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CONTROL SYSTEM ENGINEERING

A DATA DRIVEN APPROACH TO THE ABSOLUTE STABILIZATION PROBLEM AND ITS EXTENSION TO NON EUCLIDEAN NORMS

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To my supervisor
Thank you for following me during this thesis even though I was constantly late and imprecise, forcing me to do things the right way

To my parents
Despite making everything more difficult that should have been, thank you for supporting me during the process and not kicking me out.

To Toni
Your sandwiches gave me a good reason to come to Padova everyday.

Feedback will free your mind
and set you free
Dopethrone-Electric Wizard

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Introduction

The absolute stability problem constitutes one of the milestones in the history of nonlinear control and still represents one of the most interesting topics of in the subject, bridging the classic and well established linear system theory to the more complex and less known nonlinear one. Alongside its theoretical interest, the versatility of its formulation apply to a large number of modelling problem, justifying almost a century of research.

The recent advances in the field of contraction theory and the introduction of the stability analysis in non Euclidian norms pose a new challenge in the absolute stability theory, opening the possibility to a new view of the notion of stabilization. On the other hand, one of the most relevant subjects in control theory is the employment of data in order to reconstruct the relevant characteristics of a unknown system, and assuring desired features. On this direction, the idea of obtaining a control law for an unknown dynamical system directly from the data represents an interesting prospective.

In light of those considerations, this thesis has been dedicated to review the absolute stabilization problem and to study its extension to a data-driven approach. Starting from a classical approach based on quadratic Lyapunov functions we studied how to solve the absolute stability problem resorting uniquely to a data-based knowledge of the system. Subsequently, after considering a detailed review of the stability analysis in non Euclidean spaces, we proposed its extension to the data driven design problem.

The purpose pursued during our studies has been to obtain a novel data driven design approach, which, exploiting contractive theory results on general spaces, is able to propose a novel solution to the absolute stability problem.

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The Absolute Stabilization problem

The first chapter of the thesis is dedicated to introduce the problem of the *Absolute Stability* and the notion of *Lur'e System*, alongside to some of the most common solutions to the problem.
It will be first presented the original problem statement, together with its solution, as initially proposed by A.I. Lur'e and based on the construction of a modified quadratic Lypunov function, moving to the statement of the criteria based on frequency analysis and the ideas behind their proofs. Due to its influence in the history of absolute stability, we will also briefly review the Aizerman and Kalman conjectures.
In the second section of the chapter we will consider the well-known *S-Lemma*, which will have a fundamental role in course of the thesis, both in formulating the classical solution of the absolute stability problem in a data-driven fashion and in the extension of Lyapunov stability analysis to non-Euclidean spaces. Given its importance in the nonlinear control field and in the results presented in the following chapters it will be presented a complete proof of the lemma.

1.1 Lur'e systems and problem formulation

Historically, the problem of *absolute stability* and the concept *Lur'e system* was introduced for the first time by A.I. Lur'e in V.N. Postnikov in 1944 in the paper [1], where it is studied the global asymptotically stability of the origin for a third order linear system in feedback connec-

tion to a non linear transformation of the system output. The problem has been then formalized and extended to systems of arbitrary order in 1951 by Lur'e in his first book [2].
The book of Lur'e is considered to be the first book to be entirely devoted to the topic of nonlinear control [3].
Formally, a Lur'e system is described by the following equations

$$\begin{cases} \dot{x}(t) = Ax(t) + B\varphi(y(t)) \\ y(t) = Cx(t) + r\xi(t) \\ \dot{\xi}(t) = -\varphi(y(t)) \end{cases} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$, $r \in \mathbb{R}$.
The nonlinearty, which is the function $\varphi(\cdot)$, can be indeed both linear or nonlinear. In its first formulation it has been assumed to be time-invariant and to satisfy the so-called *sector conditions*:

$$\begin{aligned} 0 \le y\varphi(y) \ &\le \kappa y^2 \quad \forall y \in \mathbb{R}\backslash\{0\} \\ \varphi(0) &= 0 \end{aligned} \tag{1.2}$$

for some $\kappa > 0$.
The sector condition has originally been introduced to model the friction effect affecting the actuators dynamics in a mechanical engine, from which it follows the positiveness and bound conditions on the product $y\varphi(y)$, see [2].
As shown later in the introduction of the *Circle Criterion*, the problem can be extended to the case of time-varying nonlinearities.
Even if will be not explicitly mentioned we will always assume the nonlinearity to be at least Lipschitz continuous, in order to satisfy the conditions of existence and uniqueness of solution in the ODE in (1.1). Such requirement will be still assumed fulfilled in the extension to the time-varying and multi-dimensional case.
As already mentioned, a Lur'e system can be essentially described as a system divided in two main blocks: a linear time invariant subsystem and a function of the system output, the latter considered as a linear combination of the state variables and the derivative of the function itself, related together by a feedback connection.
The *nonlinearity* models a part of the dynamic which doesn't have a simple analytical form or some uncertainties present in the model description.
Examples of dynamics modeled by *Lur'e type* systems can be found in the analytical descriptions of linear system for which the actuator limitations introduce some non linear features, when some part of the dynamics cannot be approximated with sufficient precision in the range of values in which the linear system operates or in dynamics affected by friction.
The solution of the absolute stability problem consists in finding conditions on the linear time

invariant system, and on the sector constraint (1.2), in order to guarantee that the origin is globally asymptotically stable for all nonlinearities that satisfy (1.2).
The problem was initially solved by A.I. Lur'e and V.N. Postnikov using a modified Lypunov function composed by a quadratic term of the state variables x plus a weighted integral of the nonlinearity

$$V(x,\xi) = \frac{1}{2}x^T Hx + \beta \int_0^y \varphi(\zeta)\, d\zeta \tag{1.3}$$

with H positive definite, $\beta > 0$; by condition (1.2) the integral term is assured to be non-negative. Exploiting the controllability of the system together with some others ad hoc adjustments for the given problem considered in [1] it was possible to prove the time derivative of (1.3) to be negative definite.
A Lypunov function of type (1.3) is also known a *Lur'e-Postnikov* function.

Remark.
Frequently it is considered the slightly different definition of the sector condition (1.2)

$$\kappa_1 \leq \frac{\varphi(y)}{y} \leq \kappa_2 \quad \forall t \in \mathbb{R}\backslash\{0\}, \quad \varphi(0) = 0 \tag{1.4}$$

for some real κ_1, κ_2. *However, as we will show in Chapter 4,* (1.4) *can always be rewritten in the same form of* (1.2).
We point out moreover, for reason that will be clarified in the study of the S-Lemma in the next section, that (1.4) *can be rewritten using the quadratic form*

$$(\varphi(y) - \kappa_1 y)(\kappa_2 y - \varphi(y)) \geq 0 \tag{1.5}$$

1.2 Stability criteria

The implications, both theoretical and applicative, that the introduction of the absolute stability concept had in the field control theory rapidly attracted the interest of a large number of researcher from mathematicians to engineers. For over a decade the problem of extending the applications of a Lypunov function in the Lur'e-Postnikov form has been investigated, leading to numerous hypotheses and conjectures, of which the most famous has been the ones by Aizerman and Kalman.[4] [3].
Besides the employment of the classical Lyapunov approaches, some of the most outstanding results to guarantee the absolute stability of Lur'e-type systems were found in the frequency domain, which connected the existence of Lyapunov functions to some conditions related to the linear time invariant system transfer function, avoiding the explicit construction of such Lyapunov function.

The first frequency domain version was discovered and published by the Romanian mathematician V.M Popov in 1961[5].

Theorem 1.1 (Popov Criterion)**.**
Consider a Lur'e system of the form (1.1) *with* φ *satisfying the sector condition* (1.2)*, and let* $h(s)$ *be the transfer function of its linear time invariant part, namely* $h(s) = c(sI - A)^{-1}B$*. Assume the following conditions holds*

(1.) A is Hurwitz stable

(2.) (A,B) is controllable, (A,C) is observable

Then system (1.1) *is absolute stable if there exists* $m > 0$ *such that* $Re(1 + j\omega m)h(jw) > 0 \; \forall \; \omega \in \mathbb{R}$.

The theorem was proven by using a Lur'e-Postnikov type function of the form

$$V(x) = x^T P x + \alpha(y - C^T x)^2 + \beta \int_0^y \varphi(\zeta)\, d\zeta \tag{1.6}$$

where α, β are positive real numbers, and $P = P^T > 0$ is a positive definite matrix.
In the same article [5] Popov proved that the existence of a function (1.6) is sufficient to assure that the conditions of his theorem are satisfied. The necessity of Popov conditions for the existence of Lur'e Postnikov function, which can be reduced on the existence of a negative definite matrix and a vector satisfying a linear quadratic equality, was proved in 1962 by V.A. Yakubovich and extended by R.Kalman the same year [6] [7].
The solution of such matrix equality were linked to a frequency condition on the transfer function of the linear system, as formalized by the celebrated Kalman-Yakubovic-Popov lemma:

Theorem 1.2. *[Kalman-Yakubovic-Popov Lemma]*
Given a number $\gamma > 0$ *two vectors* $B \in \mathbb{R}^n$ *and* $C \in \mathbb{R}^{1\times n}$ *and an* $n \times n$ *Hurwitz matrix A, if the pair (A,B) is controllable, then a symmetric matrix* P *and a vector* q *satisfying*

$$\begin{aligned} -qq^T &= PA - A^T P \\ \sqrt{\gamma} q &= PB - C^T \end{aligned} \tag{1.7}$$

exist if and only if

$$2\gamma + Re[C^T(i\omega I - A)^{-1}b] \geq 0 \;\forall \omega \in \mathbb{R} \tag{1.8}$$

Moreover the set $\{x : x^T P x = 0\}$ *is the unobservable subspace of the pair* (C, A)

It is possible to prove in fact that the the conditions of the Kalman-Yakubovich-Popov lemma implies the one of the Popov criterion, and assure the existence of a Lur'e Postnikov

Lyapunov function of the type (1.6) (a detailed proof can be found in [8][9]).
Besides the outstanding result of linking the existence of a Lur'e-Postnikov Lyapunov function to a system of linear matrix equality, the applications of the Kalman-Yakubovic-Popov exceed the one related to the absolute stability problem: indeed, it is possible to prove that the conditions of Theorem (1.2) are equivalent to the passivity property[8][10].[1]
Lastly we give the statement of the circle criterion, which represents an extension of the previous results to the time varying case.

Theorem 1.3. *[Circle Criterion]*
Consider system

$$\begin{cases} \dot{x}(t) = Ax(t) + b\varphi(y(t)) \\ y(t) = Cx(t) + r\xi \\ \dot{\xi} = -\varphi(t, y(t)) \end{cases} \tag{1.9}$$

with $\varphi(t, y(t))$ satisfying sector condition (1.4) *for some μ_1, μ_2 If the following condition holds*

(i) the matrix A has no purely imaginary eigenvalues.

(ii) exists $\kappa_0 \in [\kappa_1, \kappa_2]$ such that the linear system obtained by setting $\varphi(t, y(t) = \kappa_0 y$ is Hurwitz stable.

(iii) $Re[(\kappa_2 C(i\omega I_n - A)^{-1}B - 1)(1 - \kappa_1 C(i\omega I_n - A)^{-1}B)] < 0 \ \forall\omega \in [-\infty, +\infty]$

Then then the origin is exponential stable for system (1.9).
Moreover, under conditions (i-ii) condition (iii) is necessary and sufficient for the existence of a quadratic Lypunov function $V(x) = x^T Px$ with time derivative $\dot{V}(x) < 0$ for all systems (1.9) *with $\varphi(t, y)$ satisfying* (1.4) *for some μ_1, μ_2.*

The circle criterion (1.3), proved the same year by Yakubovich and E.N. Rosenwasser in [11] and [12], follows the idea that time-varying nonlinearities can be better treated by truncating the integral part in (1.3): such an idea, already considered by Rosenwasser in [13], is related to the fact that the time derivative of the integral component in the Lur'e-Postinikov function, when considered a time varying function, would introduce a time varying component in the time derivative of $V(x)$

$$\frac{d}{dt}\Big(\int_0^y \varphi(\zeta, t)\, d\zeta \Big) = \frac{dy}{dt}\varphi(y) + \int_0^y \frac{\partial \varphi(\zeta, t)}{\partial t}\, d\zeta \tag{1.10}$$

which would make the problem of more difficult.

[1]The property of passivity can shortly be introduced by stating that in a passive system the product of the system input and output $u^T y$ is always non-negative. Intuitively a passive system is one that dissipate power.

As a matter of fact, in the stability analysis presented in the next chapters, which will be focused on the absolute stability problem for time varying nonliearities, we will find a stabilizing solution resorting to quadratic Lyapunov functions.

Remark. *The formulations of the Lur'e system adopted in the previous results and statements have been chosen for historical reasons.*
However, in the course of the thesis we will consider the following different formulation of Lur'e system with respect to the one already presented

$$\begin{cases} \dot{x}(t) = Ax(t) + B\omega(t) \\ y(t) = Cx(t) \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{1.11}$$

Nonetheless (1.11) *and* (1.1) *are equivalent: indeed, if we rewrite the Lur'e system* (1.1) *in the augmented state* $\tilde{x} = \begin{pmatrix} x & \xi \end{pmatrix}^T$*, defining the matrices*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} B \\ -1 \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} C & r \end{pmatrix} \tag{1.12}$$

we obtain that the descriptions the two system are equivalent.
It is easy to note, that controllability and observability properties of system (1.1) *are preserved.*

1.3 Aizerman and Kalman conjectures

Right after that the formulation of a sufficient condition to absolute stability has appeared, the natural interest arose about defining also the necessary conditions.
The research in this direction led to a series of hypothesis and counterexamples for which the aforementioned Aizerman and Kalman conjectures represent the most influential ones.
Proposed in 1949 in [14] the Aizerman's hypothesis suggested that a Lur'e system of the form (1.1) is absolute stable for all nonlinearities satisfying the sector condition (1.2) if it is stable for all linear systems obtained by replacing the nonlinearity with a linear output feedback of the form $\varphi(y) = \mu y \ \forall \ \mu \in [0, \kappa]$.
The hyphotesis however was proven to be false by 1958 by V. Pliss which constructed a third order system counterexample[15]. However, under the stronger assumptions that the Lur'e system considered is positive, the conjecture holds true[16]. We will show an equivalent proof of the last statement in chapter 4.
In 1957 Rudolf Kalman tried to extend Aizerman's hypothesis to a rigid condition on the nonlinearity: he suggested that if the derivative of the function $\varphi(y)$ lies on the sector $[0, \kappa]$ and the system is asymptotically stable for all $\varphi(y) = \mu y, \ \mu \in [0, \kappa]$ then the Lur'e system (1.1)

is absolute stable.
Kalman's conjecture has proved to be true for system system up to the third order, while for systems of higher order there are effective methods for construction of counterexamples[17][18].

1.4 S-Lemma

The *S-Lemma* (also known as *S-procedure*) is one of the most popular and important result in the field of nonlinear control theory. The procedure, introduced by M.A Aizerman in [19] and already used in the early days of the absolute stability theory development, was formally stated and proved as we know nowadays by V.A. Yakubovich in his seminal paper [20].
The objective of the S-Lemma is to find conditions which guarantee that a sign condition on a quadratic function implies another sign condition to a second quadratic function.
In the absolute stability problem such question arises naturally when the first quadratic inequality is the sector condition expressed as a quadratic inequality (1.5), while the second one is the time derivative of a quadratic Lyapunov function: indeed, consider the a Lur'e system described by equations (1.11) and a quadratic Lyapunov function in the state variables $V(x) = x^T Px$ where $P = P^T > 0$; the time derivative of $V(x)$ is given by

$$\begin{aligned} \dot{V}(x,\omega) &= \dot{x}^T Px + X^T P\dot{x} = (Ax + B\omega)^T Px + x^T P(Ax + B\omega) \\ &= \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} A^T P + PA & PB \\ B^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \end{aligned} \tag{1.13}$$

where we have neglected the time dependency of $x(t)$ and $\omega(t)$ for notational reasons.
We will call the quadratic form induced by $\dot{V}(x,\omega)$ as $\mathcal{F}(x,\omega)$. Let $\mathcal{G}(x,\omega) = (\omega - \kappa_1 y)(\kappa_2 y - \omega)_{|y=Cx}$.
We observe that

$$\begin{aligned} \mathcal{G}(x,\omega) &= \kappa_2 \omega y - \omega^2 - \kappa_1\kappa_2 y^2 + \kappa_1 y\omega \\ &= \omega\kappa_2 Cx - \omega^2 - \kappa_1\kappa_2 (Cx)(Cx) + \kappa_1 Cx\omega \\ &= \omega\kappa_2 Cx - \omega^2 - \kappa_1\kappa_2 x^T C^T (Cx) + \kappa_1 x^T C^T \omega \\ &= \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} -\kappa_1\kappa_2 C^T C & \kappa_1 C^T \\ \kappa_2 C & -1 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \geq 0 \quad for\ all\ (x,\omega) \end{aligned}$$

where we have exploited the fact that Cx is a scalar, so that $Cx = (Cx)^T = x^T C^T$.
For proving the stability of the system we need to detemine wheter

$$\dot{V}(x,\omega) < 0 \quad for\ all \quad (x,\omega) \quad s.t. \quad (\omega - \kappa_1 Cx)(\kappa_2 Cx - \omega) \geq 0 \tag{1.14}$$

that is equivalent to

$$\{(x,\omega) : \mathcal{F}(x,\omega) < 0\} \subseteq \{(x,\omega) : \{\mathcal{G}(x,\omega) \geq 0\} \tag{1.15}$$

In such context the S-Lemma proves that (1.15) is equivalent to having that $\exists\ \tau \geq 0$ such that

$$\mathcal{F}(x) + \tau\mathcal{G}(x) < 0 \tag{1.16}$$

In fact, the implication (1.14) its true if and only if (1.16) has solution, as proven in Theorem (1.7).
Before proceeding in the statement and the proof of the S-Lemma for a single quadratic constraint we give the definition of *quadratic form* and mention two preliminary results.

Definition. *A real quadratic form is a polynomial with only terms with degree two*

$$q_A(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \quad a_{ij} \in \mathbb{R}\ x_i \in \mathbb{R}\ \forall\ i,j = 1,\ldots,n \tag{1.17}$$

Equivalently a real quadratic form can be expressed as

$$\begin{aligned} q_A : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto x^T A x \end{aligned}$$

for some real matrix A

Theorem 1.4 (Dines)**.**
Let us consider $\mathbb{R}^{\mathbb{N}}$, and assume $\mathcal{F}(x)$, $\mathcal{G}(x)$ to be quadratic forms on $\mathbb{R}^{\mathbb{N}}$.
Consider the map

$$\begin{aligned} \eta : X &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (\mathcal{F}(x), \mathcal{G}(x)) \end{aligned} \tag{1.18}$$

Then, the image $\eta(\mathbb{R}^{\mathbb{N}})$ of $\mathbb{R}^{\mathbb{N}}$ through the map η, is a convex cone, i.e.
$\forall\ x, y\ \in \eta(\mathbb{R}^{\mathbb{N}})$ *and* $\forall\ a, b$ *positive scalars, the sum* $ax + by$ *belongs to* $\eta(\mathbb{R}^{\mathbb{N}})$

Theorem 1.5 (Hahn Banach)**.**
Let Q and P be two convex sets in $\mathbb{R}^n$ *s.t.* $Q \cap P = \emptyset$*; then there exist* $h \in \mathbb{R}^n$*,* $h \neq 0$ *and* $c \in \mathbb{R}$ *for which* $h^T q \leq c$*,* $\forall q \in Q$ *and* $h^T p \geq c$*,* $\forall\, p \in P$*.*

Theorem 1.6. *[S-Lemma for non-strict inequalities]*
Consider $\mathcal{F}(x)$ *and* $\mathcal{G}(x)$ *to be two real quadratic quadratic forms defined on a* $\mathbb{R}^N$ *and define*

$$\mathcal{F}_0^- := \{x \in X : \mathcal{F}(x) \leq 0\} \qquad \mathcal{G}_0^+ := \{x \in X : \mathcal{G}(x) \geq 0\}$$

Assume moreover there exist $\tilde{x} \in X$ *s.t* $\mathcal{G}(\tilde{x}) > 0$*.*
Then the following conditions are equivalent

(i) $\mathcal{F}_0^- \supseteq \mathcal{G}_0^+$

(ii) $\exists\, \tau \geq 0\ s.t.\ \mathcal{F}(x) + \tau\mathcal{G}(x) \leq 0 \quad \forall\, x \in \mathbb{R}^N$

Proof.
$(ii) \Rightarrow (i)$ is an obvious implication: in fact if $\exists\tau \geq 0$ such that $\mathcal{F}(x) \leq -\tau\mathcal{G}(x)$ for all $x \in \eta(\mathbb{R}^{\mathbb{N}})$, if x is such that $\mathcal{G}(x) \geq 0$ than consequently $\mathcal{F}(x) \leq 0$
$(i) \Rightarrow (ii)$: Let us start by showing that if (i) is true than there exist τ_1 and τ_2 s.t. $\tau_1\mathcal{F}(x) - \tau_2\mathcal{G}(x) \geq 0\ \forall\ x \in P$, where $P = \eta(\mathbb{R}^N)$ and η is the map defined as in Theorem (1.4).
We already know from the same theorem that P is a convex cone.
Consider now the set $Q \subset \mathbb{R}^2$

$$Q := \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 > 0,\ \eta_2 \geq 0\}$$

It straightforward to observe that $P \cap Q = \emptyset$. Indeed assume by contradiction that exist $\eta^0 = (\eta_1^0, \eta_2^0) \in P \cap Q$: since η_0 belongs to P exist x that satisfies $\mathcal{F}(x) = \eta_1^0$, $\mathcal{G}(x) = \eta_2^0$. Belonging to Q, $\eta_2^0 \geq 0$, which implies $\eta_1^0 \leq 0$ by hypothesis (i). This means η_0 cannot belong to Q, giving a contradiction.
Since P and Q are convex sets, by Theorem (1.5) there exist an straight line that strictly separates them, which means that exist τ_1 and τ_2, $(\tau_1, \tau_2) \neq (0, 0)$ such that :

$$\tau_1\eta_1 - \tau_2\eta_2 \geq 0 \quad \forall(\eta_1, \eta_2) \in P \tag{1.19}$$

$$\tau_1\eta_1 - \tau_2\eta_2 \leq 0 \quad \forall(\eta_1, \eta_2) \in Q \tag{1.20}$$

Take now $(\eta_1^1, \eta_2^1) = (1, 0) \in Q$ and $(\eta_1^2, \eta_2^2) = (\epsilon, 1) \in Q$, with $\epsilon > 0$; by substituting them in (1.20) we obtain

$$\tau_1\eta_1^1 - \tau_2\eta_2^1 = \tau_1 \leq 0 \Rightarrow \tau_1 \leq 0$$

$$\tau_1\eta_1^2 - \tau_2\eta_1^2 = \epsilon\tau_1 - \tau_2 \leq 0$$

where the last inequality is satisfied for ϵ arbitrary small, so taking the limit $\epsilon \to 0^+$ we obtain $\tau_2 \geq 0$.
We prove by contradiction that $\tau_1 \neq 0$. If we assume $\tau_1 = 0$ then $\tau_2 \neq 0$ (τ_1,τ_2 cannot be both zero) and since $\tau_2 \geq 0$ then we conclude that $\tau_2 > 0$. By hypothesis there exists $\tilde{x}$ such that $\mathcal{G}(\tilde{x}) > 0$.
Take $(\mathcal{F}(\tilde{x}), \mathcal{G}(\tilde{x})) \in P$. Then

$$0 \leq \tau_1 \mathcal{F}(\tilde{x}) - \tau_2 \mathcal{G}(\tilde{x}) = -\tau_2 \mathcal{G}(\tilde{x}) < 0 \quad for\ \tilde{\eta} \in P \tag{1.21}$$

that is a contradiction. Lastly, dividing by (1.19) by τ_1 and calling $-\frac{\tau_2}{\tau_1} = \tau$ (observe that divide for a negative number change the verse of the inequality) we obtain that

$$\eta_1 + \tau\eta_2 \leq 0 \quad \forall\ (\eta_1, \eta_2) \in P$$

which is equivalent to $(i) \Rightarrow (ii)$. □

By extending Theorem (1.6) we obtain the S-Lemma corresponding to *strict inequalities*, which will be used in Chapter 3.

Theorem 1.7. *[S-Lemma for strict inequalities]*
Consider $\mathcal{F}(x)$ and $\mathcal{G}(x)$ to be two real quadratic forms defined on $\mathbb{R}^N$ and define

$$\mathcal{F}^- := \left\{x \in X : \mathcal{F}(x) < 0\right\} \qquad \mathcal{G}_0^+ := \left\{x \in X : \mathcal{G}(x) \geq 0\right\}$$

Assume moreover there exist $\tilde{x} \in \mathbb{R}^N$ s.t $\mathcal{G}(\tilde{x}) > 0$.
Then the following conditions are equivalent

(i) $\mathcal{F}^- \supseteq \mathcal{G}_0^+$

(ii) $\exists\ \tau \geq 0\ s.t.\ \mathcal{F}(x) + \tau\mathcal{G}(x) < 0 \quad \forall\ x \in \mathbb{R}^N \setminus \{0\}$

Proof. $(ii) \Rightarrow (i)$: as in the previous theorem.
$(i) \Rightarrow (ii)$: define the set S = $\{$x : $\|x\| = 1,\ \mathcal{G}(x) \geq 0\}$, which is bounded, closed and non empty since there exists $\tilde{x} \in \mathbb{R}^N$ such that $\mathcal{G}(\tilde{x}) > 0$. From hypothesis (i) it follows that $\mathcal{F}(x) < 0$ for $x \in S$. Therefore, by Weierstrass theorem since S is compact and $\mathcal{F}$ is continuous, we argue that $\sup_{x \in S} \mathcal{F}(x) = e < 0$. Hence, $\mathcal{F}(x) - e \leq 0\ \forall\ x \in S$. If we choose $x \neq 0$ such that $\mathcal{G}(x) \geq 0$, then, $\frac{x}{\|x\|} = 1 \in S$, and so $\mathcal{F}(x/\|x\|) - e \leq 0$, i.e. $\mathcal{F}(x) - e\|x\|^2 \leq 0$ for x s.t. $\mathcal{G}(x) \geq 0$.
That is, $(\mathcal{F}(x) - e\|x\|^2)$ is a quadratic form which assumes nonpositive values for all $x\ s.t.\ \mathcal{G}(x) \geq 0$. By the previous theorem, exists $\tau \geq 0$ that satisfies $\mathcal{F}(x) - e\|x\|^2 + \tau\mathcal{G}(x) \leq 0$, $\forall\ x \in \mathbb{R}^N$. Observing that $e\|x\|^2 < 0$ for all $x \neq 0$ we have $\mathcal{F}(x) - \tau\mathcal{G}(x) \leq e\|x\|^2 < 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$ □

Remark 1.

Theorems (1.6) *and* (1.7) *can be extended to the case of generic quadratic functions, i.e.* $\mathcal{F}(x) = x^T A_f x + b_f^T x + c_f$, $\mathcal{G}(x) = x^T A_g x + b_g^T x + c_g$, *as showed in [21]*

Remark 2.

Theorems (1.6) *and* (1.7) *can be extended to the case in which multiple quadratic constraints are considered: consider the following conditions*

(i) $\mathcal{F}_0^- \subseteq \bigcap_{i=1}^h \mathcal{G}_{0i}^+$

(ii) $\exists\ \tau_i \geq 0\ i = 1, \ldots, h\ s.t.\ \mathcal{F}(x) - \sum_{i=1}^n \mathcal{G}_i(x) \geq 0\ \forall x \in X$

we have that $(ii) \Rightarrow (ii)$ *, where* $\mathcal{F}_0^-$ *is the same as the previous theorems and* $\mathcal{G}_{0i}^+ = \{x \in X : \mathcal{G}_i(x) \geq 0\}$.

The implication showed above has immediate proof; however the inverse implication is not true in general, since the map

$$\begin{aligned} \eta :\, & X \longrightarrow \mathbb{R}^h \\ & x \longmapsto \eta(x) = (\mathcal{F}(x), \mathcal{G}_1(x), \ldots, \mathcal{G}_h(x)) \end{aligned}$$

can be proven to have convex image only for $h = 2$*. For* $h > 2$ *the proof of Theorem* (1.6) *fails when it comes to find a straight line which separates the sets* P *and* Q*.*

2
Data-Driven Control

In this chapter we will be present survey on the Data-Driven approach to control design for linear time invariant systems.
Despite not being directly used in the the stability analysis to Lur'e systems presented in the next chapters, the results considered here have the purpose of introducing the subject of direct data-driven control and of developing the intuition behind this approach. We will present the *Willems Fundamental Lemma* in its state-space form and we will discuss its role in obtaining a representation of both open and closed loop dynamics of a linear time invariant system by exploiting a collection of sample data from the system itself. We will show and how to relate such results to the design of a stabilizing controller.
Finally we will consider the case in which the data are affected by disturbances, and we will discuss sufficient conditions under which is still possible to assure stability of the closed loop.
Given the illustrative nature of the chapter, we will omit the proofs of the presented results.

2.1 Introduction: Direct Data-Driven control

Learning from data is an essential component in engineering: the ubiquitous presence of disturbances, modeling errors and uncertainties require the constant adjustment from updating new data of the employed model, and consequently of the implemented control law, in order to guarantee that the behaviour of the controlled system satisfies the required performances. As matter

of facts, part of the effort in the control design phase relies in guaranteeing the stability of the controlled system in case its dynamics differs from the the nominal one, due to the presence of unknown dynamics. Some of the most remarkable fields based on data-dependent control design can be found in adaptive and model reference control [22] [23].
Moreover, despite the existing results in the field of control theory which resort to of a mathematical model of the system, the case in which the actual system is not known and requires to be identified from collected samples are not uncommon.
Nowadays, fields like deep and statistical learning, which usually exploit contexts where the amount of data available is humongous for obtaining a description of the data-generative model, have obtained considerable performance and relevance in several applications [24][25].
As a natural consequence, data-driven paradigms have been widely pursued the direction of control design One of the most recognized and prolific paradigm is the identification field [26].
In the system identification paradigm a system dynamics estimate is obtained using only samples collected during some experiments previous to the design phase. In such paradigm, also defined as *indirect method* [27], the synthesis of the controller is performed after the estimation of the system dynamics.
Different approaches, such as Reinforcement Learning, relies on the employment of Machine Learning tools, in which the controller learn an optimal policy by recurring to a trial and error oriented method [28].
However, such methods present the major drawback to be time consuming and could suffer for the large number of iteration required to reach desired system model, in particular in the case of systems of large dimension or with dynamics which does not allow a simple mathematical representation. Moreover, despite being well established paradigms, the possibility of a wrong identification of the system to be controlled has a non trivial impact on its stability.
The aim of the *direct* data-driven approach is then to overcome such drawbacks, by obtaining the controller directly from the data, avoiding the system identification phase.
In recent years this field of study has achieved many relevant results, which span from control design to optimal and nonlinear control[29] [30],[31] to model predictive control[32] and set invariance [33].
The indirect data driven control finds its cornerstone in the *Willems et al's fundamental lemma*, which states that all trajectories that can be generated by a linear system can be represented by a linear combination of a finite set of the system trajectories, provided that the input used to generate such trajectories is sufficiently exciting. By a modeling point of view this allows to represent the open loop dynamics with a singular measured trajectory, which can be considered equivalent to be able to reconstruct the system from the data, and consequently it can be considered a sufficient condition for the effectiveness of the system identification procedure[34].
Nonetheless, what appears surprising is the fact, by exploiting a sufficiently exciting single in-

put/output trajectory, it is also possible to represent the closed loop dynamics of the system in presence of a state feedback input.
As pointed out in the next section, this allows to obtain a design method for a stabilizing feedback matrix which does not require the knowledge of the internal state of the system, and so allows to neglect the estimation part. We refer to this control paradigm as *model free* design.
As we will see in the following, under suitable assumptions, the model free design can be extended to the case of noisy measured trajectories, giving robustness to the procedure, which, as pointed out above, is a necessary requirement in control .

2.2 State space Willems Fundamental Lemma and data-driven representation

Before presenting the main result of the chapter, the *State Space Fundamental Lemma*, we introduce the notation used in the following.
Consider a signal $z : \mathbb{Z} \to \mathbb{R}^n$, we denote by $z_{[k,k+T]}$, $k \in \mathbb{Z}$, $T \in \mathbb{N}$ the restriction in vectorized form of z to the interval $[k, k+T] \cap \mathbb{Z}$, i.e.

$$z_{[k,k+T]} = \begin{bmatrix} z(k) \\ \vdots \\ z(k+T) \end{bmatrix}$$

We define *Hankel matrix* associated to the signal $z_{[i,j]}$ as

$$\mathcal{Z}_{i,t,N} = \begin{bmatrix} z(i) & z(i+1) & \dots & z(i+N-1) \\ z(i+1) & z(i+2) & \dots & z(i+N) \\ \vdots & \vdots & & \vdots \\ z(i+t-1) & z(i+t) & \dots & z(i+t+N-2) \end{bmatrix} \tag{2.1}$$

where the first subscript denotes the time at which the first sample of the signal is taken, the second one the number of samples per each column, and the last one the number of signal samples per each row. We say that the sequence $z_{[0,T-1]} \in \mathbb{R}^n$ is exciting of order L if its Hankel matrix of depth L the matrix

$$\mathcal{Z}_{0,L,T-L+1} = \begin{bmatrix} z(0) & z(1) & \dots & z(T-L) \\ z(1) & z(2) & \dots & z(T-L+1) \\ \vdots & \vdots & & \vdots \\ z(L-1) & z(L) & \dots & z(T-1) \end{bmatrix}$$

has full row rank nL.
We observe that, in order to have a sequence that is $z_{[0,T-1]}$ persistently exciting of order L, the length of the sequence must satisfy $T \geq (n+1)L - 1$.
As showed in the next theorem, the notion of persistent excitability plays a central role in representing system trajectories by means of input/output samples: in fact a persistent exciting input signal guarantee that the description obtained by means of the input/output samples carries enough information on the internal dynamics to be equivalent to the analytic one, at least for trajectories with length T.
For the moment we will focus on the fundamental lemma and its application for the case of discrete time systems. A the end of the chapter we will briefly discuss how to extend the result presented to the continuous time case.

Theorem 2.1 (State Space Fundamental Lemma)**.** *[35][36]*
Consider the following linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(t) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{2.2}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$. Assume moreover that the pair (A, B) is controllable.
Consider the following input/state/output samples measured during an experiment

$$U_{0,t,T-t+1} = \begin{bmatrix} u_m(0) & u_m(t-1)(1) & \dots & u_m(T-t) \\ u_m(1) & u_m(1) & \dots & u_m(T-t+1) \\ \vdots & \vdots & & \vdots \\ u_m(t-1) & u_m(t) & \dots & u_m(T-1) \end{bmatrix}$$

$$Y_{0,t,T-t+1} = \begin{bmatrix} y_m(0) & y_m(t-1)(1) & \dots & y_m(T-t) \\ y_m(1) & y_m(1) & \dots & y_m(T-t+1) \\ \vdots & \vdots & & \vdots \\ y_m(t-1) & y_m(t) & \dots & y_m(T-1) \end{bmatrix}$$

$$X_{0,T-t+1} = \begin{bmatrix} x_m(0), \dots, x_m(T-1) \end{bmatrix}$$

where the subscript "m" indicates that we are considering measured signals.
The following statements hold:

(a) if the input $u_{m,[0,T-1]}$ is persistently exciting of order $n+t$, then

$$\mathit{rank} \begin{bmatrix} U_{0,t,T-t+1} \\ \hline X_{0,t,T-t+1} \end{bmatrix} = n + tm \tag{2.3}$$

(b) if $u_{m,[0,T-1]}$ is persistently exciting of order $n+t$, then any t-long input-output trajectory of system (2.2) *can be represent by*

$$\left[\begin{array}{c} u_{0,t-1} \\ \hline y_{0,t-1} \end{array}\right] = \left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline Y_{0,t,T-t+1} \end{array}\right] g \tag{2.4}$$

for some $g \in \mathbb{R}^{T-t+1}$.
On the other hand, any $g \in \mathbb{R}^{T-t+1}$ the linear combination

$$\left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline Y_{0,t,T-t+1} \end{array}\right] g \tag{2.5}$$

is a t-long input/output trajectory of system (2.2)

Theorem (2.1) shows that for a sufficient long sequence, it is possible to represent every system trajectory using only a measured trajectory, that is, a basis for all trajectories of the LTI system (2.2) is formed by time-shifts of a single measured trajectory, given that the respective input signal is persistently exciting[34].
We show now, as outlined in the introduction of the chapter, how the relationship between excitability and linear representation obtained in the Fundamental Lemma can be exploited to give a data-dependent (or *model free*) representation of both open loop and closed loop trajectories.

Theorem 2.2. *[35] Consider again the system* (2.2) *and take $t = 1$; assume moreover to have collected the following sampled data matrices*

$$\begin{aligned} X_{0,T} &= \left[x_m(0), \dots, x_m(T-1)\right] \\ X_{1,T} &= \left[x_m(1), \dots, x_m(T)\right] \\ U_{0,1,T} &= \left[u_m(0), \dots, u_m(T-1)\right] \end{aligned} \tag{2.6}$$

where $u_m[0, T-1]$ persistently exciting of order $n+1$ so that the following condition holds

$$rank \left[\begin{array}{c} U_{0,1,T} \\ \hline X_{0,T} \end{array}\right] = n + m \tag{2.7}$$

Then, system (2.2) *admits an open loop representation of the form*

$$x(k+1) = X_{1,T} \left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline Y_{0,t,T-t+1} \end{array}\right]^{\dagger} \left[\begin{array}{c} u(k) \\ x(k) \end{array}\right] \tag{2.8}$$

where $\dagger$ denotes the right-inverse.
Under the same hypothesis, consider again system (2.2) *with applied input $u = Kx$, for some*

matrix K.

Then the closed loop dynamics can be rewritten as

$$x(k+1) = X_{1,T} G_K x(k) \tag{2.9}$$

where $G_K \in \mathbb{R}^{T \times n}$ satisfies

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,1,T} \\ \hline X_{0,T} \end{bmatrix} G_K \tag{2.10}$$

In particular we have $K = U_{0,1,T} G_K$

Notice that theorem (2.2) gives a design method for finding a stabilizing state feedback matrix K. In fact, by considering Lyapunov stability conditions, the origin is asymptotically stable for system (2.2) if and only if $\exists P \in \mathbb{R}^{N \times N} \succ 0$ such that

$$(A + BK)P(A + BK)^T - P \prec 0 \tag{2.11}$$

for which, by using the data driven representation given by Theorem (2.2), is equivalent to

$$X_{1,T} G_K P G_K^T X_{1,T}^T - P \prec 0 \tag{2.12}$$

As showed in [35], by introducing the change of variables $Q \coloneqq G_K P$ and rewriting condition (2.11) as

$$\begin{cases} X_{1,T} Q P^{-1} Q^T 1, T^T - P \prec 0 \\ X_{0,T} Q = P \\ K = U_{0,1,T} Q (X_{0,T} Q)^{-1} \end{cases} \iff \begin{cases} X_{1,T} Q (X_{0,T} Q)^{-1} Q^T 1, T^T - X_{0,T} Q \prec 0 \\ X_{0,T} Q \succ 0 \\ K = U_{0,1,T} Q (X_{0,T} Q)^{-1} \end{cases} \tag{2.13}$$

the problem can be reduced to find a matrix Q by means of a linear matrix inequality, as summarized in the next theorem.

Theorem 2.3. *Consider the discrete time system* (2.2)*, and assume to have access to the sampled data* (2.6) *which satisfy condition* (2.7)*. Then for any matrix $Q \in \mathbb{R}^{T \times n}$ which satisfies*

$$\begin{pmatrix} X_{0,T} Q & X_{1,T} Q \\ X_{1,T}^T Q^T & X_{0,T} Q \end{pmatrix} \prec 0 \tag{2.14}$$

the states feedback matrix obtained as $K = U_{0,1,T} Q (X_{0,T} Q)^{-1}$ stabilizes system (2.2)*.*

We observe that the data driven approach to the stabilization problem of LTI system can be obtained directly by the data exploiting a linear matrix inequality, for which the solution can be found by efficient algorithms.

Guarantee the efficiency in solve the Lyapunov stability problem will be of central interest in the extension of data driven control to the stabilization of Lur'e systems presented in the next chapters.

Remark. *We observe that for control design purpose in order to obtain a data driven description of a continuous time system closed loop dynamics it is sufficient to choose a input sequence persistently exciting in the a discrete time sense.*
In fact, consider the continuous time system

$$\begin{aligned} x(k+1) &= Ax(t) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{2.15}$$

by sampling system (2.15) *with arbitrary sampling period* Δ *we can collect the following input/state samples*

$$\begin{aligned} X_{0,T} &= \left[x_m(0), x_m(\Delta)\ldots, x_m(\Delta(T-1))\right] \\ X_{1,T} &= \left[\dot{x}_m(\Delta), \ldots, \dot{x}_m(\Delta T)\right] \\ U_{0,1,T} &= \left[u_m(0), u_m(\Delta), \ldots, x_m(\Delta T-1)\right] \end{aligned} \tag{2.16}$$

One can notice, as discussed in [35], that if the sequence $u_m(0), u_m(\Delta), \ldots, x_m(\Delta T-1)$ *is persistently exciting of order* $n+1$ *(in the discrete time sense), then the application of the zero-order hold signal obtained from the input samples ensures that the condition*

$$rank \begin{bmatrix} U_{0,t,T-t+1} \\ \hline X_{0,t,T-t+1} \end{bmatrix} = n + tm \tag{2.17}$$

is satisfied, allowing the same closed loop data-driven representation as the one obtained for discrete time systems.
Similarly on what done for discrete time case, the controller synthesis can be then obtained by using the continuous time Lyapunov matrix equation for stability

$$(A+BK)^T P + P(A+BK) \prec 0$$

for which, applying the data driven representation we obtain

$$X_{1,T} G_K P + P G_K^T X_{1,T}^T \prec 0$$

In complete analogy with the discrete time case it follows that any matrix Q *that satisfies*

$$\begin{cases} X_{1,T} Q + Q^T X_{1,T}^T \prec 0 \\ X_{0,T} Q \prec 0 \end{cases}$$

is such that the matrix $K = U_{0,1,T} Q (X_{0,T} Q)^{-1}$ is a stabilizing feedback gain.

2.2.1 Controller design in presence of measurements disturbances

As discussed in the introduction of the chapter, to obtain a control law which is able to counteract the effect of modeling uncertainties is a necessity in control. In a data driven framework, since the control law is obtained by means of combination of state/output sampled data, the notion of robustness translates into stability guarantees in the case of the samples of the measured system trajectories are corrupted by noise.
Given the vastness of results in this subject we will limit ourselves to a basic case which will help the intuition on the results presented in the in the next chapter.
Consider a linear time invariant system of the form

$$\begin{aligned} x(k+1) &= Ax(tk) + Bu(k) + \omega(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{2.18}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and $\omega(\cdot)$ is an unknown measurement noise, for which it is not assumed any particular statistics.
In presence of the signal $\omega(\cdot)$ the state measurements becomes equal to

$$\begin{aligned} Z_{0,T} &= X_{0,T} + W_{0,T} \\ Z_{1,T} &= X_{1,T} + W_{1,T} \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} W_{0,T} &= \begin{bmatrix} \omega_m(0), \dots, \omega_m(T-1) \end{bmatrix} \\ W_{1,T} &= \begin{bmatrix} \omega_m(1), \dots, \omega_m(T) \end{bmatrix} \end{aligned} \tag{2.20}$$

are the (unknown) noise measurements.
Before presenting the noise counterpart of theorem (2.3), we give the following assumptions

Assumption 1. *The matrices*

$$\left[\begin{array}{c} U_{0,1,T} \\ \hline Z_{0,T} \end{array} \right], \quad Z_{1,T} \tag{2.21}$$

have full row rank.

Assumption 2.

$$R_{0,T} R_{0,T}^T \preceq \gamma Z_{1,T} Z_{1,T}^T \tag{2.22}$$

for some $\gamma > 0$, where $R_{0,T} := AW_{0,T} - W_{1,T}$.

Theorem 2.4. *Suppose assumptions 1 and 2 hold. Then, if there exists $Q \in \mathbb{R}^{n \times T}$ and $\alpha > 0$*

such that

$$\begin{pmatrix} Z_{0,T}Q - \alpha Z_{1,T}Z_{1,T}^T & Z_{1,T}Q \\ Z_{1,T}^T Q^T & Z_{0,T}Q \end{pmatrix} \prec 0$$
$$\begin{pmatrix} I_T & Q \\ Q^T & Z_{0,T}Q \end{pmatrix} \prec 0 \tag{2.23}$$

with $\gamma < \alpha^2/(4+2\alpha)$*.Then* $K = U_{0,1,T}Q(Z_{0,T}Q)^{-1}$ *is a stabilizing controller*

We spend few words on Assumption 1 and 2. Both of them are equivalent to requiring that the loss of information caused by the presence of noise is small enough.
We can in fact observe that Assumption 1 is the noise case equivalent of condition (2.7), which allow the possibility of represent the closed loop dynamics. However, it is intuitive that such condition is not sufficient to guarantee that the stability of the closed loop, since such assumption can be verified even in presence of a noise that is so large that the data does not carry any useful information.
In this prospective, Assumption 2 plays the role of a matrix equivalent signal to noise ratio condition. Indeed, even if one can notice that, when Assumption 1 holds, Assumption 2 is always satisfied for a sufficiently large γ. However, in order to get closed loop stability it is necessary to restrict the magnitude of γ, as showed in the previous theorem. In such sense, the value of α represents a way to quantify the required SNR to guarantee the stability of the closed loop.
More refined approaches to deal with the presence of measurement noise can be implemented exploiting Lyapunov stability theory [37] or the S-Lemma [38].

3

Multivariate Lur'e System and Data-Driven absolute stabilization

In this chapter we will consider the problem of the absolute stabilization of a multivariate Lur'e system, which is a natural multi-input/multi-output extension of the problem considered in chapter one.
It will however assume that the dynamics of the system to stabilize is partially unknown. Inspired by the approaches illustrated in chapter 2, we will resort to a data driven representation of the system dynamics to synthesize a stabilizing feedback controller. The solution of the problem will be obtained by exploiting a quadratic Lypunov function together with the strictly inequality S-Lemma. Such result will be extended to the case in which the internal dynamics is completely unknown by including the action of the nonlinearity in the controller design. Finally we will consider the problem in the case of the presence of disturbances in the data collecting phase. We will refer to [39],with the exception of the second theorem presented which is an original formulation inspired by the same article.

3.1 Problem Framework

Consider a Lur'e system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + L\omega(t) \\ y(t) = Hx(t) \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the internal state of the system, $y(t) \in \mathbb{R}^p$ is the output, $u(t) \in \mathbb{R}^m$ is the control input and $\varphi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^q$ is a memoryless and possible time-varying function, which in line with Chapter 1, will be called *nonlinearity*. Consequently, we have $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $L \in \mathbb{R}^{n \times q}$, $H \in \mathbb{R}^{p \times n}$.
We require moreover that the nonlinearity satisfies the following quadratic inequality

$$\begin{pmatrix} y^T & \omega^T \end{pmatrix} \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} y \\ \omega \end{pmatrix} \geq 0 \tag{3.2}$$

for all $\omega = \varphi(t, y)$ and for all pairs (t, y) in $\mathbb{R} \times \mathbb{R}^p$.
We assume $\hat{Q} = \hat{Q}^T \in \mathbb{R}^{p \times p}$, $\hat{S} \in \mathbb{R}^{p \times q}$ and $\hat{R} \prec 0 \in \mathbb{R}^{q \times q}$. In the course of the analysis we will sometimes also require the constraint to be regular, which means we will assume the existence of a pair $(\bar{t}, \bar{y})$ such that (3.2) holds with strict inequality.
Note that considering $\hat{R} \prec 0$ implies $\varphi(t, 0) = 0 \ \forall \ t \in \mathbb{R}$, assuring the origin to be an equilibrium: in fact evaluating (3.2) in $x = 0$ we have

$$\begin{aligned} &\begin{pmatrix} y^T & \omega^T \end{pmatrix} \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} y^T \\ \omega^T \end{pmatrix}_{\Big|_{x=0}} = \begin{pmatrix} Hx^T & \omega^T \end{pmatrix} \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} Hx^T \\ \omega^T \end{pmatrix}_{\Big|_{x=0}} \\ &= \omega^T \hat{R} \omega_{|_{x=0}} = \varphi(t, 0)^T \hat{R} \varphi(t, 0) \geq 0 \end{aligned} \tag{3.3}$$

which under the assumption $R \prec 0$ can satisfied if and only if $\varphi(t, 0) = 0 \ \forall \ t \ \in \mathbb{R}$.
We observe that (3.2) represents an extension of the quadratic constraints (1.5) considered in chapter one: in fact, if we consider $K_1, K_2 \in \mathbb{R}^{q \times p}$ the function $\varphi(t, y)$ satisfies the (multi-dimensional) sector constraint (see [10], definition 2.6) if the inequality $(\varphi(t, y) - K_1 y)^T (K_2 y - \varphi(t, y)) \geq 0$, is fulfilled for any $(t, y) \in \mathbb{R} \times \mathbb{R}^p$. In such case, we have $\hat{Q} = -K_2^T K_1 - K_1^T K_2$, $\hat{S} = K_1^T + K_2^T$ and $\hat{R} = -2I_q$.
Condition (3.2) can be used to model more general constraints on the function $\varphi(t, y)$: for example by taking $\hat{Q} = -\ell I_n$, $\hat{S} = 0$ and $\hat{R} = -I_q$ for $\ell \geq 0$ it is possible to model *norm bounded nonlinearities*, which satisfies the inequality $\varphi(t, y)^T \varphi(t, y) \leq \ell y^T y$. Constraints of this type are considered in robust nonlinear control and absolute stability theory[40],[41].

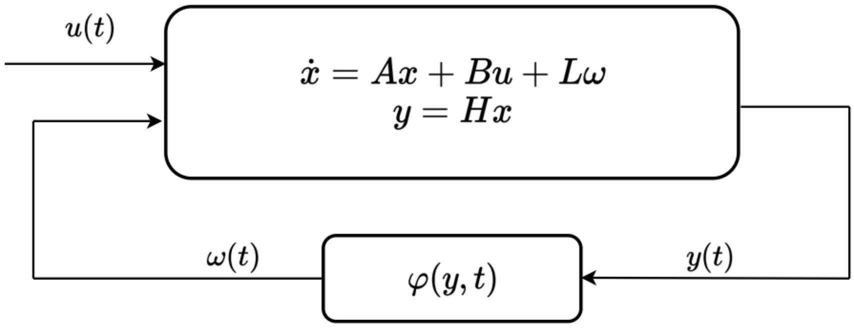


Figure 3.1: Block scheme of the multivariate Lur'e system (3.1)

In this framework we will assume that the matrices A and B, which regulate respectively the state and the input dynamics in equation (3.1), are unknown, whereas we assume to know the matrix L, which regulates the effect of the nonlinearity in the state dynamics, and the matrix H, which regulates the output dynamics.

3.2 Data-Driven absolute stabilization

Following the ideas presented in chapter 2, to overcome the lack of knowledge in the dynamics, we exploit an analogous representation of the system based on data: it is in fact assumed we are able to collect the following matrices containing data-samples from an experiment conducted previous to the controller synthesis

$$\begin{aligned} U_0 &:= [u(0) \ldots u(T-1)] \\ X_0 &:= [x(0) \ldots x(T-1)] \\ X_1 &:= [\dot{x}(0) \ldots \dot{x}(T-1)] \\ F_0 &:= [\omega(0) \ldots \omega(T-1)] \end{aligned} \tag{3.4}$$

where X_0, $X_1 \in \mathbb{R}^{n \times T}$, $U_0 \in \mathbb{R}^{m \times T}$ and $L_0 \in \mathbb{R}^{q \times T}$.
We remark that having access to F_0 implies that the action of the non linear block must to be physical detached from the state dynamics, as illustrated in figure (3.1). In order to guarantee the possibility of describe the system dynamics using the data, we make the assumption that the the matrix

$$W_0 = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \in \mathbb{R}^{(m+n) \times T} \tag{3.5}$$

is full row rank.
We point out however that, differently from the linear case, there is no guarantee that such requirement is met, even when the applied input is a persistently exciting signal of order equal or

greater than $n + m$. As consequence, the rank condition on (3.5) is considered to be an hard assumption.
The problem to be solved is then to find, under the assumption that W_0 has full row rank, a state feedback controller assuring the global asymptotically stability of the origin for all nonlinearities $\varphi(t, y)$ which obey condition (3.2). In order to solve the absolute stability problem of system (3.1), we consider a quadratic Lyapunov function of the type

$$V(x) = x^T P x \tag{3.6}$$

When the applied control input is $u(t) = Kx(t)$, the time derivative of $V\big(x(t)\big)$ is

$$\begin{aligned}\dot{V}(x, \omega) &= \dot{x}^T P x + X^T P \dot{x} = \big((A + BK)x + L\omega\big)^T P x + x^T P((A + BK)x + L\omega) \\ &= \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} (A + BK)^T P + P(A + BK) & PL \\ L^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix}\end{aligned} \tag{3.7}$$

We need $\dot{V}(x, \omega)$ to be negative $\forall\ (x, \omega) \neq 0$ with ω satisfying conditions (3.1).
We observe the matrix obtained in (3.7) represents the state-feedback analogous of the one encountered in the discussion about the application of the S-lemma (see (1.13)).
Since the matrices A and B are unknown, similarly to what we did in the linear case, we exploit the collected data (3.4). By the assumption that W_0 is full row rank, for each $K \in \mathbb{R}^{n\times m}$ there exists a matrix $G \in \mathbb{R}^{T\times n}$ such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \tag{3.8}$$

The following representation of the closed loop dynamics holds

$$\begin{aligned} A + BK &= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \\ &= AX_0G + BU_0G = (AX_0 + BU_0 + LF_0 - LF_0)G \\ &= (X_1 - LF_0)G = X_LG \end{aligned} \tag{3.9}$$

where we exploit $X_1 = AX_0 + BU_0 + LF_0$.
By using the representation (3.9) the time derivative (3.7) becomes

$$\dot{V}(x, \omega) = \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} (X_LG)^T P + PX_LG & PL \\ L^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \tag{3.10}$$

We reformulate the constraints (3.2) in order to make explicit the dependence on the system state, for which we obtain

$$\begin{aligned}
&\begin{pmatrix} y^T & \omega^T \end{pmatrix} \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} y^T \\ \omega^T \end{pmatrix} = \begin{pmatrix} Hx^T & \omega^T \end{pmatrix} \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} Hx \\ \omega \end{pmatrix} = \\
&\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} H^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} = \\
&\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} Q & S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \geq 0
\end{aligned} \tag{3.11}$$

where we have defined the matrices $Q = Q^T = H^T \hat{Q} H \in \mathbb{R}^{n \times n}$, $S = H^T \hat{S} \in \mathbb{R}^{n \times p}$ and $R = \hat{R} \in \mathbb{R}^{q \times q}$. With some abused of notation we identify with 0 the matrices containing all zero elements of adequate dimension; we will apply the same notation through the course of the chapter.
The relationship between $\dot{V}(x, \omega)$ in (3.7) and its data driven representation (3.10) lead to the following stability theorem

Theorem 3.1. *Consider a Lur'e system of the form* (3.1)*, with ω satisfying the sector conditions* (3.2).
Under the data-driven representation (3.9)*, the origin is a globally asymptotically stable equilibrium of system* (3.1) *if exist two matrices $P = P^T \succ 0 \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}^{T \times n}$ such that*

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} (X_L G)^T P + P X_L G & PL \\ L^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} < 0 \tag{3.12}$$

for all (x, ω) such that

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} Q & S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \geq 0$$

Therefore, the problem of finding a stabilizing matrix K such that the origin is globally asymptotically stable reduces to finding two matrices P and G for which Theorem (3.1) is satisfied.
Before presenting the next theorem, in which it is displayed a solution to the problem, we remind the definition of *Schur Complement* and its use in characterize negative definite matrices, which will be the key of the proof.

Definition. *Let $M \in \mathbb{R}^{n \times n}$ be a 2×2 block matrix*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A and D are respectively $p \times p$ and $q \times q$ matrices, with $n = p + q$, and consequently B and C are respectively $p \times q$ and $q \times p$ matrices.
Assuming D invertible, the Schur complement of the matrix M is defined as

$$M/D := A - BD^{-1}C$$

Proposition. *For any 2×2 symmetric block matrix **M***

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

if C is invertible, then $M \prec 0$ if and only if $C \prec 0$ and $A - B^T C^{-1} B \prec 0$.

We are now ready to state the following theorem.

Theorem 3.2. *Consider a Lur'e system of the form* (3.1) *with nonlinearity ω satisfying the sector conditions* (3.2) *and assume that the matrix W_0 defined in* (3.5) *has full row rank. Assume moreover that the constraint* (3.2) *to be regular.*
We have the following cases

(i) if $Q \succeq 0$, the two matrices P, G satisfying Theorem (3.1) *exist if and only if there exists a matrix $Y \in \mathbb{R}^{T \times n}$ which satisfies*

$$\begin{pmatrix} Y^T X_L + X_L Y & L + X_0 Y S & X_0 Y Q^{1/2} \\ (L + X_0 Y S)^T & R & 0 \\ (X_0 Y Q^{1/2})^T & 0 & -I \end{pmatrix} \prec 0 \tag{3.13}$$

(ii) if $Q = 0$, the two matrices P, G satisfying Theorem (3.1) *exist if and only if there exists a matrix $Y \in \mathbb{R}^{T \times n}$ which satisfies*

$$\begin{pmatrix} Y^T X_L + X_L Y & L + X_0 Y S \\ (L + X_0 Y S)^T & R \end{pmatrix} \prec 0 \tag{3.14}$$

(iii) if $Q \preceq 0$, the two matrices P, G satisfying Theorem (3.1) *exist if there exists a matrix $Y \in \mathbb{R}^{T \times n}$ which satisfies the same matrix inequality* (3.14)*. In such a case regularity of the constraint* (3.2) *is not required*

In all three cases the stabilizing matrix K is given by $K = U_0 Y (X_0 Y)^{-1}$

Proof. As stated in Theorem (3.1) the desired condition is that the inequality

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} (X_L G)^T P + P X_L G & PL \\ L^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} < 0 \tag{3.15}$$

is fulfilled for all (x, ω) such that

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} Q & S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \geq 0 \tag{3.16}$$

We know by the strict-inequality S-Lemma (Theorem (1.7)) that the inequality (3.15) is satisfied for all pairs (x, ω) for which (3.16) holds if and only if there exists $\tau \geq 0$ such that

$$\begin{pmatrix} (X_L G)^T P + P X_L G + \tau Q & PL + \tau S \\ L^T P + \tau S^T & \tau R \end{pmatrix} \prec 0 \tag{3.17}$$

We note that τ must be greater than zero: indeed, since a if $\tau = 0$ then matrix in (3.17) cannot be negative definite having the lower diagonal block equal to zero.
It is therefore possible remove τ from the inequality by diving by τ itself and rename, with some abuse of notation, $\frac{P}{\tau} = P$

$$\tau \begin{pmatrix} (X_L G)^T \frac{P}{\tau} + \frac{P}{\tau} X_L G + Q & L\frac{P}{\tau} + S \\ P^T \frac{P}{\tau} + S^T & R \end{pmatrix} \prec 0 \tag{3.18}$$

$$\begin{pmatrix} (X_L G)^T P + P X_L G + Q & PL + S \\ L^T P + S^T & R \end{pmatrix} \prec 0 \tag{3.19}$$

where diving by τ does not change the sign of the inequality since it is a strictly positive number.
Case 1: Assume $Q \geq 0$
Consider the matrix (3.19). By Schur complement, that matrix is negative definite if and only if (3.19)

$$\begin{pmatrix} (X_L G)^T P + P X_L G & PL + S & Q^{1/2} \\ L^T P + S^T & R & 0 \\ (Q^{1/2})^T & 0 & -I \end{pmatrix} \prec 0 \tag{3.20}$$

where $Q^{1/2}$ is the square root of Q, that exists because $Q \geq 0$.
By right and left multiplying for $diag(P^{-1}, I, I)$ we obtain

$$\begin{pmatrix} P^{-1} G^T X_L^T + X_L G P^{-1} & L + P^{-1} S & P^{-1} Q^{1/2} \\ L^T + S^T P^{-1} & R & 0 \\ (Q^{1/2})^T P^{-1} & 0 & -I \end{pmatrix} \prec 0 \tag{3.21}$$

By defining $Y = GP^{-1}$, we obtain that $X_0 P = X_0 G P^{-1} = P^{-1}$ which allows us to substitute P^{-1} with $X_0 Y$ and reducing the original problem in the two variables G and P to a problem in one variable Y, which lead to (3.13). Finally,considering that relation $K = U_0 G$, given (3.8), and that $G = YP$ we obtain $K = U_0 G = U_0 Y P = U_0 Y (X_0 Y)^{-1}$.

Case 2: assume $Q = 0$; the proof is identical to case 1 up to condition (3.19), which now must hold without the matrix Q, i.e.

$$\begin{pmatrix} (X_L G)^T P + P X_L G & PL + S \\ L^T P + S^T & R \end{pmatrix} \prec 0 \tag{3.22}$$

By right and left multiplying by $diag(P^{-1}, I)$ and using the same change of variables as in case 1 we obtain the statement in (3.14). The equivalence $K = U_0 Y (X_0 Y)^{-1}$ s given again by the change of variables adopted

Case 3: assume $Q \preceq 0$; it is straightforward to observe that condition if condition (3.22) holds, than

$$\begin{aligned} &\begin{pmatrix} (X_L G)^T P + P X_L G + Q & PL + S \\ L^T P + S^T & R \end{pmatrix} = \\ &\begin{pmatrix} (X_L G)^T P + P X_L G & PL + S \\ L^T P + S^T & R \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \prec 0 \end{aligned} \tag{3.23}$$

since $Q \preceq 0$. Using again the same change of variables we obtain that (3.14) is a sufficient condition to (3.23) to hold. □

Remark. *In the proof case 3 we have neglected the regularity hypothesis: that is a consequence of the sufficiency implication in Theorem* (1.7)*. In fact, by calling $\mathcal{F}(x, \omega)$ the quadratic form obtained from matrix* (3.10) *and $\mathcal{G}(x, \omega)$ the one from the last equality in* (3.11) *we are exploiting the following implication*

$$\begin{aligned} &if\ \exists \tau \geq 0\ s.t\ \mathcal{F}(x, \omega) + \tau \mathcal{G}(x, \omega) < 0\ \forall (x, \omega) \neq 0 \\ &then \quad \mathcal{G}(x, \omega) > 0 \Rightarrow \mathcal{F}(x, \omega) < 0 \end{aligned} \tag{3.24}$$

which is an obvious consequence. The regularity assumption must be valid to prove that the opposite implication holds, that is:

$$\begin{aligned} &if\ \mathcal{G}(x, \omega) > 0 \Rightarrow \mathcal{F}(x, \omega) < 0 \\ &then\ \exists \tau \geq 0\ s.t\ \mathcal{F}(x, \omega) + \tau \mathcal{G}(x, \omega) < 0\ \forall (x, \omega) \neq 0 \end{aligned} \tag{3.25}$$

as showed in the proof of the S-Lemma.

3.2.1 Relaxing prior knowledge in the system dynamics

We notice that the previous solution to the absolute stabilization problem rely on the prior knowledge of the matrices H and L. Indeed, to apply the strict-inequality S-Lemma, we had to make explicit the relationship between the nonlinearity and the state, while in order exploit the data-driven formulation (3.9) it is necessary describe the influence of the nonlinearity on the state represented by L.

In the case the nonlinear function $\varphi(t, y)$ can be measured for each instant t, then is possible to include the additional information in the design of the controller, discarding the prior knowledge on the matrix and L.

Such assumption appears reasonable under the hypothesis that the non linear block is physically detached from linear time invariant system, which has been already considered to assure the possibility of collect sample data of such a block.

In this case the feedback input becomes then

$$u(t) = Kx(t) + M\omega(t) \tag{3.26}$$

where K and M are the feedback gains to be designed.

We reformulate then assumption (3.5) by consider the case in which the matrix

$$\Psi = \begin{bmatrix} X_0 \\ F_0 \\ U_0 \end{bmatrix} \in \mathbb{R}^{(m+n)\times T} \tag{3.27}$$

has full row rank.

Under assumption (3.27) for any matrix $[K \; M] \in \mathbb{R}^{m\times(n+q)}$ exists a matrix $G = [G_1 \; G_2] \in \mathbb{R}^{T\times(n+q)}$, where G_1 has n columns and G_2 has q columns such that

$$\begin{bmatrix} I_n & 0_{n\times q} \\ 0_{q\times n} & I_q \\ K & M \end{bmatrix} = \begin{bmatrix} X_0 \\ F_0 \\ U_0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \end{bmatrix} \tag{3.28}$$

from which follows the representation holds

$$\begin{aligned} \begin{bmatrix} A + BK | L + BM \end{bmatrix} &= \begin{bmatrix} A & L \end{bmatrix} + B \begin{bmatrix} K & M \end{bmatrix} = \begin{bmatrix} A & L & B \end{bmatrix} \begin{bmatrix} X_0 & 0_{n\times q} \\ 0_{q\times n} & I_q \\ K & M \end{bmatrix} \\ &= \begin{bmatrix} A & L & B \end{bmatrix} \begin{bmatrix} X_0 \\ F_0 \\ U_0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \end{bmatrix} = X_1 \begin{bmatrix} G_1 & G_2 \end{bmatrix} \end{aligned} \tag{3.29}$$

where we have exploited $X_1 = AX_0 + BU_0 + LF_0$.
In the next theorem we address how to find a stabilizing controller $[K\ M]$ by using a the data driven description of the closed loop dynamics depicted above; we will show the proof only for the case in which $Q \geq 0$, since similarly to the previous proof the others are similar.
Before giving the statement of the theorem, we rewrite the on the time derivative of the quadratic Lyapunov function $V(x) = x^T Px$ under the input $u = Kx + M\omega$

$$\begin{aligned}\dot{V}(x,\omega) &= \dot{x}^T Px + X^T P\dot{x} \\ &= ((A+BK)x + (L+BM)\omega)^T Px + x^T P((A+BK)x + (L+BM)\omega) \\ &= \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} (A+BK)^T P + P(A+BK) & P(L+BM) \\ (L+BM)^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix}\end{aligned} \tag{3.30}$$

which, under the representation (3.29), becomes

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} G_1^T X_1^T P + PX_1G_1 & PX_1G_2 \\ G_2^T X_1^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \tag{3.31}$$

for which the desired condition is that (3.31) is negative for all $(x\ \omega)$, with ω satisfying the sector conditions; similar to what done in the previous chapter we summarise such considerations in the following theorem

Theorem 3.3. *Consider a Lur'e system of the form* (3.1) *with ω satisfying the sector conditions* (3.2)*, and consider a quadratic Lyapunov function $x^T Px$ for some,$P \in \mathbb{R}^{n\times n}$, $P = P^T \succ 0$. Under the data-driven representation* (3.29) *the origin is a globally asymptotically stable equilibrium of system* (3.1) *if exist two matrices $P \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{T\times n}$ such that*

$$\dot{V}(x,\omega) = \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} G_1^T X_1^T P + PX_1G_1 & PX_1G_2 \\ G_2^T X_1^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} < 0 \tag{3.32}$$

for all (x,ω) such that

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} Q & S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \geq 0$$

We are now ready to present the next result

Theorem 3.4. *Consider a Lur'e system of the form* (3.1) *and assume that the matrix Ψ in* (3.27) *has full row rank. Assume moreover the constraint* (3.2) *to be regular and that $Q \geq 0$.*
Then there exist two matrices P, G satisfying Theorem (3.3) *if and only there exist $Y_1 \in$*

$\mathbb{R}^{T\times n}$,$Y_2 \in \mathbb{R}^{T\times q}$ *and* $W \in \mathbb{R}^{n\times n}$ *such that the conditions*

$$\begin{pmatrix} Y_1^T X_1^T + X_1 Y_1 & X_1 Y_2 + WS & WQ^{1/2} \\ Y_2^T X_1^T + S^T W & R & 0 \\ (Q^{1/2})^T W & 0 & -I \end{pmatrix} \prec 0 \tag{3.33}$$

$$\begin{bmatrix} X_0 Y_1 - W & X_0 Y_2 \\ F_0 Y_1 & F_0 Y_2 - I_q \end{bmatrix} = 0 \tag{3.34}$$

are satisfied.

Moreover,the stabilizing controllers K and M are given respectively by $K = U_0 Y_1 (X_0 Y_1)^{-1}$ and $M = U_0 Y_2$.

Proof. Similarly to the previous proof, we apply the non strict S-Lemma to the quadratic in (3.31), from which we obtain the condition

$$\begin{pmatrix} G_1^T X_1^T P + P X_1 G_1 + \tau Q & P X_1 G_2 + \tau S \\ G_2^T X_1^T P + \tau S^T & \tau R \end{pmatrix} \prec 0 \tag{3.35}$$

Using the same argument used for the proof of point (i) in Theorem (3.2), we can assume τ to be positive, and so remove from the formulation.

By the Schur complement, we argue that (3.35) implies

$$\begin{pmatrix} G_1^T X_1^T P + P X_1 G_1 & P X_1 G_2 + S & Q^{1/2} \\ G_2^T X_1^T P + S^T & R & 0 \\ (Q^{1/2})^T & 0 & -I \end{pmatrix} \prec 0 \tag{3.36}$$

By right and left multiplying by $diag(P^{-1}, I, I)$ we finally obtain a result similr to the one in (3.21)

$$\begin{pmatrix} P^{-1} G_1^T X_1^T + X_1 G_1 P^{-1} & X_1 G_2 + P^{-1} S & P^{-1} Q^{1/2} \\ G_2^T X_1^T + S^T P^{-1} & R & 0 \\ (Q^{1/2})^T P^{-1} & 0 & -I \end{pmatrix} \prec 0 \tag{3.37}$$

By defining $W := P^{-1}$, $Y_1 := G_1 P^{-1}$ and $Y_2 := G_2$ we obtain the first part of the statement. We now consider the condition

$$\begin{bmatrix} I_n & 0_{n\times q} \\ 0_{q\times n} & I_q \end{bmatrix} = \begin{bmatrix} X_0 \\ F_0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \end{bmatrix} = \begin{bmatrix} X_0 G_1 & X_0 G_2 \\ F_0 G_1 & F_0 G_2 \end{bmatrix} \tag{3.38}$$

imposed in (3.28). By taking the transpose and multiplying both matrices on the left by the block diagonal matrix $diag(P^{-1}, I_p)$ we obtain

$$\begin{aligned}\begin{bmatrix} P^{-1} & 0_{n\times q} \\ 0_{q\times n} & I_q \end{bmatrix} &= \begin{bmatrix} P^{-1}G_1^T X_0^T & P^{-1}G_1^T F_0^T \\ G_2^T X_0^T & G_2^T F_0^T \end{bmatrix} \\ \begin{bmatrix} W & 0_{n\times q} \\ 0_{q\times n} & I_q \end{bmatrix} &= \begin{bmatrix} Y_1^T X_0^T & Y_1^T F_0^T \\ Y_2^T X_0^T & Y_2^T F_0^T \end{bmatrix}\end{aligned} \tag{3.39}$$

which follows by recalling that we defined $P^{-1} = W$. By subtracting to each terms the left hand side of the equation, and taking the transpose again we obtain the constraint on equation (3.33). Finally, we have $K = U_0 G_1 = U_0 Y_1 P = U_0 Y_1 (X_0 Y_1)^{-1}$ and $M = U_0 Y_2$ □

3.3 Stabilization in presence of disturbed data

We now examine now the case in which the system is affected by disturbances during the process of collecting data. Hence, we introduce the presence of noise in system (3.1), namely we focus on a system described by the equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + L\omega(t) + Ed(t) \\ y(t) = Hx(t) \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{3.40}$$

where $d(t) \in \mathbb{R}^s$ is an unknown signal which represent the disturbances affecting the dynamics during the data-collecting phase, while $E \in \mathbb{R}^{n\times s}$ is a known matrix that models how the disturbance enters the system state.
In presence of the disturbance $d(t)$, alongside the data collected in (3.4)

$$\begin{aligned} U_0 &:= [u(0) \, ... \, u(T-1)] \\ X_0 &:= [x(0) \, ... \, x(T-1)] \\ X_1 &:= [\dot{x}(0) \, ... \, \dot{x}(T-1)] \\ F_0 &:= [\omega(0) \, ... \, \omega(T-1)] \end{aligned} \tag{3.41}$$

we must consider matrix containing the noise samples

$$D_0 := [d(0) \, ... \, d(T-1)]$$

which is unknown.
Accordingly we have $X_1 = AX_0 + BU_0 + LF_0 + ED_0$.

We observe that assuming the matrix W_0 defined in (3.5) to be still full row rank, so that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

Then the closed loop representation (3.9) becomes

$$\begin{aligned} A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \\ = AX_0G + BU_0G = (AX_0 + BU_0 + LF_0 - LF_0 + ED_0 - ED_0)G \\ = (X_1 - LF_0 - ED_0)G = (X_L - ED_0)G \end{aligned} \tag{3.42}$$

and, in result of that, the data-driven representation of the time derivative of $V(x) = x^T Px$ in (3.10) translates to

$$\dot{V}(x, \omega) = \begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} G^T(X_L - ED_0)^T P + P(X_L - ED_0)G & PL \\ L^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} < 0 \tag{3.43}$$

Retracing the same steps as in Theorem (3.2) we obtain that the closed loop system is absolute stable if there exists $Y \in \mathbb{R}^{T\times n}$ for which the two following matrix inequalities are satisfied, respectively for the cases of $Q \geq 0$

$$\begin{pmatrix} Y^T(X_L - ED_0)^T + (X_L - ED_0)Y & L + X_0YS & X_0YQ^{1/2} \\ (L + X_0YS)^T & R & 0 \\ X_0(YQ^{1/2})^T & 0 & -I \end{pmatrix} \prec 0 \tag{3.44}$$

and in the case $Q = 0$, $Q \leq 0$

$$\begin{pmatrix} Y^T(X_L - ED_0)^T + (X_L - ED_0)Y & L + X_0YS \\ (L + X_0YS)^T & R \end{pmatrix} \prec 0 \tag{3.45}$$

However such condition cannot be actually checked, due to the presence of the unknown vector D_0.

To get rid from the dependence D_0 we introduce the following condition on the disturbance matrix:

$$D_0 \in \mathcal{D} := \{D \in \mathbb{R}^{n\times T} : DD^T \prec \Delta\Delta^T\} \tag{3.46}$$

where Δ is some known matrix.

We make the following observation.

Remark. *For every matrices $\Gamma \in \mathbb{R}^{k\times n}$ and $\Theta \in \mathbb{R}^{n\times T}$, where k is arbitrary, and for every*

$\epsilon > 0$, $D_0 \in \mathcal{D}$ *we have that*

$$(\epsilon \Gamma D_0 - \Theta^T)(\epsilon \Gamma D_0 - \Theta^T)^T = \epsilon^2 \Gamma D_0 D_0^T - \epsilon \Gamma \Theta - \epsilon \Theta^T D_0 \Gamma^T + \Theta^T \Theta \geq 0 \tag{3.47}$$

From this it follows that

$$\epsilon^2 \Gamma D_0 D_0^T \Gamma^T + \Theta^T \Theta \geq \epsilon \Gamma \Theta + \epsilon \Theta^T D_0 \Gamma^T \tag{3.48}$$

finally, dividing by ϵ *and exploiting condition* (3.46) *we obtain*

$$\epsilon \Gamma \Delta \Delta^T \Gamma^T + \epsilon^{-1} \Theta^T \Theta \geq \Gamma D_0 \Theta + \Theta^T D_0 \Gamma^T \tag{3.49}$$

Before state the solution of the absolute stabilization problem with noisy measurements we give the noise equivalent of Theorem (3.1)

Theorem 3.5. *Consider a Lur'e system of the form* (3.1)*, with* ω *satisfying the sector conditions* (3.2).
Under the data-driven representation (3.9)*, the origin is a globally asymptotically stable equilibrium of system* (3.1) *if exist two matrices* $P = P^T \succ 0 \in \mathbb{R}^{n \times n}$ *and* $G \in \mathbb{R}^{T \times n}$ *such that*

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} ((X_L - ED_0)G)^T P + P(X_L - ED_0)G & PL \\ L^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} < 0 \tag{3.50}$$

for all (x, ω) *such that*

$$\begin{pmatrix} x^T & \omega^T \end{pmatrix} \begin{pmatrix} Q & S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} \geq 0$$

We now give the solution to the absolute stability problem in the case of disturbed collected data. Again, since the proofs of the three items follow the same arguments, we will present only the one for the first case.

Theorem 3.6. *Consider a Lur'e system of the form* (3.40) *nonlinearity* ω *satisfying the sector conditions* (3.2) *and assume that the matrix* W_0 *defined in* (3.5) *has full row rank. Assume moreover that the constraint* (3.2) *to be regular.*
We have following cases

(i) if $Q \geq 0$*, the two matrices* P*,* G *satisfying Theorem* (3.5) *exist if and only if there exists a*

matrix $Y \in \mathbb{R}^{T\times n}$ *and a scalar* $\epsilon > 0$ *which satisfies*

$$\begin{pmatrix} Y^T X_L + X_L Y & L + X_0 Y S & X_0 Y Q^{1/2} & Y^T \\ (L + X_0 Y S)^T & R & 0 & 0 \\ X_0 (Y Q^{1/2})^T & 0 & -I & 0 \\ Y & 0 & 0 & -\epsilon I \end{pmatrix} \prec 0 \tag{3.51}$$

(ii) if $Q = 0$*, the two matrices* P*,* G *satisfying Theorem* (3.5) *exist if and only if there exists a matrix* $Y \in \mathbb{R}^{T\times n}$ *and a scalar* $\epsilon > 0$ *which satisfies*

$$\begin{pmatrix} Y^T X_L + X_L Y & L + X_0 Y S & Y^T \\ (L + X_0 Y S)^T & R & 0 \\ Y & 0 & -\epsilon I \end{pmatrix} \prec 0 \tag{3.52}$$

(iii) if $Q \preceq 0$*,the two matrices* P*,* G *satisfying Theorem* (3.5) *exist if there exists a matrix* $Y \in \mathbb{R}^{T\times n}$ *and a scalar* $\epsilon > 0$*which satisfies the same matrix inequality above* (3.14)*. In such case the regularity of constraint* (3.2) *is not required*

In all three cases the stabilizing matrix K *is given by* $K = U_0 Y (X_0 Y)^{-1}$

Proof. We start by rewriting the condition from stability in presence of disturbed data from (3.44)

$$\begin{aligned} &\begin{pmatrix} Y^T X_L^T - Y^T D_0^T E^T + X_L Y - E D_0 Y & L + X_0 Y S & X_0 Y Q^{1/2} \\ (L + X_0 Y S)^T & R & 0 \\ X_0 (Y Q^{1/2})^T & 0 & -I \end{pmatrix} = \\ &\begin{pmatrix} Y^T X_L^T + X_L Y & L + X_0 Y S & X_0 Y Q^{1/2} \\ (L + X_0 Y S)^T & R & 0 \\ X_0 (Y Q^{1/2})^T & 0 & -I \end{pmatrix} + \begin{pmatrix} -Y^T D_0^T E^T - E D_0 Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.53}$$

By calling $\Gamma = -E$, we have $-Y^T D_0^T E^T - E D_0 Y = \Gamma D_0 Y + Y^T D_0^T \Gamma^T$.
Exploiting the matrix inequality (3.49) we obtain

$$\begin{pmatrix} -Y^T D_0^T E^T - E D_0 Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} \epsilon^{-1} Y Y^T + \epsilon E \Delta \Delta^T E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.54}$$

which leads to

$$
\begin{pmatrix}
Y^T X_L^T - Y^T D_0^T E^T + X_L Y - E D_0 Y & L + X_0 Y S & X_0 Y Q^{1/2} \\
(L + X_0 Y S)^T & R & 0 \\
X_0 (Y Q^{1/2})^T & 0 & -I
\end{pmatrix} \preceq
\begin{pmatrix}
Y^T X_L^T + X_L Y + \epsilon^{-1} Y Y^T + \epsilon E \Delta \Delta^T E & L + X_0 Y S & X_0 Y Q^{1/2} \\
(L + X_0 Y S)^T & R & 0 \\
X_0 (Y Q^{1/2})^T & 0 & -I
\end{pmatrix} \tag{3.55}
$$

By Schur complement the matrix (3.55) is negative definite if and only if

$$
\begin{pmatrix}
Y^T X_L + X_L Y & L + X_0 Y S & X_0 Y Q^{1/2} & Y^T \\
(L + X_0 Y S)^T & R & 0 & 0 \\
X_0 (Y Q^{1/2})^T & 0 & -I & 0 \\
Y & 0 & 0 & -\epsilon I
\end{pmatrix} \prec 0 \tag{3.56}
$$

holds true. □

4

Non polynomial S-Lemma and absolute stability in non Euclidean spaces

In this chapter we will discuss the extension of the S-Lemma to non Euclidean spaces, i.e. finite dimension vector spaces (e.g. $\mathbb{R}^n$) equipped with general ℓ_p norms, and how the result applies to stability analysis of a single-input/single-output Lur'e system when the Lyapunov function considered can be non quadratic.
We will give an introduction of the mathematical tools used to study stability in non Euclidean spaces, alongside with a detailed derivation of the derivation of the non polynomial S-Lemma, followed by its application to absolute stability.
We consider as references [42] [43] [44] [45].

4.1 Mathematical Preliminaries

We start our analysis by giving the definitions of *Log norms* and *pairings*, which will be essential in extending the classical Lyapunov stability theory to non Euclidean spaces.

4.1.1 Log Norms and Pairings on normed spaces

Definition. *Let $||\cdot||$ denote both a norm on $\mathbb{R}^n$ or an operator norm on matrices, depending on the context. The log norm of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as*

$$\mu(A) := \lim_{h\to 0^+} \frac{||I_n + hA|| - 1}{h} \tag{4.1}$$

Remark. *A matrix norm is a vector norm in the space of matrices. The matrix norm induced by a (finite) vector space ℓ_p norm is defined as*

$$\sup_{x\neq 0_n} \frac{||Ax||_p}{||x||_p}$$

where 0_n identifies the vector in which all entries are zeros.

The log-norm, despite its name, does not represent a norm, since it can assume negative values. The log norm can be considered to represent a measure on the action of the matrix A when interpreted as an operator on $\mathbb{R}^n$. More precisely, a one side derivative of the map $||\cdot||$ applied to the point I in the direction of A [46].

Remark. *Beside the applications considered in the course of this chapter and the following one, the log-norm provides an upper bound to the trajectories of an autonomous system of the form*

$$\dot{x}(t) = Ax(t)$$

Where A is an $n \times n$ matrix. In fact, for a given norm, it holds

$$||x(t)|| \leq e^{\mu(A)} ||x(0)||$$

Moreover, we have that

$$\mu(A) = min\{b \in \mathbb{R} : ||x(t)|| \leq e^{b} ||x(0)|| \; for\ all\ t > 0, ||x(0)|| \in \mathbb{R}^n\}$$

Definition. *A pairing on $\mathbb{R}^n$ is a map $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying the following properties*

i) (Subadditivity and continuity of first argument) : $[\![x_1+x_2, y]\!] \leq [\![x_1, y]\!]+[\![x_2, y]\!] \quad \forall x_1, x_2, y \in \mathbb{R}^n$ and $[\![\cdot,\cdot]\!]$ is continuous in the first argument,

ii) (Weak homogeneity) : $[\![\alpha x, y]\!] = [\![x, \alpha y]\!] = \alpha[\![x, y]\!] \quad \forall x, y \in \mathbb{R}^n, \alpha \geq 0$; moreover $[\![-x, -y]\!] = [\![x, y]\!]$,

iii) (Positive definiteness) : $[\![x, x]\!] > 0 \quad \forall x \neq 0_n$, *iv) (Cauchy-Schwarz inequality) :* $|[\![x,x]\!]| \le [\![x,x]\!]^{1/2}[\![y,y]\!]^{1/2} \quad \forall x, y \in \mathbb{R}^n$

A pairing $[\![\cdot,\cdot]\!]$ is said to be *compatible* with a norm $||\cdot||$ if $[\![x,x]\!] = ||x||^2$ for all $x \in \mathbb{R}^n$.

The notion of pairing can be though as a weakened version of a inner product: indeed, it allows to connect the product of two vector to a general ℓ_p norm, overcoming the fact that general ℓ_p spaces do not admit inner products induced by a norm.

In the following we will consider pairings satisfying certaing properties, i.e. the Lumer's equality and the curve norm derivative. Such properties will be the key in the derivation of the non polynomial S-lemma and in the application of non quadratic Lyapunov functions.

Definition (Non polynomial 2-forms)**.**
Given a pairing $[\![\cdot,\cdot]\!]$, compatible with the ℓ_p norm $||\cdot||$, $p \neq 2$ and a matrix $P \in \mathbb{R}^{n\times n}$ we define a non polynomial 2-form as

$$p(x) = [\![Px, x]\!]$$

Definition. *A pairing $[\![\cdot,\cdot]\!]$ satisfies the Lumer's equalities if $\forall\ A \in \mathbb{R}^{n\times n}$ it holds*

$$\mu(A) = \sup_{||x||=1} [\![Ax, x]\!] = \sup_{x\neq 0_n} \frac{[\![Ax, x]\!]}{||x||^2} \tag{4.2}$$

Definition. *We say that the pairing $[\![\cdot,\cdot]\!]$ satisfies the the curve norm derivative formula if for every differentiable curve $x : (a,b) \to \mathbb{R}^n$ and for almost every $t \in (a,b)$ the right upper Dini derivative of $||x(t)||^2$ satisfies*

$$D^+||x(t)||^2 := \limsup_{h\to 0^+} \frac{||x(t+h)||^2 - ||x(t)||^2}{h} = 2[\![\dot{x}(t), x(t)]\!] \tag{4.3}$$

The right upper Dini derivative represents a generalization of the notion of derivative used to study a continue but not differentiable function: indeed the right upper Dini derivative is well defined for almost all functions, even for function that are not conventionally differentiable. Obviously the upper Dini derivative coincides with the usual time derivative in the case of continuously differentiable functions.
In the following it will be used to study a Lyapunov function of the type $||x(t)||_p^2$.
In the table in Figure 1 we report some examples of pairings together with their compatible norms and log-norms; it can be shown that the pairings presented satisfy both the Lumer's inequality and the curve norm derivative, and we will refer to them as *strong pairings*. As we can observe in Figure (4.1) a general ℓ_p norm is continuous but not everywhere differentiable in the usual sense, which requires the notion of differentiability inducted by the Dini derivative.

Norm	Weak pairing	Log norms and Lumer's equality
$\|x\|_2 = \sqrt{x^\top x}$	$[\![x,y]\!]_2 = x^\top y$	$\mu_2(A) = \frac{1}{2}\lambda_{\max}(A + A^\top) = \max_{\|x\|_2=1} x^\top A x$
$\|x\|_p = \left(\sum_i \lvert x_i\rvert^p\right)^{1/p}$, $1 < p < \infty$	$[\![x,y]\!]_p = \|y\|_p^{2-p}(y \circ \lvert y\rvert^{p-2})^\top x$	$\mu_p(A) = \max_{\|x\|_p=1} (x \circ \lvert x\rvert^{p-2})^\top A x$
$\|x\|_1 = \sum_i \lvert x_i\rvert$	$[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i\neq j} \lvert a_{ij}\rvert\right) = \sup_{\|x\|_1=1} \operatorname{sign}(x)^\top A x$; $\mu_1^+(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i\neq j} a_{ij}^+\right) = \sup_{\|x\|_1=1, x\geq \mathbb{0}_n} \operatorname{sign}(x)^\top A x$
$\|x\|_\infty = \max_i \lvert x_i\rvert$	$[\![x,y]\!]_\infty = \max_{i\in I_\infty(y)} x_i y_i$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j\neq i} \lvert a_{ij}\rvert\right) = \sup_{\|x\|_\infty=1} \max_{i\in I_\infty(x)} (Ax)_i x_i$; $\mu_\infty^+(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j\neq i} a_{ij}^+\right) = \sup_{\|x\|_\infty=1, x\geq \mathbb{0}_n} \max_{i\in I_\infty(x)} (Ax)_i x_i$

Figure 4.1: Table of norms, strong pairings, and log norms for ℓ_p norms.
Here the symbol "$\circ$" denotes the Hadamard (or entrywise) product between vectors, $|y|$ the entrywise absolute value of the vector y and $I_\infty(x)$ is the set of indices s.t. the entry of vector x have max absolute value, i.e. $I_\infty(x) = \{i \in \{1, ...n\} : |x_i| = ||x||_\infty\}$

4.2 Non-polynomial S-Lemma

Consider a strong pairing $[\![\cdot,\cdot]\!]$ on $\mathbb{R}^n$ compatible with a norm $||\cdot||$ on vectors and log norm $\mu(\cdot)$ on matrices.
Consider also a family of s + 1 matrices $P_0, \dots, P_s \in \mathbb{R}^{n\times n}$ from which we define the functions $p_i(x) = [\![P_i x, x]\!]$, $i = 0, ..., s$.
Given a constant vector $\rho \in \mathbb{R}^s$, we define the primal optimization problem

$$\begin{aligned} \alpha = \sup_{x\in\mathbb{R}^n} \quad & p_0(x) \\ \text{subject to} \quad & ||x|| = 1, \quad p_1(x) \leq \rho_1, ..., p_s(x) \leq \rho_s \end{aligned} \tag{4.4}$$

Another optimization that we will see being related with the previous one is

$$
\begin{aligned}
\beta = \inf_{\tau \in \mathbb{R}^s} \qquad & \mu(P_0 - \sum_{j=1}^{s} \tau_j P_j) + \tau^\top \rho \\
\text{subject to} \qquad & \tau \geq 0
\end{aligned}
\tag{4.5}
$$

where the inequality $\tau \geq 0$ has to be intended elementwise, i.e. $\tau_i \geq 0$ $i = 1, \ldots, s$.

Theorem 4.1. *[Non polynomial S-Lemma]*
Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, compatible with the strong pairing $[\![\cdot,\cdot]\!]$. Given $P_0, \ldots, P_s \in \mathbb{R}^n$ and $\rho \in \mathbb{R}^s$.
For an arbitrary.pairing $[\![\cdot,\cdot]\!]$, let α and β be respectively the supremum in (4.4) *and the infimum in* (4.5)*. Then $\alpha \leq \beta$.*

Proof.
For each vector $x \in \mathbb{R}^n$ satisfying the constraints in the primal optimization problem (4.4) and for all $\tau \geq 0$ we have

$$
\begin{aligned}
[\![P_0 x, x]\!] &= [\![P_0 x - \sum_{j=1}^{s} \tau_j P_j x + \sum_{j=1}^{s} \tau_j P_j x, x]\!] \\
&\leq [\![P_0 x - \sum_{j=1}^{s} \tau_j P_j x, x]\!] + [\![\sum_{j=1}^{s} \tau_j P_j x, x]\!] \\
&\leq [\![(P_0 - \sum_{j=1}^{s} \tau_j P_j)x, x]\!] + \sum_{j=1}^{s} \tau_j [\![P_j x, x]\!] \\
&\leq \mu\big(P(\tau)\big) + \sum_{j=1}^{s} \tau_j \rho_j
\end{aligned}
\tag{4.6}
$$

Where in the first inequality we have exploited the subadditivity property of the pairings, while in the second one we have written $[\![\sum_{j=1}^{s} \tau_j P_j x, x]\!] \leq \sum_{j=1}^{s} \tau_j [\![P_j x, x]\!]$ by using weak homogeneity and subadditivity again. Finally the last inequality comes from the fact that the domain of primal problem is the unit sphere, and so for all x such that $\|x\| = 1$

$$
[\![P_0 x - \sum_{j=1}^{s} \tau_j P_j x, x]\!] \leq \sup_{\|x\|=1} [\![P_0 x - \sum_{j=1}^{s} \tau_j P_j x, x]\!] = \mu(P_0 x - \sum_{j=1}^{s} \tau_j P_j)
$$

while $\sum_{j=1}^{s} \tau_j [\![P_j x, x]\!] \leq \sum_{j=1}^{s} \tau_j \rho_j = \tau^T \rho$ comes from the inequality constraints. By taking supremum of $[\![P_0 x, x]\!]$ over all the feasible points $x \in \mathbb{R}^n$ on the left and hand side and the infimum over all $\tau \geq 0$ on the right hand side in the last inequality of (4.6) we obtain the claim and concludes the proof. □

Remark. *If the pairing* $[\![\cdot,\cdot]\!]$ *is linear in its first argument, then the optimization problem* (4.5) *is the Lagrangian dual problem of* (4.4).
Indeed, we can show this fact by defining the Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ *as*

$$\begin{aligned} L(x,\tau) &= -p_0(x) + \sum_{j=1}^{s} \tau_j (p_j(x) - \rho_j) \\ &= -[\![P_0 x, x]\!] + \sum_{j=1}^{s} \tau_j [\![P_j x, x]\!] - \sum_{j=1}^{s} \tau_j \rho_j = -[\![P(\tau)x, x]\!] - \tau^T \rho \end{aligned} \tag{4.7}$$

where we have exploited the linearity hypothesis of the pairing and defined $P(\tau) = P_0 - \sum_{j=1}^{s} \tau_j P_j$.
If we consider now the Lagrangian dual function, for a fixed τ *we have*

$$\begin{aligned} g(\tau) &= \inf_{x\in\mathbb{R}^n, \|x\|=1} L(x,\tau) = \inf_{x\in\mathbb{R}^n, \|x\|=1} -\tau^T\rho - [\![P(\tau)x, x]\!] \\ &= -\tau^T\rho + \inf_{x\in\mathbb{R}^n \|x\|=1} -[\![P(\tau)x, x]\!] = -\tau^T\rho - \sup_{x\in\mathbb{R}^n \|x\|=1} [\![P(\tau)x, x]\!] \\ &= -\tau^T\rho - \mu\big(P(\tau)\big) \end{aligned} \tag{4.8}$$

where we have taken the infimum over the the domain of the primal problem, i.e. the unit sphere $\|x\| = 1$*, and in the last step we have exploited Lumer's equality* (4.2).
Finally, by taking the Lagrangian dual problem of (4.4)

$$\begin{aligned} \gamma = \inf\ & L(\tau) \\ \textit{subject to } & \tau \geq 0 \end{aligned} \tag{4.9}$$

we obtain

$$\sup_{\tau\in\mathbb{R}^s, \tau\geq 0} g(\tau) = \sup_{\tau\in\mathbb{R}^s, \tau\geq 0} -\tau^T\rho - \mu\big(P(\tau)\big) = \inf_{\tau\in\mathbb{R}^s, \tau\geq 0} \tau^T\rho + \mu\big(P(\tau)\big) \tag{4.10}$$

which shows that $\gamma = \beta$.
A review of the notions of Lagrangian function and Lagrangian dual function, alongside with a slight more detailed derivation of (4.7) *and* (4.10) *can be found in Appendix A.*

The result in theorem (4.1) can be strengthened in the case of Metzler matrices and ℓ_1 norm, as stated in the next lemma, whose proof is omitted.

Lemma 4.1. *Consider the primal optimization problem*

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{sup} && [\![P_0 x, x]\!] \\ &\textit{subject to} && \|x\|_1 = 1, x > 0_n \quad [\![P_1 x, x]\!] \leq \rho_1, ..., [\![P_s x, x]\!] \leq \rho_s \end{aligned} \tag{4.11}$$

Assume that the matrices P_0, $-P_1, \ldots, -P_s$ are Metzler. Then the supremum α in (4.11) *and the solution β to its dual problem* (4.5) *coincide.*

4.2.1 Connection with the classical S-Lemma

In the case of $p = 2$, i.e. $[\![\cdot, \cdot]\!]$ is an inner product and considering the number of constraints to be $s = 1$ and $\rho_1 = 0$, under the further assumption that exists $\tilde{x}$ such that $[\![P_1 \tilde{x}, \tilde{x}]\!] < 0$ the Non-Polynomial S-Lemma is equivalent to the strict inequalities S-Lemma (Theorem (1.7)).
We start by observing that in Theorem (1.6) and Theorem (1.7) the quadratic form which describes the sector condition is non negative, which is equivalent to consider the quadratic form induced by $-[\![P_1 x, x]\!]$, so that the constraint $p_1(x) = [\![P_1 x, x]\!] \leq 0$ becomes $-p_1(x) = [\![P_1 x, x]\!] \geq 0$. Since the constraint $-[\![P_1 x, x]\!] > 0$ is satisfied in one point $\tilde{x}$, it is also satisfied for some x inside the unit sphere $\|x\|_2 = 1$, due to the homogeneity of the inner product: indeed, if $-[\![P_1 \tilde{x}, \tilde{x}]\!] > 0$ then $-\frac{1}{\|\tilde{x}\|_2^2}[\![P_1 \tilde{x}, \tilde{x}]\!] = -[\![P_1 \frac{\tilde{x}}{\|\tilde{x}\|_2}, \frac{\tilde{x}}{\|\tilde{x}\|_2}]\!] > 0$. That is, the existence of a solution inside the unit sphere guarantee that the problem (4.4) is feasible and that the supremum α exists. Since set of feasible points in (4.4) is a compact non-empty set and $[\![\cdot, \cdot]\!]$ is continuous being an inner product, from Weierstrass theorem we can assure the existence of a maximum.
As consequence the supremum in the primal optimization problem can be substituted with a maximum.
We observe that again for the homogeneity of the inner product for each x s.t. $-p_1(x) = -[\![P_1 x, x]\!] > 0$ we have $[\![P_0 x, x]\!] - \alpha\|x\|_2^2 = p_0(x) - \alpha\|x\|_2^2 \leq 0$. From the non strict inequalities S-Lemma (Theorem (1.6)) we know that exists $\tau^* \geq 0$ such that $p_0(x) - \alpha\|x\|_2^2 + \tau^*(-p_1(x)) = p_0(x) - \alpha\|x\|_2^2 - \tau^* p_1(x) \leq 0$ for all x.
Hence we obtain

$$\mu_2(P(\tau)) = \sup_{\|x\|_2 = 1} (p_0(x) - \tau^* p_1(x)) \leq \alpha$$

On the other hand we know from the Non-Polynomial S-Lemma that $\mu_2(P(\tau)) = \beta \geq \alpha$ for all $\tau \geq 0$, implying $\alpha = \beta$.
In the case of the classical S-Lemma the condition searched on the objective function $[\![P_0 x, x]\!]$ is negative for all x satisfying the constraint $-[\![P_1 x, x]\!] \geq 0$. This turns out to be equivalent to

require that the supremum of $[\![P_0x, x]\!]$ over the set of feasible points is negative, that is

$$\begin{aligned}\alpha = \sup_{x\in\mathbb{R}^n} \quad & p_0(x)\\ \text{subject to} \quad & \|x\| = 1, \quad -p_1(x) \geq 0\end{aligned}$$

is negative.
Since in the case of $p = 2$ with $s = 1$ and $\rho_1 = 0$, we have $\alpha = \beta$, it follows that the quadratic form induced by $[\![P_0x, x]\!]$ is negative over the set of x such that $-[\![P_1x, x]\!] \geq 0$ if and only if exists τ^* such that

$$\begin{aligned}\beta = \inf_{\tau\in\mathbb{R}} \quad & \mu\big(P(\tau)\big)\\ \text{subject to } & \tau \geq 0\end{aligned}$$

is negative.
We observe that

$$\begin{aligned}\mu(P(\tau)) &= \sup_{\|x\|=1} (P(\tau)) = \sup_{\|x\|=1} [\![(P_0 - \tau P_1)x, x]\!]\\ &= \sup_{\|x\|=1} [\![P_0x, x]\!] - \tau[\![P_1x, x]\!] = \sup_{\|x\|=1} [\![P_0x, x]\!] + \tau[\![-P_1x, x]\!]\end{aligned}$$

where in the last equality we have exploited the linearity of the inner product. This implies that $\beta < 0$ is equivalent to the existence of a $\tau^* \geq 0$ such that

$$\sup_{\|x\|=1} [\![P_0x, x]\!] - \tau^*[\![P_1x, x]\!] < 0$$

i.e. there exists τ^* such that the quadratic form induced by

$$[\![P_0x, x]\!] - \tau^*[\![P_1x, x]\!] = [\![P_0x, x]\!] + \tau^*[\![-P_1x, x]\!]$$

is negative for all x, which is the statement of the non strict inequalities S-Lemma.

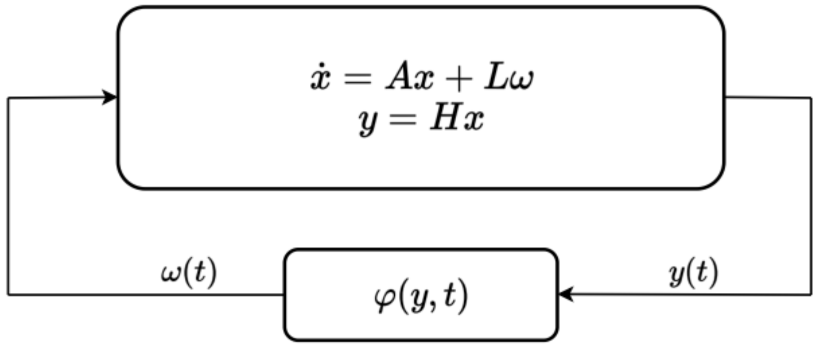


Figure 4.2: Block scheme of an autonomous Lur'e system (4.12)

4.3 Sufficient conditions for stability using pairings

Consider an autonomous Lur'e system in of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + L\omega(t) \\ y(t) = Cx(t) \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{4.12}$$

with $A \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$.
We will consider both the cases in which the function $\varphi(y,t)$ satisfies the constraint

$$\kappa_1 \le \frac{\varphi(t,y)}{y} \le \kappa_2 \quad \forall y \neq 0 \tag{4.13}$$

Equivalently, by using the notation $\varphi(t,y) = \omega(t)$ we have that (4.13) is equivalent to

$$\kappa_1 y^2 \le \omega y \le \kappa_2 y^2 \tag{4.14}$$

for all $t \in \mathbb{R}$.
We allow the possibility of $\kappa_1 = +\infty$ or $\kappa_2 = -\infty$, disregarding however the trivial case in which both equalities hold at the same time.
However, without loss of generality, through the course of the chapter we will restrict ourselves to the case in which $\kappa_1 = 0$.

$$0 \le \omega y \le \kappa y^2 \tag{4.15}$$

with $\kappa > 0$.

We show in fact that its always possible to trace back constraints (4.14) to (4.15): indeed, by defining the matrices $A' = A + \kappa_1 BC$, $B' = B$, we obtain the following dynamic description of $x(t)$ as a function of $v(t) := \omega(t) - \kappa_1 y(t)$

$$\begin{aligned}\dot{x} &= A'x + Lv = Ax + \kappa_1 LCx + L(\omega - \kappa_1 y) \\ &= Ax + L\omega + \kappa_1 Ly - \kappa_1 y) = Ax + L\omega\end{aligned}$$

for which the sector and slope constraints can be rewritten as

$$0 \leq \ \omega y - \kappa_1 y^2 \ \leq \kappa_2 y^2 - \kappa_1 y^2 \longrightarrow 0 \leq vy \leq \kappa y^2$$

where we have defined $\kappa = \kappa_2 - \kappa_1$.
In the case $\kappa_1 = -\infty$ we can apply similar transformations by denoting $v(t) := \kappa_2 y(t)$, $A' = A - \kappa_1 LC$, $L' = L$, $\kappa = +\infty$.
We then rewrite the constraints on the nonlinearity in the quadratic forms

$$\begin{aligned}&\omega(\kappa^{-1}\omega - y) \leq 0 \\ &\Delta\omega(\kappa^{-1}\Delta\omega - \Delta y) \leq 0\end{aligned} \tag{4.16}$$

which are equivalent, to (4.15).
We next prove that fact. Observe that

$$\omega(\kappa^{-1}\omega - y) = \omega^2\kappa^{-1} - \omega y \leq 0 \Rightarrow \kappa_1\omega y \geq \omega^2 \tag{4.17}$$

which implies that $\omega y \geq 0$ is greater than zero and hence that ω and y have same sign. By dividing by ω and multiplying by y both sides of (4.17) we obtain the second inequality in (4.15).
On the other hand, if it holds

$$0 \leq \ \omega y \ \leq \kappa y^2$$

it is straightforward observe that ω and y have the same sign. Subtracting κy^2 from each side we obtain

$$\omega y - \kappa y^2 = (\omega - \kappa y)y \ \leq 0$$

Since y and ω have the same sign we can divide by y and multiply by ω without change the direction of the inequality. Finally, we divide by $\kappa > 0$, obtaining (4.16).
We define now the *Lyapunov condition*, which guarantees sufficient condition for exponential stability

Definition. *Fixed $c > 0$, consider* (4.12) *the* ***Lyapunov condition*** *holds, respectively for the* *sector constraints and the slope condition, if*

$$[\![Ax + L\omega, x]\!] \leq -c\|x\|^2 \qquad \textit{for all } \omega \in \mathbb{R} \textit{ and } x \in \mathbb{R}^n \textit{ s.t. } \omega(\kappa^{-1}\omega - Cx) \leq 0 \tag{4.18}$$

If condition (4.18) holds for a strong pairing $[\![\cdot, \cdot]\!]$ it is possible to use $V(x) = \|x\|^2$ as Lyapunov function, providing the following estimate

$$\|V\big(x(t)\big)\| \leq c\|x_0\|e^{-ct}$$

Indeed, if condition (4.18) hold we have

$$D^+(\|x(t)\|^2) = 2[\![\dot{x}, x]\!] = 2[\![Ax + L\omega, x]\!] \leq 2\big(- c\|x(t)\|^2\big) = -2cV(x) \tag{4.19}$$

Where we have exploited the curve norm derivative formula in (4.3). We observe moreover that, since the Lyapunov function obtained by means of non Euclidean norms is in general only continuous, but not everywhere differentiable, the time derivative of $V(x)$ holds only in the Dini derivative sense.

Remark.
Since $\varphi(t, y) \equiv 0$ respect the sector constraints (4.14)*, the Lyapunov condition* (4.18) *can hold if and only if $[\![Az, z]\!] \leq -c\|z\|^2$ for all $z \in \mathbb{R}^n$. Since for every strong pairings the Lumer's equality* (4.2) *is satisfied, the latter inequality is equivalent to $\mu(A) \leq -c < 0$. Consequently, a necessary condition for having Lyapunov inequality* (4.18) *is $\mu(A) < -c$. Note that this condition implies A is an Hurwitz matrix, since, when Lumer's equality holds, the log norm is an upper bound on the spectral abscissa.*

4.4 Stability analysis via pairings and Non-Polynomial S-Lemma

In order to apply the Non Polynomial S-Lemma to the stability analysis of system (4.12) we aim to obtain a description of the Lyapunov inequality using non-polynomial 2-forms associated to a pairing.
We start by defining augmented state z and the matrices P_0 and P_1

$$P_0 = \begin{pmatrix} A + cI_n & L \\ 0_{1\times n} & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0_{n\times n} & 0_{n\times 1} \\ -C & \kappa^{-1} \end{pmatrix} \quad z = \begin{pmatrix} x \\ \omega \end{pmatrix} \in \mathbb{R}^{n+1}$$

We have

$$P_0 z = \begin{pmatrix} (A + cI_n)x + L\omega \\ 0 \end{pmatrix} \quad \text{and} \quad P_1 z = \begin{pmatrix} 0_n \\ \kappa^{-1}\omega - Cx \end{pmatrix}$$

We will use the matrices P_0, P_1, together with the augmented state space x to represent the *Lyapunov inequality* and the *sector constraint* using the pairings, as showed by the following lemma.

Lemma 4.2. *Consider $p \in [1, \infty)$; the following equivalences hold:*

i) ω and Cx satisfy the sector condition $\omega(\kappa^{-1}\omega - Cx) \leq 0$ if and only if $[\![P_1 z, z]\!] \leq 0$

ii) for a fixed $c > 0$, the inequality $[\![Ax + L\omega, x]\!] \leq -c\|x\|^2$ holds if and only if $[\![P_0 z, z]\!] \leq 0$

If $p = \infty$ we have instead:

i) ω and Cx satisfy the sector condition $\omega(\kappa^{-1}\omega - Cx) \leq 0$ implies $[\![P_1 z, z]\!] \leq 0$; the converse implication holds when $|\omega| > \|z\|_\infty^2$

ii) if $|\omega| < \|x\|_\infty$ or $x = 0$, for a fixed $c > 0$ the inequality $[\![Ax + L\omega, x]\!] \leq -c\|x\|_\infty$ holds if and only if $[\![P_0 z, z]\!] \leq 0$

Proof. The proof of the lemma comes directly form the inspection of the product $[\![P_0 z, z]\!]_p$ and by the definitions of the strong pairings in figure (4.1.1). We present the proof only for the case $p \neq 1, \infty$.

i) The proof come directly by evaluating $[\![P_1 z, z]\!]$

$$\begin{aligned}
[\![P_1 z, z]\!]_p &= \|z\|_p^{2-p}(z \circ |z|^{p-2})^T P_1 z \\
&= \|z\|_p^{2-p}\begin{bmatrix} sign(x) \circ |x|^{p-1} & sign(\omega)|\omega|^{p-1}\end{bmatrix}\begin{bmatrix} O_n \\ -Cx + \kappa^{-1}\omega\end{bmatrix} \\
&= \|z\|_p^{2-p}|\omega|^{p-1} sign(\omega)(-Cx + \kappa^{-1}\omega)
\end{aligned} \tag{4.20}$$

where $\|z\|_p^{2-p}|\omega|^{p-1} sign(\omega)(-Cx + \kappa^{-1}\omega) \leq 0$ if and only if $\omega(\kappa^{-1}\omega - Cx) \leq 0$.
ii) Consider two vectors $x^1,\ x^2 \in \mathbb{R}^n$ and a scalar ω^1. Define

$$z^1 = \begin{bmatrix} x^1 \\ 0 \end{bmatrix} \quad z^2 = \begin{bmatrix} x^2 \\ \omega \end{bmatrix}$$

$z^1,\ z^2 \in \mathbb{R}^{n+1}$.
We starting by proving the that following equivalence

$$[\![z^1, z^2]\!]_p \leq 0 \iff [\![x^1, x^2]\!]_p \leq 0 \tag{4.21}$$

holds.
In fact, by evaluating the pairing between$[\![z^1, z^2]\!]_p$ one can notice

$$\begin{aligned}
[\![z^1, z^2]\!]_p &= \|z^2\|_p^{2-p}\left[sign(z^2) \circ |z^2|^{p-1}\right] z^1 \\
&= \|z^2\|_p^{2-p}\left[sign(x^2) \circ |x^2|^{p-1} \quad sign(\omega)\omega\right]\begin{bmatrix} x^1 \\ 0 \end{bmatrix} \\
&= \|z^2\|_p^{2-p}(sign(x^2) \circ |x^2|^{p-1})^T x^1 \\
&= k[\![x^1, x^2]\!]_p \leq 0
\end{aligned}$$

for some $k > 0$.
We show now that the inequality $[\![Ax + L\omega, x]\!]_p \leq -c\|x^2\|_p^2$ is equivalent to $[\![Ax + cI_n + L\omega, x]\!]_p \leq 0$
Indeed, by the subadditivity property of the pairings it holds

$$[\![(A + cI_n)x + L\omega, x]\!]_p \leq \quad [\![Ax + L\omega, x]\!]_p + c[\![x, x]\!]_p$$

From this fact it follows

$$[\![Ax + L\omega, x]\!]_p \leq -c[\![x, x]\!]_p \Rightarrow \quad [\![(A + cI_n)x + L\omega, x]\!]_p \leq 0$$

Calling $x^1 = (A + cI)x + L\omega$, $x^2 = x$ and $\omega^2 = \omega$ and considering z^1, z^2 defined as above, by exploiting the equivalence (4.21) we obtain the statement. □

We finally show how to exploit the representations showed in the last lemma together with the Non-polynomial S-Lemma to obtain a procedure that allow us to guarantee the exponential stability of the origin when the norm employed in the analysis is an arbitrary ℓ_p norm.

Theorem 4.2. *Consider a Lur'e system with sector constraint defined by (A, L, C, κ), such as in* (4.12). *Fix $c > 0$ and a norm ℓ_p, $p \in [1, \infty]$, with log norm μ_p and a compatible strong pairing $[\![\cdot, \cdot]\!]_p$. If $p = \infty$ then also assume $\kappa\|C\|_\infty < 1$. The following statements hold:*

i) the Lyapunov inequality (4.15) *holds if*

$$[\![P_0 z, z]\!]_p \leq 0 \; for \; all \; z \in \mathbb{R}^{n+1} \; s.t. \; [\![P_1 z, z]\!]_p \leq 0 \tag{4.22}$$

furthermore for $p < \infty$ (4.18) *and* (4.22) *are equivalent*

ii) the Lyapunov inequality holds if

$$\exists \tau \geq 0 \; s.t. \; \mu_p(P(\tau)) \leq 0, \; where \; P(\tau) = P_0 - \tau P_1 = \begin{pmatrix} A + cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} \tag{4.23}$$

furthermore for $p = 2$ or if $p = 1$, A Metzler and L, C are non negative, (4.18) *and* (4.23) *are equivalent.*

Proof.
$i)$: in the case of $p < \infty$ the proof comes directly from lemma (4.2).
In the case of $p = \infty$ we start by observing that, whenever the sector conditions (4.15) are satisfied, we have

$$\omega(\omega\kappa^{-1} - \kappa Cx) \leq 0 \Rightarrow \omega^2 \leq \kappa Cx\omega \Rightarrow |\omega| \leq \kappa|Cx|$$

By the assumption $\kappa||C||_\infty < 1$, we obtain $|\omega| \leq \kappa|Cx| \leq \kappa||C||_\infty||x||_\infty$, which implies $|\omega| < ||x||_\infty$, unless $\omega = ||x|| = 0$.
By the previous Lemma we have that $[\![P_0z, z]\!]_\infty \leq 0$ implies the Lyapunov condition (4.18), while $[\![P_1z, z]\!]_\infty \leq 0$ implies the sector condition (4.15). However (4.22) is only sufficient for $p = \infty$.
$ii)$ The proof comes directly by taking as primal problem the supremum of $[\![P_0z, z]\!]$ over the set $||z|| = 1,\ [\![P_1z, z]\!] \leq 0$, and then using the upper bound given by from Theorem (4.1).
In the case of $p = 2$ the equality between the optimal solution of the primal problem (4.4) and the dual formulation (4.5) comes from the considerations made in section 4.2.1, from which we obtain $\alpha = \beta$.
In the case of $p = 1$ the equality between the optimal values of the primal and dual problem come from Lemma (4.1) by noticing that P_0 and $-P_1$ are Metzler when A Metzler and L, C are non negative. □

For $p = 2$ condition (4.23) can be expressed in analytical form.
As showed in figure (4.1), for an arbitrary square matrix M, $\mu_2(M) = \lambda_{max}(M^s)$, where M^s is the symmetrized matrix obtained from M, i.e. $M^s := \frac{M+M^T}{2}$ and λ_{max} its greatest eigenvalue (which is always real since M^s is symmetric). That is, condition $\mu_2\big(P(\tau)\big) \leq 0$ holds if the symmetrized version of $P(\tau)$ is negative-semidefinite.
In the next corollary we state, without proof, an equivalent analytical condition to $\mu_2\big(P(\tau)\big) \leq 0$, which will be exploited later in chapter 5.

Corollary 4.1.
Inequality (4.23) *with $p = 2$ and $\kappa < \infty$ holds if and only if*

$$\exists\, \tau > 0,\ c > 0 \quad s.t. \quad \frac{A + A^T}{2} + cI + \frac{\kappa}{4\tau}(L + \tau C^T)(L^t + \tau C) \leq 0 \qquad (4.24)$$

We notice that we must have $\tau \neq 0$ coherently with the results obtained in the course of chapter 3.

Theorem 2 can be generalized to the weighted norm case, by introducing a new variable $\tilde{x} = Rx$. Calling $[\![\cdot,\cdot]\!]_{p,R}$ the pairings of norm ℓ_p applied together with the change of variable $\tilde{x} = Rx$, the Lyapunov condition (4.18) holds for $[\![\cdot,\cdot]\!]_{p,R}$ if and only if it holds for $[\![\cdot,\cdot]\!]_p$ with matrices $\tilde{A} = RAR^{-1},\ \tilde{L} = RL,\ \tilde{C} = CR^{-1}$.
Indeed, this could be seen by noticing that the change of variables $\tilde{x} = Rx$ is equivalent to consider a system whose dynamics is described by the matrices above

$$\begin{cases} \dot{\tilde{x}} = \dot{Rx} = RAR^{-1}Rx + RL\omega = \tilde{A}\tilde{x} + \tilde{L}\omega \\ y = CRx = \tilde{C}x \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{4.25}$$

so that if the Lyapunov condition (4.18) holds for system (4.12) with $[\![\cdot,\cdot]\!]_{p,R}$, it must hold for system (4.25) with $[\![\cdot,\cdot]\!]_p$.
Introducing as $\tilde{R} = diag(R, 1)$ we have the following corollary:

Corollary 4.2. *Consider* $[\![\cdot,\cdot]\!]_{p,R}$ *where* $1 \le p \le \infty$*, if* $p = \infty$ *assume* $\kappa\|CR^{-1}\|_\infty < 1$*; the Lyapunov condition* (4.18) *holds if there exist* $c > 0,\ \tau \ge 0$ *s.t.*

$$\mu_{p,\tilde{R}}(P(\tau)) = \mu_p \begin{pmatrix} RAR^{-1} + cI_n & RL \\ \tau CR^{-1} & -\tau\kappa^{-1} \end{pmatrix} \le 0 \tag{4.26}$$

In synthesis, we can guarantee that system (4.12) is asymptotically stable if

$$\inf_{c,\tau} \quad \mu_{p,R} \begin{pmatrix} A + cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} \quad \text{subject to} \quad c > 0, \quad \tau \ge 0$$

is non positive.
Lastly, we provide a sufficient condition for ensure that condition (4.26) is satisfied for an arbitrary $p \in [1, \infty]$ and for some arbitrary *diagonal* matrix R.
We define as now $\lceil A \rceil$ the Metzler version of A, i.e.

$$(\lceil A \rceil)_{i.j} = \begin{cases} a_{ii} & \text{if } i = j \\ |a_{ij}| & \text{if } i \ne j \end{cases}$$

and $|L|$, $|C|$ the vectors with elements that are the absolute values of the entries of L and C.
Using a weighted norm allows us to state the following corollary

Lemma 4.3. *Consider the Metzler matrix* $\Omega(\kappa) = \lceil A \rceil + \kappa|L||C|$ *If* $\alpha(\Omega(\kappa)) < -c$*, where* $\alpha(A)$ *is the maximum real part of the eigenvalues of A, then for every* $p \in [1, \infty], \tau > 0$ *there* *exist a diagonal matrix* $R = R(p, \tau) \succ 0$ *such that* (4.26) *holds; in the case of* $p = \infty$*, it holds* $\kappa \|CR^{-1}\|_\infty < 1$

Remark. *Lemma* (4.3) *shows that the Aizerman conjecture is true when we consider positive systems, i.e. when* A *is a Metzelr matrix and* L *and* C *are positive vectors. Indeed, we recall from Chapter 1 that the conjecture state that a Lur'e system is absolute stable if the linear version of the system obtained by substituting the nonlinearity* $\varphi(y)$ *with* $\gamma y, \gamma \in [0, \kappa]$ *is stable, or equivalently if the matrix* $A + L\gamma y = A + \gamma LC$ *is Hurwitz for all* $\gamma \in [0, \kappa]$.
In the case of a positive system $\Omega(\kappa) = \lceil A \rceil + \kappa |L||C|$ *coincides with* $A + \kappa LC$. *In fact if* $\lceil A \rceil + \gamma |L||C| = A + \gamma LC$ *is Hurwitz for all* $\gamma \in [0, \kappa]$ *by the previous corollary there exists a matrix* R *such that* $\mu_{p,R}((P(\tau)) \leq 0$*, assuring exponential stability and consequently proving that the Aizerman hypothesis is correct for positive systems.*
On the other hand, for general systems, the Hurwitz condition on the matrix$\lceil A \rceil + \kappa |L||C|$ *does not imply the Hurwizness property on* $A + \kappa LC$*, meaning that there could exist systems for which* $A + \gamma LC$ *is Hurwitz for all* $\gamma \in [0, \kappa]$*, but such that* $\alpha(\Omega(\kappa)) \geq 0$*, so that the linear version of the Lur'e system absolute stability is attained for all* $\gamma \in [0, \kappa]$ *but not by the actual system.*

5

Data Driven absolute stabilization via non polynomial Lyapunov function

In this chapter we will consider the application of the stability results in non Euclidean spaces introduced in the previous chapter to the problem of designing a stabilizing feedback controller for a system whose dynamics is partially unknown.
We will perform the stability analysis using the ℓ_1 and ℓ_∞ norms, which better suit the design problem due to the existence of a computable closed form for their respective log-norms.
Then, drawing inspiration from the approach used in chapter 3, we will extend these design approaches to the case in which the system dynamics is given by a data-driven representation.
Finally, we will show how to solve the design problem by means of a linear programming techniques and discuss how the introduction of a data-based approach does not change the problem structure, increasing nonetheless the number of computations required.

5.1 Problem Framework

Consider a single-input/single-output Lur'e system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + L\omega(t) \\ y(t) = Cx(t) \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{5.1}$$

with $A \in \mathbb{R}^{n \times n}$, B and $L \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$.
We assume that the nonlinearity $\varphi(y, t) = \omega(t)$ satisfies the sector constraints

$$0 \leq \ \omega y \ \leq \kappa y^2 \tag{5.2}$$

We consider again the assumptions made in chapter 3, in which we suppose to have no prior knowledge of the matrices A and B, whereas we assume that the matrices L and C are known. We assume moreover we are able to collect the following samples from an experiment conducted before the controller synthesis

$$\begin{aligned} U_0 &:= [u(0) \ldots u(T-1)] \\ X_0 &:= [x(0) \ldots x(T-1)] \\ X_1 &:= [\dot{x}(0) \ldots \dot{x}(T-1)] \\ F_0 &:= [\omega(0) \ldots \omega(T-1)] \end{aligned} \tag{5.3}$$

where it holds that $X_1 = AX_0 + BU_0 + LF_0$.
The problem considered is to find a state feedback controller capable of stabilize the nonlinear dynamics for all functions $\varphi(t, y)$ such that conditions (5.2) are satisfied using the prior knowledge on the matrices C and L and of the collected data (5.3) from the system.
Again, by following the same approach of chapter 3, we will make the assumption that the matrix

$$W_0 = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \in \mathbb{R}^{(1+n) \times T} \tag{5.4}$$

is full row rank.
Applying as input $u = Kx$ for some $K \in \mathbb{R}^{1 \times n}$ system (5.1) becomes then

$$\begin{cases} \dot{x}(t) = (A + BK)x(t) + L\omega(t) \\ y(t) = Cx(t) \\ \omega(t) = \varphi(t, y(t)) \end{cases} \tag{5.5}$$

We remark that since system (5.5) is an autonomous system with state matrix $(A + BK)$, the stability results obtained in the previous chapter still hold.

Tracing back the considerations of chapter 3, to obtain a data driven representation of the system dynamics we can exploit the data samples (5.3) and the full row rank assumption on W_0, obtaining the representation

$$\begin{aligned} A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \\ = AX_0G + BU_0G = (AX_0 + BU_0 + LF_0 - LF_0)G \\ = (X_1 - LF_0)G = X_LG \end{aligned} \tag{5.6}$$

where we recall that $X_L \triangleq X_1 - LF_0$.

The existence of a matrix $G \in \mathbb{R}^{T \times n}$ such that for any matrix $K \in \mathbb{R}^{1 \times n}$ the following equality holds

$$W_0 G = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G = \begin{bmatrix} K \\ I_n \end{bmatrix} \tag{5.7}$$

is assured by the rank condition on W_0.

Using the data-based representation exposed above, we provide a design method for a stabilizing controller based on the exponential stability results presented in the previous chapter for norms ℓ_1 and ℓ_∞.

Theorem 5.1. *(ℓ_1 norm)*
Consider system (5.5) *for some $K \in \mathbb{R}^{1 \times n}$, and the sector constraints* (5.2)*,ì with $\kappa < +\infty$.*
Suppose moreover that Assumption 1 holds, and define $\tau^* = \kappa \sum_{i=1}^{n} |L_i|$.
Then origin is exponentially stable for system (5.6) *if there exists a matrix $G \in \mathbb{R}^{T \times n}$ such that*

$$\begin{cases} X_0 G = I_n \\ \displaystyle\sum_{\substack{i=1 \\ i \neq j}}^{n} |(X_L G)_{ij}| + (X_L G)_{jj} + \tau^* |C_j| < 0 \quad \forall\ j = 1, .., n \end{cases} \tag{5.8}$$

Moreover the stabilizing controller is given by $K = U_0 G$.

Proof. Consider the matrix

$$P(\tau) = \begin{pmatrix} A + BK + cI & L \\ \tau C & -\tau \kappa^{-1} \end{pmatrix} \tag{5.9}$$

We know from Theorem 4.2 that, for a chosen norm ℓ_p, the closed loop feedback is exponen-

tially stable if exist $c > 0$ and $\tau \geq 0$ such that $\mu_p(P(\tau)) \leq 0$.
According to the table in Figure (4.1) we have $\mu_1(P(\tau)) \leq 0$ if

$$\max_{j \in 1,\dots,n+1} P(\tau)_{jj} + \sum_{i \neq j} |P(\tau)_{ij}| \leq 0 \tag{5.10}$$

If we consider the last column of $P(\tau)$ in (5.9), condition (5.10) yields

$$\sum_{i=1}^{n} |L_i| - \tau\kappa^{-1} \leq 0 \iff \tau \geq \kappa \sum_{i=1}^{n} |L_i| \tag{5.11}$$

where L_i is the i-th element of the column vector L. Hence (5.11) holds for all $\tau \geq \tau^*$.
Consider now condition (5.10) for $j = 1, \dots, n$. We have that such condition holds if $\exists\ c > 0$, $\tau \geq 0$ and $K \in \mathbb{R}^{1\times n}$ such that

$$\sum_{i \neq j} |(A + BK)_{ij}| + (A + BK)_{jj} + c + |\tau C_j| \quad \leq 0 \tag{5.12}$$

Since we need that $\tau \geq \tau^*$, this condition hold only if holds for $\tau = \tau^*$.

If we consider now representation (5.6), the, (5.12) becomes equivalent to finding a matrix $G \in \mathbb{R}^{T\times n}$ satisfying $X_0 G = I_n$ such that for all $j = 1, \dots, n$

$$\sum_{i \neq j} |(X_L G)_{ij}| + (X_L G)_{jj} + \tau^* |C_j| \quad < 0 \tag{5.13}$$

In fact, choosing c as

$$c^* = -\min_j \Big\{ \sum_{i \neq j} |X_L G|_{ij} + (X_L G)_{jj} + +\tau^* |C_j| \Big\} > 0$$

we obtain that the sum in (5.10) is non positive for all j.
Finally, we observe that by definition of G we have $K = U_0 G$. □

Theorem 5.2. *(ℓ_∞ norm)*
Assume that the same hypotheses of the previous theorem hold true; moreover assume $\kappa \|C\|_\infty <$ 1.
Then system (5.5) *is asymptotically stable if there exist a matrix $G \in \mathbb{R}^{T\times n}$ such that*

$$\begin{cases} X_0 G = I_n \\ \sum_{\substack{j=1 \\ j \neq i}}^{n} |(X_L G)_{ij}| + (X_L G)_{ii} + |L_j| < 0 \quad \forall\ i = 1, .., n \end{cases} \tag{5.14}$$

The stabilizing controller is given by $K = U_0 G$.

Proof. Consider again

$$P(\tau) = \begin{pmatrix} A + BK + cI & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} \tag{5.15}$$

for some $c > 0$ and $\tau \geq 0$.
According to the table in Figure (4.1) we have $\mu_\infty(P(\tau)) \leq 0$ if

$$\max_{i \in 1,\dots n+1} P(\tau)_{ii} + \sum_{j \neq i} |(P(\tau))_{ij}| \leq 0 \tag{5.16}$$

Similarly to the previous theorem we start by considering the last row of $P(\tau)$ for which (5.16) yields

$$\sum_{i=1}^{n} |\tau C_i| - \tau\kappa^{-1} \leq 0 \iff \kappa \sum_{i=1}^{n} |C_i| \leq 1 \quad \text{or} \quad \tau = 0 \tag{5.17}$$

where we have used the fact that $\tau \geq 0$ and where C_i is the i-th element of the row vector C. That is, we can always assure that the last row of matrix (5.15) has negative matrix measure accordingly to the ℓ_∞ log norm.
Consider (5.16) for $j < n + 1$: we have that such condition holds if $\exists\ c > 0$ and $K \in \mathbb{R}^{1 \times n}$ such that

$$\sum_{j \neq i} |(A + BK)_{ij}| + (A + BK)_{ii} + c + |L_i| \quad \leq 0 \tag{5.18}$$

If we consider again representation (5.6) the problem becomes equivalent to finding a matrix $G \in \mathbb{R}^{T \times n}$ satisfying $X_0 G = I_n$ such that for all $i = 1, \dots, n$

$$\sum_{j \neq i} |(X_L G)_{ij}| + (X_L G)_{ii} + |L_j| \quad < 0 \tag{5.19}$$

In fact, similarly to the previous theorem, if condition (5.19) is satisfied, we can choose

$$c^* = -\min_i \Big\{ \sum_{j \neq i} |X_L G|_{ij} + (X_L G)_{ii} + |L_i| \Big\} > 0$$

in order that (5.16) is non positive for all i.
Finally by definition of G we have $K = U_0 G$. □

The representation used in the previous theorem can be applied to Corollary 4.3 when it is considered $\Omega(\kappa) = \lceil X_L G \rceil + \kappa |L||C|$.
Then we can state the following corollary, which is a data driven version of Corollary 4.3

Corollary 5.1. *Consider X_L defined as in Theorem 5.1 and 5.2, if exist G $\in \mathbb{R}^{T\times n}$ such that to $X_0G = I_n$ and $\alpha(\Omega(\kappa)) < -c$ for some $c > 0$, then for every $p \in [1, \infty], \tau > 0$ there exist a diagonal matrix $R = R(p,\tau) \succ 0$ such that*

$$\mu_{p,R}\begin{pmatrix} X_LG + cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} \leq 0 \tag{5.20}$$

In the case of $p = \infty$, it holds moreover that $\kappa\|CR^{-1}\|_\infty < 1$.

Remark. *From the representation* (5.6)*, the existence of a matrix G such that $\alpha(\Omega(\kappa)) < -c$ is equivalent to the existence of a diagonal matrix $R = R(p,\tau) \succ 0$ and $c > 0, \tau > 0$ such that*

$$\mu_{p,R}\begin{pmatrix} A + BK + cI_n & L \\ \tau C & -\tau\varkappa^{-1} \end{pmatrix} \leq 0 \tag{5.21}$$

which ensures that the origin is exponentially stable for the closed loop system.

5.2 Numerical Examples

Example (Application of Theorems 5.1 and 5.2)**.**
Consider a dynamical systems described as in (5.1)*, with*

$$A = \begin{pmatrix} -2.1 & 1.2 \\ -1.9 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 0.95 & 0.91 \end{pmatrix}, \quad L = \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}$$

and $\omega = 0.3y + 0.1\sin(y)^2$; we note that the eigenvalues of A are -0.9 and -0.2, making A Hurwitz, while as κ we can simply choose 1*.*
It is worth noting that even if the matrix A is Hurwitz the system is unstable, in fact, taking as $x_0 = [4, 5]$ the state of the system, and the output, diverge, as shown in figure (5.2)*.*

In order to obtain a data driven representation of the system, we collect the following input/state pairs of length $T = 6$, where we took as input $u = \sin(t) + \sin(3t/4)$ and used a sample time of $S_t = 0.1$

$$U_0 = \begin{bmatrix} 0 & 0.1748 & 03481 & 0.5186 & 0.6849 & 0.8457 \end{bmatrix}$$
$$X_0 = \begin{bmatrix} 0 & 0.0004 & 0.0028 & 0.0092 & 0.0214 & 0.0408 \\ 0 & 0.0091 & 0.0375 & 0.0867 & 0.1581 & 0.2527 \end{bmatrix}$$

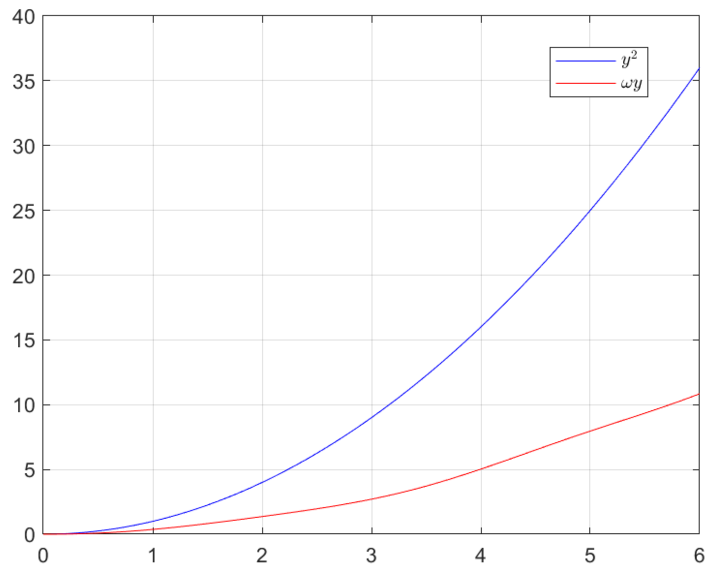


Figure 5.1: Comparison between κy^2 and ωy

We can observe that

$$W_0 = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$$

is full row rank.
We can solve the problem using both Theorem 5.1 and Theorem 5.2, and obtain the following matrices

$$G_1 = 10^3 \times \begin{bmatrix} 0 & 0 \\ 1.8638 & -0.3538 \\ -1.7950 & 0.2704 \\ 0.5804 & -0.0683 \end{bmatrix} \qquad G_\infty = 10^3 \times \begin{bmatrix} 0 & 0 \\ 1.8638 & -0.3293 \\ -1.7950 & 0.2579 \\ 0.5804 & -0.0654 \end{bmatrix}$$

which lead respectively to the controllers $K_1 = [1.9, -3.1]$ *and* $K_\infty = [1.9, -1.7]$.
We can see by the phase diagrams below that both method lead to an asymptotically stable closed loop system.

Example.
To stress how the norm used in the stability analysis could lead to different stability outcomes, we show some examples in which the existence of a solution to the stabilization problem using the Non Polynomial S-Lemma depends on the chosen norm.
Disregarding the data driven representation, we firstly propose a problem for which a stabilizing feedback matrix which makes the origin exponential stable exists for the ℓ_∞ *norm, but not for the* ℓ_1 *norm, i.e. we can find* $K \in \mathbb{R}^{1\times n}$ *such that* $\mu_\infty(P(\tau)) \le 0$ *but not* $\mu_1(P(\tau)) \le 0$, *where*

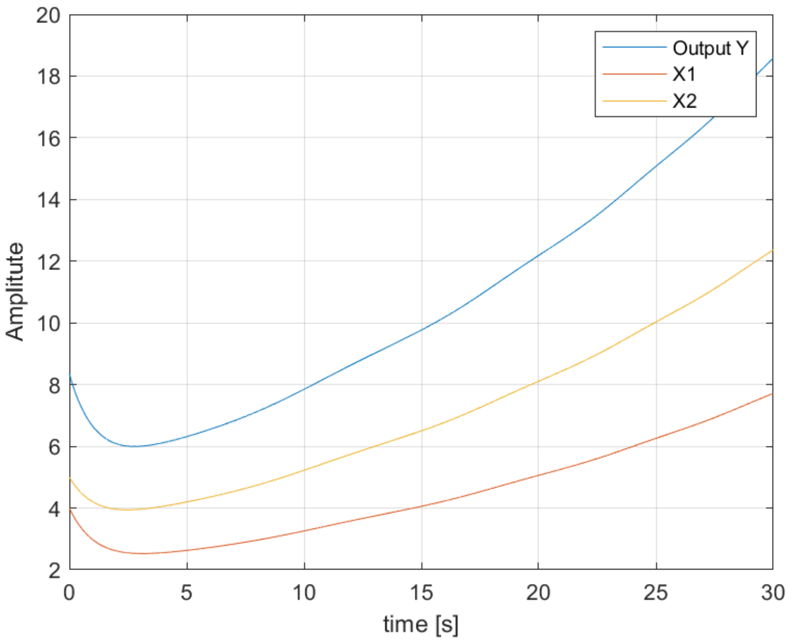


Figure 5.2: Output of the free response obtained using $x_0 = [4, 5]$

we remind that the matrix $P(\tau)$ is defined as

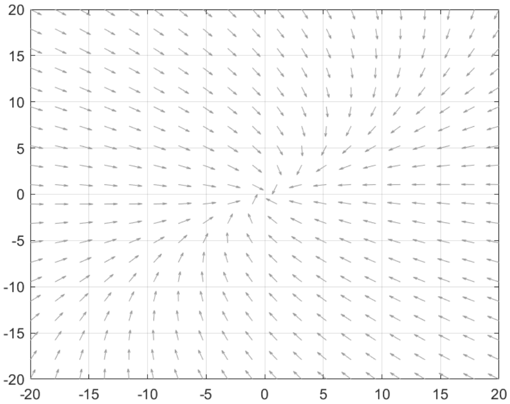
$$P(\tau) = \begin{pmatrix} A + BK + cI & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix}$$

for some $c > 0$. Consider in fact a Lur'e system of the type (5.1) *with*

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0.5 \end{pmatrix}, \qquad L = \begin{pmatrix} 0.23 \\ 0.58 \\ 0.4 \end{pmatrix}$$

and $\kappa = 0.9$. By applying as input $u = Kx$ we obtain

$$\begin{aligned} A + BK &= \begin{pmatrix} -2 & 0 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ k_1 & k_2 & k_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 1 \\ k_1 & -3 + k_2 & 1 + k_3 \\ 1 + k_1 & 1 + k_2 & -1 + k_3 \end{pmatrix} \end{aligned} \tag{5.22}$$



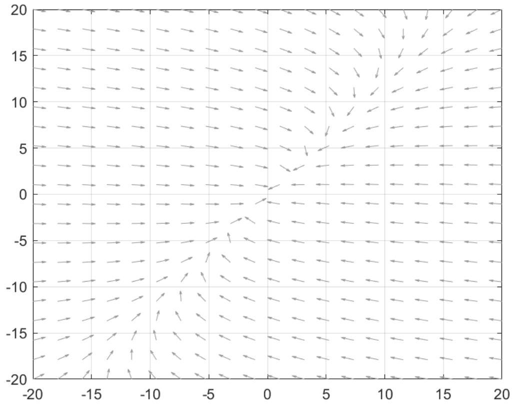


Figure 5.3: Phase diagrams for closed loop systems using ℓ_1 and ℓ_∞ in the analysis

We know by the definition of ℓ_1 log-norm that $\mu_1(P(\tau)) \le 0$ if

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} |(A + BK)_{ij}| + (A + BK)_{jj} + \tau^* |C_j| < 0 \tag{5.23}$$

for all $j = 1, \dots, n$.

By simply inspecting the condition (5.23) *obtained with $A + BK$ as above, we can see that there is no matrix K such that $\mu_1\big(P(\tau)\big) \le 0$. Since we need to have*

$$-2 + |k_1 + 1| + |k_1| + \tau^* < 0$$

where $\tau^* = \kappa \sum_{i=1}^{3} |L_i| = 1.089$, and k_1 is the first element of the matrix K. We can see that the sum is always positive for all values of k_1.

Let us consider now the analysis with norm ℓ_∞. Again, by applying the definition of the ℓ_∞ log norm to the matrix $P(\tau)$, we have $\mu_\infty(P(\tau)) \le 0$ for $\tau = 0$ if

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |(A + BK)_{ij}| + (A + BK)_{ii} + |L_j| \le 0$$

for all $i = 1 \dots n$, which translates into

$$\begin{cases} -2 + |1| + 0.23 < 0 \\ |k_1| - 3 + k_2 + |1 + k_3| + 0.8 < 0 \\ |1 + k_1| + |1 + k_2| - 1 + k_3 + 0.4 < 0 \end{cases}$$

Where we have not considered the summation $\tau(\sum_{i=1}^{n} |C_i| - \kappa^{-1})$, becouse we take $\tau = 0$.

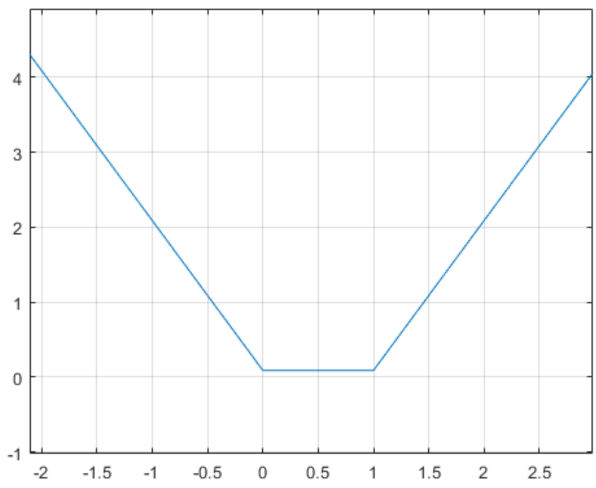


Figure 5.4: Summation (5.23) in function of k_1

Choosing $K = [-1, -1, -1]$ *we have all the inequalities satisfied, and so we can guarantee the exponential stability of the origin.*
We propose now a case in which the design of a stabilizing controller lead exponential stability when considering the analysis with the ℓ_1 *norm, but not when considering the* ℓ_2 *norm.*
Consider a Lur'e system of the type (5.1) *with*

$$A = \begin{pmatrix} -1.3 & 0 & 0 \\ 1 & -0.2 & 0 \\ 0 & 1 & -0.3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}, \quad L = \begin{pmatrix} 0.2 \\ 0.1 \\ 0 \end{pmatrix}$$

with $\kappa = 0.2$.
To prove that system above is not exponentially stabilizable by means of a state feedback matrix designed using the ℓ_2 *norm we have exploited corollary* (4.1).
In order to prove that there does not exist $K \in \mathbb{R}^{1x3}$ *such that the matrix inequality*

$$\frac{(A+BK)+(A+BK)^T}{2} + cI + \frac{\kappa}{4\tau}(L+\tau C^T)(L^t+\tau C) \preceq 0 \tag{5.24}$$

holds, we have used the numerical solver CVX [47]. However we can give a more intuitive explanation behind this fact. The last term in (5.24) *can be rewritten as*

$$\frac{\kappa}{4\tau}(L+\tau C^T)(L^T+\tau C) = \frac{\kappa}{4\tau}LL^T + \frac{\kappa}{4}LC + \frac{\kappa}{4}C^TL + \frac{\kappa}{4}C^TC$$

Despite the presence of the term $\frac{\kappa}{4}LC + \frac{\kappa}{4}C^T L$, which could be both positive and negative semidefinite, the terms $\frac{\kappa}{4\tau}LL^T$, $\frac{\kappa}{4}C^T C$ are both positive semidefinite and possess eigenvalues sufficiently large for making the overall matrix summation positive semidefinite. Since the action of the controller does not influence the totality of the dynamics (due to the zero element in the first and second entry of the matrix B), it is not possible to obtain exponential stability of the origin i.e. to make the summation negative semidefinite.
However, following the same steps as the previous examples, it is possible to show that, choosing the controller $K = [0, -1, 0]$ the origin is exponentially stable in the sense of ℓ_1 log norm approach.

Remark. *The Non Polynomial S-Lemma provides only a sufficient condition on the solution to the exponential stabilization problem, with the exception of positive systems, for which the stability analysis made using the ℓ_1 norm, for which the condition imposed by the Non Polynomial S-Lemma are also necessary. As consequence an exponential stabilizing controller, at least for general norms ℓ_p, could exist in the case it doesn't exist K such that $\mu_p(P(\tau)) \leq 0$. We point out, moreover, that in the last example the analysis has been made considering as quadratic Lyapunov function the identity I_n; the results found does not exclude the existence of a stabilizing matrix K when we consider as Lyapunov function $V(x) = x^T Rx$ for a generic symmetric positive definite matrix R.*
To find classes of problems for which the the condition imposed by the Non Polynomial S-Lemma are also necessary, alongside with classes of problems for which the Non Polynomial S-Lemma better suits the design of controllers for exponential stabilization with respect the classical methods in the ℓ_2 norm is still an open problem.

5.3 Computational Analysis

One important feature of the data driven approach presented in chapter 2 and chapter 3 is that the the stabilizing solution can be found by means of a linear matrix inequality, which can be solved by using some convex program solvers, like the aforementioned CVX. Moreover, we noticed that the dual problem (4.5) introduced for the statement of the Non Polynomial S-Lemma in Chapter 4 is a convex problem.
As a consequence, it is of our interest to understand if the design of a stabilizing feedback matrix can still be obtained by means of a convex program, and, in such a case, if the introduction of the data-based approach does not changes the structure of the solution design. We dedicate the next section to answer this questions and to study the convexity of the problem both in a model based and in a data driven approaches, providing an actually implementable solution to the problem.

5.3.1 Problem Convexity

We start our analysis by considering the original formulation of the optimization problem

$$\inf_{c,\tau} \ \mu_{p,R}\begin{pmatrix} A + cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} \quad \text{subject to} \quad c > 0, \quad \tau \geq 0 \tag{5.25}$$

where we assume the weigh matrix R to be equal to the identity. We show that the problem is convex in in $\tau \geq 0$ and in $c > 0$.
To do so we exploit the properties of subadditivity and positive homogeneity of the log-norm (see [48]), i.e.

Subadditivity

$$\forall \ P_1, P_2 \in \mathbb{R}^{n\times n}$$
$$\mu(P_1 + P_2) \leq \mu(P_1) + \mu(P_2)$$

Positive homogeneity

$$\forall \ P, \in \mathbb{R}^{n\times n} \text{ and } \forall \ a \in \mathbb{R} \quad \mu(aP) = |a|\mu(sgn(a)P)$$

Observe that for $\forall \ \lambda \in [0,1]$

$$\begin{pmatrix} A + [\lambda c_1 + (1-\lambda)c_2]I_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} =$$
$$\begin{pmatrix} \lambda A + \lambda c_1 I_n & \lambda L \\ \lambda\tau C & -\lambda\tau\kappa^{-1} \end{pmatrix} + \begin{pmatrix} (1-\lambda)A + (1-\lambda)c_2 I_n & (1-\lambda)L \\ (1-\lambda)\tau C & -(1-\lambda)\tau\kappa^{-1} \end{pmatrix}$$

By calling

$$P(\lambda c_1 + (1-\lambda)c_2) = \begin{pmatrix} A + [\lambda c_1 + (1-\lambda)c_2]I_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix}$$

and

$$\lambda P(c_1) = \begin{pmatrix} \lambda A + \lambda c_1 I_n & \lambda L \\ \lambda\tau C & -\lambda\tau\kappa^{-1} \end{pmatrix},$$
$$(1-\lambda)P(c_2) = \begin{pmatrix} (1-\lambda)A + \lambda(1-\lambda)c_2 I_n & (1-\lambda)L \\ (1-\lambda)\tau C & -(1-\lambda)\tau\kappa^{-1} \end{pmatrix}$$

If we consider the log-norm as a function of the parameter c (i.e. $\mu(c)$), since $\lambda > 0$, $(1-\lambda) > 0$

we have

$$\mu(\lambda P(c_1) = \mu|\lambda|(P(sgn(\lambda)c_1) = \lambda P(c_1)$$
$$\mu((1-\lambda)P(c_2) = |1-\lambda|\mu(P(sgn(1-\lambda)c_2) = (1-\lambda)P(c_2)$$

We have

$$\mu(\lambda c_1 + ((1-\lambda)c_2)) = \mu(P(\lambda c_1 + ((1-\lambda)c_2))) = \mu(\lambda P(c_1) + (1-\lambda)P(c_2))$$
$$\leq \mu P(\lambda c_1) + \mu((1-\lambda)P(c_2)) = \lambda\mu(P(c_1)) + (1-\lambda)\mu(P(c_2)) = \lambda\mu((c_1) + (1-\lambda)\mu((c_2))$$

The convexity in $\tau \geq 0$ can be proven similarly.
We observe moreover that $c > 0$ and $\tau \geq 0$ define convex sets.

We consider now the problem of stabilizing the system in (5.1) by introducing the state feedback matrix $K \in \mathbb{R}^{1\times n}$ as variable in the optimization problem (5.5), that can be translated into the optimization problem

$$\begin{aligned} &\inf_{c,\tau,K} \quad \mu_p\begin{pmatrix} A+BK+cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} \\ &\text{subject to} \quad c > 0,\ \tau \geq 0 \end{aligned} \tag{5.26}$$

We use the same argument as above to show that the problem is still convex in K.
We start again by considering the log-norm as function of K, $\mu(K) = \mu(P(K))$

$$P(K) = \begin{pmatrix} A+BK+cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix}$$

$$P(\lambda K_1 + (1-\lambda)K_2) = \begin{pmatrix} A+\lambda BK_1 + (1-\lambda)BK_2 cI_n & L \\ \tau C & -\tau\kappa^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda A + \lambda BK_1 + c\lambda I_n & \lambda L \\ \lambda\tau C & -\lambda\tau\kappa^{-1} \end{pmatrix} + \begin{pmatrix} (1-\lambda)A + (1-\lambda)BK_2 + c(1-\lambda)I_n & (1-\lambda)L \\ (1-\lambda)\tau C & -(1-\lambda)\tau\kappa^{-1} \end{pmatrix}$$

$$= \lambda P(K_1) + (1-\lambda)P(K_2)$$

Using again subadditivity and positive homogeneity of the log-norm we obtain

$$\mu(\lambda K_1 + (1-\lambda)K_2) = \mu(P(\lambda K_1 + (1-\lambda)K_2)) = \mu(\lambda P(K_1) + (1-\lambda)P(K_2))$$
$$\leq \lambda\mu(P(K_1)) + (1-\lambda)\mu(P(K_2)) = \lambda\mu(K_1) + (1-\lambda)\mu(K_2)$$

That proves the convexity in K.
By using the same argument as above, it is possible to show that the data-driven representation

of problem (5.26)

$$\inf_{c,\tau,G} \mu_p \begin{pmatrix} X_L G + cI_n & L \\ \tau C & -\tau \kappa^{-1} \end{pmatrix} \tag{5.27}$$

$$\text{subject to} \quad c > 0,\ \tau \geq 0,\ X_0 G = I_n$$

is still convex in the matrix G.
We notice moreover that the constraint introduced by the equality $X_0 G = I_n$ is a linear constraint, and so it is convex.

Remark. *It is possible to show that the optimization problem* (5.25) *is not convex in the weight matrix R, making the research for a weighted non polynomial Lyapunov function which gives exponential stability hard to solve.*

5.3.2 Linear programming formulation

In this section we show how the solution of the absolute stability problem presented in the last sections can be obtained by means of a linear programming approach.
Since the problems in ℓ_1 and ℓ_∞ norm are similar we will make our analysis considering the ℓ_1 norm, adding some considerations on the adaptations to the ℓ_∞ norm of the results proposed at the end of the section.
We start by observing that in Theorems (5.1) we have been able of getting rid of the presence of the constant $c > 0$ and the optimization variable $\tau \geq 0$, reducing the stability problem to finding a state-feedback matrix such that

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} |(A + BK)_{ij}| + (A + BK)_{jj} < -|\tau^* C_j| \tag{5.28}$$

The state-feedback matrix K which minimize each sum and assure exponential stability of the origin with maximum rate of convergence can be found then by solving the following optimization problem

$$\begin{aligned} \underset{K\in\mathbb{R}^{1xn}}{\text{minimize}} \quad & \sum_{j=1}^{n} \Big(\sum_{i\neq j} \Big(|(A + BK)_{ij}| + (A + BK)_{jj} \Big) \\ \text{subject to} \quad & \sum_{j=1}^{n} |(A + BK)_{ij}| + (A + BK)_{jj} \ < -|\tau^* C_j| \quad \forall\ j = 1, \dots, n \end{aligned} \tag{5.29}$$

Indeed, since introduction of the inequality (5.28) in the constraints of the problem guarantees that the conditions for exponential stability are met, the minimization of the overall summation is equivalent to minimize each single summation that satisfy the constraints. We observe that

$|C_j|$ are known, since we assume to know the vector $C \in \mathbb{R}^n$.
In order to transform problem (5.29) into a linear programming problem we associate to each element $|(A+BK)_{ij}|$ in the sum an auxiliary variable t_{ij} that for all $i, j = 1, \ldots, n$ $i \neq j$ must satisfy

$$
\begin{aligned}
t_{ij} &\geq (A+BK)_{ij} = (a_{ij} + b_i k_j) \\
t_{ij} &\geq -(A+BK)_{ij} = -(a_{ij} + b_i k_j)
\end{aligned}
$$

where a_{ij}, b_j, k_j are the entries of the matrices A, B and K.
After the introduction of the auxiliary variables, we obtain the following optimization problem

$$
\begin{aligned}
\underset{\substack{k_j,\, t_{ij} \\ i,j=1\ldots n\ i\neq j}}{\text{minimize}} \quad & \sum_{j=1}^{n} \Big(\sum_{\substack{i=1 \\ i\neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j) \Big) \\
\text{subject to} \quad & t_{ij} \geq (a_{ij} + b_i k_j) \\
& t_{ij} \geq -(a_{ij} + b_i k_j) \\
& \sum_{\substack{i=1 \\ i\neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j) \ < -|\tau^* C_j| \quad \forall\ j = 1, \ldots, n
\end{aligned}
\tag{5.30}
$$

The constraints guarantee that $t_{ij} \geq max((A+BK)_{ij}, -(A+BK)_{ij}) = |(A+BK)_{ij}|$; at the optimal solution it must hold the condition must be satisfied with an equality, for otherwise t_{ij} could further decrease.
We observe however that the optimization problem presented in (5.30) differs from a standard linear program due to the presence of the strict inequality in the constraints.
In order to avoid the presence of the strict inequalities we can introduce the variable $c_j = |\tau^* C_j| + \epsilon$, with $\epsilon > 0$ arbitrary small, so that the constraint becomes $\sum_{\substack{i=1 \\ i\neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j) \ \leq -c_j$.
Rewriting the problem constraints as

$$
\begin{aligned}
& -t_{ij} + b_i k_j \leq -a_{ij} \\
& -t_{ij} - b_i k_j \leq a_{ij} \\
& \sum_{\substack{i=1 \\ i\neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j) \ \leq -c_j \quad \forall\ j = 1, \ldots, n
\end{aligned}
$$

we obtain the following linear program formulation of problem (5.29)

$$\begin{aligned}
\underset{k_j,\, t_{ij}}{\text{minimize}} \quad & \sum_{j=1}^{n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j)\Big) \\
\text{subject to} \quad & -t_{ij} + b_i k_j \leq -a_{ij} \\
& -t_{ij} - b_i k_j \leq a_{ij} \\
& \sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j) \;\leq -c_j \quad \forall\ j = 1, \ldots, n
\end{aligned} \tag{5.31}$$

To give a better intuition behind the transformation of problem (5.29) into the linear program (5.30) we show the procedure for a general system of dimension two.

Example. *Given the matrices*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \qquad K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

we obtain the optimization problem

$$\begin{aligned}
\underset{K\in\mathbb{R}^{1\times n}}{\textit{minimize}} \quad & \sum_{j=1}^{2}\Big(\sum_{\substack{i=1\\ i\neq j}}^{2} \Big(|(a_{ij} + b_i k_j)_{ij}| + ((a_{jj} + b_j k_j))_{jj} \Big) \\
\textit{subject to} \quad & \sum_{\substack{i=1\\ i\neq j}}^{2} \Big(|(a_{ij} + b_i k_j)| + ((a_{jj} + b_j k_j)) \;< -|\tau^* C_j| \quad \forall\ j = 1 \ldots n
\end{aligned}$$

Defining the augmented decision variable $x = (k_1, k_2, t_{12}, t_{21})$*, and the constants* $c_j = |\tau^* C_j| + \epsilon,\ j = 1, 2$ *the optimization problem becomes*

$$\begin{aligned}
\underset{\substack{k_j,\, t_{ij}\\ i,j=1\ldots n\ i\neq j}}{\textit{minimize}} \quad & t_{12} + a_{22} + b_2 k_2 + t_{21} + a_{11} + b_1 k_1 \\
\textit{subject to} \quad & t_{12} \geq a_{12} + b_2 k_1, \quad t_{12} \geq -(a_{12} + b_2 k_1) \\
& t_{21} \geq a_{21} + b_1 k_2, \quad t_{21} \geq -(a_{21} + b_1 k_2) \\
& (a_{11} + b_1 k_1) + t_{12} \leq -c_1, \quad t_{21} + (a_{22} + b_2 k_2) \leq -c_2
\end{aligned} \tag{5.32}$$

We rewrite the constraints as

$$\begin{aligned} -t_{12} + b_2 k_1 &\leq -a_{12}, & -t_{12} - b_2 k_1 &\leq a_{12} \\ -t_{21} + b_1 k_2 &\leq -a_{21}, & -t_{21} - b_1 k_2 &\leq a_{21} \\ b_1 k_1 + t_{12} &\leq -c_1 - a_{11}, & t_{21} + b_2 k_2 &\leq -c_2 - a_{22} \end{aligned}$$

Defining

$$F = \begin{pmatrix} b_2 & 0 & -1 & 0 \\ -b_2 & 0 & -1 & 0 \\ 0 & b_1 & 0 & -1 \\ 0 & -b_1 & 0 & -1 \\ b_1 & 0 & 1 & 0 \\ 0 & b_2 & 0 & 1 \end{pmatrix} \qquad h = \begin{pmatrix} a_{12} \\ -a_{12} \\ a_{21} \\ -a_{21} \\ -c_1 - a_{11} \\ -c_2 - a_{22} \end{pmatrix} \qquad x = \begin{pmatrix} k_1 \\ k_2 \\ t_{12} \\ t_{21} \end{pmatrix}$$

$$m = \begin{pmatrix} b_1 & b_2 & 1 & 1 \end{pmatrix} \qquad n = a_{11} + a_{22}$$

$$c_1 = |\tau^* C_1| + \epsilon \quad c_2 = |\tau^* C_2| + \epsilon$$

we obtain the following linear program

$$\begin{aligned} &\underset{x}{\textit{minimize}} \quad m^T x + n \\ &\textit{subject to} \quad Fx \leq h \end{aligned} \tag{5.33}$$

where x is the augmented variable defined before.

We consider now the case in which the stability analysis is made using the ℓ_∞ norm. We start by observing that he problem objective is to find a state-feedback matrix $K \in \mathbb{R}^{1\times n}$ such that the condition

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |(A+BK)_{ij}| + (A+BK)_{ii} + |L_i| < 0$$

is satisfied for all $i = 1, \dots, n$.
Similarly to what done in the case of the ℓ_1, norm we can rewrite such condition as

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |(A+BK)_{ij}| + (A+BK)_{ii} < -|L_i|$$

Defining $l_i = |L_i| + \epsilon$, for $\epsilon > 0$ arbitrary small, we can substitute the strict inequalities with non-strict ones.
Using the same augmented objective function used for the analysis in the ℓ_1 norm we obtain the

optimization problem

$$
\begin{aligned}
\underset{\substack{k_j, t_{ij} \\ i,j=1\ldots n\ i \neq j}}{\text{minimize}} \quad & \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ i \neq j}}^{n} t_{ij} + a_{ii} + b_i k_j \Big) \\
\text{subject to} \quad & t_{ij} \geq (a_{ij} + b_i k_j) \\
& t_{ij} \geq -(a_{ij} + b_i k_j) \\
& \sum_{\substack{j=1 \\ j \neq i}}^{n} (t_{ij} + a_{ii} + b_i k_i) \ \leq -l_i \quad \forall\ i = 1, \ldots, n
\end{aligned}
\tag{5.34}
$$

Rewriting again the constraints we obtain

$$
\begin{aligned}
\underset{\substack{k_j, t_{ij} \\ i,j=1\ldots n\ i \neq j}}{\text{minimize}} \quad & \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} t_{ij} + a_{ii} + b_i k_j \Big) \\
\text{subject to} \quad & - t_{ij} + b_i k_j \leq -a_{ij} \\
& - t_{ij} - b_i k_j \leq a_{ij} \\
& \sum_{\substack{j=1 \\ j \neq i}}^{n} (t_{ij} + a_{ii} + b_i k_i) \ \leq -l_i \quad \forall\ i = 1, \ldots, n
\end{aligned}
\tag{5.35}
$$

that is a linear program.

Remark. *Due to the structure of the problem, the minimization of the rate of convergence in the ℓ_1 norm can be obtained in an alternative way. One can notice in fact that, when the analysis is made by using the ℓ_1 norm, then for each $j = 1, \ldots, n$ the summation*

$$
\sum_{\substack{j=1 \\ j \neq i}}^{n} |(A + BK)_{ij}| + (A + BK)_{jj} = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij} + b_i k_j| + (a_{jj} + b_j k_j)
$$

only depend on the j-th element of the matrix K, that is, the problem can be solved separately for each column, obtaining n disjointed unconstrained optimization problems

$$
\underset{k_j \in \mathbb{R}}{\textit{minimize}} \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij} + b_i k_j| + (a_{jj} + b_j k_j)
$$

for $j = 1, \ldots, n$ In fact, if exists j such that the minimum obtained is a positive value, we can conclude that does not exist K which makes the origin exponential stable.

Rewriting the problem using the auxiliary variables t_{ij} we obtain

$$\begin{aligned} \underset{\substack{k_j \\ t_{ij}\ i=1,\dots,n}}{\textit{minimize}} \quad & \sum_{\substack{i=1 \\ i \neq j}}^{n} t_{ij} + (a_{jj} + b_j k_j) \\ \textit{subject to} \quad & t_{ij} \geq (a_{ij} + b_i k_j) \\ & t_{ij} \geq -(a_{ij} + b_i k_j) \end{aligned}$$

for each $j = 1, \dots, n$

Finally, we point out that the problem can be reformulated such that the optimization consists in research of the minimum norm feedback matrix K which stabilize the system

$$\begin{aligned} \underset{K \in \mathbb{R}^{1xn}}{\text{minimize}} \quad & \|K\|_2^2 \\ \text{subject to} \quad & \sum_{\substack{i=1 \\ i \neq j}}^{n} |(A + BK)_{ij}| + (A + BK)_{jj} + \tau^* |C_j| < 0 \quad \forall j = 1, ..., n \end{aligned} \tag{5.36}$$

Since that both the objective function $\|K\|_2^2$ and the constraints are convex, the problem can be solved again by means of a convex program.
By following a similar approach to the previous optimization problem and by introducing auxiliary variables and constraints, it is possible to transform the problem to the following one

$$\begin{aligned} \underset{\substack{k_j, t_{ij} \\ i,j=1,\dots,n\ i \neq j}}{\text{minimize}} \quad & \|K\|_2^2 \\ \text{subject to} \quad & \sum_{\substack{i=1 \\ i \neq j}}^{n} t_{ij} + (A + BK)_{jj} \leq -c_j \\ & t_{ij} \geq (a_{ij} + b_i k_j), \quad t_{ij} \geq -(a_{ij} + b_i k_j) \\ & \forall\ j = 1, ..., n \end{aligned}$$

Similar considerations holds in the ℓ_∞ norm substituting the row and column indices and the constants c_j with l_i, giving the optimization problem

$$\begin{aligned} \underset{\substack{k_j, t_{ij} \\ i,j=1,\dots,n\ i \neq j}}{\text{minimize}} \quad & \|K\|_2^2 \\ \text{subject to} \quad & \sum_{\substack{j=1 \\ j \neq i}}^{n} t_{ij} + (A + BK)_{ii} \leq -l_j \\ & t_{ij} \geq (a_{ij} + b_i k_j), \quad t_{ij} \geq -(a_{ij} + b_i k_j) \\ & \forall\ i = 1, ..., n \end{aligned} \tag{5.37}$$

5.3.3 Data-driven representation

We now show how the formulation of the optimization problems proposed in the previous section changes using a data-driven approach.
We start by considering again the optimization problem (5.29)

$$\begin{aligned}
\underset{K \in \mathbb{R}^{1\times n}}{\text{minimize}} \quad & \sum_{j=1}^{n} \Big(\sum_{\substack{i=1 \\ i \neq j}}^{n} \Big(|(A+BK)_{ij}| + (A+BK)_{jj} \Big) \\
\text{subject to} \quad & \sum_{\substack{i=1 \\ i \neq j}}^{n} |(A+BK)_{ij}| + (A+BK)_{jj} \; < -|\tau^* C_j| \quad \forall\, j = 1, \dots, n
\end{aligned}$$

Using representation (5.6), where $X_L \in \mathbb{R}^{n\times T}$ and $G \in \mathbb{R}^{T\times n}$, the problem becomes

$$\begin{aligned}
\underset{G \in \mathbb{R}^{T\times n}}{\text{minimize}} \quad & \sum_{j=1}^{n} \Big(\sum_{\substack{i=1 \\ i \neq j}}^{n} \Big(|(X_L G)_{ij}| + (X_L G)_{jj} \Big) \\
\text{subject to} \quad & \sum_{\substack{i=1 \\ i \neq j}}^{n} |(X_L G)_{ij}| + (X_L G)_{jj} \; < -|\tau^* C_j| \quad \forall\, j = 1, \dots, n \\
& \sum_{s=1}^{T} X_0 G = I_n
\end{aligned} \tag{5.38}$$

Similarly to the solution obtained in the model based case, we can reformulate (5.38) to a linear program by introducing the auxiliary variables

$$\begin{aligned}
t_{ij} &\geq (X_L G)_{ij} = \sum_{h=1}^{T} (X_L)_{ih}\, G_{hj} \\
t_{ij} &\geq -(X_L G)_{ij} = -\sum_{h=1}^{T} (X_L)_{ih}\, G_{hj}
\end{aligned}$$

from which we obtain

$$
\begin{aligned}
\underset{\substack{G\in\mathbb{R}^{T\times n}\\ t_{ij}\ i,j=1\dots n\ i\neq j}}{\text{minimize}} \quad & \sum_{j=1}^{n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{jh}\,G_{hj}\Big) \\
\text{subject to} \quad & t_{ij} \geq \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \\
& t_{ij} \geq -\sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \\
& \sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{jh}\,G_{hj} \ < -|\tau^* C_j| \quad \forall\ j = 1,\dots,n \\
& \sum_{s=1}^{T}(X_0)_{is}G_{s_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\end{aligned}
\tag{5.39}
$$

Again, if we rewrite constraints of the problem as

$$
\begin{aligned}
& -t_{ij} + \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \leq 0 \\
& -t_{ij} - \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \leq -0 \\
& \sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{jh}\,G_{hj} \ < -c_j \quad \forall\ j = 1,\dots,n \\
& \sum_{s=1}^{T}(X_0)_{is}G_{s_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\end{aligned}
$$

with c_j is defined in the previous section, we obtain the linear program

$$\begin{aligned}
\underset{\substack{G \in \mathbb{R}^{T\times n} \\ t_{ij}\ i,j=1\dots n\ i\neq j}}{\text{minimize}} \quad & \sum_{j=1}^{n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{jh}\,G_{hj}\Big) \\
\text{subject to} \quad & -t_{ij} + \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \leq 0 \\
& -t_{ij} - \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \leq -0 \\
& \sum_{\substack{i=1\\ i\neq j}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{jh}\,G_{hj} \ < -c_j \quad \forall\ j = 1,\dots,n \\
& X_0 G = I_n
\end{aligned} \tag{5.40}$$

Using similar arguments, the linear program obtained when the stability analysis is made on ℓ_∞ is

$$\begin{aligned}
\underset{\substack{G \in \mathbb{R}^{T\times n} \\ t_{ij}\ i,j=1\dots n\ i\neq j}}{\text{minimize}} \quad & \sum_{i=1}^{n}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{ih}\,G_{hi}\Big) \\
\text{subject to} \quad & -t_{ij} + \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \leq 0 \\
& -t_{ij} - \sum_{h=1}^{T}(X_L)_{ih}\,G_{hj} \leq -0 \\
& \sum_{\substack{j=1\\ j\neq i}}^{n} t_{ij} + \sum_{h=1}^{T}(X_L)_{ih}\,G_{hi} \ < -l_j \quad \forall\ i = 1,\dots,n \\
& \sum_{s=1}^{T}(X_0)_{is}G_{s_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\end{aligned} \tag{5.41}$$

Example (continuation of the previous example)**.**

Consider again the case of a general two dimensional system presented in the last section. Adopting a data-based approach the optimization problem (5.32) *becomes*

$$\begin{aligned} \underset{G,\, t_{ij}}{\textit{minimize}} \quad & \sum_{j=1}^{2} \Big(\sum_{j \neq i} t_{ij} + \sum_{h=1}^{T} (X_L)_{jh}\, G_{hj} \Big) \\ \textit{subject to} \quad & t_{12} \geq \sum_{h=1}^{T} (X_L)_{1h}\, G_{h2} \quad t_{12} \geq -\sum_{h=1}^{T} (X_L)_{1h}\, G_{h2} \\ & t_{21} \geq \sum_{h=1}^{T} (X_L)_{2h}\, G_{h1} \quad t_{21} \geq -\sum_{h=1}^{T} (X_L)_{2h}\, G_{h1} \\ & \sum_{i \neq j} t_{ij} + \sum_{h=1}^{T} (X_L)_{jh}\, G_{hj} \ < -|\tau^+ C_j| \quad \forall\, j = 1, \ldots, n \\ & \sum_{s=1}^{T} (X_0)_{is} G_{s_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{aligned} \tag{5.42}$$

We can rewrite the constraints as

$$\begin{aligned} & -t_{12} + \sum_{h=1}^{T} (X_L)_{1h}\, G_{h2} \leq 0 \quad -t_{12} - \sum_{h=1}^{T} (X_L)_{1h}\, G_{h2} \leq 0 \\ & -t_{21} + \sum_{h=1}^{T} (X_L)_{2h}\, G_{h1} \leq 0 \quad -t_{21} - \sum_{h=1}^{T} (X_L)_{2h}\, G_{h1} \leq 0 \\ & t_{ij} + \sum_{h=1}^{T} (X_L)_{jh}\, G_{hj} < -|\tau^* C_j| \quad i, j = 1, 2 \quad i \neq j \end{aligned}$$

We assume that $T = 3$, (that is, the minimum sample length for which the matrix W_0 defined in (5.4) *can be full row rank).*

As done in the previous example, we define

$$F = \begin{pmatrix} 0 & 0 & 0 & X_{11}^L & X_{12}^L & X_{13}^L & -1 & 0 \\ 0 & 0 & 0 & -X_{11}^L & -X_{12}^L & -X_{13}^L & -1 & 0 \\ X_{21}^L & X_{22}^L & X_{23}^L & 0 & 0 & 0 & 0 & -1 \\ X_{11}^L & X_{12}^L & X_{13}^L & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & X_{21}^L & X_{22}^L & X_{23}^L & 0 & 1 \end{pmatrix} \qquad h = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -c_1 \\ -c_2 \end{pmatrix}$$

$$c_1 = |\tau^* C_1| + \epsilon \quad c_2 = |\tau^* C_2| + \epsilon$$

and obtain the linear program in the augmented objective variable x, given by the vectorization *of the matrix G and the auxiliary variables t_{ij}*

$$
\begin{aligned}
&\underset{x}{\textit{minimize}} && \sum_{j=1}^{2}\Big(\sum_{i\neq j} t_{ij} + \sum_{h=1}^{T}(X_L)_{jh}\, G_{hj}\Big) \\
&\textit{subject to} && Fx \leq 0 \\
& && X_0 G = I_n
\end{aligned}
\tag{5.43}
$$

We conclude that the introduction of the data driven approach does not modify the structure of the problem.

Nonetheless, we spend some words on how the introduction of a data-based model impact the number of computations required.

Consider again the previous examples: to solve the problem we introduced an augmented decision variable composed by the original decision variables k_j, the component of the controller to be synthesized, and the auxiliary variables t_{ij}, which are used to avoid the presence of absolute values and obtain a linear optimization problem. Despite being presented for a two dimensional system such procedure can be generalized for a system of arbitrary dimension.

In the model based approach the dimension of the augmented decision variable is n^2, since we have n decision variables for the controller entries and one for each non diagonal term of the matrix $A+BK$, i.e. n^2-n (since each term $(A+BK)_{ij},\ i\neq j$ appears in absolute values, and requires an auxiliary variable). On the other hand, in the data-based approach, the augmented decision variable is composed by the entries of the matrix G and the auxiliary variables. While we can see that the number of auxiliary variables t_{ij} remains equal to n^2-n, since we consider again the absolute value of each non diagonal term of $X_L G$, which has dimension $n\times n$, the vectorization of the matrix G require Tn variables (we recall $G\in\mathbb{R}^{T\times n}$), where $T \geq n+)$ is the length of the data samples. That is, the augmented optimization variables pass from having n^2 elements to $Tn + (n^2-n) \geq 2n^2$.

We can make a similar considerations on the matrix F, which is used to the describe the linear constraints in the optimization problem, when considered the augmented optimization variable. In the model-based approach the matrix F has dimension $(2n^2-n)\times(n^2+n)$, where the number of rows $2n^2-n$ corresponds to the $2(n^2-n)$ constraints used to model the presence of an absolute value (each non diagonal term of the matrix $A+BK$ has been considered twice in the constraints of the auxiliary variables), plus n constraints to guarantee that each summation on the columns (or the rows) of the matrix $P(\tau)$ to be negative.

In the data-based approach the dimension of the matrix F becomes $(2n^2-n)\times(n^2+Tn-n)$, due to the fact that number of constraints (which defines the number of rows of F) remain the same, but the augmented increases in dimension, having at least $2n^2$ components.

Moreover in the data based approach we must add the additional constraint $X_0 G = I_n$ to guarantee the effectiveness of the representation (5.6), which consists in a linear system of n^2 equations.

We conclude that, applying a data-driven approach to the solution of the exponential stability problem using the ℓ_1 or ℓ_∞ norm does not change the structure of the problem, increasing however the number of computations required.

As the last observation we show a data driven version of problems (5.36) and (5.37), which consists in finding the minimal norm $K \in \mathbb{R}^{1\times n}$ for which exponential stability is assured.

These are, in the ℓ_1 case

$$
\begin{aligned}
\underset{G\in\mathbb{R}^{Txn}}{\text{minimize}} \quad & ||U_0G||_2^2 \\
\text{subject to} \quad & \sum_{i\neq j} |(X_LG)_{ij}| + (X_LG)_{jj} + \leq -c_j \quad \forall j = 1, ..., n \\
& X_0G = I_n
\end{aligned}
\tag{5.44}
$$

and in the ℓ_∞ case

$$
\begin{aligned}
\underset{G\in\mathbb{R}^{Txn}}{\text{minimize}} \quad & ||U_0G||_2^2 \\
\text{subject to} \quad & \sum_{j\neq i} |(X_LG)_{ij}| + (X_LG)_{ii} \leq -l_i \quad \forall i = 1, ..., n \\
& X_0G = I_n
\end{aligned}
\tag{5.45}
$$

Since the constraints are convex, by introducing some auxiliary variable its possible to see that these are both convex problems.

Conclusions

In this thesis we studied how to solve the absolute stability problem for Lur'e system recurring to a data driven approach, first considering the classical Lyapunov stability theory, then proposing a novel solution using the notion contractivity for general ℓ_p spaces. We have studied the effect of the introduction of a data-based approach on its efficiency, observing that a data driven formulation of the problem does not change its convexity, increasing however the computational buden required to solve the problem.
In both analysis we have proposed a solution that can be actually be implemented by means of LMI or convex programs. In particular, we have proposed a solution which aims to maximize the rate of convergence using a linear program.
Nonetheless, the research in this field is far from being exhausted. The application of Lyapunov stability theory to general ℓ_p spaces has the potentiality to establish a new paradigm of stability analysis and control design. However, in order to materialize the possibility of application of the non polynomial 2-forms in control system theory it is necessary to comprehend if there exist classes of problem which the classical stability methods in the Euclidian space provide poorer performances than the ones obtained by studying the problem with general ℓ_p spaces.
On the other hand, with specific regards to the results we proposed, our data driven extension lacks a formal design procedure which guarantee robustness in the presence of disturbances. Such arguments will be the subject of future research.

A
Notions in Convex Optimization

We provide here a review of some definitions and procedures used in formulating or solving convex optimization problems. We consider as references [49], [50].

A.1 Optimization problems

An optimization problem is a problem of the form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \le 0,\ i = 1, \dots, m \\ & h_i(x) = 0,\ i = 1, \dots, p \end{aligned}$$

which, in words, consists in minimize the value of $f_0(x)$ among all x that satisfy the the conditions $f_i(x) \le 0,\ i = 1, \dots, m$ and $h_i(x) = 0,\ i = 1, \dots, p$. The function $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called the *objective* or *cost* function, while the inequalities $f_i(x) \le 0$ and the equalities $h_i(x) = 0$ are called he *constraints* of the problem, while the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$ are the corresponding inequality and equality functions. In the case there are no constraints the problem is said to be *unconstrained*. We will refer to x as the *optimization variable*. The points where the optimization problem is defined is called the *domain* of the problem, and it is defined as

$$\mathcal{D} := \bigcap_{i=1}^{m} \mathbf{dom} f_i \cap \bigcap_{i=1}^{p} \mathbf{dom} h_i \tag{A.1}$$

where $\mathbf{dom} f_i$ and $\mathbf{dom} h_i$ are the points where the functions f_i $i = 1 \dots m$ and h_i $i = 1 \dots p$ are defined.
A point $x \in \mathbb{R}^n$ is said to be *feasible* if satisfies the equality and inequality constraints of the problem; similarly, an optimization problem is said to be feasible if there exists at least one feasible point.
We define the *optimal value* as

$$p^* = inf\{f_0(x) | f_i(x) \leq 0 \; i = 1, \dots, m; \; h_i(x) = 0 \; i = 1, \dots, p\}$$

which is allowed to take as values $\pm\infty$. In the case in which the problem is infeasible we will assume $p^* = +\infty$.
A class of particularly relevant optimization problems consists in the *convex optimization problems*, that is, problems in which both the cost function and the inequality constraint functions are convex, while the equality constraint functions are affine.
Before presenting the standard form of a convex optimization problem, we review the definitions given above.

Definition (Convex Set)**.** *A set $C \subseteq \mathbb{R}^n$ is said to be Convex if the line segment between any two points of C lies in C, i,e. :*

$$\forall \; c_1, \; c_2 \in C, \forall \; \theta \in [0, 1] \quad \theta c_1 + (1 - \theta) c_2 \; \in C$$

Definition (Convex Function)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Convex if its domain is a convex set and if for all $x_1, x_2 \in dom \; f$ the line segment between $(x_1, f(x_1))$ and $(x_2, f(x_2))$, lies above the graph of f i.e. :*

$$\forall \; x_1, \; x_2 \in dom \; f, \; \forall \; \theta \in [0, 1] \quad f(\theta x_1 + (1 - \theta) x_2) \leq \theta f(x_1) + (1 - \theta) f(x_2)$$

We provide some noteworthy examples of convex sets, which will be useful in defining a convex optimization problem in standard form.

Example.
The unitary open ball in $\mathbb{R}^n$: $\mathcal{B} = \{x : ||x|| < 1\}$ is a convex set.
In fact, if x and y belongs to $\mathcal{B}$ we have, for all $\theta \in [0, 1]$

$$||\theta x + (1 - \theta) y|| \leq \theta ||x|| + (1 - \theta) ||y|| < \theta + 1 - \theta = 1 \tag{A.2}$$

Example.
Sublevels sets of a convex function $C_\alpha = \{x \in \ dom\ f : f(x) \leq \alpha\}$ *are convex sets.*
The proof is immediate by observing that for all x, y *in* $dom\ f$ *and for all* $\theta \in [0, 1]$ *the sum* $\theta x + (1 - \theta)y \ \in \ dom\ f$, *and*

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) \leq \alpha$$

Example.
Hyperplanes $\mathcal{H} = \{x : a^T x = b\}$, *with* $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$ *are (obviously) convex sets.*

A convex optimization problem is a problem of the form

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0 \ i = 1, \dots, m \\ & a_i^T x = b_i, \ i = 1, \dots, p \end{aligned} \tag{A.3}$$

where $f_0, f_1, \dots, f_m$ are convex functions.
We can notice that the set of all feasible point is a convex set, since the domain of the problem $\cap_{i=0}^{m} dom\ f_i$ is convex and the feasible set is the intersection of m level set of convex functions and p hyperplanes, which are convex sets.
Convex optimization problems are of particular interest for several reasons: first of all if the solution of the problem exists, any local minimum is also a global minimum; moreover if the objective function is strictly convex, the optimal solution, whenever it exists, is unique.
Another remarkable reason behind the interest in convex optimization problem is the fact that they can be solved by efficient algorithms.

Remark. *Concave problems are of the type*

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\textit{maximize}} \quad & f_0(x) \\ \textit{subject to} \quad & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & a_i^T x = b_i \quad i = 1, \dots, p \end{aligned} \tag{A.4}$$

where the objective function $f_0(x)$ *is concave and the inequality functions* $f_i(x)$ $i = 1, \dots, m$ *are convex: as a matter of fact it is immediate so observe that problem* (A.4) *is equivalent to problem* (A.3)*, if we consider to minimize the convex objective function* $-f_0(x)$*. All the properties presented for convex problems also hold for concave problems.*

A.1.1 Linear optimization problems

In the case of which the objective and constraints functions are all affine the problem is referred as a *linear program*.
A linear program is expressed as

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x + d \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned} \tag{A.5}$$

where $G \in \mathbb{R}^{m \times n}$, $h \in \mathbb{R}^m$ and $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^m$. In this framework the inequality $Gx \leq h$ must be interpreted elementwise, i.e. $(Gx)_i \leq h_i \ \forall \ i = 1, \dots m$.

Remark. *A linear program is said to be in standard form if is written as*

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\textit{minimize}} \quad & c^T x + d \\ \textit{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{A.6}$$

A.1.2 Quadratic optimization problems

An optimization problem in which the objective function is a quadratic function and the constraints are affine is called *quadratic program*

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & 1/2 x^T P x + q^T x + c \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned} \tag{A.7}$$

where P is a symmetric $n \times n$ matrix, $G, A \in \mathbb{R}^{n \times n}$ and $q, h, b \in \mathbb{R}^n$.

A.2 Duality

Duality is a principle which allows to obtain a lower bond on the optimization problem considered by exploiting the *Lagrangian function* and the *Lagrangian dual function*.
We start by considering again a general optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0,\ i = 1, \dots, m \\ & h_i(x) = 0,\ i = 1, \dots, p \end{aligned} \tag{A.8}$$

where we assume the domain $\mathcal{D} = \bigcap_{i=0}^{m} f_i \ \cap \ \bigcap_{i=0}^{p} h_i$ to be non-empty; we do not make the assumption that problem (A.8) is convex. We now give the notions of *Lagrangian function* and *Lagrangian dual function*.

Definition (Lagrangian Function)**.**
The Lagrangian function associated with problem (A.8) *is defined as*

$$L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \longrightarrow \mathbb{R}$$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

where $dom\ L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$*. The vectors* λ *and* ν *are called Lagrange multiplier associated with problem* (A.8)*.*

Definition (Lagrangian Dual Function)**.**
The Lagrangian dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ *is defined as*

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

As already mentioned, the Lagrangian dual function yield a lower bound on the optimal value $p^* = inf\{f_0(x) | f_i(x) \leq 0\ i = 1, \dots, m;\ h_i(x) = 0\ i = 1, \dots, p\}$: indeed, chosen an arbitrary $\lambda \geq 0$, for any feasible $\tilde{x}$ we have

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{m} \nu_i h_i(\tilde{x}) \leq 0$$

Since on a feasible point $f_i(\tilde{x}) \leq 0$ and $h_i(\tilde{x}) = 0$, then

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{m} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$

from which we obtain

$$g(\lambda,\nu) = \inf_{x\in\mathcal{D}} L(x,\lambda,\nu) \leq \inf_{x\in\mathcal{D}} L(x,\lambda,\nu) \leq f_0(\tilde{x}) \leq p^*$$

that is, $g(\lambda,\nu)$ is always smaller than the oprimal value p^*.
The *Lagrangian Dual Problem* consists in finding the optimal multipliers λ^* and ν^* which minimize the gap between the optimal value p^* solution of the primal optimization problem (A.8), and the lower bound obtained by the Lagrangian dual function. Equivalently the Lagrangian Dual problem can be written as

$$\begin{aligned} &\text{maximize} \quad g(\lambda,\nu) \\ &\text{subject to} \quad \lambda \geq 0 \end{aligned} \tag{A.9}$$

It can be seen that the Lagrangian function is always concave, and since the constraint of problem (A.9) are convex, the Lagrangian Dual problem is a convex optimization problem.

Remark (On the Lagrangian function in the proof of Non-Polynomial S-Lemma)**.**
In the proof of the Non-Polynomial S-Lemma we have considered as Lagrangian function

$$L(x,\tau) = -[\![P_0x,x]\!] + \sum_{j=1}^{s}\tau_j[\![P_jx,x]\!] - \sum_{j=1}^{s}\tau_j\rho_j \tag{A.10}$$

The reason behind negative sign in the Lagrangian function relies on the fact that the primal problem considered in (4.4) *is a maximization problem instead of minimization one*

$$\begin{aligned} &\sup_{x\in\mathbb{R}^n} \quad p_0(x) \\ &\textit{subject to} \quad \|x\| = 1, \quad p_1(x) \leq \rho_1, ..., p_s(x) \leq \rho_s \end{aligned} \tag{A.11}$$

Since the Lagrangian function has been defined in relation to minimization problems, in order to be coherent with the definition we must consider the equivalent problem of minimizing $-p_0(x)$ which allows us to consider the usual definition of Lagrangian function and leads to the Lagrangian function considered in (4.7).

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