UNIVERSITÀ DEGLI STUDI DI PADOVA

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Generalized symmetries and spontaneous symmetry breaking

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Abstract

The fundamental concept of global symmetry in quantum field theory has been generalized in many different directions in the last few years. Some of these generalizations include: higher form symmetries that only act on operators supported on manifolds of dimension greater than 0; higher group symmetries, where the associativity law for the composition of symmetries is modified by the higher form terms; non-invertible symmetries, where groups are replaced by more general mathematical structures such as fusion or n-categories. Most of the properties of the standard global symmetries admit an analogue in these generalized settings: they lead to selection rules for correlation functions, they can be spontaneously broken, they can be gauged, they admit 't Hooft anomalies. In this thesis, after a description of some these generalized definitions of symmetries, the focus will be on spontaneous symmetry breaking. By an extension of Goldstone theorem, it can be proven that spontaneous breaking of a p-form symmetry implies the existence of massless p-form excitation in the spectrum (Goldstone boson); the standard case corresponds to p=0, where the boson is a scalar. A similar results also holds for non-invertible symmetries. Finally, an argument will be presented showing that, for $p > 0$, the mass of a p-form Goldstone mode remains exactly massless even if the spontaneously broken p-form symmetry is only an emergent IR symmetry, i.e. that the mass does not receive corrections due to UV perturbations. This phenomenon represents a major qualitative difference between higher (p>0) form and the standard (p=0) symmetries.

Contents

Chapter 1

Introduction

Symmetries are a fascinating and omnipresent topic in any area of physics. The presence of symmetries in a given theory leads to interesting results such as conservation rules, relations between correlators, constraints on the renormalization group (RG) flow and classification of phases. Often times one only looks at symmetries acting on point-like degrees of freedom, which are called ordinary symmetries. However the zoology of symmetries is much larger and contains phenomena beyond ordinary symmetries which are called generalized symmetries [1–6]. Generalized symmetries were originally studied in their gauged versions in the context of topological quantum field theories [7, 8].

In this thesis we will study a sub-class of these generalized symmetries called higher form symmetries [9]. Higher form symmetries are characterized by having charged objects defined on manifolds of dimension greater than 0. In particular, if the charged objects are operators defined on p dimensional manifolds, we call the symmetry a p-form symmetry. In our study we will only focus on higher form symmetries that are invertible, that is they admit an inverse operator. The case of non-invertible symmetries [10, 11], which do not admit an inverse operator, is beyond the scope of this thesis.

We will study the main properties of higher form symmetries and the consequences one can derive from their presence in a given theory. We will work with symmetries that are present in the UV, but also higher form symmetries that are explicitly broken in the UV [12] but emerge in the IR [13, 14]. We will study the connection between the higher form symmetries of a given theory and some exact claims we may make about the low energy dynamics of said theory.

We will find that many properties of ordinary symmetries can be generalized to higher form symmetries, however some interesting results are valid only for higher form symmetries [15].

In particular we focus on the spontaneous symmetry breaking (SSB) of higher form symmetries, the appearance of Goldstone bosons [16] and the implications on the RG flow. We will consider examples both in high energy and condensed matter physics.

Chap.2 gives a summary of the main properties of ordinary symmetry, both in classical and quantum mechanics, and fixes some important notation in the thesis. Chap.3 gives a schematic definition of higher form symmetries with the intent of providing an intuitive picture of what is formally discussed later. In Chap.4 we make an explicit example using free Maxwell theory [17]. We find a 1-form symmetry and identify the charged operators, which are the Wilson loops. We study the action of the 1-form symmetry on said operators. We give a physical understanding

of the Wilson loops operators [18] and, using the electrostatic potential, we make a connection between the phases of the theory and the spontaneous breaking of the 1-form symmetry. Chap.5 is dedicated to the study of a more general free p-form gauge theory and its charged operators, namely the Wilson cycles. We make explicit calculations for the expectation values of the Wilson cycle operators. We provide a renormalization scheme for said operators and as a by-product of our calculation we obtain a generalized version of Mermin-Wagner theorem. In Chap.6 we analyse the presence of higher form symmetry in $SU(N)$ Yang-Mills theory [19, 20]. We find a discrete 1-form symmetry associated to the center of the group $SU(N)$ and we make explicit examples for $SU(2)$ and $SU(3)$. In Chap.7 we provide a generalization of the Goldstone boson theorem [21] which applies to higher-form symmetries and implies the existence of a massless particle in the spontaneously broken phase of the symmetry. The section concludes with a heuristic example of why higher form symmetries are exact at low energies. Chap.8 is dedicated to building the definition of topological order [22]. We provide two equivalent definitions, one macroscopic and one microscopic. We underline some important properties of topological order and their physical interpretations. In the last part of the chapter we give a review of category theory, which is the mathematical formalism describing topological order. In Chap.9 we use the knowledge acquired in Chap.8 to state the holographic principle of topological order [23] in 1 higher dimension. Staring from a theory in d spatial dimensions, this principle assigns a new theory in $d+1$. We provide an example of this machinery, and then use this holographic principle to construct a formal argument stating that higher form symmetries are exact at low energies. Finally in Chap.10 we study a lattice model (originally introduced in [15]) that exemplifies and validates all the knowledge acquired so far.

Chapter 2

Recap on 0-form symmetries

Following the same path taken in [3] let us start by briefly reviewing ordinary global symmetries, which from now will be referred to as 0-form symmetries.

2.1 Symmetries at classical level

Given a certain action $\mathcal{S}[\Phi]$ depending on the set of fields Φ, we define a (internal) symmetry as a map $G(\Phi)$ that sends the set of fields Φ into a new set Φ' such that the action is invariant, i.e. $\mathcal{S}[\Phi'] = \mathcal{S}[\Phi]$.

If the map G is continuous we may take the transformation to be infinitesimal and write $\Phi' = \Phi + \alpha\delta\Phi + O(\alpha^2)$ where α is the parameter of the transformation and it is not dependent on space-time coordinates.

At this point Noether's theorem guarantees the existence of a 1-form which is conserved if the equations of motion are respected,

$$J \coloneqq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_k)} \delta\Phi_k \, dx^\mu \quad \text{such that} \quad \partial_\mu J^\mu = d \star J = 0, \tag{2.1}$$

where Φ_k indicates a single field within the set Φ and k runs over all fields. The last equation implies that if we integrate $\star J$ over a $D-1$ dimensional manifold Σ_{D-1} the resulting object $Q \coloneqq \int_{\Sigma_{D-1}} \star J$ is topological and does not depend on small deformations of Σ_{D-1}. In particular we usually pick $\Sigma_{D-1} = \mathbb{R}^d$ to be the whole space at fixed time and we have

$$Q(t) = \int_{\mathbb{R}^d} J_0 \, d^d x = \int_{\mathbb{R}^d} \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi_k)} \delta\Phi_k \, d^d x, \tag{2.2}$$

which is the conserved charge of the symmetry, i.e. $\dot{Q}(t) = 0$. If we shift to the Hamiltonian formalism we can define the conjugated momentum of Φ_k as $\pi_k \coloneqq \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi_k)}$ and alongside it we have the canonical Poisson brackets,

$$\begin{aligned} \{\Phi_i(t,x), \pi_j(t,y)\} &= \delta_{ij}\delta^d(x-y), \\ \{\Phi_i(t,x), \Phi_j(t,y)\} &= \{\pi_i(t,x), \pi_j(t,y)\} = 0. \end{aligned} \tag{2.3}$$

We can rewrite the charge as

$$Q(t) = \int_{\mathbb{R}^d} \pi_k \delta\Phi_k d^d x. \tag{2.4}$$

and we can regard $Q(t)$ as the generator of the transformation on the fields since

$$\delta\Phi_k(x,t) = \frac{\delta Q(t)}{\delta\pi_k(x,t)} = -\{Q(t), \Phi_k(x,t)\}. \tag{2.5}$$

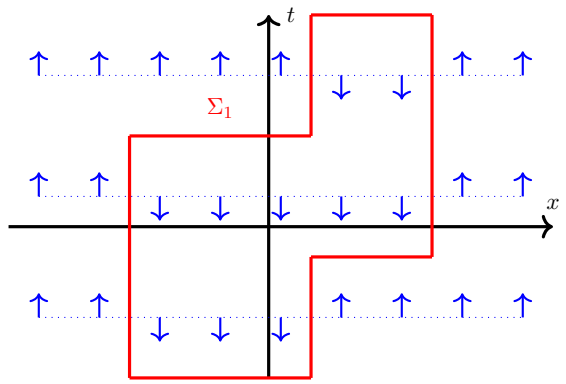
Then a symmetry transformation can be expressed using the differential operator $U(\alpha, \bullet) \coloneqq \exp(-\alpha\{Q(t), \bullet\})$ acting on the fields

$$\Phi_k^{'}(x,t) = U(\Phi_k(x,t)) = \Phi_k(x,t) + \alpha\{Q(t), \Phi_k(x,t)\} + O(\alpha^2). \tag{2.6}$$

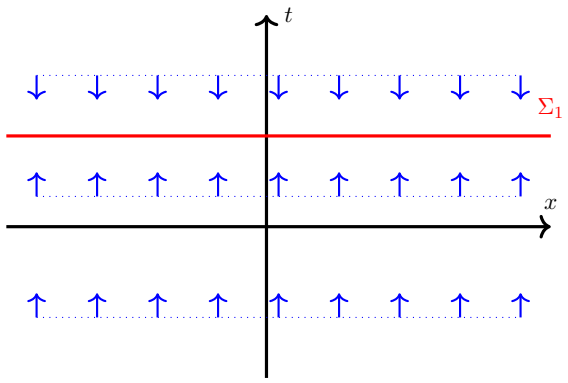
Let us now discuss what happens when making different choices for the manifold Σ_{D-1}. Physically speaking Σ_{D-1} identifies a locus of spacetime where the symmetry is applied, namely as we move across Σ_{D-1} the symmetry transformation acts on the degrees of freedom of the theory. The general case is when Σ_{D-1} is the boundary of an open patch $\mathcal{P}$ of spacetime, and thus the degrees of freedom within $\mathcal{P}$ are transformed (Fig[2.1(a)]). Now consider the limiting case where we stretch the patch $\mathcal{P}$ to infinity and leave only a piece of the boundary of $\mathcal{P}$ at fixed time. The result of this procedure is represented in Fig[2.1(b)] for a classical $D = 1+1$ spin chain model with global $U(1)$ symmetry, where the symmetry transformation corresponds to rotation of the spin by π. Formally we can realize Fig[2.1(b)] by acting the symmetry operator $U(\alpha = \pi, \bullet)$ at fixed time. We conclude that if Σ_{D-1} is codimension 1 in time we may interpret the symmetry transformation as a differential operator acting on the degrees of freedom. On the other hand we may stretch $\mathcal{P}$ to infinity, and leave a piece of boundary with fixed space coordinate, this would lead to Fig[2.1(c)]. Fig[2.1(c)] represents a domain wall with fixed locus. We conclude that if Σ_{D-1} is codimension 1 in space the resulting object of the symmetry transformation has a defect-like structure. If we wish to realize Fig[2.1(c)] formally we need to construct a symmetry patch operator, that is a fixed time operator that applies the symmetry transformation only on a part of space. To this purpose consider the charge operator

$$Q_{x_0}(t) = \int_{x_0}^{\infty} J_0 dx, \tag{2.7}$$

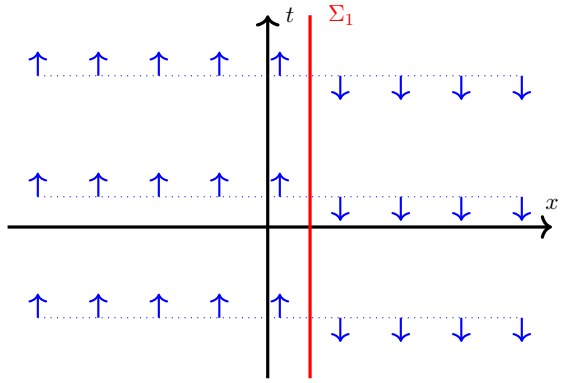
and define the patch symmetry operator $U_{x_0}(\alpha, \bullet) \coloneqq \exp(-\alpha\{Q_{x_0}(t), \bullet\})$, which only transforms spins on the right of x_0. Fig[2.1(c)] is realised by applying $U_{x_0}(\alpha = \pi, \bullet)$. At this point we must make an important clarification. We said that Fig[2.1(c)] represents a defect-like configuration, in this case a domain wall. If we were considering only the $\mathbb{Z}_2$ subgroup of $U(1)$ then domain walls would be well defined defects of the theory, but if we consider the full $U(1)$ global symmetry domain walls are not stable and do not represent proper defects. The true defects for $U(1)$ would be vortices. Thus we must be careful since even though for discrete symmetries applying a symmetry transformation on a codimension 1 in space manifold produces a stable defects, the same procedure for continuous symmetries does not lead to true defects but only defect-like configurations.



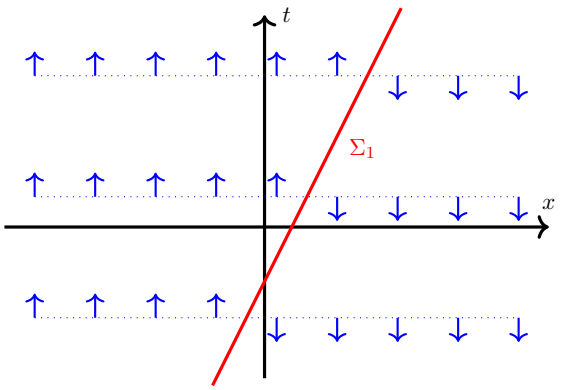
(a) The symmetry transformation applied on patch $\mathcal{P}$ whose boundary defines Σ_1.



(b) The symmetry transformation as an operator U acting at fixed time.



(c) The symmetry transformation as a defect-like configuration with fixed locus.



(d) The symmetry transformation as a defect-like configuration with moving locus.

Figure 2.1: 2.1(b): The symmetry acts at fixed time via the operator U and it maps one ground state into another. 2.1(c): Σ_1 is codimension 1 in space and the symmetry transformation acts only on spins to the right of Σ_1, we obtain a kink-like configuration with fixed locus. 2.1(d): Σ_1 has non zero projection both in space and time and the we obtain a defect-like configuration with moving locus.

2.2 Canonical quantization and operator formalism

Once a classical theory is established we may quantise using the canonical quantization map, which relates fields with operators operators and Poisson brackets with commutators,

$$\begin{cases} \Phi_k(x,t) \\ \pi_k(x,t) \\ \{\bullet,\bullet\} \end{cases} \longmapsto \begin{cases} \hat{\Phi}_k(x,t) \\ \hat{\pi}_k(x,t) \\ -i[\bullet,\bullet] \end{cases} . \tag{2.8}$$

Then all the relations seen above admit a quantum partner. Choosing again Σ_{D-1} to be $\mathbb{R}^d$ at fixed time, we find the charge operator $\hat{Q}(t)$ whose expectation value are preserved by time

evolution, and Eq(2.5) becomes

$$\delta\hat{\Phi}_k(x,t) = i[\hat{Q}(t), \Phi_k(x,t)]. \tag{2.9}$$

The action of the symmetry can be expressed using the quantum analog of Eq(2.6), which is

$$\hat{\Phi}'_k(x,t) = \hat{U}^\dagger\hat{\Phi}_k(x,t)\hat{U} = \hat{\Phi} + i\alpha[\hat{Q}(t), \hat{\Phi}_k(x,t)] + O(\alpha^2), \tag{2.10}$$

where $\hat{U}(t) \coloneqq e^{-i\alpha Q(t)}$. In this section we distinguished between classical and quantum object using the hat symbol. We will now drop this notation and the nature of the various objects we are considering will be specified when needed.

2.3 Discrete symmetries

Discrete symmetries are similar to continuous ones but they lack the differentiable structure. This means that the group parameter α labeling the possible transformations $G(\Phi)$ takes values in a discrete set. In this case there is no conserved current or charge, but we can still define an operator $Z(\Sigma_{D-1}, \alpha)$ which acts the transformation on a $D-1$ dimensional manifold of the system. The interpretation is then analogous to the continuous case as seen in Fig.[2.1] (this time the system would simply be the 1 dimensional Ising model with the usual $\mathbb{Z}_2$ symmetry).

Let us now list the important features of 0-symmetries so that we may explicitly see what we have to generalize. A 0-form symmetry is characterized by the following:

1. A transformation acting non trivially on local operators (defined on zero dimensional manifolds), such as the fields $\Phi(x)$,
2. (continuous only) A 1-form conserved current J, such that $d \star J = 0$,
3. (continuous only) A topological charge Q obtained by integrating $\star J$ over a $D-1$ dimensional manifold Σ_{D-1},
4. An operator defined on Σ_{D-1} which acts the transformation on the system.

Chapter 3

Higher form symmetries

In order to define higher form symmetries we simply increase the characteristic dimensions of 0-form symmetries. In a space-time of dimension D we define a p-form symmetry as a transformation leaving the action invariant with the following properties:

1. The transformation acts non trivially on extended operators supported on manifold of dimension p (this claim will soon be clarified)
2. (continuous only) There exist a conserved $p+1$-form current J_p, such that $d \star J_p = 0$,
3. (continuous only) There exist a topological charge Q_p obtained by integrating $\star J_p$ over a $D-(p+1)$ dimensional manifold $\Sigma_{D-(p+1)}$,
4. There exist an operator defined on $\Sigma_{D-(p+1)}$ which acts the transformation on the system.

Once we establish these conditions the rest should be identical to the 0-form case. In particular for continuous symmetries we can define $U_p \coloneqq e^{i\alpha Q_p}$ and the formalism of Noether's theorem holds the same. [1]

Let us now clarify what we mean by claim 1. As we have seen in Fig.[2.1], for a given 0-form symmetry, applying the operator (or defect) U we can classify the points in our space-time according to their position with respect to Σ_1. In Fig.[2.1(b)] we differentiate between above and below Σ_1, in Fig.[2.1(c)] between left and right, and if Σ_1 were to be closed we would differentiate between the inside and the outside of Σ_1. For a given D dimensional space-time and a $D-1$ dimensional manifold Σ_{D-1} this is always true since in order for a point to move from one side of Σ_{D-1} to the other we must cross Σ_{D-1} and thus apply the transformation upon that point, then the space is classified by U.

If now instead we act the symmetry U on a $D-2$ dimensional manifold Σ_{D-2} then the task of classifying points becomes senseless. Since now any point on one side of Σ_{D-2} can be moved to the other side without crossing Σ_{D-2}, then the only points which are "classified" are those that sit precisely on Σ_{D-2}. We can think of the action of U on points to be "trivial" since it does not differentiate points into subsets. On the other hand if we consider 1 dimensional manifolds, such as loops, then the symmetry can classify between loops that concatenate Σ_{D-2} and loops that do not. It is impossible to move from one type of loop to the other without crossing Σ_{D-2}

[1] In the examples presented later it will often be easier to start from claim 2, find a conserved 2-form current and then reconstruct the transformation from there.

and the space of loops is non trivially classified by the U. For a simple example of this logic consider Fig.[3.1], where we have a $D = 2 + 1$ system (without distinction between space and time).

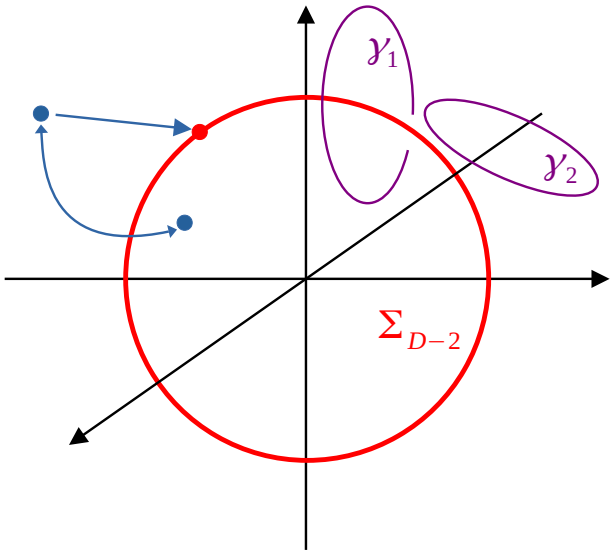


Figure 3.1: Σ_{D-2} is a ring centered on the origin and aligned vertically. Any point can move from the inside to the outside of Σ_{D-2} without crossing it. The only action on points happens when they belong to Σ_{D-2} (red dot in figure). Loops are classified via their topology. In this case U acts on γ_1 but not on γ_2 and the two loops cannot be deformed into each other without crossing Σ_{D-2}. The action of U is non trivial on loops and provides a proper classification.

Chapter 4

Free $U(1)$ gauge theory in $D = 3 + 1$

Let us present a simple example of a 1-form symmetry in a context which should be rather familiar, that is Maxwell theory on $\mathbb{R}^{3+1}$.

Before diving into the calculations we need to make some remarks on the topology of space-time. From the perspective of homology $\mathbb{R}^{3+1}$ is trivial, however we may consider $\mathbb{R}^{3+1}$ as the limit of 4 dimensional disk $\mathbb{D}^4$. $\mathbb{D}^4$ has a non trivial relative homology with respect to its boundary S^3. This implies that if consider the relative homology of $\mathbb{R}^{3+1}$, with respect to the sphere at infinity $S^1_\infty = \partial\mathbb{R}^{3+1}$, we have something non trivial. Namely $H^n(\mathbb{R}^{3+1}, \partial\mathbb{R}^{3+1}) \neq \varnothing$. The non trivial elements of $H^1(\mathbb{R}^{3+1}, \partial\mathbb{R}^{3+1})$ are lines stretching to infinity. Throughout this section every time we talk about about closed and exact form, we intend it in terms of the relative cohomology.

Now consider a 1-form gauge field A defined on $\mathbb{R}^{3+1}$, with the following action,

$$S = \int_{\mathbb{R}^{3+1}} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} d^4x = \int_{\mathbb{R}^{3+1}} -\frac{1}{2} F \wedge \star F = -\frac{1}{2}\langle F, F\rangle, \tag{4.1}$$

where F is the usual electromagnetic tensor $F = dA$. This theory is invariant under a $U(1)$ gauge symmetry that shifts A by an exact form, $A \mapsto A + d\lambda$. We require the gauge field λ and its exterior derivative to vanish at the boundary,

$$\lambda|_{S^1_\infty} = d\lambda|_{S^1_\infty} = 0 \tag{4.2}$$

The equation of motion for the theory are

$$d \star F = 0 \quad \text{and} \quad dF = 0. \tag{4.3}$$

Which are respectively Maxwell's equations and Bianchi identity. These equations imply the presence of two conserved 2-form currents $\star F$ and F, and we would like to verify that these currents can be related to 1-form symmetries.

We will focus on the case of $\star F$. We consider a closed 2 dimensional manifold M at fixed time. We define the charge $Q_e \coloneqq \int_M \star F$ and the operator $U \coloneqq e^{-i\alpha Q_e}$. We fix the temporal gauge $A_0 = 0$ and the CCR are $[A_i(x), F_{0j}(y)] = i\delta_{ij}\delta^{(3)}(x - y)$.

The action of the symmetry is expressed via the commutator

$$\begin{aligned}
[Q_e(t), A_i(t,x)] &= \left[\int_M \frac{1}{4}\epsilon^{\alpha\beta}{}_{jk} F_{\alpha\beta}(z)\, dz^j \wedge dz^k, A_i(x)\right] \\
&= \int_M \frac{1}{4}\epsilon^{\alpha\beta}{}_{jk}[F_{\alpha\beta}(z), A_i(x)]\, dz^j \wedge dz^k \\
&= \int_M \frac{1}{4}\epsilon^{0l}{}_{jk}[F_{0l}(z), A_i(x)] + \epsilon^{l0}{}_{jk}[F_{l0}(z), A_i(x)]\, dz^j \wedge dz^k \\
&= -\int_M \frac{1}{2} i\epsilon^{0l}{}_{jk}\delta_{li}\delta^{(3)}(x-z)\, dz^j \wedge dz^k \\
&= -\int_M \frac{1}{2} i\epsilon^{0}{}_{ijk}\delta^{(3)}(x-z)\, dz^j \wedge dz^k \\
&= -i\hat{M}_i(\mathbb{R}^3).
\end{aligned} \tag{4.4}$$

Where $\hat{M}_i(\mathbb{R}^3)$ is the i component of the Poincare dual of M with respect to $\mathbb{R}^3$ and not to the full space $\mathbb{R}^{3+1}$. Recalling Eq.[2.9] we identify $\delta A_i = \hat{M}_i(\mathbb{R}^3)$, and the symmetry can be written explicitly as

$$A \mapsto A' = A + \alpha\hat{M}(\mathbb{R}^3). \tag{4.5}$$

This is exactly what was expected from Cap.[3], the symmetry acts only on local operators defined on M. In particular notice that $d\hat{M} = 0$ which confirms that the transformation is a symmetry of the action, however we also want $\hat{M}$ not to be exact otherwise Eq(4.5) corresponds to a trivial gauge transformation and not a global symmetry. As a consistency check let us apply Noether's theorem and verify that the conserved charge is the one we expect.

$$\begin{aligned}
J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_j)}\delta A_j \to \tilde{Q}_e &\coloneqq \int_{\mathbb{R}^3} J^0 d^3x \\
&= \int_{\mathbb{R}^3}\int_M F^{0i}(x)\frac{1}{2}\epsilon_{0ijk}\delta^{(3)}(x-z)\, dz^j \wedge dz^k\, d^3x \\
&= \int_M F^{0i}(z)\frac{1}{2}\epsilon_{0ijk}\, dz^j \wedge dz^k \\
&= \int_M \star F.
\end{aligned} \tag{4.6}$$

Thus the charge coming from Noether's theorem is the one we started from.

In this simple case we see the natural emergence of the 1-form aspect of the symmetry. In fact even though Noether's theorem requires the conserved current to be a 1-form the fact that the variation of the field is a Poincare dual allows us to rewrite the conserved charge as the integral of a 2-form over the chosen manifold M making explicit the 1-form nature of the symmetry.

The application of the symmetry on the field operators is consistent but rather uninteresting, since it shifts the fields only if they are evaluated on M. More interesting results can be gathered if we apply the symmetry on extended operators such as the Wilson loops. So let us recall the form of the Wilson loop,

$$W_\gamma \coloneqq e^{i\oint_\gamma A_\mu(x)dx^\mu} = e^{i\oint_\gamma A}, \tag{4.7}$$

Where γ is a closed loop. Now we evaluate the action of the symmetry onto W_γ,

$$U^\dagger W_\gamma U = W_\gamma \exp\left(\alpha\left[\oint_\gamma A, \int_M \star F\right]\right), \tag{4.8}$$

where we used the BCH formula and the fact that the commutator $[A, \star F]$ commutes with both A and $\star F$. In particular we may take γ to be a closed curve in space with the same time coordinate as M. Then the commutator has already been calculated,

$$\left[\oint_\gamma A, \int_M \star F\right] = i \oint_\gamma \hat{M}_i(\mathbb{R}^3)\, dx^i = iI(\gamma, M), \tag{4.9}$$

where $I(\gamma, M)$ is the intersection number between γ and M. Looking back at Eq.[4.8] we see that the 1-form symmetry acts on the Wilson loops by adding a phase and in particular we can now classify loops via a topological "charge", which is $I(\gamma, M)$.

It is important to notice that if we want our Wilson loop to be properly charged under the 1-form symmetry then we must require γ to be non trivial i.e. γ should not be the boundary of a surface, otherwise we would find

$$\oint_{\gamma=\partial\Sigma} \hat{M}(\mathbb{R}^3) = \int_\Sigma d\hat{M}(\mathbb{R}^3) = 0 \tag{4.10}$$

A similar treatment can be carried out for the 1-form symmetry coming from $dF = 0$. This symmetry would yield the magnetic flux as the conserved charge and the charged objects would be 't Hooft loops.

In this example we have seen that a generalised symmetry is realised when we shift A by a 1-form that is closed but not exact, and the charged object turn out to be the Wilson loops. We will see that this is a more general structure which appears when we construct symmetries of p-form gauge theories.

4.1 Wilson loops and symmetry breaking

Wilson loops are interesting objects to study in a QFT since we may gain information about the phase of the theory by looking at their expectation value. We classify the phases of a QFT according to the following criteria.

- Unbroken phase
  In this phase the Wilson loop operator follows an area law,

  $$|\langle W_\gamma \rangle_0| \sim e^{-T_A \cdot Area(\Sigma)}, \tag{4.11}$$

  where $\langle ... \rangle_0 = \langle 0|...|0\rangle$ is the vacuum expectation value and Σ is the smallest surface whose boundary is γ. In this phase expectation values of large loops always tend to zero leaving the associated 1-form symmetry unbroken and we expect the theory to exhibit confinement.

- Broken Phase

In this phase the Wilson loop follows a perimeter law

$$|\langle W_\gamma \rangle_0| \sim e^{-T_P \cdot Perimeter(\gamma)}. \tag{4.12}$$

In this situation the theory is not expected to exhibit confinement and even though the expectation value decays with the size of the loop one can show that through renormalization it is always possible to set the constant T_P to 0 giving a finite $v.e.v.$ also to infinitely large loops. Then the 1-form symmetry is spontaneously broken.

Notice that we consider only the asymptotic behavior of W_γ because we are interested in SSB and thus we only care about what happens in IR regime of the theory, where we have long distances and large loops.

4.2 Physical interpretation of Wilson loops

Let us justify better why the area an perimeter law for the Wilson loop correspond respectively to the confined and de-confined phases. To do so let us provide a physical interpretation of the Wilson loop. Consider the term in the exponential of a Wilson loop W_γ. Note that using Poincare duality one can write

$$\oint_\gamma A = \int_{\mathbb{R}^{3+1}} A \wedge \hat{\gamma} = \int_{\mathbb{R}^{3+1}} A \wedge \star J = \int_{\mathbb{R}^{3+1}} d^4x A_\mu J^\mu, \tag{4.13}$$

where $\hat{\gamma}$ is the Poincare dual of γ and $J = \star\hat{\gamma}$, which we may write explicitly as

$$J^\mu(x) = \oint_\gamma \delta^D(x-y)\, dy^\mu. \tag{4.14}$$

The evaluation of the the expectation value of a Wilson loop is done by inserting W_γ in the path integral. This can also be viewed as shift of the action $S(A)$,

$$S(A) \to S(A) + \oint_\gamma A = S(A) + \int_{\mathbb{R}^{3+1}} d^4x A_\mu J^\mu. \tag{4.15}$$

Then one can interpret the evaluation of the expectation value of W_γ as the the evaluation of the path integral in the presence of an external source $J^\mu(x)$. From Eq(4.14) we see that $J^\mu(x)$ is the current associated to a probe particle created at a point $P \in \gamma$, transported along γ, and the annihilated at P. Another interpretation is that $J^\mu(x)$ represents a particle anti-particle pair created at $P \in \gamma$, transported along γ and then annihilated on the opposite side of γ. This last interpretation proves most useful to us.

Consider a rectangular loop C defined at $y = z = 0$, with $x \in [0, r]$ and $t \in [0, T]$, where r and T are arbitrary. Now let us evaluate the value of the Wilson loop in a static configuration,

namely $\partial_t A_\mu = 0$. For clarity in this calculation we set $x = (t, x, y, z)$ and $\mu = t, x, y, z$. We find

$$\begin{aligned}\oint_C A &= \int_0^r A_x(0,x)dx + \int_0^T A_t(t,r)dt + \int_r^0 A_x(T,x)dx + \int_T^0 A_t(t,0)dt \\ &= \int_0^r A_x(0,x)dx - \int_0^r A_x(T,x)dx + \int_0^T A_t(t,r)dt - \int_0^T A_t(t,0)dt \\ &\overset{(a)}{=} \int_0^T A_t(t,r)dt - \int_0^T A_t(t,0)dt \\ &= (A_t(t,r) - A_t(t,0))T \overset{(b)}{=} V(r)T,\end{aligned} \tag{4.16}$$

where in (a) we used $A(0,x) = A(T,x)$ and in (b) we identified the static potential $V(r) = A_t(t,r) - A_t(t,0)$. Given that W_C is associated to a particle anti-particle pair being transported along C, one can conclude that $V(r)$ represents the static potential between the two probe particles. This calculation was done classically, but the interpretation is still valid at quantum level provided we make the correct adjustments. At quantum level we expect

$$\langle W_C \rangle \sim e^{iV_q(r)T} \to e^{-V_q(r)\tau}, \tag{4.17}$$

where $V_q(r)$ is the effective quantum potential, and we performed an analytic continuation to Euclidian spacetime $T \to i\tau$. Then we way distinguish between different behaviors,

- $V_q(r) \sim r$ as $r \to \infty$. In this case the potential grows linearly with the distance between the particle. This is a confinement phase. In this phase, according to Eq(4.17), one would expect $\langle W_C \rangle \sim e^{-r\tau}$. Note that $r\tau$ is the area of the Wilson loop C, thus the confinement phase corresponds to an area law.

- $V_q(r) \sim k$ as $r \to \infty$, where k is a constant. In this case the potential does not grows linearly with the distance between the particle. This is a de-confinement phase. In this phase, according to Eq(4.17), one would expect $\langle W_C \rangle \sim e^{-k\tau}$. The expectation values of W_C varies only with τ and thus this is a perimeter law.
  - ⋆ An interesting of sub-case is when $V_q(r) \sim 1/r$ as $r \to \infty$. This behavior of the potential corresponds to a Coulomb phase. In this case considering C as a square loop, i.e. $r \sim \tau$ we have $\langle W_C \rangle \sim e^{-\frac{\tau}{r}} \sim e^{-\text{constant}}$. Thus naively one would expect the Wilson loop not to vanish even for large loops. This intuition would however be wrong. In fact as it was argued in [24, 25], the asymptotic behavior of the Wilson loop is upper bounded by a perimeter law. The proves were given on the lattice but it is reasonable to believe that the results generalize in the continuum. Thus even in the Coulomb phase we have that $\langle W_C \rangle$ must vanish for large loops. We expect that in the Coulomb phase the Wilson loop follows a perimeter law. Later on we will make an explicit calculation for Maxwell's in $D = 3 + 1$ and indeed we will see that the Wilson loop follows a perimeter law.

Let us anticipate that later on we will provide a formal renormalization scheme for the Wilson loop. This scheme is necessary to treat the Wilson loop as a proper order parameter. In particular we will use

$$W_\gamma \quad \xrightarrow{\text{renorm}} \quad e^{T_p \cdot Perimeter(\gamma)} W_\gamma, \tag{4.18}$$

with T_p a constant to fix. After the renormalization the connection between the static potential and the behavior of the Wilson loop is summarized by Table 4.1.

Potential	**Wilson loop**	**Phase**
$V_q(r) \to r$	$\langle W_\gamma \rangle \to 0$	Confinement
$V_q(r) \to$ constant	$\langle W_\gamma \rangle \to$ constant	De-onfinement
$V_q(r) \to 1/r$	$\langle W_\gamma \rangle \to$ constant	Coulomb

Table 4.1: The possible asymptotic behaviors of the quantum potential $V_q(r)$, the associated behaviors of the Wilson loop operator after renormalization and the possible phases of the theory.

Chapter 5

p-form gauge theories

Consider a more general p-form gauge theory defined on X, a $D = d+1$ dimensional manifold, where the gauge field is a p-form A with field strength $F \coloneqq dA$. We fix the boundary conditions for A as

$$A|_{\partial X} = \bar{A}. \tag{5.1}$$

A gauge transformation corresponds to shifting A by a closed form $d\gamma$ such that the boundary conditions are unaltered,

$$A \to A + d\gamma \qquad \text{with} \qquad d\gamma|_{\partial X} = 0. \tag{5.2}$$

Notice that we do not put any constraint on the components of $d\gamma$ that are orthogonal to ∂X, in this sense the gauge transformation is not strictly vanishing at the boundaries and it may still change the value of $\star A|_{\partial X}$.

In order to calculate expectation values we need to fix a gauge and in this case we will use

$$d^\dagger A = 0. \tag{5.3}$$

Recalling the representation of $d^\dagger \sim \star d \star$ we can verify that Eq(5.3) is the generalization of the Lorentz gauge to higher form fields[1]. The full action for this theory in Euclidian signature is then given by

$$\begin{aligned} S(A) &= \frac{1}{2g^2}\int_X \left(dA \wedge \star dA + \frac{1}{\alpha} d^\dagger A \wedge \star d^\dagger A\right) + S_{gh} \\ &= \underbrace{\frac{1}{2g^2}\langle dA, dA\rangle}_{\text{free action}} + \underbrace{\frac{1}{2g^2\alpha}\langle d^\dagger A, d^\dagger A\rangle}_{\text{gauge fixing term}} + S_{gh}, \end{aligned} \tag{5.4}$$

where S_{gh} is the ghost term which we will omit in the following since it will not play any role in the calculations.

As we have seen in the case of Maxwell theory this action allows for a generalized p-form global symmetry that shifts A by a p-form σ

$$A \to A + \beta\hat{\sigma} \quad \text{with} \quad d\hat{\sigma} = d^\dagger\hat{\sigma} = 0. \tag{5.5}$$

[1] This particular gauge choice is a subset of a larger class of the type $d^\dagger A = f$.

The requirements that $\hat{\sigma}$ is both closed and coclosed are necessary in order to keep the free action and the gauge fixing term invariant. The conserved current for such symmetry can be written by generalizing Eq(2.1) as is shown in [26].

$$\star J = \delta A \wedge \frac{\delta \mathcal{L}}{\delta(dA)} = \frac{1}{g^2}\hat{\sigma} \wedge \star dA, \tag{5.6}$$

where the variation of $\mathcal{L}$ with respect to dA is implicitly defined in the equation

$$\delta \mathcal{L} \coloneqq \delta A \wedge \frac{\delta \mathcal{L}}{\delta A} + \delta dA \wedge \frac{\delta \mathcal{L}}{\delta(dA)}. \tag{5.7}$$

In Eq(5.6) we neglected the contribution coming from the gauge fixing term since the conserved charge is a property of the free theory and should thus be independent on the gauge parameter. Integrating $\star J$ over a $D-1$ dimensional manifold Σ yields the conserved charge, which is a generalization of the electric flux seen for the Maxwell case,

$$Q = \frac{1}{g^2}\int_\Sigma \hat{\sigma} \wedge \star dA \overset{(a)}{=} (-1)^{D-1}\int_\sigma \star dA, \tag{5.8}$$

where (a) only holds if $\hat{\sigma}$ is the Poincare dual of σ with respect to Σ and not X. The generalization of the Wilson loop is the Wilson cycle

$$W_C = e^{i\int_C A}, \tag{5.9}$$

where C is a p dimensional closed submanifold of X, and the charge of W_C under the global p-form symmetry is again the intersection number $I(C,\sigma)$. The Wilson cycle admits analogous behaviors to those seen for the Wilson loop. In particular we distinguish between two phases for very large loops:

1. $\log|\langle W_C\rangle_0| \sim -Vol_C$, which corresponds to spontaneously broken phase where the Wilson cycles can be renormalized to have non zero expectation value even for large cycles.

2. $\log|\langle W_C\rangle_0| < -Vol_C$, which is the unbroken phase where the expectation of W_C value goes to zero for large cycles.

We stress again that if we want the Wilson cycles to be charged under the symmetry depicted in Eq(5.5) we must require that C belongs to a non trivial class of $H_p(X,\partial X)$, otherwise Stoke's theorem combined with the closeness of σ prevents $\int_C A$ from receiving any meaningful contribution.

We will now move on to the explicit computation and renormalization of $|\langle W_C\rangle_0|$ for this simple free theory. We will follow an analogous treatment to [17]. We will start by investigating the classical behavior of A. The equations of motion are given by minimizing the action the

under the shift $A \to A + \epsilon\delta A$, with $\delta A \in \Omega_D(X)$, yielding

$$\begin{aligned} S(A+\epsilon\delta A) &= S(A) + \frac{\epsilon}{g^2}(\langle dA, d\delta A\rangle + \frac{1}{\alpha}\langle d^\dagger A, d^\dagger \delta A\rangle) + O(\epsilon^2) \\ &\overset{(a)}{=} S(A) + \frac{\epsilon}{g^2}(\langle d^\dagger dA, \delta A\rangle + \frac{1}{\alpha}\langle dd^\dagger A, \delta A\rangle) + O(\epsilon^2) \\ &= S(A) + \frac{\epsilon}{g^2}\langle (d^\dagger d + \frac{1}{\alpha} d^\dagger d)A, \delta A\rangle + O(\epsilon^2) \\ &\to \langle (d^\dagger d + \frac{1}{\alpha} dd^\dagger)A, \delta A\rangle = 0 \to (d^\dagger d + \frac{1}{\alpha} dd^\dagger)A = 0. \end{aligned} \tag{5.10}$$

where in (a) we were allowed to treat $d^\dagger$ as the adjoint of d since δA has Dirichlet boundary conditions. Now we choose for simplicity the Feynman gauge $\alpha = 1$ so we have that the kinematical operator is simply

$$d^\dagger d + \frac{1}{\alpha} dd^\dagger|_{\alpha=1} = \Delta, \tag{5.11}$$

which admits an inverse Δ^{-1}. Now we expand A around a classical solution $\Delta A_{cl} = 0$,

$$A = A_{cl} + A_q, \tag{5.12}$$

and rewrite the boundary conditions in Eq(5.1) as

$$A_{cl}|_{\partial X} = \bar{A} \quad , \quad A_q|_{\partial X} = 0. \tag{5.13}$$

With this choice of boundary the IR behavior of A is controlled by the classical configuration while the quantum fluctuations vanish at the boundary, $A_q \in \Omega^p_D(X)$. The action becomes

$$\begin{aligned} S(A_{cl}+A_q) &= \frac{1}{2g^2}\langle d(A_{cl}+A_q), d(A_{cl}+A_q)\rangle + \frac{1}{2g^2}\langle d^\dagger(A_{cl}+A_q), d^\dagger(A_{cl}+A_q)\rangle \\ &\overset{(a)}{=} S(A_{cl}) + \frac{1}{g^2}(\langle dA_{cl}, dA_q\rangle + \langle d^\dagger A_{cl}, d^\dagger A_q\rangle) \\ &+ \frac{1}{2g^2}(\langle dA_q, dA_q\rangle + \langle d^\dagger A_q, d^\dagger A_q\rangle) \\ &= S(A_{cl}) + \frac{1}{g^2}\langle \Delta A_{cl}, A_q\rangle + \frac{1}{2g^2}\langle A_q, \Delta A_q\rangle \\ &\overset{(b)}{=} S(A_{cl}) + \frac{1}{2g^2}\langle A_q, \Delta A_q\rangle, \end{aligned} \tag{5.14}$$

where in (a) we defined $S(A_{cl})$ which contains all the contributions coming from the classical configuration and in (b) we used the equations of motion to remove the mixed term between A_{cl} and A_q. Now we are ready to move forward and evaluate the expectation value of a Wilson cycle. We work directly in Euclidian signature ,[2] and rewrite the Wilson cycle using the Poincare dual

[2]Recall that the Wilson cycle is not modified under Wick rotation.

of C with respect to X, i.e.$\int_C A = \int_X A \wedge \hat{C}$.

$$\begin{aligned}\langle W_C\rangle_0 &= \int \mathcal{D}A\, e^{-S(A)+i\int_X A\wedge\hat{C}} \\ &= \int \mathcal{D}A_q\, e^{-S(A_{cl})-\frac{1}{2g^2}\langle A_q,\Delta A_q\rangle+i\int_X A_{cl}\wedge\hat{C}+i\int_X A_q\wedge\hat{C}} \\ &\overset{(a)}{=} e^{-\tilde{S}(A_{cl},C)}\int \mathcal{D}A_q\, e^{-\frac{1}{2g^2}\langle A_q,\Delta A_q\rangle+i\int_X A_q\wedge\hat{C}},\end{aligned} \tag{5.15}$$

where in (a) we reabsorbed $i\int_X A_{cl}\wedge\hat{C}$ in $\tilde{S}(A_{cl},C)$ and even though this term is explicitly dependent on the cycle C, we can safely ignore since it only contributes a phase and we are interested in the behavior of the $|\langle W_C\rangle|$. Now we shift A_q in order to cancel the mixed term between A_q and $\hat{C}$,

$$A_q \to A_q + (-1)^{p(D-p)} i g^2 \Delta^{-1}\star\hat{C}. \tag{5.16}$$

Such shift is allowed as long as it does not change the boundary conditions in Eq(5.13), and this can be achieved by choosing a C that does not intersect ∂X. Eq(5.15) becomes

$$\begin{aligned}\langle W_C\rangle_0 =& e^{-\tilde{S}(A_{cl},C)}\int \mathcal{D}A_q\, e^{-\frac{1}{2g^2}\langle A_q,\Delta A_q\rangle-\frac{g^2}{2}\langle\star\hat{C},\Delta^{-1}\star\hat{C}\rangle} \\ =& N e^{-\tilde{S}(A_{cl},C)}\,\frac{1}{\sqrt{\det\Delta}}\, e^{-\frac{g^2}{2}\langle\hat{C},\Delta^{-1}\hat{C}\rangle},\end{aligned} \tag{5.17}$$

where we first used the fact that the laplacian and the Hodge dual operator commute, and then we took advantage of Eq(B.18). The factor N is simply a multiplicative constant coming from the integration over A_q. For our purposes, the only relevant term in Eq(5.17) is

$$e^{-\frac{g^2}{2}\langle\hat{C},\Delta^{-1}\hat{C}\rangle}, \tag{5.18}$$

since it dictates the behavior of $|\langle W_C\rangle|$ as a function of C. We need to evaluate the scalar product in the exponential of Eq(5.18),

$$\begin{aligned}\langle\hat{C},\Delta^{-1}\hat{C}\rangle &= \int_X \hat{C}\wedge\star\Delta^{-1}\hat{C} \\ &= (-1)^{p(D-p)}\int_X \Delta^{-1}\star\hat{C}\wedge\hat{C} \\ &= (-1)^{p(D-p)}\int_C \Delta^{-1}\star\hat{C}.\end{aligned} \tag{5.19}$$

Δ^{-1} is an operator that maps $\Omega^p(X)$ into itself and we can write explicitly as,

$$(\Delta^{-1})^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_p}(x) = \int_X d^D z \underbrace{\Delta_G(x-z)\delta^{\mu_1\dots\mu_p}_{\nu_1\dots\mu_p}}_{\text{Feynman propagator}}, \tag{5.20}$$

where $\delta^{\mu_1\dots\mu_p}_{\nu_1\dots\mu_p}$ is the generalised Kronecker delta and $\Delta_G(x-y)$ is the Green function of the Laplacian, which in flat space can be written as

$$\Delta_G(x-y) = \int \frac{d^D k}{(2\pi)^D} e^{ik(x-y)}\frac{1}{k^2}. \tag{5.21}$$

Now we take the integral representation of $\hat{C}$, as seen in Eq(B.28), and we obtain

$$\begin{aligned}(-1)^{p(D-p)}\int_C \Delta^{-1}\star\hat{C} &= \frac{1}{(p!)^2}\int_C dy^{\nu_1}\wedge\cdots\wedge dy^{\nu_p}\int_C dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\\ &\cdot\int_X d^Dz\sqrt{|g(z)|}\Delta_G(x-z)\delta^{(D)}(z-y)\delta^{\alpha_1\ldots\alpha_p}_{\nu_1\ldots\mu_p}\delta_{\alpha_1\ldots\alpha_p\mu_1\ldots\mu_p}\\ &= \int_C d^py\sqrt{|\lambda_C(y)|}\int_C d^px\Delta_G(x-y)\\ &\overset{(a)}{=} Vol_C\int_C d^px\Delta_G(x-y_0),\end{aligned} \tag{5.22}$$

where in the last step we separated the integrals using the symmetry of $\Delta_G(x-y)$ and we fixed y_0 as a reference point on C. Putting all together we have

$$|\langle W_C\rangle_0|\sim\exp\Big[-\frac{g^2}{2}Vol_C\int_C d^px\Delta_G(x-y_0)\Big]. \tag{5.23}$$

It is clear that the integral in x is responsible for determining whether or not SSB occurs. If $\int_C d^px\Delta_G^{-1}$ produces a term which is independent of C as we take large cycles then the only contribution to the behavior of $|\langle W_C\rangle|$ comes from Vol_C. We may redefine the Wilson cycle as

$$W_C^{'} = W_C\, e^{\frac{g^2}{2}\int_C vol_C}, \tag{5.24}$$

and $W^{'}$ would have expectation value different from zero in the limit of large loop, thus causing SSB. The renormalization in Eq(5.24) can be reproduced at the level of the Euclidian action by the addition of geometrical counterterm

$$S_{CT} = -\frac{g^2}{2}\int_X vol_C\wedge\hat{C}. \tag{5.25}$$

However if $\int_X\Delta_G^{-1}$ produces a term that grows with the size of C then a counterterm as in Eq(5.25) is not enough to cancel the divergence in the exponential of $|\langle W_C\rangle|$ and we have that the Wilson cycle has zero expectation value for large cycles, thus preventing SSB. As an example let us go back to Maxwell theory where $D=4$ and $p=1$, then we have $\Delta_G(x-y_0)=\frac{1}{4\pi}\frac{1}{|x-y_0|^2}$. We take C to be a ring S^1 of radius R and use polar coordinates around the center of the ring. We denote by θ the angle between x and y_0.

$$\begin{aligned}Vol_C\int_{S^1}dx\frac{1}{4\pi}\frac{1}{|x-y_0|^2} &= \frac{R}{2}\int_{0+\epsilon}^{2\pi-\epsilon}d\theta\frac{R}{2R^2-2R^2\cos\theta}\\ &= \frac{1}{4}\int_{0+\epsilon}^{2\pi-\epsilon}d\theta\frac{1}{1-\cos\theta},\end{aligned} \tag{5.26}$$

where we added an ultraviolet regulator ϵ to avoid the divergence caused by overlapping x and y_0. The angular cutoff ϵ should be derived from a more rigorously defined spatial ultraviolet cutoff a. Using only a and the other characteristic length of the problem R, the most natural option is then is $\epsilon=\frac{a}{R}$, so that now ϵ corresponds to the infinitesimal angle whose arc length is the cutoff a. With this choice of regulator we find

$$\frac{1}{4}\int_{0+\epsilon}^{2\pi-\epsilon}d\theta\frac{1}{1-\cos\theta} = \frac{1}{2}\frac{1+\cos\epsilon}{\sin\epsilon}\sim\frac{R}{a}+O(\epsilon). \tag{5.27}$$

The first term of Eq(5.27) is the divergent contribution which is however proportional to the perimeter of S^1, then we may renormalize it using a counterterm as seen in Eq(5.25), resulting in a symmetry broken phase.

To conclude this chapter we note that there exists a special case of Eq(5.22) which provides a lot of insight in a more general version of the Mermin-Wagner theorem. Consider the situation where X is D dimensional box of size L and C a p dimensional copy of X such that $\partial C \subseteq \partial X$. Note that C is not closed but relatively closed, which is allowed given that the gauge transformation vanishes at ∂X according to Eq(5.2). Eq(5.22) then becomes, choosing $y_0 = 0$

$$\begin{aligned} Vol_C \int_C d^p x \Delta_G(x - y_0) &= L^p \int_C d^p x \Delta_G(x) \\ &= L^p \int_C d^p x \int \frac{d^D k}{(2\pi)^D} e^{ikx} \frac{1}{k^2} \\ &= L^p \int d^{D-p} k_\perp \frac{1}{(2\pi)^{D-p}} \frac{1}{k_\perp^2}, \end{aligned} \tag{5.28}$$

where the integration over the directions parallel to C produces Dirac deltas and the integral in momentum space is the restricted to only the modes that are perpendicular to C. Now if we let $L \to \infty$ we move deep in the IR where the value of k is arbitrarily small, but the last integral in Eq(5.28) becomes IR divergent if $D \leq p + 2$. Since the IR divergence cannot be renormalized the only possibility is to have Wilson cycle with vanishing expectation value, thus defining a lower critical dimension for p-form global symmetries.

Eq(5.28) was derived for Maxwell's theory, however its applicability can be extended. In fact the divergence of $\langle W_C \rangle_0$ is caused by a IR contribution, and thus dominated by the IR behavior of the theory.

Assume to start with a more complex theory, including more massless or massive fields and interaction terms. Now integrate out the massive degrees of freedom to obtain a low energy effective field theory. This IR theory will contain interaction terms between massless fields, including A. However because of the $U(1)$ p-form symmetry all interaction term between A and other fields must be proportional to dA. This implies that in perturbation theory the couplings between A and other fields must be proportional to one power of the momentum k_μ of A. If we work at very low energies, namely in the limit $k_\mu \to 0$, then all interaction terms are negligible since their couplings are very small. Then any extra correction to Maxwell's theory would produce subleading contributions in Eq(5.28). This implies that the divergence of Eq(5.28) cannot be avoided by any theory that flows towards Maxwell's theory. Recalling that A can be interpreted as the Goldstone boson of the p-form symmetry, then this logic corresponds to the standard statement that Goldstone bosons are free particle deep in the IR [27].

Chapter 6

$SU(N)$ center symmetry

So far we considered $U(1)$ gauge theories and the associated $U(1)$ 1-form symmetries, said symmetries were connected to the conservation of the electric flux. These example were pedagogical and gave us the opportunity to do explicit calculation. Now we turn to the case of $SU(N)$ gauge theories in $D = 4$ Euclidean spacetime $\mathbb{R}^4$, with τ indicating the Euclidean time and $x = (\tau, \vec{x})$. $SU(N)$ gauge theories are non abelian and thus show phenomena of IR confinement and UV asymptotic freedom. In this case the classification of phases provided by the Wilson loop is of more practical interest. We will see that for $SU(N)$ we still find a 1-form symmetry, which however is a discrete finite symmetry associated to the center $\mathbb{Z}_N$ of $SU(N)$ and it thus called a center symmetry. For this chapter the main references are [[2, 4, 28]. In $SU(N)$ gauge theories the degrees of freedom are Lie algebra valued 1-form gauge field which can be written on patches as

$$A = A^a \lambda_a = A^a_\mu \lambda_a dx^\mu, \tag{6.1}$$

where the set $\{\lambda_a\}$ indicates the generators of the Lie algebra $su(N)$. The gauge group acts on A as

$$A(x) \mapsto \Omega(x) A(x) \Omega^\dagger(x) + \frac{i}{g} \Omega(x) d\Omega^\dagger(x), \quad \Omega(x) \in SU(N), \tag{6.2}$$

with g being the gauge coupling constant. The gauge covariant field strength can be written as

$$G = dA - igA \wedge A, \tag{6.3}$$

and the gauge invariant action is

$$S = -\frac{1}{2} \int Tr[G \wedge \star G]. \tag{6.4}$$

The Wilson line operators are given by

$$W_\gamma(x, y) = \mathcal{P} e^{i \int_x^y A}, \tag{6.5}$$

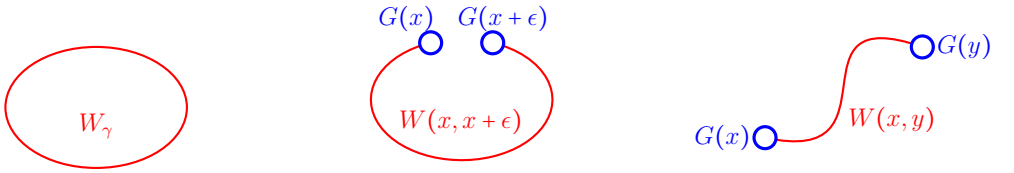
where $\mathcal{P}$ indicates the path order, and the integral is take over a line γ connecting x and y. Under a gauge transformation we have

$$W_\gamma(x, y) \mapsto \Omega(y) W_\gamma(x, y) \Omega^\dagger(x), \tag{6.6}$$

and thus the gauge invariant Wilson loop operator in a representation R of the gauge group is

$$W_{\gamma,R} = Tr_R\big[\mathcal{P}e^{i\oint_\gamma A}\big]. \tag{6.7}$$

It is important to note that the Wilson loop in $SU(N)$ theories is dependent in the representation R that we pick upon taking the trace. Differently from the case of $U(1)$, the action is not invariant under the shift of A by a closed form, since now we have a quadratic term within the field strength and thus $SU(N)$ gauge theories do not have a $U(1)$ 1-form symmetry. This is somehow expected, since in $SU(N)$ gauge theories Wilson loop operators, which would be charged under a hypothetical 1-form symmetry, can be opened in a gauge invariant way. To do so we can insert two copies of the field strength on a two points of the loop γ which are close to each other, then we may open γ without breaking gauge invariance. This would lead to a Wilson line operator with field strength operators on the boundary. Physically this procedure correspond to starting and ending the Wilson line on the charged particles of the theory (for example in $SU(3)$ these particles would be the gluons). This procedure is referred to as screening of Wilson lines and it is depicted in Fig[6.1]. If a Wilson loop can be opened then it can be shrunk to a point, thus it cannot be charged under a 1-form symmetry since this screening effect would imply that the 1-form symmetry charge is not conserved.



(a) A Wilson loop defined on γ in the adjoint representation.

(b) Inserting two copies of the field strength very close to each other one can open the loop.

(c) The loop can be stretched into a Wilson line between x and y.

Figure 6.1: The screening of a Wilson loop operator in the adjoint representation. Starting from a Wilson loop W_γ one can insert copies of the field strength and open a hole in the loop. Stretching the loop leads to the gauge invariant operator $Tr\big[G(y)W(x,y)G(x)\big]$. The procedure never breaks gauge invariance.

However it is not true that any Wilson loop can be screened by dynamical particles. In fact the dynamical particles, represented by the field strength, are always taken in the adjoint representation. This means that Wilson loops can be opened by applying multiple copies of the field strength on a point if and only if their representation is "similar" to the adjoint one (this statement will be made more clear in the case of $SU(2)$). For example, Wilson loops in the fundamental representation cannot be screened by dynamical particles and thus are good candidate to be charged object of a 1-form symmetry. Note that this argument is only valid for pure Yang-Mills theories. If we introduce matter fields, which are in the fundamental representation, then also Wilson loops in the fundamental representation can be screened by matter particles. Thus they cannot be charged objects of an hypothetical 1-form symmetry. Let us see how to construct said symmetry for pure Yang-Mills theories.

As in [29] we begin by compactifying our spacetime along one direction. For simplicity we will compactify along the time direction and the new spacetime manifold is $S^1 \times \mathbb{R}^3$. We denote the circumference of S as β so that $\tau \sim \tau + \beta$. Now consider a standard gauge transformation associate to a group element $\Omega(x)$. When we loop around S^1 the gauge transformation must come back to itself as we complete the cycle,

$$\Omega(\tau + \beta, \vec{x}) = \Omega(\tau, \vec{x}). \tag{6.8}$$

By construction the Yang-Mills action is invariant under such transformation. Now let us modify of Eq(6.8). Consider a local transformation $\tilde{\Omega}(x) \in SU(N)$ which upon looping around S^1 comes back to itself up to some element h in the center of $SU(N)$,

$$\tilde{\Omega}(\tau + \beta, \vec{x}) = h\tilde{\Omega}(\tau, \vec{x}). \tag{6.9}$$

The center of $SU(N)$ is $\mathbb{Z}_N$ and recall that an element $h \in \mathbb{Z}_N$ acts in a trivial way on any element of the group $SU(N)$ and the algebra $su(N)$. In particular for $SU(N)$ we have that the elements of the center are the identity element times an N-th root of 1, namely $h \in e^{i\frac{2\pi\mathbb{Z}}{N}}\mathbb{1}$. Different representations of the $SU(N)$ can be classified according to the action the the generator of the center, denoted as g_R. In general one has

$$g_R = e^{i\frac{2\pi}{N}I_R(N)}\mathbb{1}, \tag{6.10}$$

where $I_R(N) \in \mathbb{Z} \bmod N$ is a number that depends on the representation called the N-ality of the representation ([30]). For example the adjoint representation has $I_R(N) = 0$, while the fundamental has $I_R(N) = 1$. Given a representation R an element h_R of the center is represented as

$$h_R = e^{i\frac{2\pi\mathbb{Z}}{N}I_R(N)}\mathbb{1}. \tag{6.11}$$

Eq(6.11) defines the action of the center of $SU(N)$ in a representation R.

Locally $\tilde{\Omega}(x)$ acts exactly as a gauge transformation so that the action is invariant, but because of the global structure $\tilde{\Omega}(x)$ is not a true gauge transformation. In fact $\tilde{\Omega}(x)$ acts on Wilson loops along the time direction, these are usually called Polyakov loops.

$$\begin{aligned} W_{\gamma,R} = Tr_R\big[\mathcal{P}e^{i\int_0^\beta A_0(\tau,\vec{x})d\tau}\big] &\mapsto Tr_R\big[\tilde{\Omega}(0,\vec{x})\mathcal{P}e^{i\int_0^\beta A_0(\tau,\vec{x})d\tau}\tilde{\Omega}^\dagger(\beta,\vec{x})\big] \\ &= Tr_R\big[\tilde{\Omega}(0,\vec{x})\mathcal{P}e^{i\int_0^\beta A_0(\tau,\vec{x})d\tau}\tilde{\Omega}^\dagger(0,\vec{x})h^\dagger\big] \\ &= Tr_R\big[h^\dagger\mathcal{P}e^{i\int_0^\beta A_0(\tau,\vec{x})d\tau}\big] \\ &= e^{i\frac{2\pi\mathbb{Z}}{N}I_R(N)}Tr_R\big[\mathcal{P}e^{i\int_0^\beta A_0(\tau,\vec{x})d\tau}\big] \\ &= e^{i\frac{2\pi\mathbb{Z}}{N}I_R(N)}W_{\gamma,R}. \end{aligned} \tag{6.12}$$

The Polyakov loop is then charged with $\mathbb{Z}_N$ charge under the action $\tilde{\Omega}(x)$, which represents our center symmetry. Now we will show that this center symmetry can be realized as a shift symmetry of the field A, in a similar fashion to what we saw for Maxwell theory.

The compactification along the time direction we did previously was arbitrary and could be done along any direction. We will now construct a more general form of Eq(6.9). We wish to

find a local transformation $\tilde{\Omega}(x)$ that picks up an element of the center when we loop around a non trivial cycle. To do so consider the following ansatz,

$$\tilde{\Omega}(x) = e^{i\int_0^x a}\Omega(x), \tag{6.13}$$

where $\Omega(x)$ is a true gauge transformation respecting Eq(6.8) and a is a $\mathbb{Z}_N$ 1-form gauge field. $\mathbb{Z}_N$ gauge fields are characterized by two properties, they are closed, so $da = 0$ and their holonomies are quantized, namely $\oint_\gamma a \in \frac{2\pi\mathbb{Z}}{N} \bmod 2\pi$. In a general spacetime X, a is an elements of first cohomology group with $\mathbb{Z}_N$ periods, noted as $H^1(X;\mathbb{Z}_N)$. When we loop around a cycle $\tilde{\Omega}(x)$ picks up a phase corresponding to an element of the center of $SU(N)$. Eq(6.13) represents the center symmetry we were looking for. Let us see how $\tilde{\Omega}(x)$ acts on the fields A. Recalling the behavior of A under a gauge transformation (Eq(6.2)) we have

$$\begin{aligned}
A &\mapsto \tilde{\Omega}(x)A\tilde{\Omega}^\dagger(x) + i\tilde{\Omega}(x)d\tilde{\Omega}^\dagger(x) \\
&= \Omega(x)A\Omega^\dagger(x) + ie^{i\int_0^x a}\Omega(x)d\big(e^{-i\int_0^x a}\Omega^\dagger(x)\big) \\
&= \Omega(x)A\Omega^\dagger(x) + i\Omega(x)d\Omega^\dagger(x) + ie^{i\int_0^x a}de^{-i\int_0^x a}\mathbb{1} \\
&= \underbrace{\Omega(x)A\Omega^\dagger(x) + i\Omega(x)d\Omega^\dagger(x)}_{\text{gauge transformation of } A} + \underbrace{a\mathbb{1}}_{\text{shift by a } \mathbb{Z}_N \text{ field}} .
\end{aligned} \tag{6.14}$$

Thus we have that up to an irrelevant gauge transformation the center symmetry acts on A via a shift by a $\mathbb{Z}_N$ gauge fields. In short we can represent the action of global center symmetry as

$$A \mapsto A + a\mathbb{1}, \quad a \in H^1(X;\mathbb{Z}_N). \tag{6.15}$$

Wilson loop operators are charged under the center symmetry with $\mathbb{Z}_N$ charge. It is crucial to note that the charge of the Wilson loop is dependent on the specific representation that is chosen upon taking the trace in W_γ, R. This is because even if the center of $SU(N)$ is $\mathbb{Z}_N$ for any representation, the elements of the center actually depend on the specific representation. We will see examples of this for $SU(2)$ and $SU(3)$. We will find that Wilson loops in the adjoint representation, which can be screened by dynamical particles, have zero charge under the center symmetry. While Wilson loops in the fundamental representation, which cannot be screened by particles, have non zero charge.

6.1 $SU(2)$ center symmetry

Previously we talked about representations that are "similar" to the adjoint one and that can be screened by dynamical particles. In the case of $SU(2)$ we can make this classification more rigorous. Recall that in $SU(2)$ representations can be classified according to the value of the quadratic Casimir operator $C = \sum_{a=1}^3 (\lambda_a)^2 = j(j+1)\mathbb{1}$, which corresponds to the $SU(2)$ spin $j \in \mathbb{Z}/2$. For example the fundamental representation has spin $j = 1/2$, while the adjoint representation has spin $j = 1$. All dynamical particles in $SU(2)$ Yang-Mills theory are in the adjoint and thus have spin $j = 1$. Recall that starting from the $j = 1$ representation one can obtain all other integer spin representation by tensor multiplication. On the other hand given only an integer representation it is impossible to get an half integer representation. To open a

Wilson loop in a gauge invariant way we need to match it with an operator that has the same spin. Then we conclude Wilson loops with integer spin can be opened by applying multiple copies of the field strength, while Wilson loops with half-integer spin cannot be opened in Yang-Mills theory.

As an example consider the fundamental representation of $SU(2)$ with $j = 1/2$. The generators of Lie algebra are the Pauli matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.16}$$

The center $\mathbb{Z}_2$ of $SU(2)$ is represented by the set $\{\mathbb{1}, -\mathbb{1}\} = \{e^{i\pi n}\mathbb{1} | \, n \in \mathbb{Z}\}$. According to the transformation property of the Wilson under the center symmetry we have

$$W_{\gamma,R=Fun} \mapsto e^{i\pi n} W_{\gamma,R=Fun}. \tag{6.17}$$

Thus Wilson loops in the fundamental representation are charged under the center symmetry, as it is expected since they cannot be screened.

Now consider the adjoint representation where the generator are proportional to the structure constants

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.18}$$

The center of the group is still $\mathbb{Z}_2$ but in the adjoint the center is represented by only the identity element $\mathbb{1}$. This means that Wilson loops in the adjoint representation cannot pick up any phase under the center symmetry, thus their charge is zero. This result stems from the fact that Wilson loop in the adjoint representation can in fact be screened by dynamical particles.

6.2 $SU(3)$ center symmetry

For $SU(3)$ Yang-mills theory we do not have a simple classification as the one provided by the spin on $SU(2)$. Nonetheless we can use the N-ality to understand if a Wilson loop is charged or not under the center symmetry. Let us make explicit examples using the fundamental and adjoint representation. In the fundamental representation the generator of the Lie algebra are the Gell-Mann matrices and the center $\mathbb{Z}_3$ is represented as $\{e^{i\frac{2\pi n}{3}}\mathbb{1} | \, n \in \mathbb{Z}\}$. While in the adjoint representation the generators are given by the structure constants times i, and the center is represented by only the identity element $\mathbb{1}$. As for $SU(2)$ we conclude that Wilson loops in the adjoint representation are have zero under the center symmetry, as expected since they can screened by charged gluons.

For general groups $SU(N)$ it possible to achieve a formula for the charge of a Wilson loop ([4]) in any representation. Recall how the symmetry can be regarded as a shift of the gauge field A as in Eq(6.15). Then we may write

$$W_{\gamma,R} \mapsto e^{iI_R(N) \oint_\gamma a} W_{\gamma,R}. \tag{6.19}$$

Chapter 7

Generalized Goldstone boson theorem

Analogously to 0-form symmetry case, also when we are dealing with a p-form symmetry we have a version of the Goldstone theorem guaranteeing the presence of a massless mode in our spectrum. The proof we present is an adaptation of what can be found in [21]. For this section we will consider Wilson loop operators W_C defined on fixed time loops C. Then W_C is interpreted as a fixed time operator acting on the Hilbert space of the theory. As for ordinary symmetries the key feature is the presence of non trivial expectation values for charged operator. In the broken phase we will take the expectation value of W_C to be a constant different form zero even for large cycles (after renormalization that is),

$$\langle W_C \rangle_0 \coloneqq K. \tag{7.1}$$

Now pick a $D-p-1$ dimensional manifold M that defines the conserved current J^μ_M and the conserved charge Q_M. We have that $\langle [Q_M, W_C] \rangle_0$ and $\langle [J^\mu_M(y), W_C] \rangle_0$ cannot vanish everywhere and in particular if M and C are manifolds defined at the same fixed time we have

$$\langle [Q_M, W_C] \rangle_0 = I(M,C)K \neq 0. \tag{7.2}$$

Before we move on with the proof of the theorem let us recall the form of the differential phase space for a multiparticle state with n particles

$$d\Pi_n = \prod_{j \in n} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{E_j}, \tag{7.3}$$

where E_j are the energies of the particles in the state. We can then construct the completeness relation

$$\sum_n \int d\Pi_n |n\rangle\langle n| = \mathbb{1}, \tag{7.4}$$

where the sum over n spans all possible states of the theory and each state $|n\rangle$ is an eigenvector of the momentum operator $\hat{P}$ with eigenvector p_n.

Let us now find an alternative representation for $\langle [J^\mu_M(y), W_C]\rangle_0$,

$$\begin{aligned}
\langle [J^\mu_M(y), W_C]\rangle_0 &= \sum_n \int d\Pi_n \Big[\langle 0|J^\mu_M(y)|n\rangle\langle n|W_C|0\rangle - \langle 0|W_C|n\rangle\langle n|J^\mu_M(y)|0\rangle \Big] \\
&= \sum_n \int d\Pi_n \Big[\langle 0|J^\mu_M(0)|n\rangle\langle n|W_C|0\rangle e^{ip_n y} - \langle 0|W_C|n\rangle\langle n|J^\mu_M(0)|0\rangle e^{-ip_n y} \Big] \\
&= \int d^4p \sum_n \int d\Pi_n \Big[\langle 0|J^\mu_M(0)|n\rangle\langle n|W_C|0\rangle e^{ipy} - \langle 0|W_C|n\rangle\langle n|J^\mu_M(0)|0\rangle e^{-ipy} \Big] \delta^{(4)}(p - p_n) \\
&\overset{(a)}{=} \int \frac{d^4p}{(2\pi)^3} \Big[\rho^\mu_{MC}(p) e^{ipy} - \tilde{\rho}^\mu_{MC}(p) e^{-ipy} \Big],
\end{aligned} \tag{7.5}$$

where in (a) we defined the two spectral functions

$$\begin{aligned}
\rho^\mu_{MC}(p) &\coloneqq \sum_n \int d\Pi_n \langle 0|J^\mu_M(0)|n\rangle\langle n|W_C|0\rangle \delta^{(4)}(p - p_n) \\
\tilde{\rho}^\mu_{MC}(p) &\coloneqq \sum_n \int d\Pi_n \langle 0|W_C|n\rangle\langle n|J^\mu_M(0)|0\rangle \delta^{(4)}(p - p_n).
\end{aligned} \tag{7.6}$$

Using Lorentz covariance and the fact that all momenta in the spectrum must reside in the positive forward light cone, and thus have $p_0 > 0$, we can decompose the two spectral functions as

$$\begin{aligned}
\rho^\mu_{MC}(p) &= p^\mu \rho(-p^2)\theta(p_0) F(M, C) \\
\tilde{\rho}^\mu_{MC}(p) &= p^\mu \tilde{\rho}(-p^2)\theta(p_0) \tilde{F}(M, C),
\end{aligned} \tag{7.7}$$

where $F(M,C)$ and $\tilde{F}(M,C)$ are topological factors accounting for the relative structure of the two manifolds M and C. Notice that the charged operator W_C is not self adjoint but rather

$$W_C^\dagger = e^{-i\int_C A} = e^{i\int_{-C} A} = W_{-C}, \tag{7.8}$$

where $-C$ stands for the manifold C taken with opposite orientation. Taking the adjoint of Eq(7.5) and requiring that it properly matches with the spectral representation of $\langle [W_{-C}, J^\mu_M(y)]\rangle_0$ yields the following conditions

$$\rho^*(-p^2) = \tilde{\rho}(-p^2) \qquad\qquad F^*(M, C) = \tilde{F}(M, -C). \tag{7.9}$$

Notice moreover that in the definition of the spectral functions, Eq(7.7), we see that they differ only by the order with which we insert W_C and $J^\mu_M(y)$. The topological factor, which only cares about the geometry of the problem, should be unaltered by such order and thus it reasonable to take $F(M,C) = \tilde{F}(M,C)$. To satisfy the condition in Eq(7.9) we could consider $F(M,C) = iI(M,C)$, which for now is just a guess, but as we will see it is the correct choice in

order to reproduce Eq(7.2). Going back to Eq(7.5) we now have

$$\begin{aligned}\langle [J^\mu_M(y), W_C]\rangle_0 &= I(M,C)\frac{\partial}{\partial y_\mu}\int \frac{d^4p}{(2\pi)^3}\Big[\rho(-p^2)\theta(p_0)e^{ipy} + \tilde{\rho}(-p^2)\theta(p_0)e^{-ipy}\Big] \\ &= I(M,C)\frac{\partial}{\partial y_\mu}\int \frac{d^4p}{(2\pi)^3}\int d\mu^2\Big[\rho(\mu^2)\theta(p_0)e^{ipy} + \tilde{\rho}(\mu^2)\theta(p_0)e^{-ipy}\Big]\delta(p^2+\mu^2) \\ &= I(M,C)\frac{\partial}{\partial y_\mu}\int d\mu^2\Big[\rho(\mu^2)\Delta_+(y,\mu^2) - \tilde{\rho}(\mu^2)\Delta_+(-y,\mu^2)\Big],\end{aligned} \tag{7.10}$$

where we defined

$$\Delta_+(y,\mu^2) = \int \frac{d^4p}{(2\pi)^3}\delta(p^2+\mu^2)\theta(p_0)e^{ipy}. \tag{7.11}$$

As can be seen in [31] the Δ_+ function constitutes part of the arbitrary time commutator for a generic scalar theory, and in particular thanks to locality we may infer that it is a symmetric function of y if y is spacelike. Now recall that for higher form symmetries the current $J^\mu_M(y)$ is proportional to the Poincare dual of M, which has support only in a neighborhood of M itself. Consider a point y which is both spacelike and far away from M, then the commutator $[J^\mu_M(y), W_C]$ must vanish by locality and taking advantage of the symmetry of $\Delta_+(y,\mu^2)$ we have

$$\begin{aligned}0 &= I(M,C)\frac{\partial}{\partial y_\mu}\int d\mu^2\Big[\rho(\mu^2) - \tilde{\rho}(\mu^2)\Big]\Delta_+(y,\mu^2) \\ &\to \rho(\mu^2) - \tilde{\rho}(\mu^2) = 0.\end{aligned} \tag{7.12}$$

The final expression for the representation of the commutator is

$$\langle [J^\mu_M(y), W_C]\rangle_0 = I(M,C)\frac{\partial}{\partial y_\mu}\int d\mu^2\rho(\mu^2)\Big[\Delta_+(y,\mu^2) - \Delta_+(-y,\mu^2)\Big]. \tag{7.13}$$

At this point the procedure to obtain the Goldstone theorem is completely analogous to [21]. Take the derivative of Eq(7.13) with respect to y^μ, the LHS vanish since $J^\mu_M(y)$ is conserved and for the RHS use

$$\Box_y\Delta_+(y,\mu^2) = \mu^2\Delta_+(y,\mu^2), \tag{7.14}$$

to arrive at the condition

$$\mu^2\rho(\mu^2) = 0. \tag{7.15}$$

Then take the 0 component of Eq(7.13) at $t = 0$ and integrate it over all space, the LHS becomes Eq(4.5), while for the RHS we have

$$\begin{aligned}\int d^4y\frac{\partial}{\partial y_0}\int d\mu^2\rho(\mu^2)\Big[\Delta_+(y,\mu^2) - \Delta_+(-y,\mu^2)\Big] &= i\int d^3y\delta^{(3)}(\vec{y})\int d\mu^2\rho(\mu^2) \\ &= i\int d\mu^2\rho(\mu^2).\end{aligned} \tag{7.16}$$

Matching both sides we end up with

$$I(M,C)K = iI(M,C)\int d\mu^2\rho(\mu^2), \tag{7.17}$$

leading us to the condition

$$\rho(\mu^2) = -iK\delta(\mu^2), \tag{7.18}$$

which is exactly the contribution arising from the presence of massless particle in the spectrum.

7.1 Robustness under RG flow

The topological nature of higher form symmetries grants an extra feature with respect to their ordinary counterparts. We anticipate the final result of a later chapter which is

*Emergent higher form symmetries are **exact** under renormalization group flow.*

To understand this property we must first define what it means for a symmetry to be **emergent**. Assume we start with an UV theory characterised by a symmetry group G and a Lagrangian $\mathcal{L}(\Phi_l, \Phi_h)$ describing the dynamics of two sets of fields where,

$$\begin{aligned} \Phi_l &= \{\text{fields with mass} < \Lambda_{IR}\} \\ \Phi_h &= \{\text{fields with mass} > \Lambda_{IR}\}. \end{aligned} \tag{7.19}$$

If we are working at energies scales smaller then Λ_{IR} then we can integrate out the heavy degrees of freedom obtaining an effective action describing only the light fields

$$\mathcal{L}_{eff}(\Phi_l) = \mathcal{L}_{rel}(\Phi_l) + \sum_{n=1}^{\infty} \frac{1}{\Lambda_{IR}^n} \hat{O}_n(\Phi_l), \tag{7.20}$$

where we have singled out $\mathcal{L}_{rel}$, which contains only relevant operators, from an infinite sum of irrelevant local operators. Now in order to get a usable description of our theory we can approximate $\mathcal{L}_{eff}$ by stopping the expansion at a given order q thus obtaining

$$\mathcal{L}_{IR}^q = \mathcal{L}_{rel}(\Phi_l) + \sum_{n=1}^{q} \frac{1}{\Lambda_{IR}^n} \hat{O}_n(\Phi_l). \tag{7.21}$$

$\mathcal{L}_{IR}^q$ describes the low energy properties of our theory and in particular it may have a larger symmetry group than the original $\mathcal{L}_{UV}$. Let us say that the symmetry group of $\mathcal{L}_{IR}^q$ is $G \otimes G'$, then we would call G' an emergent symmetry present only in the low energy description at order q. To this new symmetry we can associate a conserved current such that $\partial_\mu j^\mu = 0$. However emergent symmetries are in general not exact, that is they are actual symmetries of the Lagrangian only if our IR description stops at order q, if instead now we turn on irrelevant operators of order greater than q we are likely to encounter some operators that are charged under under G' and the emergent symmetry would be explicitly broken. The current conservation would then be only approximate

$$\partial_\mu j^\mu \sim \frac{\hat{O}_k}{\Lambda_{IR}^k} \quad \text{with} \quad k > q. \tag{7.22}$$

As a concrete example recall that the Standard Model, regarded as low energy effective field theory, has an emergent $U(1)$ 0-form symmetry corresponding to lepton number conservation,

which is explicitly broken as soon as we turn on the dimension 5 Weinberg operator. This is what generally happens in the case of ordinary symmetries. Let now assume that the emergent symmetry $G^{'}$ in $\mathcal{L}^{q}_{IR}$ is a p-form symmetry. Upon turning on irrelevant operators of order $k > q$ we see that the symmetry cannot be explicitly broken since all the operators in the expansion [7.20] are local and the p-form symmetry cannot act on them. This means that as long as we are at energy scales smaller than Λ_{IR} where $\mathcal{L}_{eff}(\Phi_l)$ provides a good description of the dynamics then the higher form symmetry is exact and the associated current is always conserved. The only way that we can break the higher form symmetry is by going to energy scales above Λ_{IR} where we have to reconstitute the full $\mathcal{L}_{UV}(\Phi_l, \Phi_h)$. Because of the presence of new degrees of freedom the higher form symmetry is not guaranteed anymore and in fact it will be explicitly broken by operators depending on Φ_h. The situation is summarized in Table[7.1] As a simple

$E < \Lambda_{IR}$	$E > \Lambda_{IR}$
$\partial_{\mu_1} j^{\mu_1 \cdots \mu_p} = 0$	$\partial_{\mu_1} j^{\mu_1 \cdots \mu_p} \sim \hat{O}(\Phi_h)$

Table 7.1: Behavior of the conserved current for an emergent p-form symmetry.For energy scales below Λ_{IR} the current is always closed and the symmetry is exact. For energies above Λ_{IR} the conservation of the current will be broken by operators depending on the heavy degrees of freedom.

example consider QED with a massive fermion ψ of mass m. Differently from Maxwell's theory QED does not have a $U(1)$ 1-form symmetry since it contains an interaction term of the type $e\bar{\psi}\not{A}\psi$. Said term is not invariant if we shift A by a closed form.

Now let us work at energies $E < m$. In this regime we can integrate out the fermion and obtain and effective theory for the field A. This effective theory will be contain Maxwell's term and correction terms proportional to higher orders of F, namely

$$\mathcal{L}_{eff} = -\frac{1}{2}\langle F, F\rangle + \frac{1}{\Lambda^2}\hat{O}(F^4) + \dots . \tag{7.23}$$

The first correction to Maxwell theory would yield the Euler-Heisenberg Lagrangian, but nonetheless, because of gauge invariance all the operators in $\mathcal{L}_{eff}$ are functions of the field strength F. It is clear that no matter how long our expansion the effective Lagrangian will always admit a 1-form symmetry that shifts A by a closed form, thus we say the at low energies this 1-form symmetry is exact. However if now we raise the energy scale above the mass of the electron, than $\mathcal{L}_{eff}$ is not a good description anymore and we must turn to its UV completion which is QED. In short for QED the 1-form symmetry is present and exact only at energies $E < m$.

This is an expected result. In fact, similarly to the $SU(N)$ case, in QED all Wilson loops can be screened by dynamical charged fermions. If the Wilson loops can be screened then they cannot be charged under a 1-form symmetry, since the screening effect would not preserve the charge of the symmetry.

Chapter 8

Topological order

As we stated previously the presence of a p-form symmetry implies the existence of a symmetry operators, defined on a $D-p-1$ manifold Σ (Fig[2.1]), whose correlation function are are independent on small deformations of Σ. We call such operator a topological defect operators. One can flip this logic and state that **the presence of a topological defect operators in a theory implies the existence of a symmetry within the theory**. This means that to probe a symmetry one needs not to write the action of the symmetry on the fields (or more in general the degrees of freedom). It is enough to look at the operators in our theory and establish whether some of them are topological defect operators. In order to study higher form symmetries and theories with topological defect operators, it proves useful to introduce to formalism of **topological order** ([23, 32]). In this chapter we will discuss the definition of topological orders, alongside their properties and classification. At the end of the chapter we will review the theory of fusion categories, which is capable of describing generalized symmetries including higher form symmetries, categorical symmetries and non invertible symmetries. Fusion categories are the mathematical framework of topological order. Topological order will then be used to discuss the behavior of higher form symmetries under RG flow.

Topological order is at its core a set of equivalence classes of gapped Hamiltonians, so before giving its definition we must understand the concept of gapped Hamiltonians and specify an equivalence relation for them.

8.1 Equivalence of gapped Hamiltonians

Consider an Hamiltonian H for a finite system of N degrees of freedom. Denote as $|\psi_0\rangle$ the ground state of H with energy E_0. Assume there exists a set of excited states $\{|\psi_k\rangle\}$ that have a small energy difference ϵ with respect to $|\psi_0\rangle$. All the other excited states of H have energy greater than $E_0+\Delta$ where Δ is called the gap and $\Delta > \epsilon$. If in the thermodynamic limit $N \to \infty$, $N/V \to$ constant[1], we have that $\epsilon \to 0$ but Δ tends to a constant different than zero, then we say that H is gapped.

In the thermodynamic limit the Hilbert space spanned by ψ_0 and the set $\{|\psi_k\rangle\}$ is called the ground state sub-Hilbert space $\mathcal{H}_{GS}$ and the dimension of $\mathcal{H}_{GS}$ is called the ground state

[1] we denote with V the volume of space.

degeneracy and denoted by N_{GS}.

Now consider the space of all possible Hamiltonians denoted as **H**, and within it consider two gapped Hamiltonians H and $H^{'}$. We say that H and $H^{'}$ are equivalent to each other if they can be smoothly deformed into each other without closing the gap. More formally consider a parameter $s \in [0.1]$ and path of Hamiltonians $H(s)$, then we have

$$\begin{cases} H(0) = H \\ H(1) = H^{'} \\ H(s) \text{ is gapped } \forall s \end{cases} \iff H \sim H^{'}, \tag{8.1}$$

where $\sim$ indicates the equivalence relations. One may then define the equivalence class of H as

$$[H] = \{ H^{'} \in \mathbf{H} \mid H^{'} \sim H \}. \tag{8.2}$$

In condensed matter physics it is strongly believed that, excluding first order phase transitions, the closing of the gap Δ is in a one to one correspondence with phase transitions. This can be justified by observing expectation values of observables $\langle O \rangle(\Delta)$ are dependent on the gap Δ. $\langle O \rangle(\Delta)$ can develop a non perturbative behavior with respect to Δ, such as a $1/\Delta$ dependence. Then as we take $\Delta \to 0$ we have that $\langle O \rangle(\Delta)$ would diverge signaling the presence of phase transition.

According to this principle when we move between Hamiltonians in the same class $[H]$, we encounter no phase transition, while if we wish to exit $[H]$ we must go trough a phase transition. This implies that **Hamiltonians in the same equivalence class correspond to the same phase of matter**. Practically speaking, checking all possible paths between two Hamiltonians and verifying that the gap never closes is a complex task. We would prefer a more operational definition. To this point, one can prove that is if two Hamiltonians can be smoothly deformed into each other they must be connect by a **local unitary evolution** (or simply LU) operator $U(s)$. That is, consider a parameter λ and an hermitian operator $\tilde{H}(\lambda) = \sum_i O(\lambda)$, where $O_i(\lambda)$ are local operators, then $U(s)$ is defined as follows,

$$U(s) = \mathcal{P} e^{i \int_0^s \tilde{H}(\lambda) d\lambda}, \tag{8.3}$$

where $\mathcal{P}$ is the path ordering. Physically speaking, $U(s)$ represents the "time evolution" operator associated to a system with Hamiltonian $\tilde{H}(\lambda)$. One can then construct the Hamiltonian path as $H(s) = U(s) H U^{\dagger}(s)$, for a given initial Hamiltonian H. The equivalence relation between Hamiltonian can be restated as

$$H \sim H^{'} \iff H^{'} = U(1) H U^{\dagger}(1). \tag{8.4}$$

The operator $U(s)$ also provides a map for the ground states of equivalent Hamiltonians. If $|\psi\rangle$ is the ground state of H, the ground state $|\psi(s)\rangle$ of $H(s) = U(s) H U^{\dagger}(s)$ is given by $|\psi(s)\rangle = U(s) |\psi\rangle$. One can then construct an equivalence relation for the ground states as

$$|\psi\rangle \sim |\psi^{'}\rangle \iff |\psi^{'}\rangle = U(1) |\psi\rangle . \tag{8.5}$$

This equivalence naturally extends to all states in the ground state sub-Hilbert space $\mathcal{H}_{GS}$. According to our previous principle this means that the ground state sub-Hilbert spaces of H

and $H^{'}$, denoted as $\mathcal{H}_{GS}$ and $\mathcal{H}^{'}_{GS}$ are equivalent if and only if they represent the same phase of matter.

Combining Eq(8.2) and Eq(8.4) we have

$$[H] = \{ H^{'} \in \mathbf{H} \,|\, H^{'} = UHU^{\dagger} \}, \tag{8.6}$$

where U is a local unitary evolution. If every Hamiltonian in $[H]$ admits a well defined thermodynamic limit then we call the equivalence class $[H]$ a **gapped quantum liquid**.

8.2 Definition and properties of topological orders

Now we are ready to give the definition of topological order.

*A **topological order** is a stable gapped quantum liquid.*

The term stable refers to the fact the ground state degeneracy is stable against any local perturbation in the thermodynamic limit, namely we have that for large N and any local operator O

$$|\psi_i\rangle, |\psi_j\rangle \in \mathcal{H}_{GS} \implies \langle\psi_i| O |\psi_j\rangle = C_O \delta_{ij}. \tag{8.7}$$

Among topological orders we can distinguish two macro categories, which are trivial and non trivial topological orders. We are mostly interested in non trivial topological orders as they present many interesting physical phenomena. Non trivial topological orders, also called topologically ordered phases, are phases of matter with a manifest pattern of **long range entanglement** (LRE). We will formalize the meaning of LRE later, but intuitively if we have LRE it means that regions of space are that are far away from each other are not independent. For example we will see that creating an excitation in a region of space must necessarily influence regions that are infinitely far away. There are two ways to define non trivial topological order. The first definition is macroscopic and underlines the connection between non trivial topological order and the topology of space. the second one is a microscopic definition and it makes manifest the presence of LRE, while also providing a quantum number to classify topological orders.

8.2.1 Macroscopic definition

A non trivial topological order is one whose ground state degeneracy N_{GS} is dependent on the topology of space.

Moreover the number of ground states is expected to grow with the number of non trivial cycles of space. In 2 spatial dimensions a non trivial topological order will have different values of N_{GS} depending on whether we put it on disk, an annulus or a torus. For example,

- The toric code model has $N_{GS} = 1$ on the sphere, but $N_{GS} = 4$ on the torus. The toric code model is a non trivial topological order (A).

- The Ising model in the SSB phase has two possible ground states, and this independently on the topology of the space. The Ising model in the SSB phase is a trivial topological order.

From this definition one can argue that the spectrum of a non trivial topological order contains **topological excitations**. A topological excitation is one that cannot be created or annihilated by any local operator. Let us see why this should be the case in an example.

Consider a non trivial topological order in 2 spatial dimensions. Put this topological order on a disk and assume that there exists only one ground state configuration $|\psi_0\rangle$. Now consider the same theory but on an annulus. According to our definition we expect the theory to develop at least one new ground state configuration $|\psi_1\rangle$. This new configuration should be connected with the non trivial topology of the annulus. At this point one can make a simple ansatz for $|\psi_1\rangle$. We expect $|\psi_1\rangle$ to be identical to $|\psi_0\rangle$ in most of the annulus except for a line σ that wraps around the hole of the annulus (Fig[8.1(a)]). This is the simplest way one can make a new configuration which is connected to the topology of space. Now let us assume we have a local operator O_δ defined on a portion δ of σ. O_δ can open the non trivial line in $|\psi_1\rangle$ (Fig[8.1(b)]). That means the state $|\phi\rangle = O_\delta |\psi_1\rangle$ is characterized by an open string configuration where it differs form $|\psi_0\rangle$. The state $|\phi\rangle$ is necessarily an excited state. That is because if $|\phi\rangle$ were a ground state we would violate the stability of the topological order in the thermodynamic limit. The excitation in $|\phi\rangle$ cannot be localized on the open line δ since δ is a piece of σ, and σ is topological and can be deformed. The excitations in $|\phi\rangle$ are therefore localized on the two points that form boundary of δ (Fig[8.1(c)]).

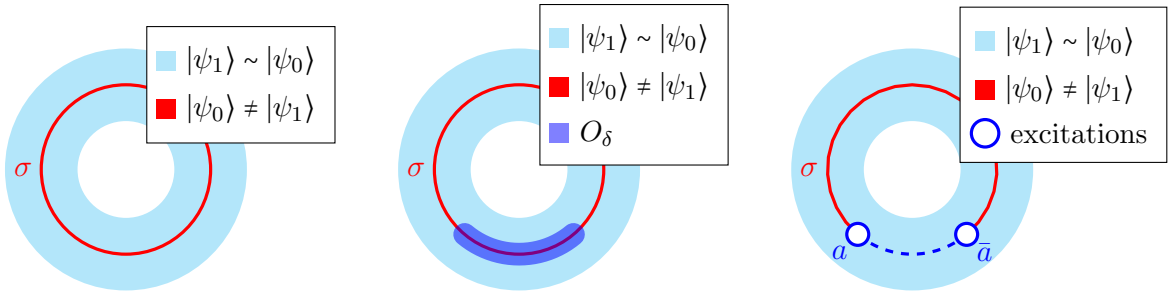
We just showed in this example that the definition of topological order implies the existence of excitations that must be created in pairs by local operators. We may call these excitation a and $\bar{a}$. If we wish to create only one excitation we must apply O_δ, and then move either a or $\bar{a}$ to infinity using some other operator. This implies that to create a single a we must use a non local operator, and thus a (and similarly $\bar{a}$) is a topological excitation.

Topological excitations are a key feature of topological order in any dimension. Since in 2 spatial dimensions topological excitations carry with them a non local string, they are perfect candidates to be anyons, either abelian or non-abelian. Anyons obey braid statics, and thus the presence of an non local line attached to a topological excitation implies a braiding structure. Topological orders can also contain non topological excitations, but these excitations are often considered less interesting. If we care about topological excitations alone we need to describe how said excitations braid with others, and we need to specify what happens when two topological excitation meet each other at a point, namely when they fuse. The mathematical structure that contains these information is called a **fusion category**.

8.2.2 Microscopic definition

As we said previously a non trivial topological order exhibits LRE. To understand the meaning of this sentence we must first fix a criteria that defines LRE. To do we will use the idea if **Von Neumann entanglement entropy**, or simply entanglement entropy.

Assume that the space X is connected. Begin by partitioning X into two patches A and B such that $A \cup B = X$ and $A \cap B = \varnothing$. The total Hilbert space is $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, with $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ the Hilbert space of the patches A and B. Now consider a mixed state on X represented by



(a) The ground state $|\psi_1\rangle$ is locally identical to $|\psi_0\rangle$ in most of the annulus. However on the non trivial line σ we have that $|\psi_1\rangle$ develops a configuration that distinguishes it from $|\psi_0\rangle$.

(b) Applying O_δ on $|\psi_0\rangle$ opens the non trivial line σ, and creates the state $|\phi\rangle$.

(c) The state $|\phi\rangle$ is characterized by the point like excitations localized on the boundary of the string.

Figure 8.1: Having a topologically non trivial state implies that a local operator can only create excitations in pairs, a and $\bar{a}$. Then to create a single excitation we must push one of the two to infinity. This implies that the excitation a and $\bar{a}$ are topological excitations.

the density matrix $\rho = \sum_i \eta_i |\psi_i\rangle \langle\psi_i|$, with $\eta_i > 0$ and $\sum_i \eta_i = 1$. Let us trace the density matrix over B to obtain a density matrix for the patch A, namely $\rho_A = Tr_B \rho$. Then the entanglement entropy of the state ρ and the patch A is

$$S_A = Tr_A \rho_A \ln \rho_A. \tag{8.8}$$

The entanglement entropy gives information on how the subsystem in A is influenced by what happens in B. One can prove that the entanglement entropy of A is equal to that of B, that is

$$S_A = Tr_A \rho_A \ln \rho_A = Tr_B \rho_B \ln \rho_B = S_B. \tag{8.9}$$

Eq(8.9) implies that the entanglement entropy must depended on features that A and B have in common. Geometrically A and B are independent of each other, except for the common boundary they must share. Then one can safely assume that S_A only depends on the boundary ∂A of A, in fact via direct computation one can find

$$S_A \propto Area(\partial A), \tag{8.10}$$

where $Area(\partial A)$ is the volume of the boundary of ∂A. This behavior is called an area law. This name comes form the case of 3 dimensional space where ∂A is a surface. In a theory with local interaction the area law of S_A has a simple interpretation. Since the interactions are local the degrees of freedom in the bulk of A and B are invisible to each other. On the other hand A and B can still influence each other via the common boundary. The strength of this mutual influence should be proportional to the number of interactions and thus to the area of the boundary.

The complete behavior of S_A can generally be parameterized as

$$S_A = \alpha Area(\partial A) - \gamma + O\left(\frac{1}{Area(\partial A)}\right), \tag{8.11}$$

where α and γ are positive constants. For our discussion the most interesting contribution to S_A is the constant γ, which is called the **topological entanglement entropy** (TEE). Physically if the area law of S_A underlines the interactions at the boundary of A, a non zero value of γ implies that the degrees of freedom in the bulk of A can somehow influence what happens outside of A. Then we say that a state with $\gamma \neq 0$ has long range entanglement.

A non trivial topological order is one whose ground states have a non zero topological entanglement entropy.

An operational formula for calculating γ was constructed in [33] and [34] approximately at the same time using a clever partition of space. Moreover in 2 spatial dimensions it was show that the TEE of the ground state of a theory is related to properties of the excitations in the theory, in particular

$$\gamma = \ln \sqrt{\sum_a d_a^2}, \tag{8.12}$$

where d_a is the **quantum dimension** of a particle-like excitation of type a and the sum runs over all types of particle-like excitations in the theory. The quantum dimension of a particle a is the vacuum expectation value of the topological defect U_a associated to a. That is, consider looping the particle a around a loop γ. This procedure would result in a topological defect U_α supported on γ. The quantum dimension is the weight associated to shrinking U_α to nothing, namely $d_a = \langle U_a \rangle$, where the expectation value is taken in a ground state [35]. Another way to understand the quantum dimension is to look at the wave function amplitude of a ground state [33]. Say we have a ground state $|\Phi\rangle$ given by

$$|\Phi\rangle = \sum_{\text{conf}} \Phi(\text{conf}) \, |\text{conf}\rangle \, , \tag{8.13}$$

where $\Phi(\text{conf})$ denotes the quantum amplitude of a given configuration $|\text{conf}\rangle$, and the sum runs over all possible configurations. Then the quantum dimension of a is defined as follows

$\Phi\Big(\text{conf} +$  $\Big) = d_a \Phi(\text{conf}).$ (8.14)

An important remark is that, given a state with TEE γ applying a local unitary evolution does not change the value of γ. This implies that for all Hamiltonians in the same topological order γ is the same. The only way to charge γ would be to go trough a phase transition and move to different topological order. Thus γ can viewed as a quantum number that classifies topological orders. In particular if $\gamma = 0$ we call the topological order a trivial topological order. Let us give a couple of examples to clarify this point.

- The toric code model has four topological excitations e, m, f and $\mathbb{1}$ (A.1), all with quantum dimensions $d = 1$. It is a non trivial topological order with $\gamma = \ln 2$.

- the Ising model in the SSB phase with fixed boundary condition has two possible ground states. Both of these ground states are trivial product state, that is they admit the following decomposition,

$$|\psi\rangle = \bigotimes_i |\psi_i\rangle \, , \tag{8.15}$$

where $|\psi_i\rangle$ is the state of single lattice site. Trivial product states have $\gamma = 0$, thus this is a trivial topological order.

To summarize let us list the most important features that are commonly used to detect an non trivial topological order

1. It is an equivalence class of gapped Hamiltonians, all representing the same phase of matter.
2. The ground state degeneracy depends on the topology of space.
3. There are topological excitations with possibly fractional statistics, described by a fusion category.
4. The ground state has non zero topological entanglement entropy.

Topological order is a powerful tool the study generalized discrete symmetries, since in the SSB phase of a discrete symmetry we expect to have gapped symmetry defects. Unfortunately topological order is not well equipped to study continuous symmetries. That is because in the SSB phase of continuous symmetry we have gapless Goldstone bosons. By definition a gapless theory cannot be described by topological order.

In the following chapters we will use topological order to discuss some properties of higher form symmetries. Formally speaking the discussion only works for discrete symmetries, however we will implicitly assume that the theory of topological order admits a generalization that can also describe continuous symmetries.

8.3 Review of category theory

In this section we will focus on some key concepts in category theory to fix some useful notation. We will not dive deep into the details of the theory and consider only the aspects of categories we will make use of. For a more formal review of categories we suggest [36], [37].

In order to define a category $\mathcal{C}$ we must specify two components. A class of objects, which we will call C_0, and a set of morphisms called C_1 between objects in C_0. The morphisms in C_1 must admit a composition rule such that given $A, B, C \in C_0$ and two morphisms $f : A \to B$ and $g : B \to C$ we may compose them to obtain a third morphism $g \circ f : A \mapsto C_1$, as seen in Fig[8.2]. Moreover the morphisms must be associative and there must exist an identity morphisms $1_A : A \mapsto A$ which does not affect the action of other morphisms from or to A.

There is a wide range of different types of categories one can construct by requiring extra properties. In order to describe phenomena related to symmetries we are mostly interested in so called fusion categories. Given a category $\mathcal{C}$ we say $\mathcal{C}$ is a **fusion category** if it respect the following properties.

1. **Monoidal**: There exists a functor (a map between categories) noted as $\otimes : \mathcal{C} \times \mathcal{C} \mapsto \mathcal{C}$, and called a tensor product, which respects some constraints called the pentagon and triangle identity. The tensor product is equipped with a unit object $\mathbb{1}$ such that

$$\mathbb{1} \otimes A \sim A \otimes \mathbb{1} \sim A, \tag{8.16}$$

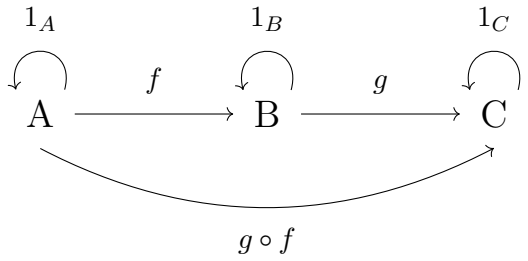


Figure 8.2: Graphical representation of a category, with a composed morphism $g \circ f$ and identity morphisms $1_A, 1_B, 1_C$

.

where the symbol ~ signals the presence of an isomorphism between two objects. The important point is that $\otimes$ defines the concept of multiplication in $\mathcal{C}$, which we can use to understand what happens when two symmetries defects merge into each other.

2. **Linear**: Given two objects A and B, the morphisms between A and B compose a vector space.
3. **Rigid**: To each object in $A \in \mathcal{C}$ we can associate a dual A^* and an evaluation (ev_A) and a co-evaluation (coev_A) map ([37])

$$A \otimes A^* \xrightarrow{\mathrm{ev}_A} \mathbb{1}, \quad \mathbb{1} \xrightarrow{\mathrm{coev}_A} A^* \otimes A, \tag{8.17}$$

such that they satisfy the following identities

$$\begin{aligned} (\mathrm{ev}_A \otimes \mathbb{1}_A) \circ (\mathbb{1}_A \otimes \mathrm{coev}_A) &= \mathbb{1}_A, \\ (\mathbb{1}_{A^*} \otimes \mathrm{ev}_A) \circ (\mathrm{coev}_A \otimes \mathbb{1}_{A^*}) &= \mathbb{1}_{A^*}. \end{aligned} \tag{8.18}$$

4. **Semisimple**: There exists a biproduct $\oplus : C_0 \times C_0 \mapsto C_0$, called direct sum, which allows us to sum two objects in C_0. Each $A \in C_0$ admits a decomposition

$$A = \bigoplus_n X_n, \tag{8.19}$$

where X_n are simple objects, which are objects that cannot be decomposed further. This property introduces the idea of sum within $\mathcal{C}$ and most crucially it will allow us to interpret complex topological defects as the superposition of simple excitations.
5. $\mathcal{C}$ posses only a finite number of simple objects. This is important since X_n represent the fundamental topological excitations in our theory and we expect to have only a finite amount of them[2].

Using property 1 and 4 of we may write

$$X_a \otimes X_b = \bigoplus_k N_{ab}^k X_k, \tag{8.20}$$

[2]It important to remark that for continuous symmetries we would have an infinite amount of topological excitations. Throughout this paper we will assume that the theory of fusion categories admits an extension to the case where we have infinitely many simple objects.

where the coefficients N_{ab}^c are called fusion multiplicities. The physical interpretation of Eq.[8.20] is to assign a certain weight N_{ab}^c, to the process in which two defects a and b merge to produce a third one c. In order to have a complete description of a fusion category we need to extrapolate some more information coming from complex fusion diagrams. Consider a diagram describing the process in which a defect a splits into three defects b, c and d. This process could happen through a variety of repeated fusion channels which are not independent from each other. We can connect two different channels using a tensor like object know as the F-symbol (or $6j$ symbol). The F-symbol must respect a consistency condition known as the pentagon identity which involves the commutation of complex fusion diagrams [38].

Using categories we can describe the "scattering" between 1-dim topological defects, but we would like to enrich this structure to include also topological defects of higher dimensions. To do so consider that when two n dimensional defects merge the result is a $n-1$ dimensional junction, and now we could take two $n-1$ junctions and merge them to form a $n-2$ junction. The process can be iterated until we get to 0 dimensional junctions (points). We need a structure that not only includes fusion rules for defects but also for junctions of defects and junction of junctions of defects. The appropriate formalism is given by so called **n-categories**. In order to construct an n-category $\mathcal{C}_n$ we must specify $n+1$ components:

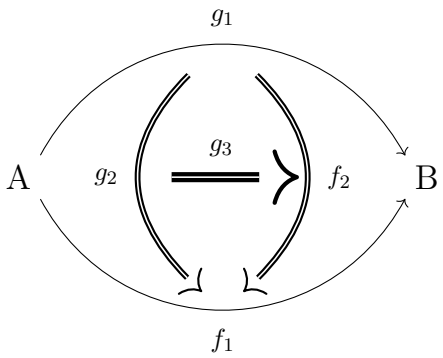
1. A family of objects $|C|$ (also called 0-morphisms).
2. A set of 1-morphisms C_1 between objects. That is for $g_1 \in C_1$ we have $A, B \in |C|$ such that $g_1 : A \mapsto B$.
3. A set of 2-morphisms C_2 between 1-morphisms. That is for $g_2 \in C_2$ we have $h_1, f_1 \in C_1$ such that $g_2 : h_1 \mapsto f_1$.

$\vdots$

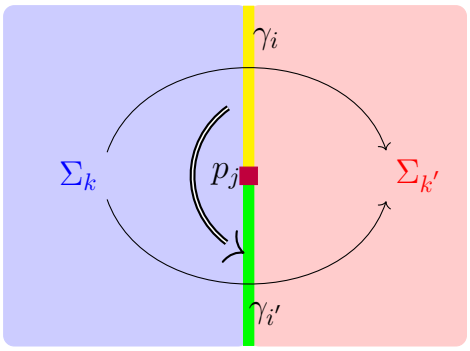
n. A set of n-morphisms C_n (also called top morphisms) between (n-1)-morphisms. That is for $g_n \in C_n$ we have $h_{n-1}, f_{n-1} \in C_{n-1}$ such that $g_n : h_{n-1} \mapsto f_{n-1}$.

In Fig[8.3(a)] we present a graphical depiction of a 3-category.

To understand how n-categories are applied in a physical scenario consider a theory in $D \geq 3$ which admits the following excitations. N_2 types of extended excitations denoted as the set $\{\Sigma_k\}_{k=1,\dots,N_2}$ and defined on 2 dimensional manifolds. N_1 types of extended excitations denoted as the set $\{\gamma_k\}_{k=1,\dots,N_1}$ and defined on 1 dimensional manifolds. N_0 types of 0 dimensional excitations denoted as $\{p_k\}_{k=1,\dots,N_0}$ and defined on points. It is often the case that the boundary of a Σ_k excitation will contain a γ_i excitation, and the boundary of a γ_i excitation will contain a p_j excitation. As an example consider what we did for $SU(N)$ where in order to have a gauge invariant Wilson line we had to put field strength operators on the boundary of the Wilson line, Fig[6.1]. Consider now the situation where two 2 dimensional excitation Σ_k and $\Sigma_{k'}$ share a common boundary. On the common boundary we have an excitation γ_i. Imagine to sit on Σ_k and to start moving towards $\Sigma_{k'}$. At a certain point we will cross γ_i, which can be regarded as an indicator of the fact that Σ_k is ending and $\Sigma_{k'}$ is beginning. Then the excitation γ_i can be associated to an isomorphism mapping Σ_k to $\Sigma_{k'}$. In a similar fashion two excitation γ_i and $\gamma_{i'}$ could meet at a point where we have a p_j excitation. Moving from γ_i to $\gamma_{i'}$ we must cross p_j and thus we may associate to p_j an isomorphism mapping from γ_i to $\gamma_{i'}$. This construction can



(a) Graphical representation of a 3-category. g_1 and f_1 are 1-morphisms, g_2 and f_2 are 2-morphisms while g_3 is a 3-morphism.



(b) A theory with extended excitations represented by a 2-category. The 1-dimensional excitations γ_i, $\gamma_{i'}$ act as 1-morphisms between Σ_k and $\Sigma_{k'}$. While the 0 dimensional excitation p_j acts as a 2-morphism between γ_i and $\gamma_{i'}$.

Figure 8.3: n-categories are used to describe the interplay between extended excitations of different dimensions. The 0-morphisms (objects) are the highest dimensional excitation while the top morphisms are 0 dimensional excitations.

summarized by 2-category as in Fig[8.3(b)].

8.3.1 The Drinfeld center

Notice how in the definition of fusion category the order of the tensor product is relevant. That is, given two object A and B in a fusion category C, we have that C is not generally equipped with an isomorphism $\eta : A \otimes B \mapsto B \otimes A$. If such an isomorphism were to exist in C for all A and B we would call C a **braided fusion category**, and η would be called the braiding morphism. Staring from a monoidal category C it is possible to construct a braided monoidal category called the **Drinfeld center** of C and denoted as $\mathcal{Z}(C)$. We will now give a recipe to construct $\mathcal{Z}(C)$. We will not be too rigorous and we will present many results without proof. If one is interested in a more complete treatment all the calculations can be found in [39].

1. The objects of $\mathcal{Z}(C)$ are pairs $(X, \eta_{X,\bullet})$ where X is an object in C and $\eta_{X,\bullet}$ is a morphism defined for X such that

$$\eta_{X,Y} : X \otimes Y \mapsto Y \otimes X \quad \forall Y \text{ objects in } C. \tag{8.21}$$

   $\eta_{X,\bullet}$ must respect the condition $\eta_{X,Y\otimes Z} = (\mathbb{1}_Y \otimes \eta_{X,Z}) \circ (\eta_{X,Y} \otimes \mathbb{1}_Z)$, alongside some naturality conditions involving commuting diagrams.
2. A morphism $f : (X, \eta_{X,\bullet}) \mapsto (Y, \eta_{Y,\bullet})$ in $\mathcal{Z}(C)$ can be constructed from morphism f in C by requiring the following property.

$$(\mathbb{1}_Z \otimes f) \circ \eta_{X,Z} = \eta_{Y,Z} \circ (f \otimes \mathbb{1}_Z) \quad \forall Z \text{ objects in } C. \tag{8.22}$$

3. The tensor product in $\mathcal{Z}(C)$ is given by

$$(X, \eta_{X,\bullet}) \otimes (Y, \eta_{Y,\bullet}) = (X \otimes Y, \eta_{X\otimes Y,\bullet}), \tag{8.23}$$

where $\eta_{X\otimes Y,\bullet} = (\eta_{X,\bullet} \otimes \mathbb{1}_Y) \circ (\mathbb{1}_X \otimes \eta_{Y,\bullet})$.

In the specific case of a category associated to a group G the Drinfeld center is constructed using the elements of the center of the group $Z(G)$. Let us make this statement more explicit. Consider a vector space V over a field K equipped with a linear action of a group G, that is a map $\phi : G \times V \mapsto V$ that respects the group structure. Then we call V a **G-module**. Saying that V is a G-module is equivalent to saying that V admits a representation of the group G, namely we can define an homomorphism $\rho : G \mapsto GL(V)$ called a representation. Physically speaking recall that if G is a symmetry of the theory, then when acting on a vector space V the representation of the group is fixed by the symmetry charges present in V. This means that each G-module and each representation is associated to a mix of symmetry charges on V. Given two G-modules V and W one can define a map between them called a **G-homomorphism** $F : V \mapsto W$ such that

$$F(g(v)) = g(F(v)), \tag{8.24}$$

where $v \in V$, $F(v) \in W$ and $g(\bullet) \in G$ indicates the action of a group element on either V or W. One can prove that if we consider G-modules as objects and G-homomorphisms as morphisms between objects then we may construct a well defined monoidal category called the **representation category of** G, in short $\mathcal{R}ep_G$. In particular given two G-module V and W, the tensor product $V \otimes W$ is still a G-module and the action of a group element $g \in G$ is given by

$$g(v \otimes w) = g(v) \otimes g(w), \tag{8.25}$$

with $v \in V$ and $w \in W$. $\mathcal{R}ep_G$ is the category containing all possible representations of G. If G is a symmetry of a physical theory, then $\mathcal{R}ep_G$ can be used to describe the behavior of the symmetry charges. Unfortunately $\mathcal{R}ep_G$ is not a fusion category for continuous groups G, as it does not have a finite number of simple objects.

On the other hand if G is a discrete finite group then $\mathcal{R}ep_G$ is a well defined fusion category and each G-module can be decomposed into a direct sum of simple G-modules (these would be the simple objects in the definition of fusion category). This result is known as **Maschke's theorem**. To construct the Drinfeld center $\mathcal{Z}(\mathcal{R}ep_G)$ we proceed as follows. Consider an element $z \in Z(G)$, and define the map $\Phi_z : V \otimes W \mapsto W \otimes V$ as

$$\Phi_z : v \otimes w \mapsto z(w) \otimes v. \tag{8.26}$$

One can prove that the pairs (V, Φ_z) have all the properties needed to be defined as objects of the Drinfeld center of $\mathcal{R}ep_G$. Notice how the objects (V, Φ_z) depend both on the choice of the G-module V and the specific group element $z \in Z(G)$.

As an example, let us build explicitly the Drinfeld center in the case of a simple group. Consider $\mathbb{Z}_2 = \{\mathbb{1}, e\}$ where the group composition rule implies $e \otimes e = \mathbb{1}$. From this one can deduce that $\Phi_e \otimes \Phi_e = \Phi_{\mathbb{1}}$. Given a vector space V, it is easy to convince ourselves that $\mathbb{Z}_2$ has two simple modules, i.e. two ways to represent $\mathbb{Z}_2$ on a vector space V. The simple representations are characterized by the action of the group elements on V. Note that the

element $\mathbb{1}$ always acts trivially on V. On the other hand the element e could be represented by a trivial transformation, or by a multiplication by a -1. The simple module where e acts trivially is called the trivial module, and denoted by $\mathbb{C}_{triv}$. The simple module where e acts as a multiplication by a minus sign is called the sign module, and denoted as $\mathbb{C}_{sign}$. There are no other simple modules. All other modules V can be expressed as direct sum of $\mathbb{C}_{triv}$ and $\mathbb{C}_{sign}$. Moreover from Eq(8.26) we have $\mathbb{C}_{sign} \otimes \mathbb{C}_{sign} = \mathbb{C}_{triv}$.

$\mathbb{Z}_2$ is abelian and thus equal to its center, $Z(\mathbb{Z}_2) = \{\mathbb{1}, e\}$. To study the Drinfeld center of $\mathcal{R}ep_{\mathbb{Z}_2}$ we can analyze its simple objects. The simple objects of $\mathcal{Z}(\mathcal{R}ep_G)$ are given by considering the pair (V, Φ_z) when V is a simple object (or simple module) of $\mathcal{R}ep_G$. Thus the simple objects of $\mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2})$ are

$$\underbrace{(\mathbb{C}_{triv}, \Phi_{\mathbb{1}})}_{I}, \quad \underbrace{(\mathbb{C}_{triv}, \Phi_e)}_{E}, \quad \underbrace{(\mathbb{C}_{sign}, \Phi_{\mathbb{1}})}_{M}, \quad \underbrace{(\mathbb{C}_{sign}, \Phi_e)}_{F}, \tag{8.27}$$

where we assigned a name to each simple object. The set of simple objects in Eq(8.27) is actually reminiscent of a very important physical theory. To see this let us look at the fusion rules of said objects. Recalling Eq(8.23) we have

$$\begin{aligned}
&E \otimes E = (\mathbb{C}_{triv}, \Phi_e) \otimes (\mathbb{C}_{triv}, \Phi_e) = (\mathbb{C}_{triv} \otimes \mathbb{C}_{triv}, \Phi_e \otimes \Phi_e) = (\mathbb{C}_{triv}, \Phi_{\mathbb{1}}) = I,\\
&M \otimes M = (\mathbb{C}_{sign}, \Phi_{\mathbb{1}}) \otimes (\mathbb{C}_{sign}, \Phi_{\mathbb{1}}) = (\mathbb{C}_{sign} \otimes \mathbb{C}_{sign}, \Phi_{\mathbb{1}} \otimes \Phi_{\mathbb{1}}) = (\mathbb{C}_{triv}, \Phi_{\mathbb{1}}) = I,\\
&F \otimes F = (\mathbb{C}_{sign}, \Phi_e) \otimes (\mathbb{C}_{sign}, \Phi_e) = (\mathbb{C}_{sign} \otimes \mathbb{C}_{sign}, \Phi_e \otimes \Phi_e) = (\mathbb{C}_{triv}, \Phi_{\mathbb{1}}) = I,\\
&E \otimes M = (\mathbb{C}_{triv}, \Phi_e) \otimes (\mathbb{C}_{sign}, \Phi_{\mathbb{1}}) = (\mathbb{C}_{triv} \otimes \mathbb{C}_{sign}, \Phi_e \otimes \Phi_{\mathbb{1}}) = (\mathbb{C}_{sign}, \Phi_e) = F.
\end{aligned} \tag{8.28}$$

Eq(8.28) corresponds to the fusion rules for the anyons in the toric code model, which has $\mathbb{Z}_2$ gauge symmetry (A). This means that physically $\mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2})$ is the category that describes the excitations of a $\mathbb{Z}_2$ gauge theory. We denote the fusion category of a $\mathbb{Z}_2$ gauge theory as $\mathcal{G}au_{\mathbb{Z}_2}$. Then we have

$$\mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2}) = \mathcal{G}au_{\mathbb{Z}_2}. \tag{8.29}$$

One can show that Eq(8.29) generalizes to $\mathbb{Z}_N$ as

$$\mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_N)} = \mathcal{G}au_{\mathbb{Z}_N}. \tag{8.30}$$

That is the Drinfeld center of $\mathcal{R}ep_{\mathbb{Z}_N}$ corresponds to a $\mathbb{Z}_N$ gauge theory.

For our purposes there exists another important category one can assign to a group G. Begin by considering a map f_V that assigns a vector space V_g to each element $g \in G$. The family of vector spaces $\{V_g\}_{g\in G}$ is called a **G-graded vector space**. One can also denote a G-graded vector space $\{V_g\}_{g\in G}$ simply as V_G, where V_G is a vector space that admits the following decomposition

$$V_G = \bigoplus_{g\in G} V_g. \tag{8.31}$$

Given two G-graded vector spaces V_G and W_G consider the collection of maps $\{f_g : V_g \to W_g\}_{g\in G}$. We denote the set $\{f_g\}$ simply as f_G and we call it a **grade preserving morphism**. One can prove that if we consider G-graded vector spaces as objects and grade preserving morphisms as morphisms between objects then we may construct a well defined monoidal category called the

vector category of G, in short $\mathcal{V}ec_G$. The tensor product between two G-graded vector spaces V_G and W_G is defined as $V_G \otimes W_G = \{(V_G \otimes W_G)_g\}_{g\in G}$ where the components are

$$(V_G \otimes W_G)_g = \sum_{hk=g} V_h \otimes W_k, \quad h,k \in G \tag{8.32}$$

The action of a grade preserving morphism f_G on $V_G \otimes W_G$ is

$$f_g : (V_G \otimes W_G)_g \to f_g((V_G \otimes W_G)_g) = \sum_{hk=g} f_h(V_h) \otimes f_k(W_k). \tag{8.33}$$

From Eq(8.31) one can read that the simple objects of $\mathcal{V}ec_G$ are G-graded vector spaces of the form $V_G = \cdots \oplus \mathbb{0} \oplus \mathbb{0} \oplus V_g \oplus \mathbb{0} \oplus \mathbb{0} \oplus \ldots$, where $\mathbb{0}$ denotes the null vector space. To avoid a cumbersome notation we will simply write the simple objects of $\mathcal{V}ec_G$ as $V_G = V_g$. Notice how the simple objects are in a one to one correspondence with the group elements. This can help us build an physical understanding of $\mathcal{V}ec_G$. Consider a theory that admits a symmetry G. Assume this theory is defined on a vector space on M which admits a representation ρ of G. Consider an open patch $\mathcal{P}$ in M. Now select a group element $g \in G$ and apply the symmetry transformation associated to g on patch $\mathcal{P}$. Denote as V_g the vector space after the symmetry transformation. We may repeat this process for all $g \in G$ and build a G-graded vector space given as $V_G = \{V_g\}_{g\in G}$. Physically speaking we know that applying a symmetry transformation on an open patch $\mathcal{P}$ generates defects on $\partial\mathcal{P}$. This implies that the vector spaces V_g are characterized by having a symmetry defect on $\partial\mathcal{P}$. Then the G-graded vector space V_G can be interpreted as a set describing all possible symmetry defects that we may place in $\partial\mathcal{P}$. the simple objects of $\mathcal{V}ec_G$ are V_g and they correspond to vector spaces with a unique type of symmetry defect on ∂P. Now consider a new open patch $\mathcal{P}'$. If now we move the symmetry defects from $\partial\mathcal{P}$ to $\partial\mathcal{P}'$ we get a new G-graded vector space W_G. This implies that a grade preserving morphism physically correspond to moving the defects around. Given these observation, we can understand that for a theory with symmetry G, the category $\mathcal{V}ec_G$ describes the behavior of the symmetry defects. As for $\mathcal{R}ep_G$, if G is finite and discrete then $\mathcal{V}ec_G$ is fusion category, rather then just a monoidal category.

Now we wish to construct the Drinfeld center $\mathcal{Z}(\mathcal{V}ec_G)$. To do so consider a representation ρ of G on V_G, such that for $q \in G$ we have

$$\rho_q(V_g) \in V_{q^{-1}gq}. \tag{8.34}$$

That is for a vector $v_g \in V_g$ we have

$$\rho_q(v_g) = q(v)_{q^{-1}gq}, \quad q(v)_{q^{-1}gq} \in V_{q^{-1}gq}. \tag{8.35}$$

Given ρ we may define a braiding morphism $\Phi(\rho) : V_G \otimes W_G \mapsto W_G \otimes V_G$ such that for $v_g \in V_g$ and $w_g \in W_g$ we have

$$\Phi(\rho) : (v\otimes w)_g = \sum_{hk=g} v_h \otimes w_k \to \sum_{hk=g} w_k \otimes \rho_k(v_h) = \sum_{hk=g} w_k \otimes k(v)_{k^{-1}hk} \in \sum_{hk=g} W_h \otimes V_k = (W_G \otimes V_G)_g, \tag{8.36}$$

where in the second-last step we used $k(k^{-1}hk) = hk = g$. One can prove that the pairs $(V_G, \Phi(\rho))$ have all the properties needed to be defined as objects of the Drinfeld center of $\mathcal{V}ec_G$. Note how

$\Phi(\rho)$ only depends on the particular representation ρ that we choose on V_G. Once again as an example let us study the case of $G = \mathbb{Z}_2 = \{\mathbb{1}, e\}$. Then the simple objects of $\mathcal{V}ec_{\mathbb{Z}_2}$ are denoted as $V_{\mathbb{1}}$ and V_e. From the tensor product rule in Eq(8.32) one finds $V_e \otimes V_e = V_{\mathbb{1}}$. As we did for $\mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2})$, let us study $\mathcal{Z}(\mathcal{V}ec_{\mathbb{Z}_2})$ by looking at its simple objects. We already mentioned that $\mathbb{Z}_2$ has only 2 simple representations, which are characterized by the action of the e element. We have the trivial representation *triv*, and the sign representation *sign*. Then the simple objects of $\mathcal{Z}(\mathcal{V}ec_{\mathbb{Z}_2})$ are given by composing the simple objects of $\mathcal{V}ec_G$ with the braiding morphisms of the simple representations, that is

$$\underbrace{(V_{\mathbb{1}}, \Phi(triv))}_{\tilde{I}}, \quad \underbrace{(V_{\mathbb{1}}, \Phi(sign))}_{\tilde{E}}, \quad \underbrace{(V_e, \Phi(triv))}_{\tilde{M}}, \quad \underbrace{(V_e, \Phi(sign))}_{\tilde{F}}, \tag{8.37}$$

where once again we gave a name to each simple object. The fusion rules can be obtained by carefully using the tensor product rules. One would find

$$\tilde{E} \otimes \tilde{E} = \tilde{M} \otimes \tilde{M} = \tilde{F} \otimes \tilde{F} = \tilde{I}, \quad \tilde{E} \otimes \tilde{M} = \tilde{F}. \tag{8.38}$$

The fusion rules for $\mathcal{Z}(\mathcal{V}ec_{\mathbb{Z}_2})$ are identical to those of $\mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2})$ (Eq(8.28)). This implies that these two categories are actually the same. Both of them describe the behavior of anyon excitations in a $\mathbb{Z}_2$ gauge theory. In conclusion we have

$$\mathcal{Z}(\mathcal{V}ec_{\mathbb{Z}_2}) = \mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_2}) = \mathcal{G}au_{\mathbb{Z}_2}. \tag{8.39}$$

Also in this case one can extend this statement to $\mathbb{Z}_N$, namely

$$\mathcal{Z}(\mathcal{V}ec_{\mathbb{Z}_N}) = \mathcal{Z}(\mathcal{R}ep_{\mathbb{Z}_N}) = \mathcal{G}au_{\mathbb{Z}_N}. \tag{8.40}$$

The Drinfeld center is an interesting structure in itself, but as we will show later it has a very powerful physical interpretation. In fact using the Drinfeld center one can define a holographic principle mapping theories in dimension D to theories in $D + 1$.

Chapter 9

Holographic approach

In this section we will discuss the holographic principle of topological order [40]. The holographic approach proved to be a powerful tool to classify generalized symmetries and identify gapless phases of matter [41]. In particular it is believed to be the most general approach to unify the theory concerning higher form symmetries, categorical symmetries, non invertible symmetries, etc. We will first present a general recipe to obtain the principle starting from the operator algebra and then we will apply said recipe to a simple case, namely a 1 dimensional spin chain with $\mathbb{Z}_2$ symmetry, as was shown in [42]. Finally we will take advantage of the holographic principle to study the exactness of emergent higher form symmetries [15]. As a clarification in the following section we will use the term *local operator* not only for point like operators, but for any operator supported on a locus that does not grow as system size in the thermodynamic limit. Operators that do grow as system size will be referred to as *global operators*.

Before we can state the principle we need to fix some preliminary knowledge. In particular we need to understand the concept of **non-invertible gravitational anomaly**[1]([43]). Consider a theory defined on spacetime manifold X, which we equip with a non dynamical background metric $g_{\mu\nu}$. Denote the partition function of the theory as $Z(g_{\mu\nu})$, where the partition function is given as a functional of the metric $g_{\mu\nu}$ on X. Now apply a diffeomorphism π of X which is not connected to the identity. π is often called a large diffeomorphism since it acts on the whole spacetime X. π acts on the metric as follows, $g_{\mu\nu} \xrightarrow{\pi} g^{\pi}_{\mu\nu}$. We say that a theory has a gravitational anomaly if

$$Z(g^{\pi}_{\mu\nu}) \neq Z(g_{\mu\nu}). \tag{9.1}$$

It is possible that the anomalous term in $Z(g^{\pi}_{\mu\nu})$ can be canceled by adding to the action some new degrees of freedom described by a local Lagrangian density. in this case we say that the anomaly is invertible.

On the other, it is sometimes the case that the partition function of a theory on X is actually dependent on specific boundary conditions we impose on our degrees of freedom [44, 45]. For example on a ring imposing periodic or anti-periodic boundary condition would lead respectively to a non twisted and twisted partition function. In said cases our theory does not have a unique partition function, but rather a set of partition function $\{Z_i\}$ which form a vector space. Each partition function Z_i can be associated to a specific boundary condition that generates it. Now

[1]Note that, despite the name, this anomaly is not directly connected to a theory of gravity.

apply a diffeomorphism π of X, and assume that π mixes different partition functions, namely

$$Z_i(g^{\pi}_{\mu\nu}) = R_i^j Z_j(g_{\mu\nu}), \tag{9.2}$$

with R_i^j a matrix acting on the space of partition functions. In this situation one cannot cancels the anomaly by adding new degrees of freedom to Z_i, and thus we call this type of anomaly **non-invertible gravitational anomaly**.

It is important to note that a non-invertible gravitational anomaly implies a particular property when we one works on a lattice. Consider a theory with non-invertible gravitational on a spatial lattice. Denote the Hilbert space of the theory as $\mathcal{H}$. Due to the non-invertible gravitational anomaly one finds that $\mathcal{H}$ is a **non local Hilbert space**, namely

$$\mathcal{H} \neq \bigotimes_i \mathcal{V}_i, \tag{9.3}$$

where $\mathcal{V}_i$ is the Hilbert space of a single lattice site and $\otimes$ is the standard tensor product. We are not able to provide a rigorous argument for the non locality of the Hilbert space, however one can justify this property in a heuristic manner.

Consider a theory on X, described by an action $S(\phi)$, with a set of partition function $\{Z_i\}$. As we said the different partition functions are given by different boundary conditions we may impose. To each boundary condition, labeled by the index i, we associate a characteristic function $\Psi_i(\phi)$ that imposes the specific boundary condition. That is

$$\Psi_i(\phi) = \begin{cases} 1 & \text{if } \phi \text{ respects the boundary condition associated to } \Psi_i \\ 0 & \text{if } \phi \text{ does not respect the boundary condition associated to } \Psi_i \end{cases} \tag{9.4}$$

At the level of path integral, after performing a Wick rotation, one may write

$$Z_i = \int D\phi\, \Psi_i(\phi)\, e^{-S_E(\phi)}, \tag{9.5}$$

where $S_E(\phi)$ is the Euclidian action. The functional integral in Eq(9.5) is then restricted to field configurations ϕ that respect the boundary condition imposed by $\Psi_i(\phi)$. Note that $\Psi_i(\phi)$ can generically be a non local term that acts on all of X.

Now we would like to replace $\Psi_i(\phi)$ with a new term in the action. In particular consider the term $\delta S_i(\phi) = -\ln \Psi_i(\phi)$ and notice that

$$\delta S_i(\phi) = \begin{cases} 0 & \text{if } \phi \text{ respects the boundary condition associated to } \Psi_i \\ \infty & \text{if } \phi \text{ does not respect the boundary condition associated to } \Psi_i \end{cases} . \tag{9.6}$$

Then we may rewrite Z_i as

$$Z_i = \int D\phi\, e^{-S_E(\phi) - \delta S_i(\phi)}. \tag{9.7}$$

The term $\delta S_i(\phi)$ assigns an infinite action to all field configurations that do not respect the boundary condition associated to $\Psi_i(\phi)$. This ensures that all filed configuration that do not respect the appropriate boundary condition are irrelevant in the evaluation of the path integral.

Eq(9.7 implies that our original theory with a given boundary condition can be described by a total Euclidian action $S_{tot}(\phi) = S_E(\phi) + \delta S_i(\phi)$. Note that we expect $S_i(\phi)$ to be a non local contribution since also $\Psi_i(\phi)$ is non local.

This implies that the action S_{tot} that leads to Z_i must inherently contain a non local term given by δS_i. If the action S_{tot} contains a non local term then it is reasonable to think that also the Hamiltonian H contains a non local term. Recall that the Hilbert $\mathcal{H}$ is spanned by the eigenvector of H. If H contains a non local operator then we expect its eigenstate to have a non local structure, with long range entanglement. From here we may conclude that the Hilbert space cannot be local.

Practically speaking this last property proves rather useful. in fact one often probes the presence of a non-invertible gravitational anomaly by observing that Hilbert space is non local. We will have an example of this later on.

Given these preliminary observations we may state the holographic principle of topological order.

***Statement 1**: A QFT in d spatial dimension with non-invertible gravitational anomaly described by a fusion n-category $\mathcal{C}_{boundary}$ can be realized as the boundary of a $d+1$ dimensional topological order described by a braided fusion n-category $\mathcal{C}_{bulk}$. In particular the bulk category is obtained as the Drinfeld center of the boundary category, $\mathcal{C}_{bulk} = \mathcal{Z}(\mathcal{C}_{boundary})$.*

Notice how according to the holographic principle the boundary uniquely determines the bulk while the opposite is not true. Indeed we may have multiple d dimensional theories corresponding to the same $d+1$ topological order, and in such a situation we would say that the d dimensional theories are **holo-equivalent**. This property proves crucial if one wishes to obtain a classification of theories based on their (generalized) symmetries [46–48].

9.0.1 From algebra to category

Start by considering a non anomalous quantum field theory defined on d dimensional spatial manifold Σ which admits a global symmetry, we denote such theory as QFT_{sym}. The global symmetry in QFT_{sym} will be described by a symmetry transformation realized trough a fixed time operator U_C defined on a (relatively) closed submanifold $C \subseteq \Sigma$. We can use U_C to construct the algebra of local symmetric operators (LSO for short)

$$\mathcal{A}_{LSO} = \{ O \mid [O, U_C] = 0\}, \tag{9.8}$$

and identify the symmetric sub-Hilbert space given as the set of state with null total charge

$$\mathcal{V}_{sim} = \{ |\psi\rangle \mid U_C|\psi\rangle = |\psi\rangle\}. \tag{9.9}$$

$\mathcal{V}_{sym}$ is not a local Hilbert space [45] and thus QFT_{sym} when restricted to $\mathcal{V}_{sym}$ becomes a theory with non-invertible gravitational anomaly that we denote as QFT_{ano}. Notice moreover how if we restrict our selves to states in $\mathcal{V}_{sym}$ and operators in $\mathcal{A}_{LSO}$ the symmetry operator becomes practically trivial, $U_C \sim \mathbb{1}$, so it might seem as if the symmetry is not present anymore. To recover information regarding the symmetry within $\mathcal{V}_{sym}$ we can switch our focus and study **transparent patch operators** (or t-patch operators for short). t-patch operators are defined on an open sub-manifold of Σ, also called a patch[2], and they are characterized by a transparency

[2] On the lattice a t-patch operator could also be called a tensor network operator.

condition, namely given a t-patch operator O_P defined on a patch P we have

$$[O_P, O] = 0 \quad \forall O \in \mathcal{A}_{LSO} \tag{9.10}$$

if the boundary of the patch ∂P is "far away" from O. The concept of far away here is not defined formally, but starting from a finite system of characteristic length L we may assume the existence of another length scale $L_{fa} < L$ such that if two objects are separated by a distance greater than L_{fa} then we say that they are far away from each other. The transparency condition in Eq(9.10) implies that the bulk of t-patch operator is invisible within $\mathcal{A}_{LSO}$. If $O \in \mathcal{A}_{LSO}$ is completely contained in P or completely outside P then O and O_P will have trivial commutation relation, which means that the action of O_p on operators in $\mathcal{A}_{LSO}$ is non trivial only along ∂P. Notice however that even if the bulk of a t-patch operator is invisible to operators in $\mathcal{A}_{LSO}$ it might still be non trivial when acting O_P on states in $\mathcal{V}_{sym}$. This observation leads to the following classification of t-patch operators.

- **Empty bulk t-patch operators** or **patch charge operators**: We say that a t-patch operator O_P has empty bulk if the action of O_P is trivial on any point that does not belong to ∂P. Notice how this implies that O_P does not depend on the patch P but only depends on the boundary ∂P.

- **Patch symmetry operator**: We say that a t-patch operator with non empty bulk is a patch symmetry operator. Patch symmetry operators do explicitly depend on the choice of P and not only of ∂P.

Now we would like to understand how from the algebra of these t-patch operators we can extrapolate a fusion category that will eventually lead to the holographic principle. To do so we need to isolate some specific operators within the set of t-patch operators. In particular we assume the existence of a **charge transporter operator**, which is an operator that transport symmetry charges around in Σ. For example for 0-form symmetry the charged objects are point-like excitation. Define $c(x)$ as the operator that creates a symmetry charge excitation around x, then a charge transporter operator $T_c(x \to y)$ would be defined on an open path (a patch) connecting x and y and would act as follows,

$$T_c(x \to y)c(x) = c(y). \tag{9.11}$$

$T_c(x \to y)$ does not create or annihilate any charge and thus must commute with the global charge operator U_C[3]. Moreover from Eq(9.11) we can see the action of $T_c(x \to y)$ is independent on the specific path connecting x and y and thus $T_c(x \to y)$ has empty bulk. Once we have identified $T_c(x \to y)$ we can inquire what happens when two symmetry excitations are fused together, this allows to establish the fusion rules and the values of the F-symbols which in turn fixes the fusion category of the symmetry charges, also called **representation category** which we denote as $\mathcal{R}$. Note that given what we have seen in Sec.8.3 we expect $\mathcal{R} = \mathcal{R}ep_G$. We will see an example later on with $G = \mathbb{Z}_2$ that confirms this intuition. Moreover using $T_c(x \to y)$

[3]Note that if one can associate a state $|c(x)\rangle$ to the operator $c(x)$, then $T_c(x \to y)$ can be acted directly on the state, namely $T_c(x \to y)\,|c(x)\rangle = |c(y)\rangle$.

we can also obtain information regarding the braiding properties of the charge excitations. To complete the study of our symmetry we must also incorporate information regarding the symmetry defects. Consider a symmetry transformation applied on a patch P in Σ. We will describe such transformation using the patch symmetry operator U_P, which is simply the global symmetry operator restricted to patch P. The bulk of U_P is transparent to operators in $\mathcal{A}_{LSO}$ but acts non trivially on states in $\mathcal{V}_{sym}$. The application of U_P generates a defect whose support is ∂P. Let us now consider a patch $P^{'}$ adjacent to P as shown in Fig[9.1). Applying $U_{P'}$ will move the defect from the boundary of P to the boundary of $P \cup P^{'}$. We find that $U_{P'}$ can be interpreted as a **defect transporter operator** T_d, that is an operator that moves defects,

$$U_{P'} = T_d(\partial P \to \partial(P \cup P^{'})). \tag{9.12}$$

As for charges once we have T_d we may acquire information regarding the fusion and braiding properties of defects so to find the fusion category describing their behavior in $\mathcal{V}_{sym}$, we call said category the **transformation category** which we denote as $\mathcal{T}$. Note that given what we argued in Sec.8.3 we expect $\mathcal{T} = \mathcal{V}ec_G$. We will an example later on with $G = \mathbb{Z}_2$ that will confirm this intuition. If we are equipped both with T_c and T_d we can also fuse charges with defects, and this may lead to new types of excitations in the spectrum. Whether we use the representation category $\mathcal{R}$, or the transformation category $\mathcal{T}$ depends on which phase of the theory we are working in. In the SSB phase charges will condense on the ground state and thus the excitations are given by the defects and described by $\mathcal{T}$. In the symmetric phase defects will condense leaving charges as the only excitations described by $\mathcal{R}$.

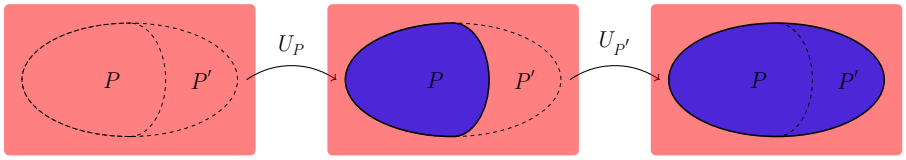


Figure 9.1: Gluing together two patch symmetry operators in 2 spatial dimensions. Applying U_P generates a defect (black continuous line) supported on ∂P and subsequently applying $U_{P'}$ moves the support of the defect to $\partial(P \cup P^{'})$.

9.0.2 The topological order in $d+1$

Now consider a topological order defined on $d+1$ dimensional space X with boundaries whose excitations are described by a braided fusion category $\mathcal{M}$. In order to realize the holographic principle we need to construct $\mathcal{M}$ in such a way that when we restrict the excitations to one of the boundaries of X and add boundary conditions we recover the same category describing the excitations in QFT_{ano}. The correct choice to construct $\mathcal{M}$ is by taking the Drinfeld center of $\mathcal{R}$ [40], written as $\mathcal{Z}(\mathcal{R})$. This procedure will also eliminate any ambiguity since as we will see later $\mathcal{R}$ and $\mathcal{T}$ have the same center,

$$\mathcal{M} = \mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\mathcal{T}), \tag{9.13}$$

so from the point of view of the topological order in 1 higher dimension it does not matter if we express the theory in terms of charges or defects. At this point we have effectively realized

the holographic principle and found the topological order in 1 higher dimension which leads to Fig[9.2]. Nonetheless we do not have to stop here. In fact we may ask if there is a holographic picture to describe states outside $\mathcal{V}_{sym}$ which have total charge different from zero.

Denote as $\mathcal{V}_n$ the sub-Hilbert space containing states with total charge n, in particular $\mathcal{V}_0 = \mathcal{V}_{sym}$. States in $\mathcal{V}_n$ can be regarded as having n symmetry charge excitations of charge 1. Within $\mathcal{V}_n$ a single charge cannot be created or destroyed by any local symmetric operator. Consider a local symmetric operator O_{LSO}. Assume that acting O_{LSO} on a state in $\mathcal{V}_n$ creates a pair of symmetry charge excitations with opposite charges. The action of O_{LSO} is independent on the specific value of n. The excitations created by O_{LSO} are the same symmetry charge excitations one could create on a state in $\mathcal{V}_{sym}$. This implies that no matter the symmetric sub-Hilbert space, symmetry charge excitations must be described by the fusion category $\mathcal{R}$.

We can then describe states outside $\mathcal{V}_{sym}$ by adding a new boundary $\tilde{\Sigma}$, which is a copy of Σ, to X and placing all charged states on this new boundary. For bookkeeping reasons we will use $\tilde{\mathcal{R}}$ (instead of simply $\mathcal{R}$ or $\mathcal{T}$) to indicate the fusion category on $\tilde{\Sigma}$. At this point we need a crucial observation. Within X we define a bulk Hamiltonian as a sum of local symmetric operators,

$$H_{bulk} = \sum_{O \in \mathcal{A}_{LSO}} O. \tag{9.14}$$

Some of the local operators in $\mathcal{A}_{LSO}$ might be extended operators connecting the two boundaries Σ and $\tilde{\Sigma}$ of the bulk X, we call these operators inter-boundary operators and we denote their algebra as $\mathcal{A}_{inter} \subseteq \mathcal{A}_{LSO}$. It is important to note that operators in $\mathcal{A}_{inter}$ cannot transport symmetry charges from $\tilde{\Sigma}$ to Σ or vice versa, since such operators would map chargeless states into charged ones and would automatically be non symmetric. This implies the following statement,

Statement 2: *A theory QFT_{sym} with an exact symmetry admits a stack decomposition of the type*
$QFT_{ano} \boxtimes_{\mathcal{M}} \tilde{\mathcal{R}}$ (Fig[9.3]) *where in particular,*

- *QFT_{ano} is a theory with non invertible gravitational anomaly describing the behavior of states within the symmetric sub-Hilbert space $\mathcal{V}_{sym}$.*
- *The bulk category $\mathcal{M}$ is the Drinfeld center of $\tilde{R}$.*
- *The bulk Hamiltonian does not contain any extended operator transferring charge from $\tilde{\mathcal{R}}$ to QFT_{ano}.*

The symbol $\boxtimes_{\mathcal{M}}$ indicates the stacking operation, that is connecting two boundary theory via a bulk described by $\mathcal{M}$. Practically speaking the statement above constitutes a possible way to probe the existence of an exact symmetry. If a theory admits a stack decomposition then it must have an exact symmetry, while if it does not admit such decomposition then the symmetry is missing or possibly approximate.

Finally we may put an extra requirement on our stack decomposition. Assume that the bulk $\mathcal{M}$ and the boundary $\tilde{\mathcal{R}}$ have a very large (possibly infinite) energy gap, meaning that creating excitations in $\mathcal{M}$ and $\tilde{\mathcal{R}}$ requires a large amount of energy. If we work at low energies all the

Figure 9.2: The anomalous theory QFT_{ano} realized as the boundary of a bulk theory described by $\mathcal{M}$

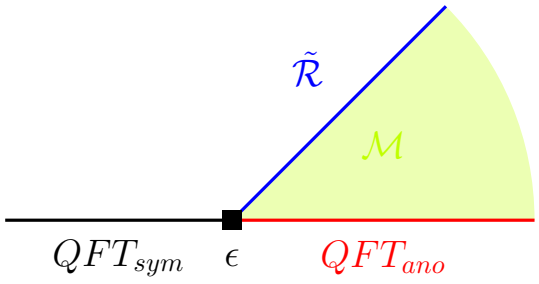


Figure 9.3: The stack decomposition of QFT_{sym}. The black box is a graphical representation of the isomorphism that maps the theory into its stack decomposition $\epsilon : QFT_{sym} \mapsto QFT_{ano} \boxtimes_{\mathcal{M}} \tilde{\mathcal{R}}$.

dynamical degrees of freedom of $QFT_{ano} \boxtimes_{\mathcal{M}} \tilde{\mathcal{R}}$ reside exclusively on the boundary QFT_{ano} while $\mathcal{M}$ and $\tilde{\mathcal{R}}$ play the role of a trivial background. This leads us to the following statement

Statement 3: *Given a symmetric theory QFT_{sym} in d spatial dimensions its low energy properties within the symmetric sub-Hilbert space can be exactly low energy simulated by the boundary of a $d+1$ topological order whose bulk excitations have a large energy gap.*

Where to *exactly low energy simulate* a theory means to find another theory with equivalent correlation function in the low energy limit.

9.0.3 An example with $\mathbb{Z}_2$ global symmetry

Let us now apply this machinery to a spin chain defined on a 1 d lattice with $\mathbb{Z}_2$ global 0-form symmetry that we studied in [41]. We make a small chance in notation to align with the standard literature of $\mathbb{Z}_2$ models. We will use e instead of c to denote symmetry charges, and we will use m instead of d to denote symmetry defects. We denote as N the total number of spins and on each lattice site i the spin operators are represented by the Pauli matrices X_i, Y_i, Z_i, and the spins are labeled by eigenvalues of Z_i,

$$|\uparrow\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i , \quad |\downarrow\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i . \tag{9.15}$$

We denote the Hilbert space of a single site as $\mathcal{V}_i = \text{span}\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$, and the total Hilbert space is $\mathcal{H} = \bigotimes_i^N \mathcal{V}_i$. We adopt the standard notation for which the tensor product of single spin states can be written as a single big ket, for example in the case of the state with all spin up we have

$$\bigotimes_i^N |\uparrow\rangle = \underbrace{|\ldots\uparrow\uparrow\uparrow\uparrow\ldots\rangle}_{N \text{ times}} . \tag{9.16}$$

The $\mathbb{Z}_2$ symmetry transformation is represented by a global operator U given as

$$U = \bigotimes_i^N X_i = X_1 \dots X_N. \tag{9.17}$$

Applying U on Z_i results in a minus sign and thus when acting on state U flips all spins in the state,

$$U Z_i U^\dagger = -Z_i \quad \longrightarrow \quad U\,|\dots\downarrow\downarrow\uparrow\downarrow\dots\rangle = |\dots\uparrow\uparrow\downarrow\uparrow\dots\rangle\,. \tag{9.18}$$

We have that $U^2 = \mathbb{1}$ which confirms that U represents a $\mathbb{Z}_2$ symmetry. The symmetric sub-Hilbert space $\mathcal{V}_{sym}$ is given by vectors $|\Psi\rangle \in \mathcal{H}$ which are invariant under the action of U, i.e. $U|\Psi\rangle = |\Psi\rangle \to |\Psi\rangle \in \mathcal{V}_{sym}$. Starting from a generic state $|\Phi\rangle \in \mathcal{H}$ we can construct a state belonging to the symmetric sub-Hilbert as

$$|\Phi\rangle \in \mathcal{H} \quad \longrightarrow \quad |\Phi\rangle + U|\Phi\rangle = |\Psi\rangle \in \mathcal{V}_{sym}. \tag{9.19}$$

Eq(9.19) implies that states in $\mathcal{V}_{sym}$ are written as a linear combination of states in $\mathcal{H}$ and thus $\mathcal{V}_{sym}$ does not admit a tensor decomposition in terms of single lattice sites, which results, as expected, in a non-invertible gravitational anomaly when we restrict our theory to state in $\mathcal{V}_{sym}$. The algebra of local symmetric operators is constructed as

$$\mathcal{A}_{LSO} = \{O\,|\,[O,U] = 0\}. \tag{9.20}$$

As seen previously using $\mathcal{A}_{LSO}$ we identify the t-patch operator defined via Eq(9.10). In 1 spatial dimension a patch P is an open string from site i to site j of the lattice, and if we pick $i < j$ the boundary ∂P is formed by the links $\langle i-1, i\rangle$ and $\langle j, j+1\rangle$. Let us define the two string operators

$$\begin{aligned} &Z_{\mathrm{str}_{ij}} = Z_i Z_j \quad \text{with}\, i < j, \\ &X_{\mathrm{str}_{ij}} = X_{i+1} X_{i+2} \dots X_{j-1} X_j \quad \text{with}\, i < j, \end{aligned} \tag{9.21}$$

whose algebraic structure is

$$\begin{aligned} &Z_{\mathrm{str}_{ij}} Z_{\mathrm{str}_{jk}} = Z_{\mathrm{str}_{ik}}, \\ &X_{\mathrm{str}_{ij}} X_{\mathrm{str}_{jk}} = X_{\mathrm{str}_{ik}}, \\ &Z_{\mathrm{str}_{ij}} X_{\mathrm{str}_{kl}} = -X_{\mathrm{str}_{kl}} Z_{\mathrm{str}_{ij}} \quad \text{if} \quad i < k < j < l, \\ &Z_{\mathrm{str}_{ij}} X_{\mathrm{str}_{kl}} = X_{\mathrm{str}_{kl}} Z_{\mathrm{str}_{ij}} \qquad \text{else.} \end{aligned} \tag{9.22}$$

The algebra of t-patch operators is then

$$\mathcal{A}_{\text{t-patch}} = \left\{ X_i, Z_i Z_{i+1}, \{Z_{\mathrm{str}_{ij}}\}, \{X_{\mathrm{str}_{ij}}\} \right\}, \tag{9.23}$$

where $\{Z_{\mathrm{str}_{ij}}\}$ and $\{X_{\mathrm{str}_{ij}}\}$ are respectively the sets of all $Z_{\mathrm{str}_{ij}}$ and all $X_{\mathrm{str}_{ij}}$ operators. Notice that even though $Z_{\mathrm{str}_{ij}}$ is a t-patch operator defined on a string connecting i and j, the bulk of the string is trivial and $Z_{\mathrm{str}_{ij}}$ acts only at the boundaries of the string, thus we classify $Z_{\mathrm{str}_{ij}}$ as a patch charge operator. The operator $X_{\mathrm{str}_{ij}}$ is the global symmetry operator U restricted to a an open string between i and j (we referred to it as U_p in the previous chapter). The bulk

of $X_{\text{str}_{ij}}$ is non trivial and thus $X_{\text{str}_{ij}}$ is patch symmetry operator. The global operator U is capable of measuring the total symmetry charge of a state while the patch symmetry operator $X_{\text{str}_{ij}}$ measures the symmetry charge present between sites i and j. The next step is the look for the charge/defect transporter operators within $\mathcal{A}_{\text{t-patch}}$, as we will now see these are the string operators defined in Eq(9.21). To study explicitly the behaviour of symmetry charges let us define a reference state for charges $\left|\Psi^e_{ref}\right\rangle \in \mathcal{V}_{sym}$ as

$$\left|\Psi^e_{ref}\right\rangle = \bigotimes_i^N \frac{|\uparrow\rangle_i + |\downarrow\rangle_i}{\sqrt{2}}. \tag{9.24}$$

Notice how $\left|\Psi^e_{ref}\right\rangle$ is the state with zero charge everywhere on the lattice as one can check by applying $X_{\text{str}_{ij}}$,

$$X_{\text{str}_{ij}} \left|\Psi^e_{ref}\right\rangle = \left|\Psi^e_{ref}\right\rangle \quad \forall i,j. \tag{9.25}$$

Acting with $Z_{\text{str}_{ij}}$ on $\left|\Psi^e_{ref}\right\rangle$ generates a state with two symmetry charges on sites i and j,

$$Z_{\text{str}_{ij}} \left|\Psi^c_{ref}\right\rangle = \cdots \otimes \underbrace{\frac{|\uparrow\rangle_i - |\downarrow\rangle_i}{\sqrt{2}}}_{\text{charge on site } i} \otimes \cdots \otimes \underbrace{\frac{|\uparrow\rangle_j - |\downarrow\rangle_j}{\sqrt{2}}}_{\text{charge on site } j} \otimes \dots. \tag{9.26}$$

Let now apply a second string operator on top of the first one

$$Z_{\text{str}_{jk}} Z_{\text{str}_{ij}} \left|\Psi^c_{ref}\right\rangle \overset{(a)}{=} Z_{\text{str}_{ik}} \left|\Psi^c_{ref}\right\rangle = \cdots \otimes \underbrace{\frac{|\uparrow\rangle_i - |\downarrow\rangle_i}{\sqrt{2}}}_{\text{charge on site } i} \otimes \cdots \otimes \underbrace{\frac{|\uparrow\rangle_k - |\downarrow\rangle_k}{\sqrt{2}}}_{\text{charge on site } k} \otimes \dots, \tag{9.27}$$

where in (a) we used the first line of Eq(9.22). Applying $Z_{\text{str}_{jk}}$ has moved the symmetry charge from site j to site k and thus we can conclude that $Z_{\text{str}_{ij}}$ can be interpreted as our charge transporter operator

$$Z_{\text{str}_{ij}} = T_e(i \to j) \quad i < j. \tag{9.28}$$

If we want to move the charge backward we have to use the adjoint of $T_e(i \to j)$,

$$T_e(j \to i) = [T_e(i \to j)]^\dagger \quad i < j. \tag{9.29}$$

Now that we have identified T_e we may identify the representation category by studying the fusion properties of charges, for example consider the operator

$$T_e(-\infty \to i) T_e(-\infty \to i) = \mathbb{1}, \tag{9.30}$$

which represents the situation where we start with two charges at $-\infty$ and we move them separately to site i, where then they fuse. Eq(9.30) implies the following fusion rule

$$e \otimes e = \mathbf{1}, \tag{9.31}$$

where we used e to denote the symmetry charge and $\mathbf{1}$ to denote the trivial excitation[4]. Eq(9.31) alone is not enough to fix the representation category as we also need the values of the F-symbols,

[4]Recall from our review of category theory that a fusion category always admits a simple object $\mathbf{1}$ whose fusion and braiding rules are trivial, in this context we call $\mathbf{1}$ the trivial excitation.

which are coefficients associated with the non associativity of the fusion process of 3 excitations, namely for 3 excitations labeled a, b, c we have

$$(a \otimes b) \otimes c = F(a,b,c)\, a \otimes (b \otimes c). \tag{9.32}$$

As a proof of concept let us compute $F(e,e,e)$. Start from a state with 3 symmetry charges on sites 1, 2 and 3 denoted as $|e_1, e_2, e_3\rangle$ and apply to T_e to fuse the charges together

$$\begin{aligned} T_e(3\to 1) \quad \overbrace{T_e(1\to 2)} \quad &= Z^\dagger_{\mathrm{str}_{23}} Z_{\mathrm{str}_{12}} \\ &= Z_2 Z_3 Z_1 Z_2 \\ &= Z_1 Z_2 Z_3 Z_2 \\ &= Z_{\mathrm{str}_{12}} Z^\dagger_{\mathrm{str}_{23}} \\ &= T_e(1\to 2) \quad \underbrace{T_e(3\to 2)} \quad . \end{aligned} \tag{9.33}$$

$$\overbrace{}^{\text{fuse } e_1 \text{ and } e_2 \text{ first}} \qquad \underbrace{}_{\text{fuse } e_2 \text{ and } e_3 \text{ first}}$$

We have that changing the order of the fusion does not change the final result and thus $F(e,e,e) = 1$. In a similar fashion one can see that all F-symbols are 1 (the computation can be found in [42]). Given the fusion rules and the values of the F-symbol one can uniquely fix the representation category which in this case turns out to be $\mathcal{R} = \mathcal{R}ep_{\mathbb{Z}_2}$ with simple objects $\{e, \mathbf{1}\}$. Now we move onto the study of defects. As we did for charges to make the computation explicit let us define a reference state for defects $\left|\Psi^m_{ref}\right\rangle \in \mathcal{V}_{sym}$ which is the state with no domain wall,

$$\left|\Psi^m_{ref}\right\rangle = \frac{|\ldots\uparrow\uparrow\uparrow\uparrow\ldots\rangle + |\ldots\downarrow\downarrow\downarrow\downarrow\ldots\rangle}{\sqrt{2}}. \tag{9.34}$$

Acting $X_{\mathrm{str}_{ij}}$ on $\left|\Psi^m_{ref}\right\rangle$ flips all the spins between sites i and j and generates two point-like defects on the links $\langle i, i+1\rangle$, $\langle j, j+1\rangle$. Acting with a second patch symmetry operator and using the second line in Eq(9.22) we have

$$X_{\mathrm{str}_{jk}} X_{\mathrm{str}_{ij}} \left|\Psi^m_{ref}\right\rangle = X_{\mathrm{str}_{ik}} \left|\Psi^m_{ref}\right\rangle, \tag{9.35}$$

where $X_{\mathrm{str}_{ik}} \left|\Psi^m_{ref}\right\rangle$ is a state with defects on the links $\langle i, i+1\rangle$ and $\langle k, k+1\rangle$. We find, as we anticipated in the previous chapter, that the patch symmetry operator $X_{\mathrm{str}_{ij}}$ acts as a defect transporter operator and Eq(9.12) in this specific case becomes

$$X_{\mathrm{str}_{ij}} = T_m(i\to j) \quad i < j. \tag{9.36}$$

Once again to move backward we need to use the adjoint

$$T_m(j\to i) = [T_m(i\to j)]^\dagger \quad i < j. \tag{9.37}$$

As we did for charges using $T_m(i\to j)$ we can deduce the fusion rule and the values of the F-symbols which turn out to be

$$m \otimes m = \mathbf{1}, \quad F(a,b,c) = 1 \text{ with } a,b,c = \mathbf{1}, m. \tag{9.38}$$

Now we can fix the transformation category uniquely as $\mathcal{T} = \mathcal{V}ec_{\mathbb{Z}_2}$ with simple objects $\{\mathbf{1}, m\}$. Now that we have both the representation category and the transformation category we may establish the topological order in 2 dimension, namely

$$\mathcal{M} = \mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\mathcal{T}) = \mathcal{G}au_{\mathbb{Z}_2}. \tag{9.39}$$

the category $\mathcal{G}au_{\mathbb{Z}_2}$ describes a $\mathbb{Z}_2$ gauge theory with four topological excitations $\mathbf{1}, e, m, f$ (the simplest example would be the 2d toric code model [49]). $\mathbf{1}, e$ and m are the same bosonic excitations that we already encountered while f is a point-like fermion excitation given by the bound state of e and m, $f = e \otimes m$. The excitation f can actually be observed directly in the 1 dimensional model. Consider the following operator

$$T_f(i \to j) = T_e(i \to j) T_m(i \to j) \quad i < j. \tag{9.40}$$

$T_f(i \to j)$ transports the bound state f from site i to site j. f has fermionic self statistic whose origin lies in the mutual statistic of T_e and T_m which is expressed in the third and fourth line of Eq(9.22). Start by considering a state with two f excitations, one on site 1 and the other on site 3, denoted as $|f_1, f_3\rangle$. The exchange of the two particles in $|f_1, f_3\rangle$ can be realized by applying the operator $T_f(2 \to 3) T_f(3 \to 1) T_f(1 \to 2)$, but now notice

$$\begin{aligned} T_f(2 \to 3) T_f(3 \to 1) T_f(1 \to 2) &= T_f(2 \to 3) [T_f(1 \to 3)]^\dagger T_f(1 \to 2) \\ &= Z_2 Z_3 X_3 (Z_1 Z_3 X_2 X_3)^\dagger Z_1 Z_2 X_2 \\ &= Z_2 Z_3 X_3 X_3 X_2 Z_3 Z_1 Z_1 Z_2 X_2 \\ &= Z_2 \underbrace{X_2 Z_2 X_2}_{-Z_2} \\ &= -Z_2 Z_2 = -\mathbb{1}, \end{aligned} \tag{9.41}$$

Thus exchanging two f particles results in a minus sign implying that f is a fermion. Now that we know the exact shape of the $d+1$ topological order we may apply the stack decomposition $QFT_{sym} = QFT_{ano} \boxtimes_{\mathcal{M}} \tilde{\mathcal{R}}$ as represented in Fig[9.4]. For clarity we denote the sites on the boundary $\tilde{\mathcal{R}}$ with an extra tilde. In Fig[9.4] we also represented the charge transporter operator $T_e(\tilde{i} \to j)$ as a black continuous string, which moves a symmetry charge from $\tilde{\mathcal{R}}$ to QFT_{ano}. Notice how only one end of the string resides on the boundary QFT_{ano} and thus when applied on a state $|\Psi\rangle \in \mathcal{V}_{sym}$ we have $T_e(\tilde{i} \to j) |\Psi\rangle = Z_j |\Psi\rangle$, and since $Z_j \notin \mathcal{A}_{LSO}$ we conclude

$$T_e(\tilde{i} \to j) \notin H_{bulk} = \sum_{O \in \mathcal{A}_{LSO}} O. \tag{9.42}$$

This is concrete example of why for a true symmetry we must not have a bulk operator transporting charge from one boundary to an other.

9.0.4 Emergent symmetries and holography

Now we are going to apply the holographic approach to formally argue why in most cases emergent higher form symmetries are exact at low energies. Let us start from a UV theory, denoted as QFT_{UV}, described by Hamiltonian H_{UV} defined on a d dimensional space Σ. We

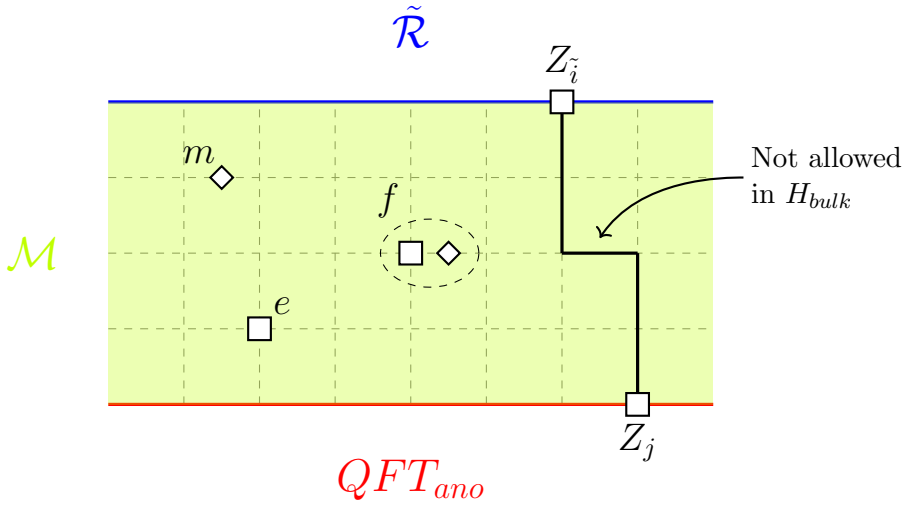


Figure 9.4: The stack decomposition of a theory with $\mathbb{Z}_2$ global symmetry. The boundary QFT_{ano} contains all states in $\mathcal{V}_{sym}$ while the bulk is a $\mathbb{Z}_2$ gauge theory with excitations $\mathbf{1}$, e, m and $f = e \otimes m$. We also represented the string operator $T_e(\tilde{i} \to j) = Z_{\tilde{i}} Z_j$ (black continuous line) which is not allowed in H_{bul} since it maps charged states in chargeless ones and is this not a symmetric operator.

assume QFT_{UV} has a symmetry described by the unitary operator U_{UV} with $[H_{UV}, U_{UV}] = 0$ that also defines the UV algebra of local symmetric operator

$$\mathcal{A}^{UV}_{sym} = \{O \mid [O, U_{UV}] = 0\}. \tag{9.43}$$

We assume the H_{UV} has a well defined built in energy cut off Λ (such as the mass of particular excitation) which splits the energy scale into UV and IR. The Hilbert space $\mathcal{H}_{UV}$ of the UV theory is local and can be constructed using the eigenvectors of the Hamiltonian

$$H_{UV}|\psi_n\rangle = E_n|\psi_n\rangle. \tag{9.44}$$

We can now define a set projectors $\{P_i\}$ which explicitly splits the Hilbert space in high and low energy states.

$$P_i|\psi_n\rangle = \begin{cases} 0 & \text{if} \quad E_n < \Lambda \\ |\psi_n\rangle & \text{if} \quad E_n > \Lambda \end{cases} \quad \forall P_i \in \{P_i\}, \tag{9.45}$$

and alongside $\{P_i\}$ we can construct the low energy Hilbert space $\mathcal{H}_{IR}$ as

$$\mathcal{H}_{IR} = \{\, |\psi\rangle \,|\, P_i|\psi\rangle = 0 \ \forall P_i \in \{P_i\},\ |\psi\rangle \in \mathcal{H}_{UV} \,\}. \tag{9.46}$$

All operators allowed in the IR description of our theory must commute with the projectors in $\{P_i\}$ otherwise they could cause mixing between states in the IR and states in the UV. This implies that the local symmetric operators allowed in the IR form a smaller algebra than $\mathcal{A}_{UV}$ since we must also impose a commutation condition with the projectors $\{P_i\}$ and this defines the low energy algebra of local symmetric operators,

$$\mathcal{A}^{IR}_{sym} = \{O \mid [O, P_i] = 0 \ \forall P_i \in \{P_i\} \ , \ O \in \mathcal{A}^{UV}_{sym}\}. \tag{9.47}$$

If we restrict our selves to states in $\mathcal{H}_{IR}$ and operators in $\mathcal{A}_{IR}$ we obtain the low energy description of our original theory QFT_{UV}, we call this new theory QFT_{IR}. It is important to notice that within QFT_{IR} the projectors $\{P_i\}$ are equivalent to zero, $P_i \sim 0_{IR}$, since the commute with all low energy symmetric operators and annihilate all low energy states. At this point one may witness the emergence of new symmetries. In fact since $\mathcal{A}_{IR} \subseteq \mathcal{A}_{UV}$ it is plausible that the low energy algebra admits a new non trivial operator $U_{IR} \neq U_{UV}, \mathbb{1}_{IR}$ such that

$$[U_{IR}, O] = 0 \quad \forall O \in \mathcal{A}^{IR}_{sym}, \tag{9.48}$$

then U_{IR} would define a new exact emergent symmetry of QFT_{IR}. Using U_{IR} one could define t-patch operators of QFT_{IR} and construct the topological charges and defects of the low energy theory. Unfortunately this is often not case. In fact the extra commutation relation $[O, P_i] = 0$ in the definition of $\mathcal{A}^{IR}_{sym}$ projects out some operators in $\mathcal{A}^{UV}_{sym}$. Even if $\mathcal{A}^{IR}_{sym}$ is a smaller algebra then $\mathcal{A}^{UV}_{sym}$, it is not guaranteed that $\mathcal{A}^{IR}_{sym}$ admits a new symmetry operator U_{IR}. However one may define a hierarchy within $\mathcal{A}^{IR}_{sym}$ and separate the relevant operators from those that would be considered small corrections. If all operators that do not commute with U_{IR} are small corrections then, in the spirit of effective field theory, one may ignore the non-commuting operators and state that QFT_{IR} has an approximate symmetry described by U_{IR}. This is generally what happens for 0-form symmetries. Now let us visualize the situation using the holographic principle, in particular we want to take advantage of ***Statement 2*** in section 9.0.2. If QFT_{IR} has an exact emergent symmetry we should be able to construct a well defined stack decomposition $QFT_{ano} \boxtimes_{\mathcal{M}} \tilde{\mathcal{R}} \simeq QFT_{IR}$ (Fig[9.3]) where the Hilbert space of QFT_{ano} is $\mathcal{V}^{IR}_{sym}$, namely the low energy symmetric sub-Hilbert space defined by U_{IR}

$$\mathcal{V}^{IR}_{sym} = \{|\Psi\rangle \mid U_{IR}|\Psi\rangle = |\Psi\rangle\,,\ |\Psi\rangle \in \mathcal{H}_{IR}\}. \tag{9.49}$$

The Hilbert space of the boundary $\tilde{\mathcal{R}}$ is given by all the remaining states in $\mathcal{H}_{IR}$. The symmetry excitations on both boundaries are described by a fusion category $\mathcal{R}$ while the excitations in the bulk are described by the Drinfeld center of $\mathcal{R}$. So far it may seem as if the stack decomposition is always possible even if U_{IR} is not an exact symmetry, however we need to take a careful look at what happens in the bulk. If U_{IR} is not an exact symmetry then the charge conservation is only approximate and chargeless states can time evolve into charged ones. From this holographic point of view this means that we may have bulk operators transferring charge for $\tilde{\mathcal{R}}$ to QFT_{ano}. The symmetry (and the associated charge conservation rule) is present only if we ignore the charge transporter operators by treating them as small perturbations.

For 0-form symmetries the charged objects are point-like and the charge transferring operators are defined on a string connecting $\tilde{\mathcal{R}}$ to QFT_{ano} as shown in Fig[9.5(a)], we denote this of type operators as $S(\tilde{R} \to QFT_{ano})$. The inter-boundary string operators need not to grow as the size of the system so they are admitted operators in $\mathcal{A}_{IR}$. Given the IR projectors $\{P_i\}$, it is possible that

$$\big[S(\tilde{R} \to QFT_{ano}), P_i\big] \neq 0 \implies S(\tilde{R} \to QFT_{ano}) \notin A^{IR}_{sym}. \tag{9.50}$$

In this case $S(\tilde{R} \to QFT_{ano})$ cannot appear in the IR Hamiltonian of the bulk H^{IR}_{bulk}. By ***Statement 2*** we conclude that the stack decomposition and thus the 0-form symmetry would be exact. However Eq(9.50) is often not realized and we have that $S(\tilde{R} \to QFT_{ano}) \in A^{IR}_{sym} \implies S(\tilde{R} \to QFT_{ano}) \in H^{IR}_{bulk}$. In this case the stack decomposition and the symmetry would not

be exact not exact. We conclude that in most cases emergent 0-form symmetries are always approximate.

On the other hand the charged objects of higher form symmetries are defined on non trivial (relatively) closed manifolds. Let us assume that the space Σ on which we defined the original theory does not have any punctures[5]. Then non trivial cycles must grow as the size of the system. This implies that for higher form symmetries a potential charge transferring operator, denoted as $\hat{O}_{\mathcal{M}}$, should be defined on a global manifold connecting the two boundaries of $\mathcal{M}$, as in Fig[9.5(b)]. The charge transferring operator would then be a non-local operator, growing as the size of the system, and it would not be allowed in $\mathcal{A}_{IR}$. Since $\hat{O}_{\mathcal{M}} \notin \mathcal{A}_{IR}$, then it cannot be an element of $\mathcal{A}^{IR}_{sym}$ nor it can appear in H^{IR}_{bulk}. **We conclude that the stack decomposition, and thus the symmetry, is exact**.

If instead Σ does have a puncture that we may define a charged object by wrapping a cycle around the puncture as in Fig[9.5(a)]. this would be a local operator defined on a local cycle. $\hat{O}_{\mathcal{M}}$

meaning that once again the symmetry would only be approximate. Actually at this point we should make a comment originally discussed in [15]. Assume to be working on a lattice, then if the density the lattice punctures is small we can hope that via a coarse graining procedure they will disappear from the IR description of our theory. This means that if we work at low energies and long distances the punctures are masked and higher form symmetry are protected once again.

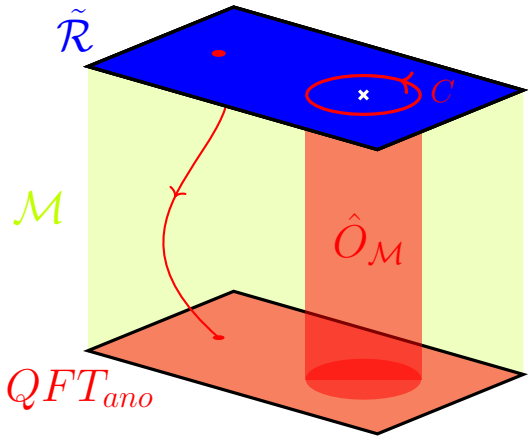
9.0.5 Symmetry, dual symmetry and categorical symmetry

Developing the concepts of higher form symmetry and holographic principle opens up a window to discuss dual and **categorical symmetries** and their connection to the topological order in 1 higher dimension. As we have seen above (Eq(9.22)) a symmetry U can be described via an algebra t-patch operators $\mathcal{A}_{t-patch}$ which encodes all information about the behavior of charges and defects. Using $\mathcal{A}_{t-patch}$ we can derive the representation category $\mathcal{R}$ (describing charges) and the transformation category $\mathcal{T}$ (describing defects).

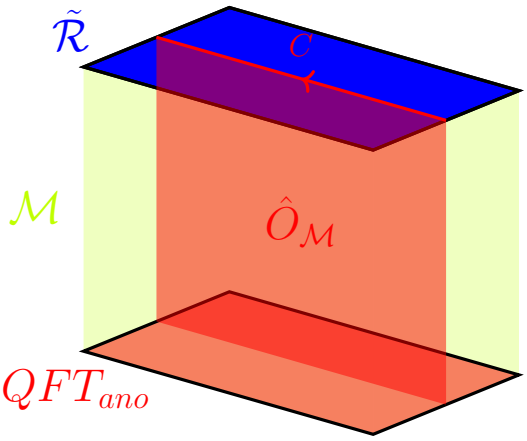
We say that U^{dual} is the dual symmetry of U if the charges of U^{dual} are the defects of U, and the defects of U^{dual} are the charges of U.

Notice how this definition is only valid if we include higher form symmetries. In fact assume to have a p-form symmetry described by a group $G^{(p)}$ in d dimensional space at fixed time, the charges of the $G^{(p)}$ are p dimensional objects while the defects are $d-p-1$ dimensional objects, then the dual symmetry will necessarily be a $(d-p-1)$-form symmetry $\tilde{G}^{d-p-1}$ with $(d-p-1)$ dimensional charges and p dimensional defects. From the point point of view of the categories it means that the representation category of the U^{dual} is the transformation category of U, $\mathcal{R}^{dual} = \mathcal{T}$, while the transformation category of U^{dual} is the representation category of U, $\mathcal{T}^{dual} = \mathcal{R}$. The representation and transformation categories are fully determined by the algebra of t-patch operators, and said algebra is fully determined by the algebra of local symmetric operators, then it follows that the algebras describing a symmetry and its dual must

[5] In the lattice formulation no punctures means no missing sites.



(a) The boundary $\tilde{\Sigma}$ on which we defined $\tilde{\mathcal{R}}$ with a puncture.



(b) The boundary $\tilde{\Sigma}$ on which we defined $\tilde{\mathcal{R}}$ without punctures. Here C is relatively closed and can be the support of a charged operator of higher form symmetries.

Figure 9.5: Graphical representation of $QFT_{ano}\boxtimes_{\mathcal{M}}\tilde{\mathcal{R}}$ in $d = 2$ space dimensions. 9.5(a): Charged objects of 0-form symmetries are defined on points of the upper boundary, and thus they can always be transported to the lower boundary QFT_{ano} via the use of line operator in $\mathcal{M}$. This line operator does not grow as the size of system and thus is allowed in the set of local symmetric operators used to construct the bulk Hamiltonian. The injection of charged states in QFT_{ano} implies that conservation law associated with U_{IR} is not valid and thus U_{IR} is at best an approximate emergent symmetry. Moreover if Σ (and thus $\tilde{\Sigma}$) suffers form a puncture (white cross in Fig[9.5(a)]) then also higher form symmetries are not safe. We can loop around the puncture and form a non trivial cycle C. On C we may defines a charged operator of the higher form symmetry U_{IR} which should reside on the $\tilde{\mathcal{R}}$ boundary of $\mathcal{M}$. Now one can construct the bulk operator $\hat{O}_{\mathcal{M}}$ defined on the red cylinder which transports the charged object to QFT_{ano}. $\hat{O}_{\mathcal{M}}$ is a local operator and thus allowed by the bulk Hamiltonian. This spoils the stack decomposition and thus the higher form symmetry. 9.5(b): In this case $\tilde{\Sigma}$ does not have any puncture and the only non trivial cycles are relatively closed. This means that the charged object must stretch all the way to the boundary of $\tilde{\Sigma}$ thus growing as the size of the system. The bulk operator $\hat{O}_{\mathcal{M}}$ transporting charged objects will have to be non local and will not be allowed in the Hamiltonian describing $\mathcal{M}$. We conclude that the stack decomposition and thus the emergent symmetry U_{IR} is exact.

be isomorphic to each other

$$\mathcal{A}_{LSO} \overset{i}{\longmapsto} \mathcal{A}^{dual}_{LSO}, \tag{9.51}$$

where the isomorphism i is a duality map, such as the spin-domain wall duality in the 1d Ising model [32]. A way to understand dual symmetries using the holographic approach was presented in [47]. Given two different symmetries described by the representation n-categories $\mathcal{R}$ and $\mathcal{R}'$ in the respective symmetric sub-Hilbert spaces we say that $\mathcal{R}$ and $\mathcal{R}'$ are dual to each other if:

- They correspond to the same topological order in 1 higher dimension,

$$\mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\mathcal{R}') = \mathcal{M}. \tag{9.52}$$

- Stacking them trough $\mathcal{M}$ produces a trivial topological order described by $n\mathcal{V}ec$,

$$\mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{R}' = n\mathcal{V}ec. \tag{9.53}$$

To see why the above statement is correct we will work backward in the $\mathbb{Z}_2$ global symmetry case. We will first construct the slab $\mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{R}^{dual} = \mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{T}$ and then argue why said slab must have trivial topological order. The boundary R of the slab can be obtained by working in the symmetric sub-Hilbert and imposing that we are in the symmetric phase i.e. the $\mathbb{Z}_2$ defects condense and the topological excitations are given by the symmetry charges e and described by $\mathcal{R} = \mathcal{R}ep_{\mathbb{Z}_2}$. On top of the $\mathcal{R}$ boundary we place the topological order $\mathcal{M} = \mathcal{G}au_{\mathbb{Z}_2}$ which admits both e and m excitations, and finally we close the sandwich with the top boundary $\mathcal{R}^{dual} = \mathcal{T} = \mathcal{V}ec_{\mathbb{Z}_2}$ which is obtained by working in the symmetric sub-Hilbert in the SSB phase where the charges condense and the topological excitations are only defects described by $\mathcal{T}$. The final picture, Fig[9.6], is that of a $2d$ slab where the upper and lower boundary conditions correspond to imposing the condensation of e and m respectively. Because of the boundary conditions all non trivial excitations within $\mathcal{M}$ condense on one of the boundaries and thus the only excitation in $\mathcal{M}$ is the trivial one $\mathbf{1}$. A theory with only trivial topological excitations $\mathbf{1}$ is necessarily a trivial topological order described by the fusion category $\mathcal{V}ec$. We have presented an explicit example using $\mathbb{Z}_2$ but the logic can be generalized to any symmetry.

Symmetry and dual symmetry are often not manifest together at the same time even if they essentially contain the same information as can be deduced from Eq(9.51). Nonetheless a the theory described by Hamiltonian H with symmetry U also has the dual symmetry U^{dual} since using the duality isomorphism i defined in Eq(9.51) we can map H to an Hamiltonian H^{dual} where the symmetry U^{dual} is explicit. To make manifest the fact that the theory has both symmetry and dual symmetry we say that the theory has a **categorical symmetry**,

A **categorical symmetry** *is an isomorphic class of algebras of local symmetric operators.*

From the point of view of the holographic approach if two theories have isomorphic algebras of local symmetric operators they also have isomorphic algebras of t-patch operators. This implies that the two theories contain the same topological excitation, and thus they are described by the same topological order in 1 higher dimension. This means that if two theories are **holo-equivalent** (they are different boundaries of the same bulk) then they must have the same

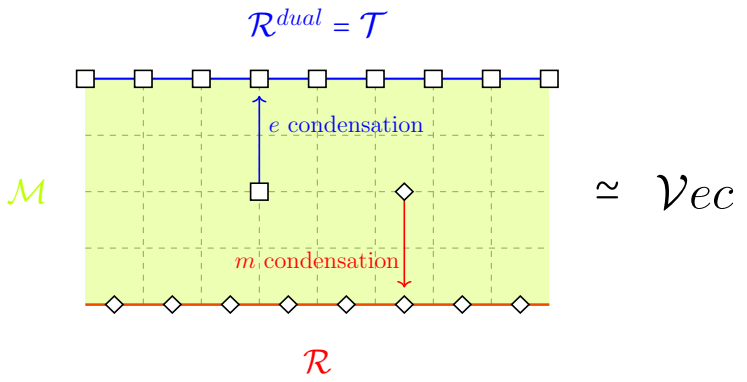


Figure 9.6: The slab given by $\mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{T}$. The lower boundary is induced by the condensation of m excitations and described by $\mathcal{R}ep_{\mathbb{Z}_2}$, while the upper boundary is induced by e condensation and described by $\mathcal{V}ec_{\mathbb{Z}_2}$. Since all excitations condense on one of the two boundaries the bulk itself cannot contain any topological excitations and is thus a trivial topological order $\mathcal{V}ec$.

categorical symmetry (for a more complete discussion of **categorical symmetry** consult [41]).

Chapter 10

Emergent symmetries in a quantum rotor system

Now we wish to construct an Hamiltonian model to use an example of the machinery we just provided. We will work on a d dimensional hypercubic space lattice defined at fixed time. First we need to fix some conventions for discrete exterior calculus on the lattice.

10.1 Discrete exterior calculus

In order to obtain a general description we may unify all the fundamental structures of the lattice under a single class, which we will call p-cells. A p-cell, noted as c_p, is defined as the fundamental p dimensional object of the lattice obtained by composing together 2^p sites, so that a 0-cell is a single site, a 1-cell is a link, a 2-cell is a plaquette and so on. To each site of the lattice we assign a system of Cartesian coordinates represented by the base vectors $\mu_1, \ldots, \mu_d$, as in Fig[10.1]. The direction of the base vectors fixes a canonically positive orientation for the p-cells in our lattice. If we want to identify a particular p-cell c_p we must specify a coordinate x on the lattice and a set of indices depending on the dimension of c_p. The indices are necessary since to each site we can assign multiple p-cells and thus we need to specify which one we are referring to.

$$c_p(\vec{x})_{\mu_1\mu_2\ldots\mu_p} = \{\vec{x}\} \cup \{\vec{x}+\hat{\mu}_i | 1 \leq i \leq p\} \cup \{\vec{x}+\hat{\mu}_i+\hat{\mu}_j | 1 \leq i < j \leq p\} \cup \cdots \cup \{\vec{x}+\hat{\mu}_1+\hat{\mu}_2+\ldots\hat{\mu}_p\}. \quad (10.1)$$

The set $\mu_1, \mu_2, \ldots, \mu_p$ is ordered from smallest to largest but we can drop this constraint by imposing that c_p be completely antisymmetric i.e. $c_p(\vec{x})_{\ldots\mu\nu\ldots} = -c_p(\vec{x})_{\ldots\nu\mu\ldots}$, with $-c_p$ indicating the same c_p with opposite orientation. For a graphical representation see Fig[10.1]. We can construct a boundary operator that maps a p-cell c_p into a set of (p-1)-cells which are the boundaries of c_p.

$$\partial c_p(\vec{x})_{\mu_1\ldots\mu_p} = \sum_{k=1}^{p} (-1)^{k+1} [c_{p-1}(\vec{x}+\hat{\mu}_k)_{\mu_1\ldots\overset{\times}{\mu}_k\ldots\mu_p} - c_{p-1}(\vec{x})_{\mu_1\ldots\overset{\times}{\mu}_k\ldots\mu_p}], \quad (10.2)$$

where the cross above μ_k means that we are excluding that particular index.

To our hypercubic lattice we can associate a dual lattice whose sites have coordinates $\tilde{x} = x + \frac{1}{2}(\mu_1 + \cdots + \mu_d)$. Using the dual lattice we may define a discrete version of the Hodge dual which maps a p-cell c_p into a (d-p)-cell $\tilde{c}_{d-p}$ in the dual lattice,

$$\star c_p(x)_{\mu_1 \dots \mu_p} = \epsilon_{\mu_1 \dots \mu_p \mu_{p+1} \mu_d} \tilde{c}_{d-p}(\tilde{x} - \hat{\mu}_{p+1} - \cdots - \hat{\mu}_d)_{\mu_{p+1} \dots \mu_d}. \tag{10.3}$$

The indices in Eq(10.3) are not contracted and as seen in the continuous case we have $\star \star c_p = (-1)^{p(d-p)} c_p$.

Moreover we may map a p-cell c_p to a set of $(p+1)$-cells whose boundaries contain c_p, this is achieved using the coboundary operator which we may write as (notice the analogy with the $d^\dagger$ operator seen for differential forms)

$$\delta c_p = (-1)^{d(p+1)+1} \star \partial \star c_p, \tag{10.4}$$

or more explicitly

$$\delta c_p(x)_{\mu_1 \dots \mu_p} = \sum_\nu c_{p+1}(\vec{x})_{\nu \mu_1 \dots \mu_p} - c_{p+1}(\vec{x} - \hat{\nu})_{\nu \mu_1 \dots \mu_p} \tag{10.5}$$

When writing a quantity as $O_{c_p}(x)_{\mu_1 \dots \mu_p}$ we simply mean to indicate a quantity defined on the p-cell identified by the coordinate x and the indices $\mu_1, \dots, \mu_p$. Finally we can define a discrete version of the exterior derivative

$$(dO)_{c_{p+1}} = \sum_{c_p \in \partial c_{p+1}} O_{c_p}, \tag{10.6}$$

and alongside it we have a discrete version of Stoke's theorem [50]. Consider Σ_{p+1} as the union of a set of $(p+1)$-cells, then Stoke's theorem states

$$\sum_{c_{p+1} \in \Sigma_{p+1}} dO_{c_{p+1}} = \sum_{c_p \in \partial \Sigma_{p+1}} O_{c_p} \tag{10.7}$$

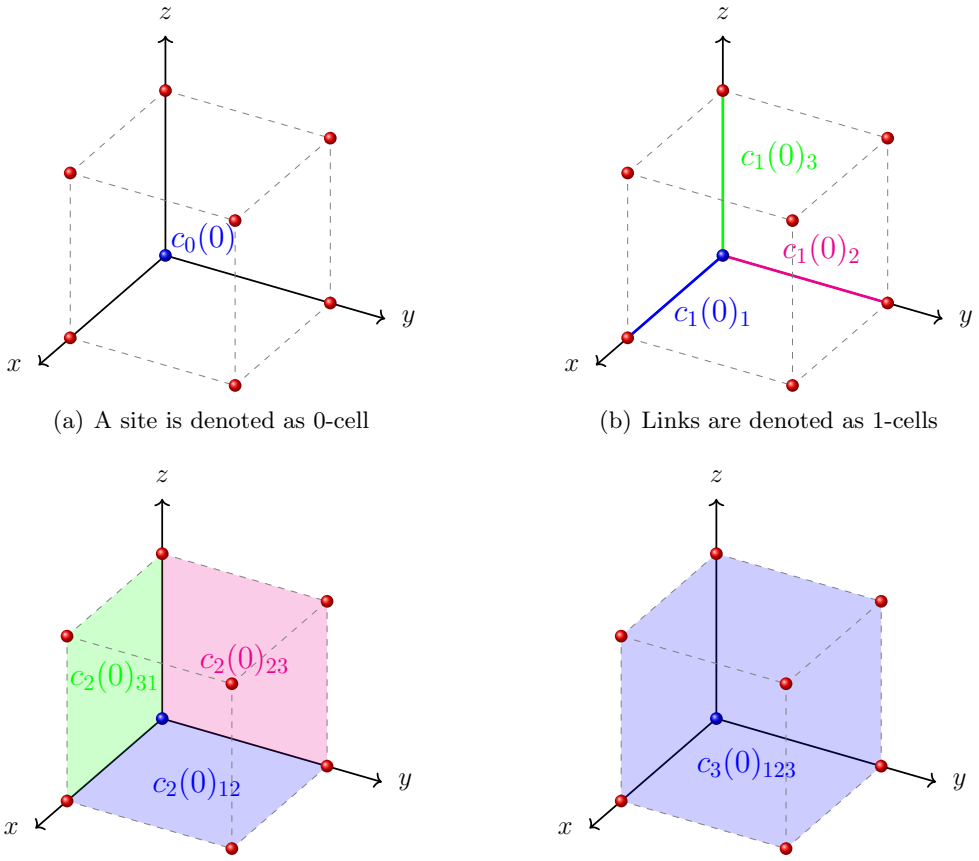
10.2 The UV theory

The model we are about to showcase was discussed in depth in [15], here we simply report a summary of the most important results. Consider a set of $U(1)$ quantum rotors each placed on the geometrical center of a p-cell in our d dimensional lattice. We view each rotor as a particle rotating around an infinitesimal circle perpendicular to a z axis (the choice of the axis is arbitrary and we pick the z axis for simplicity). The position of each particle is encoded in the angle Θ_{c_p}. Each particle carries an angular momentum $L^z_{c_p}$ along the z axis, which is also the conjugated momentum of Θ_{c_p}.

$$\left[\Theta_{c_p}, L^z_{c'_p}\right] = i\delta_{c_p, c'_p} \quad \longrightarrow \quad L^z_{c'_p} = -i\frac{\partial}{\partial \Theta_{c_p}}, \tag{10.8}$$

where δ_{c_p, c'_p} is a Kronecker delta which is different from zero only when c_p and c'_p are the same p-cell. Notice that since Θ_{c_p} is an angle then the eigenvalues of $L^z_{c_p}$ are integers. The raising and lowering operator for $L^z_{c_p}$ can be realized as

$$L^+_{c_p} = (L^-_{c_p})^\dagger = (L^+_{-c_p})^\dagger = e^{i\Theta_{c_p}}, \tag{10.9}$$



(a) A site is denoted as 0-cell

(b) Links are denoted as 1-cells

(c) Links are denoted as 1-cells

(d) Cubes are denoted as 3-cells

Figure 10.1: Possible types of p-cells centered around a site of coordinate $\vec{x} = (0,0,0)$ (blue site in figure). By changing the coordinate $\vec{x}$ and the indices of the p-cell we can cover all the fundamental structures of the lattice.

as can be checked via direct computation, in fact

$$\left[L^z_{c_p}, L^+_{c_p}\right] = L^+_{c_p}, \qquad \left[L^z_{c_p}, L^-_{c_p}\right] = -L^-_{c_p}, \tag{10.10}$$

and applying on a $L^z_{c_p}$ eigenstate $|j\rangle$ we have

$$L^z_{c_p} L^+_{c_p}|j\rangle = (L^+_{c_p} L^z_{c_p} + L^+_{c_p})|j\rangle = (j+1)L^+_{c_p}|j\rangle. \tag{10.11}$$

The model we consider is described by the following Hamiltonian

$$H_{UV} = \underbrace{\frac{U}{2}\sum_{c_{p-1}} \rho^2_{c_{p-1}}}_{①} \underbrace{- U \sum_{c_{p+1}} W^\dagger_{c_{p+1}}}_{②} + \underbrace{\frac{K}{2}\sum_{c_p}(L^z_{c_p})^2}_{③} + \underbrace{\frac{J}{2}\sum_{c_p} L^+_{c_p} + h.c.}_{④}, \tag{10.12}$$

where we defined the two operators

$$\rho_{c_{p-1}} = \sum_{c_p \in \delta c_{p-1}} L^z_{c_p}, \qquad W^\dagger_{c_{p+1}} \coloneqq W^\dagger[\partial c_{p+1}] = \prod_{c_p \in \partial c_{p+1}} L^+_{c_p}. \tag{10.13}$$

Upon using Eq(10.5) we can rewrite $\rho_{c_{p-1}}$ as

$$\rho_{c_{p-1}}(\vec{x})_{\mu_1 \dots \mu_p} = \sum_\nu L^z_{c_p}(\vec{x})_{\nu\mu_1\dots\mu_p} - L^z_{c_p}(\vec{x} - \hat{\nu})_{\nu\mu_1\dots\mu_p}, \tag{10.14}$$

meaning that $\rho_{c_{p-1}}$ behaves as discrete divergence, measuring the change in angular momentum as we cross c_{p-1}. We deduce that term ① in Eq(10.12) is a kinetic term quantifying the energy stored in the spatial variation of $L^z_{c_p}$.

It is also important to clarify the expression of the $W^\dagger_{c_{p+1}}$ operator, since despite the initial appearance this operator contains both raising and lowering operators. This is because as we move across the boundary c_{p+1} half of the p-cells will be travelled with positive orientation while the other half will have negative orientation and, as can be read form Eq(10.9), applying the raising operator on $-c_p$ is equivalent to applying the lowering operator on c_p.

As a last preliminary observation note that applying $L^+_{c_p}$ changes the value of the divergence $\rho_{c_{p-1}}$ on the boundary of c_p. We can interpret such a variation as having topological charges on ∂c_p. Now let consider Σ_{c_p} as the the union of a set of p-cell. Applying the raising operator on Σ_{c_p} will produce topological charges on $\partial\Sigma_{c_p}$. If Σ_{c_p} is a cycle in our lattice then it will have no boundary and thus it will produce no charge detectable by $\rho_{c_{p-1}}$. This means that we can define the the discrete Wilson cycle $W^\dagger[C_p] = \prod_{c_p \in C_P} L^+_{c_p}$, with C_p a p-cycle, and we have

$$\left[\rho_{c_{p-1}}, W^\dagger[C_p]\right] = 0, \tag{10.15}$$

and this of course also valid in the special case of $C_p = \partial c_{p+1}$. Now we are ready to study the emergent symmetries of this model.

10.2.1 Low energy theory for $J = 0$

Let us start by considering the case of $J = 0$. In this situation we have that $\rho_{c_{p-1}}$ commutes with H_{UV} since it trivially commutes with ① and ③, and by Eq(10.15) it also commutes with ②. This means that the value of $\rho_{c_{p-1}}$ does not change under time evolution and moreover that we can diagonalize H_{UV} and $\rho_{c_{p-1}}$ at the same time. We may use $\rho_{c_{p-1}}$ to characterize the eigenstates of H_{UV}, and noticing that the eigenvalues of $\rho_{c_{p-1}}$ are integers we have

$$\begin{aligned} \mathcal{I}_0 &= \{\, |\psi_n\rangle \,|\, H_{UV}|\psi_n\rangle = E_n|\psi_n\rangle \,,\, \rho_{c_{p-1}}|\psi_n\rangle = 0 \;\forall c_{p-1}\} \\ \mathcal{I}_k(c_{p-1}) &= \{\, |\psi_n\rangle \,|\, H_{UV}|\psi_n\rangle = E_n|\psi_n\rangle \,,\, \rho_{c_{p-1}}|\psi_n\rangle = k_{c_{p-1}}|\psi_n\rangle \,\}, \end{aligned} \tag{10.16}$$

with $k_{c_{p-1}}$ integers. The term ① in H_{UV} implies that states in $\mathcal{I}_0$ and states in $\mathcal{I}_k(c_{p-1})$ are separated by an energy difference of order $\Delta E \sim U(k_{c_{p-1}})^2$. We interpret this energy difference as the energy associate to the gapped topological charges generated by applying $L^+_{c_p}$. In the limit were U is very large we can work at energy scales lower than U and construct an effective theory with no topological charges. The low energy Hilbert space is constructed using $\rho_{c_{p-1}}$ as

$$\mathcal{V}_{IR} = \{|\psi\rangle \,|\, \rho_{c_{p-1}}|\psi\rangle = 0\,\} \tag{10.17}$$

and we can recognize that the $\rho_{c_{p-1}}$ act as the projector we defined in Eq(9.46)[1]

In the low energy theory the operator $L^{+}_{c_p}$ is not allowed since it excites topological charges with energy of order U but as argued before $W^\dagger[C_p]$ excites no charges and thus it is potentially allowed in the IR. We need to estimate energy generated by applying $W^\dagger[C_p]$ on state to understand if can use it in the IR description. In the case $K \ll 1$ we may neglect term ③ in H_{UV}. We have that $W^\dagger[C_p]$ commutes with H_{UV} and thus the energy associated to $W^\dagger[C_p]$ is zero and all $W^\dagger[C_p]$ are allowed in the IR. When instead $K \sim 1$ we have that $W^\dagger[C_p]$ receives an energy contribution from ③. In particular note that if a certain c_p does not belongs to C_p then $L^z_{c_p}$ and $W^\dagger[C_p]$ commute while if c_p belongs to C_p we have the following commutation relation

$$\begin{aligned}\Big[(L^z_{c_p})^2, W^\dagger[C_p]\Big] &= L^z_{c_p}\Big[L^z_{c_p}, W^\dagger[C_p]\Big] + \Big[L^z_{c_p}, W^\dagger[C_p]\Big]L^z_{c_p} \\ &\overset{(a)}{=} (-1)^{O_{c_p}[C_p]}\Big[L^z_{c_p} W^\dagger[C_p] + W^\dagger[C_p]L^z_{c_p}\Big] \\ &\overset{(b)}{=} (-1)^{O_{c_p}[C_p]}\Big[\Big((W^\dagger[C_p]L^z_{c_p} + (-1)^{O_{c_p}[C_p]}W^\dagger[C_p]\Big) + W^\dagger[C_p]L^z_{c_p}\Big] \\ &= W^\dagger[C_p] + (-1)^{O_{c_p}[C_p]}2W^\dagger[C_p]L^z_{c_p},\end{aligned} \tag{10.18}$$

where in (a) we defined O_{c_p}, which is a quantity measuring the orientation of c_p, namely

$$O_{c_p}[C_p] = \begin{cases} 0 & \text{if } c_p \text{ has positive orientation in } C_p \\ 1 & \text{if } c_p \text{ has negative orientation in } C_p \end{cases}, \tag{10.19}$$

and in (b) we used the commutation relation to move $W^\dagger[C_p]$ to the left. Now we take Eq(10.18) and sum over all $c_p \in C_p$. We expect the oscillating term, proportional to $(-1)^{O_{c_p}}$, to a be subleading as it changes sign while we loop around C_p. For example if we take $L^z_{c_p}$ to be a constant along C_p, then we find that the oscillating term vanishes since any cycle has the same number of positively and negatively oriented p-cells.

$$\left[\sum_{c_p \in C_p}(L^z_{c_p})^2, W^\dagger[C_p]\right] \sim \sum_{c_p} W^\dagger[C_p] = |C_p| W^\dagger[C_p], \tag{10.20}$$

with $|C_p|$ being the number of p-cells forming C_p. Using Eq(10.20) we can estimate the energy variation generated by applying $W^\dagger[C_p]$ on low energy eigenstate.

$$\begin{aligned} H_{UV}\, W^\dagger[C_p]|\psi_n\rangle &= W^\dagger[C_p]H_{UV}|\psi_n\rangle + \left[\frac{K}{2}\sum_{c_p}(L^z_{c_p})^2, W^\dagger[C_p]\right]|\psi_n\rangle \\ &\sim W^\dagger[C_p]\left(E_n + \frac{K}{2}|C_p|\right)|\psi_n\rangle. \end{aligned} \tag{10.21}$$

Now it is clear that if $K \sim 1$ we have that $W^\dagger[C_p]$ produces a cycle excitation with energy $E_{C_p} \sim \frac{K}{2}|C_p|$, then in the low energy theory we only allow operators producing excitations such

[1]Notice that the $\{\rho_{c_{p-1}}\}$ are not actually projectors since $\rho^2_{c_{p-1}} \neq \rho_{c_{p-1}}$. Nonetheless, from a practical point of view, they still play the same role as the set $\{P_i\}$ defined in Eq(9.45) as they can be used to discriminate between the low and high energy states.

that $E_{C_p} \ll U$ which corresponds to having small p-cycles so that $|C_p| \ll 2U/K$. We denote the sets of all allowed IR p-cycles as $\mathcal{Z}_p$. Finally we can construct the algebra of IR allowed operators given by

$$\mathcal{A}_{IR} = \{L^z_{c_p}, W^\dagger[C_p] \,\text{with}\, C_p \in \mathcal{Z}_p\}. \tag{10.22}$$

10.2.2 Low energy theory with $J \neq 0$

If new we turn on the term ④ in H_{UV} it might seem as all our effort is lost. In fact $\rho_{c_{p-1}}$ does not commute with the Hamiltonian and thus time evolution will cause states in $\mathcal{I}_0$ and $\mathcal{I}_k(c_{p-1})$ to mix, which means that $\rho_{c_{p-1}}$ is not a good projector anymore and does not help us identify a low energy sub-Hilbert space. Fortunately there is fix to this problem. Let us start by assuming that our Hamiltonian can be written as

$$H = H_0 + H_1, \tag{10.23}$$

where H_0 describes a theory with a well defined low energy cutoff and an associated projector, while H_1 is a perturbation term with relatively small couplings. We denote the energy eigenstates of H_0 as $|\psi_n\rangle$ while we use $|\tilde{\psi_n}\rangle$ for the eigenstates of H. In [51] it was found that there exists a local unitary operator, referred to as U_{LU}, acting as follows

$$\begin{aligned} \hat{O} &\longrightarrow \hat{O}' \coloneqq U_{LU}\hat{O}U^\dagger_{LU}, \\ |\phi\rangle &\longrightarrow |\phi'\rangle \coloneqq U_{LU}|\phi\rangle. \end{aligned} \tag{10.24}$$

which preserves the locality of $\hat{O}$, and approximately maps eigenstates of H_0 into eigenstates of H. More precisely we have

$$U_{LU}|\psi_n\rangle = \sum_m c_{nm}|\tilde{\psi}_m\rangle, \qquad \tilde{E}_m - E_n < \delta, \tag{10.25}$$

where $\tilde{E}_m$ are the energies of the states included in the sum and δ is a small quantity. We understand that U_{LU} is capable of preserving the energy scale of a given eigenstate. Moreover it is straightforward using unitarity to show that

$$\langle\phi|\hat{O}|\phi\rangle = \langle\phi'|\hat{O}'|\phi'\rangle. \tag{10.26}$$

By assumption the low energy states of H_0 satisfy $\langle\phi|\mathcal{P}|\phi\rangle = 0$, but then using Eq(10.25) and Eq(10.26) we have that the low energy states of H must satisfy $\langle\phi'|\mathcal{P}'|\phi'\rangle = 0$. This means that also H admits a well defined low energy sub-Hilbert analogous the one of H_0, and by constructing the fattened projectors $\mathcal{P}' = U_{LU}\mathcal{P}U'_{LU}$ we can identify such space as

$$\mathcal{V}'_{IR} = \{\,|\psi\rangle \,|\, \langle\psi|\mathcal{P}'|\psi\rangle = 0\,\}. \tag{10.27}$$

Back to our model, we wish to study the case where $J \ll 1$ but strictly different from 0. We can identify a low energy space by dressing $\rho_{c_{p-1}}$ as $\rho'_{c_{p-1}}$ and all the states in the IR will respect $\langle\rho'_{c_{p-1}}\rangle = 0$. In the IR theory all allowed operators must commute with $\rho'_{c_{p_1}}$ and will be obtained by dressing the operators in Eq(10.22) yielding

$$\mathcal{A}'_{IR} = \{L^{z'}_{c_p}, W^{\dagger'}[C_p] \,\text{with}\, C_p \in \mathcal{Z}_p\}. \tag{10.28}$$

Now we need to construct the low energy Hamiltonian an for this purpose we chose the general form used in [15],

$$H_{IR} = Uk\sum_{c_p}(L^{z'}_{c_p})^2 - U\sum_{c_{p+1}} W'_{c_{p+1}} + U\sum_{C_p\in\mathcal{Z}_p}\epsilon_{C_p}W'[C_p], \tag{10.29}$$

with $k = \frac{K}{U}$ and ϵ_{C_p} a loop dependent coupling constant. The exact value of ϵ_{C_p} is unknown but we can take it to be proportional to $(\frac{J}{U})^{|C_p|}$. This is because the operators $L^{\pm}_{c_p}$ appearing in H_{UV} are multiplied by J. When building $W'[C_p]$ in the effective Hamiltonian we use $L^{\pm}_{c_p}$ multiple times. Each $L^{\pm}_{c_p}$ will carry a J factor and thus $W'[C_p]$ should be accompanied by coupling proportional to $J^{|C_p|}$, then dimensional analysis imposes us to divide by $U^{|C_p|}$. Let us now assume that our lattice has no missing sites, and thus no punctures. Under this assumption all non trivial p-cycles must grow as the size of the system, like we have seen in Fig[9.5(b)]. That is because the only non trivial p-cycles are relatively closed, and thus they must stretch all the way to the boundary of space. Then in the thermodynamic limit we have $|C_p| \to \infty$ for all non trivial p-cycle, which means that they will disappear from H_{IR} since $\epsilon_{C_p} \to 0$. This result is quite nice since non trivial loops are non local operators and thus we would not admit them in the Hamiltonian. In general if $|C_p| \to \infty$, then $W'[C_p] \notin \mathcal{A}'_{IR}$, $W'[C_p]$ would not be local. Then we would need to remove $W'[C_p]$ by hand. However in this case we need not require that these loops are absent since their coupling constants already take care of them. The final form form the IR is

$$H^{Th}_{IR} = Uk\sum_{c_p}(L^{z'}_{c_p})^2 - U\sum_{c_{p+1}} W'^{\dagger}_{c_{p+1}} + U\sum_{C_p\in\mathcal{B}_p}\epsilon_{C_p}W'^{\dagger}[C_p], \tag{10.30}$$

with $\mathcal{B}_p$ being the set of trivial p-cycles i.e. boundaries, with $|C_p| \ll 2U/K$. The Hamiltonian in Eq(10.30) admits a new p-form symmetry that adds a phase to $W'[C_p]$. Consider the following transformation

$$L^{+'}_{c_p} \mapsto e^{i\alpha\Gamma_p}L^{+'}_{c_p} \implies W'^{\dagger}[C_p] \mapsto e^{i\alpha\sum_{c_p\in C_p}\Gamma_p}W'^{\dagger}[C_p], \tag{10.31}$$

with $(d\Gamma)_{c_{p+1}} = 0$. Using the discrete Stoke's theorem (Eq(10.7)) $\sum_{C_p=\partial M_{p+1}}\Gamma_{c_p} = \sum_{M_{p+1}}(d\Gamma)_{c_{p+1}} = 0$ so all the trivial cycles in H^{Th}_{IR} are are not charged and thus H^{Th}_{IR} is left invariant. To construct the operator of this symmetry start by considering $V_{c_p} = \exp[i\alpha L^{z'}_{c_p}]$ and

$$\left[L^{z'}_{\tilde{c}_p}, L^{+'}_{c_p}\right] = \delta_{c_p,\tilde{c}_p}L^{+'}_{c_p} \implies V_{\tilde{c}_p}L^{+'}_{c_p}V^{\dagger}_{\tilde{c}_p} = e^{i\alpha\delta_{c_p,\tilde{c}_p}}L^{+'}_{c_p}. \tag{10.32}$$

Note that Eq(10.32) is correct only if we assume that c_p and $\tilde{c}_p$ are taken with the same orientation. In the case of opposite orientation the phase in Eq(10.32) picks up an extra minus sign[2]. Now consider a cycle in the dual lattice $\hat{\Sigma}_{d-p}$ and its dual $\Sigma_p = \star\hat{\Sigma}_{d-p}$. Define the symmetry operator $U(\Sigma_p)$ as

$$U(\Sigma_p) = \prod_{c_p\in\Sigma_p}V_{c_p} = e^{i\alpha\sum_{c_p\in\Sigma_p}L^{z'}_{c_p}} \implies U(\Sigma_p)L'^{+}_{c_p}U^{\dagger}(\Sigma_p) = e^{i\alpha\Gamma_p}L'^{+}_{c_p}, \tag{10.34}$$

[2]Explicitly, assume c_p negatively oriented and define $-c_p \coloneqq \hat{c}_p$. Note that $L^{+'}_{c_p} = L^{-'}_{\hat{c}_p}$ and thus Eq(10.32) becomes

$$\left[L^{z'}_{\tilde{c}_p}, L^{-'}_{\hat{c}_p}\right] = -\delta_{\hat{c}_p,\tilde{c}_p}L^{-'}_{\hat{c}_p} \implies V_{\tilde{c}_p}L^{-'}_{\hat{c}_p}V^{\dagger}_{\tilde{c}_p} = e^{-i\alpha\delta_{\hat{c}_p,\tilde{c}_p}}L^{-'}_{\hat{c}_p} = e^{-i\alpha\delta_{\hat{c}_p,\tilde{c}_p}}L^{+'}_{c_p}. \tag{10.33}$$

where Γ_{c_p} is now explicitly

$$\Gamma_{c_p} = \begin{cases} 0 & \text{if} \quad c_p \notin \Sigma_p = \star\hat{\Sigma}_{d-p} \\ (-1)^{O_{c_p}[\Sigma_p]} & \text{if} \quad c_p \in \Sigma_p = \star\hat{\Sigma}_{d-p} \end{cases}. \tag{10.35}$$

Eq(10.35) is a discrete realization of the Poincare dual of $\hat{\Sigma}_{d-p}$. Now we can recognize that the charge of the Wilson cycles $\sum_{c_p \in C_p} \Gamma_{c_p}$ corresponds to the number of p-cells that C_p and Σ_p have in common, which can also be understood as the intersection number between C_p and $\hat{\Sigma}_{d-p}$ as is represented in Fig[10.2]. Using this observation we can define $\star L^{z'}_{\hat{c}_{d-p}} \coloneqq L^{z'}_{\star\hat{c}_{d-p}}$ and write $U(\Sigma_p)$ in terms of dual lattice

$$U(\hat{\Sigma}_{d-p}) = e^{i\alpha \sum_{\hat{c}_{d-p} \in \hat{\Sigma}_{d-p}} \star L^{z'}_{\hat{c}_{d-p}}}. \tag{10.36}$$

We found that the IR admits a p-form $U(1)$ exact symmetry which behaves precisely as the one that we saw for the p-form Maxwell theory.

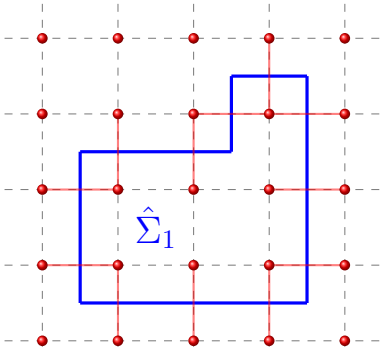
10.2.3 Going into deep IR

Since the p-form symmetry is exact we can use it in H^{Th}_{IR} to classify phases of the theory and to potentially define a new energy scale $\Lambda_{deep-IR}$ for even lower energies. Start by observing that H^{Th}_{IR} admits two phases depending on the parameters J and K. For $J = 0$ and $K \neq 0$ the state of lowest energy i.e. the vacuum satisfies $L^{z'}_{c_p}|0\rangle = 0$ for all c_p which implies

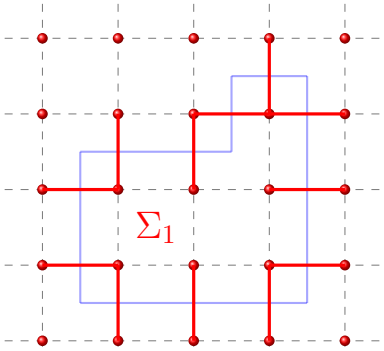
$$0 = \langle 0|\Big[L^{z'}_{c_p}, W'^{\dagger}[C_p]\Big]|0\rangle = \langle 0|(-1)^{O_{c_p}} W'^{\dagger}[C_p]|0\rangle = 0, \tag{10.37}$$

so that we have $W'^{\dagger}[C_p]|0\rangle = 0$. In this case there is no SSB. While for $K = 0$ and $J \neq 0$ the vacuum must minimize the energy contribution coming from $W'^{\dagger}[C_p]$. Recall that $W'^{\dagger}[C_p]$ is product of unitary operators and thus unitary itself, then the eigenvalues of $W'^{\dagger}[C_p]$ are all phases and the minimal eigenvalue is -1. The vacuum must then satisfy $(-W'^{\dagger}[C_p])|0\rangle = |0\rangle$, which implies that we are now in the SSB phase for the 1-form generalized symmetry, and somewhere in the middle for $J \neq 0$ and $K \neq 0$ we must have a phase transition. If now we restrict ourselves to the SSB phase we expect the theory to develop new gapped topological defects. The energy gap of these excitations can be utilized as low energy cutoff $\Lambda_{deep-IR}$ and we may try construct a $deep - IR$ theory describing states with energy smaller than $\Lambda_{deep-IR}$. In the $deep - IR$ we have no symmetry charges nor defects and the only excitations we expect to encounter are the massless Goldstone bosons associated to the SSB phase of the $U(1)^{(p)}$ symmetry. We may attempt to construct projectors able to count the number of topological defects in a given state. Consider a $p+1$ cycle C_{p+1} and decompose it into the $p+1$ cells that constitute it. Applying $W'^{\dagger}_{c_{p+1}} = W'^{\dagger}[\partial c_{p+1}]$ on each $c_{p+1} \in C_{p+1}$ corresponds to wrapping a p dimensional (potentially) charged object around a $p+1$ dimensional cycle. If C_{p+1} contains a symmetry defect then this procedure spits out the winding number $Q[C_{p+1}]$,

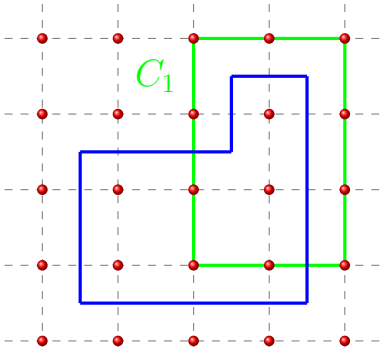
$$\prod_{c_{p+1} \in C_{p+1}} W'^{\dagger}[\partial c_{p+1}]\,|\psi\rangle = e^{2\pi i Q[C_{p+1}]}\,|\psi\rangle\,, \tag{10.38}$$



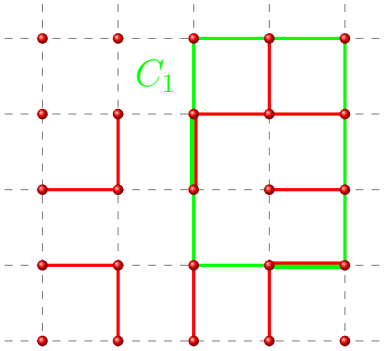
(a) A 1-cycle $\hat{\Sigma}_1$ in the dual lattice.



(b) the set of 1-cells Σ_1 represents the dual of $\hat{\Sigma}_1$.



(c) Intersections between $\hat{\Sigma}_1$ and a 1-cycle C_1 in the original lattice.



(d) Common 1-cells between Σ_1 and C_1.

Figure 10.2: 10.2(a), 10.2(b): Graphical representation of a 1-cycle $\hat{\Sigma}_1$ of the dual lattice and its dual Σ_1 in a 2 dimensional spatial lattice. 10.2(c), 10.2(d): The intersection number between $\hat{\Sigma}_1$ and C_1 is equal to the number of common 1-cells between Σ_1 and C_1.

where $Q[C_{p+1}]$ represents the number of topological defects wrapped by C_{p+1}. To keep things concrete consider 2 examples:

- The abelian Higgs model in $d = 2$ is described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2}|F_{\mu\nu}|^2 + |D_\mu\phi|^2 - \lambda(\phi^2 - v^2)^2, \tag{10.39}$$

  where ϕ is a complex scalar field, A_μ is a $U(1)$ gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative is given by

$$D_\mu\phi = (\partial_\mu - ieA_\mu)\phi. \tag{10.40}$$

  This model admits vortices solutions which can be probed by taking the 0 dimensional field ϕ and wrapping it around a 1 dimensional loop containing a vortex. in fact for the vortex solution we have that ϕ has the following behavior

$$\phi(r,\theta) = ve^{i\theta}(1 - g(r)), \tag{10.41}$$

where r is the distance form the center of the vortex, θ is the angular coordinate around the vortex, and $g(r) \sim e^{-mr}$, with m a constant. Then if we wrap ϕ around a cycle γ that encircles the vortex we would find

$$\phi \longrightarrow e^{2\pi i}\phi, \tag{10.42}$$

where in this case the winding number is one since γ encircles only one defect.

- In $d = 3$ Maxwell with charged matter, a monopole configuration can be measured measured by taking the 1 dimensional Wilson loop and wrapping it around a 2 dimensional closed surfaces containing a monopole.

We know that on the vacuum $Q[C_{p+1}]\,|0\rangle = 0$ while for a state $|\psi\rangle$ with at least one topological defect we have $Q[C_{p+1}]|\psi\rangle = N_d|\psi\rangle$ with $N_d \neq 0$. Since C_{p+1} is trivial we have $C_{p+1} = \partial O_{p+2}$ and we may also characterize defects via a defect density operator $(\star\rho)_{c_{p+2}}$ defined implicitly as

$$Q[C_{p+1} = \partial O_{p+2}] = \sum_{c_{p+2}\in O_{p+2}} (\star\rho)_{c_{p+2}} \tag{10.43}$$

We can connect $Q[C_{p+1}]$ to our original variable Θ_{c_p} by rewriting Eq(10.38)

$$\begin{aligned}\prod_{c_{p+1}\in C_{p+1}} W'^{\dagger}[\partial c_{p+1}] &= \prod_{c_{p+1}\in C_{p+1}} e^{i\sum_{c_p\in c_{p+1}}\Theta_{c_p}} \\ &= \prod_{c_{p+1}\in C_{p+1}} e^{i(d\Theta)_{c_{p+1}}} \\ &= e^{i\sum_{c_{p+1}\in C_{p+1}}(d\Theta)_{c_{p+1}}},\end{aligned} \tag{10.44}$$

and comparing to Eq(10.38) we find

$$2\pi Q[C_{p+1}] = \sum_{c_{p+1}\in C_{p+1}} (d\Theta)_{c_{p+1}} \mod 2\pi. \tag{10.45}$$

For later purposes we take advantage of the identity $x \bmod p = x - p\lfloor\frac{x}{p}\rfloor$, where $\lfloor\bullet\rfloor$ is the Gauss symbol[3], and define

$$F_{c_{p+1}} = (d\Theta)_{c_{p+1}} \mod 2\pi = (d\Theta)_{c_{p+1}} + w_{c_{p+1}}, \tag{10.47}$$

with $w_{c_{p+1}} = -2\pi\lfloor\frac{(d\Theta)_{c_{p+1}}}{2\pi}\rfloor$. Comparing Eq(10.45) and Eq(10.43) and using the discrete Stoke's theorem leads to

$$(\star\rho)_{c_{p+2}} = \frac{1}{2\pi}(dF)_{c_{p+2}}. \tag{10.48}$$

Unfortunately for us $Q[C_{p+1}]$ does not constitute a good $deep-IR$ projector as along as we work on the lattice. This is because, while working on the lattice, $Q[C_{p+1}]$ is not directly contained in $\mathcal{A}'_{IR}$. In fact $\mathcal{A}'_{IR}$ contains products of $W'^{\dagger}[c_{p+1}]$ which are proportional to $e^{2\pi i Q[C_{p+1}]}$. Since $Q[C_{p+1}]$ (and $(\star\rho)_{c_{p+2}}$) only appears as the exponential of products of $W'^{\dagger}[\partial c_{p+1}]$ we have that $Q[C_{p+1}] \sim Q[C_{p+1}] + 1$. Thus defects are invisible to lattice operators. To see the effects of defects we need the continuum limit.

[3]The Gauss symbol of a number x is the biggest integer smaller or equal to x, for example

$$\lfloor 1.5\rfloor = 1, \quad \lfloor 2\rfloor = 2, \quad \lfloor\pi\rfloor = 3. \tag{10.46}$$

10.2.4 Continuum limit

Even though we failed to obtain a $deep-IR$ projector in the lattice formulation we may still define a very low energy theory by using a phenomenological approach in the continuum limit. Firstly let us discuss how to obtain the continuum limit. We express the fundamental quantities as integral over p-cells of globally defined p-forms,

$$L'^{z}_{c_p} = \int_{c_p} \tilde{L}_p, \quad \Theta'_{c_p} = \int_{c_p} \tilde{\Theta}_p, \quad F'_{c_{p+1}} = \int_{c_{p+1}} \tilde{F}_{p+1}, \tag{10.49}$$

or more explicitly in components

$$\begin{aligned} L'^{z}_{c_p(\vec{x})}(t)_{i_1\dots i_p} &= \int_{c_p(\vec{x})} \tilde{L}(\vec{y},t)_{i_1\dots i_p}\, dy^{i_1}\wedge\cdots\wedge dy^{i_p}, \\ \Theta'_{c_p(\vec{x})}(t)_{i_1\dots i_p} &= \int_{c_p(\vec{x})} \tilde{\Theta}(\vec{y},t)_{i_1\dots i_p}\, dy^{i_1}\wedge\cdots\wedge dy^{i_p}, \\ F'_{c_{p+1}}(\vec{x})(t)_{i_1\dots i_{p+1}} &= \int_{c_{p+1}(\vec{x})} \tilde{F}(\vec{y},t)_{i_1\dots i_{p+1}}\, dy^{i_1}\wedge\cdots\wedge dy^{i_{p+1}}, \end{aligned} \tag{10.50}$$

where the spatial indices $i_1\dots i_p$ are not summed over. All operation defined in the discrete formalism carry out naturally in the continuum, in particular for the exterior derivative we have

$$(d\Theta')_{c_{p+1}} = \sum_{c_p\in\partial c_{p+1}} \tilde{\Theta}_{c_p} = \sum_{c_p\in\partial c_{p+1}} \int_{c_p} \tilde{\Theta}_p = \int_{\partial c_{p+1}} \tilde{\Theta}_p = \int_{c_{p+1}} d\tilde{\Theta}_p. \tag{10.51}$$

Then we can write

$$\tilde{F}_{c_{p+1}} = d\tilde{\Theta}_p + \tilde{\omega}_{p+1} \tag{10.52}$$

We will now show that the fields $\tilde{\Theta}_p$ and $\tilde{L}_p$ are not canonical and they need to be rescaled. To see this let us derive the commutation relations of $\tilde{\Theta}_p$ and $\tilde{L}_p$ from those of their discrete counterparts in Eq(10.8). Firstly we need to rewrite the δ_{c_p,c'_p} in the continuum, and to this end consider the following quantity

$$\int_{c'_p} \star\hat{c}_p, \tag{10.53}$$

where $\hat{c}_p$ is the Poincare dual (Eq(B.28)) of c_p and $\star$ denotes the Hodge dual (Eq(B.9)) both in the continuum and taken with respect to the space at fixed time. Since two p-cells are either totally separated or totally overlapping the result of Eq(10.53) is different from zero if and only if c_p and $c_{p'}$ are the same p-cell. By using the explicit expression for $\star\hat{c}_p$ and taking advantage of the contraction properties of the Levi-Civita tensor we find

$$\int_{c'_p} \star\hat{c}_p = \int_{c'_p}\int_{c_p} p!\,\delta^{j_1}_{[i_1}\dots\delta^{j_p}_{i_p]}\,\delta^d(\vec{x}-\vec{y})\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,dy^{j_1}\wedge\cdots\wedge dy^{j_p}, \tag{10.54}$$

and if c_p and c'_p are the same p-cell we have

$$\int_{c'_p} \star\hat{c}_p = a^p\delta^{d-p}(0) = a^p\Lambda_{UV}^{d-p} \approx a^{2p+1-d}\Lambda_{UV}, \tag{10.55}$$

where Λ_{UV} is the ultraviolet cutoff of the original theory and we used $\delta(0) = \Lambda_{UV}$ and $\Lambda_{UV} \sim 1/a^4$. We may then write

$$\left[\Theta'_{c_p}, L'^{z}_{c'_p}\right] = i\delta_{c_p,c'_p} = \frac{i}{a^{2p+1-d}\Lambda_{UV}} \int_{c'_p} \star \hat{c}_p = \begin{cases} i & \text{if} \quad c_p = c'_p \\ 0 & \text{if} \quad c_p \neq c'_p \end{cases} . \tag{10.56}$$

On the other hand the same commutation relations can be written using the continuous p-forms as

$$\left[\Theta'_{c_p}, L'^{z}_{c'_p}\right] = \int_{c'_p}\int_{c_p} \left[\tilde{\Theta}(\vec{x},t)_{i_1\dots i_p}, \tilde{L}(\vec{y},t)_{j_1\dots j_p}\right] dx^{i_1}\wedge\dots\wedge dx^{i_p}\, dy^{j_1}\wedge\dots\wedge dy^{j_p},. \tag{10.57}$$

Matching Eq(10.57) and Eq(10.56) requires the following relation

$$\left[\tilde{\Theta}(\vec{x},t)_{i_1\dots i_p}, \tilde{L}(\vec{y},t)_{j_1\dots j_p}\right] = \frac{i}{a^{2p+1-d}\Lambda_{UV}} p!\, \delta^{j_1}_{[i_1}\dots\delta^{j_p}_{i_p]}\, \delta^d(\vec{x}-\vec{y}), \tag{10.58}$$

which implies that $\tilde{\Theta}_p$ and $\tilde{L}_p$ are not canonical due to the factor preceding the deltas. The canonical variables can be obtained via a rescaling

$$\tilde{\Theta}_p = a^{\frac{d}{2}-p-1}\frac{1}{\sqrt{\Lambda_{UV}}}\Theta_p, \quad \tilde{L}_p = a^{\frac{d}{2}-p}\frac{1}{\sqrt{\Lambda_{UV}}}L_p, \quad \tilde{F}_{p+1} = a^{\frac{d}{2}-p-1}\frac{1}{\sqrt{\Lambda_{UV}}}F_{p+1}. \tag{10.59}$$

where we defined the variables Θ_p and L_p such that

$$\left[\Theta(\vec{x},t)_{i_1\dots i_p}, L(\vec{y},t)_{j_1\dots j_p}\right] = ip!\, \delta^{j_1}_{[i_1}\dots\delta^{j_p}_{i_p]}\, \delta^d(\vec{x}-\vec{y}). \tag{10.60}$$

Now we are ready to take the $deep-IR$ limit of Eq(10.29). First let us recall that ϵ_{C_p} is a small quantity. The contributions proportional are expected to be small, and we thus we choose to work at zero order in ϵ_{C_p}. Recall the Wilson cycle operator is given as

$$W'^{\dagger}_{c_{p+1}} = e^{iF'_{c_{p+1}}} = 1 + iF'_{c_{p+1}} - \frac{1}{2}(F'_{c_{p+1}})^2 + \dots. \tag{10.61}$$

We have seen that in the continuum limit $F_{c_{p+1}}$ is proportional to an actual exterior derivative and thus to a power of the momentum. If we work at very low energy we can safely ignore terms with power greater than 2 in Eq(10.61) as they are irrelevant contributions. Moreover the term linear in $F_{c_{p+1}}$ cancels as it is not self adjoint. Our candidate Hamiltonian for the $deep-IR$ regime is then (for later convenience we applied the rescale $k \to k/2$)

$$H_{deep-IR} = \frac{Uk}{2}\sum_{c_p}(L'^{z}_{c_p})^2 + \frac{U}{2}\sum_{c_{p+1}}(F'_{c_{p+1}})^2. \tag{10.62}$$

To obtain the continuum limit we will work piece by piece, starting from the term quadratic in $L'^{z}_{c_p}$. If the lattice spacing a is a small quantity we can approximate the integral in Eq(10.50) as

$$L'^{z}_{c_p(\vec{x})}(t)_{i_1\dots i_p} = \int_{c_p(\vec{x})} \tilde{L}(\vec{y},t)_{i_1\dots i_p}\, dy^{i_1}\wedge\dots\wedge dy^{i_p} \approx \tilde{L}(\vec{x},t)_{i_1\dots i_p} a^p. \tag{10.63}$$

[4] The correct expression would be $\Lambda = \frac{2\pi}{a}$ but here we ignore the 2π factor for simplicity.

Moreover the sum over p-cells can be split into a sum over lattice coordinates and one over indices, $\sum_{c_p} = \sum_{\vec{x}\in\text{Lat}} \sum_{i_1<\cdots<i_p}$, then the first term of $H_{deep-IR}$ becomes

$$\begin{aligned}
\frac{Uk}{2}\sum_{c_p}(L'^{z}_{c_p})^2 &\approx \frac{Uk}{2}\sum_{\vec{x}\in\text{Lat}}\sum_{i_1<\cdots<i_p}^{d}(\tilde{L}(\vec{x},t)_{i_1\dots i_p})^2\, a^{2p} \\
&= \frac{Uk}{2}\sum_{\vec{x}\in\text{Lat}} a^{2p}\frac{1}{p!}\sum_{i_1\dots i_p}^{d}(\tilde{L}(\vec{x},t)_{i_1\dots i_p})^2 \\
&= \frac{Uk}{2}\sum_{\vec{x}\in\text{Lat}} a^{2p}a^{d-2p}\frac{1}{\Lambda_{UV}}\frac{1}{p!}\sum_{i_1\dots i_p}^{d}(L(\vec{x},t)_{i_1\dots i_p})^2 \\
&= \frac{1}{2}\frac{kU}{\Lambda_{UV}}\sum_{\vec{x}\in\text{Lat}} a^{d}\frac{1}{p!}\sum_{i_1\dots i_p}^{d}(L(\vec{x},t)_{i_1\dots i_p})^2,
\end{aligned} \tag{10.64}$$

where now we define $1/g^2 = kU/\Lambda_{UV}$ and $|L(\vec{x},t)|^2 = \frac{1}{p!}\sum_{i_1\dots i_p}^{d}(L(\vec{x},t)_{i_1\dots i_p})^2$, and observe that if a is small scale $\sum_{\vec{x}\in\text{Lat}} a^d \approx \int d^dx$, so that our final expression reads

$$\frac{Uk}{2}\sum_{c_p}(L'^{z}_{c_p})^2 \approx \frac{1}{2g^2}\int d^dx\frac{|L(\vec{x},t)|^2}{p!}. \tag{10.65}$$

The same procedure can be applied for the second term,

$$\begin{aligned}
\frac{U}{2}\sum_{c_{p+1}}(F'_{c_{p+1}})^2 &\approx \frac{U}{2}\sum_{\vec{x}\in\text{Lat}}\sum_{i_1<\cdots<i_{p+1}}^{d}(\tilde{F}(\vec{x},t)_{i_1\dots i_{p+1}})^2 a^{2p+2} \\
&= \frac{U}{2}\sum_{\vec{x}\in\text{Lat}} a^{2p+2}\frac{1}{(p+1)!}\sum_{i_1\dots i_{p+1}}^{d}(\tilde{F}(\vec{x},t)_{i_1\dots i_{p+1}})^2 \\
&= \frac{U}{2}\sum_{\vec{x}\in\text{Lat}} a^{2p+2}a^{d-2p-2}\frac{1}{\Lambda_{UV}}\frac{1}{(p+1)!}\sum_{i_1\dots i_{p+1}}^{d}(F(\vec{x},t)_{i_1\dots i_{p+1}})^2 \\
&= \frac{1}{2k}\frac{kU}{\Lambda_{UV}}\sum_{\vec{x}\in\text{Lat}} a^{d}\frac{1}{(p+1)!}|F(\vec{x},t)|^2 \\
&\approx \frac{1}{2kg^2}\int d^dx\frac{1}{(p+1)!}|F(\vec{x},t)|^2.
\end{aligned} \tag{10.66}$$

The final expression for $H_{deep-IR}$ in the continuum is

$$H_{deep-IR} = \frac{1}{2g^2}\int d^dx\frac{|L(\vec{x},t)|^2}{p!} + \frac{1}{k}\frac{|F(\vec{x},t)|^2}{(p+1)!}. \tag{10.67}$$

Alongside $H_{deep-IR}$ we have to impose conditions for absence of charge and defects which in the continuum become a Gauss-like law

$$\partial_j L(\vec{x},t)_{ji_1\dots i_{p-1}} = 0, \tag{10.68}$$

and the Bianchi-like identity

$$\frac{1}{2\pi}dF_{p+1}(\vec{x},t) = 0 \tag{10.69}$$

Eq(10.69) implies that F_{p+1} is a closed form, but since it is not exact it may have non zero winding numbers if we wrap around a non trivial cycle. Moreover the integrals of $\tilde{F}_{p+1}$ (and by extension those of F_{p+1}) are quantized. To see this start from the discrete description and consider a $p+1$ cycle C_{p+1}

$$\sum_{c_{p+1}\in C_{p+1}} F'_{c_{p+1}} = \sum_{c_{p+1}\in C_{p+1}} w'_{c_{p+1}} = -2\pi \sum_{c_{p+1}\in C_{p+1}} \lfloor \frac{d\Theta'_{c_{p+1}}}{2\pi} \rfloor \in 2\pi\mathbb{Z}, \tag{10.70}$$

and rewrite the above quantity in the continuum

$$\sum_{c_{p+1}\in C_{p+1}} F'_{c_{p+1}} = \sum_{c_{p+1}\in C_{p+1}} \int_{c_{p+1}} \tilde{F}_{p+1} = \int_{C_{p+1}} \tilde{F}_{p+1} \in 2\pi\mathbb{Z}. \tag{10.71}$$

We then have that both $\tilde{F}_{p+1}$ and $\tilde{w}_{p+1}$ are elements of the $(p+1)$th de-Rham cohomology group of space Σ with 2π periods,

$$\tilde{F}_{p+1} \in H^{p+1}(\Sigma; 2\pi\mathbb{Z}), \quad \tilde{w}_{p+1} \in H^{p+1}(\Sigma; 2\pi\mathbb{Z}). \tag{10.72}$$

$H_{deep-IR}$ with the aforementioned constraints describes the dynamics of a p-form massless boson. To see this let us consider a simple case. Fix $d = 3$ and $p = 1$ and define the electric and magnetic fields as follows

$$\begin{aligned} L(\vec{x},t)_i &= E(\vec{x},t)_i, \\ F(\vec{x},t)_{ij} &= \epsilon_{ijk} B(\vec{x},t)_k. \end{aligned} \tag{10.73}$$

$H_{deep-IR}$ is then

$$H_{deep-IR} = \frac{1}{2g^2} \int d^d x |E(\vec{x},t)|^2 + \frac{1}{k} |B(\vec{x},t)|^2, \tag{10.74}$$

and the dynamical constraints are then equivalent to

$$\begin{aligned} \partial_i E(\vec{x},t)_i &= 0 \quad \text{no electric charge}, \\ \partial_i B(\vec{x},t)_i &= 0 \quad \text{no magnetic charge}. \end{aligned} \tag{10.75}$$

Using Eq(10.60) we can derive the commutation relations for the electric and magnetic fields. We find

$$[E(\vec{x},t)_i, B(\vec{y},t)_j] = i\epsilon_{ijk} \frac{\partial}{\partial x^k} \delta^3(\vec{x}-\vec{y}), \tag{10.76}$$

which are exactly the commutation relations of a free Maxwell theory in the temporal gauge. We may then safely conclude that Eq(10.74) describes a massless photon moving at the "speed of light" $c = \frac{1}{\sqrt{k}}$, now regarded as the Goldstone boson arising in the SSB phase of associated 1-form symmetry.

Chapter 11

Conclusions

The properties of higher form symmetries were successfully analyzed. In the case of $U(1)$ gauge p-form theory, following the calculation done in [17], we found a generalized version of Mermin-Wagner theorem which implies that continuous p-form symmetries cannot be spontaneously broken in $D < 2 + p$, while discrete p-form symmetries cannot be spontaneously broken in $D < p + 1$. The Goldstone boson theorem given in [21] was generalized to continuous p-form symmetries. This last theorem implies (in the SSB phase) the existence of a massless mode which for a p-form symmetry is a p-form gauge field. In the case of Maxwell theory in $D = 3 + 1$ we identified said Goldstone boson with the photon.

We were able to obtain a renormalization scheme for the Wilson cycle operators. This lead to the definition the renormalized Wilson loop operators, which may have an expectation value different than zero even for large loops. This renormalization allowed us to classify the symmetric and SSB phases of higher form symmetries. In the case of $SU(N)$ Yang-Mills theories [4], we found a 1-form center symmetry, associated to the impossibility of screening some types of Wilson loop operators. We saw that this 1-form symmetry can be used to classify the confined and de-confined phases [18].

We gave an extensive definition of topological order, which we now understand as a phase of matter with long range entanglement in the ground state [22]. Studying the concept of topological order in 1 higher dimension we were able to construct an alternative definition of symmetry based on a stack decomposition [23].

Using the holographic principle of topological order we obtained a formal argument stating that emergent higher form symmetries are exact at low energies. This argument was based on the idea that in a space without puncture the charged objects of higher form symmetries must grow as the size of the system. Thus in the thermodynamic limit said operators are non local and excluded form the Hamiltonian of the system.

The lattice model [15] we studied in Chap.10 proved to be an excellent concrete example of many of the properties mentioned above. In particular, we found that the emergent p-form symmetry is exact at low energies. We verified that even if the p-form symmetry is emergent, it still manages to provide constraints on the low energy dynamics of the theory. In fact in the $deep - IR$ regime we found the Goldstone boson associated to the SSB phase of said emergent p-form symmetry.

Given what we have seen, there are many directions in which a future research could take

us. Let us mention a few.

In this thesis we have overlooked an important aspect of symmetries, that is we never explicitly mentioned symmetry anomalies, such as 't Hooft and mixed 't Hooft anomalies. These anomalies are given when, starting from a global symmetry, we cannot turn the global symmetry into a gauge symmetry. That is, 't Hooft anomalies are obstructions to gauging global symmetries [52]. These types of anomalies are preserved by RG flow and thus constrain the IR dynamics of a given UV theory. Moreover, anomalies can also be used to classify theories [46]. It would be interesting to carefully study p-form symmetries, either UV or IR emergent, that present 't Hooft anomalies, and explicitly derive the constraints on the low energy dynamics.

In the case of exact emergent higher form symmetries, it would be of interest to dive deeper in the calculations and see explicitly how these emergent symmetries influence the RG flow, as for instance in Sec.10.2.3 we only considered the SSB phase of the emergent p-form symmetry. It would be interesting to reiterate the same calculation in the symmetric phase to see what happens to our theory in the $deep - IR$.

Explicitly breaking a p-form symmetry is not a trivial task. From what we have seen one would expect to have to introduce some new degrees of freedom or some non trivial finite cycles in space. It would be interesting to conduct a rigorous analysis of the necessary conditions for the explicit breaking a p-symmetry. This may lead to unique results, such as those found in [12].

From the mathematical point of view, the theory of fusion categories, on which we based our holographic principle, is well defined only when the number of simple objects is finite. As we said above, this setup is not well suited to describe continuous symmetries. It would be ideal to find a generalization of fusion categories capable of incorporating an infinite number of simple objects, and thus of describing continuous symmetries.

Appendices

Appendix A

The toric code model

Consider a 2 dimensional square lattice, with spin $\frac{1}{2}$ degrees of freedom on each link. We denote the lattice links as c_1. On each link define the spin operators Z_{c_1} and X_{c_1}, such that

$$\begin{aligned} X_{c_1} Z_{c_1'} &= -Z_{c_1'} X_{c_1} \quad \text{if} \quad c_1' = c_1, \\ X_{c_1} Z_{c_1'} &= Z_{c_1'} X_{c_1} \quad \text{if} \quad c_1' \neq c_1, \\ X X_{c_1}^2 &= Z_{c_1}^2 = \mathbb{1}. \end{aligned} \tag{A.1}$$

for each lattice site we denote as $|\uparrow\rangle_{c_1}$ the eigenstate of Z_{c_1} with eigenvalue 1. We will call $|\uparrow\rangle_{c_1}$ the "spin-up" configuration. We denote as $|\downarrow\rangle_{c_1}$ the eigenstate of Z_{c_1} with eigenvalue -1. We will call $|\downarrow\rangle_{c_1}$ the "spin-down" configuration. Note that

$$X_{c_1} |\uparrow\rangle_{c_1} = |\downarrow\rangle_{c_1}, \quad X_{c_1} |\downarrow\rangle_{c_1} = |\uparrow\rangle. \tag{A.2}$$

For each lattice site c_0 denote as $star(c_0)$ the set of links that contain c_0 in their boundary, namely $star(c_0) = \{c_1 | c_0 \in \partial c_1\}$. Then we may define the projector $A(c_0)$ as

$$A(c_0) = \prod_{c_1 \in star(c_0)} Z_{c_1}. \tag{A.3}$$

For each lattice plaquette c_2 denote as $bdy(c_2)$ the set of links belonging to the boundary of c_2, namely $bdy(c_2) = \{c_1 | c_1 \in \partial c_2\}$. The we may define the projector $B(c_2)$ as

$$B(c_2) = \prod_{c_1 \in bdy(c_2)} X_{c_1}. \tag{A.4}$$

The Hamiltonian of the toric code model is defined as

$$H_{tor} = -U_e \sum_{c_0} A(c_0) - U_m \sum_{c_2} B(c_2). \tag{A.5}$$

Consider a spin flip transformation $S_f : Z_{c_1} \mapsto -Z_{c_1}$. if we apply S_f to all links H_{tor} is left invariant. We conclude that H_{tor} has a global $\mathbb{Z}_2$ symmetry. But actually we can do better. Consider a plaquette c_2 and apply S_f only on the links in $bdy(c_2)$. Any set $bdy(c_2)$ has either zero or two common links with any other set $star(c_0)$. This implies that after the mapping all

$A(c_0)$, and thus also H_{tor}, are left invariant. We found that if apply S_f locally on the links in $bdy(c_2)$, H_{tor} is invariant. We conclude that H_{tor} **admits a $\mathbb{Z}_2$ gauge symmetry**. Then the toric code model is a $\mathbb{Z}_2$ gauge model.

Now we would like to study the ground state and the excitation of this model. Since $A(c_0)$ and $B(c_2)$ are projectors their eigenvalues are ± 1. The ground states are characterized by the condition.

$$A(c_0)\,|\Psi\rangle = B(c_2)\,|\Psi\rangle = |\Psi\rangle\,, \tag{A.6}$$

for any c_0 and c_2. Let us start by finding the possible eigenstate of the $A(c_0)$ projectors with eigenvalue 1.

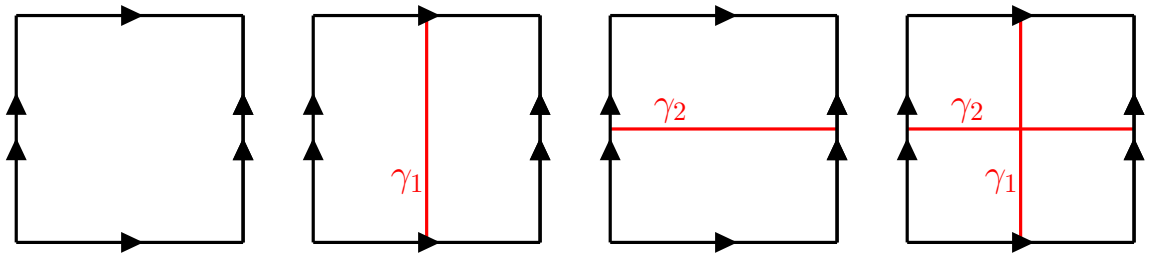
First of all note that to be an eigenstate of all $A(c_0)$ operators a state must also be an eigenstate of all Z_{c_1} operators. This implies that in an eigenstate of $A(c_0)$ a lattice link c_1 cannot be in a superposition of $|\uparrow\rangle_{c_1}$ and $|\downarrow\rangle_{c_1}$. Each lattice link must either be in the spin-up or the spin-down configuration. $A(c_0)$ acts on the spin contained in $star(c_0)$. If the number of spin-up configurations in $star(c_0)$ is even then $A(c_0)$ has eigenvalue 1, while if the number of spin-up configuration in $star(c_0)$ is odd $A(c_0)$ will have eigenvalue -1. Now consider the reference state $|AD\rangle$, which is defined as the state where all links in the lattice are in the spin-down configuration. $|AD\rangle$ is an eigenstate of all $A(c_0)$ projectors with eigenvalue +1.

Consider an open string L_1 made of links, and call a_0 and b_0 the two sites that constitute the boundary of L_1. Starting from $|AD\rangle$ let us flip the spins along L_1. After the spin flip the number spin-up configurations in $star(a_0)$ and $star(b_0)$ is 1. Thus this new state is a -1 eigenstate of $A(a_0)$ and $A(b_0)$, and cannot be a ground state. One can check graphically that the number of spin-up configurations in $starc_0$ is still even if $c_0 \neq a_0, b_0$. This is an excited state with energy $2U_e$ as one can read from the Hamiltonian. The excitations of this state are localized on a_0 and b_0.

Now consider a closed cycle C_1 made of links. Starting again form $|AD\rangle$, flip all the spins along C_1. The state obtained after the flip is denoted as $|C_1\rangle$. One can check graphically that after the spin flip the number of spin-up configurations in $star(c_0)$ is even (either 0 or 2) for any site c_0. This implies that $|C_1\rangle$ is +1 eigenstate of all $A(c_0)$. This procedure is independent on the specific cycle C_1. We conclude that the +1 eigenstates of the $A(c_0)$ operators are all the states that have closed spin-up cycles.

Now we need to find the +1 eigenstates of all $B(c_2)$ operators. from Eq(A.2) we deduce that the action of $B(c_2)$ flips all spin configurations in $bdy(c_2)$. We stress again that any set $bdy(c_2)$ has either zero or two common links with any other set $star(c_0)$. This implies that in the state $B(c_2)\,|C_1\rangle$ the number of spin-up configurations in any set $star(c_0)$ is still even. Then $B(c_2)\,|C_1\rangle$ is again a +1 eigenstate of all $A(c_0)$, and thus $B(c_2)\,|C_1\rangle$ is a new configuration of closed spin-up cycles. From this observation we deduce that the action of $B(c_2)$ on $|C_1\rangle$ can lead to one of two things. Either $B(c_2)$ creates a new trivial spin-up cycle separated from C_1, or $B(c_2)$ deforms C_1 into a new closed cycle $\tilde{C}_1$. Denote as $[C_1]_B$ the set of spin-up cycles that can be obtained starting from C_1 and acting with any operator $B(c_2)$ any number of times. We call $[C_1]_B$ the B-class of C_1. Then the +1 eigenstates of all $A(c_0)$ and all $B(c_2)$ operators, which are also the ground states of the Hamiltonian, are written as

$$|\Psi\rangle_{C_1} = \sum_{\tilde{C}_1 \in [C_1]_B} \left|\tilde{C}_1\right\rangle. \tag{A.7}$$



(a) $|\Psi\rangle_{Triv}$ (b) $|\Psi\rangle_{\gamma_1}$ (c) $|\Psi\rangle_{\gamma_2}$ (d) $|\Psi\rangle_{\gamma_1\cup\gamma_2}$

Figure A.1: The four grounds states of the toric code model on a torus. The torus is represented as a square with opposite sides identified. Each ground state is characterized by lines of spin-up configurations that wrap around the trivial cycles of the torus. The line non trivial spin-up cycles are represented as red line.

It is easy to convince ourselves that $|\Psi\rangle_{C_1}$ is an actual ground state of this models. In fact $|\Psi\rangle_{C_1}$ is super position of closed spin-up cycles configurations and thus it is a +1 eigenstate of all $A(c_0)$. Moreover we have $B(c_2)\,|\Psi\rangle_{C_1} = |\Psi\rangle_{C_1}$, since the action of $B(c_2)$ simply reshuffles the states in the sum of Eq(A.7). The number of inequivalent B-classes, and thus the ground states depends on the topology of space. For example let us consider what happens if we put this model on a sphere. In the case all cycles are trivial. The action of the $B(c_2)$ operators can map any spin-up cycle in any other spin-up cycle, thus we only have 1 ground state, corresponding to the B-class of the trivial cycle, which we denote as $[Triv]_B$.

On the other hand if put the model on a torus the number of B-classes increases. We denote the two non trivial cycles of the torus as γ_1 and γ_2. Staring from the state $|AD\rangle$ we can flip all spins along γ_1. This would lead to a +1 eigenstate of all $A(c_0)$ operators. The action of $B(c_2)$ can deform γ_1 but can never open γ_1. This means that a configuration with a spin-up cycle on γ_1 cannot be transformed into a configuration with only trivial spin-up cycles. Thus $[\gamma_1]_B \neq [Triv]_B$. This implies the existence of a new ground state $|\Psi\rangle_{\gamma_1}$. Repeating this logic for other cycles we find that on the torus there 4 inequivalent B-classes and thus 4 ground state, which are

$$|\Psi\rangle_{Triv}\,, \quad |\Psi\rangle_{\gamma_1}\,, \quad |\Psi\rangle_{\gamma_2}\,, \quad |\Psi\rangle_{\gamma_1\cup\gamma_2}\,. \tag{A.8}$$

All these ground states can be graphically represented in using their non trivial spin-up cycles.

At this point one may ask why there are no ground states associated to looping multiple times around the same non trivial cycle. That is because the action of the $B(c_2)$ operators can transform two copies of the same non trivial cycle into a trivial one. Formally denote as $|2\gamma_1\rangle$ the state with two spin-up cycles around γ_1, then one would find $|2\gamma_1\rangle \in [Triv]_B$. The same logic can be repeated for γ_2.

A.1 Anyon excitations

Now that we have the ground states of the theory we may study the types of excitations. There are three types of non trivial excitations in the toric code model.

1. The first type of excitation is associated to charge in the eigenvalues of the $A(c_0)$ operators. Consider an open string L_1 of the lattice, and denote as a_0 and b_0 the sites composing the boundary of L_1. Consider the following operator defined on L_1,

$$X_{L_1}(a_0, b_0) = \prod_{c_1 \in L_1} X_{c_1}, \tag{A.9}$$

When acting on a ground state $X_{L_1}(a_0, b_0)$ flips all the spins along L_1. This causes the eigenvalues of $A(a_0)$ and $A(b_0)$ to change from +1 to −1. Thus $X_{L_1}(a_0, b_0)$ creates an excited state with excitation on a_0 and b_0. We denote this type of excitations as e excitations. Notice that if we want only one e excitation we must apply $X_{L_1}(a_0, b_0)$ and then push one of the two excitation to infinity. This process requires the use of a non local operator and thus the e excitation are topological excitation. each e excitation carries energy U_e as one can read from the Hamiltonian. Note that $X_{L_1}(a_0, b_0)$ can also be interpreted as the operator that transports the e excitations around, in fact we have

$$X_{L_1}(b_0, c_0) X_{L_1}(a_0, b_0) = X_{L_1}(a_0, c_0) \implies X_{L_1}(b_0, c_0) = T_e(b_0 \to c_0), \tag{A.10}$$

where $T_e(a_0 \to b_0)$ is by definition the operator that moves an e excitation from site a_0 to site b_0. Now that we are equipped with $T_e(a_0 \to b_0)$ we can deduce the fusion rules for the e excitation. Denote as $|\Psi\rangle$ one of the possible ground states and consider

$$T_e(b_0 \to a_0) X_{L_1}(a_0, b_0) |\Psi\rangle = X_{C_1}(a_0, a_0) |\Psi\rangle, \tag{A.11}$$

where $X_{C_1}(a_0, a_0)$ is an operator that flips the spins along a cycle C_1. As we observed before, flipping all the spins around a cycle does not charge the eigenvalues of any $A(c_0)$ or $B(c_2)$, and thus the state $X_{C_1}(a_0, a_0) |\Psi\rangle$ is still a ground state. This last observation implies that when two e excitations are transported to the same site they fuse into no excitation. Thus we obtain the fusion rule

$$e \otimes e = \mathbb{1}, \tag{A.12}$$

where $\mathbb{1}$ is the trivial excitation, representing the absence of excitations. Moreover using $T_e(a_0 \to b_0)$ one can prove that e is a boson.

2. The second type of excitation is associated to charge in the eigenvalues of the $B(c_2)$ operators. Consider an open string P_1 in the dual lattice. The boundary P_1 is composed by two plaquettes which we denote as a_2 and b_2. Consider the following operator defined on P_1,

$$Z_{P_1}(a_2, b_2) = \prod_{c_1 \in P_1} Z_{c_1}. \tag{A.13}$$

The action of $Z_{P_1}(a_2, b_2) \prod_{c_1 \in P_1} Z_{c_1}$ causes the eigenvalues of $B(a_2)$ and $B(b_2)$ to change from +1 to −1. Thus $Z_{P_1}(a_2, b_2)$ creates an excited state with excitation on a_2 and b_2. We call this type of excitations the m excitation. Just like the e excitations, also the m excitations are topological excitations. Each m excitation carries energy U_m as one can

read from the Hamiltonian. Also for the m excitations the operator $Z_{P_1}(a_2, b_2)$ can be interpreted as a transporter operator. Namely

$$Z_{P_1}(a_2, b_2) = T_m(a_2 \to b_2), \tag{A.14}$$

where $T_m(a_2 \to b_2)$ is by definition the operator that moves m from plaquette a_2 to plaquette b_2. Using $T_m(a_2 \to b_2)$ we can find the fusion rules of the m excitations and we would find

$$m \otimes m = \mathbb{1}. \tag{A.15}$$

One can also prove that m is a boson. It is interesting to note that even though m and e are bosons they have a non trivial mutual statistic. That is, using $T_m(a_2 \to b_2)$ and $T_e(a_2 \to b_2)$ one can prove that exchanging e and m produces a minus sign.

3. The last type of excitation come from making a bound state of e and m. We call this excitations f and by definition we have

$$e \otimes m = f. \tag{A.16}$$

Also f is topological excitation, and because of the mutual statistic between e and m we find that f is fermion. Each f excitation carries the energy $U_e + U_m$.

To summarize the toric code model has 4 types of excitations, namely $\mathbb{1}$, e, m and f. Using the transporter operators one can deduce all the fusion rules for said excitations and find

$$\begin{aligned} &e \otimes e = \mathbb{1}, && m \otimes e = f, && f \otimes e = m, \\ &e \otimes m = f, && m \otimes m = \mathbb{1}, && f \otimes m = e, \\ &e \otimes f = m, && m \otimes f = e, && f \otimes f = \mathbb{1}. \end{aligned} \tag{A.17}$$

These fusion rule can be associated to fusion category, and moreover using the transporter operators one can consider complex fusion processes and evaluate the F-symbols, which in this case are all 1. Fixing the fusion rules and the F-symbol fixes the fusion category, and this particular category describes the excitations of the toric code model. **Since the toric code model is $\mathbb{Z}_2$ gauge theory we denote its associated category as $\mathcal{G}au_{\mathbb{Z}_2}$.**

Appendix B

Review of differential forms

Consider a D dimensional orientable smooth manifold X. Define T_pX and T_p^*X to be respectively the tangent and cotangent space of X at point p. For simplicity we will take canonical bases both in T_pX and T_p^*X, namely for a vector $v \in T_pX$ we have $v = v^\mu \partial_\mu$ and for a one form $W \in T_p^*X$ we have $W = W_\mu dx^\mu$ in such a way that $dx^\mu(\partial_\nu) = \delta^\mu_\nu$.
Then we can define a tensor of type (n, m) as a map T,

$$T : \underbrace{T_p^*X \otimes \cdots \otimes T_p^*X}_{\text{n times}} \otimes \underbrace{T_pX \otimes \cdots \otimes T_pX}_{\text{m times}} \mapsto \mathbb{R}. \tag{B.1}$$

Writing T in terms of the canonical base we have,

$$T = T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_n} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_m}, \tag{B.2}$$

with the action of T given by

$$T(W_1 \dots W_n; v_1 \dots v_m) = T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} W_{1\mu_1} \dots W_{n\mu_n} v_1^{\nu_1} \dots v_m^{\nu_m}, \tag{B.3}$$

where $W_1, \dots, W_n$ are one forms and $v_1, \dots, v_m$ are vectors.

Now we proceed by smoothly assigning to each point $p \in X$ a tensor T, thus constructing a tensor field. We denote by $\mathcal{T}^n_m$ the set of all tensor fields of type (n, m). On a open patch with coordinates x^μ a k-form A on X is defined as a totally screw symmetric tensor field of type $(0, k)$ and can be explicitly written as

$$A = \frac{1}{k!} A_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}, \tag{B.4}$$

where $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$ is the antisymmetric part of the canonical base of $\mathcal{T}^0_k$, $dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k}$, and the $\frac{1}{k!}$ factor is a consequence of the antisymmetrization. Denote $\Omega^k(X)$ the set of all k-forms defined on X. The wedge product between forms is defined as a bilinear map $\wedge : \Omega^k(x) \times \Omega^p(X) \mapsto \Omega^{k+p}(X)$ that acts as follows,

$$\wedge : (A, B) \mapsto A \wedge B = \frac{1}{k!p!} A_{\mu_1 \dots \mu_k} B_{\nu_1 \dots \nu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p}, \tag{B.5}$$

with the property

$$A \wedge B = (-1)^{kp} B \wedge A. \tag{B.6}$$

The exterior derivative of a k-form is defined as a map $d : \Omega^k(X) \mapsto \Omega^{k+1}(X)$ acting as follows,

$$d : A \mapsto dA = \frac{1}{k!} \partial_\alpha A_{\mu_1 \dots \mu_k} dx^\alpha \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}. \tag{B.7}$$

and by antisymmetry we have that $d^2 = 0$. We say that $A \in \Omega^k(X)$ is closed on X if $dA = 0$, while it is exact on X if $A = dB$ where $B \in \Omega^{k-1}(X)$. We denote the set of all exact k-form as $d\Omega^{k-1}(X)$. Moreover the Leibniz rule for the exterior derivative is

$$dA \wedge B = dA \wedge B + (-1)^k A \wedge dB, \tag{B.8}$$

with $A \in \Omega^k(X)$.

If X admits a well defined metric tensor g the Hodge dual of a k-form is defined as a map $\star : \Omega^k(X) \mapsto \Omega^{D-k}(X)$, acting as follows,

$$\star : A \mapsto \star A = \frac{\sqrt{|g|}}{k!(D-k)!} A_{\mu_1 \dots \mu_k} \epsilon^{\mu_1 \dots \mu_k}_{\nu_{k+1} \dots \nu_D} dx^{\nu_{k+1}} \wedge \dots \wedge dx^{\mu_D}, \tag{B.9}$$

where $|g| = |\det g|$ is the determinant of the metric tensor of X, and $\epsilon_{\mu_1 \dots \mu_D}$ is the Levi-Civita symbol. The Hodge dual acting on a k-forms satisfies the property,

$$\star\star = (-1)^{k(D-k)+q}, \tag{B.10}$$

where q is factor which is 0 if X is Euclidian ($\det g = 1$) and 1 if X is Lorentzian ($\det g = -1$). The Hodge dual can be used to define the volume element on X as

$$vol_X = \star 1 = \frac{\sqrt{|g|}}{D!} \epsilon_{\nu_1 \dots \nu_D} dx^{\nu_1} \wedge \dots \wedge dx^{\mu_D} = \sqrt{|g|} dx^D, \tag{B.11}$$

and we may also construct the induced volume element on a d dimensional hypersurface M in X,

$$\begin{aligned} vol_M &= \frac{\sqrt{|\lambda_M|}}{d!} n_1^{\mu_1} \dots n_d^{\mu_{D-d}} \epsilon_{\mu_1 \dots \mu_{D-d} \nu_1 \dots \nu_d} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_d} \\ &= \sqrt{|\lambda_M|} dx^1 \wedge \dots \wedge dx^d = \sqrt{|\lambda_M|} d^d x \end{aligned} \tag{B.12}$$

where $\lambda_M = g|_M$ and $n_1^{\mu_1} \dots n_d^{\mu_{D-d}}$ is a positively oriented orthonormal base for $(T_p M)^\perp$, the normal subspace of M. Then we write the full volume of M as

$$Vol_M = \int_M vol_M. \tag{B.13}$$

In the following we will be interested in SSB (spontaneous symmetry breaking), which is a phenomenon concerning the asymptotic behavior of fields in the IR and in particular their boundary conditions. For this reason we will work on spacetimes X such that $\partial X \neq \varnothing$ and we will usually distinguish between two types of boundary conditions. Given a k-form A defined

on X, we say that A has Dirichlet boundary conditions if the components of A parallel to ∂X vanish at ∂X,

$$A|_{\partial X} = 0. \tag{B.14}$$

We denote by $\Omega^k_D(X) \subset \Omega^k(X)$ the subset of k-forms satisfying Eq(B.14). On the other hand we will say that A has Neumann boundary conditions if the components of A orthogonal to ∂X vanish at ∂X,

$$\star A|_{\partial X} = 0, \tag{B.15}$$

and the corresponding subset is denoted by $\Omega^k_N(X)$. Notice that if we consider A to be a k-form gauge field then we require the respective gauge transformation not to change the boundary conditions. Thus is if $A \in \Omega^k_D(X)$ then the gauge transformation can still act on the boundary ∂X as long as it only changes the components of A which are orthogonal to ∂X. The opposite holds of $A \in \Omega^k_N(X)$.

Once we have defined differential forms it is natural to construct quantities by integrating forms over submanifolds of X. In particular if $A \in \Omega^k(X)$ we can integrate A over a k dimensional submanifold M, obtaining

$$\int_M A. \tag{B.16}$$

Using the Hodge dual we may define a scalar product $\langle \cdot, \cdot \rangle : \Omega^k(X) \times \Omega^k(X) \mapsto \mathbb{R}$ given by

$$\langle A, B \rangle = \langle B, A \rangle = \int_X A \wedge \star B, \tag{B.17}$$

and via direct calculation one can check that

$$\langle \star A, B \rangle = (-1)^{k(D-k+q} \langle A, \star B \rangle. \tag{B.18}$$

Alongside the scalar product we also define the adjoint of the exterior derivative as $\langle A, dB \rangle = \langle d^\dagger A, B \rangle$. Notice that if $\partial X \neq \varnothing$ then $d^\dagger$ cannot be well defined as the adjoint of d on any $A, B \in \Omega^k(X)$ since upon applying the Leibniz rule we find that the boundary term might not vanish. For this reason we take $d^\dagger$ to be the adjoint of d only for scalar products involving k-forms that have either Dirichlet or Neumann boundary conditions. When acting on k-forms $d^\dagger$ admits the representation $d^\dagger = (-1)^{Dk+D+1+q} \star d \star$, where q is the same factor defined in Eq(B.10). Analogously to the exterior derivative we say that a k-form A is coclosed if $d^\dagger A = 0$, while it is coextact if $A = d^\dagger B$ with $B \in \Omega^{k+1}(X)$. We denote the set of all coextact forms as $d^\dagger \Omega^{k+1}(X)$.

Finally the Hodge laplacian is given by $\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d$, and a straightforward computation shows that

$$[\Delta, d] = [\Delta, d^\dagger] = 0. \tag{B.19}$$

B.1 Homology, cohomology and Poincare duality

Before moving on we will briefly set up the concepts of homology and cohomology. We will be very pragmatical and work directly using the language of manifolds and differential forms.

Consider the usual D dimensional space X and denote by $B_n(X)$ the boundary group, i.e. the group of n dimensional submanifolds which are boundaries of some $n+1$ dimensional submanifold.

Denote by $Z_n(X)$ the cycle groups, i.e. the group of n dimensional closed submanifolds of X, and observe that $B_n(X) \subseteq Z_n(X)$. The n^{th} homology group is defined as

$$H_n(X) \coloneqq Z_n(X)/B_n(X), \tag{B.20}$$

that is, elements of $H_n(X)$ are distinguished by the fact that they are cycles but not boundaries.

For the case of (De Rham) cohomology, denote by $B^n(X) \subset \Omega^n(X)$ the coboundary group, i.e. the group of exact n-forms defined on X. Denote by $Z^n(X) \subset \Omega^n(X)$ the cocycle group, i.e. the group of closed n-forms defined on X, and observe that $B^n(X) \subseteq Z^n(X)$. The n^{th} (De Rham) cohomology group is defined as

$$H^n(X) \coloneqq Z^n(X)/B^n(X), \tag{B.21}$$

that is, elements of $H^n(X)$ are distinguished by the fact that they are closed forms that are not exact. The crucial result that we will often use is that in the case $\partial X = 0$, there exists an isomorphism

$$H_n(X) \cong H(X)^{D-n} \tag{B.22}$$

which connects a n dimensional closed submanifold η with its $D-n$ Poincare dual $\hat{\eta}$ so that for any closed $(D-n)$-form w we have

$$\int_X w \wedge \hat{\eta} = \int_{\eta \subset X} w. \tag{B.23}$$

Note that in Eq(B.23) we considered X to be the reference space for the Poincare dual of η, but in general the reference space could be submanifold of X. Then we will say that $\hat{\sigma}$ is the Poincare dual of σ with respect to Σ if

$$\int_\Sigma w \wedge \hat{\sigma} = \int_\sigma w, \tag{B.24}$$

where $\Sigma \subset X$ is d dimensional manifold and $\hat{\sigma} \in \Omega^{d-n}(X)$. If instead $\partial X \neq \varnothing$ then the isomorphism takes on a different form,

$$H^n(X, \partial X) \cong H_{D-n}(X), \tag{B.25}$$

where $H^n(X, \partial X)$ is the relative cohomology group, which is the group of n-form that vanish at the boundary, and

$$H_n(X, \partial X) \cong H^{D-n}(X), \tag{B.26}$$

where $H_n(X, \partial X)$ is the relative homology group, that is the group of submanifolds whose boundaries are on ∂X. We may generalize the notion of Poincare duality to open manifolds and open forms by including p-currents in our description. p-currents are the generalization of distributions. A p-current on X is a linear functional defined on the space of smooth $(D-p)$-forms of compact support. In particular, given a submanifold $\eta \subset X$ its Poincare dual with respect to X, can be regarded as p-current acting on a $D-p$ form ω as

$$\hat{\eta}(\omega) = \int_\eta \omega = \int_X \omega \wedge \hat{\eta} \tag{B.27}$$

where ω is a general $(D-p)$-form and $\hat{\eta}$ has support in a tubular neighborhood of η and we may chose to represent the components of $\hat{\eta}$ as

$$\hat{\eta}_{\mu_1 \dots \mu_{D-p}}(x) = \frac{1}{p!} \int_\eta \epsilon_{\nu_1 \dots \nu_p \mu_1 \dots \mu_{D-p}} \delta^{(D)}(x-y) dy^{\nu_1} \wedge \dots \wedge dy^{\nu_p}. \tag{B.28}$$

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