}{dt}\ . \tag{B.60}$$

Eq. (B.60) admits a solution in terms of a path-ordered integral, as seen before,

$$\tilde{g}(t) = \left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right)\tilde{g}(a), \tag{B.61}$$

from which

$$g(t) = \Omega\left(\alpha(t)\right)\left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right)\Omega^{-1}\left(\alpha(a)\right) g_0. \tag{B.62}$$

From this expression it's clear how the path-ordered integral transforms under a gauge transformation:

$$\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s) \longmapsto \Omega\left(\alpha(t)\right)\left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right)\Omega^{-1}\left(\alpha(a)\right)\ , \tag{B.63}$$

it is often said that the path-ordered integral transforms homogeneously.
If G is a matrix group and α is a closed loop, i.e. $\alpha(a) = \alpha(b)$, it is clear that the function

$$W_\alpha[A] := tr\left(\mathcal{P}\mathsf{exp} - \oint_\alpha ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right) \tag{B.64}$$

is gauge invariant, it is called the Wilson loop.
At this point we can give a precise definition of what is meant by parallel transport, intuitively we want to render substantial the concept of a horizontal curve in P, in such a way that a vector field on P would be transported from a fibre to another without being "rotated" along the fibre.
Let $\alpha : [a, b] \to \mathcal{M}$ be a curve in $\mathcal{M}$; the parallel transport along α is the map

$$\tau : \pi^{-1}\left(\{\alpha(a)\}\right) \to \pi^{-1}\left(\{\alpha(b)\}\right), \quad p \mapsto \alpha^{\uparrow}(b), \tag{B.65}$$

where $\alpha^{\uparrow}$ is the unique horizontal lift of α which passes through p when $t = a$.
A special case is when α is a closed curve, i.e a loop, in $\mathcal{M}$. In general the horizontal lift of a loop has not to be closed, therefore we get a non-trivial map from $\pi^{-1}\left(\{\alpha(a)\}\right)$ onto itself given by

$$p \longmapsto p\left(\mathcal{P}\mathsf{exp} - \oint_\alpha ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right) . \tag{B.66}$$

It is clear from (B.66) that we can associate an element of G (which is given by the path-ordered integral) to each loop in $\mathcal{M}$, that is we have a natural map from the loop space of $\mathcal{M}$ into G. The subgroup of G whose elements are obtained in this way is called the *holonomy group* of the bundle at the point $\alpha(0) \in \mathcal{M}$.
We see therefore that the word "holonomy" is referring to loops, while in LQG terminology it is referring to a curve (more precisely to an edge).

Appendix C

Symplectic Geometry

The goal of this section is to introduce the mathematical tools used in Hamiltonian Mechanics and to present the Poisson structure of the theory.

C.1 Symplectic Algebra

The study of Hamiltonian Mechanics is based on the fundamental concept of symplectic manifold. To achieve that, we enlight first what we mean by symplectic structure on a vector space.

A symplectic tensor is an antisymmetric covariant 2-tensor which is non degenerate.

A couple (V, ω) where V is a vector space and $\omega \in \bigwedge_2 V$ is a symplectic tensor is said symplectic vector space.
To give an example let V be a $2n$-dimensional vector space, we denote a basis of V with $\{v_1, w_1, ..., v_n, w_n\}$, whose dual basis of V^* is given by $\{v^1, w^1, ..., v^n, w^n\}$. Let $\omega \in \bigwedge_2 V$ be given by

$$\omega = \sum_{j=1}^{n} v^j \wedge w^j, \tag{C.1}$$

then ω is symplectic, in fact

$$\begin{aligned} \omega\left(v_i, w_j\right) &= -\omega\left(w_j, v_i\right) = \delta_{ij}, \\ \omega\left(v_i, v_j\right) &= -\omega\left(w_i, w_j\right) = 0, \end{aligned} \tag{C.2}$$

for each $1 \leq i, j \leq n$. Then, if we choose a vector $v = \sum_i \left(a^i v_i + b^i w_i\right) \in V$ such that $\omega(v, w) = 0,\ \forall w \in V$ we have that:

$$\begin{aligned} 0 &= \omega(v, v_j) = -b^j, \\ 0 &= \omega(v, w_j) = a^j, \end{aligned} \tag{C.3}$$

for $1 \leq j \leq n$, thus $v = 0$ and ω is non-degenerate.
The example above is very important, because one can show that if (V, ω) is a symplectic vector space then the dimension of V is even and it exists a basis of V with respect to which ω has the form given

by (4.1).
Let's consider now a subspace $W \subset V$, the symplectic complement of W is the subspace

$$W^{\perp} = \{v \in V \mid \omega(v, w) = 0, \ \forall w \in W\}. \tag{C.4}$$

In general, it's not true that $W \cap W^{\perp} = \{0\}$, in fact, if $dimW = 1$ then $W \subseteq W^{\perp}$ because ω is antisymmetric.
Then we have the following classification of subsets of V:

- W is symplectic if $W \cap W^{\perp} = \{0\}$;

- W is isotropic if $W \subseteq W^{\perp}$;

- W is coisotropic if $W^{\perp} \subseteq W$;

- W is Lagrangian if $W = W^{\perp}$.

From these definitions it is not difficult to show the following properties:

(i) $dimW + dimW^{\perp} = dimV$;

(ii) $\left(W^{\perp}\right) = W$;

(iii) W symplectic $\Leftrightarrow \omega|_{W\times W}$ non-degenerate;

(iv) W isotropic $\Leftrightarrow \omega|_{W\times W} = 0$;

(v) W Lagrangian $\Leftrightarrow \omega|_{W\times W} = 0$ and $dimV = 2dimW$.

C.2 Symplectic Manifolds

At this point we can generalise these considerations to the context of differentiable manifolds, where the role of the vector space V is naturally given by $T_x\mathcal{M}$, and the symplectic tensor ω is now expressed in terms of differential forms. More precisely:
a symplectic form on a differentiable manifold $\mathcal{M}$ is a 2-form $\omega \in A^2(\mathcal{M})$ which is closed and non-degenerate, therefore ω_p is a symplectic tensor $\forall p \in \mathcal{M}$.

A symplectic manifold is a pair $(\mathcal{M}, \omega)$ where $\mathcal{M}$ is a differentiable manifold and $\omega \in A^2(\mathcal{M})$ is a symplectic form.
For each point $p \in \mathcal{M}$ we have that $(T_p\mathcal{M}, \omega_p)$ is a symplectic vector space, therfore a symplectic manifold has even dimension ($dimT_p\mathcal{M} = dim\mathcal{M}$).
Not every differentiable manifold admits a symplectic structure, in fact it is possible to show that $H^2\left(\mathcal{M}\right) \neq 0$ if $\mathcal{M}$ is symplectic and compact, this result comes from the fact that starting from ω one can build a volume form on $\mathcal{M}$, which is given by $\Omega_\omega := \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\omega^n$ (where $\omega^n := \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}}$),

then by applying Stoke's theorem one can prove it.

Now we deal with an important type of functions between symplectic manifolds:
a symplectomorphism (or canonical transformation) between symplectic manifolds $(\mathcal{M}, \omega)$ and $\left(\tilde{\mathcal{M}}, \tilde{\omega}\right)$ is a diffeomorphism $F : \mathcal{M} \to \tilde{\mathcal{M}}$ such that $F^*\tilde{\omega} = \omega$.

As one may expect, the types of subspaces of a symplectic vector space admit a generalization to the case of symplectic manifolds, this is done at the level of fibres.
If $(\mathcal{M}, \omega)$ is a symplectic manifold and $\mathcal{N}$ is another differentiable manifold, an immersion $F : \mathcal{M} \to \mathcal{N}$ is called symplectic (isotropic, coisotropic, Lagrangian, respectively) if $dF_p\left(T_p\mathcal{N}\right)$ is a symplectic (isotropic, coisotropic, Lagrangian, respectively) subspace of $T_{F(p)}\mathcal{M}$ for each $p \in \mathcal{N}$.
Therefore, if F is symplectic then $F^*\omega$ is a symplectic form on $\mathcal{N}$, in fact

$$\left(F^*\omega\right)_p(v_1, v_2) = \omega_{F(p)}\left(dF_p(v_1), dF_p(v_2)\right), \ \ \forall v_1, v_2 \in T_p\mathcal{N}. \tag{C.5}$$

We saw at the beginning of the section that it is always possible to find a basis on a symplectic vector space such that (C.1) holds. Thanks to the Darboux theorem it is possible to generalise this result to the context of differentiable manifolds:
let $(\mathcal{M}, \omega_0)$ be a $2n$-dimensional symplectic manifold, then for each $p \in \mathcal{M}$ is possible to find a local chart (V, φ) in p with $\varphi = \left(x^1, y^1, ..., x^n, y^n\right)$ such that

$$\omega_0|_V = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{C.6}$$

The local coordinates $\left(x^1, y^1, ..., x^n, y^n\right)$ are called Darboux coordinates.
An important application frequently used in physics concern the cotangent bundle, in fact, on a cotagent bundle exists a canonical symplectic form.
Let $\mathcal{M}$ be a differentiable manifold and define a canonical one-form $\theta \in A^1\left(T^*\mathcal{M}\right)$ on the cotangent bundle $\pi : T^*\mathcal{M} \to \mathcal{M}$, called the tautologic form, by

$$\theta_\xi = \pi^*\xi \quad \in T^*_\xi\left(T^*\mathcal{M}\right), \ \ \forall \xi \in T^*\mathcal{M}. \tag{C.7}$$

Let's choose a local chart (U, φ) on $\mathcal{M}$, with coordinates $\varphi = \left(x^1, ..., x^n\right)$, if $p \in U$ we can write $\xi_p = \xi_i dx^i|_p$ and a local chart $\left(\pi^{-1}(U), \tilde{\varphi}\right)$ on $T^*\mathcal{M}$ is given by $\tilde{\varphi}\left(\xi_p\right) = \left(x^1, ..., x^n, \xi_1, ..., \xi_n\right)$.
These coordinates induce the local frame on $T\left(T^*\mathcal{M}\right)$ given by $\left\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \xi^1}, ..., \frac{\partial}{\partial \xi^n}\right\}$ and the respective dual frame on $T^*\left(T^*\mathcal{M}\right)$ given by $\left\{dx^1, ..., dx^n, d\xi_1, ..., d\xi_n\right\}$.
In these coordinates the projection $\pi : T^*\mathcal{M} \to \mathcal{M}$ is represented by $\varphi \circ \pi \circ \tilde{\varphi}^{-1}(x, \xi) = x$, thus we have that $d\pi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}$ and $d\pi\left(\frac{\partial}{\partial \xi_i}\right) = 0$
$\forall i = 1, ..., n$, but then $\pi^* dx^j = dx^j$ (with a slightly abuse of notation) and so in local coordinates the tautologic one-form reads:

$$\theta_\xi = \xi_i dx^i|_\xi. \tag{C.8}$$

Starting from (C.8) we can consider the 2-form $\omega \in A^2\left(T^*\mathcal{M}\right)$ defined by:

$$\omega = d\theta. \tag{C.9}$$

This form is closed, being exact, and in local coordinates is given by

$$\omega = d\xi_j \wedge dx^i, \tag{C.10}$$

and so it is clearly non-degenerate (as already seen in the case of symplectic vector spaces), then it defines a symplectic form on $T^*\mathcal{M}$.
A word about notation, usually in physics $\mathcal{M}$ is the configuration space, $T^*\mathcal{M}$ is then the phase space, local coordinates on $T^*\mathcal{M}$ are denoted with $\left(q^i, p_i\right)$ rather than $\left(x^i, \xi_i\right)$.

C.3 Hamiltonian Fields and Poisson structure

We address at this point the topic of Hamiltonian fields and Poisson structure.
We recall a general result:
let V, W be two finite-dimensional (real) vector spaces and $\Phi : V \times W \to \mathbb{R}$ a bilinear form, then Φ is non-degenerate if and only if the linear applications $\Phi^\flat : V \to W^*$ and $\Phi^\sharp : W^* \to V$ are isomorphisms. This means that $\omega^\flat : T\mathcal{M} \to T^*\mathcal{M}$ is the isomorphism induced by ω given by $\omega^\flat(v) = \omega(v, \cdot)$ and $\omega^\sharp = \left(\omega^\flat\right)^{-1}$. A similar result holds in the context of (pseudo-)Riemannian manifolds, where the isomorphisms are induced by the metric.

For each function $f \in C^\infty(\mathcal{M})$ the Hamiltonian vector field associated to f is defined by:

$$X_f := -\omega^\#(df), \tag{C.11}$$

in other terms $X_f \lrcorner \omega = -df$[1], or equivalently $\omega(X_f, Y) = -df(Y) = -Y(f)$ for each $Y \in \mathcal{T}(\mathcal{M})$.
Viceversa, a vector field $X \in \mathcal{T}(\mathcal{M})$ is Hamiltonian if exists a function $f \in C^\infty(\mathcal{M})$ such that $X = X_f$, and is locally Hamoltanian if each $p \in \mathcal{M}$ has a neighbourhood on which X is Hamiltonian.

Furthermore, $X \in \mathcal{T}(\mathcal{M})$ is symplectic if ω is invariant on the flow of X, that is $\mathcal{L}_X \omega = 0$.
Finally, a Hamiltonian system is a triple $(\mathcal{M}, \omega, H)$ where $(\mathcal{M}, \omega)$ is a symplectic manifold and $H \in C^\infty(\mathcal{M})$ is a function called the Hamiltonian of the system.
From the definition of Hamiltonian vector field we deduce that $X_f \lrcorner \omega$ is a closed one-form. From this, and from the fact that $\mathcal{L}_X \omega = d(X \lrcorner \omega) + X \lrcorner (d\omega)$, we see that a vector field is symplectic if and only if is locally Hamiltonian.
Let's see now what a Hamiltonian vector field associated to $f \in C^\infty(\mathcal{M})$ looks like in Darboux coordinates: if (U, φ) is a local chart and $\varphi = \left(q^1, p^1, ..., q^n, p^n\right)$ we have that:

$$\begin{aligned} X_f|_U &= \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right) \\ &= \left(\partial_{p_i} f\right) \partial_{q^i} - \left(\partial_{q^i} f\right) \partial_{p_i} , \end{aligned} \tag{C.12}$$

which follows from the definition of X_f and $\omega|_U = dp_i \wedge dq^i$.

[1] the contraction of $\omega \in A^n(\mathcal{M})$ by a vector field $X \in \mathcal{T}(\mathcal{M})$ is the function $\iota_X : A^n(\mathcal{M}) \to A^{n-1}(\mathcal{M})$ defined by $\iota_X(\omega)(Y_1, Y_2, ..., Y_{k-1}) := \omega(Y_1, Y_2, ..., Y_{k-1}, X)$, we use the notation $X \lrcorner \omega$ in place of $\iota_X(\omega)$.

In the context of symplectic manifolds it is possible to introduce a structure on the space $C^\infty(\mathcal{M})$ that turns it into a Lie algebra, we are talking about Poisson brackets:
let $(\mathcal{M}, \omega)$ be a symplectic manifold, f and $g \in C^\infty(\mathcal{M})$, the Poisson bracket between f and g is the function $\{f, g\} \in C^\infty(\mathcal{M})$ defined by the following equivalent formulas:

$$\{f, g\} = \omega(X_f, X_g) = -df(X_g) = -X_g(f). \tag{C.13}$$

From (4.14) it si possible to show that the Poisson bracket is $\mathbb{R}$-linear, antisymmetric and it satisfies the Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \tag{C.14}$$

furthermore, $X_{\{f,g\}} = [X_f, X_g]$, so there is a homomorphism between the Lie algebra of Hamiltonian vector fields and the Lie algebra of $C^\infty(\mathcal{M})$.
Using Darboux coordinates we have:

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}. \tag{C.15}$$

In particular, then, the Poisson brackets between the coordinate functions (thought as functions on the domain $U \subset \mathcal{M}$ of the local chart) are given by:

$$\{q^i, q^j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, q^j\} = \delta_i^j. \tag{C.16}$$

In addition to this, we can notice that the set of symplectic vector fields is a Lie subalgebra of $\mathcal{T}(\mathcal{M})$, this is a simple consequence of
$\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X \quad \forall X, Y \in \mathcal{T}(\mathcal{M})$; also, thanks to the Poisson brackets we have that $[X_f, X_g] = X_{\{f,g\}}$ and so the set of Hamiltonian vector fields is a Lie subalgebra of the symplectic vector fields.
There's an interesting relation with the first cohomologic group of $\mathcal{M}$, $H^1(\mathcal{M})$, in fact we can summarize what we have found in the following way:

- X symplectic $\Leftrightarrow X \lrcorner \omega$ is closed;

- X Hamiltonian $\Leftrightarrow X \lrcorner \omega$ is exact;

it's clear then that the quotient of symplectic vector fields by the subspace of the Hamiltonian vector fields is isomorphic, as a vector space (not as a Lie algebra), to $H^1(\mathcal{M})$.
Therefore, if $H^1(\mathcal{M}) = 0$ each local Hamiltonian vector field (then symplectic) is globally Hamiltonian.

C.4 Symplectic structure of T^*G

Our next goal is to achieve the symplectic structure of T^*G, since it plays an important role in the phase space of LQG. In order to do so we have to mention a few results concerning the Poisson structure on $L(G)^*$.
We begin with the definition of a Poisson manifold:
a Poisson manifold is a pair $(\mathcal{M}, \{\ ,\ \})$, where $\mathcal{M}$ is a differentiable manifold and $\{\ ,\ \}$ is a Lie algebra structure on $C^\infty(\mathcal{M})$ which satisfies the following Leibniz rule:

$$\{f, gh\} = g\{f, h\} + \{f, g\}h, \quad \forall f, g, h \in C^\infty(\mathcal{M})\,. \tag{C.17}$$

A function between Poisson manifolds $\Phi : \mathcal{M} \to \mathcal{N}$ is called a Poisson function if

$$\Phi^*\{f, g\}_{\mathcal{N}} = \{\Phi^* f, \Phi^* g\}_{\mathcal{M}} \quad \forall f, g \in C^\infty(\mathcal{N})\,. \tag{C.18}$$

We notice that the function $\{f, \cdot\} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is a derivation and so defines a vector field $X_f \in \mathcal{T}(\mathcal{M})$, given by:

$$X_f(g) = \{f, g\}, \tag{C.19}$$

which recalls the Hamiltonian vector field associated to f (if $\mathcal{M}$ is a symplectic manifold).
Furthermore, we have that for each $p \in \mathcal{M}$:

$$\{f, g\}(p) = (X_f(g))(p) = dg_p(X_f) = -df_p(X_g)\,, \tag{C.20}$$

it is then clear that $\{f, g\}(p)$ depends linearly on dg_p and df_p.
Now, every element of $T_p^*\mathcal{M}$ can be written as df_p with $f \in C^\infty(\mathcal{M})$, then it exists a unique tensor field (or bivector) $\Pi \in \mathcal{T}\left(\bigwedge^2 T\mathcal{M}\right)$ such that:

$$\{f, g\} = \Pi(df, dg) \quad \forall f, g \in C^\infty(\mathcal{M}) \tag{C.21}$$

The tensor field $\Pi \in \mathcal{T}\left(\bigwedge^2 T\mathcal{M}\right)$ is called Poisson tensor of $(\mathcal{M}\{\ ,\ \})$, is a sort of analogue of a symplectic form $\omega \in A^2(\mathcal{M})$.
Starting from the definition of Lie derivative of a tensor field it's not diffiuclt to show that $\mathcal{L}_{X_f}\Pi = 0$, thanks to the Jacobi identity of $\{,\}$.
In particular Π defines a (vertical) morphism of vector fibre bundles
$\Pi^\flat : T^*\mathcal{M} \to T\mathcal{M}$ defined by:

$$\left\langle \beta, \Pi^\flat(\alpha) \right\rangle := \Pi(\alpha, \beta) \quad \forall \alpha, \beta \in A^1(\mathcal{M})\,, \tag{C.22}$$

and also $\Pi^\flat \circ df = X_f$, in fact:

$$\langle dg, \Pi^\flat(df) \rangle = \Pi(df, dg) = \{f, g\} = X_f(g). \tag{C.23}$$

In the case where $(\mathcal{M}, \omega)$ is a symplectic manifold we have that $\Pi^\flat = \omega^\sharp$ and there is a one-to-one

corrispondence between symplectic forms and Poisson tensors.
Let's see at this point the expression in local coordinates of the Poisson tensor.
Let (U, φ) be a local chart , $\varphi = \left(x^1, ..., x^n\right)$, we have that:

$$\Pi|_U = \frac{1}{2}\Pi^{ij}\partial_i \wedge \partial_j, \tag{C.24}$$

from (C.22) follows that

$$\{f, g\} = \Pi^{ij}\partial_i f \partial_j g, \tag{C.25}$$

in particular:

$$\Pi^{ij} = \left\{x^i, x^j\right\} \quad \text{and} \quad \{f, g\} = \left\{x^i, x^j\right\}\partial_i f \partial_j g. \tag{C.26}$$

Furthermore, by eq. (C.22), every bivector field Π on $\mathcal{M}$ defines a bilinear antisymmetric map $\{\ ,\ \} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ which satisfies the Leibniz rule. It also satisfies the Jacobi identity, thus yielding a Poisson structure, if and only if

$$\Pi\left(df, d\left(\Pi\left(dg, dh\right)\right)\right) + \Pi\left(dg, d\left(\Pi\left(dh, df\right)\right)\right) + \Pi\left(dh, d\left(\Pi\left(df, dg\right)\right)\right) = 0\ , \tag{C.27}$$

for all $f, g, h \in C^\infty(\mathcal{M})$. In local coordinates it reads

$$\Pi^{il}\partial_l \Pi^{jk} + \Pi^{jl}\partial_l \Pi^{ki} + \Pi^{kl}\partial_l \Pi^{ij} = 0\ . \tag{C.28}$$

In practice, a Poisson manifold can be seen also as a pair $(\mathcal{M}, \Pi)$ where Π is a Poisson tensor on $\mathcal{M}$.
To reach the Poisson structure of $L(G)^*$ we focus on a vector space V in place of a differentiable manifold $\mathcal{M}$.
So, let V be a real vector space, take the (algebrical) dual V^*, we can identify V with TV and, consequently, V^* with TV^*.
Clearly, to each bivector field Π on V^* corresponds a map $V^* \to \bigwedge^2 V^*$ and Π is linear if this map is linear. In this case $\forall v, w \in V$ the function $\xi \mapsto \Pi_\xi(v, w)$ is linear and corresponds to an element of V (because $(V^*)^* \cong V$), it is clear then that this element is associated to the pair (v, w), we can therefore introduce a bilinear function $[\, ,\,] : V \times V \to V$ defined by:

$$\langle \xi\, , [v, w]\rangle = \Pi_\xi(v, w), \quad \forall \xi \in V^*. \tag{C.29}$$

If, in addition to this, Π satisfies equation (C.28) we can see how this condition reflects on the bilinear map $[\ ,\ ]$: the result is that $[\, ,\,]$ has to satisfy the Jacobi identity.
In particular, if $f \in V^*$, we have that $df \in T^*(V^*) = (T(V^*))^* \cong (V^*)^* \cong V$, from which:

$$\Pi_\mu\left(df, dg\right) = \langle \mu, [df(\mu), dg(\mu)]\rangle\, , \quad \forall \mu \in V^*. \tag{C.30}$$

To conclude, Poisson structures on V^* whose Poisson tensor is linear are in one-to-one corrispondence with Lie algebra structures on V.

What we have found turns useful when we analyze the case $V = T_eG \cong L(G)$, thanks to what we have shown is possible to introduce a Lie-Poisson structure on $L(G)^*$, in the following way:
let $\{e^1, ..., e^n\}$ be a basis of $L(G)^*$, the respective basis of $L(G)$ is given by $\{e_1, ..., e_n\}$, in this way we have that:

$$\begin{aligned} &v = v^i e_i, \quad w = w^j e_j, \;\; \xi = \xi_k e^k \\ &[v, w] = [v^i e_i, w^j e_j] = v^i w^j [e_i, e_j] = v^i w^j c^k_{ij} e_k, \end{aligned} \tag{C.31}$$

where c^k_{ij} are the structure constants with respect to the base $\{e_k\}$.
The Poisson structure is given by (C.28), which in coordiantes reads:

$$\Pi\left(e_i, e_j\right) = \left\langle \xi_k e^k, c^l_{ij} \right\rangle = \xi_k c^l_{ij} \delta^k_l = \xi_k c^k_{ij}, \tag{C.32}$$

thus we have the following expressions:

$$\Pi\left(\xi\right) = \frac{1}{2} \xi_k c^k_{ij} \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} \tag{C.33}$$

and

$$\left\{f, g\right\}\left(\xi\right) = \xi_k c^k_{ij} \frac{\partial f}{\partial \xi_i}(\xi) \frac{\partial g}{\partial \xi_j}(\xi). \tag{C.34}$$

We observe now that on $L(G)$ we can introduce an inner product defined by $\langle A, B\rangle := -\mathsf{tr}\left(AB\right)$, in this way it is possible to show that $L(G) \cong L(G)^*$, the isomorphism being given by $i : L(G) \to L(G)^*, \;\; A \mapsto \left(i(A)\right)(B) = \langle A, B\rangle$.
In this sense, we can introduce a Poisson structure on $L(G)$.
Let's consider the case $L(G) = \mathsf{su}(2)$: we choose the basis of $\mathsf{su}(2)$ given by $\{\tau_1, \tau_2, \tau_3\}$, where $\tau_i = -\frac{i}{2}\sigma_i$ (σ_i are the usual Pauli matrices), in this way the structure constants are $c^k_{ij} = \epsilon^k_{ij}$.
Called L^i the coordinates of the vector $\mathbf{L} = L^i \tau_i$, we have that:

$$\begin{aligned} \left\{L^i, L^j\right\}\left(\mathbf{L}\right) &= L^k \epsilon^{mn}_k \frac{\partial L^i}{\partial L^m} \frac{\partial L^j}{\partial L^n} \\ &= L^k \epsilon^{mn}_k \delta^j_m \delta^j_n = \epsilon^{ij}_k L^k. \end{aligned} \tag{C.35}$$

We shall go on to deal with T^*G.
First of all, we show that $TG \cong G \times L(G)$, that is, TG is a trivial vector bundle whose fibre is $L(G)$, in order to do so we prove that the map

$$\begin{aligned} \chi_l \;:\; G \times L(G) &\longrightarrow \;\; TG \\ (a\;,\;X) \;\; &\longmapsto \;\; \chi_l(a, X) := \left(L_a\right)_* X \end{aligned} \tag{C.36}$$

is a vector bundle isomorphism. We recall that $l_a : G \to G$, $b \mapsto ab$ and $\left(l_a\right)_*(b) : T_bG \to T_{ab}G$. We proceed with the proof:
the map χ_l can be factorized as follows

$$
\begin{array}{ccc}
G \times T_eG & \xrightarrow{s_0 \times j} & TG \times TG \\
\big\downarrow \chi_l & & \big\downarrow \wr \\
TG & \xleftarrow{(\mu)_*} & T(G \times G)
\end{array}
\tag{C.37}
$$

where $\mu : G \times G \to G$ is the multiplication in G, $s_0 : G \times TG$ is the null section and $j : T_eG \to TG$ is the natural inclusion.
To see that (4.36) commutes it is sufficient to observe that:

$$
(a, X) \xmapsto{s_0 \times j} ((a, 0), (e, X)) \overset{\sim}{\mapsto} ((a, e), (0, X)) \xmapsto{\mu_*} (a, (l_a)_* X) \tag{C.38}
$$

where the last one is true since $\mu : G \times \{e\} \to G$, $(a, e) \mapsto ae = l_a(e)$, therefore $\mu_*(X)$ is given by $(l_a)_*(X)\ \forall X \in T_eG$.
Since each map is smooth it follows that χ_l is smooth, furthermore we have that

$$
\begin{array}{ccc}
G \times T_eG & \xrightarrow{\chi_l} & TG \\
\big\downarrow \pi_1 & & \big\downarrow \pi \\
G & \xrightarrow{id_G} & G
\end{array}
\tag{C.39}
$$

as one immeditely verifies, thus χ_l is a vertical morphism of vector bundles.
Finally, χ_l is bijective because $(l_a)_*$ is an isomorphism and it is linear since $(l_a)_*$ is linear, therefore χ_l is an isomorphism.
In a similar way it is possible to show that $T^*G \cong G \times L(G)^*$, knowing that $L(G) \cong L(G)^*$ we get the following result: $T^*G \cong G \times L(G)$.
At this point we aim to find the symplectic structure on T^*G.
We work with local coordinates, let $(g, p_g) \in T^*G$ where $p_g = p_\mu dg^\mu$; we denote with $\{e_\alpha\}$ a basis in $L(G)$ and with $\{\varepsilon^\alpha\}$ the respective basis in $L(G)^*$. It is easy to show that l_g defines left-invariant vector fields e^L_α and left-invariant one-forms ε^α_L on G in the following way:

$$
\begin{aligned}
e^L_\alpha(g) &:= l_{g*} e_\alpha \\
\varepsilon^\alpha_L(g) &:= l^*_{g^{-1}} \varepsilon^\alpha.
\end{aligned}
\tag{C.40}
$$

For the moment we denote with:

$$
L^\alpha_\beta(g, h) := \frac{\partial (gh)^\alpha}{\partial g^\beta}, \tag{C.41}
$$

then we can write the field e^L_α as:

$$
e^L_\alpha(g) = L^\mu_\alpha(g, e) \frac{\partial}{\partial g^\mu}, \tag{C.42}
$$

analogously we get;

$$
\varepsilon^\alpha_L(g) = L^\alpha_\mu \left(g^{-1}, g\right) dg^\mu. \tag{C.43}
$$

This basis, as already shown for TG, allows us to introduce a canonical local trivialisation:

$$\begin{array}{rccc} \lambda \ : & T^*G & \longrightarrow & G \times L(G)^* \\ & (g, p_g = p_\mu dg^\mu) & \longmapsto & \left(g, \pi^L = l_g^*|_e p_g = \pi^L_\mu \varepsilon^\mu\right) \end{array} \tag{C.44}$$

where $\pi^L_\mu = p_g\left(e^L_\mu\right) = p_\nu L^\nu_\mu(g,e)$.
A basis in $T^*_x\left(T^*G\right)$, where $x = (g^\alpha, \varepsilon^\alpha_L) \in T^*G$ is given by:

$$\left\{\varepsilon^\alpha_L := L^\alpha_\mu\left(g^{-1}, g\right) dg^\mu, \;\; \varepsilon^L_\mu := d\pi^L_\mu\right\}. \tag{C.45}$$

We know that on T^*G there is a canonical one-form, i.e. the tautological one-form $\theta = p_\alpha dg^\alpha$, from which we obtain the symplectic form

$$\omega = d\theta = dp_\alpha \wedge dg^\alpha. \tag{C.46}$$

In our case we have then $\theta = \pi^L_\mu \varepsilon^\mu_L$ and

$$\omega = \varepsilon^L_\mu \wedge \varepsilon^\mu_L - \frac{1}{2}\pi^L_\mu f^\mu_{\alpha\beta}\varepsilon^\alpha_L \wedge \varepsilon^\beta_L, \tag{C.47}$$

where $f^\mu_{\alpha\beta}$ are the structure constants of the Lie algebra $L(G)$ in the basis $\{e^L_\alpha\}$. We recall also that since ε^μ_L are left-invariant one-forms on a Lie group G, they satisfy the Cartan-Maurer equation $d\varepsilon^\mu_L = -\frac{1}{2}\varepsilon^\alpha_L \wedge \varepsilon^\beta_L$.
The Hamiltonian vector field X_A associated to a function $A \in C^\infty\left(T^*G\right)$ is given by the equation $X_A \lrcorner \omega = -dA$, we can split the components as follows:

$$\begin{aligned} X^\mu_A &:= \varepsilon^\mu_L\left(X_A\right) = -dA\left(e^\mu_L\right) \\ \left(X_A\right)_\alpha &:= \varepsilon^L_\alpha\left(X_A\right) = dA\left(e^L_\alpha\right) + \pi^L_\mu f^\mu_{\alpha\beta} dA\left(e^\beta_L\right). \end{aligned} \tag{C.48}$$

Let X_B be the Hamiltonian vector field associated to $B \in C^\infty\left(T^*G\right)$, the Poisson bracket between A and B is the function given by $\{A, B\} = \omega\left(X_A, X_B\right)$, in coordinates it reads:

$$\{A, B\} = dA\left(e^L_\alpha\right)\frac{\partial B}{\partial \pi^L_\alpha} - \frac{\partial A}{\partial \pi^L_\alpha} dB\left(e^L_\alpha\right) + \frac{\partial A}{\partial \pi^L_\alpha}\pi^L_\mu f^\mu_{\alpha\beta}\frac{\partial B}{\partial \pi^L_\beta}. \tag{C.49}$$

In particular, we obtain that:

$$\begin{aligned} &\{g^\alpha, g^\beta\} = 0\,, \qquad \{g^\alpha, \pi^L_\nu\} = L^\alpha_\nu(g,e) \\ &\text{and} \qquad \{\pi^L_\mu, \pi^L_\nu\} = \pi^L_\alpha f^\alpha_{\mu\nu}. \end{aligned} \tag{C.50}$$

We want now rewrite these relations in a more compact and slightly different form.
From Lie group theory it is known that to each element of the Lie algebra $A \in T_eG$ corresponds a left-invariant vector field $L^A \in L(G)$, clearly we have that $L^A(g) = (l_g)_* A$. If we consider a matrix Lie group we can introduce a coordinate system by defining $x^{ij}(g) := g^{ij}$. In this coordinate system we can write L^A_g as follows:

$$L^A_g = \sum_{i,j=1}^{n} (gA)^{ij}\frac{\partial}{\partial x^{ij}}, \tag{C.51}$$

where $L_g^A x^{ij} = (gA)^{ij}$ are the components of L^A in the chosen coordinates (recall eq. (A.3)).
Identifying $L(G)^*$ with $L(G)$ we can associate to each $\pi^L \in L(G)^*$ an element $\overrightarrow{L} \in L(G)$. Furthermore, the components $L^\alpha_\beta(g,e)$ defined in (C.40) become now $(gA)^{ij}$ as seen before, in particular, we choose $A = \tau^i$, where $\{\tau^i\}$ is a basis of $T_eG \cong L(G)$.
Finally, considering the case $G = SU(2)$, $T_eG = \mathsf{su}(2)$, we obtain a clear expression of the Poisson parentheses in (C.49):

$$\begin{aligned} &\{g^{ij}, g^{kl}\} = 0\ , \qquad \{g^{ij}, L^k\} = \left(g\tau^k\right)^{ij} \\ &\text{and} \qquad \{L^i, L^j\} = L^k \epsilon_k^{ij}. \end{aligned} \tag{C.52}$$

In a more compact way, and taking into account the fact that in LQG to each link of a boundary graph is associated a $SU(2)$ element, we can write an equivalent expression of the latter:

$$\begin{aligned} &\{U_l, U_{l'}\} = 0\ , \qquad \{U_l, L^i_{l'}\} = \delta_{ll'} U_l \tau^i \\ &\text{and} \qquad \{L^i, L^j\} = \delta_{ll'} \epsilon_k^{ij} L^k_l. \end{aligned} \tag{C.53}$$

More precisely, let Γ be the boundary graph coming from the 2-complex associated to Δ^* and let $G = SU(2)$ the gauge group. We denote with Γ^i the i-dimensional element of Γ (nodes, links, triangles, tetrahedra).
The gauge potential corresponds to the pull-back (by a local section) of the principal connection on the principal fibre bundle whose structure group is $G = SU(2)$, it is approximated on the links of Γ by the holonomy:

$$\begin{aligned} \Gamma^1 &\longrightarrow G \\ (x,y) &\longmapsto g_{(x,y)} = \left(\mathcal{P}\mathsf{exp} - \int_a^b ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right), \end{aligned} \tag{C.54}$$

where the curve $\alpha : [a,b] \to \mathcal{M}$ is the link between the spacetime points x and y.
The configuration space and the phase space are given by, respectively:

$$Q = G^L \quad \text{and} \quad M = T^* G^L \tag{C.55}$$

where L is the number of links in Γ.
Using the identifications $T^*G^L \cong (T^*G)^L$ and $T^*G \cong G \times L(G)^* \cong G \times L(G)$, as seen before, we have that the momentum conjugate to the gauge potential is given by the map

$$\begin{aligned} \Gamma^1 &\longrightarrow L(G) \\ (x,y) &\longmapsto A_{(x,y)}, \end{aligned} \tag{C.56}$$

We have seen that local gauge transformations act on the holonomy in the following way:

$$g'_{(x,y)} = h_x g_{(x,y)} h_x^{-1}, \tag{C.57}$$

this expression defines an action of G^N on Q (N is the number of nodes in Γ).

Thanks to the identifications above the lift of this action on $M = T^*Q = G^L \times L(G)^L$ is given by:

$$A'_{(x,y)} = \mathsf{Ad}(h_x)A_{(x,y)}. \tag{C.58}$$

Appendix D

Haar Measure

In this section we briefly introduce the concept of measure with the goal of defining a Haar measure on a Lie group G, specifically we are interested in $G = SU(2)$. The presence of a Haar measure allows us to introduce a scalar product on $L_2\left(SU(2)\right)$ and, consequently, on $L_2\left(SU(2)^L\right)$, which is the starting point to construct the Hilbert space of LQG.

D.1 Positive Measure

Therefore, we begin by giving the definition of a σ-algebra: let X be a set and $\mathfrak{M}$ a collection of subsets of X, $\mathfrak{M}$ is a σ-algebra if

$$\begin{aligned} &(i)\ \ X \in \mathfrak{M} \\ &(ii)\ \ A \in \mathfrak{M} \Rightarrow A^c \in \mathfrak{M} \\ &(iii)\ \ A = \bigcup_{n=1}^{\infty} A_n, \ \ A_n \in \mathfrak{M}\ \forall n \Rightarrow A \in \mathfrak{M}, \end{aligned} \tag{D.1}$$

if it is so, X is said measurable space and the elements of $\mathfrak{M}$ are called measurable sets.
In the case where X is a topological space the following result holds: it exists a minimal σ-algebra $\mathfrak{B}$ in X such that each open set in X belongs to $\mathfrak{B}$, the elements of $\mathfrak{B}$ are called Borel sets of X. This fact is true since there is a theorem which states that for each collection of subsets of X is always possible to find a σ-algebra that contains it, then, in particular, if (X, τ) is a topological space, τ is a topology on X and so a collection of open subsets of X.
We move on now and give the definition of measure: a (positive) measure is a function μ defined on a σ-algebra $\mathfrak{M}$,

$$\mu : \mathfrak{M} \to [0, \infty] \tag{D.2}$$

which is additive measurable, i.e. if $\{A_i\}$ is a countable family of disjoint sets of $\mathfrak{M}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=0}^{\infty} \mu\left(A_i\right). \tag{D.3}$$

D.2 Left-Invariant Measure on a group

We consider a compact Lie group G, since we are interested in the case $G = SU(2)$, which is compact. We recall that if G is a Lie group then the multiplication is a smooth function, that is

$$\begin{aligned} m: \; & G \times G \to G \\ & (g,h) \mapsto gh \end{aligned} \tag{D.4}$$

is smooth, which means that the left and right multiplication are smooth too:

$$\begin{aligned} l_g: \; & G \to G \\ & h \mapsto gh \qquad \forall g,h \in G, \end{aligned} \tag{D.5}$$

$$\begin{aligned} r_g: \; & G \to G \\ & h \mapsto hg \qquad \forall g,h \in G. \end{aligned} \tag{D.6}$$

In addition to this l_g and r_g are invertible with smooth inverse, that is they are diffeomorphisms $\forall g \in G$.
At this point we can tell what we mean by a Haar measure on a group G: a left Haar measure on a group G is a positive measure μ_H^l on the Borel σ-algebra in G with the following properties:

(i) is locally finite, that is each point in G possesses a neighbourhood with finite measure;

(ii) is left-invariant, i.e.:

$$\mu\,(gE) = \mu(E) \tag{D.7}$$

$\forall g \in G$ and for each Borel set $E \subset G$, where $gH = \{gh | h \in H\}$, $H \subset G$.
Now that we have defined what a Haar measure is, let's see how we can contruct one on a Lie group G, obviously differential forms will be involved, since they play a crucial role in integration theory on a differentaible manifold.

We have already seen that a left-invariant one-form on G has to satisfy the Cartan-Maurer equation:

$$d\omega^\alpha + \frac{1}{2}\sum_{\beta,\gamma} C^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma = 0, \;\; \forall \alpha = 1, ..., n = dimG \tag{D.8}$$

where $\{\omega^1, \omega^2, ..., \omega^n\}$ is a basis of $L^*(G)$, dual of the basis $\{E_1, E_2, ..., E_n\}$ of $L(G) \cong T_eG$, such that $\omega^\alpha(E_\beta) = \delta^\alpha_\beta$.
We can define in a natural way a n-form on G by means of

$$\eta := f\,\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n, \quad f \in C^\infty(G). \tag{D.9}$$

We want to prove that $\eta \in A^n(G)$ is a volume form on G, that is, a nowhere vanishing n-form on G, and also that η is left-invariant, i.e.:

$$l_g^* \eta = \eta, \quad \forall g \in G. \tag{D.10}$$

We begin with the latter:

$$\begin{aligned}\left(l_g^* \eta\right)_h = l_g^* \left(f\, \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\right)_h = f(gh)\ \left(l_g^* \omega^1\right)_h \wedge \left(l_g^* \omega^2\right)_h \wedge \cdots \wedge \left(l_g^* \omega^n\right)_h = \\ = f(gh)\, \omega^1_{gh} \wedge \omega^2_{gh} \wedge \cdots \wedge \omega^n_{gh} = \eta_{gh}\end{aligned} \tag{D.11}$$

which is true since ω^α are left-inariant $\forall \alpha = 1, ..., n$.
Now, let $v_1, v_2, ..., v_n \in T_g G$ linearly independent, we know that we can express them as $v_i = l_{g_*} A_i$, for a certain $A_i \in T_e G$, therefore:

$$\begin{aligned}\eta\left(v_1, v_2, ..., v_n\right) &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \left(v_1, v_2, ..., v_n\right) \\ &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \left(l_{g_*} A_1, l_{g_*} A_2, ..., l_{g_*} A_n\right) \\ &= l_g^* \left(\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\right) \left(A_1, A_2, ..., A_n\right) \\ &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \left(A_1, A_2, ..., A_n\right) = \mathsf{det} A,\end{aligned} \tag{D.12}$$

where A is the matrix of the basis change from $\{E_1, E_2, ..., E_n\}$ to $\{A_1, A_2, ..., A_n\}$. Since $\mathsf{det} A \neq 0$, because A is invertible, we have that η is nowhere vanishing on G, thus η is a volume form on G.
Integrating functions against this form we obtain a Haar measure.
We are also interested under which circumstances a Haar measure is right-invariant, to see this we proceed as follows:
let μ be a Haar measure on G, we define a new measure $r_g(\mu)$ by
$r_g(\mu)(E) := \mu\left(r_g(E)\right)$, $\forall g \in G$, $E \subset G$ and r_g is the right multiplication by g. It is not difficult to show that $r_g(\mu)$ is left-invariant, in fact:

$$\begin{aligned}r_g(\mu)\left(l_h E\right) &= \mu\left(r_g l_h E\right) = \mu\left(l_h r_g E\right) \\ &= \mu\left(r_g E\right) = r_g(\mu)\left(E\right),\end{aligned} \tag{D.13}$$

since l_h and r_g commute $\forall g, h \in G$, and so $r_g(\mu)$ is given by a left-invariant n-form. However the n-form which describes $r_g(\mu)$ could be different from the n-form that describes μ, $r_g(\mu)$ and μ could clearly differ by a multiplicative constant.
Therefore, $\forall g \in G$, it exists a constant $\chi(g)$ such that $r_g(\mu) = \chi(g)\mu$, the function $\chi : G \to \mathbb{R}$ is said modular function of G.
A group G is called unimodular if $\chi(g) = 1$, $\forall g \in G$. It follows that if G is unimodular we have that $r_g(\mu) = \mu$ and so μ is right-invariant, too.

Using the description of a Haar measure in terms of differential forms it is possible to show that if G is a connected Lie group then G is unimodular if and only if $\mathsf{det}\mathrm{Ad}_g = 1$ $\forall g \in G$ or, equivalently, if and only if $\mathrm{tr}\left(\mathrm{ad}_X\right) = 0, \forall X \in T_e G \cong L(G)$.
To prove this result we begin by showing the equivalence mentioned above: since $e^{\mathrm{tr}(\mathrm{ad}_X)} = \mathsf{det} e^{\mathrm{ad}_X} = \mathsf{det}\mathrm{Ad}_{e^X}$, then $\mathrm{tr}(\mathrm{ad}_X) = 0$ if and only if $\mathsf{det}\mathrm{Ad}_{e^X} = 1$, $\forall X \in T_e G$. Furthermore, since G is connected we have that $g = e^{X_1} e^{X_2} \cdots e^{X_m}$ for a certain number of $X_i \in T_e G$, thus

$$
\begin{aligned}
\mathsf{det}\mathrm{Ad}_g &= \mathsf{det}\left(\mathrm{Ad}_{e^{X_1}e^{X_2}\cdots e^{X_m}}\right) \\
&= \mathsf{det}\left(\mathrm{Ad}_{e^{X_1}}\mathrm{Ad}_{e^{X_2}}\cdots\mathrm{Ad}_{e^{X_m}}\right) \\
&= \mathsf{det}\left(\mathrm{Ad}_{e^{X_1}}\right)\mathsf{det}\left(\mathrm{Ad}_{e^{X_2}}\right)\cdots\mathsf{det}\left(\mathrm{Ad}_{e^{X_m}}\right) = 1,
\end{aligned}
\tag{D.14}
$$

where the second equality comes from the fact that Ad is a (linear) representation of G.
To prove the proposition we make the following observation:
let $C_g : G \to G$, $h \mapsto ghg^{-1}$ be the conjugate action, recalling that $\mathrm{Ad}_g : T_eG \to T_eG$ is defined by $\mathrm{Ad}_g := (C_g)_*$, we introduce the measure $C_g\mu$ defined by $(C_g\mu)(E) := \mu(C_gE)$ and we show that it is left-invariant:

$$
\begin{aligned}
(C_g\mu)(E) &= \mu(C_gE) = \mu\left(l_g \circ r_{g^{-1}}E\right) \\
&= \mu\left(r_{g^{-1}}E\right) = \left(r_{g^{-1}}\mu\right)(E).
\end{aligned}
\tag{D.15}
$$

Therefore, since $r_g\mu$ is left-invariant $\forall g \in G$, $C_g\mu$ is left-invariant, too.
In addition to this, μ is C_g-invariant if and only if μ is right-invariant, as it is evident from (D.15).
Let's see at this point how we can express in terms of differential forms the requirement of C_g invariance.
Since $C_g\mu$ is left-invariant it is sufficient to prove it for $e \in G$, doing so we obtain:

$$
\begin{aligned}
\left(C_g^*\mu\right)_e(E_1, E_2, ..., E_n) &= \mu_e\left(C_{g_*}E_1, C_{g_*}E_2, ..., C_{g_*}E_n\right) \\
&= \mu_e\left(\mathrm{Ad}_gE_1, \mathrm{Ad}_gE_2, ..., \mathrm{Ad}_gE_n\right) \\
&= f(e)\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\left(\mathrm{Ad}_gE_1, \mathrm{Ad}_gE_2, ..., \mathrm{Ad}_gE_n\right) \\
&= f(e)\mathsf{det}\mathrm{Ad}_g = \mathsf{det}\mathrm{Ad}_g\ \mu(E_1, E_2, ..., E_n),
\end{aligned}
\tag{D.16}
$$

it is then clear that $C_g^*\mu = \mu$ if and only if $\mathsf{det}\mathrm{Ad}_g = 1$. The proof is ended.
In particular, we have that compact groups are unimodular, because if G is compact it exists an inner product with respect to which ad_X is antisymmetric and so ad_X is traceless. The reason for this is that if G is compact it is always possible to find a basis $\{E_1, E_2, ..., E_n\}$ in $L(G)$ such that:

$$
\mathrm{tr}\left(E_\alpha E_\beta\right) = -c\,\delta_{\alpha\beta}, \quad \text{with } c > 0,
\tag{D.17}
$$

by the way, the fact that c is positive implies that, in gauge theories, the kinetic term in the Lagrangian is positive.
An inner product in $L(G)$ is then given by:

$$
\begin{aligned}
\tilde{\mathrm{tr}}\ : L(G) \times L(G) &\longrightarrow \quad \mathbb{R} \\
(A\ , B) \quad &\longmapsto \quad \tilde{\mathrm{tr}}(A, B) := \mathrm{tr}(AB)\ ,
\end{aligned}
\tag{D.18}
$$

one can easily verify that $\tilde{\mathrm{tr}}$ is real, symmetric and independent from the choice of a basis in $L(G)$.
We observe that

$$
\begin{aligned}
\mathrm{tr}\left(\mathsf{ad}_{E_\alpha}(E_\beta)\,E_\gamma\right) &= \mathrm{tr}\left(C^\delta_{\alpha\beta}E_\delta E_\gamma\right) = C^\delta_{\alpha\beta}\mathrm{tr}\left(E_\delta E_\gamma\right) \\
&= -c\,C^\delta_{\alpha\beta}\delta_{\delta\gamma} \equiv -c\,C_{\alpha\beta\gamma}.
\end{aligned}
\tag{D.19}
$$

We know that $C_{\alpha\beta\gamma}$ is antisymmetric in the first two indices, however it is possible to show that it is also antisymmetric in the last two, that is, $C_{\alpha\beta\gamma}$ is completely antisymmetric, in fact:

$$
\begin{aligned}
-c\, C_{\alpha\beta\gamma} = \operatorname{tr}\left([E_\alpha, E_\beta]\, E_\gamma\right) &= \operatorname{tr}\left(E_\alpha E_\beta E_\gamma - E_\beta E_\alpha E_\gamma\right) \\
&= \operatorname{tr}\left(E_\beta E_\gamma E_\alpha - E_\beta E_\alpha E_\gamma\right) \\
&= \operatorname{tr}\left(E_\beta\, [E_\gamma, E_\alpha]\right) \\
&= -c\, C_{\beta\gamma\alpha},
\end{aligned} \tag{D.20}
$$

which implies $C_{\beta\gamma\alpha} = C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma}$.
The fact that $C_{\alpha\beta\gamma}$ is completely antisymmetric shows that ad_X is antisymmetric with respect to the inner product given by $\tilde{\mathrm{tr}}$.
In the case of interest, $G = SU(2)$ is compact and then unimodular, i.e. the Haar measure seen before is left-invariant and right-invariant, specifically, it is gauge invariant for a gauge transformation taking place on a node of the 2-complex.

Appendix E

Hilbert Space

From the previous discussion on the Poisson structure of T^*G we deduce that is possible, by means of canonical quantization, to promote the Poisson brackets to a commutator and to assign to each U_l and L_l the role of operators acting on a Hilbert space, with the due specifications.
We have seen that, in a compact form, U_l represents coordinates in G and L_l is its conjugate momentum. This fact suggests to deal with the following Hilbert space:

$$\mathcal{H}_\Gamma = L_2\left[SU(2)^L\right], \tag{E.1}$$

where L is the number of links in the boundary graph Γ.
The states are then the wave functions $\psi\left(U_l\right)$ of L group elements U_l.
On $\mathcal{H}_\Gamma$ is defined a scalar product compatible with the Haar measure:

$$\langle\psi|\phi\rangle = \int_{SU(2)^L} dU_l \overline{\psi\left(U_l\right)}\phi\left(U_l\right). \tag{E.2}$$

For what concerns the operators U_l and L_l the following results hold: the operator U_l is simply defined by

$$\left(U_l\psi\right)\left(U_l'\right) := \psi\left(U_l U_l'\right), \tag{E.3}$$

it acts then as a multiplicative operator.
Recalling now that L_l^i is a left-invariant vector field on G we can prove that it coincides with the vector field on G induced by the right multiplication $\delta_g : G \to G$, $\delta_g(h) = hg$. In fact, taking into account the results found previously on induced vector fields (see eq. (B.7)) we have that:

$$X_g^A = l_{g*}\left(A\right) = L_g^A, \tag{E.4}$$

in particular, $\forall f \in C^\infty\left(G\right)$,

$$L_g^A(f) = \frac{d}{dt} f\left(g \mathsf{exp} t A\right)|_{t=0} \tag{E.5}$$

since $\delta_{\mathsf{exp}(tA)}$ is the flow of the induced vector field X^A which coincides with L^A. In this manner it's easy to see that the field L_l^i acts on the wave functions in the following way [1]

[1] the factor $-i$ comes from the commutation relations $[\tau_i\,,\tau_j] = i\epsilon_{ij}^k\tau_k$

$$\left(J^{i}\psi\right)(U) := -i\frac{d}{dt}\psi\left(Ue^{t\tau_i}\right)|_{t=0}, \tag{E.6}$$

where we have used the exponential of a matrix since $SU(2)$ is a matrix group. If we finally insert the multiplicative constants we can write L_l^i as follows:

$$L_l^i := (8\pi\hbar G)\, J_l^i. \tag{E.7}$$

Having realised the canonical quantisation we want to know, at this point, how to build gauge-invariant wave function. This requirement is necessary because in LQG physical states contain geometrical informations, for instance, in 3D we know that at each node of the dual triangulation are located tetrahedra, whose properties (volume, surfaces of its faces,...) are invariant under rotations. We have seen how the holonomy transforms under an active gauge transformation:

$$U_l \mapsto \Lambda_{s_l} U_l \Lambda_{t_l}^{-1} \tag{E.8}$$

where s_l and t_l specify the points in spacetime that bound the chosen link. Gauge-invariant states under these transformations must then satisfy

$$\psi\left(\Lambda_{s_l} U_l \Lambda_{t_l}^{-1}\right) = \psi\left(U_l\right), \quad \text{with} \quad \Lambda_{\mathrm{n}} \in SU(2). \tag{E.9}$$

If we consider a wave function on the (2-dimensional) bounding graph then a gauge transformation acts on every node of that graph, that is, for each node we have three gauge transformations. For rhis reason we shall introduce an operator $C_{\mathrm{n}}^i := L_{l_1}^i + L_{l_2}^i + L_{l_3}^i$ which has to satisfy the following property;

$$C_{\mathrm{n}}^i \psi = 0. \tag{E.10}$$

More precisley, the condition of gauge invariance of ψ can be rewritten following this reasoning: say we choose a node **n** which we consider a target of three links, in this way a gauge transformation will produce the same element Λ_{t_l} for each of the three links. Now, in general, we know that (dropping the constants):

$$\left(L_{l_j}^i \psi\right)\left(U_{l_j}\right) := -i\frac{d}{dt}\psi\left(U_{l_j} e^{t\tau_i}\right)|_{t=0}, \tag{E.11}$$

with $j = 1, 2, 3$ indexing the three links convergent on **n**. Since we are focusing on the node **n** we can keep the gauge transformations on the other nodes as generic, that is $\Lambda_{s_{l_j}}$ generic. Then, it follows that

$$\begin{aligned} \left(L_{l_j}^i \psi\right)\left(\Lambda_{s_{l_j}} U_{l_j}\right) &= -i\frac{d}{dt}\psi\left(\Lambda_{s_{l_j}} U_{l_j} e^{t\tau_i}\right)|_{t=0} \\ &= i\frac{d}{dt}\psi\left(\Lambda_{s_{l_j}} U_{l_j} e^{-t\tau_i}\right)|_{t=0}. \end{aligned} \tag{E.12}$$

To conclude, since $SU(2)$ is compact and simply connected, every element of $SU(2)$ can be written as the exponential of an element of $\mathsf{su}(2)$, we understand that the condition (6.9) is translated into the following requirement:

$$L_{l_1}^i + L_{l_2}^i + L_{l_3}^i = 0, \quad i = 1, 2, 3. \tag{E.13}$$

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Covariant Loop Quantum Gravity, An Introduction and some Mathematical Tools

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Abstract

The aim of this thesis is to give a presentation of Loop Quantum Gravity in its covariant form, also known as spinfoam approach, and to present the basic mathematical tools to access the theory.

Contents

Chapter 1

Introduction

1.1 Historical Overview

Quantum gravity is the search for a theory that aims to merge two well established theories of the twenthieth century: General Relativity and Quantum Mechanics. The need for such a theory came clear early, in lue of the fact that the gravitational field, being a field, was expected to be quantized. Along the years, three main lines of research have been established:

- **The covariant line of research** is the attempt to build the theory as a quantum field theory of the fluctuations of the metric over a flat Minkowski space, this approach eventually led to string theory in the late eighties.

- **The canonical line of research** is the attempt to construct a quantum theory in which the Hilbert space carries a representation of the operators corresponding to the full metric, or some functions of the metric, without background metric to be fixed. The formal equations of the quantum theory were then written down by Wheeler and DeWitt in the middle sixties, but turned out to be too ill-defined. A well defined version of the same equations was successfully found only in the late eighties, with loop quantum gravity.

- **The sum over historiers line of research** is the attempt to use some version of Feynman's functional integral quantization to define the theory, leading eventually to the spin foam approach.

In this introducion we focus solely on the last two approaches and we try to sketch the evoultion of loop quantum gravity.
In the early thirties attempts are made in order to apply the quantization method of gauge theories to the linearized Einstein field equations. Later in the decade, Bronstein realizes that field quantization techniques must be generalized in such a way as to be applicable in the absence of a background geometry, in sharp contrast to the approach used in quantum electrodynamics [1].
At the beginning of the fifties starts the development of the "flat space quantization" of the gravitational field. The idea is to quantize the small fluctations around the Minkowski metric, that is, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. This idea represents the birth of the covariant approach.
On the other hand, Bergmann starts its program of phase space quantization of non linear field theories and problems raised by systems with constraints are studied too [2]. The canonical approach to quantum gravity is born.

Later in the decade, Charles Misner introduces the "Feynman quantization of general relativity" $Z = \int \exp(iS[g])dg$ [3], and so the three lines of research are established.
In 1959 Dirac has completely unraveled the canonical structure of GR [4], and two years later, Arnowit, Deser and Misner complete the so-called ADM formulation of GR [5], namely its hamiltonian version in appropriate variables, which greatly simplify the hamiltonian formulation and make its geometrical reading transparent. Following the ADM methods, in 1962 Peres writes the Hamilton-Jacobi formulation of GR [6]:

$$G^2(q_{ab}q_{cd} - \frac{1}{2}q_{ac}q_{bd})\frac{\delta S(q)}{\delta q_{ac}}\frac{\delta S(q)}{\delta q_{bd}} + \det(q)R[q] = 0\ ,$$

which will lead to the Wheeler-DeWitt equation.
In 1967 Bryce DeWitt publishes the "Einstein-Schrödinger equation" [7]:

$$\left((\hbar G)^2(q_{ab}q_{cd} - \frac{1}{2}q_{ac}q_{bd})\frac{\delta}{\delta q_{ac}}\frac{\delta}{\delta q_{bd}} + \det(q)R[q]\right)\Psi(q) = 0\ ,$$

which is known as the "Wheeler-DeWitt equation". This equation comes with the so-called "problem of time" in quantum gravity, because the time variable disappears. To be fair, the time coordinate already disappears in the classical Hamilton-Jacobi form of GR, thus the fact that physical obsservables are coordinate independent is a genuine feature of any formulation of GR. But in the quantum context there is no single spacetime, as there is no trajectory for a quantum particle, and the very concepts of space and time become fuzzy.
In the seventies, Hawking announces the derivation of black hole radiation [8] and he states that a Schwarzschild black hole of mass M emits thermal radiation at the temperature

$$T = \frac{\hbar c^3}{8\pi k G M}\ ,$$

opening a new field of research in "black hole thermodynamics" and leading to the understanding of the statistical origin of the black hole entropy, which, for a Schwarzschild black hole, reads

$$S_{BH} = \frac{1}{4}\frac{c^3}{\hbar G}A\,,$$

where A is the area of the black hole surface. Later in the decade, the Hawking radiation is rederived in a number of ways, strongly reinforcing its credibility.
In 1986 the connection formulation of GR is developed by Abhay Ashtekar [9](as opposed to the metric formulation), semplifying the canonical analysis in the sense that the constraints take a simpler form. Furthermore, the theory now takes the form of a $SU(2)$-theory, since the structure constants associated to the Poisson structure of the Ashtekar variables coincide with the structure constants of the su(2) algebra. In addition to this, there is a geometric interpretation of the "Ashtekar electric field", namely, the field conjugate to the Ashtekar connection, in terms of area elements.
In 1988, Ted Jacobson and Lee Smolin find loop-like solutions to the Wheeler-DeWitt equation formulated in the connection formulation [10], that is, they present a large class of exact solutions to the hamiltonian constraint written in terms of Wilson loops. Based on these results, and on knot theory, the canonical approach gets new blood, and "loop quantum gravity" gets started. Let's birefly summarize this important step and explain where the word "loop" in LQG comes from. The Jacobson-Smolin solutions are not physical states of quantum gravity, since they fail to satisfy the second equation of canonical quantum gravity (the first being the Wheeler-DeWitt equation), which demands states to be invariant under 3d diffeomorphisms. Then, soon afterwards, Smolin starts to think that since loops

up to diffeomorphisms mean knots, knots could play a role in quantum gravity. The solutions are written in terms of Wilson loops

$$\psi(\gamma) = \mathrm{Tr}\left[\mathcal{P}e^{\int_\gamma A}\right]\psi(A)dA\,, \tag{1.1}$$

moving from the connection representation to the loop representation means considering the "loop transform"

$$\psi_\gamma[A] = \mathrm{Tr}\left[\mathcal{P}e^{\int_\gamma A}\right]\,, \tag{1.2}$$

where dA is a diffeomorphism-invariant measure on the space of connections (constructed by Ashtekar and Lewandowski). This transform maps the $\psi(A)$ representation of quantum states in A space to the representation $\psi(\gamma)$ of quantum states in γ space, that is, in loop space. The advantages of moving to the loop basis are: $\psi(\gamma)$ depending only on the knot class of the loop γ solve the diffeomorphism constraint, that is, there is one independent solution for each knot; all such states where the loop does not self-intersect are exact solutions of all equations of quantum gravity (the partial result obtained by Smolin was that not self-intersecting loops gave rise to solution of the Hamiltonin constraint only). Later on, Jorge Pullin realizes that all solutions without nodes (intersections between loops) correspond to 3-geometries with zero volume, meaning therefore that nodes are essential to describe physical quantum geometry [11]. In 1995 the spin network orthonormal basis on the Hilbert space of loop quantum gravity is found [12], and a main main physical result is obtained: the computation of the eigenvalues of area and volume.
In 1996 the Bekenstein-Hawking black hole entropy is computed within loop quantum gravity [13], as well as within string theory.

1.1.1 The Problems Addressed

There are three major theoretical and conceptual problems that the theory addresses:

- **Quantum geometry**: What is a physical "quantum space"? That is, what is the mathematics that describes the quantum spacetime metric? LQG predicts that any measured physical area must turn out to be quantized and given by the spectrum (4.58).
- **Ultraviolet divergences of quantum field theory**: This is a major open problem in non-gravitational contexts. But it is a problem physically related to quantum gravity because the ultraviolet divergences appear in the calculations as effects of ultra-short trans-Planckian modes of the field. If physical space has a quantum discreteness at small scale, these divergences should disappear. In LQG the ultraviolet divergences are not present since there is a natural cut-off due to the discretized spectrum of the area, nevertheless infrared divergences could possibly arise by considering greater values of the spins, these are called "spikes". Interestingly, when one considers the theory with the presence of a cosmological constant, it can be shown that this provides an upper limit for the greatest value of a spin, thus resolving the problem of infrared divergences.
- **General covariant quantum field theory**: Loop gravity "takes seriously" general relativity, and explores the possibility that the symmetry on which general relativity is based (general

covariance) holds beyond the classical domain. Since standard quantum field theory is defined on a metric manifold, this means that the problem is to find a radical generalization of quantum field theory, consistent with full general covariance and with the physical absence of a background metric structure. In other words, loop gravity, before being a quantum theory of general relativity, is the attempt to define a general covariant quantum field theory.

1.1.2 Open Problems

- **Consistency**: With the cosmological constant, the transition amplitudes are finite at all orders and the classical limit of each converges to the truncation of classical limit of GR on a finite discretization of spacetime; in turn, these converge to classical GR when the discretization is refined. This gives a coherent approximation scheme. However the approximation scheme may go wrong if the quantum part of the corrections that one obtains refining the discretization is large. These can be called "radiative corrections", since they are somewhat similar to standard QFT radiative corrections: possibly large quantum effects effects that appear taking the next order in the approximation. It is not sufficient for these radiative corrections to be finite, for the approximation to be viable, they must also be small. Since the theory includes a large number, the ratio of the cosmological constant scale over the Planck scale (or over the observation scale), these radiative corrections a priori could be large.

- **Completeness**: The matter sector of the theory has not been sufficiently developed. In addition to this, the q-deformed version of the theory, that is, based on the quantum group $SU(2)_q$ (a one-parameter deformation of the representations of $SU(2)$), is very little developed. This version is utilized in order to introduce the cosmological constant but it's not clear if one can obtain the results of the $\Lambda = 0$ theory.

Chapter 2

Classical General Relativity

This chapter is devoted to a formulation of classical General Relativity more suitable for the discretization and the consequent quantization.

2.1 Tetrad-Connection formalism

As already carefully explained in the Appendix, which we refer to, it is possible to express the Einstein-Hilbert action in terms of the (co)tetrads and a Lorentz connection. Briefly we recall the main formulas. Tetrads are such that:

$$g_{\mu\nu}(x) = e^I_\mu(x) e^J_\nu(x) \eta_{IJ} \tag{2.1}$$

The metric is not affected if the tetrads undergo a local $SO(3,1)$ transformation; the Lorentz connection associated to this gauge invariance is a one-form with values in the Lie algebra $\mathrm{sl}(2,\mathbb{C})$, therefore it is antisymmetric:

$$\omega^{IJ}_\mu = -\omega^{JI}_\mu \tag{2.2}$$

The curvature of the connection is given by:

$$F^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J \ , \tag{2.3}$$

if the connection is torsionless then it can be shown to be unique, namely, the spin connection, or the Levi-Civita connection. In terms of these objects the Einstein-Hilbert action reads:

$$S[e] = \frac{1}{2} \int e^I \wedge e^J \wedge F^{KL} \epsilon_{IJKL}. \tag{2.4}$$

In order to rewrite the action in a more succint form we introduce the Hodge dual in the Minkowski space, that is, $F^\star_{IJ} := \star F_{IJ} := \frac{1}{2}\epsilon_{IJKL} F^{KL}$. Furthermore the 2-form $\Sigma^{IJ} := e^I \wedge e^J$ is called the Plebanski 2-form, and, suppressing contracted indices we get:

$$S[e] = \int e \wedge e \wedge F^\star \, . \tag{2.5}$$

We point out a difference between the Einstein-Hilbert action written in terms of the metric and the action written in terms of tetrads. In fact, if we write both in terms of tetrads we see that:

$$
\begin{aligned}
S_{EH}[e] &= \frac{1}{2}\int |\mathrm{det}e| R[e] d^4x \,, \\
S_T[e] &= \frac{1}{2}\int (\mathrm{det}e) R[e] d^4x \,.
\end{aligned}
\tag{2.6}
$$

The difference then amounts by a sign factor $s := \mathrm{sgn}(\mathrm{det}e)$. Therefore, when moving to the quantum case, where one takes the path-integral over tetrads, this sign translates to two different terms, namely:

$$
e^{-\frac{i}{\hbar}S_{EH}[g]} \qquad \text{and} \qquad e^{+\frac{i}{\hbar}S_{EH}[g]} \,. \tag{2.7}
$$

These two terms will reappear when dealing with the classical limit.
We can regard (2.5) also as a function of a tetrad and a Lorentz connection as independent fields, namely:

$$
S[e,\omega] = \int e \wedge e \wedge F[\omega]^{\star} \,, \tag{2.8}
$$

performing the variation with respect to the connection gives the torsionless condition and the variation with respect to the tetrad yelds Einstein equations. This polynomial action is referred to as the "tetrad-Palatini" action. It is possible to add another term respecting the given symmetries, this term has the form $\int e \wedge e \wedge F := \int e_I \wedge e_J \wedge F^{IJ}$. If we add this term with a coupling constant $1/\gamma$ (γ is known as the "Barbero-Immirzi constant") we get the following action:

$$
S[e,\omega] = \int e \wedge e \wedge F^{\star} + \frac{1}{\gamma}\int e \wedge e \wedge F \,. \tag{2.9}
$$

It can be shown that the second term has no effect on the equation of motion, because it vanishes when the connection is torsionless. Seeking a more compact form we observe that:

$$
\begin{aligned}
S[e,\omega] &= \int e \wedge e \wedge \left(F^{\star} + \frac{1}{\gamma}F\right) \\
&= \int e \wedge e \wedge \left(\star + \frac{1}{\gamma}\right) F \\
&= \int \left(\star\,(e \wedge e) + \frac{1}{\gamma} e \wedge e\right) \wedge F \,,
\end{aligned}
\tag{2.10}
$$

renaming the term in parentheses by $B := \left(\star\,(e \wedge e) + \frac{1}{\gamma} e \wedge e\right)$ we finally get:

$$
S[e,\omega] = \int B \wedge F \,. \tag{2.11}
$$

From this equation we can read out that, on a $t = 0$ boundary, B is the derivative of the action with respect to $\partial\omega/\partial t$, therefore B is the momentum conjugate to the connection. More precisely, reintroducing the dimensionful constant $\frac{1}{8\pi G}$ in front of the action and going to a time gauge where the restriction of $\star\,(e \wedge e)$ on the boundary vanishes, the momentum is the 2-form on the boundary with values on $\mathrm{sl}\,(2,\mathbb{C})$, that is:

$$\Pi = \frac{1}{8\gamma\pi G} B\,. \tag{2.12}$$

2.2 Linear Simplicity Constraint

We consider now a spacelike boundary surface Σ, this is characterized by a vector which is normal to all the tangent vectors in Σ, we can write it as:

$$n_I = \epsilon_{IJKL} e^J_\mu e^K_\nu e^L_\rho \frac{\partial x^\mu}{\partial \sigma^1}\frac{\partial x^\nu}{\partial \sigma^2}\frac{\partial x^\rho}{\partial \sigma^3}\,, \tag{2.13}$$

where $\{\sigma^i\}$, $i = 1, 2, 3$ are the coordinates of the point $\sigma \in \Sigma$ and $x^\mu(\sigma)$ indicates the embedding of the boundary Σ into spacetime. By choosing a specific n_I we can focus on a fixed-time surface where $n_I = (1, 0, 0, 0)$. By doing so, the pull-back on Σ of the 2-form B can be decomposed into its electric $K^I = n_J B^{IJ}$ and magnetic $L^I = n_J\,(\star B)^{IJ}$ parts. Since B is antisymmetric, L^I and K^I do not have components normal to Σ, i.e. $n_I K^I = n_I L^I = 0$ and so they are three-dimensional vectors in Σ. In the gauge where $n_I = (1, 0, 0, 0)$ they are given by:

$$K^i = B^{i0}, \qquad L^i = \frac{1}{2}\epsilon^i_{jk} B^{jk}\,. \tag{2.14}$$

Now, from the definition of B we have that:

$$n_I B^{IJ} = n_I \left(\star\,(e\wedge e) + \frac{1}{\gamma} e\wedge e\right)^{IJ} = n_I \left(\epsilon^{IJ}_{KL} e^K \wedge e^L + \frac{1}{\gamma} e^I \wedge e^J\right), \tag{2.15}$$

on the boundary we have $n_I e^I|_\Sigma = 0$, therefore

$$n_I B^{IJ} = n_I\,(\star e\wedge e)^{IJ}\,. \tag{2.16}$$

Analogously:

$$n_I\,(B^\star)^{IJ} = n_I \left(\left(\frac{1}{\gamma} e\wedge e\right)^\star\right)^{IJ} = \frac{1}{\gamma} n_I\,(\star e\wedge e)^{IJ} = \frac{1}{\gamma} n_I B^{IJ}\,. \tag{2.17}$$

In conclusion, by definition of K^I and L^I we can notice that:

$$\vec{K} = \gamma \vec{L}\,. \tag{2.18}$$

This equation is called “linear symplicity constraint” and turns out to be a fundamental feature of covariant loop quantum gravity, indeed, it completely determines the dynamics of the theory.

2.3 Hamilton Function and Boundary Term

In writing the action on a compact region of spacetime we have to add a boundary term if we want to have a well-defined Hamilton function. In General Relativty, Gibbons and Hawkings have shown that the boundary term is given by:

$$S_{EH\ boundary} = \int_{\Sigma} k^{ab} q_{ab} \sqrt{q} d^3\sigma \,, \tag{2.19}$$

where k^{ab} is the extrinsic curvature of the boundary, q_{ab} is the three-metric induced by the embedding, q its determinant and σ are coordinates on the boundary. In the case of pure gravity without cosmological constant the Ricci scalar vanishes on the solution of the Einstein equations, therefore the bulk action vanishes and the Hamilton function is given by the boundary term:

$$S_{EH}[q] = \int_{\Sigma} k^{ab} q_{ab} \sqrt{q} d^3\sigma \,. \tag{2.20}$$

Notice that the Hamilton function is a functional of the boundary 3-metric, while the action is a functional of the 4-metric. Indeed, the Hamilton function represents a non-trivial functional to compute, because the extrinsic curvature $k^{ab}[q]$ is determined by the bulk solution singled out by the boundary intrinsic geometry, therefore it is going to be non-local. Knowing the general dependence of k^{ab} from q is equivalent to knowing the general solution of the Einstein equations.

2.4 ADM variables and Ashtekar variables

In order to approach a Hamiltonian formulation of General Relativity we introduce the so-called ADM variables and later on the Ashtekar variables.
The ADM variables are obtained by defining the following fields:

$$\begin{aligned} q_{ab} &= g_{ab} \,, \\ N_a &= g_{a0} \,, \\ N &= (-g_{00})^{-\frac{1}{2}} \,, \end{aligned} \tag{2.21}$$

where $a, b = 1, 2, 3$. N and N_a are called Lapse and Shift functions, q_{ab} is the three-metric. In these variables the line element reads

$$ds^2 = -(N^2 - N_a N^a) dt^2 + 2N_a dx^a dt + q_{ab} dx^a dx^b \,, \tag{2.22}$$

and the extrinsic curvature of a $t = constant$ surface is given by

$$k_{ab} = \frac{1}{2N} (\dot{q}_{ab} - D_{(a} N_{b)}) \tag{2.23}$$

where the dot indicates the derivative with respect to t and D_a is the covariant derivative of the three-metric. The action in terms of this variables takes the form

$$S[N, \overrightarrow{N}, q] = \int dt \int d^3x \sqrt{q} N \left(k_{ab} k^{ab} - k^2 + R[q] \right) \,, \tag{2.24}$$

where $k = k_a^a$ and $\sqrt{q} = \sqrt{\det q}$. From (2.24) one can read out the Lagrangian and the conjugate variables, thus obtaining an action written in hamiltonian form with the presence of constraints. Moving to the Ashtekar variables accounts for a simplification of these constraints and a better comprehension of the geometrical picture. Essentially, instead of dealing with tetrads on spacetime, we can introduce tetrads on each $t = constant$ surface. By doing so we have:

$$q_{ab}(x) = e_a^i(x) e_b^j(x) \delta_{ij} \,, \tag{2.25}$$

where q_{ab} is the 3-metric and $i, j = 1, 2, 3$ are flat indices. We can define also the triad version of the extrinsic curvature by:

$$k_i^a e_b^i := k_{ab} \,. \tag{2.26}$$

In this way, we can consider the following connection:

$$A_a^i = \Gamma_a^i[e] + \beta k_a^i \,, \tag{2.27}$$

where $\Gamma_a^i[e]$ is the torsionless spin connection of the triad and β is an arbitrary parameter, and the so-called "Ashtekar electric field":

$$E_i^a(x) = \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} e_b^j e_c^k \,, \tag{2.28}$$

that is, the inverse of the triad multiplied by its determinant. What's remarkable about these two fields is that they satisfy the following Poisson brackets:

$$\left\{ A_a^i(x) \,,\, A_a^i(y) \right\} = 0 \tag{2.29}$$

and

$$\left\{ A_a^i(x) \,, E_j^b(y) \right\} = \beta \delta_a^b \delta_j^i \delta^3(x, y) \,. \tag{2.30}$$

Therefore A_a^i and E_i^a are canonically conjugate variables. This simplifies the canonical analysis, i.e. the expressions of the constraints are easier to read. From a geometrical point of view there is an important feature which has a counterpart also in the quantum theory. In fact, the field E_i^a has an interpretation in terms of the area element: choosing a two-surface S in a $t = constant$ hypersurface we have that:

$$A_S = \int_S d^2\sigma \sqrt{E_i^a n_a E_i^b n_b} \,. \tag{2.31}$$

Then, by introducing the 2-form

$$E^i = \frac{1}{2} \epsilon_{abc} E^{ai} dx^b dx^c \,, \tag{2.32}$$

we can write

$$A_S = \int_S |E| \,. \tag{2.33}$$

Now, in the limit where the surface is small, the quantity

$$E^i_S = \int_S E^i = \frac{1}{2}\epsilon^i_{jk} \int_S e^j \wedge e^k \,, \tag{2.34}$$

is a vector normal to the surface, whose length is the area of the surface. Therefore we can say that the momentum conjugate to the connection represents an area element, this fact still holds in the quantum case.

Chapter 3

Discretization

3.1 Lattice QCD and Regge Calculus

Let's consider a $SU(2)$ Yang-Mills theory in four dimensions. The Yang-Mills field is known to be the connection whose components are $A^i_\mu(x)$, where i is an index in the Lie algebra su(2). Explicitly we can write the connection as:

$$A(x) = A^i_\mu(x)\tau_i dx^\mu \,, \tag{3.1}$$

where τ_i provide a basis of su(2). In order to discretize such a theory Wilson suggests to fix a cubic lattice with N vertices connected by E edges, this of course breaks the Lorentz invariance of the theory, recovered only in a suitable limit. We call a the length of the lattice edges, this is determined by the flat metric. Then we associate to each oriented edge a group variable $U_\mathbf{e} \in SU(2)$ in the following way:

$$U_\mathbf{e} = \mathcal{P}e^{\int_\mathbf{e} A} \,, \tag{3.2}$$

where $\mathcal{P}e$ stands for the path-ordered exponential (see Appendix). The idea is then to use a discrete set of group variables in place of the continuous variable A. Starting from this group variables instead of the algebra variables it is possible to calculate physical quantities in the limit where $N \to \infty$ and $a \to 0$. Under a gauge transformation the group elements $U_\mathbf{e}$ transform "homogeneously", that is:

$$U_\mathbf{e} \to \lambda_{s_\mathbf{e}} U_\mathbf{e} \lambda^{-1}_{t_\mathbf{e}} \,, \tag{3.3}$$

where $s_\mathbf{e}$ and $t_\mathbf{e}$ are the initial and final vertices of the edge $\mathbf{e}$ (source and target), $\lambda_\mathbf{v}$ is an element of $SU(2)$. Therefore a gauge transformation can be thought as an element of $SU(2)^V$, where V is the number of vertices. Gauge transformations take place at each vertex.
From eq. (3.3) it is straightforward to see that if we take the ordered product of four group elements around a face $\mathbf{f}$

$$U_\mathbf{f} = U_{\mathbf{e}_1} U_{\mathbf{e}_2} U_{\mathbf{e}_3} U_{\mathbf{e}_4} \tag{3.4}$$

and we consider its trace, we get a gauge invariant quantity. In addition to this $U_\mathbf{f}$ is a discrete version of the connection, since it is the holonomy of the connection on the loop given by a square. Wilson has shown that the discrete action

$$S = \beta \sum_{\mathbf{f}} \mathrm{Tr} U_{\mathbf{f}} \; + \; c.c. \tag{3.5}$$

approximates the continuous action in the limit where a is small.

The Hamiltonian formulation of the discretized theory is particularly important, since it is going to have a counterpart in the quantum realm. We focus on a boundary, say spacelike, the hamiltonian coordinates are given by the group elements U_l on the boundary edges, called "links". The canonical configuration space is therefore $SU(2)^L$, where L is the number of links. The corresponding phase space is the cotangent space $T^*SU(2)^L$, the Poisson structure of this space is carefully explained in the Appendix. We denote the conjugate momentum of U_l by $L_l^i \in \mathrm{su}(2)$. The Poisson brackets are then given by:

$$\begin{aligned} &\{U_l \, , \, U_{l'}\} = 0 \, , \\ &\{U_l \, , \, L^i_{l'}\} = \delta_{ll'} U_l \tau^i \, , \\ &\left\{L^i_l \, , \, L^j_{l'}\right\} = \delta_{ll'} \epsilon^{ij}_{k} L^k_l \, , \end{aligned} \tag{3.6}$$

(no summation over l). The Hilbert space of the discrete theory can therefore be represented by states $\psi\left(U_l\right)$, i.e. functions on the configuration space. The space of these functions carries a natural scalar product which is invariant under the gauge tranformations on the boundary, this is given by the $SU(2)$ Haar measure:

$$(\phi \, , \, \psi) = \int_{SU(2)} dU_l \overline{\phi\left(U_l\right)} \psi\left(U_l\right) \, . \tag{3.7}$$

The boundary gauge transformations act at the nodes of the boundary and transform the states as follows

$$\psi\left(U_l\right) \rightarrow \psi\left(\lambda_{s_l} U_l \lambda^{-1}_{t_l}\right) \, , \quad \lambda_n \in SU(2). \tag{3.8}$$

We move on and introduce Regge calculus now. Tullio Regge introduced a discretization of General Relativity called "Regge calculus". We can summarize it as follows: a d-simplex is a generalization of a triangle or a tetrahedron to arbitrary dimensions, more precisely, it is the convex hull of its $d+1$ vertices. These vertices are connected by $d(d+1)/2$ line segments whose length L_s fully specify the shape of the simplex, i.e. its geometry.
A Regge space $(M \, , L_s)$ in d dimensions is a d-dimensional metric space obtained by gluing d-simplices along matching boundary $(d-1)$-simplices. For example, in two dimensions we can obtain a surface by gluing triangles, bounded by segments, which meet at points. In three dimensions we chop space into tetrahedra, bounded by triangles, in turn bounded by segments, which meet at points. In four dimensions we chop spacetime into 4-simplices, bounded by tetrahedra, in turn bounded by triangles, in turn bounded by segments, which meet at points. These structures are called triangulations. We can legitimately ask how curvature arises in a Regge space, since all these geometrical objects are flat. We consider the simplest case, that is $d = 2$ dimensions: it is easy to see that if we glue triangles around a common vertex, curvature arises in terms of a deficit angle, that is, the sum of all the angles insisting on a given vertex does not add up to 2π. In formulas:

$$\delta_P\left(L_s\right) = 2\pi - \sum_{\mathbf{t}} \theta_{\mathbf{t}}\left(L_s\right) \tag{3.9}$$

This fact admits a generalization to higher dimensions: gluing flat d-simplices can generate curvature on the $(d-2)$-simplices (sometimes called "hinges"). Now, a Riemannian manifold $(M\,,g)$ can be approximated arbitrarly well by a Regge manifold, in fact, for any $(M\,,g)$ and any ϵ we can find a $(M\,,L_s)$ with sufficiently many simplices such that, for any two points $x, y \in M$, the difference between the Riemannian distance and the Regge distance is smaller than ϵ. In order to dicretize General Relativity we need a discretized version of the action, the Regge action is defined as:

$$S_M\left(L_s\right) = \sum_h A_h\left(L_s\right)\delta\left(L_s\right) \ , \tag{3.10}$$

where the sum is over the hinges and A_h is the $(d-2)$-volume of the hinge h. Remarkably, this action converges to the Einstein-Hilbert action when the Regge manifold $(M\,,L_s)$ converges to the Riemann manifold $(M\,,g)$. The Regge action can be also rewritten as a sum over the d-simplices $\mathbf{v}$ of the triangulation: from (3.9) and (3.10) we have that

$$S_M\left(L_s\right) = 2\pi\sum_h A_h\left(L_s\right) - \sum_{\mathbf{v}} S_{\mathbf{v}}\left(L_s\right) \ , \tag{3.11}$$

where the action of a d-simplex is

$$S_{\mathbf{v}}\left(L_s\right) = \sum_h A_h\left(L_s\right)\theta_h\left(L_s\right) \ . \tag{3.12}$$

3.2 Discretization in 3D

The discretization used in LQG differs from the Regge one, because essentially lengths are constrained by inequalities (think about a triangle for instance), and it's difficult to implement a configuration space with such constraints. It is preferable then to consider also the "dual" of a triangulation, in three dimensions this is simply obtained by replacing each tetrahedra by a vertex sitting at its center, each face (a triangle) by an edge coming off the vertex and puncturing the triangle. Therefore, adjacent tetrahedra are replaced by vertices connected by edges. The dual of the triangulation Δ is denoted as Δ^* and the set of vertices, edges and faces is called a "2-complex" (denoted with $\mathcal{C}$). Thus, we are going to discretize classical GR on a 2-complex. One word about the boundary: if we consider a compact region of spacetime, the discretization Δ will induce a discretization of the boundary, formed by the boundary segments and the boundary triangles of Δ. Moving to Δ^* we realize that the boundary is formed now by the end points of the edges dual to the boundary triangles, which are called nodes, and the boundary of the faces dual to the boundary segments, together they form the graph Γ of the boundary. The boundary graph, by construction, is at the same time the boundary of the 2-complex and the dual of the boundary of the triangulation:

$$\Gamma = \partial\left(\Delta^*\right) = \left(\partial\Delta\right)^* \ . \tag{3.13}$$

Now, in 3 dimensions the gravitational field is described by a tetrad field $e^i = e^i_a dx^a$ and a $SO(3)$ connection $\omega^i_j = \omega^i_{aj} dx^a$, where $a, b, .. = 1, 2, 3$ are spacetime indices and $i, j = 1, 2, 3$ are internal

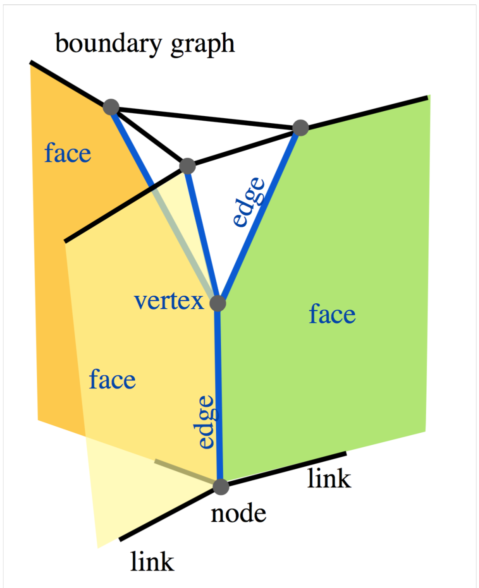

Figure 3.1: A 2-complex

indices. We discretize the connection as in Yang-Mills theory, that is, by assigning a $SU(2)$ element $U_{\mathbf{e}}$ to each edge $\mathbf{e}$ of the 2-complex. We discretize the triad by associating a vector L_s^i of $\mathbb{R}^3$ to each segment s of the original triangulation:

$$\begin{aligned} \omega &\longrightarrow U_{\mathbf{e}} = \mathcal{P}\exp \int_{\mathbf{e}} \omega \in SU(2) \\ e &\longrightarrow L_s^i = \int_s e^i \in \mathbb{R}^3 \,, \end{aligned} \tag{3.14}$$

in the LQG jargon $U_{\mathbf{e}}$ is called the "holonomy" (of the connection along the edge). The Einstein-Hilbert action can be approximated in terms of these objects. We have seen that under a gauge transformation the holonomy transforms "well", that is, as

$$U_{\mathbf{e}} \mapsto R_{s_e} U_{\mathbf{e}} R_{t_e}^{-1} \tag{3.15}$$

whereas the algebra values L^i apparently don't follow this rule. Nevertheless, it is possible to give a gauge equivalent definition of L^i in such a way that it transforms as the holonomy, as shown in [22]. The discretization approximates well the continuum theory when the curvature is small at the scale of the triangulation and the segments are straight lines. We notice that the norm of vector L_s^i associated with the segment s is the length of the segment, i.e.

$$L_s^2 = |\overrightarrow{L_s}|^2 \,. \tag{3.16}$$

Since each face $\mathbf{f}$ of the 2-complex corresponds to a segment $s = s_{\mathbf{f}}$ of the triangulation, we can view L_s^i as associated with the face: $L_{\mathbf{f}}^i = L_{s_{\mathbf{f}}}^i$. Furthermore, since $\mathbb{R}^3$ equipped with the usual cross product is isomorphic (as a Lie algebra) to su(2) we can express $L_{\mathbf{f}}^i$ as an element of su(2), that is:

$$L_{\mathbf{f}} = L_{\mathbf{f}}^i \tau_i \,. \tag{3.17}$$

Summarizing, the variables of the discretized theory are:

• An $SU(2)$ group element $U_{\mathbf{e}}$ for each edge $\mathbf{e}$ of the 2-complex;
• An su(2) algebra element $L_{\mathbf{f}}$ for each face $\mathbf{f}$ of the 2-complex.

In the four-dimensional case the approach will be the same, group and algebra variables associated to edges and faces.
Having discretized space, we need a discrete version of the action. The idea is to mimic the Regge action, knowing that in three dimensions curvature arises around segments and so the volume of the hinge is the length of the segment. The deficit angle is replaced by the curvature written in terms of the holonomy, as already remarked earlier we have curvature around a segment if the group element $U_{\mathbf{f}} = U_{\mathbf{e}_1} \cdots U_{\mathbf{e}_n}$ is different from the identity ($\mathbf{e}_1, \dots, \mathbf{e}_n$ being the edges bounding the face $\mathbf{f}$). In this way, we can write the discretized action as follows:

$$\mathcal{S} = \frac{1}{8\pi G} \sum_{\mathbf{f}} \mathrm{Tr}\left(L_{\mathbf{f}} U_{\mathbf{f}}\right) , \tag{3.18}$$

Performing the variation of the action with respect to $L_{\mathbf{f}}$ and setting it to zero gives $U_{\mathbf{f}} = \mathbb{1}$, that is, flatness, which is equivalent to the continuous Einstein equations in three dimensions.

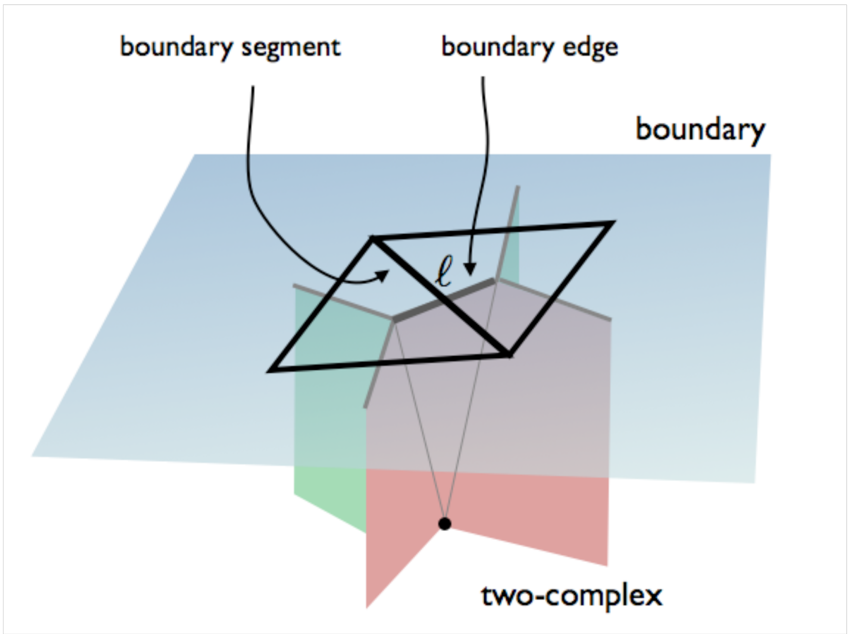

Figure 3.2: Boundary graph

We specify on the boundary now. On the boundary there are two kinds of variables: the group elements U_l of the boundary edges, namely, the links, and the algebra elements L_s of the boundary segment s. Notice that there is precisely one boundary segment s per each link l, and the two cross. We can therefore rename L_s as L_l whenever l is the link crossing the boundary segment s. In this way, the boundary variables are formed by a pair $(L_l\, , U_l) \in \mathrm{su}(2) \times SU(2)$ for each link l of the graph Γ. Therefore on the boundary we have a pair of conjugate variables at each link, the Poisson brackets are the ones already written in (3.6), only with a factor $8\pi G$ on the RHS coming from the action.

3.3 Discretization in 4D

Moving to the four-dimensional case we consider a triangulation Δ made of 4-simplices, the corresponding dual triangulation Δ^* has the following properties: a vertex is dual to a 4-simplex, an edge is dual to a tetrahedron, a face is dual to a triangle (for instance a face in the (x, y) plane is dual to a triangle in the (z, t) plane). Therefore, we have that a face of the 2-complex that touches the boundary is dual to a boundary triangle, this in turn is dual to a boundary link l. Geometrically, this link is the intersection of the face with the boundary. Thus, a boundary link l is obviously a boundary edge (by definition), but is also associated with a face $\mathbf{f}$ touching the boundary. From these considerations follows that we discretize the connection and the triad as

$$\begin{aligned} \omega &\longrightarrow U_{\mathbf{e}} = \mathcal{P}\text{exp} \int_{\mathbf{e}} \omega \in SL(2,\mathbb{C}) \\ e &\longrightarrow B_{\mathbf{f}} = \int_{\mathbf{t_f}} B \quad \in \text{sl}(2,\mathbb{C})\,, \end{aligned} \tag{3.19}$$

where $B = \Big((e \wedge e)^* + \frac{1}{\gamma} (e \wedge e) \Big)$ is the 2-form defined in the action, and $\mathbf{t_f}$ is the triangle dual to the face $\mathbf{f}$.
The variables of the discretized theory are then:

- a group element $U_{\mathbf{e}}$ for each edge $\mathbf{e}$ of the 2-complex;
- an algebra element $B_{\mathbf{f}}$ for each face $\mathbf{f}$ of the 2-complex.

Pretty much the same as seen in the three-dimensional case. Therefore we call U_l the group elements associated with the boundary edges l, that is, the links of the boundary graph Γ, and B_l are the elements of a face bounded by the link l. There is a remarkable geometric interpretation of B_l: consider a triangle lying on the boundary, choose the tetrad field in the time gauge, that is, $e^0 = dt$ and $e^i = e^i_a dx^a$, the pull-back of $(e \wedge e)^*$ on the boundary vanishes and we are left with

$$L^i_{\mathbf{f}} = \frac{1}{2\gamma} \epsilon^i_{\ jk} \int_{\mathbf{t_f}} e^j \wedge e^k\,. \tag{3.20}$$

In the approximation in which the metric is constant on the triangle it follows then that the norm of $L^i_{\mathbf{f}}$ is proportional to the area of the triangle:

$$|L_{\mathbf{f}}| = \frac{1}{\gamma} A_{\mathbf{t_f}}\,. \tag{3.21}$$

Here we see an analogy between the vector $\overrightarrow{L_{\mathbf{f}}}$ and the vector $\overrightarrow{E_S}$ defined in terms of the Ashtekar variables.

Chapter 4

Quantization

4.1 3D Theory

In order to define the quantum theory two features are needed:

• a boundary Hilbert space that describes the quantum states of the boundary geometry;
• the transition amplitude for these boundary states; in the small $\hbar$ limit the transition amplitude must reproduce the exponential of the Hamilton function.

4.1.1 Hilbert Space

To construct the Hilbert space and the transition amplitude we proceed as follows: first, we discretize the classical theory; then, we study the quantum theory that corresponds to the discretized theory; finally we discuss the continuum limit.
We recall from the previous chapter that the discrete boundary geometry is described by a pair of variables for each link of the graph Γ: $(U_l\,, L_l) \in SU(2) \times \mathrm{su}(2)$. We are seeking the quantum version of these phase space variables, i.e. we are looking for operators U_l and L_l satisfying the quantum version of the Poisson brackets seen earlier:

$$\left[U_l\,,\, L^i_{l'}\right] = i\,(8\pi\hbar G)\,\delta_{ll'} U_l \tau^i \tag{4.1}$$

For this purpose we consider, as the Hilbert space, the space of square integrable functions on $SU(2)^L$:

$$\mathcal{H}_\Gamma = L_2\left[SU(2)^L\right]\,. \tag{4.2}$$

States are therefore wavefunctions $\psi(U_l)$ of L group elements U_l. The scalar product compatible with the $SU(2)$ structure is given by the group-invariant measure, that is, the Haar measure:

$$\langle\phi|\psi\rangle = \int_{SU(2)^L} dU_l \overline{\phi\left(U_l\right)} \psi\left(U_l\right)\,. \tag{4.3}$$

By doing so, U_l can be seen simply as a multiplicative operator acting as $U_{l'}\left(\psi(U_l)\right) = \psi\left(U_{l'}U_l\right)$.
Furthermore, as showed in the Appendix, on the Lie group $SU(2)$ it is defined a left-invariant vector

field, whose components are:

$$\left(J^i\psi\right) := -i\frac{d}{dt}\psi\left(Ue^{t\tau_i}\right)|_{t=0}\,. \tag{4.4}$$

Then, to get the correct operator satysfing (4.1) it is sufficient to scale the left-invariant vector field with the appropriate dimensionful factor:

$$L_l^i := (8\pi\hbar G)\, J_l^i\,. \tag{4.5}$$

One important consequence is that length is quantized. In fact, we recall that $L_s = |\overrightarrow{L_{\mathbf{f}}}|$, with $\mathbf{f}$ being the face dual to the segment s. This means that on the boundary we have $L_l = |\overrightarrow{L_l}|$ where l is the link crossing the boundary segment s. Therefore, since $\overrightarrow{J_l}$ is the generator of $SU(2)$, $|\overrightarrow{J_l}|^2$ is the $SU(2)$ Casimir, its eigenvalues are $j(j+1)$, j being an half-integer. Then we get the following spectrum for the operator L_l:

$$L_{j_l} = 8\pi\hbar G\sqrt{j_l(j_l+1)}\,, \tag{4.6}$$

for half-integers j_l.
We go on now with the definition of the boundary Hilbert space. Since the theory must be invariant under $SU(2)$ gauge transformations (taking place at nodes), we have to take that into account. Then, the gauge-invariant states must satisfy

$$\psi\left(U_l\right) = \psi\left(\Lambda_{s_l} U_l \Lambda_{t_l}^{-1}\right)\,, \quad \Lambda_{\mathrm{n}} \in SU(2)\,. \tag{4.7}$$

We can write equivalently

$$\overrightarrow{C_{\mathrm{n}}}\psi = 0 \tag{4.8}$$

for every node n of the boundary graph, where $\overrightarrow{C_{\mathrm{n}}}$ is the generator of $SU(2)$ transformations at the node n, i.e. :

$$\overrightarrow{C_{\mathrm{n}}} = \overrightarrow{L_{l_1}} + \overrightarrow{L_{l_2}} + \overrightarrow{L_{l_3}} = 0\,; \tag{4.9}$$

where l_1, l_2, l_3 are the three links emerging from the node n. This relation is called gauge constraint. From a geometrical standpoint the interpretation of this equation is straightforward: l_1, l_2, l_3 are three links that cross three segments which in turn bound a triangle, then, the condition (4.9) can be read as the closure condition satisfied by every triangle (since L_{l_i} represents the length of the segment s_i). It is worth noting that a similar result was obtained by Roger Penrose in 1971 [21], in his "spin-geometry theorem". Penrose observed that if we consider the operators $\overrightarrow{L}_l$, which are not gauge invariant, we can define a gauge invariant operator, called "Penrose metric operator", by

$$G_{ll'} = \overrightarrow{L}_l \cdot \overrightarrow{L}_{l'}\,, \tag{4.10}$$

where l and l' share the same source. The Casimir operators of $SU(2)$ are then given by

$$A_l^2 = \overrightarrow{L}_l \cdot \overrightarrow{L}_l\,. \tag{4.11}$$

The theorem states that the equations (4.10), (4.11) and (4.9) (which generalizes to a node with arbitrary valence), are sufficient to guarantee the existence of a flat polyhedron, such that the area of its faces is A_l and where $G_{ll'}$ is given by $G_{ll'} = A_l A_{l'} \cos\theta_{ll'}$, $\theta_{ll'}$ being the angle between the normals to the faces l and l'. More precisley, there exists a 3×3 metric tensor g_{ab}, $a, b = 1, 2, 3$ and normal to the faces $\overrightarrow{n}_l$, such that

$$G_{ll'} = g_{ab} n_l^a n_{l'}^b \tag{4.12}$$

and the length of these normals is equal to the area of the face. In conclusion, the algebraic structure of the momentum operators in $\mathcal{H}_\Gamma$ determine the existence of a metric at each node and therefore endows each quantum of space with a geometry. It is curious that these results reappered more than 20 years later in LQG.

Proceeding with the construction of the boundary Hilbert space, we consider the subspace of $\mathcal{H}_\Gamma$ where (4.7) is verified, which is a proper subspace, we call it $\mathcal{K}_\Gamma$ and write it as

$$\mathcal{K}_\Gamma = L_2\left[SU(2)^L/SU(2)^N\right]_\Gamma . \tag{4.13}$$

Clearly L indicates the number of links, N is the number of nodes and the subscript Γ denotes the fact that the pattern of the $SU(2)^N$ transformations is dictated by the structure of the graph Γ. Let's study the structure of $\mathcal{K}_\Gamma$. On this Hilbert space the length operators L_l are gauge-invariant, furthermore, they form a complete commuting set. This means that a basis of $\mathcal{K}_\Gamma$ is given by the normalized eigenvectors of these operators, which we indicate as $|j_l\rangle$. An element of this basis is therefore determined by assigning a spin j_l to each link l of the graph. A graph with a spin assigned to each link is called a "spin network". The spin network states $|j_l\rangle$ form a basis of $\mathcal{K}_\Gamma$, this is called a *spin-network basis* and spans the quantum states of geometry.

More concretely, we can make use of the Peter-Weyl theorem to get a more intuitive picture of what's going on. In fact, we know that the Wigner matrices D^j_{mn} provide an orthogonal basis for the spin-j representation, that is:

$$\int dU \overline{D^{j'}_{m'n'}(U)} D^j_{mn}(U) = \frac{1}{d_j}\delta^{jj'}\delta_{mm'}\delta_{nn'} , \tag{4.14}$$

where $d_j = 2j+1$ is the dimension of the j representation and dU is the $SU(2)$ Haar measure. In other words, the Hilbert space $L_2[SU(2)]$ can be decomposed into a sum of finite dimensional subspaces of spin j, spanned by the basis states formed by the matrix elements of the Wigner matrices $D^j(U)$. This matrix is a map from the Hilbert space $\mathcal{H}_j$ to itself, therefore we can see $D^j(U)$ as an element of $\mathcal{H}_j \otimes \mathcal{H}_j^*$. Since we know that $\mathcal{H}_j \cong \mathcal{H}_j^*$, for notational convenience we omit the asterisk. All together this reads as

$$L_2[SU(2)] = \bigoplus_j (\mathcal{H}_j \otimes \mathcal{H}_j) . \tag{4.15}$$

Having L links it is straightforward to consider the following:

$$L_2\left[SU(2)^L\right] = \otimes_l \left[\oplus_j (\mathcal{H}_j \otimes \mathcal{H}_j)\right] = \oplus_{j_l} \otimes_l (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}) . \tag{4.16}$$

The two Hilbert spaces associated with a link can be seen as belonging to the two ends of the link, because each transforms according to the gauge transformation at one end. In order to see what's going on a node we can regroup the Hilbert spaces $\mathcal{H}_j$ in such a way that

$$L_2\left[SU(2)^L\right] = \oplus_{j_l} \otimes_n \left(\mathcal{H}_j \otimes \mathcal{H}_{j'} \otimes \mathcal{H}_{j''}\right) \,, \tag{4.17}$$

where j, j', j'' are the spins coming out from the node n. Next, we want the space of gauge-invariant states, thus we should restrict to the invariant part of the spaces transforming at the same node, that is:

$$L_2\left[SU(2)^L/SU(2)^N\right] = \oplus_{j_l} \otimes_n \mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right) \,. \tag{4.18}$$

From $SU(2)$ representation theory it is known that $\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right)$ does not exist unless the sum of three spins is an integer and the three spins satisfy the triangular inequality:

$$|j_1 - j_2| < j_3 < j_1 + j_2 \,. \tag{4.19}$$

If this condition holds then the invariant space is one-dimensional:

$$\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right) = \mathbb{C} \,. \tag{4.20}$$

Therefore

$$L_2\left[SU(2)^L/SU(2)^N\right] = \oplus_{j_l}\mathbb{C} \tag{4.21}$$

where the sum is restricted to the j_l that satisfy the triangular inequalities. Since spins are associated to the lengths of the sides of a triangle, and these are known to satisfy the triangular inequality, the resemblance with the geometrical picture holds nicely.
Then, a generic quantum state in loop quantum gravity is a superposition of spin-network states:

$$|\psi\rangle = \sum_{j_l} \mathcal{C}_{j_l} |j_l\rangle \,. \tag{4.22}$$

Summarizing, the spin network states $|j_l\rangle$:

- are an eigenbasis of all lengths operators;
- span the gauge-invariant Hilbert space;
- have a simple geometric interpretation: they just say how long the boundary links are.

Next, we would like to write the spin-network states $|j_l\rangle$ in the $\psi\left(U_l\right)$ representation, that is, compute the spin-network wavefunctions:

$$\psi_{j_l}\left(U_l\right) = \langle U_l | j_l\rangle \,. \tag{4.23}$$

This can be done explicitly by solving the eigenvalue equation for the length operators L_l

$$L_l \psi_{j_l}\left(U_l\right) = L_{j_l} \psi_{j_l}\left(U_l\right) \,. \tag{4.24}$$

It is possible to write a generic state $\psi(U) \in L_2[SU(2)]$ as a linear combination in the basis provided by the Wigner matrices, as follows:

$$\psi(U) = \sum_{jmn} \mathcal{C}_{jmn} D^{j}_{mn}(U)\,. \tag{4.25}$$

Therefore, in our case, a state $\psi\left(U_l\right) \in L_2[SU(2)^L]$ can be written as

$$\psi\left(U_l\right) = \sum_{j_i, m_i, n_i} \mathcal{C}_{j_1\cdots j_L m_1\cdots m_L n_1\cdots n_L} D^{j_1}_{m_1 n_1}\left(U_{l_1}\right) \cdots D^{j_L}_{m_L n_L}\left(U_{l_L}\right)\,, \tag{4.26}$$

where $i = 1, \ldots, L$. A state invariant under a $SU(2)$ transformation must be invariant if we act with a transformation Λ_{n} taking place at the node n. This in turn acts on the three group elements of the three links that meet at the node. Since the Wigner matrices are representation matrices, the gauge transformation acts on the three corresponding indices, for this reason we have that, for the state to be invariant, $\mathcal{C}_{j_1\cdots j_L m_1\cdots m_L n_1\cdots n_L}$ must be invariant when acted upon by a group transformation on the three indices corresponding to the same node. From representation theory it is known that, up to normalization, it exists only one invariant tensor with three indices in three $SU(2)$ representations, it is called the Wigner $3j$-symbol and is denoted as

$$\iota^{m_1 m_2 m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\,. \tag{4.27}$$

In this way, we can express any invariant state in the triple tensor product of representations of $SU(2)$ as

$$\iota^{m_1 m_2 m_3} = \sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle\,. \tag{4.28}$$

Going on, a gauge-invariant state must then have the form

$$\psi\left(U_l\right) = \sum_{j_1\cdots j_L} \mathcal{C}_{j_1\cdots j_L} \iota_1^{m_1 m_2 m_3} \cdots \iota_N^{m_{L-2} m_{L-1} m_L} D^{j_1}_{m_1 n_1}\left(U_{l_1}\right) \cdots D^{j_L}_{m_L n_L}\left(U_{l_L}\right) \tag{4.29}$$

where all the indices are contracted between the intertwiner ι and the Wigner matrices D. Don't let confuse yourself if you don't see any n-indices contracted, because the pattern of contraction is dictated by the structure of the graph (and so, broadly speaking, m's and n's are interchangeable, it is just the notation of Wigner matrices that keeps them different).
Seeking a more compact form we write a generic gauge-invariant state as

$$\psi\left(U_l\right) = \sum_{j_l} \mathcal{C}_{j_l} \psi_{j_l}\left(U_l\right)\,, \tag{4.30}$$

where

$$\psi_{j_l}\left(U_l\right) = \iota_1^{m_1 m_2 m_3} \cdots \iota_N^{m_{L-2} m_{L-1} m_L} D^{j_1}_{m_1 n_1}\left(U_{l_1}\right) \cdots D^{j_L}_{m_L n_L}\left(U_{l_L}\right) \tag{4.31}$$

are the orthogonal states labeled by a spin associated with each link. These are the spin-network wavefunctions. We can write them more compactly as

$$\langle U_l | j_l \rangle = \psi_{j_l}\left(U_l\right) = \bigotimes_{\mathrm{n}} \iota_{\mathrm{n}} \cdot \bigotimes_{l} D^{j_l}\left(U_l\right)\,. \tag{4.32}$$

4.1.2 Transition Amplitude

The next goal is to write down the transition amplitude of the three-dimensional theory. The transition amplitude is a function of the boundary states, therefore we assume that a triangulation Δ is fixed and we consider a boundary, which means considering the boundary graph $\Gamma = (\partial\Delta)^*$. We denote the transition amplitude expressed in terms of the "coordinates" as $W_\Delta(U_l) = \langle W_\Delta | U_l \rangle$ and the transition amplitude in terms of the "momenta" as $W_\Delta(j_l) = \langle W_\Delta | j_l \rangle$.
Notice that the transition matrix between the two basis is given precisely by the spin-network states. To compute the transition amplitude W_Δ of the theory discretized on the 2-complex, dual to Δ, we use the Feynman path integral. The amplitude is given by the integral over all classical configurations weighted by the exponential of the (discretized) classical action:

$$W_\Delta(U_l) = \mathcal{N} \int dU_{\mathbf{e}} \int dL_{\mathbf{f}} e^{\frac{i}{8\pi\hbar G} \sum_{\mathbf{f}} \mathrm{Tr}[U_{\mathbf{f}} L_{\mathbf{f}}]} , \tag{4.33}$$

where $\mathcal{N}$ is a normalization factor. Reabsorbing factors on the overall constant $\mathcal{N}$ and performing the integral over the momenta we obtain

$$W_\Delta(U_l) = \mathcal{N} \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \delta(U_{\mathbf{f}}) . \tag{4.34}$$

To compute this integral, we expand the delta function over the group in representations using

$$\delta(U) = \sum_j d_j \mathrm{Tr} D^{(j)}(U) , \tag{4.35}$$

where $d_j = 2j + 1$ is the dimension of the spin-j representation. Therefore (4.34) turns out to be

$$\begin{aligned} W_\Delta(U_l) &= \mathcal{N} \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \left(\sum_j d_j \mathrm{Tr} D^j(U_{\mathbf{f}}) \right) \\ &= \mathcal{N} \sum_{\mathbf{f}} \left(\prod_{\mathbf{f}} d_{j_{\mathbf{f}}} \right) \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \mathrm{Tr}\left(D^{j_{\mathbf{f}}}(U_{1\mathbf{f}}) \cdots D^{j_{\mathbf{f}}}(U_{n\mathbf{f}}) \right) , \end{aligned} \tag{4.36}$$

where $U_{\mathbf{f}} = U_{1\mathbf{f}} \cdots U_{n\mathbf{f}}$. Now, if we focus our attention on one edge in particular, we notice that an edge bounds precisely three faces (because an edge is dual to a triangle, which is bounded by three segments, and segments are dual to faces). Therefore each $dU_{\mathbf{e}}$ integral is of the form

$$\int dU D^{j_{l_1}}_{m_1 n_1}(U) D^{j_{l_2}}_{m_2 n_2}(U) D^{j_{l_3}}_{m_3 n_3}(U) , \tag{4.37}$$

but, since the Haar measure is invariant on both sides, the result must be invariant in both set of indices. As we have seen before, there is only one such object, the Wigner $3j$-symbol, then

$$\begin{aligned}\int dU D^{j_{l_1}}_{m_1 n_1}(U) D^{j_{l_2}}_{m_2 n_2}(U) D^{j_{l_3}}_{m_3 n_3}(U) &= \iota^{m_1 m_2 m_3} \iota^{n_1 n_2 n_3} \\ &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \end{aligned} \tag{4.38}$$

where, again, we emphasize that the m and n indices are dictated by the structure of the graph Γ. Thus, what we obtain in the end is nothing but $3j$-symbols contracted among themselves. More precisley, we observe that each edge produce two $3j$-symbols which we can view as located at the two ends of the edge, since their indices are contracted at the end (on a vertex). At each vertex there are four edges, therefore four $3j$-symbols contracted among themselves. The contraction must be $SU(2)$-invariant, so we are looking for an object which involves four $3j$-symbols and is invariant under a $SU(2)$ transformation, it turns out it exists and it is called the Wigner $6j$-symbol:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} := \sum_{m_a, n_a} \prod_{a=1}^{6} g_{m_a n_a} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_5 \\ n_1 & m_4 & m_3 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_6 \\ n_2 & n_4 & m_6 \end{pmatrix} \begin{pmatrix} j_3 & j_5 & j_6 \\ n_3 & n_5 & n_6 \end{pmatrix}, \tag{4.39}$$

where

$$g_{mn} = \sqrt{2j+1} \begin{pmatrix} j & j & 0 \\ m & n & 0 \end{pmatrix} = \delta_{m,-n} (-1)^{j-m}. \tag{4.40}$$

After integrating over all internal edge-group variables, the group variables of the boundary are left. We can integrate these as well contracting with a boundary spin network state, obtaining [14]

$$W_\Delta(j_l) = \mathcal{N}_\Delta \sum_{j_\mathbf{f}} \prod_{\mathbf{f}} (-1)^{j_\mathbf{f}} d_{j_\mathbf{f}} \prod_{\mathbf{v}} (-1)^{J_\mathbf{v}} \{6j\}, \tag{4.41}$$

where the sum is over the association of a spin to each face, respecting the triangular inequalitites at all edges, $J_\mathbf{v} = \sum_{a=1}^{6} j_a$, and j_a are the spin of the faces adjacent to the vertex $\mathbf{v}$ (a vertex of the 2-complex is adjacent to six faces).

We can see the connection with general relativity in the classical limit (the continuum limit will be discussed in the four-dimensional case, which is more interesting). If we consider a single tetrahedron whose sides have length $L_a = j_a + 1/2$, it is possible to show [15] that, in the large j limit we have

$$\{6j\} \underset{j\to\infty}{\sim} \frac{1}{\sqrt{12\pi V}} \cos\left(S + \frac{\pi}{4}\right). \tag{4.42}$$

Thus, by using the well known relation $e^{i\alpha} = \cos\alpha + i\sin\alpha$ we get the following

$$\{6j\} \underset{j\to\infty}{\sim} \frac{1}{2\sqrt{-12i\pi V}} e^{iS} + \frac{1}{2\sqrt{12i\pi V}} e^{-iS}. \tag{4.43}$$

We see therefore that two terms with opposite phase enter here, this is precisely the discussion we were addressing when dealing with the tetrad action.
If we consider only large spins we can disregard quantum discreteness and the sum over the spins is approximated by an integral over lengths in a Regge geometry. This is a discretization of a path

integral over geometries of the exponential of the Einstein-Hilbert action. Therefore (4.38) is a concrete implementation of the path-integral "sum over geometries" formal definition of quantum gravity:

$$Z \sim \int D[g] e^{\frac{i}{\hbar}\int \sqrt{-g}R} . \tag{4.44}$$

The next discussion addresses a topic that will be generalised to the four-dimensional case and it is particularly relevant. Consider a triangulation formed by a single tetrahedron τ, the boundary graph has again the shape of a tetrahedron, since we have four vertices obtained as the end points of the four edges puncturing the four faces of the original tetrahedron. The amplitude is then a function of the variables of the links of the graph. Let's label with $a, b = 1, 2, 3, 4$ the nodes of the graph and denote with $U_{ab} = U_{ba}^{-1}$ the boundary group elements. The transition amplitude is then a function $W(U_{ab})$. Notice that we have already constructed the 2-complex, which consists of the four edges puncturing the four faces and of the boundary links, therefore it is made of six faces (obtained by connecting the vertex sitting inside τ with the six boundary links). Using (4.34) and dropping the normalization we get:

$$W(U_{ab}) = \int dU_a \prod_{ab} \delta\left(U_a U_{ab} U_b^{-1}\right) . \tag{4.45}$$

Once this integrals are performed we obtain:

$$W(U_{ab}) = \delta(U_{12}U_{23}U_{31})\, \delta(U_{13}U_{34}U_{41})\, \delta(U_{23}U_{34}U_{42}) \tag{4.46}$$

Notice that each sequence of U_{ab} inside the deltas corresponds to an independent closed loop in the boundary graph. The interpretation of this amplitude is therefore immediate: the amplitude forces the connection to be flat on the boundary (by the very definition of the delta). More precisely, it is the three-dimensional connection which is flat, not the two-dimensional one living on the boundary. We can think of it as having a spacetime reference frame on each face that can be parallel transported along the boundary in such a way that any closed loop gives unity. In other words, $W(U_{ab})$ is just the gauge-invariant version of $\prod_{ab} \delta(U_{ab})$.
Notice in particularly that:

$$\langle W|\psi\rangle = \int dU_{ab} W(U_{ab})\, \psi(U_{ab}) = \int dU_a \psi\left(U_a U_b^{-1}\right) , \tag{4.47}$$

from which we read that W projects on the flat connections, averaged over the gauge orbits.
To achieve the important result we are aiming for we would like to see if everything is still consistent, that is, we know that the same amplitude in the spin representation is given by:

$$W(j_{ab}) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} , \tag{4.48}$$

therefore, we expect to obtain the same result by considering

$$W(j_{ab}) = \int dU_{ab} \psi_{j_{ab}}(U_{ab})\, W(U_{ab}) . \tag{4.49}$$

Thus, by inserting the definitions we get

$$W\left(j_{ab}\right)=\int dU_{ab}\int dU_a\prod_{ab}\delta\left(U_aU_{ab}U_b^{-1}\right)\otimes_a\iota_a\cdot\prod_{ab}D^{j_{ab}}\left(U_{ab}\right)\,. \tag{4.50}$$

Performing the integral we obtain

$$W\left(j_{ab}\right)=\int dU_a\prod_{ab}\otimes_a\iota_a\cdot\prod_{ab}D^{j_{ab}}\left(U_a\right)D^{j_{ab}}\left(U_b^{-1}\right)\,. \tag{4.51}$$

It is possible to show that the overall result of this integral is:

$$W\left(j_{ab}\right)=\mathrm{Tr}\left[\otimes_a\iota_a\right]\,, \tag{4.52}$$

which coincides precisely with the $6j$-symbol (since it is the invariant contraction of four $3j$-symbols). This result tells us that the $6j$-symbol can be thought as the Fourier transform of the gauge-invariant delta functions on flat connections, in the Hilbert space associated with the tetrahedral graph. This can be written in the notation

$$W\left(j_{ab}\right)=\psi_{j_{ab}}\left(\mathbb{1}\right)\,, \tag{4.53}$$

or, by using the projector $P_{SU(2)}$ on the $SU(2)$ invariant part of a function, the vertex amplitude can be written as

$$\langle\psi_{\mathbf{v}}|W_{\mathbf{v}}\rangle=\left(P_{SU(2)}\psi_{\mathbf{v}}\right)\left(\mathbb{1}\right)\,, \tag{4.54}$$

where ψ_v is a state in the boundary of a vertex.

We can summarize the properties of the amplitude by pointing out the following features:

1. **Superposition principle**: this is the basic principle of quantum mechanics, the amplitude is given by the sum of elemetary amplitudes, that is, by a Feynman's sum over the possible paths σ:

$$\langle W|\psi\rangle=\sum_\sigma W(\sigma)\,. \tag{4.55}$$

2. **Locality**: the elementary amplitudes can be seen as products of amplitudes associated with spacetime points (in QFT the product is expressed as the exponential of an integral on spacetime):

$$W(\sigma)\sim\prod_{\mathbf{v}}W_{\mathbf{v}}\,. \tag{4.56}$$

3. **Local euclidean invariance**: the $6j$-symbol can be written as the projection on the $SU(2)$-invariant part of the state on the boundary graph of the vertex, i.e.

$$W_{\mathbf{v}}=\left(P_{SU(2)}\psi_{\mathbf{v}}\right)\left(\mathbb{1}\right)\,. \tag{4.57}$$

These properties will be found also in the 4-dimensional theory.

4.2 4D Theory

Following the same line of reasoning of the previous section and recalling the results obtained in the four-dimensional discretization we are ready to face the four-dimenisonal quantization.

4.2.1 Hilbert Space

The boundary Hilbert space we are interested in is obtained in the same manner of the three-dimensional one: the variables $B_l \in \mathrm{sl}(2, \mathbb{C})$ and $U_l \in SL(2, \mathbb{C})$ become operators in the quantum theory, the states are given by $\psi(U_l)$, functions on $SL(2, \mathbb{C})^L$ and the operator $B_l \in \mathrm{sl}(2, \mathbb{C})$ is realized as the generator of $SL(2, \mathbb{C})$ transformations. We recall that B on the boundary is split into its electric and magnetic parts, and these are constrained by $\overrightarrow{K} = \gamma \overrightarrow{L}$, therefore we expect this condition continues to hold, at least in the classical limit. Keeping this constraint in the quantum case has crucial consequences, it completley determines the dynamics of LQG.
Furthermore, we recall from eq. (3.21) that $|L_{\mathbf{f}}| = \frac{1}{\gamma} A_{\mathbf{t_f}}$ and this, together with eq (4.6), which now reads

$$L_{j_l} = 8\pi\hbar G\gamma\sqrt{j_l(j_l+1)}\,, \tag{4.58}$$

suggests that the scale of LQG is given by $L^2_{\mathrm{loop}} = 8\pi\hbar G\gamma$. Then it can be stated, that, since the value of the Barbero-Immirzi constant γ is of order unity ($\gamma \sim 0.274067$ is the value fixed by the Bekenstein-Hawking entropy) the scale of LQG is of the same order of the Planck scale ($L^2_{\mathrm{Planck}} = \hbar G$).
To begin with, we are interested in irreducible unitary representations of $SL(2, \mathbb{C})$, these are labeled by a positive real number p and a non-negative half-integer k. The Hilbert space $V^{(p,k)}$ of the (p, k) representation decomposes into irreducibles representations of $SU(2) \subset SL(2, \mathbb{C})$ as follows:

$$V^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j\,, \tag{4.59}$$

where $\mathcal{H}_j$ is the $2j + 1$-dimensional space that carries the spin j irreducible representation of $SU(2)$. Therefore, we can choose a basis of states $|p, k; j, m\rangle$, with $j = k, k + 1, \ldots$ and $m = -j, \ldots, j$. The quantum numbers (p, k) are related to the two Casimir operators of $SL(2, \mathbb{C})$ by

$$\begin{aligned} |\overrightarrow{K}|^2 - |\overrightarrow{L}|^2 &= p^2 - k^2 + 1\,, \\ \overrightarrow{K} \cdot \overrightarrow{L} &= pk\,, \end{aligned} \tag{4.60}$$

where j and m are the quantum numbers of $|\overrightarrow{L}|^2$ and L_z respectively. Now, taking into account the linear simplicity constraint for large quantum numbers means that

$$\begin{aligned} |\overrightarrow{K}|^2 - |\overrightarrow{L}|^2 &= \left(\gamma^2 - 1\right) |\overrightarrow{L}|^2\,, \\ \overrightarrow{K} \cdot \overrightarrow{L} &= \gamma |\overrightarrow{L}|^2\,, \end{aligned} \tag{4.61}$$

and so, by means of (4.59) we get

$$\begin{aligned} p^2 - k^2 + 1 &= \left(\gamma^2 - 1\right) j(j+1)\,, \\ pk &= \gamma j(j+1)\,. \end{aligned} \tag{4.62}$$

In the large quantum numbers limit we then obtain

$$\begin{aligned} p^2 - k^2 + 1 &= \left(\gamma^2 - 1\right) j^2 \,, \\ pk &= \gamma j^2 \,, \end{aligned} \tag{4.63}$$

which is solved by

$$\begin{aligned} p &= \gamma k \,, \\ k &= j \,. \end{aligned} \tag{4.64}$$

The first of these two equations is a restriction on the set of unitary representations, whereas the second one picks out a subspace within each representation (the lowest one).
Thus, the states that satisfy these relations have the form

$$|p.k; j, m\rangle = |\gamma j, j; j, m\rangle \,. \tag{4.65}$$

Clearly these states are in one-to-one correspondence with the states in the representations of $SU(2)$. It is legit then to introduce a map Y_γ defined by

$$\begin{aligned} Y_\gamma \;:\quad \mathcal{H}_j &\longrightarrow V^{(p=\gamma j,\, k=j)} \\ |j; m\rangle &\longmapsto |\gamma j, j; j, m\rangle \,, \end{aligned} \tag{4.66}$$

and all the vectors in the image of this map satisfy the linear simplicity constraint, in the sense that

$$\langle Y_\gamma \psi | \overrightarrow{K} - \gamma \overrightarrow{L} | Y_\gamma \phi \rangle = 0 \,, \tag{4.67}$$

holds in the large j limit. For this reason, we assume that the states of the four-dimensional theory are constructed from the states $|\gamma j, j; j, m\rangle$ alone.
The map Y_γ can be extended to a map from functions over $SU(2)$ to functions over $SL(2,\mathbb{C})$, namely

$$\begin{aligned} Y_\gamma \;:\qquad L_2[SU(2)] &\longrightarrow F[SL(2,\mathbb{C})] \\ \psi(h) = \sum_{jmn} c_{jmn} D^{(j)}_{mn}(h) &\longmapsto \psi(g) = \sum_{jmn} c_{jmn} D^{(\gamma j, j)}_{mn}(g) \,, \end{aligned} \tag{4.68}$$

This is the way to map $SU(2)$ spin-networks into $SL(2,\mathbb{C})$ spin-networks.
The physical states of quantum gravity are thus, essentially, $SU(2)$ spin-networks. This fact is consistent with the classical theory expressed in terms of the Ashtekar variables, which form the same kinematical phase space of a $SU(2)$ Yang-Mills theory.
Following the same line of reasoning of the three-dimensional case, we would like to find the gauge-invariant states. In order to do this we decompose the Hilbert space as

$$L_2\left[SU(2)^L\right] = \otimes_l \left[\oplus_j \left(\mathcal{H}_j \otimes \mathcal{H}_j\right)\right] = \oplus_{j_l} \otimes_l \left(\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}\right) \tag{4.69}$$

and so

$$L_2\left[SU(2)^L / SU(2)^N\right] = \oplus_{j_l} \otimes_n \mathrm{Inv}_{SU(2)} \left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}\right) \tag{4.70}$$

where clearly we have an additional factor due to the fact that now each edge is bounded by four faces, not three. The space $\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}\right)$ is not one-dimensional in general, it turns out that linearly independent invariant tensors in this space can be constructed as follows:

$$\iota_k^{m_1m_2m_3m_4} = \begin{pmatrix} j_1 & j_2 & k \\ m_1 & m_2 & m \end{pmatrix} g_{mn} \begin{pmatrix} k & j_3 & j_4 \\ n & m_3 & m_4 \end{pmatrix}, \tag{4.71}$$

for any k that satisfies the triangular relations both with j_1, j_2 and j_3, j_4, more precisely:

$$\max[|j_1 - j_2|, |j_3 - j_4|] \le k \le \min[j_1 + j_2, j_3 + j_4]\,. \tag{4.72}$$

We denote these states with $|k\rangle$ and the invariant subspace as

$$\mathcal{K}_{j_1\dots j_4} := \mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}\right)\,, \tag{4.73}$$

whose dimension is therefore

$$\dim[\mathcal{K}_{j_1\dots j_4}] = \min[j_1 + j_2, j_3 + j_4] - \max[|j_1 - j_2|, |j_3 - j_4|] + 1\,. \tag{4.74}$$

It follows that a generic gauge-invariant state is a linear combination

$$\psi\left(U_l\right) = \sum_{j_l k_{\mathbf{n}}} \mathcal{C}_{j_l k_{\mathbf{n}}} \psi_{j_l k_{\mathbf{n}}}\left(U_l\right)\,, \tag{4.75}$$

of the orthogonal states

$$\psi_{j_l k_{\mathbf{n}}}\left(U_l\right) = \iota_{k_1}^{m_1m_2m_3m_4}\cdots\iota_{k_N}^{m_{L-3}m_{L-2}m_{L-1}m_L} D^{j_1}_{m_1n_1}\cdots D^{j_L}_{m_Ln_L}\,. \tag{4.76}$$

The difference from the three-dimensional case is that the spin-networks of the four-dimensional case are labeled not only by spins, but also by an intertwine quantum number k associated to each node n. Using a more compact notation we denote the spin-network wave functions as

$$\psi_{j_l k_{\mathbf{n}}}\left(U_l\right) = \langle U_l | j_l, k_{\mathbf{n}}\rangle = \bigotimes_{\mathbf{n}} \iota_{k_{\mathbf{n}}} \cdot \bigotimes_{l} D^{j_l}\left(U_l\right)\,. \tag{4.77}$$

At a classical level, this residual geometric freedom at each node is described by the space of possible shapes of a tetrahedron with fixed areas, which is a two-dimensional space (coordinatized for instance by two opposite dihedral angles). This space can also be seen as the space of quadruplets of vectors satisfying the closure relation, with given areas, up to global rotations, the counting of the dimension gives: $4 \times 3 - 4 - 3 - 3 = 2$.
An observable on this space is given by the volume V of the tetrahedron, which is given by

$$V^2 = \frac{2}{9}\epsilon_{ijk}E^iE^jE^k\,. \tag{4.78}$$

where the operator $\overrightarrow{E}$ is associated with each link and it is given by

$$\overrightarrow{E}_l = 8\pi\gamma\hbar G\overrightarrow{L}_l\,. \tag{4.79}$$

The matrix elements of V can be computed in the $|k\rangle$ basis, then by diagonalization of this matrix it is possible to obtain the eigenvalues v and their correspondent eigentates $|v\rangle$ of the volume in each Hilbert space $\mathcal{K}_{j_1 \ldots j_4}$. In this basis, the spin-network states can be written as

$$\psi_{j_l v_{\mathbf{n}}}(U_l) = \langle U_l | j_l, v_{\mathbf{n}} \rangle = \bigotimes_{\mathbf{n}} \iota_{v_{\mathbf{n}}} \cdot \bigotimes_{l} D^{j_l}(U_l) \ . \tag{4.80}$$

To summarize, the Hilbert space associated with the boundary graph Γ is given by

$$\mathcal{H}_\Gamma = L_2[SU(2)^L / SU(2)^N] \tag{4.81}$$

and spin-network states are denoted by $|\Gamma, j_l, v_{\mathbf{n}}\rangle$, where j_l is a spin associated with each link of the graph and $v_{\mathbf{n}}$ is a volume eigenvalue associated with each node of the graph.
This formalism is referred to as "spinfoam", where "foam" refers to a 2-complex and "spin" is obviuosly associated to the spin representation sitting on each edge.

4.2.2 Transition Amplitude

To complete the description of the full theory we need to write down the transition amplitude. First of all, we give an alternative form of the amplitude which will be more suitable for the four-dimensional case. We start from

$$Z = \int dU_{\mathbf{e}} \prod_{\mathbf{f}} \delta\left(U_{\mathbf{e}_1} \cdots U_{\mathbf{e}_n}\right) \ . \tag{4.82}$$

At this point, we introduce two group variables per each edge $\mathbf{e}$, that is, $U_{\mathbf{e}} = g_{\mathbf{ve}} g_{\mathbf{ev}'}$, where $g_{\mathbf{ev}} = g_{\mathbf{ve}}^{-1}$ is a variable associated with each couple vertex-edge. Thus, we can write

$$Z = \int dg_{\mathbf{ve}} \prod_{\mathbf{f}} \delta\left(g_{\mathbf{ve}} g_{\mathbf{ev}'} g_{\mathbf{v}'\mathbf{e}'} g_{\mathbf{e}'\mathbf{v}''} \cdots\right) \ . \tag{4.83}$$

Then we regroup the $g_{\mathbf{ev}}$ variables in a different way, namely we define $h_{\mathbf{vf}} = g_{\mathbf{ev}} g_{\mathbf{ve}'}$, where $\mathbf{e}$ and $\mathbf{e}'$ are the two edges coming out from the vertex $\mathbf{v}$ and bounding the face $\mathbf{f}$.

Clearly, the amplitude takes the form

$$Z = \int dh_{\mathbf{vf}} dg_{\mathbf{ve}} \prod_{\mathbf{f}} \delta\left(g_{\mathbf{ve}} g_{\mathbf{ev}'} g_{\mathbf{v}'\mathbf{e}'} g_{\mathbf{e}'\mathbf{v}''} \cdots\right) \prod_{\mathbf{vf}} \delta\left(g_{\mathbf{e}'\mathbf{v}} g_{\mathbf{ve}} h_{\mathbf{vf}}\right) \ . \tag{4.84}$$

This can be reorganised as a transition amplitude where a delta function glues the group element around each face:

$$Z = \int dh_{\mathbf{vf}} \prod_{\mathbf{f}} \delta\left(h_{\mathbf{f}}\right) \prod_{\mathbf{v}} A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) \ , \tag{4.85}$$

where $h_{\mathbf{f}} := \prod_{\mathbf{v} \in \partial \mathbf{f}} h_{\mathbf{vf}}$ is a group variable associated with a face.

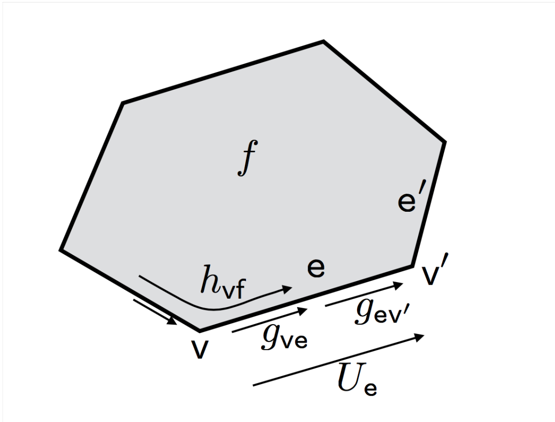

Figure 4.1: Splitting of the group elements

Furthermore, the vertex amplitude is defined by

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) := \int dg_{\mathbf{ve}} \prod_{\mathbf{f}} \delta\left(g_{\mathbf{e'v}} g_{\mathbf{ve}} h_{\mathbf{vf}}\right) . \tag{4.86}$$

The $SU(2)$ integrals in a vertex are $n = 4$, that is, one group element for each of the $n = 4$ edges coming out of the vertex. However, if one thinks about it, there is one redundant integral, because after integrating $n-1$ group variables the result is not affected by the last integration. We denote this fact by

$$\int_{SU(2)^n} dg'_{\mathbf{ve}} := \int_{SU(2)^{(n-1)}} dg_{\mathbf{ve_1}} \cdots dg_{\mathbf{ve_{n-1}}} . \tag{4.87}$$

In three dimensions this observation does not change anything: performing the last integral gives unity, since the volume of $SU(2)$ is just one, but in the four-dimensional case this turns out to be crucial, because $SL(2,\mathbb{C})$ is non-compact. If we expand the delta function in representations we get

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) = \sum_{j_{\mathbf{f}}} \int dg'_{\mathbf{ve}} \prod_{\mathbf{f}} \left(2j_{\mathbf{f}}+1\right) \mathrm{Tr}_{j_{\mathbf{f}}}\left[g_{\mathbf{e'v}} g_{\mathbf{ve}} h_{\mathbf{vf}}\right] , \tag{4.88}$$

where $\mathrm{Tr}_j(U) := \mathrm{Tr}\left[D^j(U)\right]$. Therefore the vertex amplitude is a function of one $SU(2)$ variable per face around the vertex. We can also picture this by drawing a sphere around the vertex, the intersection between this sphere and the 2-complex is a graph, $\Gamma_{\mathbf{v}}$. The vertex amplitude is then a function of the states in

$$\mathcal{H}_{\Gamma_{\mathbf{v}}} = L_2\left[SU(2)^6/SU(2)^4\right]_{\Gamma_{\mathbf{v}}} , \tag{4.89}$$

where $\Gamma_{\mathbf{v}}$ is the complete graph with four nodes and represents the boundary graph of the vertex. Therefore, we can express the transition amplitude in the following way:

$$W(h_l) = \int dh_{\mathbf{vf}} \prod_{\mathbf{f}} \delta(h_{\mathbf{f}}) \prod_{\mathbf{v}} A_{\mathbf{v}}(h_{\mathbf{vf}}) \,, \tag{4.90}$$

where the vertex amplitude is given by

$$A_{\mathbf{v}}(h_{\mathbf{vf}}) = \mathcal{N} \sum_{j_{\mathbf{f}}} \int dg'_{\mathbf{ve}} \prod_{\mathbf{f}} (2j_{\mathbf{f}}+1) \operatorname{Tr}_{j_{\mathbf{f}}} [g_{\mathbf{e'v}} g_{\mathbf{ve}} h_{\mathbf{vf}}] \,,. \tag{4.91}$$

We are ready now to treat the four-dimensional case.
We notice that the form of the transition amplitude is the same as in (4.84), since this only reflects the superposition principle, therefore the dynamics is contained in the vertex amplitude. The vertex amplitude in turn must be $SL(2,\mathbb{C})$-invariant in the four-dimensional case, but $A_{\mathbf{v}}(h_{\mathbf{vf}})$ can be regarded only as a function of $SU(2)$ group elements living on the graph of a node (which is on the boundary of a 4-simplex). To obtain the analogue of (4.87) in the four-dimensional case then we have to replace the $SU(2)$ integrals with $SL(2,\mathbb{C})$ ones and to map the $SU(2)$ group elements into the $SL(2,\mathbb{C})$ ones. In order to do that we make use of the Y_γ map, as follows:

$$A_{\mathbf{v}}(\psi) = \left(P_{SL(2,\mathbb{C})} Y_\gamma \psi\right)(\mathbb{1}) \tag{4.92}$$

which, more expilicitly, it reads

$$A_{\mathbf{v}}(h_{\mathbf{vf}}) = \mathcal{N} \sum_{j_{\mathbf{f}}} \int dg'_{\mathbf{ve}} \prod_{\mathbf{f}} (2j_{\mathbf{f}}+1) \operatorname{Tr}_{j_{\mathbf{f}}} \left[Y_\gamma^\dagger g_{\mathbf{e'v}} g_{\mathbf{ve}} Y_\gamma h_{\mathbf{vf}}\right] \,, \tag{4.93}$$

where the trace is given by

$$\operatorname{Tr}_j \left[Y_\gamma^\dagger g Y_\gamma h\right] = \operatorname{Tr}_j \left[Y_\gamma^\dagger D^{(\gamma j, j)}(g) Y_\gamma D^{(j)}(h)\right] = \sum_{mn} D^{(\gamma j, j)}_{jm, jn}(g) D^{(j)}_{nm}(h) \,. \tag{4.94}$$

The vertex amplitude is then a function of one $SU(2)$ variable per face around the vertex. As seen before, we can picture a small sphere around a vertex, obtaining a graph $\Gamma_{\mathbf{v}}$, the vertex amplitude becomes thus a function of the states in

$$\mathcal{H}_{\Gamma_{\mathbf{v}}} = L_2 \left[SU(2)^{10}/SU(2)^5\right]_{\Gamma_{\mathbf{v}}} \,. \tag{4.95}$$

The graph $\Gamma_{\mathbf{v}}$ is the complete graph with five nodes.

4.2.3 Continuum Limit

We have seen above the equations that describe the theory on a given graph Γ, obtained from a given 2-complex $\mathcal{C}$. This is a theory with a finite number of degrees of freedom, beacuse it corresponds to a truncation of classical general relativity, which is a theory with an infinte number of degrees of freedom. The full theory is approximated by choosing increasingly refined complexes $\mathcal{C}$ and $\Gamma = \partial\mathcal{C}$, where the refinement is chosen in relation to the desired precision, in analogy with a finite order in

perturbation theory in QED. More precisely: let Γ' be a subgraph of Γ, namely, a graph formed by a subset of nodes and links of Γ, then, there is a subspace $\mathcal{H}_{\Gamma'} \subset \mathcal{H}_{\Gamma}$ which is isomorphic to the loop-gravity Hilbert space of the graph Γ'. Indeed, this is formed by all the states $\psi(U_l) \in \mathcal{H}_{\Gamma}$ which are independent of the group elements U_l associated with the links l that are in Γ but not in Γ'. Equivalently, $\mathcal{H}_{\Gamma'}$ is the linear span of the spin-network states characterized by $j_l = 0$ for any l that is in Γ but not in Γ'.
Therefore, if we define the theory on Γ we have at our disposal a subset of states that captures the theory defined on the smaller graph Γ', in this way, the step from Γ' to Γ is a refinement of the theory. More precisely, the continuum limit can be defined by

$$Z(h_l) = \lim_{\mathcal{C}\to\infty} Z_{\mathcal{C}}(h_l)\,, \tag{4.96}$$

which is well defined in the sense of nets, because two-complexes form a partially ordered set with upper bound. Nevertheless, there is not a unique notion of limit at the present time, and it is often said that the approximation is good when the discretized theory approximates the continuum theory in the classical context, that is, when the degree of accuracy of the triangulation meets the desired expectations.
When dealing with the continuum limit it is natural then to ask what happens to the tranistion amplitude when refining the triangulation. The simplest case to analyze is considering a single tetrahedron τ and adding a point P inside it, then joining P to the four vertices of τ. In this way the original tetrahedron has been split into four smaller tetrahedra. If we call Δ_1 the original triangulation and Δ_4 the new one, it's clear that, when dealing with the respective 2-complexes, the refinement produces a "bubble", as shown in figure.

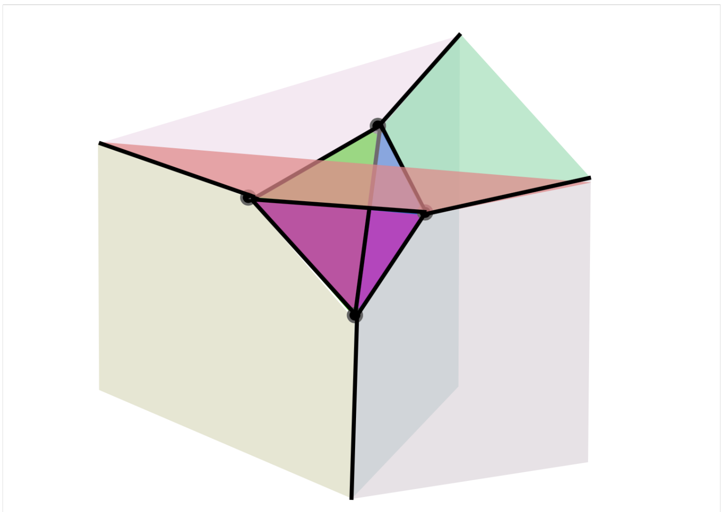

Figure 4.2: The graph Δ_4^*

Starting from (4.45) it is possible to compute the amplitude of this triangulation W_{Δ_4}. It can be shown [16] that the relation between W_{Δ_4} and the original W_{Δ_1} amounts to an infinite factor multipling the latter.
The appearance of the divergence is a manifestation of the standard quantum field theory divergences. It is strictly connected to the existence of the bubble. To see that this is the case, reconsider the same calculation in the spin representation. From eq. (4.41):

$$W_{\Delta_4}(j_{ab}) = \sum_{j_{ab}} \prod_{ab} d_{j_{ab}} \prod_{a} \{6j\} \,. \tag{4.97}$$

In general, in a sum like this the range of summation of the j_{ab} is restricted by the triangular identities. Since the boundary faces have finite spins, the only possibility for an internal face to have a large spin is to be adjacent, at each edge, to at least one other face with a large spin. In other words, a set of faces with arbitrary large spins cannot have boundaries. Therefore to have a sum which is not up to a maximum spin by the triangular identities the only possibility is to have a set of faces that form a surface without boundaries in the two complex. That is, a bubble. All this is very similar to the ultraviolet divergences in the Feynman expansion of a normal quantum field theory, where divergences are associated to loops, because the momentum is conserved at the vertices. Here, divergences are associated to bubbles, because angular momentum is conserved on the edges. A Feynman loop is a closed set of lines where arbitrary high momentum can circulate. A spinfoam divergence is a closed set of faces, that can have arbitrarily high spin. Notice however that in spite of the formal similarity there is an important difference in the physical interpretation of the two kinds of divergences. The Feynman divergences regards what happens at very small scale. On the contrary, the spinfoam divergences concern large spins, namely large geometries. Therefore they are not ultraviolet divergences, they are infrared. A way to get rid of these divergences is by considering the so-called "Turaev-Viro" amplitude, in which, instead of considering the group $SU(2)$, one chooses the group $SU(2)_q$ (q being a parameter), i.e. a one-parameter deformation of the algebra of the representations of $SU(2)$. The Turaev-Viro amplitude is given by:

$$W_q(j_l) = w_q^p \sum_{j_{\mathbf{f}}} \prod_{j_{\mathbf{f}}} (-1)^{j_{\mathbf{f}}} d_q(j_{\mathbf{f}}) \prod_{\mathbf{v}} (-1)^{J_{\mathbf{v}}} \{6j\}_q \,. \tag{4.98}$$

The remarkable fact, is that the dimension d_j^q has a maximum value [17], this finiteness makes the amplitude finite.
Furthermore, the parameter q can be put in relation with the cosmological constant $q = e^{i\sqrt{\Lambda}\hbar G}$ as shown in [18], thus relating the finiteness of the amplitude to the presence of the cosmological constant.

4.3 Classical Limit

The classical limit in covariant LQG is studied on the basis of the so-called coherent states: these are similar to wave packets in quantum mechanics, i.e. states in which both position and momentum are minimally spread. Geometrically, a tetrahedron is uniquely determined by giving six numbers, that is, the lengths of its sides, but we have seen before that a state associated to a node (and therefore to a tetrahedron) is characterized only by five numbers: four areas and the volume. In a sense, the geometry of the tetrahedron is fuzzy, in the same way angular momentum is, in quantum mechanics. Let's consider then a node n, we have a Hilbert space $\mathcal{H}_{\mathbf{n}}$, a basis of states is given by $|\iota_k\rangle$ defined in (4.70). It can be shown that these states are eigenstates of $\overrightarrow{L}_1 \cdot \overrightarrow{L}_2$, that is, they diagonalize the dihedral angle θ_{12} between the faces 1 and 2. We would like to find, given a classical tetrahedron, a quantum state whose dihedral angles are minimally spread around the classical variables. These states are called “intrinsic coherent states”.

4.3.1 Intrinsic Coherent States

We recall that a tetrahedron is characterized by four vectors $\overrightarrow{E}_a$ (one per each face) whose length is the area of the correspondent face; in the quantum theory these are quantized and they are given by

$$\overrightarrow{E}_a = 8\pi G\hbar\gamma \overrightarrow{L}_a\,. \tag{4.99}$$

From the following commutation relations

$$\left[L_a^i, L_b^j\right] = i\delta_{ab}\epsilon_k^{ij} L_a^k \tag{4.100}$$

it is possible to show that the commutation relations between two dihedral angles are given by

$$\left[\overrightarrow{E}_1 \cdot \overrightarrow{E}_2, \overrightarrow{E}_1 \cdot \overrightarrow{E}_3\right] = i8\pi G\hbar\gamma \overrightarrow{E}_1 \cdot (\overrightarrow{E}_2 \times \overrightarrow{E}_3)\,. \tag{4.101}$$

Then, from this, it follows the Heisenberg relation

$$\Delta(\overrightarrow{E}_1 \cdot \overrightarrow{E}_2) \cdot \Delta(\overrightarrow{E}_1 \cdot \overrightarrow{E}_2) \geq \frac{1}{2} 8\pi G\hbar\gamma |\langle \overrightarrow{E}_1 \cdot (\overrightarrow{E}_2 \times \overrightarrow{E}_3)\rangle|\,, \tag{4.102}$$

where $\langle A\rangle = \langle\iota|A|\iota\rangle$ and $\Delta A = \sqrt{\langle\iota|A^2|\iota\rangle - (\langle\iota|A|\iota\rangle)^2}$. Thus, we are aiming for states whose dispersion is small compared with their expectation value, that is

$$\frac{\Delta(\overrightarrow{E}_a \cdot \overrightarrow{E}_b)}{|\overrightarrow{E}_a||\overrightarrow{E}_b|} \ll 1 \qquad \forall a, b\,. \tag{4.103}$$

The first step is to consider $SU(2)$ coherent states. We start from a state of fixed total angular momentum j, $|j, m\rangle \in \mathcal{H}_j$ is then a basis of these states. Then, because $[L_x, L_y] = iL_z$, we have the Heisenber relation

$$\Delta L_x \Delta L_y \geq \frac{1}{2}|\langle L_z\rangle|\,, \tag{4.104}$$

which is satisfied by every state. A state that saturates this inequality can be shown to be given by $|j, j\rangle$.
Furthermore, there is an entire family of coherent states which can be obtained starting from the state $|j, j\rangle$, namely, by rotating the state by means of a matrix $R \in SO(3)$:

$$|j, \overrightarrow{n}\rangle = D_{\overrightarrow{n}}(R)|j, j\rangle\,, \tag{4.105}$$

where $\overrightarrow{n}$ is the direction obtain by starting from the z-axis and then applying the rotation. These coherent states can be expanded in terms of eigenstates of L_z as follows

$$|j, \overrightarrow{n}\rangle = \sum_m \phi_m(\overrightarrow{n})|j, m\rangle\,, \tag{4.106}$$

where $\phi_m(\overrightarrow{n}) = \langle j, m|D(R)|j, j\rangle = D^{(j)}(R)^j_m$.
One of the most important properties of the coherent states is that they provide a resolution of the identity, that is,

$$\mathbb{1}_j = \frac{2j+1}{4\pi}\int_{S^2} d^2\overrightarrow{n}|j, \overrightarrow{n}\rangle\langle j, \overrightarrow{n}|\,. \tag{4.107}$$

By means of these coherent states it is possible to describe a "coherent" tetrahedron, whose faces are described by coherent states. More precisely, let's consider the coherent state

$$|j_1, \overrightarrow{n}_1\rangle \otimes |j_2, \overrightarrow{n}_2\rangle \otimes |j_3, \overrightarrow{n}_3\rangle \otimes |j_4, \overrightarrow{n}_4\rangle \in \mathcal{H}_1 \otimes \cdots \mathcal{H}_4\,, \tag{4.108}$$

which is still a coherent state, since tensor products of coherent stantes are coherent, and project it down to its invariant part by means of

$$P\ :\ \mathcal{H}_1 \otimes \cdots \mathcal{H}_4 \to \mathrm{Inv}(\mathcal{H}_1 \otimes \cdots \mathcal{H}_4)\,. \tag{4.109}$$

Thus, we denote this coherent state as

$$||j_a, \overrightarrow{n}_a\rangle := P(|j_1, \overrightarrow{n}_1\rangle \otimes |j_2, \overrightarrow{n}_2\rangle \otimes |j_3, \overrightarrow{n}_3\rangle \otimes |j_4, \overrightarrow{n}_4\rangle)\,, \tag{4.110}$$

which is then the element of $\mathcal{H}_\Gamma$ that describes a semicalssical tetrahedron. More precisely, the projection can be explicitly implemented by the following

$$||j_a, \overrightarrow{n}_a\rangle = \int_{SO(3)} dR(|j_1, R\overrightarrow{n}_1\rangle \otimes |j_2, R\overrightarrow{n}_2\rangle \otimes |j_3, R\overrightarrow{n}_3\rangle \otimes |j_4, R\overrightarrow{n}_4\rangle)\,, \tag{4.111}$$

which can be translated in a $SU(2)$ integral as

$$||j_a, \overrightarrow{n}_a\rangle = \int_{SU(2)} dh(D^{j_1}(h)|j_1, \overrightarrow{n}_1\rangle \otimes D^{j_2}(h)|j_2, \overrightarrow{n}_2\rangle \otimes D^{j_3}(h)|j_3, \overrightarrow{n}_3\rangle \otimes D^{j_4}(h)|j_4, \overrightarrow{n}_4\rangle\,. \tag{4.112}$$

These states are also referred to as the "Livine-Speziale coherent intertwiners", since they are associated to a tetrahedron which is in turn associated to a node. It can be shown that these states can be expanded in any intertwiner basis:

$$||j_a, \overrightarrow{n}_a\rangle = \sum_k \Phi_k(\overrightarrow{n}_a)|\iota_k\rangle\,, \tag{4.113}$$

where the coefficients $\Phi_k(\overrightarrow{n}_a) = \iota^{m_1m_2m_3m_4}\psi_{m_1}(\overrightarrow{n}_1)\cdots\psi_{m_4}(\overrightarrow{n}_4)$, for large j, have the form
$\Phi_k(\overrightarrow{n}_a) \sim e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}}e^{ik\psi}$, i.e. they are concentrated around a single value k_0 which determines the value of the corresponding dihedral angle, and have a phase such that, when changing basis to a different intertwined basis, we still obtain a state concentrated around the same value.

For large j, these states satisfy the following properties

$$\langle\iota(n_l)|E_a \cdot E_b|\iota(n_l)\rangle \sim j_a j_b \vec{n}_a \cdot \vec{n}_b \tag{4.114}$$

and

$$\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|\vec{E}_a||\vec{E}_b|} \ll 1\,, \tag{4.115}$$

the last one proves that these are in fact coherent states.
Putting everything together, that is, combining coherent intertwiners at each node, we can define a coherent state in $\mathcal{H}_\Gamma$, which can be thought as a "wave packet" peaked on a classical triangulated geometry:

$$\psi_{j_l,\vec{n}_{s_l},\vec{n}_{t_l}}(U_l) = \otimes_l D^{(j_l)}(U_l) \cdot \otimes_{\mathbf{n}} \iota_{\mathbf{n}}(\vec{n}_l)\,. \tag{4.116}$$

4.3.2 Spinors

Coherent states provide a tool to perform the classical limit, but to reach that goal we need to exploit their relation with spinors. Spinors are the elements of the fundamental representation of $SU(2)$, namely, $\mathcal{H}_{\frac{1}{2}} = \mathbb{C}^2$, that coincides with the fundamental representation of $SL(2,\mathbb{C})$. We denote a spinor $\mathbf{z} \in \mathbb{C}^2$ by

$$\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} = z^A = |z\rangle\,. \tag{4.117}$$

The spinor $\mathbf{n} = (1,0)$ is the eigenvector of L_z with eigenvalue $\frac{1}{2}$ and unit norm, then, we can identify it with the state $|j=\frac{1}{2}, m=\frac{1}{2}\rangle$, which is a coherent state. Since all the coherent states in the $j=\frac{1}{2}$ representation are obtained by rotating $|\frac{1}{2},\frac{1}{2}\rangle$ and since rotation preserves the norm of a spinor, it follows that all normalized spinors $\mathbf{n}$ describe coherent states in the fundamental representation, that is,

$$|\mathbf{n}\rangle = |\frac{1}{2}, \vec{n}\rangle\,. \tag{4.118}$$

Now, with each spinor $\mathbf{n} \in \mathbb{C}^2$ we can associate a three-dimensional real vector by

$$\vec{n} = \langle\mathbf{n}|\vec{\sigma}|\mathbf{n}\rangle\,, \tag{4.119}$$

therefore we have that

$$\langle\mathbf{n}|\vec{L}|\mathbf{n}\rangle = \langle\mathbf{n}|\frac{\vec{\sigma}}{2}|\mathbf{n}\rangle = \frac{1}{2}\vec{n} = j\vec{n}\,. \tag{4.120}$$

Normalized spinors are coherent states for the normalized three-vector they define. This result can be extended to any representation, because the tensor product of coherent states is a coherent state.

Thus, it makes sense to consider the followng state

$$|j, \mathbf{n}\rangle = \underbrace{\mathbf{n} \otimes \cdots \otimes \mathbf{n}}_{2j}, \tag{4.121}$$

which coincides with the spin-j representation, and is precisely the coherent state $|j, \overrightarrow{n}\rangle$ that satisfies

$$\langle j, \mathbf{n}| \overrightarrow{L} |j, \mathbf{n}\rangle = j \overrightarrow{n} \,. \tag{4.122}$$

Armed with spinors we can look for a different realization of the spin-j representation, namely, the finite-dimensional vector space $\mathcal{H}_j$ can be realized as the space of the totally symmetric polynomial functions $f(\mathbf{z})$ of degree $2j$. In order to see this, we recall that the spin-j representation space $\mathcal{H}_j$ can be realized by symmetric tensors $y^{A_1 A_2 \ldots A_{2j}}$ with $2j$ indices. Therefore, the corresponding polynomial function of $\mathbf{z}$ is simply

$$f(\mathbf{z}) = y^{A_1 A_2 \ldots A_{2j}} z_{A_1} \cdots z_{a_{2j}} \,. \tag{4.123}$$

This function satisfies the homogeneity condition

$$f(\lambda \mathbf{z}) = \lambda^{2j} f(\mathbf{z}) \tag{4.124}$$

and the $SU(2)$ action on these functions is given by

$$(Uf)(\mathbf{z}) = f(U^T \mathbf{z}) \,. \tag{4.125}$$

We would like to see how coherent states look in this representation. For spin $1/2$, a coherent state is represented by the linear function

$$f_{\mathbf{n}}(\mathbf{z}) \sim n^A z_A \sim \langle \mathbf{z} | \mathbf{n} \rangle \,, \tag{4.126}$$

up to normalization. If we take the symmetrized tensor product of this state with itself $2j$-times, we obtain the coherent state in the j representation in the following form (including normalization)

$$f^{(j)}_{\mathbf{n}}(\mathbf{z}) = \sqrt{\frac{2j+1}{\pi}} \langle \mathbf{z} | \mathbf{n} \rangle^{2j} \,. \tag{4.127}$$

This relaization of $SU(2)$ representation spaces turns out to be very useful to relate $SU(2)$ representations with $SL(2, \mathbb{C})$ unitary representations. We have seen previously that these representations were given by $V^{(p,k)}$, but again we would like to write these spaces in terms of functions of spinors $f(\mathbf{z})$, with $\mathbf{z} \in \mathbb{C}^2$. The representation (p, k) is defined on the space of the homogeneous functions of spinors that have the property

$$f(\lambda \mathbf{z}) = \lambda^{(-1+ip+k)} \overline{\lambda}^{(-1+ip-k)} f(\mathbf{z}) \tag{4.128}$$

and the $SL(2, \mathbb{C})$ action reads

$$g f(\mathbf{z}) = f(g^T \mathbf{z}) \,. \tag{4.129}$$

The transition between the canonical basis and the spinor basis can be shown to be given by

$$f_m^j(\mathbf{z}) = \langle \mathbf{z}|p,k;j,m\rangle = \sqrt{\frac{2j+1}{\pi}}\langle \mathbf{z}|\mathbf{z}\rangle^{ip-1-j} D^j_{mk}(g(\mathbf{z})) \tag{4.130}$$

where

$$g(\mathbf{z}) = \begin{pmatrix} z_0 & \overline{z}_1 \\ z_1 & \overline{z}_0 \end{pmatrix} . \tag{4.131}$$

In these representations the scalar product between two functions is given by an integral in spinor space, that is, if f and g are functions of spinors, we have:

$$\langle f|g\rangle = \int \overline{f} g d\Omega \, , \tag{4.132}$$

where

$$d\Omega = \frac{i}{2}(z^0 dz^1 - z^1 dz^0) \wedge (\overline{z}^0 d\overline{z}^1 - \overline{z}^1 - d\overline{z}^0) \, . \tag{4.133}$$

These spinor representations are particurarly convenient because the Y_γ map takes a particurarly simple form in this language. Since the embedding of $\mathcal{H}_j$ in $V^{(p,k)}$ is given by

$$f(\mathbf{z}) \mapsto \langle \mathbf{z}|\mathbf{z}\rangle^{-1+ip-k} f(\mathbf{z}) \, , \tag{4.134}$$

we have then

$$Y_\gamma f(\mathbf{z}) = \langle \mathbf{z}|\mathbf{z}\rangle^{-1+(i\gamma-1)j} f(\mathbf{z}) \, . \tag{4.135}$$

This allows us to write the action of the Y_γ map on the coherent states:

$$\langle \mathbf{z}|Y_\gamma|j, \overrightarrow{n}\rangle = \sqrt{\frac{2j+1}{\pi}} \langle \mathbf{z}|\mathbf{z}\rangle^{-1+(i\gamma-1)j} \langle \mathbf{z}|\mathbf{n}\rangle^{2j} \, , \tag{4.136}$$

which we write also as

$$\langle \mathbf{z}|Y_\gamma|j, \overrightarrow{n}\rangle = \frac{\sqrt{2j+1}}{\sqrt{\pi}\langle \mathbf{z}|\mathbf{z}\rangle} e^{j[(i\gamma-1)\ln\langle \mathbf{z}|\mathbf{z}\rangle + 2\ln\langle \mathbf{z}|\mathbf{n}\rangle]} \, . \tag{4.137}$$

At this point, we would like to rewrite the amplitude in terms of spinors. In order to do so, we first recall that

$$A_{\mathbf{v}}\left(h_{\mathbf{vf}}\right) = \sum_{j_{\mathbf{f}}} \int_{SL(2,\mathbb{C}} dg'_{\mathbf{ve}} \prod_{\mathbf{f}} \left(2j_{\mathbf{f}}+1\right) \mathrm{Tr}_{j_{\mathbf{f}}} \left[Y_\gamma^\dagger g_{\mathbf{e'v}} g_{\mathbf{ve}} Y_\gamma h_{\mathbf{vf}} \right] \, , \tag{4.138}$$

which can be written, dropping the subscript $\mathbf{v}$ and labeling the edges emerging from the vertex with

$a, b = 1, \ldots, 5$ and the faces adjacent to the vertices as ab,

$$A_{\mathbf{v}}\left(h_{ab}\right) = \sum_{j_{ab}} \int_{SL(2,\mathbb{C}} dg'_a \prod_{ab} \left(2j_{ab}+1\right) \mathrm{Tr}_{j_{ab}} \left[Y_\gamma^\dagger g_a^{-1} g_b Y_\gamma h_{ab}\right] . \tag{4.139}$$

The trace in the last equation can be written inserting two resolutions of the identity in terms of coherent states

$$\mathrm{Tr}_j \left[Y_\gamma^\dagger g g' Y_\gamma h\right] = \int_{S^2} d\vec{n}\, d\vec{m} \langle j, \vec{m}| Y_\gamma^\dagger g g' Y_\gamma | j, \vec{n}\rangle \langle j, \vec{n}| h | j, \vec{m}\rangle . \tag{4.140}$$

The first matrix element can be expressed in terms of spinors:

$$\langle j, \vec{m}| Y_\gamma^\dagger g g' Y_\gamma | j, \vec{n}\rangle = \int_{\mathbb{C}^2} d\Omega \langle Y_\gamma j, \vec{m}| g\mathbf{z}\rangle \langle g'^\dagger \mathbf{z} | Y_\gamma j, \vec{n}\rangle . \tag{4.141}$$

Using (4.133) and introducing the notation

$$\mathbf{Z} = g\mathbf{z}\,, \quad \mathbf{Z}' = g'^\dagger \mathbf{z}\,, \tag{4.142}$$

we obtain

$$\langle j, \vec{m}| Y_\gamma^\dagger g g' Y_\gamma | j, \vec{n}\rangle = \frac{2j+1}{\pi} \int_{\mathbb{C}^2} \frac{d\Omega}{\langle \mathbf{Z}|\mathbf{Z}\rangle \langle \mathbf{Z}'|\mathbf{Z}'\rangle} e^{j\, S(\mathbf{n},\mathbf{m},\mathbf{Z},\mathbf{Z}')} , \tag{4.143}$$

where

$$S(\mathbf{n},\mathbf{m},\mathbf{Z},\mathbf{Z}') := \ln \frac{\langle \mathbf{Z}|\mathbf{m}\rangle^2 \langle \mathbf{Z}'|\mathbf{n}\rangle^2}{\langle \mathbf{Z}|\mathbf{Z}\rangle \langle \mathbf{Z}'|\mathbf{Z}'\rangle} + i\gamma \ln \frac{\langle \mathbf{Z}|\mathbf{Z}\rangle}{\langle \mathbf{Z}'|\mathbf{Z}'\rangle} . \tag{4.144}$$

We would like to insert this result in the expression of the amplitude. In order to do so, we choose a coherent state in $\mathcal{H}_{\Gamma_{\mathbf{v}}}$, that is, we pick a quadruplet of normalized vector $\vec{n}_{ab}$ for each node of $\Gamma_{\mathbf{v}}$, these define a state $|j_{ab}, \vec{n}_{ab}\rangle$. Therefore the amplitude takes the form:

$$A_{\mathbf{v}}\left(j_{ab}, \vec{n}_{ab}\right) \equiv \langle A_{\mathbf{v}} | j_{ab}, \vec{n}_{ab}\rangle = \int_{SL(2,\mathbb{C})} dg'_a \prod_{ab} \left(2j_{ab}+1\right) \langle j_{ab}, \mathbf{n}_{ab}| Y_\gamma^\dagger g_a^{-1} g_b Y_\gamma | j_{ba}, \mathbf{n}_{ba}\rangle . \tag{4.145}$$

Now, using the result in (4.137), we get

$$A_{\mathbf{v}}\left(j_{ab}, \vec{n}_{ab}\right) = \mu(j_{ab}) \int_{SL(2,\mathbb{C})} dg'_a \int_{\mathbb{C}^2} \frac{d\Omega_{ab}}{|\mathbf{Z}_{ab}||\mathbf{Z}_{ba}|} e^{\sum_{ab} j_{ab}\, S(\mathbf{n}_{ab},\mathbf{n}_{ba},\mathbf{Z}_{ab},\mathbf{Z}_{ba})} , \tag{4.146}$$

where $\mu(j_{ab}) = \prod_{ab} \frac{(2j_{ab}+1)^2}{\pi}$ and $\mathbf{Z}_{ab} = g_a \mathbf{z}_{ab}$ and $\mathbf{Z}_{ba} = g_b \mathbf{z}_{ab}$.
In order to perform the classical limit we have to take the limit of large quantum numbers, that is, when j_{ab} are large. In this limit, the integral in (4.146) can be computed using the saddle-point approximation, which, in d dimensions takes the form:

$$\int_{\mathbb{R}^d} dx^d g(x) e^{jf(x)} = \left(\frac{2\pi}{j}\right)^{\frac{d}{2}} (\det H_2 f)^{-\frac{1}{2}} g(x_0) e^{jf(x_0)} \left[1 + o\left(\frac{1}{j}\right)\right] , \tag{4.147}$$

where $H_2 f$ is the Hessian of f at the saddle point x_0, which is the point where the gradient of f

vanishes. Now, if f is real and negative, a large j gives a narrow gaussian around the maximum of f; if f is imaginary, when j is large, the exponential oscillates very rapidly and the integral is canceled out except for the points where the derivative of f vanishes.
For this reason, we start from the real part of the action, this is given by:

$$\mathrm{Re}[S] = \sum_{ab} \log \frac{|\langle \mathbf{Z}_{ab}|\mathbf{n}_{ab}\rangle|^2\,|\langle \mathbf{Z}_{ba}|\mathbf{n}_{ba}\rangle|^2}{\langle \mathbf{Z}_{ab}|\mathbf{Z}_{ab}\rangle\langle \mathbf{Z}_{ba}|\mathbf{Z}_{ba}\rangle}\,. \tag{4.148}$$

The maximum is obtained when the logarithm vanishes, that is, when

$$\mathbf{n}_{ab} = e^{i\phi_{ab}} \frac{\mathbf{Z}_{ab}}{|\mathbf{Z}_{ab}|} \quad \mathbf{n}_{ba} = e^{i\phi_{ba}} \frac{\mathbf{Z}_{ba}}{|\mathbf{Z}_{ba}|}\,, \tag{4.149}$$

which, by definition of $\mathbf{Z}$, turns into

$$g_a^{-1}\mathbf{n}_{ab} = \frac{|\mathbf{Z}_{ba}|}{|\mathbf{Z}_{ab}|} e^{i\theta_{ab}} g_b^{-1}\mathbf{n}_{ba}\,. \tag{4.150}$$

At this point, we look at the extrema of the action under a variation of the spinor variables $\mathbf{z}_{ab}$. The explicit calculation gives

$$g_a\mathbf{n}_{ab} = \frac{|\mathbf{Z}_{ba}|}{|\mathbf{Z}_{ab}|} e^{i\theta_{ab}} g_b\mathbf{n}_{ba}\,. \tag{4.151}$$

Next, we consider a variation with respect to the group elements g_a and the spinor variables $\mathbf{z}_{ab}$. The former variation gives the action of the algebra elements, therefore the saddle-point equations for the group elements give the vanishing of the action of an infinitesimal $SL(2,\mathbb{C})$ transformation. This action can be decomposed into boosts and rotations, but in the relevant representations these are proportional, and so the needed invariance is only under rotations. In lue of (4.149), this can be moved from the variables $\mathbf{Z}$ (which contain the group elements) to the normals, thus obtaining [19]

$$\sum_b j_{ab}|\mathbf{n}_{ab}\rangle = 0\,. \tag{4.152}$$

This equation shows exactly the closure conditions for the normal at each of the boundary nodes of the vertex graph. This is remarkable, because the initial set of normals is arbitrary; then, the dynamics suppreses all the possible sets of $\mathbf{n}_{ab}$ unless these satisfy the closure constraint at each node. Therefore, the normals define a proper tetrahedron τ_a at each node a of the vertex graph. We have then five tetrahedra in the vertex graph (which is the complete graph with five nodes), that is, one for each boundary node. These tetrahedra are three-dimensional objects, we can think of them as lying in a common three-dimensional surface Σ of Minkowski space, left invariant by the $SU(2)$ action. Now, a vector in Σ defines a surface in Σ to which it is normal, then, a Lorentz transformation can act on this surface and move it to an arbitrary (spacelike) surface. In terms of spinors, this action is given by the action an element of $SL(2,\mathbb{C})$ on the spinor associated to the surface. This reasoning allows to interpret (4.152) in the following way: there are five Lorentz transformations g_a that rotate the five tetrahedra τ_a in such a way that the b face of the tetrahedron τ_a is parallel to the a face of the tetrahedron τ_b. The value of the action at the saddle point can be shown to be given by [19]

$$S = i\gamma \sum_{ab} j_{ab}\Theta_{ab}\,, \tag{4.153}$$

where Θ_{ab} is the difference between the Lorentz transformations to the opposite sides of adjacent tetrahedra, that is, it is the dihedral angle between two tetrahedra. We recall that γj_{ab} is the area of the boundary faces of the 4-simplex, in units where $8\pi G\hbar = 1$, therefore, S on the critical point is the Regge action of the 4-simplex having the boundary geometry determined by the 10 areas j_{ab}.

4.3.3 Classical Limit versus Continuum Limit

The classical limit is obtained when considering a fixed triangulation and then taking the large-j limit of the transition amplitude, whereas the continuum limit is obtained by refining the 2-complex $\mathcal{C}$. The two procedures are obviously not equivalent, but the strategy to obtain the Hamilton function of General Relativity from the transition amplitude involves both. Indeed, one can perform the classical limit in the first place, thus obtaining the Regge Hamilton function, and then perform the continuum limit by considering more refined discretizations. The latter limit is known to converge to the General Relativity Hamilton function as mentioned earlier.
Now, the regimes where the classical limit is good in quantum gravity are those involving scales L that are much larger than the Planck scale:

$$L \gg L_{\text{Planck}} \,. \tag{4.154}$$

The regimes where the truncation is good are suggested by the Regge approximation, that is, the deficit angles have to be small. This happens when the scale of the discretization is small with respect to the curvature scale $L_{\text{curvature}}$:

$$L \ll L_{\text{curvature}} \,. \tag{4.155}$$

Therefore a triangulation with few cells, and, correspondingly, a two-complex with few vertices, provide an approximation in the regimes (determined by the boundary data) where the size of the cells considered is small with respect to the curvature scale (of the classical solution of the Einstein's equation determined by the given boundary data).
Refining the triangulation leads to including shorter length-scale degrees of freedom. But the physical scale of a spinfoam configuration is not given by the graph or the two complex. It is given by the size of its geometrical quantities, which is determined by the spins (and intertwiners). The same triangulation can represent both a small and a large size of spacetime. A large chunk of nearly flat spacetime can be well approximated by a coarse triangulation, while a small chunk of spacetime where the curvature is very high requires a finer triangulation. In other words, triangulations do not need to be uselessly fine, they need to be just as fine as to to capture the relevant curvature.

4.3.4 Extrinsic Coherent States

We would like to build, for practical applications, states which are coherent both in the intrinsic and extrinsic geometry, since we recall that the extrinsic curvature is the variable conjugate to the 3-metric in the ADM variables. In order to introduce extrinsic coherent states, we recall that a wave packet in quantum mechanics peaked on the phase space point (q, p) is of the form

$$\langle x|q,p\rangle \equiv \psi_{q,p}(x) = e^{-\frac{(x-q)^2}{2\sigma^2} + \frac{i}{\hbar}px} \,. \tag{4.156}$$

Its Fourier transform is proportional to

$$\langle k|q,p\rangle \sim e^{-\frac{(k-p/\hbar)^2}{2/\sigma^2}+iqk}\,. \tag{4.157}$$

We can rewrite this state also as

$$\psi_{q,p}(x) = e^{-\frac{(x-z)^2}{2\sigma^2}}\,, \tag{4.158}$$

where z is the complex variable given by

$$z = q - i\frac{\sigma^2}{\hbar}p\,. \tag{4.159}$$

We need to find the analogue of this state in $\mathcal{H}_\Gamma$. Starting from $L_2[SU(2)]$, we notice that a state peaked on group variables is given by a delta function:

$$\psi(U) = \delta(Uh^{-1})\,, \tag{4.160}$$

$\psi(U)$ is a state sharp on the element $h \in SU(2)$. This state is, on the other hand, completely spread in the conjugate variable since

$$\delta(U) = \sum_j d_j \mathrm{Tr}_j[U]\,. \tag{4.161}$$

It is possible to obtain a state peaked on the value $j = 0$ by adding an exponential factor, more precisely,

$$\psi_{h,0}(U) = \sum_j d_j e^{-tj(j+1)} \mathrm{Tr}_j[Uh^{-1}] \tag{4.162}$$

is a state peaked on $U = h$ and $j = 0$. By complexifing the group variable it is possible to get a state peaked on a generic $j \neq 0$, in analogy with the wave packet seen before, where, in that case, the factor needed was $e^{ipx/\hbar}$. A complexification of $SU(2)$ is given by $SL(2,\mathbb{C})$, for this reason we consider the following state:

$$\psi_H(U) = \sum_j d_j e^{-tj(j+1)} \mathrm{Tr} D^{(j)}[UH^{-1}]\,, \tag{4.163}$$

where $H \in SL(2,\mathbb{C})$ is given by

$$H = e^{it\frac{E}{l_0^2}} h\,, \tag{4.164}$$

with $h \in SU(2)$ and $E \in \mathrm{su}(2)$. This state can be regarded as a wave packet peaked both on the group variable and its conjugate, since it is possible to show that:

$$\frac{\langle\psi_H|U|\psi_H\rangle}{\langle\psi_H|\psi_H\rangle} = h \quad , \quad \frac{\langle\psi_H|\vec{E}|\psi_H\rangle}{\langle\psi_H|\psi_H\rangle} = \vec{E}\,. \tag{4.165}$$

In order to generalize these states to spin-network states, it is necessary to make them invariant under $SU(2)$ at the nodes, therefore, an extrinsic coherent state on a graph Γ is labeled by a $SL(2,\mathbb{C})$ variable H_l associated with each link and is given by

$$\psi_{H_l}(U_l) = \int_{SU(2)} dh_{\mathbf{n}} \prod_l \sum_{j_l} d_{j_l} e^{-tj_l(j_l+1)} \mathrm{Tr} D^{(j_l)}[U_l h_{s_l} H_l^{-1} h_{t_l}^{-1}] \,. \tag{4.166}$$

Extrinsic coherent states represent the ideal tools when studying cosmology. More precisely, if we write the Hamilton function associated to a homogeneous and isotropic geometry, i.e. the one associated to the Friedmann-Lemaître metric, and then write down the expected form of the transition amplitude, it is possible to obtain the same behaviour starting from two extrinsic coherent states and then performing the classical limit [20] (large spins and saddle point).

Appendices

Appendix A

Lie Algebra

A.1 Left-Invariant Vector Fields

In this section we introduce the basic notions in order to address the standard formulation of a gauge theory from a mathemathical perspective. In addition to this we add some important issues concerning the quantization in LQG and a different formulation of General Relativity based on the so called tetrad fields.

Let's consider a Lie group G, a vector field $X \in \mathcal{T}(G)$ is called left-invariant if

$$l_{g*} X_{g\prime} = X_{gg\prime} \quad \forall g, g\prime \in G \tag{A.1}$$

where l_g is the left multiplication by g , i.e $l_g : G \to G$, $h \mapsto gh$.
We denote by $L(G)$ the vector space of left-invariant vector fields, one can easily show that it is a Lie subalgebra of $\mathcal{T}(G)$, this is due to the fact that X as in (1) is the field l_g-correlated to itself and so the Lie bracket of two left-invariant vector fields is still left-invariant.
In addition to this, one remarkable property is that $L(G)$ is isomorphic to T_eG, the latter being known as the Lie algebra of the Lie group G. The map that does the job is given by $i : T_eG \longrightarrow L(G)$, $A \mapsto L^A$, where $L^A \in L(G)$ is defined by ${L_g}^A := l_{g*}A$.
An important feature of left-invariant vector fields is that they are complete, in the sense that if $X \in L(G)$ then its integral curve is defined everywhere on $\mathbb{R}$, that is we have $\sigma^X : \mathbb{R} \longrightarrow G$ such that ${\sigma_*}^X(\frac{d}{dt}) = X$.
This fact allows us to define a map from T_eG to G called the *exponential map*, in the following way: first of all, the unique integral curve $t \mapsto \sigma^{L^A}(t)$ of $L^A \in L(G)$ such that $\sigma^{L^A}(0) = e$ and ${\sigma_*}^{L^A}(\frac{d}{dt})_0 = A$ is denoted by $t \mapsto \mathsf{exp}tA$, where $A \in T_eG$; then the *exponential map* is the map $\mathsf{exp} : T_eG \longrightarrow G$ defined by $\mathsf{exp} := \mathsf{exp}tA|_{t=1}$.
If we consider the case $G = GL(n,\mathbb{R})$ we can find a useful expression for a left-invariant vector field, introducing coordinates on $GL(n,\mathbb{R})$. First of all, we choose a coordinate system on $GL^+(n,\mathbb{R})$, which is the connected component containing matrices whose determinant is positive, then, in a neighbourhood of the identity we define:

$$x^{ij}(g) := g^{ij}, \quad g \in GL^+(n,\mathbb{R}), \quad i,j = 1,...,n\ . \tag{A.2}$$

Now, let $A \in T_eG \cong M(n,\mathbb{R})$, we get:

$$L_g^A = \sum_{i,j=1}^{n} \left(L^A x^{ij}\right)_g \left(\frac{\partial}{\partial x^{ij}}\right)_g \tag{A.3}$$

and furthermore

$$(L^A x^{ij})_g = \frac{d}{dt}\left(x^{ij}(g\mathsf{exp}tA)\right)_{t=0}, \tag{A.4}$$

where we used the definition of integral curve:

$$X(f)_p = \frac{d}{dt}\left(f(\sigma^X(t))\right)_{t=0}, \tag{A.5}$$

where $\sigma^X(0) = p$.
Since $A \in M(n, \mathbb{R})$ is a matrix , it is possible to consider the curve $t \mapsto e^{tA}$ in $GL^+(n, \mathbb{R})$, where e^{tA} is the exponential of matrix defined by means of a series. Clearly, the tangent vector to this curve in $t = 0$ is the matrix A, furthermore, the curve defines a one-parameter subgroup of $GL^+(n, \mathbb{R})$, thus, because every one-parameter subgroup is necessariely of the form $\mathsf{exp}(tA)$ we have that:

$$e^{tA} = \mathsf{exp}tA, \quad \forall t \in \mathbb{R}, \quad \forall A \in T_e G. \tag{A.6}$$

By means of (A.6) we can rewrite (A.4) as:

$$\begin{aligned}(L^A x^{ij})_g = \frac{d}{dt}\left(x^{ij}(ge^{tA})\right)_{t=0} &= \sum_{k=1}^{n} \frac{d}{dt}\left(e^{tA}\right)^{kj}|_{t=0} \\ &= \sum_{k=1}^{n} g^{ik} A^{kj} = (gA)^{ij}, \end{aligned} \tag{A.7}$$

thus we get the following expression for a left-invariant vector field on a matrix group:

$$L_g^A = \sum_{i,j=1}^{n} (gA)^{ij} \left(\frac{\partial}{\partial x^{ij}}\right)_g . \tag{A.8}$$

This expression will reappear later on, when we'll introduce the Poisson structure of T^*G, choosing $G = SU(2)$.

A.2 Left-Invariant One-Forms

We can move now to left-invariant one-forms. An n-form $\omega \in A^n(G)$ is left-invariant if

$$l_g^* \left(\omega_{g\prime}\right) = \omega_{g^{-1}g\prime}\,, \quad \forall g, g\prime \in G. \tag{A.9}$$

We have seen that $T_eG \cong L(G)$ via the map $i(A) = L^A$, we can therefore expect that $T_e^*G \cong L^*(G)$, that is to each $d \in T_e^*G$ is associated a left-invariant one-form λ^d defined by

$$\lambda_g^d := l_{g^{-1}}^*(d) \ \in T_e^*G \quad \forall g \in G. \tag{A.10}$$

It is possible to find an explicit relation between a left-invariant vector field and a left-invariant one-form, by contracting the latter with the previous one, as follows:

$$\begin{aligned} \left\langle \lambda^d, L^A \right\rangle_g &= l_{g^{-1}}^*(d)\left(L_g^A\right) = l_{g^{-1}}^*(d)\left(l_{*_g}(A)\right) \\ &= d\left(l_{*_g}^{-1} \circ l_{*_g}(A)\right) = \langle d, A\rangle \qquad \forall g \in G. \end{aligned} \tag{A.11}$$

We have seen how $L(G)$ is a Lie subalgebra of $\mathcal{T}(G)$, is there a similar result for the dual space $L^*(G)$, thought as the dual vector space? Let $\{E_1, ..., E_n\}$, $n = dimG$, a base of $L(G)$, then we have:

$$[E_\alpha, E_\beta] = \sum_{\gamma=1}^{n} C_{\alpha\beta}^{\gamma} E_\gamma, \tag{A.12}$$

where $C_{\alpha\beta}^{\gamma}$ are the srtucture constants of G with respect to the chosen basis. Therefore in $L^*(G)$ we have the corrispondent dual basis $\{\omega^1, ..., \omega^n\}$ of $L^*(G)$ which, by definiton, is such that: $\langle \omega^\alpha, E_\beta\rangle := \delta_\beta^\alpha$. We see at this point that, in general, given two vector fields, there is a natural way of obtaining a third one by combining the two via the commutator; on the other hand, there is not a natural way of doing the same thing using one-forms in place of vector fields. Nevertheless, we know that, given a one-form, we can obtain a two-form either by taking the differential or by making the external product of two one-forms. In the following we will use both these operations to find an equation that every left-invariant one-form satisfies. Taking the differential of a left-invariant one-form we get:

$$d\omega^\alpha(E_\beta, E_\gamma) = E_\beta\left(\langle \omega_\alpha, E_\gamma\rangle\right) - E_\gamma\left(\langle \omega^\alpha, E_\beta\rangle\right) - \langle \omega^\alpha, [E_\beta, E_\gamma]\rangle = -C_{\alpha\beta}^{\gamma}, \tag{A.13}$$

taking the external product of two left-invariant one-forms we get:

$$\omega^\delta \wedge \omega^\epsilon(E_\beta, E_\gamma) = \omega^\delta \otimes \omega^\epsilon(E_\beta, E_\gamma) - \omega^\epsilon \otimes \omega^\delta(E_\gamma, E_\beta) = \delta_\beta^\delta \delta_\gamma^\epsilon - \delta_\gamma^\delta \delta_\beta^\epsilon, \tag{A.14}$$

joining these two results we finally obtain:

$$d\omega^\alpha + \frac{1}{2} \sum_{\beta,\gamma=1}^{n} C_{\alpha\beta}^{\gamma} \omega^\beta \wedge \omega^\gamma = 0. \tag{A.15}$$

This equation is called the Cartan-Maurer equation and it is always satisfied by a left-invariant one-form. We will find again this equation when we will deal with T^*G.

A.3 Cartan-Maurer Form

Next, we are going to deal with the so called Cartan-Maurer form which we'll find in the context of gauge fields. The Cartan-Maurer form is the $L(G)$-valued one-form which assigns to each $v \in T_gG$ the left-invariant vector field on G whose element in g is precisely v. If we denote with $\langle \Xi, v \rangle$ the left-invariant vector field then we have:

$$\langle \Xi, v \rangle (g\prime) := l_{g\prime_*} \left(l_{g^{-1}*} v \right) \quad \forall v \in T_gG, \tag{A.16}$$

in particular, that means that:

$$\langle \Xi, L_g^A \rangle (g\prime) = L_{g\prime}^A, \tag{A.17}$$

furthermore, because $L(G) \cong T_eG$ we can associate to L_g^A the element $A \in T_eG$, to get $\langle \Xi, L_g^A \rangle = A$. The Cartan-Maurer form is clearly left-invariant.
In the special case in which $G = GL(n, \mathbb{R})$ we have seen that $L_g^A = (gA)^{ij} \left(\frac{\partial}{\partial x^{ij}} \right)$, thus:

$$\delta^{ij} = \left(\langle \Xi, L_g^{\mathbb{1}} \rangle \right)^{ij} = \Xi^{ik} \left(L_g^{\mathbb{1}} \right)^{kj} = \Xi^{ik} g^{kj}, \tag{A.18}$$

where $\mathbb{1} \in T_eG$ is the identity matrix. From (A.18) we can deduce that:

$$\Xi_g^{ij} = \sum_{k=1}^{n} \left(g^{-1} \right)^{ik} \left(dx^{kj} \right)_g. \tag{A.19}$$

The expressions for L_g^A and Ξ_g found in a coordinate system as in (A.2) are still valid for a general Lie matrix group.

A.3.1 Gauge Transofrmations and Cartan-Maurer Form

Let's end this section with an example, which turns out to be useful in the following.
Let $U \in \mathcal{M}$ be an open set, $\Omega : U \to G$, $\mathcal{M}$ is a m-dimensional differentiable manifold (in Yang-Mills theories it represents spacetime), G is the gauge group and so Ω is meant to be a gauge function which assigns to each point of $\mathcal{M}$ a gauge transformation. Obviously, on G is present the Cartan-Maurer form, then we can consider the pull-back of Ξ on $\mathcal{M}$ through Ω. If G is a matrix group, using coordinates as in (2.2) we can find an expression of $\Omega^*\Xi$ in the following way:

$$
\begin{aligned}
\left\langle (\Omega^*\Xi)^{ij}_p \, , \left(\frac{\partial}{\partial x^\mu}\right)_p \right\rangle &= \left\langle \Xi^{ij}, \Omega_* \left(\frac{\partial}{\partial x^\mu}\right)\right\rangle_{\Omega(p)} \\
&= \left\langle \sum_{k=1}^{n} \left(\Omega(p)^{-1}\right)^{ik} \left(dx^{kj}\right)_{\Omega(p)}, \Omega_* \left(\frac{\partial}{\partial x^\mu}\right)_{\Omega(p)} \right\rangle \\
&= \sum_{k=1}^{n} \left(\Omega(p)^{-1}\right)^{ik} \Omega_* \left(\frac{\partial}{\partial x^\mu}\right)_p \left(x^{kj}\right) \\
&= \sum_{k=1}^{n} \left(\Omega(p)^{-1}\right)^{ik} \frac{\partial}{\partial x^\mu} x^{kj} \left(\Omega(p)\right),
\end{aligned} \tag{A.20}
$$

for each p belonging to a local chart whose domain is U.
Therefore we obtain the expression:

$$
(\Omega^*\Xi)^{ij}_p = \sum_{\mu=1}^{m} \sum_{k=1}^{n} \left(\Omega^{-1}(p)\right)^{ik} \frac{\partial}{\partial x^\mu} \Omega^{kj}(p) \left(dx^\mu\right)_p, \tag{A.21}
$$

which very often appears in the more succinct form:

$$
\Omega^*\Xi = \Omega^{-1} d\Omega. \tag{A.22}
$$

Appendix B

Principal Fibre Bundles

In this section we are going to explore the idea of a principal fibre bundle, the reason for that is because it's the main structure in guauge theories.

B.1 Principal Fibre Bundles

The idea of a principal fibre bundle is that of a fibre bundle in which fibres are diffeomorphic to a Lie group G and on which the same group G acts in such a way to "move points along the fibres".

The preliminary definition we have to give is that of a $G-$ *bundle*: a bundle $(E, \pi, \mathcal{M})$ is a $G-$ *bundle* if E is a G-space (i.e. there is a G-action on E) and if $(E, \pi, \mathcal{M})$ is isomorphic to the bundle $(E, \rho, E/G)$ where E/G is the space of the orbits of the G-action on E and ρ is the projection on the orbit, to summarize we say that the following diagram has to commute:

$$\begin{array}{ccc} E & \xrightarrow{u} & E \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\rho} \\ \mathcal{M} & \xrightarrow{\simeq} & E/G \end{array} \tag{B.1}$$

The fact that $(E, \pi, \mathcal{M})$ and $(E, \rho, E/G)$ are isomorphic means that the fibres of E are the orbits of the G-action on E. If the action of G on E is free, that is, if $\forall p \in E$ we have $\{g \in G \mid pg = p\} = \{e\}$, then $(E, \pi, \mathcal{M})$ is said principal G-bundle and G is called the structure group of the bundle. The fact that the action of G is free implies that every orbit is homeomorphic to G, therefore it makes sense to say that $(E, \pi, \mathcal{M})$ is a fibre bundle with fibre G.

A principal fibre bundle which turns out to be very useful is the bundle of frames of a m-dimensional differentiable manifold $\mathcal{M}$. Let be $x \in \mathcal{M}$, $(b_1, ..., b_m)$ a basis of vectors in $T_x\mathcal{M}$, the total space $\mathcal{B}(\mathcal{M})$ of the bundle of frames is defined as the set of all frames at each point of $\mathcal{M}$, the projection $\pi : \mathcal{B}(\mathcal{M}) \to \mathcal{M}$ is defined by the function which sends each frame to the point it is attached. It is possible to introduce a free right action of $GL(m, \mathbb{R})$ on $\mathcal{B}(\mathcal{M})$ given by:

$$(b_1, ..., b_m)\, g := \left(\sum_{j_1=1}^{m} b_{j_1} g_{j_1 1}, ..., \sum_{j_m=1}^{m} b_{j_m} g_{j_m m} \right) \ \forall g \in GL(m, \mathbb{R}). \tag{B.2}$$

This action is clearly free, as one can verifies; the action corresponds to a change of basis in $T_x\mathcal{M}$, $x \in \mathcal{M}$. In addition to this, $\mathcal{B}(\mathcal{M})$ can be endowed with a differentiable structure, as follows: let $U \subset \mathcal{M}$ be a domain of a local chart on $\mathcal{M}$, whose coordinates we denote by $(x^1, ..., x^m)$, then each basis $b = (b_1, ..., b_m)$ of $T_x\mathcal{M}$, $x \in U$, can be written as

$$b_i = \sum_{j=1}^{m} b_i^j \left(\frac{\partial}{\partial x^j}\right)_x, \ i = 1, ..., m, \tag{B.3}$$

for a certain $b_i^j \in GL(m, \mathbb{R})$. We can therefore define the following map:

$$\begin{aligned} h\ : U \times GL(m,\mathbb{R}) &\longrightarrow \ \pi^{-1}(U) \\ (\,x\,,\,g\,) \qquad &\longmapsto \left(\sum_{j_1=1}^{m} g_1^{j_1}\,(\partial_{j_1})_x\,, ..., \sum_{j_m=1}^{m} g_1^{j_m}\,(\partial_{j_m})_x\right) \end{aligned} \tag{B.4}$$

and use $\left(x^1, ..., x^m; g_i^j\right)$ as coordinates in $\mathcal{B}(\mathcal{M})$. In this way, $\mathcal{B}(\mathcal{M})$ becomes a $m + m^2$ differentiable manifold.

Having said what is meant by a G-principal bundle, now we have to say what we mean by a principal map, i.e. a map between principal bundles. A bundle map (u, f) between a pair of G-principal bundles $(P, \pi, \mathcal{M})$ and $(P', \pi', \mathcal{M}')$ is said a principal map if $u : P \to P'$ is G-equivariant, that is, $u(pg) = pg \ \forall p \in P$, $\forall g \in G$; in other words the orbit O_p is sent to the orbit $O'_{u(p)}$, thus preserving the fibre structure of P and P'.
It is possible to generalize this definition to the case of a pair of principal bundles with different structure groups, say G and G', the requirement of G-equivariance is now implemented by adding a group homomrphism
$\Lambda : G \to G'$ and demanding that $u(pg) = u(p)\Lambda(g) \ \forall p \in P$, $\forall g \in G$.
This last property is important when we will deal with the so called spin connection: we will consider a principal bundle with structure group given by $G' = SO(3)$, the base space $\mathcal{M}$ will be a 3-dimensional Riemannian manifold and $G = \mathsf{Spin}(3, \mathbb{R})$, that is, the double cover of $SO(3)$, which coincides with SU(2), the universal cover of SO(3). Clearly, SO(3) and SU(2) are homomorphic, in addition to this we know also that they have isomorphic Lie algebras, i.e. $\mathsf{su}(2) \cong \mathsf{so}(3)$.
In the same fashion, if $\mathcal{M}$ is a Lorentzian manifold, $G' = SO(3, 1)$ and $G = \mathsf{Spin}(3, 1) \cong SL(2, \mathbb{C})$ and $SO(3, 1)$ is homomorphic to $SL(2, \mathbb{C})$. When we'll introduce the connection on a principal bundle we'll see, as an application, the case of the principal bundle of orthonormal frames, which turns out to be a SO(3,1)-bundle, and then take the pull-back of the connection on the principal bundle of spinorial frames (hence the name, spin connection) which turns out to be a Spin(3,1)-bundle. We can safely do that because there is a principal map between these two principal bundles.

B.2 Tetrads

We go on now introducing an object that will find place in the action of the GR and also in the Poisson structure of LQG.

We shall consider the bundle of orthonormal frames, denoted as $\mathcal{O}(\mathcal{M})$, which can be seen as a sub-bundle of $\mathcal{B}(\mathcal{M})$, whose structure group $GL(m, \mathbb{R})$ has been reduced to $O(m.\mathbb{R})$ (this can be seen by introducing the concept of associated bundle). However, $\mathcal{O}(\mathcal{M})$ is a principal bundle in its own right. We choose a local orthonormal frame of $T\mathcal{M}$, or equivalently, a local section of $\mathcal{O}(\mathcal{M})$, denoted as $\{e_1, ..., e_m\}$, this local frame is called $m - bein$ and each e_i is a *tetrad*. If we consider $T^*\mathcal{M}$ instead of $T\mathcal{M}$, each e^i is called a $co - tetrad$. The requirement of orthonormality reads:

$$\delta_{ij} = g_{\mu\nu}(x)e_i^{\mu}(x)e_j^{\nu}(x), \tag{B.5}$$

in the case of a Riemannian manifold, if $\mathcal{M}$ is a Lorentzian manifold we have instead

$$\eta_{ij} = g_{\mu\nu}(x)e_i^{\mu}(x)e_j^{\nu}(x), \tag{B.6}$$

where x belongs to the local domain of the m-bein.

B.3 Connection

In order to write the action of GR in another form we need to replace the metric with tetrads and connection (on a principal bundle). To introduce the idea of a connection we can follow this way of reasoning: we seek a vector field on a principal bundle P that lets us move from one fibre to another and not along the fibre. Now, in general, if G is a Lie group which acts on a differentiable manifold $\mathcal{M}$ by means of a right action $\delta : \mathcal{M} \times G \to \mathcal{M}$, $(p, g) \mapsto \delta(p, g) =: \delta_g(p)$, it is possible to define a vector field $X^A \in \mathcal{T}(\mathcal{M})$ induced by the action of the one-parameter subgroup $t \mapsto \mathsf{exp}(tA)$, $A \in T_eG$ (i.e. restricting the right action δ to those elements of G that can be written as the exponential of an element in T_eG). The vector field X^A is defined as follows:

$$X_p^A(f) := \frac{d}{dt} f\left(p\mathsf{exp}(tA)\right)|_{t=0}, \tag{B.7}$$

where $f \in C^{\infty}(\mathcal{M})$ and $pg := \delta_g(p)$. In other words, the curve through p given by $t \mapsto p\mathsf{exp}(tA)$ is the integral curve of X^A. The flux of X^A, denoted as ϕ_t^A is thus given by $\phi_t^A(p) = p\mathsf{exp}(tA) = \delta_{\mathsf{exp}(tA)}(p)$, that is $\phi_t^A = \delta_{\mathsf{exp}(tA)}$.
If, in lieu of $\mathcal{M}$, we consider a principal bundle P, where, as we know, is defined a right action of G, we can write down the vector field induced by this action. At this point, it is possible to show that the map $\iota : L(G) \to \mathcal{T}(P)$, $A \mapsto X^A$ is a Lie algebra homomorphism, that is $X^{[A,B]} = \left[X^A, X^B\right]$.
However, the vector fields X_p^A point along the fibre $\forall A \in T_eG$, that's because the right action moves a point along the fibre, by its very definition. In this context we say that X_p^A is a *vertical* vector, in the sense that it belongs to the vertical subspace V_pP of T_pP, which is defined by $V_pP := \{\tau \in T_pP | \pi_*\tau = 0\}$, from which it is clear that τ points along the fibre. Now, the map $A \mapsto X^A$ is an isomorphism of $L(G)$ onto V_pP because it is linear, injective (beacuse the action of G on P is free) and for dimensional reasons $dimV_pP = dimG = dimL(G)$.
Intuitively, then, it is justified the following definition of connection:
a connection on a prinicipal bundle $G \to P \to \mathcal{M}$ is an assignement to each point $p \in P$ of a subspace H_pP of T_pP such that

$$
\begin{aligned}
&(a)\ \ T_pP \cong V_pP \oplus H_pP \ \ \forall p \in P \\
&(b)\ \ \delta_{g_*}(H_pP) = H_{pg}P \ \ \forall g \in G,\ \forall p \in P;
\end{aligned} \tag{B.8}
$$

this means that a connection is first of all a k-dimensional distribution on P, with $k = dimP - dimG$. We can therefore split a vector $\tau \in T_pP$ into two components, horizontal and vertical: $\tau = ver(\tau) + hor(\tau)$. The condition (b) guarantees that this operation is compatible with the right action on P, in the sense that: $\delta_{g_*}(\tau) = \delta_{g_*}hor(\tau) + \delta_{g_*}ver(\tau) = hor(\delta_{g_*}\tau) + ver(\delta_{g_*}\tau)$.
There is also an equivalent definition of a connection, less intuitive but which is very used to find explicit expressions in which the connection is involved. The alternative definition goes as follows: a connection can be associated to a $L(G)$-valued one-form on P in the following way, if $\tau \in T_pP$ we define

$$
\omega_p(\tau) := \iota^{-1}\left(ver(\tau)\right), \tag{B.9}
$$

where $\iota : L(G) \to V_pP$ is the isomorphism introduced before. From this definition it follows that:

$$
\begin{aligned}
&(i)\ \ \omega_p(X^A) = A,\ \forall p \in P,\ \forall A \in L(G), \\
&(ii)\ \ \delta_g^*\omega = \mathsf{Ad}_{g^{-1}}\omega,\ \ i.e.\ \left(\delta_g^*\omega\right)_p(\tau) = \mathsf{Ad}_{g^{-1}}\left(\omega_p(\tau)\right),\ \forall \tau \in T_pP,
\end{aligned} \tag{B.10}
$$

where we recall that $C_g : G \to G$, $h \mapsto hgh^{-1}$ (C being the conjugate action), and $\mathsf{Ad}_g := dC_g$.
In particular, we notice that $\tau \in H_pP$ if and only if $\omega_p(\tau) = 0$. From this last equation it is clear that the connection is a sort of constraint on the space of vector fields on P, thought in this way, its counterimage $\omega_p^{-1}(0)\ \ \forall p \in P$ is exactly a distribution of horizontal vector fields.

B.4 Yang-Mills Fields and Gauge Transformations

At this point, we can establish the relation between a connection ω on a principal bundle, thought as a $L(G)$-valued one-form on P and the so called Yang-Mills fields, often introduced as functions on spacetime.
Usually, a Yang-Mills field is denoted as A_μ^a, where μ is a spacetime index and a is a Lie algebra index, therefore the following expression is meaningful:

$$
A(x) = \sum_{\mu=1}^{m}\sum_{a=1}^{dimG} A_\mu^a(x) E_a \left(dx^\mu\right)_x \tag{B.11}
$$

where $\{E_1, ..., E_{dimG}\}$ is a basis of $L(G)$, thus, locally, a Yang-Mills field corresponds to a $L(G)$-valued one-form. More precisely, let $\sigma : U \subset \mathcal{M} \to P$ be a local section of the principal bundle $G \to P \to \mathcal{M}$ on which is present also a connection one-form ω. We define the local σ-representative of ω as $\omega^U := \sigma^*\omega$, which is then a $L(G)$-valued one-form on U. Let then $h : U \times G \to \pi^{-1}(U) \subset P$ be the local trivialisation of P induced by σ, that is, $h(x,g) := \sigma(x)g$. As a consequence, it can be shown that if $(\alpha, \beta) \in T_{(x,g)}(U \times G) \cong T_xU \oplus T_gG$ we have that $h^*\omega$ can be written in terms of ω^U as follows:

$$
\left(h^*\omega\right)_{(x,g)}(\alpha,\beta) = \mathsf{Ad}_{g^{-1}}\left(\omega_x^U(\alpha)\right) + \Xi_g(\beta), \tag{B.12}
$$

where Ξ is the Cartan-Maurer form. Therefore we notice that, locally, a connection one-form ω is split into the sum of a $L(G)$-valued one-form on spacetime and a $L(G)$-valued one-form on the structure group G.

Next, we know that YM fields are subjected to local gauge transformations, where with gauge transformations we mean a principal automorphism of $G \to P \to \mathcal{M}$; if $\phi : P \to P$ is such a map then $\phi^*(\omega)$ is still a $L(G)$-valued one-form and $\phi^*(\omega)$ is called the gauge transform of ω. This one is the so called active version of gauge transformations. We can ask ourselves how ω^U changes if we choose a different section σ, that is we address the same issue adopting a passive view. Let's consider then two local sections of P, $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$, where $U_1, U_2 \subset \mathcal{M}$, $U_1 \cap U_2 \neq \emptyset$.
We call $A_\mu^{(1)}$ and $A_\mu^{(2)}$ the local representatives of ω with respect to σ_1 and σ_2. Then, if $\Omega : U_1 \cap U_2 \to G$ is the unique (because the action of G on P is free) local gauge function such that $\sigma_2(x) = \sigma_1(x)\Omega(x)$ we have from (B.12) that

$$A_\mu^{(2)}(x) = \mathsf{Ad}_{\Omega(x)^{-1}}\left(A_\mu^{(1)}(x)\right) + \left(\Omega^*\Xi\right)_\mu(x), \tag{B.13}$$

in the case where G is a matrix group we can write

$$A_\mu^{(2)}(x) = \Omega(x)^{-1}A_\mu^{(1)}(x)\Omega(x) + \Omega(x)^{-1}\partial_\mu\Omega(x). \tag{B.14}$$

How it reads a gauge transformations if instead we adopt an active view?
The answer is easy, if $\sigma : U \to P$ is a local section, $A := \sigma^*(\omega)$ and $\phi : P \to P$ is an automorphism of $G \to P \to \mathcal{M}$, we can consider the transformation $A \mapsto \sigma^*(\phi^*\omega) = (\phi \circ \sigma)^*\omega$. Comparing now A_μ with $A_\mu^{(1)}$ and $\sigma^*(\phi^*\omega)$ with $A^{(2)}$ it is clear that

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega(x)^{-1} + \Omega(x)\partial_\mu\Omega(x)^{-1}. \tag{B.15}$$

B.5 Analogies between $(\mathcal{M}, g)$ and $\mathcal{O}(\mathcal{M})$

B.5.1 Linear Connection and Connection one-form

At this point we can specialize to the case of the bundle of orthonormal frames $\mathcal{O}(\mathcal{M})$ on a differentiable manifold $\mathcal{M}$, where $\mathcal{M}$ is thought as spacetime. We have already established the relation between the metric g of a (pseudo-)Riemannian manifold and the tetrads e_i, we could ask then which is the relation between the Levi-Civita connection (defined on $(\mathcal{M}, g)$) and the corrispondent connection one-form on $\mathcal{O}(\mathcal{M})$.
In order to do that we begin by recalling that, if $\mathcal{M}$ is a differentiable manifold, a linear connection on $\mathcal{M}$ is a map:

$$\begin{array}{rccc} \nabla : & \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) & \longrightarrow & \mathcal{T}(\mathcal{M}) \\ & (X\ ,\ Y) & \longmapsto & \nabla_X Y \end{array} \tag{B.16}$$

which is $\mathbb{R}$-linear in the second argument, C^{∞}-linear in the first one and that satisfies the Leibniz rule. Using local coordinates $\left(x^1, ..., x^m\right)$ on $U \subset \mathcal{M}$ we have that $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$ (summation of repeated indices implied), Γ^k_{ij} are called the Christoffel symbols of the linear connection ∇ and determine uniquely the connection. There is another equivalent definition of a linear connection, that states that a linear connection ω on $\mathcal{M}$ is a $\mathcal{T}(\mathcal{M})$-valued one-form, locally it is represented by a matrix of one-forms as follows: $(\omega)^k_j = \Gamma^k_{ij}dx^i$. In the context of (pseudo-)Riemmanian manifolds it is known that it exists a unique linear connection symmetric and compatible with the metric, called the Levi-Civita connection; in this case it is possible to show that the relation between Γ^k_{ij} and $(\omega)^k_j$ is given by:

$$g\left(\nabla_X e_i, e_j\right) = \omega^k_i(X)\eta_{kj} = \omega_{ij}(X), \tag{B.17}$$

where e_i are the tetrads. Let's see why it is so: from $\nabla_X e_i = \omega^k_i(X)e_k$ follows that $\nabla_\rho e_i = \omega^k_j(\partial_\rho)e_k$, thus

$$(\nabla_\rho e_i)^\mu = \partial_\rho e^\mu_i + \Gamma^\mu_{\rho\sigma}e^\sigma_i = \omega^k_j(\partial_\rho)e^\mu_k, \tag{B.18}$$

because $g_{\mu\nu}e^\mu_k e^\nu_j = e^\mu_k e_{j\mu} = \eta_{kj}$. Finally we get:

$$\omega_{ij}\left(\partial_\rho\right) = \left(\partial_\rho e^\mu_i\right)e_{j\mu} + \Gamma^\mu_{\rho\sigma}e^\sigma_i e_{j\mu}. \tag{B.19}$$

On the other hand we have that

$$\begin{aligned} g\left(\nabla_\rho e_i, e_j\right) = g_{\mu\nu}\left(\nabla_\rho e_i\right)^\mu e^\nu_j = g_{\mu\nu}\left(\partial_\rho e^\mu_i + \Gamma^\mu_{\rho\sigma}e^\sigma_i\right)e^\nu_j = \\ = \left(\partial_\rho e^\mu_i\right)e_{j\nu} + \Gamma^\mu_{\rho\sigma}e^\sigma_i e_{j\mu} \end{aligned} \tag{B.20}$$

from which it is clear that (B.17) holds.
Obviously, it is possible to write the Christoffel symbols in terms of the tetrads, it is sufficient to substitute in $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)$ the expression $g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}$, in doing so we find an expression of ω_{ij} solely in terms of e_i.
The conclusion is that we deal with the bundle of orthonormal frames on $\mathcal{M}$ in place of a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$. In physics, the metric represents the gravitational field and the independent components of $g_{\mu\nu}$ are 10, that equals the sum of the indepenent components of η_{ij} (6) and e_i (4).
The connection one-form ω on $\mathcal{O}(\mathcal{M})$ is often called *spin connection*, because one has in mind the G-principal bundle with $G = Spin(3,1) \cong SL(2,\mathbb{C})$, i.e. the universal cover of $SO(3,1)$. However, we know that homomorphic Lie groups have isomorphic Lie algebras, in this case $\mathsf{so}(3,1) \cong \mathsf{sl}(2,\mathbb{C})$, then the connection one-form doesn't change.

B.5.2 Torsion and Curvature

At this point we want to introduce another very important concept: the curvature of a connection. In particular, we shall focus our attention on the Levi-Civita connection, which, as remarked earlier, is the unique linear connection on a Riemannian manifold that is compatible with the metric and symmetric (i.e with null torsion).

Let's begin therefore by recalling how it is defined the torsion of a linear connection ∇:

$$\tau : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M}), \quad \tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \tag{B.21}$$

it is easily verified that τ is $C^\infty(\mathcal{M})$-linear in all the variables and so it can be regarded as a tensor field $\tau \in \mathcal{T}_1^2(\mathcal{M})$, furthermore τ is antisymmetric.
We have already established the relation between a linear connection on $\mathcal{M}$ and a connection one-form on the bundle of orthonormal frames, we would like to accomplish the same goal regarding the torsion. In order to do so we choose local coordinates $\{x^1, ..., x^m\}$, the torsion in coordinates reads:

$$\tau^\gamma_{\alpha\beta} = (\nabla_\alpha \partial_\beta)^\gamma - (\nabla_\beta \partial_\alpha)^\gamma - [\partial_\alpha, \partial_\beta]^\gamma = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}, \tag{B.22}$$

furthermore, in terms of the tetrads e_i, we have

$$\tau(\partial_\alpha, \partial_\beta) = \tau^\gamma_{\alpha\beta}\partial_\gamma = \tau^\gamma_{\alpha\beta} e^i_\gamma e_i = T^i_{\alpha\beta}, \tag{B.23}$$

where we have defined $T^i_{\alpha\beta} := \tau^\gamma_{\alpha\beta} e^i_\gamma = \left(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\right) e^i_\gamma$. It happens that $T^i_{\alpha\beta}$ are the components of the following 2-form:

$$T^i := \Gamma^\gamma_{\mu\nu} e^i_\gamma dx^\mu \wedge dx^\nu. \tag{B.24}$$

Thus, the information on the torsion τ is equally contained in the 2-forms T^i, $i = 1, ..., 4$. At this point we seek an expression of T^i purely in terms of the tetrads and the connection one-form ω on $\mathcal{O}(\mathcal{M})$.
Firstly, we observe that $dx^\mu = e^\mu_j e^j$, $dx^\nu = e^\nu_k e^k$, then:

$$T^i = \Gamma^\gamma_{\mu\nu} e^i_\gamma e^\nu_k \left(e^j \wedge e^k\right), \tag{B.25}$$

recalling eq. (B.19) we notice that:

$$\Gamma^\gamma_{\mu\nu} e^i_\gamma e^\nu_k = \omega^i_k(\partial_\mu) - (\partial_\mu e^\alpha_k) e^i_\alpha, \tag{B.26}$$

from which

$$\begin{aligned} T^i &= \omega^i_k(\partial_\mu) e^\mu_j \left(e^j \wedge e^k\right) - (\partial_\mu e^\alpha_k) e^i_\alpha e^\mu_j \left(e^j \wedge e^k\right) = \\ &= e^\mu_j \omega^i_k(\partial_\mu) \left(e^k \wedge e^j\right) + \left(\partial_\mu e^i_\alpha\right) e^\alpha_k e^\mu_j \left(e^j \wedge e^k\right), \end{aligned} \tag{B.27}$$

where we used the fact that $(\partial_\mu e^\alpha_k) e^i_\alpha = \partial_\mu \left(e^\alpha_k e^i_\alpha\right) - e^\alpha_k \partial_\mu e^i_\alpha = \partial_\mu \left(\eta^i_k\right) - \partial_\mu e^i_\alpha e^\alpha_k$.
Now the last steps:

$$e^\mu_j \omega^i_k(\partial_\mu) e^k = \omega^i_j, \tag{B.28}$$

in fact $e^\mu_j \omega^i_k(\partial_\mu) e^k_\nu = \omega^i_j(\partial_\nu)$ is true since $e^\mu_j e^k_\nu = \eta^k_j$, as one can easily verify.
Finally we observe that:

$$de^i = d\left(e^i_\alpha dx^\alpha\right) = \partial_\mu e^i_\alpha dx^\mu \wedge dx^\alpha = \partial_\mu e^i_\alpha e^\mu_j e^\alpha_k \left(e^j \wedge e^k\right), \tag{B.29}$$

thus we can rewrite T^i as follows:

$$T^i = de^i + \omega^i_j \wedge e^j. \tag{B.30}$$

Eq. (B.30) is the expression of the torsion of the connection one-form ω on $\mathcal{O}(\mathcal{M})$.
From the Riemannian geometry it is known that it exists a unique linear connection which is compatible with the metric and symmetric.
The fact that ∇ is compatible with the metric is already implicitly contained in the fact that ω_{ij} is antisymmetric, in fact from $g(\nabla_X e_i, e_j) = \omega_{ij}$ we deduce that $0 = X(g(e_i, e_j)) = \nabla_X (g(e_i, e_j)) = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j)$, thus ω_{ij} being antisymmetric (a necessary condition since ω is a $\mathsf{sl}(2,\mathbb{C})$-valued one-form) is equivalent to the fact that ∇ is compatible with the metric.
If, in addition to this, $\tau(X,Y) = 0 \quad \forall X, Y \in \mathcal{T}(\mathcal{M})$ then ∇ is the Levi-Civita connection, equivalently, if $T^i = 0 \quad \forall i = 1, ..., 4$ then ω is the Levi-Civita connection one-form on $\mathcal{O}(\mathcal{M})$.

It is known from GR that the curvature of the Levi-Civita connection is related to the gravitational force; in gauge theories, similarly, the curvature of a connection one-form is associated with the gauge interaction. In the tetrad-connection formalism GR resembles a gauge theory, for this reason we attempt to establish a relation between the Riemann tensor $R \in \mathcal{T}^3_1(\mathcal{M})$ defined by

$$R(X,Y,Z) := R_{XY}(Z,W) := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{B.31}$$

and the analogous of the curvature of a connection one-form.
Let's give then the definition of curvature in this context: if ω is a k-form on P, the exterior covariant derivative of ω is the horizontal $(k+1)$-form defined by:

$$D\omega := d\omega \circ hor, \tag{B.32}$$

that is, $D\omega(X_1, X_2, ..., X_{k+1}) = d\omega(horX_1, ..., horX_k)$, $\forall X_1, ..., X_{k+1}$ vector fields on P.
If ω is a connection one-form on P the curvature 2-form of ω is defined by

$$G := D\omega. \tag{B.33}$$

We mention a very important result: if $G = D\omega$ is the curvature 2-form of ω then, $\forall p \in P$, we have that:

$$G_p(X,Y) = d\omega_p(X,Y) + [\omega_p(X), \omega_p(Y)] \quad \forall X, Y \in \mathcal{T}(\mathcal{P}), \tag{B.34}$$

where [,] denotes the Lie brackets in $L(G)$.
Choosing a basis $\{E_1, ..., E_{dimG}\}$ of $L(G)$ we obtain that $\omega = \omega^a E_a$ and then:

$$G^a = d\omega^a + \frac{1}{2} C^a_{bc} \omega^b \wedge \omega^c, \tag{B.35}$$

where C^a_{bc} are the structure contants of $L(G)$ with respect ot the basis $\{E_1, ..., E_{dimG}\}$.

As already done previously, we can find what the curvature looks like when we consider its pullback by a local section. Let then $\sigma : U \to P$ be a local section, $A := \sigma^*\omega$ the local representative of ω, $F := \sigma^* G$ the local representative of G. From the properties of the pull back it follows that:

$$F^a = dA^a + \frac{1}{2} C^a_{bc} A^b \wedge A^c. \tag{B.36}$$

Introducing local coordinates on $U \subset \mathcal{M}$ we can write:

$$F^a_{\mu\nu} = \frac{1}{2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \right), \tag{B.37}$$

which is the familiar expression of the field strength found in gauge theories. It's not difficult, in this context, to prove the Bianchi identity $DG = 0$, in fact, from $DG_p(X, Y, Z) = dG_p(horX, horY, horZ)$ it's enough to apply the definition of external differential and notice that $G_p(horX, horY) = 0 \quad \forall X, Y \in \mathcal{T}(P)$. At this point, we choose as a basis of $\mathsf{sl}(2, \mathbb{C})$ the set $\{E_{IJ}\}$ of the antisymmetric matrices such that $\omega = \omega^{IJ} E_{IJ}$ and:

$$[E_{KL}, E_{MN}] = (\eta_{KM}\eta_{LN} - \eta_{LM}\eta_{KN})^{IJ} E_{IJ}. \tag{B.38}$$

Then we notice that, since $(\eta_{KM}\eta_{LN})^{IJ} = (\eta_{KM})^I_P (\eta_{LN})^{PJ} = \eta^I_K \eta_{MP} \eta^P_L \eta^J_N = \eta^I_K \eta_{ML} \eta^J_N$, we have $(\eta_{KM}\eta_{LN})^{IJ} \omega^{KM} \wedge \omega^{MN} = \omega^I_M \wedge \omega^{MJ}$, for this reason the curvature can be written as:

$$F^{IJ} = d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ}. \tag{B.39}$$

Now we want to find the relation between R and F. Starting from (B.39) it is possible to show that (after long and painful calculations)

$$R^\mu_{\nu\rho\sigma} = e^\mu_I e^J_\nu F^I_{J\rho\sigma}, \tag{B.40}$$

or, equivalently,

$$F^{IJ} = e^I_\mu e^J_\nu R^{\mu\nu}_{\rho\sigma} dx^\rho \wedge dx^\sigma. \tag{B.41}$$

We recall that the Ricci tensor is obtained by contracting the first and the third indeces of the Riemann tensor:

$$R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} = e^\mu_I e^J_\nu F^I_{J\mu\sigma}, \tag{B.42}$$

from which it follows that the Ricci scalar is given by:

$$R = g^{\nu\sigma} R_{\nu\sigma} = g^{\nu\sigma} e^\mu_I e^J_\nu F^I_{J\mu\sigma} = e^\mu_I e^\sigma_J F^{IJ}_{\mu\sigma} = \left(F^{IJ}\right)_{IJ}. \tag{B.43}$$

The last equality is due to the fact that:

$$F^{IJ} = \frac{1}{2} F^{IJ}_{\mu\sigma} dx^\mu \wedge dx^\sigma = \frac{1}{2} F^{IJ}_{\mu\sigma} e^\mu_K e^\sigma_L \left(e^K \wedge e^L\right) = \frac{1}{2} F^{IJ}_{KL} \left(e^K \wedge e^L\right) \tag{B.44}$$

where, again, the last equality holds by the very definition of differential form, F^{IJ}_{KL} are the componenents of the 2-form F^{IJ} in the basis of the cotetrads e^I. With the intention of writing the Einstein-Hilbert action we observe that:

$$\mathsf{det}\left(g_{\mu\nu}\right) = \mathsf{det}\left(e^T \eta e\right)_{\mu\nu} = \mathsf{det}^2(e)\mathsf{det}(\eta) = -\mathsf{det}^2(e), \tag{B.45}$$

from which it follows that $g := \mathsf{det}\left(g_{\mu\nu}\right) = -\mathsf{det}^2(e) =: -e^2$ and so $\sqrt{-g} = |e|$. Finally we notice that

$$\begin{aligned}\epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} &= \frac{1}{2}\epsilon_{IJKL} F^{KL}_{MN} e^I \wedge e^J \wedge e^M \wedge e^N = \\ &= \frac{1}{2}\epsilon_{IJKL}\epsilon^{IJMN} F^{KL}_{MN}|e|d^4x = \\ &= -\left(\delta^M_K \delta^N_L - \delta^M_L \delta^N_K\right) F^{KL}_{MN}|e|d^4x = \\ &= -2|e|R d^4x,\end{aligned} \tag{B.46}$$

thanks to the fact that

$$\begin{aligned}e^I \wedge e^J \wedge e^M \wedge e^N &= e^I_\mu e^J_\nu e^M_\rho e^N_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \\ &= e^I_\mu e^J_\nu e^M_\rho e^N_\sigma \varepsilon^{\mu\nu\rho\sigma} d^4x = \\ &= \epsilon^{IJMN}|e|d^4x.\end{aligned} \tag{B.47}$$

If we define now $Tr\left(e \wedge e \wedge F\right) := \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}$ we can write the Einstein-Hilbert action S_{EH} as follows:

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R = -\frac{1}{32\pi G}\int Tr\left(e \wedge e \wedge F\right). \tag{B.48}$$

In this form S_{EH} is a functional of e and ω.

B.6 Holonomy

In order to approach the definition of holonomy it is necessary to explain what is meant by parallel transport on a principal bundle. The idea is to find a curve that lets us move from one fiber to the other, we have already seen that $\pi_* : H_pP \to T_{\pi(p)}\mathcal{M}$ is an isomorphism, then for each vector field $X \in \mathcal{T}\left(\mathcal{M}\right)$ it exists a unique vector field on P, denoted as $X^\uparrow$ such that, $\forall p \in P$, we have

$$\begin{aligned}&(a)\ \pi_*\left(X^\uparrow_p\right) = X_{\pi(p)} \\ &(b)\ ver\left(X^\uparrow_p\right) = 0\ ,\end{aligned} \tag{B.49}$$

$X^\uparrow$ is called the horizontal lift of X. Intuitively the picture is quite clear: the integral curve on $\mathcal{M}$ of X is lifted to a curve on P which does precisely the job we are looking for. Indeed, we can define the horizontal lift of a curve $\alpha : [a,b] \to \mathcal{M}$ as the curve $\alpha^\uparrow : [a,b] \to P$ such that $\pi\left(\alpha^\uparrow(t)\right) = \alpha(t), \forall t \in [a,b]$ and that is horizontal, i.e. $ver\left[\alpha^\uparrow_*\left(\frac{d}{dt}\right)\right] = 0$. It is possible to show that $\forall p \in \pi^{-1}\{\alpha(a)\} \subset P$ it exists a unique horizontal lift of α such that $\alpha^{\uparrow}(a) = p$. As usual, it is useful to find an explicit expression for $\alpha^{\uparrow}$, which will naturally contain the connection one-form ω. In order to do so we can follow this line of reasoning: let's suppose that $\beta : [a,b] \to P$ is a lift of α (not necessarily horizontal), that is, $\pi(\beta(t)) = \alpha(t)\ \forall t \in [a,b]$ (thus the vector field $\beta_* \left(\frac{d}{dt}\right)$ will have nonzero vertical and horizontal components). Then, it exists a unique function $g : [a,b] \to G$ such that $\alpha^{\uparrow}(t) = \beta(t)g(t)$ (because, again, the action of G on P is free). It is worth considering the following factorisation:

$$\begin{array}{llll} [a,b] \xrightarrow{\beta\times g} & P \times G & \xrightarrow{\delta} & P \\ t & \mapsto (\beta(t), g(t)) & \mapsto & \beta(t)g(t) \end{array} \tag{B.50}$$

in this way (see [1], page 265, for further specifications), since $\omega\left(\alpha_*\left(\frac{d}{dt}\right)\right) = 0$, we find that

$$0 = \mathsf{Ad}_{g(t)^{-1}}\left(\omega_{\beta(t)}\left(\beta_*\left(\frac{d}{dt}\right)\right)\right) + \Xi_{g(t)}\left(g_*\left(\frac{d}{dt}\right)\right), \tag{B.51}$$

from which it follows that, for a matrix Lie group G,

$$0 = g(t)^{-1}\omega_{\beta(t)}\left(\beta_*\left(\frac{d}{dt}\right)\right)g(t) + g(t)^{-1}\frac{dg}{dt}. \tag{B.52}$$

This is the differential equation that determines the function $g(t)$, which turns the (general) lift β into a horizontal lift; $g(t)$ clearly depends on ω. Sometimes the function $g(t)$ is also called the parallel transport matrix.

Our next goal is to find $g(t)$, namely to resolve the differential equation (3.52). Before doing so, we have to make a choice on the function β, because up to now it is a generic lift of α. A natural choice is given by a local section of P, $\sigma : U \to P$, we recall that σ is needed also to have a local representative of ω, i.e. to deal with a Yang-Mills field. Let then be $\beta(t) := \sigma(\alpha(t))$, from which $\beta_*\left(\frac{d}{dt}\right) = \sigma_*\left(\alpha_*\left(\frac{d}{dt}\right)\right)$, then $\omega_{\beta(t)}\left(\beta_*\left(\frac{d}{dt}\right)\right) = (\sigma^*\omega)_{\alpha(t)}\left(\alpha_*\left(\frac{d}{dt}\right)\right)$ and $\sigma^* = \omega^U$ was named A (the Yang-Mills field).

With this notation eq. (3.52) becomes:

$$0 = \sum_{\mu=1}^{m} g(t)^{-1}A_\mu(\alpha(t))\,g(t)\frac{dx^\mu(\alpha(t))}{dt} + g(t)^{-1}\frac{dg(t)}{dt}, \tag{B.53}$$

where x^μ are local coordinates on $U \subset \mathcal{M}$. Choosing initial conditions on $t \mapsto g(t)$ as $g(a) = g_0 \in G$ we get:

$$g(t) = g_0 - \int_a^t ds A_\mu(\alpha(s))\,\dot{\alpha}^\mu(s)g(s), \tag{B.54}$$

which admits a solution in terms of the path-ordered integral:

$$\begin{aligned} g(t) = \left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu(\alpha(s))\,\dot{\alpha}^\mu(s)\right)g_0 := \Bigg(1 - \int_a^t ds A_\mu(\alpha(s))\,\dot{\alpha}^\mu(s) \\ + \int_a^t ds_1 \int_a^{s_1} ds_2 A_{\mu_1}(\alpha(s))\,A_{\mu_2}(\alpha(s))\,\dot{\alpha}^{\mu_1}(s)\dot{\alpha}^{\mu_2}(s) + \ldots\Bigg)g_0. \end{aligned} \tag{B.55}$$

Finally, we can conclude that the horizontal lift $\alpha^{\uparrow}$ is expressed, locally, in terms of σ by:

$$\alpha^{\uparrow}(t) = \sigma\left(\alpha(t)\right)\left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right) g_0. \tag{B.56}$$

A word about terminology, in LQG the function $g(t)$ is often referred to as the holonomy, though matematically speaking that's an abuse of language, as we shall see later.
Summarizing, we have seen that, in order to move from one fiber to another, we have to lift a curve on the base manifold in an horizontal fashion. The result of this operation depends on the connection one-form (even in the context of Riemannian manifolds the parallel transport depends on the Levi-Civita connection), since we are interested in a local expression we use a local section to lift the curve and to pullback the connection.

It is very important to know how the function $g(t)$ changes if A is subjected to an active gauge transformation, i.e. when $A_\mu(x) \mapsto \Omega(x)A_\mu(x)\Omega(x)^{-1} + \Omega\partial_\mu\Omega(x)^{-1}$. To see this, we shall consider Eq. (B.53) (adopting a compact notation):

$$0 = g^{-1}\Omega A_\mu \Omega^{-1} g\dot{\alpha}^\mu + g^{-1}\Omega\partial_\mu\Omega^{-1}g\dot{\alpha}^\mu + g^{-1}\Omega\Omega^{-1}\frac{dg}{dt}\ , \tag{B.57}$$

now we rewrite

$$\begin{aligned} g^{-1}\Omega\Omega^{-1}\frac{dg}{dt} =& g^{-1}\Omega\frac{d}{dt}\left(\Omega^{-1}g\right) - g^{-1}\Omega\left(\frac{d}{dt}\Omega^{-1}\right)g \\ =& g^{-1}\Omega\frac{d}{dt}\left(\Omega^{-1}g\right) - g^{-1}\Omega\partial_\mu\Omega^{-1}g\dot{\alpha}^\mu, \end{aligned} \tag{B.58}$$

inserting (B.58) back into (B.57) we get:

$$0 = g^{-1}\Omega A_\mu\Omega^{-1}g\dot{\alpha}^\mu + g^{-1}\Omega\frac{d}{dt}\left(\Omega^{-1}g\right), \tag{B.59}$$

denoting with $\tilde{g}(t) = \Omega^{-1}\left(\alpha(t)\right)g(t)$, finally we obtain:

$$0 = \tilde{g}^{-1}(t)A_\mu\left(\alpha(t)\right)\tilde{g}(t)^{-1} + \tilde{g}(t)^{-1}\frac{d\tilde{g}(t)}{dt}\ . \tag{B.60}$$

Eq. (B.60) admits a solution in terms of a path-ordered integral, as seen before,

$$\tilde{g}(t) = \left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right)\tilde{g}(a), \tag{B.61}$$

from which

$$g(t) = \Omega\left(\alpha(t)\right)\left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right)\Omega^{-1}\left(\alpha(a)\right)g_0. \tag{B.62}$$

From this expression it's clear how the path-ordered integral transforms under a gauge transformation:

$$\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s) \longmapsto \Omega\left(\alpha(t)\right)\left(\mathcal{P}\mathsf{exp} - \int_a^t ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right)\Omega^{-1}\left(\alpha(a)\right)\ , \tag{B.63}$$

it is often said that the path-ordered integral transforms homogeneously.
If G is a matrix group and α is a closed loop, i.e. $\alpha(a) = \alpha(b)$, it is clear that the function

$$W_\alpha[A] := tr\left(\mathcal{P}\mathsf{exp} - \oint_\alpha ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right) \tag{B.64}$$

is gauge invariant, it is called the Wilson loop.
At this point we can give a precise definition of what is meant by parallel transport, intuitively we want to render substantial the concept of a horizontal curve in P, in such a way that a vector field on P would be transported from a fibre to another without being "rotated" along the fibre.
Let $\alpha : [a, b] \to \mathcal{M}$ be a curve in $\mathcal{M}$; the parallel transport along α is the map

$$\tau : \pi^{-1}\left(\{\alpha(a)\}\right) \to \pi^{-1}\left(\{\alpha(b)\}\right), \quad p \mapsto \alpha^\uparrow(b), \tag{B.65}$$

where $\alpha^\uparrow$ is the unique horizontal lift of α which passes through p when $t = a$.
A special case is when α is a closed curve, i.e a loop, in $\mathcal{M}$. In general the horizontal lift of a loop has not to be closed, therefore we get a non-trivial map from $\pi^{-1}\left(\{\alpha(a)\}\right)$ onto itself given by

$$p \longmapsto p\left(\mathcal{P}\mathsf{exp} - \oint_\alpha ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right) \ . \tag{B.66}$$

It is clear from (B.66) that we can associate an element of G (which is given by the path-ordered integral) to each loop in $\mathcal{M}$, that is we have a natural map from the loop space of $\mathcal{M}$ into G. The subgroup of G whose elements are obtained in this way is called the *holonomy group* of the bundle at the point $\alpha(0) \in \mathcal{M}$.
We see therefore that the word "holonomy" is referring to loops, while in LQG terminology it is referring to a curve (more precisely to an edge).

Appendix C

Symplectic Geometry

The goal of this section is to introduce the mathematical tools used in Hamiltonian Mechanics and to present the Poisson structure of the theory.

C.1 Symplectic Algebra

The study of Hamiltonian Mechanics is based on the fundamental concept of symplectic manifold. To achieve that, we enlight first what we mean by symplectic structure on a vector space.

A symplectic tensor is an antisymmetric covariant 2-tensor which is non degenerate.

A couple (V, ω) where V is a vector space and $\omega \in \bigwedge_2 V$ is a symplectic tensor is said symplectic vector space.
To give an example let V be a $2n$-dimensional vector space, we denote a basis of V with $\{v_1, w_1, ..., v_n, w_n\}$, whose dual basis of V^* is given by $\{v^1, w^1, ..., v^n, w^n\}$. Let $\omega \in \bigwedge_2 V$ be given by

$$\omega = \sum_{j=1}^{n} v^j \wedge w^j, \tag{C.1}$$

then ω is symplectic, in fact

$$\begin{aligned} \omega\left(v_i, w_j\right) &= -\omega\left(w_j, v_i\right) = \delta_{ij}, \\ \omega\left(v_i, v_j\right) &= -\omega\left(w_i, w_j\right) = 0, \end{aligned} \tag{C.2}$$

for each $1 \leq i, j \leq n$. Then, if we choose a vector $v = \sum_i \left(a^i v_i + b^i w_i\right) \in V$ such that $\omega(v, w) = 0,\ \forall w \in V$ we have that:

$$\begin{aligned} 0 &= \omega(v, v_j) = -b^j, \\ 0 &= \omega(v, w_j) = a^j, \end{aligned} \tag{C.3}$$

for $1 \leq j \leq n$, thus $v = 0$ and ω is non-degenerate.
The example above is very important, because one can show that if (V, ω) is a symplectic vector space then the dimension of V is even and it exists a basis of V with respect to which ω has the form given

by (4.1).
Let's consider now a subspace $W \subset V$, the symplectic complement of W is the subspace

$$W^{\perp} = \{v \in V \mid \omega(v, w) = 0, \ \forall w \in W\}. \tag{C.4}$$

In general, it's not true that $W \cap W^{\perp} = \{0\}$, in fact, if $dimW = 1$ then $W \subseteq W^{\perp}$ because ω is antisymmetric.
Then we have the following classification of subsets of V:

- W is symplectic if $W \cap W^{\perp} = \{0\}$;
- W is isotropic if $W \subseteq W^{\perp}$;
- W is coisotropic if $W^{\perp} \subseteq W$;
- W is Lagrangian if $W = W^{\perp}$.

From these definitions it is not difficult to show the following properties:

(i) $dimW + dimW^{\perp} = dimV$;

(ii) $\left(W^{\perp}\right) = W$;

(iii) W symplectic $\Leftrightarrow \omega|_{W \times W}$ non-degenerate;

(iv) W isotropic $\Leftrightarrow \omega|_{W \times W} = 0$;

(v) W Lagrangian $\Leftrightarrow \omega|_{W \times W} = 0$ and $dimV = 2dimW$.

C.2 Symplectic Manifolds

At this point we can generalise these considerations to the context of differentiable manifolds, where the role of the vector space V is naturally given by $T_x\mathcal{M}$, and the symplectic tensor ω is now expressed in terms of differential forms. More precisely:
a symplectic form on a differentiable manifold $\mathcal{M}$ is a 2-form $\omega \in A^2(\mathcal{M})$ which is closed and non-degenerate, therefore ω_p is a symplectic tensor $\forall p \in \mathcal{M}$.

A symplectic manifold is a pair $(\mathcal{M}, \omega)$ where $\mathcal{M}$ is a differentiable manifold and $\omega \in A^2(\mathcal{M})$ is a symplectic form.
For each point $p \in \mathcal{M}$ we have that $(T_p\mathcal{M}, \omega_p)$ is a symplectic vector space, therfore a symplectic manifold has even dimension ($dimT_p\mathcal{M} = dim\mathcal{M}$).
Not every differentiable manifold admits a symplectic structure, in fact it is possible to show that $H^2\left(\mathcal{M}\right) \neq 0$ if $\mathcal{M}$ is symplectic and compact, this result comes from the fact that starting from ω one can build a volume form on $\mathcal{M}$, which is given by $\Omega_\omega := \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\omega^n$ (where $\omega^n := \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}}$),

then by applying Stoke's theorem one can prove it.

Now we deal with an important type of functions between symplectic manifolds:
a symplectomorphism (or canonical transformation) between symplectic manifolds $(\mathcal{M}, \omega)$ and $\left(\tilde{\mathcal{M}}, \tilde{\omega}\right)$ is a diffeomorphism $F : \mathcal{M} \to \tilde{\mathcal{M}}$ such that $F^*\tilde{\omega} = \omega$.

As one may expect, the types of subspaces of a symplectic vector space admit a generalization to the case of symplectic manifolds, this is done at the level of fibres.
If $(\mathcal{M}, \omega)$ is a symplectic manifold and $\mathcal{N}$ is another differentiable manifold, an immersion $F : \mathcal{M} \to \mathcal{N}$ is called symplectic (isotropic, coisotropic, Lagrangian, respectively) if $dF_p\left(T_p\mathcal{N}\right)$ is a symplectic (isotropic, coisotropic, Lagrangian, respectively) subspace of $T_{F(p)}\mathcal{M}$ for each $p \in \mathcal{N}$.
Therefore, if F is symplectic then $F^*\omega$ is a symplectic form on $\mathcal{N}$, in fact

$$\left(F^*\omega\right)_p(v_1, v_2) = \omega_{F(p)}\left(dF_p(v_1), dF_p(v_2)\right), \quad \forall v_1, v_2 \in T_p\mathcal{N}. \tag{C.5}$$

We saw at the beginning of the section that it is always possible to find a basis on a symplectic vector space such that (C.1) holds. Thanks to the Darboux theorem it is possible to generalise this result to the context of differentiable manifolds:
let $(\mathcal{M}, \omega_0)$ be a $2n$-dimensional symplectic manifold, then for each $p \in \mathcal{M}$ is possible to find a local chart (V, φ) in p with $\varphi = \left(x^1, y^1, ..., x^n, y^n\right)$ such that

$$\omega_0|_V = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{C.6}$$

The local coordinates $\left(x^1, y^1, ..., x^n, y^n\right)$ are called Darboux coordinates.
An important application frequently used in physics concern the cotangent bundle, in fact, on a cotagent bundle exists a canonical symplectic form.
Let $\mathcal{M}$ be a differentiable manifold and define a canonical one-form $\theta \in A^1\left(T^*\mathcal{M}\right)$ on the cotangent bundle $\pi : T^*\mathcal{M} \to \mathcal{M}$, called the tautologic form, by

$$\theta_\xi = \pi^*\xi \quad \in T^*_\xi\left(T^*\mathcal{M}\right), \quad \forall \xi \in T^*\mathcal{M}. \tag{C.7}$$

Let's choose a local chart (U, φ) on $\mathcal{M}$, with coordinates $\varphi = \left(x^1, ..., x^n\right)$, if $p \in U$ we can write $\xi_p = \xi_i dx^i|_p$ and a local chart $\left(\pi^{-1}(U), \tilde{\varphi}\right)$ on $T^*\mathcal{M}$ is given by $\tilde{\varphi}\left(\xi_p\right) = \left(x^1, ..., x^n, \xi_1, ..., \xi_n\right)$.
These coordinates induce the local frame on $T\left(T^*\mathcal{M}\right)$ given by $\left\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \xi^1}, ..., \frac{\partial}{\partial \xi^n}\right\}$ and the respective dual frame on $T^*\left(T^*\mathcal{M}\right)$ given by $\left\{dx^1, ..., dx^n, d\xi_1, ..., d\xi_n\right\}$.
In these coordinates the projection $\pi : T^*\mathcal{M} \to \mathcal{M}$ is represented by $\varphi \circ \pi \circ \tilde{\varphi}^{-1}(x, \xi) = x$, thus we have that $d\pi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}$ and $d\pi\left(\frac{\partial}{\partial \xi_i}\right) = 0$
$\forall i = 1, ..., n$, but then $\pi^*dx^j = dx^j$ (with a slightly abuse of notation) and so in local coordinates the tautologic one-form reads:

$$\theta_\xi = \xi_i dx^i|_\xi. \tag{C.8}$$

Starting from (C.8) we can consider the 2-form $\omega \in A^2\left(T^*\mathcal{M}\right)$ defined by:

$$\omega = d\theta. \tag{C.9}$$

This form is closed, being exact, and in local coordinates is given by

$$\omega = d\xi_j \wedge dx^i, \tag{C.10}$$

and so it is clearly non-degenerate (as already seen in the case of symplectic vector spaces), then it defines a symplectic form on $T^*\mathcal{M}$.
A word about notation, usually in physics $\mathcal{M}$ is the configuration space, $T^*\mathcal{M}$ is then the phase space, local coordinates on $T^*\mathcal{M}$ are denoted with $\left(q^i, p_i\right)$ rather than $\left(x^i, \xi_i\right)$.

C.3 Hamiltonian Fields and Poisson structure

We address at this point the topic of Hamiltonian fields and Poisson structure.
We recall a general result:
let V, W be two finite-dimensional (real) vector spaces and $\Phi : V \times W \to \mathbb{R}$ a bilinear form, then Φ is non-degenerate if and only if the linear applications $\Phi^\flat : V \to W^*$ and $\Phi^\sharp : W^* \to V$ are isomorphisms. This means that $\omega^\flat : T\mathcal{M} \to T^*\mathcal{M}$ is the isomorphism induced by ω given by $\omega^\flat(v) = \omega(v, \cdot)$ and $\omega^\sharp = \left(\omega^\flat\right)^{-1}$. A similar result holds in the context of (pseudo-)Riemannian manifolds, where the isomorphisms are induced by the metric.

For each function $f \in C^\infty(\mathcal{M})$ the Hamiltonian vector field associated to f is defined by:

$$X_f := -\omega^\# (df), \tag{C.11}$$

in other terms $X_f \lrcorner \omega = -df$[1], or equivalently $\omega(X_f, Y) = -df(Y) = -Y(f)$ for each $Y \in \mathcal{T}(\mathcal{M})$.
Viceversa, a vector field $X \in \mathcal{T}(\mathcal{M})$ is Hamiltonian if exists a function $f \in C^\infty(\mathcal{M})$ such that $X = X_f$, and is locally Hamoltanian if each $p \in \mathcal{M}$ has a neighbourhood on which X is Hamiltonian.

Furthermore, $X \in \mathcal{T}(\mathcal{M})$ is symplectic if ω is invariant on the flow of X, that is $\mathcal{L}_X\omega = 0$.
Finally, a Hamiltonian system is a triple $(\mathcal{M}, \omega, H)$ where $(\mathcal{M}, \omega)$ is a symplectic manifold and $H \in C^\infty(\mathcal{M})$ is a function called the Hamiltonian of the system.
From the definition of Hamiltonian vector field we deduce that $X_f \lrcorner \omega$ is a closed one-form. From this, and from the fact that $\mathcal{L}_X\omega = d(X \lrcorner \omega) + X \lrcorner (d\omega)$, we see that a vector field is symplectic if and only if is locally Hamiltonian.
Let's see now what a Hamiltonian vector field associated to $f \in C^\infty(\mathcal{M})$ looks like in Darboux coordinates: if (U, φ) is a local chart and $\varphi = \left(q^1, p^1, ..., q^n, p^n\right)$ we have that:

$$\begin{aligned} X_f|_U &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right) \\ &= \left(\partial_{p_i} f\right) \partial_{q^i} - \left(\partial_{q^i} f\right) \partial_{p_i} \,, \end{aligned} \tag{C.12}$$

which follows from the definition of X_f and $\omega|_U = dp_i \wedge dq^i$.

[1]the contraction of $\omega \in A^n(\mathcal{M})$ by a vector field $X \in \mathcal{T}(\mathcal{M})$ is the function $\iota_X : A^n(\mathcal{M}) \to A^{n-1}(\mathcal{M})$ defined by $\iota_X(\omega)(Y_1, Y_2, ..., Y_{k-1}) := \omega(Y_1, Y_2, ..., Y_{k-1}, X)$, we use the notation $X \lrcorner \omega$ in place of $\iota_X(\omega)$.

In the context of symplectic manifolds it is possible to introduce a structure on the space $C^\infty(\mathcal{M})$ that turns it into a Lie algebra, we are talking about Poisson brackets:
let $(\mathcal{M}, \omega)$ be a symplectic manifold, f and $g \in C^\infty(\mathcal{M})$, the Poisson bracket between f and g is the function $\{f, g\} \in C^\infty(\mathcal{M})$ defined by the following equivalent formulas:

$$\{f, g\} = \omega(X_f, X_g) = -df(X_g) = -X_g(f). \tag{C.13}$$

From (4.14) it si possible to show that the Poisson bracket is $\mathbb{R}$-linear, antisymmetric and it satisfies the Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \tag{C.14}$$

furthermore, $X_{\{f,g\}} = [X_f, X_g]$, so there is a homomorphism between the Lie algebra of Hamiltonian vector fields and the Lie algebra of $C^\infty(\mathcal{M})$.
Using Darboux coordinates we have:

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}. \tag{C.15}$$

In particular, then, the Poisson brackets between the coordinate functions (thought as functions on the domain $U \subset \mathcal{M}$ of the local chart) are given by:

$$\{q^i, q^j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, q^j\} = \delta_i^j. \tag{C.16}$$

In addition to this, we can notice that the set of symplectic vector fields is a Lie subalgebra of $\mathcal{T}(\mathcal{M})$, this is a simple consequence of
$\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X \quad \forall X, Y \in \mathcal{T}(\mathcal{M})$; also, thanks to the Poisson brackets we have that $[X_f, X_g] = X_{\{f,g\}}$ and so the set of Hamiltonian vector fields is a Lie subalgebra of the symplectic vector fields.
There's an interesting relation with the first cohomologic group of $\mathcal{M}$, $H^1(\mathcal{M})$, in fact we can summarize what we have found in the following way:

- X symplectic $\Leftrightarrow X \lrcorner \omega$ is closed;

- X Hamiltonian $\Leftrightarrow X \lrcorner \omega$ is exact;

it's clear then that the quotient of symplectic vector fields by the subspace of the Hamiltonian vector fields is isomorphic, as a vector space (not as a Lie algebra), to $H^1(\mathcal{M})$.
Therefore, if $H^1(\mathcal{M}) = 0$ each local Hamiltonian vector field (then symplectic) is globally Hamiltonian.

C.4 Symplectic structure of T^*G

Our next goal is to achieve the symplectic structure of T^*G, since it plays an important role in the phase space of LQG. In order to do so we have to mention a few results concerning the Poisson structure on $L(G)^*$.
We begin with the definition of a Poisson manifold:
a Poisson manifold is a pair $(\mathcal{M}, \{\,,\,\})$, where $\mathcal{M}$ is a differentiable manifold and $\{\,,\,\}$ is a Lie algebra structure on $C^\infty(\mathcal{M})$ which satisfies the following Leibniz rule:

$$\{f, gh\} = g\{f, h\} + \{f, g\}h, \quad \forall f, g, h \in C^\infty(\mathcal{M})\,. \tag{C.17}$$

A function between Poisson manifolds $\Phi : \mathcal{M} \to \mathcal{N}$ is called a Poisson function if

$$\Phi^*\{f, g\}_{\mathcal{N}} = \{\Phi^* f, \Phi^* g\}_{\mathcal{M}} \quad \forall f, g \in C^\infty(\mathcal{N})\,. \tag{C.18}$$

We notice that the function $\{f, \cdot\} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is a derivation and so defines a vector field $X_f \in \mathcal{T}(\mathcal{M})$, given by:

$$X_f(g) = \{f, g\}, \tag{C.19}$$

which recalls the Hamiltonian vector field associated to f (if $\mathcal{M}$ is a symplectic manifold).
Furthermore, we have that for each $p \in \mathcal{M}$:

$$\{f, g\}(p) = (X_f(g))(p) = dg_p(X_f) = -df_p(X_g)\,, \tag{C.20}$$

it is then clear that $\{f, g\}(p)$ depends linearly on dg_p and df_p.
Now, every element of $T_p^*\mathcal{M}$ can be written as df_p with $f \in C^\infty(\mathcal{M})$, then it exists a unique tensor field (or bivector) $\Pi \in \mathcal{T}\left(\bigwedge^2 T\mathcal{M}\right)$ such that:

$$\{f, g\} = \Pi(df, dg) \quad \forall f, g \in C^\infty(\mathcal{M}) \tag{C.21}$$

The tensor field $\Pi \in \mathcal{T}\left(\bigwedge^2 T\mathcal{M}\right)$ is called Poisson tensor of $(\mathcal{M}\{\,,\,\})$, is a sort of analogue of a symplectic form $\omega \in A^2(\mathcal{M})$.
Starting from the definition of Lie derivative of a tensor field it's not diffiuclt to show that $\mathcal{L}_{X_f}\Pi = 0$, thanks to the Jacobi identity of $\{,\}$.
In particular Π defines a (vertical) morphism of vector fibre bundles
$\Pi^\flat : T^*\mathcal{M} \to T\mathcal{M}$ defined by:

$$\left\langle \beta, \Pi^\flat(\alpha) \right\rangle := \Pi(\alpha, \beta) \quad \forall \alpha, \beta \in A^1(\mathcal{M})\,, \tag{C.22}$$

and also $\Pi^\flat \circ df = X_f$, in fact:

$$\langle dg, \Pi^\flat(df) \rangle = \Pi(df, dg) = \{f, g\} = X_f(g). \tag{C.23}$$

In the case where $(\mathcal{M}, \omega)$ is a symplectic manifold we have that $\Pi^\flat = \omega^\sharp$ and there is a one-to-one

corrispondence between symplectic forms and Poisson tensors.
Let's see at this point the expression in local coordinates of the Poisson tensor.
Let (U, φ) be a local chart , $\varphi = \left(x^1, ..., x^n\right)$, we have that:

$$\Pi|_U = \frac{1}{2}\Pi^{ij}\partial_i \wedge \partial_j, \tag{C.24}$$

from (C.22) follows that

$$\{f, g\} = \Pi^{ij}\partial_i f \partial_j g, \tag{C.25}$$

in particular:

$$\Pi^{ij} = \left\{x^i, x^j\right\} \text{ and } \{f, g\} = \left\{x^i, x^j\right\}\partial_i f \partial_j g. \tag{C.26}$$

Furthermore, by eq. (C.22), every bivector field Π on $\mathcal{M}$ defines a bilinear antisymmetric map $\{\ ,\ \} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ which satisfies the Leibniz rule. It also satisfies the Jacobi identity, thus yielding a Poisson structure, if and only if

$$\Pi(df, d(\Pi(dg, dh))) + \Pi(dg, d(\Pi(dh, df))) + \Pi(dh, d(\Pi(df, dg))) = 0\ , \tag{C.27}$$

for all $f, g, h \in C^\infty(\mathcal{M})$. In local coordinates it reads

$$\Pi^{il}\partial_l\Pi^{jk} + \Pi^{jl}\partial_l\Pi^{ki} + \Pi^{kl}\partial_l\Pi^{ij} = 0\ . \tag{C.28}$$

In practice, a Poisson manifold can be seen also as a pair $(\mathcal{M}, \Pi)$ where Π is a Poisson tensor on $\mathcal{M}$.
To reach the Poisson structure of $L(G)^*$ we focus on a vector space V in place of a differentiable manifold $\mathcal{M}$.
So, let V be a real vector space, take the (algebrical) dual V^*, we can identify V with TV and, consequently, V^* with TV^*.
Clearly, to each bivector field Π on V^* corresponds a map $V^* \to \bigwedge^2 V^*$ and Π is linear if this map is linear. In this case $\forall v, w \in V$ the function $\xi \mapsto \Pi_\xi(v, w)$ is linear and corresponds to an element of V (because $(V^*)^* \cong V$), it is clear then that this element is associated to the pair (v, w), we can therefore introduce a bilinear function $[\,,\,] : V \times V \to V$ defined by:

$$\langle \xi\, , [v, w]\rangle = \Pi_\xi(v, w), \quad \forall \xi \in V^*. \tag{C.29}$$

If, in addition to this, Π satisfies equation (C.28) we can see how this condition reflects on the bilinear map $[\ ,\ ]$: the result is that $[\,,\,]$ has to satisfy the Jacobi identity.
In particular, if $f \in V^*$, we have that $df \in T^*(V^*) = (T(V^*))^* \cong (V^*)^* \cong V$, from which:

$$\Pi_\mu(df, dg) = \langle \mu, [df(\mu), dg(\mu)]\rangle\ , \quad \forall \mu \in V^*. \tag{C.30}$$

To conclude, Poisson structures on V^* whose Poisson tensor is linear are in one-to-one corrispondence with Lie algebra structures on V.

What we have found turns useful when we analyze the case $V = T_eG \cong L(G)$, thanks to what we have shown is possible to introduce a Lie-Poisson structure on $L(G)^*$, in the following way:
let $\{e^1, ..., e^n\}$ be a basis of $L(G)^*$, the respective basis of $L(G)$ is given by $\{e_1, ..., e_n\}$, in this way we have that:

$$\begin{aligned} &v = v^i e_i, \quad w = w^j e_j, \ \ \xi = \xi_k e^k \\ &[v, w] = [v^i e_i, w^j e_j] = v^i w^j [e_i, e_j] = v^i w^j c_{ij}^k e_k, \end{aligned} \tag{C.31}$$

where c_{ij}^k are the structure constants with respect to the base $\{e_k\}$.
The Poisson structure is given by (C.28), which in coordiantes reads:

$$\Pi\left(e_i, e_j\right) = \left\langle \xi_k e^k, c_{ij}^l \right\rangle = \xi_k c_{ij}^l \delta_l^k = \xi_k c_{ij}^k, \tag{C.32}$$

thus we have the following expressions:

$$\Pi\left(\xi\right) = \frac{1}{2}\xi_k c_{ij}^k \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} \tag{C.33}$$

and

$$\{f, g\}\left(\xi\right) = \xi_k c_{ij}^k \frac{\partial f}{\partial \xi_i}(\xi) \frac{\partial g}{\partial \xi_j}(\xi). \tag{C.34}$$

We observe now that on $L(G)$ we can introduce an inner product defined by $\langle A, B\rangle := -\mathsf{tr}\left(AB\right)$, in this way it is possible to show that $L(G) \cong L(G)^*$, the isomorphism being given by $i : L(G) \to L(G)^*$, $A \mapsto \left(i(A)\right)(B) = \langle A, B\rangle$.
In this sense, we can introduce a Poisson structure on $L(G)$.
Let's consider the case $L(G) = \mathsf{su}(2)$: we choose the basis of $\mathsf{su}(2)$ given by $\{\tau_1, \tau_2, \tau_3\}$, where $\tau_i = -\frac{i}{2}\sigma_i$ (σ_i are the usual Pauli matrices), in this way the structure constants are $c_{ij}^k = \epsilon_{ij}^k$.
Called L^i the coordinates of the vector $\mathbf{L} = L^i \tau_i$, we have that:

$$\begin{aligned} \{L^i, L^j\}\left(\mathbf{L}\right) &= L^k \epsilon_k^{mn} \frac{\partial L^i}{\partial L^m}\frac{\partial L^j}{\partial L^n} \\ &= L^k \epsilon_k^{mn} \delta_m^j \delta_n^j = \epsilon_k^{ij} L^k. \end{aligned} \tag{C.35}$$

We shall go on to deal with T^*G.
First of all, we show that $TG \cong G \times L(G)$, that is, TG is a trivial vector bundle whose fibre is $L(G)$, in order to do so we prove that the map

$$\begin{aligned} \chi_l \ : \ G \times L(G) &\longrightarrow \ TG \\ (a\ ,\ X) \quad &\longmapsto \ \chi_l(a, X) := \left(L_a\right)_* X \end{aligned} \tag{C.36}$$

is a vector bundle isomorphism. We recall that $l_a : G \to G$, $b \mapsto ab$ and $\left(l_a\right)_*(b) : T_bG \to T_{ab}G$. We proceed with the proof:
the map χ_l can be factorized as follows

$$
\begin{array}{ccc}
G\times T_eG & \xrightarrow{s_0\times j} & TG\times TG \\
\Big\downarrow \chi_l & & \Big\downarrow \wr \\
TG & \xleftarrow{(\mu)_*} & T(G\times G)
\end{array}
\tag{C.37}
$$

where $\mu : G \times G \to G$ is the multiplication in G, $s_0 : G \times TG$ is the null section and $j : T_eG \to TG$ is the natural inclusion.
To see that (4.36) commutes it is sufficient to observe that:

$$
(a,X) \xmapsto{s_0\times j} ((a,0),(e,X)) \overset{\sim}{\mapsto} ((a,e),(0,X)) \xmapsto{\mu_*} (a,(l_a)_*X) \tag{C.38}
$$

where the last one is true since $\mu : G \times \{e\} \to G$, $(a, e) \mapsto ae = l_a(e)$, therefore $\mu_*(X)$ is given by $(l_a)_*(X)$ $\forall X \in T_eG$.
Since each map is smooth it follows that χ_l is smooth, furthermore we have that

$$
\begin{array}{ccc}
G\times T_eG & \xrightarrow{\chi_l} & TG \\
\Big\downarrow \pi_1 & & \Big\downarrow \pi \\
G & \xrightarrow{id_G} & G
\end{array}
\tag{C.39}
$$

as one immeditely verifies, thus χ_l is a vertical morphism of vector bundles.
Finally, χ_l is bijective because $(l_a)_*$ is an isomorphism and it is linear since $(l_a)_*$ is linear, therefore χ_l is an isomorphism.
In a similar way it is possible to show that $T^*G \cong G \times L(G)^*$, knowing that $L(G) \cong L(G)^*$ we get the following result: $T^*G \cong G \times L(G)$.
At this point we aim to find the symplectic structure on T^*G.
We work with local coordinates, let $(g, p_g) \in T^*G$ where $p_g = p_\mu dg^\mu$; we denote with $\{e_\alpha\}$ a basis in $L(G)$ and with $\{\varepsilon^\alpha\}$ the respective basis in $L(G)^*$. It is easy to show that l_g defines left-invariant vector fields e^L_α and left-invariant one-forms ε^α_L on G in the following way:

$$
\begin{aligned}
e^L_\alpha(g) &:= l_{g*}e_\alpha \\
\varepsilon^\alpha_L(g) &:= l^*_{g^{-1}}\varepsilon^\alpha.
\end{aligned}
\tag{C.40}
$$

For the moment we denote with:

$$
L^\alpha_\beta(g,h) := \frac{\partial (gh)^\alpha}{\partial g^\beta}, \tag{C.41}
$$

then we can write the field e^L_α as:

$$
e^L_\alpha(g) = L^\mu_\alpha(g,e)\frac{\partial}{\partial g^\mu}, \tag{C.42}
$$

analogously we get;

$$
\varepsilon^\alpha_L(g) = L^\alpha_\mu\left(g^{-1},g\right)dg^\mu. \tag{C.43}
$$

This basis, as already shown for TG, allows us to introduce a canonical local trivialisation:

$$\begin{array}{rccc} \lambda \ : & T^*G & \longrightarrow & G \times L(G)^* \\ & (g, p_g = p_\mu dg^\mu) & \longmapsto & \left(g, \pi^L = l_g^*|_e p_g = \pi^L_\mu \varepsilon^\mu\right) \end{array} \tag{C.44}$$

where $\pi^L_\mu = p_g\left(e^L_\mu\right) = p_\nu L^\nu_\mu(g,e)$.
A basis in $T^*_x\left(T^*G\right)$, where $x = (g^\alpha, \varepsilon^\alpha_L) \in T^*G$ is given by:

$$\left\{\varepsilon^\alpha_L := L^\alpha_\mu\left(g^{-1}, g\right) dg^\mu, \ \ \varepsilon^L_\mu := d\pi^L_\mu\right\}. \tag{C.45}$$

We know that on T^*G there is a canonical one-form, i.e. the tautological one-form $\theta = p_\alpha dg^\alpha$, from which we obtain the symplectic form

$$\omega = d\theta = dp_\alpha \wedge dg^\alpha. \tag{C.46}$$

In our case we have then $\theta = \pi^L_\mu \varepsilon^\mu_L$ and

$$\omega = \varepsilon^L_\mu \wedge \varepsilon^\mu_L - \frac{1}{2}\pi^L_\mu f^\mu_{\alpha\beta} \varepsilon^\alpha_L \wedge \varepsilon^\beta_L, \tag{C.47}$$

where $f^\mu_{\alpha\beta}$ are the structure constants of the Lie algebra $L(G)$ in the basis $\{e^L_\alpha\}$. We recall also that since ε^μ_L are left-invariant one-forms on a Lie group G, they satisfy the Cartan-Maurer equation $d\varepsilon^\mu_L = -\frac{1}{2}\varepsilon^\alpha_L \wedge \varepsilon^\beta_L$.
The Hamiltonian vector field X_A associated to a function $A \in C^\infty\left(T^*G\right)$ is given by the equation $X_A \lrcorner \omega = -dA$, we can split the components as follows:

$$\begin{aligned} X^\mu_A &:= \varepsilon^\mu_L\left(X_A\right) = -dA\left(e^\mu_L\right) \\ \left(X_A\right)_\alpha &:= \varepsilon^L_\alpha\left(X_A\right) = dA\left(e^L_\alpha\right) + \pi^L_\mu f^\mu_{\alpha\beta} dA\left(e^\beta_L\right). \end{aligned} \tag{C.48}$$

Let X_B be the Hamiltonian vector field associated to $B \in C^\infty\left(T^*G\right)$, the Poisson bracket between A and B is the function given by $\{A, B\} = \omega\left(X_A, X_B\right)$, in coordinates it reads:

$$\{A, B\} = dA\left(e^L_\alpha\right)\frac{\partial B}{\partial \pi^L_\alpha} - \frac{\partial A}{\partial \pi^L_\alpha} dB\left(e^L_\alpha\right) + \frac{\partial A}{\partial \pi^L_\alpha}\pi^L_\mu f^\mu_{\alpha\beta}\frac{\partial B}{\partial \pi^L_\beta}. \tag{C.49}$$

In particular, we obtain that:

$$\begin{aligned} &\{g^\alpha, g^\beta\} = 0\,, \qquad \{g^\alpha, \pi^L_\nu\} = L^\alpha_\nu(g,e) \\ &\text{and} \qquad \{\pi^L_\mu, \pi^L_\nu\} = \pi^L_\alpha f^\alpha_{\mu\nu}. \end{aligned} \tag{C.50}$$

We want now rewrite these relations in a more compact and slightly different form.
From Lie group theory it is known that to each element of the Lie algebra $A \in T_eG$ corresponds a left-invariant vector field $L^A \in L(G)$, clearly we have that $L^A(g) = (l_g)_* A$. If we consider a matrix Lie group we can introduce a coordinate system by defining $x^{ij}(g) := g^{ij}$. In this coordinate system we can write L^A_g as follows:

$$L^A_g = \sum_{i,j=1}^{n} (gA)^{ij}\frac{\partial}{\partial x^{ij}}, \tag{C.51}$$

where $L_g^A x^{ij} = (gA)^{ij}$ are the components of L^A in the chosen coordinates (recall eq. (A.3)).
Identifying $L(G)^*$ with $L(G)$ we can associate to each $\pi^L \in L(G)^*$ an element $\overrightarrow{L} \in L(G)$. Furthermore, the components $L^{\alpha}_{\beta}(g,e)$ defined in (C.40) become now $(gA)^{ij}$ as seen before, in particular, we choose $A = \tau^i$, where $\{\tau^i\}$ is a basis of $T_eG \cong L(G)$.
Finally, considering the case $G = SU(2)$, $T_eG = \mathsf{su}(2)$, we obtain a clear expression of the Poisson parentheses in (C.49):

$$\begin{aligned} &\{g^{ij}, g^{kl}\} = 0\ , \qquad \{g^{ij}, L^k\} = \left(g\tau^k\right)^{ij} \\ &\text{and} \qquad \{L^i, L^j\} = L^k \epsilon_k^{ij}. \end{aligned} \tag{C.52}$$

In a more compact way, and taking into account the fact that in LQG to each link of a boundary graph is associated a $SU(2)$ element, we can write an equivalent expression of the latter:

$$\begin{aligned} &\{U_l, U_{l'}\} = 0\ , \qquad \{U_l, L^i_{l'}\} = \delta_{ll'} U_l \tau^i \\ &\text{and} \qquad \{L^i, L^j\} = \delta_{ll'} \epsilon_k^{ij} L_l^k. \end{aligned} \tag{C.53}$$

More precisely, let Γ be the boundary graph coming from the 2-complex associated to Δ^* and let $G = SU(2)$ the gauge group. We denote with Γ^i the i-dimensional element of Γ (nodes, links, triangles, tetrahedra).
The gauge potential corresponds to the pull-back (by a local section) of the principal connection on the principal fibre bundle whose structure group is $G = SU(2)$, it is approximated on the links of Γ by the holonomy:

$$\begin{aligned} \Gamma^1 &\longrightarrow G \\ (x,y) &\longmapsto g_{(x,y)} = \left(\mathcal{P}\mathsf{exp} - \int_a^b ds A_\mu\left(\alpha(s)\right)\dot{\alpha}^\mu(s)\right), \end{aligned} \tag{C.54}$$

where the curve $\alpha : [a,b] \to \mathcal{M}$ is the link between the spacetime points x and y.
The configuration space and the phase space are given by, respectively:

$$Q = G^L \quad \text{and} \quad M = T^*G^L \tag{C.55}$$

where L is the number of links in Γ.
Using the identifications $T^*G^L \cong (T^*G)^L$ and $T^*G \cong G \times L(G)^* \cong G \times L(G)$, as seen before, we have that the momentum conjugate to the gauge potential is given by the map

$$\begin{aligned} \Gamma^1 &\longrightarrow L(G) \\ (x,y) &\longmapsto A_{(x,y)}, \end{aligned} \tag{C.56}$$

We have seen that local gauge transformations act on the holonomy in the following way:

$$g'_{(x,y)} = h_x g_{(x,y)} h_x^{-1}, \tag{C.57}$$

this expression defines an action of G^N on Q (N is the number of nodes in Γ).

Thanks to the identifications above the lift of this action on $M = T^*Q = G^L \times L(G)^L$ is given by:

$$A'_{(x,y)} = \mathsf{Ad}(h_x) A_{(x,y)}. \tag{C.58}$$

Appendix D

Haar Measure

In this section we briefly introduce the concept of measure with the goal of defining a Haar measure on a Lie group G, specifically we are interested in $G = SU(2)$. The presence of a Haar measure allows us to introduce a scalar product on $L_2(SU(2))$ and, consequently, on $L_2\left(SU(2)^L\right)$, which is the starting point to construct the Hilbert space of LQG.

D.1 Positive Measure

Therefore, we begin by giving the definition of a σ-algebra: let X be a set and $\mathfrak{M}$ a collection of subsets of X, $\mathfrak{M}$ is a σ-algebra if

$$\begin{aligned} &(i)\ \ X \in \mathfrak{M} \\ &(ii)\ \ A \in \mathfrak{M} \Rightarrow A^c \in \mathfrak{M} \\ &(iii)\ \ A = \bigcup_{n=1}^{\infty} A_n, \quad A_n \in \mathfrak{M}\ \forall n \Rightarrow A \in \mathfrak{M}, \end{aligned} \tag{D.1}$$

if it is so, X is said measurable space and the elements of $\mathfrak{M}$ are called measurable sets.
In the case where X is a topological space the following result holds: it exists a minimal σ-algebra $\mathfrak{B}$ in X such that each open set in X belongs to $\mathfrak{B}$, the elements of $\mathfrak{B}$ are called Borel sets of X. This fact is true since there is a theorem which states that for each collection of subsets of X is always possible to find a σ-algebra that contains it, then, in particular, if (X, τ) is a topological space, τ is a topology on X and so a collection of open subsets of X.
We move on now and give the definition of measure: a (positive) measure is a function μ defined on a σ-algebra $\mathfrak{M}$,

$$\mu : \mathfrak{M} \rightarrow [0, \infty] \tag{D.2}$$

which is additive measurable, i.e. if $\{A_i\}$ is a countable family of disjoint sets of $\mathfrak{M}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=0}^{\infty} \mu\left(A_i\right). \tag{D.3}$$

D.2 Left-Invariant Measure on a group

We consider a compact Lie group G, since we are interested in the case $G = SU(2)$, which is compact. We recall that if G is a Lie group then the multiplication is a smooth function, that is

$$\begin{aligned} m:\ & G \times G \to G \\ & (g,h) \mapsto gh \end{aligned} \tag{D.4}$$

is smooth, which means that the left and right multiplication are smooth too:

$$\begin{aligned} l_g:\ & G \to G \\ & h \mapsto gh \qquad \forall g,h \in G, \end{aligned} \tag{D.5}$$

$$\begin{aligned} r_g:\ & G \to G \\ & h \mapsto hg \qquad \forall g,h \in G. \end{aligned} \tag{D.6}$$

In addition to this l_g and r_g are invertible with smooth inverse, that is they are diffeomorphisms $\forall g \in G$.
At this point we can tell what we mean by a Haar measure on a group G: a left Haar measure on a group G is a positive measure μ_H^l on the Borel σ-algebra in G with the following properties:

(i) is locally finite, that is each point in G possesses a neighbourhood with finite measure;

(ii) is left-invariant, i.e.:

$$\mu\,(gE) = \mu(E) \tag{D.7}$$

$\forall g \in G$ and for each Borel set $E \subset G$, where $gH = \{gh | h \in H\}$, $H \subset G$.
Now that we have defined what a Haar measure is, let's see how we can contruct one on a Lie group G, obviously differential forms will be involved, since they play a crucial role in integration theory on a differentaible manifold.

We have already seen that a left-invariant one-form on G has to satisfy the Cartan-Maurer equation:

$$d\omega^\alpha + \frac{1}{2}\sum_{\beta,\gamma} C^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma = 0, \;\; \forall \alpha = 1,...,n = dimG \tag{D.8}$$

where $\{\omega^1, \omega^2, ..., \omega^n\}$ is a basis of $L^*(G)$, dual of the basis $\{E_1, E_2, ..., E_n\}$ of $L(G) \cong T_eG$, such that $\omega^\alpha(E_\beta) = \delta^\alpha_\beta$.
We can define in a natural way a n-form on G by means of

$$\eta := f\,\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n, \quad f \in C^\infty(G). \tag{D.9}$$

We want to prove that $\eta \in A^n(G)$ is a volume form on G, that is, a nowhere vanishing n-form on G, and also that η is left-invariant, i.e.:

$$l_g^* \eta = \eta, \quad \forall g \in G. \tag{D.10}$$

We begin with the latter:

$$\begin{aligned}\left(l_g^* \eta\right)_h = l_g^* \left(f\, \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\right)_h = f(gh)\ \left(l_g^* \omega^1\right)_h \wedge \left(l_g^* \omega^2\right)_h \wedge \cdots \wedge \left(l_g^* \omega^n\right)_h = \\ = f(gh)\, \omega^1_{gh} \wedge \omega^2_{gh} \wedge \cdots \wedge \omega^n_{gh} = \eta_{gh}\end{aligned} \tag{D.11}$$

which is true since ω^α are left-inariant $\forall \alpha = 1, ..., n$.
Now, let $v_1, v_2, ..., v_n \in T_g G$ linearly independent, we know that we can express them as $v_i = l_{g*} A_i$, for a certain $A_i \in T_e G$, therefore:

$$\begin{aligned}\eta\left(v_1, v_2, ..., v_n\right) &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \left(v_1, v_2, ..., v_n\right) \\ &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \left(l_{g*} A_1, l_{g*} A_2, ..., l_{g*} A_n\right) \\ &= l_g^* \left(\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\right) \left(A_1, A_2, ..., A_n\right) \\ &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \left(A_1, A_2, ..., A_n\right) = \mathsf{det} A,\end{aligned} \tag{D.12}$$

where A is the matrix of the basis change from $\{E_1, E_2, ..., E_n\}$ to $\{A_1, A_2, ..., A_n\}$. Since $\mathsf{det} A \neq 0$, because A is invertible, we have that η is nowhere vanishing on G, thus η is a volume form on G.
Integrating functions against this form we obtain a Haar measure.
We are also interested under which circumstances a Haar measure is right-invariant, to see this we proceed as follows:
let μ be a Haar measure on G, we define a new measure $r_g(\mu)$ by
$r_g(\mu)(E) := \mu\left(r_g(E)\right)$, $\forall g \in G$, $E \subset G$ and r_g is the right multiplication by g. It is not difficult to show that $r_g(\mu)$ is left-invariant, in fact:

$$\begin{aligned}r_g(\mu)\left(l_h E\right) &= \mu\left(r_g l_h E\right) = \mu\left(l_h r_g E\right) \\ &= \mu\left(r_g E\right) = r_g(\mu)\left(E\right),\end{aligned} \tag{D.13}$$

since l_h and r_g commute $\forall g, h \in G$, and so $r_g(\mu)$ is given by a left-invariant n-form. However the n-form which describes $r_g(\mu)$ could be different from the n-form that describes μ, $r_g(\mu)$ and μ could clearly differ by a multiplicative constant.
Therefore, $\forall g \in G$, it exists a constant $\chi(g)$ such that $r_g(\mu) = \chi(g)\mu$, the function $\chi : G \to \mathbb{R}$ is said modular function of G.
A group G is called unimodular if $\chi(g) = 1$, $\forall g \in G$. It follows that if G is unimodular we have that $r_g(\mu) = \mu$ and so μ is right-invariant, too.

Using the description of a Haar measure in terms of differential forms it is possible to show that if G is a connected Lie group then G is unimodular if and only if $\mathsf{det}\mathrm{Ad}_g = 1$ $\forall g \in G$ or, equivalently, if and only if $\mathrm{tr}\left(\mathrm{ad}_X\right) = 0, \forall X \in T_e G \cong L(G)$.
To prove this result we begin by showing the equivalence mentioned above: since $e^{\mathrm{tr}(\mathrm{ad}_X)} = \mathsf{det} e^{\mathrm{ad}_X} = \mathsf{det}\mathrm{Ad}_{e^X}$, then $\mathrm{tr}(\mathrm{ad}_X) = 0$ if and only if $\mathsf{det}\mathrm{Ad}_{e^X} = 1$, $\forall X \in T_e G$. Furthermore, since G is connected we have that $g = e^{X_1} e^{X_2} \cdots e^{X_m}$ for a certain number of $X_i \in T_e G$, thus

$$
\begin{aligned}
\mathsf{det}\mathrm{Ad}_g &= \mathsf{det}\left(\mathrm{Ad}_{e^{X_1}e^{X_2}\cdots e^{X_m}}\right) \\
&= \mathsf{det}\left(\mathrm{Ad}_{e^{X_1}}\mathrm{Ad}_{e^{X_2}}\cdots\mathrm{Ad}_{e^{X_m}}\right) \\
&= \mathsf{det}\left(\mathrm{Ad}_{e^{X_1}}\right)\mathsf{det}\left(\mathrm{Ad}_{e^{X_2}}\right)\cdots\mathsf{det}\left(\mathrm{Ad}_{e^{X_m}}\right) = 1,
\end{aligned}
\tag{D.14}
$$

where the second equality comes from the fact that Ad is a (linear) representation of G.
To prove the proposition we make the following observation:
let $C_g : G \to G$, $h \mapsto ghg^{-1}$ be the conjugate action, recalling that $\mathrm{Ad}_g : T_eG \to T_eG$ is defined by $\mathrm{Ad}_g := (C_g)_*$, we introduce the measure $C_g\mu$ defined by $(C_g\mu)(E) := \mu(C_gE)$ and we show that it is left-invariant:

$$
\begin{aligned}
(C_g\mu)(E) = \mu(C_gE) &= \mu\left(l_g \circ r_{g^{-1}}E\right) \\
&= \mu\left(r_{g^{-1}}E\right) = \left(r_{g^{-1}}\mu\right)(E).
\end{aligned}
\tag{D.15}
$$

Therefore, since $r_g\mu$ is left-invariant $\forall g \in G$, $C_g\mu$ is left-invariant, too.
In addition to this, μ is C_g-invariant if and only if μ is right-invariant, as it is evident from (D.15).
Let's see at this point how we can express in terms of differential forms the requirement of C_g invariance.
Since $C_g\mu$ is left-invariant it is sufficient to prove it for $e \in G$, doing so we obtain:

$$
\begin{aligned}
\left(C_g^*\mu\right)_e(E_1, E_2, ..., E_n) &= \mu_e\left(C_{g_*}E_1, C_{g_*}E_2, ..., C_{g_*}E_n\right) \\
&= \mu_e\left(\mathrm{Ad}_gE_1, \mathrm{Ad}_gE_2, ..., \mathrm{Ad}_gE_n\right) \\
&= f(e)\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\left(\mathrm{Ad}_gE_1, \mathrm{Ad}_gE_2, ..., \mathrm{Ad}_gE_n\right) \\
&= f(e)\mathsf{det}\mathrm{Ad}_g = \mathsf{det}\mathrm{Ad}_g\ \mu(E_1, E_2, ..., E_n),
\end{aligned}
\tag{D.16}
$$

it is then clear that $C_g^*\mu = \mu$ if and only if $\mathsf{det}\mathrm{Ad}_g = 1$. The proof is ended.
In particular, we have that compact groups are unimodular, because if G is compact it exists an inner product with respect to which ad_X is antisymmetric and so ad_X is traceless. The reason for this is that if G is compact it is always possible to find a basis $\{E_1, E_2, ..., E_n\}$ in $L(G)$ such that:

$$
\mathrm{tr}\left(E_\alpha E_\beta\right) = -c\,\delta_{\alpha\beta}, \quad \text{with } c > 0,
\tag{D.17}
$$

by the way, the fact that c is positive implies that, in gauge theories, the kinetic term in the Lagrangian is positive.
An inner product in $L(G)$ is then given by:

$$
\begin{aligned}
\tilde{\mathrm{tr}}\ : L(G) \times L(G) &\longrightarrow \quad \mathbb{R} \\
(A\ , B) \qquad &\longmapsto \quad \tilde{\mathrm{tr}}(A, B) := \mathrm{tr}(AB)\ ,
\end{aligned}
\tag{D.18}
$$

one can easily verify that $\tilde{\mathrm{tr}}$ is real, symmetric and independent from the choice of a basis in $L(G)$.
We observe that

$$
\begin{aligned}
\mathrm{tr}\left(\mathsf{ad}_{E_\alpha}(E_\beta)\,E_\gamma\right) = \mathrm{tr}\left(C_{\alpha\beta}^{\delta}E_\delta E_\gamma\right) &= C_{\alpha\beta}^{\delta}\mathrm{tr}\left(E_\delta E_\gamma\right) \\
&= -c\,C_{\alpha\beta}^{\delta}\delta_{\delta\gamma} \equiv -c\,C_{\alpha\beta\gamma}.
\end{aligned}
\tag{D.19}
$$

We know that $C_{\alpha\beta\gamma}$ is antisymmetric in the first two indices, however it is possible to show that it is also antisymmetric in the last two, that is, $C_{\alpha\beta\gamma}$ is completely antisymmetric, in fact:

$$
\begin{aligned}
-c\, C_{\alpha\beta\gamma} = \mathrm{tr}\left([E_\alpha, E_\beta]\, E_\gamma \right) &= \mathrm{tr}\left(E_\alpha E_\beta E_\gamma - E_\beta E_\alpha E_\gamma \right) \\
&= \mathrm{tr}\left(E_\beta E_\gamma E_\alpha - E_\beta E_\alpha E_\gamma \right) \\
&= \mathrm{tr}\left(E_\beta \left[E_\gamma, E_\alpha \right] \right) \\
&= -c\, C_{\beta\gamma\alpha},
\end{aligned}
\tag{D.20}
$$

which implies $C_{\beta\gamma\alpha} = C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma}$.
The fact that $C_{\alpha\beta\gamma}$ is completely antisymmetric shows that ad_X is antisymmetric with respect to the inner product given by $\tilde{\mathrm{tr}}$.
In the case of interest, $G = SU(2)$ is compact and then unimodular, i.e. the Haar measure seen before is left-invariant and right-invariant, specifically, it is gauge invariant for a gauge transformation taking place on a node of the 2-complex.

Appendix E

Hilbert Space

From the previous discussion on the Poisson structure of T^*G we deduce that is possible, by means of canonical quantization, to promote the Poisson brackets to a commutator and to assign to each U_l and L_l the role of operators acting on a Hilbert space, with the due specifications.
We have seen that, in a compact form, U_l represents coordinates in G and L_l is its conjugate momentum. This fact suggests to deal with the following Hilbert space:

$$\mathcal{H}_\Gamma = L_2\left[SU(2)^L\right], \tag{E.1}$$

where L is the number of links in the boundary graph Γ.
The states are then the wave functions $\psi\left(U_l\right)$ of L group elements U_l.
On $\mathcal{H}_\Gamma$ is defined a scalar product compatible with the Haar measure:

$$\langle\psi|\phi\rangle = \int_{SU(2)^L} dU_l\overline{\psi\left(U_l\right)}\phi\left(U_l\right). \tag{E.2}$$

For what concerns the operators U_l and L_l the following results hold: the operator U_l is simply defined by

$$\left(U_l\psi\right)\left(U_l'\right) := \psi\left(U_lU_l'\right), \tag{E.3}$$

it acts then as a multiplicative operator.
Recalling now that L_l^i is a left-invariant vector field on G we can prove that it coincides with the vector field on G induced by the right multiplication $\delta_g : G \to G$, $\delta_g(h) = hg$. In fact, taking into account the results found previously on induced vector fields (see eq. (B.7)) we have that:

$$X_g^A = l_{g*}\left(A\right) = L_g^A, \tag{E.4}$$

in particular, $\forall f \in C^\infty\left(G\right)$,

$$L_g^A(f) = \frac{d}{dt}f\left(g\mathsf{exp}tA\right)|_{t=0} \tag{E.5}$$

since $\delta_{\mathsf{exp}(tA)}$ is the flow of the induced vector field X^A which coincides with L^A. In this manner it's easy to see that the field L_l^i acts on the wave functions in the following way [1]

[1]the factor $-i$ comes from the commutation relations $[\tau_i\,,\tau_j] = i\epsilon_{ij}^k\tau_k$

$$\left(J^i\psi\right)(U) := -i\frac{d}{dt}\psi\left(Ue^{t\tau_i}\right)|_{t=0}, \tag{E.6}$$

where we have used the exponential of a matrix since $SU(2)$ is a matrix group. If we finally insert the multiplicative constants we can write L^i_l as follows:

$$L^i_l := (8\pi\hbar G)\, J^i_l. \tag{E.7}$$

Having realised the canonical quantisation we want to know, at this point, how to build gauge-invariant wave function. This requirement is necessary because in LQG physical states contain geometrical informations, for instance, in 3D we know that at each node of the dual triangulation are located tetrahedra, whose properties (volume, surfaces of its faces,...) are invariant under rotations. We have seen how the holonomy transforms under an active gauge transformation:

$$U_l \mapsto \Lambda_{s_l} U_l \Lambda^{-1}_{t_l} \tag{E.8}$$

where s_l and t_l specify the points in spacetime that bound the chosen link. Gauge-invariant states under these transformations must then satisfy

$$\psi\left(\Lambda_{s_l} U_l \Lambda^{-1}_{t_l}\right) = \psi\left(U_l\right), \quad \text{with} \quad \Lambda_{\mathrm{n}} \in SU(2). \tag{E.9}$$

If we consider a wave function on the (2-dimensional) bounding graph then a gauge transformation acts on every node of that graph, that is, for each node we have three gauge transformations. For rhis reason we shall introduce an operator $C^i_{\mathrm{n}} := L^i_{l_1} + L^i_{l_2} + L^i_{l_3}$ which has to satisfy the following property;

$$C^i_{\mathrm{n}}\psi = 0. \tag{E.10}$$

More precisley, the condition of gauge invariance of ψ can be rewritten following this reasoning: say we choose a node **n** which we consider a target of three links, in this way a gauge transformation will produce the same element Λ_{t_l} for each of the three links. Now, in general, we know that (dropping the constants):

$$\left(L^i_{l_j}\psi\right)\left(U_{l_j}\right) := -i\frac{d}{dt}\psi\left(U_{l_j}e^{t\tau_i}\right)|_{t=0}, \tag{E.11}$$

with $j = 1, 2, 3$ indexing the three links convergent on **n**. Since we are focusing on the node **n** we can keep the gauge transformations on the other nodes as generic, that is $\Lambda_{s_{l_j}}$ generic. Then, it follows that

$$\begin{aligned}\left(L^i_{l_j}\psi\right)\left(\Lambda_{s_{l_j}} U_{l_j}\right) &= -i\frac{d}{dt}\psi\left(\Lambda_{s_{l_j}} U_{l_j} e^{t\tau_i}\right)|_{t=0} \\ &= i\frac{d}{dt}\psi\left(\Lambda_{s_{l_j}} U_{l_j} e^{-t\tau_i}\right)|_{t=0}.\end{aligned} \tag{E.12}$$

To conclude, since $SU(2)$ is compact and simply connected, every element of $SU(2)$ can be written as the exponential of an element of $\mathsf{su}(2)$, we understand that the condition (6.9) is translated into the following requirement:

$$L^i_{l_1} + L^i_{l_2} + L^i_{l_3} = 0, \quad i = 1, 2, 3. \tag{E.13}$$

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