Università degli Studi di Padova

DIPARTIMENTO DI FISICA E ASTRONOMIA 'GALILEO GALILEI'
Corso di Laurea in Fisica

Linear and Weakly Non-linear Analysis of the Stability of the Boundary Layer in Transonic Flow

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Academic Year 2013/2014

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1 Abstract

In this thesis we analize linear and weakly nonlinear behavior of perturbations in a boundary layer with transonic free stream velocity. In the matter of linear analysis, we deduce the dispersion relation which relates wave number and frequency for normal perturbations. An affine transformation that reduces this transonic equation onto the subsonic one is found. This transformation is used to find governing equations in the supersonic limit. The aim is to extend the Tollmien-Schlichting waves theory toward supersonic regimes. In the matter of weakly nonlinear analysis, we find an equation for amplitude of Tollmien-Schlichting waves. The derivation of this equation takes into account both the linear displacement of the wave and the nonlinear process of growth of the amplitude.

2 Introduction

Currently, significant attention in aerospace industry is given to drag reduction of passenger aircraft, which comes mainly from:

1. Wave drag, which is due to losses in the shock waves. The phenomenon is described on the basis of compressibility effects, therefore it is independent on viscosity. Although shock waves are tipically associated with supersonic speed, they can form at transonic aircraft speeds on areas of wings where the local airflow experiences an acceleration above sonic speed;

2. The induced drag, which is due to trailing vortices behind a wing of finite span. The pressure below the wing is greater than above. On a wing of finite span, this pressure causes air to flow around the wing tip. This airflow causes vortices along the wing trailing edge. These vortices create a downwash region behind the wing (fig. 1) which reduces the effectiveness of the wing to generate lift and changes the effective relative airflow. This modified condition tilts the total aerodynamic force rearwards and its component parallel to the free stream is the induced drag (fig. 2). Still, this effect is not related to viscosity;

3. Viscous drag is the sum of friction and form drags, which are due to viscous interaction between fluid and surface. Form drag generates when the fluid flow separates from a wing surface. Because of an ineffective pressure recovery in the separation region, an adverse pressure difference along with a loss of lift take place. Friction depends substantially on boundary layer configuration and viscosity.

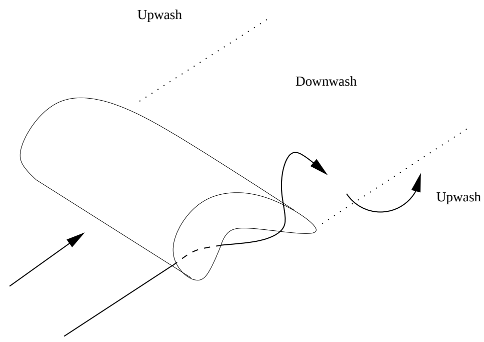

Figure 1: Trailing vortices

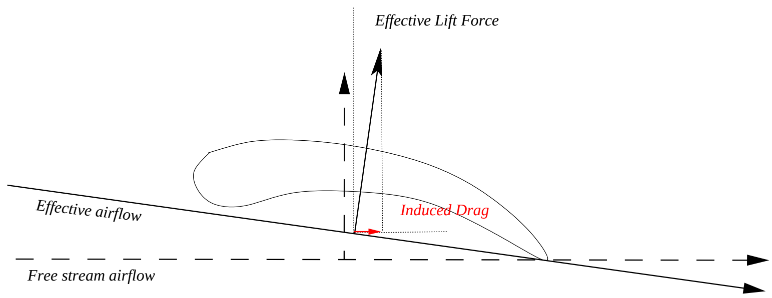

Figure 2: Effective airflow and induced drag

Let us focus on the third source of drag, which involves viscous effects. What basically happens is that an initially laminar boundary layer, due to external disturbances, presents some instabilities. Turbulence is the result of their amplification. The consequence is a rise in skin friction. Furthermore, a rise of the external pressure can lead to separation, which results in form drag. At present the main challenge in passenger aircraft research is a delay of the laminar-turbulent transition in the boundary layer. Before continuing to intoduce the problem, let us take a historical digression into separation.

Separation is a fluid dynamic phenomenon that influences the bahaviour of a wide variety of liquid and gas flows. Figures (3) show the difference between a theoretical attached flow, predicted via Euler equations , and the real flow visualization by Taneda (1956) for a circular cylinder in a water tank. Clearly, the Euler equation cannot predict the wakes which develops behind the cylinder. In case of an incompressible flow, the incompressible steady Navier-Stokes equations in the nondimensional form are:

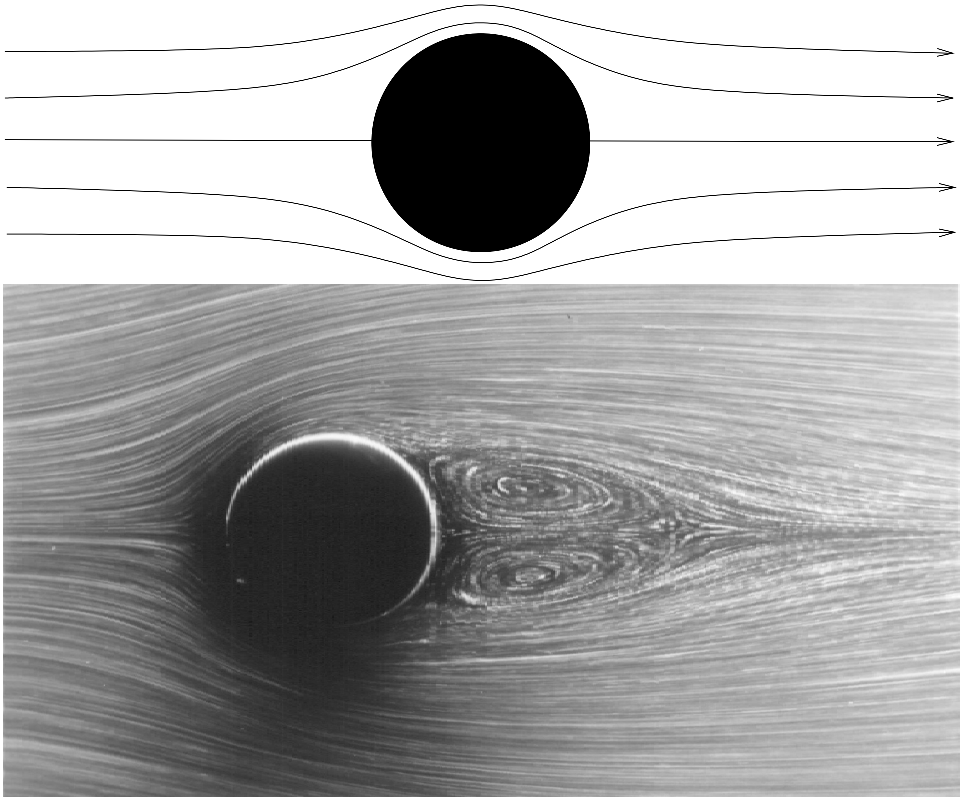

Figure 3: Theoretical streamline pattern and experimental visualization by Taneda ($R_e = 26$)

$$\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{R_e}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{cases}$$

where $R_e = \frac{V_\infty a}{\nu}$, being a the radius of the cylinder. Dealing with fluids with an extremely small viscosity, these equations reduce to the Euler equations, being the viscous term neglectible, and predict a fully attached flow. However, such flows cannot be observed in practice except for some special cases. In particular, the flow past a cylinder assumes an attached form only if $R_e < 6$. The actual flow shown in figure (3) corresponds to $R_e = 26$. Further increase of the Reynolds number results in an extension of the eddies and a loss of symmetry, but the flow never returns to an attached form. The first model of a separated flow was developed by Helmholtz (1868) and Kirchhoff (1869). The major conclusion of this inviscid theory is that Euler equations allow for a family of separated flow solutions where the position of the separation point remains a free parameter. The dilemma is how to find the location of the separation point. Prandtl (1904) introduced the concept of boundary layer. His idea is that for large Reynold numbers most of the flow can be treated as inviscid and there always exists a thin boundary layer developing along the wall where the flow is viscous in nature. Matematically, the second derivative with respect to y is large in this region and viscous terms are still present in the equations. The behavior of the boundary layer depends on the pressure distribution along the wall. If pressure decreases downstream (favourable pressure gradient) the boundary layer stays more likely attached to the wall. On the other hand, with adverse pressure gradient the boundary layer tends to separate from the body surface. This is because the velocity in the boundary layer drops towards the wall and the closer a fluid particle is to the wall the smaller its kinetic energy is. Indeed while the pressure rise in the outer flow may be quite significant, the fluid particles inside the boundary layer may not be able to get over it. That causes the fluid particles near the wall to stop and then turn back to form a reverse flow region characteristic of separated flows (figure (4)). A

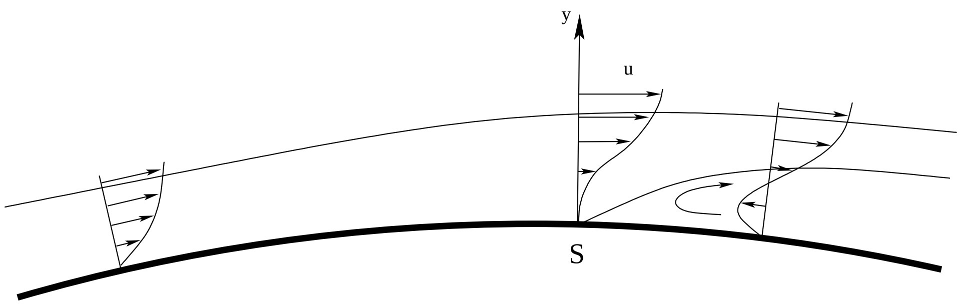

Figure 4: Boundary layer separation

mathematical analysis of the separation has lead to the development of the triple-deck theory. Interestingly enough this theory is also applicable to the description of the laminar-turbulent transition.

Coming back to laminar-turbulent transition, in aerodynamic flows this transition follows a classical scenario when turbulence develops as a result of amplification of instability modes. In the flow past a swept wing, two modes of instability are observed: cross-flow vortices and Tollmien-Schlichting waves. The former dominate the transition process on a wing with larger sweep angle, tipical of long-distance passenger carriers. The latter prevails in the case of smaller sweep angles, characteristic of regional aircrafts. In this project the main attention is with the Tollmien-Schlichting waves. When studying those waves there are a receptivity problem and a stability one. Although in this project we deal with a stability problem, we take a short digression into the receptivity theory.

The receptivity theory is a branch of fluid dynamics the importance of which has been highlighted by various experimental observations. It was observed that the same aerodynamic model tested in two different wind tunnels presented a different transition point, despite reproducing the principal similarity parameters, the Reynolds and Mach numbers. This difference is due to apparently less important factors, like difference in the quality of the flow in the test section, level of turbulence in the oncoming flow, acustic noise in the test section, smoothness of the wind tunnel and the model surface, etc. Basically, the quiter the wind tunnel, the longer the boundary layer stays laminar. This can be understood only considering the interaction between the boundary layer and the surrounding enviroment. The analysis of possible forms of interaction is the subject of receptivity theory. Some disturbances easily penetrate into the boundary layer and turn into instability modes; others not. In the former category are acoustic waves, free stream turbulence, local and distributed wall rourhness, etc. These perturbations have to satisfy rather restrictive resonance conditions in order to amplify and trigger the non-linear effects, characteristic of the transition process.

Finally the stability theory, which is the approach used in the second part of this project. We basically disregard how the instability has been generated and we focus on describing Tollmien-Schlichting waves. The Hydrodynamic Stability theory is concerned with understanding how and why transition occurs. Reynolds (1883) was the first to investigate the laminar-turbulent transition process in the Hagen-Poiseuille flow in a circular tube. Reynold observed that that flow suddenly develops unsteadiness for $R_e > 13000$. Figure (5) shows the difference between the laminar flow and the turbulent one. The latter is significantly more complicated and no mathematical description of the phenomenon is present at the moment. Transition in the boundary layer flow on a flat plate was first observed by Burgers (1924) and later in more detail studied by Drygen (1947) and Klebanoff & Tidstrom (1959). They found that near the leading edge of the flat plate the flow is laminar and well described by the Blasius solution. However, at a certain point the unsteadiness given by Tollmien-Schlichting waves superimpose on the steady Blasius flow (figure (6)). Typical situtation is that the initial amplitude of these waves is too small to cause noticeable changes in the velocity field. Nevertheless, they grow downstream and exists a second point where transition happens.

The behaviour of Tollmien-Schlichting waves in subsonic flows is well known. Two different approaches, depending on whether the parallel flow approximation is considered or not, are possible. When dealing with large Raynolds numbers, the rate of change of the longitudinal velocity in the lon-

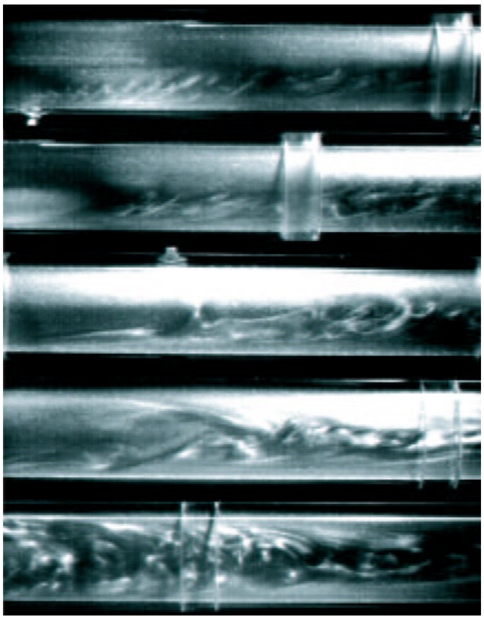

Figure 5: Transition in a pipeline

gitudinal direction is much smaller than in the trasversal direction and the lateral velocity is such small that may be neglected. Under these assumptions, namely the independence of the longitudinal velocity on the longitudinal coordinate and the cancellation of the lateral velocity, parallel stability theory can be used (see (3.6)). On the other hand, when considering the non-parallel effects, the boundary layer assumes a different conformation, consisting on many layers. In the more general case there are five layers shown in figure (7):

1. Potential flow zone (V): by means of this region the free stream condition can be attained;
2. Main part of the boundary layer (IV): it is the continuation of the Prandtl boundary layer;
3. Critical layer (III) : a singularity that occurs where the longitudinal velocity equals the phase speed of the perturbation;
4. Wall layer (I): here the viscous effects are dominant;
5. Inviscid adjustment zone (II).

In this project we consider the case of coincident critical and wall layers, situation that occurs on the lower branch of the neutral curve (later in

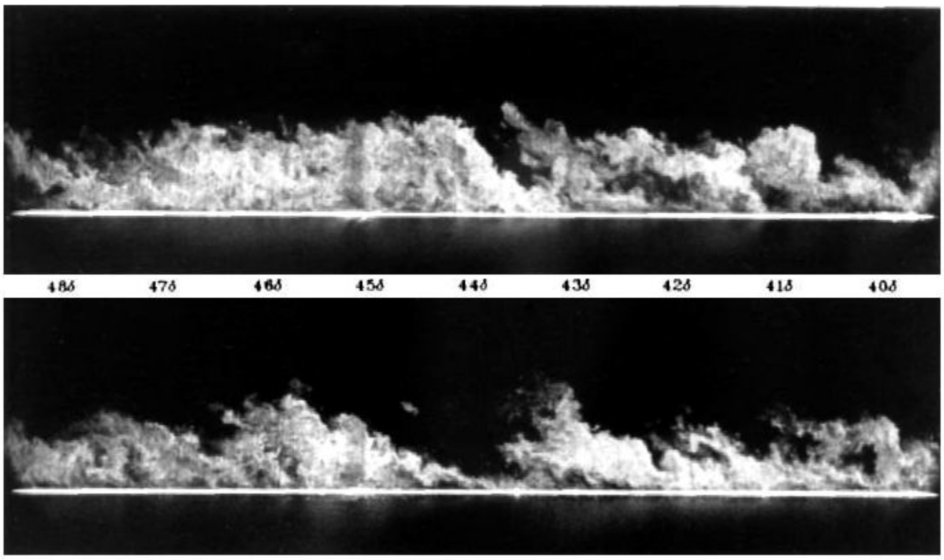

Figure 6: In this image, acetone droplet scattering is used to visualize streamwise cross-sections of a turbulent flat plate boundary layer at Mach 2.82 in streamwise wall-normal planes. The Reynolds number based on momentum thickness is about 82,000. The flow is from right to left, and the horizontal scale indicates the distance from the acetone injection point.

the discussion more details about the neutral curves). In this case we deal with a Triple-Deck structure studied in section (4): an external potential flow zone, the continuation of the boundary layer and a boundary sublayer, where viscous effects are relevant and displacement of the stream lines takes place. Let us now explain what the neutral curve is. In the classical stability theory, perturbations named normal modes are considered. Namely, the perturbations superimposed on the stationary state are periodic in time and longitudinal coordinate. Two parameters, wavenumber and frequency, characterize the periodicity respectively in space and time. Given that the frequency is always real and positive, different values of the wavenumber lead to different situations: amplification, damping or conservation of perturbations. The latter is possible only if the wavenumber is real. The neutral curve is a collection of points in which both wavenumber and frequency are real.

The goal in this project is to analyze how the well known subsonic theory of boundary layer instability near the lower branch of neutral curve modifies when moving to transonic flows and when inspecting the supersonic limit. In the first part we recall the Blasius boundary layer, the parallel flow stability, Triple-Deck and Tollmien-Schlichting waves theories for a subsonic

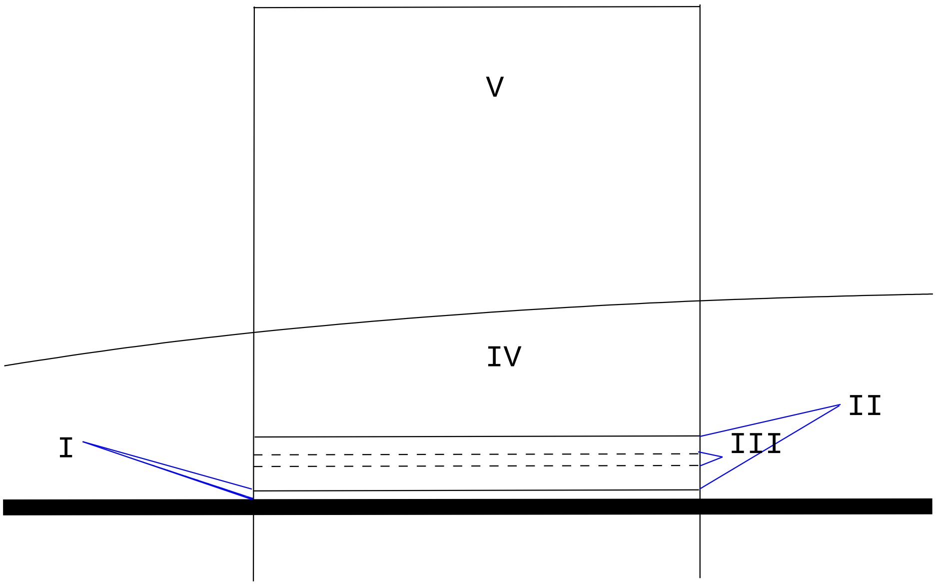

Figure 7: Five-zoned structure

free stream velocity. Subsequently, transonic Triple-Deck and Tollmien-Schlichting waves theories are discussed. At this point our research follows two directions.

First, we perform linear analysis. We end up with a dispersion relation which relates wavenumber and frequency. We found an affine transformation which reduces the transonic equation to the subsonic one. The utility of this transformation is dual. Not only does it allow to easly hand over all the well known subsonic results, for example the neutral values, but also it conveys the dependence of wavenumber and frequency on the Mach number (to be more precise on the deviation from Mach number equals to one). Making use of that result, we can explore the supersonic limit simply assuming that the Karman-Guderley parameter tends to infinity.

Second, we study the nonlinear evolution of a Tollmien-Schlichting wave for a transonic free stream velocity. Supposing that at a certain point a Tollmien-Schlichting wave has frequency close to the neutral value and its amplitude is known, the problem is to determine the wave parameters downstream of this point. This process is given by a linear displacement of the wave combined with a nonlinear process of growth of the amplitude. Having the two processes a different characteristic longitudinal lenght, the multi-scale method has to be used. The aim is to work out a nonlinear equation for amplitude of Tollmen-Schlichting waves. This equation has a term which comes from the propagation of the wave in the inhomogeneous flow accompained by an increase in the growth rate of the wave and a term which comes from the nonlinear growth of the wave amplitude in a field with constant parameters of the undisturbed flow. What this equations shows is which terms accelerate or retard the growth of the Tollmien-Schlichting wave amplitude. Regarding practical applications, the hope is that the process is retarded, so that transition to turbulence is delayed. We conclude the present project with some suggestions for further research on these topics.

3 Prandtl Boundary Layer

Let us consider a two-dimensional steady flow past a flat plate of lenght L aligned with the oncoming flow. We further assume that the flow is incompressible, i.e. the density ρ and the dynamic viscosity coefficient ν are constant all over the flow. Dimensional variables are always denoted by hat. We can place the origin of our cartesian coordinate system at the leading edge of the flat plate, with $\hat{x}$-axis lying on the flat plate. With

Figure 8: Problem layout

velocity components denoted by $\hat{u}$, $\hat{v}$ and pressure by $\hat{p}$, the Navier-Stokes equations, assuming that the body force is negligible, are written in the following form:

$$\left\{ \begin{aligned} \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} &= -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{x}} + \nu\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right) \\ \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} &= -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{y}} + \nu\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right) \\ \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} &= 0 \end{aligned} \right. \tag{3.1}$$

Since the problem is symmetric, we can focus on the upper half plane. The boundary conditions for this problem are the impermeability condition on the plate surface

$$\hat{v} = 0 \text{ at } \hat{y} = 0 \ , \ \hat{x} \in [0, L] \tag{3.2}$$

the no-slip condition on the same

$$\hat{u} = 0 \text{ at } \hat{y} = 0 \ , \ \hat{x} \in [0, L] \tag{3.3}$$

and the free stream condition

$$\left.\begin{array}{l} \hat{u} \to V_\infty \\ \hat{v} \to 0 \\ \hat{p} \to p_\infty \end{array}\right\} \text{ as } \hat{x}^2 + \hat{y}^2 \to \infty \tag{3.4}$$

where V_∞ and p_∞ are respectively velocity and pressure in the free stream. Applying the following transformation

$$\hat{x} = Lx \quad \hat{y} = Ly \quad \hat{u} = V_\infty x \quad \hat{v} = V_\infty v \quad \hat{p} = p_\infty + \rho V_\infty^2 p \tag{3.5}$$

equation (3.1) can be written in the nondimensional form.

$$\left\{\begin{array}{rcl} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} & = & -\frac{\partial p}{\partial x} + \frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} & = & -\frac{\partial p}{\partial y} + \frac{1}{R_e}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & = & 0 \end{array}\right. \tag{3.6}$$

where $R_e = \frac{V_\infty L}{\nu}$ and the boundary conditions become

$$u = v = 0 \text{ at } y = 0 \ , \ x \in [0, 1] \tag{3.7}$$

$$\left.\begin{array}{l} u \to 1 \\ v \to 0 \\ p \to 0 \end{array}\right\} \text{ as } x^2 + y^2 \to \infty \tag{3.8}$$

3.1 Large Reynolds Number Flows

When dealing with fluids like air which have a small viscosity and in addition with high free stream velocities, which is the case of an airplane motion, we shall assume that $R_e \to \infty$. Looking at equation (3.6), at first it seems that we can disregard the viscous terms, being proportional to R_e^{-1}. However, disregarding those terms we obtain the Euler equations, which are solvable considering only the free stream condition and the impermeability condition on the plate. In this problem, being the flat plate infinitely thin, we have simply that

$$u = 1 \quad v = 0 \quad p = 0$$

everywhere in the plane and there is no way to satisfy the no-slip condition on the flat plate surface. Ludwig Prandtl, in his talk at the 3^{rd} International Mathematics Congress which took place in Heidelberg in 1904, showed how high Reynolds number flows should be treated and put forward the idea of singular perturbation, which later became one of the most important concepts in modern applied mathematics and mathematical physics. He introduced the concept of boundary layers, which are regions where a rapid change occurs in the value of a variable. Indeed, we have to introduce a region in the proximity of the plate surface where, in order to satisfy the no-slip condition, the horizontal velocity should go from 1 to 0. Mathematically, the occurrence of boundary layers is associated with the presence of a small parameter multiplying the highest derivative in the governing equation of a process. A perturbative expansion using an asymptotic expansion in the small parameter leads to differential equations of lower order than the original ones, so that the number of necessary boundary conditions is reduced and one of the initial conditions is not satisfied. The solution consists in introducing an expansion, in terms of a new streched variable, valid within a layer adiacent to the boundary where that condition is not satisfied.

3.2 Boundary layer over a flat plate

We apply the idea of boundary layers to the flat plate problem (3.6), (3.7), (3.8). The idea suggested in subsection (3.1) is to consider two regions:

- Outer region

  In the outer region both coordinates x and y are of the order of the flat plate lenght, i.e. they are order one quantities. Hence, there is no inspection of the boundary layer and the equations turn out to be the Euler equations, as in subsection (3.1).

- Inner region

  Since the outer solution does not satisfy the no-slip condition we introduce a small region with a scaled vertical coordinate where the velocity u experiences a ripid variation.

See figure (9) for a scatch of the two regions.

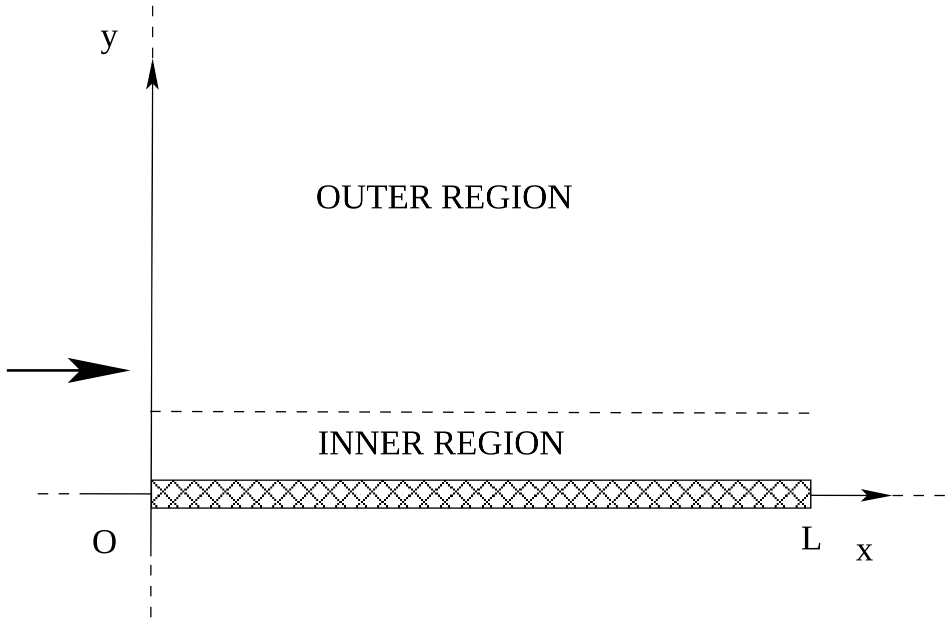

Figure 9: Scatch of inner and outer regions

3.2.1 Outer region

In this region we assume that

$$x = O(1) \qquad y = O(1) \qquad R_e \to \infty \tag{3.9}$$

and we can seek the following expansions

$$\begin{cases} u(x, y; R_e) = u_0(x, y) + ... \\ v(x, y; R_e) = v_0(x, y) + .. \\ p(x, y; R_e) = p_0(x, y) + .. \end{cases} \tag{3.10}$$

Substituting (3.10) into the Navier-Stokes equations (3.6) we end up with the following

$$\begin{cases} u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} & = & -\frac{\partial p_0}{\partial x} \\ u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} & = & -\frac{\partial p_0}{\partial y} \\ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} & = & 0 \end{cases} \tag{3.11}$$

which are the Euler equations. They do not involve the second order derivatives of velocity components and therefore they cannot be solved with the entire set of boundary conditions (3.7) and (3.8). According with the inviscid theory, Euler equations are compatible with the free stream conditions

$$\left.\begin{array}{l} u_0 \to 1 \\ v_0 \to 0 \\ p_0 \to 0 \end{array}\right\} \text{ as } x^2 + y^2 \to \infty \tag{3.12}$$

and the impermeability condition

$$v_0 = 0 \text{ at } y = 0 \text{ , } x \in [0, 1] \tag{3.13}$$

An infinitely thin flat plate does not produce any perturbation in an inviscid flow and, indeed, by direct substitution one can easily verify that the solution is

$$u_0 = 1 \qquad v_0 = 0 \qquad p_0 = 0 \tag{3.14}$$

3.2.2 Inner region

In this region the x coordinate is still order one, given the fact that the boundary layer extends along the entire flat plate surface. On the other hand, we write that

$$y = \delta(R_e)Y \text{ with } \delta(R_e) \to 0 \text{ as } R_e \to \infty \tag{3.15}$$

where Y is an order one quantity. The limit procedure is

$$x = O(1) \qquad Y = \delta^{-1}y \qquad R_e \to \infty \text{ .} \tag{3.16}$$

Correspondingly, the leading order terms of the asymptotic expansions of u, v and p in this region will be sought in the form

$$\left\{\begin{array}{l} u(x, y; R_e) = U_0(x, Y) + ..., \\ v(x, y; R_e) = \sigma(R_e)V_0(x, Y) + .., . \\ p(x, y; R_e) = \chi(R_e)P_0(x, Y) + ..., \end{array}\right. \tag{3.17}$$

where we know that u decreases from 1 to 0 and therefore is an order one quantity, but we do not have any information about v and p in advance. Let us start substituting (3.17) into (3.6).

- Continuity equation:

$$\frac{\partial U_0}{\partial x} + \frac{\sigma}{\delta}\frac{\partial V_0}{\partial Y} = 0$$

Prandtl suggested the Principle of Least Degeneration, consisting of retaining the largest number of terms in equations. This ensures that the boundary layer solutions contain rapidly varying functions. Hence, we have to choose

$$\sigma = \delta. \tag{3.18}$$

We can also prove that this choice is the only possible. Indeed, if $\delta >> \sigma$ the continuity equation degenerates to

$$\frac{\partial U_0}{\partial x} = 0$$

which, according to the fact that $u = 1$ at the trailing edge of the flat plate, has the solution

$$U_0 = 1$$

like if we were still dealing with the inviscid region. If, on the other hand, $\delta << \sigma$ the equation becomes

$$\frac{\partial V_0}{\partial Y} = 0$$

With $V_0 = 0$ on the plate surface, the solutions is

$$V_0 = 0$$

which means that the asymptotic expansion for v in (3.17) does not really have a term with σ larger that δ.

- Longitudinal momentum equation:

$$U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial Y} = -\chi \frac{\partial P_0}{\partial x} + \frac{1}{R_e} \frac{\partial^2 U_0}{\partial x^2} + \frac{1}{\delta^2 R_e} \frac{\partial^2 U_0}{\partial Y^2}$$

The second term on the right hand side is small compared to any left hand side terms, which are all order one. The principle of least degeneration suggests to set

$$\delta = R_e^{-1/2}. \tag{3.19}$$

Again we can verify this condition. If $\delta^2 >> 1$ the fluid would appear inviscid and it would be impossible to satisfy the no-slip condition on the flat plate surface. On the other hand, if $\delta^2 << 1$ the equation degenerates to

$$\frac{\partial^2 U_0}{\partial Y^2} = 0 \tag{3.20}$$

with boundary conditions $U_0(Y = 0) = 0$, which is the no-slip condition, and $U_0(Y = \infty) = 1$, which is the matching condition with the outer region. We have not mentioned before that when solving the equations in the inner region we need one more condition which is the matching condition with the outer region, in this case expressed by:

$$\lim_{Y\to\infty} U_0(x,Y) = \lim_{y\to 0} u_0(x,y).$$

Clearly (3.20) has no solution. Finally, the equation is:

$$U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial Y} = -\chi\frac{\partial P_0}{\partial x} + \frac{\partial^2 U_0}{\partial Y^2}. \tag{3.21}$$

- Lateral momentum equation:

$$U_0\frac{\partial V_0}{\partial x} + V_0\frac{\partial V_0}{\partial Y} = -R_e\chi\frac{\partial P_0}{\partial Y} + \cancel{\frac{1}{R_e}\frac{\partial^2 V_0}{\partial x^2}} + \frac{\partial^2 V_0}{\partial Y^2} \tag{3.22}$$

If $\chi \sim 0$ equation (3.21) reduces to

$$U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial Y} = \frac{\partial^2 U_0}{\partial Y^2} \tag{3.23}$$

If $\chi = O(1)$ or $\chi >> 1$ equation (3.22) becomes

$$\frac{\partial P_0}{\partial Y} = 0 \tag{3.24}$$

which, with boundary condition $P_0(Y=\infty)=0$, produces

$$P_0 = 0$$

everywhere inside the boundary layer and again (3.21) reduces to (3.23). It is an important conclusion that, being the pressure perturbation equal to zero outside the boundary layer, the pressure does not change across the boundary layer.

We can conclude that the full set of equations is:

$$\begin{cases} U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial Y} = \frac{\partial^2 U_0}{\partial Y^2} \\ P_0 = 0 \\ \frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial Y} = 0 \end{cases} \tag{3.25}$$

We need now to formulate the boundary conditions. Let us start with the momentum equation. In order to formulate the boundary conditions, we have first to evaluate the type of the equation. The latter is determined by the higher order derivatives in the equation considered, i.e.

$$U_0\frac{\partial U_0}{\partial x} = \frac{\partial^2 U_0}{\partial Y^2}$$

This is a parabolic equation, among which is the heat equation, which describes the heat transfer for a metallic rod.

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

where T is the temperature, t the time and a a positive constant. The boundary conditions needed to make unique the solution to this problem, i.e. the temperature distribution at time t_0, are:

- Initial temperature distribution at t_0
- Thermal condition at the rod ends for any $t \in [0, t_0]$. There is no dependence of the solution on later times.

In the same way, to solve the momentum equation we need to express:

- Initial condition for U_0 at the trailing edge of the flat plate
- Boundary conditions for U_0 on the plate surface and at the outer edge of the boundary layer

These conditions are the free stream condition at the trailing edge:

$$U_0 = 0 \text{ at } x = 0 \text{ , } Y \in (0, \infty) \tag{3.26}$$

the no-slip condition

$$U_0 = 0 \text{ at } Y = 0 \text{ , } x \in [0, 1] \tag{3.27}$$

and the matching condition with the outer region

$$U_0 = 1 \text{ at } Y = \infty \tag{3.28}$$

Finally, the continuity equation needs a condition for V_0 which comes from the impermeability condition:

$$V_0 = 0 \text{ at } Y = 0 \text{ , } x \in [0, 1] \tag{3.29}$$

The boundary layer problem consists in solving (3.25) with boundary conditions (3.26)-(3.29).

3.3 The Blasius solution

We said that the momentum equation is parabolic. Therefore, at any point x the solution for U_0 does not depend on the following points and it is like if the flat plate was semi-infinite. This is an important consideration, as in

the case of a semi-infinite flat plate we do not have any characteristic lenght and we can expect a solution in the self-similar form. Matematically we are looking for an affine transformation.
Let us suppose that

$$U_0 = F(x, Y) \qquad V_0 = G(x, Y) \tag{3.30}$$

are solutions to our problem. We look for an affine transformation that leaves (3.25) with boundary conditions (3.26)-(3.29) unchanged. We write

$$U_0 = A\tilde{U}_0 \qquad V_0 = B\tilde{V}_0 \qquad x_0 = C\tilde{x} \qquad y = D\tilde{y} \tag{3.31}$$

where A, B, C and D are positive constants. Substitution into (3.25) and into the boundary conditions (3.26)-(3.29) gives:

$$\begin{cases} \frac{A^2}{C}\tilde{U}_0\frac{\partial \tilde{U}_0}{\partial \tilde{x}} + \frac{AB}{D}\tilde{V}_0\frac{\partial \tilde{U}_0}{\partial \tilde{Y}} = \frac{A}{D^2}\frac{\partial^2 \tilde{U}_0}{\partial \tilde{Y}^2} \\ \frac{A}{C}\frac{\partial \tilde{U}_0}{\partial \tilde{x}} + \frac{B}{D}\frac{\partial \tilde{V}_0}{\partial \tilde{Y}} = 0 \end{cases} \tag{3.32}$$

$$\begin{cases} A\tilde{U}_0 = 1 \text{ at } \tilde{x} = 0 \\ \tilde{U}_0 = \tilde{V}_0 = 0 \text{ at } \tilde{Y} = 0 \\ A\tilde{U}_0 = 1 \text{ at } \tilde{Y} = \infty \end{cases} \tag{3.33}$$

To ensure that the equations and the boundary conditions remain unchanged we have to set

$$\frac{A^2}{C} = \frac{AB}{D} = \frac{A}{D^2} \qquad \frac{A}{C} = \frac{B}{D} \qquad A = 1$$

Solving these equations we find the following

$$A = 1 \qquad B = \frac{1}{\sqrt{C}} \qquad D = \sqrt{C} \tag{3.34}$$

with C remaining arbitrary. Since this problem coincides with the original one, also

$$\tilde{U}_0 = F(\tilde{x}, \tilde{Y}) \text{ , } \tilde{V}_0 = G(\tilde{x}, \tilde{Y})$$

is a solution to the boundary layer problem. Using (3.34) we can write that

$$U_0 = F(\tfrac{x}{C}, \tfrac{Y}{\sqrt{C}}) \qquad V_0 = \frac{1}{\sqrt{C}}G(\tfrac{x}{C}, \tfrac{Y}{\sqrt{C}}) \tag{3.35}$$

Being C an arbitrary value, it may be considered as an additional independent variable which, in particular, could be chosen to coincide with x. Therefore:

$$U_0(x, Y) = F(1, \eta) \qquad V_0(x, Y) = \frac{1}{\sqrt{x}}G(1, \eta) \qquad \eta = \frac{Y}{\sqrt{x}}$$

The distributions for U_0 and V_0 across the boundary layer are reduced to those at the trailing edge of the plate. Let us call those functions at the trailing edge $f(Y)$ and $g(Y)$

$$U_0(x,Y) = f(\eta) \qquad V_0(x,Y) = \frac{1}{\sqrt{x}} g(\eta) \qquad \eta = \frac{Y}{\sqrt{x}} \tag{3.36}$$

In order to find f and g we have to substitute (3.36) into (3.25) and into the boundary conditions (3.26)-(3.29), keeping in mind that we expect those equations to be dependent only on η and not on x or Y separately. This is because f and g depend only on η and therefore the equations cannot contain x or Y separately. The derivatives are

$$\begin{array}{lll} \frac{\partial \eta}{\partial Y} = x^{-1/2} & \frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{\eta}{x} & \frac{\partial U_0}{\partial x} = -\frac{1}{2}x^{-1}\eta f' \\ \frac{\partial U_0}{\partial Y} = x^{-1/2} f' & \frac{\partial^2 U_0}{\partial Y^2} = x^{-1} f'' & \frac{\partial V_0}{\partial Y} = x^{-1} g' \end{array}$$

The problem takes the following form

$$\begin{cases} -\frac{1}{2}\eta f f' + g f' = f'' \\ -\frac{1}{2}\eta f' + g' = 0 \end{cases} \tag{3.37}$$

$$\begin{cases} f(0) = g(0) = 0 \\ f(\infty) = 1 \end{cases} \tag{3.38}$$

where the first in (3.38) comes from the impermeability and no-slip conditions while the second comes from both the initial and the matching conditions. We can simplify the problem by writing the continuity equation as

$$g' = \frac{1}{2}\eta f' = \frac{1}{2}(\eta f)' - \frac{1}{2} f \tag{3.39}$$

If we introduce $\varphi(\eta)$ such that

$$\varphi'(\eta) = f(\eta) \text{ and } \varphi(0) = 0 \tag{3.40}$$

we can integrate from 0 to η equation (3.39) getting

$$g = \frac{1}{2}\eta\varphi' - \frac{1}{2}\varphi \tag{3.41}$$

When substituting into the momentum equation we have the Blasius equation

$$\varphi''' + \frac{1}{2}\varphi\varphi'' = 0 \tag{3.42}$$

with boundary conditions deduced by (3.38)

$$\varphi(0) = \varphi'(0) = 0 \qquad \varphi'(\infty) = 1 \tag{3.43}$$

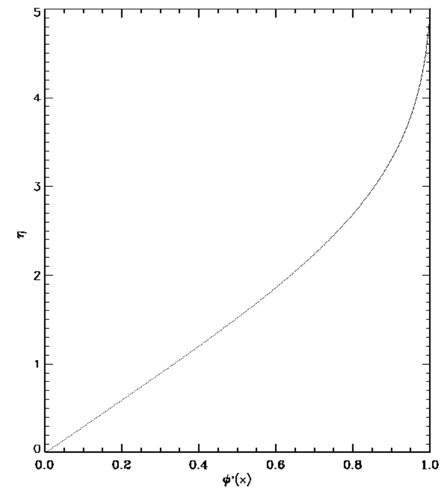

Figure 10: Solutionto the Blasius problem. This is the velocity profile $U_0(\eta)$.

When φ is numerically calculated, the velocity components in the boundary layer are, from (3.36) with (3.40) and (3.41)

$$U_0 = \varphi'(\eta) \qquad V_0 = \frac{1}{2\sqrt{x}}(\eta\varphi' - \varphi) \tag{3.44}$$

The results of the numeric calculation of φ are shown in (3.4). Figure (10) shows the longitudinal velocity profile. For future references we need to study the asymptotic behavior of $\varphi(\eta)$ at the upper edge of the boundary layer and near the wall.

- Beheavior near the wall, $\eta \to 0$

  In this limit we can write that

  $$\varphi(\eta) = \varphi(0) + \varphi'(0)\eta + \frac{1}{2}\varphi''(0)\eta^2 + ...$$

  However, according with (3.43) we have that

  $$\varphi(\eta) = \frac{1}{2}\lambda\eta^2 + ... \text{ where } \lambda = \varphi''(0) = 0.33 \tag{3.45}$$

  is known from the numerical solution (see (3.4)).

- Large values $\eta \to \infty$

  Due to the second expression in (3.43) we have that, in this limit:

  $$\varphi(\eta) = \eta + ...$$

For the next term let us try a power function

$$\varphi(\eta) = \eta + A\eta^{\alpha} + ...$$

Of course the second term must be smaller than the first one, therefore $\alpha < 1$. Substituting into the Blasius Equation (3.42)

$$A\alpha(\alpha-1)(\alpha-2)\eta^{\alpha-3} = -\frac{1}{2}A\alpha(\alpha-1)\eta^{\alpha-1}$$

and we have to set $A\alpha(\alpha-1) = 0$, as the right hand side is much greater than the left hand side. The only possible choice, according with the constraint $\alpha < 1$, is

$$\alpha = 0$$

Finally we have that

$$\varphi(\eta) = \eta + A + ... \tag{3.46}$$

with

$$A = -1.78 \tag{3.47}$$

from the numerical calculation (see (3.4)).

The last consideration about the Blasius boundary layer is about the displacement thickness which can be expressed in the following way:

$$\begin{aligned} \frac{d\delta}{dx} &= \lim_{\eta\to\infty} \frac{V_0}{U_0} = \lim_{\eta\to\infty} \frac{\eta\varphi'(\eta)-\varphi}{2\sqrt{x}\varphi'(\eta)} = \\ &= \lim_{\eta\to\infty} \frac{\not\eta - \not\eta - A}{2\sqrt{x}} = -\frac{A}{2\sqrt{x}} \end{aligned} \tag{3.48}$$

where we have used the asympthotic expansion at $\eta \to \infty$ and the boundary condition $\varphi'(\infty) = 1$. Integrating with $\delta(0) = 0$ we have

$$\delta(x) = -Ax^{1/2} \tag{3.49}$$

i.e. a progressive displacement of the stream lines going downstream. This is due to the fluid deceleration in the boundary layer. Indeed, looking at figure (11) we can say that the fluid volume flux through AA' and BB' must be the same, i.e.

$$\int_A^{A'} udY = \int_B^{B'} udY$$

Moreover, we know that on any line parallel to the flat plate u decreases monotolically as moving downstream. Indeed, at fixed Y, an increase of x results in a decrease of η which leads to a decrease of u, as shown in figure (10).

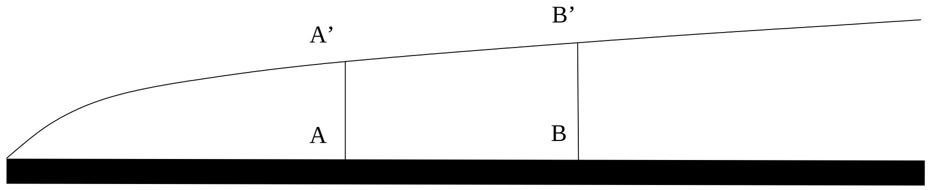

Figure 11: Displacement of the stream lines

3.4 Numerical solution to the Blasius equation

We recall that we have to solve

$$\varphi'''(x) = -\frac{1}{2}\varphi(x)\varphi''(x) \tag{3.50}$$

with boundary conditions

$$\varphi(0) = \varphi'(0) = 0 \text{ and } \varphi'(\infty) = 1 \tag{3.51}$$

First of all we reduce equation (3.50) to a system of first order differential equations. We call $\varphi(x) = f_0(x)$ and then

$$\begin{cases} f_0'(x) = f_1(x) \\ f_1'(x) = f_2(x) \\ f_2'(x) = -\frac{1}{2}f_0(x)f_2(x) \end{cases} \tag{3.52}$$

with boundary conditions

$$f_0(0) = 0 \qquad f_1(0) = 0 \qquad f_1(\infty) = 1 \tag{3.53}$$

3.4.1 Runge Kutta method

Let an initial value problem be specified as follows

$$\dot{y} = f(t, y) \text{ with } y(t_0) = y_0$$

The numerical solution using the forth order Runge Kutta method consists of choosing a step-size $h > 0$ and defining

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$t_{n+1} = t_n + h$$

where

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

being k_1 the increment based on the slope at the beginning of the interval (Euler method) using $\dot{y}$, k_2 and k_3 the increments based on the slope at the midpoint of the interval using respectively $\dot{y} + \frac{1}{2}hk_1$ and $\dot{y} + \frac{1}{2}hk_2$ and finally k_4 the increment based on the slope at the end of the interval using $\dot{y} + hk_3$.

3.4.2 Runge Kutta for the Blasius Problem

Coming back to the Blasius problem (3.52) and (3.53) we can see that we miss an initial condition for the third of equations (3.52). The only possibility is to guess a value for $f_2(0)$ and change it untill the boundary condition $f_1(\infty) = 1$ is satisfied. Shooting methods are used in these situations. Our aim here is neither to calculate with extreme precision the value of the velocity nor to develop a vast programme to do so. Hence, in the programme that is presented here the initial value for $f_2(0)$ comes from some runnings with different values. A noticeable consideration is that already for values like $x \sim 5$ we are in a good approximation of ∞. This is the reason for which the programme will evaluate all the quatities for $x \in [0, 5]$ with boundary condition at $x = \infty$ evaluated at $x = 5$.

3.4.3 The programme

```
function effe,z,i
a=0.
if i eq 0 then a=z[1]
if i eq 1 then a=z[2]
if i eq 2 then a=-0.5*z[0]*z[2]
return,a
end

function runge4,x
common costanti,N,step,y
h=step/2.
t1=dindgen(N)
t2=dindgen(N)
t3=dindgen(N)
```

```
k1=dindgen(N)
k2=dindgen(N)
k3=dindgen(N)
k4=dindgen(N)
for i=0,N-1 do begin
k1[i]=step*effe(y, i)
t1[i]=y[i]+0.5*k1[i]
endfor
for i=0,N-1 do begin
k2[i]=step*effe(t1, i)
t2[i]=y[i]+0.5*k2[i]
endfor
for i=0,N-1 do begin
k3[i]=step*effe(t2, i)
t3[i]=y[i]+k3[i]
endfor
for i=0,N-1 do k4[i]= step*effe(t3, i)

for i=0,N-1 do y[i]+=(k1[i]+2*k2[i]+2*k3[i]+k4[i])/6.0
plots,[y[1],x],color=0,psym=3
;print,y
end

common costanti,N,step,y
N=3
step=0.01
y=dindgen(N)
massimo=5.
y[0]=0.
y[1]=0.
y[2]=0.3360781
j=0
loadct,5
set_plot, 'z'
device, set_resolution=[640,680]
plot,[0.,0.],back=255,color=0,yrange=[0,5],xrange=[0,1], $
ytitle='!7g !N',xtitle='!7u!A,!N!3(x)',charsize=1.2, $
charthick=2,thick=0,xthick=2,ythick=2
;plots,[j*step,y[1]],color=0,psym=3
for j=0,massimo/step do a=runge4(j*step)
print,y
```

```
write_png, 'Blasius.png',tvrd()
end
```

3.4.4 Results

The guessed value is

$$f_2(0) = \varphi''(0) = 0.3360781$$

This is what we call $\lambda = 0.33$ in (3.45). We end up with the velocity profile in figure (12). The values for f_1, f_2 and f_3 at infinity are

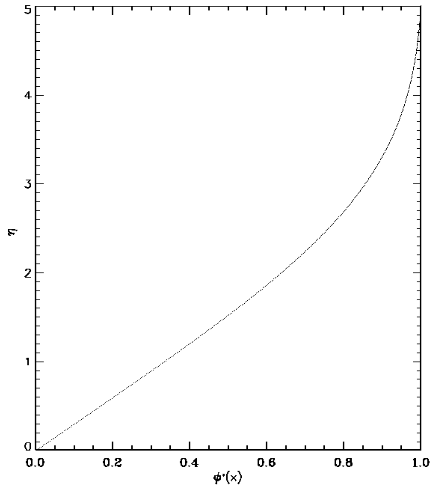

Figure 12: Velocity profile

```
IDL> .GO
LOADCT: Loading table STD GAMMA-II
       3.3264852       1.0000015     0.015318755
```

We note that f_1 is quite close to 1 as desired. From the first value, using the asymptotic expansion (3.46), we can write that

$$3.32 = 5 + A$$

being therefore able to evaluate $A = -1.78$ as in (3.47).

3.5 Parallel stability of the Boundary Layer

Lut us suppose that

$$U_0(x, Y) = f(\eta)\ ,\ V_0(x, Y) = g(\eta)$$

is a solution to the boundary layer problem. However, the existence of the solution does not guarantee that the corresponding flow can actually exist in Nature. For this to happen the flow has to be stable, namely, if the basic state is superimposed by a perturbation

$$\begin{cases} U_0 = f(\eta) + \epsilon u'(t,x,Y) \\ V_0 = g(\eta) + \epsilon v'(t,x,Y) \\ P_0 = \epsilon p'(t,x,Y) \end{cases}$$

of small amplitude ϵ, then the perturbation has to extinguish with time returning the solution to its basic state.
The idea is to use the Orr-Sommerfeld two-dimensional equation which describes parellel flows, like the Poiseuille and the Shear ones. A parallel flow has longitudinal velocity dependent only on y and has no vertical velocity component. If the steady solution is:

$$\begin{cases} u(x,y) = U(y) \\ v(x,y) = 0 \\ p(x,y) = 0 \end{cases}$$

and we superimpose a normal mode perturbation as follows

$$\begin{cases} u(x,y,t) = U(y) + \epsilon \bar{u}(Y) e^{i(kx-\omega t)} \\ v(x,y,y) = \epsilon \bar{v}(Y) e^{i(kx-\omega t)} \\ p(x,y,t) = \epsilon \bar{p}(Y) e^{i(kx-\omega t)} \end{cases}$$

Substituting these into the full Navier-Stokes equations, Orr and Sommerfeld obtained the following equation:

$$\frac{1}{ikR_e}\left(\ddot{\ddot{\bar{v}}} - 2k^2\ddot{\bar{v}} + k^4\bar{v}\right) = (U - \frac{\omega}{k})\left(\ddot{\bar{v}} - k^2\bar{v}\right) - \ddot{U}\bar{v} \tag{3.54}$$

This equation has to be solved numerically, finding the complex phase $c = \frac{\omega}{k} = c_r + ic_i$ for each pair of real R_e and k. Note that R_e is positive and we can consider k positive as well, without any loss of generality. Of primary interest are points were $c_i = 0$, in which case the perturbations are waves with constant amplitude propagating with velocity c_r. The locus of such points is the (R_e, k)-plane called neutral curve for which the perturbations neither grow or decay. This allows to separate the region of instability where $c_i > 0$ from the region of stability where $c_i < 0$.

3.6 Parallel approximation

The Blasius solution shows that the longitudinal velocity component u depends not only on y but also on x and that the lateral velocity component v is non-zero. Therefore, the flow is not parallel. However, if the Reynolds number is large enough, than the rate of change of u in the x-direction is much smaller than in the y-direction (see (3.16) and (3.19))

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \sim R_e^{-1/2}$$

and furthermore the lateral velocity comoponent is small (see (3.17), (3.18) and (3.19))

$$v \sim R_e^{-1/2} \tag{3.55}$$

Being guided by the results of the instability analysis for the channel flow, we can expect the wave lenght l of the normal mode perturbations to be comparable with the characteristic lenght scale across the boundary layer, i.e.

$$l \sim R_{ec}{}^{-1/2}$$

under the assumption that the critical Reynolds number is large. We now choose a position x_* and assume that the velocity profile can be considered frozen in a vicinity of x_*. In this position, the displacement thickness of the boundary layer is

$$\delta_* = -A x_*^{1/2}$$

Let us now rescale in the following way:

$$x = x_*^{1/2}\tilde{x} \qquad Y = x_*^{1/2}\tilde{Y} \qquad t = x_*^{1/2}\tilde{t}$$

Consequently, the Reynolds number becomes

$$R_{e*} = \frac{V_\infty L_*}{\nu} \text{ with } L_* = \sqrt{\frac{\nu x_*}{V_\infty}} \tag{3.56}$$

and the new similarity variable in x_* is

$$\eta = \frac{Y}{\sqrt{x_*}} = \tilde{Y}$$

According with (3.55) the flow functions in the basic laminar flow are (variables are now written without $\sim$)

$$\begin{cases} U_0 = \varphi'(Y) \\ V_0 = 0 \\ P_0 = 0 \end{cases} \tag{3.57}$$

Note that in this parallel flow approximation, the stream function ψ which satisfies the conditions

$$U_0 = \frac{\partial \psi}{\partial Y} \qquad V_0 = -\frac{\partial \psi}{\partial x}$$

coincides with

$$\psi(Y) = \varphi(Y)$$

Let us now introduce a small perturbation to the stream function

$$\psi(x, Y, t) = \varphi(Y) + \epsilon \bar{\psi}(Y) e^{i\alpha(x-ct)} \tag{3.58}$$

Therefore, the Orr-Sommerfeld equation is

$$\frac{i}{\alpha R_{e*}} (\bar{\psi}''''(Y) - 2\alpha^2 \bar{\psi}''(Y) + \quad \alpha^4 \bar{\psi}) + (\varphi'(Y) - c)(\bar{\psi}''(Y) - \alpha^2 \bar{\psi}(Y)) + \\ -\varphi'''(Y)\bar{\psi}(Y) = 0 \tag{3.59}$$

with boundary conditions on the wall (from the impermeability and no-slip conditions for V_0)

$$\bar{\psi}(0) = \bar{\psi}'(0) = 0 \tag{3.60}$$

and in the free stream

$$\bar{\psi}(\infty) = \bar{\psi}'(\infty) = 0 \tag{3.61}$$

This is an eigenvalue problem for the parameter c when α and R_{e*} are given. The parameter c is complex, in general. A point of the neutral curve is obtained if for real α also c becomes real and therefore $\omega = \alpha c$ is real. The initial condition to integrate the problem will be given at ∞, since the asymptotic behavior at ∞ is simple. Since from (3.43)

$$\varphi'(\infty) = 1$$

and from (3.42) evaluated at ∞ with (3.43)

$$\varphi'''(\infty) = 0$$

equation (3.59) becomes

$$\frac{i}{\alpha R_{e*}} \left(\bar{\psi}''''(Y) - 2\alpha^2 \bar{\psi}''(Y) + \alpha^4 \bar{\psi}\right) + (1 - c)(\bar{\psi}''(Y) - \alpha^2 \bar{\psi}(Y)) = 0 \tag{3.62}$$

which is a linear equation with constant coefficients. Hence, four complementary solutions of this equation may be sought in the form

$$\bar{\psi}_i = e^{\lambda_i Y} \text{ at } Y = \infty$$

Solving the equation we have the following solutions

$$\lambda_{1,2} = \pm\alpha$$

$$\lambda_{3,4} = \pm\gamma \text{ with } \gamma = \sqrt{\alpha^2 + i\alpha R_{e*}(1-c)}$$

In order to satisfy the free stream boundary condition (3.61) we have to consider only λ_2 and λ_4, therefore:

$$\bar{\psi}(Y) = Ae^{-\alpha Y} + Be^{-\sqrt{\alpha^2+i\alpha(1-c)}Y} \tag{3.63}$$

3.7 Numerical solution of the Orr-Sommerfeld equation

The numerical method to solve the Orr-Sommerfeld equation is a shooting method. The values of R_{e*} and α real are given. The shooting consists of finding which values of c_r and c_i are such that, integrating from a large value $Y = Y_1$ toward $Y = 0$, the solution satisfies the boundary condition (3.60) on the wall. Figure (13) shows the neutral curve. The critical value of R_{e*_c}, at which the Blasius boundary layer looses stability, is found to be

$$R_{e*_c} = 518.0$$

with the corresponding wave number $\alpha_c = 0.303$.
In the following section the non-parallel effect will be taken into account. As anticipated in the Introduction, we deal with the lower branch of the neutral curve, where there is no distinction between the wall layer and the critical layer, i.e. the critical layer is not considered.

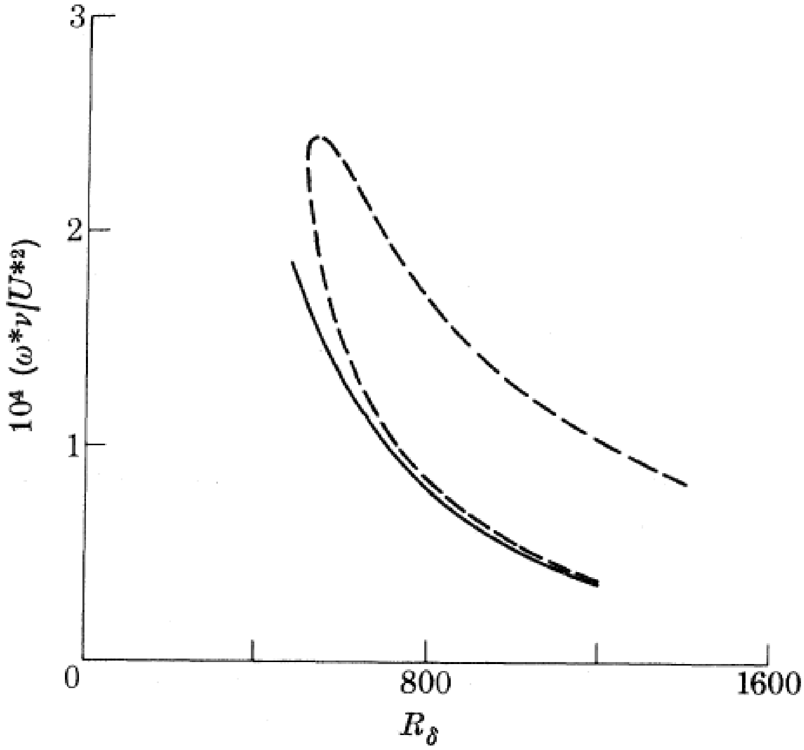

Figure 13: (- - -) Parallel flow theory numerical solution for the neutral curve

4 Subsonic Triple-Deck Theory

The aim is to understand how a small perturbation $O(\epsilon)$, occurring at distance x_0 and interesting a region $O(\delta)$ along the x-axis, changes the boundary layer structure. Let us place the origin of the coordinate system in x_0. Considering a subsonic flow regime, i.e. $M_\infty = 0$, we are dealing with the incompressible unsteady Navier-Stokes equations, here expressed in their adimensional form:

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{R_e}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{cases} \tag{4.1}$$

4.1 Perturbation of the Boundary Layer

This region is the continuation of the normal boundary layer, whose quantities and equations are shown in section (3). However, a small perturbation $O(\epsilon)$ is added. Keeping in mind that $\Delta x = O(\delta)$ and $\Delta u = O(\epsilon)$ and therefore $\Delta t = \frac{\Delta x}{\Delta u} = O(\frac{\delta}{\epsilon})$, the scaling for the coordinates is the following:

$$x = \delta x_* \quad y = R_e^{-1/2} Y \quad t = \frac{\delta}{\epsilon} t_* \tag{4.2}$$

We can seek the following expansions:

$$\begin{cases} u(x,y,t) &= U_0(Y) + \epsilon\bar{u}(x_*, Y, t_*) + ... \\ v(x,y,t) &= \chi\bar{v}(x_*, Y, t_*) + ... \\ p(x,y,t) &= \sigma\bar{p}(x_*, Y, t_*) + ... \end{cases}$$

The following considerations follow from the principle of least degeneration. From the continuity equation $\chi = \frac{\epsilon}{\delta R_e^{1/2}}$. Although from the first momentum equation we obtain that $\sigma = \epsilon$, the second momentum equation shows that $\bar{p} = 0$. Hence, the pressure perturbation is order ϵ^2 and the equations are:

$$\begin{cases} U_0(Y)\frac{\partial \bar{u}}{\partial x_*} + \bar{v}\frac{dU_0}{dY} = 0 \\ \frac{\partial \bar{u}}{\partial x_*} + \frac{\partial \bar{v}}{\partial Y} = 0 \end{cases}$$

The first equation is written as $U_0^2 \frac{\partial}{\partial Y}\left(\frac{\bar{v}}{U_0}\right) = 0$ and the solution for $\bar{u}$ and $\bar{v}$ leads to:

$$\begin{cases} u(x,y,t) &= U_0(Y) + \epsilon A(x_*, t_*)\frac{dU_0}{dY} + ... \\ v(x,y,t) &= -\frac{\epsilon}{\delta R_e^{1/2}}\frac{\partial A}{\partial x_*} U_0(Y) + ... \\ p(x,y,t) &= O(\epsilon^2) + ... \end{cases} \tag{4.3}$$

The limit for $Y \to \infty$, given that in this limit $U_0 = 1$, produces:

$$\begin{cases} u = 1 + O(\epsilon^2) \\ v = -\frac{\epsilon}{\delta R_e^{1/2}} \frac{\partial A}{\partial x_*} + \dots \\ p = O(\epsilon^2) \end{cases} \tag{4.4}$$

The limit for $Y \to 0$, given that in this limit $U_0 = \lambda Y$, produces:

$$\begin{cases} u = \lambda Y + \epsilon A(x_*, t_*) + \dots \\ v = -\frac{\epsilon}{\delta R_e^{1/2}} \frac{\partial A}{\partial x_*} \lambda Y + \dots \\ p = O(\epsilon^2) \end{cases} \tag{4.5}$$

Since at Y=0 we have that $v = 0$ but $u = \epsilon A(x_*, t_*)\lambda$, we need to introduce a sublayer in order to satisfy the no-slip condition.

4.2 Viscous Sublayer

In the sublayer the velocity scale is ϵ and, being the velocity linear in Y near the wall, we expect the scaling for y to be order $\epsilon R_e^{-1/2}$. Hence, the scaling is:

$$x = \delta x_* \quad y = \epsilon R_e^{-1/2} \breve{y} \quad t = \frac{\delta}{\epsilon} t_* \tag{4.6}$$

and we seek the following expansions, where v is due to a least degeneration of the continuity equation:

$$\begin{cases} u(x,y,t) &=& \epsilon u_*(x_*, \breve{y}, t_*) + \dots \\ v(x,y,t) &=& \frac{\epsilon^2}{\delta R_e^{1/2}} v_*(x_*, \breve{y}, t_*) + \dots \\ p(x,y,t) &=& \epsilon^2 p_*(x_*, \breve{y}, t_*) + \dots \end{cases}$$

Imposing the following equality between the time derivative and the viscous term in the Navier-Stokes equation:

$$\frac{\partial u}{\partial t} \sim \frac{1}{R_e} \frac{\partial^2 u}{\partial y^2} \to \frac{\epsilon}{\delta} \sim \frac{1}{r_e} \frac{1}{\left(\epsilon R_e^{-1/2}\right)^2} \Rightarrow \epsilon^3 = \delta$$

The Navier-Stokes equations turn out to be:

$$\begin{cases} \frac{\partial u*}{\partial t_*} + u_* \frac{\partial u_*}{\partial x_*} + v_* \frac{\partial u_*}{\partial \breve{y}} = -\frac{\partial p_*(x_*, t_*)}{\partial x_*} + \frac{\partial^2 u_*}{\partial \breve{y}^2} \\ \frac{\partial u_*}{\partial x_*} + \frac{\partial v*}{\partial \breve{y}} = 0 \end{cases} \tag{4.7}$$

The boundary conditions are the impermeability and no-slip conditions on the wall's surface and a matching condition with region 2.

4.3 Region outside the boundary layer

It is known that the unperturbed flow solution is:

$$\begin{cases} u(x,y) = u_0 = 1 \\ v(x,y) = v_0 = 0 \\ p(x,y) = p_0 = 0 \end{cases}$$

The scaling for the coordinates is the following:

$$x = \delta x_* \quad y = \eta y_* \quad t = \tfrac{\delta}{\epsilon} t_* \tag{4.8}$$

where x_*, y_* and t_* are order one quantities. The addition of a perturbation according with the lower limit we found above produces:

$$\begin{cases} u(x,y,t) &= 1 + \epsilon^2 u_1(x,y,t) \\ v(x,y,t) &= \omega v_1(x,y,t) \\ p(x,y,t) &= \epsilon^2 p_1(x,y,t) \end{cases} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.1) without considering the viscous terms, being the external region an inviscid one:

$$\begin{cases} \frac{\epsilon^2}{\delta}\frac{\partial u_1}{\partial t_*} + \frac{\epsilon}{\delta}(1+\epsilon u_1)\frac{\partial u_1}{\partial x_*} + \frac{\epsilon\omega}{\eta} v_1 \frac{\partial u_1}{\partial y_*} &= -\frac{\epsilon}{\delta}\frac{\partial p_1}{\partial x_*} \\ \frac{\epsilon^2}{\delta}\frac{\partial v_1}{\partial t_*} + \frac{\omega}{\delta}(1+\epsilon u_1)\frac{\partial v_1}{\partial x_*} + \frac{\omega^2}{\eta} v_1 \frac{\partial v_1}{\partial y_*} &= -\frac{\epsilon}{\eta}\frac{\partial p_1}{\partial y_*} \\ \frac{\epsilon}{\delta}\frac{\partial u_1}{\partial x_*} + \frac{\epsilon}{\eta}\frac{\partial v_1}{\partial y_*} &= 0 \end{cases}$$

The least degeneration principle gives $\eta = \delta$ and $\omega = \epsilon^2$. Disregarding higher order terms the equations become:

$$\begin{cases} \frac{\partial u_1}{\partial x_*} = -\frac{\partial p_1}{\partial x_*} \\ \frac{\partial v_1}{\partial x_*} = -\frac{\partial p_1}{\partial y_*} \\ \frac{\partial u_1}{\partial x_*} + \frac{\partial v_1}{\partial y_*} = 0 \end{cases}$$

From these equation we can obtain an equation (Laplace equation) for pressure perturbation:

$$\frac{\partial^2 p_1(x_*, y_*, t_*)}{\partial x_*^2} + \frac{\partial^2 p_1(x_*, y_*, t_*)}{\partial y_*^2} = 0 \tag{4.10}$$

with boundary conditions:

$$p_1 \to 0 \text{ as } y_* \to \infty \qquad \frac{\partial p_1}{\partial y_*} = -\frac{\partial v_1}{\partial x_*} \text{ at } y_* = 0 \tag{4.11}$$

where the second equation will be obtained by matching with the boundary layer solution.

4.4 Matching

We can now consider (4.5) and (4.4) in the middle tier and match with the other solutions, in order to find the boundary conditions for the problem and the order of ϵ in R_e. First, u in the limit $Y \to 0$ with $Y = \epsilon \breve{y}$ has to be matched with u in the sublayer:

$$\lambda\epsilon\left(\breve{y} + A(x_*, t_*)\right) = \epsilon u_*(x_*, \breve{y} \to \infty, t_*)$$

which closes the system in the sublayer. Then, we look at the limit $Y \to \infty$ for which

$$v = -\frac{\epsilon}{\delta R_e^{1/2}} \frac{\partial A}{\partial x_*} = \epsilon^2 v_1(x_*, y_*, t_*)$$

which means that $\epsilon = R_e^{-1/8}$. We are ready to express the whole problem with all the needed boundary conditions.

4.5 Canonical form of the Triple-Deck problem

See figure (14) for a scatch of the three tiers. We can disregard the middle tier and write the problem involving only the external region and the sublayer.

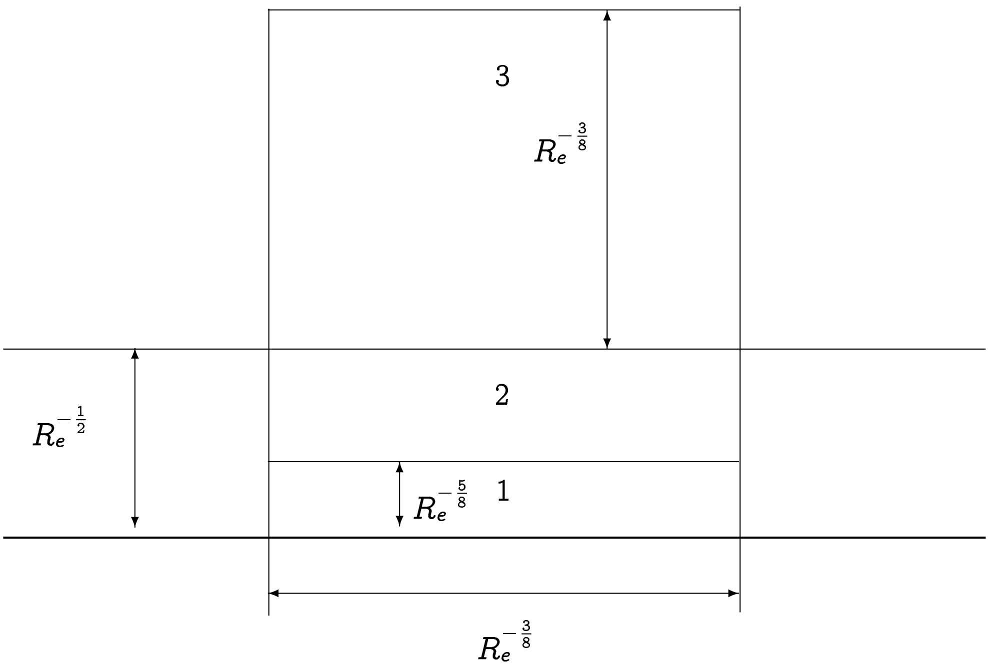

Figure 14: The triple-deck structure

4.5.1 External region

The independent variables in the upper tier are scaled as

$$\begin{cases} x = R_e^{-3/8} x_* \\ y = R_e^{-3/8} y_* \\ t = R_e^{-1/4} t_* \end{cases} \tag{4.12}$$

The velocity components and pressure are represented in this region by the asymptotic expansions

$$\begin{cases} u(x,y,t) = 1 + R_e^{-1/4} u_1(x_*, y_*, t_*) \\ v(x,y,t) = R_e^{-1/4} v_1(x_*, y_*, t_*) \\ p(x,y,t) = R_e^{-1/4} p_1(x_*, y_*, t_*) \end{cases} \tag{4.13}$$

Substituting this into the Navier-Stokes equations, we have

$$\begin{cases} \frac{\partial u_1}{\partial x_*} = -\frac{\partial p_1}{\partial x_*} \\ \frac{\partial v_1}{\partial x_*} = -\frac{\partial p_1}{\partial y_*} \\ \frac{\partial u_1}{\partial x_*} + \frac{\partial v_1}{\partial y_*} = 0 \end{cases} \tag{4.14}$$

Equations (4.14) can be reduced to a simple equation for pressure:

$$\frac{\partial^2 p_1(x_*, y_*, t_*)}{\partial x_*^2} + \frac{\partial^2 p_1(x_*, y_*, t_*)}{\partial y_*^2} = 0 \tag{4.15}$$

It has to be solved with boundary conditions:

$$p_1 \to 0 \text{ as } y_* \to \infty \qquad \frac{\partial p_1}{\partial y_*} = \frac{d^2 A(x_*, t_*)}{dx_*^2} \text{ at } y_* = 0 \tag{4.16}$$

4.5.2 Viscous sublayer

The independent variables in the viscous sublayer are scaled as

$$\begin{cases} x = R_e^{-3/8} x_* \\ y = R_e^{-5/4} \breve{y} \\ t = R_e^{-1/4} t_* \end{cases} \tag{4.17}$$

The velocity components and pressure are represented in this region by the asymptotic expansions

$$\begin{cases} u(x,y,t) = R_e^{-1/8} u_*(x_*, \breve{y}, t_*) \\ v(x,y,t) = R_e^{-3/8} v_*(x_*, \breve{y}, t_*) \\ p(x,y,t) = R_e^{-1/4} p_1(x_*, 0, t_*) \end{cases} \tag{4.18}$$

Where the expression for p takes into account that the pressure does not change in the boundary layer and sublayer, therefore it is equal to the pressure at the bottom of the external part. The Navier-Stokes equations:

$$\begin{cases} \frac{\partial u*}{\partial t_*} + u_* \frac{\partial u_*}{\partial x_*} + v_* \frac{\partial u_*}{\partial \breve{y}} = -\frac{\partial p_1(x_*,0,t_*)}{\partial x_*} + \frac{\partial^2 u_*}{\partial \breve{y}^2} \\ \frac{\partial u_*}{\partial x_*} + \frac{\partial v*}{\partial \breve{y}} = 0 \end{cases} \tag{4.19}$$

And the boundary conditions:

$$u_* = v_* = 0 \text{ at } \breve{y} = 0 \qquad \text{and} \quad u_* = \lambda\left(\breve{y} + A(x_*, t_*)\right) \text{ at } \breve{y} = \infty \tag{4.20}$$

5 Subsonic Tollmien-Schlichting waves theory

Starting from the steady flow, now we add a time-dependent perturbation. It will lead to linearized equations. The steady flow solution of (4.19), (4.20) is:

$$\begin{cases} u_* = \lambda \breve{y} \\ v_* = p_1 = A = 0 \end{cases} \tag{5.1}$$

5.1 Sublayer

Now we add to (5.1) small unsteady perturbations:

$$\begin{cases} u_* = \lambda \breve{y} + \epsilon u'(x_*, \breve{y}, t_*) \\ v_* = \epsilon v'(x_*, \breve{y}, t_*) \\ p_1 = \epsilon p'(x_*, \breve{y}, t_*) \\ A = \epsilon A'(x_*, \breve{y}, t_*) \end{cases} \tag{5.2}$$

Substituting into (4.19) and disregarding order ϵ^2 terms, the linearized equations are:

$$\begin{cases} \frac{\partial u'}{\partial t_*} + \lambda \breve{y} \frac{\partial u'}{\partial x_*} = -\frac{\partial p'}{\partial x_*} + \frac{\partial^2 u'}{\partial \breve{y}^2} \\ \frac{\partial u'}{\partial x_*} + \frac{\partial v'}{\partial \breve{y}} = 0 \end{cases} \tag{5.3}$$

with boundary conditions:

$$\begin{cases} u' = v' = 0 \text{ at } \breve{y} = 0 \\ u' = \lambda A' \text{ at } \breve{y} = \infty \end{cases} \tag{5.4}$$

We seek the solution to (5.3), (5.4) in the normal modes:

$$\begin{cases} u' = e^{i(kx_* - \omega t_*)}\bar{u}(\breve{y}) \\ v' = e^{i(kx_* - \omega t_*)}\bar{v}(\breve{y}) \\ p' = e^{i(kx_* - \omega t_*)}\bar{p}(\not{\breve{y}}) \\ A' = e^{i(kx_* - \omega t_*)}\bar{A}(\not{\breve{y}}) \end{cases} \tag{5.5}$$

where ω and k are real and positive and pressure p' and displacement function A' are independent on the vertical coordinate. The choice of the negative sign is due to a downstream propagating perturbation. As soon as we perform the substitution into (5.2) we obtain:

$$\begin{cases} u_* = \lambda\breve{y} + \epsilon e^{i(kx_* - \omega t_*)}\bar{u}(\breve{y}) \\ v_* = \epsilon e^{i(kx_* - \omega t_*)}\bar{v}(\breve{y}) \\ p_* = \epsilon e^{i(kx_* - \omega t_*)}\bar{p} \\ A_* = \epsilon e^{i(kx_* - \omega t_*)}\bar{A} \end{cases} \tag{5.6}$$

Plugging (5.6) into the Navier-Stokes equations (4.19) yields

$$\begin{cases} -i\omega\bar{u}(\breve{y}) + ik\bar{u}(\breve{y})\lambda\breve{y} + \lambda\bar{v}(\breve{y}) = -ik\bar{p} + \ddot{\bar{u}}(\breve{y}) \\ ik\bar{u}(\breve{y}) + \dot{\bar{v}}(\breve{y}) = 0 \end{cases} \tag{5.7}$$

and the boundary conditions (4.20) take the form

$$\begin{cases} \bar{u}(0) = \bar{v}(0) = 0 \\ \bar{u}(\infty) = \lambda\bar{A} \end{cases} \tag{5.8}$$

Furthermore, by evaluating (5.7) at $\breve{y} = 0$ we can deduce that

$$\ddot{\bar{u}}(0) = ik\bar{p} \quad \text{and} \quad \dot{\bar{v}}(0) = 0 \tag{5.9}$$

Joining the two equations (5.7) we work out an equation for $\bar{u}$ which is the following

$$\dddot{\bar{u}}(\breve{y}) - i(k\lambda\breve{y} - \omega)\dot{\bar{u}}(\breve{y}) = 0 \tag{5.10}$$

$$\begin{cases} \bar{u}(0) = 0 \\ \bar{u}(\infty) = \lambda\bar{A} \\ \ddot{\bar{u}}(0) = ik\bar{p} \end{cases} \tag{5.11}$$

Let us now solve this differential equation. Setting $\dot{\bar{u}}(\breve{y}) = f(\breve{y})$ and rearranging:

$$\frac{1}{(ik\lambda)^{2/3}}\frac{d^2 f}{d\breve{y}^2} - (ik\lambda)^{1/3}\left(\breve{y} - \frac{\omega}{k\lambda}\right) f = 0 \tag{5.12}$$

The idea is to rewrite this equation in such a way that leads to the well-known Airy equation. Clearly, after the following transformation:

$$\zeta = (ik\lambda)^{1/3}\left(\breve{y} - \tfrac{\omega}{k\lambda}\right) \qquad d\zeta = (ik\lambda)^{1/3} d\breve{y}$$

the resulting equation is the Airy equation:

$$\frac{d^2 f(\breve{y}(\zeta))}{d\zeta^2} - \zeta f(\breve{y}(\zeta)) = 0 \tag{5.13}$$

Its general solution is

$$f(\breve{y}(\zeta)) = AAi(\zeta) + BBi(\zeta)$$

where functions $Ai(\zeta)$ and $Bi(\zeta)$ are plotted in figure (15). We can deduce

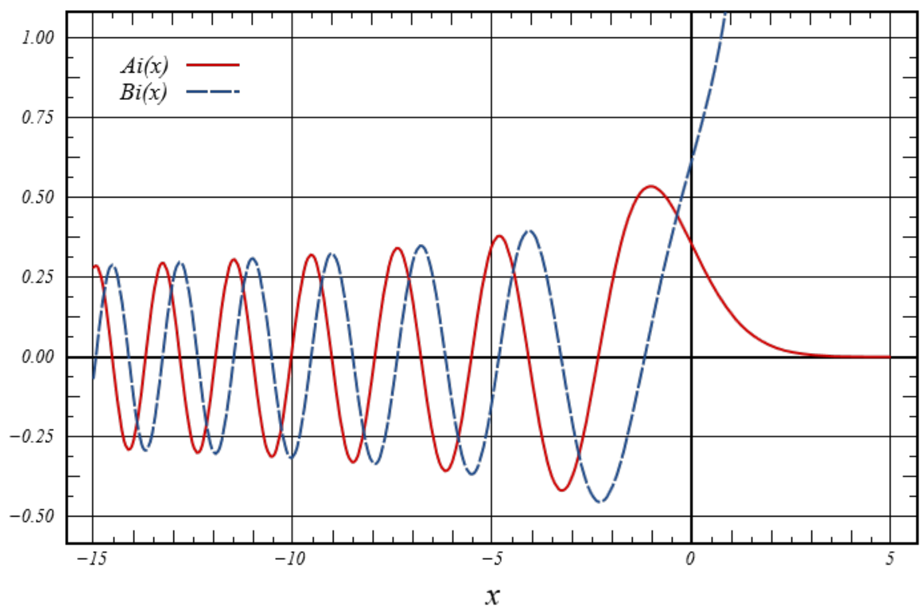

Figure 15: Airy functions of first and second kind

some information from the boundary conditions:

1. From $\bar{u}(\infty) = \lambda\bar{A}$ we know that $\dot{\bar{u}}(\infty) = 0$. Therefore, the solution has to be bounded at infinity, requirement that forces $B = 0$. Therefore, the equation turns out to be:

$$\frac{d\bar{u}(\breve{y})}{d\breve{y}} = (ik\lambda)^{1/3} AAi(\zeta) \tag{5.14}$$

2. Now we consider that $\ddot{\bar{u}}(0) = ik\bar{p}$. Basically, we derive the previous equation and we evaluate it at $\breve{y} = 0$. Hence,

$$A(ik\lambda)^{2/3}Ai'(\zeta_0) = ik\bar{p}, \tag{5.15}$$

where $\zeta_0 = \zeta(\breve{y} = 0) = -(ik\lambda)^{1/3}\frac{\omega}{k\lambda}$.

3. The last condition is $\bar{u}(\infty) = \lambda\bar{a}$. We need to integrate (5.14):

$$\begin{aligned} \int_0^\infty \frac{d\bar{u}}{d\breve{y}} d\breve{y} &= \bar{u}(\infty) - \cancel{\bar{u}(0)} = \bar{u}(\infty) = \\ &= A(ik\lambda)^{1/3} \int_{y'=0}^{y'=\infty} Ai(\zeta')dy' = \\ &= A\int_{\zeta_0}^{\infty} Ai(\zeta')d\zeta' = \lambda\bar{A} \end{aligned}$$

Calling $\kappa = \int_{\zeta_0}^{\infty} Ai(\zeta')d\zeta' = \lambda\bar{A}$, we finally have:

$$\lambda\bar{A} = A\kappa. \tag{5.16}$$

Eliminating A in (5.15) and (5.16) we have a relation between $\bar{A}$ and $\bar{p}$, which is:

$$\frac{\lambda\bar{A}}{\kappa}(ik\lambda)^{2/3}Ai'(\zeta_0) = ik\bar{p} \tag{5.17}$$

Now, analizing the esternal region, we will be able to find $\bar{p}$ as function of $\bar{A}$. This will lead to the large Reynolds number version of the Orr-Sommerfeld equation.

5.2 External region

We can write:

$$\begin{cases} p_1 = \epsilon p_1'(x_*, y_*, t_*) = \epsilon e^{i(kx_* - \omega t_*)}\bar{p_1}(y_*) \\ A = \epsilon A(x_*, y_*, t_*) = \epsilon e^{i(kx_* - \omega t_*)}\bar{A} \end{cases} \tag{5.18}$$

where A is the same as in (5.6) and, since the pressure does not change across the middle tier, we can state that $\bar{p_1}(0) = \bar{p}$. Equation (4.15) becomes:

$$\ddot{\bar{p_1}}(y_*) - k^2\bar{p_1}(y_*) = 0 \tag{5.19}$$

$$\begin{cases} \bar{p_1}(\infty) = 0 \\ \dot{\bar{p_1}}(0) = -k^2\bar{A} \end{cases} \tag{5.20}$$

The solution to this problem is:

$$\bar{p_1}(y_*) = k\bar{A}e^{-ky_*} \tag{5.21}$$

This gives the expression for $\bar{p}$ in the sublayer, since

$$\bar{p_1}(0) = \bar{p} = k\bar{A} \tag{5.22}$$

5.3 Large Reynolds numbers Orr-Sommerfeld equation

Plugging (5.22) into (5.17) we find the dispersion relation:

$$\frac{\lambda^{5/3}}{k^{4/3}} Ai'(\zeta_0) = e^{i\frac{\pi}{6}} \int_{\zeta_0}^{\infty} Ai(q)dq \tag{5.23}$$

Let us now eliminate the dependence on λ by setting:

$$k = \tilde{k}\lambda^{5/4} \qquad \omega = \tilde{\omega}\lambda^{3/2}$$

This turns (5.23) into:

$$Ai'(\zeta_0) = (ik)^{1/3} k \int_{\zeta_0}^{\infty} Ai(q)dq \tag{5.24}$$

As said earlier, this equation represents the large Reynolds number version of the Orr-Sommerfeld equation. In section (5.4) we shall see that it has an infinite countable number of roots which all originate from $\tilde{\omega} = 0$. Furthermore, all of them, except the first one, remain in the second quadrant for all $\tilde{\omega}$. It means that the corresponding perturbations in the boundary layer decay with x. On the other hand, the first root crosses the real axis in the complex plane k at

$$(\tilde{\omega}_* = 2.29797, \tilde{k}_* = 1.00049) \tag{5.25}$$

and then stays in the third quadrant for any other $\tilde{\omega}$. This root is the Tollmien-Schlichting wave. In the following subsection the numerical solution is explained.

5.4 Numerical solution to the Orr-Sommerfeld high Reynolds number equation

We show here how to solve equation (5.24), which is addressed as the high Reynolds number Orr-Sommerfeld equation. We start by considering this equation for ω real and k complex, with the aim to find the solution for k real, corresponding to the neutral curve. In this case the equation is

$$Ai'(z_0) - (ik)^{1/3}|k| \int_{z_0}^{\infty} Ai(z)dz = 0 \tag{5.26}$$

with

$$z_0 = -\frac{i\omega}{(ik)^{2/3}} \tag{5.27}$$

Let us start considering small values of ω, namely $\omega \to 0$. We must have $k \to 0$ if we want z_0 to be finite. The equation is

$$Ai'(z_0) = 0$$

which has an infinite countable number of zeros. We list here the first five zeros:

$$z_0^{(1)} = -1.01879...$$
$$z_0^{(2)} = -3.24819...$$
$$z_0^{(3)} = -4.82009...$$
$$z_0^{(4)} = -6.16330...$$
$$z_0^{(5)} = -7.37217...$$

The idea is to find the value of z_0 which solves (5.26) via the Newton method with starting point $z_0^{(i)}$ and small ω, say $\omega = 0.01$. The pocedure is as follows. Let us call the left hand side of (5.26) $\varphi(z_0)$ after it has been arranged as exclusively dependent on $z_0 = -\frac{i\omega}{(ik)^{2/3}}$

$$\varphi(z_0) = Ai'(z_0) - \left(-\frac{i\omega}{z_0}\right)^{1/2} \left|-i\left(-\frac{i\omega}{z_0}\right)^{1/2}\right| \int_{z_0}^{\infty} Ai(z)dz = 0$$

However, when calculating $\varphi(z_0^{(i)})$ with $\omega = 0.01$ we have $\varphi(z_0^{(i)}) \neq 0$. The starting point is supposed to be reasonably close to the real root, therefore we can write:

$$\varphi(z_0^{(i)} + \delta z_0) = \varphi(z_0^{(i)}) + \varphi'(z_0^{(i)})\delta z_0$$

The requirement for this to be zero produces

$$\delta z_0 = -\frac{\varphi(z_0^{(i)})}{\varphi'(z_0^{(i)})}$$

This is the Newton method, and it converges after a few iterations. Once found the first value, we increase ω and we apply again the above procedure, with the new root as starting point. The derivative of $\varphi(z_0)$ has a quite long expression

$$\varphi'(z_0) = \left\{z_0 - \left(-\tfrac{i\omega}{z_0}\right)^{\frac{1}{2}} \left|-i\left(-\tfrac{i\omega}{z_0}\right)^{\frac{3}{2}}\right|\right\} Ai(z_0) +$$
$$+ \left\{\tfrac{1}{2z_0}\left(-\tfrac{i\omega}{z_0}\right)^{\frac{1}{2}} \left|-i\left(-\tfrac{i\omega}{z_0}\right)^{\frac{3}{2}}\right| - \left(-\tfrac{i\omega}{z_0}\right)^2 \tfrac{3i}{2z_0} sign\left|-i\left(-\tfrac{i\omega}{z_0}\right)^{\frac{3}{2}}\right|\right\} \int_{z_0}^{\infty} Ai(z)dz$$

A programme in MATLAB has been developed in order to find the first five roots of (5.26). The code is as follows:

```
function [ x, ex ] = newton( x0, omega, tol, nmax )
if nargin == 3
        tol = 1e-6;
        nmax = 1e1;
```

```
    elseif nargin == 4
        nmax = 1e1;
    elseif nargin ~= 5
        error('newton: invalid input parameters');
    end

    f= @(z0) airy(1,z0)-(-i*omega/z0)^0.5*abs(-i*(-i*...
    df= @(z0) (z0+(-i*omega/z0)^0.5*abs(-i*(-i*...

    x(1) = x0 - (f(x0)/df(x0));
    ex(1) = abs(x(1)-x0);
    k = 2;
    while (ex(k-1) >= tol) && (k <= nmax)
        x(k) = x(k-1) - (f(x(k-1))/df(x(k-1)));
        ex(k) = abs(x(k)-x(k-1));
        k = k+1;
    end
end

function [ int ] = integraleairy( z )
% Calculates the integral of the airy function from z to infinity
f=@(x) airy(x);
int=1/3-integral(f,0,z);
end

syms z0 k
syms omega real
k(1)=0;
syms count int
count=0;
%z0=-1.018792;
%z0=-3.24819;
%z0=-4.82009;
%z0=-6.16330;
z0=-7.37218;
for omega=0.1:0.2:30,
    count=count+1;
    [x ,ex]=Newton(z0,omega,1.0*10^(-6),20);
    z0=x(numel(x));
    k(count)=(-i^(1/3)*omega/x(numel(x)))^(3/2);
end
```

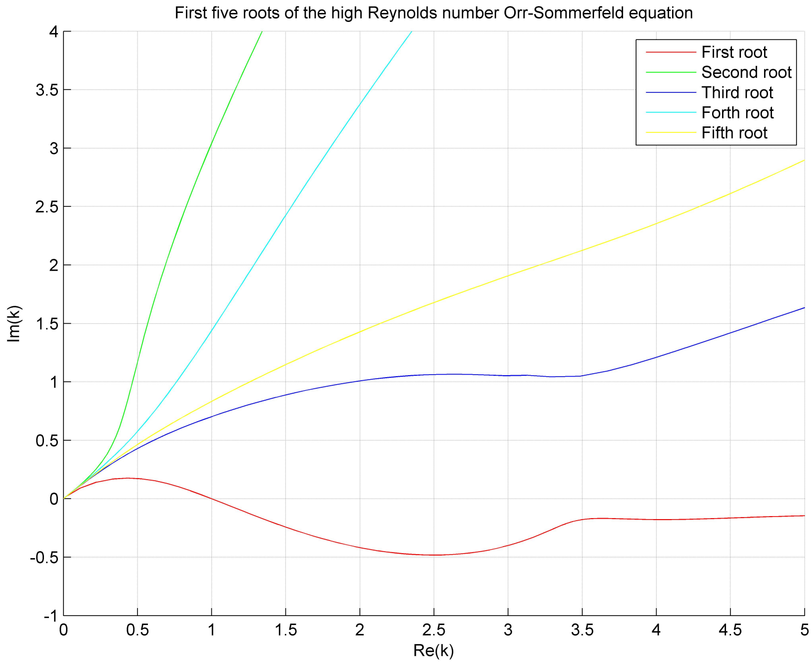

Figure 16: The first five roots

```
plot(k,'b');
end
```

Figure (16) shows the trajectories of the first five roots as ω changes from zero to large values. All the roots originate at $\omega = 0$ from the coordinate origin, and all of them, except the first one, remain in the first quadrant for all $\omega > 0$, indicating that the corresponding perturbations in the boundary layer have the form

$$e^{i(kX-\omega T)} = e^{-Im(k)X} e^{i(Re(k)X-\omega T)}$$

and therefore decay with X. The beheavior of the first root is different. It crosses the real axis at

$$(\omega_*, k_*) = (2.29797, 1.00049)$$

This root represents the Tollmien-Schlichting wave. For $\omega \in (0, \omega*)$ the perturbation decays, for $\omega > \omega_*$ it grows.

6 Compressible Navier-Stokes equations

When dealing with transonic regime we have to consider full compressible Navier-Stokes equations:

$$\left\{\begin{array}{rcl}
\hat\rho\left(\frac{\partial\hat u}{\partial\hat t}+\hat u\frac{\partial\hat u}{\partial\hat x}+\hat v\frac{\partial\hat u}{\partial\hat y}\right) & = & -\frac{\partial\hat p}{\partial\hat x}+\frac{\partial}{\partial\hat x}\left[\hat\mu\left(\frac{4}{3}\frac{\partial\hat u}{\partial\hat x}-\frac{2}{3}\frac{\partial\hat v}{\partial\hat y}\right)\right]+\\
 & + & \frac{\partial}{\partial\hat y}\left[\hat\mu\left(\frac{\partial\hat u}{\partial\hat y}+\frac{\partial\hat v}{\partial\hat x}\right)\right]\\
\hat\rho\left(\frac{\partial\hat v}{\partial\hat t}+\hat u\frac{\partial\hat v}{\partial\hat x}+\hat v\frac{\partial\hat v}{\partial\hat y}\right) & = & -\frac{\partial\hat p}{\partial\hat y}+\frac{\partial}{\partial\hat y}\left[\hat\mu\left(\frac{4}{3}\frac{\partial\hat v}{\partial\hat y}-\frac{2}{3}\frac{\partial\hat u}{\partial\hat x}\right)\right]+\\
 & + & \frac{\partial}{\partial\hat x}\left[\hat\mu\left(\frac{\partial\hat u}{\partial\hat y}+\frac{\partial\hat v}{\partial\hat x}\right)\right]\\
\hat\rho\left(\frac{\partial\hat\rho}{\partial\hat t}+\hat u\frac{\partial\hat\rho}{\partial\hat x}+\hat v\frac{\partial\hat\rho}{\partial\hat y}\right) & = & \hat u\frac{\partial\hat p}{\partial\hat x}+\hat v\frac{\partial\hat p}{\partial\hat y}+\frac{1}{Pr}\left[\frac{\partial}{\partial\hat x}\left(\hat\mu\frac{\partial\hat h}{\partial\hat x}\right)+\right.\\
 & + & \left.\frac{\partial}{\partial\hat y}\left(\hat\mu\frac{\partial\hat h}{\partial\hat y}\right)\right]+\hat\mu\left(\frac{4}{3}\frac{\partial\hat u}{\partial\hat x}-\frac{2}{3}\frac{\partial\hat v}{\partial\hat y}\right)\frac{\partial\hat u}{\partial\hat x}+\\
 & + & \hat\mu\left(\frac{4}{3}\frac{\partial\hat v}{\partial\hat y}-\frac{2}{3}\frac{\partial\hat u}{\partial\hat x}\right)\frac{\partial\hat v}{\partial\hat y}+\hat\mu\left(\frac{\partial\hat u}{\partial\hat y}+\frac{\partial\hat v}{\partial\hat x}\right)^2\\
\frac{\partial\hat\rho}{\partial\hat t}+\frac{\partial\hat\rho u}{\partial\hat x}+\frac{\partial\hat\rho v}{\partial\hat y} & = & 0
\end{array}\right. \tag{6.1}$$

Considering the gas as perfect, the state equation is

$$\hat h=\frac{\gamma}{\gamma-1}\frac{\hat p}{\hat\rho}$$

The inviscid flow can be described by the full unsteady potential form:

$$\left(\hat a^2-\hat\varphi_{\hat x}^2\right)\hat\varphi_{\hat x\hat x}-2\hat\varphi_{\hat x}\hat\varphi_{\hat y}\hat\varphi_{\hat x\hat y}+\left(\hat a^2-\hat\varphi_{\hat y}^2\right)\hat\varphi_{\hat y\hat y}-2\hat\varphi_{\hat x}\hat\varphi_{\hat x\hat t}-2\hat\varphi_{\hat y}\hat\varphi_{\hat y\hat t}-\hat\varphi_{\hat t\hat t}=0 \tag{6.2}$$

combined with the Bernoulli equation:

$$\hat\varphi_{\hat t}+\frac{\hat\varphi_{\hat x}+\hat\varphi_{\hat y}}{2}+\frac{\hat a^2}{\gamma-1}=\frac{V_\infty^2}{2}+\frac{\hat a_\infty^2}{\gamma-1} \tag{6.3}$$

We perform the following transformation in order to express the equations in their adimensional form:

$$\begin{array}{cccc}
\hat x=Lx & \hat y=Ly & \hat u=V_\infty u & \hat v=V_\infty v\\
\hat\rho=\rho_\infty\rho & \hat p=p_\infty+\rho_\infty V_\infty^2 p & \hat h=V_\infty^2 h & \hat\mu=\mu_\infty\mu\\
\hat t=L/V_\infty t & \hat\varphi=V_\infty L\varphi & \hat a=a_\infty a &
\end{array} \tag{6.4}$$

This yields the nondimensional equations are:

$$
\left\{
\begin{array}{rcl}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &=& -\frac{\partial p}{\partial x}+\frac{1}{R_e}\left\{\frac{\partial}{\partial x}\left[\mu\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\right)\right]+\right. \\
&+& \left.\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\} \\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) &=& -\frac{\partial p}{\partial y}+\frac{1}{R_e}\left\{\frac{\partial}{\partial y}\left[\mu\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right)\right]+\right. \\
&+& \left.\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\} \\
\rho\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right) &=& u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+\frac{1}{R_e}\left\{\frac{1}{Pr}\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial h}{\partial x}\right)+\right.\right. \\
&+& \left.\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right)\right]+\mu\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial x}+ \\
&+& \left.\mu\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial y}+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\} \\
\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y} &=& 0
\end{array}
\right.
\tag{6.5}
$$

with state equation assuming the form

$$
h=\frac{1}{(\gamma-1)M_\infty^2}\frac{1}{\rho}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}
$$

The potential equations are:

$$
\left\{
\begin{array}{rl}
\left(\frac{a^2}{M_\infty^2}-\varphi_x^2\right)\varphi_{xx} & -2\varphi_x\varphi_y\varphi_{xy}+\left(\frac{a^2}{M_\infty^2}-\varphi_y^2\right)\varphi_{yy} \\
& -2\varphi_x\varphi_{xt}-2\varphi_y\varphi_{yt}-\varphi_{tt}=0 \\
\varphi_t+\frac{\varphi_x+\varphi_y}{2} & +\frac{a^2}{(\gamma-1)M_\infty^2}=\frac{1}{2}+\frac{1}{(\gamma-1)M_\infty^2}
\end{array}
\right.
\tag{6.6}
$$

Here the nondimensional parameters are calculated as

$$
M_\infty=\frac{V_\infty}{a_\infty}\quad a_\infty=\sqrt{\gamma\frac{p_\infty}{\rho_\infty}}\quad R_e=\frac{\rho_\infty V_\infty L}{\mu_\infty}
\tag{6.7}
$$

7 Compressible Boundary Layer

Again we consider a flat plate aligned with the oncoming flow. Its motion is considered to be steady and two-dimensional. We already know that the scaling in the boundary layer is:

$$
x=O(1)\quad y=R_e^{-1/2}Y\quad R_e\to\infty
\tag{7.1}
$$

and the solution to the Navier-Stokes equations may be sought in the form of the asymptotic expansions

$$
\begin{array}{ll}
u(x,y;R_e)=U_0(x,Y)+\ldots & v(x,y;R_e)=R_e^{-1/2}V_0(x,Y)+\ldots \\
\rho(x,y;R_e)=\rho_0(x,Y)+\ldots & p(x,y;R_e)=R_e^{-1/2}p_0(x,Y)+\ldots \\
h(x,y;R_e)=h_0(x,Y)+\ldots & \mu(x,y;R_e)=\mu_0(x,Y)+\ldots
\end{array}
\tag{7.2}
$$

Substitution into (6.5) leads to the classical boundary layer equations which can be solved with the free stream conditions $U_0 = 1$ and $h_0 = h_\infty$ at the leading edge of the flat plate ($x = 0$) as well as at the outer edge of the boundary layer ($Y = \infty$) and the impermeability and no-slip conditions $U_0 = V_0 = 0$ on the plate surface ($Y = 0$), supplemented with a thermal condition (e.g. the wall temperature is known $h_0 = F(x)$ or the wall is thermally isolated $\frac{\partial h_0}{\partial Y} = 0$). We highlight here the expression for:

$$h_\infty = \frac{\gamma}{\gamma - 1}\frac{p_\infty}{\rho_\infty}\frac{1}{V_\infty^2} = \frac{1}{(\gamma - 1)M_\infty^2}$$

What we need to proceed is to know that the solution remains smooth when the trailing edge of the plate is approached, therefore we can assume valid the following expansions:

$$\left\{\begin{array}{c} U_0(x,Y) = U_{00}(Y) + (1-x)U_{01}(Y) + ... \\ h_0(x,Y) = h_{00}(Y) + (1-x)h_{01}(Y) + ... \\ \rho_0(x,Y) = \rho_{00}(Y) + (1-x)\rho_{01}(Y) + ... \\ \mu_0(x,Y) = \mu_{00}(Y) + (1-x)\mu_{01}(Y) + ... \end{array}\right\} \text{ as } x - 1 \to 0^- \qquad (7.3)$$

The leading order terms exhibit the following behaviour near the plate surface

$$\left\{\begin{array}{c} U_{00}(Y) = \lambda Y + ... \\ h_{00}(Y) = h_w + ... \\ \rho_{00}(Y) = \rho_w + ... \\ \mu_{00}(Y) = \mu_w + ... \end{array}\right\} \text{ as } Y \to 0 \qquad (7.4)$$

where λ, h_w, ρ_w and μ_w are positive constants representing the dimensionless skin friction, enthalpy, density and viscosity on the wall surface.

8 Transonic Triple-Deck Theory

Before starting, we will introduce some notation. We will label with u, m and l the quantities in the upper, medium and lower tier respectively. Indeed, as expected, the structure will be the same as in the subsonic incompressible case, although with different scalings and expansions for the solution to the Navier-Stokes equation. We will start from the upper deck, where the full unsteady potential equation holds. It will allow us to work out all the quantities in the upper tier. Being in a transonic flow regime means that:

$$M_\infty^2 = 1 + ...$$

8.1 The upper tier

The equation is (6.6). We consider that $M_\infty^2 = 1 + \alpha m$. We choose the following scaling:

$$x = \delta X \quad y = \delta\alpha^{-1/2} y_u \quad t = \lambda T \tag{8.1}$$

and we write the potential in the following form:

$$\varphi = \delta X + R_e^{-1/2}\delta^{1/3}\alpha^{-1/2}\varphi_{1u} + \ldots \tag{8.2}$$

Hence,

$$\begin{aligned} u = \varphi_X = 1 + R_e^{-1/2}\delta^{-2/3}\alpha^{-1/2}\tfrac{\partial\varphi_{1u}}{\partial X} + \ldots \\ v = \varphi_{y_u} = R_e^{-1/2}\delta^{-2/3}\tfrac{\partial\varphi_{1u}}{\partial y_u} + \ldots \\ \varphi_T = R_e^{-1/2}\delta^{1/3}\lambda^{-1}\alpha^{-1/2}\tfrac{\partial\varphi_{1u}}{\partial T} + \ldots \\ \varphi_{TT} = R_e^{-1/2}\delta^{1/3}\lambda^{-2}\alpha^{-1/2}\tfrac{\partial^2\varphi_{1u}}{\partial T^2} + \ldots \end{aligned} \tag{8.3}$$

Plugging into the Bernoulli equation (6.6) we obtain:

$$a^2 = 1 - M_\infty^2(\gamma - 1)\alpha^{-1/2}R_e^{-1/2}\left[\frac{\delta^{1/3}}{\lambda}\frac{\partial\varphi_{1u}}{\partial T} + \frac{\delta^{1/3}}{\delta}\frac{\partial\varphi_{1u}}{\partial X}\right] + \ldots$$

In order to make the derivative with respect to T a second order term, we set $\lambda = \delta^{2/3}$. Then:

$$\begin{aligned} a^2 &= 1 - M_\infty^2(\gamma-1)\alpha^{-1/2}R_e^{-1/2}\delta^{-2/3}\tfrac{\partial\varphi_{1u}}{\partial X} + \ldots \\ \varphi_{XX} &= R_e^{-1/2}\delta^{-2/3}\delta^{-1}\alpha^{-1/2}\tfrac{\partial^2\varphi_{1u}}{\partial X^2} + \ldots \\ \varphi_{y_u y_u} &= R_e^{-1/2}\delta^{-2/3}\delta^{-1}\alpha^{1/2}\tfrac{\partial^2\varphi_{1u}}{\partial y_u^2} + \ldots \\ \varphi_{XT} &= R_e^{-1/2}\delta^{-2/3}\lambda^{-1}\alpha^{-1/2}\tfrac{\partial^2\varphi_{1u}}{\partial X\partial T} + \ldots \\ \varphi_{Ty_u} &= R_e^{-1/2}\delta^{-2/3}\lambda^{-1}\tfrac{\partial^2\varphi_{1u}}{\partial y_u\partial T} + \ldots \end{aligned} \tag{8.4}$$

Finally, the potential equation becomes:

$$\begin{aligned} \tfrac{1-M_\infty^2}{M_\infty^2}R_e^{-1/2}\delta^{-5/3}\alpha^{-1/2}\tfrac{\partial^2\varphi_{1u}}{\partial X^2} + \tfrac{1}{M_\infty^2}R_e^{-1/2}\delta^{-5/3}\alpha^{1/2}\tfrac{\partial^2\varphi_{1u}}{\partial y_u^2} + \\ -2R_e^{-1/2}\delta^{-4/3}\alpha^{-1/2}\tfrac{\partial^2\varphi_{1u}}{\partial X\partial T} = 0 \end{aligned}$$

It may be rearranged using the fact that $M_\infty \sim 1$ and $1 - M_\infty^2 = -\alpha m$.

$$\begin{aligned} -mR_e^{-1/2}\delta^{-5/3}\alpha^{1/2}\tfrac{\partial^2\varphi_{1u}}{\partial X^2} + R_e^{-1/2}\delta^{-5/3}\alpha^{1/2}\tfrac{\partial^2\varphi_{1u}}{\partial y_u^2} + \\ -2R_e^{-1/2}\delta^{-4/3}\alpha^{-1/2}\tfrac{\partial^2\varphi_{1u}}{\partial X\partial T} = 0 \end{aligned}$$

In order to avoid further degeneration, we have to set:

$$\alpha = \delta^{1/3}$$

As soon as we consider the viscous-inviscid interaction where $R_e^{-1/2}\delta^{-4/3}\alpha^{-1/2} = 1$ we can conclude that:

$$\delta = R_e^{-1/3} \quad \lambda = R_e^{-2/9} \quad 1 - M_\infty^2 = R_e^{-1/9} m \tag{8.5}$$

And the equation for φ_{1u} assumes the following simple form:

$$m\frac{\partial^2 \varphi_{1u}}{\partial X^2} + 2\frac{\partial^2 \varphi_{1u}}{\partial X \partial T} = \frac{\partial^2 \varphi_{1u}}{\partial y_u^2} \tag{8.6}$$

We already know the asymptotic expansion for u and v from (8.3). Using the equation of state and the Poisson adiabat equation, we can write that:

$$\begin{array}{c} \frac{\hat{p}-p_\infty}{\rho_\infty V_\infty^2} = p = \frac{1}{\gamma M_\infty^2}\left[\left(\frac{\hat{\rho}}{\rho_\infty}\right)^\gamma - 1\right] = \frac{1}{\gamma M_\infty^2}\left(\rho^\gamma - 1\right) \\ \frac{\hat{\rho}}{\rho_\infty} = \rho = \left(\frac{\hat{a}}{a_\infty}\right)^{2/(\gamma-1)} = a^{2(\gamma-1)} \end{array} \tag{8.7}$$

Therefore, using the first of (8.4), the expansion $(1+x)^\alpha = 1 + \alpha x + ...$ and $M_\infty^2 \sim 1$, we obtain:

$$\begin{cases} \rho = 1 - \alpha^{-1/2} R_e^{-1/2} \delta^{-2/3} \frac{\partial \varphi_{1u}}{\partial X} + ... \\ p = -\alpha^{-1/2} R_e^{-1/2} \delta^{-2/3} \frac{\partial \varphi_{1u}}{\partial X} + ... \end{cases} \tag{8.8}$$

We can conclude that the problem in the upper deck is the following:

$$x = R_e^{-1/3} X \quad y = R_e^{-5/18} y_u \quad t = R_e^{-2/9} T \tag{8.9}$$

$$\begin{cases} u(x,y,R_e) = 1 + R_e^{-2/9} u_{1u}(X, y_u, T) + ... \\ v(x,y,R_e) = R_e^{-5/18} v_{1u}(X, y_u, T) + ... \\ p(x,y,R_e) = R_e^{-2/9} p_{1u}(X, y_u, T) + ... \\ \rho(x,y,R_e) = 1 + R_e^{-2/9} \rho_{1u}(X, y_u, T) + ... \end{cases} \tag{8.10}$$

Where we notice that

$$p_{1u} = \rho_{1u} = -u_{1u} = -\frac{\partial \varphi_{1u}}{\partial X}$$

$$v_{1u} = \frac{\partial \varphi_{1u}}{\partial y_u}$$

This allow us to get an equation for p_{1u} by differentiating (8.6) with respect to X:

$$m\frac{\partial^2 p_{1u}}{\partial X^2} + 2\frac{\partial^2 p_{1u}}{\partial X \partial T} = \frac{\partial^2 p_{1u}}{\partial y_u^2} \tag{8.11}$$

We need two boundary conditions to solve this equation. The first is

$$p_{1u} \to 0 \text{ as } y_u \to \infty \tag{8.12}$$

and the second one comes from differentiating $v_{1u} = \frac{\partial \varphi_{1u}}{\partial y_u}$ with respect to X:

$$\frac{\partial v_{1u}}{\partial X} = -\frac{\partial p_{1u}}{\partial y_u} \text{ at } y_u = 0 \tag{8.13}$$

The last calculation in the upper deck before proceeding is about the slope angle, which is defined as follows:

$$\theta = \frac{v}{u} = \frac{O(R_e^{-5/18})}{O(1)} = O(R_e^{-5/18}) \tag{8.14}$$

8.2 The Lower Deck

The thickness of the sublayer can be estimated using the following balance in the longitudinal momentum equation:

$$\frac{\partial u}{\partial t} \sim \frac{1}{R_e}\frac{\partial^2 u}{\partial y^2}$$

We find that

$$\Delta y = R_e^{-1/2}\Delta t^{-1/2} = O(R_e^{-11/18})$$

and the right scaling turns out to be:

$$x = R_e^{-1/3}X \quad y = R_e^{-11/18}y_l \quad t = R_e^{-2/9}T \tag{8.15}$$

The form of the asymptotic expansions for the velocity components may be found taking into account:

- Being the velocity u linear in y at the bottom of the boundary layer, we expect $y = O(\Delta u)R_e^{-1/2}y_l$. Hence,

  $$O(\Delta u) = R_e^{-1/9}$$

- The slope angle does not change across the middle boundary layer, therefore

  $$\theta = \frac{v}{u} = \frac{\Delta v}{\Delta u} = O(R_e^{-5/18})$$

  Using the previous statement:

  $$\Delta v = O(R_e^{-7/18})$$

- We know from (7.4) the form of the expansion for h, ρ and μ near the wall.

- The pressure does not change across the boundary layer, therefore it is espected to be order $R_e^{-2/9}$.

Finally, the expansions are:

$$\begin{cases} u(x,y;R_e) = R_e^{-1/9} u_{1l}(X,y_l,T) + ..., \\ v(x,y;R_e) = R_e^{-7/18} v_{1l}(X,y_l,T) + ..., \\ p(x,y;R_e) = R_e^{-2/9} p_{1u}(X,0,T) + ..., \\ \rho(x,y;R_e) = \rho_w + ..., \\ h(x,y;R_e) = h_w + ..., \\ \mu(x,y;R_e) = \mu_w + ... \end{cases} \tag{8.16}$$

Plugging into the Navier-Stokes equations (6.5) we get the following equations to the leading order:

$$\begin{cases} \rho_w \left(\frac{\partial u_{1l}}{\partial T} + u_{1l} \frac{\partial u_{1l}}{\partial X} + v_{1l} \frac{\partial u_{1l}}{\partial y_l} \right) = -\frac{\partial p_{1u}(X,0,T)}{\partial X} + \mu_w \frac{\partial^2 u_{1l}}{\partial y_l^2}, \\ \frac{\partial u_{1l}}{\partial X} + \frac{\partial v_{1l}}{\partial y_l} = 0. \end{cases} \tag{8.17}$$

The boundary conditions are the impermeability and the no-slip conditions on the wall surface and the matching condition for u at the outer edge of the sublayer.

8.3 The Middle Tier

The middle tier is known to have the following scalings:

$$x = R_e^{-1/3} X, \quad y = R_e^{-1/2} y_m, \quad t = R_e^{-2/9} T \tag{8.18}$$

and we can seek the following expansions:

$$\begin{cases} u(x,y;R_e) = U_0(x,y_m) + R_e^{-1/9} u_{1m}(X,y_m,T) + R_e^{-2/9} u_{2m}(X,y_m,T) + ... \\ v(x,y;R_e) = R_e^{-5/18} v_{1m}(X,y_m,T) + R_e^{-7/18} v_{2m}(X,y_m,T) + ... \\ p(x,y;R_e) = R_e^{-2/9} p_{1u}(X,0,T) + ... \\ \rho(x,y;R_e) = \rho_0(x,y_m) + R_e^{-1/9} \rho_{1m}(X,y_m,T) + ... \end{cases} \tag{8.19}$$

We are interested only in u and v, since our task is to perform tha matching with the other solutions. For perturbation of this form, the viscous effects are manifested only in the sublayer, therefore we can plug (8.31) into the inviscid Navier-Stokes equations, getting, like in (4.3)

$$\begin{cases} u_{1m} = A(T,X)\frac{dU_0}{dy_m} \\ v_{1m} = -\frac{\partial A}{\partial X}U_0(Y_m) \end{cases}$$

The limits we are interested in are the following:

$$\lim_{y_m \to \infty} v = -R_e^{-5/18}\frac{\partial A}{\partial X} \tag{8.20}$$

Comparing with (8.10) we have

$$v_{1u}(X,0,T) = -\frac{\partial A}{\partial X} \tag{8.21}$$

which turns out (8.13) to be:

$$\frac{\partial p_{1u}}{\partial y_u} = \frac{\partial^2 A}{\partial X^2} \text{ at } y_u = 0 \tag{8.22}$$

The other limit is:

$$\lim_{y_m \to 0} u = \lambda(y_m + R_e^{-1/9}A) = \lambda R_e^{-1/9}(y_l + A) \tag{8.23}$$

The matching condition with the sublayer is the following:

$$u_{1l}(X,\infty,T) = \lambda(y_l + A(X,T)) \tag{8.24}$$

8.4 Canonical Form

See figure (17) for a scatch of the tiered structure. In the upper deck (8.9) and (8.10) hold

$$x = R_e^{-1/3}X \quad y = R_e^{-5/18}y_u \quad t = R_e^{-2/9}T \tag{8.25}$$

$$\begin{cases} u(x,y,R_e) = 1 + R_e^{-2/9}u_{1u}(X,y_u,T) + ... \\ v(x,y,R_e) = R_e^{-5/18}v_{1u}(X,y_u,T) + ... \\ p(x,y,R_e) = R_e^{-2/9}p_{1u}(X,y_u,T) + ... \\ \rho(x,y,R_e) = 1 + R_e^{-2/9}\rho_{1u}(X,y_u,T) + ... \end{cases} \tag{8.26}$$

and we need to solve (8.27)

$$m\frac{\partial^2 p_{1u}}{\partial X^2} + 2\frac{\partial^2 p_{1u}}{\partial X \partial T} = \frac{\partial^2 p_{1u}}{\partial y_u^2} \tag{8.27}$$

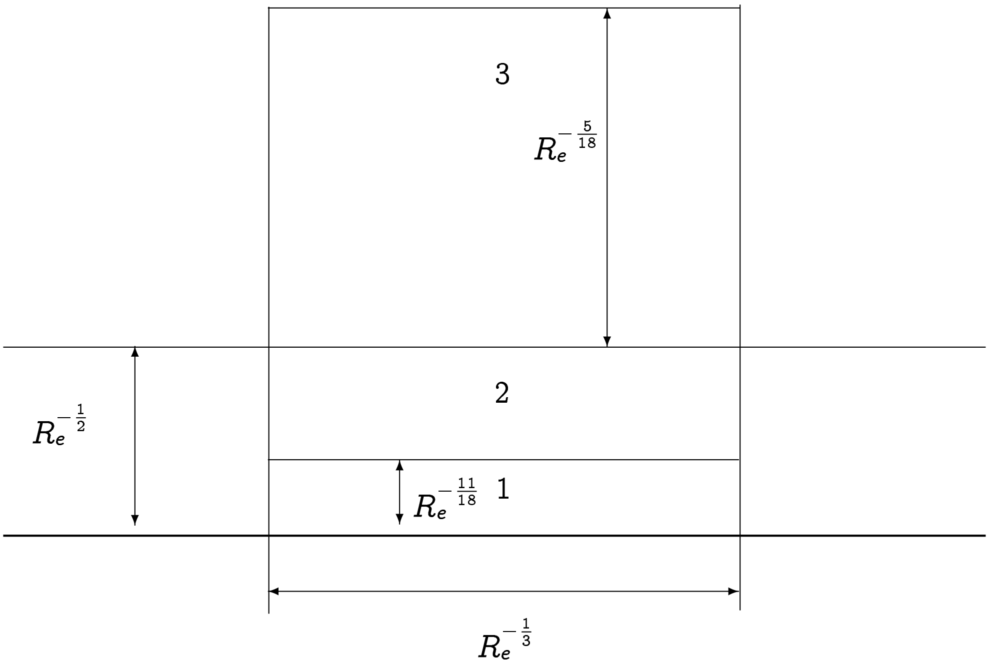

Figure 17: Triple-deck structure

with boundary conditions (8.28) and (8.29)

$$p_{1u} \to 0 \text{ as } y_u \to \infty \tag{8.28}$$

$$\frac{\partial p_{1u}}{\partial y_u} = \frac{\partial^2 A}{\partial X^2} \text{ at } y_u = 0 \tag{8.29}$$

In the viscous sublayer

$$x = R_e^{-1/3} X \quad y = R_e^{-11/18} y_u \quad t = R_e^{-2/9} T \tag{8.30}$$

and

$$\begin{cases} u(x, y; R_e) = R_e^{-1/9} u_{1l}(X, y_l, T) + ... \\ v(x, y; R_e) = R_e^{-7/18} v_{1l}(X, y_l, T) + ... \\ p(x, y; R_e) = R_e^{-2/9} p_{1u}(X, 0, T) + ... \end{cases} \tag{8.31}$$

After the following transformation

$$\rho = \rho_w \tilde{\rho} \quad \mu = \mu_w \tilde{\mu}$$

the problem in the sublayer takes the form:

$$\begin{cases} \frac{\partial u_{1l}}{\partial T} + u_{1l} \frac{\partial u_{1l}}{\partial X} + v_{1l} \frac{\partial u_{1l}}{\partial y_l} = -\frac{\partial p_{1u}(X,0,T)}{\partial X} + \frac{\partial^2 u_{1l}}{\partial y_l^2} \\ \frac{\partial u_{1l}}{\partial X} + \frac{\partial v_{1l}}{\partial y_l} = 0 \end{cases} \tag{8.32}$$

with boundary conditions

$$u_{1l} = v_{1l} = 0 \text{ at } y_l = 0 \tag{8.33}$$

and (8.34)

$$u_{1l}(X, \infty, T) = \lambda(y_l + A(X,T)) \tag{8.34}$$

8.5 Corrispondence to the subsonic regime

As little exercise, useful in showing that what obtained is right, consists in rewriting equation (8.27) in the subsonic limit $m \to -\infty$. We must end up with equation (4.15). Comparing equations (8.9) and (8.10) with (4.12) and (4.13) we have:

$$\begin{cases} X = R_e^{-1/24} x_* \\ y_u = R_e^{-7/72} y_* \\ T = R_e^{-1/36} t_* \\ p_{1u} = R_e^{-1/36} p_1 \end{cases} \tag{8.35}$$

Since $M_\infty^2 = 1 + R_e^{-1/9} m$, in the subsonic regime we have $m \sim -R_e^{-1/9}$. Using this and (8.35) in (8.27) we end up with (4.15).
Other relations which could be useful are:

$$A_T = R_e^{-1/72} A_S \tag{8.36}$$

which comes from the correspondence

$$\frac{\partial p_{1u}}{\partial y_u} = \frac{\partial^2 A_T}{\partial X^2}$$

And:

$$k_T = R_e^{1/24} k_S \qquad \omega_T = R_e^{1/36} \omega_S \tag{8.37}$$

which comes from the correspondence between the exponentials in the normal mode form.

9 Transonic Tollmien-Schlichting waves theory

The derivation of the neutral perturbation equation in the boundary layer is exactly the same as in the subsonic regime, being (4.19) analogous to (8.32).

$$\frac{\lambda \bar{A}}{\kappa}(ik\lambda)^{2/3} Ai'(\zeta_0) = ik\bar{p}_{1u}(0) \tag{9.1}$$

where $\zeta_0 = -(ik\lambda)^{1/3}\frac{\omega}{k\lambda}$. What is different is the equation for pressure in the upper tier (8.27) (to be compared to (4.15)). Again:

$$\begin{cases} p_{1u} = \epsilon e^{i(kX-\omega T)}\bar{p}_{1u}(y_u) \\ A = \epsilon e^{i(kX-\omega T)}\bar{A} \end{cases} \tag{9.2}$$

and (8.27) becomes

$$\ddot{\bar{p}}_{1u}(y_u) + (k^2m - 2k\omega)\bar{p}_{1u}(y_u) = 0$$

$$\begin{cases} \bar{p}_{1u}(\infty) = 0 \\ \dot{\bar{p}}_{1u}(0) = -k^2\bar{A} \end{cases}$$

Now we distinguish two different cases:

1. $k^2m - 2k\omega < 0$

   The solution to the equation, according to the boundary conditions, is:

   $$\bar{p}_{1u}(y_u) = \frac{k^2\bar{A}}{\sqrt{2k\omega - k^2m}}e^{-\sqrt{2k\omega-k^2m}\ y_u}$$

   Therefore:

   $$\bar{p}_{1u}(0) = \frac{k^2\bar{A}}{\sqrt{2k\omega - k^2m}} \tag{9.3}$$

2. $k^2m - 2k\omega > 0$

   Now

   $$\bar{p}_{1u}(y_u) = \alpha e^{i\sqrt{k^2m-2k\omega}\ y_u} + \beta e^{-i\sqrt{k^2m-2k\omega}\ y_u}$$

   Looking at this expression is evident that it is impossible to satisfy the condition $\bar{p}_{1u}(\infty) = 0$ which seemed to be reasonable in the subsonic regime. Hence, in the transonic regime we have to allow the perturbation to propagate at infinity. However, we need a boundary condition in place of the no any more valid one.

   As we said in the introduction, in this paper we are dealing with a stability problem and not a receptivity one. Therefore, we can only allow a perturbation which goes from the boundary layer toward infinity and not vice versa.

   Let us fix $x = 0$ and write:

   $$p_{1u} = \epsilon\{\alpha e^{i\{\sqrt{k^2m-2k\omega}\ y_u-\omega T\}} + \beta e^{-i\{\sqrt{k^2m-2k\omega}\ y_u-\omega T\}}\}$$

We choose $\beta = 0$ in order to have an upward propagating perturbation. This leads to the following, after using the boundary condition for the first derivative evaluated in 0:

$$\bar{p}_{1u}(y_u) = \frac{ik^2\bar{A}}{\sqrt{k^2m - 2k\omega}} e^{i\sqrt{k^2m-2k\omega}\ y_u}$$

Therefore:

$$\bar{p}_{1u}(0) = \frac{ik^2\bar{A}}{\sqrt{k^2m - 2k\omega}} \tag{9.4}$$

The dispersion relations, obtained by substituting in (9.1), are the following:

1. $k^2m - 2k\omega < 0$

$$\sqrt{2k\omega - k^2m}\ \frac{\lambda^{5/3}}{k^{7/3}} Ai'(\zeta_0) = e^{i\frac{\pi}{6}} \int_{\zeta_0}^{\infty} Ai(q)dq \tag{9.5}$$

2. $k^2m - 2k\omega > 0$

$$\sqrt{k^2m - 2k\omega}\ \frac{\lambda^{5/3}}{k^{7/3}} Ai'(\zeta_0) = e^{i\frac{\pi}{3}} \int_{\zeta_0}^{\infty} Ai(q)dq \tag{9.6}$$

The dependence on λ can be hidden as in subsection (5.3). From now on, we concentrate on the firs case, for which $k^2m - 2k\omega < 0$.

9.1 Subsonic limit

This is a way to check that everything is right. Indeed, as soon as we perform the limit for $m \to -\infty$ we are in the sunsonic regime and we must obtain equation (5.23). Since

$$M_\infty = 1 + R_e^{-1/9} m$$

we have to set

$$m = -R_e^{1/9} \tag{9.7}$$

in order to have $M_\infty = 0$. We have to consider the fact that in the subsonic regime tha scaling is different and (8.37) holds.

$$2k_T\omega_T = 2R_e^{5/72} k_S\omega_S$$

$$k^2m = -R_e^{14/72} k_S^2$$

$$\zeta_0 = -(ik_T\lambda)^{1/3} \frac{\omega_T}{k_T\lambda} = -(ik_S\lambda)^{1/3} \frac{\omega_S}{k_S\lambda}$$

Hence, $2k_s\omega_S$ is neglectible and equation (9.5) takes the form:

$$\frac{\lambda^{5/3}}{k_S^{4/3}} Ai'(\zeta_0) = e^{i\frac{\pi}{6}} \kappa$$

which is exactly the awaited equation.

10 Affine transformation and behaviors of k and ω with respect to m

One possibility is to solve numerically equation (9.5). There is an easier way, though. As soon as we find a transformation which enables to write (9.5) in the same form as (5.23) we can use everything already known about that equation. The transformation is:

$$\begin{cases} k = A\tilde{k} \\ \omega = B\tilde{\omega} \end{cases} \tag{10.1}$$

Remember that the variables with $\sim$ are those in the subsonic equations. What we basically want is that:

$$\begin{cases} \frac{\sqrt{2k\omega - mk^2}}{k^{7/3}} = \frac{1}{\tilde{k}^{4/3}} \\ \zeta_0(k,\omega) = \zeta_0(\tilde{k},\tilde{\omega}) \end{cases}$$

From the second equation we have:

$$\frac{A^{1/3}B}{A} = 1 \Rightarrow B = A^{2/3}$$

Hence:

$$\begin{cases} k = A\tilde{k} \\ \omega = A^{2/3}\tilde{\omega} \end{cases} \tag{10.2}$$

The second equation leads to:

$$2A^{5/3}\frac{\tilde{\omega}}{\tilde{k}} - mA^2 = A^{14/3}$$

Multiplying for A^{-2}:

$$A^{8/3} - 2\frac{\tilde{\omega}}{\tilde{k}}A^{-1/3} + m = 0$$

Setting $x = A^{1/3} > 0$ we obtain:

$$x^8 - 2\frac{\tilde{\omega}}{\tilde{k}}\frac{1}{x} + m = 0 \tag{10.3}$$

What we are interested in is evaluating the neutral point (5.25) for different values of m (with the aim to study the limit $m \to \infty$). The equation (10.3) may be written as

$$f(x;m) = x^8 - 2\frac{\tilde{\omega}_*}{\tilde{k}_*}\frac{1}{x} + m = 0 \tag{10.4}$$

with ($\tilde{\omega}_* = 2.29797, \tilde{k}_* = 1.00049$). Keeping in mind that $x > 0$, we observe that:

$$f(x \to 0; m) = -\infty$$

$$f(\infty; m) = \infty$$

and there is no stationary point for $x > 0$, since the only one possible

$$f'(x; m) = 8x^7 + 2\frac{\tilde{\omega}_*}{\tilde{k}_*}\frac{1}{x^2} = 0 \Rightarrow x = -\left(\frac{1}{4}\frac{\tilde{\omega}_*}{\tilde{k}_*}\right)^{1/9}$$

is for a negative value of x. Therefore, there is only one solution ammitted for x. A programm (language IDL) has been developed in order to work out x for different values of m (see the following subsection). Once x is found, $A = x^3$ and the transformation (10.2) are defined. See figure (18).

10.1 Transormation programme

```
function func,x
common constant,m
return,x^8-2*(2.29797/1.00049)/x+m
end

function funcprime,x
common constant,m
return,8*x^7+2*(2.29797/1.00049)/x^2
end


function newtonmethod,func,funcprime,x0
IF Size(func, /Type) NE 7 THEN BEGIN
print, 'String argument required'
RETURN, -1
ENDIF
IF Size(funcprime, /Type) NE 7 THEN BEGIN
print, 'String argument required'
RETURN, -1
ENDIF
tolerance = 10.0^(-7)
epsilon = 10.0^(-14)

maxIterations = 20
haveWeFoundSolution = 0
```

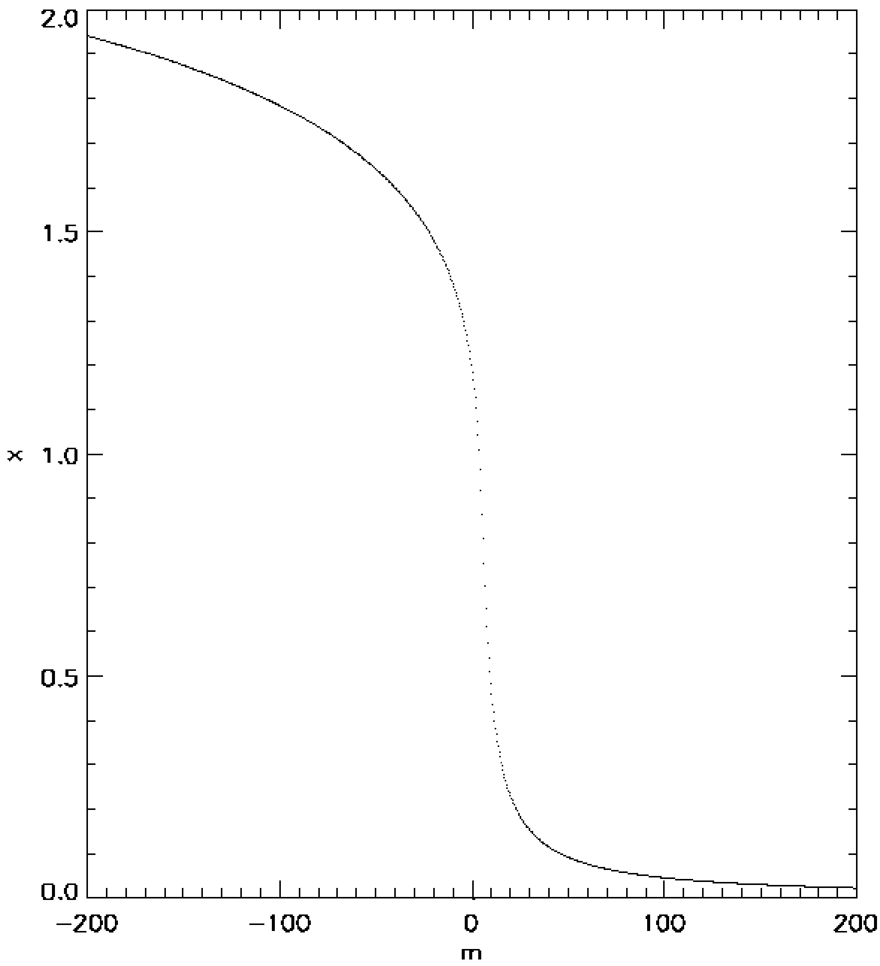

Figure 18: x as function of m

```
for i=1,maxIterations do begin

    y = Call_Function(func,x0)
    yprime = Call_Function(funcprime,x0)

    if abs(yprime) lt epsilon then begin
        print,'NEWTON>>WARNING: denominator is too small'
        break
    endif
```

```
    x1 = x0 - y/yprime

    if abs(x1 - x0)/abs(x1) lt tolerance then begin
        haveWeFoundSolution = 1
        break
    endif

    x0 = x1

endfor

if haveWeFoundSolution  then begin
    print,'NEWTON>>The root is:',x1
endif else begin
    print,'NEWTON>>Warning: Not able to find solution to within the desired toler
    print,'NEWTON>>The last computed approximate root was',x1
endelse
return,x1
end

function find,enter
common constant,m
m=enter

r=(dindgen(5000)+0.01)/4999
f=func(r)
step=dindgen(n_elements(f)-1)
x=dindgen(n_elements(f)-1)
for I=1,n_elements(f)-1 do begin
step[I-1]=f[I]*f[I-1]
endfor
d=where(step lt 0, n)
;print,n,format='(I0," zeros")'
if n eq 1 then begin
;print,(r[d]+r[d+1])/2
a=newtonmethod('func','funcprime',(r[d]+r[d+1])/2)
endif else begin
a=newtonmethod('func','funcprime',2.)
endelse
return,a
end
```

```
loadct,5
set_plot, 'z'
device, set_resolution=[640,680]
a=-200.0
b=200.0
plot,[0,0],back=255,color=0,xrange=[a,b],yrange=[0,2],xtitle='m',ytitle='x',charsize=1.2,
for I=a,b,0.5 do begin
plots,[I,find(I)],color=0,psym=3
endfor
write_png, 'filename.png',tvrd()
end
```

10.2 Subsonic limit

In the previous section we found that through the following affine transformation for ω and k

$$\begin{cases} k = A\tilde{k}, \\ \omega = A^{2/3}\tilde{\omega}, \end{cases} \tag{10.5}$$

with A satisfacting the equation

$$A^{8/3} - 2\frac{\tilde{\omega}}{\tilde{k}}\frac{1}{A^{1/3}} + m = 0, \tag{10.6}$$

we can describe the Tollmien-Schlicting waves theory in the transonic regime. The purpose in this section is to verify that, when performing the subsonic limit $m \to -\infty$, everything is in agreement with the well known equations. The idea is to find what order with respect to m are k and ω. After that, it will be possible to evaluate which terms of the transonic equations to keep and which ones becomes neglectible.
When $m \to -\infty$ only $A \to \infty$ can balance its growth, indeed

- If $A \to 0$ we have
$$2\frac{\tilde{\omega}}{\tilde{k}}\frac{1}{A^{1/3}} = -|m|$$
which is impossible, being all the quatities in the left hand side positive;

- If $A \to \infty$ we have
$$A^{8/3} = |m| \Rightarrow A = |m|^{3/8} \tag{10.7}$$
that is what we actually want.

Given (10.7) we can explicitly express (10.5) as

$$\begin{cases} k = |m|^{3/8}\tilde{k} \\ \omega = |m|^{1/4}\tilde{\omega} \end{cases} \tag{10.8}$$

Let us first check that the condition $k^2 m - 2k\omega < 0$ is still satisfied. It is immediatly proven by substitution

$$-|m|^{7/4}\tilde{k}^2 - 2|m|^{5/4}\tilde{k}\tilde{\omega} < 0$$

Remember now that k and ω were introduced with the normal modes beheaving like

$$e^{i(kX-\omega T)}$$

The need for kX and ωT to be order one, leads to

$$X \sim \tfrac{1}{k} \sim |m|^{-3/8} \qquad T \sim \tfrac{1}{\omega} \sim |m|^{-1/4} \tag{10.9}$$

What we want to do now is to analize the consequences of this beheavior on the governing equations (8.27) to (8.34). Let us start from the external region evaluating the order of all the terms.

$$m\frac{\partial^2 p_{1u}}{\partial X^2} + 2\frac{\partial^2 p_{1u}}{\partial X \partial T} = \frac{\partial^2 p_{1u}}{\partial y_u^2}$$

with boundary conditions (8.28) and (8.29)

$$\begin{cases} p_{1u} \to 0 \text{ as } y_u \to \infty \\ \frac{\partial p_{1u}}{\partial y_u} = \frac{\partial^2 A}{\partial X^2} \text{ at } y_u = 0 \end{cases}$$

Analizing the first equation we can forget about p_{1u} which is present in each term.

- $$m\frac{\partial^2}{\partial X^2} \sim \frac{m}{X^2} \sim -\frac{|m|}{|m|^{-3/4}} \sim -|m|^{7/4}$$

- $$2\frac{\partial^2}{\partial X \partial T} \sim \frac{X}{T} \sim \frac{1}{|m|^{-3/8}|m|^{-1/4}} = |m|^{5/8}$$

  which is negelectible when compared to the previous term

- $$\bar{p}_{1u}(y_u) = \frac{k^2 \bar{A}}{\sqrt{2k\omega - k^2 m}} e^{-\sqrt{2k\omega - k^2 m}\ y_u}$$

We want

$$y_u\sqrt{2k\omega - k^2 m} = O(1)$$

Therefore

$$y_u^{-1} \sim \sqrt{2k\omega - k^2 m} = \sqrt{2|m|^{5/8}\tilde{k}\tilde{\omega} + |m|^{7/4}\tilde{k}^2} \sim |m|^{7/8}$$

Namely

$$y_u \sim |m|^{-7/8} \tag{10.10}$$

and it follows that

$$\frac{\partial^2}{\partial y_u^2} \sim y_u^{-2} \sim |m|^{7/4}$$

The first conclusion is that the equation for pressure turns out to be

$$|m|\frac{\partial^2 p_{1u}}{\partial X^2} + \frac{\partial^2 p_{1u}}{\partial y_u^2} = 0 \tag{10.11}$$

We want to retain the boundary condition as well

$$\frac{\partial p_{1u}}{\partial y_u} = \frac{\partial^2 A}{\partial X^2} \Rightarrow \frac{p_{1u}}{|m|^{-7/8}} \sim \frac{A}{|m|^{-3/4}} \tag{10.12}$$

Let us now inspect the sublayer, where the pressure perturbation is of the same order as in the outer region. This means that from (10.12) we have

$$p_{1l}(X,T) = p_{1u}(X,0,T) \sim A|m|^{-1/8} \tag{10.13}$$

Here we recall the boundary sublayer equations

$$\begin{cases} \frac{\partial u_{1l}}{\partial T} + u_{1l}\frac{\partial u_{1l}}{\partial X} + v_{1l}\frac{\partial u_{1l}}{\partial y_l} = -\frac{\partial p_{1u}(X,0,T)}{\partial X} + \frac{\partial^2 u_{1l}}{\partial y_l^2} \\ \frac{\partial u_{1l}}{\partial X} + \frac{\partial v_{1l}}{\partial y_l} = 0 \end{cases}$$

with boundary conditions

$$\begin{cases} u_{1l} = v_{1l} = 0 \text{ at } y_l = 0 \\ u_{1l}(X,\infty,T) = \lambda(y_l + A(X,T)) \end{cases}$$

Our will to maintain the second boundary consition leads to

$$u_{1l} \sim y_l \sim A$$

We clearly have

$$u_{1l} \sim \frac{X}{T} \sim |m|^{-1/8}$$

Thus,

$$u_{1l} \sim y_l \sim A \sim |m|^{-1/8} \tag{10.14}$$

It follows that (10.13) becomes

$$p_{1l} \sim |m|^{-1/4} \tag{10.15}$$

Finally, from the continuity equation, that we clearly want to be still balanced, we have that

$$v_{1l} \sim \frac{u_{1l} y_l}{X} \sim |m|^{1/8} \tag{10.16}$$

The collection of all the quantities found so far is

$$\begin{cases} X \sim |m|^{-3/8} \\ T \sim |m|^{-1/4} \\ y_l \sim |m|^{-1/8} \\ u_{1l} \sim |m|^{-1/8} \\ v_{1l} \sim |m|^{1/8} \\ p_{1l} \sim |m|^{-1/4} \end{cases} \tag{10.17}$$

The analysis of the momentum equation shows that

$$\frac{\partial u_{1l}}{\partial T} \sim u_{1l}\frac{\partial u_{1l}}{\partial X} \sim v_{1l}\frac{\partial u_{1l}}{\partial y_l} \sim \frac{\partial p_{1u}(X,0,T)}{\partial X} \sim \frac{\partial^2 u_{1l}}{\partial y_l^2} \sim |m|^{1/8}$$

and therefore it stays unchanged. As we hoped since the beginning, the final strucure completely matches with that in the subsonic regime. This is fair enought when performing the following transformation:

$$\begin{cases} X = |m|^{-3/8}\hat{X} \\ T = |m|^{-1/4}\hat{T} \\ y_l = |m|^{-1/8}\hat{y}_l \\ y_u = |m|^{-7/8}\hat{y}_u \\ u_{1l} = |m|^{-1/8}\hat{u}_{1l} \\ v_{1l} = |m|^{1/8}\hat{v}_{1l} \\ p_{1l} = |m|^{-1/4}\hat{p}_{1l} \\ A = |m|^{-1/8}\hat{A} \end{cases} \tag{10.18}$$

Indeed now the equations are

$$\frac{\partial^2 \hat{p}_{1u}}{\partial \hat{X}^2} + \frac{\partial^2 \hat{p}_{1u}}{\partial \hat{y}_u^2} = 0$$

with boundary conditions

$$\begin{cases} \hat{p}_{1u} \to 0 \text{ as } \hat{y}_u \to \infty \\ \frac{\partial \hat{p}_{1u}}{\partial \hat{y}_u} = \frac{\partial^2 \hat{A}}{\partial \hat{X}^2} \text{ at } \hat{y}_u = 0 \end{cases}$$

and

$$\begin{cases} \frac{\partial \hat{u}_{1l}}{\partial \hat{T}} + \hat{u}_{1l}\frac{\partial \hat{u}_{1l}}{\partial \hat{X}} + \hat{v}_{1l}\frac{\partial \hat{u}_{1l}}{\partial \hat{y}_l} = -\frac{\partial \hat{p}_{1l}}{\partial \hat{X}} + \frac{\partial^2 \hat{u}_{1l}}{\partial \hat{y}_l^2} \\ \frac{\partial \hat{u}_{1l}}{\partial \hat{X}} + \frac{\partial \hat{v}_{1l}}{\partial \hat{y}_l} = 0 \end{cases}$$

with boundary conditions

$$\begin{cases} \hat{u}_{1l} = \hat{v}_{1l} = 0 \text{ at } \hat{y}_l = 0 \\ \hat{u}_{1l}(\hat{X}, \infty, \hat{T}) = \lambda(\hat{y}_l + \hat{A}) \end{cases}$$

which are the same as in the subsonic theory. Once proved that everything works as expected in the limit $m \to -\infty$ let us move to the supersonic limit, which is the actual subject of our research.

10.3 Supersonic limit

We want to find A such that

$$A^{8/3} - 2\frac{\tilde{\omega}}{\tilde{k}}\frac{1}{A^{1/3}} + m = 0 \tag{10.19}$$

when $m \to \infty$. We rearrange this equation in the following form

$$A^{1/3}m = \frac{2\tilde{\omega}}{\tilde{k}} - A^3$$

Clearly we want $A^{1/3}m = O(1)$, being $\frac{2\tilde{\omega}}{\tilde{k}} = O(1)$ the only term which can balance the growth of m. It means that

$$A^{1/3} = \frac{\alpha}{m} \Rightarrow A = \left(\frac{\alpha}{m}\right)^3 \text{ and } A^3 = \left(\frac{\alpha}{m}\right)^9 \tag{10.20}$$

Plugging (10.20) into (10.19) we have

$$\alpha = \frac{2\tilde{\omega}}{\tilde{k}} - \left(\frac{\alpha}{m}\right)^9 = \frac{2\tilde{\omega}}{\tilde{k}} - \left(\frac{\frac{2\tilde{\omega}}{\tilde{k}} - \left(\frac{\alpha}{m}\right)^9}{m}\right)^9$$

Considering the asymptotic expansions we have that

$$\alpha = \frac{2\tilde{\omega}}{\tilde{k}} - \frac{1}{m^9}\left(\frac{2\tilde{\omega}}{\tilde{k}}\right)^9 + O(m^{-18}) \tag{10.21}$$

Now we are ready to express the transformation

$$\begin{cases} k = \left(\frac{2\tilde{\omega}}{\tilde{k}m}\right)^3 \left\{1 - \frac{3}{m^9}\left(\frac{2\tilde{\omega}}{\tilde{k}}\right)^8\right\} \tilde{k} + O(m^-21) \\ \omega = \left(\frac{2\tilde{\omega}}{\tilde{k}m}\right)^2 \left\{1 - \frac{2}{m^9}\left(\frac{2\tilde{\omega}}{\tilde{k}}\right)^8\right\} \tilde{\omega} + O(m^-20) \end{cases} \tag{10.22}$$

A calculation which makes use of (10.22) and will be useful later is

$$\sqrt{2k\omega - 2k^2m} = km^{1/2}\sqrt{\frac{2\omega}{km} - 1} = \tilde{k}\left(\frac{2\tilde{\omega}}{\tilde{k}}\right)^7 m^{-7} \tag{10.23}$$

This is the beheavior of the longitudinal and time coordinates

$$X \sim m^3$$
$$T \sim m^2$$

Let us start from the upper deck where we know that

$$p_{1u}(y_u) = \frac{k^2A}{\sqrt{2k\omega - k^2m}} e^{-\sqrt{2k\omega - k^2m}\ y_u}$$

Regarding the exponent we can consider that:

- If $\sqrt{2k\omega - k^2m}\ y_u >> 1$ then we have $p_{1u}(y_u) = 0$, which is not an interesting solution.
- If $\sqrt{2k\omega - k^2m}\ y_u << 1$ then we have

  $$p_{1u}(y_u) = \frac{k^2A}{\sqrt{2k\omega - k^2m}} \sim \frac{m^{-6}A}{m^{-7}} = Am \tag{10.24}$$

  Nevertheless, the boundary condition

  $$\frac{\partial p_{1u}}{\partial y_u} = \frac{\partial^2 A}{\partial X^2}$$

  gives

  $$y_u \sim \frac{p_{1u}X^2}{A} \sim \frac{Am^7}{A} = m^7$$

  which exactly produces $\sqrt{2k\omega - k^2m}\ y_u = O(1)$, in conflict with the first assumption.
- From the previous item we deduce that $y_u \sim m^7$ satisfies both the preservation of the exponent and the boundary condition.

We can notice that the choice of A is arbitrary so far and we can deduce it by analizing the boundary sublayer. From the boundary condition

$$u_l = \lambda(y_l + A)$$

we can write that

$$u_l \sim \frac{X}{T} \sim m \sim y_l \sim A \tag{10.25}$$

Therefore, from (10.24)

$$p_{1u} \sim p_{1l} \sim m^2 \tag{10.26}$$

From the continuity equation

$$v_l \sim \frac{y_l u_l}{X} \sim \frac{1}{m} \tag{10.27}$$

We perform the following transformation

$$\begin{cases} X = m^3 \hat{X} \\ T = m^2 \hat{T} \\ y_l = m \hat{y}_l \\ y_u = m^7 \hat{y}_u \\ u_{1l} = m \hat{u}_{1l} \\ v_{1l} = m^{-1} \hat{v}_{1l} \\ p_{1l} = m^2 \hat{p}_{1l} \\ A = m \hat{A} \end{cases} \tag{10.28}$$

Now the equations are

$$\frac{\partial^2 \hat{p}_{1u}}{\partial \hat{X}^2} + 2 \frac{\partial^2 \hat{p}_{1u}}{\partial \hat{X} \partial \hat{T}} = \frac{1}{m^9} \frac{\partial^2 \hat{p}_{1u}}{\partial \hat{y}_u^2} \tag{10.29}$$

with boundary conditions

$$\begin{cases} \hat{p}_{1u} \to 0 \text{ as } \hat{y}_u \to \infty \\ \frac{\partial \hat{p}_{1u}}{\partial \hat{y}_u} = \frac{\partial^2 \hat{A}}{\partial \hat{X}^2} \text{ at } \hat{y}_u = 0 \end{cases}$$

and

$$\begin{cases} \frac{\partial \hat{u}_{1l}}{\partial \hat{T}} + \hat{u}_{1l} \frac{\partial \hat{u}_{1l}}{\partial \hat{X}} + \hat{v}_{1l} \frac{\partial \hat{u}_{1l}}{\partial \hat{y}_l} = -\frac{\partial \hat{p}_{1l}}{\partial \hat{X}} + \frac{\partial^2 \hat{u}_{1l}}{\partial \hat{y}_l^2} \\ \frac{\partial \hat{u}_{1l}}{\partial \hat{X}} + \frac{\partial \hat{v}_{1l}}{\partial \hat{y}_l} = 0 \end{cases}$$

with boundary conditions

$$\begin{cases} \hat{u}_{1l} = \hat{v}_{1l} = 0 \text{ at } \hat{y}_l = 0 \\ \hat{u}_{1l}(\hat{X}, \infty, \hat{T}) = \lambda(\hat{y}_l + \hat{A}) \end{cases}$$

We can precisely calculate (10.24)

$$p_{1u}(y_u) = \frac{\bar{A}\tilde{k}^2}{2\tilde{\omega}} e^{-\tilde{k}\left(\frac{2\tilde{\omega}}{\tilde{k}}\right)^7 y_u}$$

Therefore,

$$p_{1u}(X, y_u, T) = \frac{\bar{A}\tilde{k}^2}{2\tilde{\omega}} e^{-\tilde{k}\left(\frac{2\tilde{\omega}}{\tilde{k}}\right)^7 y_u} e^{i(kX - \omega T)}$$

with k and ω given by (10.22). Equation (10.29) shows that the determination of the wave speed becomes inviscid at first order and governed by its left hand side.

11 Nonlinear equation for amplitude of Tollmien-Schlichting waves in a boundary layer with transonic free stream velocity

Let us recall some results and remarks obtained so far. The primordial stage of the transition to turbulence is associated with the appearance of Tollmien-Schlichting waves in the boundary layer, either spontaneously under the influence of disturbances or artificially by using oscillating devices.
In section (4) we first analized the stability of the boundary layer under the assumption of parallel flow. For small perturbations in a parallel flow, the structure of a Tollmien-Schlicthing wave is described by the Orr-Sommerfeld equation. It shows that the perturbation either increases or decreases depending on the flow parameters. Of particular interest is the borderline case of neutral waves with constant amplitude.
However, the boundary layer flow is not parallel. Only for large Reynolds numbers it becomes asymptotically parallel. Therefore, we need to revise stability theory for such flows. The first consequence is the development of a triple-tired structure. The second consequence concerns the presence of two different lenght scales along the surface of the flat plate for a Tollmien-Schlichting wave propagating in such a weakly non-parallel flow: the wavelenght and a lenght associated with the variation of the flow downstream. Given the presence of these two different typical lenghts, it is natural to use the method of multiscale expansion.
Mathematically, the problem consists of finding eigensolutions to the linearized Navier-Stokes equations. It was found that such solutions exist and are proportional to

$$E = e^{i(\Theta(X)-\beta\tau)}$$

where the frequency β is constant, to be more precise it is independent of the spatial variable, and X is the fast longitudinal coordinate. The amplitude is found to be dependent on the slow longitudinal coordinate. The stability boundary is determined by the condition that the derivative of the amlplitude vanishes. Further downstream, the relative Reynolds number increases, while the oscillation frequency remains the same. The result is a growth in amplitude. Its further evolution cannot be studied without allowance for the non-linear effects.

11.1 Formulation of the problem

As usual, we consider the two-dimensional flow past a flat plate parallel to the oncoming flow. The free stream velocity is considerated to be transonic. We consider a point O on the flat plate at distance L from the leading edge.

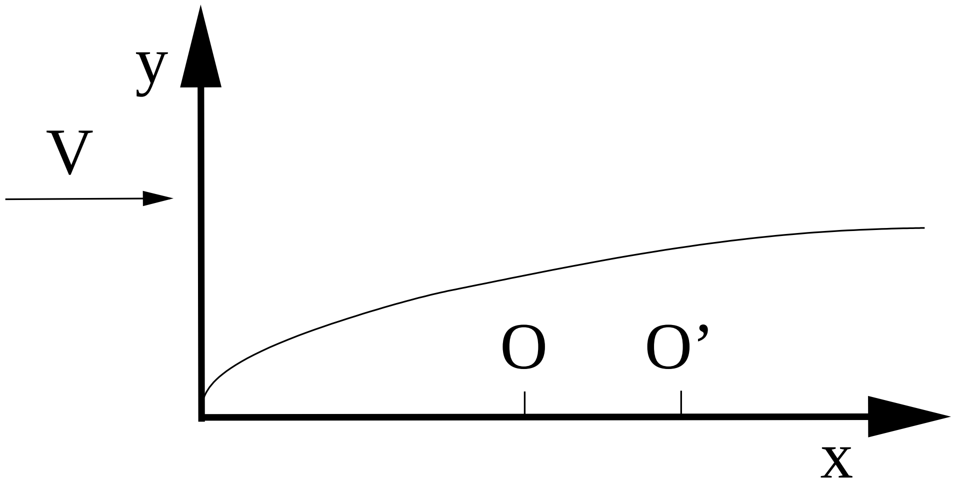

Figure 19: Scatch of the problem

As we said in our previous analysis, for large Reynolds numbers a boundary layer is formed.
Suppose that a Tollmien-Schlichting wave has been formed and its frequency is close to its neutral value at point O. Furthermore, suppose that, always at point O, the wave amplitude in known. Dealing again with the lower branch of the neutral curve, a triple-decked structure has to be considered. The aim is to determine the wave parameters downstream of point O, for example at point O' (see figure (19)).
As usual we introduce a Cartesian coordinate system with origin at the leading edge of the flat plate. The problem is treated in its nondimensional form. We consider the displacement of the wave from O to O', i.e. from the point $x = 1$ to the point $x = 1 + \Delta x$. We divide this process into two stages: linear displacement of the wave from O to O' and non-linear process of growth of the amplitude at point O'. We proceed in the following way.

1. Linear displacement of the wave from O to O'.

   In accordance with what seen hitherto, namely the linear theory of boundary layer stability, the perturbations are proportional to

   $$e^{i(\alpha x - \omega t)}$$

   where the oscillation frequency ω is real and α in general is complex. Regarding the point O, the immaginary part of α vanishes, being on the neutral curve. Remember that

   $$x = O(R_e^{-1/3}) \tag{11.1}$$

Therefore

$$\alpha_r = (R_e^{1/3}) \tag{11.2}$$

Given that the oscillation frequency ω does not change from O to O', we have that at point O'

$$\frac{\alpha_i}{\alpha_r} = O(\Delta x) \tag{11.3}$$

2. Nonlinear process of growth of the amplitude at point O'

   From this point of view the flow is no longer non-parallel and the non-linear stability theory of parallel flows can be used. According to this theory, the wave amplitude does not increase to infinity but tends to a limit, which is of order $(\alpha_i/\alpha_r)^{1/2}$. The distance $\Delta x'$ needed for this transition process satisfies

$$\alpha_i \Delta x' = O(1) \tag{11.4}$$

These effects are compatible and are manifested simultaneously if $\Delta x = \Delta x'$. From equations (11.2), (11.3) and (11.4) we have that this requirement is

$$\Delta x' = O(\alpha_i)^{-1} = O(\alpha_r)^{-1} O(\Delta x)^{-1} = O(\Delta x)$$

and we have

$$O(\Delta x) = O(\alpha_r)^{-1/2} = R_e^{-1/6} \tag{11.5}$$

What happens to the longitudinal variable is that it has two different lenght scales. In the multiscale theory we distinguish between a fast and a slow variable. The slow variable is

$$X = R_e^{1/6}(x-1)$$

The fast variable is, from (11.1)

$$x_* = R_e^{1/3}(x-1)$$

Let us write the fast variable and time in the following way

$$x_* = R_e^{1/3}\alpha(x-1) = R_e^{1/3}(\alpha_0 + ...)(x-1) \tag{11.6}$$

$$t_* = R_e^{2/9}\omega t = R_e^{2/9}(\omega_0 + ...)t \tag{11.7}$$

where only the leading order in the asymptotic expansion for wavenumber and frequency, given by the linear stability theory, can be considered. Writing those variables in this format allows to write the perturbations as proportional to

$$e^{i(x_* - t_*)} = e^{i\xi}$$

All the way through the following research, the time derivative is

$$\frac{\partial}{\partial t} = R_e^{2/9}\omega_0\frac{\partial}{\partial t_*}$$

and the longitudinal coordinate derivative, due to the two-scale theory, becomes

$$\partial_x = R_e^{1/6}\partial_X + R_e^{1/3}\alpha_0\partial_{x_*}$$

Finally the transversal variable, which behaves according to the Triple-Deck theory widely treated in the previous chapters. Our analysis, therefore, is divided in three parts, one for each layer. We start from the viscous wall layer, where the starting point is the Blasius boundary layer solution in the vicinity of the wall surface. Subsequently, we move to the main part of the boundary layer and the exterior potenzial flow. The equations for the first three orders are derived and an equation for the amplitude is inspected.

11.2 Viscous Wall Layer

Remember that the unperturbed steady flow in a boundary layer is described by the Blasius solution. In terms of $\psi(x, Y)$, the stream function such that

$$u = \partial_y\psi \qquad v = -\partial_x\psi$$

the Blasius solution may be written as follows

$$\psi(x, Y) = R_e^{-1/2}x^{1/2}f(\eta) \tag{11.8}$$

with

$$\eta = Yx^{-1/2} \tag{11.9}$$

The variable $Y = R_e^{1/2}y$ is order one in the main part of the boundary layer and the function f is the solution to the following boundary-value problem

$$f''' + \tfrac{1}{2}ff'' = 0 \qquad f(0) = f'(0) = 0 \qquad f'(\infty) = 1$$

We have already shown that when moving towards the wall, i.e. $\eta \to 0$, the following expansion holds

$$f = \frac{1}{2}\lambda\eta^2 + O(\eta^5) \tag{11.10}$$

with $\lambda = 0.33206....$ Therefore, in the boundary sublayer we have

$$\psi(x, Y) = R_e^{-1/2}x^{1/2}\frac{1}{2}\lambda\left(Yx^{-1/2}\right)^2 = \frac{1}{2}\lambda R_e^{-1/2}Y^2x^{-1/2} \tag{11.11}$$

We want to rewrite (11.11) in the variables $y_* = R_e^{11/18} y$, which is of order unity in the viscous wall layer, and X. We have

$$y_* = R_e^{11/18} y = R_e^{11/18} R_e^{-1/2} Y = R_e^{1/9} Y \quad \Rightarrow \quad Y = R_e^{-1/9} y_*$$

and

$$x = 1 + R_e^{-1/6} X$$

The expression (11.11) becomes

$$\begin{aligned} \psi(X, y_*) &= \tfrac{1}{2}\lambda R_e^{-1/2} R_e^{-2/9} y_*^2 (1 + R_e^{-1/6} X)^{-1/2} = \\ &= \tfrac{1}{2}\lambda R_e^{-13/18} y_*^2 \left(1 - \tfrac{1}{2} R_e^{-1/6} X + O(R_e^{-1/3})\right) = \\ &= \tfrac{1}{2}\lambda y_*^2 R_e^{-13/18} - \tfrac{1}{4}\lambda y_*^2 X R_e^{-16/18} + O(R_e^{-19/18}) \end{aligned}$$

The relative amplitude of the perturbations is of order $R_e^{-3/36}$ and, therefore, the unsteady solution to the Navier-Stokes equations can be sought in the form

$$\begin{aligned} \psi(X, x_*, y_*, t_*) &= \tfrac{\lambda}{2} y_*^2 R_e^{-26/36} + R_e^{-29/36} \psi_1(X, x_*, y_*, t_*) + \\ &+ \; R_e^{-32/36} \left(\psi_2(X, x_*, y_*, t_*) - \tfrac{\lambda}{4} y_*^2 X\right) + \\ &+ \; R_e^{-35/36} \psi_3(X, x_*, y_*, t_*) + O(R_e^{-38/36}) \end{aligned} \tag{11.12}$$

The first unsteady term has order $O(R_e^{-3/36})$ relative to the main steady term of the expansion. In order to find the expansion for pressure, we have to recall that in the Triple-Deck theory it is known to have order $R_e^{-2/9}$ which has to be multiplied for the relative amplitude of perturbations $R_e^{-3/36}$. It yields

$$p = R_e^{-11/36} p_1 + R_e^{-14/36} p_2 + R_e^{-17/36} p_3 + O(R_e^{-21/36}) \tag{11.13}$$

All the coefficients p_i are functions of t_*, x_*, X and y_*.
Remember that, from the triple-deck theory, when

$$\begin{cases} u = R_e^{-4/36} u_* \\ v = R_e^{-14/36} v_* \\ p = R_e^{-8/36} p_* \end{cases} \tag{11.14}$$

the equations in the viscous sublayer are

$$\begin{cases} \omega_0 \partial_{t_*} u_* + \alpha_0 u_* \partial_{x_*} u_* + v_* \partial_{y_*} u_* = -\alpha_0 \partial_{x_*} p_* + \partial_{y_*}^2 u_* \\ \partial_{y_*} p_* = 0 \end{cases} \tag{11.15}$$

As soon as we write u, v and p as in (11.14), the equation is analogous to (11.15) with the following modification for the x_* derivative

$$\alpha_0\partial_{x_*} = R_e^{-6/36}\partial_X + \alpha_0\partial_{x_*}$$

Therefore (11.15) becomes

$$\begin{cases} \omega_0\partial_{t_*}u_* & + \quad \alpha_0 u_*\{R_e^{-6/36}\partial_X + \alpha_0\partial_{x_*}\}u_* + v_*\partial_{y_*}u_* = \\ & = \quad -\alpha_0\{R_e^{-6/36}\partial_X + \alpha_0\partial_{x_*}\}p_* + \partial^2_{y_*}u_* \\ \partial_{y_*}p_* & = \quad 0 \end{cases} \tag{11.16}$$

Let us calculate u, v and p considering up to the third order terms

$$\begin{aligned} u \quad &= \quad \partial_y\psi = R_e^{11/18}\partial_{y_*}\psi = R_e^{-4/36}\left\{\lambda y_* + R_e^{-3/36}\partial_{y_*}\psi_1 + \right. \\ &+ \quad \left. R_e^{-6/36}\left(\partial_{y_*}\psi_2 - \tfrac{\lambda}{2}Xy_*\right) + R_e^{-9/36}\partial_{y_*}\psi_3\right\} = R_e^{-4/36}u_* \end{aligned}$$

$$\begin{aligned} v \quad &= \quad \partial_x\psi = -\left\{R_e^{1/3}\alpha_0\partial_{x_*} + R_e^{1/6}\partial_X\right\}\psi = \\ &= \quad -R_e^{-14/36}\left\{R_e^{-3/36}\alpha_0\partial_{x_*}\psi_1 + R_e^{-6/36}\alpha_0\partial_{x_*}\psi_2 + R_e^{-9/36}\left(\alpha_0\partial_{x_*}\psi_3 + \partial_X\psi_1\right)\right\} \end{aligned}$$

$$p = R_e^{-8/36}\left\{R_e^{-3/36}p_1 + R_e^{-6/36}p_2 + R_e^{-9/36}p_3\right\} = R_e^{-8/36}p_*$$

Let us call $\epsilon = R_e^{-3/36}$ and substitute the expressions for u_*, v_* and p_* into equation (11.16).

$$\begin{gathered} \epsilon\omega_0\partial^2_{t_*y_*}\psi_1 + \epsilon^2\omega_0\partial^2_{t_*y_*}\psi_2 + \epsilon^3\omega_0\partial^2_{t_*y_*}\psi_3 + \alpha_0\left[\lambda y_* + \epsilon\partial_{y_*}\psi_1 + \epsilon^2(\partial_{y_*}\psi_2\right. \\ \left.-\tfrac{\lambda}{2}Xy_*) + ..\right]\cdot\left[\epsilon\partial^2_{x_*y_*}\psi_1 + \epsilon^2\partial^2_{x_*y_*}\psi_2 + \epsilon^3(\partial^2_{Xy_*}\psi_1 + \partial^2_{x_*y_*}\psi_3)\right] - \alpha_0\left[\epsilon\partial_{x_*}\psi_1\right. \\ \left.+\epsilon^2\partial_{x_*}\psi_2 + \epsilon^3(\partial_{x_*}\psi_3 + \partial_X\psi_1)\right]\cdot\left[\lambda + \epsilon\partial^2_{y_*}\psi_1 + \epsilon^2(\partial^2_{y_*} - \tfrac{\lambda X}{2})\psi_2 + ...\right] = \\ -\epsilon\alpha_0\partial_{x_*}p_1 - \epsilon^2\alpha_0\partial_{x_*}p_2 - \epsilon^3\alpha_0\partial_{x_*}p_3 + \epsilon\partial^3_{y_*}\psi_1 + \epsilon^2\partial^3_{y_*}\psi_2 + \epsilon^3\partial^3_{y_*}\psi_3 \end{gathered}$$

and

$$\partial_{y_*}p_1 = \partial_{y_*}p_2 = \partial_{y_*}p_3 = 0$$

Collecting the same order terms we end up with the following equations with $i = 1, 2, 3$

$$\begin{gathered} \omega_0\partial^2_{t_*y_*}\psi_i + \alpha_0\lambda\left(y_*\partial^2_{x_*y_*}\psi_i - \partial_{x_*}\psi_i\right) + \alpha_0\partial_{x_*}p_i = \partial^3_{y_*}\psi_i + g_i \\ \partial_{y_*}p_i = 0 \\ g_1 = 0 \\ g_2 = \alpha_0\left(\partial_{x_*}\psi_1\partial^2_{y_*}\psi_1 - \partial_{y_*}\psi_1\partial^2_{x_*y_*}\psi_1\right) \end{gathered} \tag{11.17}$$

$$g_3 = \alpha_0 \left(\partial_{x_*}\psi_2 \partial^2_{y_*}\psi_1 + \partial_{x_*}\psi_1 \partial^2_{y_*}\psi_2 - \partial_{y_*}\psi_1 \partial^2_{x_* y_*}\psi_2 - \partial_{y_*}\psi_2 \partial^2_{x_* y_*}\psi_1 \right) +$$
$$\frac{1}{2}\lambda X \left(y_* \partial^2_{x_* y_*}\psi_1 - \alpha_0 \partial_{x_*}\psi_1 \right) + \lambda \left(\partial_X \psi_1 - y_* \partial^2_{X y_*}\psi_1 \right) - \partial_X p_1$$

The boundary conditions for the equations are given by the impermeability and no-slip conditions

$$\psi_i(y_* = 0) = \partial_{y_*}\psi_i|_{y_*=0} = 0 \tag{11.18}$$

and the condition of matching to the main part of the boundary layer.

11.3 Main Part of the Boundary Layer

In the main part of the boundary layer, the variable $Y = R_e^{1/2} y$ is of order one. The steady solution is described by (11.8). We now expand in terms of the variable X

$$x = 1 + R_e^{-1/6} X$$

$$\psi(X,Y) = R_e^{-1/2}(1+R_e^{-1/6})^{1/2} f\left(\frac{Y}{(1+R_e^{-1/6}X)^{1/2}} \right) = R_e^{-1/2}(1+\frac{1}{2}R_e^{-1/6}X$$
$$+\frac{1}{8}R_e^{-2/6}X^2+...)f(Y(1-\frac{1}{2}R_e^{-1/6}X-\frac{3}{8}R_e^{-2/6}X^2+...)) = R_e^{-1/2}(1+\frac{1}{2}R_e^{-1/6}X$$
$$+\frac{1}{8}R_e^{-2/6}X^2+...)f(Y-[\frac{1}{2}R_e^{-1/6}X+\frac{3}{8}R_e^{-2/6}X^2+...]Y)$$

Using the following expansion for f

$$f(Y-[\frac{1}{2}R_e^{-1/6}X+\frac{3}{8}R_e^{-2/6}X^2+...]Y) = f(Y)+$$
$$f'(Y)(-\frac{1}{2}R_e^{-1/6}X-\frac{3}{8}R_e^{-2/6}X^2)Y+\frac{1}{2}f''(Y)Y^2(-\frac{1}{2}R_e^{-1/6}X-\frac{3}{8}R_e^{-2/6}X^2)^2$$

Finally, we have

$$\psi(X,Y) = R_e^{-18/36} f(Y) + R_e^{-24/36} f_1(Y) X + R_e^{-30/36} f_2(Y) X^2 + ... \tag{11.19}$$

where

$$f_1(Y) = \frac{1}{2}(f - Yf') \qquad f_2(Y) = -\frac{1}{8}(f - Yf' - Y^2 f'')$$

Now we have to represent the unsteady solution. From (11.12) the perturbations move into the main deck. We can find their magnitude in the following way

$$u = \partial_y \psi(..., y_*) = \partial_y \Psi(..., Y)$$

Given that

$$y_* = R_e^{11/18} y \qquad Y = R_e^{1/2} y$$

we can conclude that

$$\Psi(..., Y) = R_e^{4/36}\psi(..., y_*)$$

Therefore, the unsteady solution in the main part of the boundary layer is

$$\begin{aligned}\psi = R_e^{-18/36} f(Y) + R_e^{-24/36} f_1(Y)X + R_e^{-25/36}\Psi_1 + R_e^{-28/36}\Psi_2 + \\ + R_e^{-30/36} f_2(Y)X^2 + R_e^{-31/36}\Psi_3 + O(R_e^{-34/36})\end{aligned} \tag{11.20}$$

From the triple-deck theory we know that the pressure has the same expression as in the sublayer

$$p = R_e^{-11/36} P_1 + R_e^{-14/36} P_2 + R_e^{-17/36} P_3 + O(R_e^{-20/36}) \tag{11.21}$$

All the functions Ψ_i and P_i are dependent on t_*, x_*, X and Y. The next step is the substitution of (11.20) and (11.21) into the inviscid Navier-Stokes equations. The calculation is performed in the following way.

$$u = \partial_y \psi = R_e^{1/2} \partial_Y \psi = R_e^{18/36} \partial_Y \psi$$

$$v = -\partial_x \psi = -\{R_e^{1/6} \partial_X + R_e^{1/3} \alpha_0 \partial_{x_*}\}\psi$$

The calculation produces

$$\begin{array}{cccc} u = f'(Y) + & R_e^{-6/36} f_1'(Y)X + & R_e^{-7/36}\partial_Y \Psi_1 + & R_e^{-10/36}\partial_Y \Psi_2 + \\ (u1) & (u2) & (u3) & (u4) \\ +R_e^{-12/36} f_2'(Y)X^2 + & R_3^{-13/36}\partial_Y \Psi_3 + & \dots & \\ (u5) & (u6) & & \end{array}$$

$$\begin{array}{ccc} v = -R_e^{-13/36}\alpha_0 \partial_{x_*}\Psi_1 & -R_e^{-16/36}\alpha_0 \partial_{x_*}\Psi_2 & -R_e^{-19/36}(\partial_X \Psi_1 + \alpha_0 \partial_{x_*}\Psi_3) \\ (v1) & (v2) & (v3) \\ -R_e^{-22/36}\partial_X \Psi_2 & -R_e^{-25/36}\partial_X \Psi_3 * & \dots \\ (v4) & (v5) & \end{array}$$

When derivating u with respect to x, i.e.

$$\partial_x = R_e^{6/36}\partial_X + R_e^{12/36}\alpha_0 \partial_{x_*}$$

we note that only the terms (u3), (u4) and (u6) produce a result and each stems two terms, due to the X and x_* derivatives. When deriving u with respect to y all the terms are retained, even if multiplied for $R_e^{18/36}$.

Keeping in mind that in the boundary layer $\partial_Y P_i = 0$, the equation which needs to be calculated is the following

$$u \ \partial_x u + v \ \partial_y u = -\partial_x p \tag{11.22}$$

The following tables help us in calculating only the terms which are present in the first, second and third approximations, instead of calculating all the terms.

u		$\partial_x u$	
$(u1)$	1	$\alpha_0 \partial_{x_*}(u3)$	$R_e^{5/36}$
$(u2)$	$R_e^{-6/36}$	$\alpha_0 \partial_{x_*}(u4)$	$R_e^{3/36}$
$(u3)$	$R_e^{-7/36}$	$\partial_X(u3)$	$R_e^{-1/36}$
		$\alpha_0 \partial_{x_*}(u6)$	
$(u4)$	$R_e^{-10/36}$	$\partial_X(u4)$	$R_e^{-4/36}$
$(u5)$	$R_e^{-12/36}$	$\partial_X(u6)$	$R_e^{-7/36}$
$(u6)$	$R_e^{-13/36}$		

v		$\partial_y u$	
$(v1)$	$R_e^{-13/36}$	$\partial_Y(u1)$	$R_e^{18/36}$
$(v2)$	$R_e^{-16/36}$	$\partial_Y(u2)$	$R_e^{12/36}$
$(v3)$	$R_e^{-19/36}$	$\partial_Y(u3)$	$R_e^{11/36}$
$(v4)$	$R_e^{-22/36}$	$\partial_Y(u4)$	$R_e^{8/36}$
$(v5)$	$R_e^{-25/36}$	$\partial_Y(u5)$	$R_e^{6/36}$
		$\partial_Y(u6)$	$R_e^{5/36}$

$\partial_x p$	
$\alpha_0 \partial_{x_*}(p3)$	$R_e^{-1/36}$
$\alpha_0 \partial_{x_*}(p2)$	$R_e^{-2/36}$
$\partial_X(p1)$	$R_e^{-5/36}$
$\alpha_0 \partial_{x_*}(p3)$	
$\partial_X(p2)$	$R_e^{-8/36}$
$\partial_X(p3)$	$R_e^{-11/36}$

Now we can collect the terms of our interest.

- $O(R_e^{5/36})$

$$f'(Y)\alpha_0\partial^2_{x_*Y}\Psi_1 - \alpha_0\partial_{x_*}\Psi_1 f''(Y) = 0$$

It can be written as

$$\cancel{f'(Y)^2}\partial_Y\left(\frac{\alpha_0\partial_{x_*}\Psi_1}{f'(Y)}\right) = 0$$

Therefore

$$\Psi_1 = f'(Y)\tilde{A}_1(x_*, X, t_*)$$

For simplicity in future calculations we write

$$\Psi_1 = \frac{f'(Y)}{\lambda}A_1(x_*, X, t_*)$$

- $O(R_e^{2/36})$

The equation is the same as before.

$$\Psi_2 = \frac{f'(Y)}{\lambda}A_2(x_*, X, t_*)$$

- $O(R_e^{-1/36})$ At this order the solution is

$$\Psi_3 = \frac{f'(Y)}{\lambda}A_3(x_*, X, t_*) + \frac{X}{2\lambda}(f' - Yf'')A_1$$

We can express these equations in a compact form:

$$\Psi_i = \frac{f'(Y)}{\lambda}A_i(x_*, X, t_*) + G_i \quad \partial_Y P_i = 0 \tag{11.23}$$

where $G_1 = G_2 = 0$ and $G_3 = \frac{X}{2\lambda}(f' - Yf'')A_1$. We can now formulate a boundary condition for the wall layer considering the limit $Y = 0$. At $Y = 0$ we have that $\Psi_i = YA_i$. The boundary condition is

$$\psi_i = A_i y_* + ... \text{ as } y_* \to \infty \tag{11.24}$$

11.4 Exterior potential flow

In the main part of the boundary layer we have that the transversal velocity is

$$v = -R_e^{-13/16}\alpha_0\partial_{x_*}\Psi_1 - R_e^{-16/36}\alpha_0\partial_{x_*}\Psi_2 + R_e^{-19/36}(\partial_X\Psi_1 + \alpha_0\partial_{x_*}\Psi_3)$$

Kepoing in mind that $f'(\infty) = 1$, we can work out the exression for v in this limit.

$$v = -R_e^{-13/36}\frac{\alpha_0}{\lambda}\partial_{x_*}A_1 - R_e^{-16/36}\frac{\alpha_0}{\lambda}\partial_{x_*}A_2 +$$

$$-R_e^{-19/36}\left(\frac{\alpha_0}{\lambda}\partial_{x_*}A_3+\frac{1}{\lambda}\partial_X A_1+\frac{\alpha_0}{2\lambda}X\partial_{x_*}A_1\right)$$

Therefore, always remembering the triple-deck expansions obtained earlier, we represent the solution in the potential flow in the form

$$\begin{cases} u = 1 + R_e^{-11/36}u_1 + R_e^{-14/36}u_2 + R_e^{-17/36}u_3 \\ v = R_e^{-13/36}v_1 + R_e^{-16/36}v_2 + R_e^{-19/36}v_3 \\ p = R_e^{-11/36}p_1^o + R_e^{-14/36}p_2^o + R_e^{-17/36}p_3^o \end{cases} \tag{11.25}$$

Being $u \sim R_e^{-1/4}$, $v \sim R_e^{-5/18}$ and $p \sim R_e^{-1/4}$ in the external region, we write

$$\begin{cases} u = 1 + R_e^{-8/36}\left\{R_e^{-3/36}u_1 + R_e^{-6/36}u_2 + R_e^{-9/36}u_3\right\} \\ v = R_e^{-10/36}\left\{R_e^{-3/36}v_1 + R_e^{-6/36}v_2 + R_e^{-9/36}v_3\right\} \\ p = R_e^{-8/36}\left\{R_e^{-3/36}p_1^o + R_e^{-6/36}p_2^o + R_e^{-9/36}p_3^o\right\} \end{cases} \tag{11.26}$$

The governing equation for pressure in the external potential flow is the following

$$2\partial_{tx}^2 p + mR_e^{-1/9}\partial_x^2 p = \partial_y^2 p - \partial_t^2 p \tag{11.27}$$

This is the small perturbed pressure equation, whose derivation is in appendix. Given that

$$t = \omega_0 R_e^{-2/9}t_*$$
$$y = R_e^{-5/18}y_0$$
$$x = R_e^{-1/3}\alpha_0 x_*$$
$$x = 1 + R_e^{-1/6}X$$

and consequently

$$\partial_x = R_e^{1/3}\partial_{x_*} + R_e^{1/6}\partial_X$$
$$\partial_t = \omega_0 R_e^{2/9}\partial_{t_*}$$
$$\partial_y = R_e^{5/18}\partial_{y_0}$$

equation (11.27) becomes

$$\begin{array}{ccc} 2\omega_0\alpha_0 R_e^{20/36}\frac{\partial^2 p}{\partial t_*\partial x_*}+ & 2\omega_0 R_e^{14/36}\frac{\partial^2 p}{\partial t_*\partial X}+ & m\omega_0^2 R_e^{20/36}\frac{\partial^2 p}{\partial x_*^2}+ \\ (1) & (2) & (3) \\ +2m\alpha_0 R_e^{14/36}\frac{\partial^2 p}{\partial x_*\partial X}+ & mR_e^{8/36}\frac{\partial^2 p}{\partial X^2} = & R_e^{20/36}\frac{\partial^2 p}{\partial y_0^2}+ \\ (4) & (5) & (6) \\ -\omega_0^2 R_e^{16/36}\frac{\partial^2 p}{\partial t_*^2} & & \\ (7) & & \end{array} \tag{11.28}$$

Given that pressure is present in each term, the expansion we have to plug into (11.28) is

$$p_1^o + R_e^{-3/36} p_2^o + R_e^{-5/36} p_3^o$$

In the following table the resulting terms

	p_1^o	p_2^o	p_3^o
(1)	$R_e^{20/36}$	$R_e^{17/36}$	$R_e^{14/36}$
(2)	$R_e^{14/36}$	$R_e^{11/36}$	$R_e^{8/36}$
(3)	$R_e^{20/36}$	$R_e^{17/36}$	$R_e^{14/36}$
(4)	$R_e^{14/36}$	$R_e^{11/36}$	$R_e^{8/36}$
(5)	$R_e^{8/36}$	$R_e^{5/36}$	$R_e^{2/36}$
(6)	$R_e^{20/36}$	$R_e^{17/36}$	$R_e^{14/36}$
(7)	$R_e^{16/36}$	$R_e^{13/36}$	$R_e^{10/36}$

Collection of the same order terms yields

$$2\omega_0 \alpha_0 \partial_{t_* x_*}^2 p_i^o + m\alpha_0^2 \partial_{x_*}^2 p_i^o = \partial_{y_0}^2 p_i^o + R_i \tag{11.29}$$

with

$$R_1 = R_2 = 0$$

and

$$R_3 = -2\omega_0 \partial_{t_* X}^2 p_1^o - 2m\alpha_0 \partial_{x_* X}^2 p_1^o$$

The boundary conditions for this problem are analogous to those formulated in the previous sections. Namely, the condition at infinity

$$p_i^o \to 0 \text{ as } y_0 \to \infty \tag{11.30}$$

and the matching condition

$$\partial_y p = -\partial_x v \to R_e^{5/18} \partial_{y_0} p = -R_e^{1/3} \alpha_0 \partial_{x_*} v - R_e^{1/3} \partial_X v \text{ at } y_0 = 0$$

Using the expression for v presented at the beginning of the present section, we have

$$R_e^{-1/36} \partial_{y_0} p_1^o + R_e^{-4/36} \partial_{y_0} p_2^o + R_e^{-7/36} \partial_{y_0} p_3^o =$$

$$= R_e^{-1/36} \frac{\alpha_0^2}{\lambda} \partial_{x_*}^2 A_1 + R_e^{-4/36} \frac{\alpha_0^2}{\lambda} \partial_{x_*}^2 A_2 + R_e^{-7/36} \cdot$$

$$\cdot \left(\frac{\alpha_0^2}{\lambda} \partial_{x_*}^2 A_3 + \frac{\alpha_0}{\lambda} \partial_{x_* X}^2 A_1 + \frac{\alpha_0^2}{2\lambda} X \partial_{x_*}^2 A_1 \right) + R_e^{-7/36} \frac{\alpha_0}{\lambda} \partial_{x_* X}^2 A_1 + \dots$$

We can conclude that the boundary conditions at $y_0 = 0$ are

$$\partial_{y_0} p_i^o = \frac{\alpha_0^2}{\lambda} \partial_{x_0}^2 A_i + r_i \tag{11.31}$$

with

$$r_1 = r_2 = 0$$

$$r_3 = \frac{2\alpha_0}{\lambda} \partial_{x_* X}^2 A_1 + \frac{\alpha_0^2}{2\lambda} X \partial_{x_*}^2 A_1$$

Remember that

$$p_i = p_i^o(x_*, X, y_0 = 0, t_*) \tag{11.32}$$

11.5 Equation for amplitude - First order

The free interaction problem is given by equations (11.17) with boundary conditions (11.18), (11.24) and equation (11.29) with boundary conditions (11.30), (11.31). Namely,

$$\omega_0 \partial_{t_* y_*}^2 \psi_1 + \alpha_0 \lambda \left(y_* \partial_{x_* y_*}^2 \psi_1 - \alpha_0 \partial_{x_*} \psi_1 \right) + \alpha_0 \partial_{x_*} p_1 = \partial_{y_*}^3 \psi_1 \tag{11.33}$$

with boundary conditions

$$\psi_1(y_* = 0) = \partial_{y_*} \psi_1|_{y_*=0} = 0 \qquad \psi_1 = A_1 y_* + ... \text{ as } y_* \to \infty \tag{11.34}$$

and equation for pressure

$$2\omega_0 \alpha_0 \partial_{t_* x_*}^2 p_1^o + m \alpha_0^2 \partial_{x_*}^2 p_1^o = \partial_{y_0}^2 p_1^o \tag{11.35}$$

with boundary conditions

$$\partial_{y_0} p_1^o = \tfrac{\alpha_0^2}{\lambda} \partial_{x_0}^2 A_1 \qquad p_1^o \to 0 \text{ as } y_0 \to \infty \tag{11.36}$$

We seek the solution to this problem in the form of periodic functions of the variable

$$\xi = x_* - t_*$$

We set

$$\psi_1 = \Phi_1(X, y_*) e^{i\xi} + \bar{\Phi}_1(X, y_*) e^{-i\xi} \tag{11.37}$$

It follows from the second of (11.34) that

$$A_1 = a_1(X) e^{i\xi} + \bar{a}_1(X) e^{-i\xi} \tag{11.38}$$

We have already shown in (9) the solution to (11.35)

$$p_1^o(y_0 = 0) = p_1 = \frac{\alpha_0^2 a_1(X)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}} \frac{e^{i\xi}}{\lambda} + \frac{\alpha_0^2 \bar{a}_1(X)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}} \frac{e^{-i\xi}}{\lambda} \tag{11.39}$$

Let us focus our attention on the first term, neglecting the complex conjugate. We found that, when solving (11.33) we end up with equation (5.13). Remember that

$$z = (i\alpha_0\lambda)^{1/3}\{y_* - \frac{\omega_0}{\alpha_0\lambda}\} = \Theta y_* + z_0$$

being

$$\Theta = (i\alpha_0\lambda)^{1/3} \quad \text{and} \quad z_0 = -\frac{\Theta\omega_0}{\alpha_0\lambda}$$

When derivating we have

$$\partial_z = \Theta\partial_{y_*}$$

In equation (5.13) we can write f as

$$f = \partial_{y_*}u = \Theta^2\partial_z^2\Phi_1(z)$$

Therefore, the Airy equation is

$$\Phi_1^{(IV)}(z) - z\Phi_1^{(II)}(z) = 0 \tag{11.40}$$

From boundary conditions (5.8) and (5.11) it follows that

$$v(0) = -\partial_{x_*}\psi_1 = -i\alpha_0\Phi_1(z_0)e^{i\xi} = 0 \Rightarrow \Phi_1(z_0) = 0$$

$$u(0) = \partial_{y_*}\psi_1|_{y_*=0} \sim \Phi_1^{(I)}(z_0) = 0$$

$$u(\infty) = a_1(X)e^{i\xi} \Rightarrow \Theta\partial_z\Phi_1 = a_1$$

$$\ddot{u}(0) = \Theta^3\Phi_1^{(III)}(z_0)e^{i\xi} = i\alpha_0 p_1$$

We can rearrange the boundary conditions as follows

$$\begin{cases} \Phi_1 = 0 \text{ at } z = z_0 \\ \Phi_1^{(I)} = 0 \text{ at } z = z_0 \\ \Phi_1^{(III)} = \frac{1}{\lambda^2}\frac{\alpha_0^2 a_1(X)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}} \text{ at } z = z_0 \\ \Phi_1^{(I)} = \frac{a_1(X)}{\Theta} \text{ at } z = \infty \end{cases} \tag{11.41}$$

We know from (5.13) that the solution to this problem is

$$\Theta^2\Phi_1^{(II)}(z) = AAi(z)$$

We can write

$$\Phi_1^{(II)}(z) = B(X)Ai(z)$$

as well as

$$\Phi_1(z) = B(X)\varphi_1(z) \quad \text{with} \quad \varphi_1^{(II)} = Ai(z) \tag{11.42}$$

Finally, from the boundary conditions (11.41)

$$\begin{cases} \varphi_1(z_0) = \varphi_1^{(I)}(z_0) = 0 \\ a_1(X) = \frac{\lambda^2}{\alpha_0^2} B(X) Ai'(z_0) \sqrt{2\alpha_0\omega_0 - \alpha_0^2 m} \end{cases} \tag{11.43}$$

Note that $B(X)$ is the amplitude of the Tollmien-Schlivhting wave and we know from the previous section that the nontrivial solution is for

$$\alpha_0 = A \cdot 1.001 \cdot \lambda^{5/4}$$

$$\omega_0 = A^{2/3} \cdot 2.299 \cdot \lambda^{3/2}$$

where we remind that A is the solution to

$$A^{8/3} - 2\frac{2.299}{1.001}\lambda^{1/4} A^{-1/3} + m = 0$$

11.6 Equation for amplitude - Second order

The equations for the second order approximation are

$$\begin{aligned} \omega_0 \partial^2_{t_* y_*} \psi_2 + \alpha_0 \lambda \left(y_* \partial^2_{x_* y_*} \psi_2 - \alpha_0 \partial_{x_*} \psi_2 \right) + \alpha_0 \partial_{x_*} p_2 = \\ \partial^3_{y_*} \psi_2 + \alpha_0 \left(\alpha_0 \partial_{x_*} \psi_1 \partial^2_{y_*} \psi_1 - \partial_{y_*} \psi_1 \partial^2_{x_* y_*} \psi_1 \right) \end{aligned} \tag{11.44}$$

with boundary conditions

$$\psi_2(y_* = 0) = \partial_{y_*} \psi_2|_{y_*=0} = 0 \qquad \psi_2 = A_2 y_* + \ldots \text{ as } y_* \to \infty \tag{11.45}$$

and equation for pressure

$$2\omega_0 \alpha_0 \partial^2_{t_* x_*} p_2^o + m\alpha_0^2 \partial^2_{x_*} p_2^o = \partial^2_{y_0} p_2^o \tag{11.46}$$

with boundary conditions

$$\partial_{y_0} p_2^o = \frac{\alpha_0^2}{\lambda} \partial^2_{x_0} A_2 \qquad p_2^o \to 0 \text{ as } y_0 \to \infty \tag{11.47}$$

The equation for the flow function is inhomogeneous. On the basis of the term on its right-hand side, we represent the function ψ_2 in the form

$$\psi_2 = B\bar{B}h + B^2 \varphi_2 e^{2i\xi} + \bar{B}^2 \bar{\varphi}_2 e^{-2i\xi} \tag{11.48}$$

We set

$$A_2 = a_2(X) e^{2i\xi} + \bar{a}_2(X) e^{-2i\xi} \tag{11.49}$$

The equation for pressure is exactly the same as in the previous approximation, therefore

$$p_2 = 2\frac{\alpha_0^2 a_1(X)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}}\frac{e^{2i\xi}}{\lambda} + 2\frac{\alpha_0^2 \bar{a}_1(X)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}}\frac{e^{-2i\xi}}{\lambda} \tag{11.50}$$

From the first of (11.45) with (11.50) we deduce that

$$h(z_0) = h'(z_0) = 0 \quad \text{and} \quad \varphi_2(z_0) = \varphi_2'(z_0) = 0 \tag{11.51}$$

Let us start by analizing the governing equation for $h(z)$. It is done by substituting $\psi_2 = B\bar{B}h(z)$ and (11.37) along with (11.42) into equatione (11.44). The only terms involved are:

$$\partial^3_{y_*}\{B\bar{B}h(z)\} = \partial_{y_*}\psi_1\partial^2_{x_*y_*}\psi_1 - \partial_{x_*}\psi_1\partial^2 y_*\psi_1$$

After substitution

$$\Theta^3 B\bar{B}h^{(III)}(z) = \Theta[B\varphi_1^{(I)}e^{i\xi} + \bar{B}\bar{\varphi}_1^{(I)}e^{-i\xi}]i\alpha_0\Theta[B\varphi_1^{(I)}e^{i\xi} - \bar{B}\bar{\varphi}_1^{(I)}e^{-i\xi}] +$$
$$-\Theta^2 i\alpha_0[B\varphi_1 e^{i\xi} - \bar{B}\bar{\varphi}_1 e^{-i\xi}][B\varphi_1^{(II)}e^{i\xi} + \bar{B}\bar{\varphi}_1^{(II)}e^{-i\xi}]$$

After multiplication, terms proportional to $e^{\pm i\xi}$ are discarded. Finally:

$$h^{(III)}(z) = \frac{i\alpha_0}{\Theta}[\bar{\varphi}_1\varphi_1^{(II)} - \varphi_1\bar{\varphi}_1^{(II)}] \tag{11.52}$$

We can see that

$$h^{(III)}(z) = \frac{i\alpha_0}{\Theta}[\bar{\varphi}_1\varphi_1^{(I)} - \varphi_1\bar{\varphi}_1^{(I)}]^{(I)}$$

and therefore

$$\begin{aligned} h^{(II)}(\infty) - \cancel{h^{(II)}(z_0)} = \tfrac{i\alpha_0}{\Theta}[\bar{\varphi}_1(\infty)\varphi_1^{(I)}(\infty) - \varphi_1(\infty)\bar{\varphi}_1^{(I)}(\infty) + \\ -\bar{\varphi}_1(z_0)\cancel{\varphi_1^{(I)}(z_0)} + \cancel{\varphi_1(z_0)}\bar{\varphi}_1^{(I)}(z_0)] = 0 \Rightarrow h^{(II)}(\infty) = 0 \end{aligned} \tag{11.53}$$

Here we used the fact that Stuart (see [11]) showed that the solution to this equation is not unique and we can choose $h^{(II)}(\infty) = 0$. Now we move to the remaining part of ψ_2, namely $\psi_2 = B^2\varphi_2 e^{2i\xi}$. In the right-hand side of (11.44) we use $\psi_1 = B\varphi_1 e^{i\xi}$. Clearly, only terms with phase $e^{2i\xi}$ in the right-hand side are considered. Keep in mind that

$$y_* = \frac{z - z_0}{\Theta}$$

Equation (11.44) is

$$\begin{aligned} \cancel{-2i\omega_0 B^2\varphi_2^{(I)}} + \alpha_0\lambda\left(\tfrac{z-\cancel{z_0}}{\cancel{\Theta}}2iB^2\varphi_2^{(I)}\cancel{\Theta} - 2iB^2\varphi_2\right) + i\alpha_0 p_2 = \\ = \Theta^3 B^2\varphi_2^{(III)} + \alpha_0 iB^2\left(\varphi_1\Theta^2\varphi_1^{(II)} - \varphi_1^{(1)}\Theta^2\varphi_1^{(I)}\right) \end{aligned} \tag{11.54}$$

It can be rearranged as

$$2i\lambda\alpha_0[z\varphi_2^{(I)} - \varphi_2] + \tfrac{i\alpha_0}{B^2}p_2 = \Theta^3\varphi_2^{(III)} + \\ +\alpha_0 i\Theta^2(\varphi_1\varphi_1^{(II)} - (\varphi_1^{(I)})^2) \tag{11.55}$$

We can formulate an additional boundary condition by evaluating the previous expression at $z = z_0$. From the first condition in (11.45) it follows that

$$\varphi_2(z_0) = \varphi_2^{(I)}(z_0) = 0$$

as well as we knew in the first order approximation that

$$\varphi_1(z_0) = \varphi_1^{(I)}(z_0) = 0$$

Therefore, the equation becomes:

$$\varphi_2^{(III)}(z_0) = \frac{i\alpha_0}{B^2}\bar{p}_2$$

From (11.50)

$$\varphi_2^{(III)}(z_0) = \frac{1}{\lambda}\frac{4i\alpha_0^3}{B^2}\frac{a_2(X)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}}$$

Furthermore, from the second in (11.45) we can deduce that

$$\psi_2^{(I)}(\infty) = A_2 \Rightarrow B^2\varphi_2^{(I)}(\infty) = a_2(X)$$

Finally, the boundary condition turns out to be

$$\varphi_2^{(III)}(z_0) = 4\left(\frac{\alpha_0}{\lambda}\right)^2 \frac{\Theta}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}}\varphi_2^{(I)}(\infty) \tag{11.56}$$

We can now derivate (11.53) with respect to z. We get

$$\varphi_2^{(IV)} - z\varphi_2^{(II)} = \frac{i\alpha_0}{\Theta}(\varphi_1^{(I)}\varphi_1^{(II)} - \varphi_1\varphi_1^{(III)}) \tag{11.57}$$

We can conclude that the problem in the second approximation is the following

(11.58)

$$h^{(III)}(z) = \frac{i\alpha_0}{\Theta}[\bar{\varphi}_1\varphi_1^{(II)} - \varphi_1\bar{\varphi}_1^{(II)}]$$

$$h^{(I)}(z_0) = h^{(II)}(\infty) = 0$$

$$\varphi_2^{(IV)} - z\varphi_2^{(II)} = \frac{i\alpha_0}{\Theta}(\varphi_1^{(I)}\varphi_1^{(II)} - \varphi_1\varphi_1^{(III)})$$

$$\varphi_2(z_0) = \varphi_2^{(I)}(z_0) = 0 \quad \text{and} \quad \varphi_2^{(III)}(z_0) = 4\left(\frac{\alpha_0}{\lambda}\right)^2 \frac{\Theta}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}}\varphi_2^{(I)}(\infty)$$

11.7 Equation for amplitude - Third order

The equations are:

$$\omega_0 \partial^2_{t_* y_*} \psi_3 + \alpha_0 \lambda \left(y_* \partial^2_{x_* y_*} \psi_3 - \alpha_0 \partial_{x_*} \psi_3 \right) + \alpha_0 \partial_{x_*} p_3 = \partial^3_{y_*} \psi_3 + \alpha_0$$
$$\cdot \left(\alpha_0 \partial_{x_*} \psi_2 \partial^2_{y_*} \psi_1 + \alpha_0 \partial_{x_*} \psi_1 \partial^2_{y_*} \psi_2 - \partial_{y_*} \psi_1 \partial^2_{x_* y_*} \psi_2 - \partial_{y_*} \psi_2 \partial^2_{x_* y_*} \psi_1 \right)$$
$$+ \tfrac{1}{2} \alpha_0 \lambda X \left(y_* \partial^2_{x_* y_*} \psi_1 - \alpha_0 \partial_{x_*} \psi_1 \right) + \lambda \left(\partial_X \psi_1 - y_* \partial^2_{X y_*} \psi_1 \right) - \partial_X p_1 \tag{11.59}$$

and

$$\partial_{y_*} p_3 = 0 \tag{11.60}$$

with boundary conditions

$$\psi_3(y_* = 0) = \partial_{y_*} \psi_3|_{y_* = 0} = 0 \quad \psi_3 = A_3 y_* + ... \text{ as } y_* \to \infty \tag{11.61}$$

and equation for pressure

$$2\omega_0 \alpha_0 \partial^2_{t_* x_*} p^o_3 + m \alpha_0^2 \partial^2_{x_*} p^o_3 = \partial^2_{y_0} p^o_3 - 2\omega_0 \partial^2_{t_* X} p^o_1 - 2m\alpha_0 \partial^2_{x_* X} p^o_1 \tag{11.62}$$

with boundary conditions

$$p^o_3 \to 0 \text{ as } y_0 \to \infty \tag{11.63}$$

and

$$\partial_{y_0} p^o_3 = \frac{\alpha_0^2}{\lambda} \partial^2_{x_0} A_3 + \frac{2\alpha_0}{\lambda} \partial^2_{x_* X} A_1 + \frac{\alpha_0^2}{2\lambda} X \partial^2_{x_*} A_1 \tag{11.64}$$

The solution to the problem of the third approximation can be represented in the form

$$\psi_3 = \Phi_3 e^{i\xi} + \bar{\Phi}_3 e^{-i\xi} + F_3 e^{3i\xi} + \bar{F}_3 e^{-3i\xi} \tag{11.65}$$

Let us focus on the $e^{\pm i\xi}$ terms. As before, we can write:

$$A_3 = a_3(X) e^{i\xi} + c.c. \tag{11.66}$$

and

$$p^o_3 = p^o_3(y_0) e^{i\xi} + c.c. \tag{11.67}$$

The latter, when evaluated at $y_0 = 0$ yields

$$p_3 = p_3(X) e^{i\xi} + c.c. \tag{11.68}$$

We start by evaluating expression (11.59).

$$
\begin{aligned}
&-i\omega_0\Phi_3^{(I)}\Theta e^{i\xi} + i\alpha_0\lambda\left(\tfrac{z-z_0}{\Theta}\Phi_3^{(I)}\Theta - \Phi_4\right)e^{i\xi} + i\alpha_0 p_3(X)e^{i\xi} = \\
&= \Theta^3\Phi_3^{(III)}e^{i\xi} + i\alpha_0\Theta^2\left\{\left(2B^2\varphi_2 e^{2i\xi} - c.c.\right)\left(B\varphi_1^{(II)}e^{i\xi} + c.c.\right)\right. \\
&+ \left(B\varphi_1 e^{i\xi} - c.c.\right)\left(B\bar{B}h^{(II)} + 2B^2\varphi_2^{(II)}e^{2i\xi} + c.c.\right) + \\
&- \left(B\varphi_1^{(I)}e^{i\xi} + c.c.\right)\left(2B^2\varphi_2^{(I)}e^{2i\xi} - c.c.\right) + \\
&- \left.\left(B\bar{B}h^{(I)} + 2B^2\varphi_2^{(I)}e^{2i\xi} + c.c.\right)\left(B\varphi_1^{(I)}e^{i\xi} - c.c.\right)\right\} + \\
&+\tfrac{1}{2}i\alpha_0\lambda XB\left(y_*\Theta\varphi_1^{(I)} - \varphi_1\right)e^{i\xi} + \lambda\tfrac{dB}{dX}(\varphi_1 - y_*\varphi_1^{(I)}\Theta)e^{i\xi} - \partial_X p_1
\end{aligned}
\tag{11.69}
$$

We are interested in $e^{i\xi}$ terms, and therefore we have

$$
\begin{aligned}
&\Theta^3\left(z\Phi_3^{(I)} - \Phi_3\right) + i\alpha_0 p_3 = \Theta^3\Phi_3^{(III)} + i\alpha_0 B^2\bar{B}\Theta^2 + \\
&\left\{2\varphi_2\bar{\varphi}_1^{(II)} + \varphi_1 h^{(II)} - \bar{\varphi}_1 + \varphi_2^{(II)} - \varphi_2^{(I)}\bar{\varphi}_1^{(I)} - h^I\varphi_1^{(I)}\right\} + \\
&+\tfrac{1}{2}\Theta^3 XB(y_*\Theta\varphi_1^{(I)} - \varphi_1) + \lambda\tfrac{dB}{dX}(\varphi_1 - y_*\Theta\varphi_1^{(I)}) - \partial_X p_1
\end{aligned}
\tag{11.70}
$$

From the first of equations (11.61) we can deduce that

$$
\Phi_3(z_0) = \Phi_3^{(I)}(z_0) = 0
$$

This condition, as well as the analogous ones for φ_1 and φ_2 and their first derivatives, enable to evaluate (11.69) at $z = z_0$

$$
\Phi_4^{(III)}(z_0) = \frac{i\alpha_0}{\Theta^3}p_4 + \frac{1}{\Theta^3}\partial_X p_1 = \frac{1}{\lambda}p_4 + \frac{1}{\Theta^3}\partial_X p_1 \tag{11.71}
$$

We can finally derivate equation (11.69) with respect to z , getting an equation for the third approximation function Φ_3.

$$
\begin{aligned}
\Phi_3^{(IV)} - z\Phi_3^{(II)} = -B^2\bar{B}\tfrac{i\alpha_0}{\Theta}&\left\{\varphi_1 h^{(III)} - h^{(I)}\varphi_1^{(II)} + 2\varphi_2\bar{\varphi}_1^{(III)} + \right. \\
-2\bar{\varphi}_1^{(I)}\varphi_2^{(II)} &+ \left.\varphi_2^{(I)}\bar{\varphi}_1^{(II)} - \bar{\varphi}_1\varphi_2^{(III)}\right\} - \tfrac{\Theta}{2}BXy_*Ai(z) + \\
&+\tfrac{dB}{dX}\tfrac{\lambda}{\Theta^2}y_*Ai(z)
\end{aligned}
\tag{11.72}
$$

where the Airy functions come from the fact that $\varphi_1^{(II)}(z) = Ai(z)$. It is now time to evaluate properly the boundary condition (11.71). p_1 is expressed in (11.39). We have to solve (11.62). The homogeneus equation for p_3 is

$$
2\omega_0\alpha_0\partial^2_{t_*x_*}p_3^o + m\alpha_0^2\partial^2_{x_*}p_3^o = \partial^2_{y_0}p_3^o
$$

which has the following solution

$$
p_3^o = Ae^{-\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}y_0}
$$

We can seek the solution to (11.62) in the following form

$$p_3^o = A(y_0)e^{-\sqrt{2\alpha_0\omega_0-\alpha_0^2 m}y_0}$$

Plugging it into (11.62) yields to the following differential equation for $A(y_0)$

$$A''(y_0) - 2\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}A'(y_0) = -2i\frac{(\omega_0 - m\alpha_0)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}}\frac{\alpha_0^2}{\lambda}\frac{da_1(X)}{dX} \quad (11.73)$$

The first integration is simple. The second integration requires to multiply the whole equation for the factor

$$e^{-2\sqrt{2\alpha_0\omega_0-\alpha_0^2 m}y_0}$$

Let us call $\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}$ by s. The final result is the following

$$A(y_0) = \frac{i\alpha_0^2}{2\lambda}\frac{(\omega_0 - m\alpha_0)}{s}\frac{da_1(X)}{dX}\left\{\frac{2y_0}{s} + \frac{1}{s^2}\right\} + c_1 + c_2 e^{2sy_0} \quad (11.74)$$

The constant c_2 must vanish, due to the requirement that the solution has to be bounded at infinity. The constant c_1 comes from condition (11.64).

$$c_1 = \frac{i\alpha_0}{\lambda}\frac{da_1(X)}{dX}\frac{3m\alpha_0^2 - 7\alpha_0\omega_0}{2s^3} + \frac{\alpha_0^2}{\lambda s}\Phi_3^{(I)}(\infty) + \frac{\alpha_0}{2\lambda s}Xa_1(X)$$

where we used the fact that $a_3 = \Phi_3^{(I)}(\infty)$. What we are interested in is p_3, which is

$$p_3 = A(y_0 = 0)$$

Therefore,

$$p_4 = \frac{i\alpha_0}{\lambda}\frac{da_1(X)}{dX}\frac{m\alpha_0^2 - 3\alpha_0\omega_0}{2s^3} + \frac{\alpha_0^2}{\lambda s}\Phi_3^{(I)}(\infty) + \frac{\alpha_0}{2\lambda s}Xa_1(X) \quad (11.75)$$

It is by simply plugging into (11.71) that we obtain the following condition

$$\Phi_3^{(III)}(z_0) = \frac{\alpha_0^2}{\lambda^2}\frac{\Phi_3^{(I)}(\infty)}{\sqrt{2\alpha_0\omega_0 - \alpha_0^2 m}} + \frac{BX}{2}Ai'(z_0) + \frac{i}{\alpha_0}\frac{dB}{dX}Ai'(z_0)\frac{2\alpha_0^2 m - 5\alpha_0\omega_0}{2\alpha_0\omega_0 - \alpha_0^2 m} \quad (11.76)$$

We can conclude that the third approximation problem consists of equation (11.72) with boundary condition (11.76).

12 Conclusions

In this thesis we analized linear and weakly nonlinear behavior of perturbations in a boundary layer with transonic free stream velocity. The problem structure is the same as in the case of subsonic free stream velocities: the boundary layer is composed of three layers, which form the so-called triple-deck structure. The first layer is the exterior potential flow zone. It is situated in the potential flow region outside the boundary layer and it serves to convert the perturbations in the form of streamlines into perturbations of pressure. The importance of this layer is situated in the fact that it shows that the boundary layer comes into interaction with external inviscid flow. For this reason, this interaction is also termed *viscous-inviscid interaction*. The middle tier of the interactive structure is the main part of the boundary layer, which represents a continuation of the conventional boundary layer developing along the plate. Pressure does not experience variations across this deck and all the streamlines are parallel to each other, meaning that they simply carry the deformation produced by the displacement effect of the viscous sublayer. The latter, indeed, takes place in a region which is comprised of the stream filaments immediately adjacent to the wall. It is the viscous wall layer. The viscous effects are important here and, owing to the the slow motion of fluid here, the flow exhibits high sensitivity to pressure variations along the wall. Even a small variation of pressure along the wall may cause significant changes in particles' speed. Despite having the same structure, triple-decks with subsonic and transonic free stream velocities show different thicknesses of the three layers and different equations. To be more precise, the only equation which differs is the pressure equation in the upper tier.

Having a mathematical solution of the Navier-Stokes equations does not guarantee that the configuration actually exists in Nature. Indeed, for this to happen the flow has to be stable. Namely, if the stationary basic flow is superimposed by a perturbation of small amplitude, then the perturbation has to extinguish with time returning the solution to its basic state. This is why stability theories have been developed. Their aim, given some parameters which describe the configuration of the particular problem, is to predict whether the perturbations will grow or not. Particular is the case of the neutral curve, a set of points in which the perturbation stays unchanged. It is a borderline between a region where the perturbations grow and another where perturbations decay. In this type of problems, the perturbations are regarded in the form of normal modes, namely periodic in the longitudinal coordinate and time. The periodicity parameters are, respectively, wavenumber and frequency. The neutral curve given by the

non-parallel stability analysis of the Blasius boundary layer was presented with the intent to show that two branches are present: the upper and the lower branch. Even if we did not scrutinize the structure of the critical layer, we stressed the fact that on the upper branch a five-zoned boundary layer is present, owing to a distinction between the wall layer and the critical one. On the other hand, these layers are coincident on the lower branch of the neutral curve and this is the part we dealt with.
In the matter of linear analysis, we deduced the dispersion relation which relates the wave number with the frequency of the normal perturbations. Without any lost of generality, the frequency can be considered real. Depending on the sign, or vanishing, of the wave number immaginary part we can predict the behavior of the perturbations. The equation in question shows a dependence on the parameter m, which is related to the Mach number as follows

$$M_\infty^2 = 1 + R_e^{-1/9} m$$

At this point, we were able to find an affine transformation which reduces the transonic equation onto the well-known subsonic one. The importance of this transformation is double. On one hand it enables to use the well-known and widely studied results from the subsonic analysis for any value of m. On the other hand it expresses the dependence on m for the wave number and the frequency. Furthermore, given that the wave number is inversely proportional to the longitudinal coordinate and the frequency does the same with time, we were able to determine the dependence on m for any quantities involved in the problem, such as coordinates, pressure, velocities and displacement function. Firstly, we led m to minus infinity. In this limit we expected to reduce to the subsonic case, and this is exactly what we found. The equations turn out to be the same as in the subsonic analysis. Secondly, we led m to infinity, in the will to extend the Tollmien-Schlichting waves theory toward supersonic regimes. The most important result we obtained in this direction, is that the upper deck equation for pressure has the second derivative with respect to the trasversal coordinate multiplied for a small parameter which vanishes for large m. This means that the determination of the wave speed becomes inviscid to first order. Essentially any disturbance moves downstream with the speed of the slowest sound wave in the free stream. Further research has to be developed in this direction, in order to better understand how a Tollmien-Schlichting wave supersonic theory could be formulated.
In the matter of weakly non-linear analysis, we wanted to seek an equation for amplitude of Tollmien-Schlichting waves. Starting from a point in which the frequency of the wave is close to its neutral value and the amplitude is known, the problem is to determine the wave parameters downstream of

this point. From the linear analysis, we are aware of the behavior of the wave downstream. Because of the growth of the local Raynolds number and a conservation of the frequency, the wave ceases to be neutral and begins to grow in amplitude. However, its futher evolution cannot be studied without allowance for the non-linear effects. Therefore, we divide this process into two stages: a linear displacement of the wave, which can be treated using the linear theory of boundary layer stability studied earlier, and a non-linear process of growth of the amplitude. The latter process can be analyzed on the basis of the non-linear stability theory of parallel flows, namely, neglecting the non-parallel nature of the boundary layer. These two processes are compatible and are manifested simultaneously. They have a different lenght scale as compared to the typical lenght given by the wavelenght. For this reason we used the multiple-scales theory, distinguishing between slow and fast longitudinal coordinates. Under these assumptions, we were able to obtain the equations for amplitude in the first three orders of approximation. The condition for the existence of a solution to this problem (see [8]) is the required equation for amplitude:

$$\frac{dB}{dX} = \chi X B + \kappa B^2 \bar{B}$$

The calculation of the coefficients χ and κ goes beyond the scopes of this project and will be subject of future research. The solution to this equation depends only on the initial wave amplitude. Let us analyze the terms on the right-hand side of this equation. The term proportional to X comes from the linear growth of the wave, while the nonlinear term stems from the non-linear phenomena. Clearly, we expect the real part of χ to be positive, since the boundary layer's being non-parallel accelerates the growth of the Tollmien-Schlichting amplitude. On the other hand, we expect the real part of κ to be negative, given the fact that in the subsonic flow regime (see [6]) the non-linear term retards the growth of the Tollmien-Schlichting amplitude. Any phenomenon which retards the growth of the instability waves helps in delaying transition.

13 Appendix - Small perturbed pressure equation

Let us start regarding the unsteady nonlinear Euler system of equations

$$\begin{cases} \partial_{\hat{t}}\hat{\rho} + \partial_{\hat{x}}(\hat{\rho}\hat{u}) + \partial_{\hat{y}}(\hat{\rho}\hat{v}) = 0 \\ \hat{\rho}(\partial_{\hat{t}}\hat{u} + \hat{u}\partial_{\hat{x}}\hat{u} + \hat{v}\partial_{\hat{y}}\hat{u}) = -\partial_{\hat{x}}\hat{p} \\ \hat{\rho}(\partial_{\hat{t}}\hat{v} + \hat{u}\partial_{\hat{x}}\hat{v} + \hat{v}\partial_{\hat{y}}\hat{v}) = -\partial_{\hat{y}}\hat{p} \\ \hat{\rho}(\partial_{\hat{t}}\hat{h} + \hat{u}\partial_{\hat{x}}\hat{h} + \hat{v}\partial_{\hat{y}}\hat{h}) = -\partial_{\hat{t}}\hat{p} + \hat{u}\partial_{\hat{x}}\hat{p} + \hat{v}\partial_{\hat{y}}\hat{p} \end{cases} \tag{13.1}$$

with state equation

$$\hat{h} = \frac{\gamma}{\gamma - 1}\frac{\hat{p}}{\hat{\rho}}$$

We consider non-dimensional coordinates x, y and t and small perturbed flow functions u, p, ρ and h, represented as

$$\hat{x} = Lx$$

$$\hat{t} = \frac{L}{V_\infty}t$$

$$\hat{u} = V_\infty(1 + u)$$

$$\hat{p} = p_\infty + \rho_\infty V_\infty^2 p$$

$$\hat{\rho} = \rho_\infty(1 + \rho)$$

$$\hat{h} = h_\infty + V_\infty^2 h$$

We consider a subsonic flow, i.e. with $M_\infty < 1$, where the entropy stays unchanged and

$$\frac{\hat{p}}{\hat{\rho}^\gamma} = \frac{\hat{p}_\infty}{\hat{\rho}_\infty^\gamma}$$

The substitution into this parity gives the solution

$$\rho = M_\infty^2 p$$

To determine u, p, ρ and h we substitute the expansions into the Euler equations. the substitution into the continuity equation gives the linear equation

$$\partial_t\rho + \partial_x(\rho + u) + \partial_y v = 0$$

The linearized momentum equations may be written as

$$\partial_t u + \partial_x u = -\partial_x p$$

$$\partial_t v + \partial_x v = -\partial_y p$$

To obtain a solution for pressure we shall eliminate all functions except pressure perturbation from our equations. Differentiating the linear continuity equation and first momentum equation with respect to x and the second one with respect to y we obtain

$$\partial^2_{tx}\rho + \partial^2_x\rho + \partial^2_x u + \partial^2_{xy} v = 0$$

$$\partial^2_{tx}u + \partial^2_x u = -\partial^2_x p$$

$$\partial^2_{ty}v + \partial^2_{xy} = -\partial^2_y p$$

Let us substitute the term $\partial^2_x u$ from the second equation and $\partial^2_{xy} v$ from the third to the first one

$$\partial^2_{tx}\rho + \partial^2_x\rho - \partial^2_{tx}u - \partial^2_x p - \partial^2_y p - \partial^2_{ty} = 0$$

From the continuity equation we have

$$-\partial_t(\partial_x u + \partial_y v) = -\partial_t(-\partial_t\rho - \partial_x\rho)$$

It yields

$$M^2_\infty\partial^2_{tx}p + (M^2_\infty - 1)\partial^2_x p - \partial^2_y p + \partial_t(\partial_t\rho + \partial_x\rho) = 0$$

and then

$$M^2_\infty\partial^2_{tx}p + (M^2_\infty - 1)\partial^2_x p - \partial^2_y p + M^2_\infty\partial^2_t p + M^2_\infty\partial^2_{tx}p = 0$$

The small perturbed pressure equation has the form

$$2M^2_\infty\partial^2_{tx}p + (M^2_\infty - 1)\partial^2_x p - \partial^2_y p + M^2_\infty\partial^2_t p = 0$$

When dealing with a transonic regime, in which we write

$$M^2_\infty = 1 + R_e^{-1/9}m + ...$$

the equation becomes

$$2\partial^2_{tx}p + R_e^{-1/9}m\partial^2_x p - \partial^2_y p + \partial^2_t p = 0$$

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