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THE FUNDAMENTAL COSTACK

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1 Acknowledgements

a chi ha le mani sempre aperte
e un solo volto
e nessun nome

a chi gioca sulla spiaggia
o riposa, tra le braccia
del sole, accarezzato
dalla risacca

a chi conserva il proprio cuore
in casse d'acero
e pur di darlo via
a chiunque l'abbia amato
continua a sfregiarlo
con corde di metallo

a chi misura gli interstizi
tra le pietre della piazza
per incastrarle
negli spazi tra le costole,
si adagia per terra
e si commuove un po'
se amanti immaginari
esplodono tra la gente
in fiumi di sangue e fango
che li sfiorano appena

a chi ha lasciato le ossa
nelle arterie del mondo
e le ritrova ogni notte
nel prossimo corpo di plastica
per offrirle a sorrisi ignoti
e mai piú rivisti

a chi urla, e scuote la terra
e ride della morte e di sé,
ché devi alleggerire il mondo
per poterlo stringere tra le mani

a chi mi ha donato la carta, e la parola:
prometto che l'inchiostro
lo troveró da me

2 Introduction

A classical result in algebraic topology, known as the *Seifert-Van Kampen theorem*, provides a way to recover $\pi_1(X, x)$, the fundamental group of a topological space X based at the point $x \in X$, knowing the fundamental groups of open subsets that cover X and satisfying some additional properties. To get rid of the dependence on the base point, one considers the fundamental groupoid $\Pi_1(X)$: it is possible to prove a version of the Seifert-Van Kampen theorem for $\Pi_1(X)$ (see [Bro06]), which might be restated in terms of *2-colimits* by saying that for any open cover $\{U_i\}_{i\in I}$ of X closed by taking finite intersections, we have that the natural functor

$$2\varinjlim_{i\in I} \Pi_1(U_i) \to \Pi_1(X)$$

is an equivalence of categories (see [Pir15]). This condition is easily seen to be dual to the one in the definition of a stack: hence we say that $\Pi_1 : Op(X) \to \mathbf{Cat}$ is a *costack* of groupoids on X.

In general, if $\mathbf{C}$ is a 2-category which admits small 2-colimits, a costack is a 2-functor

$$C : \mathbf{Op}(X) \to \mathbf{C}$$

such that, for any open subset U of X the 1-cell

$$2\varinjlim_{i\in I} C(U_i) \to C(U)$$

is an equivalence whenever $\{U_i\}_{i\in I}$ is an open cover of U.

The main goal of our work is to study the properties of Π_1 as a costack. First, after recalling some preliminaries about the general theory of 2-categories, following [KS05] we give the definition of a *1-final functor* and prove a theorem that relates this notion to 2-limits (as one classically relates final functors to limits), and this will imply immediately that, when X is locally 1-connected, the *costalk* (the dual construction for costacks corresponding to the stalk of a stack) of Π_1 at each point $x \in X$ (i. e. $2\varprojlim_{x\in U} \Pi_1(U)$) is equivalent to the terminal category $\mathbf{Pt}$; then, inspired by the work of Bredon on cosheaves in [Bre12], we give the definition of a *locally trivial costack* and prove that any locally trivial costack of Cauchy complete categories (and in particular of groupoids) on a locally 1-connected space must be equivalent to Π_1.

3 Notation and conventions

Before going into the definitions, let us fix some notations about 2-categories.

Let **C** be a 2-category. As usual, we will denote by $Ob(\mathbf{C})$ the class of objects of **C**, and, for any $x, y \in Ob(\mathbf{C})$, by $\mathbf{Hom_C}(x, y)$ the category with objects the 1-cells $f : x \to y$ and morphisms, for any two parallel 1-cells $x \overset{f}{\underset{g}{\rightrightarrows}} y$ transformations $\alpha : f \Rightarrow g$, which will usually be represented by diagrams

$$x \underset{g}{\overset{f}{\Downarrow \alpha}} y \ .$$

For horizontally composable 2-cells α and β, we indicate by $\beta \circ \alpha$ their horizontal composition, and for vertically composable 2-cells γ and δ, we indicate by $\delta \bullet \gamma$: this can be visualized in the diagrams

$$x \underset{g}{\overset{f}{\Downarrow \alpha}} y \underset{i}{\overset{h}{\Downarrow \beta}} z \ = \ x \underset{i \circ g}{\overset{h \circ f}{\Downarrow \beta \circ \alpha}} z$$

$$x \underset{h}{\overset{f}{\underset{\Downarrow \delta}{\overset{\Downarrow \gamma}{\xrightarrow{g}}}}} y \ = \ x \underset{h}{\overset{f}{\Downarrow \delta \bullet \gamma}} z \ .$$

Notice that we use the same symbol for composition of 1-cells and horizontal composition of 2-cells: this is a small abuse of notation, but it is motivated by the fact that composition of 1-cells can be seen as a particular care of composition of 2-cells, like it is shown below

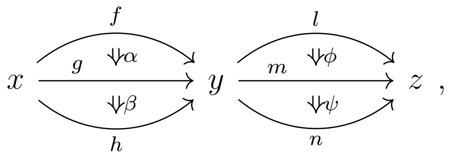
$$x \xrightarrow{f} y \xrightarrow{g} z \ = \ x \underset{f}{\overset{f}{\Downarrow id_f}} y \underset{g}{\overset{g}{\Downarrow id_g}} z \ = \ x \underset{g \circ f}{\overset{g \circ f}{\Downarrow id_{g \circ f}}} z \ .$$

For the same reason, in cases like

$$x' \xrightarrow{h} x \underset{g}{\overset{f}{\Downarrow \alpha}} y \xrightarrow{i} y'$$

we will write $\alpha \circ i$ and $h \circ \alpha$ instead of $\alpha \circ id_i$ and $id_h \circ \alpha$.

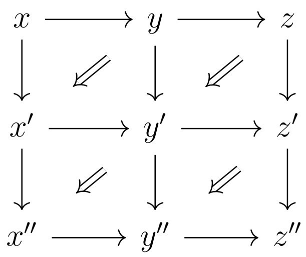
In a situation like



there are two ways of perfoming the composition of these 2-cells, namely $(\psi \circ \beta) \bullet (\phi \circ \alpha)$ and $(\beta \bullet \alpha) \circ (\psi \bullet \phi)$. Since by the definition of a 2-category the composition $c_{xyz} : \mathbf{Hom_C}(x, y) \times \mathbf{Hom_C}(y, z) \to \mathbf{Hom_C}(x, z)$ is a functor, we get that

$$\begin{aligned}(\psi \circ \beta) \bullet (\phi \circ \alpha) &= c_{xyz}(\beta, \psi) \bullet c_{xyz}(\alpha, \phi) \\ &= c_{xyz}((\beta, \psi) \bullet (\alpha, \phi)) \\ &= c_{xyz}(\beta \bullet \alpha, \psi \bullet \phi) \\ &= (\beta \bullet \alpha) \circ (\psi \bullet \phi),\end{aligned}$$

so the two ways turn out to be equal. This formula is called the "interchange law". In more general situations like, for example,



one can prove again (for a reference, see [Pow90]) that the result of the composition of these 2-cells is independent of the choice of the order.

We now mention a list of basic definitions concerning 2-categories.

In a 2-category **C**, two arrows $x \underset{g}{\overset{f}{\rightleftarrows}} y$ constitute an *adjoint pair* when there exist 2-cells

$$\eta : id_y \Rightarrow f \circ g, \epsilon : g \circ f \Rightarrow id_x$$

such that the following equalities between 2-cells hold:

$$(id_f \circ \epsilon) \bullet (\eta \circ id_f) = id_f, (\epsilon \circ id_g) \bullet (id_g \circ \eta) = id_g.$$

When η and ϵ are invertible, the adjunction is called an *equivalence*, and x and y are said to be *equivalent.* For example, in **Cat**, the 2-category of small categories, it is well known that two categories C and D are equivalent if and only if there exists a functor $F : C \to D$ which is fully faithful and essentially surjective. Keeping in mind this classical example, it is natural to think that, working with 2-categories, one might be interested to characterize objects up to equivalence, and not up to isomorphism.

A 2-category $\mathbf{C}$ is said to be *small* if $Ob(\mathbf{C})$ is a set.

Given a 2-category $\mathbf{C}$, we say that $\mathbf{C}'$ is a 2-*subcategory* of $\mathbf{C}$ if $Ob(\mathbf{C}') \subseteq Ob(\mathbf{C})$ and, for any two $x, y \in Ob(\mathbf{C}'$, $\mathbf{Hom}_{\mathbf{C}'}(x, y)$ is a subcategory of $\mathbf{Hom}_{\mathbf{C}}(x, y)$, and $\mathbf{C}'$ is *full* if $\mathbf{Hom}_{\mathbf{C}'}(x, y) = \mathbf{Hom}_{\mathbf{C}}(x, y)$.

For a 2-category, there are two possible ways of constructing an "opposite", namely reversing 1-cells or 2-cells. If $\mathbf{C}$ is a 2-category, we will indicate by $\mathbf{C}^{op}$ and by $\mathbf{C}^{co}$ respectively the 2-category obtained by reversing 1-cells and the one obtained by reversing 2-cells: formally, one has that $Ob(\mathbf{C}^{op}) = Ob(\mathbf{C}^{co}) = \mathbf{C}$ and, for any two $x, y \in Ob(\mathbf{C})$

$$\mathbf{Hom}_{\mathbf{C}^{op}}(x, y) = \mathbf{Hom}_{\mathbf{C}}(y, x),\ \mathbf{Hom}_{\mathbf{C}^{co}}(x, y) = \mathbf{Hom}_{\mathbf{C}}(x, y)^{op}.$$

$\mathbf{C}^{op}$ and $\mathbf{C}^{co}$ are called respectively the *opposite* and the *co-opposite* 2-category of $\mathbf{C}$.

4 Cauchy complete categories

We recall in this section the definition of a Cauchy complete category (also called Karoubi complete or idempotent complete), since this notion will play a central role in the proof of the most relevant result of our work. We will present a of classical results about such categories, and then prove an important proposition that will give an equivalent condition for 2-functors with values in the 2-category of Cauchy complete categories to be equivalent.

4.1 Definition and Cauchy completion

Definition 1. *A category C is Cauchy complete if every idempotent morphism in C is split, that is for any morphism $e : x \to x$ in C such that $e \circ e = e$, there exist and morphisms* $y \underset{i}{\overset{r}{\leftrightarrows}} x$ *such that $r \circ i = id_y$ and $e = i \circ r$.*

Example 1. *1. Any (co)complete category is Cauchy complete, because one can prove that this definition is equivalent to requiring that C has all absolute (co)limits, meaning that C admits all the (co)limits that are preserved by any functor;*

2. any groupoid is Cauchy complete, since for an invertible endomorphism $e : x \to x$, $e \circ e = e$ implies $e = id_x$.

We define **CauCat** to be the full 2-subcategory of **Cat** whose objects are Cauchy complete small categories. There is a canonical way of assigning a Cauchy complete category to a category C, called its *Cauchy completion*, and denoted by $\widehat{C}$. The following proposition is taken by [BRD94, Proposition 6.5.9].

Proposition 1. *Every small category C can be embedded as a full subcategory in a Cauchy complete small category $\widehat{C}$. Moreover,*

(1) given a functor $F : C \to D$ where D is Cauchy complete, F extends uniquely (up to isomorphism) as a functor $\widehat{F} : \widehat{C} \to D$,

(2) given another functor $G : C \to D$, its extension $\widehat{G} : \widehat{C} \to D$, and a natural transformation $\alpha : F \Rightarrow G$, α extends uniquely as a transformation $\widehat{\alpha} : \widehat{F} \Rightarrow \widehat{G}$,

(3) the inclusion $C \hookrightarrow \widehat{C}$ is an equivalence if and only if C is Cauchy complete.

Proof. We only recall how $\widehat{C}$ is defined. Consider the Yoneda embedding $h_C : C \to \mathbf{Hom_{Cat}}(C^{op}, \mathbf{Set})$. For any idempotent $e : x \to x$ in C, $h(e)$ is idempotent, and in this case it can be shown that

$$eq(h(e), id_{h(x)}) \cong coeq(h(e), id_{h(x)}) \cong x_e$$

and that $h(e) = i_e \circ r_e$, where $x_e \underset{i_e}{\overset{r_e}{\leftrightarrows}} x$ are the natural morphisms of equalizers and coequalizers (a proof of these facts can be found in [BRD94, Proposition 6.5.4]). Set $\widehat{C}$ the full subcategory of $\mathbf{Hom_{Cat}}(C^{op}, \mathbf{Set})$ whose objects are the objects of $h(C)$ union all the x_e defined has before, hence the category obtained by formally adding to C objects that make any idempotent split, and clearly the Yoneda embedding factorizes through a full embedding $C \hookrightarrow \widehat{C}$. □

We see that, in particular, Proposition 1 states that we get a 2-functor

$$\widehat{(-)} : \mathbf{Cat} \to \mathbf{CauCat},$$

left 2-adjoint to the inclusion 2-functor $\mathbf{CauCat} \hookrightarrow \mathbf{Cat}$.

For any small category C, we set

$$\mathbf{Hom_{Cat}}(C, \mathbf{Set}) = \mathbf{Rep_{Set}}(C)$$

and we call it the the category of ***Set**-valued representations of* C. Notice that there is an abuse of notation in $\mathbf{Hom_{Cat}}(C, \mathbf{Set})$ (which will appear again later), since **Set** is not a small category, but with this we mean obviously the category of functors $F : C \to \mathbf{Set}$ and natural transformations between them.

We recall now a classical result (see for example [BRD94, Theorem 6.5.11]) about Cauchy completions.

Proposition 2. *Given two small categories C, D, the following conditions are equivalent:*

(1) the categories $\boldsymbol{Rep_{Set}}(C^{op})$ and $\boldsymbol{Rep_{Set}}(D^{op})$ are equivalent;

(2) $\widehat{C}$ and $\widehat{D}$ are equivalent.

In particular, given a small category C, the categories $\boldsymbol{Rep_{Set}}(C^{op})$ and $\boldsymbol{Rep_{Set}}(\widehat{C}^{op})$ are equivalent.

Proof. Since the full proof uses many results about Cauchy completions that we do not treat here, we only recall that, if

$$\varphi : \mathbf{Rep}_{\mathbf{Set}}(C^{op}) \to \mathbf{Rep}_{\mathbf{Set}}(D^{op})$$

is an equivalence, then the embedding

$$\varphi_{|_{\widehat{C}}} : \widehat{C} \hookrightarrow \mathbf{Rep}_{\mathbf{Set}}(D^{op})$$

factors through an equivalence $\widehat{C} \simeq \widehat{D}$. □

4.2 Some results

Lemma 1. *Let C be a small category. Then we have the equivalence $\widehat{C}^{op} \simeq \widehat{C^{op}}$, which is 2-functorial on C.*

Proof. For any small category C, since a small category is Cauchy complete if and only if its opposite is Cauchy complete, we have

$$\begin{aligned}
\mathbf{Hom}_{\mathbf{CauCat}}(\widehat{C}^{op}, D) &\cong \mathbf{Hom}_{\mathbf{CauCat}^{co}}(\widehat{C}, D^{op}) \\
&\simeq \mathbf{Hom}_{\mathbf{Cat}^{co}}(C, D^{op}) \\
&\cong \mathbf{Hom}_{\mathbf{Cat}}(C^{op}, D) \\
&\simeq \mathbf{Hom}_{\mathbf{CauCat}}(\widehat{C^{op}}, D)
\end{aligned}$$

thus, by 2-Yoneda lemma, we get an equivalence $\widehat{C}^{op} \simeq \widehat{C^{op}}$ which is 2-functorial on C. □

Remark 1. *Lemma 1 and Proposition 2 imply that we have an equivalence (2-functorial on C) given by the composition*

$$\boldsymbol{Rep}_{\boldsymbol{Set}}(\widehat{C}) \cong \boldsymbol{Rep}_{\boldsymbol{Set}}((\widehat{C}^{op})^{op}) \simeq \boldsymbol{Rep}_{\boldsymbol{Set}}(C).$$

Lemma 2. *Let $F : C \to D$ be an equivalence between small Cauchy complete categories. Then $(\circ F^{op})_{|_{\widehat{D}}} : \widehat{D} \to \widehat{C}$ induces a functor $G : D \to C$ which is a quasi inverse of F.*

Proof. First notice that, since $C \simeq \widehat{C}$ and $D \simeq \widehat{D}$, $(\circ F^{op})_{|_{\widehat{D}}}$ induces indeed a functor $G : D \to C$.

If $x \in C$, then (here, with an abuse of notations, we indicate by F and G the respective functors between the Yoneda embeddings of C and D induced by F and G)

$$\begin{aligned}
G \circ F(\mathrm{Hom}_C(-, x)) &= G((\mathrm{Hom}_D(-, F(x))) \\
&= \mathrm{Hom}_D(F(-), F(x)) \\
&\cong \mathrm{Hom}_C(-, x)
\end{aligned}$$

since F is fully faithful, and, if $y \in D$,

$$\begin{aligned} F \circ G(\mathrm{Hom}_D(-, y)) &= F((\mathrm{Hom}_D(F(-), y)) \\ &\cong F(\mathrm{Hom}_D(F(-), F(x))) \\ &\cong F(\mathrm{Hom}_C(-, x)) \\ &= \mathrm{Hom}_D(-, F(x)) \\ &\cong \mathrm{Hom}_D(-, y) \end{aligned}$$

since F is essentially surjective, and this concludes the proof. □

5 2-(co)limits

In this section we give a brief review on 2-limits and 2-colimits, giving an explicit description of these constructions for 2-functors $F : I \to \mathbf{Cat}$, where I is a small category. Afterwards we will prove two useful results: the first one is about *1-final functors*, which provides a way to compute 2-limits easily in some special cases, and the other one is a rather technical proposition that will be used many times later to show that some 2-functors are costacks.

5.1 2-(co)limits

Lax functors are intuitively functors up to a 2-cell: we follow the same idea to give the definition of lax (and 2-)limits. For this part we won't give any proof, since they are rather technical and useless for the purpose of our work.

The next definition is taken from [BRD94, chapter 7].

Definition 2. *Let $\boldsymbol{C}$, $\boldsymbol{D}$ be 2-categories, with $\boldsymbol{C}$ small. For every object $x \in Ob(\boldsymbol{D})$, we consider $\Delta_x : \boldsymbol{C} \to \boldsymbol{D}$ the constant strict 2-functor on x, assigning to x to each object of $\boldsymbol{C}$, id_x to each morphism of $\boldsymbol{C}$ and id_{id_x} to each transformation of $\boldsymbol{C}$. The lax limit of a lax functor $F : \boldsymbol{C} \to \boldsymbol{D}$, if it exists, is a pair (l, π) where l is an object of $\boldsymbol{D}$ and $\pi : \Delta_l \Rightarrow F$ is a lax natural transformation such that the functor*

$$\boldsymbol{Hom_D}(x, l) \to \boldsymbol{Hom}_{l\mathfrak{F}(\boldsymbol{C},\boldsymbol{D})}(\Delta_x, F)$$

given by the composition with π is an isomorphism of categories, for each object $x \in Ob(\boldsymbol{D})$. Replacing "lax" by "2-" or "strict", and $l\mathfrak{F}(\boldsymbol{C}, \boldsymbol{D})$ by $p\mathfrak{F}(\boldsymbol{C}, \boldsymbol{D})$ or $2\mathfrak{F}(\boldsymbol{C}, \boldsymbol{D})$, we get respectively the corresponding notions of 2-limits and of strict 2-limits. Lax colimits are just lax limits in $\boldsymbol{D}^{op}$.

Since we will mostly use lax limits and colimits of 2-functors $F : I \to \mathbf{Cat}$ with I a small (1-)category, we now give an explict description of these constructions following [Was04, Appendix A].

First we will deal with Grothendieck construction of the category $\int_I F$ for any 2-functor $F : I \to \mathbf{Cat}$, which will turn out to be its lax colimit; the 2-colimit of F will be given by a localization of it, and when I in particular is filtrant, the description ofthe 2-colimit will be very simple.

Definition 3. *Let $F : I \to \boldsymbol{Cat}$ be a 2-functor. The category $\int_I F$ is defined by*

$$\begin{aligned} Ob(\int_I F) &= \coprod_{i \in I} Ob(F(i)) \\ &= \{(i, x) \,|\, i \in I \; and \, x \in Ob(F(i))\} \end{aligned}$$

$$Hom_{\int_I F}((i,x),(j,y))$$
$$= \{(s,f)\,|\,(s\,:\,i \to j) \in Mor(I),\ (f\,:\,F(s)(x) \to y) \in Mor(F(j))\}$$

Composition of two morphisms $(s,f) : (i,x) \to (j,y)$, $(t,g) : (j,y) \to (k,z)$ *is defined such that the following diagram is commutative*

$$\begin{array}{ccc} F(t \circ s)(x) & \longrightarrow & z \\ {\scriptstyle \Phi^F(s,t)_x}\downarrow & & \uparrow{\scriptstyle g} \\ F(t) \circ F(s)(x) & \xrightarrow[F(t)(f)]{} & F(t)(y) \end{array}$$

hence we have set

$$(t,g) \circ (s,f) = (t \circ s, g \circ F(t)(f) \circ \Phi^F(s,t)_x).$$

One checks easily that the composition of morphisms is associative and that, for any object (i,x), the morphism $(id_i, \Phi^F(i))$ has the property of identity morphism, so that $\int_I F$ is a well defined category.

For each $i \in I$ we have a natural functor

$$\sigma_i : F(i) \to \int_I F$$

that maps an object $x \in F(i)$ to (i,x) and a morphism f of $F(i)$ to $(id_i, f \circ \Phi^F(i))$. Note that for any morphism $s : i \to j$ we have $\sigma_j(F(s)(x)) = (j, F(s)(x))$, thus we get a morphism

$$(s, id_{F(s)(x)}) : \sigma_i(x) = (i,x) \to (j, F(s)(x)) = \sigma_j(F(s)(x)).$$

Set $(\Theta^\sigma_s)_x = (s, id_{F(s)(x)})$ for $x \in Ob(F(i))$. One checks easily that these morphisms define a natural transformation $\Theta^\sigma_s : \sigma_i \Rightarrow \sigma_j \circ F(s)$, so that $\sigma : F \Rightarrow \Delta_{\int_I F}$ defines a lax natural transformation, and it can be shown that each lax natural transformation $F \Rightarrow \Delta_C$, for a category C, factors uniquely through σ, implying (with some additional details about modifications tht we will not treat in detail) that the Grothendieck construction on F is in fact its lax colimit. One can show that the localization $\int_I F[\mathcal{S}^{-1}]$ with respect to the set

$$\mathcal{S} = \{(s,f) : (i,x) \to (j,y)\,|\, f : F(s)(x) \to y \text{ is an isomorphism}\},$$

with 2-natural transformation $\sigma' : F \Rightarrow \Delta_{\int_I F[\mathcal{S}^{-1}]}$ defined by

$$\sigma'_i = Q \circ \sigma_i : F(i) \to \int_I F[\mathcal{S}^{-1}]$$
$$\Theta^{\sigma'}_s = Q \circ \Theta^\sigma_s : \sigma_i \Rightarrow \sigma_j \circ F(s),$$

where $Q : \int_I F \to \int_I F[\mathcal{S}^{-1}]$ is the localization functor, is the 2-colimit of F, and thus will be indicated by $2 \varinjlim_{i \in I} F(i)$.

In the case in which I is a filtrant preordered set (as for example $\mathbf{Op}(X)$ for a topological space X), one can show that the hom-sets in the 2-colimit are given by the simple formula

$$\mathrm{Hom}_{2 \varinjlim_{i \in I} F(i)}((i,x),(j,y)) = \varinjlim_{\substack{i \to k \\ j \to k}} \mathrm{Hom}_{F(k)}(\epsilon_{ik}(x), \epsilon_{jk}(y)).$$

We now begin with the description of 2-limits in **Cat**. First, for a 2-functor $F : I \to \mathbf{Cat}$, we define F-admissible pairs.

Definition 4. *A F-admissible pair (x, θ^x) is the following data;*

(1) an object $x_i \in Ob(F(i))$ for any $i \in I$;

(2) a morphism $\theta^x_s : x_j \to F(s)(x_i)$ for any $(s : i \to j)$ in I,

such that the two following conditions hold

(A) for any $i \in I$ we have $\Phi^F(i)_{x_i} \circ \theta^x_{id_i} = id_{x_i}$ as visualized by

$$\begin{array}{ccc} x_i & \xrightarrow{\theta^x_{id_i}} & F(id_i)(x_i) \\ & {\scriptstyle id_{x_i}} \searrow \quad \swarrow {\scriptstyle \Phi^F(i)_{x_i}} & \\ & x_i & \end{array}$$

(B) and for any two composable morphisms $s : i \to j$, $t : j \to k$ the equation

$$F(t)(\theta^x_s) \circ \theta^x_t = \Phi^F(s,t)_{x_i} \circ \theta^x_{tos}$$

holds as visualized by the following diagram

$$\begin{array}{ccc} x_k & \xrightarrow{\theta^x_t} & F(t)(x_j) \\ {\scriptstyle \theta_{tos}} \downarrow & & \downarrow {\scriptstyle F(t)(\theta^x_s)} \\ F(t \circ s)(x_i) & \xrightarrow{\Phi^F(s,t)_{x_i}} & F(t) \circ F(s)(x_i) \end{array}$$

A strictly F-admissible pair is a F-admissible pair (x, θ^x) such that all the morphisms of (B) are isomorphisms.

Let (x, θ^x), (y, θ^y) be two F-admissible pairs. A morphism of F-admissible pairs $\varphi : (x, \theta^x) \to (y, \theta^y)$ is given by a family of morphisms $\varphi_i : x_i \to y_i$ in $F(i)$, indexed by $i \in I$, satisfying for any $s : i \to j$ of $Mor(I)$ the equation

$$F(s)(\varphi_i) \circ \theta^x_s = \theta^y_s \circ \varphi_j$$

as visualized by

$$\begin{array}{ccc} x_j & \xrightarrow{\theta^x_s} & F(s)(x_i) \\ {\scriptstyle \theta_{t\circ s}}\downarrow & & \downarrow{\scriptstyle F(s)(\varphi_i)} \\ y_j & \xrightarrow{\theta^y_s} & F(s)(y_i) \end{array}$$

A morphism of strictly F-admissible pairs is a morphism of the underlying F-admissible pairs. Composition and identities are given in the obvious ways. We are now ready for the next definition.

Definition 5. *We will denote by dF/dI the category in which the objects are F-admissible pairs, and the morphisms are the ones defined above.*

For each $i \in I$ there is a natural functor

$$\pi_i : \frac{dF}{dI} \to F(i)$$

that projects an object (x, θ^x) to x_i and a morphism $\varphi : (x, \theta^x) \to (y, \theta^y)$ to φ_i.

Let $s : i \to j$ be a morphism in $Mor(I)$ and (x, θ^x) an object in $Ob(dF/dI)$. Then we have a morphism

$$\theta^x_s : \pi_j(x, \theta^x) = x_j \to F(s)(x_i) = F(s) \circ \pi_i(x, \theta^x).$$

Put $(\Theta^\pi_s)_{x,\theta^x} = \theta^x_s$. The definition of morphisms of F-admissible pairs immediately implies that Θ^π_s defines actually a natural transformation $\Theta^\pi_s : \pi_j \Rightarrow F(s) \circ \pi_i$ and one checks that π is a 2-natural tranformation.

Proposition 3. *Let $\Delta_C \Rightarrow F$ be a 2-natural transformation. Then it factors uniquely through dF/dI.*

Proposition 3 and the verification of some other details show that dF/dI is in fact a lax limit of F.

Theorem 1. *Let $F : I \to$ **Cat** be a 2-functor. Then F admits a 2-limit, which is given the full subcategory d^SF/dI of dF/dI consisting of strictly F-admissible pairs.*

We have the following useful formulas.

Proposition 4. *Let $F : I \to$ **Cat** be a 2-functor. Then we have the following canonical isomorphisms*

(i)

$$\boldsymbol{Hom}_{\boldsymbol{Cat}}(2\varinjlim_{i\in I} F(i), C) \cong 2\varprojlim_{i\in I} \boldsymbol{Hom}_{\boldsymbol{Cat}}(F(i), C),$$

(ii)

$$\boldsymbol{Hom}_{\boldsymbol{Cat}}(C, 2\varprojlim_{i\in I} F(i)) \cong 2\varprojlim_{i\in I} \boldsymbol{Hom}_{\boldsymbol{Cat}}(C, F(i)).$$

Note that these formulas can be used to define 2-limits and 2-colimits in any 2-category by only using 2-limits in **Cat**, which are rather simple objects, and this definition will coincide with the one give in Definition 2.

The formulas have an interesting corollary that will be used later.

Corollary 1. *Let (L, R), $L : \boldsymbol{C} \to \boldsymbol{D}$ and $R : \boldsymbol{D} \to \boldsymbol{C}$, a 2-adjoint couple. Then L commutes (up to equivalence) with 2-colimits and R commutes (up to equivalence) with 2-limits.*

Proof. We only write down the proof for the commutativity of L with 2-colimits of 2-functors $F : I \to \mathbf{C}$ where I is a small category, since this will be the only meaningful case for us.

We have, 2-functorially on $x \in Ob(\mathbf{D})$, the following equivalence

$$\begin{aligned}
\mathbf{Hom}_{\mathbf{D}}(L(2\varinjlim_{i\in I} F(i)), x) &\simeq \mathbf{Hom}_{\mathbf{C}}(2\varinjlim_{i\in I} F(i), R(x)) \\
&\cong 2\varprojlim_{i\in I} \mathbf{Hom}_{\mathbf{C}}(F(i), R(x)) \\
&\simeq 2\varprojlim_{i\in I} \mathbf{Hom}_{\mathbf{D}}(L \circ F(i), x) \\
&\cong \mathbf{Hom}_{\mathbf{D}}(2\varinjlim_{i\in I} L \circ F(i), x)
\end{aligned}$$

so, by the 2-Yoneda lemma, we get the thesis. □

5.2 1-connected categories and 1-(co)final functors

Now we will introduce (following [KS05]) the notion of a *1-final functor*, and prove a result which is an analogue for 2-limits of what happens in the 1-categorical setting for final functors and limit: this will be useful later to give an explicit calculation of the *costalk* of the fundamental groupoid.

First, we recall the definitions of a *connected category* and of the set of *connected components* of a category.

Definition 6. *A category C is connected if it is non-empty and for any pair of objects $x, y \in C$, there is a finite sequence of objects $\{x_0 = x, x_1, \ldots, x_n = y\}$ such that at least one of the sets $Hom_C(x_j, x_{j+1})$ or $Hom_C(x_{j+1}, x_j)$ for any $j \in \{0, \ldots, n-1\}$.*

Example 2. *(1) A groupoid is connected if and only if it is equivalent to a group.*

(2) $\Pi_1(X)$ is connected if and only if X is path connected.

Definition 7. *For a small category I, we denote by $\pi_0(I)$ the set of equivalence classes of objects of I by the equivalence relation generated by the relation $x \sim y$ if $Hom_I(x, y) \neq \emptyset$.*

Example 3. $\pi_0(\Pi_1(X)) = \pi_0(X)$.

We see immediately (for example, [KS05, Corollary 2.4.4] that for any small category I,

$$\varinjlim_{i \in I} \Delta_S \cong \coprod_{i \in \pi_0(I)} S,$$

where $\Delta_S : I \to \mathbf{Set}$ is the constant functor with value the set S, so

$$\pi_0(I) \cong \varinjlim_{i \in I} \Delta_{\{pt\}}$$

and a category is connected if and only if, for any constant functor $\Delta_S : I \to \mathbf{Set}$, the natural morphism

$$S = \Delta_S(i) \to \varinjlim_{i \in I} \Delta_S$$

is an isomorphism for any $i \in I$.

This remark leads to the following generalization.

Definition 8. *A small non-empty category I is simply connected (or 1-connected) if for any category $\boldsymbol{C}$ and any functor $\alpha : I \to \boldsymbol{C}$ such that $\alpha(u)$ is an isomorphism for any $u \in Mor(I)$, $\varinjlim \alpha$ exists in $\boldsymbol{C}$ and $\alpha(i) \to \varinjlim \alpha$ is an isomorphism for any $i \in I$.*

It is easy to see that the definition of a 1-connected category may have been given just by asking the condition for functors $\alpha : I \to \mathbf{Set}$ and clearly we could have given the definition exchanging colimits with limits, since if a functor $\alpha : I \to \mathbf{C}$ such that $\alpha(u)$ is an isomorphism for any $u \in Mor(I)$ admits colimit and $\alpha(i) \to \varinjlim \alpha$ are isomorphisms, taking their inverses $\alpha(i) \leftarrow \varinjlim \alpha$ we get that $\varinjlim \alpha$ is a limit for α.

Lemma 3. *(1) A 1-connected category is connected.*

(2) A category I is 1-connected if and only if I^{op} is 1-connected.

Proof. 1) is an immediate consequence of [KS05, Corollary 2.4.5].

For 2), if I is 1-connected and $\alpha : I^{op} \to \mathbf{C}$ is such that $\alpha(u)$ is an isomorphism for any $u \in Mor(I^{op})$, then $\alpha^{op} : I \to \mathbf{C}^{op}$ has the same property, so it admits colimit in $\mathbf{C}^{op}$ and $\alpha^{op}(i) \to \varinjlim \alpha^{op}$ are isomorphisms. Passing to $\mathbf{C}$, this means that α admits limit and $\alpha(i) \leftarrow \varinjlim \alpha$ are isomorphisms, so we are done. □

We prove two easy facts about 1-connected categories, the first of which provides a class of examples.

Lemma 4. *Let I be a small (co)filtrant category. Then I is 1-connected.*

Proof. Suppose that I is filtrant. First we note that, if $\alpha : I \to \mathbf{C}$ is a functor such that $\alpha(u)$ is an isomorphism for any $u \in Mor(I)$, then, for any two parallel morphisms $i \overset{u_1}{\underset{u_2}{\rightrightarrows}} j$, $\alpha(u_1) = \alpha(u_2)$: in fact, since I is filtrant, we have that there exists a morphism $v : j \to k$ such that

$$\begin{aligned} \alpha(v) \circ \alpha(u_1) &= \alpha(v \circ u_1) \\ &= \alpha(v \circ u_2) \\ &= \alpha(v) \circ \alpha(u_2), \end{aligned}$$

so $\alpha(u_1) = \alpha(u_2)$ since $\alpha(v)$ is an isomorphism. Fixing $k \in I$, for any $i \in I$ there is an isomorphism $\sigma_i : \alpha(i) \to \alpha(k)$ because, being I filtrant, there exist $i' \in I$ and two morphisms $i \xrightarrow{u} i' \xleftarrow{v} k$, so we can define $\sigma_i = \alpha(v)^{-1} \circ \alpha(u)$.

The definition of the σ_i is independent from the choice of the object i' since, for another i'' and couple of morphisms $i \xrightarrow{u'} i'' \xleftarrow{v'} k$, surely there is $j \in I$ and $i' \xrightarrow{s} j \xleftarrow{t} i''$ that add up to the non necessarily commutative diagram

$$\begin{array}{ccccc} & & i' & & \\ & \overset{u}{\nearrow} & \downarrow s & \overset{v}{\nwarrow} & \\ i & & j & & k \\ & \underset{u'}{\searrow} & \uparrow t & \underset{v'}{\swarrow} & \\ & & i'' & & \end{array}$$

which becomes commutative after applying α and, since α maps every morphism to an isomorphism, we gat that $\sigma_i = \alpha(v)^{-1} \circ \alpha(u) = \alpha(v')^{-1} \circ \alpha(u')$; this in particular shows that, for any morphism $i \to j$ in I, the diagram

$$\begin{array}{ccc} \alpha(i) & \xrightarrow{\sigma_i} & \alpha(k) \\ \downarrow & \nearrow_{\sigma_j} & \\ \alpha(j) & & \end{array}$$

commutes, so the σ_i's add up to a natural isomorphism $\sigma : \alpha \to \Delta_{\alpha(k)}$, which implies that $\varinjlim \alpha \cong \alpha(k)$, so we have the thesis.

If I is cofiltrant, then I^{op} is filtrant, so 1-connected, and then I must be 1-connected. □

Lemma 5. *Let I be a small groupoid. Then I is 1-connected if and only if it is equivalent to* ***Pt****.*

Proof. Obviously if I is equivalent to **Pt**, then it is 1-connected.

Suppose now that I is a 1-connected groupoid. Since 1-connectedness implies connectedness, I is equivalent to a groupoid J with one object j, and J is still 1-connected. If $\alpha : J \to \mathbf{C}$ is a functor, then clearly $\alpha(u)$ is an isomorphism for any $u \in Hom_J(j,j)$, so $\sigma_j : \alpha(j) \to \varinjlim \alpha$ is an isomorphism and for any $u \in Hom_J(j,j)$, $\sigma_j \circ \alpha(u) = \sigma_j$ so $\alpha(u) = 1_{\alpha(j)}$ which means that $\alpha = \Delta_{\alpha(j)}$. We have shown that any functor with domain J must be constant, so $Fct(J, \mathbf{C}) \cong \mathbf{C}$ for any category $\mathbf{C}$, which implies that $J \cong \mathbf{Pt}$ by the 2-Yoneda Lemma. □

This last result has an immediate consequence which gives a geometric interpretation of the definition of a 1-connected category, and in some sense explains why it is natural to borrow this idea from topology to give a name to the categories having the properties in the definition.

Corollary 2. *Let X be a topological space. Then $\mathbf{\Pi}_1(X)$ is 1-connected if and only if X is simply connected.*

Proof. For the previous lemma, $\mathbf{\Pi}_1(X)$ is 1-connected if and only if it is equivalent to **Pt**, but this happens if and only if X is path connected and, for any $x \in X$, $\pi_1(X, x) \cong \mathbf{1}$, i.e. if and only if X is simply connected. □

Recall that, if $\varphi : J \to I$ is a functor between small categories and $i \in I$, we define $i \downarrow \varphi$ and $\varphi \downarrow i$ respectively by

$$Ob(i \downarrow \varphi) = \{(j, s) \,|\, j \in J, s : i \to \varphi(j)\},$$

$$\mathrm{Hom}_{i\downarrow\varphi}((j_1,s),(j_2,t)) = \{f \in \mathrm{Hom}_J(j_1,j_2) \,|\, t = \varphi(f)\circ s\}$$

and

$$Ob(\varphi\downarrow i) = \{(j,s) \,|\, j\in J, s:\varphi(j)\to i\}\,,$$

$$\mathrm{Hom}_{\varphi\downarrow i}((j_1,s),(j_2,t)) = \{f \in \mathrm{Hom}_J(j_1,j_2) \,|\, s = t\circ\varphi(f)\}$$

The next definition is the natural analogue of the one of a cofinal functor, using 1-connected instead of connected categories.

Definition 9. *(i) We say that a functor $\varphi : J \to I$ is 1-cofinal if the category $i \downarrow \varphi$ if 1-connected for any $i \in I$.*

(ii) We say that a functor $\varphi : J \to I$ is 1-final if the functor $\varphi^{op} : J^{op} \to I^{op}$ is 1-cofinal, or equivalently, if the category $\varphi \downarrow i$ is 1-connected for any $i \in I$.

We will see in a moment that the notion of 1-final functors is strictly related to 2-limits. First we note that, in general, if $\varphi : J \to I$ is a functor and $F : I \to \mathbf{Cat}$ is a 2-functor, then we have naturally a functor $\Phi : 2\varprojlim F \to 2\varprojlim F\circ\varphi$ coming, by universal property, from the functors $2\varprojlim F \to F(\varphi(j))$. The next result (which is an adaptation of [KS05, Proposition 19.2.3]) will be the most important of this part.

Theorem 2. *Let $\varphi : J \to I$ be a 1-final functor and $F : I \to \mathbf{Cat}$ a 2-functor. Then $\Phi : 2\varprojlim F \to 2\varprojlim F\circ\varphi$ is an equivalence of categories.*

Proof. We will construct a quasi inverse Ψ. We recall that an object in $2\varprojlim F\circ\varphi$ is a family $X = \{(X_j, f_u)\}_{j\in J, u\in Mor(J)}$ where $X_j \in F(\varphi(j))$ and, for $u : j_1 \to j_2$, $f_u : F(\varphi(u))(X_{j_1}) \to X_{j_2}$ is an isomorphism in $F(\varphi(j_2))$. To define $\Psi(X)$, we will use a family of functors $\beta^i : \varphi\downarrow i \to F(i)$ defined in the following way: we associate to any $(j,u)\in\varphi\downarrow i$ $\beta^i(j,u) = F(u)(X_j)$ and to any morphism $v : (j_1,u_1) \to (j_2,u_2)$ $\beta^i(v) = F(u_2)(f_v)\circ\Phi^F(u_2,\varphi(v))(X_{j_1})$. If $v : (j_1,u_1)\to(j_2,u_2)$, $v' : (j_2,u_2)\to(j_3,u_3)$ then

$$
\begin{aligned}
\beta^i(v' \circ v) &= F(u_3)(f_{v' \circ v}) \circ \Phi^F(u_3, \phi(v' \circ v))(X_{j_1}) \\
&= F(u_3)(f_{v'}) \circ (F(u_3) \circ F(v'))(f_v) \circ \\
&\circ F(u_3)(\Phi^F(\phi(v'), \phi(v))(X_{j_1})) \circ \Phi^F(u_3, \phi(v' \circ v))(X_{j_1}) \\
&= F(u_3)(f_{v'}) \circ \Phi^F(u_3, \phi(v'))(X_{j_2}) \circ \\
&\circ F(u_2)(f_v) \circ \Phi(u_3, \phi(v'))(F(v)(X_{j_1}))^{-1} \circ \\
&\circ F(u_3)(\Phi^F(\phi(v'), \phi(v))(X_{j_1})) \circ \Phi^F(u_3, \phi(v' \circ v))(X_{j_1}) \\
&= F(u_3)(f_{v'}) \circ \Phi^F(u_3, \phi(v'))(X_{j_2}) \circ F(u_2)(f_v) \circ \\
&\circ \Phi^F(u_2, \phi(v))(X_{j_1})
\end{aligned}
$$

where the second equality holds because of condition (B) for objects of a 2-limit and because $F(u_3)$ is functor, the third because $\Phi^F(u_3, \varphi(v'))$ is a natural transformation and the fourth because of condition (2F2) for F, again by condition (B) for objects of a 2-limit $\beta^i(1_{(j,u)}) = 1_{\beta^i(j,u)}$, so β^i is indeed a functor. By definition, $\beta^i(v)$ is an isomorphism for any $v : (j_1, u_1) \to (j_2, u_2)$, so, since by hypothesis $\varphi \downarrow i$ is 1-connected, $\varprojlim \beta^i$ exists in $F(i)$ and, if $v : i_1 \to i_2$, $\varprojlim \beta^{i_1} \cong \beta^{i_1}(j, u)$ for every $(j, u) \in \varphi \downarrow i$, then $F(v)(\varprojlim \beta^{i_1}) \cong F(v)(\beta^{i_1}(j, u)) = (F(v) \circ F(u))(X_j) \cong F(v \circ u)(X_j) \cong \beta^{i_2}(j, v \circ u) \cong \varprojlim \beta^{i_2}$, so $\Psi(X) = \left\{(\varprojlim \beta^i, g_v)\right\}_{i \in I, v \in Mor(I)}$, where $g_v : F(v)(\varprojlim \beta^{i_1}) \to \varprojlim \beta^{i_2}$ is the isomorphism found before, defines an object in $2\varprojlim F$. One can similarly define $\Psi(f)$ if $f : X \to Y$ is a morphism in $2\varprojlim F$ and show that Ψ is a quasi inverse of Φ. $\square$

Corollary 3. *Let $\varphi : J \to I$ be a final functor, with J cofiltrant, and let $F : I \to$ **Cat** be a 2-functor. Then $\Phi : 2\varprojlim F \to 2\varprojlim F \circ \varphi$ is an equivalence of categories.*

Proof. $\varphi : J \to I$ is final and J is cofiltrant, by [KS05, Proposition 3.2.2], $\varphi \downarrow i$ is cofiltrant, and then 1-connected. This means that φ is 1-final, so, by the previous theorem, the proof is complete. $\square$

Corollary 4. *Let $\boldsymbol{C}$ be a 2-category, $\varphi : J \to I$ be a 1-final functor and $F : I \to \mathbf{C}$ is a 2-functor. If $2\varprojlim F$ exists in $\boldsymbol{C}$, then $\Phi : 2\varprojlim F \to 2\varprojlim F \circ \varphi$ is an equivalence in$\boldsymbol{C}$.*

Proof. By the univeral property of the 2-limits and by the previous theorem, for any $x \in \mathbf{C}$ we have

$$\begin{aligned}\mathbf{Hom}_{\mathbf{C}}(x, 2\varprojlim F) &\cong 2\varprojlim \mathbf{Hom}_{\mathbf{C}}(x, F)\\ &\simeq 2\varprojlim \mathbf{Hom}_{\mathbf{C}}(x, F\circ\varphi)\\ &\cong \mathbf{Hom}_{\mathbf{C}}(x, 2\varprojlim F\circ\varphi),\end{aligned}$$

hence we conclude by applying the 2-Yoneda lemma. □

5.3 (co)Limits indexed by 2-colimits

Proposition 5. *Let I be a small category and $F : I \to \boldsymbol{Cat}$. For a functor $G : 2\varinjlim F \to C$, set*

$$G_{|_{F(i)}} : \; F(i) \longrightarrow 2\varinjlim F \xrightarrow{G} C$$

for any $i \in Ob(I)$, where the left hand arrow is the natural functor.

(i) If $\varinjlim_{2\varinjlim F} G$ exists in C, then $\varinjlim_{i\in I}\varinjlim_{F(i)} G_{|_{F(i)}}$ exists and there is a natural isomorphism in C

$$\varinjlim_{i\in I}\varinjlim_{F(i)} G_{|_{F(i)}} \to \varinjlim_{2\varinjlim F} G.$$

(ii) If $\varprojlim_{(2\varinjlim F)^{op}} G$ exists in C, then $\varprojlim_{i\in I^{op}}\varprojlim_{F(i)^{op}} G_{|_{F(i)}}$ exists and there is a natural isomorphism in C

$$\varprojlim_{i\in I^{op}}\varprojlim_{F(i)^{op}} G_{|_{F(i)}} \leftarrow \varprojlim_{(2\varinjlim F)^{op}} G.$$

Proof. Suppose that $\varinjlim_{F(i)} G_{|_{F(i)}}$ exists in C for any $i \in Ob(I)$. Then for any morphism $i \to j$ in I there is a commutative diagram in C

$$\begin{array}{ccc}
\varprojlim_{F(i)} G_{|_{F(i)}} & & \\
\downarrow & \searrow & \\
& & \varinjlim_{2 \varinjlim F} G. \\
& \nearrow & \\
\varprojlim_{F(j)} G_{|_{F(j)}} & &
\end{array}$$

By universal property, we get a natural morphism in C

$$\varinjlim_{i \in I} \varinjlim_{F(i)} G_{|_{F(i)}} \to \varinjlim_{2 \varinjlim F} G.$$

To prove that it is an isomorphism, let us consider the morphism in **Set**

$$\begin{aligned}
\varprojlim_{i \in I} \varprojlim_{F(i)} \mathrm{Hom}_C(G_{|_{F(i)}}, x) &\cong \mathrm{Hom}_C\left(\varinjlim_{i \in I} \varinjlim_{F(i)} G_{|_{F(i)}}, x \right) \\
&\leftarrow \mathrm{Hom}_C\left(\varinjlim_{2 \varinjlim F} G, x \right) \\
&\cong \varprojlim_{2 \varinjlim F} \mathrm{Hom}_C(G, x)
\end{aligned}$$

obtained by applying the contravariant functor $\mathrm{Hom}_C(-, x)$ for $x \in Ob(C)$: thus, it suffices to prove part (ii) to get the whole proof, assuming $C = \mathbf{Set}$.

For any morphism $i \to j$ in I there is a commutative diagram of sets

$$\begin{array}{ccc}
\varprojlim_{F(i)^{op}} G_{|_{F(i)}} & & \\
\uparrow & \nwarrow & \\
& & \varprojlim_{(2 \varinjlim F)^{op}} G. \\
& \swarrow & \\
\varprojlim_{F(j)^{op}} G_{|_{F(j)}} & &
\end{array}$$

By universal property, we get a natural morphism in C

$$\varprojlim_{i \in I^{op}} \varprojlim_{F(i)^{op}} G_{|_{F(i)}} \leftarrow \varprojlim_{(2 \varinjlim F)^{op}} G.$$

This morphism is in fact an isomorphism, by the two lemmas below. □

Lemma 6. *Let $F : I \to \boldsymbol{Cat}$ be a lax functor. Let $G : F \Rightarrow \Delta_{\boldsymbol{Set}}$ be a 2-natural transformation (where we denote by $\Delta_{\boldsymbol{Set}}$ the constant 2-functor with value $\boldsymbol{Set}$) and $\widetilde{G} : \int_{i \in I} F(i) \to \boldsymbol{Set}$ its unique factorization. There is a natural isomorphism in $\boldsymbol{Set}$*

$$\varprojlim_{i \in I^{op}} \varprojlim_{F(i)^{op}} G_{|_{F(i)}} \leftarrow \varprojlim_{(\int_{i \in I} F(i))^{op}} \widetilde{G}.$$

If F is a 2-functor, we may replace $\int_{i \in I} F(i)$ by $2 \varinjlim_{i \in I} F(i)$.

Proof. The elements on the right hand side are families $\{X_{i,a} \in G_i(a)\}$ where $i \in I$ and $a \in F(i)$ such that for any $s : i \to j$, $b \in F(j)$ and $f : F(s)(a) \to b$ we have $G_j(f)((\rho_s(X_{i,a}))) = X_{j,b}$ where $\rho_s : G_i \Rightarrow G_j \circ F(s)$ is the natural transformation from the data of G. These conditions are clearly the same as asking that for every $s : i \to j$ we have $\rho_s(X_{i,a}) = X_{j,F(s)(a)}$ and for every morphism $f : a \to a'$ in $F(i)$ we have $G_i(f)(X_{i,a}) = X_{i,a'}$ which are exactly the families onthe left hand side.

The last part follows from the lemma below, since, when F is a 2-functor, $2 \varinjlim_{i \in I} F(i)$ is obtained by a localization of $\int_{i \in I} F(i)$. □

Lemma 7. *Let C be a category and S a family of morphisms in C. Then the localization functor $Q : C \to C[S^{-1}]$ is cofinal.*

Proof. Recall that $Q : C \to C[S^{-1}]$ is cofinal if and only if $\pi_0(p \downarrow Q) \cong pt$ for any $p \in Ob(C[S^{-1}]) = Ob(C)$.

As $(p, id_p) \in Ob(p \downarrow Q)$, it is enough to show that any morphism $f : p \to r$ in $C[S^{-1}]$ is connected to id_p. Recall that a morphism $f : p \to r$ in $C[S^{-1}]$ is given by a sequence of objects $p_0 = p, p_1, \dots, p_n = r$ and morphisms $f_0, \dots, f_{n-1}$ in C, where either $f_i : p_i \to p_{i+1}$ of $f_i : p_{i+1} \to p_i$ (and in the latter cae $f_i \in S$). Note that every subsequence is a morphism from p to p_k in $C[S^{-1}]$, therefore defines an object of $p \downarrow Q$. Hence the sequence defining f connects f to id_p in $p \downarrow Q$. □

6 Costacks

In this section we first recall some facts about cosheaves and then give the definition and the basic properties of a costack with values in a 2-cocomplete 2-category, i.e. a 2-category which admits all small 2-colimits, even though in most part of our work we will only deal with **Cat**-valued costacks.

From here on, for a topological space X, we will indicate by $\mathbf{Op}(X)$ the category whose objects are open subsets of X with morphisms given by the inclusions, and by $\mathbf{Op}(X)_x$ its full subcategory whose objects are the open subsets of X containing x.

6.1 Cosheaves

Definition 10. *(i) A C-valued pre-cosheaf on X is a functor*

$$\mathcal{C} : \boldsymbol{Op}(X) \to C.$$

(ii) The morphisms corresponding to inclusions $U \subset V$ are often denoted by $\epsilon_{UV} : \mathcal{C}(U) \to \mathcal{C}(V)$ and are called extension morphisms.

(iii) Let C be a cocomplete category. A C-valued pre-cosheaf on X is a functor $\mathcal{C}$ is called a cosheaf if for any open cover $\{U_i\}_{i\in I}$ stable by finite intersections the natural morphism

$$\varinjlim_{i\in I} \mathcal{C}(U_i) \to \mathcal{C}(\bigcup_{i\in I} U_i)$$

is an isomorphism.

Since the empty set $\emptyset$ is an initial object in $\mathbf{Op}(X)$, then $\mathcal{C}(\emptyset)$ is an initial object in C for any C-valued cosheaf $\mathcal{C}$.

One may rephrase the cosheaf condition by requiring that for any open subset $U \subseteq X$ and any open cover $\{U_i\}_{i\in I}$ of U, the natural sequence

$$\coprod_{i,j\in I} \mathcal{C}(U_i \cap U_j) \rightrightarrows \coprod_{i\in I} \mathcal{C}(U_i) \longrightarrow \mathcal{C}(U)$$

is exact.

Lemma 8. *Let $\mathcal{C}$ be a C-valued pre-cosheaf, with C a cocomplete category. Then $\mathcal{C}$ is a cosheaf if and only if* *(i)* $\mathcal{C}$ *commutes with filtrant colimits, i.e. the morphism*

$$\varinjlim_{i\in I} \mathcal{C}(U_i) \to \mathcal{C}(\bigcup_{i\in I} U_i)$$

is an isomorphism for any open cover stable by finite unions,

(ii) for any pair of open sets $U, V \subset X$*, the commutative diagram given by the extension morphisms*

$$\begin{array}{ccc} \mathcal{C}(U\cap V) & \longrightarrow & \mathcal{C}(V) \\ \downarrow & & \downarrow \\ \mathcal{C}(U) & \longrightarrow & \mathcal{C}(U\cup V) \end{array}$$

is cocartesian.

Proof. Recall that the diagram in (ii) is cocartesian if and only if the sequence

$$\mathcal{C}(U\cap V) \rightrightarrows \mathcal{C}(U) \coprod \mathcal{C}(V) \longrightarrow \mathcal{C}(U\cup V)$$

is exact, hence we only have to prove that (i) and (ii) imply the cosheaf condition. By a standard result on colimits (see for example [KS05, Lemma 3.2.8], we have that for any open cover $\{U_i\}_{i\in I}$ stable by finite intersection one has

$$\varinjlim_{i\in I} \mathcal{C}(U_i) \cong \varinjlim_{\substack{J\subseteq I \\ J finite}} \varinjlim_{j\in J} \mathcal{C}(U_j)$$

where J is such that $\{U_j\}_{j\in J}$ is stable by finite intersections. As the set of finite subsets of I, ordered by inclusion, is stable by finite unions, it follow that $\mathcal{C}$ is a cosheaf whenever the morphism

$$\varinjlim_{i\in I} \mathcal{C}(U_i) \to \mathcal{C}(\bigcup_{i\in I} U_i)$$

is an isomorphism for all open covers which are stable by finite unions and for those which are finite and stable by finite intersections. By induction, we may consider only finite open covers of the form $\{U, V, U\cap V\}$, hence we find (ii). □

Remark 2. *Unlike sheaves, cosheaves are not preserved by forgetful functors, since in general these do not commute with colimits. However, if a forgetful functor admits a left adjoint, the latter commutes with colimits, hence it preserves cosheaves. In particular, a cosheaf of (abelian) groups does not define an underline cosheaf of sets, but the pre-cosheaf of (abelian) groups freely generated by a cosheaf of sets is a cosheaf of (abelian) groups.*

There are natural examples of pre-cosheaves with values in **Set**:

(1) the trivial pre-cosheaf t_X assigning to any open subset $U \subseteq X$ U itself, which is easily seen to be a cosheaf of sets;

(2) the terminal pre-cosheaf pt^X assigning to any open subset of X the terminal set pt;

(3) the initial pre-cosheaf $\emptyset^X$ assigning to any open subset of X the empty set $\emptyset$, which is a cosheaf;

(4) the connected components pre-cosheaf $\#_X$, which assigns to each open subset of X the set of its connected components;

(5) the 0-th homotopy pre-cosheaf $\pi_{0,X}$ which assigns to each open subset of X the set of its path connected components. Note that this is a cosheaf. Note that this is a cosheaf. Indeed, we have to show that, for any open cover $\{U_i\}_{i\in I}$ of an open subset $U \subseteq X$, the natural morphism

$$coeq\Big(\coprod_{i,j\in I} \pi_0(U_i \cap U_j) \rightrightarrows \coprod_{i\in I} \pi_0(U_i) \Big) \to \pi_0(U)$$

is an isomorphism in **Set**. It is clearly surjective, because for every path connected component C of U there exists some $i \in I$ such that $U_i \cap C \neq \emptyset$. For the injectivity, consider two points $x \in U_i$ and $y \in U_j$ lying in the same connected component of U, i.e. such that there exists a continous map $\gamma : [0,1] \to U$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Using Lebesgue's Number Lemma and the fact that the image of γ is compact, one gets that there exists a finite open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of $\gamma([0,1])$ and paths γ_i, $i \in \{1, \dots, m\}$ such that each γ_i lies entirely in some $U_j \in \mathcal{U}$, $\gamma_i(1) \in U_j \cap U_k$ for some $U_j, U_k \in \mathcal{U}$ and $\gamma = \gamma_m \circ \cdots \circ \gamma_1$: this implies that x and y define the same element in the coequalizer, so that the natural map is injective.

(6) since $\pi_{0,X}$ is a cosheaf, by Remark 2, we get that the pre-cosheaf of abelian groups $H_{0,X}$ assigning to each open subset its 0-th homology group is a cosheaf, since $H_0(U)$ is the free abelian group on the set $\pi_0(U)$.

Notice that $\pi_{0,X}$ and $\#_X$ do not coincide in general, but they do when the space X is locally path connected, and in particular they are both equal to t_X when X is totally disconnected.

Definition 11. *Let $\mathcal{C}$ be a C-valued pre-cosheaf on a space X, with C complete. For any $x \in X$, we define the costalk of $\mathcal{C}$ at x, and indicate it by $\mathcal{C}_x$, by*

$$\mathcal{C}_x = \varprojlim_{U \in \mathbf{Op}(X)_x} \mathcal{C}(U).$$

We see easily that

(1) for any $x \in X$

$$\begin{aligned}(t_X)_x &= \bigcap_{U \in \mathbf{Op}(X)_x} U \\ &= \left\{ y \in X \mid x \in \overline{\{y\}} \right\}\end{aligned}$$

(2) if X is locally path connected, since the inclusion of the subcategory of path connected open neighbourhoods of x in the category of all open neighbourhood of x is final, for any $x \in X$ we get

$$(\pi_{0,X})_x \cong pt.$$

Example 4. *Note that, if X is not locally path connected, it's not necessarily true that $(\pi_{0,X})_x \cong pt$ for any $x \in X$: take for example*

$$X = \{(x, y) \in [0, 1] \times [0, 1] \mid x = 1/n \text{ or } y = 1/n \text{ for some} n \geq 1\} \cup \{(0, 0)\} \subseteq \mathbb{R}^2$$

with the induced topology, which is essentially a union of two orthogonal comb spaces. We see that any open ball centered on the point $x = (0, 0)$ intersected with X has exactly two path connected components, one given by the single point x and the other by a piece of grid, so that $(\pi_{0,X})_x$ is isomorphic to a set with two points, since the inclusion of the family of those open neighbourhoods of x in the category of all open neighbourhoods of x is final.

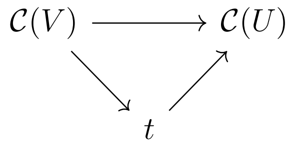
Remark 3. *Again, contrary to what happens for sheaves of sets, it's not necessarily true that a morphism of cosheaves is an isomorphism if and only it induces an isomorphism on any costalk. For example, consider the natural morphism $p : t_X \to \pi_{0,X}$ given, for any U open subset of X, by the usual quotient map. Then, if X is locally path connected and T1 (i.e. if singletons in X are closed), we get that the diagram*

$$\begin{array}{ccc} (t_X)_x & \xrightarrow{\sim} & \{x\} \\ \downarrow{\scriptstyle p_x} & & \downarrow{\scriptstyle id} \\ (\pi_{0,X})_x & \xrightarrow{\sim} & \{x\} \end{array}$$

is commutative, so p_x must be an isomorphism, where by p_x we mean the morphism induced by p on the costalks. But clearly we see that, unless X is totally disconnected, t_X and $\pi_{0,X}$ cannot be isomorphic.

We now give the definition of a *locally trivial pre-cosheaf*, coming essentially from [Bre12]: the author introduced locally zero pre-cosheaves while struggling with a way to deal with the bad local behaviour of cosheaves (as we have just seen, for example, morphisms are not completely characterized by how they act on costalks); later we will generalize this definition to pre-costacks and study some properties of costacks satisfying this additional requirement.

Definition 12. *Let $\mathcal{C}$ be a C-valued pre-cosheaf, where C is a category with a terminal object t. We say that $\mathcal{C}$ is locally trivial if for any open subset $U \subseteq X$ and any $y \in U$ there is a neighbourhood $V \subseteq U$ of y such that we have a commutative diagram*



Notice that if $\mathcal{C}$ takes values in **Set**, the commutative diagram in the definition means that there exists an element $c_{VU} \in \mathcal{C}(U)$ such that $\epsilon_{VU}(a) = c_{VU}$ for every $a \in \mathcal{C}(V)$. Furthermore, for any open subset $\emptyset \neq U \subseteq X$, $\mathcal{C}(U) \neq \emptyset$, since one gets in particular from the definition that there exists at least a map $pt \to \mathcal{C}(U)$.

Lemma 9. *Let $\mathcal{C}$ be a locally trivial C-valued pre-cosheaf, with C a complete category. Then, for any $x \in X$, $\mathcal{C}_x \cong t$, where t is a terminal object for C.*

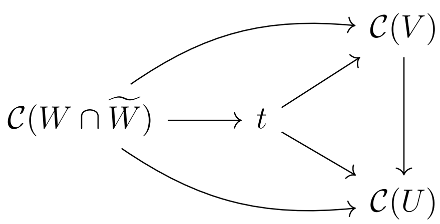
Proof. By the definition of a locally trivial pre-cosheaf, for any open subset $U \subseteq X$ there exists a morphism

$$t \to \mathcal{C}(U).$$

We will show that these morphisms add up to a natural transformation

$$\Delta_t \Rightarrow \mathcal{C}_{|\mathbf{Op}(X)_x}.$$

Since $\mathcal{C}$ is locally trivial, for any morphism $(V \subseteq U) \in \mathbf{Op}(X)_x$, there exist $(W \subseteq V), (\widetilde{W} \subseteq U) \in \mathbf{Op}(X)_x$ such that the diagram



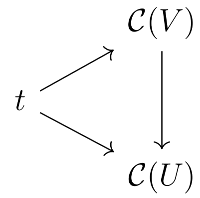
commutes. The existence of

$$t \to \mathcal{C}(W \cap \widetilde{W})$$

implies that the unique morphism

$$\mathcal{C}(W \cap \widetilde{W}) \to t$$

is an epimorphism, so

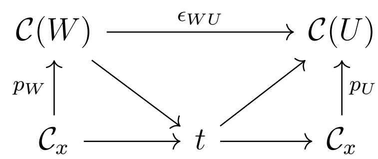


commutes, and thus we get the desired arrow $t \to \mathcal{C}_x$.

To show that this is an isomorphism, we only need to prove that the composistion

$$\mathcal{C}_x \to t \to \mathcal{C}_x$$

is equal to $id_{\mathcal{C}_x}$. Again by the local triviality of $\mathcal{C}$, for any $U \in \mathbf{Op}(X)_x$ there exists $(W \subseteq U) \in \mathbf{Op}(X)_x$ such that the diagram, in which the vertical arrows are the one of the natural transformation given in the definition of a limit,



commutes, so, since $\epsilon_{WU} \circ p_W = p_U$, by the universal property of limits the proof is done. □

Remark 4. *Note that the converse of the statement in Lemma 9 is not necessarily true: if we take the cosheaf t_X with X a T1 space, we get that for any $x \in X$ $(t_X)_x \cong pt$, but t_X is locally trivial if and only if X has the discrete topology.*

Lemma 10. *(1) the cosheaf of sets $\pi_{0,X}$ is locally trivial if and only if X is locally path connected.*

(2) the pre-cosheaf of sets $\#_X$ is locally trivial if and only if X is locally connected (in which case $\#_X$ is a cosheaf).

Proof. We only give a proof of (1), since the proof of (2) is analogous.

If X is locally path connected, then for any open subset $U \subseteq X$ and any $y \in U$ there is a neighbourhood $V \subseteq U$ of y such that V is path connected, so that $\pi_0(V) \cong pt$, and in particular $\pi_{0,X}$ must be locally trivial.

This implication is essentially [Spa89, Exercise 2.A.1]. Suppose now that $\pi_{0,X}$ is locally trivial and let $U \subseteq X$ be an open subset, with S a path connected component of U. For any $x \in S \subseteq U$, there exists $V_x \subseteq U$ open neighbourhood of x such that the map $\pi_0(V_x) \to \pi_0(U)$ is trivial, which means that for each pair of points in V_x there exists a path U connecting them, hence $V_x \subseteq S$, so S must be open. □

Proposition 6. *Let X be a locally path connected space, and $\mathcal{C}$ a cosheaf of sets on X. Then $\mathcal{C}$ is locally trivial if and only if $\mathcal{C} \cong \pi_{0,X}$.*

Proof. One implication is given by Lemma 10.

Let $\mathcal{C}$ be a locally trivial cosheaf on a locally path connected space X. For any open subset $U \subseteq X$, one has that $\mathcal{C}(U) \cong \coprod_{i \in I} \mathcal{C}(U_i)$, where the U_i's are the path connected components of U, which are open. Hence, to prove that $\mathcal{C} \cong \pi_{0,X}$ we just need to show that for any path connected open subset $V \subseteq X$, $\mathcal{C}(V)$ is a singleton.

Let V be a path connected open subset of X. Since $\mathcal{C}$ is locally trivial, there exists an open cover $\{V_i\}$ of V such that, for every $i \in I$, there exists an element $c_i \in \mathcal{C}(V)$ such that $\epsilon_{V_i V}(a) = c_i$ for any $a \in \mathcal{C}(V_i)$. Since the morphism

$$\coprod_{i \in I} \mathcal{C}(V_i) \to \mathcal{C}(V)$$

induced by the extensions is a coequalizer map, it is surjective, so one gets that $\mathcal{C}(V) = \{c_i\}_{i \in I}$: we will show that these c_i's are in fact the same element, so that $\mathcal{C}(V) \cong pt$.

Let V_i and V_j be two elements of the open cover of V, and suppose that $V_{ij} = V_i \cap V_j \neq \emptyset$, so one has that $\mathcal{C}(V_{ij}) \neq \emptyset$ because $\mathcal{C}$ is locally trivial, and for any $a \in \mathcal{C}(V_{ij})$

$$\begin{aligned} c_i &= \epsilon_{V_i V}(\epsilon_{V_{ij} V_i}(a)) \\ &= \epsilon_{V_{ij} V}(a) \\ &= \epsilon_{V_j V}(\epsilon_{V_{ij} V_j}(a)) \\ &= c_j. \end{aligned}$$

Suppose now that $V_{ij} = \emptyset$, and consider two points $x \in V_i$ and $y \in V_j$: since V is path connected there exists a continous map $\gamma : [0,1] \to V$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Arguing as in the proof that $\pi_{0,X}$ is a cosheaf, one gets that there exists a finite open cover $\mathcal{V} = \{V_i = V_1, \dots, V_j = V_n\} \subseteq \{V_i\}_{i\in I}$ of $\gamma([0,1])$ and paths γ_i, $i \in \{1, \dots, m\}$ such that each γ_i lies entirely in some $V_j \in \mathcal{V}$, $\gamma_i(1) \in V_j \cap V_k$ for some $V_j, V_k \in \mathcal{V}$ and $\gamma = \gamma_m \circ \cdots \circ \gamma_1$: since $\gamma([0,1])$ is connected, surely there exists some $V_k \in \mathcal{V}$, with $k \neq 1$, such that $V_i \cap V_k \neq \emptyset$, so by applying the same reasoning as above for a finite number of times, one gets that $c_i = c_j$. □

6.2 Costacks

Let X be a topological space, and denote by $\mathbf{Op}(X)$ the 2-category obtained by trivially enriching with identity 2-cells the small site of its open subsets and inclusion morphism. Note that for any open cover $\{U_i\}_{i\in I}$, we have that $2\varinjlim_{i\in I} U_i = \bigcup_{i\in I} U_i$.

Definition 13. *A $\boldsymbol{C}$-valued pre-costack on X is a 2-functor*

$$C : \boldsymbol{Op}(X) \to \boldsymbol{C}.$$

A morphism of pre-costacks is a 2-natural transformation of 2-functors and a transformation of morphisms of pre-costacks is just a modification of 2-natural transformations of 2-functors: thus, we have defined the 2-category of $\mathbf{C}$-valued pre-costacks on X, which we will denote by $\mathbf{PCoSt}(\mathbf{C}_X)$. We will denote by $\epsilon_{UV} : C(U) \to C(V)$ the extension 1-cell in $\mathbf{C}$ associated to the open inclusion $U \subseteq V$.

Definition 14. *A $\boldsymbol{C}$-valued pre-costack C is called a costack if for any open cover $\{U_i\}_{i\in I}$ stable by finite intersections the natural 1-cell in $\boldsymbol{C}$*

$$2\varinjlim_{i\in I} C(U_i) \to C(\bigcup_{i\in I} U_i)$$

is an equivalence.

We denote by $\mathbf{CoSt}(\mathbf{C}_X)$ the full sub 2-category of $\mathbf{PCoSt}(\mathbf{C}_X)$ whose objects are costacks.

Let C be a costack of categories. Then, dualizing the proof of Proposition 19.3.4. on [KS05], we obtain that, similarly to the definition of a cosheaf, the costack condition means that, for any open subset $U \subset X$ and any open cover $\{U_i\}_{i\in I}$ of U, the natural sequence given by the extension functors

$$\coprod_{i,j,k\in I} C(U_{ijk}) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \coprod_{i,j\in I} C(U_{ij}) \rightrightarrows \coprod_{i\in I} C(U_i) \longrightarrow C(U)$$

is exact, i.e. $C(U)$ is a 2-colimit of the diagram on the left, where we set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. For reasons that will be clear later, the condition expressed on the above diagram will be referred as "Seifert - van Kampen condition".

Example 5. *Given a cosheaf of sets $\mathcal{C}$ on X, it defines a costack of categories by trivially enriching with identity arrows the set $\mathcal{C}(U)$, for any open subset $U \subset X$. In particular, we observe that the initial cosheaf $U \mapsto \emptyset$ defines an initial object in the 2-category of costacks of categories.*

Proposition 7. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two 2-categories, with $\boldsymbol{C} \subseteq \boldsymbol{D}$, and let C be a $\boldsymbol{D}$-valued costack. Suppose that the inclusion 2-functor admits a 2-left adjoint $G : \boldsymbol{D} \to \boldsymbol{C}$. Then the $\boldsymbol{C}$-valued pre-costack $G \circ C$ is a costack.*

Proof. This is clear because 2-left adoints commute (up to equivalence) with 2-colimits. □

Remark 5. *In the special case of $M^{-1} : \boldsymbol{Cat} \to \boldsymbol{Grpd}$, since $M^{-1}(C) \cong C$ whenever C is a groupoid and since a 2-colimit of groupoids in $\boldsymbol{Cat}$ is already a groupoid, we get that a pre-costack of groupoids is a costack if and only if it is a costack of categories.*

As for cosheaves, one has:

Lemma 11. *Let C be a pre-costack of categories. Then C is a costack if and only if*

1. *C commutes up to equivalence with filtrant 2-colimits, i.e. the morphism*
$$2\varinjlim_{i \in I} C(U_i) \to C(\bigcup_{i \in I} U_i)$$
is an equivalence for any open cover stable by finite unions,

2. *for any pair of open subsets $U, V \subseteq X$, the 2-cell in $\boldsymbol{Cat}$*
$$\begin{array}{ccc} C(U \cap V) & \longrightarrow & C(V) \\ \downarrow & \swarrow & \downarrow \\ C(U) & \longrightarrow & C(U \cup V) \end{array}$$
is 2-cocartesian.

Proof. Since 2-colimits of categories over a filtrant small category are equivalent to colimits (see [Pir15]), the proof goes exactly as for cosheaves. □

Consider now the strict 2-functor $\Pi_{1,X}$ defined by taking any open subset $U \subseteq X$ to its fundamental groupoid, with obvious extensions. The next corollary justifies why we called the costack condition the "Seifert - van Kampen condition".

Corollary 5. $\Pi_{1,X}$ *is a costack (of categories and) of groupoids.*

Proof. We prove that $\Pi_{1,X}$ is a costack of groupoids, which by remark implies that it is a costack of categories.

By the version of the Seifert- van Kampen theorem for the fundamental groupoid proven in [Bro06, 6.7.2], for any pair of open subsets $U, V \subseteq X$ one has that

$$\begin{array}{ccc} \Pi_1(U \cap V) & \longrightarrow & \Pi_1(V) \\ \downarrow & & \downarrow \\ \Pi_1(U) & \longrightarrow & \Pi_1(U \cup V) \end{array}$$

is cocartesian thus, since Π_1 commutes with filtrant colimits ([Pir15, Theorem 2.5]), we get the thesis. □

From now on, we will denote by $\Pi_{1,X}$ the restriction of $\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$ to $\mathbf{Op}(X)$ and call it the *fundamental costack of* X.

6.3 Cosheaf of connected components of a costack

In section 1 we have defined the set of connected components of a small category I, $\pi_0(I)$, and we have seen that $\pi_0(I) \cong \varinjlim_{i \in I} \Delta_{\{pt\}}$; using this we will define the pre-cosheaf of the *connected components of a pre-costack.*

Clearly, $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ defines a functor, but what can we say about its behaviour on the 2-cells of $\mathbf{Cat}$?

Lemma 12. *Let $F, G : I \to J$ be two functors between small categories, and let $\alpha : F \Rightarrow G$ be a natural transformation. Then $\pi_0(F) = \pi_0(G)$, i.e. π_0 is a strict 2-functor, where **Set** is seen as the sub 2-category of **Cat** of small discrete categories.*

Proof. Since α is a natural transformation, we know that, for any $i \in I$, there exists a morphism $\alpha(i) : F(i) \to G(i)$ in J, so $\mathrm{Hom}_J(F(i), G(i)) \neq \emptyset$ and $[F(i)] = [G(i)]$, where with the square brackets we indicate the equivalence class of an object in the set of connected components. Thus, for any $i \in I$

$$\pi_0(F)([i]) = [F(i)] = [G(i)] = \pi_0(G)([i]),$$

which concludes the proof. □

The previous lemma assures us that, if C is a pre-costack of categories, the composition $\pi_0 \circ C : \mathbf{Op}(X) \to \mathbf{Set}$ is a well defined 1-functor, so the following definition makes sense.

Definition 15. *Let C be a pre-costack of small categories. The pre-cosheaf of sets $\pi_0 \circ C$ is called the pre-cosheaf of connected components of C, and is denoted by $\pi_0(C)$.*

Proposition 8. *If C is a costack, then $\pi_0(C)$ is a cosheaf.*

Proof. Let $U \subseteq X$ be an open subset and $\{U_i\}_{i \in I}$ an open cover of U stable by finite intersections. By applying π_0 to the functor

$$2 \varinjlim_{i \in I} C(U_i) \to C(U)$$

and by the universal property of colimits, we get a commutative diagram

$$\begin{array}{ccc} \varinjlim_{i \in I} \pi_0(C(U_i)) & \longrightarrow & \pi_0(C(U)) \\ \downarrow & \nearrow & \\ \pi_0(2 \varinjlim_{i \in I} C(U_i)) & & \end{array}$$

where the diagonal arrow is an isomorphism since C is a costack. By Proposition 5, one gets

$$\begin{aligned} \varinjlim_{i \in I} \pi_0(C(U_i)) &\cong \varinjlim_{i \in I} \varinjlim_{C(U_i)} \Delta_{\{pt\}} \\ &\cong \varinjlim_{2 \varinjlim_{i \in I} C(U_i)} \Delta_{\{pt\}} \\ &\cong \pi_0(2 \varinjlim_{i \in I} C(U_i)), \end{aligned}$$

so $\pi_0(C)$ is indeed a cosheaf. □

Example 6. *We have that $\pi_0(\Pi_{1,X}) = \pi_{0,X}$, since $Ob(\Pi_1(U)) = U$ and $Hom_{\Pi_1(U)}(x, y) \neq \emptyset$ if and only if there exists a path in U from x to y: in particular one recovers that $\pi_{0,X}$ is a cosheaf of sets.*

6.4 Costalks

Dualizing the usual construction of the stalk at a point $x \in X$ of a pre-stack on X, in this section we will define the *costalk* of a pre-costack, and then an explicit calculation for the costalks of the Fundamental groupoid.

Definition 16. *Let C be a $\boldsymbol{C}$-valued pre-costack, where $\boldsymbol{C}$ is 2-complete. The costalk of C at a point $x \in X$, denoted by C_x, is the 2-limit*

$$2 \varprojlim_{\boldsymbol{Op}(X)_x} C.$$

Let us give an explicit description of the costalk of the fundamental groupoid at a generic point $x \in X$. Following [Was04, Appendix A] we get that, in the category $(\mathbf{\Pi}_{1,X})_x = 2 \varprojlim_{\mathbf{Op}(X)_x} \mathbf{\Pi}_{1,X}$, the objects are couples $((y_U)_{x \in U}, \gamma^{\mathbf{y}})$ where

1. $y_U \in U$
2. $\gamma^{\mathbf{y}} = (\gamma^{\mathbf{y}}_{UV})$, with $\gamma^{\mathbf{y}}_{UV} : \epsilon_{UV}(y_U) = y_U \to y_V$ an homotopy class of paths in V $\forall$ $(U \subseteq V) \in \mathbf{Op}(X)_x$

with the following conditions:

- $(\gamma^{\mathbf{y}}_{UU} : y_U \to y_U) = id_{y_U}$ $\forall$ $U \in \mathbf{Op}(X)_x$
- $\gamma^{\mathbf{y}}_{VW} \circ \epsilon_{VW}(\gamma^{\mathbf{y}}_{UV}) = \gamma^{\mathbf{y}}_{UV}$ $\forall$ $(U \subseteq V \subseteq W) \in \mathbf{Op}(X)_x$;

a morphism $\varphi : ((y_U)_{x \in U}, \gamma^{\mathbf{y}}) \to ((z_U)_{x \in U}, \gamma^{\mathbf{z}})$ in $(\mathbf{\Pi}_{1,X})_x$ is given by a collection of classes of paths $\varphi_U : y_U \to z_U$ in U such that $\gamma^{\mathbf{z}}_{UV} \circ \epsilon_{UV}(\varphi_U) = \varphi_V \circ \gamma^{\mathbf{y}}_{UV}$ $\forall$ $(U \subseteq V) \in \mathbf{Op}(X)_x$.

Suppose that X is locally path connected. Then it follows immediately from the definition that, if $((y_U), \gamma^{\mathbf{y}}) \in (\mathbf{\Pi}_{1,X})_x$, $\forall$ $U \in \mathbf{Op}(X)_x$ y_U lies in the path connected component U_x of U which contains x: in fact, by definition we now that $\gamma^{\mathbf{y}}_{U_x U}$ is an homotopy class of paths from y_{U_x} to y_U, so $y_U \in U_x$.

Definition 17. *A topological space X is said to be locally 1-connected if each point $x \in X$ has a fundamental system of simply connected neighbourhoods.*

Corollary 6. *Let X be a locally 1-connected space. Then for any $x \in X$ $(\mathbf{\Pi}_{1,X})_x$ is equivalent to $\boldsymbol{Pt}$.*

Proof. Let $\mathbf{Op}^s(X)_x$ be the category of simply connected open neighbourhoods of x, and $\iota : \mathbf{Op}^s(X)_x \to \mathbf{Op}(X)_x$ the inclusion functor. If $U, V \in \mathbf{Op}^s(X)_x$ then $U \cap V$ is in $\mathbf{Op}(X)_x$, and since X is locally 1-connected, there a exists $W \in \mathbf{Op}^s(X)_x$, with $W \subseteq U \cap V$, so $\mathbf{Op}^s(X)_x$ is cofiltrant.

Furthermore, ι is final: in fact, for any $U \in \mathbf{Op}(X)_x$, being X locally 1-connected, $\iota \downarrow U$ is non-empty, and for any two objects $(V \subseteq U)$, $(V' \subseteq U) \in \iota \downarrow U$ there exists $W \in \mathbf{Op}^s(X)_x$, $W \subseteq V \cap V'$ so $(W \subseteq U) \in \iota \downarrow U$. By Corollary 2 and since $\mathbf{\Pi}_{1,X} \simeq \Delta_{\mathbf{Pt}}$ when restricted to $\mathbf{Op}^s(X)_x$, we have the thesis. □

Example 7. *In general, if X is not locally 1-connected, it's not necessarily true that $(\mathbf{\Pi}_{1,X})_x \simeq$ **Pt** for any $x \in X$. Consider again the space X given by the union of two orthogonal comb spaces, as defined in Example 4: X is clearly not locally 1-connected, since it is not locally path connected. We claim that, if $x = (0, 0)$, $(\mathbf{\Pi}_{1,X})_x$ is not equivalent to **Pt**.*

To prove this, first consider the inclusion

$$\iota : \mathcal{B} = \{B_n = B(x, 1/n) \cap X \mid n \geq 1\} \hookrightarrow Op(X)_x$$

where by $B(x, 1/n)$ we mean the open ball in $\mathbb{R}^2$ centered in x and with radius $1/n$. It easy to show that ι is final and that $\mathcal{B}$ is cofiltrant (it follows from the obvious observation that $B_n \cap B_m = B_{max(n,m)}$), so by Corollary 3

$$(\mathbf{\Pi}_{1,X})_x \simeq 2 \varprojlim_{\mathcal{B}} \Pi_{1,X}.$$

*Hence, to show that $(\mathbf{\Pi}_{1,X})_x$ is not equivalent to **Pt** it is sufficient to find two objects in $2\varprojlim_{\mathcal{B}} \Pi_{1,X}$ that are not isomorphic. We may take, for example, the objects $((x_U), id_x)$ and $((y_n), \gamma^y)$, where $x_U = x$ for any U and $((y_n), \gamma^y)$ is a couple in which $y_n = (1/(n+1), 1/(n+1))$ and, for each inclusion $B_m \subseteq B_n$, with $m \geq n$, γ^y is given by homotopy classes of paths $\gamma_{m,n} : [0, 1] \to B_n$ given by composition $\gamma_{n+1,n} \circ \cdots \circ \gamma_{m,m-1}$, where for any $i = n+1, \ldots, m$ $\gamma_{i,i-1} : [0, 1] \to B_n$ is defined by*

$$\gamma_{i,i-1}(t) = \begin{cases} (1-2t)(1/(i+1), 1/(i+1)) + 2t(1/(i+1), 1/i) & \text{if } t \in [0, 1/2], \\ (2-2t)(1/(i+1), 1/i) + (2t-1)(1/i, 1/i) & \text{if } t \in [1/2, 1]. \end{cases}$$

It is straightforward that these two objects cannot be connected by any morphism in $2\varprojlim_{\mathcal{B}} \Pi_{1,X}$ since x and the y_n's are not in the same path connected component.

6.5 Stack of representations of a costack

Since costacks are the dual notion of stacks, it comes to mind that one might check the costack condition by verifying the stack condition after applying a contravariant hom functor: this statement will be made more precise later in a lemma which is an easy consequence of the 2-Yoneda lemma, but before going into it, we recall the definition of a stack and prove an easy fact that will be used to prove the main result about locally trivial costacks of the last section.

Recall that, for a topological space X and a 2-category $\mathbf{C}$, a $\mathbf{C}$-valued pre-stack on X is just a 1-contravariant 2-functor $F : \mathbf{Op}(X)^{op} \to \mathbf{C}$.

Definition 18. *A $\boldsymbol{C}$-valued pre-stack F is called a costack if for any open cover $\{U_i\}_{i\in I}$ stable by finite intersections the natural 1-cell in $\boldsymbol{C}$*

$$F(\bigcup_{i\in I} U_i) \to 2\varprojlim_{i\in I} F(U_i)$$

is an equivalence.

As one usually does for sheaves, we will call restrictions the functors $\rho_{UV} : F(V) \to F(U)$ corresponding to inclusions $U \subset V$.

Let F be a stack of categories. Then one can show that (for example, see [KS05, Proposition 19.3.4]), similarly to the definition of a sheaf, the stack condition means that, for any open subset $U \subset X$ and any open cover $\{U_i\}_{i\in I}$ of U, the natural sequence given by the restriction functors

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_{ij}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k\in I} F(U_{ijk})$$

is exact, i.e. $F(U)$ is a 2-limit of the diagram on the right, where we set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Let F be a pre-stack of categories, and $x, y \in Ob(F(U))$, with $U \subset X$ an open subset of X. For $V \subseteq U$, we set

$$\mathcal{H}om_{F(U)}(x,y)(V) = \mathrm{Hom}_{F(V)}(\rho_{VU}(x), \rho_{UV}(y)).$$

One may easily verify that $\mathcal{H}om_{F(U)}(x,y)$ defines a presheaf of sets called the *internal hom-presheaf* associated to F, which is a sheaf when F is a stack (for a proof, see [KS05, Proposition 19.4.5 (i)]).

Proposition 9. *Let $\alpha : F \Rightarrow G$ be a morphism of stacks. Then α is an equivalence if and only if for any open $U \subset X$ $\alpha_U : F(U) \to G(U)$ is essentially surjective and the morphisms*

$$\alpha_x : \mathcal{H}om_{F(U)}(a,b)_x \to \mathcal{H}om_{F(U)}(\alpha_U(a), \alpha_U(b))_x$$

induced by α on the stalks of the respective internal hom-sheaves are isomorphisms for any $x \in X$.

Proof. This is an easy consequence of Lemma 17 and the well known fact that a morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphism on each stalk. □

For a $\mathbf{C}$-valued pre-costack C on X, we denote by $\mathbf{Hom_C}(C, Q)$ the composition of C with the contravariant 2-functor $\mathbf{Hom_C}(-, Q) : \mathbf{C} \to \mathbf{Cat}$, and by $\circ\epsilon_{VU}$ the restriction associated to the open inclusion $V \subseteq U$, where $\epsilon_{VU} : C(V) \to C(U)$ is the corresponding extension.

Lemma 13. *A* $\boldsymbol{C}$*-valued pre-costack* C *on* X *is a costack if and only if for any object* $Q \in Ob\boldsymbol{C}$ *the pre-stack of categories* $\boldsymbol{Hom_C}(C, Q)$ *is a stack.*

Proof. Let $U \subseteq X$ be an open subset and and $\{U_i\}_{i\in I}$ an open cover of U stable by finite intersections. For any $Q \in \mathrm{Ob}\mathbf{C}$ the 1-cell

$$2\varinjlim_{i\in I} C(U_i) \to C(U)$$

induces a functor

$$\mathbf{Hom_C}(C(U), Q) \to \mathbf{Hom_C}(2\varinjlim_{i\in I} C(U_i), Q) \cong 2\varprojlim_{i\in I} \mathbf{Hom_C}(C(U_i), Q).$$

By the 2-Yoneda lemma, the composition of the two arrows above is an equivalence if and only if C is a costack. □

To a pre-costack of categories, we can associate a particular pre-stack of this kind which will turn out to be very useful afterwards.

Definition 19. *If* C *is a pre-stack of categories, we set*

$$\boldsymbol{Hom_{Cat}}(C, \boldsymbol{Set}) = \boldsymbol{Rep_{Set}}(C)$$

and we call it the pre-stack of ***Set****-valued representations of* C*.*

We now give the definition of the *Cauchy completion* of a pre-costack, which will turn out to be very useful later.

Definition 20. *Let* C *be a pre-costack of categories. We will call the Cauchy completion of* C*, and denote it by* $\widehat{C}$*, the pre-costack of Cauchy complete categories given by the composition of* C *with the 2-functor* $\widehat{(-)} : \boldsymbol{Cat} \to \boldsymbol{CauCat}$ *sending a small category to its Cauchy completion.*

Remark 6. *Since* $\widehat{(-)} : \boldsymbol{Cat} \to \boldsymbol{CauCat}$ *is 2-left adjoint to the inclusion* $\boldsymbol{CauCat} \to \boldsymbol{Cat}$*, by Proposition 7 we get that if* C *is a costack, then so will be* $\widehat{C}$*.*

The next result will be a generalization of Proposition 2 for pre-costacks.

Proposition 10. *Let* C_1, C_2 *be two pre-costacks of categories. Then* $\widehat{C_1}$ *and* $\widehat{C_2}$ *are equivalent if and only if* $\boldsymbol{Rep_{Set}}(C_1)$ *and* $\boldsymbol{Rep_{Set}}(C_2)$ *are equivalent.*

Proof. Let now $\varphi : \widehat{C_1} \to \widehat{C_2}$ be an equivalence of pre-costacks, then clearly it induces an equivalence

$$\circ\varphi : \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_2}) \to \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_1}).$$

By the equivalence in Remark 1, we get that the bottom horizontal arrow in the diagram

$$\begin{array}{ccc} \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_2}) & \xrightarrow{\circ\varphi} & \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_1}) \\ \big\downarrow{\simeq} & & \big\downarrow{\simeq} \\ \mathbf{Rep}_{\mathbf{Set}}(C_2) & \longrightarrow & \mathbf{Rep}_{\mathbf{Set}}(C_1) \end{array}$$

is a well defined equivalence of pre-stacks.

Suppose now that we have an equivalence $\psi : \mathbf{Rep}_{\mathbf{Set}}(C_1) \to \mathbf{Rep}_{\mathbf{Set}}(C_2)$, so, for any open subset $U \subseteq X$, an equivalence of categories

$$\psi(U) : \mathbf{Rep}_{\mathbf{Set}}(C_1(U)) \to \mathbf{Rep}_{\mathbf{Set}}(C_2(U))$$

which restricts, by the arguments of the proof of Proposition 2, to the equivalence of categories $\psi(U)_{|\widehat{C_1(U)}^{op}} : \widehat{C_1(U)}^{op} \to \widehat{C_2(U)}^{op}$: we have to show that these functors add up to a 2-natural transformation, and by applying to it the 2-functor $(-)^{op}$ we will get what we need.

Lemma 2 implies that, if $\varphi(U)$ is a quasi inverse of $\psi(U)_{|\widehat{C_1(U)}^{op}}$, then $\psi(U)_{|\widehat{C_1(U)}^{op}} \cong (\circ\varphi(U))_{|\widehat{C_1(U)}^{op}}$, and this implies that $\psi(U) \cong \circ\varphi(U)$ because they coincide on the subcategory of the representable functors of $\mathbf{Rep}_{\mathbf{Set}}(\widehat{C_1(U)})$ and, since they are equivalences, they commute with colimits, and it is well known (for example, see [ML13, Ch III, §7, Theorem 1]) that every presheaf is a colimit of representables. Since ψ is a 2-natural transformation, by pasting 2-cells we get that, for any open inclusion $V \subseteq U$, the square

$$\begin{array}{ccc} \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_1(U)}) & \xrightarrow{\circ\varphi(U)} & \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_2(U)}) \\ \big\downarrow{\circ\widehat{\epsilon^1_{VU}}} & & \big\downarrow{\circ\widehat{\epsilon^2_{VU}}} \\ \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_1(V)}) & \xrightarrow{\circ\varphi(V)} & \mathbf{Rep}_{\mathbf{Set}}(\widehat{C_2(V)}) \end{array}$$

commutes up to natural isomorphism, thus so does

$$\begin{array}{ccc} \widehat{C_2(V)}^{op} & \xrightarrow{\varphi(V)} & \widehat{C_1(V)}^{op} \\ \big\downarrow{\widehat{\epsilon^2_{VU}}^{op}} & & \big\downarrow{\widehat{\epsilon^1_{VU}}^{op}} \\ \widehat{C_2(U)}^{op} & \xrightarrow{\varphi(U)} & \widehat{C_1(U)}^{op} \end{array}$$

and the coherences of φ follow again from the coherences of ψ. □

Lemma 14. *Let C be a pre-costack of categories. Then $\widehat{C}$ is a costack if and only if $\boldsymbol{Rep}_{\boldsymbol{Set}}(C)$ is a stack. In particular, a pre-costack C of Cauchy complete categories is a costack if and only if $\boldsymbol{Rep}_{\boldsymbol{Set}}(C)$ is a stack.*

Proof. This is an obvious consequence of the theorem in [BRD94, 6.5.11] since the completion of a Cauchy complete category is the category itself. $\square$

6.6 The functor ν

It is well known that the monodromy functor

$$\mu : \mathbf{LcSh}_X \longrightarrow \mathbf{Rep}_{\mathbf{Set}}(\Pi_{1,X})$$

defines an equivalence of stacks when the base space X is locally 1-connected, called the *monodromy equivalence*, and one may explicitely describe its quasi inverse

$$\nu : \mathbf{Rep}_{\mathbf{Set}}(\Pi_{1,X}) \longrightarrow \mathbf{LcSh}_X,$$

as it is done for example in [Spa89, Exercise 6.F.1]. In this section we define this ν for the representations of a general pre-costack and show that, if it is a costack, ν has image in the stack of sheaves on X.

Let C be a pre-costack of small categories and D a complete category. Given a functor $F : C(X) \to D$, the assignement

$$X \supseteq U \mapsto \nu(F)(U) = \varprojlim_{C(U)} F_{|U}$$

defines a D-valued presheaf on X.

Proposition 11. *If C is a costack, then $\nu(F)$ is a sheaf.*

Proof. Let $U \subseteq X$ be an open subset and $\{U_i\}_{i\in I}$ an open cover of U stable by finite intersections, and consider the natural morphism in D

$$\nu(F)(U) \to \varprojlim_{i\in I} \nu(F)(U_i).$$

By definition, $\nu(F)$ is a sheaf if and only if this morphism is an isomorphism.

By universal property, it factors as

$$\begin{array}{ccc} \varprojlim_{C(U)} F_{|U} & \longrightarrow & \varprojlim_{i\in I} \varprojlim_{C(U_i)} F_{|U_i} \\ \uparrow & \overset{\cong}{\nearrow} & \\ \varprojlim_{2 \varinjlim_{i\in I} C(U_i)} F_{|U} & & \end{array}$$

where the isomorphism is given by Proposition 5, and the arrow on the left is induced by

$$2 \varinjlim_{i \in I} C(U_i) \to C(\bigcup_{i \in I} U_i)$$

so it is an isomorphism since C is a costack, hence we get the thesis. $\square$

Let $\mathrm{PSh}(D_X)$ denote the category of D-valued presheaves on X. It is easy to check that the assignement

$$(F : C(X) \to D) \mapsto \nu(F)$$

defines a functor

$$\mathbf{Hom_{Cat}}(C(X), D) \to \mathrm{PSh}(D_X)$$

compatible with the restriction functors. Hence one gets a functor of pre-stacks on X

$$\nu : \mathbf{Hom_{Cat}}(C, D) \to \mathbf{PSh}(D_X).$$

Corollary 7. *If C is a costack, then ν factors through the stack of D-valued sheaves $\boldsymbol{Sh}(D_X)$. In particular, we get*

$$\nu : \boldsymbol{Rep_{Set}}(C) \to \boldsymbol{Sh}_X.$$

7 Locally trivial costacks

We propose in this section the definition of a *locally trivial pre-costack*, which is a categorical counter part of the one given by Bredon [Bre12], and prove some results implying that, if X is locally 1-connected, the Cauchy completion of a locally trivial costack is equivalent to the Fundamental costack on X.

7.1 Locally trivial costacks and connectedness

Definition 21. *We say that a pre-costack $P : \boldsymbol{Op}(X) \to \boldsymbol{Cat}$ is locally trivial if for any open subset $U \subseteq X$ and any $y \in U$ there is a neighbourhood $V \subseteq U$ of y such that we have a quasi commutative diagram*

$$\begin{array}{ccc} P(V) & \longrightarrow & P(U) \\ & \searrow \quad \nearrow & \\ & \boldsymbol{Pt} & \end{array}$$

which means that there exists an object $c_{VU} \in P(U)$ such that the functor $\epsilon_{V,U} : P(V) \to P(U)$ is isomorphic to the constant functor $\Delta_{c_{VU}}$.

Example 8. *The fundamental costack $\Pi_{1,X}$ gives an example of locally trivial costack, when the space X is locally 1-connected.*

Lemma 15. *Let P be a locally trivial pre-costack on X. Then $\pi_0(P)$ is a locally trivial pre-cosheaf.*

Proof. Since C is locally trivial for any open subset $U \subseteq X$ and any $y \in U$ there is a neighbourhood $V \subseteq U$ of y such that the diagram

$$\begin{array}{ccc} P(V) & \longrightarrow & P(U) \\ & \searrow \quad \nearrow & \\ & \mathbf{Pt} & \end{array}$$

commutes up to natural isomorphism. By applying π_0 one gets a strictly commutative diagram

$$\begin{array}{ccc} \pi_0(P)(V) & \longrightarrow & \pi_0(P)(U) \\ & \searrow \quad \nearrow & \\ & pt & \end{array}$$

which concludes the proof. $\square$

Corollary 8. *Let X be a locally path connected space, and let C be a locally trivial costack on X. Then $\pi_0(C) \cong \pi_{0,X}$. In particular, this means that if $U \subseteq X$ is a path connected open subset, then $C(U)$ is a connected category.*

Proof. Since C is a locally trivial costack, by Lemma 15 and Proposition 8 $\pi_0(C)$ is a locally trivial cosheaf, but any locally trivial cosheaf on a locally path connected space is isomorphic to $\pi_{0,X}$ by Proposition 6, so we get the thesis. □

7.2 Representations of a locally trivial costack

Recall that we defined the functor of stacks

$$\nu : \mathbf{Rep}_{\mathbf{Set}}(C) \longrightarrow \mathbf{Sh}_X.$$

We show that, for locally trivial costacks, the target is indeed $\mathbf{LcSh}_X$.

For the rest of the section we assume that X is locally path connected.

Lemma 16. *Let C be a locally trivial costack on X. Then, for any functor $F : C(U) \to$ **Set**, the sheaf $\nu(F)$ on U is locally constant.*

Proof. Let $y \in U$ and let $V \subseteq U$ be an open neighbourhood of y such that the functor $\epsilon_{VU} : C(V) \to C(U)$ is equivalent to a constant functor $\Delta_{c_{VU}}$: we will show that $\nu(F)_{|V}$ is constant, which implies that $\nu(F)$ is locally constant.

Let $W \subseteq V$ be an open subset and let $\{W_k\}_{k\in J}$ be its open connected components, then we have

$$
\begin{aligned}
\nu(F)(W) &= \varprojlim_{C(W)} F \circ \epsilon_{WU} \\
&\cong \varprojlim_{C(W)} F \circ \epsilon_{VU} \circ \epsilon_{WV} \\
&\cong \varprojlim_{C(W)} F \circ \Delta_{c_{VU}} \circ \epsilon_{WV} \\
&\cong \varprojlim_{C(W)} F \circ \Delta_{c_{VU}} \\
&\cong \varprojlim_{C(W)} \Delta_{F(c_{VU})} \\
&\cong \varprojlim_{2\varinjlim_{k\in J} C(W_k)} \Delta_{F(c_{VU})} \\
&\cong \varprojlim_{k\in J} \varprojlim_{C(W_k)} \Delta_{F(c_{VU})} \\
&\cong \prod_{k\in J} \varprojlim_{C(W_k)} \Delta_{F(c_{VU})} \\
&\cong \prod_{k\in J} F(c_{VU})
\end{aligned}
$$

where the last isomorphism is given by the fact that, for the previous lemma, $C(W_k)$ is connected. □

We use the two lemmas of this section to prove another result about locally trivial costacks, which is a generalization of the well-known result about the representations of the Fundamental Groupoid of a space; our proof is somewhat inspired by the proof of Theorem 5.7 in [Tre09].

Theorem 3. *Let C be a locally trivial costack on X. Then the functor of stacks*

$$\nu : \boldsymbol{Rep_{Set}}(C) \longrightarrow \boldsymbol{LcSh}_X$$

is an equivalence.

Proof. First we show that, for any open $U \subseteq X$, $\nu(U)$ is fully faithful.

We show that, for any pair $F, G \in \mathbf{Rep_{Set}}(C(U))$,

$$\mathcal{H}om_{\mathbf{Rep_{Set}}(C(U))}(F, G) \xrightarrow{\ \nu\ } \mathcal{H}om_{\mathbf{LcSh}_U}(\nu(F), \nu(G))$$

is an isomorphism of sheaves by proving that it induces an isomorphism on any stalk.

Since X is locally connected and $\nu(F), \nu(G)$ are locally constant sheaves, then $\mathcal{H}om_{\mathbf{LcSh}_U}(\nu(F), \nu(G))$ must be locally constant and for any $x \in U$, if $V \subseteq U$ is a connected open neighbourhood of x which trivializes C (so restricted to which $\nu(F)$ and $\nu(G)$ are constant), we have

$$\begin{aligned}\mathcal{H}om_{\mathbf{LcSh}_U}(\nu(F), &\nu(G))_x \cong \\ &\cong \mathrm{Hom}_{\mathbf{LcSh}_V}(\nu(F)_{|V}, \nu(G)_{|V}) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(\nu(F)(V), \nu(G)(V)) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(F(c_{VU}), G(c_{VU})).\end{aligned}$$

Consider now $\mathcal{H}om_{\mathbf{Rep}_{\mathbf{Set}}(C(U))}(F, G)$: if V is the same open as before, we have that

$$\begin{aligned}\mathcal{H}om_{\mathbf{Rep}_{\mathbf{Set}}(C(U))}(F, &G)(V) = \\ &= \mathrm{Hom}_{\mathbf{Rep}_{\mathbf{Set}}(C(V))}(F \circ \epsilon_{VU}, G \circ \epsilon_{VU}) \\ &\cong \mathrm{Hom}_{\mathbf{Rep}_{\mathbf{Set}}(C(V))}(\Delta_{F(c_{VU})}, \Delta_{G(c_{VU})}) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(F(c_{VU}), G(c_{VU}))\end{aligned}$$

and $\mathcal{H}om_{\mathbf{Rep}_{\mathbf{Set}}(C(U))}(F, G)$ is locally constant, since if $W \subseteq V$ and $\{W_k\}_{k \in J}$ are its connected components

$$\begin{aligned}\mathcal{H}om_{\mathbf{Rep}_{\mathbf{Set}}(C(U))}(F, G)(W) &= \\ &\cong \prod_{k \in J} \mathrm{Hom}_{\mathbf{Rep}_{\mathbf{Set}}(C(V))}(F \circ \epsilon_{W_k U}, G \circ \epsilon_{W_k U}) \\ &\cong \prod_{k \in J} \mathrm{Hom}_{\mathbf{Set}}(F(c_{VU}), G(c_{VU})),\end{aligned}$$

so, in conclusion,

$$\begin{aligned}\mathcal{H}om_{\mathbf{Rep}_{\mathbf{Set}}(C(U))}(F, &G)_x \cong \\ &\cong \mathcal{H}om_{\mathbf{Rep}_{\mathbf{Set}}(C(U))}(F, G)(V) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(F(c_{VU}), G(c_{VU})) \\ &\cong \mathcal{H}om_{\mathbf{LcSh}_U}(\nu(F), \nu(G))_x\end{aligned}$$

hence $\nu(U)$ is fully faithful.

To see that $\nu(U)$ is also essentially surjective, we begin by showing that constant sheaves on U are in its essential image, and then we glue some of them to get the full proof.

Let $S_U \in \mathbf{LcSh}_U$, $S \in \mathbf{Set}$, be a constant sheaf, and let $\Delta_S : C(U) \to \mathbf{Set}$ be the constant functor with value S; then, $\forall\ V \subseteq U$

$$\begin{aligned}\nu(\Delta_S)(V) &= \varprojlim_{C(V)} \Delta_S \circ \epsilon_{VU} \\ &= \varprojlim_{C(V)} \Delta_S \\ &\cong \varprojlim_{2\varinjlim_{i\in I} C(V_i)} \Delta_S \\ &\cong \varprojlim_{i\in I} \varprojlim_{C(V_i)} \Delta_S \\ &\cong \prod_{i\in I} \varprojlim_{C(V_i)} \Delta_S \\ &\cong \prod_{i\in I} S \cong S_U(V)\end{aligned}$$

where the V_i's are the connected components of V and the last but one isomorphism is given by the fact that $C(V_i)$ is a connected category.

Let $\mathcal{F} \in \mathbf{LcSh}_U$, and $\{U_i\}_{i\in I}$ an open covering of U such that $\mathcal{F}_{|U_i}$ is constant: then, for what we have shown before, we can find a functor $F_{U_i} : C(U_i) \to \mathbf{Set}$ such that $\nu(F_{U_i}) \cong \mathcal{F}_{|U_i}$, and since $\forall\ i,j \in I\ \nu(F_{U_i} \circ \epsilon_{U_{ij}U_i}) \cong \mathcal{F}_{|U_i|U_{ij}} = \mathcal{F}_{|U_{ij}} \cong \nu(F_{U_{ij}})$, then $F_{U_i} \circ \epsilon_{U_{ij}U_i} \cong F_{U_{ij}}$, and so, being $\mathbf{Rep}_{\mathbf{Set}}(C)$ a stack, we obtain a functor $F : C(U) \to \mathbf{Set}$ such that $\nu(F) = \mathcal{F}$. □

Corollary 9. *Let X be a locally 1-connected space and C a locally trivial costack on X. Then $\widehat{C} \simeq \Pi_{1,X}$.*

Proof. In light of the previous theorem, since C and $\Pi_{1,X}$ are both locally trivial, they must have equivalent Cauchy completions; the proof is finished by noticing the simple fact that any groupoid is Cauchy complete. □

The content of the corollary may be restated by saying that, on a locally 1-connected space X, the fundamental costack, up to equivalence, is the *unique locally trivial costack of groupoids*, hence it is uniquely determined by the properties of being locally trivial and taking values in **Grpd**.

In [Pir15], with further assumptions on the space X, is proved the following interesting result:

Theorem 4. *Let X be a topological space such that any open subset $U \subseteq X$ has an open cover $\{U_i\}_{i\in I}$ such that, for any $i,j,k \in I$, $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$ are 1-connected. Then $\Pi_{1,X}$ is 2-terminal, that is, for any costack C of groupoids on X, we have an equivalence of categories*

$$\boldsymbol{Hom}_{\boldsymbol{CoSt}(\boldsymbol{Grpd}_X)}(C, \Pi_{1,X}) \simeq \boldsymbol{Pt}.$$

Note that the hypothesis in Theorem 4 implies that X is locally 1-connected, so if C is a locally trivial costack on a space X satisfying those properties, then $\widehat{C}$ is 2-terminal.

8 Appendix: 2-categories and 2-functors

In this appendix we will deal with the generalities of the theory of *2-categories*.

We recall briefly that a 2-category is just a category enriched over **Cat**, which is essentialy the data of a class of objects (or 0-cells), a set of morphisms (or 1 cells) for any two objects and a set of transformations (or 2-cells) for any two parallel morphisms, with two different compositions and identities for 1-cells and 2-cells (called respectively horizontal and vertical) satisfying some compatibilities: topological spaces, with continous maps and homotopies, and (small) categories, with functors and natural transformations, constitute the most famous examples of 2-categories.

This definition could lead one to think that the theory of 2-categories is just a special case of the more general *enriched category theory*, but in fact this point of view appears to be too strict, and does not allow one to include many things that appear naturally in different parts of mathematics (we will not explain this sentence in detail here, but a motivating example that one should keep in mind for our context is the theory of *stacks*). In particular, one is somehow forced to consider *lax functors* and *pseudofunctors* (or 2-functors) as morphisms between two 2-categories, and the corresponding notions of *lax limits* and *pseudolimits* (2-limits), instead of the usual enriched (or strict) functors and limits. To be short, 2-functors could be described as functors respecting compositions and identities only up to coherent invertible 2-cells, and they appear to be "the right notion" of morphism for the so called *bicategories*. Even if we won't touch the theory of bicategories, we will follow the "philosophy" arising from it (i.e., *the right notions are the ones defined up to equivalence*) and mention *biadjointness* of 2-functors, *bilimits* and a "weak" version of the enriched Yoneda Lemma.

There are different references in the literature: an informal introduction to the theory is [BM09, A 2-Categories Companion], but for definitions and examples we will mostly draw on [BR94] and the unpublished notes [Was].

8.1 2-functors, 2-natural transformations, modifications

Definition 22. *Let $\boldsymbol{C}$, $\boldsymbol{D}$ be two 2-categories. A lax functor $F : \boldsymbol{C} \to \boldsymbol{D}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ consists of*

(1) for any object $x \in Ob(\boldsymbol{C})$, an object $F(x) \in Ob(\boldsymbol{D})$,

(2) for any morphism $f : x \to y$ of $\boldsymbol{C}$, a morphism $F(f) : F(x) \to F(y)$ of $\boldsymbol{D}$,

(3) for any transformation $\alpha : f \Rightarrow g$ of $\boldsymbol{C}$, a transformation $F(\alpha) : F(f) \Rightarrow F(g)$ of $\boldsymbol{D}$,

(4) for any object $x \in Ob(\boldsymbol{C})$, a transformation of $\boldsymbol{D}$

$$\Phi^F(x) : F(id_x) \Rightarrow id_{F(x)},$$

(5) for any three objects $x, y, z \in Ob(\boldsymbol{C})$ and any two morphisms $f : x \to y$ and $g : y \to z$, a transformation

$$\Phi^F(f, g) : F(g \circ f) \Rightarrow F(g) \circ F(f),$$

such that the following axioms are verified:

(i) for any morphism $f : x \to y$, we have $F(id_f) = id_{F(f)}$,

(ii) for any two horizontally composable transformations $\alpha : f \Rightarrow f'$ and $\alpha' : f' \Rightarrow f''$ we have $F(\alpha' \bullet \alpha) = F(\alpha') \bullet F(\alpha)$,

(iii) for any two morphisms $x \underset{f'}{\overset{f}{\rightrightarrows}} y$ and $y \underset{g'}{\overset{g}{\rightrightarrows}} z$ and transformations $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow g'$ we have

$$(F(\beta) \circ F(\alpha)) \bullet \Phi = \Phi \bullet F(\beta \circ \alpha),$$

as visualized by

$$\begin{array}{ccc} F(g \circ f) & \xRightarrow{F(\beta\circ\alpha)} & F(g' \circ f') \\ {\scriptstyle \Phi(f,g)}\Downarrow & & \Downarrow{\scriptstyle \Phi(f',g')} \\ F(g) \circ F(f) & \xRightarrow[F(\beta)\circ F(\alpha)]{} & F(g') \circ F(f') \end{array}$$

(iv) for any morphism $f : x \to y$ the equation

$$(F(f) \circ \Phi^F(x)) \bullet \phi^F(id_x, f) = id_{F(f)} = (\Phi^F(y) \circ F(f)) \bullet \phi^F(f, id_y)$$

holds,

(v) for any four objects $x, y, z, w \in Ob(\boldsymbol{C})$ and any three morphisms $f : x \to y$, $g : y \to z$, $h : z \to w$ we have

$$(F(h) \circ \Phi^F(f, g)) \bullet \Phi^F(g \circ f, h) = \\ (\Phi^F(g, h) \circ F(f)) \bullet \Phi^F(g, h \circ g)$$

which can be visualized in the following commutative diagram

$$\begin{array}{ccc} F(h \circ g \circ f) & \xrightarrow{\Phi^F(g\circ f,h)} & F(h) \circ F(g \circ f) \\ {\scriptstyle \Phi^F(f,h\circ g)} \Downarrow & & \Downarrow {\scriptstyle F(h)\circ\Phi^F(f,g)} \\ F(h \circ g) \circ F(f) & \xrightarrow[\Phi^F(g,h)\circ F(f)]{} & F(h) \circ F(g) \circ F(f) \end{array}$$

*A lax 1-contravariant functor from **C** to **D** is a lax functor $F : \boldsymbol{C}^{op} \to \boldsymbol{D}$.*

*A lax 2-contravariant functor from **C** to **D** is a lax functor $F : \boldsymbol{C}^{co} \to \boldsymbol{D}$.*

A 2-functor (or pseudofunctor) is a lax functor in which the transformations in (4) and (5) are all invertible.

A 2-functor with strict identity is a 2-functor such that all transformations in (4) are identities.

*A strict 2-functor (or **Cat** functor) is a 2-functor in which all transformations involved are identities.*

Notice that if I is a category, then it can always be seen as a 2-category by defining, for any two $x, y \in I$, $\mathbf{Hom}_I(x, y)$ to be the discrete category on the set $\mathrm{Hom}_I(x, y)$, i.e. the category with objects the elements of $\mathrm{Hom}_I(x, y)$ and with only identities for morphisms; this 2-category is said to be obtained by enriching I with identity transformations. If I is a small site and $\mathbf{C}$ is a 2-category, then a 1-contravariant 2-functor $F : I^{op} \to \mathbf{C}$ is called a *pre-stack* on I with values in $\mathbf{C}$ (it is easily seen that, for the case $\mathbf{C} = \mathbf{Cat}$, this corresponds exactly to the definition of pre-stack given in [KS05]). We begin now to look at a long list of examples of 2-functors that we will use later.

Consider a 2-category $\mathbf{C}$. For any object $x \in Ob(\mathbf{C})$, we define a strict 2-functor $\mathbf{Hom}_{\mathbf{C}}(-, x) : \mathbf{C}^{op} \to \mathbf{Cat}$, called the *contravariant Yoneda embedding of* x in the following way:

(a) for any object $y \in Ob(\mathbf{C})$, the category

$$\mathbf{Hom}_{\mathbf{C}}(-, x)(y) = \mathbf{Hom}_{\mathbf{C}}(y, x),$$

(b) for any 1-cell $f : y \to z$, a functor

$$\mathbf{Hom}_{\mathbf{C}}(f, x) : \mathbf{Hom}_{\mathbf{C}}(z, x) \to \mathbf{Hom}_{\mathbf{C}}(y, x)$$

defined by

$$(g : z \to x) \mapsto (g \circ f : y \to x), (\alpha : g \Rightarrow g') \mapsto (\alpha \circ f : g \circ f \Rightarrow g' \circ f)$$

(c) for any 2-cell $\alpha : f \Rightarrow f'$, a natural transformation

$$\mathbf{Hom_C}(\alpha, x) : \mathbf{Hom_C}(f, x) \Rightarrow \mathbf{Hom_C}(f', x)$$

defined by

$$\mathbf{Hom_C}(\alpha, x)_g = g \circ \alpha : g \circ f \Rightarrow g \circ f'.$$

We see that the one defined in (c) is indeed a natural transformation because

$$\begin{aligned}(\beta \circ f') \bullet (g \circ \alpha) = &\\ &= (\beta \bullet g) \circ (f' \bullet \alpha)\\ &= \beta \circ \alpha\\ &= (g' \bullet \beta) \circ (\alpha \bullet f)\\ &= (g' \circ \alpha) \bullet (\beta \circ f)\end{aligned}$$

by the interchange law, and it can be easily shown that $\mathbf{Hom_C}(-, x)$ verifies all the other axioms of a strict 2-functor. One can define the strict 2-functor $\mathbf{Hom_C}(x, -) : \mathbf{C} \to \mathbf{Cat}$ in the obvious way, and it is called the *covariant Yoneda embedding of* x.

Another example of strict 2-functor is given by $(-)^{op} : \mathbf{Cat}^{co} \to \mathbf{Cat}$, defined by

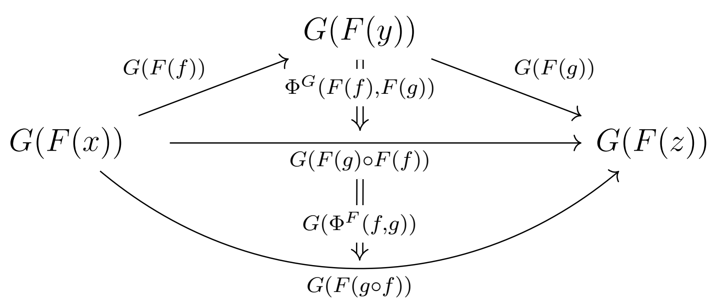
1. for any category C, $(-)^{op}(C) = C^{op}$,
2. for any functor $F : C \to D$,

$$F^{op} = op \circ F \circ op : C^{op} \to C \to D \to D^{op},$$

   where by *op* is the obvious functor from any category to its opposite.
3. for any natural transformation $\alpha : F \Rightarrow G$, $\alpha^{op} : G^{op} \Rightarrow F^{op}$ defined by $\alpha^{op}(f) = op(\alpha(op(f)))$ for any morphism f in C^{op}.

If $\mathbf{C}'$ is a 2-subcategory of $\mathbf{C}$, we get an obvious strict 2-functor $\mathbf{C}' \to \mathbf{C}$, often called inclusion 2-functor.

Let $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{E}$ be lax functors. One can easily define the composition $G \circ F : \mathbf{C} \to \mathbf{E}$ by giving as $\Phi^{G \circ F}(f, g)$, for any two morphisms $f : x \to y$, $g : y \to z$ in $\mathbf{C}$, the composition of the 2-cells



.

It is easy to check that this 2-cell verifies the axioms for a lax functor, and that the composition defined in this way is associative and has an identity.

Definition 23. *Let $F, G : \boldsymbol{C} \to \boldsymbol{D}$ be two lax functors. A lax natural transformation $\alpha : F \Rightarrow G$ is given by*

(1) for any object $x \in Ob(\boldsymbol{C})$, a morphism $\alpha_x : F(x) \to G(x)$,

(2) for any morphism $f : x \to y$ in $\boldsymbol{C}$, a natural transformation

$$\Theta_f^\alpha : G(f) \circ \alpha_x \Rightarrow \alpha_y \circ F(f)$$

as visualized by the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_x} & G(x) \\ {\scriptstyle F(f)}\big\downarrow & \Swarrow \Theta_f^\alpha & \big\downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow[\alpha_y]{} & G(y) \end{array}$$

such that the following axioms are satisfied

(i) for any object $x \in Ob(\boldsymbol{C})$, we have

$$(\alpha_x \circ \Phi^F(x)) \bullet \Theta_{id_x}^\alpha = \Phi^G(x) \circ \alpha_x,$$

as visualized by the following diagram of transformations in $\boldsymbol{D}$

$$\begin{array}{ccc} G(id_x) \circ \alpha_x & \xRightarrow{\Theta_{id_x}^\alpha} & \alpha_x \circ F(id_x) \\ {\scriptstyle \Phi^G(x)\circ\alpha_x} \searrow & & \swarrow {\scriptstyle \alpha_x \circ \Phi^F(x)} \\ & \alpha_x & \end{array}$$

(ii) for any three objects $x, y, z \in Ob(\boldsymbol{C})$ and morphisms $f : x \to y$, $g : y \to z$, we have

$$(\Theta_g^\alpha \circ F(f)) \bullet (G(g) \circ \Theta_f^\alpha) \bullet (\Phi^G(f,g) \circ \alpha_x) = \\ = (\alpha_z \circ \Phi^F(f,g)) \bullet \Theta_{g\circ f}^\alpha,$$

as visualized by

$$\begin{array}{ccc} G(g \circ f) \circ \alpha_x & \xRightarrow{\Phi^G(f,g)\circ\alpha_x} & G(g) \circ G(f) \circ \alpha_x \\ & & \Big\Downarrow {\scriptstyle G(g)\circ\Theta_f^\alpha} \\ {\scriptstyle \Theta_{g\circ f}^\alpha}\Big\Downarrow & & G(g) \circ \alpha_y \circ F(f) \\ & & \Big\Downarrow {\scriptstyle \Theta_g^\alpha \circ F(f)} \\ \alpha_z \circ F(g \circ f) & \xRightarrow[\alpha_z \circ \Phi^F(f,g)]{} & \alpha_z \circ F(g) \circ F(f) \end{array}$$

A 2-natural (or pseudonatural) transformation is a lax natural transformation such that all the 2-cells defined in (1) are invertible.

A strict 2-natural transformation (or ***Cat*** *natural transformation) is a lax natural transformation such that all the 2-cells defined in (1) are identities.*

As an example, for any morphism $f : x \to y$ in a 2-category $\mathbf{C}$, we get a 2-natural transformation

$$h_f : \mathbf{Hom_C}(-, x) \Rightarrow \mathbf{Hom_C}(-, y)$$

defined by

$$(h_f)_z = \mathbf{Hom_C}(z, f) : \mathbf{Hom_C}(z, x) \to \mathbf{Hom_C}(z, y)$$

for any object $z \in Ob(\mathbf{C})$. If $F, G, H : \mathbf{C} \to \mathbf{D}$ are lax functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ are lax natural transformations, one may define their vertical composition $\beta \bullet \alpha : F \Rightarrow H$ by attaching the 2-cells

$$\begin{array}{ccccc} F(x) & \xrightarrow{\alpha_x} & G(x) & \xrightarrow{\beta_x} & H(x) \\ {\scriptstyle F(f)}\downarrow & \swarrow\!\!\!\swarrow_{\Theta^\alpha_f} & \downarrow{\scriptstyle G(f)} & \swarrow\!\!\!\swarrow_{\Theta^\beta_f} & \downarrow{\scriptstyle H(f)} \\ F(y) & \xrightarrow[\alpha_y]{} & G(y) & \xrightarrow[\beta_y]{} & H(y) \end{array}$$

and similarly, if $F, F' : \mathbf{C} \to \mathbf{D}$ and $G, G' : \mathbf{D} \to \mathbf{E}$ are lax functors and $\alpha : F \Rightarrow F'$, $\beta : G \Rightarrow G'$ are lax natural transformations, we can define their horizontal composition $\beta \circ \alpha : G \circ F \Rightarrow G' \circ F'$ by attaching the 2-cells

$$\begin{array}{ccccc} G \circ F(x) & \xrightarrow{G(\alpha_x)} & G \circ F'(x) & \xrightarrow{\beta_{F'(x)}} & G' \circ F'(x) \\ {\scriptstyle G\circ F(f)}\downarrow & \swarrow\!\!\!\swarrow_{G(\Theta^\alpha_f)} & \downarrow{\scriptstyle G\circ F'(f)} & \swarrow\!\!\!\swarrow_{\Theta^\beta_{F'(f)}} & \downarrow{\scriptstyle G'\circ F'(f)} \\ G \circ F(y) & \xrightarrow[G(\alpha_y)]{} & G \circ F'(y) & \xrightarrow[\beta_{F'(y)}]{} & G' \circ F'(y) \end{array} .$$

Summarizing what we have just sketched (for a reference in which all the details about these facts are worked out in full generality, see [Bak]), we see that there is a 2-category in which 0-cells are (small) 2-categories, 1-cells are lax functors and 2-cells are lax natural transformations, and we will indicate it by **l2-Cat**. Other interesting 2-categories are the 2-subcategory of **l2-Cat** in which 1-cells and 2-cells are respectively 2-functors and 2-natural transformations, called **p2-Cat**, and the one in which 1-cells and 2-cells are respectively strict 2-functors and strict 2-natural transformations, called **2-Cat**.

Definition 24. *Let $F, G : \boldsymbol{C} \to \boldsymbol{D}$ be two lax functors and $\alpha, \beta : F \Rightarrow G$ two lax 2-natural transformations. A modification $\Gamma : \alpha \rightsquigarrow \beta$ consists of*

(1) for any object $x \in Ob(\boldsymbol{C})$, a transformation $\Gamma_x : \alpha_x \Rightarrow \beta_x$

such that for any two morphisms $x \overset{f}{\underset{g}{\rightrightarrows}} y$ and transformation $\gamma : f \Rightarrow g$, we have that

$$(\Gamma_y \circ F(\gamma)) \bullet \Theta_f^\alpha = \Theta_g^\beta \bullet (G(\gamma) \circ \Gamma_x)$$

which may be visualized by the commutative diagram

$$\begin{array}{ccc} G(f) \circ \alpha_x & \overset{\Theta_f^\alpha}{\Longrightarrow} & \alpha_y \circ F(f) \\ {\scriptstyle G(\gamma)\circ\Gamma_x}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \Gamma_y \circ F(\gamma)} \\ G(g) \circ \beta_x & \underset{\Theta_g^\beta}{\Longrightarrow} & \beta_y \circ F(g) \end{array}$$

Again, one can define vertical and horizontal composition of modifications just by composing the 2-cells involved in the definition; this tells us in particular that, given two small 2-categories **C** and **D**, we can define 2-categories

$$2\mathfrak{F}(\mathbf{C}, \mathbf{D}) \subset p\mathfrak{F}(\mathbf{C}, \mathbf{D}) \subset l\mathfrak{F}(\mathbf{C}, \mathbf{D})$$

which, at level of object and morpshisms are respectively $\mathbf{Hom_{2\text{-}Cat}}(\mathbf{C}, \mathbf{D})$, $\mathbf{Hom_{p2\text{-}Cat}}(\mathbf{C}, \mathbf{D})$, and $\mathbf{Hom_{l2\text{-}Cat}}(\mathbf{C}, \mathbf{D})$ and the 2-cells are modifications.

We say that two lax functors are equivalent if they are equivalent as objects of the 2-category $l\mathfrak{F}(\mathbf{C}, \mathbf{D})$. The next lemma, which we will not prove here, gives a riformulation of this definition for the case **Cat** valued lax functors that doesn't involve modifications, as a consequence to what happens in **Cat**.

Lemma 17. *Two lax functors $F, G : \boldsymbol{C} \to \boldsymbol{Cat}$ are equivalent if and only if there exists a lax natural transformation $\alpha : F \Rightarrow G$ such that $\alpha_x : F(x) \to G(x)$ is fully faithful and essentially surjective.*

8.2 Yoneda lemmas and biadjunctions

In the case of (1-)categories, the Yoneda Lemma is an extremely useful tool to recognize when two objects are isomorphic, and in general says that an object in a category is completely determined by the *set* of morphisms towards (or out of) it. In the 2-categorical setting, these morphisms constitute a category, so it surely is interesting and meaningful to try to understand how to correctly restate the Yoneda Lemma when considering these categories of morphisms up to equivalence. A direct consequence of this so called "weak" Yoneda Lemma is, as one might expect, a way to characterize when two objects in a 2-category are equivalent: we present only this result in the following lemma, which can be stated and proved in the more general context of *bicategories* (a reference can be found in [Bak]), and a "strict" version which is just a corollary of the enriched version of the classical Yoneda Lemma ([BRD94]), since we will need only these two later for our proofs.

Lemma 18. *Let $\boldsymbol{C}$ be a 2-category.*

(1) Two objects $x, y \in Ob(\boldsymbol{C})$ are equivalent if and only if the strict 2-functors $\boldsymbol{Hom_C}(-,x)$ and $\boldsymbol{Hom_C}(-,y)$ ($\boldsymbol{Hom_C}(x,-)$ and $\boldsymbol{Hom_C}(y,-)$) are equivalent in $p\mathfrak{F}(\boldsymbol{C}^{op}, \boldsymbol{Cat})$ (in $p\mathfrak{F}(\boldsymbol{C}, \boldsymbol{Cat})$);

(2) Two objects $x, y \in Ob(\boldsymbol{C})$ are isomorphic if and only if the strict 2-functors $\boldsymbol{Hom_C}(-,x)$ and $\boldsymbol{Hom_C}(-,y)$ ($\boldsymbol{Hom_C}(x,-)$ and $\boldsymbol{Hom_C}(y,-)$) are isomorphic in $\mathfrak{F}(\boldsymbol{C}^{op}, \boldsymbol{Cat})$ (in $\mathfrak{F}(\boldsymbol{C}, \boldsymbol{Cat})$).

Another notion that plays a central role in category theory is the one of adjoint functors: as before, this has a strict and a "up to equivalence" generalization for 2-categories and 2-functors.

Definition 25. *Let $L : \boldsymbol{C} \to \boldsymbol{D}$ and $R : \boldsymbol{D} \to \boldsymbol{C}$ be two 2-functors.*

We say that L is a strictly 2-left adjoint to R (or that R is strictly right 2-adjoint of L) if they are adjoint arrows in the 2-category $p\mathfrak{F}(\boldsymbol{C}, \boldsymbol{D})$ or, equivalently, if there exists an isomorphism of 2-functors

$$\boldsymbol{Hom_D}(L(-),-) \Rightarrow \boldsymbol{Hom_C}(-,R(-)).$$

We say that L is a 2-left adjoint (or biadjoint) to R if there exists a 2-natural transformation

$$\boldsymbol{Hom_D}(L(-),-) \Rightarrow \boldsymbol{Hom_C}(-,R(-))$$

which is an equivalence.

The point of view of 2-adjunctions, combined with the Yoneda Lemma, becomes useful for us for its relation with 2-limits.

8.3 Groupoids

Recall that a *groupoid* is a category in which every morphism is invertible. We indicate by **Grpd** the full 2-subcategory of **Cat** whose objects are groupoids, and hence we have a strict 2-functor **Grpd** $\to$ **Cat**; now we will define a 2-functors in the other direction, and later we will show that it is closely related to this inclusion.

Let C be a small category. A way to get a groupoid from C is to formally add inverses to any morphism in C: this construction is called a *localisation* of C with respect to the set $Mor(C)$ of all morphisms, alternatively called the *free groupoid* over C, and we will indicate it by $M^{-1}(C)$; we now briefly explain how $M^{-1}(C)$ is defined (following [Bro06]) and sketch the proof of a proposition which implies that this construction is 2-functorial.

If C is a small category, we define its *dispersion* (and indicate it by $D(C)$) to be

$$D(C) = \coprod_{f \in Mor(C)} \mathbf{2},$$

where $\mathbf{2}$ is the category with two objects, say 0 and 1, and with just a non identity morphism $0 \to 1$. Every object in $D(C)$ is of the type 0_f or 1_f, and the morphisms are identities or $(0_f \to 1_f)$, for some $f \in Mor(C)$; clearly there is a functor $P : D(C) \to C$ defined by

$$P(0_f) = dom(f), P(1_f) = tar(f), P(0_f \to 1_f) = f$$

and sending identities to corresponding identities, where $dom(f)$ and $tar(f)$ are respectively the domain and the target of f. We define $M^{-1}(\mathbf{2})$ to be the category with two objects, 0 and 1, and with two non identity morphisms $0 \leftrightarrows 1$ one inverse to the other, and

$$M^{-1}(D(C)) = \coprod_{f \in Mor(C)} M^{-1}(\mathbf{2});$$

clearly, $M^{-1}(\mathbf{2})$ and $M^{-1}(D(C))$ are both groupoids, and we get two wide inclusions (that is, surjective on the objects)

$$\mathbf{2} \hookrightarrow M^{-1}(\mathbf{2}), D(C) \hookrightarrow M^{-1}(D(C)).$$

Finally, we define $M^{-1}(C)$ by the pushout of groupoids

$$\begin{array}{ccc} Ob(D(C)) & \xrightarrow{Ob(P)} & Ob(C) \\ \downarrow & & \downarrow \\ M^{-1}(D(C)) & \longrightarrow & M^{-1}(C) \end{array}$$

where we indicate by $Ob(D(C))$ and $Ob(C)$ the discrete groupoids over the respective sets, and $Ob(P)$ is the obvious functor induced by $P : D(C) \to C$.

One can describe explicitely $M^{-1}(C)$ as the category with the same set of objects of C and with non identity morphisms given by unique representations as words of the type

$$f_n^{\epsilon_n} \cdots f_1^{\epsilon_1}$$

where $f_i \in Mor(C)$, f_i is not an identity, $\epsilon_i = \pm 1$ and for no i is true that both $f_i = f_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$. There is a canonical inclusion functor $C \hookrightarrow M^{-1}(C)$ which is the identity on the objects and sends each morphism $f \in Mor(C)$ to the corresponding word of length 1. The free groupoid has the following universal property.

Proposition 12. *Every small category C can be embedded as a full subcategory in a small groupoid $M^{-1}(C)$. Moreover,*

(1) given a functor $F : C \to D$ where D is a groupoid, F extends uniquely as a functor $M^{-1}(F) : M^{-1}(C) \to D$,

(2) given another functor $G : C \to D$, its extension $M^{-1}(G) : M^{-1}(C) \to M^{-1}(D)$, and a natural transformation $\alpha : F \Rightarrow G$, α extends uniquely as a transformation $M^{-1}(\alpha) : M^{-1}(F) \Rightarrow M^{-1}(G)$,

(3) the inclusion $C \hookrightarrow M^{-1}(C)$ is an isomorphism of categories if and only if C is already a groupoid.

Proof. If $F : C \to D$ is a functor, with D a groupoid, we can extend it uniquely to $M^{-1}(C)$ by defining

$$M^{-1}(F)(f_n^{\epsilon_n} \cdots f_1^{\epsilon_1}) = F(f_n)^{\epsilon_n} \circ \cdots \circ F(f_1)^{\epsilon_1};$$

to define $M^{-1}(\alpha)$, for any object $x \in Ob(M^{-1}(C)) = Ob(C)$ it suffices to take for $M^{-1}(\alpha)_x$ the word of length 1 α_x in $M^{-1}(D)$. □

Proposition 12 tells us in particular that

$$M^{-1}(-) : \mathbf{Cat} \to \mathbf{Grpd}$$

defines a strict 2-functor, which is a strict left 2-adjoint to the inclusion $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$.

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