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Point-Vortex Dynamics on Non-Orientable
and Multiply-Connected Surfaces

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Introduction

The Euler equations of hydrodynamics,

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, \\ \nabla \cdot v = 0, \end{cases} \quad (1)$$

are the second oldest of all PDEs. They describe the motion of a fluid that is incompressible and viscosity-free in the variables of the velocity v and the pressure p . A classical approach to solving (1) is to eliminate the pressure variable by taking the curl. In 2D, the resulting equation

$$\begin{cases} \frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = 0 \\ \omega = \nabla \times v, \end{cases} \quad (2)$$

describes the evolution of the vorticity $\omega = \nabla \times v$. The vorticity equation (2) reflects the frozen-in property of vorticity: it is neither created nor generated, but only transported along the flow. A classical ansatz to simplify the vorticity equation, introduced by Helmholtz, is to consider idealized concentrations of vorticity

$$\omega = \sum_i \Gamma_i \delta_{r_i},$$

referred to as **point-vortices**. As vorticity is transported by the flow, such point-vortices do not dissipate, but instead propagate through the flow. Their motion is a finite-dimensional Hamiltonian system, reminiscent of the N -body problem.

In this thesis, we explore geometric aspects of point-vortex dynamics. While vortex motion is a diverse, multifaceted field, we work primarily from the perspective of geometric mechanics. In particular, we adopt the geometric viewpoint of the Euler equations developed by Arnold as an infinite-dimensional Lie-Poisson system arising from reduction on the group of area-preserving diffeomorphisms.

In Chapter 1, we review the relevant geometric background for our discussion. We develop the theory of Lie-Poisson reduction and consider the case of the rigid body. We then review the Hodge decomposition of differential forms, focusing on the relevant case of one-forms on surfaces. In Chapter 2, we focus on the Lie-Poisson geometry of the group of area-preserving diffeomorphisms. We summarize Arnold's perspective on the Euler equation (1) as a reduced Euler-Poisson equation and study how the transport of vorticity arises as a consequence.

In the following chapters, we apply our machinery to the dynamics of point-vortices. In Chapter 3, we study the point-vortex system in detail. We begin with the classical case of the plane before considering point-vortices on general closed, simply-connected surfaces. We show the connection between the point-vortex system and the geometric formulation of hydrodynamics given in Chapter 2. In Chapter 4, we study the case of non-orientable surfaces. We examine first the motion of general fluids on non-orientable surfaces before focusing on the specific case of point-vortices.

We show how the orientation double cover of a non-orientable surface can be used to resolve theoretical issues involving the dynamics. In the final chapter, we study point-vortex motion on multiply-connected surfaces. Applying the results of the previous chapters, we examine the coupling between vortical and harmonic flows. We study the technical details of the point-vortex system and conclude by deriving its Hamiltonian structure.

Chapter 1

Geometric Preliminaries

This chapter introduces the mathematical setting for the Hamiltonian formulation of ideal hydrodynamics and point-vortex motion on surfaces.

We first review the Lie-Poisson bracket in a dual Lie algebra $\mathfrak{g}^*$ and its corresponding symplectic form on coadjoint orbits. We then introduce the notion of a momentum map and present Marsden and Weinstein's symplectic reduction theorem. We explore the interplay between Lie-Poisson and symplectic reduction by considering the case of the *rigid body*, which serves as a model case for our later study of ideal hydrodynamics. Our exposition of the relevant geometric mechanics follows [18] and [20], and cites several results from those sources.

We then discuss the Hodge decomposition of differential forms. We focus on the case of one-forms on closed surfaces, working in analogy with the Helmholtz decomposition of vector fields. We introduce the Green function for the Laplacian on a closed surface and review its properties. We conclude by discussing the construction of a basis of harmonic forms given a canonical basis in homology. We refer to [11] and [27] for our exposition on Hodge theory, and [6] for our discussion of the Green function.

1.1 Lie-Poisson and Symplectic Reduction

1.1.1 The Lie-Poisson Bracket and the KKS form

Let G be a finite-dimensional Lie group with associated Lie algebra $\mathfrak{g}$. Recall that the natural adjoint action of G on $\mathfrak{g}$ induces the **coadjoint action** of G on $\mathfrak{g}^*$,

$$\langle \text{Ad}_h^*(\nu), X \rangle := \langle \nu, \text{Ad}_{h^{-1}}(X) \rangle, \quad (1.1)$$

for $h \in G$, $\nu \in \mathfrak{g}^*$, and $X \in \mathfrak{g}$. This in turn descends to the infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$,

$$\langle \text{ad}_Y^*(\nu), X \rangle := -\langle \nu, \text{ad}_Y(X) \rangle, \quad (1.2)$$

for $\nu \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$. The coadjoint action of G on $\mathfrak{g}^*$ foliates $\mathfrak{g}^*$ into coadjoint orbits. As orbits of a smooth action, these coadjoint orbits are smooth, immersed submanifolds. When G is compact, they are further embedded submanifolds.

We recall the Poisson geometry of $\mathfrak{g}^*$. Given a smooth function $F \in C^\infty(\mathfrak{g}^*)$, its associated differential at a point $\bar{\nu} \in \mathfrak{g}^*$ is the unique element $\frac{\delta F}{\delta \nu} \Big|_{\bar{\nu}} \in \mathfrak{g}$ that satisfies

$$\left\langle dF \Big|_{\bar{\nu}}, \nu \right\rangle = \left\langle \nu, \frac{\delta F}{\delta \nu} \Big|_{\bar{\nu}} \right\rangle,$$

for all $\nu \in \mathfrak{g}^*$. In finite dimensions this is an instance of the canonical identification of $\mathfrak{g}$ and its bidual, since $dF|_{\bar{\nu}}$ is an element of $\mathfrak{g}^{**}$. This identification induces a natural Poisson structure on $\mathfrak{g}^*$.

Definition 1.1. *The **Lie-Poisson bracket** on a dual Lie algebra $\mathfrak{g}^*$ is the bracket given at $\bar{\nu} \in \mathfrak{g}^*$ by*

$$\{F, G\}_{\pm}(\bar{\nu}) = \pm \left\langle \bar{\nu}, \left[\frac{\delta F}{\delta \nu} \Big|_{\bar{\nu}}, \frac{\delta G}{\delta \nu} \Big|_{\bar{\nu}} \right] \right\rangle, \quad (1.3)$$

for smooth functions $F, G \in C^\infty(\mathfrak{g})^*$.

It is verified in [20] that (1.3) defines a Poisson bracket. The reason for the two possible signs of the Lie-Poisson bracket will become clear shortly. The following results, proved in [20], describe the Poisson geometry of $\mathfrak{g}^*$ under the Lie-Poisson bracket.

Proposition 1.2. *The symplectic leaves of the Poisson space $(\mathfrak{g}^*, \{\cdot, \cdot\}_+)$ are the coadjoint orbits*

$$\mathcal{O}_\nu = \{\text{Ad}_h^* \nu : h \in G\}.$$

The tangent space to a coadjoint orbit at a point $\bar{\nu} \in \mathcal{O}_\nu$ is

$$T_{\bar{\nu}} \mathcal{O}_\nu = \{\text{ad}_X^* \bar{\nu} : X \in \mathfrak{g}\}.$$

Definition 1.3. *The symplectic form σ_{KKS} induced by (1.3) on each coadjoint orbit $\mathcal{O}_\nu \subset \mathfrak{g}^*$ is known as the Kirillov-Kostant-Souriau or **KKS form**. The KKS form is given at a point $\bar{\nu} \in \mathcal{O}_\nu$ by*

$$\sigma_{KKS}(\text{ad}_X^* \bar{\nu}, \text{ad}_Y^* \bar{\nu}) = \pm \langle \bar{\nu}, [X, Y] \rangle \quad (1.4)$$

for $X, Y \in \mathfrak{g}$.

1.1.2 Lie-Poisson Reduction

Although the Lie-Poisson bracket and KKS form are defined above in terms of the algebraic and smooth structures of $\mathfrak{g}^*$, these objects can be “explained” as quotient structures arising from reduction of the canonical Poisson structure on the cotangent bundle T^*G .

The key point is that the cotangent bundle of a Lie group G is left-trivializable by the map $\Lambda : T^*G \rightarrow G \times \mathfrak{g}^*$ given as

$$\Lambda(h, \nu_h) = (h, T_e^* L_h(\nu_h)). \quad (1.5)$$

Under this trivialization, $\mathfrak{g}^*$ can be realized as the quotient of T^*G by the lifted action of G . Explicitly, G acts on T^*G by the cotangent lift of left multiplication $\Psi^L : G \times T^*G \rightarrow T^*G$ as

$$\Psi_k^L(h, \nu_h) = (kh, T_h^* L_{k^{-1}} \nu_h),$$

for $k \in G$. Using the left-trivialization (1.5), Ψ^L may be written as an action on $G \times \mathfrak{g}^*$, where it takes the form

$$\left(\Lambda \circ \Psi_k^L \circ \Lambda^{-1} \right)(h, \nu) = (kh, \nu).$$

It is thus clear that under Λ , the quotient of T^*G by the action Ψ^L is simply $T^*G/G \cong \mathfrak{g}^*$. It is shown in [20] that the action Ψ^L is Poisson, and so the quotient space $\mathfrak{g}^*$ inherits a Poisson structure from the canonical Poisson structure on T^*G . The resulting bracket in the quotient space is the minus Lie-Poisson bracket of (1.3).

The cotangent bundle T^*G can of course be right-trivialized as well, and the entire process above may be repeated using the cotangent lift of right multiplication Ψ^R . This in turn generates

the positive Lie-Poisson bracket. Thus, the two sign choices in (1.3) correspond to the two choices of trivialization of T^*G . Going forward, we suppress the notation Ψ^L and Ψ^R and simply refer to the lifted left and right action of G on T^*G . We refer to the orbit projection maps $\pi_L : T^*G \rightarrow \mathfrak{g}^*$ and $\pi_R : T^*G \rightarrow \mathfrak{g}^*$ under the two quotients as the **left and right translations to the identity**, given explicitly as

$$\pi_L((h, \nu_h)) = T_e^* L_h(\nu_h), \quad \pi_R((h, \nu_h)) = T_e^* R_h(\nu_h), \quad (1.6)$$

for $(h, \nu_h) \in T^*G \cong G \times \mathfrak{g}^*$. From the definition of the adjoint action $\text{Ad}_h = T_e(L_h \circ R_{h^{-1}})$, it is clear that the translations to the identity satisfy

$$\text{Ad}_{h^{-1}} \circ \pi_L = \pi_R, \quad \text{Ad}_h \circ \pi_R = \pi_L. \quad (1.7)$$

The trivialization of T^*G allows for the reduction of Hamiltonian systems on T^*G that are G -symmetric.

Theorem 1.4. (*Lie-Poisson Reduction, [20]*) *Let $\hat{H} \in C^\infty(T^*G)$ be a Hamiltonian function invariant under the lifted left action of G on T^*G and $X_{\hat{H}} \in \mathfrak{X}(T^*G)$ be its Hamiltonian vector field under the canonical symplectic structure on T^*G . Then there exists a unique Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ determined by the property $H \circ \pi_L = \hat{H}$. Moreover, the Hamiltonian vector field $X_H \in \mathfrak{X}(\mathfrak{g}^*)$ of H under the minus Lie-Poisson bracket in $\mathfrak{g}^*$ is π_L -related to $X_{\hat{H}}$.*

Remark 1.5. An analogous result holds for right-invariant Hamiltonians and the plus Lie-Poisson bracket together with the right translation π_R . Thus, left-invariant Hamiltonians induce Lie-Poisson dynamics under the minus bracket of $\mathfrak{g}^*$, while right-invariant Hamiltonians do the same for the plus bracket. The consequences of this duality are discussed at length in [21].

This reduction of dynamics from T^*G to $\mathfrak{g}^*$ is known as **Lie-Poisson reduction**. Quadratic Hamiltonian functions play a special role in the study of Lie-Poisson reduced systems. To elaborate on their importance, we recall the following definition from [18].

Definition 1.6. *An invertible operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is called an **inertia operator** for $\mathfrak{g}$ if the self-adjointness property*

$$\langle AX, Y \rangle = \langle AY, X \rangle$$

holds for all $X, Y \in \mathfrak{g}$.

Proposition 1.7. *There is a bijective correspondence between non-degenerate quadratic Hamiltonian functions in $\mathfrak{g}^*$, non-degenerate quadratic Hamiltonian functions in $\mathfrak{g}$, inertia operators on $\mathfrak{g}$, and left (right)-invariant metrics on G .*

Proof. By polarization, any quadratic Hamiltonian $H \in C^\infty(\mathfrak{g})$ may be written as $H(X) = \frac{1}{2} \langle AX, X \rangle$ for some self-adjoint, invertible operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$. Such A is an inertia operator, and induces a quadratic Hamiltonian in $\mathfrak{g}^*$ by $H(\nu) = \frac{1}{2} \langle \nu, A^{-1} \nu \rangle$. This correspondence is clearly bijective. To see the correspondence between inertia operators and invariant metrics, recall that each inertia operator generates an inner-product g in $\mathfrak{g}$ by

$$g(X, Y) = \langle AX, Y \rangle,$$

and left (right)-invariant metrics on G are in correspondence with inner-products in $\mathfrak{g}$ through pullback by lifted left (right) multiplication. $\square$

The following result, proved in [18], gives the form of Hamiltonian flow under the Lie-Poisson bracket. Under the minus Lie-Poisson bracket the right-hand side changes sign.

Proposition 1.8. *The Hamiltonian flow of a smooth function $H \in C^\infty(\mathfrak{g}^*)$ under the plus Lie-Poisson bracket (1.3) satisfies the following ODE in $\mathfrak{g}^*$,*

$$\dot{\nu}(t) = -\text{ad}_{\frac{\delta H}{\delta \nu}}^*|_{\nu(t)} \nu(t). \quad (1.8)$$

*This is known as the **Euler-Poisson equation** of H .*

The above proposition gives an explicit expression for the Hamiltonian flow of reduced Lie-Poisson dynamics. In particular, given a left-invariant Hamiltonian function $\hat{H} \in C^\infty(T^*G)$, Theorem 1.4 guarantees the existence of a Hamiltonian in $H \in C^\infty(\mathfrak{g}^*)$ whose Hamiltonian vector field $X_H \in \mathfrak{X}(\mathfrak{g}^*)$ is π_L -related to $X_{\hat{H}} \in \mathfrak{X}(T^*G)$. The Hamiltonian flow of X_H in $\mathfrak{g}^*$ satisfies (1.8). Moreover, if $\hat{H}$ is the kinetic energy associated to a left-invariant metric on G , then the reduced Hamiltonian H will be quadratic, and in particular the non-degenerate quadratic Hamiltonian described by Proposition 1.7.

1.1.3 Symplectic Reduction

We now consider a different reduction of Hamiltonian dynamics on a symplectic manifold. The setting is that of a Lie group action Ψ of G on a symplectic manifold (M, σ) . Recall that the action Ψ associates to each $X \in \mathfrak{g}$ an infinitesimal generator vector field $X_M \in \mathfrak{X}(M)$ by

$$X_M(m) := \left. \frac{d}{dt} \right|_{t=0} \Psi_{\exp(tX)}(m).$$

The action Ψ is called symplectic if $\Psi_h^* \sigma = \sigma$ for all $h \in G$. This descends to the infinitesimal identity

$$\mathcal{L}_{X_M} \sigma = d(i_{X_M} \sigma) = 0.$$

In other words, the infinitesimal generator fields $X_M \in \mathfrak{X}(M)$ of a symplectic action are locally Hamiltonian on M for all $X \in \mathfrak{g}$.

If the X_M vector fields are further globally Hamiltonian, which is to say if for each $X \in \mathfrak{g}$ there exists a Hamiltonian function $H_X \in C^\infty(M)$ satisfying

$$i_{X_M} \sigma = dH_X,$$

we say that the G -action Ψ is a **Hamiltonian action**.

Definition 1.9. *With $\Psi : G \times M \rightarrow M$ a Hamiltonian action on (M, σ) , a **momentum map** for Ψ is a smooth map $J : M \rightarrow \mathfrak{g}^*$ that satisfies*

$$\langle J(m), X \rangle = H_X(m),$$

for all $m \in M$ and $X \in \mathfrak{g}$.

Remark 1.10. As discussed in [20], Chapter 11, momentum maps give rise to a Hamiltonian expression of Noether's Theorem. In particular, if J is the momentum map associated to a Hamiltonian action Ψ and $H \in C^\infty(M)$ is a Hamiltonian invariant under Ψ , then J is invariant under the Hamiltonian flow of H . A momentum map can thus be understood as a conserved quantity arising from symmetries which generalizes angular and linear momentum.

The following example of a momentum map arising from the tautological inclusion of a coadjoint orbit into $\mathfrak{g}^*$ will prove useful in our later discussion of point-vortex dynamics.

Example 1.11. Let $M = \mathcal{O}_\nu$ be a coadjoint orbit in a dual Lie algebra $\mathfrak{g}^*$. The coadjoint action of G on the symplectic manifold (M, σ_{KKS}) is a symplectic action. Let us show that it is Hamiltonian, with momentum map given by the inclusion

$$i : \mathcal{O}_\nu \rightarrow \mathfrak{g}^*.$$

We first note that $X \in \mathfrak{g}$ generates the infinitesimal vector field $X_M \in \mathfrak{X}(M)$ given by $X_M(\bar{\nu}) = \text{ad}_X^* \bar{\nu}$. Indeed, calculating at the point $\bar{\nu} \in M$ shows that

$$\left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(Xt)}^* \bar{\nu} = \text{ad}_X^* \bar{\nu}.$$

Now let $H_X \in C^\infty(M)$ be the smooth function $H_X(\bar{\nu}) = \langle i(\bar{\nu}), X \rangle = \langle \bar{\nu}, X \rangle$. Let us show that dH_X and $i_{X_M} \sigma_{\text{KKS}}$ are equal by showing that they agree at $\bar{\nu} \in M$. Fixing $Y \in \mathfrak{g}$, we compare the action of these one-forms on $\text{ad}_Y^* \bar{\nu}$, which by Proposition 1.2 is a generic element of the tangent space fiber $T_{\bar{\nu}} M$. We have the calculations

$$i_{\text{ad}_Y^* \bar{\nu}}(dH_X|_{\nu=\bar{\nu}}) = (\mathcal{L}_{\text{ad}_Y^* \bar{\nu}} H_X)|_{\nu=\bar{\nu}} = (\mathcal{L}_{\text{ad}_Y^* \bar{\nu}} \langle \nu, X \rangle)|_{\nu=\bar{\nu}} = \langle \text{ad}_Y^* \bar{\nu}, X \rangle = \langle \bar{\nu}, -\text{ad}_Y X \rangle = \langle \bar{\nu}, [X, Y] \rangle,$$

and

$$i_{\text{ad}_Y^* \bar{\nu}} \left(i_{X_M} \sigma_{\text{KKS}} \right) |_{\nu=\bar{\nu}} = \sigma_{\text{KKS}}(X_M|_{\nu=\bar{\nu}}, \text{ad}_Y^* \bar{\nu}) = \sigma_{\text{KKS}}(\text{ad}_X^* \bar{\nu}, \text{ad}_Y^* \bar{\nu}) = \langle \bar{\nu}, [X, Y] \rangle.$$

So dH_X and $i_{X_M} \sigma_{\text{KKS}}$ align. The equality

$$i_{X_M} \sigma_{\text{KKS}} = dH_X = d\langle i(\nu), X \rangle,$$

is precisely the statement that i is a momentum map for the coadjoint action on $M = \mathcal{O}_\nu$.

Example 1.12. Another example of a momentum map is given by the orbit projection maps arising from the left and right trivializations of T^*G . As shown in [20], the left translation to the identity π_L given in (1.6) is the momentum map associated with the lifted right action of G on T^*G , while the right translation to the identity π_R is the momentum map associated with the lifted left action of G on T^*G .

The following property of momentum maps is immensely important in their study.

Definition 1.13. A momentum map $J : M \rightarrow \mathfrak{g}^*$ for a Hamiltonian action Ψ is called **equivariant** if for all $m \in M$ and $h \in G$, it holds that

$$J(\Psi_h(m)) = \text{Ad}_h^*(J(m)). \quad (1.9)$$

Remark 1.14. In the physical context, where a momentum map J is understood as a conserved quantity associated to a physical symmetry, equivariance communicates the idea that J changes in a “natural way” under the symmetry. For example, if a mechanical system is rotated to a new configuration, equivariant angular momentum will rotate along with it.

Example 1.15. The momentum map of Example 1.11 is equivariant by construction. Through a longer calculation, found in [20], one may show that the momentum maps of Example 1.12 are equivariant as well.

The primary results about equivariant momentum maps that we use in our discussion are the following.

Theorem 1.16. *Let $J : M \rightarrow \mathfrak{g}^*$ be an equivariant momentum map for a Hamiltonian left (right) action on the symplectic manifold (M, σ) . Then J is a Poisson map for the plus (minus) Lie-Poisson bracket on $\mathfrak{g}^*$.*

Proposition 1.17. *If J is the momentum map of a transitive, Hamiltonian action on a symplectic manifold (M, σ) then the image $J(M) \subseteq \mathfrak{g}^*$ is a coadjoint orbit, with $J^* \sigma_{KKS} = \sigma$.*

Proof. Fix $\bar{m} \in M$ and $J(\bar{m}) = \bar{\nu} \in \mathfrak{g}^*$. From equivariance, it holds that $J(\text{Orb}_G(\bar{m})) = \mathcal{O}_{\bar{\nu}}$. Transitivity gives that the orbit of $\bar{m}$ is all of M . By the above theorem J is Poisson, which gives the last result. $\square$

The following celebrated theorem of Marsden and Weinstein shows that equivariant momentum maps allow for the reduction of symplectic manifolds by certain symmetries. We denote by G_ν the stabilizer of $\nu \in \mathfrak{g}^*$ under the coadjoint action of G . For further discussion, see [14] and [20].

Theorem 1.18 (Marsden-Weinstein). *Let Ψ be a Hamiltonian, free, and proper action of a Lie group G on a symplectic manifold (M, σ) and let $J : M \rightarrow \mathfrak{g}^*$ be an equivariant momentum map for Ψ . Then for each $\nu \in \mathfrak{g}^*$, the quotient*

$$M_\nu := J^{-1}(\nu)/G_\nu$$

is a symplectic manifold with symplectic form σ_ν uniquely characterized by

$$\pi^* \sigma_\nu = i^* \sigma,$$

where $\pi : J^{-1}(\nu) \rightarrow J^{-1}(\nu)/G_\nu$ is the quotient projection and $i : J^{-1}(\nu) \rightarrow M$ is the inclusion.

1.1.4 The Rigid Body

We now explore Lie-Poisson and symplectic reduction using the example of the Euler equations of a rigid body. The rigid body is the prototypical example of a finite-dimensional Lie-Poisson system. Indeed, as we shall see in the following chapter, the Euler equations of ideal hydrodynamics¹ can be understood as an infinite-dimensional generalization of the rigid body. Our review presents only a small amount of the material found in [20] Chapter 15. The curious reader is encouraged to follow the full discussion in that source and in [14].

The dynamics considered are those of a three-dimensional solid $\mathcal{S}$ rotating about a fixed point in the absence of external forces. After fixing a reference state for $\mathcal{S}$, these can be described using the Lie group $G = SO(3)$. In particular, because $\mathcal{S}$ is taken as rigid, each particle evolves from its initial position $Y \in \mathcal{S}$ by a smooth curve of elements $t \mapsto \mathbf{R}_t \in SO(3)$ to a new position $y = y(Y, t) = \mathbf{R}_t Y$. We call such a curve $\mathbf{R}_t$ a motion of $\mathcal{S}$. Given a motion, it is possible to recover the initial position of any particle as $Y = \mathbf{R}_t^{-1} y$.

There exist two natural descriptions for the velocity of a motion. The Lagrangian velocity of a motion is

$$V_t(Y) := \frac{\partial y(Y, t)}{\partial t} = \dot{\mathbf{R}}_t Y,$$

while the Eulerian velocity is

$$v_t(y) := V_t(\mathbf{R}_t^{-1} y) = \dot{\mathbf{R}}_t \mathbf{R}_t^{-1} y.$$

¹Euler independently derived both the equations describing the motion of an unconstrained rigid body and those describing the motion of an ideal fluid. Both sets of equations were thus already dubbed ‘‘Euler equations’’ approximately 200 years before they would be unified under the banner of Euler-Arnold theory.

As $\mathbf{R} \in SO(3)$, the matrix $\dot{\mathbf{R}}_t \mathbf{R}_t^{-1}$ is skew symmetric, and so the Eulerian velocity can be written as

$$v_t(y) = \dot{\mathbf{R}}_t \mathbf{R}_t^{-1} y = \omega_t \times y,$$

where for all t the vector $\omega_t \in \mathbb{R}^3$ is the spatial angular velocity² of $\mathcal{S}$.

Together, a motion and its velocity yield a description of the rigid body dynamics with velocity phase space $TSO(3) \cong SO(3) \times \mathfrak{so}(3)$. Each element $(\mathbf{R}, \dot{\mathbf{R}}) \in TSO(3)$ describes the position and Lagrangian velocity of the object in its motion, and the motion corresponds to a path $t \mapsto (\mathbf{R}_t, \dot{\mathbf{R}}_t) \in TSO(3)$ in the phase space. The Eulerian velocity is the right translation of $\dot{\mathbf{R}}_t$ to the identity given in (1.6), adjusted to the tangent-bundle setting. Indeed, the angular velocity ω may be interpreted as an element of the Lie algebra $\mathfrak{so}(3)$ under its standard identification with $\mathbb{R}^3$ by the hat map (see [20], Chapter 9). Using the identity (1.7), the left translation of $\dot{\mathbf{R}}_t$ can be obtained from the adjoint action as

$$\Omega_t = \text{Ad}_{\mathbf{R}_t}(\omega_t) = \mathbf{R}_t^{-1} \omega_t,$$

using the expression of the adjoint action under the hat map. This vector $\Omega_t \in \mathbb{R}^3$ is called the body angular velocity of the motion.

We now show how Lie-Poisson reduction may be applied to the motion of the rigid body. Assuming constant mass density on $\mathcal{S}$ for simplicity, the total kinetic energy $K : TSO(3) \rightarrow \mathbb{R}$ of the body is

$$K(\mathbf{R}, \dot{\mathbf{R}}) = \frac{1}{2} \int_M |\dot{\mathbf{R}} Y|^2 dY.$$

This kinetic energy is invariant under the lifted left action of $SO(3)$. Indeed, a direct calculation shows that for $\mathbf{A} \in SO(3)$,

$$\begin{aligned} K(\mathbf{A}\mathbf{R}, \mathbf{A}\dot{\mathbf{R}}) &= \frac{1}{2} \int_{\mathcal{S}} |\mathbf{A}\dot{\mathbf{R}} Y|^2 dY \\ &= \frac{1}{2} \int_{\mathcal{S}} |\dot{\mathbf{R}} Y|^2 dY \\ &= K(\mathbf{R}, \dot{\mathbf{R}}), \end{aligned}$$

using the rigidity of $\mathbf{A}$.

In this way, the kinetic energy K defines a left-invariant metric on $SO(3)$. In the setting of geometric mechanics, the lowering associated with this metric

$$F : TSO(3) \rightarrow T^*SO(3),$$

is called the Legendre transform. The Legendre transform restricted to the identity of $SO(3)$ produces an inertia operator called the **moment of inertia** $\mathbb{I} : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)^*$. This inertia operator is precisely the inertia operator associated to the left-invariant metric of K guaranteed by Proposition 1.7.

Using the Legendre transform, the kinetic energy K may be lowered to a Hamiltonian function $H \in C^\infty(T^*SO(3))$ defined by $K = H \circ F$. As K is invariant under the left action of G on $TSO(3)$, H is invariant under the left action of G on $T^*SO(3)$. We can thus apply Theorem 1.4 to reduce the system to $\mathfrak{so}(3)^*$. The space $\mathfrak{so}(3)^*$ has a natural basis given by the angular momenta

$$\Pi = \mathbb{I}\Omega,$$

²Note that in the rest of this work ω is used to denote the vorticity of a fluid. To respect the classical notation for angular velocity, we use ω to denote angular velocity in this discussion.

which can be identified with vectors in $\mathbb{R}^3$ using the hat identification on Ω . The resulting reduced Hamiltonian in $\mathfrak{so}(3)^*$ is given by Proposition 1.7 as $H(\Pi) = \frac{1}{2}\langle \Pi, \mathbb{I}^{-1}\Pi \rangle$. Since $\mathbb{I}$ is symmetric, it can be diagonalized as $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$. This gives the classical form of the Hamiltonian

$$H(\Pi) = \frac{1}{2} \left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3} \right).$$

Given this Hamiltonian in $\mathfrak{so}(3)^*$, we can apply Proposition 1.8 to obtain the dynamics as the Euler-Arnold equation for H in $\mathfrak{so}(3)$,

$$\dot{\Pi}(t) = \text{ad}_{\frac{\delta H}{\delta \Pi}|_{\Pi(t)}}^* \Pi(t) = \text{ad}_{\mathbb{I}^{-1}\Pi(t)}^* \Pi(t). \quad (1.10)$$

Note that as the Hamiltonian on $T^*SO(3)$ is left-invariant, the reduced dynamics occur under the minus bracket in $\mathfrak{so}(3)$, and so the above equation has an opposite sign to Proposition 1.8. Using the fact that the coadjoint action in $\mathfrak{so}(3)$ under the hat identification is given by the cross product, (1.10) takes the form of the classical Euler equation

$$\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi = \Pi \times \Omega.$$

In this way, the dynamics of the rigid body gives a prototypical example of a finite-dimensional Lie-Poisson system. Moreover, as shown in [20], the coadjoint orbits in $\mathfrak{so}(3)$ are level sets of the function $|\Pi|^2$. Thus, the preservation of coadjoint orbits under the Euler-Poisson flow of (1.10) corresponds precisely to the conservation of a physical quantity. In the next chapter, we see the same phenomenon manifest as the transport of vorticity found in ideal hydrodynamics.

1.2 Hodge Decomposition

1.2.1 Harmonic Forms

We now leave the realm of geometric mechanics and turn to the Hodge decomposition of differential forms, focusing on the particular case of one-forms on surfaces. This material will be relevant for our later discussion of ideal fluid dynamics on multiply-connected surfaces. Throughout, we consider a closed, oriented Riemannian manifold (M, g) of dimension $\dim M = n$ with volume form μ corresponding to the metric.

For $1 \leq k \leq n$, the Hodge star operator

$$\star : \bigwedge^k(M) \rightarrow \bigwedge^{n-k}(M),$$

is defined by the condition that

$$\int_M \eta \wedge \star \gamma = \langle \eta, \gamma \rangle_{L^2}, \quad (1.11)$$

for all $\gamma, \eta \in \bigwedge^k(M)$, where $\langle \eta, \gamma \rangle_{L^2}$ is the L^2 pairing of forms defined in Appendix A.

The adjoint of the exterior derivative d under the L^2 pairing is the **codifferential**

$$\delta : \bigwedge^k(M) \rightarrow \bigwedge^{k-1}(M),$$

which acts on a k -form $\eta \in \bigwedge^k(M)$ as

$$\delta \eta = (-1)^{kn+n+1} \star d \star \eta.$$

As M is closed, it follows from Stokes theorem that the codifferential satisfies the adjoint identity

$$\langle d\eta, \gamma \rangle_{L^2} = \langle \eta, \delta\gamma \rangle_{L^2}.$$

Since $\star^2$ is a scalar multiple of the identity, it is clear that $\delta^2 = 0$, justifying its title as a differential.

Together, d and δ produce the **de Rham Laplacian**

$$\Delta_{dR} : \bigwedge^k(M) \rightarrow \bigwedge^k(M),$$

given by

$$\Delta_{dR} = d\delta + \delta d.$$

Note that on 0-forms Δ_{dR} simplifies to δd , while on n -forms Δ_{dR} simplifies to $d\delta$. We report the following properties of the de Rham Laplacian, proofs of which can be found in Appendix A.

Proposition 1.19. *The de Rham Laplacian Δ_{dR} satisfies the following properties:*

1. Δ_{dR} agrees with the Laplace-Beltrami operator on smooth functions $f \in C^\infty(M)$.
2. Δ_{dR} commutes with the Hodge star:

$$\Delta_{dR} \circ \star = \star \circ \Delta_{dR}.$$

3. Δ_{dR} is a self-adjoint operator under the L^2 inner product.
4. If η is exact and $\Delta_{dR}\eta = 0$, then $\eta = 0$.
5. $\Delta_{dR}\eta = 0$ if and only if $d\eta = 0$ and $\delta\eta = 0$.

As shown in [27], properties 3, 4, and 5 of the above induce the following decomposition.

Theorem 1.20 (Hodge decomposition). *The space of differential k -forms $\bigwedge^k(M)$ decomposes as*

$$\bigwedge^k(M) = \text{im}(d) \oplus \text{im}(\delta) \oplus \ker \Delta_{dR}. \quad (1.12)$$

This decomposition is orthogonal under the L^2 product of forms.

Elements of $\ker \Delta_{dR}$ are called **harmonic forms**. The space of harmonic k -forms is denoted $\mathcal{H}^k(M)$. The proof of the following result is technical, as discussed in [27].

Proposition 1.21. *On a closed, oriented manifold M , the space of harmonic k -forms $\mathcal{H}^k(M)$ is finite-dimensional for all $1 \leq k \leq n$.*

A Classical Interpretation

We now provide a classical interpretation of the Hodge decomposition (1.12) by focusing on the case of one-forms on surfaces. We begin by recalling the Riemannian expression of the classical vector operations.

Definition 1.22. *Let (M, g) be a Riemannian surface. The **divergence** of a vector field $X \in \mathfrak{X}(M)$ is the scalar function $\text{div}(X) \in C^\infty(M)$ satisfying*

$$\text{div}(X)\mu = \mathcal{L}_X\mu = d(i_X\mu).$$

The **scalar curl** of X is the scalar³ function $\text{curl}(X) \in C^\infty(M)$ satisfying

$$\text{curl}(X) = \star d(X^\flat).$$

The **skew-gradient** of a smooth function $f \in C^\infty(M)$ is the vector field $\nabla^\perp f \in \mathfrak{X}(M)$ given by

$$\nabla^\perp f = (\star df)^\sharp.$$

Proposition 1.23. *For $X \in \mathfrak{X}(M)$, the codifferential δ of $X^\flat$ is minus the Riemannian divergence of X .*

Proof. Using the identity $i_X \mu = \star X^\flat$ from Appendix A, we calculate

$$\text{div } X = \star \mathcal{L}_X \mu = \star d(i_X \mu) = \star d(\star X^\flat) = -\delta X^\flat.$$

□

With these identities at hand, consider the Hodge decomposition of a one-form $\gamma \in \bigwedge^1(M)$ on a surface M as

$$\gamma = df + \delta\Psi + \eta$$

for $f \in C^\infty(M)$, $\Psi \in \bigwedge^2(M)$, and $\eta \in \bigwedge^1(M)$. This corresponds to a decomposition of the vector field $\gamma^\sharp = X \in \mathfrak{X}(M)$ as

$$X = (df)^\sharp + (\delta\Psi)^\sharp + \eta^\sharp. \quad (1.13)$$

The first term in (1.13) is curl-free, since $\text{curl}((df)^\sharp) = (\star d^2 f) = 0$, while the second-term is divergence-free, since $\text{div}(\delta\Psi)^\sharp = \delta^2 \Psi = 0$. It is clear that the first term is a gradient vector field with scalar potential f . Writing Ψ as $\Psi = \star\psi$ for $\psi \in C^\infty(M)$, we see that the second term is a skew-gradient vector field, since

$$(\delta\Psi)^\sharp = (-\star d\psi)^\sharp = \nabla^\perp(-\psi).$$

In this way, we see that the first two terms of the Hodge decomposition express the classical Helmholtz decomposition of a 2D vector field into a sum of a curl-free, gradient component and a divergence-free, skew-gradient component.

The third term $\eta^\sharp$ is subtler, and does not appear in the classical Helmholtz theorem. Indeed, from item 5 of Proposition 1.19, a nontrivial harmonic one-form η raises to a vector field $\eta^\sharp$ that is both divergence-free and curl-free. On simply-connected domains, divergence-free and curl-free vector fields vanish. In fact, item 4 of Proposition 1.19 shows that any non-zero harmonic form is closed and not exact, meaning it represents non-trivial de Rham cohomology of M . The relationship between harmonic forms and de Rham cohomology is clarified by the following major theorem of Hodge.

Theorem 1.24 (Hodge). *There is an isomorphism between $\mathcal{H}^k(M)$ and $H_{dR}^k(M)$. Each cohomology class possesses a unique harmonic representative.*

Using the Hodge theorem, we can interpret the decomposition (1.20) on one-forms as a generalization of the classical Helmholtz decomposition to domains that are not simply-connected. On such multiply-connected domains each vector field decomposes into *three* components: a divergence-free component, a curl-free component, and a harmonic component that is both

³This is analogous to the classical case, in which curl returns a scalar in 2D domains, and a vector in 3D domains.

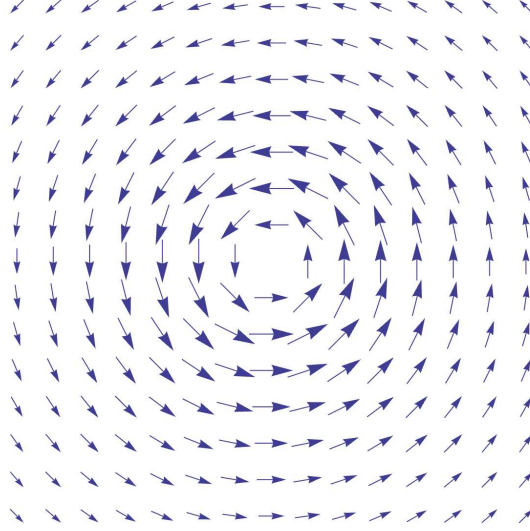


Figure 1.1: A harmonic vector field in the punctured plane. Image from [1], Licensed under CC BY-SA 3.0

divergence-free and curl-free and arises directly from the nontrivial topology of the domain M . For a multiply-connected Euclidean domain $M \subset \mathbb{R}^2$ the decomposition of one-forms

$$\bigwedge^1(M) = \text{im}(d) \oplus \text{im}(\delta) \oplus \ker \Delta_{dR}$$

corresponds to the decomposition of vector fields given in classical notation as

$$\mathfrak{X}(M) = \{\nabla f : f \in C^\infty(M)\} \oplus \{\nabla^\perp \psi : \psi \in C^\infty(M)\} \oplus \{\text{Harmonic vector fields}\}.$$

Moreover, these three subspaces are mutually orthogonal under the L^2 inner product of vector fields on M given by

$$\langle X, Y \rangle_{L^2} = \int_M g(X, Y) \mu.$$

From Proposition 1.21 it follows that the space of harmonic vector fields is finite-dimensional. Thus while the first two components of the above decomposition are infinite-dimensional spaces comprising all possible divergence-free and curl-free vector fields, the third component is smaller. Indeed, since the dimension of the first cohomology group of a closed, orientable surface is twice its genus, it follows that each “hole” in a closed, oriented surfaces induces exactly two harmonic vector fields around it. If M is not closed⁴, as in the following example, the number of harmonic vector fields may be an arbitrary finite number.

Example 1.25. (The punctured plane) The punctured plane $M = \mathbb{R}^2$ has real de Rham cohomology $H_{dR}^1(M) = \mathbb{R}$. This cohomology is generated by the angle element

$$d\theta = \frac{-ydx + xdy}{x^2 + y^2},$$

which by Theorem 1.24 is necessarily harmonic. The corresponding harmonic vector field $X \in \mathfrak{X}(M)$

⁴We do not discuss the general case of Hodge decomposition on non-compact manifolds. Example 1.25 considers the punctured plane, which can be handled with classical methods.

obtained by raising against the Euclidean metric is

$$X(x, y) = \frac{-y}{x^2 + y^2} \frac{\partial}{\partial x} + \frac{x}{x^2 + y^2} \frac{\partial}{\partial y},$$

or in polar coordinates

$$X(r, \theta) = \frac{1}{r} \frac{\partial}{\partial \theta}.$$

The vector field X is visualized in Figure 1.1. As we shall see in Chapter 3, X is exactly the fluid velocity vector field associated to a point-vortex in the plane. Since the fluid velocity of a point-vortex blows up at the position of the vortex itself, the fluid velocity field is formally defined only on the punctured plane.

Example 1.26. (The torus) The flat torus $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ has real de Rham cohomology $H_{\text{dR}}^1(M) = \mathbb{R}^2$. The cohomology is generated by the harmonic one-forms dx and dy . Raising these gives the constant harmonic vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}.$$

This matches the claim that a closed surface of genus one has two harmonic vector fields, each of which flow “around” the hole of the surface.

Let us take the Hodge decomposition of the one form $\gamma = 2 \cos^2(x) dx$ on the torus. The harmonic component of γ is its constant component, which is given by its average mode

$$\int_0^1 \int_0^1 2 \cos^2(x) dx dy = 1.$$

Writing γ as $\gamma = dx + \cos(2x) dx$, we see that the remaining component is exact with potential $f = \frac{1}{2} \sin(2x)$. Thus, γ has the Hodge decomposition

$$\gamma = d\left(\frac{1}{2} \sin(2x)\right) + dx,$$

with no co-exact component.

1.2.2 The Green Function

We now review the basic properties of the Green function for the Laplace-Beltrami operator on a Riemannian surface and discuss its use in solving the Poisson equation. The Green function will be essential in our later definition of point-vortices on surfaces. Our discussion is based on [6], [9], and [13].

Definition 1.27. Let (M, g) be a closed, oriented Riemannian surface with area form μ and total area $V = \int_M \mu$. The **Green function** for the Laplace-Beltrami operator Δ is the unique function $G_M : M \times M / \{\text{diagonal}\} \rightarrow \mathbb{R}$ characterized by the following properties:

1. $-\Delta_s G_M(s, r) = \delta_r(s) - \frac{1}{V}$,
2. $G_M(s, r) = G_M(r, s)$,
3. $\int_M G_M(s, r) \mu = 0$,
4. $G_M(s, r) + \frac{1}{2\pi} \log d(s, r)$ is bounded,

for all $s, r \in M$, where $d(s, r)$ is the geodesic distance.

Remark 1.28. The Green function G_M is singular along the diagonal $G_M(r, r)$ for $r \in M$. Item 4 of the above definition describes the nature of this singularity. It suggests that G_M is amenable to desingularization along the diagonal as

$$R(r) = \lim_{s \rightarrow r} \left(G_M(s, r) + \frac{1}{2\pi} \log d(s, r) \right).$$

The resulting regularized function is called the **Robin function**. It will be used in Chapter 3 to assign a value to the self-energy of a point-vortex on a Riemannian surface.

The following results show how the Green function allows for the inversion of the Laplace-Beltrami operator on closed surfaces.

Proposition 1.29. *On a closed surface M , the image of the Laplace-Beltrami operator $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ is $C_0^\infty(M)$, the space of mean-zero functions.*

Proof. Since M is closed, the calculation

$$\int_M \Delta f \mu = \int_M \star d \star df \mu = \int_M d \star df = \int_{\partial M} \star df = 0,$$

shows that Δ maps into the space of mean-zero functions. In order to show that each $\lambda \in C_0^\infty(M)$ is contained in the image of Δ , we consider the potential

$$\psi(s) = - \int_M G_M(s, r) \lambda(r) \mu(r).$$

Applying item 1 of Definition 1.27, we calculate

$$\Delta \psi = - \int_M \Delta_s G_M(s, r) \lambda(r) \mu(r) = \int_M \delta_r(s) \lambda(r) \mu(r) - \frac{1}{V} \int_M \lambda \mu = \lambda - \frac{1}{V} \int_M \lambda \mu.$$

The last term vanishes because λ is mean zero and so $\Delta \psi = \lambda$. □

Corollary 1.30. *On a closed surface M , the Laplace-Beltrami operator is an isomorphism between the space of smooth functions modulo constants and the space of mean-zero functions,*

$$\Delta : C^\infty(M)/\mathbb{R} \rightarrow C_0^\infty(M).$$

Proof. This follows from the above proposition together with the well-known fact that the kernel of Δ on a closed surface is the space of constant functions, see [27]. □

Definition 1.31. *On a closed surface M , we define the **inverse of the Laplacian** of a mean-zero function $\lambda \in C_0^\infty(M)$ as the potential function*

$$\Delta^{-1}(\lambda) = - \int_M G_M(s, r) \lambda(r) \mu(r),$$

unique up to a constant.

Remark 1.32. We may extend the above definition to two-forms using the Hodge star as $\Delta_{dR}^{-1} : d \bigwedge^1(M) \rightarrow \bigwedge^2(M)$,

$$\Delta_{dR}^{-1} = \star \circ \Delta^{-1} \circ \star. \tag{1.14}$$

Proposition 2.17 of the following chapter shows that this map is well-defined on the space of exact two-forms $d \bigwedge^1(M)$.

Remark 1.33. It is straightforward to extend Definition 1.27 to the case that M is compact with boundary by specifying an additional boundary condition, see [6] Chapter 4. By prescribing asymptotic behavior, one may also define the Green function on non-compact manifolds. In this case the area term $\frac{1}{V}$ in item 1 of Definition 1.27 does not appear. The general situation is thus

$$-\Delta_s G_M(s, r) = \begin{cases} \delta_r(s) - \frac{1}{V} & M \text{ compact} \\ \delta_r(s) & M \text{ non-compact} \end{cases}. \quad (1.15)$$

The only non-compact manifold discussed in this work is the Euclidean plane, on which G_M is the fundamental solution of the Laplacian. For discussion of the general non-compact case and extended commentary on the Green function in hydrodynamics, see [9].

Hodge Projections

The inverse of the Laplacian gives explicit formulas for the projection operators onto the orthogonal components of the Hodge decomposition of one-forms on a closed surface,

$$\begin{aligned} \bigwedge^1(M) &= \text{im}(d) \oplus \text{im}(\delta) \oplus \ker \Delta_{dR} \\ &= P_{\text{ex}}(M) \oplus P_{\text{coex}}(M) \oplus P_{\text{Harm}}(M). \end{aligned}$$

Proposition 1.34. *Let (M, g) be a closed Riemannian surface and $\gamma \in \bigwedge^1(M)$ be a one-form. Then the projection onto the exact and co-exact components of the Hodge decomposition of γ are*

$$P_{\text{ex}}(\gamma) = d\Delta^{-1}(\delta\gamma)$$

and

$$P_{\text{coex}}(\gamma) = \delta\Delta_{dR}^{-1}(d\gamma).$$

Proof. The one-form γ has the Hodge decomposition

$$\gamma = df + \delta\Psi + \eta, \quad (1.16)$$

for $f \in C^\infty(M)$, $\Psi \in \bigwedge^2(M)$, and $\eta \in \mathcal{H}^1(M)$. Taking the codifferential of (1.16) yields

$$\delta\gamma = \Delta f,$$

which gives $f = \Delta^{-1}(\delta\gamma)$. Similarly, after writing $\Psi = \psi\mu$ for $\psi \in C^\infty(M)$, taking the exterior derivative of (1.16) yields

$$d\gamma = d\delta\Psi = \Delta_{dR}(\Psi) = \Delta_{dR}(\star\psi) = \star\Delta\psi.$$

Since $\star^2 = 1$ on two-forms on a surface, this gives $\Delta\psi = \star d\gamma$. Inverting this yields

$$\Psi = \star\psi = \star\Delta^{-1}(\star d\gamma) = \Delta_{dR}^{-1}(d\gamma),$$

using (1.14). Inserting these expressions for f and Ψ into (1.16) then gives the result. $\square$

Corollary 1.35. *The projection onto the harmonic component of the Hodge decomposition of γ is*

$$\begin{aligned} P_{\text{Harm}}(\gamma) &= \gamma - P_{\text{ex}}(\gamma) - P_{\text{coex}}(\gamma) \\ &= \gamma - d\Delta^{-1}(\delta\gamma) - \delta\Delta_{dR}^{-1}(d\gamma). \end{aligned}$$

In the next section, we give an alternative expression for P_{Harm} in terms of a harmonic basis.

1.2.3 Harmonic Bases

We conclude our discussion by reviewing the construction of a basis of harmonic forms given a basis in homology. This will be useful for our later study of point-vortex dynamics on multiply-connected surfaces. The following material is standard, and is drawn from [11]. For a discussion focused on the context of point-vortex dynamics, see [12] and [13].

Recall that on a closed Riemannian surface (M, g) with genus $l \geq 1$, a canonical basis in homology amounts to $2l$ simple closed curves $\{a_1, \dots, a_l, b_1, \dots, b_l\}$ satisfying certain intersection conditions (see e.g. [11], Chapter 3). A canonical basis can always be chosen using the sides of the fundamental polygon of M . The existence of such a polygon is guaranteed by the Riemann uniformization theorem, applied to M as a Riemann surface under the conformal structure of g .

Example 1.36. The flat torus $M = \mathbb{T}^2$ has fundamental domain given by the square $[0, 1] \times [0, 1]$. A basis in homology for $\mathbb{T}^2$ is given by the closed curves $a = (t, 0)$ and $b = (0, t)$ for $t \in [0, 1]$.

The following identity is a consequence of the intersection properties of the homology basis.

Proposition 1.37. *Let (M, g) be a closed Riemannian surface of genus l and let $\{a_1, \dots, a_l, b_1, \dots, b_l\}$ be a canonical basis in homology. Then for closed one-forms η and θ it holds*

$$\int_M \eta \wedge \theta = \sum_{i=1}^l \oint_{a_i} \eta \oint_{b_i} \theta - \oint_{b_i} \eta \oint_{a_i} \theta. \quad (1.17)$$

Proof. See [11], Chapter 3. □

Each basis of curves in homology descends to a dual basis in cohomology via the de Rham pairing of curves and one-forms. A dual basis is given by $2l$ one-forms $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in H_{\text{dR}}^1(M)$ defined up to exact forms by the condition that

$$\int_{a_j} \alpha_i = \int_{b_j} \beta_i = \delta_{ij}, \quad \int_{b_j} \alpha_i = \int_{a_j} \beta_i = 0, \quad (1.18)$$

for all $1 \leq i, j \leq l$. Since each element of the dual basis is closed, Proposition 1.37 gives the equivalent condition

$$\int_{\Sigma} \alpha_i \wedge \beta_j = \delta_{ij}, \quad \int_{\Sigma} \alpha_i \wedge \alpha_j = \int_{\Sigma} \beta_i \wedge \beta_j = 0, \quad (1.19)$$

for all $1 \leq i, j \leq l$. As each cohomology class has a unique harmonic representative, there exists a unique basis of harmonic one-forms $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in \mathcal{H}^1(M)$ that satisfy the above relations. Thus, we may speak of *the* harmonic basis dual to the a_i and b_i curves.

The relations specifying the dual harmonic basis allow for the following expression of the harmonic component of a generic one-form on M .

Proposition 1.38. *Let (M, g) be a closed Riemannian surface with basis of harmonic forms $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in \mathcal{H}^1(M)$. Then the harmonic component of any one-form $\gamma \in \Lambda^1(M)$ is*

$$P_{\text{Harm}}(\gamma) = \sum_{i=1}^l \left(\int_M \gamma \wedge \beta_i \right) \alpha_i + \left(\int_M \alpha_i \wedge \gamma \right) \beta_i.$$

Proof. The one-form γ has the Hodge decomposition

$$\gamma = df + \delta\Psi + \eta,$$

where the harmonic component η is written in basis as

$$\eta = \sum_{i=1}^l A_i \alpha_i + B_i \beta_i.$$

It follows from the orthogonality of the Hodge decomposition under the inner-product (1.11) (together with the fact that the Hodge star maps harmonic forms to harmonic forms) that for any harmonic one-form $\theta \in \mathcal{H}^1(M)$,

$$\int_M df \wedge \theta = \langle df, \star\theta \rangle_{L^2} = 0,$$

and

$$\int_M \delta\Psi \wedge \theta = \langle \delta\Psi, \star\theta \rangle_{L^2} = 0.$$

Thus for all $1 \leq i \leq l$, it holds

$$\int_M \gamma \wedge \beta_i = \int_M \eta \wedge \beta_i = \int_M \left(\sum_{j=1}^l A_j \alpha_j + B_j \beta_j \right) \wedge \beta_i = \int_M A_i (\alpha_i \wedge \beta_i) = A_i,$$

where we have used (1.19). A similar calculation gives $\int_M \alpha_i \wedge \gamma = B_i$, which gives the result. $\square$

Example 1.39. On the flat torus, the harmonic basis dual to the homology basis $a = (t, 0)$ and $b = (0, t)$ for $t \in [0, 1]$ are simply the one-forms $\alpha = dx$ and $\beta = dy$. Applying the above Proposition to the one-form $\gamma = 2 \cos^2(x) dx$ gives exactly the results of Example 1.26.

Chapter 2

Ideal Hydrodynamics on Oriented Surfaces

This chapter reviews the geometric formulation of ideal hydrodynamics on oriented surfaces.

We begin by reviewing the Euler equations of hydrodynamics on a Riemannian manifold and justifying their status as an infinite-dimensional Lie-Poisson system, working in analogy with the case of the rigid body discussed in the previous chapter. We then discuss the Lie group of area-preserving diffeomorphisms, and review the Lie-Poisson geometry of its dual Lie algebra. This object is central to our discussion, and our review of its properties is lengthy, culminating in a summary of three different perspectives on the Lie-Poisson system at hand. Our review of ideal hydrodynamics on surfaces closely follows [5] and [14]. We cite [5] for several facts about the group of area-preserving diffeomorphisms.

After demonstrating the geometric form of the Euler equations as an Euler-Arnold flow, we review their Hamiltonian formulation by explicitly calculating the KKS form for the system, following calculations found in [21].

2.1 The Geometric Formulation of Ideal Hydrodynamics

We consider the flow of an ideal fluid on a connected Riemannian manifold (M, g) . For simplicity, we normalize the fluid density as $\rho \equiv 1$. The coordinate-free expression of the Euler equation (1) is

$$\begin{cases} \frac{\partial v}{\partial t} + \nabla_v v = -\text{grad } p, \\ \text{div } v = 0, \end{cases} \quad (2.1)$$

where ∇ is the Levi-Civita connection on M . The pressure $p \in C^\infty(M)$ and Eulerian fluid velocity vector field $v \in \mathfrak{X}(M)$ are the unknowns, varying in time. If M has boundary, we require that v be tangent to the boundary at all times.

There exist equivalent formulations of (2.1). Lowering both sides of the equation against the metric and applying the identity $(\nabla_v v)^b = \mathcal{L}_v v^b - \frac{1}{2} d|v|^2$ yields the expression

$$\begin{cases} \partial_t \nu + \mathcal{L}_v \nu = -d(p - \frac{1}{2}|v|^2) \\ \text{div } v = 0. \end{cases} \quad (2.2)$$

for the evolution of the velocity one-form $\nu = v^b$. Applying the exterior differential to this

formulation yields the coordinate-free form of the Euler vorticity equation

$$\partial_t \omega + \mathcal{L}_v \omega = 0, \quad (2.3)$$

given in terms of the vorticity two-form $\omega = d\nu$. This restates the classical result that vorticity is “frozen into the flow.” Despite their notational complexity, equations (2.1), (2.2), and (2.3) are simply the coordinate-free generalizations of (1) and (2) to the manifold setting. Written locally in Euclidean coordinates, they reduce to the classical equations.

We now review the geometric interpretation of the Euler equations as an infinite-dimensional Lie-Poisson system pioneered by Arnold in [4]. We omit the subtleties of the infinite-dimensional case and proceed in analogy with the finite-dimensional case discussed in the previous chapter, following the discussion found in [14]. For an account that considers the full infinite-dimensional details, see [10].

As with the rigid body, we consider the motion at hand as a path in a Lie group. In the case of ideal fluids, this group is $G = \text{Diff}_\mu(M)$, the infinite-dimensional group of diffeomorphisms that preserve the Riemannian density μ on M , with group operations of composition and inversion. Each element in G determines a configuration of the fluid. In particular, after fixing a reference state for each fluid particle $Y \in M$, an element $\phi \in \text{Diff}_\mu(M)$ specifies a new position for the particle as $y = \phi(Y)$. The condition that ϕ preserve the density μ corresponds to the fact that the fluid flow is incompressible, just as the unit determinant condition enforces rigidity of motion in the $G = SO(3)$ case. The coordinates Y and y provide the classical Lagrangian and Eulerian descriptions of fluid flow, respectively.

A motion of the fluid is a smooth curve of elements in G , given as $t \mapsto \phi_t \in \text{Diff}_\mu(M)$. A motion produces a family of coordinates, which is denoted

$$y = \phi(Y, t) := \phi_t(Y).$$

Since each ϕ_t is a diffeomorphism, we may invert the above to express the Eulerian position in terms of the Lagrangian position as

$$Y = \phi_t^{-1}(y).$$

There exist two descriptions of the velocity of a fluid motion. The Lagrangian fluid velocity of a motion is

$$V_t := V(Y, t) = \frac{\partial \phi(Y, t)}{\partial t},$$

while the Eulerian fluid velocity is

$$v_t := v(y, t) = V(\phi_t^{-1}(y), t) = \frac{\partial \phi(\phi_t^{-1}(y), t)}{\partial t}.$$

As in the case of the rigid body, the motion ϕ_t together with its velocity yield a description of the dynamics with velocity phase space TG . Here, the infinite-dimensional manifold TG may be formally parallelized as $TG = G \times \mathfrak{g}$, where $\mathfrak{g} = \mathfrak{X}_\mu(M)$ is the Lie algebra of divergence-free vector fields on M . The Eulerian fluid velocity

$$v_t = V_t \circ \phi_t^{-1},$$

is precisely the right translation of V_t to the identity.

We now show that the kinetic energy of the fluid is right-invariant. The total kinetic energy $K : TG \rightarrow \mathbb{R}$ is given as

$$K(\phi, \dot{\phi}) = \frac{1}{2} \int_M g(v, v) \mu,$$

where v is the Eulerian fluid velocity

$$v = \dot{\phi} \circ \phi^{-1}. \quad (2.4)$$

Note that if $\chi \in \text{Diff}_\mu(M)$ acts from the right on TG as $(\phi, \dot{\phi}) \mapsto (\phi \circ \chi, \dot{\phi} \circ \chi)$, the Eulerian velocity (2.4) is invariant, since v transforms as

$$v = \dot{\phi} \circ \phi^{-1} \mapsto (\dot{\phi} \circ \chi) \circ (\phi \circ \chi)^{-1} = \dot{\phi} \circ \chi \circ \chi^{-1} \circ \phi^{-1} = \dot{\phi} \circ \phi^{-1}.$$

Since v is right-invariant, the kinetic energy K is as well. This invariance is referred to as the **particle-relabeling symmetry** of hydrodynamics.

The right invariance of K allows us to perform a Lie-Poisson reduction of the dynamics. In particular, the kinetic energy can be viewed as a map $K : \mathfrak{g} \rightarrow \mathbb{R}$, given as

$$K(X) = \frac{1}{2} \int_M g(X, X) \mu, \quad (2.5)$$

for $X \in \mathfrak{g}$. This in turn induces an inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$, and a corresponding Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ defined by $K = H \circ A$ via an infinite-dimensional analogue¹ of Proposition 1.7. By Proposition 1.8, the resulting Hamiltonian flow in $\mathfrak{g}^*$ takes the form of the Euler-Poisson equation

$$\dot{\nu}(t) + \text{ad}^*_{-\frac{\delta H}{\delta \nu}|_{\nu(t)}} \nu(t) = 0, \quad (2.6)$$

for $\nu \in \mathfrak{g}^*$.

Remark 2.1. We follow the convention of [18] that *minus* the Hamiltonian of total kinetic energy is used in right-invariant Lie-Poisson reduced dynamics. This accounts for the two sign changes in the above equation relative to the rigid body equation (1.10). The first flipped sign stems from the use of plus Lie-Poisson bracket and the second follows from the use of the Hamiltonian $-H$. For further discussion, see [18] Chapter 1, Remark 4.16.

In the next section, we review the Lie-Poisson geometry of $\mathfrak{g}^*$. After deriving the form of the coadjoint action and the inertia operator A given above, we show that the Euler-Poisson equation (2.6) takes the form of the Euler equations (2.2) and (2.3). We further demonstrate that the preservation of coadjoint orbits under Hamiltonian flow guarantees the preservation of the vorticity, in analogy with the conservation of angular momentum found in the rigid body case.

2.2 The Lie-Poisson Geometry of $\mathfrak{X}_\mu(M)$

Let (M, g) be an oriented Riemannian surface with area form μ . In this and later Chapters, we assume all fluid domain surfaces M to be connected, without explicitly saying this each time. Motivated by the above discussion, we review the Lie-Poisson system of $G = \text{Diff}_\mu(M)$, the group of area-preserving diffeomorphisms of M , and its Lie algebra $\mathfrak{g} = \mathfrak{X}_\mu(M)$, the space of divergence-free vector fields on M .² When M has boundary, $\mathfrak{g}$ restricts to the space of divergence-free vector fields on M that are tangent to the boundary ∂M . Throughout this and subsequent chapters, the notation G and $\mathfrak{g}$ will refer to these spaces unless explicitly stated otherwise. In a similar way, the notation $\mathfrak{g}^*$ will refer to the smooth part of the dual of $\mathfrak{g}$, defined below.

As described in [5], the adjoint action of $\phi \in \text{Diff}_\mu(M)$ on $X \in \mathfrak{X}_\mu(M)$ is the geometric change of coordinates

$$\text{Ad}_\phi X = \phi_* X.$$

¹We elide over the details of the infinite-dimensional Legendre transform. For a full discussion, see [10].

²As noted in [24], it is technically more correct to consider the component of $\text{Diff}_\mu(M)$ that is isotopic to the identity.

The corresponding infinitesimal action of $\mathfrak{X}_\mu(M)$ on itself is minus the usual bracket of vector fields³

$$\mathrm{ad}_X Y = [X, Y]_{\text{fluid}} = -\mathcal{L}_X Y.$$

In order to develop our Lie-Poisson system, we must pass from $\mathfrak{g}$ to its dual. Since $\mathfrak{g}$ is infinite-dimensional, the question of its dual space is somewhat subtle. In particular, the space of continuous linear functionals on $\mathfrak{X}_\mu(M)$ cannot be identified with a space of smooth functions. There does however exist a subspace of the dual consisting of smooth linear functionals, referred to as the **smooth part of the dual**. The following definition from [5] characterizes $\mathfrak{g}^*$.

Definition 2.2. *The smooth part of the dual of $\mathfrak{g}$ can be identified with the space of one-forms modulo exact one-forms,*

$$\bigwedge^1(M)/d\bigwedge^0(M).$$

The action of a coset $[\nu] \in \bigwedge^1(M)/d\bigwedge^0(M)$ on $X \in \mathfrak{X}_\mu(M)$ is given by

$$\langle [\nu], X \rangle = \int_M \langle \nu, X \rangle \mu = \int_M i_X \mu \wedge \nu.$$

*We refer to a representative $\nu \in \bigwedge^1(M)$ of the coset $[\nu]$ as a **velocity one-form**.*

The following result verifies that the above pairing is well-defined. It will also be useful for later calculation.

Lemma 2.3. *With (M, g) a Riemannian manifold and $X \in \mathfrak{g}$, it holds*

$$\int_M \langle df, X \rangle \mu = 0$$

for all $f \in C^\infty(M)$.

Proof. This follows from the calculation

$$\langle df, X \rangle \mu = \mathcal{L}_X f \mu + f \mathcal{L}_X \mu = \mathcal{L}_X(f\mu) = d(i_X(f\mu)) = d(f(i_X \mu)).$$

If M is closed, the result follows immediately. Otherwise, we have

$$\int_M \langle df, X \rangle \mu = \int_M d(f i_X \mu) = \int_{\partial M} f(i_X \mu) = 0,$$

with the final expression vanishing because X is tangent to the boundary ∂M . $\square$

Proposition 2.4. *The coadjoint action of $\phi \in \mathrm{Diff}_\mu(M)$ on $[\nu] \in \bigwedge^1(M)/d\bigwedge^0(M)$ is given by the pullback*

$$\mathrm{Ad}_\phi^* : [\nu] \mapsto [(\phi^{-1})^* \nu].$$

Similarly, the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^$ is*

$$\mathrm{ad}_X^* : [\nu] \mapsto [-\mathcal{L}_X \nu].$$

Rather than work with cosets, it is useful to take the exterior derivative of a given velocity one-form to eliminate its exact portion. As the following proposition shows, this map is an isomorphism when M is simply-connected.

³This minus sign arises as a consequence of the fact that the kinetic energy is right rather than left-invariant in the fluid system (see [5], Chapter 1).

Proposition 2.5. *On a simply-connected surface M , the mapping*

$$d : \bigwedge^1(M)/d\bigwedge^0(M) \rightarrow d\bigwedge^1(M)$$

given by

$$[\nu] \mapsto d\nu$$

is an isomorphism.

Proof. It is immediate that the proposed mapping is well-defined on the quotient space. To see that it is an isomorphism, note that since M is simply-connected, each closed form is exact, and so $d\bigwedge^0(M) = \ker d$. By the isomorphism theorem, $\bigwedge^1(M)/d\bigwedge^0(M) \cong \bigwedge^1(M)/\ker d \cong \text{im } d$. $\square$

In classical fluid dynamics, the passage from a velocity one-form ν to the exact two-form $\omega = d\nu$ corresponds to changing variables from the velocity field v to the vorticity $\omega = \nabla \times v$ by taking the curl. This change of variables is justified precisely by the Helmholtz decomposition discussed in the previous chapter. As v is divergence-free (since the fluid is incompressible), any obstruction to the mapping $v \mapsto \omega = \nabla \times v$ being an isomorphism would be both curl-free and divergence-free. The Helmholtz decomposition states that any such vector field vanishes in a simply-connected domain.

When M is simply-connected, the above result allows for an alternate parametrization of the smooth dual $\mathfrak{g}^*$. Rather than consider a coset $[\nu]$, we instead work with $\omega = d\nu$, which we call the **vorticity two-form** associated to $[\nu]$. We define the action of a vorticity two-form on an element $X \in \mathfrak{X}_\mu(M)$ using its associated velocity one-form.

Definition 2.6. *The pairing between a vorticity two-form ω given as $\omega = d\nu$ for some $\nu \in \bigwedge^1(M)$ and a vector field $X \in \mathfrak{g}$ is*

$$\langle \omega, X \rangle := \langle \nu, X \rangle = \int_M i_X \mu \wedge \nu. \quad (2.7)$$

The coadjoint actions of G and $\mathfrak{g}$ on a vorticity two-form ω follow in a similar way. Since the Lie derivative and the pullback both commute with d , Proposition 2.4 applies to vorticity two-forms without modification.

Proposition 2.7. *The coadjoint action of $\phi \in \text{Diff}_\mu(M)$ on $\omega \in d\bigwedge^1(M)$ is the pullback*

$$\text{Ad}_\phi^* : \omega \mapsto (\phi^{-1})^* \omega.$$

The coadjoint action of $X \in \mathfrak{g}$ on $\mathfrak{g}^$ is the Lie derivative*

$$\text{ad}_X^* : \omega \mapsto -\mathcal{L}_X \omega.$$

Remark 2.8. These expressions reveal the geometric structure underlying the conservation of vorticity in ideal fluid flow. Vorticity conservation arises not out of the fluid Hamiltonian, which has not yet been utilized, but out of the underlying Lie-Poisson structure of the equations: Hamiltonian dynamics in $\mathfrak{g}^*$ evolve on coadjoint orbits, which by the above proposition are spaces of vorticity two-forms (up to equivalence by pullback). In fact, Proposition 2.7 shows that *any* Lie-Poisson reduced system on G freezes vorticity into its dynamics.

We now return to the kinetic energy function in $\mathfrak{g}$ defined in (2.5). Using our previous results, we calculate the inertia operator in $\mathfrak{g}$ generated by this energy and show that the resulting Euler-Poisson equation in $\mathfrak{g}^*$ is exactly the Euler equation (2.2).

Proposition 2.9. *The total kinetic energy $K \in C^\infty(\mathfrak{g})$*

$$K(X) = \frac{1}{2} \int_M g(X, X) \mu \quad (2.8)$$

induces the inertia operator of lowering against the metric g ,

$$A : \mathfrak{X}_\mu(M) \rightarrow \bigwedge^1(M)/d \bigwedge^0(M)$$

by

$$X \mapsto X^\flat.$$

The resulting Hamiltonian in $\mathfrak{g}^$ is*

$$H(\nu) = \frac{1}{2} \langle \nu, A^{-1} \nu \rangle = \frac{1}{2} \int_M \langle \nu, \nu^\sharp \rangle \mu. \quad (2.9)$$

Proof. The inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$ associated to the map $K \in C^\infty(\mathfrak{g})$ satisfies

$$\frac{1}{2} \langle AX, X \rangle = K(X).$$

Using the smooth pairing given in Definition 2.2 and the fact that $g(X, X) = \langle X^\flat, X \rangle$ gives the inertia operator. The resulting Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ is defined by $K = H \circ A$, which gives (2.9). $\square$

Corollary 2.10. *The Euler-Poisson equation in $\mathfrak{g}^*$ for the negative of the Hamiltonian of total kinetic energy (2.9) is*

$$[\dot{X}^\flat] + [\mathcal{L}_X X^\flat] = 0. \quad (2.10)$$

For simply-connected M , this translates to the equation of vorticity two-forms,

$$\dot{\omega} + \mathcal{L}_X \omega = 0 \quad (2.11)$$

where $\omega = dX^\flat$ is the vorticity two-form associated to X .

Proof. This is a direct application of the machinery developed in the previous sections. Using the above result, we can write elements of $\mathfrak{g}^*$ as cosets $[X^\flat]$ for $X \in \mathfrak{g}$. In accordance with Remark 2.1, we consider the Hamiltonian flow of $-H$ for the Hamiltonian (2.9). By Proposition 1.8, this induces the (plus bracket) Euler-Poisson equation

$$[\dot{X}^\flat(t)] + \text{ad}^*_{-\frac{\delta H}{\delta \nu}|_{[X^\flat(t)]}} [X^\flat(t)] = 0.$$

Proposition 2.7 gives the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ as minus the Lie derivative. Calculating the functional derivative of the Hamiltonian given in (2.9) as

$$H([X^\flat]) = \frac{1}{2} \langle [X^\flat], A^{-1} [X^\flat] \rangle$$

yields $\frac{\delta H}{\delta [X^\flat]}([X^\flat]) = A^{-1} [X^\flat] = X$. Thus, all together we have

$$[\dot{X}^\flat] = -\text{ad}^*_{-\frac{\delta H}{\delta \nu}|_{[X^\flat]}} [X^\flat] = \mathcal{L}_{-X} [X^\flat] = -\mathcal{L}_X [X^\flat],$$

which is exactly (2.10). Taking the exterior derivative to pass to the vorticity formulation then

gives (2.11). $\square$

Remark 2.11. Although (2.10) is presented modulo exact differentials, it is possible to reconstruct the scalar pressure from that equation. As noted in [16], the pressure may be recovered from (2.10) by choosing the unique representative $\nu \in [v^b]$ of the coset satisfying $\delta\nu = 0$. Expanding (2.10) out of cosets for ν gives

$$\dot{\nu} + \mathcal{L}_v \nu = dp, \quad (2.12)$$

for the scalar function $p \in C^\infty(M)$ satisfying

$$\Delta p = \delta dp = \delta(\mathcal{L}_v \nu).$$

This uniquely prescribes the pressure p . This is analogous to the situation in classical fluid dynamics, in which the pressure of an incompressible fluid is determined by its velocity via the Poisson pressure equation, written classically as

$$\Delta p = -\operatorname{div}((v \cdot \nabla)v).$$

The above result recovers the coordinate-free expressions of the Euler equations laid out at the beginning of this chapter: (2.11) is exactly the coordinate-free vorticity equation (2.3), while (2.12) in the above remark is the geometric Euler equation (2.2). In this way, we see that the Euler equations of ideal hydrodynamics arise as reduced Lie-Poisson equations in $\mathfrak{g}^*$. These reduced equations specify the Eulerian fluid velocity $v_t = X(t) \in \mathfrak{g}$ for all t . After solving for v_t , the full motion of the fluid $t \mapsto \phi_t \in \operatorname{Diff}_\mu(M)$ can be recovered by solving the reconstruction equation

$$\dot{\phi}_t = v_t \circ \phi_t,$$

after specifying some initial velocity v_0 and initial fluid configuration. Moreover, as shown in Remark 2.8, the frozenness of the vorticity into the fluid flow is a direct consequence of the conservation of coadjoint orbits in Lie-Poisson reduced dynamics.

Remark 2.12. Another insight of Arnold in his landmark paper [4] is that the Euler equation (1) possesses not only the structure of a reduced Euler-Poisson equation, but also a geodesic equation as well. As such, the motion ϕ_t is actually a geodesic in G , under the right-invariant metric defined by the total energy K . We do not explore this point further as it is not relevant for our later study of point-vortex dynamics. Nevertheless, it is a fascinating result in its own right, and discussed at length in [5] and [18].

2.2.1 An Alternate Formulation

We now show that when M is simply-connected, $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified with spaces of smooth functions. For simplicity in our presentation we assume M to be closed, although the results can be generalized to M with boundary. This section elaborates results found in [21] and [23].

We begin with the following results on $\mathfrak{X}_\mu(M)$.

Lemma 2.13. *On a closed, simply-connected surface M with symplectic structure given by the area form μ , divergence-free and Hamiltonian vector fields coincide, $\mathfrak{X}_\mu(M) = \mathfrak{X}_{Ham}(M)$.*

Proof. When the area form μ is interpreted as a symplectic form, the divergence-free condition $di_X \mu = 0$ says that $X \in \mathfrak{X}(M)$ is locally Hamiltonian. On a simply-connected symplectic manifold, locally Hamiltonian vector fields and globally Hamiltonian vector fields coincide. $\square$

Corollary 2.14. *On a closed, simply-connected manifold, each divergence-free vector field $X \in \mathfrak{X}_\mu(M)$ can be written as $-\nabla^\perp \psi$ for a scalar function $\psi \in C^\infty(M)$.*

Proof. By the above $X = X_\psi$ for some Hamiltonian function $\psi \in C^\infty(M)$. Using Lemma A.2 together with Definition 1.22, we see that

$$X = (X^\flat)^\sharp = (-\star i_X \mu)^\sharp = -(\star d\psi)^\sharp = -\nabla^\perp \psi.$$

□

The following result states the isomorphism between Hamiltonian vector fields and Hamiltonian functions, modulo constants.

Corollary 2.15. *On a closed, simply-connected manifold M , the Lie algebra $\mathfrak{g} = \mathfrak{X}_\mu(M)$ is (anti)-isomorphic to the algebra of smooth functions $C^\infty(M)/\mathbb{R}$ modulo constants under the Poisson bracket. For $\psi, \xi \in C^\infty(M)/\mathbb{R}$, it holds*

$$[X_\psi, X_\xi] = -X_{\{\psi, \xi\}}.$$

Definition 2.16. *We refer to the function space $C^\infty(M)/\mathbb{R}$ as the space of **stream functions**, with ψ a stream function for $X \in \mathfrak{X}_\mu(M)$ if $X = X_\psi$.*

The smooth dual of $\mathfrak{g}$ can also be identified with a certain space of scalar functions. From Proposition 2.5, each element of the dual is uniquely associated to a vorticity two-form ω . Each vorticity two-form is of the form $\omega = \lambda\mu$ for some scalar function $\lambda \in C^\infty(M)$. Since M is closed, it follows from Stokes theorem that such λ must be mean-zero, since

$$\int_M \lambda\mu = \int_M \omega = \int_M d\nu = \int_{\partial M} \nu = 0.$$

In fact, this mean-zero condition is also sufficient.

Proposition 2.17. *On a closed, simply-connected M , the space of vorticity two-forms $d\bigwedge^1(M)$ is isomorphic to $C_0^\infty(M)$, the space of mean-zero functions on M .*

Proof. For $\omega = \lambda\mu$, the above shows that λ is mean-zero. Now let $\lambda \in C^\infty(M)$. By Corollary 1.30, it follows that $\lambda = \Delta\psi$ for some $\psi \in C^\infty(M)$. From the agreement of the Riemannian and de-Rham Laplacians, we have

$$\lambda\mu = (\Delta\psi)\mu = \Delta_{\text{dR}}(\psi\mu) = d\star d(\psi) = d(\star d\psi).$$

So $\omega = \lambda\mu \in d\bigwedge^1(M)$ is a vorticity two-form. □

Definition 2.18. *We refer to the function space $C_0^\infty(M)$ as the space of **scalar vorticities**. We denote scalar vorticities with the symbol λ , such that a vorticity two-form is given as $\omega = \lambda\mu$.*

With $\mathfrak{g} \cong C^\infty(M)/\mathbb{R}$ and $\mathfrak{g}^* \cong C_0^\infty(M)$ identified with spaces of functions, we consider their pairing.

Lemma 2.19. *The pairing of Definition 2.2 corresponds to the L^2 pairing of the function spaces $C^\infty(M)/\mathbb{R}$ and $C_0^\infty(M)$ as*

$$\langle \lambda, \psi \rangle = \int_M (\lambda\psi)\mu.$$

Proof. With $\psi \in C^\infty(M)$ a stream function for the vector field $X \in \mathfrak{X}_\mu(M)$ and $\nu \in \bigwedge^1(M)/d\bigwedge^0(M)$ a velocity one-form associated to the vorticity $d\nu = \omega = \lambda\mu$, we calculate

$$\langle [\nu], X \rangle = \int_M i_X \mu \wedge \nu = \int_M i_{X_\psi} \mu \wedge \nu = \int_M d\psi \wedge \nu = \int_M \psi \wedge d\nu = \int_M \psi(\lambda\mu) = \int_M (\lambda\psi)\mu.$$

Since λ is mean-zero, the pairing is independent of the choice of ψ . $\square$

The infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ follows in a similar way.

Proposition 2.20. *The coadjoint action of $\psi \in C^\infty(M)/\mathbb{R}$ on $\lambda \in C_0^\infty(M)$ is*

$$\text{ad}_\psi^* \lambda = \{\psi, \lambda\},$$

the Poisson bracket of smooth functions on the symplectic manifold (M, μ) .

Proof. With $\psi \in C^\infty(M)$ a stream function for $X \in \mathfrak{X}_\mu(M)$ and vorticity $\omega = \lambda\mu$, it holds

$$\text{ad}_X^* \omega = \text{ad}_{X_\psi}^* \omega = -\mathcal{L}_{X_\psi}(\omega) = -\mathcal{L}_{X_\psi}(\lambda\mu) = -(\mathcal{L}_{X_\psi} \lambda)\mu = \{\psi, \lambda\}\mu.$$

Since constants generate zero Poisson bracket, the action is independent of the choice of ψ . $\square$

We may now consider the translation of the inertia operator induced by the Hamiltonian of total kinetic energy to this new perspective.

Proposition 2.21. *The Hamiltonian of total kinetic energy (2.9) is given on the space of scalar vorticities $\mathfrak{g}^* = C_0^\infty(M)$ as*

$$H(\lambda) = \frac{1}{2} \int_M \lambda(\Delta^{-1} \lambda) \mu,$$

with Δ^{-1} the map defined in Definition 1.31. The inertia operator corresponding to this Hamiltonian is minus the Laplacian

$$-\Delta : C^\infty(M)/\mathbb{R} \rightarrow C_0^\infty(M),$$

acting as

$$\psi \mapsto \lambda = -\Delta\psi$$

Proof. Let us first show that the correct inertia operator is the Laplacian. A stream function ψ is identified with the vector field X_ψ satisfying $i_{X_\psi} \mu = d\psi$. The inertia operator of Proposition 2.9 sends this vector field to $X_\psi^\flat$. Applying Lemma A.2 from Appendix B, we see that

$$X_\psi^\flat = -\star i_{X_\psi} \mu = -\star d\psi.$$

Taking the exterior derivative gives the vorticity two-form $\omega = -d\star d\psi$, which yields the scalar vorticity function

$$\lambda = -\star d\star d\psi = -\Delta_{dR} \psi = -\Delta\psi,$$

using the agreement of the de-Rham Laplacian and the Laplace-Beltrami operator on smooth functions. That this inertia operator is non-degenerate is exactly the statement of Corollary 1.30. Since the Hamiltonian associated to a generic inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is

$$H(\nu) = \frac{1}{2} \langle \nu, A^{-1} \nu \rangle,$$

we may apply Lemma 2.19 to find the above form for the Hamiltonian. $\square$

Applying our identifications to Corollary 2.10 then gives the following bracket form of the Euler equation.

Corollary 2.22. *The Euler-Poisson equation associated with minus the Hamiltonian (2.21) is*

$$\dot{\lambda} + \{\lambda, \psi\} = 0,$$

where

$$\psi = -\Delta^{-1}\lambda.$$

	Perspective I	Perspective II	Perspective III
$\mathfrak{g}$	Velocity field $X \in \mathfrak{X}_\mu(M)$	Velocity field $X \in \mathfrak{X}_\mu(M)$	Stream function $\psi \in C^\infty(M)/\mathbb{R}$
$\mathfrak{g}^*$	Velocity one-form $[\nu] \in \bigwedge^1(M)/d\bigwedge^0(M)$	Vorticity two-form $\omega \in d\bigwedge^1(M)$	Scalar vorticity $\lambda \in C_0^\infty(M)$
Pairing	$\langle [\nu], X \rangle = \int_M i_X \mu \wedge \nu$	$\langle \omega, X \rangle = \langle d\nu, X \rangle = \int_M i_X \mu \wedge \nu$	$\langle \lambda, \psi \rangle = \int_M \lambda \psi \mu$
Coadjoint Action	$ad_X^*[\nu] = -[\mathcal{L}_X \nu]$	$ad_X^* \omega = -\mathcal{L}_X \omega$	$ad_\psi^* \lambda = \{\psi, \lambda\}$
Inertia Operator	$\mathfrak{X}_\mu(M) \ni X \rightarrow [X^\flat] \in \bigwedge^1(M)/d\bigwedge^0(M)$	$\mathfrak{X}_\mu(M) \ni X \rightarrow dX^\flat \in d\bigwedge^1(M)$	$C^\infty(M)/\mathbb{R} \ni \psi \rightarrow -\Delta\psi \in C_0^\infty(M)$
Fluid Hamiltonian	$H([\nu]) = \frac{1}{2} \int_M g(\nu, \nu) \mu$	$H(\omega) = \frac{1}{2} \int_M (\Delta^{-1}\lambda) \omega$	$H(\lambda) = \frac{1}{2} \int_M \lambda (\Delta^{-1}\lambda) \mu$
Euler-Poisson Equation	$[\dot{X}^\flat] + \mathcal{L}_X[X^\flat] = 0$	$\dot{\omega} + \mathcal{L}_X \omega = 0$	$\dot{\lambda} + \{\lambda, \psi\} = 0.$
Restrictions	General M	Simply-connected M	Simply-connected, orientable M

Table 2.1: Three perspectives on the $\mathfrak{g}^*$ Lie-Poisson system on a closed Riemannian surface.

Table 2.1 collects the three perspectives on the Lie-Poisson space $\mathfrak{g}^*$ discussed in this chapter, presenting $\mathfrak{g}^*$ alternately as a space of velocity one-forms, vorticity two-forms, and scalar vorticity functions. We denote these formulations as perspective I, perspective II, and perspective III, respectively. Through the rest of this work we will cleanly pass between these perspectives, referencing them explicitly as such.

We emphasize that while we have presented perspective III for the case in which M is closed, it also holds for surfaces with boundary. Indeed, Lemma 2.19 and Propositions 2.20 and 2.21 can be generalized to surfaces with boundary with appropriate alterations. Similarly, although we have assumed M to be oriented in our discussion, perspectives I and II hold on non-orientable surfaces as well, as we discuss in Chapter 4.

The Hodge Decomposition of $\mathfrak{g}^*$

We now reinterpret the above results in light of the Hodge decomposition of one-forms on M ,

$$\bigwedge^1(M) = \text{im}(d) \oplus \text{im}(\delta) \oplus \ker \Delta_{dR}.$$

As Hodge decomposition requires a metric, this perspective goes strictly beyond the Lie-Poisson geometry of $\mathfrak{g}^*$ which requires only the symplectic structure of M . We do not require that M be simply-connected in this discussion, which allows for the presence of a harmonic component in the above decomposition.

From the Hodge perspective, Lemma 2.3 shows that one-forms dual to divergence-free vector fields have no exact component in their Hodge decomposition. Indeed, quotienting this decomposition by exact components induces a decomposition of $\mathfrak{g}^*$ as

$$\mathfrak{g}^* = \bigwedge^1(M)/d\bigwedge^0(M) = \text{im}(\delta) \oplus \ker \Delta_{dR}. \quad (2.13)$$

This gives a general expression for a velocity one-form $\nu \in \bigwedge^1(M)$ as

$$\nu = \delta\Psi + \eta,$$

for the stream function two-form $\Psi = \psi\mu$ and harmonic one-form $\eta \in \mathcal{H}^1(M)$. Working with the scalar stream function $\psi = \star\Psi$, this becomes

$$\nu = -\star d\psi + \eta. \quad (2.14)$$

The vorticity two-form associated to ν is

$$\omega = d\nu = -d\star d\psi = (-\Delta\psi)\mu, \quad (2.15)$$

as expected. Inverting the relation (2.15), we see that it is possible to associate a stream function to a one-form ν on a multiply-connected surface by setting

$$\psi = -\Delta^{-1}(\lambda) = -\Delta^{-1}(\star\omega) = -\Delta^{-1}(\star d\nu), \quad (2.16)$$

which is precisely the Hodge projection onto the co-exact component given in Proposition 1.34.

This clarifies the difference between ideal hydrodynamics on simply-connected surfaces and multiply-connected surfaces. On a simply-connected surface, the co-exact component (which is given by either the stream function or the vorticity) of a velocity one-form ν uniquely determines that one-form. In contrast, on a multiply-connected surface, two distinct velocity one-forms may share the same vorticity. The missing gauge freedom comes from the finite-dimensional harmonic component of $\mathfrak{g}^*$. As seen in Examples 1.25 and 1.26, this space consists of harmonic fluid flows “around” the holes in the surface M .

Indeed, by using the identifications derived in this Chapter, we may rewrite the decomposition (2.13) as

$$\mathfrak{g}^* \cong d\bigwedge^1(M) \oplus H_{\text{dR}}^1(M) \cong C^\infty(M)/\mathbb{R} \oplus H_{\text{dR}}^1(M), \quad (2.17)$$

where the first component of $\mathfrak{g}^*$ is parametrized either by vorticity two-forms or stream functions. This makes it clear that Proposition 2.5 and Corollary 2.14 are simply special cases of the Hodge decomposition in $\mathfrak{g}^*$ for simply-connected M . When M is simply-connected, the harmonic component of $\mathfrak{g}^*$ vanishes, and each velocity one-form is uniquely determined by its vorticity and stream function, as

$$\nu = -\star d\Delta^{-1}(\lambda) = -\star d\psi. \quad (2.18)$$

We emphasize that the first component of the Hodge decomposition (2.17) is not stable under the coadjoint action of G . For example, on a multiply-connected surface M , if one selects a velocity one-form $\nu = -\star d\psi$ with no harmonic component, the Hodge decomposition of its pullback by some $\phi \in \text{Diff}_\mu(M)$,

$$\text{Ad}_\phi^* \nu = (\phi^{-1})^*(\nu) = (\phi^{-1})^*(-\star d\psi)$$

may contain non-trivial harmonic components. As we discuss in Chapter 5, this has important consequences for point-vortex dynamics on multiply-connected surfaces, as it suggests that a fluid configuration of point-vortices with no initial harmonic component may generate a harmonic component over the course of its motion.

2.3 The KKS Form on a Coadjoint Orbit in $\mathfrak{g}^*$

We conclude our discussion by examining the structure of the KKS form on a coadjoint orbit in $\mathfrak{g}^*$. This allows us to give a Hamiltonian formulation of the Euler equations, which will be useful in our exploration of point-vortex dynamics. This discussion follows Section 4 of [21], generalized to the case of multiply-connected surfaces.

We consider a closed, oriented, possibly multiply-connected Riemannian surface (M, g) . The coadjoint orbits in $\mathfrak{g}^*$ are given as $\mathcal{O}_\nu = \{(\phi^{-1})^*\nu : \phi \in \text{Diff}_\mu(M)\}$ for the velocity one-form $\nu \in \mathfrak{g}^*$. We calculate at the particular element $\bar{\nu} \in \mathcal{O}_\nu$, with associated vorticity $\bar{\omega} = d\bar{\nu}$. Recall from Proposition 1.2 that the tangent space to $\mathcal{O}_\nu$ at $\bar{\nu}$ is $T_{\bar{\nu}}\mathcal{O}_\nu = \{\text{ad}_X^* \bar{\nu} : X \in \mathfrak{g}\} = \{-\mathcal{L}_X \bar{\nu} : X \in \mathfrak{g}\} = \{\mathcal{L}_X \bar{\nu} : X \in \mathfrak{g}\}$.

Theorem 2.23. *The KKS form on a coadjoint orbit in $\mathfrak{g}^*$ is given at a point $\bar{\nu} \in \mathcal{O}_\nu$ by*

$$\sigma_{\text{KKS}}(\mathcal{L}_X \bar{\nu}, \mathcal{L}_Y \bar{\nu}) = \int_M \bar{\omega}(X, Y) \mu$$

for $X, Y \in \mathfrak{g}$.

Proof. Specifying the KKS form (1.4) to the present setting yields

$$\sigma_{\text{KKS}}(\mathcal{L}_X \bar{\nu}, \mathcal{L}_Y \bar{\nu}) = \langle \bar{\nu}, [X, Y]_{\text{fluid}} \rangle = \langle \bar{\nu}, -\mathcal{L}_X Y \rangle = \int_M \langle \bar{\nu}, -\mathcal{L}_X Y \rangle.$$

We rewrite this as

$$\int_M \langle \bar{\nu}, -\mathcal{L}_X Y \rangle \mu = \int_M \langle \mathcal{L}_X \bar{\nu}, Y \rangle \mu - \int_M \mathcal{L}_X \langle \bar{\nu}, Y \rangle \mu = \int_M \langle \mathcal{L}_X \bar{\nu}, Y \rangle \mu - \int_M \langle d\langle \bar{\nu}, Y \rangle, X \rangle \mu.$$

The rightmost term vanishes by Lemma 2.3. By the Cartan formula, the remaining term gives

$$\int_M \langle \mathcal{L}_X \bar{\nu}, Y \rangle \mu = \int_M \langle d\langle \bar{\nu}, X \rangle, Y \rangle \mu + \int_M \langle i_X(d\bar{\nu}), Y \rangle \mu.$$

The first term again vanishes by Lemma 2.3, leaving only

$$\int_M \langle \mathcal{L}_X \bar{\nu}, Y \rangle \mu = \int_M \langle i_X(d\bar{\nu}), Y \rangle \mu = \int_M \langle i_X \bar{\omega}, Y \rangle \mu = \int_M \bar{\omega}(X, Y) \mu.$$

□

We now rewrite the KKS form using the Hodge decomposition of $\mathfrak{g}^*$, given in the previous section as

$$\mathfrak{g}^* = d\bigwedge^1(M) \oplus H_{\text{dR}}^1(M).$$

We make use of the following Lemma, proved in Appendix A.

Lemma A.4. *On a closed, oriented, Riemannian surface (M, g) , it holds for all $X, Y \in \mathfrak{X}(M)$ that*

$$i_X \mu \wedge i_Y \mu = \mu(X, Y) \mu.$$

Proposition 2.25. *With M and $\mathcal{O}_\nu$ as above, let $X, Y \in \mathfrak{g}$ be vector fields dual to one-forms with Hodge decomposition*

$$X^\flat = -\star d\psi + \eta \quad Y^\flat = -\star d\xi + \theta,$$

where $\psi, \xi \in C^\infty(M)$ and $\eta, \theta \in \mathcal{H}^1(M)$. Then the KKS form at a point $\bar{\nu} \in \mathcal{O}_\nu$ can be written as

$$\sigma_{\text{KKS}}(\mathcal{L}_X \bar{\nu}, \mathcal{L}_Y \bar{\nu}) = \int_M \bar{\lambda}(d\psi + \star \eta) \wedge (d\xi + \star \theta).$$

Proof. With $\bar{\omega} = \bar{\lambda}\mu$, the above lemma gives that

$$\bar{\omega}(X, Y)\mu = \lambda(i_X\mu \wedge i_Y\mu).$$

Applying the identity $i_X\mu = \star X^\flat$ from Appendix A, this becomes

$$\bar{\omega}(X, Y)\mu = \lambda(i_X\mu \wedge i_Y\mu) = \lambda(\star X^\flat \wedge \star Y^\flat),$$

which gives the result. $\square$

This result shows that the vorticity function λ serves to couple the two components in the Hodge decomposition of $\mathfrak{g}^*$. As we shall see in Chapter 5, this coupling simplifies greatly in the case that λ corresponds to point-vortices on M .

The following corollary simplifies the above result to the case of simply-connected M , adopting Perspective III of Table 2.1 to identify $\mathfrak{g}$ with the space of stream functions and $\mathfrak{g}^*$ with the space of vorticity two-forms.

Corollary 2.26. *With M and $\psi, \xi \in C^\infty(M)$ stream functions inducing Hamiltonian vector fields X_ψ and X_ξ , the KKS form at a point $\bar{\omega} \in \mathcal{O}_\omega$ takes the form*

$$\sigma_{KKS}(\mathcal{L}_{X_\psi}\omega, \mathcal{L}_{X_\xi}\omega) = \int_M \bar{\omega}(X_\psi, X_\xi)\mu = \int_M \bar{\lambda}d\psi \wedge d\xi = \int_M \{\psi, \xi\}\bar{\omega}.$$

Proof. This follows from the above and the definition of the Poisson bracket of smooth functions on M . $\square$

Remark 2.27. The formulation

$$\sigma_{KKS}(\mathcal{L}_{X_\psi}\bar{\omega}, \mathcal{L}_{X_\xi}\bar{\omega}) = \int_M \{\psi, \xi\}\bar{\omega}$$

is exactly the vorticity bracket described in Section 6 of [21].

Chapter 3

Point-Vortices on Oriented, Simply-Connected Surfaces

In this chapter, we present the Hamiltonian dynamics of point-vortices on closed, oriented, simply-connected surfaces.

We begin with a brief overview of the classical system of point-vortex dynamics in the plane and then present the framework for vortex dynamics on surfaces given in [7] and [9]. After reviewing the Hamiltonian formulation of point-vortex dynamics on surfaces, we discuss how this formulation fits into the geometric framework for ideal hydrodynamics laid out in the previous chapter. Following [21], we show that the N -vortex system represents the restriction of the infinite-dimensional Lie-Poisson system discussed in the previous chapter to a finite-dimensional coadjoint orbit of singular vorticities.

Our treatment of point-vortices is not comprehensive and focuses mainly on the perspective of geometric hydrodynamics. We do not discuss the convergence of the N -vortex dynamics towards general two-dimensional ideal fluid flow for large N , nor do we discuss the many interesting integrability results on different surfaces found for small N . For literature that captures more of the rich mathematical history of the vortex system, we refer the reader to the excellent article [3], as well as the book [26].

3.1 The Hamiltonian Formulation of Point-Vortex Dynamics

We launch our discussion with a formal definition of a point-vortex on an oriented, simply-connected surface.

Definition 3.1. *Let (M, g) be an oriented, simply-connected Riemannian surface. A **point-vortex** at the point $r_i \in M$ is a singular¹ solution of the Euler equation 2.1 with stream function*

$$\psi(r) = \Gamma G_M(r, r_i), \quad (3.1)$$

where G_M is the Green function on M .² The fluid velocity vector field associated to a point-vortex is

$$v(r) = -\nabla^\perp \psi(r). \quad (3.2)$$

¹We treat point-vortices on surfaces as singular vorticity distributions. For a treatment that presents them as currents on M , see [28].

²If M has ends, the considerations of Remark 1.33 apply (see [9]). The only non-compact surface we consider is the plane, on which G_M is simply the fundamental solution of the Laplacian.

The **circulation** Γ associated to a point-vortex is the limit

$$\Gamma = \lim_{\epsilon \rightarrow 0} \oint_{\partial C_\epsilon(r_i)} v^\flat. \quad (3.3)$$

where $C_\epsilon(r_i)$ is the geodesic disk of radius ϵ centered at r_i .

Example 3.2. (The Plane) The Green function in the plane $M = \mathbb{R}^2$ equipped with the standard metric is $G_M(r, r_i) = -\frac{1}{2\pi} \log |r - r_i|$. Consequently, the stream function associated to a point-vortex at the origin is

$$\psi(r) = -\frac{\Gamma}{2\pi} \log |r|.$$

The resulting fluid velocity vector field is given in Cartesian coordinates by

$$v(x, y) = \frac{\Gamma}{2\pi |r|^2} (-y, x),$$

where $r = (x, y)$. Outside of the singularity at the origin, this vector field is curl-free and divergence-free. Indeed, it is precisely the harmonic vector field discussed in Example 1.25. Taking the curl of v yields the Dirac scalar vorticity distribution

$$\lambda(r) = \Gamma \delta(r).$$

The integral of v around any circle C centered at the origin (or any closed curve containing the origin with winding number one) verifies

$$\Gamma = \oint_{\partial C} v \cdot dl = \int_C \lambda \mu$$

where $\mu = dx \wedge dy$ is the area element in the plane, validating (3.3).

Remark 3.3. The scalar vorticity associated to a point-vortex depends on the compactness of the fluid domain M . While the above example shows that the vorticity associated with a point-vortex in the plane is a Dirac distribution, such vorticity is not possible for compact M . Indeed, if ω is the vorticity two-form associated to a point-vortex of circulation Γ placed at $r_i \in M$ and $C_\epsilon(r_i)$ is a geodesic disk of radius ϵ centered at r_i , we have the calculation

$$0 = \int_M \omega = \int_{M/C_\epsilon} \omega + \int_{C_\epsilon} \omega = \int_{M/C_\epsilon} \omega + \int_{C_\epsilon} du^\flat = \int_{M/C_\epsilon} \omega + \int_{\partial C_\epsilon} u^\flat.$$

Taking the limit as ϵ vanishes gives that

$$\int_{M/\{r_i\}} \omega = -\Gamma.$$

Thus, on a compact surface a point-vortex induces a concentration of vorticity at its location together with a dispersed **counter-vorticity**. The resulting physical effect is somewhat counter-intuitive: in a compact domain, fluid particles flowing in the stream of a single point-vortex will experience vortical rotation in the opposite direction of the rotation of the vortex itself.

Proposition 3.4. *The vorticity two-form ω associated to a point-vortex with circulation Γ_i at position $r_i \in M$ on an oriented, simply-connected Riemannian surface (M, g) with total area $V = \int_M \mu$ is*

$$\omega = \begin{cases} (\Gamma_i \delta_{r_i} - \frac{\Gamma_i}{V}) \mu & M \text{ compact} \\ (\Gamma_i \delta_{r_i}) \mu & M \text{ non-compact} \end{cases} \quad (3.4)$$

Proof. Since the vorticity of a fluid configuration is determined by the stream function as $\omega = -\Delta\psi$, the above follows by applying Remark 1.33 to the point-vortex stream function (3.1). $\square$

Example 3.5. The Green function on the sphere $M = S^2$, understood as an embedded submanifold of $\mathbb{R}^3$ is $G_M(r, r_i) = -\frac{1}{2\pi} \log |r - r_i|$. The stream function associated to a point-vortex with circulation Γ_i placed at the north pole $r_i \in S^2$ is

$$\psi(r) = -\frac{\Gamma}{2\pi} \log |r - r_i|.$$

This induces the fluid velocity vector field $v(r) = \frac{\Gamma}{4\pi} \frac{r_i \times r}{1 - r \cdot r_i}$. Taking the curl of this vector field and restricting to the sphere S^2 yields the scalar curl

$$\lambda(r) = \Gamma \delta_{r_i}(r) - \frac{\Gamma}{4\pi},$$

displaying the expected counter-vorticity.

In order to visualize how the counter-vorticity manifests in the dynamics on the sphere (in contrast with the plane, where there is no counter-vorticity), one may stereographically project the spherical point-vortex fluid velocity field onto the plane, see [26] section 4.5.

For clarity, we list the form of the velocity one-form, stream function, and vorticity two-form associated to N point vortices on an oriented, simply-connected Riemannian surface.

Proposition 3.6. *On an oriented, simply-connected Riemannian surface (M, g) with area form μ and total area $V = \int_M \mu$, the stream function associated to N point vortices with circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ located at $r_1, \dots, r_N \in M$ is*

$$\psi = \sum_{i=1}^N \Gamma_i G_M(\cdot, r_i). \quad (3.5)$$

The resulting velocity one-form is

$$\nu = - * d\psi = - \sum_{i=1}^N \Gamma_i * dG_M(\cdot, r_i). \quad (3.6)$$

The corresponding vorticity two-form is

$$\omega = d\nu = \begin{cases} \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu & M \text{ compact}, \\ \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} \right) \mu & M \text{ non-compact}, \end{cases} \quad (3.7)$$

where

$$\Gamma = \sum_{i=1}^N \Gamma_i$$

is the **total circulation**.

We now explore the dynamics of interacting point-vortices, beginning with the case of the plane: a classical system first studied by Helmholtz in 1858, see [3].

3.1.1 Vortices in the Plane

The dynamics of N point-vortices in the plane with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ forms a Hamiltonian system

$$\Gamma_i \dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \Gamma_i \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad (3.8)$$

for the Hamiltonian function

$$H(r_1, \dots, r_N) = -\frac{1}{2\pi} \sum_{i < j} \Gamma_i \Gamma_j \log |r_i - r_j|, \quad (3.9)$$

where $r_i = (x_i, y_i)$. We show later that this Hamiltonian consists of the total energy of the fluid with the infinite self-energy terms removed.

Since the above Hamiltonian diverges at vortex collisions, where $r_i = r_j$ for $i \neq j$, the phase space for N -vortex dynamics in the plane is

$$(\mathbb{R}^2)^N \setminus \{\text{collision}\} := \{(r_1, \dots, r_N) \in (\mathbb{R}^2)^N : r_i \neq r_j \text{ for any } i \neq j\}.$$

On this phase space, Hamilton's equations (3.8) follow from the symplectic structure

$$\sigma_{\text{PV}} = \sum_i \Gamma_i dx_i \wedge dy_i.$$

The scaling by the circulations Γ_i in this symplectic form corresponds to the fact that the canonical variables in Hamilton's equations (3.8) are the scaled position variables $\Gamma_i x_i$ and $\Gamma_i y_i$ ³.

Invariance of the Hamiltonian (3.9) under translations and rotations produce the three Noether invariants

$$J_x = \sum_{i=1}^N \Gamma_i x_i, \quad J_y = \sum_{i=1}^N \Gamma_i y_i, \quad J_\theta = \frac{1}{2} \sum_{i=1}^N \Gamma_i (x_i^2 + y_i^2). \quad (3.10)$$

However these quantities do not give immediate integrability results as their Poisson commutation (under the symplectic structure given by σ_{PV}) is dependent on the total circulation Γ . As discussed in [22], this can be explained by the fact that the momentum map associated to the action of the Euclidean group $E(2)$ on the planar vortex phase space is equivariant only when $\Gamma = 0$. For an extended discussion of planar point-vortex dynamics, see [2].

Periodization and the Method of Images

The description of planar vortex motion may be extended to certain flat surfaces via periodization.

Example 3.7. (The cylinder) As described in [25], the dynamics of N vortices on the cylinder

$$\mathbb{R}^2/2\pi\mathbb{Z} = \{(x, y) \in \mathbb{R}^2 : x \sim x + 2\pi n, n \in \mathbb{Z}\}$$

with positions $(x_1, y_1), \dots, (x_N, y_N) \in \mathbb{R}^2/2\pi\mathbb{Z}$ corresponds to the dynamics of the infinite 2π -periodic lattice of vortices with positions

$$\{(x_i + 2\pi n, y_i) : k = 1, \dots, N, n \in \mathbb{Z}\}.$$

While the Hamiltonian function resulting from this periodization diverges, it may be regularized

³This reflects that point-vortices “have no inertia.” The evolution of the point-vortex system is entirely determined by the positions of the vortices. This is in contrast to the N -body problem, which has canonical variables of position and momentum.

to a convergent, finite sum. The resulting, convergent Hamiltonian is

$$H(r_1, \dots, r_N) = -\frac{1}{4\pi} \sum_{i < j} \Gamma_i \Gamma_j \log \left[\sin^2 \left(\frac{x_i - x_j}{4\pi} \right) + \sinh^2 \left(\frac{y_i - y_j}{4\pi} \right) \right]. \quad (3.11)$$

Example 3.8. (The torus) A similar periodization can be performed to model vortex motion on the flat torus

$$\mathbb{R}^2 / 2\pi\mathbb{Z}^2 = \{(x, y) \in \mathbb{R}^2 : x \sim x + 2\pi n, y \sim y + 2\pi m, n \in \mathbb{Z}\}.$$

The motion of N point vortices with positions $(x_1, y_1), \dots, (x_N, y_N) \in \mathbb{R}^2 / 2\pi\mathbb{Z}^2$ corresponds to the motion of an infinite, doubly periodic lattice in the plane (or equivalently, a $2\pi y$ -periodic lattice on the flat cylinder). As shown in [22], the Hamiltonian for this motion is

$$H(r_1, \dots, r_N) = -\frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \left[-\frac{(x_i - x_j)^2}{2\pi} + \sum_{m=-\infty}^{\infty} \log \left(\frac{\cosh(x_i - x_j - 2\pi m) - \cos(y_i - y_j)}{\cosh(2\pi m)} \right) \right]. \quad (3.12)$$

Another common generalization of the planar system comes from the method of images, which models point-vortex dynamics in domains with boundary via the introduction of image vortices.

Example 3.9. (The upper half-plane) In order to model the dynamics of N point-vortices in the upper half-plane $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ (imposing the standard boundary condition that fluid velocity be parallel to the horizontal axis), we introduce **image vortices**. The image vortex paired with the point-vortex at $r_i = (x_i, y_i)$ is placed at the position

$$r_i^* = (x_i, -y_i),$$

with the equal and opposite circulation $-\Gamma_i$. The resulting configuration describes the motion of N vortices in the half-plane. It has Hamiltonian

$$H(r_1, \dots, r_N) = -\frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \log |r_i - r_j| + \frac{1}{2\pi} \sum_{i,j} \Gamma_i \Gamma_j \log |r_i - r_j^*|.$$

A similar approach can be taken to model point-vortex motion in a disk (see [26] section 3.2).

Remark 3.10. Note that the above Hamiltonian shows that, in contrast with the case of the full plane $\mathbb{R}^2$, a single point-vortex in the half-plane will not remain static. Calculation shows that a single point-vortex in the half-plane moves horizontally parallel to the boundary with constant speed. This is an instance of **self-motion** in the point-vortex system.

3.1.2 Vortices on Simply-Connected Surfaces

We now present a generalization of the point-vortex system to general compact, oriented, simply-connected surfaces, first presented in [7].

We consider the dynamics of N vortices with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R} / \{0\}$ on a compact, oriented, simply-connected Riemannian surface (M, g) with orientation given by the area form μ . If M is not closed, we enforce that the fluid flow induced by the vortices be parallel to the boundary ∂M , as in the previous chapter.

In analogy with the plane, the phase space for N -vortex motion on M is

$$M^N \setminus \{\text{collision}\} := \{(r_1, \dots, r_N) \in M^N : r_i \neq r_j \text{ for any } i \neq j\}, \quad (3.13)$$

with the weighted symplectic structure

$$\sigma_{\text{PV}} = \sum_i \Gamma_i \mu_i. \quad (3.14)$$

The Hamiltonian for N -vortex dynamics on M is given by the total energy of the vortices. However, while the planar Hamiltonian (3.9) simply omitted the divergent self-energy terms, the general case instead **renormalizes** them using the following function.

Definition 3.11. The **Robin function** $R \in C^\infty(M)$ is the desingularization of G_M ,

$$R_M(r) = \lim_{s \rightarrow r} \left(G_M(s, r) + \frac{1}{2\pi} \log d(s, r) \right). \quad (3.15)$$

Remark 3.12. The principal effect of the Robin renormalization is to assign a finite value to the divergent point-vortex stream function at the location of the vortex itself, $\psi(r_i) = G_M(r_i, r_i)$. This finite value extends to a finite self-energy in the Hamiltonian, which in turn induces a finite velocity for the self-motion of a point-vortex. After regularization, the N -vortex stream function can be defined on all of M , including the positions of the vortices themselves.

Remark 3.13. The Robin function accounts for the self-motion due to boundary observed in Example 3.9. Self-motion may also occur due to variable curvature of the surface M . Indeed, by homogeneity it is clear that R vanishes on the sphere, the hyperboloid, and the plane. For extended discussion on this topic, see [29].

Using the Robin function, we can write the general Hamiltonian for N point-vortices,

$$H(r_1, \dots, r_N) = - \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) - \frac{1}{2} \sum_{i=1}^N \Gamma_i^2 R_M(r_i). \quad (3.16)$$

Together (3.13), (3.14), and (3.16) give the Hamiltonian formulation of point-vortex dynamics on general compact, oriented, simply-connected surfaces.

Example 3.14. Let us apply the above formulation to the sphere $M = S^2$. The phase space and symplectic structure are given as in (3.13) and (3.14). Direct calculation shows that the Robin function vanishes. This is intuitive, as the homogeneity of the sphere makes vortex self-motion impossible. The Green function on the unit sphere is (see [26])

$$G_M(r, s) = -\frac{1}{4\pi} \log(1 - r \cdot s).$$

Thus, the N -vortex Hamiltonian on the sphere is

$$H(r_1, \dots, r_N) = -\frac{1}{4\pi} \sum_{i < j} \Gamma_i \Gamma_j \log(1 - r_i \cdot r_j). \quad (3.17)$$

3.2 The Point-Vortex System as a Coadjoint Orbit

In this section, we examine the relationship between the finite-dimensional Hamiltonian system of N -vortex dynamics on the surface M and the infinite-dimensional Hamiltonian description of ideal fluid flow found in the previous chapter. In particular, we show that the Hamiltonian system of N vortices with non-zero circulations on M is a finite-dimensional coadjoint orbit in $\mathfrak{g}^*$, following an argument found in [21] and [23].

A Momentum Map for the Point-Vortex System

Consider the motion of N vortices with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ on a closed, oriented, simply-connected surface M with area form μ and total area $V = \int_M \mu$. For clarity in our discussion, we use the notation

$$(\mathcal{M}, \sigma_{\text{PV}}) := (M^N \setminus \{\text{collision}\}, \sigma_{\text{PV}})$$

to denote the phase-space for point-vortex motion with its scaled symplectic structure, given as in (3.13) and (3.14). This is in contrast with (M, μ) , which is the domain for the fluid flow together with its area form.

In order to show that $(\mathcal{M}, \sigma_{\text{PV}})$ is a (finite-dimensional) coadjoint orbit in $\mathfrak{g}^*$, we make use of Proposition 1.17 to show that it is the image of a transitive, equivariant momentum map. We begin with the following result.

Proposition 3.15. *The diagonal action of $G = \text{Diff}_\mu(M)$ on $(\mathcal{M}, \sigma_{\text{PV}})$ is Hamiltonian, with equivariant momentum map $J : \mathcal{M} \rightarrow \mathfrak{g}^*$ given by*

$$J(r_1, \dots, r_N) = \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu.$$

Proof. The action of G on M is symplectic by definition, and so it extends to a symplectic action on the symplectic manifold $(\mathcal{M}, \sigma_{\text{PV}})$. Let us show that the above J is indeed a momentum map for this action.

Fix an element $X \in \mathfrak{X}_\mu(M)$. As M is simply-connected, there exists a stream function $\psi \in C^\infty(M)$ satisfying $i_X \mu = d\psi$. As ψ is unique up to a constant, we choose the particular stream function for X normalized to be mean-zero.

The vector field $X \in \mathfrak{X}(M)$ on M induces the infinitesimal action vector field $X_{\mathcal{M}} \in \mathfrak{X}(\mathcal{M})$ given by

$$X_{\mathcal{M}}(r_1, \dots, r_N) = (X(r_1), \dots, X(r_N)).$$

We calculate

$$i_{X_{\mathcal{M}}} \sigma_{\text{PV}} = \sum_i \Gamma_i i_{X_{\mathcal{M}}(r_i)} \mu_i = \sum_i \Gamma_i d\psi(r_i) = d \sum_i \Gamma_i \psi(r_i) = d \int_M \left(\sum_i \Gamma_i \delta_{r_i} \right) \psi \mu. \quad (3.18)$$

Since our chosen stream function ψ is mean-zero, we observe that the rightmost integral is equivalent to

$$\int_M \left(\sum_i \Gamma_i \delta_{r_i} \right) \psi \mu = \int_M \left(\sum_i \Gamma_i \delta_{r_i} \right) \psi \mu - \frac{\Gamma}{V} \int_M \psi \mu = \left\langle \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu, \psi \right\rangle. \quad (3.19)$$

Adopting perspective II from Table 2.1, this rightmost term is equivalent to the pairing $\langle \omega, \psi \rangle$ for the point-vortex vorticity

$$\omega = \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu.$$

We have thus shown that

$$i_{X_{\mathcal{M}}} \sigma_{\text{PV}} = d \langle J(r_1, \dots, r_N), X \rangle,$$

for the momentum map $J : \mathcal{M} \rightarrow \mathfrak{g}^*$ given by

$$J(r_1, \dots, r_N) = \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu. \quad (3.20)$$

Moreover, J is clearly equivariant, since for $\phi \in G$ it holds

$$(\phi^{-1})^* J(r_1, \dots, r_N) = (\phi^{-1})^* \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu = \left(\sum_i \Gamma_i \delta_{\phi(r_i)} - \frac{\Gamma}{V} \right) \mu = J(\phi \cdot (r_1, \dots, r_N)).$$

□

Since the action of G on $(\mathcal{M}, \sigma_{PV})$ is transitive, we may apply Proposition 1.17 to conclude that $J(\mathcal{M})$ is a coadjoint orbit in $\mathfrak{g}^*$. As J is injective, this gives the following result.

Corollary 3.16. *Let*

$$\omega = \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu$$

be a singular point-vortex vorticity two-form describing a configuration of N point-vortices at positions $r_1, \dots, r_N \in M \setminus \{\text{collision}\}$ with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$. Then the coadjoint orbit $(\mathcal{O}_\omega, \sigma_{KKS}|_{\mathcal{O}_\omega})$ is a finite-dimensional symplectic manifold that is symplectomorphic to the N -vortex phase space $(\mathcal{M}, \sigma_{PV})$.

Remark 3.17. We note that the above calculation is precisely that of Example 1.11 from Chapter 1. The action of G on $\mathcal{M}$ is the coadjoint action on a coadjoint orbit in $\mathfrak{g}^*$, and the momentum map (3.20) is inclusion.

Restriction of the KKS Form and the Hamiltonian

The above corollary states that the σ_{KKS} restricted to the N -vortex coadjoint orbit is σ_{PV} . We now confirm this result by explicitly calculating $\sigma_{KKS}(\mathcal{L}_X \bar{\omega}, \mathcal{L}_Y \bar{\omega})$ for the case that $\bar{\omega}$ is an N -vortex vorticity two-form,

$$\bar{\omega} = \left(\sum_i \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu.$$

As we are in the case of simply-connected M , this amounts to an application of Corollary 2.26. We choose stream functions $\psi, \xi \in C^\infty(M)$ generating vector fields $X_\psi, X_\xi \in \mathfrak{g}$ and write

$$\begin{aligned} \sigma_{KKS}(\mathcal{L}_{X_\psi} \bar{\omega}, \mathcal{L}_{X_\xi} \bar{\omega}) &= \int_M \bar{\omega}(X_\psi, X_\xi) \mu \\ &= \int_M \left(\sum_i (\Gamma_i \delta_{r_i}) - \frac{\Gamma}{V} \right) \mu(X_\psi, X_\xi) \mu \\ &= \sum_i \Gamma_i \mu(X_\psi, X_\xi)|_{r=r_i} - \frac{\Gamma}{V} \int_M d\psi \wedge d\xi, \end{aligned}$$

applying Corollary 2.26 in the final step. The rightmost term vanishes by Stokes theorem, while the remaining term can be written as

$$\sum_i \Gamma_i \mu(X_\psi, X_\xi)|_{r=r_i} = \sum_i \Gamma_i \mu_{r_i}(X_\psi, X_\xi) = \sigma_{PV}(X_\psi, X_\xi),$$

where μ_{r_i} is interpreted as the area form on the i -th copy of M in the phase space $M^N \setminus \{\text{collision}\}$, as in (3.14).

We now apply the same argument to show that the point-vortex Hamiltonian (3.16) is the restriction of the general fluid Hamiltonian given in the previous chapter to the N -vortex coadjoint orbit. From Proposition 2.21, the fluid Hamiltonian is given in terms of scalar vorticity (i.e. using

Perspective III of Table 2.1) as

$$H(\lambda) = \frac{1}{2} \int_M \lambda \Delta^{-1}(\lambda) \mu = \frac{1}{2} \int_M \lambda \psi \mu.$$

We insert the N -vortex stream function (3.5),

$$\bar{\psi} = \sum_{i=1}^N \Gamma_i G_M(\cdot, r_i),$$

together with the N -vortex scalar vorticity

$$\bar{\lambda} = \left(\sum_i (\Gamma_i \delta_{r_i}) - \frac{\Gamma}{V} \right),$$

into this expression. This gives

$$\begin{aligned} H(\bar{\lambda}) &= \frac{1}{2} \int_M \bar{\lambda} \bar{\psi} \mu \\ &= \frac{1}{2} \int_M \left(\sum_i (\Gamma_i \delta_{r_i}) - \frac{\Gamma}{V} \right) \sum_{j=1}^N \Gamma_j G_M(\cdot, r_j) \mu \\ &= \frac{1}{2} \sum_{i,j} \Gamma_i \Gamma_j G_M(r_i, r_j) - \frac{1}{2} \sum_j \Gamma_j \Gamma \int_M G_M(\cdot, r_j) \mu. \end{aligned}$$

As Green functions are normalized to be mean-zero, the last term vanishes, leaving only

$$H(\bar{\lambda}) = \frac{1}{2} \sum_{i,j} \Gamma_i \Gamma_j G_M(r_i, r_j). \quad (3.21)$$

The term (3.21) indexes all point-vortex interactions, including the self interactions given by the Robin function. Using the symmetry of the Green function, the off-diagonal terms in this expression can be simplified as

$$\frac{1}{2} \sum_{i \neq j} \Gamma_i \Gamma_j G_M(r_i, r_j) = \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j). \quad (3.22)$$

The diagonal terms involve evaluation of the expression $G(r_i, r_i)$. Applying Remark 3.12, we regularize these terms using the Robin function to write

$$\frac{1}{2} \sum_i \Gamma_i^2 G_M(r_i, r_i) \Rightarrow \frac{1}{2} \sum_i \Gamma_i^2 R_M(r_i).$$

Thus the total kinetic energy is given as

$$H(\bar{\lambda}) = \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) + \frac{1}{2} \sum_i \Gamma_i^2 R_M(r_i). \quad (3.23)$$

From Remark 2.1, the fluid Hamiltonian is given by the negative of the total kinetic energy of the system. Flipping the sign of the above gives the point-vortex Hamiltonian (3.16).

Remark 3.18. The above tallying of point-vortex interactions is visualized as a square of interactions in Figure 3.1. The $G_M(r_i, r_j)$ terms for $i < j$ in (3.23) count the off-diagonal interactions in the shaded lower triangle of the square, while the Robin function terms $R_M(r_i)$

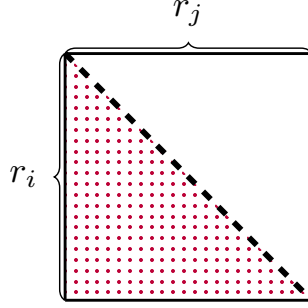


Figure 3.1: A visualization of the N -vortex Hamiltonian. The full square represents all interactions found in (3.21). The red shaded triangle represents the first term of (3.23), with $i < j$, while the dashed diagonal represents the second term of (3.23), with $i = j$ self-interaction terms.

in (3.23) count the self-interaction terms along the dashed diagonal. The simplification (3.22) amounts to folding the square along the diagonal using the symmetry of the Green function to obtain the red triangle.

Remark 3.19. Together with Corollary 3.16, the above calculations act as a bridge between this chapter and the previous chapter. They formally justify that the finite-dimensional Hamiltonian system of point-vortices that evolves on the phase space $(\mathcal{M}, \sigma_{\text{PV}})$ may be seen as a class of singular solutions of the geometric Euler equation (2.1), with the infinite-dimensional Lie-Poisson structure in $\mathfrak{g}^*$ inducing the Hamiltonian structure of the dynamics.

Chapter 4

The Non-Orientable Case

Having developed the machinery to describe ideal hydrodynamics and point-vortex dynamics on orientable surfaces, we now turn to the case of non-orientable surfaces. We begin our discussion by reviewing the relevant background on non-orientable surfaces, focusing on their orientation double covers. We then discuss the extent to which the geometric formulation of ideal hydrodynamics given in Chapter 2 carries over to non-orientable surfaces. We focus on the use of the orientation cover of a non-orientable surface as a convenient setting to describe its fluid flow.

After developing the case of general fluids, we examine point-vortex dynamics. Following [24], we review how point-vortex dynamics can be given a Hamiltonian formulation on a non-orientable surface M by considering the motion of vortex dipoles on the orientation cover $\widetilde{M}$. We provide an explicit reduction showing how this dipole formulation produces the correct Hamiltonian structure. We conclude by listing the Hamiltonians for point-vortex dynamics on well-known non-orientable surfaces.

Our discussion of general ideal fluid motion on non-orientable surfaces follows the recent publications [17] and [30]. Our discussion of point-vortex dynamics closely follows [24], the first work to offer a Hamiltonian framework for point-vortex dynamics on non-orientable surfaces.

4.1 Non-Orientable Surfaces

Before discussing fluids, we review some preliminary material on non-orientable surfaces. The central object of our discussion is the following.

Definition 4.1. *The **orientation cover** of a Riemannian manifold (M, g) is*

$$\widetilde{M} = \{(p, O_p) \mid p \in M \text{ and } O_p \text{ is an orientation of } T_p M\}.$$

The orientation cover possesses a natural fixed-point free involution $\tau : \widetilde{M} \rightarrow \widetilde{M}$,

$$\tau : (p, O_p) \mapsto (p, -O_p),$$

and a projection $\pi : \widetilde{M} \rightarrow M$,

$$\pi : (p, O_p) \mapsto p.$$

As there are only two possible orientations at each point, it is clear that $\widetilde{M}$ is a double cover of M . Indeed, we can recover M as the quotient

$$M \cong \widetilde{M}/\mathbb{Z}_2$$

under the relation generated by the involution τ . We report the following standard properties of $\widetilde{M}$, proved in [19].

Proposition 4.2. *The orientation covering $\widetilde{M}$ satisfies the following properties.*

1. *Under the projection π , the metric g on M pulls back to a metric $\widetilde{g}$ on $\widetilde{M}$.*
2. *τ is an isometry under $\widetilde{g}$.*
3. *The orientation cover $\widetilde{M}$ has a canonical orientation given as follows: At $(p, \mathcal{O}_p) \in \widetilde{M}$ a basis of $T_{(p, \mathcal{O}_p)}\widetilde{M}$ is oriented if its projection to $T_p M$ is oriented under the orientation $\mathcal{O}_p$.*
4. *The canonical orientation on $\widetilde{M}$ corresponds to the area form $\widetilde{\mu}$ associated to $\widetilde{g}$.*
5. *$\widetilde{M}$ is connected if and only if M is non-orientable.*

Note that in $\dim M = 2$, property 3 means that the orientation cover is a symplectic manifold $(\widetilde{M}, \widetilde{\mu})$ under its canonical area form $\widetilde{\mu}$.

Proposition 4.3. *In dimension two, the map τ is anti-symplectic on the symplectic manifold $(\widetilde{M}, \widetilde{\mu})$. That is,*

$$\tau^* \widetilde{\mu} = -\widetilde{\mu}.$$

Proof. As τ is an isometry, we need only calculate whether it is orientation-preserving or orientation-reversing. To this end, fix $(p, \mathcal{O}_p) = \widetilde{p} \in \widetilde{M}$ and consider an oriented, orthonormal basis for the tangent fiber $e_1, e_2 \in T_{\widetilde{p}}\widetilde{M}$. By the definition of the orientation on $\widetilde{M}$, this means that the projected basis $(T_{\widetilde{p}}\pi)e_1, (T_{\widetilde{p}}\pi)e_2 \in T_p M$ is an oriented basis for $T_p M$ under the orientation $\mathcal{O}_p$. The image of $\{e_1, e_2\}$ under $T_{\widetilde{p}}\tau$ is $\{(T_{\widetilde{p}}\tau)e_1, (T_{\widetilde{p}}\tau)e_2\}$. Projecting by π , we see that this basis satisfies

$$\{T_{\tau(\widetilde{p})}\pi(T_{\widetilde{p}}\tau)e_1, T_{\tau(\widetilde{p})}\pi(T_{\widetilde{p}}\tau)e_2\} = \{T_{\widetilde{p}}(\pi \circ \tau)e_1, T_{\widetilde{p}}(\pi \circ \tau)e_2\} = \{(T_{\widetilde{p}}\pi)e_1, (T_{\widetilde{p}}\pi)e_2\}.$$

By the definition of the orientation of $\widetilde{M}$, this is not an oriented basis for $T_{\tau(\widetilde{p})}\widetilde{M}$. So τ is orientation reversing. $\square$

Example 4.4. (The sphere) If $M = S^2$, the two possible orientations on M correspond to a choice of the exterior or interior pointing normal vector $\pm \hat{n}$. Thus,

$$\widetilde{S}^2 = \{(p, \pm \hat{n}) \mid p \in S^2\} \cong S_+^2 \sqcup S_-^2 \cong S^2 \times \mathbb{Z}_2,$$

with $\pi : S^2 \times \mathbb{Z}_2 \rightarrow S^2$ being projection onto the first slot. If we denote the symplectic form on S^2 associated with the normal $\pm \hat{n}$ as $\mu_{\pm}$, then the canonical area form on $\widetilde{S}^2$ satisfies

$$\widetilde{\mu}|_{S_+^2} = \mu_+, \quad \widetilde{\mu}|_{S_-^2} = \mu_-.$$

The involution τ swaps between the two copies of S^2 , mapping $(p, \pm \hat{n})$ to $(p, \mp \hat{n})$. This means that

$$\tau^* \widetilde{\mu}|_{S_+^2} = \mu_- \quad \text{and} \quad \tau^* \widetilde{\mu}|_{S_-^2} = \mu_+,$$

so $\tau^* \mu = -\mu$, in accordance with the above proposition.

Remark 4.5. Note that the orientation on $\widetilde{S}^2$ in the above example does not depend on the chosen orientation of $M = S^2$. The orientation of the orientation cover in general comes only from the **Riemannian density** associated to the metric g on M . While an orientable surface such as S^2 has two possible area forms compatible with its metric, both induce the same density on S^2 . For more information on densities, see [19], Chapter 16.

Remark 4.6. The above example generalizes to all orientable surfaces. In particular, if M is orientable, then

$$\widetilde{M} \cong M_+ \sqcup M_-,$$

with area form $\widetilde{\mu}$ defined by

$$\widetilde{\mu}|_{M_+} = \mu_+ \text{ and } \widetilde{\mu}|_{M_-} = \mu_-,$$

where μ_+ and μ_- are the two area forms on M associated to¹ the metric g . The involution τ interchanges the connected components of $\widetilde{M}$.

We now present the orientation covers of the three most common non-orientable surface: the real projective plane, the Möbius strip, and the Klein bottle. In contrast with the above example, and in accordance with item 5 of Proposition 4.2, these surfaces have orientation covers that are connected.

Example 4.7. The orientation cover of the real projective plane $M = \mathbb{RP}^2$ is the sphere S^2 . The involution τ is the antipodal map $p \mapsto -p$, and π is the natural quotient projection

$$\pi : S^2 \rightarrow \mathbb{RP}^2 \cong S^2 / \sim,$$

given by equivalence under τ . As τ is a reflection involution, it is anti-symplectic. In particular, regardless of how $\widetilde{M} = S^2$ is embedded into $\mathbb{R}^3$, it is clear that τ changes the direction of its normal vector, reversing its orientation.

Example 4.8. The orientation cover of the flat, infinite Möbius strip M of width π is the flat cylinder of width 2π ,

$$\widetilde{M} = \mathbb{R}^2 / 2\pi\mathbb{Z} = \{(x, y) \in \mathbb{R}^2 : x \sim x + 2\pi\}.$$

taken as a periodization of the plane as in Example 3.7. The involution τ is given in Cartesian coordinates as

$$\tau(x, y) = (x + \pi, -y).$$

As with the real projective plane, this is clearly orientation-reversing as it takes the form of a reflection, in this case over the horizontal axis.

Using τ , we can write M as the quotient

$$M = \widetilde{M} / \sim,$$

where $\sim$ is the equivalence relation induced by τ . While τ is fixed-point free, it stabilizes the horizontal axis. Thus, the x axis serves a privileged position as the “middle” of the infinite strip. This is in contrast with the infinite cylinder $\widetilde{M}$, which is homogeneous under vertical translations. We emphasize that our Möbius strip M is boundaryless and vertically infinite. This is in contrast with the typical rendering of a Möbius strip as a twisted band with finite vertical height.

Example 4.9. The orientation double cover of the flat Klein bottle M is the flat torus

$$\widetilde{M} = \mathbb{R}^2 / (2\pi\mathbb{Z} \times 2\pi\mathbb{Z}) = \{(x, y) \in \mathbb{R}^2 : (x, y) \sim (x + 2\pi m, y + 2\pi n), m, n \in \mathbb{Z}\}.$$

Just as the torus was taken as a vertical periodization of the cylinder in Example 3.8, the Klein bottle can be taken as a vertical periodization of the Möbius strip. They share the same involution

$$\tau(x, y) = (x + \pi, -y).$$

¹Note that the choice of one particular area form as positive is arbitrary and can be determined by an embedding of M in Euclidean space.

As before, M can be specified as the quotient of $\widetilde{M}$ under the τ -relation.

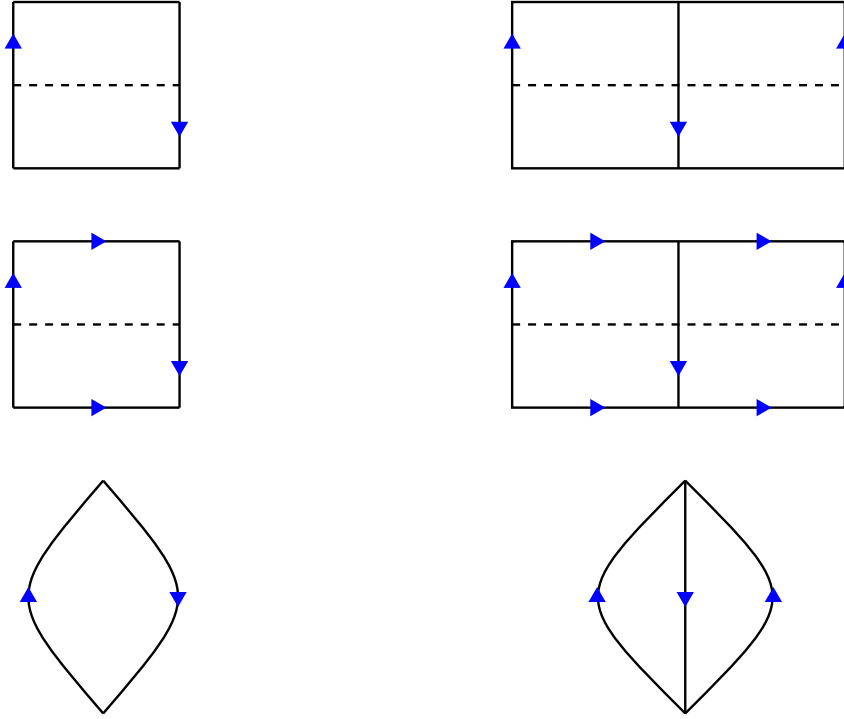


Figure 4.1: The Möbius strip, Klein bottle, and real projective plane, and their orientation covers: the cylinder, torus, and sphere.

The three examples discussed above are visualized in Figure 4.1, with the left column showing the (infinite) Möbius strip, Klein bottle, and real projective plane, and the right column showing the cylinder, torus, and sphere. Note that the sphere is viewed as a “map projection,” with the central meridian being the boundary along which the two copies of $\mathbb{RP}^2$ are glued.

Remark 4.10. The right column of Figure 4.1 shows how an orientation cover $\widetilde{M}$ can be viewed as two copies of its non-orientable partner M glued together. Since τ is orientation-reversing, these two copies of M are “mirror images” of each other. This can be seen by making a cylinder out of a paper tube. If one cuts the cylinder along a vertical line and tapes the pieces as in the top right diagram of Figure 4.1, the resulting two Möbius strips have different chiralities; one has a “left” twist and the other a “right” twist. On a mathematical level, there is no isotopy between them as embedded submanifolds of $\mathbb{R}^3$.

The involution τ induces a decomposition of sections of vector bundles on $\widetilde{M}$ over its eigenspaces. This leads to the following definitions.

Definition 4.11. A vector field $X \in \mathfrak{X}(\widetilde{M})$ is called *even* if it is invariant under τ ,

$$\tau^*X = X,$$

and *odd* if it changes sign under τ ,

$$\tau^*X = -X.$$

Definition 4.12. A k -form $\eta \in \bigwedge^k(\widetilde{M})$ is called *even* if it is invariant under τ ,

$$\tau^*\eta = \eta,$$

and **odd** if it changes sign under τ ,

$$\tau^* \eta = -\eta.$$

In the case $k = 0$, we have **even and odd functions** on $\widetilde{M}$.

We denote the space of even vector fields on $\widetilde{M}$ as

$$\mathfrak{X}^0(\widetilde{M}) = \{X \in \mathfrak{X}(\widetilde{M}) : \tau^* X = X\}$$

and the space of even k -forms on $\widetilde{M}$ as

$$\bigwedge_0^k(\widetilde{M}) = \{\eta \in \bigwedge^k(\widetilde{M}) : \tau^* \eta = \eta\}.$$

The following proposition shows that even constructions on $\widetilde{M}$ are in correspondence with standard constructions on M .

Proposition 4.13. *The space of even vector fields on $\widetilde{M}$ is isomorphic to the space of vector fields on M . Each even vector field is π -projectable, and the map*

$$\pi_* : \mathfrak{X}^0(\widetilde{M}) \rightarrow \mathfrak{X}(M)$$

is an isomorphism of vector spaces. Similarly, the space of even k -forms on $\widetilde{M}$ is isomorphic to the space of k -forms on M , with the pullback map

$$\pi^* : \bigwedge^k(M) \rightarrow \bigwedge_0^k(\widetilde{M})$$

an isomorphism of vector spaces.

Proof. This is an immediate consequence of the lifting property of covering maps applied to the double covering $\pi : \widetilde{M} \rightarrow M$, see [19]. $\square$

Remark 4.14. By definition, odd vector fields and odd k -forms on $\widetilde{M}$ do not descend to vector fields and k -forms on M . Nevertheless, there does exist an odd “partner” to the above result that gives an isomorphism between odd vector fields and k -forms on $\widetilde{M}$ and certain constructions on M . These constructions, called twisted vector fields and twisted k -forms on M , are not sections of TM and $\bigwedge^k(M)$, but instead sections of different, twisted vector bundles over M .

In general, the duality between even and odd constructions on $\widetilde{M}$ descends to duality between standard and twisted constructions on M . For clarity in our exposition, we do not explore the subject of twisted constructions on M , but instead focus on even and odd constructions on $\widetilde{M}$. For an exposition which uses twisted objects (and explores the role of the Hodge star operator in swapping between them) see [17] or [24].

When M and $\widetilde{M}$ are surfaces, the following result shows how evenness interacts with symplectic structure of $\widetilde{M}$.

Proposition 4.15. *If $\dim M = 2$, an even Hamiltonian function $\widetilde{H}$ on the symplectic manifold $(\widetilde{M}, \widetilde{\mu})$ generates an odd Hamiltonian vector field $X_{\widetilde{H}} \in \mathfrak{X}^1(\widetilde{M})$. Namely,*

$$\tau^* X_{\widetilde{H}} = -X_{\widetilde{H}}.$$

Proof. As $\widetilde{H}$ is invariant under τ , we apply Proposition 4.3 and write

$$i_{X_{\widetilde{H}}} \widetilde{\mu} = d\widetilde{H} = d(\widetilde{H} \circ \tau) = d(\tau^* \widetilde{H}) = \tau^*(d\widetilde{H}) = \tau^*(i_{X_{\widetilde{H}}} \widetilde{\mu}) = i_{\tau^* X_{\widetilde{H}}} \tau^* \widetilde{\mu} = i_{-\tau^* X_{\widetilde{H}}} \widetilde{\mu}.$$

Thus $\tau^* X_{\tilde{H}} = -X_{\tilde{H}}$. □

The correspondence between M and its orientation cover leads to two approaches to handling geometric problems on a non-orientable surface:

1. Work on M , at the cost of interchanging between standard and twisted objects (as discussed in Remark 4.14),
2. Pull back to $\tilde{M}$ to use even and odd objects, then project back down to M .

The two perspectives are of course equivalent, but we primarily adopt the second in our discussion since it is more convenient for the point-vortex case. As such, we work primarily on the orientation cover $\tilde{M}$. For an account of ideal hydrodynamics on non-orientable surfaces that explores the relationship between the two perspectives given above, see [17].

4.2 Ideal Fluids on Non-Orientable Surfaces

Having established the necessary preliminaries, we now turn to the question of ideal fluid flow on non-orientable surfaces. Our discussion is based on [15], [17], [30], and [24].

The key point is that the Euler equation (2.1) makes no reference to an orientation or area form on M , nor do the lowered expression (2.2) or the vorticity equation (2.3). These equations use only the metric on M and its induced connection, without reference to an orientation on M . Indeed, while fluid flow on a non-orientable surface may be a counterintuitive notion, it is entirely physically plausible. One might immerse a wire frame soap film bent into the shape of a Möbius band² into a soap bath and observe the evolution of the resulting film.

With some caveats, the geometric perspective on the Euler equations established in chapter 2 carries over to the non-orientable case as well. In particular perspectives I and II in Table 2.1 hold on non-orientable surfaces. While the exposition of Chapter 2 considers an oriented surface, the Lie group $G = \text{Diff}_\mu(M)$ is well-defined with respect to the Riemannian density of a non-orientable M , and the constructions of $\mathfrak{g}$ and its smooth dual $\mathfrak{g}^*$ carry over analogously. The velocity one-form $\nu = v^\flat$ is a well-defined one-form on M , and its associated vorticity $\omega = d\nu$ is a well-defined two-form. Some care must be taken, however. For example, it is not possible to integrate ω over M , since M does not possess an oriented atlas, see [30].

The primary difference with the oriented case comes in attempting to pass from perspective II to perspective III, that is in passing from the vorticity two-form ω to the vorticity scalar $\lambda = \star\omega$. Intuitively, the issue is that scalar vorticity requires a global normal vector on M to distinguish between clockwise and counter-clockwise rotation. While a point-vortex on a Möbius strip might appear to be rotating clockwise with respect to a local orientation, it may flow onward and return to the same point rotating the other way. Thus, while the magnitude of the scalar vorticity λ is fixed, its sign is not. It is thus not well-defined as a function on M .

In order to remedy this, we pass to the orientation cover $\tilde{M}$. In particular, we associate to a velocity one-form $\nu \in \Lambda^1(M)$ the lifted velocity one-form $\tilde{\nu} \in \Lambda^1(\tilde{M})$ given by $\tilde{\nu} = \pi^*\nu$. This velocity one-form has vorticity two-form $\tilde{\omega} = d\tilde{\nu}$ and scalar vorticity $\tilde{\lambda} = \star\tilde{\omega}$.

Proposition 4.16. *The velocity one-form $\tilde{\nu}$ and vorticity two-form $\tilde{\omega}$ on $\tilde{M}$ are even. That is,*

$$\tau^*\tilde{\nu} = \tilde{\nu}, \quad \tau^*\tilde{\omega} = \tilde{\omega}.$$

²Bending it into the shape of a Klein bottle may prove a more trying task.

Proof. This is a consequence of Proposition 4.13. Since $\pi \circ \tau = \pi$, we write

$$\tau^* \tilde{\nu} = \tau^* \pi^* \nu = (\pi \circ \tau)^* \nu = \pi^* \nu = \tilde{\nu}.$$

An identical argument applies to $\tilde{\omega}$. □

Corollary 4.17. *The scalar vorticity $\tilde{\lambda} \in C^\infty(\widetilde{M})$ is odd. That is,*

$$\tau^* \tilde{\lambda} = -\tilde{\lambda}.$$

Proof. Applying Proposition 4.3 together with the above, we see that

$$\tilde{\lambda} \tilde{\mu} = \tilde{\omega} = \tau^* \tilde{\omega} = \tau^* (\tilde{\lambda} \tilde{\mu}) = (\tau^* \lambda) \tau^* \tilde{\mu} = (-\tau^* \lambda) \tilde{\mu}.$$

Thus $\tau^* \tilde{\lambda} = -\tilde{\lambda}$. □

To understand the symmetries implied by the above results, we consider an example.

Example 4.18. Consider the vertical one-form ν on the flat, infinite Möbius strip M of Example 4.8 given by $\nu = \sin(x) dy$. Note that ν is π -periodic and so well-defined. Let us calculate $\tilde{\nu} = \pi^* \nu$, where $\pi : \widetilde{M} \rightarrow M$ is the covering map induced by the involution $\tau(x, y) = (x + \pi, -y)$ on the cylinder $\widetilde{M}$. For $0 \leq x \leq \pi$, this lift is

$$\tilde{\nu} = \pi^* \sin(x) dy = \sin(x) dy.$$

For $\pi \leq x \leq 2\pi$, the calculation become

$$\tilde{\nu} = \pi^* \sin(x) \pi^{-1}(dy) = \sin(x - \pi)(-dy) = \sin(x) dy.$$

Hence the lift is globally given by

$$\tilde{\nu} = \pi^* \nu = \sin(x) dy.$$

At the level of vector fields, this shows that the vector field $\tilde{\nu}^\# = \sin x \frac{\partial}{\partial y}$ on $\widetilde{M}$ is even. This vector field is visualized in the right portion of Figure 4.2, while its projection to M is visualized in the left portion. Via a similar calculation to the above, the reader is invited to check that the one-form $\tilde{\nu} = \sin^2(x) dy$ is odd.



Figure 4.2: An even vector field on the cylinder. Note that the blue arrows show the identification of the boundaries of the Möbius strip in M and $\widetilde{M}$, while the black arrows show the vector fields

Thus, we see that in an orientation covering $\pi : \widetilde{M} \rightarrow M$, fluid configurations on M lift to τ -symmetric configurations on $\widetilde{M}$. In order to work exclusively on the orientation covering $\widetilde{M}$, we must ensure that such symmetric configurations are stable and do not evolve into non-symmetric configurations under motion of the fluid. The following two propositions from [17] guarantee the stability of even fluid configurations on $\widetilde{M}$.

Proposition 4.19. *Let (M, g) be a non-orientable Riemannian surface with Riemannian density μ . Let $\pi : \widetilde{M} \rightarrow M$ be its orientation cover, with area form $\widetilde{\mu}$. The space of **even area-preserving diffeomorphisms** of $\widetilde{M}$, denoted*

$$\text{Diff}_{\widetilde{\mu}}^0(\widetilde{M}) = \{\phi \in \text{Diff}_{\widetilde{\mu}}(\widetilde{M}) : \tau \circ \phi = \phi \circ \tau\}$$

*is a subgroup of $\text{Diff}_{\widetilde{\mu}}(\widetilde{M})$. Its Lie algebra is the space of **even divergence-free vector fields** on $\widetilde{M}$,*

$$\mathfrak{X}_{\widetilde{\mu}}^0(\widetilde{M}) = \mathfrak{X}_{\widetilde{\mu}}(\widetilde{M}) \cap \mathfrak{X}^0(\widetilde{M}) = \{X \in \mathfrak{X}(\widetilde{M}) : \mathcal{L}_X \widetilde{\mu} = 0 \text{ and } \tau^* X = X\}.$$

Proposition 4.20. *There exists a Lie group isomorphism*

$$F : \text{Diff}_{\widetilde{\mu}}^0(\widetilde{M}) \longrightarrow \text{Diff}_{\mu}(M).$$

This map F induces an isomorphism of Lie algebras

$$TF : \mathfrak{X}_{\widetilde{\mu}}^0(\widetilde{M}) \rightarrow \mathfrak{X}_{\mu}(M).$$

In ideal Euler flow on the non-orientable surface M , the (coset of the) velocity one-form ν_0 describing the fluid evolves according to the Euler-Poisson equation (2.10). In particular, up to an exact form, ν_0 evolves as

$$\nu_0 \mapsto \nu_t = (\phi_t)^* \nu_0,$$

where $t \mapsto \phi_t \in \text{Diff}_{\mu}(M)$ is the motion in $G = \text{Diff}_{\mu}(M)$ discussed in Chapter 2, Section 1.

Under the projection π , the above evolution equation pulls back to

$$\widetilde{\nu}_0 \mapsto \widetilde{\nu}_t = (\widetilde{\phi}_t)^* \widetilde{\nu}_0,$$

where $\widetilde{\phi}_t$ satisfies $\pi \circ \widetilde{\phi}_t = \phi_t \circ \pi$. Clearly $\widetilde{\phi}_t$ is an even area-preserving diffeomorphism of $\widetilde{M}$. We verify that $\widetilde{\nu}_t$ is an even fluid configuration for all t by writing

$$\tau^* \widetilde{\nu}_t = \tau^* ((\widetilde{\phi}_t)^* \widetilde{\nu}_0) = (\widetilde{\phi}_t \circ \tau)^* \widetilde{\nu}_0 = (\tau \circ \widetilde{\phi}_t)^* \widetilde{\nu}_0 = (\widetilde{\phi}_t)^* (\tau^* \widetilde{\nu}_0) = (\widetilde{\phi}_t)^* \widetilde{\nu}_0 = \widetilde{\nu}_t.$$

Thus $\tau^* \widetilde{\nu}_t = \widetilde{\nu}_t$, showing that the lifted velocity one-form remains even for all time.

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\phi}_t} & \widetilde{M} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\phi_t} & M \end{array}$$

The correspondence is summarized in the above diagram. To determine the evolution of a particle in the fluid M , one may equivalently follow its trajectory on M directly or the two trajectories of its preimage on $\widetilde{M}$ under the double covering $\pi : \widetilde{M} \rightarrow M$. Since the lifted flow $\widetilde{\phi}_t$ is an even diffeomorphism for all time, the two preimage particles move synchronously and align when projected back down to M .

Remark 4.21. As noted in [24], applying this correspondence as written generates a configuration of fluid on $\widetilde{M}$ with twice as much fluid as the original configuration on M . In order to compensate for this, it is necessary to divide the total fluid energy, and hence the velocity and vorticity, by two in order to maintain conservation of energy in the map between M and $\widetilde{M}$.

4.3 Point-Vortices on Non-Orientable Surfaces

We now examine the Hamiltonian structure of point-vortices on non-orientable surfaces. Our discussion and notation closely follow [24], which first examined the Hamiltonian structure of point-vortex dynamics on non-orientable surfaces.

Reduction of Vortex Dipoles

Let M be a non-orientable surface with orientation cover $\pi : \widetilde{M} \rightarrow M$. Taking into account the discussion of the previous section, we see that a single point-vortex placed at $r_i \in M$ with circulation Γ_i lifts to a *pair* of point-vortices on $\widetilde{M}$ which, by Corollary 4.17, have equal and opposite circulations.

Definition 4.22. Let $\pi : \widetilde{M} \rightarrow M$ be an orientation covering and $\tilde{r}_i \in \widetilde{M}$. A **vortex dipole** of strength $\Gamma_i \in \mathbb{R}$ at $r_i, \tau(\tilde{r}_i) \in \widetilde{M}$ is a configuration of point-vortices with vorticity two-form

$$\tilde{\omega} = \frac{1}{2} \left(\Gamma_i \delta_{\tilde{r}_i} - \Gamma_i \delta_{\tau(\tilde{r}_i)} \right) \tilde{\mu}.$$

Remark 4.23. When considering a vortex dipole as the pullback of a point-vortex at position $r_i \in M$, there exists a freedom in how to label the two points in $\pi^{-1}(r_i)$. Following [24], we adopt a preferred lift $\tilde{r}_i \in \pi^{-1}(r_i)$ and denote the preimage of r_i as

$$\pi^{-1}(r_i) = \{\tilde{r}_i, \tau(\tilde{r}_i)\}.$$

Such a choice of preferred lift does not affect the Hamiltonian formulation of dipole dynamics.

From the discussion of the preceding section, it is clear that a configuration of vortex dipoles placed on $\widetilde{M}$ remains a configuration of vortex dipoles through the flow of the fluid. So while dipole motion describes $2N$ vortices on $\widetilde{M}$, it is parametrized by N point-vortices on $\widetilde{M}$. As the motion of N dipoles on $\widetilde{M}$ corresponds to the motion of N point-vortices on M , this suggests that the phase space for N -vortex motion on M is $\widetilde{M}^N$ (minus an appropriate diagonal to account for collisions).

We now present a formal argument to show this reduction. In particular, we show that the set of N dipole configurations represents a symplectic submanifold of $\widetilde{M}^{2N}$ that is invariant under the flow generated by the point-vortex Hamiltonian on $\widetilde{M}^{2N}$. We show this by demonstrating that the set of dipoles is the fixed set of a certain involution.

The Phase Space

Consider the motion of $2N$ point-vortices on $\widetilde{M}$. For simplicity, we omit the consideration of the diagonal collision set until the end of our argument. The phase space of $2N$ -vortex motion is then

$$\widetilde{M}^{2N} = \overbrace{\widetilde{M} \times \cdots \times \widetilde{M}}^{2N},$$

with symplectic form

$$\sigma_{\text{PV}} = \sum_i^{2N} \Gamma_i \tilde{\mu}_i.$$

If the point-vortices have equal and opposite circulations, $\Gamma_i = -\Gamma_{N+i}$, we may pair them by their circulations and write the phase space as

$$\widetilde{M}^{2N} = \overbrace{\widetilde{M}^2 \times \cdots \times \widetilde{M}^2}^N.$$

After pairing, the symplectic form may be written as

$$\sigma_{\text{PV}} = \sum_i^N \Gamma_i \sigma_{\text{pair}},$$

where σ_{pair} is the symplectic structure on $\widetilde{M}^2$ given by

$$\sigma_{\text{pair}} = \frac{1}{2}(\pi_1^* \widetilde{\mu} - \pi_2^* \widetilde{\mu}),$$

where $\pi_i : \widetilde{M}^2 \rightarrow \widetilde{M}$ for $i \in \{1, 2\}$ is projection of the pair onto the i th factor.

Let us show that the space of dipole configurations

$$D := \{((\widetilde{r}_1, \tau(\widetilde{r}_1)), \dots, (\widetilde{r}_N, \tau(\widetilde{r}_N))) \in \widetilde{M}^{2N} : (\widetilde{r}_1, \dots, \widetilde{r}_N) \in \widetilde{M}^N\} \quad (4.1)$$

is an invariant, symplectic submanifold of $(\widetilde{M}^{2N}, \sigma_{\text{PV}})$. For simplicity, we focus on a single vortex pair with phase space $(\widetilde{M}^2, \sigma_{\text{pair}})$. Our arguments carry over naturally to the larger phase space $(\widetilde{M}^{2N}, \sigma_{\text{PV}})$.

Note that the map τ extends diagonally from $(\widetilde{M}, \widetilde{\mu})$ to $(\widetilde{M}^2, \sigma_{\text{pair}})$, where it is still anti-symplectic. The vortex pair also possesses another involution: the swap $\rho : \widetilde{M}^2 \rightarrow \widetilde{M}^2$ given by

$$\rho(\widetilde{r}, \widetilde{s}) = (\widetilde{s}, \widetilde{r}).$$

Proposition 4.24. *The swap involution ρ is anti-symplectic on the point-vortex pair $(\widetilde{M}^2, \sigma_{\text{pair}})$.*

Proof. Since $\pi_1 = \pi_2 \circ \rho$, it holds that $\rho^* \pi_1^* = \pi_2^*$ and $\rho^* \pi_2^* = \pi_1^*$. Thus,

$$\rho^* \sigma_{\text{pair}} = \rho^* \frac{1}{2}(\pi_1^* \widetilde{\mu} - \pi_2^* \widetilde{\mu}) = \frac{1}{2}(\pi_2^* \widetilde{\mu} - \pi_1^* \widetilde{\mu}) = -\sigma_{\text{pair}}.$$

□

By composing τ and ρ , we obtain a third involution $\chi = \tau \circ \rho$, which acts as

$$\chi(\widetilde{r}, \widetilde{s}) = (\tau(\widetilde{s}), \tau(\widetilde{r})).$$

As the composition of anti-symplectic maps, χ is symplectic. The fixed-point set of χ is the subset of dipoles within $\widetilde{M}^2$,

$$F = \{(\widetilde{r}, \tau(\widetilde{r})) \in \widetilde{M}^2 : \widetilde{r} \in \widetilde{M}\}.$$

As the fixed-point set of an involution, F is a smooth submanifold of $\widetilde{M}^2$. The following result shows it is a symplectic submanifold.

Proposition 4.25. *On $\widetilde{M}^2$, the restriction of σ_{pair} to F is a symplectic form. The resulting symplectic manifold $(F, \sigma_{\text{pair}}|_F)$ is symplectomorphic to $(\widetilde{M}, \widetilde{\mu})$.*

Proof. With $i : F \rightarrow \widetilde{M}^2$ the inclusion map, we calculate

$$i^* \sigma_{\text{pair}} = i^* \frac{1}{2}(\pi_1^* \widetilde{\mu} - \pi_2^* \widetilde{\mu}) = \frac{1}{2}((\pi_1 \circ i)^* \widetilde{\mu} - (\pi_2 \circ i)^* \widetilde{\mu}).$$

On F , it holds that $\pi_2 \circ i = \tau \circ \pi_1 \circ i$, so we may rewrite the above as

$$i^* \sigma_{\text{pair}} = \frac{1}{2} ((\pi_1 \circ i)^* \tilde{\mu} - (\tau \circ \pi_1 \circ i)^* \tilde{\mu}) = \frac{1}{2} ((\pi_1 \circ i)^* \tilde{\mu} - (\pi_1 \circ i)^* \tau^* \tilde{\mu}) = (\pi_1 \circ i)^* \tilde{\mu}.$$

Since the map

$$\pi_1 \circ i : F \rightarrow \widetilde{M}$$

is clearly a diffeomorphism, the result follows. $\square$

Proposition 4.26. *Let $\tilde{H} \in C^\infty(\widetilde{M}^2)$ be a Hamiltonian function invariant under χ . Then the flow of $X_{\tilde{H}} \in \mathfrak{X}(\widetilde{M}^2)$ on $(\widetilde{M}^2, \sigma_{\text{pair}})$ restricts to Hamiltonian flow on $(F, \sigma_{\text{pair}}|_F)$.*

Proof. Since $\chi : \widetilde{M}^2 \rightarrow \widetilde{M}^2$ is a symplectomorphism and $\tilde{H}$ is χ -invariant, it holds

$$\chi^* X_{\tilde{H}} = X_{\chi^* \tilde{H}} = X_{\tilde{H}}.$$

This implies that χ commutes with the flow of $X_{\tilde{H}}$

$$\phi_t^{X_{\tilde{H}}} \circ \chi = \chi \circ \phi_t^{X_{\tilde{H}}}.$$

This in turn means that $F = \text{Fix } \chi$ is invariant under the flow of $X_{\tilde{H}}$, since for $\tilde{r} \in F$,

$$\chi(\phi_t^{X_{\tilde{H}}}(\tilde{r})) = \phi_t^{X_{\tilde{H}}}(\chi(\tilde{r})) = \phi_t^{X_{\tilde{H}}}(\tilde{r}),$$

so F is invariant under the flow. Infinitesimally, this means that $X_{\tilde{H}}$ is tangent to F , and so restricts to a well-defined vector field $X_{\tilde{H}}|_F \in \mathfrak{X}(F)$. The Hamiltonian property follows from the previous proposition. $\square$

The above result shows exactly what we wished for the case of one vortex dipole: the space of dipoles F within $\widetilde{M}^2$ is a symplectic submanifold invariant under the flow of certain Hamiltonians.

The generalization to N dipoles is immediate. We may extend the involution χ pairwise to each of the N dipoles in

$$\widetilde{M}^{2N} = \overbrace{\widetilde{M} \times \cdots \times \widetilde{M}}^{2N}.$$

The fixed point set of the lifted χ map is exactly the space of dipoles

$$D = \{((\tilde{r}_1, \tau(\tilde{r}_1)), \dots, (\tilde{r}_N, \tau(\tilde{r}_N))) \in \widetilde{M}^{2N} : (\tilde{r}_1, \dots, \tilde{r}_N) \in \widetilde{M}^N\}.$$

Applying the argument of Proposition 4.25 to σ_{PV} and accounting for the constant Γ_i , we see that $(D, \sigma_{\text{pair}}|_D)$ is a symplectic submanifold of $\widetilde{M}^{2N}$ that is isomorphic to $(\widetilde{M}^N, \sum_i^N \Gamma_i \tilde{\mu}_i)$. We use this isomorphism to identify D with this space, writing

$$(D, \sigma_{\text{Dipole}}) \cong (\widetilde{M}^N, \sum_i^N \Gamma_i \tilde{\mu}_i).$$

Proposition 4.26 then extends to the following result.

Proposition 4.27. *Let $\tilde{H} \in C^\infty(\widetilde{M}^{2N})$ be a Hamiltonian function invariant under the pairwise lifted involution $\chi : \widetilde{M}^{2N} \rightarrow \widetilde{M}^{2N}$. Then the flow of $X_{\tilde{H}} \in \mathfrak{X}(\widetilde{M}^{2N})$ on $(\widetilde{M}^{2N}, \sigma_{\text{PV}})$ restricts to Hamiltonian flow on $(D, \sigma_{\text{Dipole}})$.*

This almost concludes the argument that the dynamics of N vortex dipoles on $\widetilde{M}^{2N}$ may be reduced to the space of dipoles $\widetilde{M}^N$. It remains only to show that the Hamiltonian function for vortex dipole dynamics respects the above symmetry.

The Hamiltonian

By direct calculation, the Hamiltonian of N vortex dipoles with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ on $\widetilde{M}$ is found to be the unwieldy expression

$$\begin{aligned} \widetilde{H}(\widetilde{r}_1, \dots, \widetilde{r}_N) = & -\frac{1}{2} \sum_{i < j} \Gamma_i \Gamma_j G_{\widetilde{M}}(\widetilde{r}_i, \widetilde{r}_j) - \frac{1}{4} \sum_j \Gamma_j^2 R_{\widetilde{M}}(\widetilde{r}_j) \\ & - \frac{1}{2} \sum_{i < j} \Gamma_i \Gamma_j G_{\widetilde{M}}(\tau(\widetilde{r}_i), \tau(\widetilde{r}_j)) - \frac{1}{4} \sum_j \Gamma_j^2 R_{\widetilde{M}}(\tau(\widetilde{r}_j)) \\ & + \frac{1}{2} \sum_{i,j} \Gamma_i \Gamma_j G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_j)). \end{aligned} \quad (4.2)$$

Remark 4.28. Note that this map differs from the standard point-vortex Hamiltonian (3.16) in two ways. Firstly, it is divided by two in agreement with Remark 4.21. Secondly, it has domain

$$\widetilde{M}^N \setminus \{\text{collision}\} = \{(\widetilde{r}_1, \dots, \widetilde{r}_N) \in (\widetilde{M})^N : \pi(\widetilde{r}_i) \neq \pi(\widetilde{r}_j) \text{ for any } i \neq j\}$$

with the expanded collision set. The expanded collision set is the preimage of $M^N \setminus \{\text{collision}\}$ under π .

The expression (4.2) counts all possible interactions arising from the dipoles: the first line counts interactions of the $\widetilde{r}_i$ vortices with each other including self-energy interactions, the second line counts interactions of the $\tau(\widetilde{r}_i)$ vortices with each other including self-energy interactions, and the final line counts all interactions between the $\widetilde{r}_i$ vortices and the $\tau(\widetilde{r}_i)$ vortices, including when $i = j$.

Since τ is an isometry of $\widetilde{M}$, it follows that both $G_{\widetilde{M}}$ and $R_{\widetilde{M}}$ are invariant under τ . As such, the above Hamiltonian is invariant under (the lifted action of) τ , and so descends to a standard function H on $M^N \setminus \{\text{collision}\}$. This Hamiltonian is the unique function satisfying $\widetilde{H} = \pi^* H$. As shown in [24], the Hamiltonian H on M can be expressed using the following forms of the Green function and Robin function for the Laplacian on M .

Definition 4.29. With $\pi : \widetilde{M} \rightarrow M$ an orientation cover of the Riemannian surface (M, g) , the **Green function on M** satisfies

$$G_M(s, r) = G_{\widetilde{M}}(\widetilde{s}, \widetilde{r}) - G_{\widetilde{M}}(\widetilde{s}, \tau(\widetilde{r})). \quad (4.3)$$

For fixed s or fixed r , G_M pulls back to an odd function on $\widetilde{M}$. The **Robin function on M** is

$$R_M(x) = R_{\widetilde{M}}(\widetilde{x}) - G_{\widetilde{M}}(\widetilde{x}, \tau(\widetilde{x})). \quad (4.4)$$

R_M pulls back to an even function on $\widetilde{M}$.

Remark 4.30. It is shown in [24] that these expressions are independent of the chosen lift and that they verify the expected properties: G_M solves the Laplace equation and R_M is the Robin function of G_M , as given in Definition 3.11.

Proposition 4.31. The Hamiltonian function $H \in C^\infty(M^N \setminus \{\text{collision}\})$ satisfying $\widetilde{H} = \pi^* H$

for (4.2) may be written as

$$H(r_1, \dots, r_N) = - \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) - \frac{1}{2} \sum_j \Gamma_j^2 R_M(r_j). \quad (4.5)$$

Proof. This proof amounts to bean-counting of the interaction terms in (4.2), visualized in Figure 4.3. Since both $G_{\widetilde{M}}$ and $R_{\widetilde{M}}$ are invariant under τ , it is clear that the first two lines of (4.2) are equal, and sum to

$$- \sum_{i < j} \Gamma_i \Gamma_j G_{\widetilde{M}}(\widetilde{r}_i, \widetilde{r}_j) - \frac{1}{2} \sum_j \Gamma_j^2 R_{\widetilde{M}}(\widetilde{r}_j). \quad (4.6)$$

The remaining term of the third line can be written as

$$\frac{1}{2} \sum_{i,j} \Gamma_i \Gamma_j G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_j)) = \sum_{i < j} \Gamma_i \Gamma_j G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_j)) + \frac{1}{2} \sum_i \Gamma_i^2 G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_i)), \quad (4.7)$$

using the symmetry $G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_j)) = G_{\widetilde{M}}(\tau(\widetilde{r}_i), \widetilde{r}_j)$. Adding the above expressions for the sum of the first two lines and the third line and then regrouping gives the desired expression. $\square$

Remark 4.32. In Figure 4.3, the large, multi-colored triangle in the left subfigure represents all the point-vortex interactions present in (4.2). Simplifying the first two lines of (4.2) amounts to recognizing that the two smaller red triangles of the large triangle are equal and summing them. Simplifying the third line shows that the two smaller blue triangles are equal and sums them. This leaves two triangles: the red triangle represents the typical point-vortex interaction $G_{\widetilde{M}}(\widetilde{r}_i, \widetilde{r}_j)$, with its dashed hypotenuse representing standard self-interaction terms $R_{\widetilde{M}}(\widetilde{r})$. The blue triangle represents new point-vortex interactions $G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_j))$ resulting from non-orientability, with its hypotenuse representing the new type of self-interaction $G_{\widetilde{M}}(\widetilde{r}_i, \tau(\widetilde{r}_i))$. Equations (4.3) and (4.4) account for both of these triangles with their two terms. Summing over the red and blue triangle produces the third purple triangle, which represents (4.5).

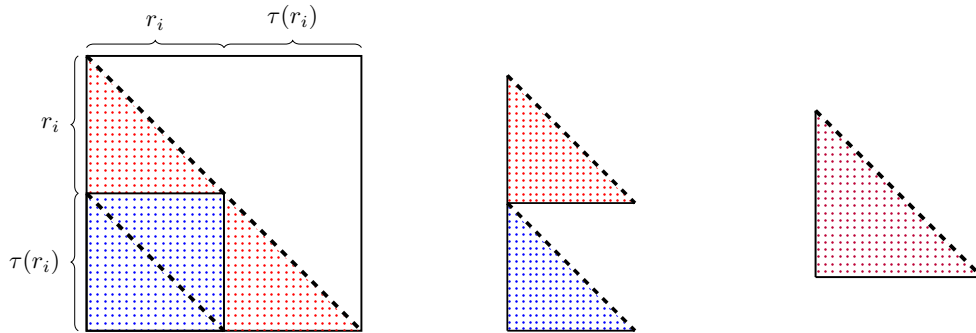


Figure 4.3: Three stages of simplification of the point-vortex Hamiltonian on a non-orientable surface. The left square represents all the terms in (4.2). The middle visual represents the simplification of these terms into the red triangle (4.6) and the blue triangle (4.7). The rightmost purple triangle represents the final simplification as (4.5).

Corollary 4.33. *The phase space for N point-vortices with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ on a non-orientable surface M is the symplectic manifold*

$$(\widetilde{M}^N \setminus \{\text{collision}\}, \sum_i^N \Gamma_i \widetilde{\mu}_i).$$

Proof. The expression (4.5) guarantees that the N -vortex Hamiltonian is a standard function on $M^N \setminus \{\text{collision}\}$. As such it pulls back to an even function on $\widetilde{M}^{2N} \setminus \{\text{collision}\}$. Even functions are invariant under the lifted action of χ on this space. Moreover, the removal of the collision set does not impact the above arguments. As such, we may apply Proposition 4.27 to reduce the system to

$$(D \setminus \{\text{collision}\}, \sigma_{\text{Dipole}}) \cong (\widetilde{M}^N \setminus \{\text{collision}\}, \sum_i^N \Gamma_i \widetilde{\mu}_i).$$

□

Hamiltonian Structure

Collecting the results of the above discussion, we obtain the following description of the Hamiltonian structure of point-vortex motion on a non-orientable Riemannian surface (M, g) .

Theorem 4.34. *Let (M, g) be a non-orientable Riemannian surface with orientation cover $\pi : \widetilde{M} \rightarrow M$. The motion of N point-vortices with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R} \setminus \{0\}$ on M is a Hamiltonian system, with phase space the symplectic manifold*

$$\left(\widetilde{M}^N \setminus \{\text{collision}\}, \sum_{i=1}^N \Gamma_i \widetilde{\mu}_i \right),$$

and Hamiltonian

$$H(r_1, \dots, r_N) = - \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) - \frac{1}{2} \sum_{i=1}^N \Gamma_i^2 R_M(r_i),$$

where G_M and R_M are the Green function and Robin function on M of Definition (4.29). This is equivalent to the Hamiltonian dynamics of N vortex dipoles on the orientation cover $\widetilde{M}$.

Through the above result, we see that non-orientability induces two key changes in the Hamiltonian formulation of point-vortex dynamics

1. The phase space is given in terms of the orientation cover $\widetilde{M}$ rather than the surface M ,
2. There exist new point-vortex interaction terms in G_M and R_M , as seen in Definition 4.29.

It is the second change that has the greatest effect on the dynamics of the point-vortex system. In particular, the additional self-interaction term present in the Robin function given in (4.4) leads to a new form of self-motion, occurring not as a result of boundaries or variable curvature, but of non-orientability. This leads to an effect quite similar to that of the image vortices discussed in Example 3.9 of the previous chapter. Indeed, the above formulation of point-vortex dynamics on M as vortex dipole dynamics on $\widetilde{M}$ can be understood as an application of the image method.

4.3.1 Examples

We conclude our discussion by providing explicit forms for the Hamiltonians for N -vortex motion on the real projective plane and the Möbius strip.

Example 4.35. (The real projective plane) Recall from Example 3.14 that the N -vortex Hamiltonian on the sphere is

$$H(r_1, \dots, r_N) = - \frac{1}{4\pi} \sum_{i < j} \Gamma_i \Gamma_j \log(1 - r_i \cdot r_j). \quad (4.8)$$

Inserting N dipoles at positions $r_1, -r_1, \dots, r_N, -r_N \in S^2$ into the above and simplifying yields

$$H(r_1, \dots, r_N) = -\frac{1}{4\pi} \sum_{i < j} \Gamma_i \Gamma_j \log \left(\frac{1 - r_i \cdot r_j}{1 + r_i \cdot r_j} \right). \quad (4.9)$$

By Theorem (4.34), this is the Hamiltonian of N -vortex motion on the real projective plane $\mathbb{RP}^2$.

Example 4.36. (The Möbius strip) Recall from Example 3.7 that the N -vortex Hamiltonian on the flat cylinder $\mathbb{R}^2/2\pi\mathbb{Z}$ is

$$H(r_1, \dots, r_N) = -\frac{1}{4\pi} \sum_{i < j} \Gamma_i \Gamma_j \log \left[\sin^2 \left(\frac{x_i - x_j}{4\pi} \right) + \sinh^2 \left(\frac{y_i - y_j}{4\pi} \right) \right]. \quad (4.10)$$

From Example 4.8, the involution on the cylinder that generates the Möbius strip is given by $\tau(x, y) = (x + \pi, -y)$. As shown in [24], placing N dipoles at positions $r_1, \tau(r_1), \dots, r_N, \tau(r_N) \in \mathbb{R}^2/2\pi\mathbb{Z}$ and simplifying yields

$$\begin{aligned} H(r_1, \dots, r_N) = & -\frac{1}{4\pi} \sum_{i < l} \Gamma_i \Gamma_j \log \left(\sin^2 \left(\frac{x_i - x_j}{2r} \right) + \sinh^2 \left(\frac{y_i - y_j}{2r} \right) \right) \\ & + \frac{1}{4\pi} \sum_{i < l} \Gamma_i \Gamma_j \log \left(\cos^2 \left(\frac{x_i - x_j}{2r} \right) + \sinh^2 \left(\frac{y_i + y_j}{2r} \right) \right) \\ & + \frac{1}{8\pi} \sum_i \Gamma_i^2 \log \left(1 + \sinh^2 \left(\frac{y_i}{r} \right) \right). \end{aligned} \quad (4.11)$$

By Theorem (4.34), this is the Hamiltonian of N -vortex motion on the Möbius strip. For an involved discussion of the resulting dynamics, see [24].

Remark 4.37. Note that the Hamiltonian for the real projective plane (4.9) does not display any self-interaction terms, while the Hamiltonian for the Möbius strip does. The final term of (4.11)

$$\frac{1}{8\pi} \sum_i \Gamma_i^2 \log \left(1 + \sinh^2 \left(\frac{y_i}{r} \right) \right)$$

is a self-interaction term arising from the non-orientability of the Möbius strip, as discussed in the previous section.

As a result of this non-trivial self-interaction, a lone point-vortex on the Möbius strip will move if it is not placed along the line $y = 0$. This is in contrast with the case of the cylinder, where no self-motion occurs. This self-interaction reflects the loss of symmetry on the Möbius strip. While the infinite cylinder is both vertically and horizontally homogeneous, the infinite Möbius strip is only horizontally homogeneous, as discussed in Example 4.8.

Chapter 5

Point-Vortices on Multiply-Connected Surfaces

We now consider the motion of ideal fluids on multiply-connected surfaces, where harmonic flows contribute to fluid motion.

We begin by reviewing the Hodge decomposition of $\mathfrak{g}^*$ in the case of a multiply-connected surface M . We then present two coordinate systems for the harmonic component of $\mathfrak{g}^*$ in terms of certain circulations, and derive expressions for the evolution equations for the harmonic component of the fluid in each coordinate system. We then specify to the case of point-vortices, where we derive the equations of motion for the finite-dimensional system of point-vortices coupled with harmonic flows. After determining the equations of motion, we derive the Hamiltonian structure underlying them. We carefully develop the phase space for point-vortex dynamics on a multiply-connected surface, and derive the symplectic structure by restricting the KKS form in $\mathfrak{g}^*$ to the N -vortex coadjoint orbit.

The study of ideal hydrodynamics on multiply-connected surfaces has seen much attention in the recent works [12], [13], [31], and [32]. The first two references focus on point-vortices, while the second two consider general fluids. Our review mainly follows the first two references, but is intended to bring together some of the discussion found in both. For a more comprehensive review of point-vortex dynamics on multiply-connected surfaces, we suggest the discussion found in [12].

5.1 The Equations of Motion

As discussed in Section 2.2, the geometric formulation of the Euler equations holds for all closed surfaces regardless of genus. When considering ideal fluid motion on a closed Riemannian surface (M, g) of genus $l \geq 1$, the fluid configuration on M is described by a velocity one-form $\nu \in \mathfrak{g}^*$ which evolves under the Euler equation

$$\begin{cases} \partial_t \nu + \mathcal{L}_\nu \nu = -d(p - \frac{1}{2}|\nu|^2) \\ \operatorname{div} \nu = 0, \end{cases} \quad (5.1)$$

where $v = \nu^\sharp$. Moreover, Perspective I of Table 2.1 applies; the evolution of ν is Hamiltonian flow in the Lie-Poisson space $\mathfrak{g}^*$ under the Hamiltonian of total kinetic energy. The principal distinction in the multiply-connected case is that ν is no longer entirely determined by its vorticity. From the geometric perspective, this reflects the fact that on a multiply-connected surface, $\mathfrak{g}^*$ is not isomorphic to the space of vorticity two-forms. Instead, as discussed in Chapter 2, the smooth

dual decomposes as

$$\mathfrak{g}^* = d \bigwedge^1(M) \oplus H_{\text{dR}}^1(M). \quad (5.2)$$

Thus, the velocity one-form ν describing the fluid configuration takes the form

$$\nu = -\star d\psi + \eta,$$

where $\psi \in C^\infty(M)$ is the stream function and $P_{\text{Harm}}(\nu) = \eta \in \mathcal{H}^1$ is a harmonic one-form.

From this decomposition, it is clear that the vorticity equation

$$\partial_t \omega + \mathcal{L}_v \omega = 0, \quad (5.3)$$

for $\omega = dv^\flat$ is not equivalent to the full Euler equation (5.1). The vorticity equation must be augmented with an additional equation describing the evolution of the harmonic component of ν . On a formal level, this is simply

$$\partial_t \eta + P_{\text{Harm}}(\mathcal{L}_v \nu) = 0 \quad (5.4)$$

where P_{Harm} is the projection operator defined in Corollary 1.35.

Together, the coupled system of (5.3) and (5.4) determine the dynamics of the two components of ν . Indeed, they are simply the projection of the Euler equation onto the two components of the space $\mathfrak{g}^*$. Note that since the harmonic component of $\mathfrak{g}^*$ is finite-dimensional, (5.4) is an ODE coupled to the PDE (5.3).

Coordinate Systems for $\mathfrak{g}^*$

While (5.4) describes the evolution of the harmonic component of the fluid on a formal level, it does not give any concrete information about the dynamics of η . We now review two coordinate systems that parametrize the finite-dimensional harmonic component of $\mathfrak{g}^*$ using certain circulations. These coordinate systems allow for an explicit expression of the coupled system of (5.3) and (5.4) in coordinates. While both coordinate systems are discussed in [12] and [31], we follow the notation of the second reference, as it is suited to point-vortex dynamics.

Remark 5.1. Before presenting the two coordinate systems for $\mathfrak{g}^*$, we provide some justification for the fact that both are given by circulations. The key observation, found in [5] and [21], is that $\mathfrak{g}^*$ tracks the total circulation of the fluid around every simple closed curve in M . Indeed, if γ is a simple closed curve bounding a region C in M , then

$$\oint_\gamma \nu = \oint_{\partial C} \nu = \int_C \omega.$$

As such the vorticity two-form keeps track of the total circulation of ν around every contractible loop. If M has non-contractible loops that bound no region (i.e. non-trivial homology), the vorticity must be supplemented by the additional data of the circulation of the fluid around these loops. These are precisely the coordinate systems discussed below.

Let (M, g) be a closed, oriented, multiply-connected Riemannian surface of genus $l \geq 1$. The first set of coordinates for $\mathfrak{g}^*$ arises from the selection of a canonical basis in homology on M . Recall from Chapter 1 that a canonical basis in homology given by curves $\{a_1, \dots, a_l, b_1, \dots, b_l\}$, determines a unique dual basis of harmonic forms $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in \mathcal{H}^1(M)$. By construction, these harmonic forms satisfy the relations

$$\int_M \alpha_i \wedge \beta_j = \int_{a_j} \alpha_i = \int_{b_j} \beta_i = \delta_{ij}, \quad (5.5)$$

and

$$\int_M \alpha_i \wedge \alpha_j = \int_M \beta_i \wedge \beta_j = \int_{b_j} \alpha_i = \int_{a_j} \beta_i = 0. \quad (5.6)$$

for all $1 \leq i, j \leq l$.

Under this basis of harmonic forms, each velocity one-form $\nu \in \mathfrak{g}^*$ may be written as

$$\nu = - * d\psi + \eta, \quad \eta = \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j). \quad (5.7)$$

From this expression, it is clear that elements of $\mathfrak{g}^*$ are parametrized by tuples of the form

$$(\omega, A_1, \dots, A_l, B_1, \dots, B_l), \quad (5.8)$$

where $\omega \in d\bigwedge^1(M)$ and $(A_1, \dots, A_l, B_1, \dots, B_l) \in \mathbb{R}^{2l}$. Projection onto the vorticity component of such a tuple is given by the exterior derivative,

$$\omega(\nu) = d\nu.$$

Projection onto the j -th harmonic coordinate is given by Proposition 1.38, which states that for all $1 \leq j \leq l$,

$$A_j(\nu) := \oint_{a_j} \eta = \int_M \nu \wedge \beta_j, \quad B_j(\nu) := \oint_{b_j} \eta = \int_M \alpha_j \wedge \nu, \quad (5.9)$$

with η the harmonic component of ν as in (5.7).

The following Proposition from [31] gives an explicit expression for the harmonic component of the Euler equation (5.4) in the coordinate system of A_j and B_j .

Proposition 5.2. *Let ν be the velocity one-form of the Euler equation (5.1) on M and $A_j = A_j(\nu)$, $B_j = B_j(\nu)$ be the coordinates for the harmonic component of ν given in (5.9). Then A_j and B_j evolve as*

$$\frac{dA_j}{dt} = - \int_M i_v \omega \wedge \beta_j, \quad \frac{dB_j}{dt} = \int_M i_v \omega \wedge \alpha_j,$$

for all $1 \leq j \leq l$.

Proof. We write

$$\frac{dA_j}{dt} = \frac{d}{dt} \int_M \nu \wedge \beta_j = \int_M \frac{\partial \nu}{\partial t} \wedge \beta_j.$$

Since M is closed and β_j is harmonic, we may eliminate the exact components of $\frac{\partial \nu}{\partial t}$. Using the Euler equation (5.1) together with the Cartan rule, we see that $\frac{\partial \nu}{\partial t}$ splits as

$$\frac{\partial \nu}{\partial t} = -\mathcal{L}_v \nu - d(p - \frac{1}{2}|v|^2) = -i_v d\nu - d(p - \frac{1}{2}|v|^2 - i_v \nu) = -i_v \omega + d(\text{something}). \quad (5.10)$$

Plugging this into the above gives the result for A_j , with an analogous calculation holding for B_j . $\square$

While the A_j and B_j coordinate system is natural in that it respects the Hodge decomposition of $\mathfrak{g}^*$, it is not well-suited to the study of point-vortex dynamics. Instead, the point-vortex system is better described using the coordinates of *total circulation* around the homology generators. Following [12], we denote these circulations using hatted coordinates

$$\hat{A}_j(\nu) := \oint_{a_j} \nu, \quad \hat{B}_j(\nu) := \oint_{b_j} \nu. \quad (5.11)$$

The total circulations (5.11) lead to a different parametrization of $\mathfrak{g}^*$ by tuples of the form

$$(\omega, \hat{A}_1, \dots, \hat{A}_l, \hat{B}_1, \dots, \hat{B}_l), \quad (5.12)$$

where $\omega \in d\bigwedge^1(M)$ and $(\hat{A}_1, \dots, \hat{A}_l, \hat{B}_1, \dots, \hat{B}_l) \in \mathbb{R}^{2l}$.

Remark 5.3. From (5.7), we see that the two coordinate systems are related in the following way:

$$\begin{aligned} \hat{A}_j(\nu) &= \oint_{a_j} \nu = \oint_{a_j} (-\star d\psi) + \eta = \oint_{a_j} \eta - \oint_{a_j} \star d\psi = A_j(\nu) - \oint_{a_j} \star d\psi, \\ \hat{B}_j(\nu) &= \oint_{b_j} \nu = \oint_{b_j} (-\star d\psi) + \eta = \oint_{b_j} \eta - \oint_{b_j} \star d\psi = B_j(\nu) - \oint_{b_j} \star d\psi. \end{aligned}$$

In order to convert a tuple of the form (5.12) into its corresponding velocity one-form $\nu \in \mathfrak{g}^*$, one uses the vorticity ω to determine the stream function ψ , and then solves the above equations for the A_j and B_j coordinates. These then produce the correct velocity one-form as in (5.7).

Remark 5.4. Without the vorticity, it is not possible to convert a tuple of hatted coordinates $(\hat{A}_1, \dots, \hat{A}_l, \hat{B}_1, \dots, \hat{B}_l)$, into a tuple of non-hatted coordinates $(A_1, \dots, A_l, B_1, \dots, B_l)$ (or vice versa). In particular, given only the total circulation of the fluid around each homology basis curve, it is not possible to determine the harmonic component η of the velocity one-form ν . The issue is that the vortical component of the fluid contributes to the circulation of the fluid around the homology basis curves, and so must be accounted for when calculating the harmonic contributions $A_j(\nu)$ and $B_j(\nu)$, as in the above remark.

The following proposition from [12] gives the expression of harmonic evolution equation (5.4) in the hatted coordinate system of $\hat{A}_j$ and $\hat{B}_j$,

Proposition 5.5. *Let ν be the velocity one-form of the Euler equation (5.1) on M and $\hat{A}_j = \hat{A}_j(\nu)$, $\hat{B}_j = \hat{B}_j(\nu)$ be the hatted coordinates for the harmonic component of ν given in (5.11). Then $\hat{A}_j$ and $\hat{B}_j$ evolve as*

$$\frac{d\hat{A}_j}{dt} = - \oint_{a_j} i_v \omega, \quad \frac{d\hat{B}_j}{dt} = - \oint_{b_j} i_v \omega,$$

for all $1 \leq j \leq l$.

Proof. Let γ be any simple closed curve in M . By bringing the time derivative inside the integral and applying an identical calculation as in (5.10), we may calculate the evolution of the circulation around γ as

$$\frac{d}{dt} \oint_{\gamma} \nu = \oint_{\gamma} \frac{\partial \nu}{\partial t} = - \oint_{\gamma} i_v \omega.$$

Taking γ to be the homology basis curves a_j and b_j then gives the result. $\square$

Propositions 5.2 and 5.5 give the expression of the formal equation of motion (5.4) describing the evolution of the harmonic flow η in the A_j, B_j and $\hat{A}_j, \hat{B}_j$ coordinates, respectively. They are the launching point for a general study of ideal hydrodynamics on multiply-connected closed surfaces. In [31] and [32], they are used to generate accurate visual simulations of fluid flow.

For our purposes, the key takeaway of these results is that they allow an immediate calculation of the equations of motion for the A_j, B_j and $\hat{A}_j, \hat{B}_j$ coordinates in the case of point-vortex dynamics.

Point-Vortices

Before determining the equations of motion of point-vortex dynamics, we provide a formal characterization of an N -vortex fluid configuration on a multiply-connected surface.

Definition 5.6. Let (M, g) be a closed, oriented, multiply-connected Riemannian surface of genus $l \geq 1$ with area form μ , total area $V = \int_M \mu$, and harmonic basis $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in \mathcal{H}^1(M)$. The velocity one-form ν associated to a fluid configuration of N point-vortices on M located at positions $r_1, \dots, r_N \in M$ with circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ is

$$\nu = - * d\psi + \eta = - \sum_{i=1}^N \Gamma_i * dG_M(\cdot, r_i) + \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j), \quad (5.13)$$

where $(A_1, \dots, A_l, B_1, \dots, B_l) \in \mathbb{R}^{2l}$ parametrizes the harmonic component of the fluid flow,

$$\eta = \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j). \quad (5.14)$$

The corresponding vorticity two-form is

$$\omega = d\nu = \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu, \quad (5.15)$$

and the stream function is

$$\psi = \sum_{i=1}^N \Gamma_i G_M(\cdot, r_i). \quad (5.16)$$

Remark 5.7. Note that the positions of the vortices do not uniquely determine the fluid velocity one-form ν associated to an N -vortex fluid configuration in (5.13). This is in line with our previous discussion; one must also supply additional circulation terms (e.g. A_j, B_j or $\hat{A}_j, \hat{B}_j$ coordinates) in order to specify ν .

Remark 5.8. From (5.13), it is clear that the harmonic component of the flow contributes to the velocity of the point-vortices. This reflects the coupling of the vortical and harmonic terms in ideal fluid flow: the harmonic flow pushes around the point-vortices, while the point-vortices affect the harmonic circulations as shown in the following results.

Using this definition, we can calculate the evolution of the A_j, B_j and $\hat{A}_j, \hat{B}_j$ coordinates in the case of N point-vortices.

Corollary 5.9. Let ν be the velocity one-form associated to a fluid configuration of N point-vortices on M as in (5.13), and let $A_j = A_j(\nu)$, $B_j = B_j(\nu)$ be the coordinates for the harmonic component of ν given in (5.9). Then A_j and B_j evolve as

$$\frac{dA_j}{dt} = \frac{\Gamma}{V} \left(\sum_{j=1}^l A_j \oint_{a_j} * \alpha_j + B_j \oint_{a_j} * \beta_j \right) + \sum_{i=1}^N \Gamma_i \beta_j(v_i) \quad (5.17)$$

$$\frac{dB_j}{dt} = \frac{\Gamma}{V} \left(\sum_{j=1}^l A_j \oint_{b_j} * \alpha_j + B_j \oint_{b_j} * \beta_j \right) - \sum_{i=1}^N \Gamma_i \alpha_j(v_i) \quad (5.18)$$

for all $1 \leq j \leq l$, where v_i is the velocity of the i -th point-vortex.

Proof. Plugging in the point-vortex vorticity (5.15) into Proposition 5.2 gives the evolution of the A_j coordinate as

$$\begin{aligned}\frac{dA_j}{dt} &= - \int_M i_v \omega \wedge \beta_j \\ &= - \int_M \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) i_v \mu \wedge \beta_j \\ &= - \int_M \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} \right) i_v \mu \wedge \beta_j + \frac{\Gamma}{V} \int_M i_v \mu \wedge \beta_j.\end{aligned}$$

Using the identities $i_v \mu \wedge \beta_j = -\mu \wedge i_v \beta_j$, and $i_v \mu = \star v^b = \star \nu$ (see Appendix A), we can simplify the above as

$$- \int_M \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} \right) i_v \mu \wedge \beta_j + \frac{\Gamma}{V} \int_M i_v \mu \wedge \beta_j = \int_M \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} \right) \mu \wedge i_v \beta_j + \frac{\Gamma}{V} \int_M \star \nu \wedge \beta_j.$$

Using the definition of A_j given in (5.9), we can write the above as

$$\frac{dA_j}{dt} = \int_M \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} \right) \mu \wedge i_v \beta_j + \frac{\Gamma}{V} A_j(\star \nu). \quad (5.19)$$

The first term in (5.19) can be written as

$$\int_M \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} \right) \mu \wedge i_v \beta_j = \sum_{i=1}^N \Gamma_i \langle \beta_j, v \rangle(r) \big|_{r=r_i} = \sum_{i=1}^N \Gamma_i \langle \beta_j, v(r_i) \rangle = \sum_{i=1}^N \Gamma_i \beta_j(v_i),$$

where $v_i := v(r_i)$ is, by definition¹, the velocity of the i -th point-vortex.

The second term in (5.19) can be simplified by using (5.13) to write

$$\star \nu = d\psi + \star \eta = d\psi + \sum_{j=1}^l A_j \star \alpha_j + B_j \star \beta_j.$$

Since $\star$ maps harmonic forms to harmonic forms, it is clear that $P_{\text{Harm}}(\star \nu) = \star \eta$. So,

$$A_j(\star \nu) = \oint_{a_j} P_{\text{Harm}}(\star \nu) = \oint_{a_j} \star \eta = \oint_{a_j} \sum_{j=1}^l A_j \star \alpha_j + B_j \star \beta_j = \sum_{j=1}^l A_j \oint_{a_j} \star \alpha_j + B_j \oint_{a_j} \star \beta_j. \quad (5.20)$$

Inserting this into (5.19) along with the previous calculation then gives the result for A_j . Similar calculations hold for the B_j coordinates. $\square$

Remark 5.10. The above result reproduces, via a different argument, the equations found in [12], Section 3.

We now perform a similar calculation to determine the evolution of the $\hat{A}_j, \hat{B}_j$ coordinates. As expected, these equations for $\hat{A}_j$ and $\hat{B}_j$ differ from the above equations by a single term.

Corollary 5.11. *Let ν be the velocity one-form of the Euler equation (5.1) on M and $\hat{A}_j = \hat{A}_j(\nu)$, $\hat{B}_j = \hat{B}_j(\nu)$ be the coordinates for the harmonic component of ν given in (5.11). Then $\hat{A}_j$*

¹Explicitly, the velocity v_i is $v_i = \nu^\sharp(r_i)$, where ν is the fluid velocity one-form of (5.13). The Robin function regularization is used in to evaluate the self-interaction term in $\nu^\sharp$, as discussed in Chapter 3.

and $\hat{B}_j$ evolve as

$$\frac{d\hat{A}_j}{dt} = \frac{\Gamma}{V} \left(\sum_{j=1}^l A_j \oint_{a_j} \star \alpha_j + B_j \oint_{a_j} \star \beta_j \right) \quad (5.21)$$

$$\frac{d\hat{B}_j}{dt} = \frac{\Gamma}{V} \left(\sum_{j=1}^l A_j \oint_{b_j} \star \alpha_j + B_j \oint_{b_j} \star \beta_j \right) \quad (5.22)$$

for all $1 \leq j \leq l$.

Proof. As in the proof of Corollary 5.9, we insert the point-vortex vorticity (5.15) into Proposition 5.5. Recalling from Appendix A the identity $i_v \mu = \star \nu$, we write

$$\begin{aligned} \frac{d\hat{A}_j}{dt} &= - \oint_{a_j} i_v \omega \\ &= - \oint_{a_j} \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) i_v \mu \\ &= - \oint_{a_j} \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \star \nu \\ &= \frac{\Gamma}{V} \oint_{a_j} \star \nu, \end{aligned}$$

where in the final step we have assumed that none of the point-vortices lie on the curve a_j . Calculating exactly as in (5.20) then gives the result for $\hat{A}_j$. A similar argument holds for $\hat{B}_j$. $\square$

Remark 5.12. In the case of zero total circulation, the hatted coordinates remain constant. This reflects the fact that the point-vortices influence the total circulation around the homology generators exactly by their net counter-vorticity. When the total circulation vanishes there is no counter-vorticity, and so the $\hat{A}_j$ and $\hat{B}_j$ terms become a constant of the motion².

Together, Corollaries 5.9 and 5.11 give the equations of motion for the harmonic component of the fluid in the case of point-vortex dynamics. We now analyze the Hamiltonian structure underlying these equations. Following [12] and [13], we show that it manifests as an augmented version of the Hamiltonian structure found in the simply-connected case.

5.2 The Hamiltonian Structure

We consider the motion of N point-vortices with circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}/\{0\}$ on M , which as before is a closed, oriented, multiply-connected Riemannian surface of genus $l \geq 1$ with area form μ , total area $V = \int_M \mu$, and basis of harmonic one-forms $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in \mathcal{H}^1(M)$, corresponding to the canonical homology basis $\{a_1, \dots, a_l, b_1, \dots, b_l\}$.

The Phase Space

From Definition 5.6, it is clear that the phase space for N -vortex dynamics on M is

$$\mathcal{M} := M^N \setminus \{\text{collision}\} \oplus \mathbb{R}^{2l} = \{(r_1, \dots, r_N) \in (M)^N : r_i \neq r_j \text{ for any } i \neq j\} \oplus \mathbb{R}^{2l}.$$

²This is unrelated to the Kelvin circulation theorem, which describes the circulation around a loop that moves along with the fluid flow.

On this phase space, the two coordinate systems discussed in the previous section descend to tuples of the form

$$(r_1, \dots, r_N, A_1, \dots, A_l, B_1, \dots, B_l),$$

and

$$(r_1, \dots, r_N, \hat{A}_1, \dots, \hat{A}_l, \hat{B}_1, \dots, \hat{B}_l).$$

The positions $r_1, \dots, r_N \in M$ parametrize the vortical component of the fluid flow by

$$\omega = \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu,$$

and the A_j, B_j or $\hat{A}_j, \hat{B}_j$ terms determine the harmonic component of the fluid flow as discussed in the previous section.

Remark 5.13. Note that level sets of the circulations in one coordinate system do not correspond to level sets of the circulations in the other. For example, if one fixes the total circulations $(\hat{A}_1, \dots, \hat{A}_l, \hat{B}_1, \dots, \hat{B}_l)$, it is clear that the resulting submanifold

$$\mathcal{S} = \{(r_1, \dots, r_N, \hat{A}_1, \dots, \hat{A}_l, \hat{B}_1, \dots, \hat{B}_l) : r_1, \dots, r_N \in M\} \subset \mathcal{M}$$

is isomorphic to M^N . In the non-hatted A_j, B_j coordinate system, $\mathcal{S}$ is parametrized with both varying vortex positions and harmonic flows. This is a result of the observations of Remark 5.3: fixing the total circulations does not uniquely determine the harmonic circulations because the vorticity contributes to the total circulation.

The map $J : \mathcal{M} \rightarrow \mathfrak{g}^*$ that takes a tuple of the form

$$p = (r_1, \dots, r_N, A_1, \dots, A_l, B_1, \dots, B_l)$$

to the velocity one-form

$$\nu = - \sum_{i=1}^N \Gamma_i * dG_M(\cdot, r_i) + \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j) \quad (5.23)$$

identifies the phase space $\mathcal{M}$ with a finite-dimensional coadjoint orbit $\mathcal{O}_\nu$ in $\mathfrak{g}^{*3}$. Similarly, at each point $p \in \mathcal{M}$, the pushforward of J identifies $T_p \mathcal{M}$ with $T_{J(p)} \mathcal{O}_\nu$, which is given by Proposition 1.2as

$$T_{J(p)} \mathcal{O}_\nu = \{\mathcal{L}_X J(p) : X \in \mathfrak{g}\}.$$

The pushforward of J thus allows us to identify tangent vectors to $\mathcal{M}$ with elements of $\mathfrak{g}$ by considering them as variations of the velocity one-form.

In the following derivation of the point-vortex symplectic structure, we use these identifications to explicitly work on the coadjoint orbit $\mathcal{O}_\nu$, identifying elements of $\mathcal{M}$ with their associated velocity one-forms in $\mathfrak{g}^*$ and tangent vectors to $\mathcal{M}$ with their associated variations in $\mathfrak{g}$.

The Symplectic Structure

We now deduce the symplectic structure for the N -vortex system by calculating it as the restriction of the KKS form to the N -vortex coadjoint orbit $\mathcal{O}_\nu$, just as we did in the simply-connected case in Chapter 3. As discussed above, we identify $\mathcal{M}$ with $\mathcal{O}_\nu$ throughout.

³Indeed, one can verify by a lengthy calculation that J is exactly the momentum map associated to the natural action of G on $\mathcal{M}$.

Lemma 5.14. *The differentials of the hatted coordinates $\hat{A}_j : \mathcal{O}_\nu \rightarrow \mathbb{R}$ and $\hat{B}_j : \mathcal{O}_\nu \rightarrow \mathbb{R}$*

$$\hat{A}_j(\nu) = \oint_{a_j} \nu, \quad \hat{B}_j(\nu) = \oint_{b_j} \nu$$

act at the point $\bar{\nu} \in \mathcal{O}_\nu$ as

$$d\hat{A}_j(X) = \frac{\Gamma}{V} \oint_{a_j} \star X^\flat, \quad d\hat{B}_j(X) = \frac{\Gamma}{V} \oint_{b_j} \star X^\flat,$$

where $X \in \mathfrak{X}_\mu(M)$ is understood as a variation of the velocity one-form $\bar{\nu} \in \mathcal{O}_\nu$.

Proof. At the point $\bar{\nu} \in \mathcal{O}_\nu$, we have the calculation

$$d\hat{A}_j|_{\bar{\nu}}(X) = (\mathcal{L}_X \hat{A}_j)|_{\bar{\nu}} = \mathcal{L}_X \oint_{a_j} \bar{\nu} = \oint_{a_j} \mathcal{L}_X \bar{\nu} = \oint_{a_j} i_X \bar{\omega},$$

where in the final step we have used the Cartan rule. The result then follows by inserting the N -vortex vorticity

$$\bar{\omega} = \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right) \mu,$$

into the above expression and calculating as in the proof of Corollary 5.11. $\square$

Proposition 5.15. *Under the $\hat{A}_j, \hat{B}_j$ coordinate system for the N -vortex coadjoint orbit $\mathcal{O}_\nu$, the KKS form takes the form*

$$\sigma_{KKS} = \sum_i^N \Gamma_i \mu_i - \frac{V}{\Gamma} \left(\sum_j^l d\hat{A}_j \wedge d\hat{B}_j \right) \quad (5.24)$$

where Γ is the total vorticity of the point-vortices. If $\Gamma = 0$, the KKS form simplifies to

$$\sigma_{KKS} = \sum_i^N \Gamma_i \mu_i. \quad (5.25)$$

Proof. We show this by explicitly calculating $\sigma_{KKS}(\mathcal{L}_X \bar{\nu}, \mathcal{L}_Y \bar{\nu})$ at the point $\bar{\nu} \in \mathcal{O}_\nu$ for $X, Y \in \mathfrak{g}$. As M is multiply-connected, this amounts to an application of Proposition 2.25, which says that if X and Y lower to one-forms with Hodge decompositions

$$X^\flat = -\star d\psi + \eta, \quad Y^\flat = -\star d\xi + \theta, \quad (5.26)$$

for $\psi, \xi \in C^\infty(M)$ and $\eta, \theta \in \mathcal{H}^1(M)$, then the KKS form on $\mathcal{O}_\nu$ acts as

$$\sigma_{KKS}(\mathcal{L}_X \bar{\nu}, \mathcal{L}_Y \bar{\nu}) = \int_M \bar{\lambda} (d\psi + \star \eta) \wedge (d\xi + \star \theta),$$

where $\bar{\lambda}$ is the scalar vorticity of $\bar{\nu}$. Inserting the N -vortex scalar vorticity

$$\bar{\lambda} = \left(\sum_{i=1}^N \Gamma_i \delta_{r_i} - \frac{\Gamma}{V} \right),$$

into the above, we see that

$$\begin{aligned}\sigma_{\text{KKS}}(\mathcal{L}_X \bar{\nu}, \mathcal{L}_Y \bar{\nu}) &= \int_M \left(\sum_i (\Gamma_i \delta_{r_i}) - \frac{\Gamma}{V} \right) (d\psi + \star\eta) \wedge (d\xi + \star\theta) \\ &= \int_M \left(\sum_i \Gamma_i \delta_{r_i} \right) \mu(X, Y) \mu - \frac{\Gamma}{V} \int_M (d\psi + \star\eta) \wedge (d\xi + \star\theta).\end{aligned}$$

The first term is exactly the term found in the simply-connected case. In Chapter 3, we found it to be equal to the standard point-vortex symplectic form $\sum_i^N \Gamma_i \mu_i$. Clearly, if $\Gamma = 0$ this is the end of our calculation.

In the case that $\Gamma \neq 0$, we consider the second term. After expanding the wedge product and applying Stokes theorem, it may be written as

$$-\frac{\Gamma}{V} \int_M (d\psi + \star\eta) \wedge (d\xi + \star\theta) = -\frac{\Gamma}{V} \int_M \star\eta \wedge \star\theta.$$

Since both $\star\eta$ and $\star\theta$ are harmonic, they are closed. Proposition 1.17 can thus be used to evaluate the integral as

$$-\frac{\Gamma}{V} \int_M \star\eta \wedge \star\theta = -\frac{\Gamma}{V} \left[\sum_j^l \oint_{a_j} \star\eta \oint_{b_j} \star\theta - \oint_{b_j} \star\eta \oint_{a_j} \star\theta \right],$$

where a_j, b_j are the canonical homology basis on M . Using Lemma 5.14 together with the Hodge decomposition (5.26), we rewrite the circulation integrals as

$$\oint_{a_j} \star\eta = \oint_{a_j} (\star\eta + d\psi) = \oint_{a_j} \star X^\flat = \frac{V}{\Gamma} d\hat{A}_j(X)$$

and

$$\oint_{a_j} \star\theta = \oint_{a_j} (\star\theta + d\xi) = \oint_{a_j} \star Y^\flat = \frac{V}{\Gamma} d\hat{A}_j(Y),$$

with a similar calculation for the b_j integrals. We then rewrite the entire second term as

$$\begin{aligned}-\frac{\Gamma}{V} \left[\sum_j^l \oint_{a_j} \star\eta \oint_{b_j} \star\theta - \oint_{b_j} \star\eta \oint_{a_j} \star\theta \right] &= -\frac{V}{\Gamma} \left[\sum_j^l \left(\frac{\Gamma}{V} \oint_{a_j} \star\eta \right) \left(\frac{\Gamma}{V} \oint_{b_j} \star\theta \right) - \left(\frac{\Gamma}{V} \oint_{b_j} \star\eta \right) \left(\frac{\Gamma}{V} \oint_{a_j} \star\theta \right) \right] \\ &= -\frac{V}{\Gamma} \left[\sum_j^l d\hat{A}_j(X) d\hat{B}_j(Y) - d\hat{B}_j(X) d\hat{A}_j(Y) \right] \\ &= -\frac{V}{\Gamma} \left(\sum_j^l d\hat{A}_j \wedge d\hat{B}_j \right) (X, Y).\end{aligned}$$

Together with the standard N -vortex symplectic form $\sum_i^N \Gamma_i \mu_i$ found earlier, this gives the result. $\square$

Remark 5.16. Notice that in the case when the total circulation Γ vanishes, the harmonic components still change in the A_j, B_j coordinates. This is a direct result of Remark 5.13. From Corollary 5.11, it is clear that the $\hat{A}_j, \hat{B}_j$ coordinates are constant in the $\Gamma = 0$ case. Nevertheless, as discussed in Remark 5.13, the resulting submanifold in $\mathcal{M}$ is parametrized with both varying vortex positions and harmonic flows. As shown in [13], the resulting Hamiltonian system can be formulated on the domain $M^N \setminus \{\text{collision}\}$, with the varying harmonic terms accounted for by an extra term in the Hamiltonian relative to the simply-connected case.

We now apply the same strategy to determine the point-vortex Hamiltonian in the multiply-connected case. As before, we identify $\mathcal{M}$ with $\mathcal{O}_\nu$ via (5.23).

Proposition 5.17. *The fluid Hamiltonian of total kinetic energy restricts to the N -vortex coadjoint orbit $\mathcal{O}_\nu$ as*

$$H(\nu) = - \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) - \frac{1}{2} \sum_{i=1}^N \Gamma_i^2 R_M(r_i) - \frac{1}{2} \int_M \eta \wedge \star \eta, \quad (5.27)$$

where η is the harmonic component of the velocity one-form ν .

Proof. Applying Perspective I of Table 2.1, we see that the fluid Hamiltonian of total kinetic energy is given by

$$H(\nu) = \frac{1}{2} \int_M g(\nu, \nu) \mu = \frac{1}{2} \langle \nu, \nu \rangle_{L^2} = \frac{1}{2} \int_M \nu \wedge \star \nu,$$

recalling the L^2 pairing of forms given in (1.11). We insert the point-vortex velocity one-form (omitting the over bar for notational simplicity)

$$\nu = - \star d\psi + \eta = - \sum_{i=1}^N \Gamma_i \star dG_M(\cdot, r_i) + \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j),$$

into this expression. This yields

$$H(\nu) = \frac{1}{2} \int_M \nu \wedge \star \nu = \frac{1}{2} \int_M (- \star d\psi + \eta) \wedge (d\psi + \star \eta).$$

As in the previous argument, the cross terms vanish by Stokes theorem, leaving

$$H(\nu) = \frac{1}{2} \int_M - \star d\psi \wedge d\psi + \frac{1}{2} \int_M \eta \wedge \star \eta. \quad (5.28)$$

Integrating by parts shows that the left term is

$$\frac{1}{2} \int_M - \star d\psi \wedge d\psi = \frac{1}{2} \int_M -d \star d\psi \wedge \psi = \frac{1}{2} \int_M \omega \psi = \frac{1}{2} \int_M \lambda \psi \mu.$$

This is exactly the form of the Hamiltonian used in Section 3.2. By the calculations of that section, this term yields the standard, simply-connected N -vortex Hamiltonian

$$\frac{1}{2} \int_M - \star d\psi \wedge d\psi = \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) + \frac{1}{2} \sum_{i=1}^N \Gamma_i^2 R_M(r_i). \quad (5.29)$$

Inserting (5.29) into (5.28) then give the result, after flipping the sign in accordance with Remark (2.1). $\square$

Remark 5.18. Note that by calculating as in (5.20), the harmonic energy term $\frac{1}{2} \int_M \eta \wedge \star \eta$ can be written explicitly in terms of the A_j, B_j coordinates for $\eta = \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j)$.

Bringing together the results of this section and transporting the above results from the setting of $\mathcal{O}_\nu$ back to the phase space $\mathcal{M}$, we have the following Hamiltonian formulation of point-vortex dynamics on a closed, multiply-connected surface.

Theorem 5.19. *Let (M, g) be a closed, oriented, multiply-connected Riemannian surface of genus $l \geq 1$ with area form μ , total area $V = \int_M \mu$, and basis of harmonic one-forms $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in$*

$\mathcal{H}^1(M)$. Then the Euler flow of N point-vortices with non-zero circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R} \setminus \{0\}$ on M is a finite-dimensional Hamiltonian system, with phase space

$$\mathcal{M} = M^N \setminus \{\text{collision}\} \oplus \mathbb{R}^{2l}$$

equipped with the symplectic form given in the $\hat{A}_j, \hat{B}_j$ coordinate system as

$$\sigma_{new} = \sum_i^N \Gamma_i \mu_i - \frac{V}{\Gamma} \left(\sum_j^l d\hat{A}_j \wedge d\hat{B}_j \right),$$

and Hamiltonian function

$$H(\nu) = - \sum_{i < j} \Gamma_i \Gamma_j G_M(r_i, r_j) - \frac{1}{2} \sum_{i=1}^N \Gamma_i^2 R_M(r_i) - \frac{1}{2} \int_M \eta \wedge \star \eta,$$

where

$$\eta = \sum_{j=1}^l (A_j \alpha_j + B_j \beta_j)$$

is the harmonic component of the fluid flow.

Through a lengthy calculation, one can verify that the above Hamiltonian system reproduces the equations of motion (5.9) and (5.11) derived in the previous section. For detailed examples of the dynamics resulting from the above Theorem, we refer the reader to Section 4 of [12], which presents a number of interesting case-studies for the above equations on tori.

Appendix A

The Hodge Star Operator

The purpose of this appendix is to formally define the Hodge star operator, and collect several identities involving its action on surfaces that we use throughout main discussion. The material presented here is standard, and can be found in [7] and [27].

Let (M, g) be a closed, oriented Riemannian manifold of dimension $\dim M = n$ with volume form μ corresponding to the metric. Recall that the metric g extends pointwise to a metric on all tensor bundles of M , and in particular to $\bigwedge^k(T^*M)$, (see [27]). By integration, this metric furnishes the L^2 inner product of k -forms $\eta, \gamma \in \bigwedge^k(M)$,

$$\langle \eta, \gamma \rangle_{L^2} := \int_M g(\eta, \gamma) \mu.$$

Definition A.1. For $1 \leq k \leq n$, the **Hodge star operator**

$$\star : \bigwedge^k(M) \rightarrow \bigwedge^{n-k}(M)$$

is the unique map satisfying

$$\int_M \eta \wedge \star \gamma = \langle \eta, \gamma \rangle_{L^2}. \quad (\text{A.1})$$

for all $\gamma, \eta \in \bigwedge^k(M)$.

In this work, we focus almost exclusively on the $n = 2$ case, when M is a surface. In this case, we have the following useful results.

Lemma A.2. Let $X \in \mathfrak{X}(M)$ be a vector field on a closed, oriented, Riemannian surface (M, g) . Then,

$$i_X \mu = \star X^\flat.$$

Proof. Let $\gamma \in \bigwedge^1(M)$ be an arbitrary one-form. Since M is a surface, the three-form $\gamma \wedge \mu$ vanishes, and so $\gamma \wedge (i_X \mu) = i_X \gamma \wedge \mu$. Hence,

$$\gamma \wedge (i_X \mu) = i_X \gamma \wedge \mu = \langle \gamma, X \rangle \mu. \quad (\text{A.2})$$

Using (A.1) together with the definition of the metric in $\bigwedge^1(M)$, we write

$$\int_M \gamma \wedge \star X^\flat = \int_M g(\gamma, X^\flat) \mu = \int_M g(\gamma^\sharp, X) \mu = \int_M \langle \gamma, X \rangle \mu.$$

Integrating (A.2), we see that $\int_M \gamma \wedge \star X^\flat = \int_M \gamma \wedge i_X \mu$. As γ is arbitrary, this gives the result. $\square$

Lemma A.3. *For all $X, Y \in \mathfrak{X}(M)$ on a closed, oriented, Riemannian surface (M, g) , it holds*

$$i_X \mu \wedge i_Y \mu = \mu(X, Y) \mu.$$

Proof. This follows from choosing $\gamma = -i_Y \mu$ and calculating as in (A.2). $\square$

If (x, y) is an isothermal coordinate system for a neighborhood $U \subset M$, the action of the Hodge star is given on U as

$$\star dx = dy, \quad \star dy = -dx \tag{A.3}$$

and

$$\star 1 = \mu, \quad \star \mu = 1.$$

Clearly $\star$ squares to the identity Id on zero-forms and two-forms, and to $-\text{Id}$ on one-forms.

From (A.3), we see that the Hodge star acts to rotate one-forms by $\frac{\pi}{2}$. Indeed, by conjugating with the musical isomorphisms of g , the Hodge star allows us to define the rotation operator on vector fields

$$J : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M),$$

by

$$J = \sharp \circ \star \circ \flat.$$

From (A.3), it follows that J satisfies

$$J \frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad J \frac{\partial}{\partial y} = -\frac{\partial}{\partial x},$$

in the neighborhood U , and that

$$J^2 = -\text{Id}.$$

The rotation operator J is thus a geometric generalization of the standard rotation, given on $\mathbb{R}^2$ by the matrix

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Remark A.4. The rotation J allows an alternate expression of the skew-gradient of a function $f \in C^\infty(M)$,

$$\nabla^\perp f = (\star df)^\sharp.$$

Using J , we can rewrite the skew-gradient as

$$\nabla^\perp f = (\star df)^\sharp = J(df^\sharp) = J(\text{grad } f),$$

where $\text{grad } f = (df)^\sharp$ is the Riemannian gradient.

Lemma A.2 extends to J in the following way.

Corollary A.5. *For vector fields $X, Y \in \mathfrak{X}(M)$, it holds*

$$\int_M g(X, Y) \mu = \int_M \mu(X, JY) \mu \tag{A.4}$$

Proof. Applying Lemma A.2 to $-JY$ yields $-i_{JY} \mu = -\star(JY)^\flat = -\star^2 Y^\flat = Y^\flat$. So,

$$\int_M g(X, Y) \mu = \int_M (i_X Y^\flat) \mu = - \int_M (i_X i_{JY} \mu) \mu = - \int_M \mu(JY, X) \mu = \int_M \mu(X, JY) \mu.$$

□

Remark A.6. By restricting the action of $\star$ and J to individual fibers, we obtain maps that we (abuse notation to) denote as $\star : T^*M \rightarrow T^*M$ and $J : TM \rightarrow TM$ which satisfy the diagram

$$\begin{array}{ccc} TM & \xrightarrow{J} & TM \\ \uparrow \sharp & & \downarrow \flat \\ T^*M & \xrightarrow{\star} & T^*M. \end{array}$$

This makes clear the way in which, on a surface, $\star$ acts as a rotation operator on one-forms. Using these fiberwise maps, (A.4) becomes the identity

$$g(X, Y) = \mu(X, JY), \quad (\text{A.5})$$

for all $X, Y \in T_m M$ and $m \in M$.

Remark A.7. The above corollary shows that M has the structure of a **Kähler manifold**, as it is equipped with the three structures of a metric g , a symplectic form μ , and an almost complex structure¹ J that are compatible in the sense of (A.5) (see [8]). This result is highly specific to the $n = 2$ case. In higher dimensions, the volume form μ is not a symplectic form, and the Hodge star is not an endomorphism on T^*M . Hence the interpretation of the Hodge star as a rotation operator, as in the above diagram, is particular to surfaces.

We conclude by supplying the proofs of the claims of Proposition 1.19. Recall that the de Rham Laplacian Δ_{dR} is defined in Chapter 1 as

$$\Delta_{dR} = d\delta + \delta d,$$

where the codifferential acts on k -forms as $\delta = (-1)^{n(k+1)+1} \star d \star$.

Proposition 1.19. *The de Rham Laplacian Δ_{dR} satisfies the following properties:*

1. Δ_{dR} agrees with the Laplace-Beltrami operator on smooth functions $f \in C^\infty(M)$.
 2. Δ_{dR} commutes with the Hodge star
- $$\Delta_{dR} \circ \star = \star \circ \Delta_{dR}.$$
3. Δ_{dR} is a self-adjoint operator under the L^2 inner product.
 4. If η is exact and $\Delta_{dR}\eta = 0$, then $\eta = 0$.
 5. $\Delta_{dR}\eta = 0$ if and only if $d\eta = 0$ and $\delta\eta = 0$.

Proof. We proceed in order.

1. Recall that the Laplace-Beltrami operator is defined by $\Delta f = -\operatorname{div}(\operatorname{grad} f)$. Proposition 1.23 says that $\operatorname{div} X = -\delta X^\flat$. Thus we may write

$$\Delta f = -\operatorname{div}(\operatorname{grad} f) = -\operatorname{div}((df)^\sharp) = \delta df = \Delta_{dR} f.$$

¹An almost complex structure on a surface M is a bundle map $J : TM \rightarrow TM$ satisfying $J^2 = -\operatorname{Id}$. In the general case, J is required to be integrable in order for M to be Kähler. On surfaces, all almost complex structures are integrable, see [8].

2. Using the fact that the Hodge star squares to $\star^2 = (-1)^{k(n-k)}$ on k -forms, one may show that $\star\delta d = d\delta\star$ by juggling exponents. Hence,

$$\star \circ \Delta_{\text{dR}} = \star d\delta + \star\delta d = \star d(-1)^{n(n+k)+1} \star d\star + \star\delta d = \delta d\star + d\delta\star = \Delta_{\text{dR}} \circ \star.$$

3. Since d and δ are adjoint operators under the L^2 inner product, we have

$$\begin{aligned} \langle \Delta_{dR}\eta, \xi \rangle &= \langle d\delta\eta, \xi \rangle_{L^2} + \langle \delta d\eta, \xi \rangle_{L^2} \\ &= \langle \delta\eta, \delta\xi \rangle_{L^2} + \langle d\eta, d\xi \rangle_{L^2} \\ &= \langle \eta, d\delta\xi \rangle_{L^2} + \langle \eta, \delta d\xi \rangle_{L^2} \\ &= \langle \eta, \Delta_{dR}\xi \rangle_{L^2}. \end{aligned}$$

4. If $\eta = df$ is harmonic, then $\Delta f = \Delta_{dR}f = \delta df = \delta\eta = 0$. So f is harmonic. As M is closed, f is constant and so $\eta \equiv 0$.

5. This follows from the calculation

$$\langle \Delta_H\eta, \eta \rangle_{L^2} = \langle d\delta\eta, \eta \rangle_{L^2} + \langle \delta d\eta, \eta \rangle_{L^2} = \langle \delta\eta, \delta\eta \rangle_{L^2} + \langle d\eta, d\eta \rangle_{L^2}.$$

□

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