Università degli studi di Padova

DEPARTMENT OF MATHEMATICS "TULLIO LEVI-CIVITA"

Master's Degree in Mathematics

Graphs encoding properties of a profinite group

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Academic year 2021-22
September 23rd, 2022

A Lara,
alla sua silenziosa ma vulcanica vitalità,
affinchè questo mio traguardo
possa essere per te d'ispirazione.

Introduction

In this master thesis, we aim to investigate the connectivity properties of some graphs that can be constructed starting from a profinite group G and a class of finite groups $\mathcal{F}$ closed under subgroups and direct products. We call *$\mathcal{F}$-groups* the groups belonging to $\mathcal{F}$ and *pro-$\mathcal{F}$-groups* the ones which are inverse limit of an inverse system of $\mathcal{F}$-groups.

In the first chapter, we start defining the *universal pro-$\mathcal{F}$-subset* of G, denoted by $I_{\mathcal{F}}(G)$, to be the set of all elements $x \in G$ such that $\langle x, y\rangle$ is a pro-$\mathcal{F}$ subgroup of G, for all $y \in G$.
After some observations about this subset, we investigate three particular cases:

- if $\mathcal{F} = \mathcal{A}$ is the family of finite abelian groups, then the universal commuting subset of G is its center,
- if $\mathcal{F} = \mathcal{N}$ is the family of finite nilpotent groups, then the universal pronilpotent subset of G is its (upper) hypercenter,
- if $\mathcal{F} = \mathcal{S}$ is the family of finite solvable groups, then the universal prosolvable subset of G is its prosolvable radical.

In the second chapter, we introduce the *non-pro-$\mathcal{F}$-graph* of G, denoted by $\Gamma_{\mathcal{F}}(G)$.
For its construction, we start considering the graph with vertex set G so that x and y are adjacent if and only if the subgroup $\langle x, y\rangle$ is not a pro-$\mathcal{F}$-group. Then, $\Gamma_{\mathcal{F}}(G)$ is the subgraph of the latter, obtained by deleting the isolated vertices, that are all elements of the universal pro-$\mathcal{F}$-subset of G.

In recent years, many authors have been studying these graphs in the finite case (see for example [9] [13] [1]), and many interesting results have turned out. Since a profinite group is a topological group that is built out of finite groups, and since many results can be extended from the finite to the profinite case, it makes sense to wonder about the properties of non-pro-$\mathcal{F}$-graphs of a profinite group.

Indeed, the connectivity of the non-$\mathcal{F}$-graph in the finite case ensures the connectivity in the profinite case, moreover, the same bound on the diameter is preserved. It follows that:

- the non-commuting graph of any profinite group is connected with a diameter of at most 2,
- the non-prosolvable graph of any profinite group is connected with a diameter of at most 2,
- the non-pronilpotent graph of any profinite group is connected with a diameter of at most 3.

In the third chapter, we introduce the *pro-$\mathcal{F}$-graph* of a profinite group, denoted by $\Lambda_{\mathcal{F}}(G)$, which is the complement of the non-pro-$\mathcal{F}$-graph of the same group. Equivalently, the pro-$\mathcal{F}$-graph of G is the graph obtained by taking as vertex set $G \setminus I_{\mathcal{F}}(G)$ and connecting two vertices if and only if they generate a pro-$\mathcal{F}$ group.

Our aim is to investigate the connectivity properties of pro-$\mathcal{F}$-graphs associated with a profinite group.
Notice that, in the definition of the pro-$\mathcal{F}$-graph, we delate the universal pro-$\mathcal{F}$-subset from the vertex set because each element would be a universal vertex (and hence, the study of the connectivity would be trivial).

Studying the connectivity of pro-$\mathcal{F}$-graphs turns out to be pretty hard. For this reason, we decide to approach the problem having a look at the *extended-pro-$\mathcal{F}$-graph* of G, denoted by $\Delta_{\mathcal{F}}(G)$. This graph has as vertex set $G \setminus \{1\}$ and the relation defining edges is the same of the one defining edges in $\Lambda_{\mathcal{F}}(G)$. Anyway, there are some troubles also in studying the connectivity of $\Delta_{\mathcal{F}}(G)$: the good behavior of finite quotients doesn't imply the good behavior of the profinite group.

For example: let $\mathcal{F} = \mathcal{A}$ to be the family of finite abelian groups; take G as the semidirect product between the group $Z_{(2)}$ of the 2-adic integers and C_2 the cyclic group of order 2, where C_2 acts on $Z_{(2)}$ with the inversion; then $\Delta_{\mathcal{A}}(G/N)$ is always connected with a diameter of at most 2 for every $N \lhd G$ open, but $\Delta_{\mathcal{A}}(G)$ turns out to be disconnected.

This is the reason why, we introduce the new concepts of *weak connectivity* and *strong connectivity*. Take $a, b \in G$ and c positive integer. For each N open normal subgroup of G, we define

$$\begin{aligned}\Omega_N(c, a, b) := \{(x_1, \ldots, x_{c+1}) \in G^{c+1} \mid\ & x_1 M = aM,\ x_{c+1} M = bM,\\ & \langle aM, x_2 M\rangle/M, \ldots, \langle x_c M, bM\rangle/M \text{ are } \mathcal{F}-\text{groups}\\ & \text{for some } M \trianglelefteq G \text{ open},\ M \le N\}.\end{aligned}$$

In other words, $\Omega_N(c, a, b)$ is the set of $(c+1)$-tuples with entries in G that project onto a path connecting aM and bM in $\Delta_{\mathcal{F}}(G/M)$, for some M open normal subgroup of G contained in N.

The extended-pro-$\mathcal{F}$-graph is said to be:

- *weakly connected* if there exists c such that:

$$\Omega_N(c, a, b) \neq \emptyset \quad \text{for every } a, b \in G,\ N \text{ open normal subgroup of } G;$$

- *strongly connected* if there exists c such that:

$$\begin{gathered}\Omega_N(c, a, b) \neq \emptyset \text{ and } \Omega_N(c, a, b) \text{ is closed}\\ \text{for every } a, b \in G,\ N \text{ open normal subgroup of } G.\end{gathered}$$

The first doesn't imply the connectivity of the graph (and this is why we call it *weak*), while the second does (and this is why we call it *strong*).

In the fourth chapter, we talk about conditions under which is not restrictive to suppose that the pro-$\mathcal{F}$-graph coincides with the extended-pro-$\mathcal{F}$ graph. For example, this is the case of pronilpotent and prosolvable graphs.

Results in the finite case, allow us to see that the pronilpotent graph of a group is not always weakly connected. In the fifth chapter, we prove a surprising and not intuitive result:

if the pronilpotent graph is weakly disconnected, then the group is virtually pronilpotent.

On the other side, thanks to the fact that the solvable graph of a finite group is always connected, we have that the prosolvable graph of a group is always weakly connected. Nobody still knows an example of a profinite group whose prosolvable graph is not strongly connected, so it keeps being an open problem. Anyway, in the sixth chapter, we construct a remarkable result which seems to suggest the likely existence of a disconnected prosolvable graph.
Let $g \in G$ and define the *solvabilizer of* g, denoted by $Sol_G(g)$, to be the set of neighbors of g in the prosolvable graph plus the unit of the group. This object has been useful in the proof of the connectivity of the solvable graph of a finite group (see [4]). While in the finite case $Sol_G(g)$ properly contains $\langle g \rangle = \langle \bar{g} \rangle$, in the sixth chapter, we show that it's possible to construct a profinite group where the equality holds.

Finally, in the seventh chapter, we talk about *profinite graphs.* These, are graphs for which the underlying set, that is the union of the vertex and edge sets, is a profinite space. For these kind of graphs it has been introduced the concept of *profinite connectivity*:

$$\textit{the profinite graph } \Gamma = \lim_{\leftarrow i \in I} \Gamma_i \textit{ is profinitely connected} \iff \textit{the finite graphs } \Gamma_i \textit{ are connected.}$$

Connectivity (in the usual sense) implies profinite connectivity but the converse doesn't hold.

Since we have found many obstacles in the study of the connectivity of pro-$\mathcal{F}$-graphs, one could ask why we have not studied the profinite connectivity instead of the usual connectivity.

Might our graphs be profinitely connected, even though not connected?

The answer is that we do not talk about profinite connectivity for pro-$\mathcal{F}$-graphs because, in general, they are not even profinite graphs. Moreover, also in the case in which a pro-$\mathcal{F}$−graph turns out to be profinite, the studying of its profinite connectivity does not lead anywhere. For example, if the prosolvable graph of a group is a profinite graph, then it is connected, and so trivially profinitely connected.

Contents

Chapter 1

Universal pro-$\mathcal{F}$-subset

Let $\mathcal{F}$ be a class of finite groups closed under subgroups and direct products.
A group is said to be

- an **$\mathcal{F}$-group** if it belongs to $\mathcal{F}$,
- a **pro-$\mathcal{F}$-group** if it is the inverse limit of an inverse system of $\mathcal{F}$-groups.

For a given profinite group G let us define the **universal pro-$\mathcal{F}$-subset** of G to be

$$I_{\mathcal{F}}(G) := \{x \in G|\ \langle x, y\rangle \text{ pro-}\mathcal{F}\text{-group for every } y \in G\}.$$

In this definition, we are considering the topological generation, which means that $\langle x, y\rangle$ is the closure of the abstract subgroup generated by the elements x and y.

We'll denote by $\mathcal{M}$ the set of all open and normal subgroups of G.

Thus, G is a compact, totally disconnected Hausdorff group, and $\mathcal{M}$ is a base of neighborhoods of the unit of the group. For every $N \in \mathcal{M}$ we define the subset $I_{N,\mathcal{F}}(G) \subseteq G$ as follows:

$$I_{N,\mathcal{F}}(G)/N := I_{\mathcal{F}}(G/N).$$

From the definition, we can describe the set $I_{N,\mathcal{F}}$ explicitly:

$$I_{N,\mathcal{F}}(G) = \{x \in G|\ xN \in I_{\mathcal{F}}(G/N)\}.$$

Hence, for any $N \in \mathcal{M}$, if we consider the natural projection to the quotient

$$\pi_N : G \to G/N,$$

we have that

$$I_{N,\mathcal{F}}(G) = \pi_N^{-1}\left(I_{\mathcal{F}}(G/N)\right).$$

Question: *Is the pro-$\mathcal{F}$-subset always a subgroup of G?*
The answer is negative, indeed if we consider

- the class $\mathcal{F}$ of finite groups in which normality is transitive,
- $G := \langle a, b, c \mid a^5 = 1, b^5 = 1, c^4 = 1, [a, b] = 1, a^c = a^2, b^c = b^3 \rangle$,

then $a, b \in I_{\mathcal{F}}(G)$ but $ab \notin I_{\mathcal{F}}(G)$ (see [14] for more datails). This means that in general the universal pro-$\mathcal{F}$-subset is not a group. But, if $I_{\mathcal{F}}(G/N)$ is a subgroup of G/N then $I_{N,\mathcal{F}}$ is a subgroup of G (which contains N).

In the following theorem, we'll show that the universal pro-$\mathcal{F}$-subset of G coincides with the intersection of all $I_{N,\mathcal{F}}$, where N ranges over the set of open normal subgroups of G.

Theorem 1.0.1. *Set* $R_{\mathcal{F}}(G) = \bigcap_{N \in \mathcal{M}} I_{N,\mathcal{F}}$. *Then* $R_{\mathcal{F}}(G) = I_{\mathcal{F}}(G)$.

Proof. We'll prove first that $I_{\mathcal{F}}(G) \subseteq R_{\mathcal{F}}(G)$. Take $x \in I_{\mathcal{F}}(G)$, then $\langle x, y \rangle$ pro-$\mathcal{F}$-group for all $y \in G$. Hence, for every $y \in G$ and every $N \in \mathcal{M}$ it holds $\langle x, y \rangle / N \cap \langle x, y \rangle \cong \langle x, y \rangle N/N \in \mathcal{F}$. This means $x \in I_{N,\mathcal{F}}(G)$ for every $N \in \mathcal{M}$. By definition, it implies $x \in R_{\mathcal{F}}(G)$. Now we'll prove the other inclusion $R_{\mathcal{F}}(G) \subseteq I_{\mathcal{F}}(G)$. If $x \in R_{\mathcal{F}}(G)$ then $x \in I_{N,\mathcal{F}}(G)$ for every $N \in \mathcal{M}$, hence $xN \in I_{\mathcal{F}}(G/N)$ for every $N \in \mathcal{M}$. This means $\langle x, y \rangle N/N \in \mathcal{F}$ for every $N \in \mathcal{M}$ and $y \in G$. Therefore $\langle x, y \rangle$ is a pro-$\mathcal{F}$-group for every $y \in G$. By definition it means $x \in I_{\mathcal{F}}(G)$. We conclude that the equality holds. □

It follows that, the universal pro-$\mathcal{F}$-subset of G is an intersection of open subsets of G, so it is closed in G (remind that open subsets of a topological group are also closed subsets).

Definition 1.0.1. *Let $\mathcal{F}$ be a family of finite groups with properties as above.*

- *$\mathcal{F}$ is said to be* ***semiregular*** *if $I_{\mathcal{F}}(H)$ is a subgroup of H for any finite group H.*
- *$\mathcal{F}$ is said to be* ***regular*** *if $I_{\mathcal{F}}(H) = \phi_{\mathcal{F}}(H)$ for every finite group H,*
  where $\phi_{\mathcal{F}}(H)$ is the intersection of all maximal $\mathcal{F}$-subgroups of H.

Notions of semiregularity and regularity can be extended to the universal pro-$\mathcal{F}$-subset of a profinite group.

Proposition 1.0.1. *If $\mathcal{F}$ is semiregular, then $I_{\mathcal{F}}(G) \leq G$ for every profinite group G.*

Proof. Since G is a profinite group we have that G/N is a finite group for every $N \in \mathcal{M}$.
By assumption $\mathcal{F}$ is semiregular, therefore $I_{\mathcal{F}}(G/N) \leq G/N$ for every $N \in \mathcal{M}$.
Fixed $N \in \mathcal{M}$, we can consider the natural projection $\pi_N : G \to G/N$, so that $I_{N,\mathcal{F}}(G) = \pi_N^{-1}\big(I_{\mathcal{F}}(G/N)\big)$ is a subgroup of G, for all $N \in \mathcal{M}$. Since intersection of subgroups is a subgroup we get $I_{\mathcal{F}}(G) \leq G$. □

Note that if $\mathcal{F}$ is semiregular, then $I_{\mathcal{F}}(G)$ is an intersection of characteristic subgroups of G, so it is characteristic.

Proposition 1.0.2. *If $\mathcal{F}$ is regular, then* $I_{\mathcal{F}}(G) = \bigcap_{K \in \mathcal{K}} K$ *for every profinite group G,*
where $\mathcal{K} := \{K \leq_{open} G \mid K/N \leq_{\mathcal{F}-max} G/N \text{ for some } N \in \mathcal{M}\}$.

Proof. Since G is a profinite group, we have that G/N is a finite group for every $N \in \mathcal{M}$.
By assumption $\mathcal{F}$ is regular, therefore $I_{\mathcal{F}}(G/N) = \phi_N(G/N)$ for every $N \in \mathcal{M}$.
Fixed $N \in \mathcal{M}$, we can consider the natural projection $\pi_N : G \to G/N$. It follows that

$$I_{N,\mathcal{F}}(G) = \pi_N^{-1}(I_{\mathcal{F}}(G/N)) = \pi_N^{-1}(\phi_{\mathcal{F}}(G/N)) = \bigcap\{H \leq G \mid \pi_N(H) \leq_{\mathcal{F}-max} G/N\}.$$

Hence, we get

$$I_{\mathcal{F}}(G) = \bigcap_{N \in \mathcal{M}} I_{N,\mathcal{F}}(G) = \bigcap\left(\bigcap_{N \in \mathcal{M}} \{K \leq G \mid K/N \leq_{\mathcal{F}-max} G/N\}\right) =$$
$$= \bigcap\{K \leq G \mid K/N \leq_{\mathcal{F}-max} G/N \ \exists N \in \mathcal{M}\} = \bigcap_{K \in \mathcal{K}} K.$$

□

1.1 Particular cases

Now that we have described the universal pro-$\mathcal{F}$-subset of a profinite group, we are ready to study this object in three particular cases:

1. when $\mathcal{F}$ is the class of finite abelian groups,
2. when $\mathcal{F}$ is the class of finite nilpotent groups,
3. when $\mathcal{F}$ is the class of finite solvable groups.

We'll analyze these cases starting from what we know in the finite case.

The easiest is the first, while the second and the third are more complicated.

1.1.1 Universal commuting subset

Let $\mathcal{A}$ be the class of finite abelian groups. By Proposition 2.1 in [2],

$$I_{\mathcal{A}}(H) = Z(H) \text{ for every } H \text{ finite group.}$$

This result can be easily extended to the profinite case:

$$I_{\mathcal{A}}(G) = \bigcap_{N \in \mathcal{M}} I_{N,\mathcal{A}}(G) = \bigcap_{N \in \mathcal{M}} \pi_N^{-1}(I_{\mathcal{A}}(G/N)) = \bigcap_{N \in \mathcal{M}} \pi_N^{-1}(Z(G/N)) = Z(G)$$

where the last equality holds because $xN \in Z(G/N)$ for all $N \in \mathcal{M}$ means that the commutator $[x, g]$ is an element of N for all $g \in G$ and for all $N \in \mathcal{M}$, therefore $[x, g] \in \bigcap_{N \in \mathcal{M}} N = 1_G$ for all $g \in G$, which means $x \in Z(G)$. Hence we have proved that:

Proposition 1.1.1. *If $\mathcal{A}$ is the class of finite abelian groups, then the universal commuting subset of the group $I_{\mathcal{A}}(G)$ coincides with the center of the group $Z(G)$, for every profinite group G.*

1.1.2 Universal pronilpotent subset

Let $\mathcal{N}$ be the class of finite nilpotent groups. By Proposition 2.1 in [2],

$$I_{\mathcal{N}}(H) = Z_\infty(H) \text{ for every } H \text{ finite group.}$$

In other words, the universal nilpotent subset of a finite group is its hypercenter.

Remind that the **hypercenter of a finite group** H is defined to be

$$Z_\infty(H) = \bigcup_i Z_i$$

where $Z_1 = Z(H)$ is the center of H and Z_{i+1} is defined recursively as $Z_{i+1}/Z_i = Z(H/Z_i)$ for all $i \geq 1$. So one could hope to generalize the definition of hypercenter from the finite to the profinite case setting

$$Z_\infty(G) := \bigcup_{n=1}^{\infty} Z_n(G)$$

where $Z_n(G)$ is the nth center of G. But since there are pronilpotent groups $G \neq 1$ with trivial center $Z(G) = Z_1(G)$ (e.g. prop-p-groups with $rank > 1$, see [22]), this definition does not yield the hoped results. The object just defined, $Z_\infty(G)$, is known as **(lower) hypercenter** of G.

Note that, if G is a profinite group, then G/N is a finite group for every $N \in \mathcal{M}$.

So, using the fact that $I_{\mathcal{N}}(G/N) = Z_\infty(G/N)$ for every $N \in \mathcal{M}$, we get that

$$I_{\mathcal{N}}(G) = \bigcap_{N\in\mathcal{M}} I_{N,\mathcal{N}}(G/N) = \bigcap_{N\in\mathcal{M}} \pi_N^{-1}\left(I_{\mathcal{N}}(G/N)\right) = \bigcap_{N\in\mathcal{M}} \pi_N^{-1}\left(Z_\infty(G/N)\right) = \bigcap_{N\in\mathcal{M}} \{x \in G|\ xN \in Z_\infty(G/N)\}.$$

This object is already known in literature as **(upper) hypercenter** of the profinite group G, it is denoted by $Z^\infty(G)$ and it has been deeply studied in [22]. Note that if G is a finite group there is no difference between its lower and upper hypercenters.

Summarizing we have proved the following proposition.

Proposition 1.1.2. *If $\mathcal{N}$ is the class of finite nilpotent groups, then the universal pronilpotent subset $I_{\mathcal{N}}(G)$ coincides with the (upper) hypercenter of the group $Z^\infty(G)$, for every profinite group G.*

Anyway, there is a characterization of the hypercenter which is the same in both the finite and the profinite case.

Theorem 1.1.1. *The hypercenter $Z^\infty(G)$ of a profinite group G is the largest subgroup of G normalizing every Sylow subgroup of G.*

Proof. Let Z be the largest subgroup of G normalizing every Sylow subgroup of G. Explicitly

$$Z := \bigcap_{S\in Syl(G)} N_G(S).$$

Take $N \lhd G$ open, then $S \cap N$ is open and normal in S. Since $S \in Syl(G)$, then $S/S \cap N \cong SN/N$ is a pro-p group. Moreover, $[G : SN] \mid [G : S]$ implies $([G : SN], p) = 1$ because $([G : S], p) = 1$.

Hence, for each $S \in Syl(G)$ and each $N \in \mathcal{M}$ we have that SN/N is a Sylow subgroup of G/N.
Moreover, we have also that SN/N is a Sylow subgroup for $N_{G/N}(SN/N) \cong N_G(S)N/N$.
This last isomorphism holds because clearly $N_G(S)N \subseteq N_G(SN)$, and, on the other hand, if we take $g \in N_G(SN)$ then $S^g \leq SN$; in particular S^g is a Sylow subgroup of SN, hence there exists $n \in G$ such that $S^g = S^n$, then $gn^{-1} \in N_G(S)$ and $g \in N_G(S)N$. For the finite group G/N is known that the hypercenter $Z_\infty(G/N)$ is the intersection of its Sylow normalizers (e.g. see [21]). Thus,

- $Z \subseteq Z^\infty(G)$ because:
  $x \in Z$ means $x \in N_G(S)$ for each $S \in Syl(G)$, hence $xN \in N_{G/N}(SN/N)$ for each $S \in Syl(G)$ and each $N \in \mathcal{M}$. This implies $xN \in Z_\infty(G/N)$ for each $N \in \mathcal{M}$, by definition of hypercenter (profinite version) we get $x \in Z^\infty(G)$.

- $Z^\infty(G) \subseteq N_G(S)N$ for each $S \in Syl(G)$ and each $N \in \mathcal{M}$ because:

  $x \in Z^\infty(G)$ implies $xN \in Z_\infty(G/N)$ for each $N \in \mathcal{M}$, therefore $xN \in N_G(S)N/N$ for each $S \in Syl(G)$ and each $N \in \mathcal{M}$. Hence, $x \in N_G(S)N$ for each $S \in Syl(G)$ and each $N \in \mathcal{M}$.

By Proposition 2.1.4 in [19], it holds that $\bigcap_{N \in \mathcal{M}} N_G(S)N = N_G(S)$.
Using this result:

$$Z = \bigcap_{S \in Syl(G)} N_G(S) \subseteq Z^\infty(G) \subseteq \bigcap_{S \in Syl(G)} \bigcap_{N \in \mathcal{M}} N_G(S)N = \bigcap_{S \in Syl(G)} N_G(S) = Z$$

Thus, we can conclude that $Z = Z^\infty(G)$ as desired. □

1.1.3 Universal prosolvable subset

Let $\mathcal{S}$ be the class of all finite and solvable groups. By Theorem 1.1 in [9],

$$I_{\mathcal{S}}(H) = R(H) \text{ for every } H \text{ finite group.}$$

In other words, the universal solvable subset of a finite group is its solvable radical.

Remind that the **solvable radical of a finite group** H is the product of all normal and solvable subgroups of H and hence is the largest (with respect to inclusion) normal and solvable subgroup of H.

Note that, if G is a profinite group, then G/N is a finite group for every $N \in \mathcal{M}$.
Using the fact that $I_{\mathcal{S}}(G/N) = R(G/N)$, we get

$$I_{\mathcal{S}}(G) = \bigcap_{N \in \mathcal{M}} I_{N,\mathcal{S}}(G) = \bigcap_{N \in \mathcal{M}} \pi_N^{-1}(I_{\mathcal{S}}(G/N) = \bigcap_{N \in \mathcal{M}} \pi_N^{-1}\left(R(G/N)\right).$$

This object is known as the **prosolvable radical** of the profinite group G, it'll be denoted by $R(G)$, and it has been deeply studied in [10]. Note that, if G is finite then $R(G)$ is the solvable radical of G.

Summarizing, we have proved that

Proposition 1.1.3. *If $\mathcal{S}$ is the class of all finite solvable groups, then the universal prosolvable subset of the group $I_{\mathcal{S}}(G)$ coincides with the prosolvable radical of the group $R(G)$, for every profinite group G.*

We can say something more about the prosolvable radical of G, giving to it another characterization.

Theorem 1.1.2. *Let G be a profinite group. Then the prosolvable radical $R(G)$ is the unique normal prosolvable subgroup which contains every normal prosolvable subgroup of G.*

Proof. Let us write R_N instead of $\pi_N^{-1}(R(G/N))$ and R instead of the prosolvable radical $R(G)$.
With these notations

$$R = \bigcap_{N \in \mathcal{M}} R_N.$$

Observe that

- $R_N \leq G$ normal for each $N \in \mathcal{M}$, because it's the preimage of $R(G/N)$ which is normal by definition of prosolvable radical. Hence, R is intersection of normal subgroups of G, so it is normal.
- Since $R \leq R_N$ for all $N \in \mathcal{M}$, we have $\pi_N(R) = RN/N \leq R(G/N) = \pi_N(R_N)$ for all $N \in \mathcal{M}$. Hence, we have RN/N solvable for all $N \in \mathcal{M}$, which implies $R \leq G$ prosolvable.
- Let K be a prosolvable and normal subgroup of G. Then $KN/N \leq G/N$ solvable for every $N \in \mathcal{M}$. By definition of solvable radical we have: $KN/N \leq R(G/N)$. Passing to the inverse images under π_N we get $KN \leq R_N$, therefore

$$K \leq \bigcap_{N \in \mathcal{M}} KN \leq \bigcap_{N \in \mathcal{M}} R_N = R.$$

We can conclude that R contains every normal prosolvable subgroup of G.

□

Chapter 2

Connectivity of the non-pro-$\mathcal{F}$-graph

Definition 2.0.1. *Given $\mathcal{F}$ class of finite groups closed under subgroups and directed products, and given G profinite group, we can consider the graph $\tilde{\Gamma}_{\mathcal{F}}(G)$ whose vertices are elements of G and $x, y \in G$ are adjacent if and only if $\langle x, y\rangle$ is not a pro-$\mathcal{F}$-group. Notice that, the universal pro-$\mathcal{F}$-subset $I_{\mathcal{F}}(G)$ is the set of isolated vertices of this graph. We define the* ***non-pro-$\mathcal{F}$-graph*** *$\Gamma_{\mathcal{F}}(G)$ of G as the subgraph of $\tilde{\Gamma}_{\mathcal{F}}(G)$ obtained by delating isolated vertices. The complement of $\Gamma_{\mathcal{F}}(G)$ is called* ***pro-$\mathcal{F}$-graph*** *of G, its vertices are elements of $G \setminus I_{\mathcal{F}(G)}$ and two different elements x, y are adjacent if and only if $\langle x, y\rangle$ is a pro-$\mathcal{F}$-group.*

We are going to investigate the connectivity of these graphs. Before proceeding, we'll remind some definitions and notations that will be useful in the sequel.

- The **distance between two vertices** in a graph is the number of edges in a shortest path connecting them. If there is no path connecting the two vertices, i.e. if they belong to different connected components, then conventionally the distance is defined as infinite. We are going to indicate the distance between two vertices x, y as $d(x, y)$.
- The **diameter of a graph** Γ is the $max_{x,y}d(x, y)$, it'll be denoted by $\delta(\Gamma)$.
- We'll say that $\mathcal{F}$ is **connected** if $\Gamma_{\mathcal{F}}(H)$ is connected for every H finite group.

Theorem 2.0.1. *If $\mathcal{F}$ is connected then $\Gamma_{\mathcal{F}}(G)$ is connected for every G profinite group.*
Moreover, if there is a bound on the diameter in the finite case then the same bound holds also in the profinite case.

Proof. Let $x, y \notin I_{\mathcal{F}}(G)$. It means that there exists an open normal subgroup N of G such that xN and yN do not belong to $I_{\mathcal{F}}(G/N)$. By assumption $\mathcal{F}$ is connected and G/N is a finite group. Let us indicate with d the distance of x and y in $\Gamma_{\mathcal{F}}(G/N)$, then there exist $z_1, \ldots, z_{d-1}$ elements of G such that $(x, z_1, \ldots, z_{d-1}, y)$ is a path connecting x and y in $\Gamma_{\mathcal{F}}(G/N)$. It means that the subgroups

$\langle xN, z_1N\rangle/N, \ldots, \langle z_{d-1}N, yN\rangle/N$ are not $\mathcal{F}$-groups. Hence $\langle x, z_1\rangle, \ldots, \langle z_{d-1}, y\rangle$ are not pro-$\mathcal{F}$-groups, whence $(x, z_1, \ldots, z_{d-1}, y)$ is a path connecting x and y in $\Gamma_{\mathcal{F}}(G)$. □

By Proposition 2.1 in [1], the non-commuting graph of a finite group is always connected and its diameter is at most 2.
By Theorem 6.2 in [9] the non-solvable graph of a finite group is always connected and its diameter is at most 2.
By Theorem 1.1 in [13] the non-nilpotent graph of a finite group is always connected and its diameter is at most 3.
Hence, from above theorems, we can state these three corollaries:

Corollary 2.0.1. *Let $\mathcal{A}$ be the class of finite abelian groups, then $\Gamma_{\mathcal{A}}(G)$ is connected and $\delta(\Gamma_{\mathcal{A}}(G) \leq 2$ for every profinite group G.*

Corollary 2.0.2. *Let $\mathcal{S}$ be the class of finite solvable groups, then $\Gamma_{\mathcal{S}}(G)$ is connected and $\delta(\Gamma_{\mathcal{S}}(G)) \leq 2$ for every profinite group G.*

Corollary 2.0.3. *Let $\mathcal{N}$ be the class of finite nilpotent groups, then $\Gamma_{\mathcal{N}}(G)$ is connected and $\delta(\Gamma_{\mathcal{N}}(G)) \leq 3$ for every profinite group G.*

Chapter 3

Connectivity of the extended-pro-$\mathcal{F}$-graph

Let G be a profinite group and $\mathcal{F}$ a family of finite groups closed under subgroups and direct products. In 2.0.1 we have defined the pro-$\mathcal{F}$-graph of G, that will be denoted by $\Lambda_{\mathcal{F}}(G)$, as the complement of the non-pro-$\mathcal{F}$-graph of G. Anyway, we can obtain $\Lambda_{\mathcal{F}}(G)$ with a direct construction.

Firstly, we consider the graph whose vertices are elements of G and two distinct vertices x, y are adjacent if and only if $\langle x, y\rangle$ is a pro-$\mathcal{F}$-group. Clearly, every element in $I_{\mathcal{F}}(G)$ is connected to every other vertex in this graph (i.e it is a universal vertex of the graph), so it makes sense to consider the more restrictive graph $\Lambda_{\mathcal{F}}(G)$, which is only defined on $G \setminus I_{\mathcal{F}}(G)$.
In this way we've recovered the pro-$\mathcal{F}$-graph of G defined in 2.0.1.

The **extended-pro-$\mathcal{F}$-graph of** G, denoted by $\Delta_{\mathcal{F}}(G)$, is the graph whose vertices are all elements of $G \setminus 1$ and two vertices are adjacent if and only if they generate a pro-$\mathcal{F}$-group.

It turns out that, studying the connectivity of the pro-$\mathcal{F}$-graph of a group is harder than we expected. For this reason, we'll start studying the connectivity properties of the extended-pro-$\mathcal{F}$-graph.

Initially, our hope would be to obtain a similar result for the extended-pro-$\mathcal{F}$-graph to the one proved for the non-pro-$\mathcal{F}$-graph in 2.0.1. Unlikely, it turns out that this is not possibile:

the good behavior of finite quotients doesn't ensure the good behavior of G.

Example 3.0.1. *Take $\mathcal{F} = \mathcal{A}$ the class of finite abelian groups.*
Let $\mathbb{Z}_{(2)}$ be the group of 2-adic integers and let $C_2 = \langle\alpha\rangle$ be the cyclic group of order 2.
Assume that $\langle\alpha\rangle$ acts on $\mathbb{Z}_{(2)}$ with the inversion. It means that for every $z \in \mathbb{Z}_{(2)}$ we have

$$z^{\alpha} = -z$$

Let's consider the group $G = \mathbb{Z}_{(2)} \rtimes \langle\alpha\rangle$. Notice that, since $Z_{(2)} = \varprojlim_{n\in\mathbb{N}} C_{2^n}$, then G is a pro-p-group that can be viewed as the inverse limit of dihedral groups of the type

$$D_{2^n} = C_{2^n} \rtimes C_2. = \langle\alpha, \beta \mid \alpha^2, \beta^{2^n}, (\beta\alpha)^2\rangle.$$

Therefore, finite quotients of G are exactly the ones described above.
If we consider $\alpha \in D_{2^n}$, then it commutes just with itself and with the involution $\gamma := \beta^{2^{n-1}}$.
This implies that, if we want to escape from α in $\Delta_{\mathcal{A}}(D^{2^n})$, we need to pass through γ. Hence there's a path of length at most 2 in the extended-commuting graph of each finite quotient of G. On the other side, α commutes just with itself in G, this means that α is an isolated vertex in the extended-commuting graph of G. Thus, we conclude that the extended-commuting graph of any finite quotients $\Delta_{\mathcal{A}}(G/N)$ is connected with diameter at most 2, but the extended-commuting graph of the group $\Delta_{\mathcal{A}}(G)$ is disconnected.

3.1 Weak and strong connectivity

We aim to understand under which assumptions we can reach the connectivity of the extended-pro-$\mathcal{F}$-graph. Take $a, b \in G$ and $c \in \mathbb{N}$. For each N open normal subgroup of G, let us define:

$$\begin{aligned}\Omega_N(c, a, b) := \{(x_1, \dots, x_{c+1}) \in G^{c+1} \mid\ & x_1 M = aM,\ x_{c+1}M = bM, \\ & \langle aM, x_2 M\rangle/M, \dots, \langle x_c M, bM\rangle/M \text{ are } \mathcal{F}-\text{groups} \\ & \text{for some } M \in \mathcal{M},\ M \leq N\}.\end{aligned}$$

It means that $\Omega_N(c, a, b)$ is the set of paths of length c connecting a, b in the graph $\Delta_{\mathcal{F}}(G/M)$ for at least one open normal subgroup M of G contained in N.

Notice that, the set $\Omega_N(c, a, b)$ is a union of closed subsets, hence it doesn't need to be closed. Therefore, it makes sense to introduce the following notations:

Definition 3.1.1. *The extended-pro-$\mathcal{F}$-graph of a profinite group, $\Delta_{\mathcal{F}}(G)$, is said to be:*

- ***c-weakly connected*** *if there exists $c \in \mathbb{N}$ such that:*

$$\Omega_N(c, a, b) \neq \emptyset \quad \textit{for every } a, b \in G,\ N \in \mathcal{M}.$$

- ***c-strongly connected*** *if there exists $c \in \mathbb{N}$ such that:*

$$\Omega_N(c, a, b) \neq \emptyset \textit{ and } \Omega_N(c, a, b) \textit{ is closed in } G^{c+1} \quad \textit{for every } a, b \in G,\ N \in \mathcal{M}.$$

Remark 3.1.1. *The weak connectivity does not imply the connectivity of the graph.*

Proof. The graph introduced in 3.0.1 is an example of extended-pro-$\mathcal{F}$-graph which is 2-weakly connected but it's not connected. □

Proposition 3.1.1. *The connectivity implies the weak connectivity of the graph.*

Proof. Let G be a profinite group, and $\mathcal{F}$ a class of finite groups such that $\Delta_{\mathcal{F}}(G)$ is connected.
We want to prove that $\Delta_{\mathcal{F}}(G)$ is weakly connected. Take $a, b \in G$ distinct and not identical.
By assumption there exists a path $(a = x_1, x_2 \dots, x_{u-1}, x_u = b)$ connecting a and b in $\Delta_{\mathcal{F}}(G)$.

Since each x_i must be not identical, and the intersection of all open normal subgroups of G is 1, there exists $L \in \mathcal{M}$ such that $x_i \notin L$ for each $i \in \{1, \dots, u\}$.
Now take $N \in \mathcal{M}$ and consider $U := L \cap N$. Note that $x_i \notin U$ for any $i \in \{1, \dots, u\}$ and $U \in \mathcal{M},\ U \leq N$. Then $(x_1U, \dots, x_uU)$ is a path in $\Delta_{\mathcal{F}}(G/U)$ connecting aU and bU, thus $\Omega_N(u, a, b) \neq \emptyset$. Since the choice of N is arbitrary, it holds for every $N \in \mathcal{M}$, that means $\Delta_{\mathcal{F}}(G)$ is weakly connected. □

Theorem 3.1.1. *The strong connectivity implies the connectivity of the graph.*

Proof. Let G be a profinite group and suppose that its extended-pro-$\mathcal{F}$-graph is c-strongly connected for some $c \in \mathbb{N}$. We want to prove that $\Delta_{\mathcal{F}}(G)$ is connected. To do that, take two distinct vertices x, y of the graph. By assumption we know that $\emptyset \neq \Omega_N(c, x, y)$ is closed for every $N \in \mathcal{M}$.
If $N_1, \dots, N_t \in \mathcal{M}$ then $N_1 \cap \dots \cap N_t \in \mathcal{M}$ and $\Omega_{N_1 \cap \dots \cap N_t}(c, x, y) \neq \emptyset$.
Thus, there exists a path $(z_1, \dots, z_{c+1})$ connecting x and y in $\Delta_{\mathcal{F}}(G/K)$ for some $K \in \mathcal{M},\ K \leq N_1 \cap \dots \cap N_t$.
Notice that, if $K \in \mathcal{M}$ is such that $K \leq N_1 \cap \dots \cap N_t$ then $K \leq N_i$ for each $i \in \{1, \dots, t\}$. This implies:

$$\emptyset \neq \Omega_{N_1 \cap \dots \cap N_t}(c, x, y) \subseteq \Omega_{N_1} \cap \dots \cap \Omega_{N_t}(c, x, y).$$

By compacteness of G:

$$\bigcap_{N \in \mathcal{M}} \Omega_N(c, x, y) \neq \emptyset.$$

Hence, there exists an element $(z_1, \dots, z_{c+1}) \in \bigcap_{N \in \mathcal{M}} \Omega_N(c, x, y)$, where $z_i \neq 1$ for each $i \in \{1, \dots, c+1\}$.
This means that for every $N \in \mathcal{M}$ there exists $K_N \in \mathcal{M},\ K_N \leq N$ for which $(z_1, \dots, z_{c+1})$ is a path in $\Delta_{\mathcal{F}}(G/K_N)$ connecting xK_N and yK_N. Let us consider the set of these open normal subgroups:

$$Q := \{K_N \mid N \in \mathcal{M}\}.$$

Notice that

$$\bigcap_{K_N \in Q} K_N \subseteq \bigcap_{N \in \mathcal{M}} N = 1,$$

therefore the family Q has trivial intersection. Since $\langle xK_N, z_1K_N \rangle / K_N, \dots, \langle z_cK_N, yK_N \rangle / K_N$ are $\mathcal{F}$-groups for every $K_N \in Q$, we get that $\langle x, z_1 \rangle, \dots, \langle z_c, y \rangle$ are all pro-$\mathcal{F}$-subgroups of G,
whence $(x, z_1, \dots, z_c, y)$ is a path of length c in $\Delta_{\mathcal{F}}(G)$ connecting x and y. □

We'll now show an example of extended-commuting graph of a profinite group, which is weakly connected but not strongly connected.

Example 3.1.1. *With the same notations used in 3.0.1, let's consider again the group*

$$G = \mathbb{Z}_{(2)} \rtimes \langle \alpha \rangle.$$

Take $\alpha \in G$ and an arbitrary element of $\mathbb{Z}_{(2)}$, for example take its topological generator 1.
Fixed an arbitrary open normal subgroup N of G, thanks to what we said in 3.0.1 we have:

$$\Omega_N(2, \alpha, 1) = \{(\alpha, \gamma, 1) \mid \{\alpha M, \gamma M, 1M\} \text{ is a path in } \Delta_{\mathcal{A}}(G/M) \text{ for some } M \in \mathcal{M},\ M \leq N\} \neq \emptyset.$$

Take $N_r \in \mathcal{M}$ *such that* $G/N_r \cong D_{2^r}$. *Then a path connecting* α *and* 1 *modulo* N_r *is of the type:*

$$(x, y, z) \text{ where } x \in \alpha N_r,\ \ y \in N_{r-1} \setminus N_r,\ \ z \in 1N_r.$$

Since G *is a topological group, every open is also closed; thus* N_r *and* N_{r-1} *are open and closed.*
This implies that $N_{r-1} \setminus N_r = N_{r-1} \cap (G \setminus N_r)$ *is an intersection of closed sets, hence it's closed in* G.
It follows that the set of paths connecting α *and* 1 *in* $\Delta_{\mathcal{A}}(G/N_r)$ *is closed.*
But, since the same set is also open, if we consider

$$\Omega_{N_r}(2, \alpha, 1) = \bigcup_{s>r} \{(x, y, z) \mid x \in \alpha N_s,\ \ y \in N_{s-1} \setminus N_s,\ \ z \in 1N_s)\}$$

we have a countable and disjoint union of opens. Hence $\Omega_{N_r}(2, \alpha, 1)$ *can't be a compact set.*
Since closed sets in a compact space are compact, it follows that $\Omega_{N_r}(2, \alpha, 1)$ *can't be closed.*
Therefore the graph $\Delta_{\mathcal{A}}(G)$ *is not strongly connected.*

Chapter 4

Connectivity of the pro-$\mathcal{F}$-graph

All we said about the connectivity of the extended-pro-$\mathcal{F}$-graph can be generalized to the pro-$\mathcal{F}$-graph, in the particular case in which $I_{\mathcal{F}}(G) = 1$ and, consequently $\Lambda_{\mathcal{F}}(G) = \Delta_{\mathcal{F}}(G)$. This is useful for our purposes because, when we'll look at finite quotients G/N (where $N \in \mathcal{M}$), we'll work with $\Delta_{\mathcal{F}}(G/N)$, that is a graph with more vertices than $\Lambda_{\mathcal{F}}(G/N)$ (indeed, $I_{\mathcal{F}}(G/N)$ does not need to be identical).

Notice that, it is not restrictive to ask $I = I_{\mathcal{F}}(G) = 1$, when the class $\mathcal{F}$ is semiregular (see 1.0.1), and for every non-identical elements $x, y \in G,\ x \neq y$, the following condition holds:

$$\langle x, y\rangle \text{ is a pro-}\mathcal{F}\text{-subgroup of } G \text{ if and only if } \langle xI, yI\rangle/I \text{ is a pro-}\mathcal{F}\text{-subgroup of } G/I. \tag{4.1}$$

In this case, we have a bijection between the connected components of $\Lambda_{\mathcal{F}}(G)$ and the connected components of $\Lambda_{\mathcal{F}}(G/I)$. Moreover, the corresponding components under this bijection have the same diameters. If the group G and the family $\mathcal{F}$ satisfy the property 4.1, then:

$$\Lambda_{\mathcal{F}}(G) \text{ is a connected graph if and only if } \Lambda_{\mathcal{F}}(G/I) \text{ is a connected graph.} \tag{4.2}$$

It follows that, studying the connectivity of $\Lambda_{\mathcal{F}}(G)$ is the same of studying the connectivity of $\Lambda_{\mathcal{F}}(G/I)$, so we are free to assume $I_{\mathcal{F}}(G) = 1$.

Notice that, if $\mathcal{F} = \mathcal{A}$ is the family of finite abelian groups, and Z is the center of G:

$$\langle xZ, yZ\rangle/Z \text{ abelian subgroup of } G/Z \text{ does not imply } \langle x, y\rangle \text{ abelian subgroup of } G. \tag{4.3}$$

Indeed it doesn't even hold in the finite case:

Example 4.0.1. *Let $G = D_4$ the dihedral group of degree 4.*

$$D_4 = \langle r, s \mid r^4 = 1,\ s^2 = 1,\ r^s = r^{-1}\rangle.$$

The center of D_4 is the subgroup generated by the involution r^2.
If we consider $D_4/\langle r^2\rangle$ we get the Klein four-group $C_2 \times C_2$ that is abelian.
Thus, even if s and r commutes modulo $Z(G)$, they don't in G (otherwise G would be abelian).

Hence, the condition 4.1 is not satisified, and we are not allowed to assume the center of the group to be trivial. This means that, for a general profinite group G, we can not work with $\Delta_{\mathcal{A}}(G)$ instead of $\Lambda_{\mathcal{A}}(G)$. Therefore, results proved above can not be generalized to the commuting graph.

Conversely, assuming $I_{\mathcal{F}}(G) = 1$ is not restrictive when $\mathcal{F} = \mathcal{S}$ is the family of finite solvable groups, or when $\mathcal{F} = \mathcal{N}$ is the family of finite nilpotent groups. We'll see that, in both cases, the condition 4.1 holds.

Theorem 4.0.1. *Assume G to be a profinite group and X a closed subgroup of G.*
If $XR(G)/R(G)$ is a prosolvable subgroup of $G/R(G)$ then X is a prosolvable subgroup of G.

For the proof of this theorem we'll use the following results:

Lemma 4.0.1 (Lemma 6.1. [10])**.**

Let G be a profinite group, and let K be a closed normal subgroup of G such that both K and G/K are prosolvable. Then G is prosolvable.

Corollary 4.0.1. *Let G be a profinite group. Then $R(G/R(G)) = 1$.*

Proof. By contradiction suppose $R(G/R(G)) > 1$.
Let K be the preimage of $R(G/R(G))$ in G with respect to $\pi\colon G \to G/R(G)$.
Then $R(G) < K$ which is closed and normal in G. Moreover both $R(G)$ and $K/R(G)$ are prosolvable.
Therefore by 4.0.1 it would mean that K is prosolvable, but it is not possibile because $K > R(G)$. □

Lemma 4.0.2. *Let G be a profinite group, and let N be a closed subgroup of G. Then $R(N) = R(G) \cap N$.*

Proof. By 1.1.2 we have that $R(G)\cap N$ is a normal and prosolvable subgroup of N, hence $R(G)\cap N \le R(N)$. By contradiction suppose that $R(G) \cap N < R(N)$. Then $R(N)/(R(G) \cap N)$ is a non trivial, closed, normal and prosolvable subgroup of $N/(R(G) \cap N)$. But $N/(R(G) \cap N) \cong R(G)N/R(G)$, thus $R(R(G)N/R(G))$ is not trivial. Since $R(R(G)N/R(G))$ is characteristic then it is normal in $G/R(G)$. Therefore $R(G/R(G))$ is non trivial, in contradiction with 4.0.1. □

Proof of the Theorem 4.0.1.

Since X is a closed subset of a profinite group G, we have that X is a profinite group.
By definition of prosolvable radical $R(X)$ is a closed prosolvable subgroup of X.
Moreover, $X/R(X) = X/(R(G) \cap X) \cong XR(G)/R(G)$ is prosolvable by assumption.
We are in the hypothesis of 4.0.1, hence X is prosolvable.

□

Theorem 4.0.2. *Assume G to be a profinite group and X a closed subgroup of G.*
If $XZ^{\infty}(G)/Z^{\infty}(G)$ is a pronilpotent subgroup of $G/Z^{\infty}(G)$ then X is a pronilpotent subgroup of G.

Proof. Set $Z := Z^{\infty}(G)$. First we'll prove the statement in the finite case.
Assume G to be finite and denote by $Y := Z_{\infty}(XZ)$ the hypercenter of XZ. Notice the $Z \leq Y$ and $XZ/Y \cong (XZ/Z)/(Y/Z)$. By assumption XZ/Z is nilpotent, so we get that the group XZ/Y is nilpotent. If XZ/Y were not identical then its center wouldn't be identical, but it contradicts the definition of hypercenter. Hence, we have $XZ = Y$ which implies that the group XZ is nilpotent.
Now let's go back to the profinite case. Since X is a closed subgroup of XZ, to prove the thesis is sufficient to show that XZ is pronilpotent. Therefore, without loss of generality we can assume that $Z \leq X$. Hence, the assumption of the theorem becomes:

$$X/Z \text{ is a pronilpotent subgroup of } G/Z,$$

i.e. XN/N is a nilpotent group for every open normal subgroup N of G that contains Z.
Let $M \in \mathcal{M}$ and define $Y \leq G$ in the following way:

$$Y/M := Z_{\infty}(G/M).$$

It means that Y is the preimage under the projection $\pi_M : G \to G/M$ of $Z_{\infty}(G/M)$ which is an open and normal subgroup of G/M. Therefore, Y is an open and normal subgroup of G which contains Z.
By hypothesis XY/Y is a nilpotent group. Notice that $M \leq Y \leq XY$ and $XY/Y \cong (XY/M)/(Y/M)$, whence XY/M is a nilpotent group. From the statement in the finite case we get that XM/M is a nilpotent group, which means that X is a pronilpotent group. □

By Theorems 4.0.1 and 4.0.2 follows that the condition 4.1 is satisfied in both cases.
This observation allows us to focus our attention on groups with:

- $R(G) = 1$, this implies that we can work with $\Delta_{\mathcal{S}}(G)$ instead of $\Lambda_{\mathcal{S}}(G)$,
- $Z^{\infty}(G) = 1$, this implies that we can work with $\Delta_{\mathcal{N}}(G)$ instead of $\Lambda_{\mathcal{N}}(G)$.

Pro-$\mathcal{F}$-graphs have been deeply studied for different choices of the class $\mathcal{F}$, but mostly in the case in which the given group is finite (and in this case we'll call the graph *$\mathcal{F}$-graph*). For example, the commuting graph and the nilpotent graph of a finite group are not always connected:

- $\Lambda_{\mathcal{N}}(A_4)$ has has 11 vertices and 5 connected components (see [7])
- by Theorem 3.1 in [11] it follows that for any prime $p \geq 3$ we have that $\Lambda_{\mathcal{A}}(S_p)$ and $\Lambda_{\mathcal{A}}(S_{p+1})$ are always disconnected.

On the contrary, by Theorem 1 in [7] the solvable graph $\Lambda_{\mathcal{S}}(H)$ of a finite group H is always connected and its diameter is at most 5. As we said before, even if we have this strong statement in the finite case, we are not allowed to extend it in the profinite case: the connectivity in the finite case just implies the weak connectivity of the prosolvable graph. Anyway, we still don't know if there exists a profinite group whose pro-$\mathcal{F}$-graph is weakly connected but not strongly connected when $\mathcal{F}$ is the class of finite solvable groups. Hence, since nobody has found yet an example of this, the existence of a prosolvable graph weakly connected but not strongly connected keeps being an open problem.

Chapter 5

Groups whose pronilpotent graphs are weakly disconnected

The pronilpotent graph $\Lambda_{\mathcal{N}}(G)$ of a profinite group G, is the graph defined in the previous chapter in which we take as $\mathcal{F}$ the class of finite and nilpotent groups. Explicitly, $\Lambda_{\mathcal{N}}(G)$ is a graph whose vertex set is $G \setminus Z^{\infty}(G)$ and two distinct elements x, y are adjacent if and only if $\langle x, y \rangle$ is a pronilpotent subgroup of G. Since the nilpotent graph of a finite group doesn't need to be always connected, then the pronilpotent graph of a profinite group doesn't need to be always weakly connected.

In this chapter, we'll prove a surprising and not intuitive result: if the pronilpotent graph of a group is weakly disconnected, i.e. it is not c-weakly connected for any $c \in \mathbb{N}$, then the group is virtually pronilpotent.

Remind that, the word *virtually* is usually used to modify a property so that it needs to hold just for a subgroup of finite index. In our case, we'll require this subgroup to be open.

Definition 5.0.1. *A profinite group G is said to be* ***virtually pronilpotent*** *if there exists an open subgroup $H \leq G$ which is pronilpotent.*

Let's denote by $\Lambda(G)$ the pronilpotent graph of a profinite group G. Thanks to the Theorem 4.0.2 we are free to assume $Z^{\infty}(G) = 1$ when we are investigating the connectivity of $\Lambda(G)$.

Let us denote by $\Delta(G)$ the extended-pronilpotent graph of G, i.e. the graph whose vertices are all elements of $G \setminus \{1\}$ and two elements x, y are adjacent if and only if $\langle x, y \rangle$ is pronilpotent.

Remind that, if we assume the hypercenter of G to be trivial then $\Lambda(G) = \Delta(G)$.

5.1 Preliminary results

This section contains some auxiliary results, that will be needed in our main proof.

Suppose H to be a finite group. From results on the commuting graph of a finite group (see Lemma 3.3 and Theorem 2.3 in [8]) it follows:

Theorem 5.1.1. *If H contains a normal and minimal subgroup N which is not abelian, then either H is almost simple or $\Delta(H)$ is connected with diameter at most* 5.

We will use this result to prove the following:

Theorem 5.1.2. *Let G be a profinite group with trivial hypercenter. If G doesn't contain any open and prosolvable subgroup then $\Delta(G)$ is* 5*-weakly connected.*

Proof. Since every open subgroup of G is not prosolvable, in particular G is not a prosolvable group. It means that there exists $M \in \mathcal{M}$ such that G/M is not a solvable group. Now let $N \in \mathcal{M}$ and define $L := N \cap M$. By assumption L is not a prosolvable group, hence there exists $A_L \in \mathcal{M}$ and $A_L \leq N$ such that L/A_L is not a solvable group. In particular, there exists $B_L \in \mathcal{M}$ such that B_L/A_L is a normal minimal non abelian subgroup of G/A_L which is contained in L/A_L. Since G/M is not solvable and $G/M \cong (G/L)/(M/L)$, we get that G/L is not solvable too. Hence, both G/L and L/A_L are not solvable, this means that G/A_L has at least two composition factors which are not abelian (one in G/L and the other one in L/A_L). Note that, if H is an almost simple group with socle group S then the group H/S is solvable (because it's a subgroup of $Aut(S)/S$ which is known to be solvable from the classification of simple groups). A consequence of this fact is that an almost simple group has just one non abelian composition factor, hence we can deduce that G/A_L is not almost simple. Therefore, by Theorem 5.1.1 we have that $\Delta(G/A_L)$ is connected with $\delta(\Delta(G/A_L)) \leq 5$. This means that $\Omega_N(5, a, b) \neq \emptyset$ for every $a, b \in G$, whence $\Delta(G)$ is 5-weakly connected. □

Proposition 5.1.1. *Let G be a profinite group with trivial hypercenter. Suppose that G contains an open and normal subgroup N such that*

1. *N is not a pronilpotent group,*
2. *$\Delta(N)$ is c-weakly connected for some $c \in \mathbb{N}$.*

Then $\Delta(G)$ is $(c + 4)$-weakly connected.

Proof. Let $x, y \in G$ not identical elements. Since G is residullay finite, there exist $M_x, M_y \in \mathcal{M}$ such that $x \notin M_x$ and $y \notin M_y$. Moreover, since N is not pronilpotent, there exists $A_N \in \mathcal{M}$ contained in N such that the group N/A_N is not nilpotent. Now define $A := M_x \cap M_y \cap A_N$. Notice that:

- $x, y \notin A$,
- N/A is not a nilpotent group.

Since xA and yA are not identical elements of G/A there are $\bar{x}A$, $\bar{y}A$ powers of prime orders of xA, yA respectively. Since N/A is not nilpotent, by Theorem 2.3 in [8], N/A has no fixed-point-free automorphisms of prime order. Hence, there exist $\beta, \gamma \in N$ such that $[\bar{x}A, \beta A] = 1$ and $[\bar{y}A, \gamma A] = 1$. Moreover, since $\Delta(N)$ is c-weakly connected, then $\Omega_A(c,\beta,\gamma) \neq \emptyset$. Whence, there exists $\tilde{A} \in \mathcal{M}$, $\tilde{A} \leq A$ such that there is a path from $\beta\tilde{A}$ to $\gamma\tilde{A}$ in $\Delta(N/\tilde{A})$ of length c. It follows that, in $G/\tilde{A}$ we get a path of length $c+4$ of the type:

$$x\tilde{A} - \bar{x}\tilde{A} - \beta\tilde{A} - \cdots - \gamma\tilde{A} - \bar{y}\tilde{A} - y\tilde{A}.$$

Therefore, $\Omega_A(c+4, x, y) \neq \emptyset$ for all non-identical $x, y \in G$, $x \neq y$, whence $\Delta(G)$ is $(c+4)$-weakly connected. □

Before proceeding we'll remind some definitions and results that will be useful in the sequel.

Definition 5.1.1. *Let G be a finite group acting transitively on a set X.*
We call G a ***Frobenius group*** *if only the identity element fixes more than one point.*
In other words, if x, y are distinct elements such that $gx = x$ and $gy = y$ then $g = 1$.

Let G be a Frobenius group acting on X and $x_0 \in X$. Let us define:

$$K := 1 \cup \{g \in G \mid g \text{ has no fixed points}\}$$

$$H := \{g \in G \mid gx_0 = x_0\}.$$

it's known that:

- K is a normal subgroup of G (Frobenius' theorem, see Lemma 6.5 and Corollary 6.6 in [12]),
- $H \cap K = 1$, $HK = G$, and so $G = K \rtimes H$,
- H acts fixed-point-free on K,
  i.e. the centralizer of H in K is the identity: $C_K(h) = 1$ for every $h \in H$,
- K is nilpotent (it was proved by J.Thompson in his PH.D. thesis in 1959).

We'll call K the **Frobenius kernel** and H the **Frobenius complement**.

Remark 5.1.1. *Let G be a Frobenius group with Frobenius kernel K and Frobenius complement H, then K and H are coprime subgroups of G, i.e. $(|K|, |H|) = 1$.*

Proof. By contradiction suppose there is a prime number p dividing both $|K|$ and $|H|$.
Set $X := P \rtimes Q$, where $P \in Syl_p(K)$ and $Q \in Syl_p(H)$. Since P is normal in X, it contains at least one element of the center, i.e. $P \cap Z(X) \neq 1$. But, if $1 \neq x \in Z(X)$ then x commutes with every elements of Q, hence the action of Q on P cannot be fixed-point-free. This is a contradiction because the action of H on K needs to be fixed-point-free. □

Definition 5.1.2. *A finite group G is a* ***2-Frobenius group*** *if it has normal subgroups A and B such that A is a Frobenius group with Frobenius kernel B and G/B is a Frobenius group with Frobenius kernel A/B.*

Remind that, given a group G we can consider the pro-$\mathcal{F}$-graph of G taking as $\mathcal{F}$ the class of finite abelian groups. This graph is known as the **commuting graph** of G: its vertices are all non-central elements of G and two vertices are adjacent if and only if they commute in G. Commuting graphs arise naturally in many different contexts and they have been intensively studied by various authors in recent years (see for example [11] [15]).

In the sequel, we are going to consider the commuting graph of G in which we take as vertices also the elements of the center (except the identity element). We'll denote this graph by $\Omega(G)$.
Note that every vertex in $Z(G)$ is connected to every other vertex in $\Omega(G)$.

Since we have much information on the commuting graph of a given group, the following observation, about the relation between $\Omega(G)$ and $\Delta(G)$, is going to be useful.

Remark 5.1.2. *Let G be a finite group, then $\Delta(G)$ is connected if and only if $\Omega(G)$ is connected. Moreover, the diameter of $\Delta(G)$ is less or equal to the diameter of $\Omega(G)$.*

Proof. One implication is trivial because obviously each arc in the commuting graph is also an arc in the nilpotent graph, hence if $\Omega(G)$ is connected then $\Delta(G)$ is connected too. On the other hand, if x and y are adjacent in $\Delta(G)$ then they generate a nilpotent group. In particular, the center of $\langle x, y\rangle$ is not trivial, take $1 \neq z \in Z(\langle x, y\rangle)$. In $\Omega(G)$ the vertex z is adjacent to every other vertex in $\langle x, y\rangle$, in particular it is adjacent to x and to y. So we have proved that every arc in the nilpotent graph produces a path, of length at most 2, in the commuting graph. We can conclude that, if $\Delta(G)$ is connected then $\Omega(G)$ is connected too and $\delta(\Delta(G)) \leq \delta(\Omega(G))$. □

Notice that, if the center of the group $Z(G)$ is not identical, then $\Omega(G)$ is clearly connected and so $\Delta(G)$ is connected too. Hence we can assume $Z(G) = 1$.
By Theorem 1.1. in [17]: If G is a finite and solvable group with trivial center, then

- $\Omega(G)$ is disconnected if and only if G is a Frobenius group or a 2-Frobenius group,
- if $\Omega(G)$ is connected then $\Omega(G)$ has diameter at most 8.

Applying Remark 5.1.2 we can state the following proposition.

Proposition 5.1.2. *If G is a finite and solvable group, then*

- *if $\Delta(G)$ is disconnected then either G is Frobenius or is 2-Frobenius,*
- *if $\Delta(G)$ is connected then $\delta(\Delta(G)) \leq 8$.*

Definition 5.1.3. *The **quaternion group** Q_8 is a non abelian group of order 8. It is given by the following presentation:*

$$\langle a, b \mid a^4 = 1,\ a^2 = b^2,\ ba = a^{-1}b\rangle.$$

Definition 5.1.4. *A **generalized quaternion group** Q_{4n} is a group of order $4n$. It is defined by the following presentation:*

$$\langle x, y \mid x^2 n = y^4 = 1,\ x^n = y^2,\ y^{-1}xy = x^{-1}\rangle.$$

Notice that for the particular case $n = 2$, we recover the quaternion group.

Proposition 5.1.3 (Coprime action on p-groups 24.4 in [5])**.**

Let $G := N \rtimes H$ with N and H two coprime subgroups, then it holds that $N = [N, H]\, C_N(H)$.

In particular:

If the action is fixed-point-free (e.g. when G is a Frobenius group) then $N = [N, H]$.

Proposition 5.1.4 (Lemma 2.1 in [15])**.**

If X is a Frobenius complement then every Sylow subgroup of X is cyclic or generalized quaternion.

As a consequence of this result we have the following lemma.

Lemma 5.1.1. *Let G be a finite group. If $C_p \times C_p$ is epimorphic image of G and p is odd, then G is neither Frobenius nor 2-Frobenius.*

Proof. Suppose that $C_p \times C_p$ is epimorphic image of G. We'll proceed by contradiction.

1. Suppose that G is Frobenius. Let us denote with K its Frobenius kernel and by H its Frobenius complement. Suppose that p divides $|H|$. Since $G = K \rtimes H$ and $(|H|, |K|) = 1$ we have that $C_p \times C_p$ is epimorphic image of H. But it can't happen because, by Proposition 5.1.4, the p-Sylow subgroups of H are cyclic. Hence, p must divide $|K|$. If we take $P \in Syl_p(K)$ then by Proposition 5.1.3 we have $P \leq K = [K, H] \leq G'$. But, this is a contradiction because on one side p^2 divides $|G/G'|$ (since $C_p \times C_p$ is epimorphic image of G) and on the other side G' contains a p-Sylow subgroup of G. Hence, we conclude that G is not Frobenius.

2. Suppose that G is 2-Frobenius. Let $A, B \leq G$ such that A is Frobenius with kernel B and G/B is Frobenius with kernel A/B. In particular, $(|A/B|, |B|) = 1$ and $(|G/A|, |A/B|) = 1$. If p divides $|A/B|$, then $C_p \times C_p$ is epimorphic image of A/B, but this can't happen because A/B is a Frobenius complement. Hence, we can suppose that p does not divide $|A/B|$. Take $N \trianglelefteq G$ such that $G/N \cong C_p \times C_p$. Since G/A is a Frobenius complement (and so $C_p \times C_p$ cannot be its epimorphic image) we deduce $A \not\leq N$. Therefore, $AN/N \cong A/(A \cap N)$ is a non trivial p-group of $G/N \cong C_p \times C_p$. This implies that C_p is epimorphic image of A. Since $(|A/B|, p) = 1$ we have that p divides $|B|$. If we take $P \in Syl_p(B)$ then we have $P \leq B = [B, A/B] \leq A'$. But, it's a contradiction because on one side $|A/A'|$ is divisible by p (because C_p is epimorphic image of A) and on the other side A' contains a p-Sylow subgroup of A. Hence we can conclude that G is not 2-Frobenius.

□

Definition 5.1.5. *Let G be a finite group, and let $H, K \trianglelefteq G$ such that $K \leq H$. Then the quotient H/K is said to be a* ***principal factor*** *of G if H/K is a minimal normal subgroup of G/K,
i.e. H/K does not contain any normal and proper subgroup of G/K.*

Lemma 5.1.2. *Let G be a finite group such that all its principal factors of odd order are cyclic.
If $C_2 \times C_2$ is epimorphic image of G, then G is neither Frobenius nor 2-Frobenius.*

Proof. As in the proof above we can procede by contradiction.
Suppose G to be Frobenius with K the kernel and H the complement. In particular $(|H|, |K|) = 1$. Suppose that 2 divides $|H|$, then $C_2 \times C_2$ is epimorphic image of H. Take M maximal H-invariant subgroup of G contained in K. By Corollary 6.2 of [12], we have that $K/M \rtimes H$ is still a Frobenius group. In particular, $H \leq Aut(K/M)$. Note that, since there are not normal subgroups between M and K we have that K/M is a principal factor of G. Moreover, since $(|H|, |K/M|) = 1$ and 2 divides $|H|$, we deduce that K/M is of odd order. By assumption it means that $K/M \cong C_p$ for some prime p. Hence, $H \leq Aut(C_p) \cong C_{p-1}$ which is cyclic. This contradicts the fact that $C_2 \times C_2$ is epimorphic image of H. Now the sequel of the proof is exactly the same as the proof in 5.1.1. □

Definition 5.1.6. *A finite group is* ***supersolvable*** *if it has an invariant normal series where all the factors are cyclic groups, i.e. the group G is supersolvable if there exists a normal series*

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \ldots H_{s-1} \trianglelefteq H_s = G$$

such that each quotient group H_{i+1}/H_i is cyclic, and each H_i is normal in G.

Definition 5.1.7. *A group is said to be* ***prosuperprosolvable*** *if it is the inverse limit of an inverse system of finite supersolvable groups. Equivalently, G is prosupersolvable if each finite quotient of G is supersolvable.*

Definition 5.1.8. *Let G be a profinite group and π a set of prime numbers.
The* ***π-core*** *of G, denoted by $O_\pi(G)$, is the group topologically generated by all normal pro-π-subgroups of G. The* ***Fitting subgroup*** *of G, denoted by $Fit(G)$, is the subgroup topologically generated by the subgroups $O_p(G)$, as p ranges over the primes.*

Remark 5.1.3. *$Fit(G)$ is the largest normal and pronilpotent closed subgroup of G.*

Proof. $Fit(G)$ is pronilpotent and normal because its generators $O_p(G)$ are normal and pro-p-groups. Suppose $N \trianglelefteq G$, closed and pronilpotent. Let P be a Sylow p-subgroup of N. Then P is characterstic in $N \trianglelefteq G$, whence $P \trianglelefteq G$. Since $O_p(G)$ is the largest, normal, pro-p-subgroup of G, we get $P \leq O_p(G)$. Therefore, for each Sylow subgroup $P \leq N$, we have $P \in Fit(G)$. Since N is pronilpotent, then it is the cartesian product of its Sylow subgroups. This implies that $N \leq Fit(G)$. □

Definition 5.1.9. *Let G be a profinite group. The* ***Frattini subgroup*** *of G, denoted by $Frat(G)$, is the intersection of all maximal closed subgroups of G.*

Remark 5.1.4. *Let G a profinite group. Suppose that H is a closed subgroup of G such that $HFrat(G) = G$. Then $H = G$.*

Proof. By contradiction suppose that H is a proper subgroup of G. Then there exists a maximal closed subgroup M of G containing H. Therefore, $G = HFrat(G) \leq MM = M$, whence $G \leq M$, which is clearly a contradiction. □

Definition 5.1.10. *A group G is said to be a* ***metabelian group*** *if its derived subgroup G' is abelian.*

Proposition 5.1.5 (Proposition 3.3. in [16])**.**

Let G a prosupersolvable group. Then $G/Frat(G)$ is profinite metabelian.

From Sylow theory in profinite groups we can easily see that the *"Frattini Argument"* keeps holding in the profinite contest.

Proposition 5.1.6 (Frattini argument)**.**

Let G be a profinite group, H a closed normal subgroup of G, and P a Sylow p-subgroup of H.
Then $G = N_G(P)H$.

Proof. We're going to follow the proof 5.2.14. in [20] of the finite case. Let $g \in G$, then $P^g \leq H$ and it is a p-Sylow subgroup of H. Hence, there exists $h \in H$ such that $P^g = P^h$. This implies that $P^{gh^{-1}} = P$, thus $gh^{-1} \in N_G(P)$, therefore $g \in N_G(P)H$. □

One application of this result, as in the finite case, is the following:

Proposition 5.1.7. *Let G be a profinite group. Suppose H to be a closed and normal subgroup of G such that $Frat(G) \leq H$, and $H/Frat(G)$ is pronilpotent. Then H is a pronilpotent group.*
In particular, $Frat(G)$ is a pronilpotent group.

Proof. Let P a Sylow p-subgroup of H. To show that H is pronilpotent, it is enough proving that P is normal in H. Set $F := Frat(G)$ and $K = PF \leq H$. Since K/F is a Sylow p-subgroup of H/F and by hypothesis H/F is pronilpotent, then K/F is characteristic in H/F, whence $K \trianglelefteq G$. Applying the Frattini argument 5.1.6:

$$G = N_G(P)K = N_G(P)F = N_G(P)$$

This shows $G = N_G(P)$ and $P \trianglelefteq G$, in particular $P \trianglelefteq H$. □

Proposition 5.1.8. *Let G be a profinite group. If G is prosupersolvable, then $G/Fit(G)$ is abelian.*

Proof. Set $F = Frat(G)$. By Proposition 5.1.5 we know G/F is metabelian, hence its derived subgroup $G'F/F$ is abelian. Now, since the derived subgroup of a profinite group is not always closed, we'll work with the closure M/F of $G'F/F$. Since M/F is the closure of a normal and abelian group, it keeps being normal and abelian. But, if M/F is abelian, then M/F is pronilpotent, and by Proposition 5.1.7 we deduce that M is pronilpotent. Therefore, since M is pronilpotent and normal in G, and $Fit(G)$ is the largest normal and pronilpotent subgroup of G, we conclude that $M \leq Fit(G)$, whence $G/Fit(G)$ is abelian. □

Theorem 5.1.3 (Theorem 2.6.2. in [23]).

Let G be a profinite group of order coprime to a prime p. Suppose that there is a a continuous automorphism α of G of order p such that $\{g \in G \mid \alpha(g) = g\} = 1$. Then G is nilpotent.

Definition 5.1.11. *Let G be a profinite group, and let $H, K \trianglelefteq G$ open such that $K \leq H$.*
We'll say that H/K is a ***principal factor*** *of G, if H/K is a minimal normal subgroup of G/K.*

5.2 Proof of the Theorem 5.2.1

With these preliminaries, we are now ready to prove the main result we talked about at the beginning of the section.

Theorem 5.2.1. *Let G be a profinite group with trivial hypercenter.*
If $\Delta(G)$ weakly disconnected, then G is virtually pronilpotent.

Proof. By contradiction suppose G is not virtually pronilpotent, explicitly every open subgroup of G is not pronilpotent. By hypothesis $\Delta(G)$ is not weakly connected for any $c \in \mathbb{N}$. Applying Theorem 5.1.2, there exists an open and prosolvable subgroup L of G. Without loss of generality I can suppose $L \trianglelefteq G$. Take $X, Y \trianglelefteq G$ open, $X, Y \leq L$, $Y \leq X$, in such a way that X/Y is a principal factor for G.
Let $a \in G$, $a \neq 1$, with order r in G/X.

- If $a^r \neq 1$ then a and a^r are adjacent in $\Delta(G)$,
- if $a^r = 1$ then a has a power a^* with prime order p.
  - Suppose that p is coprime with the order of X. In this case, since X is not pronilpotent, by Theorem 5.1.3, it does not have any fixed-point-free automorphisms of order p; it follows that, there is a non-identical element $e_{a^*} \in X$ such that $[a^*, e_{a^*}] = 1$; hence we have a path of length 2 from a to e_{a^*} crossing a^*.
  - Suppose that p divides the order of X. Then there exists a p-Sylow subgroup P of X, which is normalized by a^*. Then, we can take as e_{a^*} a non-identical element of P. In this setting, a^* and e_{a^*} don't commute but they are adjacent in $\Delta(G)$ because they generate a pro-p group.

Now, let $a, b \in G$ distinct and not identical elements.
From above, in $\Delta(G)$ we can find two paths of the type:

- $\{\alpha_1, \alpha_2, \ldots, \alpha_c\}$ such that $\alpha_1 = a$, $1 \neq \alpha_c \in X$ and $c \leq 2$,
- $\{\beta_1, \beta_2, \ldots, \beta_d\}$ such that $\beta_1 = b$, $1 \neq \beta_d \in X$ and $d \leq 2$.

Hence, in both cases, what we get is a path in $\Delta(G)$, from a non-identical element of G to a non-identical element in X. Our aim is to connect α_c and β_d, so that we get a path from a to b. Now, take $V \trianglelefteq G$ open, and such that $\alpha_i, \beta_j \notin V$ for any i, j. Observe that, X/Y is a minimal normal subgroup of G/Y, hence X/Y is the product of isomorphic and simple groups. Since, in our case, $X/Y \leq L/Y$, and L/Y is solvable, we get that also X/Y is solvable. This implies that X/Y is the product of isomorphic, simple and abelian groups. Suppose that:

$$X/Y \cong C_p^t, \quad \text{with } t \geq 2, \text{ and } p \text{ an odd prime.}$$

Notice that X/Y has $C_p \times C_p$ as epimorphic image. Take $N \trianglelefteq G$ open, and consider

$$T := V \cap Y \cap N.$$

Observe that $T \leq Y$, and it does not contain α_i, β_j for any i, j. Since $(X/T)/(Y/T) \cong X/Y$, then X/T has $C_p \times C_p$ as epimorphic image. By Lemma 5.1.1 we get that the group X/T is neither Frobenius nor 2-Frobenius. Since G is prosolvable, the group X/T is solvable. Hence, we can apply Proposition 5.1.2 to the group X/T and conclude that $\Delta(X/T)$ is connected with diameter at most 8. Therefore, we can find a path of length 8 from α_c to β_d in $\Delta(X/T)$. In particular, we've found a path, of length 12, from a to b in $\Delta(G/T)$:

$$aT = \alpha_1 T - \cdots - \alpha_c T - \cdots - \beta_d T - \cdots - \beta_1 T = bT.$$

Notice that, the choice of N in the definition of T is arbitrary. Moreover, $T \trianglelefteq G$ open, and $T \leq N$. This means that,for every $a, b \in G$ and $N \trianglelefteq G$ open, we have $\Omega_N(12, a, b) \neq \emptyset$. We can conclude that $\Delta(G)$ is 12-weakly connected, in contradiction with our hypothesis. Therefore, $X/Y \cong C_p$ and , since $X/Y \leq L/Y \leq G/Y$, we have that X/Y is a principal factor also for L. Hence, from above, we can conclude that every principal factor of odd order of L is cyclic. Now, suppose that

$$X/Y \cong C_2^t, \quad \text{with } t \geq 2,$$

Notice that, all principal factors of odd order of each finite quotient of L are cyclic, whence all principal factors of odd order of $X/Y \leq L/Y$ are cyclic. Therefore, since X/Y has $C_2 \times C_2$ as epimorphic image, we can apply Lemma 5.1.2 to the group X/Y. With the same argument used above, we can conclude that also the principal factors of even order of L are cyclic. From Definition 5.1.11 , it follows that all finite quotients of L are supersolvable, hence L is prosupersolvable. Set $F := Fit(L)$, the Fitting group of L. Notice that F is characteristic in L, which is normal in G, thus $F \trianglelefteq G$. By Proposition 5.1.8 we know that L/F is abelian and infinite (otherwise F would be an open and pronilpotent subgroup of L, and hence of G, in contradiction with our assumption). We also know that $C_p \times C_p$ can not be epimorphic image of L/F. This implies that each finite quotient of L/F is cyclic, thus L/F is procyclic. Hence, there exists $x \in L$ such that $L/F = \overline{\langle xF \rangle}$. Setting $H := \overline{\langle x \rangle}$, it follows that $L = HF$ with H abelian.
Let $N \trianglelefteq G$ open, and define

$$U := N \cap F \cap V.$$

Notice that U is normal and closed in G. Hence, $\overline{G} := G/U$ is a profinite group. For each $X \leq G$ let us denote by $\overline{X} = XU/U$. Using these notations, we have $\overline{L} = \overline{H}\,\overline{F}$, where $\overline{F}$ is finite because U is open and normal in F. Let us define

$$\overline{C} := C_{\overline{H}}(\overline{F}).$$

Note that $\overline{C}$ is closed in $\overline{H}$ and $\overline{H}/\overline{C} \leq Aut(\overline{F})$. Thus $\overline{H}/\overline{C}$ is finite, whence $\overline{C} \leq \overline{H}$ is open. Since $\overline{L}$ is infinite (otherwise U would be an open and pronilpotent subgroup of L, and hence of G) and $\overline{F}$ is finite, we deduce that $\overline{H}$ is infinite. Hence, there exists $D \leq HF = L$ open, and such that $U \leq D < C$. Take its normal core in G

$$D_G = \bigcap_{x \in G} D^x.$$

Then, we have that

- D_G is open in G, because D is open, thus has finitely many conjugates,
- $U \leq D_G < C$, because D_G is the largest closed normal subgroup of G contained in D.

Set

$$E := D_G \cap N.$$

Notice that $E \trianglelefteq G$ open. Now, we'll work with the quotient G/E. Since $E < C$ we are sure that $C/E \neq 1$. Note that $U = N \cap F \cap V \leq N$, and $U \leq D_G$ by definition of D, hence $U \leq E := D_G \cap N$. Moreover:

- $[C, H] \leq U \leq E$, since $\overline{H}$ is abelian and $\overline{C} \leq \overline{H}$,
- $[C, F] \leq U$, by definition of $\overline{C}$, hence $[C, F] \leq E$.

From these observations we get

$$1 \neq C/E \leq Z(G/E).$$

Hence $\Delta(G/E)$ is connected with diameter at most 2. Notice that the choice of N is arbitrary in the definition of $E := D_G \cap N$ and, furthermore $E \trianglelefteq G$ open such that $E \leq N$. It means that $\Omega_N(2, a, b) \neq \emptyset$ for all $a, b \in G$. It follows that $\Delta(G)$ is 2-weakly connected, in contradiction with the hypothesis of the theorem. □

Chapter 6

A group whose prosolvable graph is *almost-disconnected*

As we said before, the existence of a group whose prosolvable graph is not strongly connected is an open problem. Anyway, in this section we'll construct a profinite group whose prosolvable graph is "almost" disconnected. We've used the word "almost" because we'll not prove that this prosolvable graph is actually disconnected, but we'll notice that something different respect to the finite case happens.

Let G be a profinite group and $\mathcal{S}$ be the class of finite solvable groups. Without loss of generality suppose $R(G) = 1$. For each $x \in G$ define the **solvabilizer** of x in G to be:

$$Sol_G(x) := \{g \in G \mid \langle x, g \rangle \text{ is prosolvable}\}.$$

In other words, $Sol_G(x)$ is the set of neighbors of $x \in G$ in the prosolvable graph of G plus the unit of the group. This object has already been studied in [4] and it has been a useful tool to prove the connectivity of the solvable graph in the finite case. Notice that:

- $Sol_G(x)$, in general, is not a subgroup of G,
- $Sol_G(x)$ is a closed subset of G,
- $Sol_G(x)$ contains the closure of the cyclic group generated by x.

Proposition 6.0.1 (Corollary 3.2. in [4]).
Let H be a finite group. Then $\overline{\langle x \rangle} = \langle x \rangle \subsetneq Sol_H(x)$ for each $x \in H$.

Therefore, in the finite case, the solvabilizer of any element x properly contains the cyclic group generated by x. It means that, each vertex x is not just connected to all its powers, but there exists at least one more element y (that is not a power of x) adjacent to x.

We are going to show an example of profinite group where it's possibile to find an element for which the equality holds. Hence the general proof of the connectivity of solvable graph doesn't work anymore, and this leads us to think that the graph could be disconnected.

6.1 Construction of the group

Firstly, we'll construct recursively a sequence of finite groups as follows:

- $G_0 := A_5$ be the alternating group of degree 5.

Set $\alpha := (1\ 2\ 3)$ and $\beta := (1\ 2\ 3\ 4\ 5)$.

- $G_1 := A_5 \wr A_5 = {A_5}^{60} \rtimes A_5$
  where the wreath product is with respect the regular action of A_5.

- $M_1 := {A_5}^{60}$
  an element $m \in M_1$ is a sequence $(y_x)_{x \in A_5}$ with $y_x \in A_5$.

Let $A_5 = \bigsqcup_{1 \le i \le 20} t_i\langle\alpha\rangle$ where $T := \{1 = t_1, \dots, t_{20}\}$ is a transversal of $\langle\alpha\rangle$ in A_5.
Define

$$m_1 := (y_x)_{x \in A_5} \quad with\ y_x = \begin{cases} 1 & if\ x \notin T \\ \alpha & if\ x = t_1 \\ \beta & if\ x = t_i,\ i \neq 1 \end{cases}$$

$g_1 := m_1\alpha.$

Supposing to have defined G_t and g_t, denoting by $n_t := |G_t|$, construct:

- $G_{t+1} := A_5 \wr G_t = {A_5}^{n_t} \rtimes G_t$
  where wreath product is with respect the regular action of G_t.

- $M_{t+1} := {A_5}^{n_t}$
  an element $m \in M_{t+1}$ is a sequence $(y_x)_{x \in G_t}$ with $y_x \in A_5$.

Let $G_t = \bigsqcup_{u \in T_t} u\langle g_t\rangle$ where T_t is a transversal containing 1 of $\langle g_t\rangle$ in G_t.
Define:

$$m_{t+1} := (y_x)_{x \in G_t} \quad with\ y_x = \begin{cases} 1 & if\ x \notin T_t \\ \alpha & if\ x = 1 \\ \beta & if\ x \in T_t \setminus 1 \end{cases}$$

$g_{t+1} := m_{t+1}g_t.$

Now, let us denote with G the inverse limit of the inverse system of finite groups $\{G_t\}$ that we have just constructed. G is a profinite group and inside G there is a descending chain of open normal subgroups

$$G \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq N_3 \dots N_i \trianglerighteq N_{i+1} \trianglerighteq \dots$$

$$\text{with } G/N_t \cong G_t \quad \text{and} \quad N_t/N_{t+1} \cong A_5^{n_t} = M_{t+1}.$$

6.2 Unexpected behavior of the solvabilizer

Now, let us denote by

$$g := (g_1, g_2, g_3, \dots) \in G.$$

We want to show that $Sol_G(g) = \overline{\langle g \rangle}$.

Remark 6.2.1. *If $\alpha^*, \beta^* \in A_5$ are such that $|\alpha^*| = 3,\ |\beta^*| = 5$, then $\langle \alpha^*, \beta^* \rangle = A_5$.*

Remark 6.2.2. *Observe that the element $m_{t+1} \in M_{t+1}$ has n_t coordinates indexed on the elements of G_t. The value assumed on each coordinate depends on the the transversal T_t of $\langle g_t \rangle$ in G_t.*
Without loss of generality, we can rearrange the coordinates of m_{t+1} in such a way that those indexed on elements belonging to the same coset are adjacent.
Therefore, setting $\gamma_t = |g_t|$, we can say that:

- *the vector m_{t+1} is divided in $\frac{n_t}{\gamma_t}$ blocks,*
- *each block has γ_t coordinates ,*
- *in the first position of each block there's an element of the transversal.*

If we look at the way in which we defined the entries of m_{t+1} we get that the entries are all equal to 1 except the first coordinate of each block:

$$m_{t+1} = (\alpha, 1, \dots, 1|\beta, 1, \dots, 1| \dots |\beta, 1 \dots 1).$$

Therefore,

$$\begin{aligned} g_{t+1}^{\gamma_t} &= (m_{t+1} g^t)^{\gamma_t} = \\ &= (m_{t+1})^{g_t} (m_{t+1})^{{g_t}^2} \dots (m_{t+1})^{{g_t}^{\gamma_t - 1}} g_t^{\gamma_t} = \\ &= (m_{t+1})^{g_t} (m_{t+1})^{{g_t}^2} \dots (m_{t+1})^{{g_t}^{\gamma_t - 1}} = \\ &= (\alpha, \dots, \alpha|\beta, \dots, \beta| \dots |\beta, \dots, \beta). \end{aligned}$$

Remark 6.2.3. *From the way we constructed the elements g_t and N_t,*
for every t we have that $g_{t+1}N_{t+1} = g_t N_{t+1}$. Therefore $g_t N_t = g N_t$ for every t.

Lemma 6.2.1. *Set $\gamma_t := |g_t|$ and $\sigma := g_{t+1}^{\gamma_t}$. Then $Sol_{G_{t+1}}(\sigma) \subseteq M_{t+1}\langle g_{t+1} \rangle = M_{t+1}\langle g_t \rangle$.*

Proof. For every $z \in G_t$, let us consider the projection $\pi_z \colon M_{t+1} \to A_5,\ (y_x)_{x \in G_t} \mapsto y_z$.
Then, by the Remark 6.2.2, we have

$$\pi_z(\sigma) = \begin{cases} \alpha & if\ z \in \langle g_t \rangle \\ \beta & otherwise \end{cases}$$

Now, take $\rho := mh \in G_{t+1}$ with $m := (y_x)_{x \in G_t} \in M_{t+1}$ and $h \in G_t$.
Setting $\sigma = (x_1, \ldots, x_{n_t})$, where each $x_i \in \{\alpha, \beta\}$, we have

$$\sigma^\rho = \sigma^{(y_1,\ldots,y_{n_t})h} = ({x_1}^{y_1}, \ldots, {x_{n_t}}^{y_{n_t}})^h.$$

Therefore,

$$\pi_h(\sigma^\rho) = \alpha^{y_1}.$$

If $h \notin \langle g_t \rangle$, then

$$\pi_h\left(\langle \sigma, \sigma^\rho \rangle\right) = \langle \pi_h(\sigma), \pi_h(\sigma^\rho) \rangle = \langle \beta, \alpha^{y_1} \rangle = A_5.$$

The last equality follows from Remark 6.2.1.
Hence $\langle \sigma, \sigma^\rho \rangle$ is not solvable, thus $\langle \sigma, \rho \rangle$ is not solvable too.
This means that, if $\rho \in Sol_{G_{t+1}}(\sigma)$ then $h \in \langle g_t \rangle$, hence $\rho = mh \in M_{t+1}\langle g_t \rangle$. □

In conclusion, using the same notations of the Lemma 6.2.1, we can state:

Corollary 6.2.1. $Sol_G(g) = \overline{\langle g \rangle}$

Proof. Let $x \in Sol_G(g)$. We'll work modulo N_{t+1} :

$$\begin{aligned} xN_{t+1} \in Sol_{G/N_{t+1}}(gN_{t+1}) &= Sol_{G_{t+1}}(g_{t+1}N_{t+1}) \\ &\subseteq Sol_{G_{t+1}}(\sigma N_{t+1}) \\ &\subseteq M_{t+1}\langle g_t \rangle \subseteq N_t \langle g_t \rangle = N_t \langle g \rangle. \end{aligned}$$

It means that

$$x \in \bigcap_t N_t \langle g \rangle = \overline{\langle g \rangle}.$$

□

Therefore the group G is not prosolvable, and $\overline{\langle g \rangle}$ is not properly contained in $Sol_G(g)$.
Thanks to this last result we got a second difference from the finite case:

Theorem 6.2.1 (Theorem 1.2. in [4])**.**

Let G to be a finite group. Suppose that there exists $x \in G$ so that the elements of $Sol_G(x)$ commutes pairwise. Then G is abelian.

In our example, $Sol_G(g) = \overline{\langle g \rangle}$ is an inverse limit of an inverse system of cyclic groups, hence its elements commutes pairwise, but G is not an abelian group.

Chapter 7

Profinite graphs

7.1 Notations and preliminaries

Remind that a topological space X is a **profinite space** if it is an inverse limit $X = \varprojlim X_i$ of an inverse system of finite spaces (endowed with the discrete topology).

Recall also that a topological space is **totally disconnected** if every point in the space is its own connected component.

One can describe a profinite space in terms of internal topological properties as follows:

a topological space is profinite if and only if it is compact, Hausdorff and totally disconnected.

In this section it'll useful to describe graphs by regarding their undirected edges as pairs of inverse (directed) edges. To make this precise

Definition 7.1.1. *A **graph** Γ consists of a set X_Γ equipped with*

- *a **source map** $s : X_\Gamma \to X_\Gamma$,*
- *an **inversion map** $i : X_\Gamma \to X_\Gamma$,*

satisfying the following conditions for all $x \in X_\Gamma$:

1. $s(s(x)) = s(x)$,
2. $i(i((x)) = x$,
3. $s(x) = x \iff i(x) = x$.

The common fixed point set of these maps is called **vertex set** of Γ and it is denoted by $V(\Gamma)$. From the property 1 it follows that $s(x) \in V(\Gamma)$ for any $x \in X_\Gamma$. In particular, the vertex $s(x)$ is called the **source of** x and we define the **target of** x to be $t(x) = s(i(x))$. We'll call **inverse of** x the element $i(x)$. The set $E(\Gamma) = X_\Gamma \setminus V(\Gamma)$ is called **edge set** of Γ.

In other words, the map s associates any edge $e \in E(\Gamma)$ to its initial point, and the map t associates e to its terminal point.

$$\bullet s(e) \xrightarrow{\quad e \quad} \bullet t(e)$$

In the Definition 7.1.1, we say *graph* and not *directed graph* because, since for any edge starting from the vertex v_1 and ending to the vertex v_2 there is an edge starting from v_2 and ending to v_1, we can identify the inverse directed edges e and $i(e)$ in one undirected edge.

Example 7.1.1.

Take the abelian (undirected) graph Λ of a group G. Remind that the vertex and the edge sets are

$$V(\Lambda) = G \setminus Z(G) \quad \textit{and} \quad E(\Lambda) = \{\{x, y\} \mid [x, y] = 1\}.$$

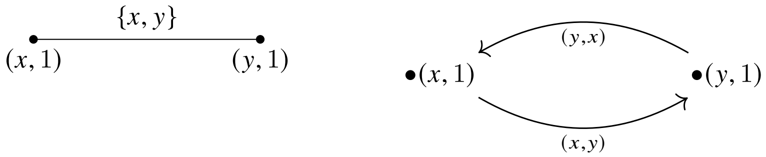
Fixed an element G, for example the identity, we can see the vertex and the edge sets as subsets of $G \times G$

$$V(\Lambda) \subseteq G \cong G \times \{1\} \subset G \times G \quad \textit{and} \quad E(\Lambda) = \{(x, y) \mid [x, y] = 1\} \subseteq G \times G$$

and we can define the source and the inversion maps as follows

$$\begin{aligned} s : & E(\Lambda) \to V(\Lambda) \\ & (x, y) \mapsto (x, 1) \end{aligned} \qquad\qquad \begin{aligned} i : & E(\Lambda) \to V(\Lambda) \\ & (x, y) \mapsto (y, x). \end{aligned}$$

The figure below shows how to interchange the two possibile ways to look at the same edge.



Definition 7.1.2. *A* ***morphism of graphs*** *is a function $f : \Gamma \to \Delta$ preserving sources and inverses. Explicitly, if we denote by s_Γ, s_Δ the two source maps, and by $i_\Gamma, i\Delta$ the two inversion maps, then*

$$s_\Delta(f(x) = f(s_\Gamma(x)) \quad \textit{and} \quad i_\Delta(f(x)) = f(i_\Gamma(x))).$$

It follows from the definition that also targets are preserved. Moreover, it should be noted that a morphism of graphs maps vertices in vertices, but edges don't need to be sent in edges.

Definition 7.1.3. *A graph Γ is said to be a* ***topological graph*** *if the underlying set X_Γ (that is $V(\Gamma) \cup E(\Gamma)$) is a topological space such that the source and the inversion maps are continuos.*

We'll be mainly interested in Hausdorff topological graphs. For such a graph the vertex set is closed (because the restriction of i and s to $V(\Gamma)$ is the identity map on $V(\Gamma)$) but edge set does not need to be closed. Thus, a compact and Hausdorff graph has a compact and Hausdorff vertex set, while the edge set does not need to be compact.

For the sake of brevity, from now on, the term *graph* will always indicate a topological graph. An abstract graph will be viewed as a topological graph with the discrete topology. By a *map of graphs* we'll mean a morphism of (topological) graphs which is a continuous function of the underlying topological spaces.

Example 7.1.2 (Cayley graph of a topological group).

*Let G be a topological group. Take X a pointed topological space, with basepoint $*$, and $\mu : X \to G$ continuous map. Moreover, let $X^{-1} = \{x^{-1} \mid x \in X\}$ be a disjoint copy of X, and set $\Delta := X \vee X^{-1}$ the topological sum of pointed spaces, in which the basepoints are identified. The **Cayley graph** of G with respect to the set and the map (X, μ) is the topological graph $\Gamma(G, X, \mu)$ whose underlying topological space is*

$$X_\Gamma := G \times \Delta$$

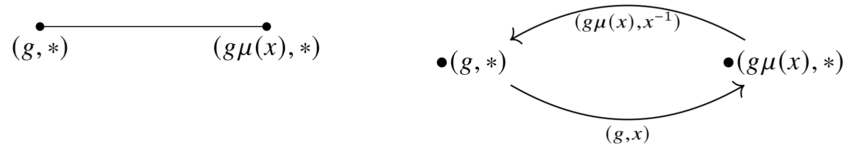
endowed with the product topology; and the source and the inversion maps are given by

$$s(g, x) := (g, *) \quad \text{and} \quad i(g, x) := (g\mu(x), x^{-1}).$$

The vertex set is $G \times \{*\}$ which is homeomorphic to G and the target map is given by

$$t(g, x) = s(i(g, x)) = (g\mu(x), *).$$

Following the Definition 7.1.1, on the left we've represented the edge with endpoints $(g, *)$ and $(g\mu(x), *)$ of the Cayley undirected graph of G associated to X and on the right we've identified the undirected edge with a pair of two inverse directed edges, thanks to the source and inversion maps.



Definition 7.1.4.

- *A **subgraph** of a graph Γ is a subset Σ which is closed under the source and the inversion maps of Γ. In this case, Σ, equipped with the restrictions of the operations of Γ and with the subspace topology, is a graph with $V(\Sigma) = V(\Gamma) \cap \Sigma$.*

- *The **product** of two graphs Γ and Δ is a graph whose underlying set is the product $\Gamma \times \Delta$ equipped with the product topology, and the source and the inversion maps are given by:*

$$s(x, y) := (s_\Gamma(x), s_\Delta(x)) \quad \text{and} \quad i(x, y) := (i_\Gamma(x), i_\Delta(y)).$$

The product of an arbitrary family of graphs is defined in the same way.

- *The **inverse limit** of an inverse system of graphs and maps of graphs, can be constructed in the usual way as a subgraph of a product of graphs; its underlying space is the inverse limit of the corresponding underlying spaces.*
  Explicitly: let $(I, \leq)$ be a directed partially ordered set. An inverse system of profinite graphs
  $$\{\Gamma_i, \phi_{i,j}, I\}$$
  over the directed poset I consists of a collection of graphs Γ_i indexed by I and maps $\phi_{i,j} : \Gamma_i \longrightarrow \Gamma_j$, whenever $i \geq j$ in such a way that
  $$\phi_{i,i} = id_i \text{ for all } i \in I$$
  $$\phi_{j,k}\phi_{i,j} = \phi_{i,k} \text{ whenever } i \geq j \geq k.$$
  The inverse limit of such a system
  $$\Gamma = \lim_{\longleftarrow i\in I} \Gamma_i$$
  is the subset of $\prod_{i\in I} \Gamma_i$ consisting of those tuples $(m_i)_{i\in I}$ with $\phi_{i,j}(m_i) = m_j$ whenever $i \geq j$.

Definition 7.1.5. *A graph Γ is said to be a **profinite graph** if it is an inverse limit of an inverse system of finite discrete graphs and maps of graphs.*

Proposition 7.1.1 (Proposition 2.1.4 in [3])**.**

Let $\Gamma = \lim_{\longleftarrow i\in I} \Gamma_i$ be a profinite graph. Then

- $V(\Gamma) = \lim_{\longleftarrow i\in I} V(\Gamma_i)$,
- *if $E(\Gamma)$ is closed, then $E(\Gamma) = \lim_{\longleftarrow i\in I} E(\Gamma_i)$.*

Similarly to the way that profinite groups and profinite spaces are characterized in terms of their topological properties, it can be shown that:

Proposition 7.1.2 (see [18])**.**

A graph is profinite if and only if its underlying topological space is totally disconnected, compact and Hausdorff, that is a profinite space.

Example 7.1.3 (Example 2.1.12 in [18])**.**

Let G be a profinite group, X a closed subset of G, and $\mu : X \longrightarrow G$ be the inclusion map.
Then the Cayley graph $\Gamma(G, X, \mu)$ is a profinite graph because its underlying topological space $X_\Gamma = G \times \Delta$ is the product of two profinite spaces (being X closed in a profinite space, it is profinite too).
Moreover, if we take the decomposition of G as an inverse limit of its finite quotients

$$G = \lim_{\longleftarrow i\in I} G_i,$$

where $G_i = G/N_i$ *and* N_i *runs over the set of open normal subgroups of* G*, then*

$$\Gamma(G, X, \mu)) = \varprojlim_{i \in I} \Gamma(G_i, X_i, \mu_i),$$

where $\pi_i : G \to G/N_i$ *is the canonical projection,* $X_i = \pi_i(X)$ *and* $\mu_i : X_i \to G_i$ *is the inclusion.*
Hence, the Cayley graph of a profinite group is the inverse limit of Cayley graphs of its finite quotients.

7.2 Profinite connectivity

In the context of profinite graphs we can introduce a new concept of connectivity.

Definition 7.2.1. *A profinite graph* Γ *is said to be* ***profinitely connected*** *if each finite continuous morphic image is connected (as an abstract graph).*
Explicitly: a profinite graph $\Gamma = \varprojlim_{i \in I} \Gamma_i$ *is profinitely connected* $\iff$ Γ_i *is connected for each* $i \in I$*.*

There's an equivalent definition of profinite connectivity:

Proposition 7.2.1 (see section 3 in [3])**.**

A profinite graph is profinitely connected if and only if it is not the union of two disjoint, non-empty, closed subgraphs.

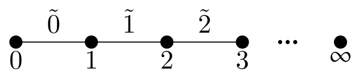
Remark 7.2.1. *The property of being profinitely connected is weaker than the property of being connected. Finite graphs, endowed with the discrete topology, are profinite graphs; in this case, the concept of connectivity and the one of profinite connectivity coincide. In general, any profinite graph in which any two vertices can be joined by a path is clearly profinitely connected, however the converse does not hold.*

Example 7.2.1 (A profinitely connected graph which is not connected as abstract graph)**.**

Let $N = \{0, 1, 2, \dots\}$*,* $\tilde{N} = \{\tilde{n} \mid n \in N\}$*, be two copies of the set of natural numbers endowed with the discrete topology. Define*

$$I = N \sqcup \tilde{N} \sqcup \{\infty\}$$

to be the one-point compactification of the space $N \sqcup \tilde{N}$*. Recall that in the topology of* I *each set* $\{n\}$*,* $\{\tilde{n}\}$ *is open (for every* $n \in N$*), and the basic open neighbourhoods of* ∞ *are the complements of finite subsets of* $N \sqcup \tilde{N}$*. Clearly,* I *is a profinite space and if we set* $V(I) = N \sqcup \{\infty\}$*,* $E(I) = \tilde{N}$ *where* $\tilde{n}$ *has as endpoints* n *and* $n+1$*, we get a profinite graph.*



We want to show that I *is profinitely connected but not connected. Consider the connected finite graph* I_n *with vertices* $V(I_n) = \{0, 1, 2, \dots, n\}$ *and edges* $E(I_n) = \{\tilde{0}, \tilde{1}, \tilde{2} \dots \widetilde{n-1}\}$

If $n \leq m$, define $\phi_{m,n} \colon I_m \to I_n$ to be the map of graphs that sends the segment $[0,n]$ identically to $[0,n]$ and the segment $[n,m]$ to the vertex n. Then $(I_n, \phi_{m,n})$ is an inverse system of graphs and

$$I = \varprojlim_{n \in N} I_n,$$

where $\infty = (n)_{n\in N}$. Hence I is a connected profinite graph, but there's no edge e of I which has ∞ as one of its endpoints; so I is not connected ad abstract graph.

7.3 Comparisons with pro-$\mathcal{F}$-graphs

It is intuitive to think that by constructing a graph, starting from a profinite group, we obtain a profinite graph. For example, we have seen that this is what happens in the case of the Cayley graph.

In the previous sections, we have run into a problem studying the connectivity of pro-$\mathcal{F}$-graphs. At first, we wished we would be able to extend the known results of the finite case to the profinite case, thanks to the good behavior of the $\mathcal{F}$-graphs of finite quotients of the profinite group. Unlikely, we have seen that this condition is not sufficient to guarantee the connectivity of the pro-$\mathcal{F}$-graph. We have talked about this problem in depth and we've found some other ways to study the connectivity of pro-$\mathcal{F}$-graphs in terms of the $\mathcal{F}$-graphs related to finite quotients. Anyway, concepts like weak and strong connectivity, even though they have been very helpful for our purposes, keep being artificial constructions.

In this last section, we have introduced a new concept of connectivity which, by definition, requires just the good behavior of the graph on finite quotients. For example, the Cayley graph of a profinite group, with respect to a a closed subset, is profinitely connected if and only if the Cayley graphs of finite quotients are connected.

Thus, one could ask why we didn't speak about the profinite connectivity before, and why we didn't try to study the profinite connectivity of the graphs (instead of the connectivity) to encode properties of profinite groups.

We did not talk about profinite connectivity because the graphs of our interest are not profinite graphs (in general). Indeed, in the construction of the pro-$\mathcal{F}$-graph, and in the one of the non-pro-$\mathcal{F}$-graph as well, the vertex set $G \setminus I_{\mathcal{F}}(G)$ is almost never compact, whence it is not a profinite space. This is because $I_{\mathcal{F}}(G)$ is not open in general; for example if G is not virtually pronilpotent, then its hypercenter $Z^{\infty}(G)$ cannot be open. In the same way, also when we speak about the extended-pro-$\mathcal{F}$-graph, the vertex set $G \setminus \{1\}$ is dense in G, so it is not closed, whence it is not a profinite space.

What happens if we consider as vertex set the whole profinite group?

Let us denote by $\Sigma_{\mathcal{F}}(G)$ the graph obtained by the pro-$\mathcal{F}$-graph of G considering also the universal vertices in $I_{\mathcal{F}}(G)$. It means that $V(\Sigma_{\mathcal{F}}(G)) = G$ and $E(\Sigma_{\mathcal{F}}(G)) = \{\{x, y\} \mid \langle x, y\rangle \text{ is a pro-}\mathcal{F}\text{-group}\}$.

Remind that we use the notation $\mathcal{M}$ to indicate the set of all open normal subgroups of G.

About the graph just introduced we have the following result.

Proposition 7.3.1. *For any profinite group G, it holds that $\Sigma_{\mathcal{F}}(G)$ is a profinite group.*

Proof. As we've done in the Example 7.1.1, we can look at $\Sigma_{\mathcal{F}}(G)$ as a subset of G^2. We need to prove that the underlying space is profinite. First, note that the vertex set of the graph is isomorphic to G, which is a profinite space by definition. So, to prove the thesis, it's enough to show that the edge set is a profinite space. For any $N \in \mathcal{M}$ we define

$$\Omega_N := \{(x, y) \in G^2 \mid \langle x, y\rangle N \text{ is a pro-}\mathcal{F}\text{-group modulo } N\}.$$

Since for any $z, w \in N$ and $(x, y) \in \Omega_N$ it holds that $(xz, yw) \in \Omega_N$, then we get that Ω_N is a union of cosets of N^2. Moreover, since N has finite index in G (and so does N^2 in G^2), then the union of cosets defining Ω_N is finite, whence Ω_N is closed in G^2. By the definition of $\Sigma_{\mathcal{F}}(G)$ its edge set is

$$E(\Sigma_{\mathcal{F}}(G)) = \{(x, y) \in G^2 \mid \langle x, y\rangle N/N \in \mathcal{F} \text{ for all } N \in \mathcal{M}\} = \bigcap_{N\in\mathcal{M}} \Omega_N.$$

Being intersection of closed subsets, the edge set of $\Sigma_{\mathcal{F}}(G)$ is closed in the profinite space G^2, whence we can conclude that $E(\Sigma_{\mathcal{F}}(G))$ is profinite. □

Hence, for a given profinite group G, we are allowed to talk about the profinite connectivity of $\Sigma_{\mathcal{F}}(G)$, but since the graph is already connected as abstract graph (because $I_{\mathcal{F}}$ belongs to the vertex set), it is also profinitely connected. So, investigating the profinite connectivity instead of the connectivity does not make sense.

Remark 7.3.1. *The same result of the Proposition 7.3.1 cannot be stated for the graph $X_{\mathcal{F}}(G)$ obtained from the non-pro-$\mathcal{F}$-graph considering also the vertices in $I_{\mathcal{F}}(G)$. In this case, the edge set would be the complement of the edge set of $\Sigma_{\mathcal{F}}(G)$) which is not closed in general. Indeed,*

$$E(X_{\mathcal{F}}(G)) = \{(x, y) \mid \text{ there exists at least one } N \in \mathcal{M} \text{ such that } \langle x, y\rangle N/N \notin F\}.$$

Hence, we can not conclude that $E(X_{\mathcal{F}}(G))$ is the inverse limit of $\{E(X_{\mathcal{F}}(G/N))\}_{N\in\mathcal{M}}$.

At the beginning of the section, we've said that the pro-$\mathcal{F}$-graph of a profinite group G, is not a profinite graph in general because $I_{\mathcal{F}}(G)$ is almost never open. We want to look at the cases in which this happens.

Consider the prosolvable graph of G. We already know that the set of its universal vertices is the prosolvable radical $R(G)$. If $R(G)$ is open, then the prosolvable graph will be a profinite graph. But, also this case turns out to be useless for our purposes. Indeed, if $R(G)$ were open then $G/R(G)$ would be finite, so the solvable graph $\Lambda_{\mathcal{S}}(G/R(G))$ would be connected. By Theorem 4.0.1 the connectivity of the prosolvable graph of G is implied by the connectivity of the prosolvable graph of $G/R(G)$, so we would conclude that

Proposition 7.3.2. *If the prosolvable graph is a profinite graph, then it is connected.*

The same observation can be done for the pronilpotent graph thanks to the Theorem 4.0.2. But, since nilpotent graphs of finite groups are not always connected we can just say that

Proposition 7.3.3. *If the pronilpotent graph $\Lambda_{\mathcal{N}}(G)$ of a profinite group G is a profinite graph, then $\Lambda_{\mathcal{N}}(G)$ is profinitely connected if and only if $\Lambda_{\mathcal{N}}(G/Z^{\infty})$ is connected.*

In other words, studying the profinite connectivity of the pronilpotent graph of G is the same of studying the connectivity of the nilpotent graph of the finite group $G/Z^{\infty}(G)$.

To conclude, we observe that we're not allowed to say something similar about the case in which the commuting graph is a profinite graph because of the bad behavior of the centre (see 4.3).

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Acknowledgments

I want to thank my supervisor Andrea Lucchini, who has been helping me a lot in writing this thesis. I'm very grateful to have had the opportunity to work with him on these interesting problems.

I want to thank my mum and my dad, thanks for your support and for never doubt on me. Without you, this would not have been possible.

Thanks to Elena and Alice, you are a second family for me, thanks to be always by my side.

Thanks to Enrico, sharing life with you makes me happy.