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LIFE INSURANCE CONTRACT VALUATION IN A STOCHASTIC MORTALITY FRAMEWORK: THEORY AND APPLICATION

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0. Introduction

The dissertation deals with the analysis of the valuation of the life insurance contracts, connecting it with credit risk modelling. Modern actuarial valuation exploits the researches and the theories developed in credit risk analysis in order to address a typical problem characterizing the pricing procedure of the claims: the trade-off between the complexity of the model in terms of fair representation of reality and its computational tractability. The fundamental link established when associating the credit and death risks is the following: the death of the contract owner, event which represents one of the main risks proper of a life insurance contract, is assimilated to the default of a company which issues bonds. This comparison allows to frame the mortality risk (expressed in terms of probability of death) as a stochastic process, giving a dynamic representation of it and exploiting the intensity-based model valuation, a scheme typically adopted for the defaultable bond pricing.

The stochastic representation of mortality is an improvement when compared to its deterministic counterpart based on mortality tables, since it leads to a more appropriate evaluation of the risk. Consequently, this way of framing the death risk results in an improvement of the valuation, nearer to the "fair value" one required by the regulation. Besides what concerns the regulatory framework (which represents an important factor influencing the insurers' business), modelling mortality as a stochastic process is important also when considering the profitability and the stability of the insurance companies. Indeed, treating as dynamic the risk of mortality allows to account for its variation, a factor that makes challenging the traditional product evaluation. Most of the life insurance policies imply to enter in long term obligations and mortality improvements represent a non-diversifiable risk: a relatively more accurate evaluation of this factor can improve the performances of the insurer. The use of stochastic processes to model mortality is treated in many works, for example in Dahl (2004) and Schrager (2006). To deal with the inclusion of the stochastic mortality framework in the actuarial valuation, the model taken as reference in this dissertation is the one of Biffis (2005), while other previous works exploited this parallelism (Artzner et al. (1995), Milevsky et al. (2001) are considered two seminal papers for the argument).

The challenging evaluation of mortality risks is only one of the difficulties that the insurance companies have to deal with in their activity: an overview of them includes at least the actual macro-economic conditions of the financial markets and the technological changes affecting the insurer's business. Those factors are discussed in Chapter 1 of the work, that also introduces the features of some traditional insurance products.

As mentioned above, the strength of this way of modelling is its possibility to associate a remarkable degree of flexibility to a high analytical tractability. The tractability is a feature linked to the structure of the stochastic processes characterizing the framework, which is known as “affine structure”. A rigorous evidence of the advantages of the affine framework in pricing of defaultable securities is given in Duffie et al. (1996), Duffie et al. (2000) and Driessen (2005). The affine structure is typical of many stochastic processes, particularly of the ones analysing interest rates dynamics (CIR process, Feller process), which can be used as a starting point for the analysis of mortality. Chapter 2 of this dissertation is based on the theorical presentation of the model explaining its basic mechanisms and features.

To show the practical use of the model introduced in the previous part of the dissertation, an application is presented in Chapter 3. When dealing with the evaluation of the insurance contracts, this kind of modelling can be applied to price many structured products, composed by a combination of insurance contracts and options. Therefore, the implementation of the model is performed focusing on a particular contract, an index-linked endowment embedding a guaranteed annuity option (GAO). This is a contract which allows at the expiration of the endowment to convert the proceeds gained from it in an annuity at a fixed rate, established at inception and named conversion rate or cash value ratio. An endowment with GAO is therefore a complex product whose evaluation is not straightforward: its dependency to many sources of risk and its long-term expiration created in the past many solvability issues to some institutions. Since the main objective of the dissertation is to show the flexibility and the power of the model analysed, the chapter follows a structure that highlights the main steps to be performed to apply it. The model starts with the calibration of the coefficients driving the processes and summarizing the evolution of mortality, interest rates and the asset to which the endowment proceeds are linked. This is a procedure which is required for every insurance contract valuation involving a stochastic modelling. Particular attention is reserved to the mortality model calibration; in many works, researchers try to empirically assess which is the more appropriate process to analyse the mortality dynamics: Luciano et al. (2009), Blackburn et. al (2013) are some examples. The first paper mentioned is considered for the choice of the appropriate process to model mortality. Then the attention shifts to the specific product’s characteristics, fundamental to understand how to act when implementing the valuation. Once defined those features, the product is priced exploiting the proprieties introduced in the dissertation and the Monte Carlo simulation. Many papers try to address the issue of the guaranteed annuity option valuation, considering different frameworks for mortality and interest rates. Seminal papers are the one of Boyle et al. (2003), who assumed the mortality deterministic, Ballotta et al. (2005), Biffis et al. (2006), who structured the mortality as a stochastic process independent from the interest rates and the market. More recent works try to better adapt the model to reality; they are the researches of Van Haastrecht et al. (2010), who analyse the index dynamics through a stochastic volatility model (but treats the mortality as deterministic), Liu et al. (2013) that allow for correlation between the mortality and the interest rates processes (but considers an endowment characterized by deterministic benefits). The valuation is then completed by a comparative statics analysis which shows the influence of some factors on the price of the product; this last section poses particular attention to the mortality process, highlighting how stochastic modelling can be useful in pricing the claim.

CHAPTER 1

1. Life insurance market

Insurances are peculiar institutions: traditionally their main role is to provide the transfer of risk in an economy, pooling and reducing it by means of diversification, but in providing this service they operate another fundamental role, the one of financial intermediaries. Indeed, those entities collect premia from the public (the population of insured), disposing in this way of huge amounts of liquidity which are invested. The proceeds coming from those investments are useful to make reimbursable the eventual claims required by the insured, to cover the expenses connected to the activity and make profits for the shareholders. Moreover, many policies' reimbursements are not fixed at inception, but are linked to the performance of some specified market variables: that is why insurance contracts can assume also some characteristics peculiar of the financial instruments, providing savings opportunities to individuals. Risk diversification and allocation improvements are the two main functions by which the insurances are considered important institutions in the economic system and those are also the ways in which they can boost economic growth[1]. In their activity those companies deal with many types of risks. The main one is the core risk, covered on behalf of the costumer (i.e. the one for which the contract is drawn-up): for example, in an annuity contract this is represented by the longevity risk (risk of outliving the resources accumulated before retirement, during the active part of the life). However, there are also other risks linked to the activity of the financial intermediary, so regarding the uncertainty involving investments. For those reasons the insurance market is quite complex, and its dynamics are understandable only when accounting for all the characteristics of the players composing it. The transformations occurring in it are thus interpreted considering the changes in the macro-economic environment (affecting investments), the variations in the trends of the core risks occurring over time and the technological changes. Moreover, as for the banking sector, a determinant role is played by the regulation which deeply influences the way in which the activity of the institution can be performed.

The chapter starts with the introduction of the main products in life insurance market, described in their functions and characteristics (Section 1.1); then the focus shifts to the main trends characterising the market (Section 1.2); finally, the attention is given to the recent regulatory changes (Section 1.3).

1 Haiss, P., Sumegi, K., 2008

1.1 Main products in life insurance market

The insurance market is essentially characterized by the formation of contracts which represent the redistribution of risk between the insured and the insurer, providing useful benefits for the first and reasonable profits for the latter. Given the nature of the element exchanged in the market (immaterial transfer of risk) the sophistication of the products has evolved over time, proposing always more particular solutions aimed at satisfying the different needs of the insured based on his age, income and education. To have a brief overview of the main contracts and their characteristics it is useful to start with the terminology that can be associated to all the agreements, and then consider them singularly describing their main characteristics.

Insurance contracts are structured to provide a benefit in response to a happening specified by the agreement. Those benefits have different structures, being fixed a priori or linked to some financial variables: the way in which they are quantified expresses one characteristic defining the contract. Providing a distinction based on benefits, it is worth to mention:

- without-profits (non-linked) contracts. They are the simplest contracts since they provide specified guaranteed benefits in return of the payments of specified premia;
- with-profit (participating) contracts. They are set up to fix a minimum guaranteed reimbursement which is then expected to be enhanced by some bonuses communicated by the insurance company and distributed in terms of cash benefit or reduction in the future premia to be paid;
- unit-linked (index-linked) insurance contracts. They present a reimbursement which is not fixed a priori but linked to the performance of a specified investment fund or index on which the premia paid are invested.

Another factor which defines the contract is the premium, sum paid in exchange of the protection from the risk. The premium payment can be:

- a lump sum payment; in this case the disbursement is performed in one time, usually at inception of the contract;
- a payment settled over time; the premium is split in different instalments. In this case there is the possibility that the policy holder fails to pay for the premium specified by the contract. If this is the case the clauses of the agreement can provide different consequences. When the policy stops to be paid voluntary and is cancelled a "surrender" is said to happen. Usually in this case the policy "lapses", it stops without paying any benefit. However, the contract can fix a "benefit on surrender"; this is a partial reimbursement guaranteed to the insured in compensation for the premia paid until the contract has been stopped. When there is no surrender and the premium

payments are stopped the contract is a “paid up policy”: it continues (is not cancelled) and the policyholder doesn’t pay the other premia; in this case the eventual benefit payment is fixed at a reduced amount with respect to the one expressed at inception (if the premia were paid regularly).

A description of the two-fundamental life-insurance contracts is now provided. There are:

- the life assurance, which pays benefits on death of the life insurer to a beneficiary. If the benefit is paid whenever death occurs the insurance is a “whole life assurance”. Instead, if the life insurance pays a benefit only when death occurs before a fixed term (the expiration of the contract) this is a “term insurance”. The general aim of the contract is to provide economic protection to the beneficiary(ies) when the policyholder dies. In the whole life case the insurance works for a long period, while in the term one it should be linked to economic needs and expirations (for example many banks require a term assurance to have a guarantee on the repayment of a mortgage in case of death of the debtor). From the insurance point of view the main risk associated to the contract is the one of mortality, increased by the possibility of adverse selection.
- the endowment, a contract to pay a benefit to the subscriber at a known date in case of survivorship of the insured. In case of death before expiration there can be a pre-specified reimbursement to a beneficiary or nothing. When there is no reimbursement a pure endowment is settled, while when the size of the reimbursement at death is the same of the one at contract’s expiration the agreement is defined as a standard endowment. Basically, the endowment is a kind of product which allows to transfer intertemporally part of the wealth accumulated up to a certain point in time. Thus, its characteristics are nearer to the one of a saving product than linked to a true insurance product.

The two already introduced can be defined as “primitive contracts[2]”, fundamental units representing payoffs that when combined can constitute the payoff of many other more complex contracts.

A third kind of product which is diffused and is important to mention is the annuity, contract structured to pay periodic benefits after a certain period, until the insured is alive. The mechanics of the annuity can be described with the definition of two periods: the accumulation phase, the one in which the premium is paid, and the liquidation phase, when the benefits are distributed. If the liquidation phase starts immediately after the end of the accumulation phase,

2 Biffis, E., 2005, p.453

the annuity is said to be an “immediate” annuity. If between the accumulation and liquidation phases there is a determinate time period, the product is classified as “deferred” annuity. Considering the payoff of the deferred annuity, it can be built up as a sum of pure endowments with different expirations[3]. The annuity contract has the aim of protecting the insured against longevity risk, the risk of outliving the resources accumulated during the working age. The adoption of the annuity as a saving instrument has been encouraged with many policies in European States (UK 1986, Italy 2004[4]) since it was considered a complementary component of the first and second pillars of the pension system[5] (i.e. the public and the compulsory – earnings related – pension accumulation). From an insurance perspective, the risk related to the annuity is the opposite to the one of the life assurances, namely the one of a reduction in mortality of the population.

As well as for other life-insurance products (but in general in all insurance markets), asymmetric information plays a relevant role for the spread of the annuities among the population, as shown in many studies. From a pure economic point of view, the model by Yaari (1965) showed that in a life-cycle multiperiod model the rational player finds optimal to invest his savings in annuities, preferring them to bonds. Conversely the popularity of annuities as a saving product, despite being increasing, is not as spread as the model forecasts. This fact, a difference between theory predictions and empirical evidences, opens the problem of the annuity puzzle which is addressed in many ways. First the real world is characterized by many frictions that are not considered in the model: the bequest motives, the preferences for liquidity and the complexity of the contract (lack of financial literacy in the population) are possible explanations for the low diffusion of the product but are not considered enough to explain the puzzle. Indeed, to have a complete overview of the puzzle it is necessary to consider the price of annuities: if they are more expensive than what is implied by the model it can be convenient to invest in bonds. The mis-pricing of the annuities can be caused by the adverse selection effect. The quantification of the latter effect is studied by Poterba (2001)[6] who computed the difference between the price applied by the insurer and the fair price of the annuities using different mortality tables regarding UK population. The first mortality table considered is referred to the entire population, the second to the voluntary annuitants while the last to the compulsory annuitants (in UK the workers were obliged to annuitize part of their liquidation benefit at retirement). The results show how the pricing of the annuity based on the voluntary

3 Biffis, E., 2005, p.453

4 Cannon, E., Tonks, I., 2008, Chapter 7

5 World Bank, 1994

6 Poterba, J, 2001, p. 260

annuitants' mortality tables is almost fair, while for the whole population mortality table it is higher. This is because the individual who annuitizes his savings statistically is exposed to a lower mortality risk (i.e. he dies older) than the population average. In numbers the results suggest that the 83% of the disparity between the annuity valuation using the population mortality table and the idealised "actuarially fair annuity" is due to adverse selection. If the most part of the population participates in the annuity market, there would be a convergency between the two mortality probabilities, and thus the problem of pricing would be almost solved, pushing then for the spread of the product.

1.2 Trends in the insurance market

The insurance market is periodically analysed by many important organisations which inspect the health of the sector in terms of volume of the business, changes in the products traded, changes and in the way of trading and main investment choices outlined by the players in the market. A detailed analysis of those arguments is presented in the OECD report on Global Insurance Market Trends (2019), in the EIOPA Consumer Trend Report (2019), focused mainly on the European market trends, and in the Swiss Re Institute Report (2018). The main trends characterising the market are summarized in this section based on those reports.

Referring to the global data analysed by Swiss Re Institute[7], for the life insurance market it can be noticed how in 2017 the total gross written premium (indicating the volume of the new business created in the period) has risen (even if marginally) in an aggregate fashion. The trend is confirmed by OECD[8] also for the year 2018. However, this increase hides many differences both in terms of the single country dynamics and in the kind of policy subscribed. Focusing on the market growth (measured in terms of gross written premia yearly variation), the life-insurance industry is rapidly expanding in the countries where the penetration is lower than the average one, so the growth of the sector is boosted by some "emerging markets". On the other hand, in the "developed markets" the volume of the business is declining or stable as shown in Figure 1.

7 Swiss Re Institute, March 2018, p.10

8 OECD, 2019, Global Insurance Market Trends, p.8

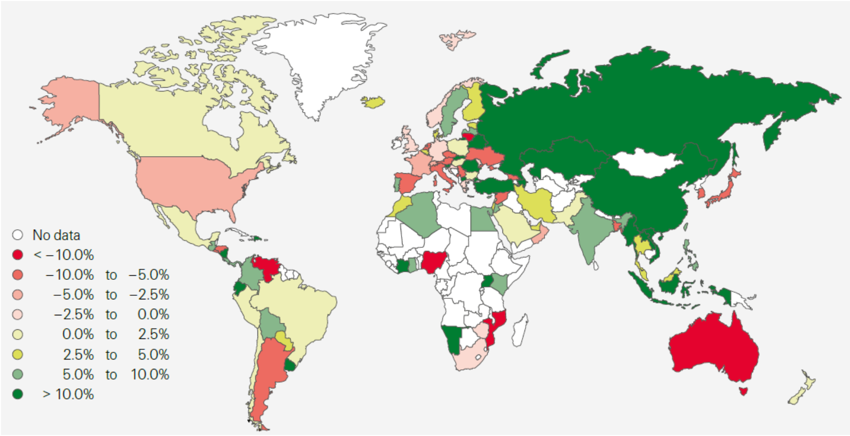

Figure 1 - Real premium growth, Life sector 2017 (Source: DataStream)

Moreover, there are also many significant variations in the performance of the different product lines: overall there is an increase in the unit-linked products subscription. If in many countries this is due to the shift from the guaranteed to the unit linked products, in others an increase in the unit linked products subscription is independent from the decline in the guaranteed one. Indeed, the OECD[9] analysis shows the presence of two distinct effects: on one hand unit linked products replaced guaranteed contracts in the insurance market, on the other those variable benefits contracts have been preferred to bank savings as investment choice. Those trends are linked to the presence of many sources of risk, that are introduced and analysed in the following paragraphs.

1.2.1 Macroeconomic environment

The first challenge is connected to the macro-economic environment, characterized by persistently low interest rates (see Figure 2). The impact of low interest rates in the conditions of life-insurance companies has been studied in many researches in early 2000. Indeed, after a period of high interest rates, from the second part of the '90s their level was going down, especially in US and Europe (which was introducing Euro in that period). Therefore, insurance companies are not new to this kind of problem. An overall view of the different challenges that a low interest rates environment creates to the business of insurances is given for instance by Holsboer (2000). More recently, since the argument became another time actual, some other papers were written. For instance, the one by Antolin et al. (2011) analysed it from a comprehensive perspective. Many reports from relevant institutions as OECD, ECB, BIS, EIOPA focused on the problem, highlighting how this is one of the most relevant issues for the insurance market.

9 OECD, 2019, Global Insurance Market Trends, p. 9

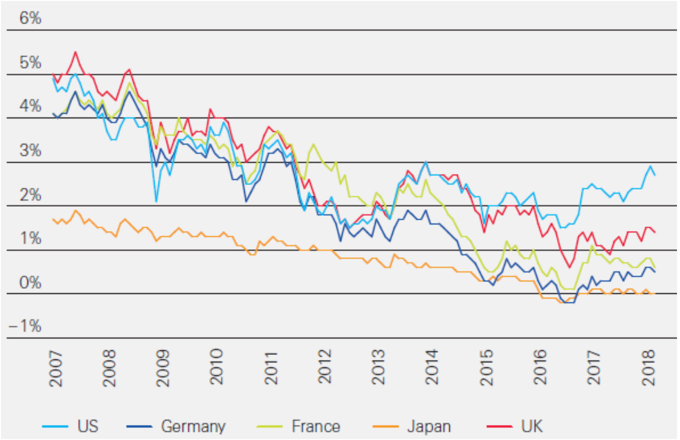

Figure 2 - Long Term Interest Rates: ten years government bond yield (Source: DataStream)

EIOPA[10] evidences how the conditions of interest rates affect both the supply and the demand for the different product lines. The main driver of the phenomenon is the search for the yield. On the supply side, insurances are better-off selling riskier products than other characterized by guaranteed returns, since the low interest rates environment makes the offer of a fixed yield risky and challenging. For instance, Siglienti (2000) studied the sensitivity of ROE to the interest rates and other variables, such as the fixed costs related to the insurance business. The analysis showed that to maintain the shareholders' value in a low rates environment, the insurance company must act on the guaranteed policies, reducing the value of the benefits implied by this class of products. Two are the ways to provide this reduction: cut the benefits associated to the new policies subscribed (as suggested by the researcher) or reduce the total volume of new guaranteed benefit contracts issued. Since in the last years the interest rates associated to fixed income securities are in many cases close to zero, to cut the benefit associated to the new policies below those values (creating a spread and thus maintaining a profit) is very difficult. Consequently, the reduction in the volume of guaranteed products offered seems to be the feasible alternative to reduce the absolute value of the exposures connected to the guaranteed benefit contracts and that is how the insurance companies acted. On the demand side the investors try to achieve higher returns with riskier investments: the non-fixed benefit contracts become the preferred products also when an individual wants to employ excess liquidity, thus exploiting the insurance contract as a pure investment.
Briefly focusing on Europe, EIOPA[11] evidences how the effect described in this last paragraph is even exacerbated, with an increase of 42% in unit and index-linked premia in 2017,

10 EIOPA, 2019, p. 11

11 EIOPA, 2019, p. 11

confirming the trend[12] registered between 2010 and 2015 (see Figures 3, 4). As a result, index and unit-linked products represent the largest life insurance line of business: this is true in terms of gross written premium, while when considering the absolute number of contracts, the one with profit participation are still the higher in quantity. In specific cases, for instance the one of Italy, the increase of unit linked products issuance is also due to fiscal incentives (tax advantages on the long-term savings plans)[13].

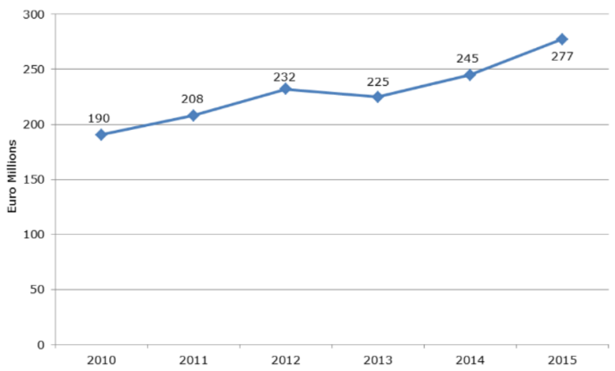

Figure 3 - Total value of GWP of unit-linked business – 2010-2015 (Source: EIOPA Solvency II Database)

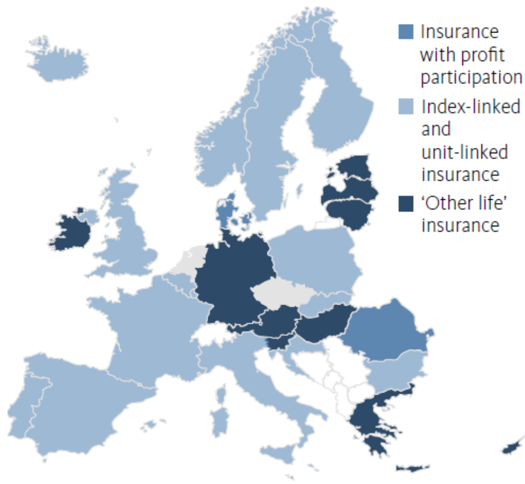

Figure 4 - Highest GWP growth life insurance lines of business per country in 2017 (Source: EIOPA 04/2017)

The current environment of protracted low interest rates poses major challenges to all the insurance companies, not only determining the change in the line of business importance but also representing a risk for the profitability and the stability of the whole sector. The issue regarding the interest rates is analysed by the ECB in a specific part of the Financial Stability Review (2015)[14] from both profits and stability perspectives.

To test the strength of the relation between profitability and interest rates the study presents a regression of ROE of insurance companies on the interest rates and other control variables: the variable of interest is shown to have a significant and relevant impact on the dependent variable. This is defined as the "income channel" of influence of interest rates on insurance conditions. As explained by OECD[15] but also in the analysis of the Swiss Re Institute[16], the drawbacks of

12 EIOPA, April 2017, p.92

13 Law 11 December 2016, n. 232 (2017 Budget Law), art. 1 (100) to (114)

14 ECB, 2015, Financial Stability Review, pp. 134-146

15 OECD, 2019, Global Insurance Market Trends, p. 15

16 Swiss Re Institute, 2017, 6, p. 24

the low interest rates environment for profitability are mainly two: the first is connected to the typical portfolio of the insurer, and the second to the reinvestment risk. The portfolio of investments for the insurer is mainly constituted by bonds, fixed income securities which are the assets most affected by the interest rates variation. Moreover, a large part of the insurance contracts has an expiration longer than the one of the bonds available in the market. Thus, the contracts subscribed in the past can represent a burden for the profitability of the company, given that they are by construction linked to past estimates of the interest rates (higher than the one characterizing the economy in the present). Strengthening this cause-effect relation, the regression presented in the ECB study shows how the negative impact of low interest rates on profitability affects more small and medium size companies. Those institutions are characterized by a higher portion of guaranteed products in their liabilities than bigger companies and in this product-line they distribute also relatively higher benefits.

On the side of the stability, the ECB (2015) defines as "balance sheet channel" the impact of low interest rates on the balance sheet via a valuation effect. The concept of balance sheet channel for insurances has been explained and commented before also by Berdin and Gruendl (2014) and is connected to the market-evaluation performed for assets and liabilities when setting the reserves and the capital required by the regulation. The researchers explain how when a market-consistent valuation of assets and liabilities is performed, low rates induce an increase in value of the fixed income securities both on the assets and on the liabilities' side. Typically, this increase is more pronounced for the liabilities since their duration (impact of variations of the yield on the security price) is on average larger than that of the assets. This increase in the liabilities value is a relevant issue for the insurance companies because it requires an increase in the own funds' accumulation. Moreover, the longer maturity of the liabilities implies also the presence of the reinvestment risk (mentioned before) which affecting profitability influences the balance sheet equilibrium. The impact of the prolonged low interest rates period on the balance sheet of insurance companies is analysed by means of a scenario analysis. The researchers present a balance sheet model of a stylized life insurer and project it 10 years ahead under different market settings (i.e. including different simulations for interest rates and stock market performances) and with different initial capital endowments. The valuation of the balance sheet is given under the market value to analyse the solvency issues. The results suggest that a prolonged period of low interest rates would markedly affect the solvency situation of life insurers, leading to relatively high cumulative probability of default for less capitalized companies. Solvency issues are analysed, more from a qualitative point of view, also in the study of the Bank for International Settlement (2017).

Despite those difficulties related to the economic environment, EIOPA[17] evidences how bonds (and in general the fixed income instruments) remain the group of securities in which the insurances invest more. Within the category there is a shift towards more illiquid assets (search for the illiquidity premium) and to lower quality securities (for the lack of safe assets – downgrading of many bonds – and the research for the higher risk premium). A minimum shift towards equity investments is also evidenced. Those investments maintained the profitability of the sector positive[18] (when this feature is analysed in terms of ROE).

1.2.2 Mortality trends

The changes in the business strategy and the pressure on profits in the life insurance sector are not only caused by the macro-conditions of the economy, but also by another important risk factor, the change in the mortality trend. Two are the most important mortality trends evidenced in the last decades: the "expansion" [19], so a systematic shift of the average mortality to older ages (mortality reduction), and the "rectangularization"[20], the concentration of the deaths around the mode. Those two trends represent two opposite effects: on one hand the expansion increases the systematic risk connected to mortality, on the other the rectangularization reduces the idiosyncratic risk. The improvement in the overall life conditions represents a challenge for the insurance sector since while the latter risk (the one reduced) can be diversified, the first one cannot be reduced owning a large portfolio of contracts. Thus, the generalized changes in the mortality must be forecasted in the best way to preserve the profits connected to the life-insurance business.

As shown in Swiss Re study on mortality[21], for what concerns those improvements they are slowing down in developed countries, and this means that mortality rates are reducing by a lower extent than in the past decades. Figure 5 shows the mortality improvements over the last 20 years. Formally the annual improvement of mortality is defined as $1-\frac{m_t}{m_{t-1}}$ where m_t is the mortality rate in the year t. It can be noticed how for many countries, as US, UK and France for example this indicator is slowly falling to zero.

The mortality improvements can be caused by the reduction of the overall mortality risk, given by the life conditions of the population (considering the diffusion of wealth, development of technology and health conditions) and by the reduction of behavioural risk, which concerns the

17 EIOPA, November 2017, Investment Behaviour Report

18 OECD, 2019, Global Insurance Market Trends, p. 22

19 MacDonald, A., Cairns, A., Gwilt, P., Miller, K., 19981

20 Wilmoth, J., Horiuchi, S., 1999

21 Swiss Re Institute, 2018, 6

lifestyle of the people. One of the causes of the decline in the mortality improvement can be given by an increase in the behavioural risk, that counterbalances the improvement of the population conditions. In other terms, if in the developed countries the lifestyle is worsening (fact that can be linked for example to the spread of wrong alimentation habits or of a sedentary lifestyle), this can partially counteract the improvement in the health and technologies.

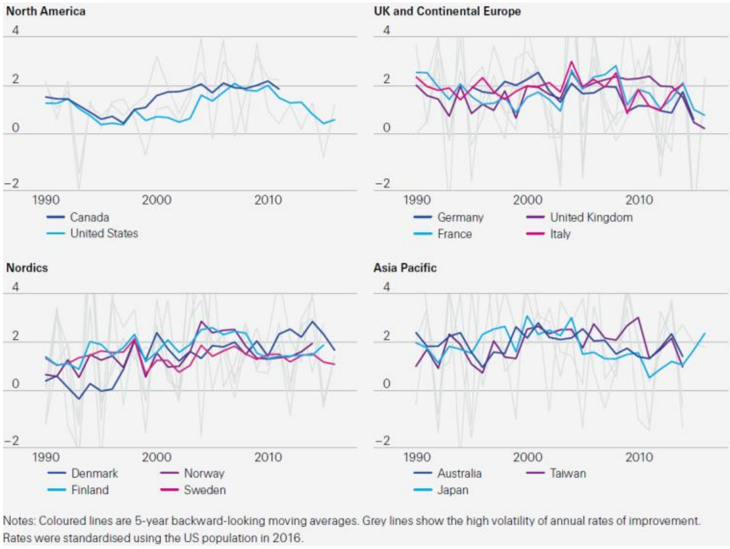

Figure 5 - 5 year moving average of annual improvements in standardised mortality rate (Sources: Human Mortality Database, Swiss Re Institute estimates)

It is however quite difficult to determine if this slowdown is connected to a real change in trend or to a mix of temporary factors (volatility characterizing mortality).

Analysing the time series, it is worth to consider the fact that the high volatility is one of the main characteristics of mortality improvements’ development. Since the analysis looks for the presence of a permanent change in the improvement of mortality it must be considered the fact that the evolution shown in the graph can be the result of this high volatility in the short period. Indeed, the presence of a permanent change in the average of the model (which represents a structural break in the time series) must be assessed in a long-time fashion. This is shown in the Swiss Re report[22] with an example of the analysis of US mortality rates: the results are sensibly different when looking at the time series from a short (few years) and a long-term (many decades) perspective. Therefore, while the mortality improvement during decades is a factor to account for in the analysis, at the moment its slowdown cannot be considered a trend which characterizes the series permanently.

22 Swiss Re Institute, 2018, 6, p.8

The right quantification of the longevity of the insured population is fundamental from an investing point of view: according to the analysis of OECD[23] each additional year of life expectancy not provisioned can add from 3% to 5% to current liabilities of a company.
The increasing life expectancy and a rising ratio of non-working age population (not included in the age 15-64 years old) to the working age one, creates also some problems from a social point of view, since this is the main factor creating a gap between the current savings for retirement and the one necessary to generate a desirable income after stop working. This is the "global retirement savings gap"[24], and it is forecasted to increase dramatically in the next years if the retirement ages are not increased and the benefits connected to retirement are not reduced. Those unpleasant measures, difficult to be taken by the politicians because unpopular, are necessary in order to make the system sustainable. The World Economic Forum[25] estimates that the gap will increase from 70 trillion in 2017 to 400 trillion by 2050. This is mostly due to the underfunded public pension system and by the gap in individual savings; the gap has many long-term impacts both on the private and on the public sector.
For what concerns the private sector the pension gap regards the corporate sponsored pension plans and does not involve the mortality issue, but it is worth to analyse it in this section, since it does contribute to the underfunding of the pension system. The corporate pension plans are agreements by which the firm retains some contributions from the salary of the workers, those contributions are invested and then liquidated at retirement. During the past those plans have been mostly issued in the form of defined benefit plans (DB), contracts which are still existing. They have the characteristic to provide a predetermined return on investment after retirement, a return based on the length of service in the company and on the employee's salary history. The investment risk in this case is bared by the employer. The performances of the market and the investments choices are thus important for the company which has to fund in this way the future repayments. The liquidation of those plans is a contingent liability for the firm (it represents a future expense), and when those investments are underfunded (for example because the guaranteed return is higher than the net return of the investments) they create a gap in the system. This gap can force employers to reduce savings and investments, damaging the firm's economic expansion.
Now the design of the plans has shifted to the defined contribution plan (DC), which does not fix the benefit (variable based on the market conditions), but only the initial contribution. In this way the investment risk shifts to the employee. This is a first measure that the firms adopted

23 OECD, 2014

24 Definition taken by: https://www.economicsonline.co.uk/Definitions/Savings_gap.html

25 World Economic Forum, 2017

in order to reduce their contingent liabilities and is a reaction to the market conditions like the one analysed for the insurances when mentioning the shift from guaranteed to unit linked products due to the interest rates environment. The issue of the private pension gap is analysed by Swiss Re[26] while for deepening the mechanisms regarding the shift from DB to DC plans a useful source can be the research of Broadbent et al. (2006).

Shifting to the public sector, the mismatching between contributions and pensions makes the system not sustainable in the long term and pushes the governments to increase their expenses, generating increasing exposures. The gap created in this case is far more serious than the private one, first for its magnitude (for instance in Italy the pension expense absorbs almost the 15% of the GDP[27]) and then because it relates to the presence of an intergenerational inequality due to the changes in the life expectancies of the last decades. Therefore, it is a structural problem which can damage the fiscal consolidation of many States. To prevent the collapse of the system, a series of structural reforms will be necessary.

In its report Swiss Re[28] notices how the increase in the pension gap represents an opportunity for private insurance companies. In anticipation to the reduced benefits from public pension schemes individuals are likely to invest more in products guaranteeing private savings at retirement, with this part of the market increasing its size. Indeed, the life insurances could play a vital role in the reduction of the pension savings gap addressing with their products the need for private voluntary savings. This is a trend which is shown to be already in place, and which is expected to be more evident in the future, also with the support of state policies (tax incentives).

1.2.3 InsurTech and technology changes

The third big challenge keeping pressure on the insurance activity and profitability is the technological development, implemented at different stages of the provision of the insurance service. Differently from the two risks analysed before, this one is not a direct consequence of insurer's activity but represents a source of change which comes from outside the insurance business. The phenomenon in its overall evolution is named "InsurTech": the detailed analysis of the argument goes beyond the scope of the dissertation: for a deeper view on it refer to OECD (2017)[29] and to the references therein. Innovations are becoming the main tool of competition

26 Swiss Re Institute, 2018, 3, p. 13

27 Data source: http://www.rgs.mef.gov.it/_Documenti/VERSIONE-I/Attivit--i/Spesa-soci/Attivita_di_previsione_RGS/2019/Rapporto-n-20.pdf

28 Swiss Re Institute, March 2018 p. 13

29 OECD, 2017

in the market, since they give the opportunity to add value to insurance products and to adapt the business to the change occurring in new generations and social norms (the Millennials or Y generation) [30]. The drivers of the entry for new technologies in the market are different and can be resumed as two main channels: different ways of analysing data (exploiting the more detailed collection of information through different sources and the ability of analysing them in an efficient way), and different ways of conducting the business (from the distribution to the way of executing the insurance contracts).

Internet, new devices, applications and more in general the new technologies, are all contributing to the possibility to collect more data from individuals.
The traditional insurance relies on the collection of a large amount of information to build up statistical indicators which are then functional to the construction of models that describe the likelihood of an event in terms of probability theory. Once determined the probability related to the happening of an event for a specific population, the business decisions (products to offer, price, market and risk management strategies) are taken. Modern data collection and analysis allows to go beyond data aggregation giving the possibility for a more detailed – in certain cases also personal – analysis of the risk, monitoring the specific behaviour of the individual. If this granularity in the analysis is for obvious reasons an improvement for the insurance business (it allows to make it more efficient), many concerns are related to the welfare purpose of insurances activity.
When data aggregation is used for actuarial aims, it can lead to potentially very high premiums for certain segments of the population and so to their cut off from the market: this ends up having a negative effect on the society overall condition[31]. The very high screening potential of the insurer when possessing a huge amount of data, can indeed induce some categories of subjects to be discriminated (in terms of premium to be paid to access to the insurer protection). Moreover, there is also a problem with the privacy of the data provider since the data collected go beyond the one necessary for the insurance porpoises, and their treatment is not clearly regulated in the insurance environment (there can be a misuse of the data collected).
The technological revolution of the last decade is not only changing the way in which the models underlying the business are set, but also how the business takes place, both on the side of intermediation and on the one of the products' distribution. The trend of the present is to shift from a relational business to an automatized one, using other networks to distribute the products – like internet or the apps – and other instruments to execute contracts – for instance the

30 Klapkiv, L., Klapkiv, J., Skłodowska, M., 2017
31 OECD, 2017, p. 27

blockchain, smart contracts and artificial intelligence (robot-advice). This new wave in the business exploits, as well as the big data analysis, the higher propension of the new generations to provide personal data and to build up technology-intermediated relations. In this way the efficiency of the execution is improved leading to simplified and faster intermediation with the costumer. The business becomes more profitable for insurers and at the same time less expensive for the service user, since there is a cut in the overall amount of costs related to it.

All those revolutions in the business are also causing many changes in the competitive environment, with big players trying to adequate to the new business and the simultaneous entry of new competitors.

1.3 Regulation

As for the banks, also for insurances the last years were full of changes directed to guarantee a homogeneous international framework for all the institutions so that they can develop in a fair way their business. The main regulatory settings introduced in the last years were the “Solvency II Directive” and the “IFRS 17 Accounting Principle”.

1.3.1 Solvency II Directive

Solvency II is a European Directive which has been issued in 2009 and whose application has become compulsory in 2016. The general purpose is to ensure the correct functioning of the insurance market, improving the stability and the solvability of the companies participating to it and letting them operate in a legal framework which is homogeneous in the European Union. The scheme of the Directive recalls the one of Basel II Directive for banks and is essentially based on three pillars.

The first pillar is the one which sets the quantitative requirements for the insurance companies to operate in the market: it is the most relevant and represents the “heart of the regulation”. Regulation sentences that to operate in a safe way the insurances need a minimum amount of solvency capital, which is calibrated on the risk of its balance sheet. This provision is resumed in the Solvency Capital Requirement (SCR) and in the Minimum Capital Requirement (MCR)[32] and responds precisely to the need that in the undertaking of insurance policies activity the company is not unprepared to face unexpected events that adversely affect its condition. Both SCR and MCR represent the capital accumulated accounting for quantifiable risks to which an insurance is exposed. Their difference is in the eligible assets that can compose them (assets defined as capital or own funds) and in the coverage they must guarantee (in terms of percentage

32 EU DIRECTIVE 2009/138/EC, Section 4, Section 5

of the balance sheet size). Indeed, the MCR can be constituted only by higher quality own funds than SCR (their detailed qualitative composition is explicated in the directive[33]), while the latter is set to cover a higher quantity of risk compared to the first one. It follows that the MCR can constitute a part of the SCR.

The insurance company has thus to compute the obligations taken with the policy holders (the liability side of the balance sheet) checking if it has the necessary resources to cover them. Those resources compose the assets of the balance sheet and essentially correspond to the premia collected from the policy issuances (then invested, with the investment generating itself risk). Moreover, those last items are integrated with the reserves present in the balance sheet (the own funds composing the other part of the liabilities of the insurer). The value of those reserves is the Net Asset Value and must be calibrated to cover the risk present in the balance sheet (both on the liabilities and on the asset sides). This amount is set coherently with the regulation provisions and updated every year (SCR) and every three months (MCR). In setting those values, a prominent issue regards the correct evaluation of all the obligations towards policyholders; on this kind of problem the regulator sentences that “the value of technical provisions should (therefore) correspond to the amount an insurance (or reinsurance) undertaking would have to pay if it transferred its contractual rights and obligations immediately to another undertaking”[34]. This provision concerns the quantification of the risk covered in place of the insured agent (liability side of the balance sheet), taking also into account the correlation between the different risks.

Other factors monitored by the first pillar are the diversification of investments and their consistency with the liabilities and with the “appetite of risk” defined by the senior manager, the profitability and the sustainability over time of products, the ability to mitigate the financial risks[35]. All those elements are mainly focused on the calibration of the risk taken by the insurance company in the financial intermediary activity and so concerning the proceeds invested (the asset side of the balance sheet).

The analysis performed in order to be compliant with the first pillar of the Directive requires an adequate set of management and analytical competences and so the definition of a corporate governance structure. This ensures that the calculation and monitoring of the MCR and SCR are central in the insurance activity and distributes the responsibilities connected to them. The principles of the corporate governance that the insurer must observe are present in the second pillar of the directive, the qualitative requirements of the insurance company.

33 EU DIRECTIVE 2009/138/EC, Section 3

34 EU DIRECTIVE 2009/138/EC, par. (55)

35 IVASS, 2016

The information on the stability of the company collected fulfilling the law requirements must be disclosed to the authority, the shareholders and the stakeholders. The way in which this is published – in terms of documents to be provided to different agents – is regulated by the third pillar of the directive, section defining the disclosure requirements.

1.3.2 IFRS 17

The second regulation provision is IFRS 17; it is more recent since it has been published in 2017 and it must be adopted in a mandatory way in 2021 replacing the previous regulation, the IFRS 4. It completes the Solvency II regulation with the issuance of a set of accounting principles regarding the main steps that an accountant makes with respect to the insurance contract when building up a balance sheet: the recognition, which represents the first inclusion in the balance sheet, the evaluation and the derecognition, so the exclusion from the balance sheet of an asset or liability which was present in the previous period report. Regulation is directed to depict a current, market-consistent valuation of the items composing the balance sheet of the insurer. It should improve the transparency in the insurance sector, and this is expected to contribute to the long-term financial stability of it: indeed, initial measurement and changes in value of the products should reflect the underlying economic phenomenon and so build trust about the relevance and the reliability of the reported figures in the financial market[36].

The IFRS builds up a principle-based approach (more resilient to changes in the economic environment) that is consistent with the accounting by other sectors. In this section the focus is mainly concerned on the products' measurement (valuation), since this is useful for the subsequent development of the research. However, the accounting standard is more extended and contains many other provisions that concern the presentation of the items in the balance sheet and in the income statement, focusing on the aim of leading to the appropriate structure of the financial statements.

After some provisions on the recognition of the insurance product, essentially consistent with the IFRS 4[37], the principles determine how the measurement of the value on initial recognition should be performed. This is overall coherent with the provisions given by the Solvency II Directive (par. 55); indeed, when compared to it, IFRS 17 reaches (even if with a different approach) similar results. Both frameworks require an explicit risk adjustment for non-financial risk in order to reflect a risk-averse and market-consistent valuation[38]. The measurement on

36 EIOPA, 2018

37 D'Onofrio, L., Micocci, M., 2018 p. 8

38 EIOPA, 2018, Section 4.6

initial recognition includes: the estimate of the future cash flows, the application of discount rates for financial risk and the risk adjustment for the non-financial risks[39].

The cash flows projections have to incorporate all available information about the amount, timing and uncertainty of those future cash flows. They have moreover to reflect the perspective of the entity. The valuation has finally to be current – reflects conditions existing at measurement date – and explicit – the components of the estimate have to be calibrated separately, unless the most appropriate measurement technique combines them. In the practical implementation all the available information must be used: about past events, present conditions and forecasts about the future, building up several scenarios describing in the most parsimonious way the possible future conditions of the value. If it is required, also sophisticated stochastic modelling can be applied. The information set to be involved in the model development depends on the availability in the market of the values needed to implement it: if there is the consideration of a market variable (i.e. one whose value is retrievable from the market) the estimation should be consistent with observable market prices at the measurement date (fair value measurement), while if not the estimation should be supported using all the available information at the measurement time.[40]

The entity has then to discount the valuation result, to reflect the time value of money and the financial risks related to those cash flows to the extent in which this is not included in the estimation of cash flow. This estimate should be consistent with observable and current market prices (if present there should be a fair value measurement) and should exclude the effect of factors that influence the market prices but do not affect the cash flows. The second adjustment has to be applied for the uncertainty concerning the cash flows that arises from non-financial factors, like the risk covered by the contract (insurance risk – the main one) the lapse risk (the one connected to an early termination of the contract) and expense risk (related to the occurrence of expenses higher than the one charged for through the premium). This adjustment should make the agent indifferent between fulfilling a liability that has a range of possible outcomes arising from non-financial risk and fulfilling a liability that will generate fixed cash flows with the same expected present value of the insurance contract. No specific estimation technique is given to perform the adjustment. At the recognition time those values constitute a specific caption of the balance sheet, the liability for remaining coverage. It is constituted by two building blocks which are functional to the accounting transparency and to the coherency in the representation of the profits: the fulfilment of the cash flows related to future service and

39 IFRS, 2017, IFRS 17, parr. 32 – 39

40 IFRS, 2017, pgg. 51 – 52 parr. B37 – B41; B44, B49

the contractual service margin. The fulfilment of the cash flows consists in the explicit and unbiased estimates of future cash flows that will arise as the insurer fulfils the contracts. From those cash flows (entry and exit) a profit is expected to arise. Those unearned profits are recorded in the "contractual service margin" a value that will be recognised as the insurer provides services in the future (amortised in the income statement year by year). At the initial recognition the fulfilment cash flows equal the opposite of the contractual service margin, so the accrued liability is equal to zero. Summarizing, the liabilities for remaining coverage represent the part of the claims' value that is connected to the future services to be provided by the insurer.

The subsequent measurement updates the value of the claims with the adjustments that reflect the happenings of the period. It involves the update of the liability for remaining coverage and the computation of the liability for incurred claims.

For what concerns the first component, it has to be updated with the estimations at the valuation moment (cash flows and discounts) and the contractual service margin created by the estimation can be amortised, being progressively recognised in the income statement.

The second component regards the cash flows related to the past services (if they are present): those flows are not generated yet but will for sure be in place, since they are related to claims already incurred [41]. The liability for incurred claims is thus linked to a service which has already been performed (does not need to be forecasted).

1.3.3 Comments

From this quick overview of the regulation, it should be clear how the regulator action conditions the way in which a life-insurance company develops its business. Therefore, regulatory requirements are crucial for the activity in the sector and influences the trends of the market as well as the risks mentioned above, namely the one related to the financial markets conditions and the risk connected to mortality and longevity. If the IFRS 17 has not been implemented yet, the Solvency II framework is already valid: thus, the direction to which the regulator pushes insurers can in part already be analysed. It has however to be noticed that it is difficult to clearly isolate the relation between regulation and market trends, since there are many factors that influence the business strategy of the companies in the same direction. For what concerns the portfolio of investments of the insurances (asset side of the balance sheet) no change can be clearly associated to the regulatory framework adjustments, since the variations in the portfolio composition show small trends which are justified by a search-for-

41 IFRS, 2017, pgg. 51 – 52 parr. B37 – B41; B44, B49

yield behaviour of the insurance companies [42]. On the product offer side (liabilities) the changes are instead significant, and in this case the regulatory change can strengthen the trend previously justified by the low yield environment. The shift to unit-linked or hybrid products and the decrease in the level of the guarantee for the contracts that include it, can indeed be motivated also because of the higher cost of the latter contracts set in the Solvency II requirements; this cost regards both the calculation of the technical provisions and of the SCR.

42 EIOPA, 2018, p. 17-18

CHAPTER 2

2. Insurance contracts valuation

The valuation of insurance products has a fundamental role in the insurance business: it constitutes the amount of proceeds which are accumulated and then invested in the market by the company and it is important to set the right Solvency Capital required by the regulator. For what concerns this field, it is the regulatory framework which issues the guidelines to be compliant with in order to provide a fair evaluation of the contracts. The rules must be implemented with mathematical modelling to achieve a quantitative measure that expresses a reliable assessment of the obligations in place at a specific moment.

In this chapter the focus is related to a class of models used in the contracts' evaluation, the intensity-based models. The starting point is the introduction of the model in conceptual terms and the presentation of the main results connected to it (Section 2.1); then the mathematical implementation is discussed in a particular setting, the affine process setting (Section 2.2); finally, the focus shifts to the connections of the mathematical model with the regulation (Section 2.3).

2.1 Credit risk modelling for actuarial valuation

The valuation models have always to deal with the trade-off between the complexity (the capacity to reflect reality in a fair way) and the computational tractability. When considering the actuarial valuation of life insurance products, the need is to consider two main sources of risk: the financial and the mortality risks. The model which is introduced to manage those risks and to provide the evaluation of the products exploits one approach typical of credit-risk modelling: the intensity-based approach, which is used to evaluate defaultable bonds. In this section the attention is given first to the original model embedding the bonds' evaluation (with default risk definition), while then the analysis is extended to the insurance securities (with the focus on mortality or longevity risk).

2.1.1 Default model

The intensity-based approach belongs to the class of the reduced form models[43] and considers the default time as a stopping-time which occurs as a total surprise. The modelling focuses on

43 Lamberton, D., Lapeyre, B., 2007, p.195

the expression of the probability of default in a precise time window, conditional to the fact that it has not yet occurred. This probability is the object of the analysis: when shrinking the time window to zero it corresponds to the "intensity of default", a quantity which itself is determined by exogenous (observable and unobservable) variables. The intensity of default can thus be interpreted as the instantaneous chance of default at a specific time t given survival up to that time[44]. More precisely, the time of default corresponds to the first jump of a Poisson process[45] with stochastic intensity: a doubly stochastic random time (see Appendix A for an introduction to the concept of doubly stochastic random time). It can be shown that this class of processes has some desirable properties, by which the evaluation of the defaultable claim can be reduced to a pricing problem in a default-free security market model with adjusted discount.

The starting point is to consider a firm default time τ. The probability of default before or at time t is indexed as $P(\tau \leq t)$. Some features of the financial market are fixed: let $(\Omega, G_t, (G_t), P)$ denote a filtered probability space, and Q correspond to the equivalent martingale measure, which exists and is unique when the market is complete and free from arbitrage. The economic background filtration (G_t) represents the information generated by an arbitrage free and complete model for non-defaultable security prices. Under the defined probability measure, the default time τ is doubly stochastic and admits a conditional intensity (γ_t) with cumulative intensity process $\Gamma_t = \int_0^t \gamma_s \, ds$. $J_t = I_{\{\tau \leq t\}}$ is the indicator associated to the default time process, an indicator which jumps to one when default occurs. The prices of the default-free securities and the intensity of default are (G_t)-adapted processes. The default-free interest rate is denoted by (r_t), and the risk-free savings account is defined as $B_t = e^{\int_0^t r_s \, ds}$. It can then be set $H_t = \sigma(\{J_s : s \leq t\})$, the filtration resuming the information on the history of default, with respect to which τ is a stopping time. $F_t = G_t \vee H_t$ is then defined as the smallest sigma-algebra containing both filtrations: it thus has information about both the history of default up to time t and the background processes influencing the financial markets. With this setting, the price of the defaultable claims traded in this market can be expressed starting from two building blocks[46].

The vulnerable claim part, a G_t-measurable promised payment X which is made at the contract expiration T if there is no default. The risk-neutral conditional expectation of the discounted actual payment equal to $X \, I_{\{\tau > T\}}$ is computed to express this value: $E^Q \left[e^{-\int_t^T r_s \, ds} X \, I_{\{\tau > T\}} \middle| F_t \right]$.

44 McNeil, A., Frey, R., Embrechts, P., 2005, p. 393

45 For an introduction to Poisson processes: Brigo, D., Mercurio, F., 2007, Appendix C

46 McNeil, A., Frey, R., Embrechts, P., 2005, p. 416

The recovery payment part, which corresponds to the payment if default occurs before the expiration of the contract. Defined $Z = (Z_t)_{t>0}$ as a G_t-adapted process, we can introduce the actual payment as $Z_\tau\ I_{\{\tau \leq t\}}$. As for the vulnerable claim part this payoff must be discounted and then computed in expectation (conditioned to the information present in the market at the time of evaluation t, F_t). Thus, its expression is $E^Q\left[\ I_{\{\tau>t\}}e^{-\int_t^\tau r_s\,ds}Z_\tau\ I_{\{\tau\leq T\}}\middle|F_t\right]$.

The two parts expressing the payments of the bond both in case of default and in case of no default are therefore two expectations conditioned to the filtration F_t. But the evaluation problem can be simplified, reformulating the conditional expectations with respect to the background filtration G_t, sub-filtration of F_t. Indeed, given that the two payoffs are G_t-measurable and the time of default is doubly stochastic, the additional information about the default history contained in F_t is not essential. Assume that all the random variables which are phrased in the following formula are integrable with respect to the risk-neutral probability measure Q, and define $R_s = r_s + \gamma_s$. Then the following relations hold:

for the vulnerable claim:

$$E^Q\left[e^{-\int_t^T r_s\,ds}X\ I_{\{\tau>T\}}\middle|F_t\right] = \ I_{\{\tau>t\}}E^Q\left[e^{-\int_t^T R_s\,ds}X\middle|G_t\right] \tag{2.1}$$

for the recovery payment part:

$$E^Q\left[\ I_{\{\tau>t\}}e^{-\int_t^\tau r_s\,ds}Z_\tau\ I_{\{\tau>t\}}\middle|F_t\right] = \ I_{\{\tau>t\}}E^Q\left[\int_t^T Z_s\gamma_s e^{-\int_t^T R_u\,du}\,ds\ \middle|G_t\right] \tag{2.2}$$

The equalities are proved in Appendix B. This formulation leads to a big advantage: the defaultable claim pricing problem is indeed reduced to the evaluation in a default-free framework. The formulas (2.1) and (2.2) represent the risk-neutral pricing of two default-free claims with different payoffs. The only difference with the non-defaultable claims formula lies in the discount factor applied to them: instead of the risk-free interest rate, it is represented by the sum of the risk-free and of the intensity of default. Consequently, (2.1) and (2.2) can be priced using the tools and the theory developed for the non-defaultable claims and therefore, under certain assumptions, those expressions are appealing from the computational point of view (see Section 2.2).

2.1.2 Mortality model

Given the advantages presented in modelling default, it is convenient to apply the formulation just introduced to the insurance contracts valuation. Moreover, this kind of analysis of the problem is coherent also with a law interpretation perspective (see Section 2.3).

If the default of the company is interpreted as the single insured's death, and the intensity of default is substituted by the intensity of mortality, the formulas in the expressions (2.1) and (2.2) can be extended to the actuarial valuation of the insurance products. With this interpretation the random time τ (which possess the same characteristics of the default time introduced before) represents the time at which death occurs for an individual. The structure of the model is almost identical to the one previously introduced, but to be applied in the different context it needs some adjustments: the intensity of mortality is indeed assumed to be not influenced by the variables determining the market performance, but by another set of independent factors. This obliges to introduce another random variable and another filtration associated to it. We suppose that the reference is made with respect to an insured aged x at time t. We fix X as the Markov process with respect to which the financial market process evolves (with a relation specified in Section 2.2) and as G_t^X the filtration generated by this process. Moreover, the intensity of mortality is defined as the process μ_t, and the cumulative intensity process as $N = \int_0^t \mu_s \, ds$. The intensity depends itself on a Markov process Y, that generates the filtration G_t^Y. Recalling the definition previously given to the intensity of default (Section 2.1.1), with mathematic notation for the intensity of mortality it can be shown[47] that it is equivalent to:

$$\mu_t = \lim_{h \to 0} P(\tau \leq t + h \,|\tau > t)$$

This variable expresses the instantaneous death probability in t conditional on survival up to time t. Therefore, the filtration G_t^Y introduced above carries the information about the likelihood of the death event and not of the actual occurrence of death. This last information is carried out by the filtration $H_t = \sigma(\{J_s : s \leq t\})$. Starting from this, it can be defined the probability of death before the contract expiration:

$$P(\tau > T | H_t) = E\left[e^{-\int_t^T \mu_s \, ds} \middle| G_t^Y \right]$$

Such a modelling for the intensity of default follows an approach which is defined as diagonal[48] (or cohort based). Indeed, the intensity of mortality expresses the evolution through time of the death probability for a fixed individual (or sample of individuals with homogeneous health status), aged x at time t: in this way the evolution of mortality is linked to the aging of the individual (belonging to a specific generation). A different interpretation of the mortality rates employed in demography studies the evolution through time of the mortality at a fixed age x for different individuals. In this case the mortality evolution is connected to a specific age and not

47 McNeil, A., Frey, R., Embrechts, P., (2005), p. 393

48 Luciano et al. (2009)

to the aging of the sample (i.e. different generations of homogeneous individuals are considered in the analysis).

As introduced above, in this model the information evolution is determined by two Markov processes: X that is related to the financial markets' performance and Y that is linked to the mortality intensity. Those two variables are assumed to be independent, so the evolution of the mortality intensity is not related with the one of the financial markets (this is reasonable, even if there are evaluation models that relax this assumption[49]). Then, given $Z = (X, Y)$ and fixed $G_t = \sigma(Z_s: 0 \leq s \leq t)$ the filtration $F_t = G_t \vee H_t$ can be defined as before: the smallest filtration including the other two. Let C be a G_t-adapted process which represents the payment to be provided by the insurance to the insured.

In this setting two basic payoffs can be defined: they are the survival and the death benefits. The survival benefit, is the one included in a contract in which the insured has the right to receive C if he survives until the end of the contract:

$$SB_t(C_T, T) = E^Q\left[e^{-\int_t^T r_s\,ds} C_T\, I_{\{\tau>T\}} \middle| F_t\right] = I_{\{\tau>t\}} E^Q\left[e^{-\int_t^T r_s+\mu_s\,ds} C_T \middle| G_t\right] \qquad (2.3)$$

Given the independence between G_t^X and G_t^Y, we assume C is G_t^X-adapted and independent from Y the expression can be transformed in:

$$SB_t(C_T, T) = I_{\{\tau>t\}} E^Q\left[e^{-\int_t^T r_s\,ds} C_T \middle| G_t^X\right] E^Q\left[e^{-\int_t^T \mu_s\,ds} \middle| G_t^Y\right]$$

The death benefit is instead comprised in a contract that assumes a payment to the beneficiary if the insured dies before the contract expiration:

$$DB_t(C_\tau, T) = E^Q\left[I_{\{\tau>t\}} e^{-\int_t^\tau r_s\,ds} C_\tau\, I_{\{\tau\leq T\}} \middle| F_t\right] = I_{\{\tau>t\}} \int_t^T E^Q\left[C_u \mu_u e^{-\int_t^u r_s+\mu_s\,ds} \middle| G_t\right] du \; (2.4)$$

With the same assumptions of the survival benefit the death benefit can be reformulated as:

$$DB_t(C_\tau, T) = I_{\{\tau>t\}} \int_t^T E^Q\left[e^{-\int_t^u r_s\,ds} C_u \middle| G_t^X\right] E^Q\left[e^{-\int_t^u \mu_s\,ds} \mu_u \middle| G_t^Y\right] du$$

It is intuitive to notice the parallelism between the expressions of the vulnerable claim (2.1) and the survival benefit, and between the one set for the recovery value (2.2) and for the death benefit. Therefore, to bring back to a risk-free framework the evaluation of a contract that links its benefits to mortality, it is needed an adjustment to the discount factor: this adjustment is the intensity of mortality (which substitutes the intensity of default in the formula).

49 Liu, X., et al. (2013)

2.1.3 Life insurance contracts valuation

A large class of insurance contracts can be priced by combining the two expressions[50] of survival and death benefits (at least all the one mentioned in Section 1.2).

The first group is composed by the life assurances. Those contracts guarantee a benefit that is defined as C and is payable in case of death when the event occurs in a specific time window, the contract period $[t, T]$. If the expiration time of the contract T is equal to a determined moment, the assurance is a term assurance while if the evaluation regards a whole life assurance contract, the expiration time T is the maximum expected life of the individual. As a result, the contract value of a life insurance with expiration T and benefit C is indexed as $A_T(C)$ and can be resumed as a death benefit, since it expresses the possibility of receiving a benefit C_τ at time of death; therefore, the following holds:

$$A_T(C) = DB_t(C_\tau; T) = I_{\{\tau>t\}} \int_t^T E^Q\left[e^{-\int_t^u r_s\,ds} C_u \middle| G_t^X\right] E^Q\left[e^{-\int_t^u \mu_s\,ds} \mu_u \middle| G_t^Y\right] du \tag{2.5}$$

The second contracts' class that is included in this evaluation scheme is the endowment one. As explained in Section 1.1 the pure endowment allows the policy holder to receive a benefit in case of survival up to time T, benefit which is defined with the notation C'. With this structure, the pure endowment essentially corresponds to a survival benefit characterized by the expiration T and by the benefit C'_T (the benefit is paid at expiration). Thus, defining it as $E_T(C')$, its value corresponds to:

$$E_T(C') = SB_t(C'_T; T) = I_{\{\tau>t\}} E^Q\left[e^{-\int_t^T r_s\,ds} C'_T \middle| G_t^X\right] E^Q\left[e^{-\int_t^T \mu_s\,ds} \middle| G_t^Y\right] \tag{2.6}$$

The other kind of endowment typically diffused in the market is the standard endowment, that reserves to the policy holder also the right to receive a benefit in case of death before the contract expiration. This second benefit is indexed with C'', and the value of the contract involving only such a reimbursement can be expressed in terms of a death benefit, as in case of the term life assurances: $DB_t(C''_\tau; T)$. Therefore, the value of the standard endowment adds the value of a contract involving a death benefit typical of a term life assurance to the one of a pure endowment. Indexing it as $E_T(C', C'')$, it corresponds to:

$$E_T(C', C'') = SB_t(C'_T; T) + DB_t(C''_\tau; T) = I_{\{\tau>t\}} E^Q\left[e^{-\int_t^T r_s\,ds} C'_T \middle| G_t^X\right] E^Q\left[e^{-\int_t^T \mu_s\,ds} \middle| G_t^Y\right] +$$
$$I_{\{\tau>t\}} \int_t^T E^Q\left[e^{-\int_t^u r_s\,ds} C_u \middle| G_t^X\right] E^Q\left[e^{-\int_t^u \mu_s\,ds} \mu_u \middle| G_t^Y\right] du \tag{2.7}$$

As explained in the Section 1.1 when the benefit is deterministic, usually it holds $C' = C''$, so the death and the survival benefits are based on the same proceeds, while when the benefit is

50 Biffis, E., 2005, pp. 452-453

not defined at inception (because it is linked to the premium investment performances) the two values are linked to the same investment.

The third kind of agreements encompassed in this scheme are annuities. Since the annuity is a sum of periodical benefits received by the policy holder until he is alive, in our framework this scheme corresponds to a series of survival benefits with different expirations. That is why, given the structure of the pure endowment value, the annuity contract evaluation can be interpreted as the value of a sum of pure endowments with different expirations, each one correspondent to the date in which the benefit is received by the contract owner. The longest endowment expiration corresponds to the maximum expected life of the individual. Fixed $a_T(C)$ as the value of the annuity contract and T as the maximum expected lifetime of the individual, it follows that:

$$a_T(C) = \sum_{i=0}^{T-1} SB_t(C_s; s) = \sum_{i=0}^{T-1} I_{\{\tau>t\}} E^Q \left[e^{-\int_t^{t+i} r_s \, ds} C'_T \middle| G_t^X \right] E^Q \left[e^{-\int_t^{t+i} \mu_s \, ds} \middle| G_t^Y \right] \qquad (2.8)$$

The formulation introduced is flexible not only because it encompasses many contract classes, but also because it works in the evaluation of both the guaranteed and the unit linked contracts. The difference between those two kinds of contract indeed lies in the definition of the structure of C, that can be fixed or variable. When this benefit is defined as a deterministic value, a guaranteed benefit contract is in place: for instance, the value of C can be $C = (1 + k)$ where k is an interest rate fixed at inception and independent on any process introduced in the model. When the analysis regards a unit or index linked contract the benefit C is instead a process whose realizations depend on the realizations of some random variables. The process of reference is typically the performance in the market of a security or of a portfolio of securities. We suppose for example that the benefit corresponds to the value of a security represented by the process S_t, and so that it holds: $C_t = S_t$. We assume then that this process is a function of the process X and is independent on Y (where the two processes are defined as above). In this case C is a G_t^X-adapted process whose value is stochastic, and therefore the valuation of a contract prescribing the payment of such a benefit is included in the model presented.

2.2 Implementation of the model: the affine processes

The conditional expectations exploiting the doubly stochastic random time properties and introduced in Section 2.1 need to be explicitly computed to be useful. The evaluation requires the introduction of some assumptions for the structure of the interest rate and the intensity of mortality.

To introduce this configuration and the properties associated to it, it is easier to start from the presentation of the results on a single generic process Ψ_t (instead of X_t or Y_t) influencing the process Λ_t (which stands of r_t or μ_t) and then extend it to the real processes of interest. The most important conditions and results are showed in this section, while shifting to Appendix C and the references therein a detailed presentation can be found [51]. Assume that the conditions fixed for the variables of interest in Section 2.1.2 hold also for the one used in the demonstration (assumptions in terms of filtration and fundamental characteristics of the variables). The aim is to compute the following generic conditional expectation, that when formulated in the suitable terms can express both the death and the survival benefits formula:

$$E^Q\left[e^{-\int_t^T \Lambda_s\,ds} g(\Psi_T)\middle|G_t\right] \tag{2.9}$$

The function $g(\Psi_T)$ expresses the final payoff related to the contract object of the evaluation. The process Ψ_t can be represented as the solution to the following stochastic differential equation:

$$\begin{cases} d\Psi_t = \delta_1(\Psi_t)dt + \sigma_1(\Psi_t)dW_t \\ \Psi_0 = \psi \end{cases} \tag{2.10}$$

with state space given by the domain $D \subseteq \mathbb{R}$. We assume that the process Ψ_t belongs to the class of the affine processes, where in this context the affinity of a process is intended in terms of strong solutions to specific stochastic differential equations in a given filtered probability space[52]. The specification of the terms $\delta_1(\Psi_t)$ and $\sigma_1^2(\Psi_t)$ is important to guarantee the affinity of the process. Indeed, both are affine functions of Ψ_t and so:

$$\delta_1(\Psi_t) = k_0 + k_1\psi_t \qquad \text{and} \qquad \sigma_1^2(\Psi_t) = h_0 + h_1\psi_t$$

k_0, k_1, h_0, h_1 are all real constants such that for all $\psi_t \subseteq D$ it holds the condition $h_0 + h_1\psi_t \geq 0$. Moreover, also the variable Λ_t is supposed to have an affine structure with respect to Ψ_t: $\Lambda_t = \lambda_0 + \lambda_1\psi_t$. As above for the other parameters, λ_0, λ_1 are both real constants. Affine processes are Markov processes with conditional characteristic function of the exponential affine form. When $g(\Psi_t) = e^{a\Psi_t}$, the assumptions introduced above yield the following expression:

$$E^Q\left[e^{-\int_t^T \Lambda_s\,ds} e^{a\Psi_t}\middle|G_t\right] = e^{\alpha(t,T)+\beta(t,T)\Psi_t} \tag{2.11}$$

51 For a rigorous demonstration: Duffie, D., Pan, J., Singleton, K., 2000

52 Biffis, E., Millossovich, P., 2006

Where the parameters $\alpha(t,T)$, $\beta(t,T)$ satisfy the following ordinary differential equations:

$$\begin{cases} \beta'(t,T) = \lambda_1 - k_1\beta(t,T) - \dfrac{1}{2}h_1\beta^2(t,T) \\ \alpha'(t,T) = \lambda_0 - k_0\beta(t,T) - \dfrac{1}{2}h_0\beta^2(t,T) \end{cases}$$

with boundary conditions: $\beta(T,T) = a$ for the first equation and $\alpha(T,T) = 0$ for the second. Those are the Ricatti Equations[53], and this formulation of the problem allows to reconduct the computation of a conditional expectation implying a stochastic differential equation to the one of a system of two ordinary differential equations. This system can be solved numerically and, in some cases (as the one of the Cox Ingersoll Ross process), presents also a closed form solution.

To extend those results to the framework of interest we need many specifications, taking as valid all the assumptions that have been presented in the Section 2.1.2. The first thing is to set the processes X_t and Y_t of the affine form, as solutions of two SDEs analogue to (2.10); moreover, the parameters of the drift and squared volatility are assumed to be affine functions of X_t in the case of the financial market process and of Y_t in the intensity of mortality model: $\delta_1(X_t) = k_0(t) + k_1(t)x$ and $\sigma_1^2(X_t) = h_0(t) + h_1(t)x$ and the same for $\delta_2(Y_t)$, $\sigma_2^2(Y_t)$. The functions $k = (k_0, k_1)$; $h = (h_0, h_1)$ are all bounded and continuous functions. When considering $Z = (X, Y)$, since the two variables are affine and independent, also Z is affine.

Now let us focus on the financial market structure. Beside the definition of the risk-free rate, the absence of arbitrage implies several restrictions on the process describing market prices to hold[54]. The risk-free interest rate is affine dependent and so shows a structure: $r_t = \rho_0(t) + \rho_1(t)X_t$. The parameters $\rho_0(t)$, $\rho_1(t)$ are bounded continuous functions such that for all $x_t \subseteq D$ it holds the condition $\rho_0(t) + \rho_1(t)x_t \geq 0$. We fix the presence of a security continuously traded, with ex-dividend price S. Defined $D_t = \int_0^t \vartheta_u S_u du$ as the instantaneous dividend process, the expression of the claim value is:

$$S_t = E^Q\left[e^{-\int_t^T r_s\, ds} S_T + \int_t^T e^{-\int_t^T r_s\, ds}\, dD_u \,\middle|\, F_t\right] \tag{2.12}$$

Let us then consider a risky security whose log-price is also affine, thus a claim with structure $S_t = e^{f(X_t)}$ where $f(X_t)$ is an affine function. The dividend yield process of the security is $\vartheta(X_t) = d_0(t) + d_1(t)x$ and is also affine. The property which must hold under the risk-neutral

53 McNeil, A., Frey, R., Embrechts, P., 2005, p. 423

54 Biffis, E., Millossovich, 2006

measure Q (to be defined as this) implies that the security S is a martingale after the deflation for the money market account, and so it must present a drift equivalent to $r - \vartheta$. To respect this property, under the security structure introduced before it is necessary to impose some conditions to the drift formulation[55]:

$$\begin{cases} k_1(t) = \rho_1(t) - d_1(t) - \frac{1}{2}h_1(t) \\ k_0(t) = \rho_0(t) - d_0(t) - \frac{1}{2}h_0(t) \end{cases}$$

All the securities composing the market must satisfy simultaneously this condition. In the simplest case it is useful to consider as security a safe zero-coupon bond with maturity T, face value 1. Its price at time t can be expressed setting in the basic model $a = 0$, thus with $g(X_T) = 1$ and applying the Ricatti equations system as before (but with boundary conditions $\beta(T,T) = 1$ and $\alpha(T,T) = 0$):

$$E^Q\left[e^{-\int_t^T r_t ds}\middle|G_t\right] = e^{\alpha(t,T)+\beta(t,T)X_t}$$

Focusing on the mortality model, the intensity of mortality μ has affine structure depending on the variable Y: $\mu_t = \eta_0(t) + \eta_1(t)Y_t$. Then recalling the definition for the probability of death to happen before the contract expiration (see Section 2.1.2) it is straightforward that this expression is encompassed in the framework of the affine process analysis (as for the zero-coupon bond with $g(Y_T) = 1$ and the same boundary conditions):

$$P(\tau > T|H_t) = E^Q\left[e^{-\int_t^T \mu_s\, ds}\middle|G_t^Y\right] = e^{\alpha(t,T)+\beta(t,T)Y_t}$$

Considering the death and survival benefits, it is clear how the formulation of the expectation in the two expressions (2.3) and (2.4) can be suitably reconnected to (2.11), for which the properties have been explained. The only one thing to be set is the benefit C_t, that can be guaranteed or stochastic: when its expression can be formulated such that $g(\Psi_T) = e^{a\Psi_T}$ the formula can be applied directly. When the formulation is different – for instance considering $g(\Psi_T) = e^{a\Psi_T}(b\Psi_T + c)$ – or the stochastic process underlying the variables changes its formulation, suitable extensions of the basic formula[56] can be used to make the mechanism work.

55 Biffis, E., 2005, p. 465

56 For details on the extension see: Biffis, E., 2005, p. 446; for an application p. 452-453

2.3 Interpretation of the model

Until now the formulation of the model has been focused on its mathematic development, analysing its tractability from a computational point of view. However, to be useful and implementable the model must be compliant with the regulatory requirements. As introduced before the regulation on the valuation of the insurance contracts is explained in the IFRS 17: the main provisions impose a discounted cash flow approach, which must be consistent with the market evaluation. The formulas introduced for the survival and death benefits can be analysed on their compliance with respect to those two fundamental principles.

The expectations taken as the basis of the model express the final reimbursement discounted by some risk factors: this is actually a discounted cash flow approach. The discount elements consider the financial and the mortality risks (the main one implied by the contract), and they can be analysed separately (explicitly) when the filtrations G_t^X and G_t^Y are assumed to be independent and C is G_t^X-adapted or independent from both the processes.

Since the expectations are taken under a risk neutral measure, the measurement is expected to be consistent with a market evaluation. As explained above when (2.3) and (2.4) have been introduced, the risk neutral valuation mechanism can be extended to securities that provide benefits contingent on mortality, provided that they are evaluated under the fictitious risk adjusted short-rate process $\mu + r$. However, when considering the risk neutral evaluation, there are two main issues connected with the structure of the insurance market.

The first one regards the calibration of the intensity of mortality rate. The lack of a deep wholesale market (absence of liquidity) is a problem when justifying the possibility of inferring the risk neutral measure with respect to the mortality risk: as a matter of fact, the lack of trade prevents the direct inference of Q from the market. Nevertheless, the market valuation of insurance products' portfolios is performed by some financial analysts using the embedded value method. This method is based on the assessment of the value of the business in place at the evaluation moment, without including any kind of forecast about the future in terms of business growth (included in the goodwill). The mechanism[57] consists in computing the sum of the shareholders' capital backing the book of assets and the value of the in-force business. The first component represents the value that theoretically can be distributed to shareholders immediately (the shareholders' net assets). The latter is the value that will be available for shareholders in the future, constituted by the future free cash flows emerging from the business already in place (the policies part of the portfolio). Those future cash flows have to be

57 To see how this method works: Tremblay, F., 2006

discounted with an appropriate rate reflecting many costs and risks: it accounts for the cost of capital, the tax liabilities and the non-diversifiable risk implied by the business. When the embedded value of a book of contracts is known, a market value of the book of policies can be retrieved, netting the discount rate used in the evaluation for its tax and cost of capital component (those components are excluded from a fair evaluation in this context). This last result should be equal to the one expressed by the computations introduced in the chapter (the aim of the formulas is to give the fair value to a contract or to a book of contracts). Therefore, the "adjusted embedded value" should be functional to calibrate the mortality risk adjustment, i.e. $E^Q\left[e^{-\int_t^T \mu_s\, ds}\middle|G_t^Y\right]$, once the financial risk component, i.e. $E^Q\left[e^{-\int_t^T r_t ds}\middle|G_t\right]$, has been retrieved[58]. This is the best way to ensure a sound theoretical basis for the rate to be estimated. However in practical applications (see for example Luciano and Vigna (2009)), the intensity of mortality process is retrieved from observed and projected mortality tables and so from historical data: this is also the method which will be used for the practical implementation of the model (Section 3.1.3 for details).

The second problem regards the difficulty of exploiting non-arbitrage type arguments in the evaluation. This issue is related to the absence of liquidity and completeness of the market and can be solved assuming that at least the endowments and the life insurances (i.e. the primitive contracts) are continuously traded in the market.
Nevertheless, even if this last hypothesis is satisfied, another fact needs to be mentioned: each single policy refers to a specific individual with his peculiar characteristics in terms of mortality. As noticed by Biffis (2005)[59], the consequence to this specificity of each policyholder is that arbitrage pricing results referring to the single contract can only approximately be scaled at a portfolio level. When considering a group of contracts, arguments as hedging and replicating strategies can be applied leading to results whose precision depends on the degree of homogeneity of the policyholders and on the dimension of the portfolio under consideration: that is why the assumption of having a homogeneous population is also necessary.

Despite for those two caveats which are solvable when keeping as valid the right assumptions, the model has many other strengths guaranteeing the valuation to be actual, and to exploit all the information about past events, present conditions and forecasts about the future. In conclusion, even if with some limitations, the model can be defined as compliant with the regulation and useful for the actuarial valuation of the insurance contracts.

58 Biffis, E., 2005, p. 454-455

59 Biffis, E., 2005, p. 454

CHAPTER 3

3. Model application

In the previous chapter we illustrated the theoretical foundations of the model object of the dissertation, showing how we can exploit the default risk modelling machinery to measure mortality risk. It is now useful to present an application of it: this allows us to show the flexibility and the power of the methodology introduced. The way of proceeding in the empirical approach follows a scheme that can be easily generalized when pricing any life insurance contract and that is built up with three steps, corresponding to the sections of this chapter. Then, Section 3.1 defines the processes composing the framework that characterizes the contract evaluation, modelling the risk free rate of interest, the mortality and the market dynamics with stochastic processes having an affine form. The calibration of the parameters constituting them is performed, based on historical and market data. Section 3.2 introduces the main characteristics of the selected contract, describing its structure useful to understand the procedure adopted for the valuation purpose. The selected product is an index linked endowment embedding a guaranteed annuity option. This is a structured contract which has been quite widespread in UK during the past years and created many solvency problems to some companies which issued it: for instance, Equitable Life (the oldest insurance company in the world) at the end of the 90s went seriously in trouble with it[60]. Indeed, the contract requires the necessity to account for many sources of risk (related to interest rates, mortality and market performances) and to forecast their evolution in the long term: that is why its pricing is particularly challenging and it is appropriate to adopt a quite sophisticated stochastic model to perform the valuation. Despite of the problems created in the past, the contract is still traded in many countries (US and Japan are two examples)[61]. Starting from the framework developed in Section 3.1 and from the analysis of the product of Section 3.2, a third section (Section 3.3) develops the valuation of the contract, with some comparative statics analysis and some comments on the advantages of this kind of modelling.

3.1 Processes calibration

The first step for the valuation of a contract is the introduction of the framework in which it is performed. This activity is the calibration and represents a fundamental step in the whole

60 Van Haastrecht et al. (2010)

61 Van Haastrecht et al. (2010)

valuation procedure: indeed, it outlines the processes selected to project the variables in the future and to solve the problems that lead to the solution of the final pricing problem. The one presented in this section is a general methodology that can be applied to all the contracts priced using the stochastic modelling. In the case examined three are the processes considered. The first two characterize every insurance contract: they resume the dynamics of the interest rate (r_t), and of the intensity of mortality (μ_t). The third process is specific of the contracts with variable benefits and expresses the performance of the index to which the final proceeds paid by the insurer is linked (S_t). To fit the process parameters, it is useful to refer to some market data or historical time series of the market data. If for the index and the interest rate dynamics this is a feasible procedure, as mentioned in Section 2.3 the calibration of the intensity of mortality is more problematic, since it cannot be deduced from market quotes, given that the trade of insurance contracts and portfolios is not frequent (the market is not liquid). Consequently, the dataset selected for this last stochastic process considers the historical data and the projections of the death probabilities of a specific generation as the source on which calibrating the parameters of interest.

3.1.1 Interest rate process

The first stochastic process calibrated is the interest rate. Interest rates modelling has always been an important topic in the academic literature since this variable represents one of the most important factors for the market dynamics. One of the most relevant models for the interest rates has been introduced by Cox, Ingersoll and Ross (1985). It is usually referred as CIR (from the authors' name), and it is represented, under the risk neutral measure Q by the following stochastic differential equation:

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t^1\,, \quad r(0) = r_0 \tag{3.1}$$

where $W_t^1, t \geq 0$ is a standard Brownian Motion and the values θ, k, r_0, σ are real positive constants. The constant θ represents the long-term average of the interest rate, to which process tends with "force" given by the coefficient k (mean reversion speed) while σ is the coefficient that resumes the volatility characterizing the process. The positivity of r_t through all the path (binding for the existence of the square root in the process expression) is guaranteed by the following condition: $2\theta k > \sigma^2$. We thus want the speed of mean reversion and the long-term average to be sufficiently large with respect to the squared volatility so that the process never reaches the value of zero[62].

62 For a proof of this condition: Lamberton et al. (2007), p.161

The expression (3.1) has an affine structure and thus can be useful for the modelling of the interest rates dynamics in our evaluation framework: that is why this is the model selected. Following Brigo and Mercurio (2007)[63], given the affine property of the process, the expectation at time t, $E_t^Q\left[e^{-\int_t^T r_t ds}\right]$, is defined in a close form solution as:

$$E_t^Q\left[e^{-\int_t^T r_s ds}\right] = A(t,T)e^{-B(t,T)r_t} \tag{3.2}$$

where $A(t,T)$, $B(t,T)$ are:

$$A(t,T) = \left(\frac{2\gamma e^{\frac{(k+\gamma)\tau}{2}}}{2\gamma+(\gamma+k)(e^{\gamma\tau}-1)}\right)^{\frac{2k\theta}{\sigma^2}} \qquad B(t,T) = \frac{2(e^{\gamma\tau}-1)}{2\gamma+(\gamma+k)(e^{\gamma\tau}-1)} \tag{3.3}$$

$\tau = T - t$, $\gamma = \sqrt{k^2 + 2\sigma^2}$ and r_t is the instantaneous rate at time t.

The calibration, useful to retrieve the parameters of (3.1) and solve properly the expression (3.2), is executed in many steps. First of all, we estimate the process exploiting the historical time series of the risk-free interest rate, following the procedure introduced by Kladivko (2007). The method is a maximum likelihood one and focuses on the estimation of the vector of parameters (k, θ, σ). The maximum likelihood approach exploits the fact that the CIR is characterized by a transition density which has a close form solution defined by Feller (1951). Fixing r_t at time t, the density of $r_{t+\Delta t}$ at time $t + \Delta t$ is:

$$p(r_{t+\Delta t}|r_t; k, \theta, \sigma, \Delta t) = ce^{-u-v}\left(\frac{v_{t+\Delta t}}{u_t}\right)^{\frac{q}{2}} I_q(2 \quad \sqrt{uv})$$

where: $c = \frac{2k}{\sigma^2(1-e^{-k\Delta t})}$; $u_t = cr_t e^{-k\Delta t}$; $v_t = cr_t$; $q = \frac{2k\theta}{\sigma^2} - 1.$

$I_q(2 \quad \sqrt{uv})$ is a modified Bessell function of the first kind and of order q. The parameters' estimation is then carried out on N observations $\{r_{t_i}, i = 1,\ 2, \ldots, N\}$, equally distributed with time intervals Δt. The likelihood function for the interest rate time series is in this case:

$$L(k, \theta, \sigma) = \prod_{i=1}^{N-1} p\left(r_{t_{i+1}}|r_{t_i}; \alpha, \Delta t\right)$$

For practical implementations the function works better with the log-transformation:

[63] Brigo and Mercurio (2007), p- 65-66

$$\ln L(k,\theta,\sigma) = \sum_{i=1}^{N-1} \ln p\left(r_{t_{i+1}} \middle| r_{t_i}; \alpha, \Delta t\right)$$

Then the log-likelihood of the CIR process is the following:

$$\ln L(k,\theta,\sigma) = (N-1)ln(c) + \sum_{i=1}^{N-1} \{-u_{t_i} - v_{t_{i+1}} + 0.5qln\left(\frac{v_{t_{i+1}}}{u_{t_i}}\right) + ln[I_q\left(2 \quad \sqrt{u_{t_i} v_{t_{i+1}}}\right)]\}$$

where $u_{t_i} = u(r_{t_i})$ and $v_{t_{i+1}} = v(r_{t_{i+1}})$. This function has then to be maximized with respect to the vector of parameters (k,θ,σ) and this enables us to find the appropriate coefficients for (3.1):

$$\left(\hat{k},\hat{\theta},\hat{\sigma}\right) = \max_{k,\theta,\sigma} \ln L(k,\theta,\sigma)$$

The solution of the problem requires a numerical optimization which can be implemented with Matlab using the command "`fminsearch`". To develop the computations the numerical optimizer requires an initial estimation of the parameters, which is important for convergency. Following Kladivko (2007), this initial estimation is performed with OLS on the discretized version of the equation. Therefore, in this case we do not exploit the distribution of the process, but we apply the Euler approximation to the process:

$$\frac{r_{t+\Delta t} - r_t}{\sqrt{r_t}} = \frac{k\theta\Delta t}{\sqrt{r_t}} - k\sqrt{r_t}\Delta t + \sigma\varepsilon_t\sqrt{\Delta t} \qquad \text{where} \qquad \varepsilon_t \sim N(0,1)$$

Since a proxy for the risk-free interest rate is needed, the data selected for the optimization is the UK Overnight Indexed Swap (OIS) with 1 moth expiration[64]; the time series is expressed in percentage points, so a preliminary operation manipulating the dataset is to set the values to real numbers (dividing the data by 100). Since the calibration performed is based on historical time series, we are now working under the physical probability measure P. The series starts at the end of 2007, ends in 2019 and is composed overall by 2960 data collected on a daily basis: as it has been common in all developed economies, the rate of interest collapsed after the financial crisis of 2007/2008 and now is stable at low levels (see Section 1.2.1 for the consequences of the low interest rates level on the insurance profitability and stability). The historical time series is plot in the Figure 6.

64 Data taken from Thomson Reuters database (30/09/2019)

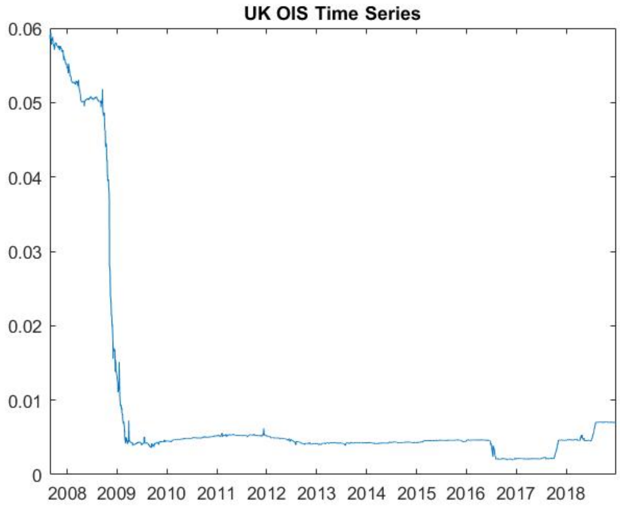

Figure 6 - Interest rates time series (Data Source: Thomson Reuters)

The code implementing the estimation procedure on Matlab is shown in Appendix D. Given the daily frequency of the data, the time interval adopted for the calibration is $\Delta t = \frac{1}{250}$ (the unit represents the year and the average trading days in a year is usually approximated to 250). The initial OLS estimates useful to implement the calibration procedure are: $\hat{k}_{OLS}$ =0.7562, $\hat{\theta}_{OLS} =$ 0.0039 and $\hat{\sigma}_{OLS}$ = 0.0322. The results are following from the numerical optimization are the following:

$\hat{k}_{ML}$	0.7938
$\hat{\theta}_{ML}$	0.0042
$\hat{\sigma}_{ML}$	0.0324

As specified above, the estimation is computed under the physical measure P since the parameters are calibrated on a historical time series. The valuation of the product is however performed under the risk-neutral measure Q: the second step is then to calibrate the process also under this probability measure. There are different ways to operate the calibration and it is worth mentioning two of them:

- the first is based on historical data, referred to a time series of bond yields for different maturities;
- the second exploits current market data and precisely the yield curve of the OIS rates.

Referring to the first method, it operates under the probability measure P and estimates the "market price of risk". Following Brigo and Mercurio (2007a)[65], the market price of risk Λ is a

65 Brigo, Mercurio (2007a), p. 65

function of the volatility σ, the speed of mean reversion k and of the parameter λ and is the parameter which differentiates the behaviour of the process between the physical and the risk neutral measure. Indeed, the risk neutral dynamics (3.1) under the physical probability P has the form[66]:

$$dr_t = (k\theta - (k + \lambda\sigma)r_t)\,dt + \sigma\sqrt{r_t}\,dZ_t\,, \quad r(0) = r_0$$

where $\{Z_t, t \geq 0\}$ is a Brownian Motion under P. Once the parameter λ has been computed via quasi maximum likelihood with the estimation based on the dynamics of the bond yields at different maturities, it is then possible to retrieve the risk neutral dynamics of the interest rate. When $\lambda = 0$ the dynamics under the risk neutral and the physical measure are the same: this is what was implicitly assumed in the Maximum Likelihood calibration procedure. For an application of the method see Duffee (2002).

The second method exploits instead the current market term-structure of the interest rates. From those rates it is possible to compute the prices of the hypothetical zero coupon bonds at different maturities: they represent the discounts associated to the risk-free rate that the market applies at different maturities. Then, once computed those market prices, the parameters k, θ and σ can be estimated exploiting the close form solution for the zero-coupon bond price implied by the CIR (3.2): the parameters are set in order to minimize the square root of the Mean Squared Error, expressed as the difference between the prices implied by the market and the one theoretically expressed by the formula.

The latter one is the method chosen for the practical estimation, based on yield curve of the UK OIS spot rates ranging from 1 month to 5 years of maturity with monthly frequency (the estimation is based on 60 different maturities)[67]. The Matlab implementation of the procedure involves the functions "`zero2disc`" to compute the discount from the yield curve and "`fminsearch`" minimizing the function "`NEST`" to optimize the values. The initial values are the one computed when using historical data to calibrate the parameters, so starting from a null market price of risk: $\hat{k}_{ML}$, $\hat{\theta}_{ML}$, $\hat{\sigma}_{ML}$. The complete code is available in Appendix D. The results are the following:

$\hat{k}$	0.5840
$\hat{\theta}$	0.0061
$\hat{\sigma}$	0.0261

66 Brigo, Mercurio (2007a), p. 65

67 Data downloaded from the Bank of England: https://www.bankofengland.co.uk/statistics/yield-curves

As we can notice from this estimation the speed of mean reversion and of the volatility are reduced, while the long- term average of the process is estimated to be higher with the interest rate expected to converge to the value of 0.0061. The changes in the parameters are relevant from a quantitative point of view. A posterior check of the condition for the positivity of the interest rates is also made and sorts out $2\theta k > \sigma^2$: the Feller condition is met.

3.1.2 Market index process

The second estimation produced is the one of the index underlying the endowment performance. The index selected is the FTSE All Share[68], whose historical time series is presented in the upper part of Figure 7. The series starts in 1964 and since data are collected on daily basis it is composed by 14994 observations: we use the historical time series to estimate the process coefficients, so our estimation procedure is performed under the real-world probability measure P. To select the stochastic process modelling the series, it is useful to compute the log-returns of the index which, as proved by the autocorrelation function in the lower part of Figure 7, are almost uncorrelated. Given uncorrelation of returns, the index performances can be fitted quite well by a Geometric Brownian Motion process: the index value shows indeed an increasing trend characterized by many drops and disturbances, expressed by the volatility in the stochastic process.

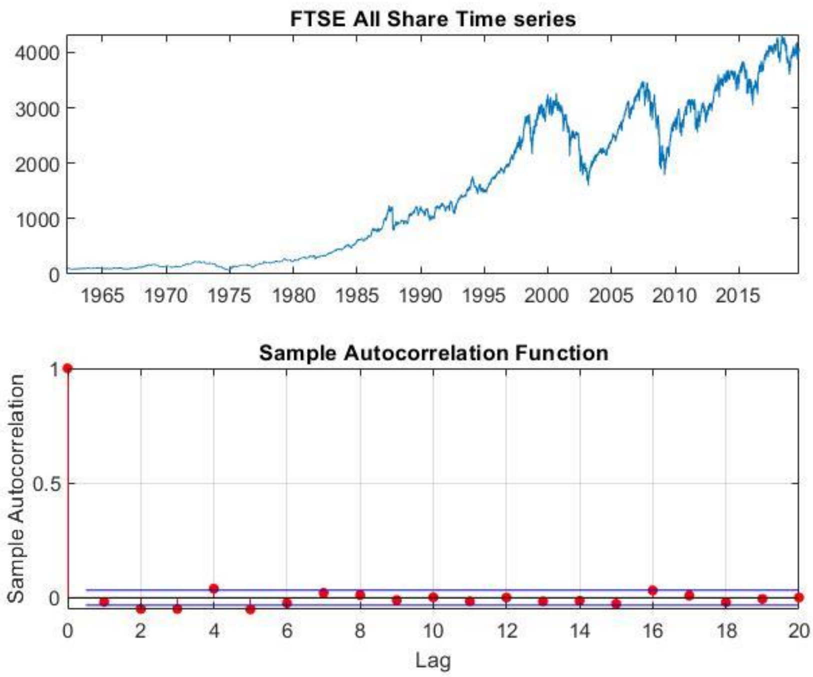

Figure 7 - Index time series and index return autocorrelation (Data Source: Thomson Reuters)

68 Data taken from Thomson Reuters database (30/09/2019)

A limitation of this way of modelling is the fact that the volatility of the return produced by the index is assumed to be constant along the whole sample, and therefore independent on the previous period volatility (returns are considered homoscedastic); this is not true, since heteroskedasticity is one of the main characteristics of the returns. However, the GBM model has a great analytical tractability and so exploiting it is a useful approximation. The process is thus the following:

$$dS_t = \mu S_t \, dt + \sigma S_t \, d\xi_t$$

where $\{\xi_t, t \geq 0\}$ is a Brownian motion process under P, σ and μ are real constants and $\sigma \geq 0$. The Brownian motion shocks influencing the path of the index are assumed to be not directly influenced by the one influencing interest rates: as for the homoskedasticity, this is a simplifying assumption that is quite strong from a theoretical point of view but is useful to make the calibration procedure and then the valuation more straightforward. Following Bjoerk (2009)[69], the stochastic differential equation admits the explicit solution:

$$S(T) = S(0)e^{\left(\left[\mu - \frac{1}{2}\sigma^2\right]T + \sigma\xi(T)\right)}$$

This equation shows how the price of the asset considered follows a log-normal distribution; once defined the log-returns of the index $X(t_i) := ln\, S(t_i) - ln\, S(t_{i-1})$, it can be shown that they are distributed as a normal and in particular that:

$$X(t_i) \sim N\left(\left[\mu - \frac{1}{2}\sigma^2\right]\Delta t, \sigma^2 \Delta t\right) \tag{3.4}$$

To perform the maximum likelihood estimation of the parameters' vector $\theta \equiv (\mu, \sigma)$ it is useful to follow Brigo et al. (2007b). Define $x_i (i = 1, \ldots, n)$ as the observations of the returns and $p(x_i; \theta)$ as the probability density function of the data. Moreover, recall that θ is the vector of parameters to be estimated. Exploiting the Markov property, we can write the likelihood function along the observations x_i as a product of transition likelihoods on single time steps between two adjacent instants (same procedure has been adopted for the interest rate process calibration). When the transition likelihood is expressed as $p(x_{i+1}|x_i; \theta)$ it results:

$$L(\theta) = \prod_{i=1}^{n-1} p(x_{i+1}|x_i; \theta)$$

But since the observations are assumed to be iid, the expression for the likelihood function results to be simplified. The transition likelihood of each observation indeed equals its

69 Bjoerk (2009), pp.67-70

probability density function, and so $p(x_{i+1}|x_i;\theta) = p(x_i;\theta)$. Thus, the log-likelihood function of the process is:

$$\ln L(\theta) = \sum_{i=1}^{N-1} \ln p(x_i;\theta)$$

Finding the maximum of the likelihood in this case is quite straightforward, since the identical distribution of the variables is normal and defined by the mean (m) and the variance (v) parameters, which can also be estimated with the well-known formulas for sample mean and variance:

$$\hat{m} = \sum_{i=1}^{N-1} \frac{x_i}{n} \qquad \hat{v} = \sum_{i=1}^{N-1} \frac{(x_i - \hat{m})^2}{n}$$

Then, given the characteristics of the returns' distribution, the parameters of the drift and of the squared diffusion can be found with a close form solution:

$$\hat{\mu} = \frac{\hat{m}}{\Delta t} + \frac{1}{2}\sigma^2 \qquad \hat{\sigma}^2 = \frac{\hat{v}}{\Delta t}$$

The values estimated for the process are then:

$\hat{\mu}$	0.0630
$\hat{\sigma}$	0.1696

The Matlab code for the estimation can be found, as well as before, in Appendix D.

As for the interest rates we need the dynamics of the process in the risk-neutral probability measure Q to perform the valuation of the contract. In this case, as a result of the Girsanov Theorem, we know that shifting from one probability measure to the other the structure of the process is changed only in its drift. Indeed, the drift doesn't correspond any more to μ (calibrated with the Maximum Likelihood) but is the risk-free interest rate r_t (process which is calibrated under the probability measure Q). The process thus takes the form:

$$dS_t = r_t S_t\, dt + \sigma S_t\, dW_t^2$$

the process $\{W_t^2, t \geq 0\}$ is now a Brownian Motion under the probability measure Q.

3.1.3 Intensity of mortality process

The last process to be calibrated is the intensity of mortality. The parameters' estimation is based on the historical data and projections about the mortality rates and the survival probabilities of a specific cohort inherent to a specific nation (the approach followed to estimate the intensity of mortality is a diagonal one as mentioned in Section 2.1.2). The database selected is the one of the UK cohort born in 1955 with data updated in 2016 by the UK government (available at www.ons.gov.uk [70]). The choice of this cohort is the solution for the presence of a trade-off between the change in mortality trends and the availability of observed data. For people born in 1955 it is intuitive that their mortality intensity should not be so far from the one of the generations that will enter in the insurance contracts in the near future; on the other hand the available data are in part observed and in part projected, so that the estimation of the parameters is not completely relying on forecasts but is based, at least in part, on real data.

Since the life table exploited expresses only the mortality rates, to perform the estimation of the mortality intensity some basic relationships regarding the main demographic statistics expressed in a mortality table are used. The mortality rate $q_{x,t}$ is the complement to one of the survival probabilities, $p_{x,t} = 1 - q_{x,t}$ which expresses the probability of an individual aged x at time t to survive until the following period (as it's common in mortality tables one period corresponds to one year). The probability of surviving n years for an individual aged x is given by the product of the periodic survival probabilities; so, the survival probability function is obtained as:

$$S_x(n) = \prod_{i=x}^{x+n-1} p_{x,i} \tag{3.5}$$

Another parameter typically characterizing the demographic condition of a population, is the central death rate, expressed as the ratio:

$$m_{x,t} = \frac{d_{x,t}}{E_{x,t}}$$

Where $d_{x,t}$ is the number of deaths in one year among the people aged x in t, and $E_{x,t}$ is the central exposed to risk: this corresponds to the size of the population exposed to risk in the middle of the interval considered, and so the part of the population still alive in the half of the interval (for details see Pitacco (2007)). As proved by Pitacco (2009) the intensity of mortality can be associated to the observed central death rates given the condition that it is considered constant within a determinate time interval. Therefore, if $\mu_t = \mu_{t+h}$ (h represents the length of the time interval) it can be set:

$$\mu_t = m_{x,t} \tag{3.6}$$

All those relations are exploited in the estimation procedure.

70https://www.ons.gov.uk/peoplepopulationandcommunity/birthsdeathsandmarriages/lifeexpectancies/bulletins/pastandprojecteddatafromtheperiodandcohortlifetables/2016baseduk1981to2066

To select the form of the process for the intensity of mortality, we refer to Luciano and Vigna (2009) who showed that the mean reverting processes (the most used until that moment for the mortality intensity expression) are not the most appropriate to simulate and fit the mortality and the survival probabilities of a population. In their research they present different non-mean reverting processes that can fit quite well the survival probabilities of a population; among them the one selected is a Feller process, characterized by two parameters:

$$d\mu_t = a\mu_t\, dt + \sigma\sqrt{\mu_t}\, dW_t^3$$

where $\{W_t^3, t \geq 0\}$ is a standard Brownian Motion and a, σ are real and positive constants. The intensity of mortality is assumed to be independent from the interest rate and the stock market: this is a reasonable assumption which follows also Section 2.1.2. Under this set-up, given the affine structure of the process it holds another expression for the survival probability $S_x(n)$ (where $n = t - T$) based on the parameters characterizing the process. Indeed, we can define[71]:

$$S_x(n) = E^Q\left[e^{-\int_t^T \mu_s\, ds}\middle| G_t^Y\right] = e^{\alpha(t,T)+\beta(t,T)\mu_t} \qquad (3.7)$$

The solution to the Ricatti equations characterizing the two functions $\alpha(t,T), \beta(t,T)$ has in this case a close form solution, and in particular:

$$\begin{cases} \beta(t,T) = \frac{1-e^{-\sqrt{a^2+2\sigma^2}t}}{c+de^{-\sqrt{a^2+2\sigma^2}t}} \\ \alpha(t,T) = 0 \end{cases} \quad \text{with} \quad \begin{cases} c = \frac{-\sqrt{a^2+2\sigma^2}+a}{2} \\ d = \frac{-\sqrt{a^2+2\sigma^2}-a}{2} \end{cases} \qquad (3.8)$$

Two are the conditions that must hold for the stochastic process to be useful for the expression of the mortality intensity: the intensity must always be positive, and the survival probability has to be decreasing while the time passes: those characteristics give sense to the interpretation of the intensity of mortality. The first constraint is satisfied in practical applications almost always, as stated by Luciano and Vigna (2009) since μ_t remains strictly positive during all its evolution (the probability of μ_t of reaching 0 is negligible). As stated by Luciano and Vigna (2009), for the survival probability to be always decreasing in t, the parameters of the stochastic process have to satisfy the following condition[72]:

$$e^{bt}(\sigma^2 + 2d^2) > \sigma^2 - dc$$

Another way of expressing the survival probability as a function of the parameters of the process equals (3.2) for the interest rates:

71 For a complete analytical proof see Apicella (2017), p. 26

72 Luciano, Vigna (2009), p. 9

$$S_x(n) = A(t,T)e^{-B(t,T)\mu_t} \tag{3.9}$$

where:

$$\begin{cases} A(t,T) = 1 \\ B(t,T) = \dfrac{2(e^{\gamma t} - 1)}{(\gamma - a)(e^{\gamma t} - 1) + 2\gamma} \end{cases}$$

in which $\gamma = \sqrt{a^2 + 2\sigma^2}$. This relation is more manageable than (3.7) from a computational point of view.

For the estimation of the parameters, (3.5) and (3.9) are exploited. The formula (3.5) is used to retrieve the survival probability function from observed and projected data and then, relating those probabilities with the expression (3.9), the parameters of the stochastic process can be calibrated: we want the survival function based on the stochastic process to show an evolution as similar as possible to the observed function. To do so we minimize the square root of the Mean Squared Error of the calibrated function: the MSE is expressed as the mean of the squared differences between the observed and the theoretical survival probabilities computed according to the specification of the model, following (3.9). As before the probability measure under which the estimation is performed is the real world one, P.

The insurer is interested in shaping the survival function of the individual starting from the time in which the contract is set. If the individual is aged x at the time of the agreement (set as time 0) the data useful for calibration are the one starting from the age of x and ending at $x + n$ (where $x + n$ is the last data available). Therefore, the cohort data exploited to calibrate the parameters of the Feller process are only part of the one available: data on the mortality of the individual when his age is from 0 to $x - 1$ are not useful. In our specific case time 0 is set when the individual is 50 (see Section 3.3 for details). Given that the individual was born in 1955, the data used are observed from 2005 to 2016 and then projected from 2017 to 2055 (last date available), ending thus at time 50 when the individual is aged 100. Thus, the data-selection is the first procedure to be performed in order to exploit the right dataset.

For the computations we exploit the Matlab function "`fmincon`", imposing as constraints the positivity of both parameters a and σ. The Mean Squared Error is computed with the function named "`SURMAX`". For the implementation code refer to Appendix D.

To perform this estimation, a guess on the mortality intensity at time 0 is required: this is indexed μ_0 and is taken following in Luciano and Vigna (2009). When we assume the intensity of mortality constant in a determinate interval of time, given formula (3.7) it can be shown[73] how $p_{x,t} = e^{-\mu_t}$. But under the hypothesis of μ_t constant in $[t, t+h]$ also (3.6) holds. Then the central mortality rate parameter can be linked to the survival probabilities of a population: $p_{x,t} = e^{-m_{x,t}}$. Thus, a valid approximation for the initial intensity of mortality can be represented by: $\mu_0 \approx -ln(p_{x,0})$.

The table shows the results of the estimation, while the plot of the two survival probabilities is presented in Figure 8. The final condition for the decreasing evolution of the survival probabilities with respect to the age is checked posteriori, and it holds.

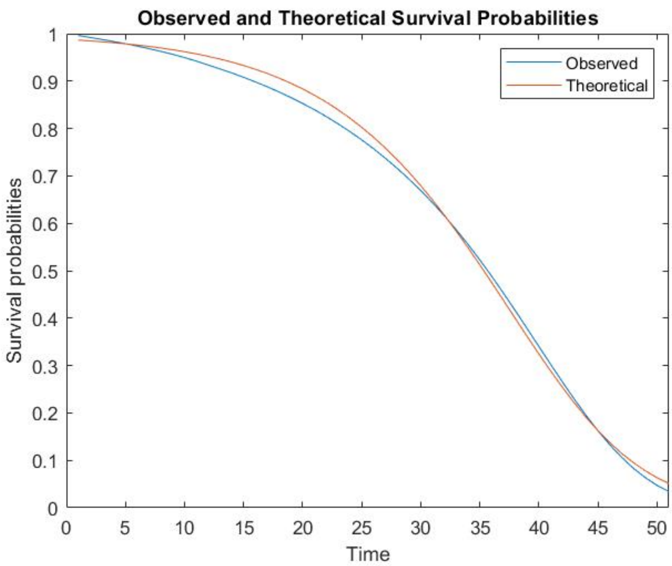

Figure 8 - Survival Probabilities: observed and theoretical (Own elaboration; observed data available at www.ons.gov.uk)

$\hat{a}$	0.1163
$\hat{\sigma}$	0.0107

Differently from the interest rate and the index dynamics, the process representing the evolution of the intensity of mortality is assumed not to change its structure under the risk neutral measure Q. This is an approximation and is necessary given the characteristics of the market, as explained in Section 2.3. Following Biffis (2005)[74], given the cohort-based approach adopted in the valuation, when we use a calibration of the process based on historical data we are mainly accounting for the random fluctuations of mortality along the expected value (a risk that can be neutralized holding a large portfolio of securities) and we are partially disregarding the possible random departures of the mortality from the expectation. This is clearly a limit given by the structure of the market and that influences the quality of the valuation, but this practical procedure is the one characterizing all the practical works I referred to for the dissertation[75].

73 For a complete analytical proof see Apicella (2017), p. 27

74 Biffis (2005), p. 454

75 Luciano, Vigna (2009), Apicella (2017), Novokreshchenova (2016), Liu (2014)

Partially counterbalancing this problem, the estimation of the parameters performed results in a survival probability function which is prudential with respect the historical data. As we can see in Figure 8 the survival probability function lies almost always above the one based on the mortality table: computing the mean of the difference between historical and estimated survival probabilities it results a negative quantity (-0.086). This partially accounts for the expansion phenomenon that otherwise wouldn't be considered in the estimation; alternatively, starting from this estimation the parameters can be modified in order to increase the expansion effect (as shown by Biffis (2005)): this kind of approach is discussed in Section 3.3.3 of the dissertation when some comparative statics experiments are implemented. Being aware of this possibility, the choice is here to maintain the estimated parameters.

3.2 Endowment embedding a GAO: contract description

The second step to be performed when evaluating a claim is the analysis of the contract structure, finding then the ideal strategy to adopt when pricing it. In our case the contract is the endowment with a guaranteed annuity option; this is an agreement which includes an endowment whose proceeds at maturity can be converted into an annuity at a fixed conversion rate.

Since the characteristics of the endowment have been introduced in Section 2.1, it is useful to focus on the description of the option embedded in the contract. The option requires some variables to be introduced.

Assume there is an endowment E_T with expiration T, whose final payoff is based on the performance of the investment in a specific index; this index is expressed by the random process S_t. Define R_T as the proceeds per unit of investment gained by the policyholder at the endowment expiration (conditioned to the fact that he is alive): $R_T = 1 + \frac{S_T - S_0}{S_0}$.

Fix $a_T(t)$ as the value of an annuity in t: this annuity pays unitary amounts, conditional on survival, at each time $T_0 < T_1 <..T_i..< T_\omega$ where ω is the maximum expected life of the individual. The payments start at $T_0 = T$ when the endowment expires, and the individual retires (we suppose to know with certainty the retirement age of the individual).

The option under analysis allows an investor to convert at a fixed rate $1/g < 1$ the proceeds from the endowment in an annuity a_T traded in the market at time T. In numerical terms, 1 unit of proceeds purchases at time T at least an amount $1/g$ of annuity. Therefore, with this agreement the owner of the contract can purchases the quantity of annuity $R_T \frac{1}{g} a_T$ investing the proceeds R_T from the contract. If the option is not exercised, the annuity is bought in the primary market at the current price, investing the amount R_T.

Let us assume the individual is rational and that the population is homogeneous in terms of death rates (i.e. there is no private information about the health state at T, so all the individuals act in the same way). The decision between the alternatives is taken at time T: it will be worth to convert the amount in annuity if and only if the cost of the annuity in the primary market is higher than the conversion factor. Thus, the option will be exercised if:

$$R_T \frac{1}{g} a_T - R_T > 0$$

Conditioning to the fact that the policy holder is alive at T, the value of the guarantee at maturity is:

$$R_T max\left[\left(\frac{a_T}{g} - 1\right), 0\right]$$

From which it follows that:

$$C(T) = \frac{1}{g} R_T max[(a_T - g), 0] \tag{3.10}$$

Then the guaranteed annuity option is essentially a call option on the annuity a_T with strike price g.

Fixed as $E_T(0)$ the value of the endowment at time of subscription (set as time 0) the total value of the contract is:

$$V(0) = E_T(0) + C(0)$$

Since the individual is alive at the time of the subscription, we have $I_{\{\tau>0\}} = 1$. Consequently, from (2.6) it is known that the value of the index linked endowment is:

$$E_T(0) = E_0^Q\left[e^{-\int_0^T r_s\, ds} e^{-\int_0^T \mu_v\, dv} R_T \middle| G_0\right]$$

Moreover from (2.8), the fair evaluation at time t of an annuity with payments equal to 1 unit of money per period is:

$$a_T(t) = \sum_{i=0}^{\omega-1} E_t^Q\left[e^{-\int_T^{T+i} r_s ds} e^{-\int_T^{T+i} \mu_v\, dv} \middle| G_t\right] \tag{3.11}$$

where ω is the maximum life expectancy of the contract owner.

The valuation of the option at time 0 will be:

$$C(0) = E_0^Q \left[e^{-\int_0^T r_s\, ds} I_{\{\tau > T\}} C(T) \middle| F_0 \right] = E_0^Q \left[e^{-\int_0^T r_s\, ds} e^{-\int_0^T \mu_v\, dv} C(T) \middle| G_0 \right] =$$

$$E_0^Q \left[e^{-\int_0^T r_s\, ds} e^{-\int_0^T \mu_v\, dv} \frac{1}{g} R_T (a_T - g)^+ \middle| G_0 \right] \tag{3.12}$$

From those formulas it is straightforward to notice which are the main risks connected to the contract: the market and the mortality variability.
The better the index is expected to perform (higher R_T), the higher the value of the endowment, and the lower the interest rate r_t the lower the intertemporal discount and so the higher the value of the endowment. Moreover, the lower the mortality intensity μ_v, the higher the probability of surviving until the contract expiration and the higher the endowment value.
The option depends on the same factors. For what concerns mortality risk, it influences both the material possibility to exercise the option (if death occurs before T the option cannot be exercised), and the expected value of the annuity (a value which determines the exercise of the option). Indeed, the market price of the annuity depends on the expected residual life of the individual in T, which depends on the evolution of the mortality intensity. Therefore, reduction of mortality is a factor which increases significantly the value of the contract. Moreover, the exercise of the option is also influenced by the economic conditions of the variables on which annuity market depends. The annuity $a_T(t)$, given its payoff structure, is connected to the interest rate variability: when the annuity contract gives the right to receive a guaranteed payment, this is negatively affected by the interest rate since its discounted value decreases the higher the rate is. Then the lower the interest rates, the higher the difference $a_T - g$ and so the more the option is in the money. Finally, it is worth emphasizing that, even if it does not influence directly the option exercise, the value of this part of the contract is proportional to the proceeds accumulated by the endowment and so it is also influenced one more time by the index performances: the better the index performs the higher the value of the option.
Despite for the number of risks to be considered, the main issue for pricing is the length of the contract. Indeed, those agreements last for many periods, and so require a long-term forecast for the annuity risk factors, a projection that is truly difficult to be performed for series as the interest rates and mortality.

Erroneous forecasts on interest rates and mortality improvements were at the basis of the mispricing of the products in UK[76]; many of the contracts were indeed singed during ‘70s, when the level of the interest rates was higher than both the decades before and were supposed not to come back to the levels of ‘50s. Therefore, the option embedded in the endowment contracts

76 For additional details: Boyle et al. (2003) Chapter 1-2

was deeply out of the money when issued; however, after a decade the interest rates started falling, and with them the options acquired value until expiration: in the end of '90s and in the first years of 2000, the interest rates were at the '50s level and so the insured found convenient to exercise the option. Furthermore, the liabilities accumulation due to the cost of the annuities was increased by the reduction in mortality for the cohorts who signed the contract. This mix of factors made Equitable Life, one of the most important insurance companies in UK, nearly collapse; many savers were penalised in that moment and some of them started a lawsuit which ended up in 2018 with a reimbursement of £1,8 billion to be shared among 261 thousand people. The people who suffered losses were however nearly 1 million[77].

3.3 Contract valuation

Once the processes characteristics and the structure of the contract is analysed, its valuation can be performed. The valuation procedure allows us to evidence which are the main features (in terms of strengths and weaknesses) of the model introduced in the Sections 2.1-2.2. The specific characteristics of the agreement (for example the length of the contract and the conversion ratio) are the result of reasonable assumptions that have the aim to make the product similar to the one traded in the market. The structure of the contract allows us to split our analysis in two parts: the first regarding the endowment and the second focused on the option, components that are then summed to obtain the total value of the contract. Those two components can be treated separately, since even if the option value depends on the endowment proceeds the converse is not true. Consequently, it is convenient to start from the endowment valuation and shift to the option only in a second moment.

3.3.1 Endowment valuation

The endowment contract is settled to start at date 0, when the individual is aged 50 and to expire in 15 years at time T, when the policyholder is 65. The final proceeds are linked to the performance of the index S_t: thus, each unit invested in the endowment will reimburse the value $R_T = 1 + \frac{S_T - S_0}{S_0}$ units at time T if the individual is alive. The valuation of this product can be performed applying the properties of the affine structure processes illustrated and commented in Section 3.1. In this specific case, the endowment value equals to:

$$E_T(0) = E^Q\left[e^{-\int_0^{15} r_s\,ds} e^{-\int_0^{15} \mu_v\,dv} R_{15} \middle| G_0\right]$$

Considering that μ_t is independent from r_t the expectation can be evaluated as:

77 https://www.theguardian.com/money/2018/jun/15/equitable-life-to-shut-down-with-surprise-18bn-policyholder-windfall

$$E^Q\left[e^{-\int_0^{15} r_s\,ds}e^{-\int_0^{15}\mu_v\,dv}R_{15}\middle|G_0\right] = E^Q\left[e^{-\int_0^{15}\mu_s\,ds}\middle|G_0^{\mu}\right]\,E^Q\left[e^{-\int_0^{15} r_s\,ds}R_{15}\middle|G_0^{r,S}\right]$$

We thus analyse the two terms separately, focusing first on the evaluation of the expectation involving the intensity of mortality. Since the Feller process is affine and in Section 3.1.3 we assumed the parameters not to change between the real-world and the risk neutral probability measure, the expectation in the formula results in the close form (3.5) where $\alpha(t,T)$ and $\beta(t,T)$ are defined as in (3.6). Once fixed $\mu_0 = -ln(p_{50,0})$ the computation is straightforward:

$$E^Q\left[e^{-\int_0^{15}\mu_s\,ds}\middle|G_0^{\mu}\right] = e^{\alpha(0,15)+\beta(0\,,15)\mu_0}$$

It is then useful to focus on the second part of the formula, involving the expectation of the discounted index return. Since the value S_0 is a known information (deterministic) the expectation to be focused on involves the future value of the index S_T. As explained in Section 3.1.2 under Q the process takes the form:

$$dS_t = r_t S_t\,dt + \sigma S_t\,dW_t^2$$

Then, to evaluate the discounted expectation of the index performance we exploit the Monte Carlo method simulating first the underlying performance and then discounting it back by the risk-free rate (which is itself stochastic). We use Monte Carlo method because the process S_T and r_t are independent (not influenced by the same variable), and the formula (2.11) is exploited only when considering the discount factor $e^{-\int_0^{15} r_s\,ds}$ alone, as it will be discussed in few lines. We consider that:

$$S_T = S_0 e^{\left(\int_0^{15} r_s\,ds - \frac{1}{2}15\sigma^2 + \sigma W_{15}^2\right)}$$

The starting point is to compute the integral $\int_0^{15} r_s\,ds$. To do so, it is useful to simulate M=50000 paths of the process r_s in the time interval $t \in [0,15]$ divided in $N = 3750$ intervals of equal length Δt: given that the market year has approximately 250 days we are taking a daily based simulation of the interest rates path along the 15 years. In order to perform this simulation, there are two options: we can adopt the Euler Approximation scheme (based on the discretization of the process) or it can be exploited the fact that the CIR process is distributed as a non-centred Chi-Squared. In this last case the simulation can be performed extracting data directly from this distribution. This second method is implementable directly in Matlab by means of the function "`cirpath`" and it is considered less time consuming and more precise than the discretization scheme: therefore, we choose to implement it.

In Figure 9 we plot the historical time series of the interest rates for the first 12 years (the dataset used for the process calibration in Section 3.1.1 is characterized by 2960 daily observations) while then for the following 15 years (3750 observations) the plot shows some of the simulations performed in order to compute numerically the stochastic integral. The solid line represents the historical time series, while the set of dashed lines some of the simulations of the interest rate series.

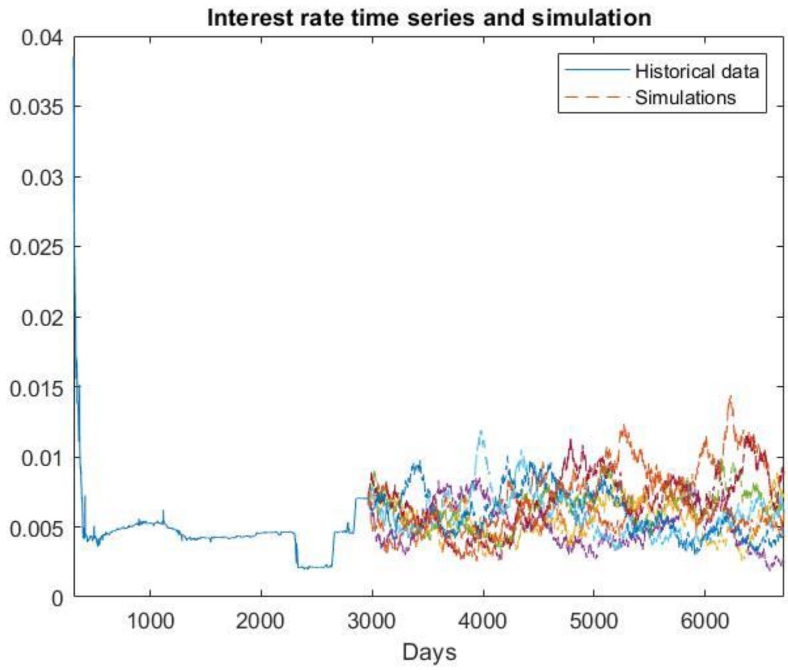

Figure 9 - Interest Rates Time Series and Simulation (Own elaboration; historical data source: Thomson Reuters)

Once we dispose of the M simulations, the integral for each of them is computed numerically by means of the trapezoidal rule; in particular:

$$\int_0^{15} r_s\, ds \approx \frac{\Delta t}{2}\left[r_0 + r_N + \sum_{i=1}^{N-1} r_i\right] \tag{3.13}$$

Once computed the integral there is another stochastic component in the value of the process S_T, the Brownian Motion shock at the end of the period, W_T^2. As well as before, to simulate this random variable there are two possibilities; the first is based on a discretization of the shock dW_t occurring in the time interval $t \in [0,15]$ and the relation $W_T^2 = \sum_{t=1}^{3750} dW_t^2$. We thus exploit the fact that in a discretized fashion $dW_t^2 = \varepsilon_{t,i}\sqrt{\Delta t}$ and $\varepsilon_{t,i} \sim N(0,1)$ and therefore, simulating the variable $\varepsilon_{t,i}$, W_T^2 is retrieved. The second way exploits the fact that we are interested in the final value W_T^2 and not on the path that produced it; so, given that $W_T^2 \sim N(0,T)$ we can extract its values straight sampling from this distribution. Once retrieved the value for the integral and for the Brownian Motion variable in the different scenarios, M values for the index S_T at the expiration of the contract are computed. When evaluating the money market discount, it can be exploited the affine structure of the interest rate process the expectation has the form of (3.2) and the variables $A(t,T)$ and $B(t,T)$ are expressed as in (3.3). Thus, given r_0 the value of the interest rate at the starting time of the contract (the last one available in our case), the valuation is performed:

$$E^Q\left[e^{-\int_0^{15} r_s\,ds}\middle|G_0^r\right] = A(0{,}15)e^{-B(0{,}15)r_0}$$

The Matlab code for the evaluation of the endowment is presented in Appendix E.

The method presented allows to compute the expectations reducing the use of the simulations (the only one is required for the numerical integration (3.13)) since they are based on a close formula determined by the parameters characterizing the process and by the actual value of the process: that is why it is time saving and more precise. To show the convenience in exploiting the affine structure of the processes and to have a proof of the convergence of the expressions we compute the expectations with an alternative method that doesn't rely on the affine structure proprieties of the processes involved in the model. In order to perform this comparison we compute the expectations in (2.6) relying on Monte Carlo method, exploiting for the numerical computation of the integrals the trapezoidal rule (3.13). Thus, M=50000 simulations of the path for the underlying index, the mortality intensity, and the interest rates are taken and each group of them constitutes a specific scenario: in this way it is possible to build up 50000 scenarios. When we simulate those different paths for each variable, different techniques are used. For the index, it can be exploited the procedure and the results adopted before, so with the computation of the closed form solution for each simulation (indeed we are interested only on the final value of the index process). For what concerns the other two processes the aim is to compute the stochastic integral for each path mentioned above using (3.13). As before the time interval $t \in [0{,}15]$ divided in $N = 3750$ intervals of equal length Δt. For the interest rates the mechanism is analogue to the one explained for $\int_0^{15} r_s\,ds$ (exploiting the non-centred Chi-Squared distribution of the process); when dealing with the Feller process there is no function which allows to simulate directly the intensity of mortality evolution in the future. Therefore, the method used is the Euler approximation, based on the discretization of the process:

$$\mu_{t,i} = \mu_{t-1,i} + a\mu_{t-1,i}\Delta t + \sigma\sqrt{|\mu_{t-1,i}|}\varepsilon_t\sqrt{\Delta t} \qquad (3.14)$$

where $\mu_{t,i}$ is the mortality intensity at time t and in scenario i. The conditions mentioned in Section 3.1.3 should guarantee that the intensity of mortality is increasing. To ensure the existence of the process along all the simulations' paths, we take the absolute value of the mortality intensity under the square root; in this way the intensity should be always positive, provided that the starting value μ_0 is positive. Once fixed $\mu_0 = -ln(p_{x,0})$, the simulations are made by recursion given $\varepsilon_{t,i} \sim N(0,1)$. A plot of part of them is shown in Figure 10. For each simulation we compute:

$$\int_0^{15} \mu_s \, ds \approx \frac{\Delta t}{2}\left[\mu_0 + \mu_N + \sum_{i=1}^{N-1} \mu_i\right]$$

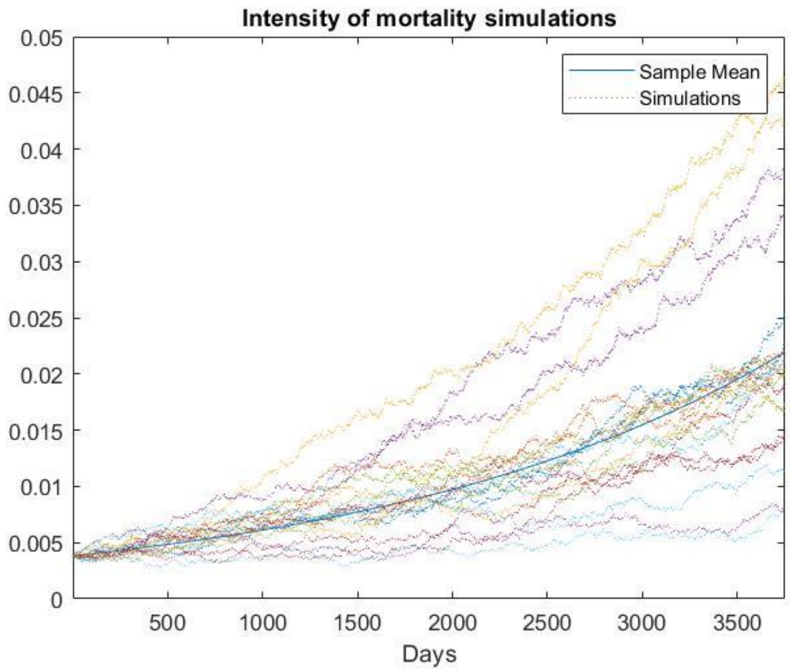

Figure 10 - Mortality Intensity Simulations (Own elaboration)

For each one of the M scenarios, the values useful for the endowment computation can be retrieved; given the independence between the intensity of mortality and the interest rates we can compute separately the two discount factors, and then multiply each other to retrieve the final endowment value. The sample mean of the mortality discount $e^{-\int_0^{15} \mu_{s,i} \, ds}$ and of the discounted value of the index $e^{-\int_0^{15} r_{\,s,i} \, ds} R_{15,i}$ are taken; in this way, an approximation for the endowment value of $E^Q\left[e^{-\int_0^{15} \mu_s \, ds}\right] E^Q\left[e^{-\int_0^{15} r_s \, ds} R_{15}\right]$ is obtained. The Matlab code for the implementation of this procedure is shown as well as before in Appendix E.

The results of the two procedures are presented in the table which shows the realizations of the integrals' expected values and of the final endowment valuation both considering the index linked case and the non-index linked one. To show the convergence of the numerical procedure we compute those values basing the numerical integration on a subset of the 50000 simulations, taking only a half of them (M=25000). The last row compares the results achieved for the two different methods and with the different number of simulations in the index linked endowment case.

	Affine Approach	Numerical Integration Approach (M=25000)	Numerical Integration Approach (M=50000)
$E^Q\left[e^{-\int_0^{15}\mu_s\,ds}\middle\|G_0^{\mu}\right]$	0.8561	0.8581	0.8575
$E^Q\left[e^{-\int_0^{15}r_s\,ds}R_{15}\middle\|G_0^{r,S}\right]$	0.9928	0.9936	0.9931
$E_T(0)$	0.8499	0.8527	0.8516
$E^Q\left[e^{-\int_0^{15}r_s\,ds}e^{-\int_0^{15}\mu_v\,dv}\middle\|G_0\right]$	0.7805	0.7826	0.7820
% DIFFERENCE		0.32%	0.20%

This difference between the two methods is almost negligible and can be motivated with the approximation given by the numerical method; indeed, the difference reduces when increasing the number of simulations. As it can be noticed by the results, a significant part of the discount is carried by the mortality intensity. The interpretation of the impact of the discount is easier when the endowment is non index linked (the result with the index linked depends on the expected market performances). Indeed, when considering the benefit deterministic case, the cost of the endowment corresponds to the price paid to have a reimbursement of 1 unit at expiration, conditioned to the fact of being alive. In this case the discount corresponds to the ~21-22% of the total sum.

3.3.2 GAO pricing

The second part of the contract, the option, is assumed to allow the individual to convert at the endowment expiration the proceeds from it into an annuity paying with yearly frequency a fraction $1/g = 1/12$ of the total benefit accumulated with the previous contract. The value of g, as for the length of the endowment contract and the other data, is presented as an assumption considering that it represents a plausible value for a real contract.

This kind of option has a value which has been discussed in Section 3.2 and is exposed in formulas (3.10)-(3.12). To price the product, it is useful to start from the annuity evaluation. The annuity which we are referring to is valuated at time 0, when the individual is aged 50, but the liquidation phase will happen at time 15, when he is aged 65, and is set to distribute one unit of money for each period from 15 on. A possible approach to valuate the contract at time

0, so $a_{15}(0)$, is to compute its expected value at time 15 (thus valuating a_{15}) and then discount it to time 0. The discount applied must account for both the intertemporal value of the money and the intensity of mortality rates (the option will be exercised if and only if the individual will be alive in T=15).

To compute the value of a_{15} we refer to Sections 2.1.3 and 3.2, by which the fair value of this contract corresponds to a sum of 44 survival benefits starting at 15 and ending in 59 with yearly frequency (3.11): the individual is assumed to be dead for sure at the age of 110. The interest rates and the intensity of mortality are thus the main variables considered and must be simulated and projected for the following 15 years. Given the hypothesis of Section 2.2 satisfied, as a result of the general formula (2.11) the expectation in (3.11) depends only on the value of the processes at the moment in which the contract starts, so at T=15. The values of interest are thus: r_{15}, μ_{15}. This property simplifies significantly the computations, since it allows to avoid simulating the path of r_t and μ_t for the 45 years following the previous T=15 (as it would be necessary with the Monte Carlo simulation). Both the simulations have been performed in a way analogous to the one exposed for the integrals numerical computation, so exploiting the Matlab function "`cirpath`" for the interest rate and the Euler approximation (3.14) for the mortality intensity. Once obtained M=50000 simulations for each process, the simulated annuity values can be priced exploiting the independency and the affinity of the processes, thus coupling each simulation of the interest rate process with one for the intensity of mortality. In this way we have 50000 scenarios each one originating one annuity value, computed with the following formula:

$$a_T = \sum_{i=0}^{45-1} E^Q\left[e^{-\int_{15}^{15+i} r_s ds} e^{-\int_{15}^{15+i} \mu_v\, dv} \middle| G_0\right]$$
$$= \sum_{i=0}^{45-1} A(15,15+i)e^{-B(15,15+i)r_{15}} \sum_{i=0}^{45-1} e^{\beta(15,15+i)\mu_{15}}$$

Given those simulated values of the annuity, for each one the option value $max[(a(T)-g),0]$ is computed. In this way we compute M values of the option involving the annuity. However, the GAO value doesn't depend only on the option valuated above: given the exercise of the right by the option holder ($a(T)-g>0$) the contract's value is proportional to the proceeds accumulated by the endowment and its value must be discounted back from time 15 to 0. Then, we act in the following way: first we exploit the simulations of the index value previously used for the endowment valuation and we multiply each index return simulated by each option value; then we discount back this partial value at time zero (exploiting the affinity of the process or numerical integration used for the endowment). Then, M values for the GAO are computed: taking the sample mean the pricing of the option should converge to its real value. The implementation with the code in Matlab is presented in Appendix E. The results can be resumed in the following table:

	Endowment	GAO	% Incidence on value	Total Contract
Index Linked	0.8499	0.0801	8.61%	0.9300
Non-Index Linked	0.7805	0.0747	8.73%	0.8552

The value of the guaranteed annuity option is: 0.0801; the price of the option is relevant in numerical terms: indeed, it represents the ~ 8.61 % of the whole contract value. The weight doesn't change much when considering the deterministic benefit contract so with a reimbursement of the fraction $1/g$ per year (~8.73). This result is reasonable, when considering for the forecast the intensity of mortality correct. When the expectation of this value is assumed to converge to the real value, a main role for the value of the option is played by the life expectancy and by the long-term path of the interest rates.

For what concerns the life expectancy issue, it can be analysed considering the age of the policyholder when the contract starts and the last updates for the life expectation for the UK inhabitants[78]. Indeed, at the age of 65 a man should expect to stay alive on average for the following 18.8 years, so almost until the age of 84. This average should be not so far from the one implied by the mortality intensity used in the analysis. Surviving almost other 19 years would allow the policyholder to recover the sum invested, also when accounting for the discount provided by the value of money.

It is then worth to analyse the interest rate influence on the GAO price. The option will be more valuable the lower the interest rate at the vesting date. It can be a useful experiment to compute which is the interest rate process value at time 15 which makes the option at the money, so the individual indifferent between exercising or not his right to convert the proceeds of the endowment in an annuity at the conversion rate g. This procedure is carried out taking as the mortality intensity value the mean of the M mortality intensities simulated for pricing originally the option and computing the interest rate discount through formula (3.3). The Matlab code can be found in Appendix E and exploits the command "`fzero`" which computes the interest rate

78https://www.ons.gov.uk/peoplepopulationandcommunity/birthsdeathsandmarriages/lifeexpectancies/bulletins/lifeexpectancyatbirthandatage65bylocalareasinenglandandwales/2015-11-04#national-life-expectancy-at-age-65 which implies a discount such that the value of the function “`AnnOpt`” is null. The result is that the interest rate which takes the option value at 0 is 0.0033. This value is far lower than the long term convergency value of the interest rate, which is 0.0061, so the option in expectation should be out of the money. However, the uncertainty embedded in the stochastic process (resumed by its diffusion part) doesn’t guarantee that the value of the process will be at the long-term average at 15. With a precise probability it can be that the interest rate is lower than 0.0033: that is what makes the option valuable. The probability for the interest rates to be at that level or lower are computed through the simulations and are thus embedded in the valuation of the option. The relevant difference between the interest rate and the long-term average makes those probabilities quite low, and so overall the interest rate is a factor which depresses the value of the option.

3.3.3 Comparative analysis

The aim of this section is to briefly analyse which are the main factors influencing the GAO value, despite of the interest rate dynamics (already analysed in the previous section): we refer for example to the conversion rate, but the focus will be in particular on the intensity of mortality process.

When setting the contracts’ characteristics an important determinant for the contract pricing is the conversion rate, indexed in Section 3.2 as g. Indeed, following (3.10) it represents the strike of the call and the overall contract price is also proportional to its inverse: it is indeed the scaling factor that transforms the endowment proceeds in the annuity periodic benefit. For example, when the conversion rate is set at the value of 10, the annuity that is purchased when exercising the option distributes each period the 10% of the total proceeds accumulated by the endowment, when it is 12 the 8%: that is why it is also called cash-value ratio.

The conversion rate influence on the option value is tested computing the GAO price for the index linked case when g variates from 9 to 14 with intervals of 0.2. This sorts out 26 values for the GAO, and they’re plot in Figure 11.

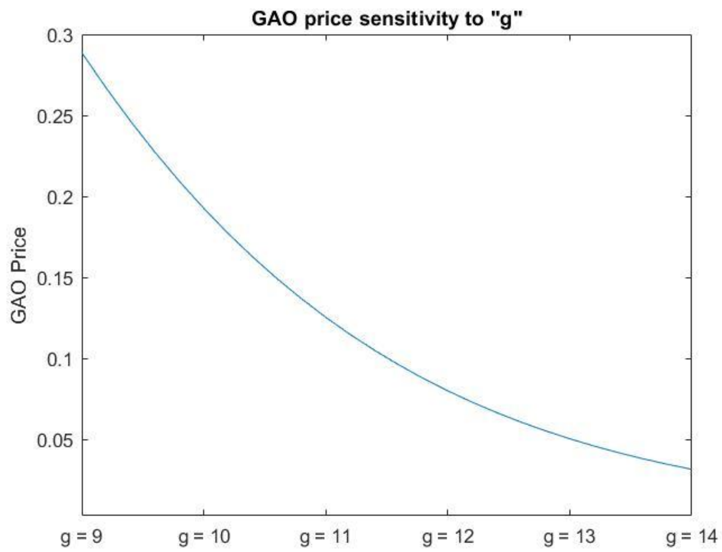

Figure 11- GAO Sensitivity to Conversion Ratio (Own elaboration)

The results are not surprising, since the value of the conversion rate deeply influences the option, whose price ranges from 0.0316 to 0.2887.

The model allows also to show the impact of the mortality intensity on the option price, and thus the importance of valuating in a precise way this quantity. To reach this aim, it can be useful to perform a comparative analysis exploiting the representation of the mortality intensity as a stochastic process. We operate in two ways:

- testing the variability of the GAO price under different mortality scenarios, embedding a variation of the mortality around the mean expected value;
- evaluating the influence of the expansion phenomenon on the contract's price, so quantifying the weight of systematic departures of mortality from the mean value.

The first analysis exploits the 50000 simulated paths for the mortality intensity, used for the valuation of the GAO. Those scenarios have been split in subsamples of 12500 scenarios, characterized by an overall decreasing intensity of mortality. In this way the discount due to mortality risk has a different weight in each subsample. The plot in Figure 12 shows the different yearly discounts $e^{-\int_0^{15} \mu_s \, ds}$ for a couple of simulations each one belonging to a different subsample: as we can see the discount gets overall heavier through time but with different speed, due to the different intensity for each scenario (the higher the intensity of mortality, the faster the variable goes to zero).

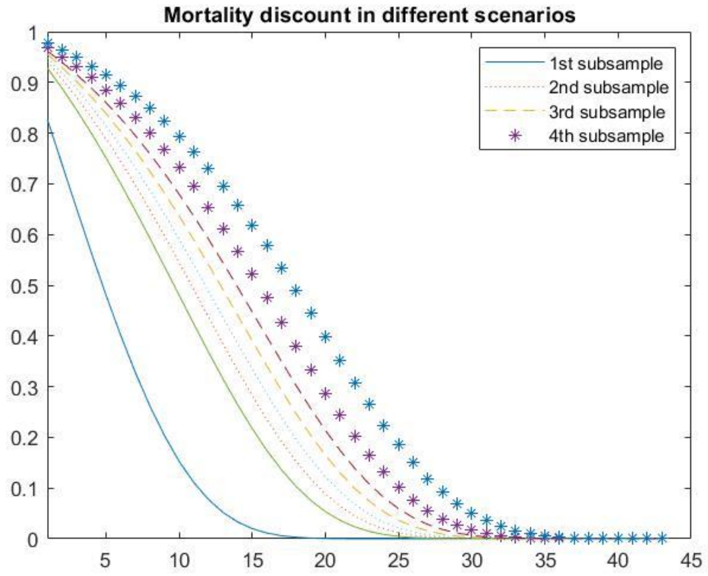

Figure 12 - Mortality discount path in different scenarios (Own elaboration)

For each subsample we operate the same valuation procedure performed with the 50000 simulations scenario. To do so we have also to select a part of the simulated interest rates, creating a subsample from the M simulations performed for the first valuation; thus we select 12500 simulations that are invariant in each mortality scenario: this makes sense since the mortality and the interest rates values are supposed to be independent one on each other and in this way the "ceteris paribus condition" is maintained. The code to analyse the impact of mortality on the option price is shown in Appendix E as well as for the others.

The results are shown in the table:

	1st Subsample	2nd Subsample	3rd Subsample	4th Subsample
Index Linked GAO	~0	~0	0.0346	0.2909
Non-Index Linked GAO	~0	~0	0.0317	0.2671

It is easy to notice how in the first two scenarios the option is expected to be almost always out of the money and that's why its value can be approximated to zero (the probabilities of the option to be exercised are negligible). The contract starts to be valuable considering the third subsample of the mortality intensity, while in the fourth one the value increases significantly, also when compared with the original endowment value. This is the impact of the mortality fluctuations under the mean on the value of the option: this kind of risk is however overall differentiable when holding a high number of contracts, since the underlying process characterizing each different policyholder mortality path is the same and is assumed to be forecasted correctly.

A different kind of analysis is performed in the second comparative statics experiment on mortality: in this case we evaluate the weight of the expansion phenomenon, a risk which cannot be differentiated since it regards the whole population. This kind of risk is mentioned both in Section 1.2.2 and in Section 3.3.1 when referring to the problems related to the presence of a risk neutral probability measure for the intensity of mortality process. The expansion regards the detachment of mortality from expected values to lower one and can be quantified when changing the parameters characterizing intensity of mortality process: intuitively, they have to result in a survival probability function which lies above the historical one, representing an overall increase decrease in mortality. It’s thus useful to recall the mortality intensity process structure:

$$d\mu_t = a\mu_t \, dt + \sigma\sqrt{\mu_t} \, dW_t^3$$

The aim of the analysis is to evaluate the contract price changing the values of a and σ to see which the effect of a reduction in mortality can be. Thus, we simulate two different scenarios:

	Drift Parameter: a	Diffusion Parameter: σ
Theoretical Estimates	0.1163	0.0107
First Scenario	0.1112	0.0095
Second Scenario	0.1078	0.0090

In the First Scenario the decrease in the drift of mortality intensity is of ∼4% while in the Second it is more significant and approaches ∼7%. The reduction in the volatility is instead of ∼11% and ∼15% respectively. The changes are higher than the one mentioned in Section 1.2.2 (the 2% improvement in the mortality rate) but the parameters are overall useful for the comparative static experiment to highlight the different sensitivity of the two parts of the contract to the mortality rate improvements. Figures 13 and 14 below show the Survival probability function and the sample mean of the simulated mortality intensities for the three different processes: as we can see to a lower a corresponds a survival probability which lies above the others and on average a lower intensity of mortality; on the other hand the volatility is reduced to allow for the convergency of the survival probabilities at late ages (in this way the assumption of life ending up at 110 is still reasonable).

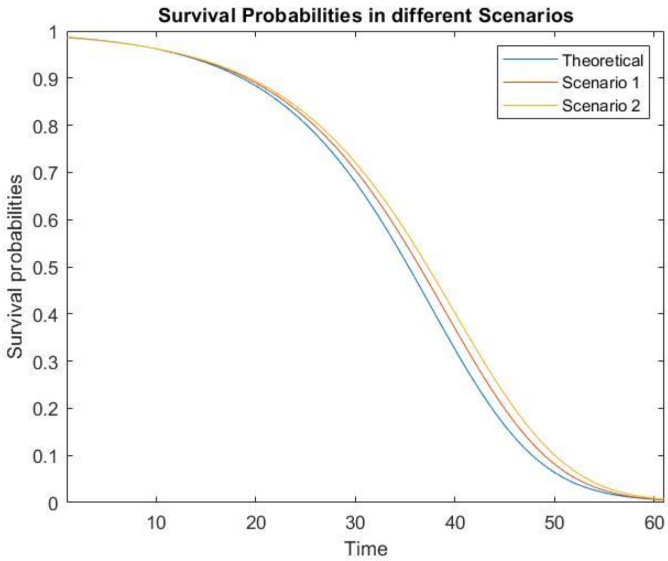

Figure 13 - Survival probability functions in different scenarios (Own elaboration)

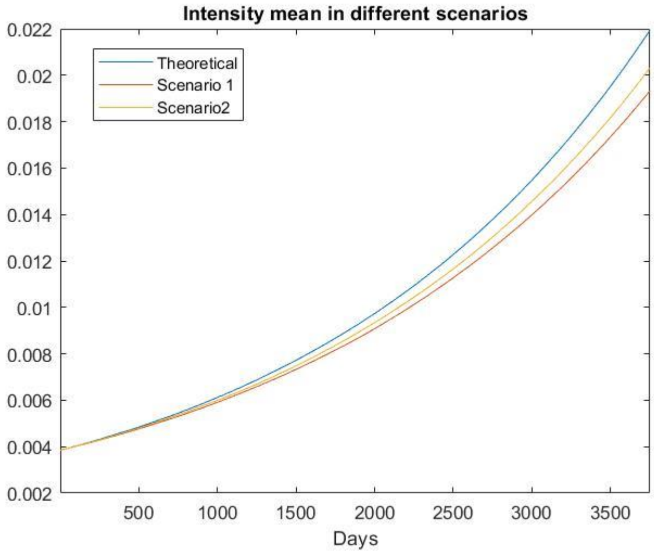

Figure 14 - Mortality intensity under different scenarios (Own elaboration)

The following table shows instead the results of the valuation (performed with the method commented in Sections 3.3.1 and 3.3.2) both for the price of the endowment and for the GAO when the contract is unit linked:

	Theoretical Measure	Scenario 1	Scenario 2
Endowment	0.8499	0.8561	0.8600
GAO	0.0801	0.0946	0.1007
Total Contract	0.9300	0.9507	0.9607
% Change in Endowment Value		0.73%	1.19%
% Change in GAO Value		18.10%	25.72%
% Change in Total Value		2.26%	3.30%
%Incidence on the Value	8.61%	9.95%	10.48%

As it can be noticed by the estimations, the expansion phenomenon influences the value of the endowment not in a really relevant way: the changes are not higher than 1.2%. The change in value is indeed related to the life expectancy discrepancy in the different scenarios: the cumulated variation of the survival probabilities function in the first 15 years accounts indeed for the 0.29% and the 0.24% of the total cumulated variation in the first and in the second scenarios respectively. This result is intuitively interpreted also referring to Figure 12, looking at the survival probabilities in the first 15 years after the contract has been signed: they do not differ much from the theoretical estimation under both the simulated scenarios. Therefore, the effect of expansion is essentially concentrated in the years following the 15th: differences in the survival probabilities are more marked when the individual becomes older. It is then easily understandable why GAO is much more sensitive to the changes in life expectancy and so to the improvements in mortality: the relative change is of 18.10%in the first scenario and of 25.72% in the second one. This variation in value rebalances also the weight of the GAO in the contract's total value, a weight which results to be overall increased (shifting to 9.95% and 10.48% respectively). The change in the process dynamics due to expansion causes overall a relevant increase in the contract's price, that variates between 2% and 3%. That is why to account for the random departure from the mean of the mortality process is so important when valuating contracts which are characterized by a long-term expiration. As shown in the example, with the stochastic modelling this operation can be performed leading to informative results.

4. Conclusions

The aim of the thesis is to show how the intensity-based models, primarily created for the valuation of defaultable bonds, can be exploited in order to price the life insurance contracts introducing the comparison between the default and the mortality risks. The calibration of the risk of mortality by means of death intensity allows to give to the valuation a stochastic mortality framework which is more flexible than the traditional way of accounting for it through deterministic tables. This kind of modelling indeed allows for scenario simulations that involve also the variability of the mortality risk and thus makes the actuarial valuation more accurate. This is a benefit for insurance companies both in terms of balance sheet stability and of business profitability: as shown in Chapter 1, which analyses the trends of the insurance market in the last years, this is really important for the insurance companies. Moreover, the valuation is compliant with the main requirements of the regulator. The gain represented by the introduction of the stochastic modelling is not penalizing the valuation with a relevant increase in the complexity of the computations: indeed, this way of modelling mortality presents some characteristics that allow the pricing process to be efficient. The model must fulfil some conditions on the structure of the processes representing mortality and the other stochastic factors; these conditions are introduced with the technical explanation of the model in Chapter 2. Another advantage of this modelling approach is represented by its own flexibility in terms of application to different products: many kinds of contract, from the primitive to the structured and more complex ones can be priced with this procedure.

To practically present the two main advantages mentioned above, an application of the model is presented pricing an index linked endowment embedding a guaranteed annuity option. Section 3 focuses on this empirical application, displaying a way of implementing the valuation that can be followed for any other kind of product. The computational tractability is presented through the pricing procedure, while the flexibility of the model is shown when analysing the impact of mortality risk on the contract, performing thus some comparative statics analysis: this exercise shows also the necessity to pay a relevant attention in calibrating mortality risk, in particular when referring to long term expiration contracts.

The model implementation presented is quite straightforward also because it benefits of some assumptions made concerning the relations between the different factors: the shocks connected to the index influencing the endowment value (and consequently the option) are indeed supposed to be not explicitly linked to the variations characterizing the interest rates and are also assumed to be homoscedastic. A relatively less strong assumption is the one regarding the absence of dependence between the mortality intensity and the interest rate time series. All those three aspects can be relaxed making more complex the structure of the processes: for example, we can include the presence of the stochastic volatility in the index performance, or we can make more complex the shocks' structure characterizing the interest rates performance, linking it with the shocks influencing the index or the mortality intensity. Those improvements have already been performed in different works, but never implemented all together, since the complexity increases, and the affinity of the processes cannot be exploited. The application of the model shows in a complete way its main strengths but also its weakness to link its computational tractability to particular assumptions.

Concluding, the dissertation offers a complete overview about the application of intensity-based models to life assurance contracts' pricing. It implements an analysis of the main points which make it suitable for this aim, looking also at the present conditions of the market and at the compliance with regulation of the method.

Appendices

Appendix A

In this Appendix, we place ourselves in the framework introduced in Section 2.1. The background filtration is G_t, while H_t is the one generated by the history of default (by the indicator J_t, which jumps to one when default happens). A random time τ is called doubly stochastic with respect to the background filtration G_t if it admits the (G_t)-conditional intensity (γ_t) and if $\Gamma_t = \int_0^t \gamma_s \, ds$, defined as the cumulative intensity process, is strictly increasing. Moreover, for all t, it must hold that:

$$P(\tau \leq t | G_\infty) = P(\tau \leq t | G_t)$$

So, given $s > t$ the information (G_s) generated by the state variable process does not contain extra information with respect to the one generated up to t, (G_t), for predicting the probability that the default occurs up to time t. The usual way in which a random time is constructed is taking a standard exponentially distributed random variable E independent of G_∞, so for which it holds $P(E \leq t | G_\infty) = 1 - e^{-t}$ for all $t > 0$. Take the intensity and the cumulative hazard rate defined as before. The random time τ is then built up as:

$$\tau := \Gamma^{-1}(E) = inf[t \geq 0 \colon \Gamma_t \geq E]$$

It can be shown[79] that this random time satisfies all the conditions set in the definition.

Appendix B

This appendix shows the main steps which allow to transform the expressions (2.1) and (2.2) in expectations conditioned to the smaller filtration $G_t \subset F_t$. To do so the assumptions made in the Section 2.1 are assumed to be valid. In addition, we refer to the probability space $(\Omega, G_t, (G_t), Q)$ adopting thus as probability measure of reference Q, defined as the risk neutral measure for the financial markets. To start it is useful to recall the two expressions:

- Vulnerable claim: $E^Q\left[e^{-\int_t^T r_s\, ds} X \, I_{\{\tau > T\}} \middle| F_t\right] = I_{\{\tau > t\}} E^Q\left[e^{-\int_t^T R_s\, ds} X \middle| G_t\right]$
- Recovery payment:

$$E^Q\left[\, I_{\{\tau>t\}} e^{-\int_t^\tau r_s\, ds} Z_\tau \, I_{\{\tau \leq T\}} \middle| F_t\right] = I_{\{\tau>t\}} E^Q\left[\int_t^T Z_s \gamma_s e^{-\int_t^s R_u\, du}\, ds \middle| G_t\right]$$

79 McNeil, A., Frey, R., Embrechts, P., (2005), p. 398

To be proved, those relations require some preliminary knowledge about the conditional expectations of the random variables[80]. The Lemmas are proved under the risk neutral framework, but the properties hold also when considering the physical (real world) probability.

1.B For every F_t-measurable random variable X there is some G_t-measurable random variable $\tilde{X}$ such that $X\, I_{\{\tau>t\}} = \tilde{X}\, I_{\{\tau>t\}}$.

From a conceptual point of view, given G_t the filtration expressing the information regarding the economic variables, before default ($I_{\{\tau>t\}}$) all the information is carried by this background filtration.

2.B For every integrable random variable X it follows that:

$$E^Q(X\, I_{\{\tau>t\}}|F_t) = I_{\{\tau>t\}} \frac{E^Q(X\, I_{\{\tau>t\}}|G_t)}{P(\tau > t|G_t)}$$

Proof. $E^Q(X\, I_{\{\tau>t\}}|F_t)$ is F_t-measurable and zero on $\tau \leq t$. For the lemma 1.B, there is a G_t-measurable random variable $\tilde{Z}$ such that it holds: $E^Q(X\, I_{\{\tau>t\}}|F_t) = I_{\{\tau>t\}}\tilde{Z}$. By the law of iterated expectations, taking the conditional expectation with respect to G_t and given $E^Q(\, I_{\{\tau>t\}}|G_t) = P(\tau > t|G_t)$ it is obtained that: $E^Q(X\, I_{\{\tau>t\}}|G_t) = P(\tau > t|G_t)\tilde{Z}$. Thus $\tilde{Z}$ is defined as: $\tilde{Z} = \frac{E^Q(X\, I_{\{\tau>t\}}|G_t)}{P(\tau>t|G_t)}$ and the condition is proved.

Starting from this evidence it follows an important consequence.

3.B We fix $s > t$. If $\tilde{X}$ is integrable and G_s-measurable, it follows that:
$E^Q(\tilde{X}\, I_{\{\tau>s\}}|F_t) = I_{\{\tau>t\}}\, E^Q(e^{-(\Gamma_s - \Gamma_t)}\tilde{X}|G_t)$

Proof. Let $X \coloneqq \tilde{X}\, I_{\{\tau>s\}}$. Since $s > t$, $X = I_{\{\tau>t\}}X$ and thus:

$$E^Q\big(\tilde{X}\, I_{\{\tau>s\}}\big|F_t\big) = E^Q\big(X\, I_{\{\tau>t\}}\big|F_t\big) = I_{\{\tau>t\}} \frac{E^Q\big(\tilde{X}\, I_{\{\tau>s\}}\big|G_t\big)}{P(\tau > t|G_t)}$$

Now we recall $P(\tau > t|G_t) = e^{-\Gamma_t}$ (definition cumulative hazard rate). Moreover, it can be shown that by the law of iterated expectations, recalling that $\tilde{X}$ is G_s-measurable:

$$E^Q\big(\tilde{X}\, I_{\{\tau>s\}}\big|G_t\big) = E^Q\big(\tilde{X}P(\tau > s|G_s)\big|G_t\big) = E^Q\big(\tilde{X}e^{-\Gamma_s}\big|G_t\big)$$

Thus, in this way, substituting the terms in the expression obtained:

80 Those Lemmas and Proofs are taken from: McNeil, A., Frey, R., Embrechts, P., (2005), p. 396 and following

$$(\tilde{X}\, I_{\{\tau>s\}}|F_t) = I_{\{\tau>t\}}\, E^Q(e^{-(\Gamma_s-\Gamma_t)}\tilde{X}|G_t)$$

Now there are the foundations to analyse the vulnerable claim expression. We define: $\tilde{X} := e^{-\int_t^T r_s\,ds}X$, a G_t measurable random variable. Using the properties in lemma 3.B, and substituting $s = T$:

$$E^Q\left[\tilde{X}\, I_{\{\tau>T\}}\middle|F_t\right] = I_{\{\tau>t\}}E^Q\left[\tilde{X}e^{-(\Gamma_T-\Gamma_t)}\middle|G_t\right]$$

Given $\Gamma_t = \int_0^t \gamma_s\, ds$, the definition of $\tilde{X}$ and of R_t (in Section 2.1) the expression becomes:

$$E^Q\left[e^{-\int_t^T r_s\,ds}X\, I_{\{\tau>T\}}\middle|F_t\right] = I_{\{\tau>t\}}E^Q\left[e^{-\int_t^T r_s\,ds}e^{-\int_t^T \gamma_s\,ds}X\middle|G_t\right] = I_{\{\tau>t\}}E^Q\left[e^{-\int_t^T R_s\,ds}X\middle|G_t\right]$$

Let us now focus on the recovery payment part. Consider in this case $\tilde{X} = e^{-\int_t^\tau r_s\,ds}Z_\tau\, I_{\{\tau\le T\}}$. Applying the Lemma 2.B, it follows that:

$$E^Q\left[\, I_{\{\tau>t\}}e^{-\int_t^\tau r_s\,ds}Z_\tau\, I_{\{\tau\le T\}}\middle|F_t\right] = I_{\{\tau>t\}}\frac{E^Q\left(\, I_{\{\tau>t\}}e^{-\int_t^\tau r_s\,ds}Z_\tau\, I_{\{\tau\le T\}}\middle|G_t\right)}{P(\tau>t|G_t)}$$

Since τ is doubly stochastic it follows that:

$P(\tau>t|G_T) = e^{-\int_0^t \gamma_s\,ds}$ and its complement is $P(\tau\le t|G_T) = 1-e^{-\int_0^t \gamma_s\,ds}$

Then the conditional density of τ given G_T is $f_{\tau|G_T} = \gamma_t e^{-\int_0^t \gamma_s\,ds}$. We put our attention on the formulation of the numerator. Applying the definition of conditional expectation to the expression the following is obtained:

$$E^Q\left(\, I_{\{\tau>t\}}e^{-\int_t^\tau r_s\,ds}Z_\tau\, I_{\{\tau\le T\}}\middle|G_T\right) = \int_t^T e^{-\int_t^s r_u\,du}\, Z_s\gamma_s e^{-\int_0^s \gamma_u\,du}ds$$

Applying the properties of the exponential variables and recalling the definition of R_t it holds that: $e^{-\int_t^s r_u\,du}e^{-\int_0^s \gamma_u\,du} = e^{-\int_0^t \gamma_u\,du}\, e^{-\int_t^s R_u\,du}$. Given the independence from s of the expression $e^{-\int_0^t \gamma_u\,du}$, it follows that:

$$\int_t^T e^{-\int_t^s r_u\,du}\, Z_s\gamma_s e^{-\int_0^s \gamma_u\,du}ds = e^{-\int_0^t \gamma_u\,du}\int_t^T Z_s\gamma_s\, e^{-\int_t^s R_u\,du}ds$$

Finally, the application of the law of iterated expectations gives that: $E^Q[E^Q(X|G_T)|G_t] = E^Q[X|G_t]$. This allows to condition the previous result to G_t, allowing to obtain:

$$E^Q\left(\, I_{\{\tau>t\}}e^{-\int_t^\tau r_s\,ds}Z_\tau\, I_{\{\tau\le T\}}\middle|G_t\right) = e^{-\int_0^t \gamma_u\,du}E^Q\left[\int_t^T Z_s\gamma_s\, e^{-\int_t^s R_u\,du}ds\,\middle|G_t\right]$$

Then now substituting this expression for the numerator on the initial one, and recalling the relation $P(\tau > t|G_T) = e^{-\int_0^t \gamma_s\, ds}$ the initial formula is recovered:

$$I_{\tau>t}E^Q\left[\int_t^T Z_s\gamma_s e^{-\int_t^T R_u\, du}\, ds \,\middle|\, G_t\right]$$

Appendix C

The result given in Section 2.2, the reduction of the computation of a conditional expectation involving stochastic processes which are solutions of a stochastic differential equation in the solution of an ordinary differential equation, follows from the Feynman-Kac formula and from its application to the affine processes. Therefore, the mechanism of the formula is the first thing to be introduced, with some preliminary notions beside it[81].

Consider a Stochastic Differential Equation with initial condition as the following:

$$\begin{cases} dX_t = \beta(t, X_t)dt + \gamma(t, X_t)dW_t \\ X_s = x_s \end{cases} \tag{1.C}$$

Denote as $X_t^{s,x}$ the solution of the equation. For a measurable function f it can be defined the function: $f(s,x) = E[g(X_T^{s,x})]$. For it the following result holds:

$$E[g(X_T)|G_s] = f(s, X_s)$$

And this is because the Ito diffusion is a Markov process. The conditional expectation is a function of the variable X_s. Moreover, the process $f(t, X_t)$ defined as solution of the conditional expectation is a martingale; the result can be shown applying the law of iterated expectations:

$$E[f(t, X_t)|G_s] = E[E[\, g(X_T)|G_t]|G_s] = E[g(X_T)|G_s] = f(s, X_s)$$

This implies that:

$$E[f(t, X_t)|G_s] = f(s, X_s)$$

To apply the Feynman-Kac formula it is then necessary to introduce the infinitesimal generator, A, an operator which associates to a $C^{1,2}$ function $f: [0, T] \times \mathbb{R} \to \mathbb{R}$ the function $Af: [0, T] \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$Af = \frac{\partial f}{\partial t} + \beta(t, x)\frac{\partial f}{\partial x} + \frac{1}{2}\gamma^2(t, x)\frac{\partial f}{\partial x^2} \tag{2.C}$$

81 The reference for the first part of the section is: Bjoerk, T., 2009 pp. 70-74; for a more rigorous discussion: Oksendal, B., 2004

Where β , γ are the drift and diffusion coefficients in (1.C). Now, applying the Ito formula to the function $f(t, X_t)$ it follows that:

$$df(t, X_t) = Af(t, X_t)dt + \frac{\partial f}{\partial x}\gamma(t, X_t)dW_t \tag{3.C}$$

and so, the infinitesimal generator (2.C) associates to f the drift part in the stochastic differential equation (1.C).

Given these preliminary statements, the Feynman-Kac theorem can be introduced. Consider a function $f: \mathbb{R} \to \mathbb{R}$, and X_t as before. The Markov property of X_t leads to:

$E[g(X_T)|F_s] = f(s, X_s)$ but given the fact that $f(s, X_s)$ is a martingale, its drift part in the Ito decomposition must be null, and this is equivalent to say that:

$$Af(t, x) = 0 \qquad \forall (t, x) \in [0, T] \times \mathbb{R}$$

Considering what has been stated until now, the following conditions on the functions f, g can be written:

$$\begin{cases} \frac{\partial f}{\partial t} + \beta(t, x)\frac{\partial f}{\partial x} + \frac{1}{2}\gamma^2(t, x)\frac{\partial f}{\partial x^2} = 0 \\ f(T, x) = g(x) \end{cases} \tag{4.C}$$

The system can be solved numerically and, in many cases, also analytically; it involves a boundary condition (condition at the terminal date T) which defines the first equation as a backward equation. Therefore, once retrieved the function $f(s, X_s)$, it is possible to express the solution to the conditional expectation $E[g(X_T)|G_s]$ where the variable X_t is the solution of the stochastic differential equation introduced above.

The Feynman-Kac Theorem can be applied also evaluating the following conditional expectation, including a discount factor:

$$e^{-r(T-t)}E[g(X_T)|G_t]$$

where r is constant. To do so, we define:

$$f(t, x) = e^{-r(T-t)}E[g(X_T)|G_t]$$

Following a reasoning analogous to the one introduced above, it can be shown that the function $f(t, X_t)$ solves the system:

$$\begin{cases} \frac{\partial f}{\partial t} + \beta(t, x)\frac{\partial f}{\partial x} + \frac{1}{2}\gamma^2(t, x)\frac{\partial f}{\partial x^2} - rf(t, x) = 0 \\ f(T, x) = g(x) \end{cases} \tag{5.C}$$

The system presented is closely related to (4.C) and has a similar analytical tractability.

The PDE (5.C) is of fundamental importance for pricing; indeed, it represents a way to retrieve the fair price of a contingent claim depending on the process X_t. The price is expressed as the expectation of the final payoff of the claim discounted by the rate r: this rate can be intended as the risk-free rate. Therefore, $f(t, X_t)$ is appropriate when expressing the fair price of the claim under the risk neutral probability measure Q which has the money market account as numeraire. Thus, given Q the risk neutral measure for the financial markets, the solution to (5.C) corresponds to the solution to the risk neutral valuation formula:

$$e^{-r(T-t)}E^Q[g(X_T)|G_t]$$

Once introduced the main results and interpretations of the Feynman-Kac formula, it is possible to move to the demonstration of the results presented in Section 2.2. We take as valid the characterization given in Section 2.1, in terms of probability space $(\Omega, G_t, (G_t), P)$, risk neutral measure Q and of the definition of Markov process Ψ_t, solution of the stochastic differential equation[82] (2.10). The task is to define the solution to the conditional expectation:

$$E^Q\left[e^{-\int_t^T \Lambda_s\, ds} g(\Psi_T)\middle|G_t\right] = f(t, \Psi_t)$$

As stated by the Feynman-Kac proposition, the function $f(t, \Psi_t)$ can be characterized in terms of the backward equation analogous to (5.C):

$$\begin{cases} \dfrac{\partial f}{\partial t} + \delta_1(\psi)\dfrac{\partial f}{\partial \psi} + \dfrac{1}{2}{\sigma_1}^2(\psi)\dfrac{\partial f}{\partial \psi^2} - \Lambda_t(\psi)f(\psi) = 0 \\ f(T, \psi) = g(\psi) \end{cases}$$

To find $f(t, \psi)$ it is useful guess a solution of the type: $f(t, \psi) = e^{\alpha(t,T)+\beta(t,T)\Psi_t}$, with $\alpha(\cdot\,, T), \beta(\cdot\,, T)$ continuously differentiable. Then define as in Section 2.2 the following functions: $g(\Psi_t) = e^{a\Psi_t}$; $\delta_1(\psi) = k_0 + k_1\psi_t$, $\sigma_1^2(\psi) = h_0 + h_1\psi_t$ (drift and squared volatility of the process are affine functions of Ψ_t); $\Lambda_t = \lambda_0 + \lambda_1\psi_t$ (Λ_t is also affine with respect to Ψ_t). Denote than as $\alpha'(\cdot\,, T), \beta'(\cdot, T)$ the derivatives of $\alpha(t, T), \beta(t, T)$ with respect to t. Given the structure of the solution, it holds that:

$$\frac{\partial f}{\partial t} = (\alpha' + \beta'\psi)f\ , \quad \frac{\partial f}{\partial \psi} = \beta f\ , \quad \frac{\partial f}{\partial \psi^2} = \beta^2 f$$

82 The reference for the second part is: McNeil, A., Frey, R., Embrechts, P., (2005), Ch. 9, Section 5

The terminal conditions are then set when keeping: $f(T,\psi) = g(\Psi_T)$, and so it results in $\alpha(T,T) = 0$; $\beta(T,T) = a$. The equation is then the following:

$$(\alpha' + \beta'\psi_t)f + (k_0 + k_1\psi_t)\beta f + \frac{1}{2}(h_0 + h_1\psi_t)\beta^2 f = (\lambda_0 + \lambda_1\psi_t)f$$

Now dividing by f and recollecting all the terms dependent on ψ_t the system of the Ricatti equations can be obtained:

$$\begin{cases} \beta'(t,T) = \lambda_1 - k_1\beta(t,T) - \frac{1}{2}h_1\beta^2(t,T) \\ \alpha'(t,T) = \lambda_0 - k_0\beta(t,T) - \frac{1}{2}h_0\beta^2(t,T) \end{cases}$$

with terminal conditions $\beta(T,T) = a$; $\alpha(T,T) = 0$.

Appendix D

```
%--------------------------------------------------------------
%CIR PARAMETERS CALIBRATION UNDER P: OLS + MAXIMUM LIKELIHOOD
%INTEREST RATES MODEL
%--------------------------------------------------------------
clear
clc

%% Loading data
%Use all the time seiries disposable of the UK 1 month
overnight OIS rate
[ir_data,text]=xlsread('OIS  S 1_5 M.xlsx');

% storing dates as string
day=datestr(datenum(text(2:size(text,1),1),'dd/mm/yyyy'));
DN=datenum(day);

save IR_DATASET.mat

%% Initial parameters estimation: OLS
V_data = ir_data(:,2);
dt=1/250;
Nobs = length(V_data);
x = V_data(1:end-1);
dx = diff(V_data);
dx = dx./x.^0.5;
regressors = [dt./x.^0.5, dt*x.^0.5];
drift = regressors\dx;
res = regressors*drift - dx;
chi = -drift(2);
teta = -drift(1)/drift(2);
sigma = sqrt(var(res, 1)/dt);
InitialParams = [chi teta sigma];
```

```
%% Parameters calibration: Maximum Likelihood, CIR_LL function

ML_CIRparams = fminsearch( @CIR_LL , InitialParams )

chi=ML_CIRparams(1);
teta=ML_CIRparams(2);
sigma=ML_CIRparams(3);

%Check for non violation of the Feller condition
COND=(2*ML_CIRparams(1)*ML_CIRparams(2))-ML_CIRparams(3)^2
%% Descriptive plot analysis
plot (DN,ir_data(:,2));
datetick('x','yyyy');
xlim([733282,737425])
title('UK OIS Time Series')

%% Store data
save IR_CIR

clear
clc
%-----------------------------------------------------------
%CIR PARAMETERS CALIBRATION UNDER Q: MSE MINIMIZATION WITH
%MARKET DATA INTEREST RATES MODEL
%-----------------------------------------------------------
%% Load useful data
[BP_data, text]=xlsread('BP_UK.xlsx');
BP=BP_data(:,2);
day=datestr(datenum(text(1:size(text,1),1),'dd/mm/yyyy'));
DN=datenum(day);
save BP_data
load IR_CIR chi teta sigma
%% Define the zero curve from the yield curve
ZeroRates = BP;
CurveDates = DN;
Settle = datenum ('28-Oct-2019');
[DiscRates,CurveDates] =
zero2disc(ZeroRates,CurveDates,Settle);
save DISC_N
%% Calibrate parameters through optimization: MSE minimization
chi0=chi;
teta0=teta;
Initialpar=[chi0,teta0];
irsol=fminsearch(@NEST, Initialpar);
chi=irsol(1)
teta=irsol(2)
COND=(2*irsol(1)*irsol(2))-sigma^2

save IR_CIR_N
```

```
%-----------------------------------------------------------
%STOCK MARKET PARAMETERS CALIBRATION
%GBM MODEL
%-----------------------------------------------------------
clear
clc
%% Loading data
%Use all the time seiries disposable of the FTSE ALL SHARE
[sm_data,text]=xlsread('FTSE ALL.xlsx');

% storing dates as string
day=datestr(datenum(text(2:size(text,1),1),'dd/mm/yyyy'));
DN=datenum(day);

save INDEX_DATASET.mat

%% Clabration of GBM on returns of the index: Maximum
Likelihood and Close form estimation

Ret=price2ret(sm_data,[],'periodic');
Ret1=Ret(end-3750:end,1);
dt=1/250;
mle_est = mle(Ret1,'distribution','normal');

sigma = sqrt(mle_est(2)^2/dt);  % sigma
mu = mle_est(1)/dt + 0.5*sigma^2;  % mu
params=[mu,sigma];
%% Descriptive plot analysis
subplot(2,1,1)
plot (DN,sm_data)
datetick('x','yyyy');
xlim([716706,737695])
title('FTSE All Share Time series')
subplot(2,1,2)
autocorr (Ret1)

save SM_GBM

%-------------------------------------------------------------
%FELLER NON-MEAN REVERTING PROCESS
%MORTALITY MODEL CALIBRATION THROUGH SURVIVAL PROBABILITIES
%-------------------------------------------------------------
clear
clc

%% Loading data
[sr_data,text]=xlsread('SUR_FUNC_UK.xlsx');
sr_data_30=sr_data(2:52,5);
sr_data_55=sr_data(2:52,10);
sr_data_23=sr_data(:,15);
sr_imp=sr_data(2:52,20);
save SR_DATASET
```

```
%% Parameters Calibration: error sqrt minimization

%Intial parameters guess
a0=0.063;
s0=0.003;

%Parameters calibration throgh fmincon (positivity constraint)
Initialpar=[a0,s0];
A=[-1,-1];
b=0;
sursol=fmincon(@SURMAX, Initialpar, A, b)

%Test for empirical goodness of parameters (plot real data vs
projected)
%Expression for the survival function with final parameters
t=[1:1:51];
a=sursol(1,1);
s=sursol(1,2);
g=sqrt(a^2+2*s^2);
lambda0=-log(sr_imp(1,1));
B=(2*(exp((g*t)-1))./((g-a)*(exp((g*t)-1))+2*g));
sur=exp(-B*lambda0);

plot(sr_data_55)
hold on
plot(sur)
xlim([0,51])
legend('Observed','Theoretical')
title ('Observed and Theoretical Survival Probabilities')
ylabel ('Survival probabilities')
xlabel ('Time')

S=sr_data_55-sur';
mean(S)
X=mean((S.^2));
sp=sqrt(X);

save MR_PAR
```

Appendix E

```
clear
clc
%-----------------------------------------------------------
%INTEREST RATES SIMULATION: 15 YEARS DAILY SIMULATION FOR
INTEREST RATES
%-----------------------------------------------------------
%% Load useful data and set the simulation parameters
load IR_CIR_N  chi teta sigma
load IR_CIR ir_data dt
t  = [dt : dt : 15];  % observation times: 15 years simulation
```

```
a  = chi;        % mean-reversion parameter
b  = teta;       % long-term mean
s  = repmat (sigma,50000,1);        % volatility; number of
simulations
r0 = ir_data(end,2);       % starting value
n = length(t);
r = nan(n,1);
M=50000;
%% Function for the simulation: cirpath
tic
for j= 1: length (s)
r(:,j) = cirpath(t,a,b,s(j),r0);
end
toc
r=r';
%Sample mean of the final interest rate
X=mean (r);

%% Plots: simulations part and sample mean
%Plots of the simulation
dataint=ir_data(:,2);
din=dataint';
matr=zeros(50000,2960);
IR_TS=[din;matr];
mat1=zeros(1,3750);
IRSIM=[mat1;r];
IRFIN=[IR_TS IRSIM];
IRFIN(IRFIN==0)=nan;
plot (IRFIN(1,:)), hold on
for i=2:5000:40001
plot (IRFIN(i,:), '--'), hold on
end
xlim([310,6710]);
legend ('Historical data', 'Simulations')
title('Interest rate time series and simulation')
xlabel ('Days')


save CIR_PATH_N

%------------------------------------------------------------
%NUMERICAL INTEGRATION OF THE STOCHASTIC INTEGRAL FOR THE
INTEREST RATES
%USING SIMULATED DATA
%------------------------------------------------------------
clear
clc
%% Load useful parameters and set the other useful variables
load CIR_PATH_N r dt s
M=50000;
%% Approximaton of the integral using the trapeziodal rule
rA=zeros (M,15);
i=[0:250:3500];
```

```
rA=r(:,i+250);
j=[1:250:3500];
rF=r(:,j+250);
rs=r;
summair=zeros(M,15);
for j=1:M
summair(j,:)=[sum(rs(j,2:249)),sum(rs(j,252:499)),sum(rs(j,502
:749)),sum(rs(j,752:999)),...
sum(rs(j,1002:1249)),sum(rs(j,1252:1499)),sum(rs(j,1502:1749))
,sum(rs(j,1752:1999))...
sum(rs(j,2002:2249)),sum(rs(j,2252:2499)),sum(rs(j,2502:2749))
,sum(rs(j,2752:2999))...
sum(rs(j,3002:3249)),sum(rs(j,3252:3499)),sum(rs(j,3502:3749))
];
end
integir= zeros(M,15);
for i= 1:M
integir(i,1)=(dt/2)*(rs(i,1)+rA(i,1)+2*summair(i,1));
for j= 2:15
integir(i,j)=(dt/2)*(rF(i,j-1)+rA(i,j)+2*summair(i,j));
end
end

for i= 1:M
cumint(i,:)=sum(integir(i,:));
end

for i= 1:M
discir15(i,:)=exp(-cumint(i,:));
end


save IR_15Y_DISCOUNT_N

%-----------------------------------------------------------
%UNDERLYING EXPECTED INDEX VALUE IN 15 YEARS
%-----------------------------------------------------------
clear
clc

%% Load useful parameters and set the other useful variables
load SM_GBM mu sigma sm_data
load IR_15Y_DISCOUNT_N cumint dt
Xzero = sm_data(end,1); % problem parameters
M=50000;
T=15;
N=3750;

%% Montecarlo simulation of the GBM
dW = sqrt(dt)*randn(M,N); % Brownian increments
W = cumsum(dW,2); % discretized Brownian path
WT=W(:,end);
```

```
%Given the presence of the risk neutral measure the drift has
to be set as the risk-free rate r
Xtrue = Xzero*exp((cumint-0.5*sigma^2*T)+(sigma*WT));
XT=mean(Xtrue(:,end));
%Expected return on the market in 15 years: the final
reimbursement
%is linked to it
PRet=((XT-Xzero)/Xzero);

save INDEX_PATH

clear
clc
%-------------------------------------------------------------
%SIMULATION OF THE INTENSITY OF MORTALITY FOR THE FOLLOWING 15
YEARS DAILY SIMULATION
%-------------------------------------------------------------
%% Load useful data and set the other variables
load MR_PAR a s lambda0
mu0=lambda0;
T = 15;
N = T*250;
dt = T/N;
M=50000;
%% Computation of the simulation with the Euler approximation
mu=zeros(50000,N);
mu (:,1) = mu0;
dW = sqrt(dt)*randn(M,N); % Brownian increments
for j = 1:M
for i=2:N
mu(j,i) = mu(j,i-1) + dt*a*mu(j,i-1) + s*sqrt(abs(mu(j,i-
1)))*dW(j,i);
end
end
X1=mean (mu);
X=mean(mu(:,end));

%% Plots to show the results
plot (X1)
hold on
for i= 1:1000:20000
plot (mu(i,:),':'), hold on
end
xlim([1,3750]);
title('Intensity of mortality simulations')
legend('Sample Mean','Simulations')
xlabel ('Days')

save MR_PATH
```

```
%-----------------------------------------------------------
%MORTALITY INTENSITY
%COMPUTATION OF THE STOCHASTIC INTEGRAL
%-----------------------------------------------------------
clear
clc

%% Load data and set the other useful elements
load MR_PATH a s M N mu dt
mA=zeros (M,15);
i=[0:250:3500];
mA=mu(:,i+250);
j=[1:250:3500];
mF=mu(:,j+250);
ms=mu;
mA=zeros (M,15);

%% Compute the integral by means of the trapeziodal rule
summamr=zeros(M,15);
for j=1:M
summamr(j,:)=[sum(ms(j,2:249)),sum(ms(j,252:499)),sum(ms(j,502
:749)),sum(ms(j,752:999)),...
sum(ms(j,1002:1249)),sum(ms(j,1252:1499)),sum(ms(j,1502:1749))
,sum(ms(j,1752:1999))...
sum(ms(j,2002:2249)),sum(ms(j,2252:2499)),sum(ms(j,2502:2749))
,sum(ms(j,2752:2999))...
sum(ms(j,3002:3249)),sum(ms(j,3252:3499)),sum(ms(j,3502:3749))
];
end

integmor= zeros(M,15);
for i= 1:M
integmor(i,1)=(dt/2)*(ms(i,1)+mA(i,1)+2*summamr(i,1));
for j= 2:15
integmor(i,j)=(dt/2)*(mF(i,j-1)+mA(i,j)+2*summamr(i,j));
end
end

for i= 1:M
cumintmor(i,:)=sum(integmor(i,:));
end

for i= 1:M
discmor15(i,:)=exp(-cumintmor(i,:));
end
%discmorfin=mean (discmor);

save MR_15_Y_DISCOUNT


%---------------------------------------------------------------
%ENDOWMENT VALUATION
%Characteristics:
```

```
%starting age: 50
%concluding age: 65
%Index-linked
%--------------------------------------------------------------
clear
clc

%% Preparing workspace with useful common variables
T = 15;
N = 3750;
dt = 15/N;

%% Mortality discount factor computation: affine farmework
exploited

load MR_PAR sr_data_55 a s
Yzero=-log(sr_data_55(1,1));
%Two equivalent methods (discussed in the text)
%1)
b=-(sqrt(a^2+s^2));
c=(b+a)/2;
d=(b-a)/2;
z=exp(b*T);
alpha=0;
beta=(1-z)/((c+(d*z))); %depends on time t
COND=z*(s^2+2*d^2)-(s^2-2*d*c)
Emu=exp(beta*Yzero);
%2)
%g=sqrt(a^2+2*s^2);
%B=(2*(exp((g*T)-1))./((g-a)*(exp((g*T)-1))+2*g));
%Emu=exp(-B*Yzero);

%% Money discount factor computation: affine farmework
exploited

load IR_CIR_N chi sigma teta
load IR_CIR ir_data
t=0;
tau=T-t;
h=sqrt(chi^2+2*sigma^2);
q=2*h+((chi+h)*(exp(tau*h)-1));
c=((2*chi*teta)/sigma^2);
alpha=((2*h*exp(((chi+h)*tau)/2))/q)^c;
beta=(2*(exp(tau*h)-1))/q;

Erate=alpha*exp(-beta*ir_data(end,2)); %apply the solution of
the expectation


%% Discount for an endowment reimbursing 1 at maturity Affine
method
EV=Erate*Emu;
```

```
%% Computation of the index linked endowment value Affine
method
load INDEX_PATH PRet Xtrue Xzero
EVF=(1+PRet)*EV;

%% Comparison with the numerical integration method
load IR_15Y_DISCOUNT_N discir15
load MR_15_Y_DISCOUNT_2 discmor15

%% Discount for an endowment reimbursing 1 at maturity
YbYdisc=discir15.*discmor15;
EV1=mean(YbYdisc);
%% Computation of the index linked endowment value Numerical
method
Ret=1+((Xtrue-Xzero)./Xzero);
Mdisc=mean (discmor15);
Ydisc=(discir15).*(1+PRet);
Idisc=mean(Ydisc);
EVF1=Idisc*Mdisc;
DIFF=(EVF1-EVF)/EVF;

%% Half of the simulations: Discount for an endowment
reimbursing 1 at maturity
discir15_2=discir15(25001:end);
discmor15_2=discmor15(25001:end);
YbYdisc_2=discir15_2.*discmor15_2;
EV2=mean(YbYdisc_2);
%% Half of the simulations: Computation of the index linked
endowment value Numerical method
Mdisc_2=mean (discmor15_2);
Ydisc_2=(discir15_2).*(1+PRet);
Idisc_2=mean(Ydisc_2);
EVF2=Idisc_2*Mdisc_2;
DIFF2=(EVF2-EVF)/EVF;


save ENDVAL


%-------------------------------------------------------------
%COMPUTATION OF THE INTEREST RATE DISCOUNTS FOR THE ANNUITY
%-------------------------------------------------------------
%% Load the useful parameters and define the other variables
clear
clc

load CIR_PATH_N chi teta sigma r
M=50000;
a   = chi;        % mean-reversion parameter
b   = teta;       % long-term mean
s   = sigma;        % volatility
t=0;
```

```
%% Numerical computation of the yearly discount for each
interest rate simulation
for T= 2:44
tau=T-t;
h=sqrt(chi^2+2*sigma^2);
q=2*h+((chi+h)*(exp(tau*h)-1));
c=((2*chi*teta)/sigma^2);
alpha(:,T)=((2*h*exp(((chi+h)*tau)/2))/q)^c;
beta(:,T)=(2*(exp(tau*h)-1))/q;
end

for j= 1:M
for i= 2:44
discir(j,i)=alpha(1,i)*exp(-beta(1,i).*r(j,end));
end
end
save DISCOUNT_INTEREST_RATE_N


%---------------------------------------------------------------
%COMPUTATION OF THE MORTALITY RATE DISCOUNTS FOR THE ANNUITY
%---------------------------------------------------------------
clear
clc

load MR_PATH mu X a s M
for T= 2:44
b=-(sqrt(a^2+s^2));
c=(b+a)/2;
d=(b-a)/2;
z=exp(b.*T);
alpha=0;
beta(:,T)=(1-z)./((c+(d.*z))); %depends on time t
end

for j= 1:M
for i= 2:44
discmr1(j,i)=exp(beta(1,i).*mu(j,end));
end
end
discmr=discmr1;


save DISCOUNT_MORTALITY_INTENSITY


%---------------------------------------------------------------
%GAO EVALUATION
%Characteristics:
%starting age: 65
%concluding age: 110 (maximum expected life of the annuitant)
%annuity conversion factor = 1/12
%---------------------------------------------------------------
```

```
clear
clc

%% Load useful data and define other variables used
load DISCOUNT_INTEREST_RATE_N
load DISCOUNT_MORTALITY_INTENSITY
g=12;

%% Annuity valuation in T=15 for each simulation: exploit the
affintiy of the processes
%Computation of the discount for each year: SB valuation for
t=[1:45]
YbYdisc=discmr.*discir;
%Annuity valuation
AnnVal=sum (YbYdisc,2);

%% Call Option Valuation
CVRatio=repmat(g,M,1);
AnnOpt=AnnVal-CVRatio;
Zann=zeros(M,1);
Vann=horzcat(AnnOpt,Zann);
Cann=(max(Vann,[],2));

%% Valuation of the GAO
load ENDVAL PRet Emu Erate Ret

GAOp=(1/g).*Erate.*Emu.*Ret.*Cann;

GAOPRICE=mean(GAOp);

GAOp1=(1/g).*Erate.*Emu.*Cann;
GAOPRICE1=mean(GAOp1);

save GAOPRICE

%-----------------------------------------------------------
%CALIBRATION OF INTEREST RATE TO KEEP OPTION ATM
%-----------------------------------------------------------
clear
clc
BEV0=0.0061;
%% Call the funciton for the computation of the GAO price in
15
BEV = fzero(@AnnOpt, BEV0);
save BEV_AV


%-----------------------------------------------------------
%MORTALITY RATES DISCOUNT INFLUENCE ON THE GAO PRICE
%-----------------------------------------------------------
clear
clc
%% Split the discounts in different subsamples
```

```
load DISCOUNT_MORTALITY_INTENSITY
CSUMMR=sum(discmr,2);
MORMAT=[discmr CSUMMR];
ORD_MOR=sortrows(MORMAT,44);
plot (ORD_MOR(1,1:end-1)); hold on
plot (ORD_MOR(13500,1:end-1), ':'); hold on
plot (ORD_MOR(26000,1:end-1), '--'); hold on
plot (ORD_MOR(38500,1:end-1), '*'); hold on
plot (ORD_MOR(7000,1:end-1)); hold on
plot (ORD_MOR(19500,1:end-1), ':'); hold on
plot (ORD_MOR(32000,1:end-1), '--'); hold on
plot (ORD_MOR(44500,1:end-1), '*');
xlim([1,45])
title ('Mortality discount in different scenarios')
legend('1st subsample', '2nd subsample', '3rd subsample', '4th
subsample')

SCOM_MOR_1=ORD_MOR(1:end-3*(end/4),1:end-1);
SCOM_MOR_2=ORD_MOR(end-3*(end/4)+1:end-(end/2),1:end-1);
SCOM_MOR_3=ORD_MOR(end-(end/2)+1:end-(end/4),1:end-1);
SCOM_MOR_4=ORD_MOR(end-((end/4)-1):end,1:end-1);

load DISCOUNT_INTEREST_RATE_N
g=12;
SCOM_MORIR_1=discir(1:end-3*(end/4),:);
%% Annuity valuation in T=15 for each simulation: exploit the
affintiy of the processes
%Computation of the discount for each year: SB valuation for
t=[1:45]
YbYdisc1=SCOM_MOR_1.*SCOM_MORIR_1;
YbYdisc2=SCOM_MOR_2.*SCOM_MORIR_1;
YbYdisc3=SCOM_MOR_3.*SCOM_MORIR_1;
YbYdisc4=SCOM_MOR_4.*SCOM_MORIR_1;

%Annuity valuation
AnnVal1=sum (YbYdisc1,2);
AnnVal2=sum (YbYdisc2,2);
AnnVal3=sum (YbYdisc3,2);
AnnVal4=sum (YbYdisc4,2);

%% Call Option Valuation
CVRatio=repmat(g,M/4,1);
AnnOpt1=AnnVal1-CVRatio;
AnnOpt2=AnnVal2-CVRatio;
AnnOpt3=AnnVal3-CVRatio;
AnnOpt4=AnnVal4-CVRatio;

Zann=zeros(M/4,1);

Vann1=horzcat(AnnOpt1,Zann);
Vann2=horzcat(AnnOpt2,Zann);
Vann3=horzcat(AnnOpt3,Zann);
Vann4=horzcat(AnnOpt4,Zann);
```

```
Cann1=(max(Vann1,[],2));
Cann2=(max(Vann2,[],2));
Cann3=(max(Vann3,[],2));
Cann4=(max(Vann4,[],2));

%% Valuation of the GAO
load ENDVAL PRet Emu Erate Ret
GAOp1=(1/g).*Erate.*Emu.*mean(Ret).*Cann1;
GAOp2=(1/g).*Erate.*Emu.*mean(Ret).*Cann2;
GAOp3=(1/g).*Erate.*Emu.*mean(Ret).*Cann3;
GAOp4=(1/g).*Erate.*Emu.*mean(Ret).*Cann4;

GAOPRICE1=mean(GAOp1);
GAOPRICE2=mean(GAOp2);
GAOPRICE3=mean(GAOp3);
GAOPRICE4=mean(GAOp4);
%% Non index linked endowment
FGAOp1=(1/g).*Erate.*Emu.*Cann1;
FGAOp2=(1/g).*Erate.*Emu.*Cann2;
FGAOp3=(1/g).*Erate.*Emu.*Cann3;
FGAOp4=(1/g).*Erate.*Emu.*Cann4;
FGAOPRICE1=mean(FGAOp1);
FGAOPRICE2=mean(FGAOp2);
FGAOPRICE3=mean(FGAOp3);
FGAOPRICE4=mean(FGAOp4);



save CS_MR_D
```

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