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Derived equivalences between diagram categories of finite posets

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University of Padua

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To Enrico,
my home in every place

Abstract

This thesis studies universal derived equivalences between categories of diagrams over finite posets. Given a finite poset X and an abelian category $\mathcal{A}$, one considers the category $\mathcal{A}^X$ of diagrams over X with values in $\mathcal{A}$, and asks when two posets X and Y give rise to derived equivalent diagram categories for every abelian category $\mathcal{A}$. After recalling the necessary background on triangulated categories, complexes, and derived categories, the thesis presents a detailed proof of a criterion, due to Ladkani, under which suitable combinatorial data on finite posets produce pairs of universally derived equivalent posets. The proof is based on the construction of explicit combinatorial data, called formulas, which give rise to functors on complexes of diagrams and induce triangulated functors on the corresponding derived categories. The final chapter develops original results aimed at producing posets that are universally derived equivalent to a given finite poset. More precisely, it studies when the order structure of a finite poset can be described in terms of a partition into two subsets and an order-preserving map between them, proves that this map is uniquely determined whenever such a description exists, and formulates algorithmic procedures for detecting these partitions and constructing the corresponding universally derived equivalent posets, together with some results on when different constructions may lead to isomorphic outputs.

Acknowledgments

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Contents

Introduction

Motivation

A classical problem in algebra is to determine when two rings are isomorphic. A more flexible question, central to representation theory, is whether two rings R and S have equivalent categories of modules, that is, when Mod-$R \simeq$ Mod-S. If R and S are isomorphic, then their module categories are certainly equivalent, but the converse does not hold in general. A classical example is that every ring R is Morita equivalent to the matrix ring $M_n(R)$ for every $n \geq 1$, so that Mod-$R \simeq$ Mod-$M_n(R)$, even though R and $M_n(R)$ are typically not isomorphic. The precise characterization of when two rings have equivalent module categories is provided by Morita theory, developed in the 1950s.

One can relax the question even further and ask when the derived categories $D(\text{Mod-}R)$ and $D(\text{Mod-}S)$ are equivalent as triangulated categories. Derived categories were introduced by Grothendieck and Verdier in the 1960s as a framework for homological algebra, and have since become a fundamental tool across algebra, algebraic geometry, and representation theory. Two rings whose module categories are equivalent are necessarily derived equivalent, but the converse fails: derived equivalence is a strictly weaker notion, yet it still preserves a rich amount of homological information.

A landmark contribution to the study of derived equivalences is due to Happel [8], who showed that a tilting module over a finite-dimensional algebra induces an equivalence between the corresponding bounded derived categories. This was subsequently generalized by Rickard, who proved that two rings are derived equivalent if and only if one can be realized as the endomorphism ring of a suitable complex, called a tilting complex, over the other. These results, which form the core of tilting theory, provide a powerful framework for establishing and detecting derived equivalences between algebras that are not equivalent at the level of modules.

A concrete and instructive example arises from quiver representations. Consider the two quivers of type A_3

$$Q_1\colon 1 \to 2 \to 3 \quad \text{and} \quad Q_2\colon 1 \to 2 \leftarrow 3.$$

The path algebras KQ_1 and KQ_2 over a field K are not isomorphic, and their module categories Mod-KQ_1 and Mod-KQ_2 are not equivalent. Indeed, this can already be seen at the level of quiver representations: Q_1 has one source, whereas Q_2 has two, so the corresponding categories have a different number of simple injective objects and therefore cannot be equivalent. Nevertheless, their derived categories are equivalent; indeed, this is a special case of the fact that any two orientations of a tree are universally derived equivalent, see [3, Corollary 1.8]. This example illustrates that derived equivalence can detect structural similarities that are invisible at the level of module categories.

For finite posets, these questions admit a particularly concrete reformulation. Given a finite poset P and a category $\mathcal{A}$, one can consider the category $\mathcal{A}^P$ of diagrams over P with values in $\mathcal{A}$: the objects are families of objects $(A_x)_{x\in P}$ in $\mathcal{A}$ together with morphisms $A_x \to A_{x'}$ for each relation $x \leq x'$, compatible with the order. When $\mathcal{A} = \text{Mod-}K$ is the category of modules over a field K, i.e. the category of K-vector spaces, this category of diagrams can be identified with the category of representations of the poset over K, and, for finite posets, with the category of modules over the incidence algebra Λ_P of P. Therefore, for posets P and P', the question of whether $D(\mathcal{A}^P)$ and $D(\mathcal{A}^{P'})$ are equivalent reduces, in the classical case of vector spaces, to the question of whether the incidence algebras Λ_P and $\Lambda_{P'}$ are derived equivalent.

However, one can ask for something stronger. Rather than fixing a field K and working only with vector spaces or modules, we ask whether the equivalence $D(\mathcal{A}^P) \simeq D(\mathcal{A}^{P'})$ holds for every abelian category $\mathcal{A}$. When this is the case, the two posets are said to be universally derived equivalent. This is a considerably stronger property: it says that the equivalence is determined entirely by the structure of the posets and does not depend on any particular choice of coefficient category. In this sense, universal derived equivalences provide a natural meeting point between homological algebra and order theory. This is the point of view of this thesis.

The central reference for this thesis is the work of Ladkani [3], who gives sufficient combinatorial conditions under which two finite posets are universally derived equivalent. More precisely, starting from two finite posets X and Y together with a family of subsets $Y_x \subseteq Y$, indexed by the elements of X and satisfying suitable compatibility conditions, he constructs two new posets on the disjoint union $X \sqcup Y$. Both posets preserve the original orders within X and within Y, but they differ in the mixed relations: in one, the additional relations go from elements of X to elements of Y, while in the other they go in the opposite direction. The main result is that these two posets are universally derived equivalent. The proof is based on the explicit construction of functors at the level of complexes of diagrams, called formulas, which are then shown to induce triangulated functors between the corresponding derived categories. In Chapter 2, we present this construction in detail and make its mechanism as explicit as possible.

Chapter 3 develops original contributions motivated by the singleton case of this construction, in which each subset Y_x consists of a single point and the data reduce to an order-preserving map $f\colon A \to B$. The singleton case provides a particularly transparent setting in which the categorical construction can be translated into concrete combinatorial questions. In other words, rather than starting from two separate posets and combining them, we start from a single poset and ask whether it can be decomposed to produce new universally derived equivalent posets. The central observation is that, given a finite poset P, one can ask whether P itself arises from this construction, that is, whether P can be decomposed into two pieces A and B in such a way that all mixed relations in P go from A to B and are determined by an order-preserving map $f\colon A \to B$. If so, one can reverse the directions of the mixed relations, obtaining a new poset universally derived equivalent to P. We call such a decomposition an admissible cut. The main result of the chapter (Theorem 3.2.2) gives a complete characterization of admissible cuts: a partition $P = A \sqcup B$ is admissible if and only if A is an ideal, B is a filter, and for every $a \in A$ the set of elements of B above a has a minimum. Moreover, whenever these conditions are satisfied, the map f is uniquely determined by the poset and the partition. This leads to an algorithmic procedure for systematically enumerating all posets universally derived equivalent to a given one via a single application of the singleton construction. We also prove that automorphisms of P, as well as automorphisms of the pieces A and B, can identify different cuts or different maps that produce isomorphic outputs, thus refining the combinatorial count. The algorithms and results are applied to concrete examples throughout the chapter.

Structure of the thesis

Chapter 1 collects the background material needed in the rest of the thesis. We recall the definition and basic properties of triangulated categories, introduce cochain complexes and the homotopy category, and construct the derived category of an abelian category via localization. The main references for this chapter are Miličić [1] and Weibel [2].

Chapter 2 contains the central mathematical result of the thesis: a detailed proof of Ladkani's criterion for universal derived equivalence (Theorem 2.2.4). The content of this chapter follows the approach of [3], but the exposition has been substantially reworked. In particular, the statement of the main theorem is reformulated so as to make the role of the combinatorial hypotheses more transparent; several intermediate lemmas are introduced in order to clarify the logical structure of the proof; a number of verifications that are only sketched or omitted in [3] are written out in detail; and a running example is used throughout the chapter to illustrate the construction concretely. The argument is developed in two stages: first, the elementary case in which the posets are $1 \to 2$ is treated in detail, since it provides the local model for the general construction; then, the formulas and natural transformations are assembled in the general case to prove the main theorem.

Chapter 3 develops some original contributions by answering questions arising from the singleton case of the construction introduced in Chapter 2, where the data reduce to an order-preserving map $f\colon A \to B$. After a preliminary counting problem for order-preserving maps, the chapter turns to its main structural point, namely Theorem 3.2.2, which characterizes exactly when a partition $P = A \sqcup B$ is an admissible cut — that is, when P arises from the construction with the mixed relations going from A to B — and shows that the order-preserving map f, whenever it exists, is uniquely determined by the poset and the partition. This leads to a reconstruction procedure and to an algorithm for detecting all admissible cuts of a finite poset. The chapter also proves new isomorphism results showing how automorphisms of the initial poset, and of the pieces A and B, may identify different cuts or different maps giving rise to isomorphic outputs, thus refining the naive combinatorial count. Finally, the algorithm is applied to a concrete example, producing several new posets universally derived equivalent to a fixed starting one, together with some structural observations on the resulting family.

In this sense, the thesis moves from exposition to exploration: Chapter 2 revisits a known construction in a more explicit way, while Chapter 3 uses it as a starting point for new combinatorial questions.

1 Preliminaries

In this chapter, we collect the background material that will be needed in the rest of the thesis. We begin by recalling the axioms and basic properties of triangulated categories, then introduce cochain complexes and the homotopy category, and finally define the derived category of an abelian category via localization. The main references for this chapter are Miličić [1] and Weibel [2].

1.1 Triangulated categories

In this section, we recall the notion of a triangulated category.

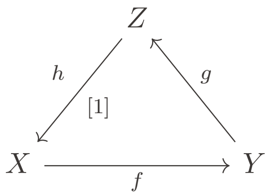
Let $\mathcal{C}$ be an additive category [1] and let $[1] : \mathcal{C} \to \mathcal{C}$ be an automorphism of the category $\mathcal{C}$, called **translation functor**, or **shift functor**.

For an object $X \in Ob\,\mathcal{C}$ we write $X[1]$ for its image under $[1]$ and for a morphism $f\colon X \to Y$, we denote by $f[1]\colon X[1] \to Y[1]$ its image under the functor $[1]$. More generally, for $n \in \mathbb{Z}$ we denote by $[n]$ the n-th iteration of $[1]$; for an object X we write $X[n]$, and for a morphism $f\colon X \to Y$ we write $f[n]\colon X[n] \to Y[n]$.

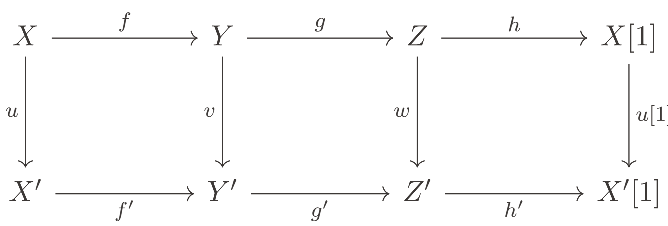
A **triangle** is a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

It can also be represented as follows.



Given two triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$, a **morphism of triangles** is a triple $\varphi = (u, v, w)$ given by the three maps $u : X \to X'$, $v : Y \to Y'$, $w : Z \to Z'$ such that the following diagram commutes.



If u, v and w are isomorphisms, then $\varphi = (u, v, w)$ is said to be an **isomorphism** of triangles.

A **triangulated category** is obtained by equipping $\mathcal{C}$ with a chosen shift functor $[1]$ and with a particular collection of triangles, called **distinguished triangles**, satisfying the following four axioms.

[1] $\mathcal{C}$ is an **additive category** if it has a zero object, it admits finite products and coproducts, $(Hom_{\mathcal{C}}(X, Y), +)$ is an abelian group $\forall X, Y \in Ob\,\mathcal{C}$ and the operation "+" is biadditive with respect to the composition of morphisms.

TR1 *(i)* Distinguished triangles are closed under isomorphisms.

(ii) For every $X \in Ob\,\mathcal{C}$, the following triangle is distinguished:

$$X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow X[1].$$

(iii) Every morphism $f\colon X \to Y$ can be completed to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

TR2 [Rotation axiom] A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

We call the latter triangle the rotation of the former triangle.

TR3 Given the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle u[1]} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & X'[1]
\end{array}$$

where the two rows are distinguished triangles and the left square commutes (i.e. $v \circ f = f' \circ u$), there exists a morphism $w\colon Z \to Z'$ making the entire diagram commute (and then becoming a morphism of triangles).

TR4 [Octahedral axiom] Given the following three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{a} Z \longrightarrow X[1]$$
$$X \xrightarrow{h} W \xrightarrow{b} K \longrightarrow X[1]$$
$$Y \xrightarrow{g} W \xrightarrow{c} J \longrightarrow Y[1]$$

if $h = g \circ f$, then there exists a distinguished triangle of the form:

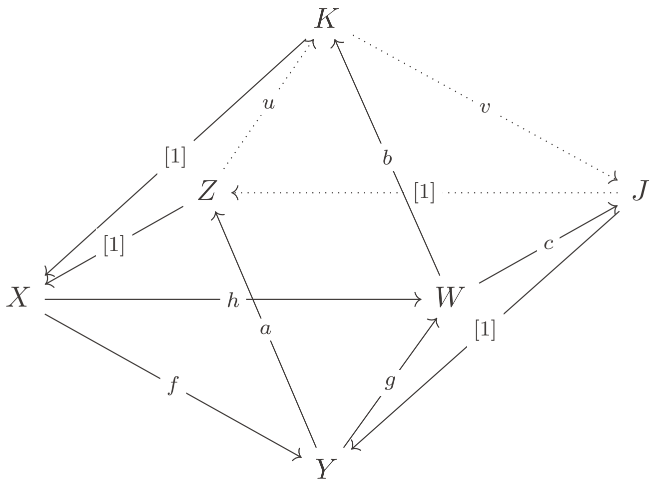
$$Z \xrightarrow{u} K \xrightarrow{v} J \longrightarrow Z[1]$$

such that the following diagram commutes.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{a} & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle id_{X[1]}} \\
X & \xrightarrow{h} & W & \xrightarrow{b} & K & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id_W} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & W & \xrightarrow{c} & J & \longrightarrow & Y[1] \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle id_J} & & \downarrow{\scriptstyle a[1]} \\
Z & \xrightarrow[u]{} & K & \xrightarrow[v]{} & J & \longrightarrow & Z[1]
\end{array}$$

.

An easy way to interpret this axiom is to picture an octahedron and place the three given distinguished triangles on it so that they appear as three faces with no edges in common. The distinguished triangle whose existence is ensured by the axiom can be viewed as a fourth face of the octahedron, sharing no edges with the previous three. The commutativity of the diagram can be interpreted as the commutativity of the corresponding octahedron.



Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. A **graded functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ together with a natural equivalence $\eta\colon F\circ[1] \Rightarrow [1]\circ F$. It means that for every $X \in Ob\,\mathcal{C}$, η_X is an isomorphism such that, for every $f\colon X \to Y$, $F(f)[1] \circ \eta_X = \eta_Y \circ F(f[1])$. By iterating η, we can obtain a natural equivalence $F(X[n]) \cong F(X)[n]$ for all $n \in \mathbb{Z}$.

A graded functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be **exact** (or **triangulated**) if it preserves distinguished triangles. It means that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

is a distinguished triangle in $\mathcal{C}$, then

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\eta_X\circ F(h)} F(X)[1]$$

is a distinguished triangle in $\mathcal{D}$.

Two triangulated categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent as triangulated categories** if there exists an exact graded functor $F\colon \mathcal{C} \to \mathcal{D}$ that is an equivalence of categories.

Definition 1.1.1 (The opposite triangulated category)

Given a triangulated category $\mathcal{C}$, we consider its opposite category $\mathcal{C}^{opp}$. We equip it with the translation functor induced by the inverse shift of $\mathcal{C}$ (so that shifting in $\mathcal{C}^{opp}$ corresponds to applying $[-1]$ in $\mathcal{C}$). Whenever

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in $\mathcal{C}$, we declare

$$Z \xrightarrow{g} Y \xrightarrow{f} X \xrightarrow{h[-1]} Z[-1]$$

to be a distinguished triangle in $\mathcal{C}^{opp}$. With these choices, $\mathcal{C}^{opp}$ satisfies the axioms of a triangulated category; a detailed proof can be found in Miličić [1, Chapter 2, Section 1.2]. We call it the **opposite triangulated category of $\mathcal{C}$**.

Distinguished triangles play in triangulated categories a role analogous to short exact sequences in abelian categories [2]. The following lemma is the analogue, in triangulated categories, of the fact that for a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

in an abelian category we have $g \circ f = 0$ (in particular, exactness at B means $\mathrm{Im}(f) = \ker(g)$).

In fact, short exact sequences in an abelian category $\mathcal{A}$ give rise to distinguished triangles in the derived category $D(\mathcal{A})$. This will be discussed in the next sections: first by introducing complexes and mapping cones (Section 1.2), and then by passing to the derived category via localization (Section 1.3).

Lemma 1.1.2 — Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a distinguished triangle. Then:

- $g \circ f = 0$;
- $h \circ g = 0$;
- $(f[1]) \circ h = 0$.

Proof. By axiom TR2, any rotation of a distinguished triangle is again distinguished. Therefore, it is enough to prove that $g \circ f = 0$; applying the same argument to the rotations yields $h \circ g = 0$ and $(f[1]) \circ h = 0$.

Consider the following triangle, which is distinguished by TR1:

$$X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow X[1].$$

Together with the distinguished triangle of the statement, we obtain the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{id_X} & X & \xrightarrow{0} & 0 & \longrightarrow & X[1] \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle id_X[1],} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & X[1], \end{array}$$

where the left square is commutative by construction. Then, by TR3, the dashed arrow can be chosen as a morphism $u \colon 0 \to Z$ so that the diagram becomes a morphism of distinguished triangles. Hence, since the middle square commutes, we obtain:

$$g \circ f = u \circ 0 = 0.$$

□

Any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

can be extended to an infinite diagram of the form

$$\dots \xrightarrow{g[-1]} Z[-1] \xrightarrow{h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{f[1]} Y[1] \xrightarrow{g[1]} \dots$$

By Lemma 1.1.2, we have $g \circ f = 0$, $h \circ g = 0$, and $f[1] \circ h = 0$. Since $[n]$ is a functor for every $n \in \mathbb{Z}$, the same relation hold for all the shifted maps, so the composition of any two consecutive arrows in the above diagram is zero.

[2] Recall that an **abelian** category is an additive category in which every morphism has a kernel and a cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel.

Let $\mathcal{C}$ be a triangulated category, $\mathcal{A}$ an abelian category, and $F\colon \mathcal{C} \to \mathcal{A}$ an additive functor. Then applying F to the above diagram yields a complex in $\mathcal{A}$ of the form:

$$\dots \xrightarrow{F(g[-1])} F(Z[-1]) \xrightarrow{F(h[-1])} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(X[1]) \xrightarrow{F(f[1])} F(Y[1]) \dots$$

An additive functor $F\colon \mathcal{C} \to \mathcal{A}$ is said to be **cohomological** if for every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact in $\mathcal{A}$. In this case, the complex obtained by applying F to the infinite diagram associated to the triangle is exact.

Example 1.1.3 (Hom-functors)

For every object X in the category $\mathcal{C}$, the covariant hom-functor [a]

$$h_X := Hom_{\mathcal{C}}(X, -)\colon \mathcal{C} \to Ab$$

is a cohomological functor on $\mathcal{C}$, where Ab denotes the category of abelian groups.

Dually, the contravariant hom-functor

$$h^X := Hom_{\mathcal{C}}(-, X)\colon \mathcal{C}^{opp} \to Ab$$

is a cohomological functor on $\mathcal{C}^{opp}$.

We refer to Miličić [1, Chapter 2, Proposition 1.4.1] for the proof.

[a] Recall that, for a fixed object X in the category $\mathcal{C}$, the hom-functors $Hom_{\mathcal{C}}(X, -)$ and $Hom_{\mathcal{C}}(-, X)$ assign to an object Y in $\mathcal{C}$ the abelian groups $Hom_{\mathcal{C}}(X, Y)$ and $Hom_{\mathcal{C}}(Y, X)$, respectively. For a morphism $f\colon Y \to Z$, the induced map $Hom_{\mathcal{C}}(X, f)\colon Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$ is given by composition $g \mapsto f \circ g$, while $Hom_{\mathcal{C}}(f, X)\colon Hom_{\mathcal{C}}(Z, X) \to Hom_{\mathcal{C}}(Y, X)$ is given by $h \mapsto h \circ f$.

Theorem 1.1.4 — Consider the following morphism between distinguished triangles.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & X'[1].
\end{array}$$

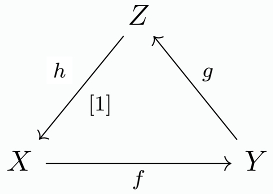
If two among u, v and w are isomorphisms, then the third one is an isomorphism as well.

Proof. (sketch) By rotating the triangles, we may reduce to the case where u and v are isomorphisms. Applying the covariant hom-functor $Hom_{\mathcal{C}}(Z', -)$ to the morphism of distinguished triangles yields a morphism between long exact sequences. Considering the five-term segment centered at $Hom_{\mathcal{C}}(Z', w)$, all the other vertical maps are isomorphisms. Hence by the 5-lemma the map $Hom_{\mathcal{C}}(Z', Z) \to Hom_{\mathcal{C}}(Z', Z')$ is an isomorphism. Therefore w admits a right inverse $a\colon Z' \to Z$ with $w \circ a = id_{Z'}$. Dually, applying $Hom_{\mathcal{C}}(-, Z)$ gives a left inverse $b\colon Z' \to Z$ with $b \circ w = id_Z$. Hence w is an isomorphism.

□

Lemma 1.1.5 — Consider the distinguished triangle depicted in the figure.

The third vertex Z is determined up to isomorphism by X, Y and the morphism $f\colon X \to Y$. We call Z a **cone** of f.



Proof. Let $f\colon X \to Y$ be a morphism. By TR1 any morphism can be completed to a distinguished triangle.

Now suppose we have two distinguished triangles completing f:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

and

$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} X[1].$$

Consider the following diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_Y} & & & & \downarrow{\scriptstyle id_{X[1]}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & X[1]. \end{array}$$

By TR3, there exists a morphism $w\colon Z \to Z'$ making the above diagram a morphism of distinguished triangles. Since the maps on the first two vertices are isomorphisms, by Theorem 1.1.4, w is an isomorphism. Therefore $Z \cong Z'$, so the third vertex is determined up to isomorphism by $f\colon X \to Y$. □

Lemma 1.1.6 — Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a distinguished triangle. If two of the objects X, Y, Z are isomorphic to 0, then so is the third.

Proof. By axiom TR2, we may rotate the triangle and assume that the two objects that are isomorphic to 0 are the first two. Since distinguished triangles are closed under isomorphism, it is enough to prove that in the triangle

$$0 \longrightarrow 0 \longrightarrow Z \longrightarrow 0[1]$$

the vertex Z is forced to be isomorphic to 0.

By TR1, the triangle

$$0 \xrightarrow{id_0} 0 \longrightarrow 0 \longrightarrow 0[1]$$

is distinguished, and, since both triangles complete the morphism $id_0\colon 0 \to 0$, by Lemma 1.1.5 $Z \cong 0$. □

Lemma 1.1.7 — Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a distinguished triangle. Then the following are equivalent:

(i) $Z \cong 0$;

(ii) f is an isomorphism.

Proof. Consider the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle id_X[1]} \\ X & \xrightarrow[f]{} & Y & \longrightarrow & Z & \longrightarrow & X[1]. \end{array}$$

The top row is distinguished by TR1, while the bottom row is distinguished by hypothesis, and the morphism of distinguished triangles exists by TR3.

(i) $\Rightarrow$ (ii): If $Z \cong 0$, then the third vertical arrow of the diagram is an isomorphism, and, since also the first one is so, then, by Theorem 1.1.4, f is an isomorphism.

(i) $\Leftarrow$ (ii): If f is an isomorphism, then the first two vertical arrows of the diagram are isomorphisms. Then, by Theorem 1.1.4, $0 \to Z$ is forced to be an isomorphism, meaning that $Z \cong 0$.

□

1.2 Cochain complexes and homotopy category

Let $\mathcal{A}$ be an additive category. A **(cochain) complex** in $\mathcal{A}$ is a family of objects $X^\bullet = (X^n)$ indexed by integers, together with morphisms $d_X^\bullet = (d_X^n)_{n\in\mathbb{Z}}$ called **differentials**, where, for every $n \in \mathbb{Z}$, we have $d_X^n \colon X^n \to X^{n+1}$ such that $d_X^{n+1} \circ d_X^n = 0$. We depict a complex as an infinite sequence of the form

$$\dots \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \dots$$

where the composition of any two consecutive arrows is zero.

Given two complexes $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$, a **morphism of complexes** $f^\bullet \colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ (also called a **cochain map**) is a family of morphisms $f^n \colon X^n \to Y^n$ such that, for every $n \in \mathbb{Z}$, we have $f^{n+1} \circ d_X^n = d_Y^n \circ f^n$, i.e. the following diagram commutes:

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{d_X^{n-2}} & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \xrightarrow{d_X^{n+1}} & X^{n+2} & \xrightarrow{d_X^{n+2}} & \dots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \downarrow{\scriptstyle f^{n+2}} & & \\ \dots & \xrightarrow[d_Y^{n-2}]{} & Y^{n-1} & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1} & \xrightarrow[d_Y^{n+1}]{} & Y^{n+2} & \xrightarrow[d_Y^{n+2}]{} & \dots \end{array}$$

Composition of morphisms of complexes is defined degreewise: if $f^\bullet = (f^n \colon X^n \to Y^n)_{n\in\mathbb{Z}}$ and $g^\bullet = (g^n \colon Y^n \to Z^n)_{n\in\mathbb{Z}}$, then $(g \circ f)^n = g^n \circ f^n$ for all $n \in \mathbb{Z}$. The identity morphism on $X^\bullet$ is given by the family $(id_{X^n})_{n\in\mathbb{Z}}$.

The category of complexes in $\mathcal{A}$, denoted by $C(\mathcal{A})$, is the category whose objects are complexes in $\mathcal{A}$ and whose morphisms are morphisms of complexes.

We define the translation functor (or shift functor) $[1] \colon C(\mathcal{A}) \to C(\mathcal{A})$ as follows: given a complex $(X^\bullet, d_X^\bullet) = (X^n, d_X^n)_{n\in\mathbb{Z}}$, its shift $(X^\bullet, d_X^\bullet)[1] = (X^\bullet[1], d_{X[1]}^\bullet)$ is the complex whose components are shifted by one degree, namely $(X[1])^n = X^{n+1}$ for every integer n, and whose differential $d_{X[1]}^n \colon (X[1])^n \to (X[1])^{n+1}$ is given by $d_{X[1]}^n = -d_X^{n+1}$.

With this choice $X[1]$ is again a complex, because the composition of two consecutive differentials is still zero.

On morphisms, the shift functor acts degreewise: if $f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ is a morphism of complexes, then $f^\bullet[1]\colon (X^\bullet[1], d_{X[1]}^\bullet) \to (Y^\bullet[1], d_{Y[1]}^\bullet)$ is defined by $(f[1])^n = f^{n+1}$ for every n. It is easy to verify that $f[1]$ is again a morphism of complexes, i.e. it commutes with the differentials of the shifted complexes. Then $[1]$ is an automorphism of $C(\mathcal{A})$, as required in the definition of a translation functor. Iterating, we obtain $X[p]$ for all integers p, and $[p]$ is an automorphism with inverse $[-p]$.

Lemma 1.2.1 — Let $\mathcal{A}$ be an additive category. The category $C(\mathcal{A})$ is additive.

Proof. Since $\mathcal{A}$ is additive, for any objects U, V, the set $Hom_{\mathcal{A}}(U, V)$ is an abelian group and composition is bilinear.

Let $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$ be complexes. By definition, a morphism of complexes $f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ is a family $(f^n\colon X^n \to Y^n)_{n\in\mathbb{Z}}$ satisfying $f^{n+1} \circ d_X^n = d_Y^n \circ f^n$ for all n. Given two such morphisms $f^\bullet, g^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$, we define their sum degreewise by $(f^\bullet + g^\bullet)^n := f^n + g^n$. Then

$$(f^\bullet + g^\bullet)^{n+1} \circ d_X^n = (f^{n+1} + g^{n+1}) \circ d_X^n = f^{n+1} \circ d_X^n + g^{n+1} \circ d_X^n =$$

$$= d_Y^n \circ f^n + d_Y^n \circ g^n = d_Y^n \circ (f^n + g^n) = d_Y^n \circ (f^\bullet + g^\bullet)^n,$$

so $f^\bullet + g^\bullet$ is again a morphism of complexes. The zero object is the zero complex

$$\ldots \to 0 \to 0 \to 0 \to 0 \to 0 \to \ldots$$

and additive inverses are given degreewise. Hence $Hom_{C(\mathcal{A})}(X, Y)$ is an abelian group. Moreover, bilinearity of composition in $\mathcal{A}$ implies bilinearity in $C(\mathcal{A})$.

Finite biproducts exist in $C(\mathcal{A})$ and are computed degreewise. We define $X^\bullet \oplus Y^\bullet$ in this way: $(X^\bullet \oplus Y^\bullet)^n := X^n \oplus Y^n$ and for the differential we define $d^n_{X^\bullet \oplus Y^\bullet} := d_X^n \oplus d_Y^n$.

The canonical inclusions and projections in each degree form morphisms of complexes:

$$i_X^\bullet\colon X^\bullet \to X^\bullet \oplus Y^\bullet, \quad i_Y^\bullet\colon Y^\bullet \to X^\bullet \oplus Y^\bullet, \quad p_X^\bullet\colon X^\bullet \oplus Y^\bullet \to X^\bullet, \quad p_Y^\bullet\colon X^\bullet \oplus Y^\bullet \to Y^\bullet;$$

and the biproduct identities

$$p_X^\bullet i_X^\bullet = id_X^\bullet, \quad p_Y^\bullet i_Y^\bullet = id_Y^\bullet, \quad p_X^\bullet i_Y^\bullet = 0, \quad p_Y^\bullet i_X^\bullet = 0 \text{ and } i_X^\bullet p_X^\bullet + i_Y^\bullet p_Y^\bullet = id_{X^\bullet \oplus Y^\bullet}$$

hold degreewise, hence in $C(\mathcal{A})$. Therefore, $C(\mathcal{A})$ has finite biproducts and bilinear composition, so it is an additive category. □

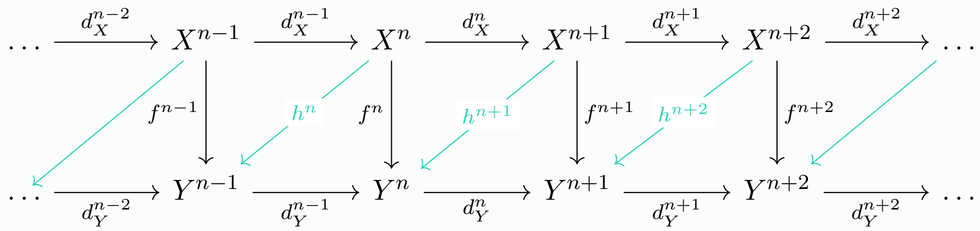
★ **Remark.** If $\mathcal{A}$ is an abelian category, then $C(\mathcal{A})$ is also abelian. Indeed, given a cochain map $f^\bullet\colon X^\bullet \to Y^\bullet$, its kernel and cokernel are computed degreewise: in each degree n we take $\ker(f^n)$ and $\operatorname{coker}(f^n)$ in $\mathcal{A}$, and the differentials are induced by the universal properties using the commutativity $f^{n+1} \circ d_X^n = d_Y^n \circ f^n$. In particular, exactness in $C(\mathcal{A})$ can be checked degree by degree.

Definition 1.2.2 (Homotopies)

Let $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$ be complexes in $C(\mathcal{A})$, and let $f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ be a morphism of complexes. We say that $f^\bullet$ is **homotopic to zero** (or **null-homotopic**) if there exists a family of morphisms $h^\bullet = (h^n\colon X^n \to Y^{n-1})_{n\in\mathbb{Z}}$ such that, for every integer n, it holds:

$$f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n.$$

We call $h^\bullet$ a **homotopy** from $f^\bullet$ to zero.



We denote by $Ht(X^\bullet, Y^\bullet)$ the set of null-homotopic morphisms from $(X^\bullet, d_X^\bullet)$ to $(Y^\bullet, d_Y^\bullet)$.

We say that two morphisms of complexes $f^\bullet, g^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ are **homotopic** if $f^\bullet - g^\bullet$ is homotopic to zero, i.e. if $f^\bullet - g^\bullet \in Ht(X^\bullet, Y^\bullet)$. In this case, we write $f^\bullet \sim g^\bullet$. Clearly, $\sim$ defines an equivalence relation on $Hom_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$.

Lemma 1.2.3 — Let $(X^\bullet, d_X^\bullet)$, $(Y^\bullet, d_Y^\bullet), (Z^\bullet, d_Z^\bullet)$ and $(W^\bullet, d_W^\bullet)$ be complexes in $C(\mathcal{A})$. Then:

(i) The set $Ht(X^\bullet, Y^\bullet)$ of null-homotopic morphisms from $(X^\bullet, d_X^\bullet)$ to $(Y^\bullet, d_Y^\bullet)$ is a subgroup of $Hom_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$.

(ii) The family $Ht(-,-)$ is stable under composition: if $f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ is null-homotopic and $g^\bullet\colon (Y^\bullet, d_Y^\bullet) \to (Z^\bullet, d_Z^\bullet)$ is any morphism of complexes, then $g^\bullet \circ f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Z^\bullet, d_Z^\bullet)$ is null-homotopic. Similarly, if $u^\bullet\colon (W^\bullet, d_W^\bullet) \to (X^\bullet, d_X^\bullet)$ is any morphism of complexes, then $f^\bullet \circ u^\bullet\colon (W^\bullet, d_W^\bullet) \to (Y^\bullet, d_Y^\bullet)$ is null-homotopic.

Proof. *(i)* The zero morphism is null-homotopic, with the zero homotopy.

If $f^\bullet, g^\bullet \in Ht(X^\bullet, Y^\bullet)$, choose homotopies $h^\bullet = (h^n)_{n\in\mathbb{Z}}$ and $k^\bullet = (k^n)_{n\in\mathbb{Z}}$ from $f^\bullet$ and $g^\bullet$ to zero, respectively. By definition $f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n$ and $g^n = d_Y^{n-1} \circ k^n + k^{n+1} \circ d_X^n$. Then, for every $n \in \mathbb{Z}$, the n-th degree of $f^\bullet + g^\bullet$ is:

$$(f+g)^n = f^n + g^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n + d_Y^{n-1} \circ k^n + k^{n+1} \circ d_X^n =$$

$$= d_Y^{n-1} \circ (h^n + k^n) + (h^{n+1} + k^{n+1}) \circ d_X^n,$$

so $f^\bullet + g^\bullet$ is null-homotopic, with homotopy $h^\bullet + k^\bullet$.

Moreover, $-f^n = d_Y^{n-1} \circ (-h^n) + (-h^{n+1}) \circ d_X^n$, so $-f^\bullet$ is homotopic to zero with homotopy $-h^\bullet$. Thus, $Ht(X^\bullet, Y^\bullet)$ is closed under additive inverses.

Hence, $Ht(X^\bullet, Y^\bullet)$ is a subgroup of $Hom_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$.

(ii) Assume $f^\bullet$ is null-homotopic, say $f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n$. Since $g^\bullet$ is a morphism of complexes, then it commutes with differentials, so $g^n \circ d_Y^{n-1} = d_Z^{n-1} \circ g^{n-1}$. Consider the composite $g^\bullet \circ f^\bullet$. In degree n it is

$$(g \circ f)^n = g^n \circ f^n = g^n \circ (d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n) = g^n \circ d_Y^{n-1} \circ h^n + g^n \circ h^{n+1} \circ d_X^n =$$

$$= d_Z^{n-1} \circ g^{n-1} \circ h^n + g^n \circ h^{n+1} \circ d_X^n.$$

This proves that $g^\bullet \circ f^\bullet$ is null-homotopic, with homotopy $\ell^\bullet = (\ell^n)_{n\in\mathbb{Z}}$, where $\ell^n = g^{n-1} \circ h^n$. The proof that, if $u^\bullet \colon (W^\bullet, d_W^\bullet) \to (X^\bullet, d_X^\bullet)$ is a morphism of complexes, then $f^\bullet \circ u^\bullet$ is null-homotopic, is analogous.

□

Definition 1.2.4 (Homotopy category)
Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ is defined as follows. Its objects are the same as those of $C(\mathcal{A})$, namely the cochain complexes in $\mathcal{A}$. Given two complexes $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$, we set

$$Hom_{K(\mathcal{A})}(X^\bullet, Y^\bullet) \coloneqq Hom_{C(\mathcal{A})}(X^\bullet, Y^\bullet)/Ht(X^\bullet, Y^\bullet).$$

By Lemma 1.2.3, null-homotopic maps are stable under composition, hence composition of morphisms of complexes descends to the quotient and is well-defined. Therefore, $K(\mathcal{A})$ is a well-defined category.

We denote by $q\colon C(\mathcal{A}) \to K(\mathcal{A})$ the canonical functor, which is the identity on objects and sends a cochain map to its homotopy class.

★ **Remark.** Two cochain maps $f^\bullet, g^\bullet \colon X^\bullet \to Y^\bullet$ become equal in $K(\mathcal{A})$ if and only if $f^\bullet - g^\bullet$ is null-homotopic, i.e., $f^\bullet \sim g^\bullet$.

Lemma 1.2.5 — The category $K(\mathcal{A})$ is additive.

Proof. (sketch) Since $Hom_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$ is an abelian group and $Ht(X^\bullet, Y^\bullet)$ is a subgroup, the quotient $Hom_{K(\mathcal{A})}(X^\bullet, Y^\bullet)$ is an abelian group. Biproducts in $K(\mathcal{A})$ are induced by the biproducts in $C(\mathcal{A})$ (the inclusions and projections are taken up to homotopy classes). The composition is biadditive, and the zero object corresponds to the zero object in $C(\mathcal{A})$. Hence $K(\mathcal{A})$ is additive. □

Lemma 1.2.6 — Let $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$ be complexes in $C(\mathcal{A})$, and let $f^\bullet \colon X^\bullet \to Y^\bullet$ be a morphism of complexes. Then the following statements are equivalent:

(i) $f^\bullet$ is null-homotopic;

(ii) $f^\bullet[1]$ is null-homotopic.

In particular, the shift functor [1] on $C(\mathcal{A})$ induces an automorphism of the homotopy category $K(\mathcal{A})$.

Proof. Let us prove that (i) and (ii) are equivalent:

$(i) \Rightarrow (ii)$: Assume $f^\bullet$ is null-homotopic. Then there exists a family of morphisms $h^\bullet = (h^n \colon X^n \to Y^{n-1})_{n\in\mathbb{Z}}$ such that, for every integer n,

$$f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n.$$

Define a family $k^\bullet = (k^n \colon (X[1])^n \to (Y[1])^{n-1})_{n\in\mathbb{Z}}$ by setting $k^n = h^{n+1}$. Using the definitions of the shifted differentials $d_{X[1]}^n = -d_X^{n+1}$ and $d_{Y[1]}^{n-1} = -d_Y^n$, we obtain

$$(f[1])^n = f^{n+1} = d_Y^n \circ h^{n+1} + h^{n+2} \circ d_X^{n+1} = (-d_{Y[1]}^{n-1}) \circ k^n + k^{n+1} \circ (-d_{X[1]}^n) =$$

$$= d_{Y[1]}^{n-1} \circ (-k^n) + (-k^{n+1}) \circ d_{X[1]}^n,$$

then $f^\bullet[1]$ is null-homotopic with homotopy $-k^\bullet$.

$\boxed{(i) \Leftarrow (ii)}$: If $f^\bullet[1]$ is null-homotopic, to show that also $f^\bullet$ is null-homotopic, we apply the same argument to the inverse shift $[-1]$.

We have proved that (i) and (ii) are equivalent. It follows that [1] preserves the subgroup $Ht(X^\bullet, Y^\bullet)$ for all $X^\bullet, Y^\bullet$, so it descends to the quotient defining $K(\mathcal{A})$. Since $[-1]$ is the inverse of [1], the induced functor on $K(\mathcal{A})$ is an automorphism.

□

Assume now that $\mathcal{A}$ is an abelian category. Since some of the arguments below involve only finite diagrams, we shall occasionally use elementwise language to simplify the exposition. This is justified by the Freyd-Mitchell embedding theorem, after restricting to the small abelian subcategory generated by the finitely many objects involved. [3]

Definition 1.2.7 (Cohomology functor)
For each $p \in \mathbb{Z}$, the **cohomology functor** $H^p : C(\mathcal{A}) \to \mathcal{A}$ is defined on objects by

$$H^p(X^\bullet) = \ker(d_X^p)/\mathrm{Im}(d_X^{p-1}).$$

This is well defined, since $d_X^p \circ d_X^{p-1} = 0$, which implies $\mathrm{Im}(d_X^{p-1}) \subseteq \ker(d_X^p)$.

For a morphism of complexes $f^\bullet : (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$, the relation $f^{p+1} \circ d_X^p = d_Y^p \circ f^p$ implies that $f^p(\ker(d_X^p)) \subseteq \ker(d_Y^p)$ and $f^p(\mathrm{Im}(d_X^{p-1})) \subseteq \mathrm{Im}(d_Y^{p-1})$. Therefore, f^p induces a morphism

$$H^p(f^\bullet) : H^p(X^\bullet) \to H^p(Y^\bullet).$$

One checks immediately that $H^p(id_{X^\bullet}) = id_{H^p(X^\bullet)}$ and that $H^p(g^\bullet \circ f^\bullet) = H^p(g^\bullet) \circ H^p(f^\bullet)$. Hence $H^p : C(\mathcal{A}) \to \mathcal{A}$ is a well-defined functor.

We now list some basic properties of the cohomology functor:

- The functor H^p is additive, because the induced map on cohomology is defined on classes by $H^p(f^\bullet)([x]) = [f^p(x)]$, hence $H^p(f^\bullet + g^\bullet) = H^p(f^\bullet) + H^p(g^\bullet)$.
- For every complex $(X^\bullet, d_X^\bullet)$,

$$H^p(X^\bullet[1]) = \ker(d_{X[1]}^p)/\mathrm{Im}(d_{X[1]}^{p-1}) = \ker(-d_X^{p+1})/\mathrm{Im}(-d_X^p) =$$

$$= \ker(d_X^{p+1})/\mathrm{Im}(d_X^p) = H^{p+1}(X^\bullet).$$

- Similarly, since $(f[1])^p = f^{p+1}$, we obtain

$$H^p(f^\bullet[1]) = H^{p+1}(f^\bullet).$$

- In particular,

$$H^p = H^0 \circ [p].$$

Lemma 1.2.8 — Let $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ be two homotopic morphisms of complexes. Then

$$H^p(f^\bullet) = H^p(g^\bullet) \quad \text{for every} \quad p \in \mathbb{Z}.$$

[3] More precisely, by Freyd-Mitchell embedding theorem, one may pass from that small abelian subcategory to a full exact embedding into a module category, where the corresponding elementwise arguments can be carried out. In particular, expressions such as $f(\ker g) \subseteq \ker h$ or $f(\mathrm{Im}\ g) \subseteq \mathrm{Im}\ h$ are understood in this sense. For the embedding theorem and the corresponding metatheorem on finite diagrams, see Mitchell [9, Chapter VI, Theorem 7.2 and Metatheorem 7.3]; for a concise modern formulation, together with an illustration via the snake lemma, see Potter, [10, §3.1.2, Theorem 3.4, Corollary 3.6, and the proof of Corollary 3.7.].

Proof. Since $f^\bullet \sim g^\bullet$, the morphism $f^\bullet - g^\bullet$ is null-homotopic. Let $h^\bullet$ be a homotopy from $f^\bullet - g^\bullet$ to zero. By definition

$$f^p - g^p = d_Y^{p-1} \circ h^p + h^{p+1} \circ d_X^p \quad \text{for every} \quad p \in \mathbb{Z}.$$

Let $x \in \ker(d_X^p)$. Then $d_X^p(x) = 0$. Hence

$$(f^p - g^p)(x) = d_Y^{p-1}(h^p(x)) \in \operatorname{Im}(d_Y^{p-1}).$$

Thus $f^p(x)$ and $g^p(x)$ represent the same class in

$$H^p(Y^\bullet) = \ker(d_Y^p)/\operatorname{Im}(d_Y^{p-1}).$$

Therefore $f^\bullet$ and $g^\bullet$ induce the same morphism

$$H^p(X^\bullet) \to H^p(Y^\bullet),$$

and hence $H^p(f^\bullet) = H^p(g^\bullet)$. □

As a consequence, the functor H^p depends only on the homotopy class of a morphism, hence it factors through the canonical quotient functor

$$q\colon C(\mathcal{A}) \to K(\mathcal{A}).$$

Therefore, for every $p \in \mathbb{Z}$, we obtain an induced additive functor, still denoted by $H^p\colon K(\mathcal{A}) \to \mathcal{A}$.

Definition 1.2.9 (Mapping cone)

Let $\mathcal{A}$ be an additive category, and let $f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ be a morphism of complexes. The **mapping cone** (or simply **cone**) of $f^\bullet$, denoted $C(f^\bullet)$, is the complex whose n-th component is

$$C(f^\bullet)^n = X^{n+1} \oplus Y^n,$$

and whose differential $d_{C(f^\bullet)}^n\colon C(f^\bullet)^n \to C(f^\bullet)^{n+1}$ is defined by

$$d_{C(f^\bullet)}^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

When $\mathcal{A}$ is a concrete additive category, for instance a category of modules, the above differential may be written as

$$d_{C(f^\bullet)}^n(x^{n+1}, y^n) = \big(-d_X^{n+1}(x^{n+1}),\ f^{n+1}(x^{n+1}) + d_Y^n(y^n)\big),$$

where (x^{n+1}, y^n) denotes an element of $C(f^\bullet)^n$.

Thus $C(f^\bullet)$ has the same underlying graded object as $X^\bullet[1] \oplus Y^\bullet$, but it is equipped with the differential

$$d_{C(f^\bullet)}^n = \begin{pmatrix} d_{X[1]}^n & 0 \\ (f^\bullet[1])^n & d_Y^n \end{pmatrix},$$

since $d_{X[1]}^n = -d_X^{n+1}$ and $(f^\bullet[1])^n = f^{n+1}$.

Lemma 1.2.10 — The mapping cone $C(f^\bullet)$, equipped with the differential $d_{C(f^\bullet)}^\bullet$, is a complex, i.e. $d_{C(f^\bullet)}^{n+1} \circ d_{C(f^\bullet)}^n = 0$ for every $n \in \mathbb{Z}$.

Proof. We compute the composition directly:

$$d^{n+1}_{C(f^\bullet)} \circ d^n_{C(f^\bullet)} = \begin{pmatrix} -d_X^{n+2} & 0 \\ f^{n+2} & d_Y^{n+1} \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^{n} \end{pmatrix} = \begin{pmatrix} d_X^{n+2} \circ d_X^{n+1} & 0 \\ -f^{n+2} \circ d_X^{n+1} + d_Y^{n+1} \circ f^{n+1} & d_Y^{n+1} \circ d_Y^n \end{pmatrix}.$$

The top-left entry vanishes because $d_X^\bullet$ is a differential, and the bottom-right entry vanishes because $d_Y^\bullet$ is a differential. For the bottom-left entry, since $f^\bullet$ is a morphism of complexes we have $f^{n+2} \circ d_X^{n+1} = d_Y^{n+1} \circ f^{n+1}$, so $-f^{n+2} \circ d_X^{n+1} + d_Y^{n+1} \circ f^{n+1} = 0$. □

The cone comes equipped with two canonical morphisms of complexes. The **canonical inclusion**

$$i_{f^\bullet} : Y^\bullet \to C(f^\bullet)$$

is defined in each degree by

$$i^n_{f^\bullet} = \begin{pmatrix} 0 \\ id_{Y^n} \end{pmatrix} : Y^n \to X^{n+1} \oplus Y^n = C(f^\bullet)^n,$$

and the **canonical projection**

$$p_{f^\bullet} : C(f^\bullet) \to X^\bullet[1]$$

is defined in each degree by

$$p^n_{f^\bullet} = \begin{pmatrix} id_{X^{n+1}} & 0 \end{pmatrix} : X^{n+1} \oplus Y^n = C(f^\bullet)^n \to X^{n+1} = (X[1])^n.$$

One verifies immediately that both $i_{f^\bullet}$ and $p_{f^\bullet}$ are morphisms of complexes: for $i_{f^\bullet}$, we have

$$d^n_{C(f^\bullet)} \circ i^n_{f^\bullet} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} \begin{pmatrix} 0 \\ id_{Y^n} \end{pmatrix} = \begin{pmatrix} 0 \\ d_Y^n \end{pmatrix} = i^{n+1}_{f^\bullet} \circ d^n_Y,$$

and for $p_{f^\bullet}$,

$$p^{n+1}_{f^\bullet} \circ d^n_{C(f^\bullet)} = \begin{pmatrix} id_{X^{n+2}} & 0 \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = \begin{pmatrix} -d_X^{n+1} & 0 \end{pmatrix} = d^n_{X[1]} \circ p^n_{f^\bullet},$$

since $d^n_{X[1]} = -d^{n+1}_X$. Moreover, $p_{f^\bullet} \circ i_{f^\bullet} = 0$.

We thus obtain the triangle associated to $f^\bullet$:

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_{f^\bullet}} C(f^\bullet) \xrightarrow{p_{f^\bullet}} X^\bullet[1].$$

We call this the **standard triangle** associated with $f^\bullet$.

In a concrete additive category, these maps are given by $y^n \mapsto (0, y^n)$ and $(x^{n+1}, y^n) \mapsto x^{n+1}$.

A triangle in $K(\mathcal{A})$ is said to be **distinguished** if it is isomorphic (in $K(\mathcal{A})$) to the standard triangle associated to some morphism of complexes.

> **Theorem 1.2.11** — Let $\mathcal{A}$ be an additive category. Then the homotopy category $K(\mathcal{A})$, equipped with the shift functor [1] and the class of distinguished triangles defined above, is a triangulated category.

For the verification of the axioms TR1–TR4 we refer to Miličić [1, Chapter 3, Section 2].

Assume now that $\mathcal{A}$ is an abelian category.

> **Definition 1.2.12** (Acyclic complex)
> A complex $(X^\bullet, d_X^\bullet)$ is said to be **acyclic** (or **exact**) if $H^p(X^\bullet) = 0$ for every $p \in \mathbb{Z}$.

Equivalently, $(X^\bullet, d_X^\bullet)$ is acyclic if $\ker(d_X^p) = \operatorname{Im}(d_X^{p-1})$ for every $p \in \mathbb{Z}$.

Definition 1.2.13 (Quasi-isomorphism)

A morphism of complexes $f^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ is called a **quasi-isomorphism** if the induced map on cohomology

$$H^p(f^\bullet)\colon H^p(X^\bullet) \to H^p(Y^\bullet)$$

is an isomorphism in the category $\mathcal{A}$ for every $p \in \mathbb{Z}$.

Since homotopic morphisms induce the same map on cohomology (Lemma 1.2.8), the notion of quasi-isomorphism depends only on the homotopy class of a morphism. Therefore, quasi-isomorphisms are naturally defined in the homotopy category $K(\mathcal{A})$: if $f^\bullet \sim g^\bullet$, then $f^\bullet$ is a quasi-isomorphism if and only if $g^\bullet$ is.

The following lemma relates quasi-isomorphisms to acyclicity of the mapping cone. It is the cohomological analogue of Lemma 1.1.7: the role of an isomorphism is played by a quasi-isomorphism, and the role of the zero object by an acyclic complex.

Lemma 1.2.14 — Let $\mathcal{A}$ be an abelian category, and let $f^\bullet\colon X^\bullet \to Y^\bullet$ be a morphism of complexes. Then $f^\bullet$ is a quasi-isomorphism if and only if the mapping cone $C(f^\bullet)$ is acyclic.

Proof. Consider the standard triangle associated to $f^\bullet$:

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_{f^\bullet}} C(f^\bullet) \xrightarrow{p_{f^\bullet}} X^\bullet[1].$$

This is a distinguished triangle in $K(\mathcal{A})$. The mapping cone $C(f^\bullet)$, viewed as an object of $K(\mathcal{A})$, is a cone of $f^\bullet$ in the triangulated sense.

Since the cohomology functors H^p are cohomological on $K(\mathcal{A})$ (see Miličić [1, Chapter 3, Corollary 2.3.2]), for every $p \in \mathbb{Z}$ we obtain a long exact sequence

$$\cdots \to H^p(X^\bullet) \xrightarrow{H^p(f^\bullet)} H^p(Y^\bullet) \xrightarrow{H^p(i_{f^\bullet})} H^p(C(f^\bullet)) \xrightarrow{H^p(p_{f^\bullet})} H^{p+1}(X^\bullet) \xrightarrow{H^{p+1}(f^\bullet)} H^{p+1}(Y^\bullet) \to \cdots$$

$\boxed{\Rightarrow}$: Assume that $f^\bullet$ is a quasi-isomorphism. Then, for every $p \in \mathbb{Z}$, both

$$H^p(f^\bullet)\colon H^p(X^\bullet) \to H^p(Y^\bullet) \quad \text{and} \quad H^{p+1}(f^\bullet)\colon H^{p+1}(X^\bullet) \to H^{p+1}(Y^\bullet)$$

are isomorphisms. Since $H^p(f^\bullet)$ is an epimorphism, exactness at $H^p(Y^\bullet)$ gives

$$\ker(H^p(i_{f^\bullet})) = \operatorname{Im}(H^p(f^\bullet)) = H^p(Y^\bullet),$$

hence $H^p(i_{f^\bullet}) = 0$. Therefore $\operatorname{Im}(H^p(i_{f^\bullet})) = 0$.

By exactness at $H^p(C(f^\bullet))$,

$$\ker(H^p(p_{f^\bullet})) = \operatorname{Im}(H^p(i_{f^\bullet})) = 0.$$

On the other hand, since $H^{p+1}(f^\bullet)$ is a monomorphism, exactness at $H^{p+1}(X^\bullet)$ gives

$$\operatorname{Im}(H^p(p_{f^\bullet})) = \ker(H^{p+1}(f^\bullet)) = 0.$$

Thus $H^p(p_{f^\bullet})$ has trivial kernel and trivial image, so necessarily $H^p(C(f^\bullet)) = 0$. Since this holds for every $p \in \mathbb{Z}$, the complex $C(f^\bullet)$ is acyclic.

$\boxed{\Leftarrow}$: Conversely, assume that $C(f^\bullet)$ is acyclic. Then $H^p(C(f^\bullet)) = 0$ for all $p \in \mathbb{Z}$. Fix $p \in \mathbb{Z}$. Exactness at $H^p(Y^\bullet)$ yields

$$\operatorname{Im}(H^p(f^\bullet)) = \ker(H^p(i_{f^\bullet})) = H^p(Y^\bullet),$$

so $H^p(f^\bullet)$ is an epimorphism (here $\ker(H^p(i_{f^\bullet})) = H^p(Y^\bullet)$ because the codomain of $H^p(i_{f^\bullet})$ is $H^p(C(f^\bullet)) = 0$, so $H^p(i_{f^\bullet})$ is the zero map).

Moreover, exactness at $H^p(X^\bullet)$ gives

$$\ker(H^p(f^\bullet)) = \mathrm{Im}(H^{p-1}(p_{f^\bullet})).$$

Since $H^{p-1}(C(f^\bullet)) = 0$, then $\mathrm{Im}(H^{p-1}(p_{f^\bullet})) = 0$. Therefore $\ker H^p(f^\bullet) = 0$, so $H^p(f^\bullet)$ is a monomorphism (here $\mathrm{Im}(H^{p-1}(p_{f^\bullet})) = 0$ because the domain of $H^{p-1}(p_{f^\bullet})$ is $H^{p-1}(C(f^\bullet)) = 0$.)

Thus $H^p(f^\bullet)$ is an isomorphism for every $p \in \mathbb{Z}$, and therefore $f^\bullet$ is a quasi-isomorphism.

□

In $K(\mathcal{A})$, the cone construction provides a canonical way to associate a distinguished triangle to any morphism. The cone associated with an isomorphism of complexes is acyclic. More generally, a morphism of complexes is a quasi-isomorphism if and only if its mapping cone is acyclic (see Lemma 1.2.14). However, quasi-isomorphisms are not, in general, invertible in $K(\mathcal{A})$. The derived category $D(\mathcal{A})$ is obtained by formally inverting quasi-isomorphisms; we will briefly recall this construction in the next section.

1.3 Localization and derived categories

As we have seen in Theorem 1.2.11, the homotopy category $K(\mathcal{A})$ of an additive category $\mathcal{A}$ is triangulated. In particular, this applies when $\mathcal{A}$ is abelian, and the cohomology functors H^p are well-defined on it. However, the homotopy category is still too rigid for many purposes: morphisms that induce isomorphisms on all cohomology groups — the quasi-isomorphisms of Definition 1.2.13 — are not in general invertible in $K(\mathcal{A})$.

The derived category $D(\mathcal{A})$ is obtained by formally inverting all quasi-isomorphisms. To make this precise, we first recall the general construction of the localization of a category with respect to a class of morphisms, and then specialize it to the class of quasi-isomorphisms.

We begin by collecting some basic properties of the class of quasi-isomorphisms. By Lemma 1.2.8, the notion of quasi-isomorphism depends only on the homotopy class of a morphism, so quasi-isomorphisms may be regarded as morphisms in $K(\mathcal{A})$.

Lemma 1.3.1 — Let $\mathcal{A}$ be an abelian category, and let S denote the class of quasi-isomorphisms in $K(\mathcal{A})$. Then:

(i) Every isomorphism in $K(\mathcal{A})$ belongs to S.

(ii) If $f^\bullet\colon X^\bullet \to Y^\bullet$ and $g^\bullet\colon Y^\bullet \to Z^\bullet$ are in S, then $g^\bullet \circ f^\bullet$ is in S.

(iii) S is compatible with the shift functor: for every $n \in \mathbb{Z}$, $f^\bullet \in S$ if and only if $f^\bullet[n] \in S$.

Proof. *(i)* If $f^\bullet$ is an isomorphism in $K(\mathcal{A})$, then in particular it induces isomorphisms on all cohomology groups, since each cohomology functor $H^p\colon K(\mathcal{A}) \to \mathcal{A}$ preserves isomorphisms.

(ii) For every $p \in \mathbb{Z}$, the induced map $H^p(g^\bullet \circ f^\bullet) = H^p(g^\bullet) \circ H^p(f^\bullet)$ is a composition of isomorphisms, hence an isomorphism.

(iii) For every $p \in \mathbb{Z}$, we have $H^p(f^\bullet[n]) = H^{p+n}(f^\bullet)$, so $f^\bullet[n]$ is a quasi-isomorphism if and only if $f^\bullet$ is.

□

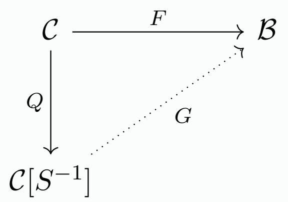
We now recall the general notion of localization, which provides a universal way to formally invert a given class of morphisms.

Definition 1.3.2 (Localization of a category)
Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. A **localization** of $\mathcal{C}$ with respect to S is a pair $(\mathcal{C}[S^{-1}], Q)$, where $\mathcal{C}[S^{-1}]$ is a category and $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ is a functor, called the **quotient functor**, satisfying:

(i) for every morphism $s \in S$, the morphism $Q(s)$ is an isomorphism in $\mathcal{C}[S^{-1}]$;

(ii) for every category $\mathcal{B}$ and every functor $F : \mathcal{C} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for all $s \in S$, there exists a unique functor $G : \mathcal{C}[S^{-1}] \to \mathcal{B}$ such that $F = G \circ Q$.

Then we obtain the following commutative diagram of functors:

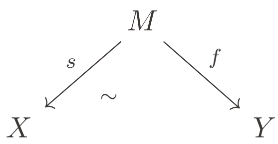


Theorem 1.3.3 — For every category $\mathcal{C}$ and every class S of morphisms in $\mathcal{C}$, the localization $(\mathcal{C}[S^{-1}], Q)$ exists and is unique up to isomorphism of categories.

Proof. We prove uniqueness. Suppose $(\mathcal{C}[S^{-1}], Q)$ and $(\mathcal{D}, Q')$ are two localizations of $\mathcal{C}$ with respect to S. Since $Q'(s)$ is an isomorphism for every $s \in S$, the universal property of $(\mathcal{C}[S^{-1}], Q)$ gives a unique functor $G\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ with $Q' = G \circ Q$. Symmetrically, we obtain $H\colon \mathcal{D} \to \mathcal{C}[S^{-1}]$ with $Q = H \circ Q'$. Then $H \circ G \circ Q = Q$, and by the uniqueness of the factorization through Q, we have $H \circ G = id_{\mathcal{C}[S^{-1}]}$. Analogously $G \circ H = id_{\mathcal{D}}$, so G is an isomorphism of categories.

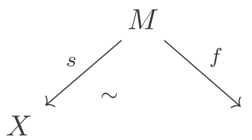
For the existence, we sketch the construction; full details can be found in Miličić [1, Chapter 1, Section 1.1].

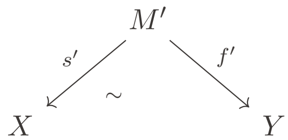
The localized category $C[S^{-1}]$ has the same objects as C. To describe its morphisms, we introduce the notion of (left) **roof**. A roof from X to Y is a diagram of the form



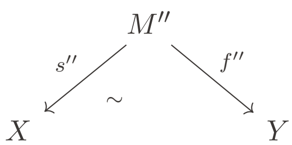
where $s \in S$. Intuitively, such a diagram represents the "fraction" $f \circ s^{-1}$, even though s need not be invertible in C.

Two roofs





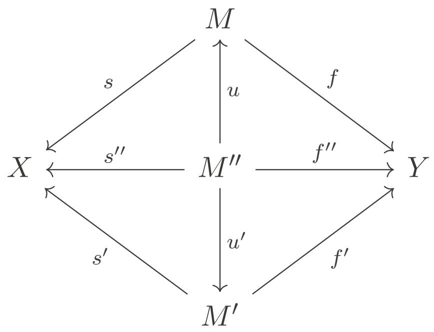
are said to be equivalent if there exists a third roof



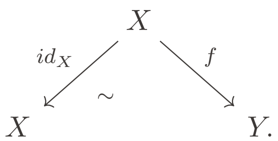
and two morphisms $u\colon M'' \to M$ and $u'\colon M'' \to M'$ such that

$$s \circ u = s'' = s' \circ u' \quad \text{and} \quad f \circ u = f'' = f' \circ u',$$

i.e. $s'' \in S$ and the following diagram commutes.



A morphism from X to Y in $C[S^{-1}]$ is an equivalence class of roofs. The quotient functor $Q\colon C \to C[S^{-1}]$ is the identity on objects and sends a morphism $f\colon X \to Y$ to the equivalence class of the roof



For this construction to yield a well-defined category, one must verify that equivalence classes of roofs can be composed associatively. We refer to Miličić [1, Chapter 1, Section 1.1] for the complete argument.

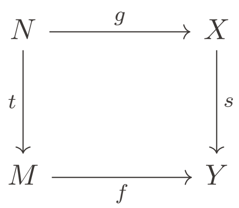
However, the situation simplifies considerably when S admits a calculus of left fractions, meaning that the following conditions hold:

<u>LF1</u> S contains all identity morphisms and is closed under composition;

<u>LF2</u> every diagram

$$M \xrightarrow{f} Y \xleftarrow{s} X$$

with $s \in S$ can be completed to a commutative square



with $t \in S$;

<u>LF3</u> if $f \circ s = g \circ s$ for some $s \in S$, then there exists $t \in S$ such that $t \circ f = t \circ g$.

When these conditions are satisfied, the composition of two roofs can be computed by using (LF2) to complete the middle square, and then the equivalence relation between roofs reduces to an easier form. Furthermore, there is an analogous notion of calculus of right fractions (with the dual conditions), and when S admits both, we can speak about calculus of fractions. □

We now apply the general localization construction to the homotopy category $K(\mathcal{A})$, taking as S the class of quasi-isomorphisms.

Definition 1.3.4 (Derived category)
Let $\mathcal{A}$ be an abelian category, and let S denote the class of quasi-isomorphisms in $K(\mathcal{A})$. The **derived category** of $\mathcal{A}$ is the localization

$$D(\mathcal{A}) := K(\mathcal{A})[S^{-1}].$$

The quotient functor $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$ sends each complex to itself and each morphism to its equivalence class. In particular, $Q(s)$ is an isomorphism in $D(\mathcal{A})$ for every quasi-isomorphism s.

Let $q_A\colon C(\mathcal{A}) \to K(\mathcal{A})$ be the canonical functor. It is standard that the same category $D(\mathcal{A})$ can equivalently be viewed as the localization of $C(\mathcal{A})$ with respect to quasi-isomorphisms. Therefore, when convenient, we shall also consider the canonical functor

$$Q_{\mathcal{A}} := Q \circ q_{\mathcal{A}}\colon C(\mathcal{A}) \to D(\mathcal{A}).$$

In particular, $Q_{\mathcal{A}}$ sends quasi-isomorphisms in $C(\mathcal{A})$ to isomorphisms in $D(\mathcal{A})$.

Concretely, $D(\mathcal{A})$ has the same objects as $K(\mathcal{A})$, namely cochain complexes over $\mathcal{A}$, while its morphisms are obtained from those of $K(\mathcal{A})$ by formally inverting quasi-isomorphisms. In the present situation, the class of quasi-isomorphisms admits a calculus of fractions (see Miličić [1, Chapter 3, Section 3]), so that morphisms in $D(\mathcal{A})$ may be represented by roofs

$$X^\bullet \xleftarrow[\sim]{\;s\;} M^\bullet \xrightarrow{\;f\;} Y^\bullet$$

in $K(\mathcal{A})$, where s is a quasi-isomorphism. We will not use this explicit description in the sequel. Instead, the functors considered in Chapter 2 will be constructed at the level of complexes and then shown, via the universal property of localization (Proposition 1.3.6), to descend to derived categories.

The derived category inherits a triangulated structure from the homotopy category. The shift functor on $D(\mathcal{A})$ is defined by $[1] \circ Q = Q \circ [1]$, which is well-defined since the shift preserves quasi-isomorphisms (Lemma 1.3.1(*iii*)).

Theorem 1.3.5 — Let $\mathcal{A}$ be an abelian category. The derived category $D(\mathcal{A})$, equipped with the shift functor [1] and the following class of distinguished triangles, is a triangulated category: a triangle in $D(\mathcal{A})$ is distinguished if and only if it is isomorphic, in $D(\mathcal{A})$, to the image under Q of a distinguished triangle in $K(\mathcal{A})$. Equivalently, the distinguished triangles in $D(\mathcal{A})$ are those isomorphic to triangles of the form

$$Q(X^\bullet) \xrightarrow{Q([f^\bullet])} Q(Y^\bullet) \xrightarrow{Q(i_{f^\bullet})} Q(C(f^\bullet)) \xrightarrow{Q(p_{f^\bullet})} Q(X^\bullet[1]),$$

where $f^\bullet$ is a morphism of complexes, $[f^\bullet]$ denotes its homotopy class in $K(\mathcal{A})$, and

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_{f^\bullet}} C(f^\bullet) \xrightarrow{p_{f^\bullet}} X^\bullet[1]$$

is the standard mapping-cone triangle associated with $f^\bullet$.

Proof. (sketch) We briefly indicate the key ideas of the proof, and we refer to Miličić [1, Chapter 3, Section 3] for a detailed proof. Axioms TR1 and TR2 follow directly from the corresponding axioms in $K(\mathcal{A})$. For TR3, the key observation is that morphisms in $D(\mathcal{A})$ can be described in terms of roofs, or equivalently as compositions of morphisms in $K(\mathcal{A})$ and formal inverses of quasi-isomorphisms. This allows one to lift the problem to $K(\mathcal{A})$, apply TR3 there, and project back to $D(\mathcal{A})$ via Q. Axiom TR4 follows by an analogous lifting argument. □

★ **Remark.** The quotient functor $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$ is an exact functor of triangulated categories: it commutes with the shift and sends distinguished triangles to distinguished triangles. This is immediate from the definition of the triangulated structure on $D(\mathcal{A})$.

The universal property of the localization gives a direct criterion for when a functor defined at the level of complexes induces a functor between derived categories, that will be used repeatedly in the rest of the thesis.

Proposition 1.3.6 — Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F\colon C(\mathcal{A}) \to C(\mathcal{B})$ be an additive functor that preserves quasi-isomorphisms, i.e., $F(s)$ is a quasi-isomorphism for every quasi-isomorphism s. Then the composition

$$Q_{\mathcal{B}} \circ F\colon C(\mathcal{A}) \to D(\mathcal{B})$$

maps quasi-isomorphisms to isomorphisms. Therefore, by the universal property of localization, there exists a unique functor $\tilde{F}\colon D(\mathcal{A}) \to D(\mathcal{B})$ such that the diagram

$$\begin{array}{ccc} C(\mathcal{A}) & \xrightarrow{F} & C(\mathcal{B}) \\ \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ D(\mathcal{A}) & \xrightarrow[\tilde{F}]{} & D(\mathcal{B}) \end{array}$$

commutes, i.e., $\tilde{F} \circ Q_{\mathcal{A}} = Q_{\mathcal{B}} \circ F$.

If moreover F is compatible with the shift functor and preserves mapping cones, then $\tilde{F}$ is a triangulated functor.

Proof. Since F preserves quasi-isomorphisms, the composition $Q_{\mathcal{B}} \circ F$ sends every quasi-isomorphism in $C(\mathcal{A})$ to an isomorphism in $D(\mathcal{B})$. Since $D(\mathcal{A})$ may also be viewed as the localization of $C(\mathcal{A})$ with respect to quasi-isomorphisms, the universal property of the localization yields a unique functor $\tilde{F}\colon D(\mathcal{A}) \to D(\mathcal{B})$ such that $\tilde{F} \circ Q_{\mathcal{A}} = Q_{\mathcal{B}} \circ F$.

For the second statement, recall that a distinguished triangle in $D(\mathcal{A})$ is isomorphic to the image under $Q_{\mathcal{A}}$ of a standard mapping-cone triangle in $C(\mathcal{A})$. If F is compatible with the shift and preserves mapping cones, then it sends mapping-cone triangles to mapping-cone triangles. Hence, after passing to the derived categories, $\tilde{F}$ sends distinguished triangles to distinguished triangles. Therefore $\tilde{F}$ is a triangulated functor. □

★ **Remark.** We can now make precise the claim of Section 1.1 that short exact sequences in an abelian category $\mathcal{A}$ give rise to distinguished triangles in $D(\mathcal{A})$. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in $\mathcal{A}$. Viewing A, B and C as complexes concentrated in degree zero, f induces a morphism of complexes $f^\bullet\colon A^\bullet \to B^\bullet$. By Definition 1.2.9, its mapping cone $C(f^\bullet)$ is the complex

$$\cdots \to 0 \to A \xrightarrow{f} B \to 0 \to \ldots$$

concentrated in degrees -1 and 0. The morphism $g\colon B \to C$ extends to a morphism of complexes $C(f^\bullet) \to C^\bullet$, which is a quasi-isomorphism by exactness of the original short exact sequence. Indeed, $H^{-1}(C(f^\bullet)) = \ker(f) = 0$ and $H^0(C(f^\bullet)) = \operatorname{coker}(f) \cong C$. Hence $C(f^\bullet) \cong C$ in $D(\mathcal{A})$. Therefore, the standard triangle associated with $f^\bullet$ yields a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$$

in $D(\mathcal{A})$.

2 Universal derived equivalences of posets

In this chapter, we consider universal derived equivalences of finite posets. Given a finite poset X and an abelian category $\mathcal{A}$, we first introduce the category of diagrams $\mathcal{A}^X$. The main goal is to prove Theorem 2.2.4, following the approach of Ladkani [3]. More precisely, starting from suitable combinatorial data, we construct explicit functors at the level of complexes of diagrams, called formulas, and show that they induce triangulated functors between the corresponding derived categories. After introducing formulas and studying their basic properties, we apply them first in the elementary case where the poset is the chain $1 \to 2$, and then in the general setting to prove the main theorem.

2.1 Posets as categories and diagram categories

Definition 2.1.1 (Poset)

A partially ordered set, also called **poset**, is a pair $(X, \leq)$, where X is a set and $\leq$ is a binary relation on X, called a **partial order**, satisfying the following properties:

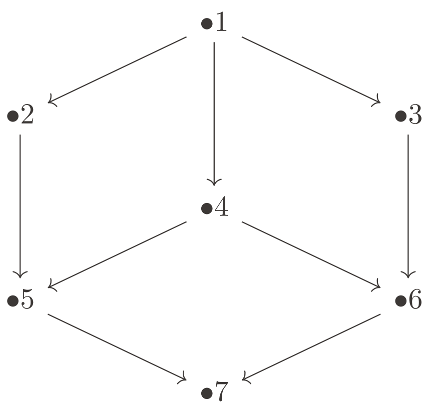
- (Reflexivity) $\forall\, x \in X$, it holds that $x \leq x$;
- (Antisymmetry) $\forall\, x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$;
- (Transitivity) $\forall\, x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

If $x \leq y$ and $x \neq y$ we also write $x < y$.

A finite poset $(X, \leq)$ can be represented graphically by a **Hasse diagram**, i.e. a quiver whose vertices are the elements of X, and in which we draw an arrow $x \to y$ iff $x < y$ and there is no $z \in X$ such that $x < z < y$.

Example 2.1.2

Consider, for example, the set $X_1 = \{1, 2, 3, 4, 5, 6, 7\}$ endowed with the following covering relations: $1 \leq 2$; $1 \leq 3$; $1 \leq 4$; $2 \leq 5$; $3 \leq 6$; $4 \leq 5$; $4 \leq 6$; $5 \leq 7$; $6 \leq 7$. We can draw the corresponding Hasse diagram on the right.



Notice that relations implied by transitivity are omitted. We denote this poset by X_1, and we will use it several times.

Definition 2.1.3 (The category of diagrams over a poset)
Given a poset X and a category $\mathcal{A}$, a **diagram** over X with values in $\mathcal{A}$ is a pair (A, r), where:

- $A = (A_x)_{x \in X}$ is a family of objects in the category $\mathcal{A}$;
- $r = (r_{xx'})_{x,x' \in X \times X, x \leq x'}$ is a family of morphisms, where every morphism $r_{xx'} \colon A_x \to A_{x'}$ is called restriction map and satisfies:
  - $r_{xx} = id_{A_x}$ for all $x \in X$;
  - $r_{xx''} = r_{x'x''} r_{xx'}$ whenever $x \leq x' \leq x''$.

Given two diagrams (A, r) and (A', r') over X, a **morphism of diagrams** $f \colon (A, r) \to (A', r')$ is a family of morphisms $(f_x \colon A_x \to A'_x)_{x \in X}$ such that, for every $x \leq x'$, the following diagram commutes; that is, $f_{x'} \circ r_{xx'} = r'_{xx'} \circ f_x$.

$$\begin{array}{ccc} A_x & \xrightarrow{f_x} & A'_x \\ {\scriptstyle r_{xx'}}\downarrow & & \downarrow{\scriptstyle r'_{xx'}} \\ A_{x'} & \xrightarrow[f_{x'}]{} & A'_{x'} \end{array}$$

The **category of diagrams** over X with values in $\mathcal{A}$, denoted by $\mathcal{A}^X$, is the category whose objects are the diagrams (A, r) and whose morphisms are morphisms of diagrams.

Since a poset X can be viewed as a category, an object in $\mathcal{A}^X$ is a functor from X to the category $\mathcal{A}$ and a morphism of diagrams is a natural transformation.

With the notation introduced above and in Section 1.2, $C(\mathcal{A}^X)$ denotes the category of complexes of diagrams over X with values in $\mathcal{A}$, while $C(\mathcal{A})^X$ denotes the category of diagrams over X with values in $C(\mathcal{A})$.

Theorem 2.1.4 ($C(\mathcal{A}^X) \simeq C(\mathcal{A})^X$) — There exists a natural equivalence $\Phi_{X,\mathcal{A}}$ between the categories $C(\mathcal{A}^X)$ and $C(\mathcal{A})^X$ such that for every additive functor $F \colon \mathcal{A} \to \mathcal{A}'$ the following diagram commutes.

$$\begin{array}{ccc} C(\mathcal{A}^X) & \xrightarrow{\Phi_{X,\mathcal{A}}} & C(\mathcal{A})^X \\ {\scriptstyle C(F^X)}\downarrow & & \downarrow{\scriptstyle C(F)^X} \\ C(\mathcal{A}'^X) & \xrightarrow{\Phi_{X,\mathcal{A}'}} & C(\mathcal{A}')^X \end{array}$$

In other words, we have a correspondence between a complex of diagrams and a diagram of complexes.

Proof. An object of $C(\mathcal{A}^X)$ is a complex $(K^\bullet, d^\bullet)$, where for each i, (K^i, r^i) is a diagram over X with values in $\mathcal{A}$, with $K^i = (K^i_x)_{x \in X}$ and $r^i = (r^i_{xy})_{x \leq y}$, and $d^i = (d^i_x \colon K^i_x \to K^{i+1}_x)_{x \in X}$ is a morphism of diagrams $(K^i, r^i) \to (K^{i+1}, r^{i+1})$. Thus, for every $x \leq y$, we have $r^{i+1}_{xy} \circ d^i_x = d^i_y \circ r^i_{xy}$.

An object of $C(\mathcal{A})^X$ is a diagram of complexes of the form $((K^\bullet_x)_{x \in X}, (r^\bullet_{xy}))$. Then a natural way to define on the objects the functor we are looking for is

$$\Phi_{X,\mathcal{A}}(K^\bullet, d^\bullet) := ((K^\bullet_x)_{x \in X}, (r^\bullet_{xy})).$$

By the commutativity relation above, $r^\bullet_{xy}$ is a morphism of complexes.

Let $f\colon (K^\bullet, d^\bullet) \to (L^\bullet, \delta^\bullet)$ be a morphism in $C(\mathcal{A}^X)$, where $f = (f^i\colon K^i \to L^i)_{i\in\mathbb{Z}}$ with $f^i = (f^i_x\colon K^i_x \to L^i_x)_{x\in X}$ for each $i \in \mathbb{Z}$.

On the morphisms we define

$$\Phi_{X,\mathcal{A}}(f) := (f_x)_{x\in X},$$

where for each $x \in X$, $f_x := (f^i_x)_{i\in\mathbb{Z}}$.

Note that $\Phi_{X,\mathcal{A}}(f)$ is defined by evaluation at each $x \in X$: the complex $(\Phi_{X,A}(K^\bullet))_x$ has terms K^i_x and differentials d^i_x and for a chain map f we have $(\Phi_{X,A}(f))_x = f_x$. In particular, evaluation commutes with differentials and chain maps, i.e. everything is defined componentwise.

Then it is immediate that $\Phi_{X,\mathcal{A}}$ is well-defined. Also the induced functors $C(F^X)$ and $C(F)^X$ act pointwise, hence the square of the statement commutes.

Define a functor $\Psi_{X,\mathcal{A}}\colon C(\mathcal{A})^X \to C(\mathcal{A}^X)$ as follows: given an object $((K^\bullet_x)_{x\in X}, (r^\bullet_{xy}))$ in $C(\mathcal{A})^X$, set, for each $i \in \mathbb{Z}$,

$$K^i := ((K^i_x)_{x\in X}, (r^i_{xy})), \text{ and } d^i := (d^i_x)_{x\in X}\colon K^i \to K^{i+1}.$$

This yields a complex $(K^\bullet, d^\bullet)$ in $C(\mathcal{A}^X)$. On morphisms, we define $\Psi_{X,\mathcal{A}}$ in this way: given $(f^\bullet_x)_{x\in X}\colon (K^\bullet_x, r^\bullet_{xy}) \to (L^\bullet_x, s^\bullet_{xy})$ in $C(\mathcal{A})^X$, define $\Psi_{X,\mathcal{A}}((f^\bullet_x)_x) = f^\bullet$, where in each degree i we set $f^i\colon K^i \to L^i$ to be the morphism in $\mathcal{A}^X$ with components $(f^i_x)_{x\in X}$. Then $f^\bullet$ is a chain map since the identities $\delta^i_x f^i_x = f^{i+1}_x d^i_x$ hold for every i.

By construction, for every $(K^\bullet, d^\bullet)$ in $C(\mathcal{A}^X)$ we have $(\Psi_{X,\mathcal{A}} \circ \Phi_{X,\mathcal{A}})(K^\bullet, d^\bullet) = (K^\bullet, d^\bullet)$ since in degree i we recover the same diagram K^i with the same restriction maps and the same differential. Similarly, for every morphism f in $C(\mathcal{A}^X)$ we obtain $(\Psi_{X,\mathcal{A}} \circ \Phi_{X,\mathcal{A}})(f) = f$, because it is defined componentwise in each degree and at each $x \in X$.

Conversely, for an object $((K^\bullet_x)_{x\in X}, (r^\bullet_{xy}))$ in $C(\mathcal{A})^X$, applying $\Psi_{X,\mathcal{A}}$ and then $\Phi_{X,\mathcal{A}}$ returns the same diagram of complexes, since in each degree i we recover the same diagram K^i and the same structure maps.

Hence $\Psi_{X,\mathcal{A}} \circ \Phi_{X,\mathcal{A}} = Id_{C(\mathcal{A}^X)}$ and $\Phi_{X,\mathcal{A}} \circ \Psi_{X,\mathcal{A}} = Id_{C(\mathcal{A})^X}$, so $\Phi_{X,\mathcal{A}}$ is an equivalence of categories.

□

Lemma 2.1.5 — Let $\mathcal{A}$ be an abelian category and X a poset. Then:

(i) Kernels are computed pointwise: for every morphism $f\colon A \to A'$ in $\mathcal{A}^X$, with $f = (f_x\colon A_x \to A'_x)_{x\in X}$, $\ker(f)$ exists and is given by a diagram $K \in \mathcal{A}^X$, defined by $K_x = \ker(f_x)$. For every $x \leq x'$, the restriction map $K_x \to K_{x'}$ is uniquely determined by the universal property of the kernel. Moreover, for every $x \in X$, we have $(\ker f)_x \cong \ker(f_x)$.

(ii) Cokernels are computed pointwise: for every morphism $f\colon A \to A'$ in $\mathcal{A}^X$, with $f = (f_x\colon A_x \to A'_x)_{x\in X}$, $\operatorname{coker}(f)$ exists and is given by a diagram $Q \in \mathcal{A}^X$, defined by $Q_x = \operatorname{coker}(f_x)$. For every $x \leq x'$, the restriction map $Q_x \to Q_{x'}$ is uniquely determined by the universal property of the cokernel. Moreover, for every $x \in X$, we have $(\operatorname{coker} f)_x \cong \operatorname{coker}(f_x)$.

(iii) Images are computed pointwise: for every morphism $f\colon A \to A'$ in $\mathcal{A}^X$,

$$(\operatorname{Im} f)_x \cong \operatorname{Im}(f_x) \text{ for all } x \in X.$$

(iv) Cohomology is computed pointwise: for every complex $K^\bullet$ in $C(\mathcal{A}^X)$ and for every $i \in \mathbb{Z}$,

$$H^i(K^\bullet)_x \cong H^i(K^\bullet_x) \text{ for all } x \in X.$$

(v) Mapping cones are computed pointwise: for any morphism of complexes $f\colon K^\bullet \to L^\bullet$ in $C(\mathcal{A}^X)$,

$$C(f)_x \cong C(f_x) \text{ for all } x \in X.$$

(vi) Acyclicity of mapping cones is characterized pointwise: for every morphism of complexes $f\colon K^\bullet \to L^\bullet$ in $C(\mathcal{A}^X)$,

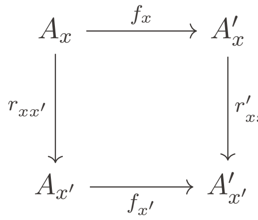
$$C(f) \text{ is acyclic } \iff C(f_x) \text{ is acyclic for all } x \in X.$$

(vii) Quasi-isomorphisms are characterized pointwise: a morphism f in $C(\mathcal{A}^X)$ is a quasi-isomorphism if and only if f_x is a quasi-isomorphism for every $x \in X$.

Proof. *(i)* Let X be a fixed poset.

Since $f = (f_x)_{x\in X}$ is a morphism of diagrams in $\mathcal{A}^X$, the diagram on the right commutes $\forall x \le x'$, i.e.

$$r'_{xx'} \circ f_x = f_{x'} \circ r_{xx'}.$$

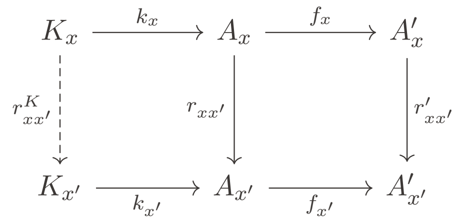


For every $x \in X$, we call (K_x, k_x) the kernel of the morphism f_x.

Since

$$f_{x'} \circ r_{xx'} \circ k_x = r'_{xx'} \circ f_x \circ k_x = r'_{xx'} \circ 0 = 0,$$

by universal property of $(K_{x'}, k_{x'})$ there exists a unique map $r^K_{xx'}\colon K_x \to K_{x'}$ such that $k_{x'} \circ r^K_{xx'} = r_{xx'} \circ k_x$, hence $k = (k_x)_{x\in X}$ is a morphism of diagrams.



By uniqueness, $r^K_{xx} = id_{K_x}$ for all $x \in X$, and $r^K_{xx''} = r^K_{x'x''} \circ r^K_{xx'}$ whenever $x \le x' \le x''$. Hence (K, r^K) is a diagram in $\mathcal{A}^X$.

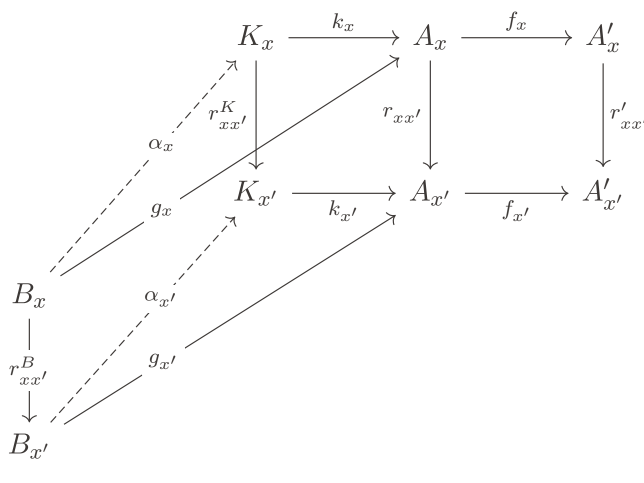
Now we claim that $(K, r^K) = ((K_x)_{x\in X}, (r^K_{xx'})_{x\le x'})$ is the kernel of $f = (f_x)_{x\in X}$.

Consider a diagram $(B, r^B) = ((B_x)_{x\in X}, (r^B_{xx'})_{x\le x'})$ and a map $g\colon B \to A$, where $g = (g_x\colon B_x \to A_x)_{x\in X}$ such that $f \circ g = 0$, i.e. $f_x \circ g_x = 0$ for every $x \in X$.

For all $x \in X$ we can use the universal property of (K_x, k_x), stating that there exists a unique map

$$\alpha_x\colon B_x \to K_x$$

such that $k_x \circ \alpha_x = g_x$. Let $\alpha = (\alpha_x)_{x\in X}$. We claim that α is a morphism in $\mathcal{A}^X$.



Since g is a morphism of diagrams, $r_{xx'} \circ g_x = g_{x'} \circ r^B_{xx'}$.

Moreover, $r_{xx'} \circ g_x = r_{xx'} \circ k_x \circ \alpha_x = k_{x'} \circ r^K_{xx'} \circ \alpha_x$ and $g_{x'} \circ r^B_{xx'} = k_{x'} \circ \alpha_{x'} \circ r^B_{xx'}$, therefore, since $k_{x'}$ is a monomorphism, we obtain $r^K_{xx'} \circ \alpha_x = \alpha_{x'} \circ r^B_{xx'}$, meaning that α is a morphism of diagrams. The uniqueness of α follows from the uniqueness of each component α_x.

(ii) The statement follows by a dual argument to (i), replacing $\ker(f_x)$ with $\mathrm{coker}(f_x)$ and using the universal property of cokernels.

(iii) By definition $\mathrm{Im} f = \ker(\mathrm{coker}(f))$, then, by the previous points, for every $x \in X$, we have:

$$(\mathrm{Im} f)_x = (\ker(\mathrm{coker} f))_x \cong \ker((\mathrm{coker} f)_x) \cong \ker(\mathrm{coker}(f_x)) = \mathrm{Im}(f_x).$$

(iv) Since by definition $H^i(K^\bullet) = \ker(d^i)/\mathrm{Im}(d^{i-1})$, by using (i), (ii) and (iii) we get

$$H^i(K^\bullet)_x = (\ker(d^i)/\mathrm{Im}(d^{i-1}))_x \cong (\mathrm{coker}(\mathrm{Im}(d^{i-1}) \hookrightarrow \ker(d^i)))_x \cong$$

$$\cong \mathrm{coker}((\mathrm{Im}(d^{i-1}))_x \hookrightarrow (\ker(d^i))_x) \cong \mathrm{coker}(\mathrm{Im}(d_x^{i-1}) \hookrightarrow (\ker(d_x^i)) \cong \ker(d_x^i)/\mathrm{Im}(d_x^{i-1}) = H^i(K_x^\bullet).$$

(v) By definition of mapping cone, $C(f)$ is the complex in $\mathcal{A}^X$ defined by

$$C(f)^n = K^{n+1} \oplus L^n, \qquad d^n_{C(f)} = \begin{pmatrix} -d_K^{n+1} & 0 \\ f^{n+1} & d_L^n \end{pmatrix}.$$

For every $x \in X$, evaluation at x sends direct sums and morphisms in $\mathcal{A}^X$ to the corresponding direct sums and morphisms in $\mathcal{A}$.

Hence

$$C(f)^n_x = (C(f)^n)_x \cong K_x^{n+1} \oplus L_x^n$$

and

$$(d^n_{C(f)})_x = \begin{pmatrix} -d_{K_x}^{n+1} & 0 \\ f_x^{n+1} & d_{L_x}^n \end{pmatrix},$$

which is the differential of the mapping cone $C(f_x)$. Therefore $C(f)_x \cong C(f_x)$ for all $x \in X$.

(vi) Using (iv) and (v), for every $i \in \mathbb{Z}$ and every $x \in X$, we have

$$H^i(C(f))_x \cong H^i(C(f)_x) \cong H^i(C(f_x)).$$

Therefore, $C(f)$ is acyclic (i.e. $H^i(C(f)) = 0$ for all i) if and only if $H^i(C(f_x)) = 0$ for all i and $x \in X$, i.e. if and only if each $C(f_x)$ is acyclic.

(vii) By Lemma 1.2.14 f is a quasi-isomorphism if and only if $C(f)$ is acyclic. By (vi), $C(f)$ is acyclic if and only if $C(f_x)$ is acyclic for every $x \in X$. Applying Lemma 1.2.14 again in $C(\mathcal{A})$, this is equivalent to f_x being a quasi-isomorphism for every $x \in X$.

□

Definition 2.1.6 (Universally derived equivalent posets)
Two posets X and Y are said to be **universally derived equivalent** if, for every abelian category $\mathcal{A}$, the categories of diagrams $\mathcal{A}^X$ and $\mathcal{A}^Y$ have equivalent derived categories (as triangulated categories), i.e.

$$D(\mathcal{A}^X) \simeq D(\mathcal{A}^Y).$$

2.2 Main theorem

Given two posets X and Y, for $y \in Y$ we define

$$[y, \bullet] := \{y' \in Y \text{ such that } y \leq y'\},$$

$$[\bullet, y] := \{y' \in Y \text{ such that } y' \leq y\}.$$

Let X and Y be two finite posets and $\{Y_x\}_{x \in X}$ a collection of subsets of Y such that the following properties hold.

(i) $\forall x \in X$, $\forall y \neq y' \in Y_x$,

$$[y, \bullet] \cap [y', \bullet] = \emptyset \quad \text{and} \quad [\bullet, y] \cap [\bullet, y'] = \emptyset;$$

(ii) $\forall x, x' \in X$ such that $x \leq x'$, we have a bijection $\varphi_{xx'} \colon Y_x \to Y_{x'}$ such that $\forall y \in Y_x$, $y \leq \varphi_{xx'}(y)$.

Now, we define two relations $\leq_+$ and $\leq_-$ on the disjoint union $X \sqcup Y$. These relations coincide with the original orders on X and on Y (i.e. if $x \leq x'$ in $[X, \leq]$, then $x \leq_+ x'$ and $x \leq_- x'$, and similarly for elements inside $[Y, \leq]$). Moreover, if $x \in X$ and $y \in Y$, we define:

$$x \leq_+ y \iff \exists\, y_x \in Y_x \text{ s.t. } y_x \leq y,$$

$$y \leq_- x \iff \exists\, y_x \in Y_x \text{ s.t. } y \leq y_x.$$

Lemma 2.2.1 — The relations $\leq_+$ and $\leq_-$ defined in the previous construction are partial orders on $X \sqcup Y$. Therefore, $(X \sqcup Y, \leq_+)$ and $(X \sqcup Y, \leq_-)$ are posets.

Proof. We first show that the element $y_x \in Y_x$ used to compare $x \in X$ with $y \in Y$ in $X \sqcup Y$ is unique (whenever it exists): for fixed $x \in X$ and $y \in Y$, there is at most one $y_x \in Y_x$ such that $y_x \leq y$. Indeed, if $y_x, y'_x \in Y_x$ satisfy $y_x \leq y$ and $y'_x \leq y$, then $y \in [y_x, \bullet] \cap [y'_x, \bullet]$, hence this intersection is nonempty; by (i) we must have $y_x = y'_x$. Analogously, there is at most one $y_x \in Y_x$ such that $y \leq y_x$.

Now, using this uniqueness, we show that $\leq_+$ is a partial order on $X \sqcup Y$. Indeed:

- For $x \in X$ we have $x \leq_+ x$ since $\leq_+$ restricts to the given order on X; similarly for $y \in Y$. It follows that $\leq_+$ is reflexive.
- To check the antisymmetry, take $a, b \in X \sqcup Y$ and assume that $a \leq_+ b$ and $b \leq_+ a$. If $a, b \in X$ or $a, b \in Y$, antisymmetry follows from antisymmetry in X or Y. Now, without loss of generality, suppose $a = x \in X$ and $b = y \in Y$. By definition of $\leq_+$, the relation $x \leq_+ y$ means that there exists $y_x \in Y_x$ s.t. $y_x \leq y$, but since by definition there are no relations from Y to X, $y \leq_+ x$ cannot occur.
- Let $a \leq_+ b$ and $b \leq_+ c$. We must show $a \leq_+ c$. If $a, b, c \in X$ or $a, b, c \in Y$, this is true because of the transitivity of X or Y. Since it is impossible that $y \leq_+ x$, with $y \in Y$ and $x \in X$, the remaining cases are two: $a, b \in X$ and $c \in Y$, and $a \in X$ and $b, c \in Y$. In the first case $a \leq_+ b$ means $a \leq b$ in X and $b \leq_+ c$ means that there exists $y_b \in Y_b$ such that $y_b \leq c$. By (ii), the bijection $\varphi_{ab} \colon Y_a \to Y_b$ satisfies $z \leq \varphi_{ab}(z)$ for all $z \in Y_a$. In particular, setting $y_a \coloneqq \varphi_{ab}^{-1}(y_b)$, we obtain

$$y_a \leq \varphi_{ab}(y_a) = y_b \leq c \Rightarrow a \leq_+ c.$$

  In the second case, where $a \in X$ and $b, c \in Y$, the conclusion $a \leq_+ c$ follows immediately from transitivity in Y, since $a \leq_+ b$ provides $y_a \in Y_a$ with $y_a \leq b \leq c$.

The proof for $\leq_-$ is analogous. □

Lemma 2.2.2 — For all $x, x', x'' \in X$ with $x \leq x' \leq x''$ we have

$$\varphi_{xx''} = \varphi_{x'x''}\varphi_{xx'}.$$

Proof. Fix $y \in Y_x$. By condition (ii) we obtain

$$y \leq \varphi_{xx'}(y) \leq \varphi_{x'x''}(\varphi_{xx'}(y)).$$

In particular, the two elements $\varphi_{xx''}(y)$ and $\varphi_{x'x''}(\varphi_{xx'}(y))$, both lying in $Y_{x''}$, belong to $[y, \bullet]$; equivalently, y belongs to the intersection between $[\bullet, \varphi_{xx''}(y)]$ and $[\bullet, \varphi_{x'x''}(\varphi_{xx'}(y))]$, forcing those elements to coincide by condition (i).

□

Example 2.2.3

Let $X \coloneqq X_1$ be the poset from Example 2.1.2. Consider the poset Y whose Hasse diagram is as follows:

$$\bullet a \longrightarrow \bullet b \longrightarrow \bullet c \longrightarrow \bullet d\,.$$

Consider the collection $\{Y_x\}_{x \in X}$ of subsets of Y defined by:

$$Y_1 = \{a\}; \quad Y_2 = Y_3 = Y_4 = \{b\}; \quad Y_5 = Y_6 = \{c\}; \quad Y_7 = \{d\}.$$

Notice that, in this particular case, all the Y_x are singletons. We can formalize these types of situations by stating that there exists a uniquely determined function $f\colon X_1 \to Y$, with $Y_x = \{f(x)\}$. Then assumption (i) is automatically satisfied, while assumption (ii) becomes $f(x) \le f(x')$ whenever $x \le x'$, that is, f is an order-preserving map.

In this singleton case, the mixed relations reduce to:

$$x \le_+ y \iff f(x) \le y,$$

$$y \le_- x \iff y \le f(x).$$

We obtain

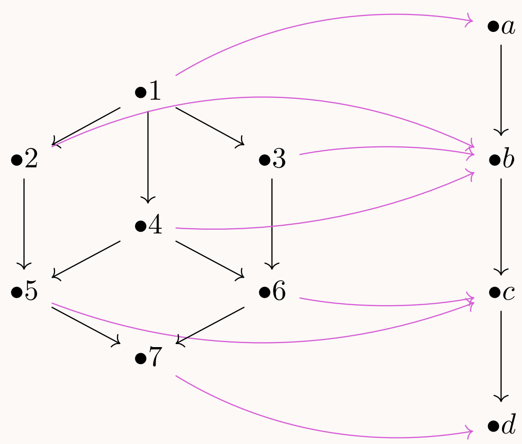
$$f(1) = a \Rightarrow 1 \le_+ a;$$

$$f(2) = f(3) = f(4) = b \Rightarrow 2 \le_+ b; 3 \le_+ b; 4 \le_+ b;$$

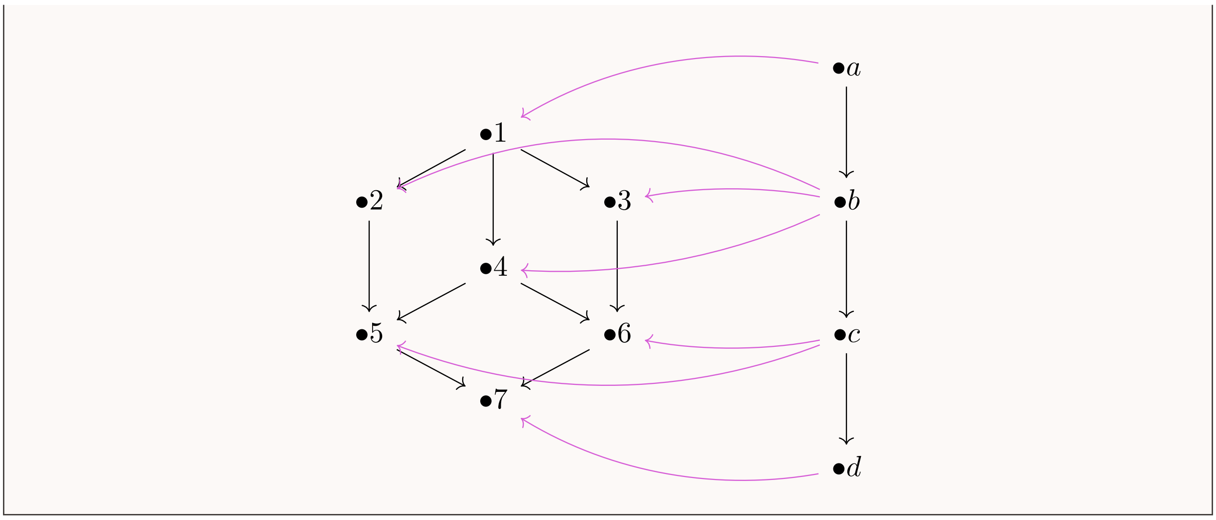
$$f(5) = f(6) = c \Rightarrow 5 \le_+ c; 6 \le_+ c;$$

$$f(7) = d \Rightarrow 7 \le_+ d.$$

Thus, $(X_1 \sqcup Y, \le_+)$ is represented by the following Hasse diagram, where the pink arrows indicate the cover relations between X and Y induced by $\le_+$.



Similarly, the Hasse diagram of $(X_1 \sqcup Y, \le_-)$ is the following.



Theorem 2.2.4 — The two posets $(X \sqcup Y, \leq_+)$ and $(X \sqcup Y, \leq_-)$ are universally derived equivalent.

Idea of the proof:
For every abelian category $\mathcal{A}$, let

$$P^+ := (X \sqcup Y, \leq_+) \qquad P^- := (X \sqcup Y, \leq_-).$$

The proof of Theorem 2.2.4 is divided into three steps.

Concretely, the goal is to show that for every abelian category $\mathcal{A}$ there is an equivalence of triangulated categories $\mathcal{D}(\mathcal{A}^{P^+}) \simeq \mathcal{D}(\mathcal{A}^{P^-})$.
The main tool is the notion of a formula, namely a combinatorial construction that associates to a diagram of complexes over a poset a new complex. We use two types of formulas:

- **formulas to a point**, inducing functors $F_{\xi,D} \colon C(\mathcal{A})^X \to C(\mathcal{A})$;
- **formulas from X to Y**, that is, Y-indexed families of formulas to a point, inducing functors $F_\xi \colon C(\mathcal{A})^X \to C(\mathcal{A})^Y$ by applying the pointwise formulas at each vertex of Y.

These functors preserve quasi-isomorphisms and therefore descend to triangulated functors on derived categories.

Subsection 2.2.2 studies the elementary case of the chain $1 \to 2$. This provides the basic local model for the general construction: two explicit formulas yield functors whose composites are naturally quasi-isomorphic to the shift functor [1].

In Subsection 2.2.3, this local construction is repeated at each vertex $x \in X$, with the subset $Y_x \subseteq Y$ playing the role of the second vertex of the chain. Using two formulas ξ^+ and ξ^-, we construct two functors

$$R^+ = F_{\xi^+} \colon C(\mathcal{A})^{P^+} \to C(\mathcal{A})^{P^-} \quad \text{and} \quad R^- = F_{\xi^-} \colon C(\mathcal{A})^{P^-} \to C(\mathcal{A})^{P^+}.$$

At vertices $y \in Y$, R^+ is the identity and R^- is the shift, while at vertices $x \in X$ both functors are defined via mapping cones. After defining the restriction maps case-by-case (inside X, inside Y, and for mixed relations), we obtain well-defined functors of diagram categories. We then verify that these assignments are compatible with identities and compositions.

Finally, using the local computations from the case $1 \to 2$, we construct two natural transformations

$$R^+R^- \Rightarrow [1] \quad \text{and} \quad [1] \Rightarrow R^-R^+,$$

whose components are quasi-isomorphisms. Passing to derived categories, we obtain

$$\tilde{R}^+\tilde{R}^- \simeq [1] \quad \text{and} \quad \tilde{R}^-\tilde{R}^+ \simeq [1].$$

Hence $\tilde{R}^+$ and $\tilde{R}^-$ are equivalences, quasi-inverse to each other up to shift, and therefore

$$D(\mathcal{A}^{P^+}) \simeq D(\mathcal{A}^{P^-}).$$

Throughout the proof, we keep in mind Example 2.2.3, which illustrates the local constructions and the different kinds of restriction maps appearing in the argument.

2.2.1 Formulas

To construct derived equivalences between diagram categories over posets, it is convenient to work with complexes and to encode the relevant functors combinatorially. For this purpose, we introduce *formulas*, which allow us to build new complexes from diagrams of complexes over a poset.

Let X be a fixed poset. Consider the category with object set $X \times \mathbb{Z}$, in which there is a unique morphism

$$(x, m) \to (x', m')$$

whenever $x \leq x'$ in X and $m \leq m'$ in $\mathbb{Z}$.

Now consider the category $\tilde{C}_X$, whose objects are finite sequences $\xi = \{(x_i, m_i)\}_{i=1}^n$, which can be regarded as finite formal sums of the generators (x, m).

Given two objects $\xi = \{(x_i, m_i)\}_{i=1}^n$ and $\xi' = \{(x'_j, m'_j)\}_{j=1}^{n'}$, a morphism $\varphi : \xi \to \xi'$ is specified by an integer matrix (c_{ji}) of size $n' \times n$, where the entry c_{ji} is as integer multiple of the morphism $(x_i, m_i) \to (x'_j, m'_j)$ whenever such a morphism exists in $\tilde{C}_X$, and is 0 otherwise.

We define the addition and the composition of morphisms in $\tilde{C}_X$ to be the usual addition and product between matrices.

The composition is well-defined. Indeed, if we have three objects $\xi = \{(x_i, m_i)\}_{i=1}^n$, $\xi' = \{(x'_j, m'_j)\}_{j=1}^{n'}$ and $\xi'' = \{(x''_k, m''_k)\}_{k=1}^{n''}$, and two morphisms $(c_{ji}) = \varphi\colon \xi \to \xi'$ and $(d_{kj}) = \psi\colon \xi' \to \xi''$, the composition $\psi \circ \varphi$ is a $n'' \times n$ matrix whose (k, i)-entry is zero unless $x_i \leq x''_k$ and $m_i \leq m''_k$. To verify the size of the matrix is trivial because of the size of the product between a $n'' \times n'$ matrix and a $n' \times n$ matrix. Now we want to verify the second feature: the entry (k, i) is given by the sum $d_{k1}c_{1i} + d_{k2}c_{2i} + \cdots + d_{kn'}c_{n'i}$, that can be different from zero only if there is at least one index r such that $d_{kr}c_{ri} \neq 0$, meaning that both d_{kr} and c_{ri} are different from zero. Since $d_{kr} \neq 0$ implies that $x'_r \leq x''_k$ and $m'_r \leq m''_k$, while $c_{ri} \neq 0$ implies that $x_i \leq x'_r$ and $m_i \leq m'_r$, by transitivity $x_i \leq x''_k$ and $m_i \leq m''_k$.

To encode the condition that morphisms shifting the second coordinate by at least 2 vanish (because squares of differentials are zero), we pass to a quotient. Let I_X be the ideal in $\tilde{C}_X$ generated by all morphisms $(x, m) \to (x, m + 2)$ for $(x, m) \in X \times \mathbb{Z}$, and set

$$C_X := \tilde{C}_X / I_X.$$

Morphisms in C_X can still be represented by integer matrices (c_{ji}) as above, viewed modulo I_X. In particular, entries with $m'_j - m_i \geq 2$ do not contribute in C_X. Hence, we will always choose representatives with $c_{ji} = 0$ whenever $m'_j - m_i \geq 2$.

Notice that both $\tilde{C}_X$ and C_X are preadditive categories, and that the natural projection functor from the first category to the second one is an additive functor.

The translation functor $[1]\colon C_X \to C_X$ is defined on objects by sending $\xi = \{(x_i, m_i)\}_{i=1}^n$ to $\xi[1] = \{(x_i, m_i + 1)\}_{i=1}^n$ and on morphisms by keeping the same coefficient matrices.

Example 2.2.5

For example, by taking $X = X_1$, where X_1 is the poset of Example 2.1.2, two possible objects in C_X can be $\xi = \{(2,0),(3,0),(5,1)\}$ and $\xi' = \{(2,1),(6,1),(5,0),(7,1)\}$, and a morphism $\varphi\colon \xi \to \xi'$ is specified by the following matrix.

$$\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ c_{31} & 0 & 0 \\ c_{41} & c_{42} & c_{43} \end{pmatrix}$$

Notice that some entries are forced to be zero because there is no morphism $(x_i, m_i) \to (x'_j, m'_j)$. For instance:

- $c_{12} = 0$ because there is no map between $(x_2, m_2) = (3,0)$ and $(x'_1, m'_1) = (2,1)$ since $x_2 = 3 \not\leq 2 = x'_1$ in X_1;
- $c_{33} = 0$ because there is no map between $(x_3, m_3) = (5,1)$ and $(x'_3, m'_3) = (5,0)$ since $m_3 = 1 > 0 = m'_3$.

Applying the translation functor, we obtain

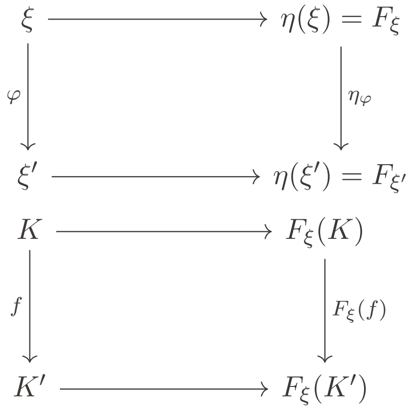
$$\xi[1] = \{(2,1),(3,1),(5,2)\} \quad \text{and} \quad \xi'[1] = \{(2,2),(6,2),(5,1),(7,2)\},$$

and the shifted morphism $\varphi[1]\colon \xi[1] \to \xi'[1]$ is represented by the same coefficient matrix as before.

Finally, for an abelian category $\mathcal{A}$ we will work with complexes of diagrams over X with values in $\mathcal{A}$. Using the natural equivalence $C(\mathcal{A}^X) \simeq C(\mathcal{A})^X$ (see 2.1.4) we identify a complex of diagrams over X with a diagram over X of complexes, and we freely switch between these viewpoints. Therefore, from now on we denote such an object simply by K, rather than $K^\bullet$.

Lemma 2.2.6 — There exists a functor $\eta\colon C_X \to Fun(C(\mathcal{A})^X, C(\mathcal{A}))$ that commutes with the shift functor [1], where $Fun(C(\mathcal{A})^X, C(\mathcal{A}))$ denotes the category of additive functors from $C(\mathcal{A})^X$ to $C(\mathcal{A})$.

Proof. We write K_x for the complex obtained by evaluating $K \in Ob(C(\mathcal{A})^X)$ at $x \in X$. For a complex K, the shifted complex $K[1]$ has $(K[1])^i = K^{i+1}$ and differential $d^i_{K[1]} = -d^{i+1}_K$, then $(K_x[m])^i = K^{i+m}_x$ and $d^i_{K_x[m]} = (-1)^m d^{i+m}_{K_x}$. We denote by $d_x[m]$ the differential on the complex $K_x[m]$, and by $r_{xx'}[m]$ the chain map $K_x[m] \to K_{x'}[m]$, obtained by shifting the restriction chain map $r_{xx'}\colon K_x \to K_{x'}$ of the diagram. For a morphism $f\colon K \to K'$ we write $f_x\colon K_x \to K'_x$ for its component at $x \in X$.



Step 1 Given an object $\xi = \{(x_i, m_i)\}_{i=1}^n$ in the category C_X, we define an additive functor $F_\xi : C(\mathcal{A})^X \to C(\mathcal{A})$ by setting

$$F_\xi(K) = \bigoplus_{i=1}^n K_{x_i}[m_i].$$

On a morphism $f \colon K \to K'$ we define

$$F_\xi(f) = \bigoplus_{i=1}^n f_{x_i}[m_i].$$

Since f is a morphism of complexes in $\mathcal{A}^X$, each component f_{x_i} is a chain map, hence so is $f_{x_i}[m_i]$, and therefore $F_\xi(f)$ is a chain map.

$F_\xi(f)$ can be viewed as a $n \times n$ diagonal matrix with entry $f_{x_i}[m_i]$ in position (i, i).

Example 2.2.7

For instance, taking $\xi = \{(2, 0), (3, 0), (5, 1)\}$ (as in 2.2.5), we obtain:

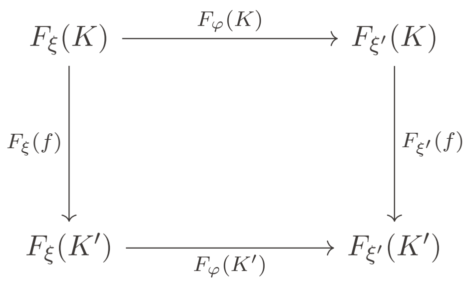
$$F_\xi(K) = K_2 \oplus K_3 \oplus K_5[1],$$

$$F_\xi(f) = f_2 \oplus f_3 \oplus f_5[1],$$

represented by the following diagonal matrix.

$$\begin{pmatrix} f_2 & 0 & 0 \\ 0 & f_3 & 0 \\ 0 & 0 & f_5[1] \end{pmatrix}$$

Step 2 Given a morphism $\varphi \colon \xi \to \xi'$ in the category C_X, we have to define a natural transformation $\eta(\varphi) := \eta_\varphi = F_\varphi \colon F_\xi \Rightarrow F_{\xi'}$ (since morphisms in the category $Fun(C(\mathcal{A})^X, C(\mathcal{A}))$ are natural transformations). This means that the following diagram commutes for every K, K' and $f \colon K \to K'$ in $C(\mathcal{A})^X$



- First consider the case where $\xi = (x, m)$ and $\xi' = (x', m')$. Then a morphism $\varphi \colon \xi \to \xi'$ is given by a 1×1 matrix (hence an integer c), which can be non-zero only if $x \leq x'$ and $m' \in \{m, m+1\}$.

  In this case $F_\xi(K) = K_x[m]$ and $F_{\xi'}(K) = K_{x'}[m']$, thus the component $\eta_\varphi(K) = F_\varphi(K)$ is a morphism $K_x[m] \to K_{x'}[m']$.

  We define

$$\eta_\varphi(K) = \begin{cases} c \cdot r_{xx'}[m] & \text{if } m' = m \text{ and } x \leq x' \\ c \cdot d_{x'}[m] \circ r_{xx'}[m] & \text{if } m' = m + 1 \text{ and } x \leq x' \\ 0 & \text{otherwise.} \end{cases}$$

Applying the first two cases to the previous diagram we obtain:

$$\begin{array}{ccc} K_x[m] & \xrightarrow{c\cdot r_{xx'}[m]} & K_{x'}[m] \\ {\scriptstyle f_x[m]}\downarrow & & \downarrow{\scriptstyle f_{x'}[m]} \\ K'_x[m] & \xrightarrow[c\cdot r'_{xx'}[m]]{} & K'_{x'}[m] \end{array} \qquad \begin{array}{ccc} K_x[m] & \xrightarrow{c\cdot d_{x'}[m]r_{xx'}[m]} & K_{x'}[m+1] \\ {\scriptstyle f_x[m]}\downarrow & & \downarrow{\scriptstyle f_{x'}[m+1]} \\ K'_x[m] & \xrightarrow[c\cdot d'_{x'}[m]r'_{xx'}[m]]{} & K'_{x'}[m+1]. \end{array}$$

Both squares commute since f is a morphism in $C(\mathcal{A})^X$, hence for every $x \le x'$ in X we have $r'_{xx'} \circ f_x = f_{x'} \circ r_{xx'}$, and for every x the map f_x is a chain map. By applying the shift, we obtain $r'_{xx'}[m] \circ f_x[m] = f_{x'}[m] \circ r_{xx'}[m]$ and $d'_{x'}[m] \circ f_{x'}[m] = f_{x'}[m+1] \circ d_{x'}[m]$. The commutativity of the remaining case is trivial since the horizontal maps are zero. Hence η_φ is natural.

To show that η is a functor, we have to check that it respects composition. Let $\varphi = (c)\colon (x, m) \to (x', m')$ and $\varphi' = (c')\colon (x', m') \to (x'', m'')$ be morphisms in C_X. We claim that for every K we have $\eta_{\varphi'\circ\varphi}(K) = \eta_{\varphi'}(K) \circ \eta_\varphi(K)$.

We distinguish three cases according to the degrees.

If $m'' = m$, then both sides reduce to $cc' \cdot r_{xx''}[m]$, using the relation $r_{xx''} = r_{x'x''} \circ r_{xx'}$.

If $m'' = m + 1$ (exactly one among φ and φ' raises the degree by 1), there are two sub-cases. If $m' = m$ and $m'' = m + 1$, the right-hand side is $c' \cdot d_{x''}[m] \circ r_{x'x''}[m] \circ c \cdot r_{xx'}[m] = cc' \cdot d_{x''}[m] \circ r_{xx'}[m]$, which equals the left-hand side. If $m' = m + 1$ and $m'' = m + 1$, then $m'' = m'$ and the right-hand side is $c' \cdot r_{x'x''}[m + 1] \circ c \cdot d_{x'}[m] \circ r_{xx'}[m] = cc' \cdot r_{x'x''}[m + 1] \circ d_{x'}[m] \circ r_{xx'}[m]$. Since $r_{x'x''}$ is a chain map, we can rewrite $r_{x'x''}[m + 1] \circ d_{x'}[m] = d_{x''}[m] \circ r_{x'x''}[m]$, and composing with $r_{xx'}[m]$ gives $cc' \cdot d_{x''}[m] \circ r_{x'x''}[m]$, which again equals the left-hand side.

If $m'' = m + 2$, then the composite $\varphi' \circ \varphi$ vanishes in C_X (since it factors through a shift of $+2$ which is zero modulo I_X), and the right-hand side also vanishes because $d_{x''}[m + 1] \circ d_{x''}[m] = 0$. Indeed, the only way to get $m'' = m + 2$ is $m' = m + 1$, in which case the composite involves $d_{x''}[m + 1] \circ r_{x'x''}[m + 1] \circ d_{x'}[m] \circ r_{xx'}[m]$; since r is a chain map this equals $d_{x''}[m + 1] \circ d_{x''}[m] \circ r_{xx'}[m] = 0$.

- Now let $\xi = \{(x_i, m_i)\}_{i=1}^n$ and $\xi' = \{(x'_j, m'_j)\}_{j=1}^{n'}$, and let $\varphi\colon \xi \to \xi'$ be represented by an integer matrix $(c_{j,i})$ with $1 \le i \le n$ and $1 \le j \le n'$. For $K \in Ob(C(\mathcal{A})^X)$,

$$F_\xi(K) = \bigoplus_{i=1}^{n} K_{x_i}[m_i] \quad \text{and} \quad F_{\xi'}(K) = \bigoplus_{j=1}^{n'} K_{x'_j}[m'_j],$$

hence $\eta_\varphi(K) = F_\varphi(K) : F_\xi(K) \to F_{\xi'}(K)$ can be written as an $n' \times n$ matrix of morphisms.

We define its (j, i)-entry by

$$(\eta_\varphi(K))_{ji} \coloneqq \eta_{(c_{ji})}(K)\colon K_{x_i}[m_i] \to K_{x'_j}[m'_j],$$

where $\eta_{(c_{ji})}(K)$ is defined above as in the singleton case.

Since naturality and compatibility with composition hold for each entry, they hold for η_φ.

Indeed, let $f\colon K \to K'$ be a morphism in $C(\mathcal{A})^X$. The morphisms $F_\xi(f)$ and $F_{\xi'}(f)$ are diagonal matrices whose diagonal entries are $f_{x_i}[m_i]$ and $f_{x'_j}[m'_j]$, respectively. Therefore the (j, i)-entry of the composition $F_{\xi'}(f) \circ \eta_\varphi(K)$ is

$$f_{x'_j}[m'_j] \circ (\eta_\varphi(K))_{ji} = f_{x'_j}[m'_j] \circ \eta_{c_{ji}},$$

while the (j,i)-entry of $\eta_\varphi(K')F_\xi(f)$ is

$$(\eta_\varphi(K'))_{ji} \circ f_{x_i}[m_i] = \eta_{c_{ji}}(K') \circ f_{x_i}[m_i].$$

These two morphisms coincide by the naturality of the singleton case $\eta_{c_{ji}}$. Hence $F_{\xi'}(f) \circ \eta_\varphi(K) = \eta_\varphi(K') \circ F_\xi(f)$.
Similarly, for composable $\varphi\colon \xi \to \xi'$ and $\varphi'\colon \xi' \to \xi''$, the (k,i)-entry of $\eta_{\varphi'\circ\varphi}(K)$ is computed by matrix multiplication as a sum over j, and each summand reduces to the singleton compatibility with composition, therefore $\eta_{\varphi'\circ\varphi}(K) = \eta_{\varphi'}(K) \circ \eta_\varphi(K)$.

Example 2.2.8

Let $\xi = \{(2,0),(3,0),(5,1)\}$ and $\xi' = \{(2,1),(6,1),(5,0),(7,1)\}$, with $X = X_1$ as above. As explained in (2.2.5), $\varphi\colon \xi \to \xi'$ is described by

$$\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ c_{31} & 0 & 0 \\ c_{41} & c_{42} & c_{43} \end{pmatrix}.$$

Hence $\eta_\varphi(K)\colon F_\xi(K) = K_2 \oplus K_3 \oplus K_5[1] \to K_2[1] \oplus K_6[1] \oplus K_5 \oplus K_7[1]$ corresponds to the matrix

$$\begin{pmatrix} c_{11}\cdot d_2 & 0 & 0 \\ 0 & c_{22}\cdot d_6 r_{36} & 0 \\ c_{31}\cdot r_{25} & 0 & 0 \\ c_{41}\cdot d_7 r_{27} & c_{42}\cdot d_7 r_{37} & c_{43}\cdot r_{57}[1] \end{pmatrix}.$$

Step 3 Finally, we check compatibility with the shift functor [1]. For every $K \in Ob(C(\mathcal{A})^X)$ we have:

$$([1]\circ F_\xi)(K) = (F_\xi(K))[1] = \left(\bigoplus_{i=1}^n K_{x_i}[m_i]\right)[1] = \bigoplus_{i=1}^n K_{x_i}[m_i+1] = F_{\xi[1]}(K).$$

On the other hand,

$$(F_\xi \circ [1])(K) = F_\xi(K[1]) = \bigoplus_{i=1}^n (K[1])_{x_i}[m_i] = \bigoplus_{i=1}^n K_{x_i}[m_i+1],$$

so $[1]\circ F_\xi = F_\xi \circ [1] = F_{\xi[1]}$.
Similarly, for a morphism $\varphi\colon \xi \to \xi'$ the natural transformation η_φ is defined entrywise in terms of restriction maps and differentials, and these constructions commute with shifting. Hence $([1]\circ\eta_\varphi)_K = (\eta_\varphi)_K[1] = (\eta_{\varphi[1]})_K$ and $(\eta_\varphi)_{K[1]} = (\eta_\varphi \circ [1])_K$, which yields $[1]\circ\eta_\varphi = \eta_\varphi\circ[1] = \eta_{\varphi[1]}$.

□

So far, for a fixed $\xi = \{(x_i,m_i)\}_{i=1}^n$, the complex $F_\xi(K) = \oplus_{i=1}^n K_{x_i}[m_i]$ carries only the differential $\oplus_{i=1}^n d_{x_i}[m_i]$. For our purposes, we need to allow more general differentials, mixing the summands through restriction maps. This is encoded by a morphism $\varphi\colon \xi \to \xi'$ in C_X via the natural transformation η_φ.

Let $\varphi = (c_{ji})\colon \xi \to \xi'$ be a morphism in C_X. We now define a morphism $\varphi^\star\colon \xi \to \xi'$ by setting

$$c^\star_{ji} \coloneqq (-1)^{m'_j - m_i} c_{ji}.$$

It means that $\varphi^\star$ is obtained from φ by multiplying each component by the sign resulting from the shift.

Definition 2.2.9 (Restriction and differentials in C_X)

Let $\varphi = (c)\colon (x, m) \to (x', m')$ be a morphism in C_X.

We call φ a **differential** if $x' = x, m' = m + 1$ and $c = 1$.

We call φ a **restriction** if $m' = m$ and $x \le x'$.

More generally, a morphism $\varphi\colon \xi \to \xi'$ (given by an integer matrix (c_{ji})) is called **restriction** if all its nonzero entries are restrictions in the above sense.

Definition 2.2.10 (The category of formulas to a point)

A **formula to a point** is a pair (ξ, D), where $\xi = \{(x_i, m_i)\}_{i=1}^n$ is an object of C_X and $D = (D_{j,i})_{1\le j,i\le n}\colon \xi \to \xi[1]$ is a morphism in C_X satisfying:

(i) $D^\star[1] \circ D = 0$ in C_X;

(ii) $D_{ji} = 0$ for all $i > j$ (i.e. D is lower triangular);

(iii) the diagonal entries D_{ii} are differentials for all $1 \le i \le n$.

Given two formulas to a point (ξ, D) and (ξ', D'), a **morphism of formulas to a point** $\varphi\colon (\xi, D) \to (\xi', D')$ is a morphism $\varphi\colon \xi \to \xi'$ in C_X such that:

- φ is a restriction;
- $D' \circ \varphi = \varphi[1] \circ D$.

Formulas to a point, together with these morphisms, form a category which we denote by $\mathcal{F}_X$.

The shift functor on C_X induces a shift functor $[1]$ on $\mathcal{F}_X$ that shifts both components: $(\xi, D)[1] \coloneqq (\xi[1], D[1])$, and for a morphism φ, $\varphi[1]$ is its image under the shift functor $[1]\colon C_X \to C_X$ (hence is represented by the same coefficient matrix of φ).

Lemma 2.2.11 — Let $D\colon \xi \to \xi[1]$ be a morphism in C_X such that $D^\star[1] \circ D = 0$ in C_X. Then, for every $K \in C(\mathcal{A})^X$, the morphism $\eta_D(K)\colon F_\xi(K) \to F_\xi(K)[1]$ is a differential, i.e

$$\eta_D(K)[1] \circ \eta_D(K) = 0.$$

Proof. Let $\xi = \{(x_i, m_i)\}_{i=1}^n$ in C_X. As we saw in the first step of the proof of Lemma 2.2.6, ξ determines a functor

$$F_\xi\colon C(\mathcal{A})^X \to C(\mathcal{A})$$
$$K \mapsto \bigoplus_{i=1}^n K_{x_i}[m_i].$$

Moreover, by the second step of the same lemma, every morphism $\varphi\colon \xi \to \xi'$ in C_X induces a natural transformation $\eta_\varphi\colon F_\xi \Rightarrow F_{\xi'}$.

In our case, $\xi' = \xi[1]$, and $\varphi = D\colon \xi \to \xi[1]$, hence we obtain a natural transformation $\eta_D\colon F_\xi \Rightarrow F_{\xi[1]}$.

Since $F_{\xi[1]}(K) = F_\xi(K)[1]$ (by the third step of lemma 2.2.6), for every $K \in C(\mathcal{A})^X$ we obtain a morphism

$$\eta_D(K)\colon F_\xi(K) \to F_\xi(K)[1].$$

To prove that $\eta_D(K)$ is a differential, we need to show that $\eta_D(K)[1] \circ \eta_D(K) = 0$.

We first prove the identity:

$$-(\eta_D(K)[1]) = \eta_{-D^\star[1]}(K). \tag{2.1}$$

Let $D = (D_{ji})_{1\le i,j\le n}$ and fix indices (j,i). Let $c := D_{ji} \in \mathbb{Z}$ be the corresponding coefficient, so that the (j,i)-component of $\eta_D(K)$ is the map

$$(\eta_D(K))_{ji} = \eta(c)(K)\colon K_{x_i[m_i]} \to K_{x_j}[m_j+1]$$

defined by Lemma 2.2.6. Since C_X is a quotient of $\tilde{C}_X$ such that shifts ≥ 2 vanish, we have $m_j + 1 - m_i \in \{0,1\}$ whenever $c \neq 0$.

Recall that the differential on a shifted complex satisfies $d_{C[1]} = -d_C[1]$. Therefore, if we view $\eta_D(K)$ as the differential on $F_\xi(K)$, the differential on the shifted complex $F_\xi(K)[1]$ is given by $-\eta_D(K)[1]$.

Now we compute the (j,i)-component case by case:

- <u>case $m_j + 1 = m_i$ (restriction):</u>

$$(-(\eta_D(K)[1]))_{ji} = -(c \cdot r_{x_ix_j}[m_i])[1] = -c \cdot r_{x_ix_j}[m_i+1].$$

- <u>case $m_j + 1 = m_i + 1$ (composite differential and restriction)</u>.

$$(-(\eta_D(K)[1]))_{ji} = -(c{\cdot}d_{x_j}[m_i]{\circ}r_{x_ix_j}[m_i])[1] = -c{\cdot}(d_{x_j}[m_i])[1]{\circ}r_{x_ix_j}[m_i{+}1] = c{\cdot}d_{x_j}[m_i{+}1]{\circ}r_{x_ix_j}[m_i{+}1].$$

Then, to show that $\eta_D(K)$ is a differential, we need $\eta_D(K)[1] \circ \eta_D(K) = 0$. This is equivalent to showing that $-(\eta_D(K)[1]) \circ \eta_D(K) = 0$.

By definition of $c^\star$, for any entry $(x_i, m_i) \to (x_j, m_j+1)$ we have $c^\star = (-1)^{(m_j+1)-m_i}c$.

To explicitly verify equation (2.1), we want to compute the (j,i)-entry of the right-hand side, $\eta_{-D^\star}(K)$. Recall that the shift functor [1] on C_X does not change the coefficient matrix. Thus, the coefficient for the (j,i)-entry of $-D^\star[1]$ is exactly $-c^\star_{ji} = -(-1)^{(m_j+1)-m_i}c$. Furthermore, this morphism evaluates from the i component of $\xi[1]$ (which has degree m_i+1) to the j component (which has degree $(m_j+1)+1 = m_j+2$). Let us apply the rules of Lemma 2.2.6 to this map:

- <u>case $m_j + 1 = m_i$ (restriction)</u>: the coefficient evaluates to $-(-1)^0c = -c$. The target degree m_j+2 is equal to the source degree m_i+1. According to lemma 2.2.6, this corresponds to a restriction map, yielding:

$$(\eta_{-D^\star}(K))_{ji} = -c \cdot r_{x_ix_j}[m_i+1].$$

- <u>case $m_j + 1 = m_i + 1$ (composite differential and restriction)</u>: the coefficient evaluates to $-(-1)^1c = c$. The target degree m_j+2 is equal to the source degree m_i+1 plus one. According to Lemma 2.2.6, this corresponds to the composite of a differential and a restriction, yielding:

$$(\eta_{-D^\star[1]}(K))_{ji} = c \cdot d_{x_j}[m_i+1] \circ r_{x_ix_j}[m_i+1].$$

Hence (2.1) holds.

By applying equation (2.1), we obtain:

$$-(\eta_D(K)[1]) \circ \eta_D(K) = \eta_{-D^\star[1]}(K) \circ \eta_D(K) = \eta_{-D^\star[1]\circ D}(K).$$

Therefore $\eta_D(K)[1] \circ \eta_D(K) = 0$, as required. In particular, the statement applies to every formula to a point (ξ, D), since condition (i) in definition 2.2.10 is precisely $D^\star[1] \circ D = 0$.

□

We introduced the category $\mathcal{F}_X$ of formulas to a point, and now we want to show that each formula (ξ, D) canonically produces a functor

$$F_{\xi,D}\colon C(\mathcal{A})^X \to C(\mathcal{A}),$$

by equipping $F_\xi(K)$ with the differential $\eta_D(K)$. Moreover, morphisms in $\mathcal{F}_X$ induce natural transformations between the corresponding functors.

Proposition 2.2.12 — There exists a functor $\eta\colon \mathcal{F}_X \to Fun(C(\mathcal{A})^X, C(\mathcal{A}))$.

Proof. We will construct this functor by adapting the functor $\eta\colon C_X \to Fun(C(\mathcal{A})^X, C(\mathcal{A}))$ constructed in Lemma 2.2.6. With a slight (but standard) abuse of notation, we will denote this new functor by the same symbol η, since its evaluation on morphisms relies entirely on the original one.

Step 1: (Action on objects.) Let (ξ, D) be a formula to a point in $\mathcal{F}_X$. We define an additive functor $F_{\xi,D}\colon C(\mathcal{A})^X \to C(\mathcal{A})$ as follows:

- For an object K in $C(\mathcal{A})^X$, we set $\mathcal{F}_{\xi,D}(K) = F_\xi(K)$ and we equip it with the differential $d_{F_{\xi,D}(K)} := \eta_D(K)$. By Lemma 2.2.11, $\eta_D(K)[1] \circ \eta_D(K) = 0$, hence $F_{\xi,D}$ is a well-defined complex in $C(\mathcal{A})$.
- For a morphism $f\colon K \to K'$ in $C(\mathcal{A})^X$, we define $F_{\xi,D}(f) := F_\xi(f)$. We must verify that $F_\xi(f)$ is a chain map between the complexes $(F_{\xi,D}(K), \eta_D(K))$ and $(F_{\xi,D}(K')), \eta_D(K'))$. This is equivalent to showing that it commutes with the differentials:

$$F_\xi(f)[1] \circ \eta_D(K) = \eta_D(K') \circ F_\xi(f).$$

Recall from Lemma 2.2.6 that $\eta_D\colon F_\xi \Rightarrow F_{\xi[1]}$ is a natural transformation. Since $F_{\xi[1]} = F_\xi \circ [1]$, by applying the morphism $f\colon K \to K'$ to the component η_D, by naturality we obtain exactly the equation written above. Thus, $F_{\xi,D}(f) = F_\xi(f)$ is a chain map, hence $F_{\xi,D}$ is well-defined on morphisms.

Step 2: (Action on morphisms.) Let $\varphi\colon (\xi, D) \to (\xi', D')$ be a morphism of formulas to a point. By definition 2.2.10, $\varphi\colon \xi \to \xi'$ is a morphism in C_X satisfying $\varphi[1] \circ D = D' \circ \varphi$. From lemma 2.2.6, φ induces a natural transformation $\eta_\varphi\colon F_\xi \Rightarrow F_{\xi'}$. We want to show that η_φ extends to a natural transformation between our new functors, $\eta_\varphi\colon F_{\xi,D} \Rightarrow F_{\xi',D'}$.

In order to do this, we need to check that for any complex K in $C(\mathcal{A})^X$, the map $\eta_\varphi(K)\colon F_{\xi,D}(K) \to F_{\xi',D'}(K)$ is a chain map. This means it must commute with the respective differentials $\eta_D(K)$ and $\eta_{D'}(K)$, i.e.

$$\eta_{D'}(K) \circ \eta_\varphi(K) = \eta_\varphi(K)[1] \circ \eta_D(K). \tag{2.2}$$

To prove this, note that, by functoriality of η on C_X (proved in Lemma 2.2.6), we have:

$$\eta_{D'}(K) \circ \eta_\varphi(K) = \eta_{D' \circ \varphi}(K),$$

which is equal to $\eta_{\varphi[1] \circ D}(K)$ because of the definition of morphism of formulas to a point. By using again the functoriality of η, we can split this, and we obtain

$$\eta_{D'}(K) \circ \eta_\varphi(K) = \eta_{\varphi[1]}(K) \circ \eta_D(K).$$

Finally, by Lemma 2.2.6, η commutes with the shift functor, meaning $\eta_{\varphi[1]}(K) = (\eta_\varphi(K))[1] = \eta_\varphi(K)[1]$. This concludes the proof of the equation 2.2, so $\eta_\varphi(K)$ is a chain map. The naturality of η_φ and the functoriality on the assignment $(\xi, D) \mapsto F_{\xi,D}$ follow immediately from the functoriality of η on C_X.

□

To illustrate how formulas to a point work in practice, we show that they can recover two standard homological constructions with two examples.

Example 2.2.13 (Evaluation at a point - zero-dimensional chain)
Let $x \in X$ be a fixed element. Consider the sequence $\xi = \{(x, 0)\}$ and the 1×1 matrix $D = (1) \colon \xi \to \xi[1]$. The pair (ξ, D) trivially satisfies the conditions of Definition 2.2.10. For any complex of diagrams $K \in Ob(C(\mathcal{A})^X)$, the induced functor yields

$$F_{\xi,D}(K) = F_\xi(K) = F_{\{(x,0)\}}(K) = K_x[0] = K_x,$$

with differential given by

$$\eta_D(K) = 1 \cdot d_x[0] = d_x.$$

Let $f \colon K \to K'$, then

$$F_{\xi,D}(f) = F_\xi(f) = F_{\{(x,0)\}}(f) = f_x.$$

Thus, this formula simply acts as the evaluation functor, extracting the complex K_x, and for morphisms it acts componentwise.

Example 2.2.14 (The mapping cone formula - one dimensional chain)
Now consider two elements $x < y$ in X. We define the sequence $\xi = \{(x, 1), (y, 0)\}$ and the morphism $D \colon \xi \to \xi[1]$ given by the matrix $D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let us verify that (ξ, D) satisfies the condition of Definition 2.2.10 to be a formula to a point:

(i) We must check that $D^\star[1] \circ D = 0$ in C_X. Recall that $c^\star_{ji} = (-1)^{m'_j - m_i} c_{ji}$. Since D maps $\xi = \{(x, 1), (y, 0)\}$ to $\xi[1] = \{(x, 2), (y, 1)\}$, we compute the signs:

- for $c_{11} \colon (x, 1) \to (x, 2)$, we have $m'_1 = 2$ and $m_1 = 1$, hence $c^\star_{11} = (-1)^{2-1} = -1$;
- for $c_{21} \colon (x, 1) \to (y, 1)$, we have $m'_2 = 1$ and $m_1 = 1$, hence $c^\star_{21} = (-1)^{1-1} = 1$;
- for $c_{22} \colon (y, 0) \to (y, 1)$, we have $m'_2 = 1$ and $m_2 = 0$, hence $c^\star_{22} = (-1)^{1-0} = -1$.

Therefore $D^\star = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. Computing the matrix product yields:

$$D^\star[1] \circ D = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that the category C_X is the quotient $\tilde{C}_X / I_X$, where I_X is generated by morphisms shifting the second coordinate by ≥ 2. The map $D^\star[1] \circ D$ goes from ξ to $\xi[2]$. The diagonal entries represent the opposites of the morphisms $(x, 1) \to (x, 3)$ and $(y, 0) \to (y, 2)$, both of which contain a $+2$ shift and thus they vanish modulo I_X. Hence $D^\star[1] \circ D = 0$ in C_X.

(ii) It is evident that D is lower triangular.

(iii) The diagonal entries are $c_{11} = 1 \colon (x, 1) \to (x, 2)$ and $c_{22} = 1 \colon (y, 0) \to (y, 1)$. Both represent shifts of exactly $+1$ on the same element from the poset, so they are differentials in C_X.

Finally, we apply this formula to a complex $K \in Ob(C(\mathcal{A})^X)$ and to a morphism $f \colon K \to K'$ in $C(\mathcal{A})^X$. We obtain:

$$F_{\xi,D}(K) = F_\xi(K) = \bigoplus_{i=1}^{2} K_{x_i}[m_i] = K_x[1] \oplus K_y[0] = K_x[1] \oplus K_y$$

and

$$F_{\xi,D}(f) = F_\xi(f) = \bigoplus_{i=1}^{2} f_{x_i}[m_i] = f_x[1] \oplus f_y \cong \begin{pmatrix} f_x[1] & 0 \\ 0 & f_y \end{pmatrix}.$$

To compute the differentials $\eta_D(K)\colon K_x[1] \oplus K_y \to K_x[2] \oplus K_y[1]$, we apply the rules established in lemma 2.2.6 to the entries of D :

- for (1, 1)-entry: $c_{11} = 1\colon (x, 1) \to (x, 2)$, with $m' = m + 1$, yields $d_x[1]$;
- for (2, 1)-entry: $c_{21} = 1\colon (x, 1) \to (y, 1)$, with $m' = m$ and $x < y$, is a restriction map, and yields $r_{xy}[1]$;
- for (2, 2)-entry: $c_{22} = 1\colon (y, 0) \to (y, 1)$, with $m' = m + 1$, yields $d_y[0] = d_y$.

Assembling all the components, we obtain the matrix

$$\eta_D(K) = \begin{pmatrix} d_x[1] & 0 \\ r_{xy}[1] & d_y \end{pmatrix}.$$

This structure corresponds to the definition of the mapping cone $C(r_{xy})$ of the restriction map $r_{xy}\colon K_x \to K_y$. Thus, for any $x < y$, the formula $F_{\xi,D}$ builds the mapping cone $C(K_x \xrightarrow{r_{xy}} K_y)$ in $C(\mathcal{A})$.

As a concrete application, we consider our running example poset X_1 defined in the Example 2.1.2. Choosing the relation $4 < 5$, the previous formula becomes $\xi = \{(4, 1), (5, 0)\}$ and we have the same matrix as before. For any complex of diagrams $K \in Ob(C(\mathcal{A})^{X_1})$, this formula yields the complex $F_{\xi,D}(K) = K_4[1] \oplus K_5$ equipped with the differential

$$\eta_D(K) = \begin{pmatrix} d_4[1] & 0 \\ r_{45}[1] & d_5 \end{pmatrix}.$$

This explicitly constructs the mapping cone $C(K_4 \xrightarrow{r_{45}} K_5)$ in $C(\mathcal{A})$ associated to the restriction map r_{45} in the poset X_1.

Lemma 2.2.15 — There exists a natural isomorphism $\epsilon\colon [1] \circ \eta \to \eta \circ [1]$.

Proof. We must construct a natural isomorphism $\epsilon_{\xi,D}$ for each object (ξ, D) in $\mathcal{F}_X$. First, we explicitly evaluate the domain and the codomain of this isomorphism on a complex $K \in Ob(C(\mathcal{A})^X)$:

- The composition $(\eta \circ [1])$ applied to (ξ, D) yields the functor $F_{\xi[1],D[1]}$. Evaluating on K yields $F_{\xi[1],D[1]}(K) = F_{\xi,D}(K[1])$.
- The composition $([1] \circ \eta)$ applied to (ξ, D) yields the functor $[1] \circ F_{\xi,D}$. Evaluating this on K shifts the resulting complex and gives $F_{\xi,D}(K)[1]$. Recall from the equation 2.1 in the proof of 2.2.11 that the differential on the shifted complex $F_{\xi,D}(K)[1]$ is $\eta_{-D^\star[1]}(K)$. Therefore, the complex $F_{\xi,D}(K)[1]$ coincides with $F_{\xi[1],-D^\star[1]}(K)$.

Thus, defining a natural isomorphism of functors $\epsilon_{\xi,D}\colon [1] \circ F_{\xi,D} \to F_{\xi,D} \circ [1]$ is equivalent to defining an isomorphism of formulas in $\mathcal{F}_X$ between $(\xi[1], -D^\star[1])$ and $(\xi[1], D[1])$.

Step 1: (Construction of the isomorphism.) Let $\xi = \{(x_i, m_i)\}_{i=1}^n$ and $D = (D_{ji})_{i,j=i}^n$. We define a morphism $I_\xi\colon \xi \to \xi$ in C_X given by the diagonal matrix whose (i, i)-entry is $(-1)^{m_i}$. Since I_ξ acts as the identity up to a sign on each component, it is its own inverse, and thus is an isomorphism in C_X.

To show that $I_\xi[1]$ defines a morphism of formulas from $(\xi[1], -D^\star[1])$ to $(\xi[1], D[1])$, it must be a restriction (that is trivial because it is diagonal) and it must commute with the differentials according to Definition 2.2.10:

$$(I_\xi[1])[1] \circ (-D^\star[1]) = D[1] \circ I_\xi[1].$$

Since applying the shift functor to morphisms in C_X does not change their coefficient matrices, this reduces to verifying the matrix equation $I_\xi \circ (-D^\star) = D \circ I_\xi$, or equivalently $-I_\xi \circ D^\star = D \circ I_\xi$.

Let us compute the (j,i)-entry on both sides. Recall that D is a morphism from ξ to $\xi[1]$, so its target degree is $m_j + 1$. By the definition of the twisted matrix, we have $D^\star_{ji} = (-1)^{(m_j+1)-m_i} Dji = -(-1)^{m_j-m_i} D_{ji}$.

- The (j,i)-entry of the left side $-I_\xi \circ D^\star$ is:

$$-(-1)^{m_j} \cdot D^\star_{ji} = -(-1)^{m_j}(-(-1)^{m_j-m_i} D_{ji}) = (-1)^{2m_j-m_i} D_{ji} = (-1)^{-m_i} D_{ji} = (-1)^{m_i} D_{ji}.$$

- The (j,i)-entry of the right side $D \circ I_\xi$ is:

$$D_{ji} \cdot (-1)^{m_i} = (-1)^{m_i} D_{ji}.$$

The two sides match, so $I_\xi[1]$ is an isomorphism in $\mathcal{F}_X$. We define $\epsilon_{\xi,D} := \eta_{I_\xi[1]}$.

Step 2: (Naturality.) To prove that ϵ is a natural transformation, we need to verify that the following diagram commutes for every morphism of formulas $\varphi\colon (\xi, D) \to (\xi', D')$ and for every $K \in Ob(C(\mathcal{A})^X)$.

$$\begin{array}{ccc} F_{\xi,D}(K)[1] & \xrightarrow{\epsilon_{\xi,D}} & F_{\xi[1],D[1]}(K) \\ \Big\downarrow{\scriptstyle \eta_\varphi[1]} & & \Big\downarrow{\scriptstyle \eta_{\varphi[1]}} \\ F_{\xi',D'}(K)[1] & \xrightarrow[\epsilon_{\xi',D'}]{} & F_{\xi'[1],D'[1]}(K) \end{array}$$

Recall that $\eta_\varphi[1] = [1] \circ \eta_\varphi = \eta_\varphi \circ [1] = \eta_{\varphi[1]}$. By functoriality of η (see Lemma 2.2.6), to check that the above square commutes, we can check that $I_{\xi'} \circ \varphi = \varphi \circ I_\xi$ in C_X. φ is represented by the matrix (c_{ji}), while I_ξ and $I_{\xi'}$ are diagonal matrices whose (i,i) and (j,j)-entries are, respectively, $(-1)^{m_i}$ and $(-1)^{m'_j}$. Then the (j,i)-entry of $\varphi \circ I_\xi$ is $c_{ji} \cdot (-1)^{m_i}$ and the (j,i)-entry of $I_{\xi'} \circ \varphi$ is $(-1)^{m'_j} \cdot c_{ji}$. By Definition 2.2.10, the morphism of formulas φ must be a restriction, so a non-zero entry can occur only when degrees match, i.e. $m'_j = m_i$, proving the commutativity of the diagram.

□

We established the algebraic structure of the functor $F_{\xi,D}$ and now we want to study its homological properties. The goal is to show that formulas induce well-defined functors on derived categories; as a first step, we show that $F_{\xi,D}$ preserves exactness.

Lemma 2.2.16 — For any formula to a point (ξ, D), the functor $F_{\xi,D}\colon C(\mathcal{A})^X \to C(\mathcal{A})$ maps short exact sequences to short exact sequences.

Proof. Let $\xi = \{(x_i, m_i)\}_{i=1}^n$ and consider a short exact sequence of complexes of diagrams in $C(\mathcal{A})^X$:

$$0 \to K' \xrightarrow{f'} K \xrightarrow{f''} K'' \to 0.$$

By Lemma 2.1.5, kernels, cokernels, and images of morphisms in $C(\mathcal{A})^X$ are computed pointwise. Consequently, exactness is a pointwise property. Thus, evaluating this sequence at any fixed point $x \in X$ yields a short exact sequence of complexes in $C(\mathcal{A})$:

$$0 \to K'_x \xrightarrow{f'_x} K_x \xrightarrow{f''_x} K''_x \to 0.$$

Next, notice that the shift functor $[m]$ is an exact functor on $C(\mathcal{A})$ (because it acts by re-indexing degrees and by changing the sign on the differential, which does not affect the exactness of the sequence). Therefore, for every element (x_i, m_i) in the formula ξ, the following sequence is still exact:

$$0 \to K'_{x_i}[m_i] \xrightarrow{f'_{x_i}[m_i]} K_{x_i}[m_i] \xrightarrow{f''_{x_i}} K''_{x_i}[m_i] \to 0.$$

Since $\mathcal{A}$ is an abelian category, the category of complexes $C(\mathcal{A})$ is also abelian (see 1.2). Moreover, in abelian categories, finite direct sums are exact. Taking the finite sum of the exact sequences above over indices $i = 1, \dots, n$, we obtain the following exact sequence:

$$0 \to \bigoplus_{i=1}^{n} K'_{x_i}[m_i] \xrightarrow{\oplus_{i=1}^{n} f'_{x_i}[m_i]} \bigoplus_{i=1}^{n} K_{x_i}[m_i] \xrightarrow{\oplus_{i=1}^{n} f''_{x_i}} \bigoplus_{i=1}^{n} K''_{x_i}[m_i] \to 0.$$

By the definition of the functor $F_{\xi,D}$ on objects and morphisms (as established in Proposition 2.2.12), the components of this sequence are exactly the images of the initial complexes and morphisms under $F_{\xi,D}$. Substituting the notation, we get:

$$0 \to F_{\xi,D}(K') \xrightarrow{F_{\xi,D}(f')} F_{\xi,D}(K) \xrightarrow{F_{\xi,D}(f'')} F_{\xi,D}(K'') \to 0.$$

This sequence is a short exact sequence in $C(\mathcal{A})$, which concludes the proof. □

Lemma 2.2.17 — For any formula to a point (ξ, D), the functor $F_{\xi,D} \colon C(\mathcal{A})^X \to C(\mathcal{A})$ maps quasi-isomorphisms to quasi-isomorphisms.

Proof. We proceed by induction on n, the length of the sequence $\xi = \{(x_i, m_i)\}_{i=1}^{n}$.

Base case ($n = 1$): Assume $\xi = \{x_1, m_1\}$. In this case, D is the 1×1 matrix (1). By definition, for any complex $K \in Ob(C(\mathcal{A})^X)$, we have $F_{\xi,D}(K) = K_{x_1}[m_1]$ equipped with the differential $d_{x_1}[m_1]$. For a morphism $f \colon K \to K'$, $F_{\xi,D}(f) = f_{x_1}[m_1]$. If f is a quasi-isomorphism in $C(\mathcal{A}^X)$, then by point (vii) of Lemma 2.1.5, its pointwise evaluation $f_{x_1} \colon K_{x_1} \to K'_{x_1}$ is a quasi-isomorphism in $C(\mathcal{A})$. Since the functor $[m_1]$ is exact and therefore preserves quasi-isomorphisms, $f_{x_1}[m_1]$ is also a quasi-isomorphism, which proves the base case.

Inductive step ($n > 1$): Assume that the claim holds for $n - 1$. Let $\xi' = \{(x_i, m_i)\}_{i=1}^{n-1}$ and let $D' = (D_{ji})_{1 \le i,j \le n-1}$ be the $(n-1) \times (n-1)$ upper-left submatrix of D.

In order to show that $(K_{x_n}[m_n], d_{x_n}[m_n])$ is a subcomplex, we can visualize the global differential $\eta_D(K)$ as a block matrix acting on the direct sum $F_{\xi,D}(K) = F_{\xi',D'}(K) \oplus K_{x_n}[m_n]$.

By Definition 2.2.10, the matrix D is lower triangular (meaning $D_{ji} = 0$ for all $i > j$), and the diagonal entries are 1. Therefore, the n-th column of D consists of zeros, apart from the diagonal entry $D_{nn} = 1$. For Lemma 2.2.6, this is the standard differential $d_{x_n}[m_n]$. Hence, the differential has the following lower-triangular block structure:

$$\eta_D(K) = \begin{pmatrix} \eta_{D'}(K) & 0 \\ R & d_{x_n}[m_n] \end{pmatrix},$$

where R is a block representing the restriction maps from the first $n - 1$ components into the n-th component.

When $\eta_D(K)$ is applied to an element lying in the n-th component (represented by a column vector with zeros in the upper block), the zero matrix in the top right ensures that the output remains entirely within the n-th component. This invariance means that $(K_{x_n}[m_n], d_{x_n}[m_n])$ is a well-defined subcomplex of $(F_{\xi,D}(K), \eta_D(K))$.

Furthermore, the canonical inclusion $\iota_K\colon K_{x_n}[m_n] \to \oplus_{i=1}^{n} K_{x_i}[m_i]$ trivially commutes with the differential due to this invariance.

Similarly, the canonical projection $\pi_K\colon \oplus_{i=1}^{n} K_{x_i}[m_i] \to \oplus_{i=1}^{n-1} K_{x_i}[m_i]$ "forgets" the bottom row of the block matrix. Since the top-left block is $\eta_{D'}(K)$, the projection strictly commutes with the differentials as well.

This gives rise to the short exact sequence of complexes in $C(\mathcal{A})$:

$$0 \to K_{x_n}[m_n] \xrightarrow{\iota_K} F_{\xi,D}(K) \xrightarrow{\pi_K} F_{\xi',D'}(K) \to 0.$$

Let $f\colon K \to K'$ be a morphism in $C(\mathcal{A})^X$. The functoriality of this construction gives a commutative diagram of short exact sequences. Passing to cohomology, this induces a commutative diagram with exact rows (the long exact sequence in cohomology):

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H^{i-1}(F_{\xi',D'}(K)) & \longrightarrow & H^i(K_{x_n}[m_n]) & \longrightarrow & H^i(F_{\xi,D}(K)) & \longrightarrow & H^i(F_{\xi',D'}(K)) & \longrightarrow \dots \\
& \downarrow{\scriptstyle H^{i-1}(F_{\xi',D'}(f))} & & \downarrow{\scriptstyle H^i(f_{x_n}[m_n])} & & \downarrow{\scriptstyle H^i(F_{\xi,D}(f))} & & \downarrow{\scriptstyle H^i(F_{\xi',D'}(f))} & \\
\dots \longrightarrow & H^{i-1}(F_{\xi',D'}(K')) & \longrightarrow & H^i(K'_{x_n}[m_n]) & \longrightarrow & H^i(F_{\xi,D}(K')) & \longrightarrow & H^i(F_{\xi',D'}(K')) & \longrightarrow \dots
\end{array}$$

Now, assume f is a quasi-isomorphism. We evaluate the vertical arrows:

(i) By Lemma 2.1.5 *(vii)*, f_{x_n} is a quasi-isomorphism, and since the shift preserves this property, $f_{x_n}[m_n]$ is a quasi-isomorphism too. Thus, the vertical maps $H^i(f_{x_n}[m_n])$ are isomorphisms for all i.

(ii) By our induction hypothesis applied to the formula of length $n-1$, $F_{\xi',D'}(f)$ is a quasi-isomorphism. Thus, the vertical map $H^i(F_{\xi',D'}(f))$ and $H^{i-1}(F_{\xi',D'}(f))$ are isomorphisms for all i.

Since the two outer vertical maps on both sides of $H^i(F_{\xi,D}(f))$ in the extended diagram are isomorphisms, we can apply the Five Lemma. It immediately follows that the middle vertical map $H^i(F_{\xi,D}(f))$ is an isomorphism for every integer i. Therefore, $F_{\xi,D}(f)$ is a quasi-isomorphism, completing the induction.

□

After all the homological preliminaries, we can now see that formulas to a point behave well with respect to localization. Since they preserve exact sequence and quasi-isomorphisms, they induce functors on the derived category.

Corollary 2.2.18 — Let (ξ, D) be a formula to a point. The functor $F_{\xi,D}\colon C(\mathcal{A})^X \to C(\mathcal{A})$ induces a triangulated functor:

$$\tilde{F}_{\xi,D}\colon D(\mathcal{A}^X) \to D(\mathcal{A}).$$

Proof. Let S_X and S denote the multiplicative classes of quasi-isomorphisms in $C(\mathcal{A})^X$ and $C(\mathcal{A})$, respectively.

Recall that, by Definition 1.3.4 and the discussion thereafter, the derived categories may also be viewed as the localizations $D(\mathcal{A}^X) = C(\mathcal{A})^X[S_X^{-1}]$ and $D(\mathcal{A}) = C(\mathcal{A})[S^{-1}]$ (see Definition 1.3.2).

Let $Q_X\colon C(\mathcal{A})^X \to D(\mathcal{A}^X)$ and $Q\colon C(\mathcal{A}) \to D(\mathcal{A})$ be the corresponding localization quotient functors.

We consider the composed functor:

$$Q \circ F_{\xi,D} \colon C(\mathcal{A})^X \to D(\mathcal{A}).$$

To apply the universal property of the localization to this composition, we must verify that it maps every morphism in S_X to an isomorphism in $D(\mathcal{A})$.

Let $s \in S_X$ be a quasi-isomorphism in $C(\mathcal{A})^X$. By Lemma 2.2.17, the functor $F_{\xi,D}$ preserves quasi-isomorphisms, meaning $F_{\xi,D}(s) \in S$. Furthermore, by the defining property of the quotient functor Q (1.3.2), it maps every element of S to an isomorphism. Thus, $(Q \circ F_{\xi,D})(s)$ is indeed an isomorphism in $D(\mathcal{A})$.

By the universal property of localization (setting the target category to $\mathcal{B} = D(\mathcal{A})$), the functor $Q \circ F_{\xi,D}$ factors uniquely through Q_X. This guarantees the existence of a unique functor $\tilde{F}_{\xi,D} \colon D(\mathcal{A}^X) \to D(\mathcal{A})$ making the following diagram commute:

$$\begin{array}{ccc} C(\mathcal{A})^X & \xrightarrow{F_{\xi,D}} & C(\mathcal{A}) \\ \downarrow{\scriptstyle Q_X} & & \downarrow{\scriptstyle Q} \\ D(\mathcal{A}^X) & \xrightarrow[\tilde{F}_{\xi,D}]{} & D(\mathcal{A}), \end{array}$$

that is, $\tilde{F}_{\xi,D} \circ Q_X = Q \circ F_{\xi,D}$.

It remains to show that $\tilde{F}_{\xi,D}$ is a triangulated functor. Recall that the triangulated structure on the derived category $D(\mathcal{A})$ is induced from the homotopy category $K(\mathcal{A})$ (see Theorem 1.3.5): a triangle in $D(\mathcal{A})$ is distinguished if and only if it is isomorphic in $D(\mathcal{A})$ to the image, under the localization functor $Q \colon K(\mathcal{A}) \to D(\mathcal{A})$, of a standard cone triangle

$$K \xrightarrow{f} L \to C_f^{\bullet} \to K[1]$$

in $K(\mathcal{A})$.

By Lemma 2.2.15 the functor $F_{\xi,D}$ is compatible with shifts. Moreover, $F_{\xi,D} \colon C(\mathcal{A})^X \to C(\mathcal{A})$ is additive and preserves short exact sequences of complexes componentwise (Lemma 2.2.16); hence it sends a short exact sequence $0 \to K \to L \to M \to 0$ in $C(\mathcal{A})^X$ to a short exact sequence $0 \to F_{\xi,D}(K) \to F_{\xi,D}(L) \to F_{\xi,D}(M) \to 0$ in $C(\mathcal{A})$.

In particular, for any morphism $f \colon K \to L$ in $C(\mathcal{A})^X$, applying $F_{\xi,D}$ to the cone construction yields a canonical isomorphism

$$F_{\xi,D}(C_f^{\bullet}) \cong C^{\bullet}_{F_{\xi,D}(f)}$$

in $C(\mathcal{A})$, and therefore the image of the cone triangle of f is a cone triangle of $F_{\xi,D}(f)$. After passing to the derived category via localization, $\tilde{F}_{\xi,D}$ sends distinguished triangles to distinguished triangles. Hence $\tilde{F}_{\xi,D}$ is a triangulated functor. □

Having understood how formulas act to produce complexes evaluating at a single point, we now generalize this concept. Instead of targeting a single complex, a general formula targets an entire category of diagrams over a new poset Y.

Definition 2.2.19 (General Formulas)
Let X and Y be two posets. A **formula from X to Y** (also referred to as general formula) is a diagram over Y with values in the category $\mathcal{F}_X$.
Explicitly, a formula from X to Y, denoted by ξ, consists of:

(i) an assignment of a formula to a point $(\xi_y, D_y) \in Ob(\mathcal{F}_X)$ for every element $y \in Y$;

(ii) a morphism of formulas to a point $\varphi_{yy'} : (\xi_y, D_y) \to (\xi_{y'}, D_{y'})$ in $\mathcal{F}_X$ for every relation $y \leq y'$ in Y, satisfying:

- $\varphi_{yy} = id_{\xi_y, D_y}$ for all $y \in Y$;
- $\varphi_{yy''} = \varphi_{y'y''} \circ \varphi_{yy'}$ for all $y \leq y' \leq y''$ in Y.

Given ξ and ξ' two formulas from X to Y, a **morphism of formulas from X to Y** $\alpha : \xi \to \xi'$ is a morphism of diagrams. It is defined by a family of morphisms in $\mathcal{F}_X$, $\alpha_y : (\xi_y, D_y) \to (\xi'_y, D'_y)$ for each $y \in Y$, such that:

- $\varphi'_{yy'} \circ \alpha_y = \alpha_{y'} \circ \varphi_{yy'}$ for every relation $y \leq y'$ in Y (where φ' denotes the morphism defining the diagram ξ').

Formulas from X to Y, together with these morphisms, form a category which we denote by $\mathcal{F}_X^Y$.

Proposition 2.2.20 — There exists a functor $\eta : \mathcal{F}_X^Y \to Fun(C(\mathcal{A})^X, C(\mathcal{A})^Y)$.

Proof. Let $\eta : \mathcal{F}_X \to Fun(C(\mathcal{A})^X, C(\mathcal{A}))$ be the functor constructed in Proposition 2.2.12. Any functor between two categories induces a functor between their respective diagram categories over a fixed poset Y via composition. Therefore, applying η componentwise induces a functor:

$$\eta^Y : \mathcal{F}_X^Y \to Fun(C(\mathcal{A})^X, C(\mathcal{A}))^Y.$$

An object $\xi \in \mathcal{F}_X^Y$ is a diagram $y \mapsto \xi_y$ in $\mathcal{F}_X$; then $\eta^Y(\xi)$ is the diagram $y \mapsto \eta(\xi_y)$ in $Fun(C(\mathcal{A})^X, C(\mathcal{A}))$, with the structure natural transformations induced by functoriality of η.
Now we want to explain why $Fun(C(\mathcal{A})^X, C(\mathcal{A}))^Y$ canonically identifies with $Fun(C(\mathcal{A})^X, C(\mathcal{A})^Y)$.
Set $\mathcal{B} := C(\mathcal{A})^X$ and $\mathcal{C} := C(\mathcal{A})$.
Define a functor $\Phi : Fun(\mathcal{B}, \mathcal{C})^Y \to Fun(\mathcal{B}, \mathcal{C}^Y)$ as follows. An object of $Fun(\mathcal{B}, \mathcal{C})^Y$ is given by functors $(F_y)_{y \in Y}$ together with a natural transformation $\tau_{y \leq y'} : F_y \Rightarrow F_{y'}$ for every relation $y \leq y'$. We define $\Phi((F_y, \tau)) = H : \mathcal{B} \to \mathcal{C}^Y$ by:

- for $K \in \mathcal{B}$, the object $H(K) \in \mathcal{C}^Y$ is the Y-diagram with

$$H(K)_y := F_y(K), \quad H(K)_{y \leq y'} := (\tau_{y \leq y'})_K;$$

- for a morphism $u : K \to L$ in $\mathcal{B}$, the morphism $H(u) : H(K) \to H(L)$ in $\mathcal{C}^Y$ is given at each y by

$$H(u)_y := F_y(u).$$

The compatibility with the arrows $y \leq y'$ is exactly the naturality of $\tau_{y \leq y'}$.

Conversely, define $\Psi : Fun(\mathcal{B}, C^Y) \to Fun(\mathcal{B}, \mathcal{C})^Y$ by sending $G : \mathcal{B} \to \mathcal{C}^Y$ to the Y-diagram (G_y, τ^G), where

$$G_y := ev_y \circ G$$

and for each relation $y \leq y'$ the natural transformation $\tau^G_{y \leq y'} : G_y \Rightarrow G_{y'}$ is defined on $K \in \mathcal{B}$ by

$$(\tau^G_{y \leq y'})_K := (G(K))_{y \leq y'} : G(K)_y \to G(K)_{y'}.$$

(That is, the structure map of the Y-diagram $G(K)$ is precisely the component at K of the required natural transformation.)

We check that Φ and Ψ are inverse to each other.

(i) Let (F_y, τ) be an object of $Fun(\mathcal{B}, \mathcal{C})^Y$, and set $H := \Phi(F_y, \tau)$.
For each y, $(\Psi(H))_y = ev_y \circ H$, hence, for every $K \in \mathcal{B}$,

$$(ev_y \circ H)(K) = ev_y(H(K)) = H(K)_y = F_y(K).$$

Thus $(\Psi(H))_y = F_y$. Moreover, for $y \leq y'$ the natural transformation in $\Psi(H)$ has component at K

$$(\tau^H_{y\leq y'})_K = H(K)_{y\leq y'} = (\tau_{y\leq y'})_K,$$

so $\tau^H_{y\leq y'} = \tau_{y\leq y'}$. Therefore $\Psi(\Phi(F_y, \tau)) = (F_y, \tau)$. The verification on morphisms is analogous (componentwise evaluation), hence $\Psi \circ \Phi = Id_{Fun(\mathcal{B},\mathcal{C})^Y}$.

(ii) Let $G\colon \mathcal{B} \to \mathcal{C}^Y$ and set $(G_y, \tau^G) := \Psi(G)$, then $H := \Phi(G_y, \tau^G)$ satisfies: for every $K \in \mathcal{B}$ and every $y \in Y$,

$$H(K)_y = G_y(K) = (ev_y \circ G)(K) = G(K)_y,$$

and for every $y \leq y'$,

$$H(K)_{y\leq y'} = (\tau^G_{y\leq y'})_K = G(K)_{y\leq y'}.$$

Hence $H(K) = G(K)$ as objects of $\mathcal{C}^Y$. Similarly, for a morphism $u\colon K \to L$ we have for each y,

$$H(u)_y = G_y(u) = ev_y(G(u)) = G(u)_y,$$

so $H(u) = G(u)$ in $\mathcal{C}^Y$. Thus $H = G$ and again the check on morphisms is componentwise. Therefore $\Phi \circ \Psi = Id_{Fun(\mathcal{B},\mathcal{C}^Y)}$.
Consequently, Φ is an isomorphism of categories (with inverse Ψ), hence

$$Fun(\mathcal{B}, \mathcal{C})^Y \cong Fun(\mathcal{B}, \mathcal{C}^Y).$$

Finally, define $\eta := \Phi \circ \eta^Y : \mathcal{F}^Y_X \to Fun(C(\mathcal{A})^X, C(\mathcal{A})^Y)$, that is the functor we were looking for.

□

Before starting the next lemma, let us make explicit the functor F_ξ associated with a general formula $\xi \in Ob(\mathcal{F}^Y_X)$. For each $y \in Y$, the component (ξ_y, D_y) is a formula to a point, hence it defines a functor

$$F_{\xi_y, D_y} \colon C(\mathcal{A})^X \to C(\mathcal{A})$$
$$K \mapsto \bigoplus_{i=1}^{n} K_{x_i}[m_i].$$

See Step 1 in the proof of Lemma 2.2.6 for explicit construction.

By Proposition 2.2.20, these component functors assemble into a functor

$$F_\xi \colon C(\mathcal{A})^X \to C(\mathcal{A})^Y.$$

For $K \in C(\mathcal{A})^X$, the object $F_\xi(K)$ is the Y-diagram whose value at y is $F_{\xi_y, D_y}(K)$. Moreover, for every relation $y \leq y'$ in Y, the morphism of formulas

$$\varphi_{yy'} \colon (\xi_y, D_y) \to (\xi_{y'}, D_{y'})$$

induces a natural transformation

$$\eta_{\varphi_{yy'}} : F_{\xi_y,D_y} \Rightarrow F_{\xi_{y'},D_{y'}},$$

and the corresponding structure map of the diagram $F_\xi(K)$ is the morphism

$$(\eta_{\varphi_{yy'}})_K : F_{\xi_y,D_y}(K) \to F_{\xi_{y'},D_{y'}}(K).$$

Finally, if $u : K \to K'$ is a morphism in $C(\mathcal{A})^X$, then $F_\xi(u)$ is defined componentwise by

$$(F_\xi(u))_y = F_{\xi_y,D_y}(u),$$

and its compatibility with the structure maps follows from the naturality of $\eta_{\varphi_{yy'}}$.

Lemma 2.2.21 — For any formula $\xi \in Ob(\mathcal{F}_X^Y)$, the induced functor F_ξ maps short exact sequences to short exact sequences and maps quasi-isomorphisms to quasi-isomorphisms.

Proof. As established in Lemma 2.1.5, exactness and quasi-isomorphisms in the functor category $C(\mathcal{A})^Y$ are characterized pointwise. Since every local component functor F_{ξ_y,D_y} preserves short exact sequences (Lemma 2.2.16) and quasi-isomorphisms (Lemma 2.2.17) in $C(\mathcal{A})$, the global functor F_ξ preserves both properties, since they are checked pointwise. □

Corollary 2.2.22 — Let $\xi \in Ob(\mathcal{F}_X^Y)$ be a formula. The functor F_ξ induces a triangulated functor

$$\tilde{F}_\xi : D(\mathcal{A}^X) \to D(\mathcal{A}^Y).$$

Proof. The proof follows exactly the localization arguments detailed in Corollary 2.2.18, applied to the global functor F_ξ. The componentwise preservation of shift, mapping cones, and quasi-isomorphisms guarantees the unique factorization through the derived categories. □

2.2.2 Application of formulas

As a first application of the theory developed so far, we consider the simplest non-trivial poset

$$\bullet 1 \longrightarrow \bullet 2$$

We focus on this specific case because it effectively demonstrates the fundamental principle underlying the main theorem and explicitly shows how standard mapping cones arise from formulas to a point.

Let us define three specific formulas to a point in $\mathcal{F}_{1\to 2}$:

(i) $\xi_1 = \{(1,1)\}, \quad D_1 = (1);$

(ii) $\xi_2 = \{(2,0)\}, \quad D_2 = (1);$

(iii) $\xi_{12} = \{(1,1),(2,0)\} \quad D_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$

Notice that the first two items are trivially formulas to a point, and we proved that the third one is a formula to a point in Example 2.2.14.

Let $\mathcal{A}$ be an abelian category and let $K \in Ob(C(\mathcal{A})^{1\to 2})$. We can view K as a diagram $K_1 \xrightarrow{r_{12}} K_2$ of chain complexes in $C(\mathcal{A})$. According to the functorial action defined in the previous sections, applying the functors associated with these three formulas yields the following complexes in $C(\mathcal{A})$:

(i) $F_{\xi_1,D_1}(K) = K_1[1];$

(ii) $F_{\xi_2,D_2}(K) = K_2$;

(iii) $F_{\xi_{12},D_{12}}(K) = C(K_1 \xrightarrow{r_{12}} K_2)$.

Now, we define two morphisms in the category $C_{1\to 2}$:

- $\varphi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} : \xi_{12} \to \xi_1$;
- $\varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \xi_2 \to \xi_{12}$.

Both are restrictions. Moreover, since the shift functor [1] on $C_{1\to 2}$ does not change the coefficient matrices of the morphisms, we can verify the commutativity condition with the differentials (i.e. $\varphi[1] \circ D = D' \circ \varphi$) via standard matrix multiplication:

$$\varphi_1 D_{12} = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} = (1) \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} = D_1 \varphi_1$$

$$D_{12}\varphi_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (1) = \varphi_2 D_2.$$

Thus, φ_1 and φ_2 are valid morphisms in $\mathcal{F}_{1\to 2}$.

By using these morphisms, we can construct two general formulas from $X := 1 \to 2$ to $Y := 1 \to 2$ (meaning two diagrams over $1 \to 2$ with values in $\mathcal{F}_{1\to 2}$):

- $\xi^- = (\xi_{12}, D_{12}) \xrightarrow{\varphi_1} (\xi_1, D_1)$;
- $\xi^+ = (\xi_2, D_2) \xrightarrow{\varphi_2} (\xi_{12}, D_{12})$.

By Corollary 2.2.22, these two formulas induce two functors

$$F_{\xi^-}, F_{\xi^+} \colon C(\mathcal{A}^{1\to 2}) \to C(\mathcal{A}^{1\to 2}).$$

We will denote them by

$$R^- := F_{\xi^-} \quad \text{and} \quad R^+ := F_{\xi^+}.$$

They induce triangulated functors

$$\tilde{R}^-, \tilde{R}^+ \colon D(\mathcal{A}^{1\to 2}) \to D(\mathcal{A}^{1\to 2}).$$

Evaluating them on the object K, we obtain the following diagrams of complexes:

$$R^-(K) = C(K_1 \xrightarrow{r_{12}} K_2) \xrightarrow{\begin{pmatrix} r_{11}[1] & 0 \end{pmatrix}} K_1[1];$$

$$R^+(K) = K_2 \xrightarrow{\begin{pmatrix} 0 \\ r_{22} \end{pmatrix}} C(K_1 \xrightarrow{r_{12}} K_2).$$

Recall that restriction maps satisfy $r_{xx} = id_{K_x}$, hence, in particular, $r_{11} = id_{K_1}$ and $r_{22} = id_{K_2}$.

Proposition 2.2.23 — There are two natural transformations

$$\epsilon^{+-} \colon R^+ \circ R^- \Rightarrow [1]$$

and

$$\epsilon^{-+} \colon [1] \Rightarrow R^- \circ R^+$$

such that for all $K = (K_1 \xrightarrow{r_{12}} K_2) \in Ob(C(\mathcal{A}^{1\to 2}))$, the maps $\epsilon^{+-}(K)$ and $\epsilon^{-+}(K)$ are quasi-isomorphisms.

Proof. We will explicitly construct the natural transformation by working at the level of formulas in $\mathcal{F}_{1\to 2}$, and then we will prove that they are quasi-isomorphisms by demonstrating an explicit chain homotopy. The proof proceeds in three main steps.

Step 1: (Composition of formulas.) We start by describing the composite formulas corresponding to the composite functors $R^+ \circ R^-$ and $R^- \circ R^+$.

We will use the notation ξ_1^- and ξ_2^- for the values of ξ^- at the vertices 1 and 2 of the poset $1 \to 2$, i.e. $\xi_1^- = (\xi_{12}, D_{12})$ and $\xi_2^- = (\xi_1, D_1)$; similarly $\xi_1^+ = (\xi_2, D_2)$ and $\xi_2^+ = (\xi_{12}, D_{12})$. Recall that shifting a formula (ξ, D) by [1] shifts every pair (x, m) in ξ to $(x, m+1)$. For the matrices, we use the convention induced by the usual shift of complexes, then in our case we have:

- $\xi_1[1] = \{(1,2)\}, \quad D_1[1] = (1)$;
- $\xi_2[1] = \{(2,1)\}, \quad D_2[1] = (1)$;
- $\xi_{12}[1] = \{(1,2),(2,1)\}$ and the off-diagonal entry, which corresponds to the restriction map, changes sign, hence $D_{12}[1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. (Concretely: (ξ_{12}, D_{12}) encodes the cone $C(K_1 \to K_2)$ with the summand $K_1[1]$ and the map induced by $K_1 \to K_2$; after shifting, that map appears with a minus sign.)

The functors $R^+ \circ R^-$ and $R^- \circ R^+$ are induced by the composition of the general formulas defining R^+ and R^-. They correspond to the composite formulas:

- $\xi^{+-} := \xi^+ \circ (\xi_1^- \to \xi_2^-)$,
- $\xi^{-+} := \xi^- \circ (\xi_1^+ \to \xi_2^+)$.

Computation of ξ^{+-} and (ξ_{121}, D_{121}):

Since $\xi_1^- = (\xi_{12}, D_{12})$ and $\xi_2^- = (\xi_1, D_1)$, this means we are applying ξ^+ to the diagram $(\xi_{12}, D_{12}) \to (\xi_1, D_1)$, i.e. to the morphism φ_1.

The value at the second vertex of ξ^+ is (ξ_{12}, D_{12}), so the second component of the composite is the cone of φ_1.Then $\xi_{121} = (\xi_{12}[1], \xi_1) = \{(1,2),(2,1),(1,1)\}$. Now we compute the differential:

$$D_{121} = \begin{pmatrix} D_{12}[1] & 0 \\ \varphi_1[1] & D_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Computation of ξ^{-+} and (ξ_{212}, D_{212}):

Since $\xi_1^+ = (\xi_2, D_2)$ and $\xi_2^+ = (\xi_{12}, D_{12})$, this means we are applying ξ^- to the diagram $(\xi_2, D_2) \to (\xi_{12}, D_{12})$, i.e. to the morphism φ_2.

The value at the first vertex of ξ^- is (ξ_{12}, D_{12}), so the first component of the composite is the cone of φ_2. Then $\xi_{212} = \{(\xi_2[1], \xi_{12})\} = \{(2,1),(1,1),(2,0)\}$.

Now we compute the differential:

$$D_{212} = \begin{pmatrix} D_2[1] & 0 \\ \varphi_2 & D_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we have:

- $(\xi_{121}, D_{121}) = \left(\{(1,2),(2,1),(1,1)\}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right)$,

- $(\xi_{212}, D_{212}) = \left(\{(2,1),(1,1),(2,0)\}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right).$

Using these objects, the composite formulas ξ^{+-} and ξ^{-+} can be expressed as the following diagrams over $1 \to 2$:

- $\xi^{+-} = (\xi_1, D_1) \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} (\xi_{121}, D_{121}),$
- $\xi^{-+} = (\xi_{212}, D_{212}) \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}} (\xi_2[1], D_2[1]).$

Step 2: (Construction of natural transformations.) Now we want to connect these compositions to the shift functor [1], that corresponds to the diagram

$$\nu = (\xi_1, D_1) \xrightarrow{(1)} (\xi_2[1], D_2[1]).$$

To relate ξ^{+-} to ν and ν to ξ^{-+}, we introduce four morphism $\alpha_1, \alpha_2, \beta_1$ and β_2 in the category $C_{1\to 2}$ that act as projections and inclusions between the 1- dimensional formulas and the 3-dimensional ones:

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : (\xi_1, D_1) \to (\xi_{212}, D_{212}), \qquad \beta_1 = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix} : (\xi_{212}, D_{212}) \to (\xi_1, D_1),$$

$$\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : (\xi_2[1], D_2[1]) \to (\xi_{121}, D_{121}), \qquad \beta_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} : (\xi_{121}, D_{121}) \to (\xi_2[1], D_2[1]).$$

These vertical maps yield the following commutative diagram of formulas in $\mathcal{F}_{1\to 2}$.

$$\begin{array}{ccccc} (\xi_1, D_1) & \xrightarrow{(1)} & (\xi_1, D_1) & \xrightarrow{\alpha_1} & (\xi_{212}, D_{212}) \\ \Big\downarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & & \Big\downarrow (1) & & \Big\downarrow (1\ 0\ 0) \\ (\xi_{121}, D_{121}) & \xrightarrow[\beta_2]{} & (\xi_2[1], D_2[1]) & \xrightarrow[(1)]{} & (\xi_2[1], D_2[1]) \end{array}$$

This commutativity induces morphisms of formulas $\xi^{+-} \to \nu$ and $\nu \to \xi^{-+}$, that induce the required natural transformations $\epsilon^{+-} \colon R^+R^- \Rightarrow [1]$ and $\epsilon^{-+} \colon [1] \Rightarrow R^-R^+$.

Step 3: (Homotopy Argument.) To conclude the proof, we must show that evaluating ϵ^{+-} and ϵ^{-+} on any complex K yields a quasi-isomorphism. By Lemma 2.1.5, quasi-isomorphisms are checked pointwise, then it is sufficient to prove that the component maps $\eta_{\alpha_i}(K)$ and $\eta_{\beta_i}(K)$ are quasi-isomorphisms in $C(\mathcal{A})$.

We will explicitly show that α_i and β_i are chain homotopy equivalences. We define the homotopies $h_1 \colon \xi_{212} \to \xi_{212}[-1]$ and $h_2 \colon \xi_{121} \to \xi_{121}[-1]$ by the matrix:

$$h_1 = h_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A standard matrix calculation leads to the following relations:

$$\beta_1\alpha_1 = (1), \qquad \alpha_1\beta_1 + (h_1[1]D_{212} + D^\star_{212}[-1]h_1) = I_3$$

and

$$\beta_2\alpha_2 = (1), \qquad \alpha_2\beta_2 + (h_2[1]D_{121} + D^\star_{121}[-1]h_2) = I_3;$$

where $D^\star_{121} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ and $D^\star_{212} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ and I_3 is the identity matrix 3×3. □

The first two equations mean that β_1 is a strict left inverse of α_1 (and similarly for β_1 and α_1). The last two equations became

$$I_3 - \alpha_1\beta_1 = h_1[1]D_{212} + D^\star_{212}[-1]h_1$$

and

$$I_3 - \alpha_2\beta_2 = h_2[1]D_{121} + D^\star_{121}[-1]h_2,$$

meaning that $\alpha_1\beta_1 \sim I_3$ with homotopy h_1 and $\alpha_2\beta_2 \sim I_3$ with homotopy h_2. Then α_1 and β_1 are homotopy equivalences, as well as α_2 and β_2. Then $\eta_{\alpha_1(K)}$, η_{α_2}, η_{β_1} and η_{β_2} are quasi-isomorphisms.

Proposition 2.2.24 — There are two natural transformations

$$\epsilon^{++} \colon R^+ \circ R^+ \to R^-$$

and

$$\epsilon^{--} \colon R^+ \circ [1] \to R^- \circ R^-$$

such that for all $K = (K_1 \xrightarrow{r_{12}} K_2) \in Ob(C(\mathcal{A}^{1\to 2}))$, the maps $\epsilon^{++}(K)$ and $\epsilon^{--}(K)$ are quasi-isomorphisms.

Proof. We work at the level of formulas in $\mathcal{F}_{1\to 2}$. As in the previous proposition, we first identify the composite formulas, then we construct morphisms of diagrams of formulas, and finally we check that the induced maps are quasi-isomorphisms.

Step 1: (Identification of the composite formulas.) Recall that the functors R^+ and R^- are induced by diagrams of formulas over the poset $1 \to 2$, denoted by ξ^+ and ξ^-. As before, we write $\xi^\pm_i$ for the value of $\xi^\pm$ at the vertex $i \in \{1, 2\}$. By the definition of composition in $\mathcal{F}_{1\to 2}$, the composites functors $R^+ \circ R^+$ and $R^- \circ R^-$ correspond to the composite diagrams:

- $\xi^{++} := \xi^+ \circ (\xi^+_1 \to \xi^+_2)$,
- $\xi^{--} := \xi^- \circ (\xi^-_1 \to \xi^-_2)$.

This is the same composition principle used in Proposition 2.2.23. We now compute the two composite formulas explicitly.

Computation of ξ^{++} and (ξ_{212}, D_{212}): Recall that ξ^+ is the diagram

$$(\xi_2, D_2) \xrightarrow{\varphi_2} (\xi_{12}, D_{12}),$$

so $\xi^+_1 = (\xi_2, D_2)$ and $\xi^+_2 = (\xi_{12}, D_{12})$. Therefore the inner diagram $(\xi^+_1 \to \xi^+_2)$ is exactly the morphism φ_2.

- The value of ξ^{++} at the first vertex is obtained by applying (ξ_2, D_2) to the inner diagram; since (ξ_2, D_2) corresponds to taking the second vertex of the inner diagram (it produces K_2 on objects), this gives

$$(\xi^{++})_1 = (\xi_{12}, D_{12}).$$

- The value of ξ^{++} at the second vertex is obtained by applying (ξ_{12}, D_{12}) to the inner diagram; since (ξ_{12}, D_{12}) is the mapping cone formula, this produces the cone of φ_2 (that we already computed in Proposition 2.2.23):

$$(\xi^{++})_2 = C(\varphi_2) = (\xi_{212}, D_{212}).$$

Finally, the arrow in the diagram ξ^{++},

$$(\xi_{12}, D_{12}) \xrightarrow{M_{++}} (\xi_{212}, D_{212}),$$

is the canonical inclusion $\xi_{12} \hookrightarrow \xi_{212} = (\xi_2[1], \xi_{12})$ inside the cone. This inclusion is represented by the matrix

$$M_{++} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\xi^{++} : (\xi_{12}, D_{12}) \xrightarrow{M_{++}} (\xi_{212}, D_{212}).$$

Computation of ξ^{--} and $(\xi_{12}[1], -D^{\star}_{12}[1])$: Recall that ξ^{-} is the diagram

$$(\xi_{12}, D_{12}) \xrightarrow{\varphi_1} (\xi_1, D_1),$$

so $\xi^-_1 = (\xi_{12}, D_{12})$ and $\xi^-_2 = (\xi_1, D_1)$. Therefore the inner diagram $(\xi^-_1 \to \xi^-_2)$ is exactly the morphism φ_1.

- The value of ξ^{--} at the first vertex is obtained by applying (ξ_{12}, D_{12}) (the cone formula) to the inner diagram, hence

$$(\xi^{--})_1 = C(\varphi_1) = (\xi_{121}, D_{121}).$$

(This is the same (ξ_{121}, D_{121}) that we already computed in Proposition 2.2.23.)

- The value of ξ^{--} at the second vertex is obtained by applying (ξ_1, D_1) to the inner diagram. Since (ξ_1, D_1) corresponds to taking the first vertex and shifting (it produces $K_1[1]$ on objects), this yields the shifted formula $\xi_{12}[1]$. On the level of differentials, the shift convention implies that the shifted differential is described by $-D^{\star}[1]$. Therefore

$$(\xi^{--})_2 = (\xi_{12}[1], -D^{\star}_{12}[1]).$$

Now we compute $-D^{\star}_{12}[1]$ explicitly.

We have $\xi_{12} = \{(1,1), (2,0)\}$ and $D_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

By definition, if $D\colon \xi \to \xi[1]$, then $(D^{\star})_{ji} = (-1)^{m'_j - m_i} D_{ji}$, where m_i is the degree in the source and m'_j is the degree in the target. Here $\xi_{12}[1] = \{(1,2), (2,1)\}$, so $(m_1, m_2) = (1, 0)$ and $(m'_1, m'_2) = (2, 1)$. Hence, we obtain the matrix

$$-D^{\star}_{12}[1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Finally, the arrow in the diagram ξ^{--},

$$(\xi_{121}, D_{121}) \xrightarrow{M_{--}} (\xi_{12}[1], -D^{\star}_{12}[1]),$$

is the canonical projection from the cone $\xi_{121} = (\xi_{12}[1], \xi_1)$ into $\xi_{12}[1]$. This projection is represented by the matrix

$$M_{--} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence

$$\xi^{--} : (\xi_{121}, D_{121}) \xrightarrow{M_{--}} (\xi_{12}[1], -D^{\star}_{12}[1]).$$

Step 2: (Construction of the natural transformations.) We now construct the morphisms of formulas that will induce the required natural transformations ϵ^{++} and ϵ^{--}. Evaluating such morphisms on $C(\mathcal{A})^{1\to 2}$ yields natural transformations between the associated functors. A morphism of formulas is given by a commutative square connecting the corresponding diagrams. We will show that the matrices α_2 and β_1 defined in the previous proposition provide exactly the maps we need.

Construction of $\epsilon^{++}\colon R^+ \circ R^+ \Rightarrow R^-$:

We need to define a morphism from the composed formula ξ^{++} to the formula ξ^-. Recall that $\xi^- = (\xi_{12}, D_{12}) \xrightarrow{\varphi_1} (\xi_1, D_1)$ with $\varphi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

At the first vertex, both formulas evaluate to (ξ_{12}, D_{12}), so we can take the identity map $id_{(\xi_{12},D_{12})} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. At the second vertex, we need a map from (ξ_{212}, D_{212}) to (ξ_1, D_1). We can use the projection into the middle component of the cone, which is $-\beta_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. This yields the following diagram:

$$\begin{array}{ccc} (\xi_{12}, D_{12}) & \xrightarrow{\quad I_2 \quad} & (\xi_{12}, D_{12}) \\ \Big\downarrow{\scriptstyle M_{++}} & & \Big\downarrow{\scriptstyle \varphi_1} \\ (\xi_{212}, D_{212}) & \xrightarrow[\quad -\beta_1 \quad]{} & (\xi_1[1], D_1[1]). \end{array}$$

Since the square commutes, it defines the natural transformation ϵ^{++}.

Construction of $\epsilon^{--}\colon R^+ \circ [1] \Rightarrow R^- \circ R^-$:

Here we need a morphism from the formula $\xi^+[1]$ to the composed formula ξ^{--}. Applying the shift functor to ξ^+ yields $\xi^+[1] = (\xi_2[1], D_2[1]) \xrightarrow{\varphi_2[1]} (\xi_{12}[1], D_{12}[1])$, where $\varphi_2[1] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

On the other hand, at the second vertex, the composed formula ξ^{--} evaluates to $(\xi_{12}[1], -D^{\star}_{12}[1])$. At the first vertex, we need a map from $(\xi_2[1], D_2[1])$ to the cone (ξ_{121}, D_{121}) We use the canonical inclusion into the middle component, which is exactly $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

At the second vertex, to connect $(\xi_{12}[1], D_{12}[1])$ to $(\xi_{12}[1], -D^{\star}_{12}[1])$, we apply the map $N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

This yields the following diagram:

$$\begin{array}{ccc} (\xi_2[1], D_2[1]) & \xrightarrow{\quad \alpha_2 \quad} & (\xi_{121}, D_{121}) \\ \Big\downarrow{\scriptstyle \varphi_2[1]} & & \Big\downarrow{\scriptstyle M_{--}} \\ (\xi_{12}[1], D_{12}[1]) & \xrightarrow[\quad N \quad]{} & (\xi_{12}[1], -D^{\star}_{12}[1]). \end{array}$$

This is a commutative square, then it defines the natural transformation ϵ^{--}.

Step 3: (Quasi-isomorphisms property.)

Finally, we show that the natural transformations ϵ^{++} and ϵ^{--} are quasi-isomorphisms. Recall that a morphism in the diagram category $C(\mathcal{A})^{1\to 2}$ is a quasi-isomorphism if and only if it is a quasi-isomorphism pointwise at each vertex of the poset. Furthermore, any homotopy equivalence of chain complexes induces a quasi-isomorphism.

For ϵ^{++}: The components of the morphism of formulas evaluated at the two vertices are I_2 and $-\beta_1$, respectively.

- At the first vertex, the identity matrix I_2 induces an isomorphism of complexes, which is in particular a quasi-isomorphism.
- At the second vertex, we must analyze the map induced by $-\beta_1$. As explicitly shown in Proposition 2.2.23, β_1 is a homotopy equivalence with right inverse α_1, satisfying the homotopy equation
$$I_3 - \alpha_1\beta_1 = h_1[1]D_{212} + D^{\star}_{212}[-1]h_1.$$
Consequently, $-\beta_1$ is also a homotopy equivalence, with right inverse $-\alpha_1$. Therefore, $-\beta_1$ induces a quasi-isomorphism.

Since ϵ^{++} is a quasi-isomorphism at both vertices, ϵ^{++} evaluates to a quasi-isomorphism for any diagram of complexes K.

For ϵ^{--}: The components of the morphism evaluated at the two vertices are α_2 and the matrix $N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

- At the first vertex, the map is given by α_2. By Proposition 2.2.23, α_2 is a homotopy equivalence with left inverse β_2, then it induces a quasi-isomorphism.
- At the second vertex, the map is given by the matrix N, which is an involution ($N^2 = I_2$). Then it induces an isomorphism between chain complexes, which trivially is also a quasi-isomorphism.

Since ϵ^{--} is a quasi-isomorphism at each vertex, it is a quasi-isomorphism.

□

The previous two propositions provide natural transformations between various composites of R^+ and R^- whose components are quasi-isomorphisms. Passing to the derived category, these maps become natural isomorphisms, yielding strong relations between the induced functors.

Corollary 2.2.25 — Let $\mathcal{A}$ be an abelian category. The functors R^+ and R^- preserve quasi-isomorphisms and therefore induce triangulated endofunctors
$$\tilde{R}^+, \tilde{R}^- : D(\mathcal{A}^{1\to 2}) \to D(\mathcal{A}^{1\to 2}).$$
Moreover, the following natural isomorphisms hold in $D(\mathcal{A}^{1\to 2})$:
$$\tilde{R}^+ \circ \tilde{R}^- \cong [1] \cong \tilde{R}^- \circ \tilde{R}^+, \quad (\tilde{R}^+)^2 \cong \tilde{R}^- \quad (\tilde{R}^-)^2 \cong \tilde{R}^+ \circ [1].$$
In particular,
$$(\tilde{R}^+)^3 \cong [1].$$

Proof. Since R^+ and R^- are defined via formulas, they send quasi-isomorphisms to quasi-isomorphisms; hence they descend to triangulated endofunctors on the derived category. We denote the induced functors by $\tilde{R}^+$ and $\tilde{R}^-$.

By Proposition 2.2.23, there are natural transformations
$$\epsilon^{+-} : R^+ \circ R^- \Rightarrow [1], \quad \epsilon^{-+} : [1] \Rightarrow R^- \circ R^+,$$
whose components are quasi-isomorphisms. After passing to $D(\mathcal{A}^{1\to 2})$, these become natural isomorphisms, hence
$$\tilde{R}^+ \circ \tilde{R}^- \cong [1] \quad \text{and} \quad [1] \cong \tilde{R}^- \circ \tilde{R}^+.$$

In particular, $\tilde{R}^+$ and $\tilde{R}^-$ are autoequivalences.

By Proposition 2.2.24, there are natural transformations

$$\epsilon^{++}\colon R^+ \circ R^+ \Rightarrow R^-, \qquad \epsilon^{--}\colon R^+ \circ [1] \Rightarrow R^- \circ R^-,$$

whose components are quasi-isomorphisms. Passing again to the derived category yields natural isomorphisms

$$(\tilde{R}^+)^2 \cong \tilde{R}^-, \quad \text{and} \quad \tilde{R}^+ \circ [1] \cong (\tilde{R}^-)^2.$$

Finally,

$$(\tilde{R}^+)^3 \cong \tilde{R}^+ \circ (\tilde{R}^+)^2 \cong \tilde{R}^+ \circ \tilde{R}^- \cong [1].$$

□

2.2.3 Proof of the main theorem

The proof of Theorem 2.2.4 can be regarded as a global version of the elementary $1 \to 2$ case treated in the previous subsection (see Proposition 2.2.23). The key idea is that the local construction appearing in that case is repeated at each vertex $x \in X$, where the role of the second endpoint is played by the subset $Y_x \subseteq Y$, and these local pieces are then assembled through suitably defined restriction maps. In this way, the case of the chain $1 \to 2$ provides the basic local model for the general construction.

Set

$$P^+ := (X \sqcup Y, \leq_+) \quad \text{and} \quad P^- := (X \sqcup Y, \leq_-).$$

We define two formulas ξ^+ and ξ^-, inducing functors

$$R^+\colon C(\mathcal{A})^{P^+} \to C(\mathcal{A})^{P^-} \quad \text{and} \quad R^-\colon C(\mathcal{A})^{P^-} \to C(\mathcal{A})^{P^+}.$$

Throughout the proof, we keep in mind the singleton example of Example 2.2.3, where $X = X_1$ and the subsets Y_x are given by $Y_x = \{f(x)\}$ for a uniquely determined order-preserving map $f\colon X_1 \to Y$. We use this example in order to make the constructions more transparent and to illustrate the different types of restriction maps.

Definition of the formulas to a point:

We begin by defining the formulas to a point associated with the elements of $X \sqcup Y$.
For each $x \in X$ and $y \in Y$, we introduce the elementary formulas to a point

$$\xi_x := ((x,0),(1)), \qquad \xi_y := ((y,0),(1)),$$

and, for every $x \in X$,

$$\xi_{Y_x} := ((y,0)_{y \in Y_x}, I),$$

where I denotes the identity matrix.

By Example 2.2.13, the formulas ξ_x and ξ_y simply evaluate a diagram at the points x and y, respectively. Thus, for a diagram K, they produce the complexes K_x and K_y. Similarly, the formula ξ_{Y_x} produces the direct sum

$$F_{\xi_{Y_x}}(K) = \bigoplus_{y \in Y_x} K_y.$$

In the running example X_1, each Y_x is a singleton, so this direct sum reduces to a single term.

For each $x \in X$, these formulas determine a local copy of the construction for the chain $1 \to 2$. More precisely, we consider the two formulas to $1 \to 2$

$$\xi_{x,Y_x} := (\xi_x \to \xi_{Y_x}) \in Ob(\mathcal{F}^{1\to 2}_{P^+}), \qquad \xi_{Y_x,x} := (\xi_{Y_x} \to \xi_x) \in Ob(\mathcal{F}^{1\to 2}_{P^-}),$$

where the first arrow is given by the column vector whose entries are all equal to 1, and the second by the row vector whose entries are all equal to 1. Evaluating these formulas on diagrams $K \in Ob(C(\mathcal{A})^{P^+})$ and $L \in Ob(C(\mathcal{A})^{P^-})$, we obtain the natural maps

$$K_x \to \bigoplus_{y \in Y_x} K_y, \qquad \bigoplus_{y \in Y_x} L_y \to L_x.$$

More explicitly, if $\{r_{xy}\}$ and $\{s_{yx}\}$ denote the restriction maps in K and L, then these morphisms are

$$\sum_{y \in Y_x} \iota_y r_{xy} \quad \text{and} \quad \sum_{y \in Y_x} s_{yx} \pi_y,$$

where ι_y and π_y are the canonical inclusions and projections.

By the discussion of the chain $1 \to 2$, the formula ξ_{12} yields the mapping cone of a morphism. We therefore define

$$\xi_x^+ := \xi_{12} \circ \xi_{x,Y_x} \qquad \xi_x^- := \xi_{12} \circ \xi_{Y_x,x}.$$

Thus, for each $x \in X$, the general construction is obtained by applying the elementary $1 \to 2$ construction to the corresponding local map.

For $y \in Y$, we simply set

$$\xi_y^+ := \xi_y \in Ob(\mathcal{F}_{P^+}), \qquad \xi_y^- := \xi_y \in Ob(\mathcal{F}_{P^-}).$$

Hence, if $K \in Ob(C(\mathcal{A})^{P^+})$ and $L \in Ob(C(\mathcal{A})^{P^-})$, the corresponding functors satisfy

$$R^+(K)_y = K_y, \qquad R^-(L)_y = L_y[1],$$

while for $x \in X$ we obtain

$$R^+(K)_x = C\left(K_x \xrightarrow{\sum_{y \in Y_x} \iota_y r_{xy}} \bigoplus_{y \in Y_x} K_y\right),$$

and

$$R^-(L)_x = C\left(\bigoplus_{y \in Y_x} L_y \xrightarrow{\sum_{y \in Y_x} s_{yx} \pi_y} L_x\right).$$

Thus, the vertices of Y are treated directly, whereas at each vertex $x \in X$ the original complex is replaced by a suitable mapping cone.

Definition of the restriction maps:

We now define the restriction maps of the diagrams $R^+(K)$ and $R^-(L)$. More precisely, after having defined the complexes attached to each vertex of P^- and P^+, we must specify, for every order relation in the target poset, the corresponding morphism between these complexes. With these definitions, the assignments defining R^+ and R^- become well-defined diagrams on $C(\mathcal{A}^{P^-})$ and $C(\mathcal{A}^{P^+})$, respectively. We distinguish several cases according to the type of relation involved.

We denote by ρ^+ the restriction maps in the diagram $R^+(K)$, and by ρ^- those in $R^-(L)$.

- **Case 1: relations inside Y**
  If $y \leq y'$ in Y, the restriction maps are the obvious ones. Indeed, by construction we have

  $$R^+(K)_y = K_y, \quad \text{and} \quad R^-(L)_y = L_y[1].$$

  Therefore we define

  $$\rho^+_{yy'}(K) := r_{yy'} \colon K_y \to K_{y'}, \qquad \rho^-_{yy'}(L) := s_{yy'}[1] \colon L_y[1] \to L_{y'}[1].$$

  In other words, on the vertices of Y, the functor R^+ keeps the original restriction maps, while R^- keeps them up to the shift.

- **Case 2: relations inside X**
  If $x \leq x'$, both $R^+(K)_x$ and $R^+(K)_{x'}$, as well as $R^-(L)_x$ and $R^-(L)_{x'}$, are mapping cones. Therefore the corresponding restriction maps are defined componentwise: on the first summand we use the restriction map $r_{xx'}[1]$, while on the second summand we use the bijection $\varphi_{xx'} \colon Y_x \to Y_{x'}$ to match the corresponding summands.

  Since $y \leq \varphi_{xx'}(y)$ for every $y \in Y_x$, the maps

  $$r_{y,\varphi_{xx'}(y)} \colon K_y \to K_{\varphi_{xx'}(y)} \quad \text{and} \quad s_{y,\varphi_{xx'}(y)} \colon L_y \to L_{\varphi_{xx'}(y)}$$

  are well-defined. We therefore define the diagonal morphisms

  $$\rho^+_{xx'}(K) \coloneqq r_{xx'}[1] \oplus \left(\bigoplus_{y \in Y_x} r_{y,\varphi_{xx'}(y)} \right) : R^+(K)_x \to R^+(K)_{x'},$$

  and

  $$\rho^-_{xx'}(L) \coloneqq \left(\bigoplus_{y \in Y_x} s_{y,\varphi_{xx'}(y)}[1] \right) \oplus s_{xx'} : R^-(L)_x \to R^-(L)_{x'}.$$

  These are well-defined chain maps, since the corresponding squares commute by functoriality of the restriction maps and by the relation $y \leq \varphi_{xx'}(y)$ for all $y \in Y_x$.

  In other words, on the X-part the restriction maps are obtained by acting separately on the two components of the cone: on the first one, we use the restriction map between the vertices x and x', while on the second one we use the bijection $\varphi_{xx'} \colon Y_x \to Y_{x'}$ to match the corresponding summands.

- **Case 3: elementary mixed relations**
  If $y_x \in Y_x$, then by definition of the orders $\leq_+$ and $\leq_-$, we have

  $$y_x \leq_- x \qquad \text{and} \qquad x \leq_+ y_x.$$

  Now we define the corresponding restriction maps.

  For $R^+(K)$, recall that

  $$R^+(K)_x = C\left(K_x \xrightarrow{\sum_{y \in Y_x} \iota_y r_{xy}} \bigoplus_{y \in Y_x} K_y \right), \qquad R^+(K)_{y_x} = K_{y_x}.$$

  Hence we define

  $$\rho^+_{y_x x}(K) \coloneqq \begin{pmatrix} 0 \\ \iota_{y_x} \end{pmatrix} : K_{y_x} \to C\left(K_x \xrightarrow{\sum_{y \in Y_x} \iota_y r_{xy}} \bigoplus_{y \in Y_x} K_y \right).$$

  In other words, this map sends K_{y_x} into the cone by placing it in the second component, namely in the direct sum $\bigoplus_{y \in Y_x} K_y$, through the canonical inclusion ι_{y_x}. Similarly, for $R^-(L)$ we have

  $$R^-(L)_x = C\left(\bigoplus_{y \in Y_x} L_y \xrightarrow{\sum_{y \in Y_x} s_{yx} \pi_y} L_x \right), \qquad R^-(L)_{y_x} = L_{y_x}[1].$$

  We define

  $$\rho^-_{x y_x}(L) \coloneqq \begin{pmatrix} \pi_{y_x}[1] & 0 \end{pmatrix} : C\left(\bigoplus_{y \in Y_x} L_y \xrightarrow{\sum_{y \in Y_x} s_{yx} \pi_y} L_x \right) \to L_{y_x}[1].$$

Thus, in the case of an elementary mixed relation, the restriction map is obtained by selecting the distinguished summand indexed by y_x: in $R^+(K)$ via the canonical inclusion into the cone, and in $R^-(L)$ via the canonical projection from the cone into its shifted first component. These are exactly the local maps appearing in the elementary construction for the chain $1 \to 2$.

- **Case 4: general mixed relations**
  It remains to treat the non-elementary mixed relations.

  Suppose first that $y \leq_- x$. By the construction of the order $\leq_-$ on $X \sqcup Y$, there exists an element $y_x \in Y_x$ such that $y \leq y_x$. Moreover, this element is unique by condition (i) on the family $\{Y_x\}_{x \in X}$ (see the beginning of the proof of Lemma 2.2.1).

  The relation $y \leq_- x$ can therefore be decomposed as

  $$y \leq y_x \leq_- x,$$

  where the first relation lies inside Y, and the second is an elementary mixed relation.

  We then define

  $$\rho^+_{yx}(K) := \rho^+_{y_x x}(K) \circ \rho^+_{y y_x}(K).$$

  Explicitly, this is the composite

  $$K_y \xrightarrow{r_{y y_x}} K_{y_x} \xrightarrow{\begin{pmatrix} 0 \\ \iota_{y_x} \end{pmatrix}} R^+(K)_x.$$

  Dually, suppose that $x \leq_+ y$. By the construction of the order $\leq_+$ on $X \sqcup Y$, there exists an element $y_x \in Y_x$ such that $y_x \leq y$. Moreover, this element is unique by condition (i) on the family $\{Y_x\}_{x \in X}$ (see the beginning of the proof of Lemma 2.2.1).

  Hence, the relation $x \leq_+$ factors as

  $$x \leq_+ y_x \leq y,$$

  where the first step is elementary mixed and the second lies inside Y.

  We define

  $$\rho^-_{xy}(L) := \rho^-_{y_x x}(L) \circ \rho^-_{x y_x}(L).$$

  Explicitly, this is the composite

  $$R^-(L)_x \xrightarrow{(\pi_{y_x}[1] \quad 0)} L_{y_x}[1] \xrightarrow{s_{y_x y}[1]} L_y[1].$$

  Therefore, every mixed restriction map is obtained by passing through the unique intermediate vertex $y_x \in Y_x$ singled out by the defining relation. This makes the construction completely local: once the elementary mixed maps are understood, all the other mixed maps are obtained by composition with the restriction maps inside Y.

With these definitions, every relation in the target poset is assigned a morphism of complexes. The identity and compatibility conditions follow from the corresponding properties of the original restriction maps and from the uniqueness of the element $y_x \in Y_x$ appearing in the mixed cases. Hence the above assignments define two formulas

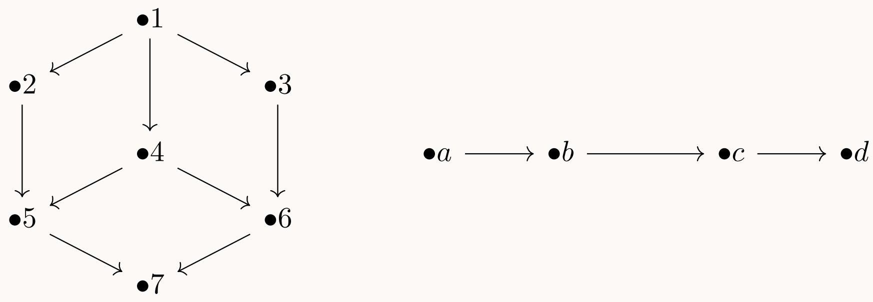
$$\xi^+ \in Ob(\mathcal{F}^{P-}_{P+}) \quad \text{and} \quad \xi^- \in Ob(\mathcal{F}^{P+}_{P-}),$$

and therefore two functors

$$R^+ : C(\mathcal{A})^{P^+} \to C(\mathcal{A})^{P^-}, \qquad R^- : C(\mathcal{A})^{P^-} \to C(\mathcal{A})^{P^+}.$$

Example 2.2.26

Before proceeding with the explicit description of the functors R^+ and R^-, we briefly recall the running example introduced in Example 2.2.3, which we will use throughout the construction. Recall that, in Example 2.2.3, we chose $X = X_1$ and Y as follows.



X_1 $\qquad\qquad$ Y

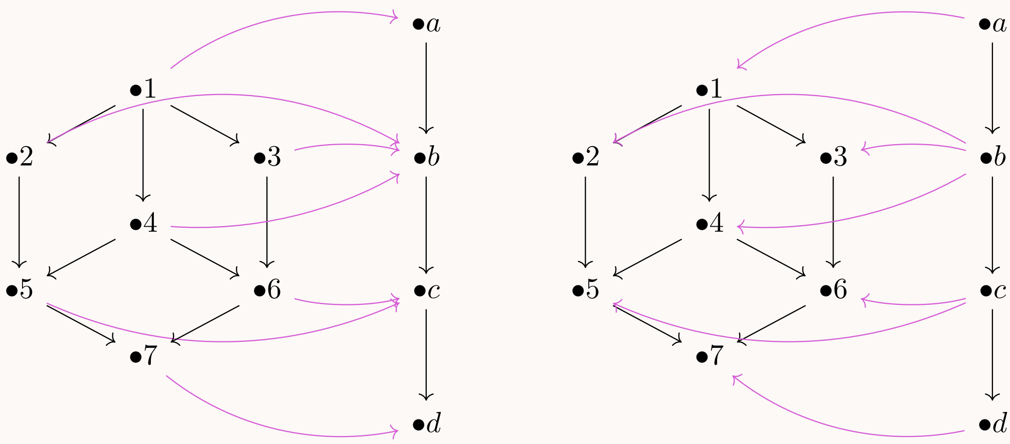
Recall also that the family $\{Y_x\}_{x\in X}$ is defined by

$$Y_1 = \{a\};\ \ Y_2 = Y_3 = Y_4 = \{b\};\ \ Y_5 = Y_6 = \{c\};\ \ Y_7 = \{d\},$$

or equivalently, that we have an order-preserving map $f\colon X_1 \to Y$ such that $Y_x = \{f(x)\}$ for every $x \in X_1$, namely

$$f(1) = a;\ \ f(2) = f(3) = f(4) = b;\ \ f(5) = f(6) = c;\ \ f(7) = d.$$

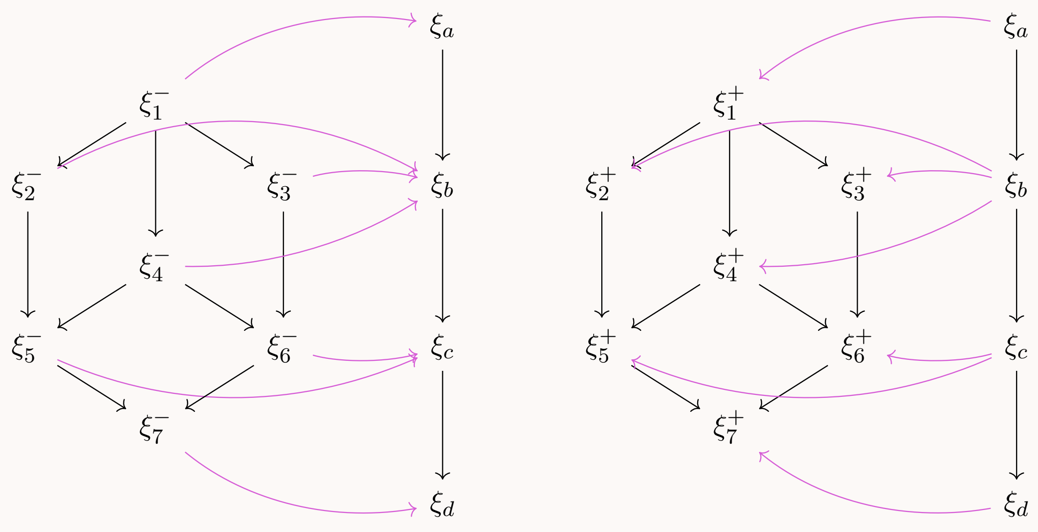
For this choice of data, the construction yields the following posets:



$P^+ = (X_1 \sqcup Y, \leq_+)$ $\qquad\qquad$ $P^- = (X_1 \sqcup Y, \leq_-)$

These are precisely the two posets whose universal derived equivalence is established by Theorem 2.2.4.

Now we describe explicitly the two general formulas ξ^+ and ξ^- on this example. Recall that $\xi^+ \in Ob(\mathcal{F}_{P^+}^{P^-})$ and $\xi^- \in Ob(\mathcal{F}_{P^-}^{P^+})$. Therefore, ξ^+ is a diagram over P^- with values in $\mathcal{F}_{P^+}$, while ξ^- is a diagram over P^+ with values in $\mathcal{F}_{P^-}$. Thus the general formula ξ^+ has the global shape of the poset P^-, and at each of its vertices it assigns a formula to a point of P^+. Dually, the general formula ξ^- has the global shape of the poset P^+, and at each of its vertices it assigns a formula to a point of P^-. Then we have the following objects:



ξ^- ξ^+

Let us describe them more explicitly: for every $y \in Y$, the value of both general formulas at the vertex y is the formula $\xi_y = ((y,0),(1))$. Thus, at the vertices a, b, c, d both ξ^+ and ξ^- assign the corresponding formula to a point

$$\xi_a = ((a,0),(1)), \quad \xi_b = ((b,0),(1)), \quad \xi_c = ((c,0),(1)), \quad \xi_d = ((d,0),(1)).$$

Now we analyze what happens for $x \in X_1$. Since each Y_x is a singleton, and we have $Y_x = \{f(x)\}$, the formulas ξ_{Y_x} reduce to the one-point formulas $\xi_{f(x)}$. Hence, the local formulas at the vertices of X_1 become the elementary two-point construction

$$\xi_x^+ = \xi_{12} \circ \xi_{x,f(x)}, \qquad \xi_x^- = \xi_{12} \circ \xi_{f(x),x},$$

where $\xi_{x,f(x)}$ is the formula to the chain $1 \to 2$ determined by the morphism $\xi_x \to \xi_{f(x)}$, and $\xi_{f(x),x}$ is the formula to the chain $1 \to 2$ determined by the morphism $\xi_{f(x)} \to \xi_x$.

Equivalently, at each vertex $x \in X_1$, the formula ξ^+ assigns the local copy of the elementary construction associated with the arrow $x \to f(x)$, while ξ^- assigns the dual local copy associated with the arrow $f(x) \to x$.

In this example, this gives the following list:

- ξ_1^+ corresponds to $1 \to a$,
- $\xi_2^+, \xi_3^+, \xi_4^+$ correspond to $2 \to b, 3 \to b, 4 \to b$,
- ξ_5^+, ξ_6^+ correspond to $5 \to c, 6 \to c$,
- ξ_7^+ corresponds to $7 \to d$,

and dually

- ξ_1^- corresponds to $a \to 1$,
- $\xi_2^-, \xi_3^-, \xi_4^-$ correspond to $b \to 2, b \to 3, b \to 4$,
- ξ_5^-, ξ_6^- correspond to $c \to 5, c \to 6$,
- ξ_7^- corresponds to $d \to 7$.

The general formulas ξ^+ and ξ^- induce the corresponding functors

$$R^+ \colon C(\mathcal{A})^{P^+} \to C(\mathcal{A})^{P^-}, \qquad R^- \colon C(\mathcal{A})^{P^-} \to C(\mathcal{A})^{P^+},$$

and describe explicitly the complexes attached to the vertices in the running example.
Let $K \in Ob(C(\mathcal{A})^{P^+})$ and $L \in Ob(C(\mathcal{A})^{P^-})$.
For the vertices of Y, the construction is immediate:

$$R^+(K)_y = K_y, \qquad R^-(L)_y = L_y[1] \qquad \text{for every } y \in Y.$$

Hence, we have

$$R^+(K)_a = K_a, \quad R^+(K)_b = K_b, \quad R^+(K)_c = K_c, \quad R^+(K)_d = K_d,$$

and

$$R^-(L)_a = L_a[1], \quad R^-(L)_b = L_b[1], \quad R^-(L)_c = L_c[1], \quad R^-(L)_d = L_d[1].$$

The direct sums appearing in the general construction reduce to a single summand. Therefore, the complexes attached to the vertices of X_1 become

$$R^+(K)_x = C(K_x \xrightarrow{r_{x,f(x)}} K_{f(x)}), \qquad R^-(L)_x = C(L_{f(x)} \xrightarrow{s_{f(x),x}} L_x).$$

Thus, on the vertices of Y, the functor R^+ leaves the original complexes unchanged, whereas R^- shifts them by [1]. On the vertices of X_1, both functors are obtained by taking the mapping cone of the unique morphism determined by the value of f at that vertex.

We next describe the restriction maps of the diagrams $R^+(K)$ and $R^-(L)$ in the running example. As above, we denote by

$$r_{uv} \colon K_u \to K_v \quad \text{and} \quad s_{uv} \colon L_u \to L_v$$

the restriction maps in $K \in Ob(C(\mathcal{A})^{P^+})$ and $L \in Ob(C(\mathcal{A})^{P^-})$, respectively. As in the general construction, we distinguish four cases.

- **Case 1: relations inside Y**

  For every relation $y \leq y'$ in Y, the restriction maps are:

  $$\rho^+_{yy'}(K) = r_{yy'}, \qquad \rho^-_{yy'}(L) = s_{yy'}[1].$$

  Then in particular, we have:

  $$\rho^+_{ab}(K) = r_{ab}, \quad \rho^+_{bc}(K) = r_{bc}, \quad \rho^+_{cd}(K) = r_{cd},$$

  $$\rho^-_{ab}(L) = s_{ab}[1], \quad \rho^-_{bc}(L) = s_{bc}[1], \quad \rho^-_{cd}(L) = s_{cd}[1].$$

  All the remaining maps inside Y are obtained by composition.

- **Case 2: relations inside X_1**

  Since f is an order-preserving map, for $x \leq x'$ in X_1 we have $f(x) \leq f(x')$. The restriction maps are diagonal block maps acting separately on the two components of the cones. For example, for the covering relation $1 \leq 2$:

$$\rho^+_{12}(K) = \begin{pmatrix} r_{12}[1] & 0 \\ 0 & r_{ab} \end{pmatrix}, \qquad \rho^-_{12}(L) = \begin{pmatrix} s_{ab}[1] & 0 \\ 0 & s_{12} \end{pmatrix}.$$

All covering relations in X_1 yield maps of the same form, with $r_{xx'}[1]$ and $r_{f(x)f(x')}$ (respectively $s_{f(x)f(x')}[1]$ and $s_{xx'}$) as diagonal entries. The restriction maps for non-covering relations are then obtained by composition.

- **Case 3: elementary mixed relations**

  The elementary mixed relations are of the form

  $$f(x) \le_- x \quad \text{and} \quad x \le_+ f(x).$$

  In each case the restriction map is either the canonical inclusion into the second component of the cone, or the canonical projection onto the shifted first component. For example, at the vertex $1 \in X_1$:

  $$\rho^+_{a1}(K) = \begin{pmatrix} 0 \\ 1_{K_a} \end{pmatrix}, \qquad \rho^-_{1a}(L) = \begin{pmatrix} 1_{L_a}[1] & 0 \end{pmatrix}.$$

  The maps at the remaining vertices of X_1 have the same form, with $K_{f(x)}$ and $L_{f(x)}$ in place of K_a and L_a.

- **Case 4: general mixed relations**

  Every remaining mixed relation is of the form

  $$y \le f(x) \le_- x \quad \text{or} \quad x \le_+ f(x) \le y.$$

  For example, the relation $a \le_- 2$ factors as $a \le b \le_- 2$. The corresponding restriction map is:

  $$\rho^+_{a2}(K) = \rho^+_{b2}(K) \circ \rho^+_{ab}(K) = \begin{pmatrix} 0 \\ 1_{K_b} \end{pmatrix} \circ r_{ab} = \begin{pmatrix} 0 \\ r_{ab} \end{pmatrix}.$$

  Dually, the relation $2 \le_+ c$ factors as $2 \le_+ b \le c$. Hence the corresponding restriction map is:

  $$\rho^-_{2c}(L) = \rho^-_{bc}(L) \circ \rho^-_{2b}(L) = s_{bc}[1] \circ \begin{pmatrix} 1_{L_b}[1] & 0 \end{pmatrix} = \begin{pmatrix} s_{bc}[1] & 0 \end{pmatrix}.$$

This completes the explicit description of $R^+(K)$ and $R^-(L)$ in the running example. In each case, the restriction maps are entirely determined by the local cone construction at each vertex X_1 and by the order-preserving map f.

Verification of commutativity:

We verify that the restriction maps satisfy the compatibility conditions

$$\rho^+_{xz} = \rho^+_{yz} \circ \rho^+_{xy} \quad \text{and} \quad \rho^-_{xz} = \rho^-_{yz} \circ \rho^-_{xy}$$

for every composable chain of relations in P^+ and P^-, respectively.
Indeed, ρ^+ denotes the system of restriction maps of the diagram $R^+(K) \in C(\mathcal{A})^{P^-}$, while ρ^- denotes the system of restriction maps of $R^-(L) \in C(\mathcal{A})^{P^+}$. We distinguish several cases according to the type of relations involved.

- **Case 1:** $y \le y' \le y''$ **in** Y**.**
  By definition, $\rho^+_{yy'}(K) = r_{yy'}$ and $\rho^+_{y'y''}(K) = r_{y'y''}$. Therefore

  $$\rho^+_{y'y''}(K) \circ \rho^+_{yy'}(K) = r_{y'y''} \circ r_{yy'} = r_{yy''} = r_{y'y''}(K),$$

  where the middle equality is the compatibility condition for the restriction maps of the diagram K.

The proof for ρ^- is analogous: since $\rho^-_{yy'}(L) = s_{yy'}[1]$ and $\rho^-_{y'y''}(L) = s_{y'y''}[1]$, we obtain

$$\rho^-_{y'y''}(L) \circ \rho^-_{yy'}(L) = s_{y'y''}[1] \circ s_{yy'}[1] = s_{yy''}[1] = \rho^-_{yy''}(L).$$

- **Case 2:** $x \leq x' \leq x''$ **in** X**.**
Recall that

$$\rho^+_{xx'}(K) = r_{xx'}[1] \oplus \bigoplus_{y \in Y_x} r_{y\,\varphi_{xx'}(y)}$$

and

$$\rho^+_{x'x''}(K) = r_{x'x''}[1] \oplus \bigoplus_{y' \in Y_{x'}} r_{y'\,\varphi_{x'x''}(y')}.$$

Since $\varphi_{xx''} = \varphi_{x'x''} \circ \varphi_{xx'}$ by Lemma 2.2.2, we can write the second map as

$$\rho^+_{x'x''}(K) = r_{x'x''}[1] \oplus \bigoplus_{y \in Y_x} r_{\varphi_{xx'}(y), \varphi_{x'x''}(y)}.$$

Hence $\rho^+_{x'x''}(K) \circ \rho^+_{xx'}(K)$ is again a diagonal map, whose first diagonal entry is

$$r_{x'x''}[1] \circ r_{xx'}[1] = r_{xx''}[1],$$

and whose y-th diagonal entry is

$$r_{\varphi_{xx'}(y)\,\varphi_{xx''}(y)} \circ r_{y\,\varphi_{xx'}(y)} = r_{y\,\varphi_{xx''}(y)}.$$

The first equality follows from the compatibility of the restriction maps in K, and the second one from the same compatibility applied inside Y-part. Therefore

$$\rho^+_{x'x''}(K) \circ \rho^+_{xx'}(K) = r_{xx''}[1] \oplus \bigoplus_{y \in Y_x} r_{y\,\varphi_{xx'}(y)} = \rho^+_{xx''}(K).$$

The proof for ρ^- is analogous. Indeed,

$$\rho^-_{xx'}(L) = \left(\bigoplus_{y \in Y_x} s_{y\,\varphi_{xx'}(y)}[1] \right) \oplus s_{xx'}$$

and

$$\rho^-_{x'x''}(L) = \left(\bigoplus_{y' \in Y_{x'}} s_{y'\,\varphi_{x'x''}(y')}[1] \right) \oplus s_{x'x''},$$

so, using again $\varphi_{xx''} = \varphi_{x'x''} \circ \varphi_{xx'}$, we obtain

$$\rho^-_{x'x''}(L) \circ \rho^-_{xx'}(L) = \rho^-_{xx''}(L).$$

- **Case 3:** $y' \leq y \leq_- x$**.**
Let $y_x, y'_x \in Y_x$ be the elements determined by the relations

$$y \leq y_x \quad \text{and} \quad y' \leq y'_x.$$

Since $y' \leq y \leq y_x$, the uniqueness part of the condition (i) implies that $y'_x = y_x$. Therefore, by the definition of the mixed restriction maps,

$$\rho^+_{y'x}(K) = \rho^+_{y_x x}(K) \circ \rho^+_{y' y_x}(K).$$

Using the compatibility inside Y, we also have

$$\rho^+_{y' y_x}(K) = \rho^+_{y y_x}(K) \circ \rho^+_{y'y}(K).$$

Hence

$$\rho^+_{y'x}(K) = \rho^+_{y_x x}(K) \circ \rho^+_{y y_x}(K) \circ \rho^+_{y'y}(K) = \rho^+_{yx}(K) \circ \rho^+_{y'y}(K),$$

which is exactly the required identity. The dual argument proves the corresponding statement for ρ^- in the case $x \leq_+ y \leq y'$.

- **Case 4:** $y_x \le_- x \le x'$ **where** $y_x \in Y_x$**.**
  Let $y_{x'} := \varphi_{xx'}(y_x) \in Y_{x'}$. By condition (ii), we have $y_x \le y_{x'}$. Moreover, $y_{x'}$ is the unique element of $Y_{x'}$ with this property. We claim that
$$\rho^+_{y_x x'}(K) = \rho^+_{xx'}(K) \circ \rho^+_{y_x x}(K).$$
  Indeed, by definition,
$$\rho^+_{y_x x}(K) = \begin{pmatrix} 0 \\ \iota_{y_x} \end{pmatrix} : K_{y_x} \to C\left(K_x \to \bigoplus_{y \in Y_x} K_y\right),$$
  while
$$\rho^+_{xx'}(K) = r_{xx'}[1] \oplus \bigoplus_{y \in Y_x} r_{y\,\varphi_{xx'}(y)}.$$
  Thus the composite $\rho^+_{xx'}(K) \circ \rho^+_{y_x x}(K)$ has zero first component, while on the second component it sends K_{y_x} into the summand indexed by $y_{x'} = \varphi_{xx'}(y_x)$ through the map $r_{y_x y_{x'}} : K_{y_x} \to K_{y_{x'}}$. In other words,
$$\rho^+_{xx'}(K) \circ \rho^+_{y_x x}(K) = \begin{pmatrix} 0 \\ \iota_{y_{x'}} r_{y_x y_{x'}} \end{pmatrix}.$$
  On the other hand, by the definition of the general mixed map,
$$\rho^+_{y_x x'}(K) = \rho^+_{y_{x'} x'}(K) \circ \rho^+_{y_x y_{x'}}(K) = \begin{pmatrix} 0 \\ \iota_{y_{x'}} \end{pmatrix} \circ r_{y_x\,y_{x'}} = \begin{pmatrix} 0 \\ \iota_{y_{x'}} r_{y_x y_{x'}} \end{pmatrix}.$$
  Therefore
$$\rho^+_{y_x x'}(K) = \rho^+_{xx'}(K) \circ \rho^+_{y_x x}(K).$$
  The proof for ρ^- is dual.

- **Case 5:** $y \le_- x \le x'$**.**
  Let $y_x \in Y_x$ be the unique element such that $y \le y_x$. Then the chain $y \le_- x \le x'$ can be refined as $y \le y_x \le_- x \le x'$. By the definition of the general mixed restriction map, we have
$$\rho^+_{yx}(K) = \rho^+_{y_x x}(K) \circ \rho^+_{y y_x}(K).$$
  On the other hand, by Case 4,
$$\rho^+_{y_x x'}(K) = \rho^+_{xx'}(K) \circ \rho^+_{y_x x}(K).$$
  Therefore
$$\rho^+_{yx'}(K) = \rho^+_{y_x x'}(K) \circ \rho^+_{y y_x}(K) = \rho^+_{xx'}(K) \circ \rho^+_{y_x x}(K) \circ \rho^+_{y y_x}(K) = \rho^+_{xx'}(K) \circ \rho^+_{yx}(K),$$
  which is exactly the required compatibility relation.

  The proof for ρ^- is dual, considering a chain $x \le x' \le_+ y$.

Therefore, in all possible cases, the restriction maps satisfy compatibility conditions with composition. Hence the assignments defined above determine well-defined diagrams
$$R^+(K) \in C(\mathcal{A})^{P^-} \quad \text{and} \quad R^-(L) \in C(\mathcal{A})^{P^+}.$$

Example 2.2.27

Before turning to the construction of the natural transformations, let us briefly illustrate the previous verification on the running example by checking two representative nontrivial compatibility relations involving general mixed relations. In P^- the relation $a \le_- 2$ factors as $a \le b \le_- 2$, so by the explicit description above we have $\rho^+_{a2}(K) = \rho^+_{b2}(K) \circ \rho^+_{ab}(K)$.

We first illustrate Case 4 on the chain $b \le_- 2 \le 5$ in P^-, where $b = f(2) \in Y_2$. By Case 3, the elementary mixed restriction map is

$$\rho^+_{b2}(K) = \begin{pmatrix} 0 \\ \mathrm{id}_{K_b} \end{pmatrix} : K_b \to C(K_2 \to K_b),$$

and by Case 2, the restriction map inside X_1 is

$$\rho^+_{25}(K) = \begin{pmatrix} r_{25}[1] & 0 \\ 0 & r_{bc} \end{pmatrix} : C(K_2 \to K_b) \to C(K_5 \to K_c),$$

where r_{bc} appears because $\varphi_{25}(b) = c$. The composite is

$$\rho^+_{25}(K) \circ \rho^+_{b2}(K) = \begin{pmatrix} r_{25}[1] & 0 \\ 0 & r_{bc} \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{K_b} \end{pmatrix} = \begin{pmatrix} 0 \\ r_{bc} \end{pmatrix}.$$

On the other hand, the relation $b \leq_- 5$ factors as $b \leq c \leq_- 5$, so the general mixed restriction map is

$$\rho^+_{b5}(K) = \rho^+_{c5}(K) \circ \rho^+_{bc}(K) = \begin{pmatrix} 0 \\ \mathrm{id}_{K_c} \end{pmatrix} \circ r_{bc} = \begin{pmatrix} 0 \\ r_{bc} \end{pmatrix}.$$

Therefore $\rho^+_{b5}(K) = \rho^+_{25}(K) \circ \rho^+_{b2}(K)$, confirming the compatibility of Case 4 on this relation. This is the point where the bijection $\varphi_{25} \colon Y_2 \to Y_5$ enters concretely: it sends b to c, and the map r_{bc} in the second component of $\rho^+_{25}(K)$ is precisely the restriction map $r_{b,\varphi_{25}(b)}$.

Analogously, the Case 5 chain $a \leq_- 2 \leq 5$ (where $a \notin Y_2$) requires an additional factorisation through $b = f(2)$.

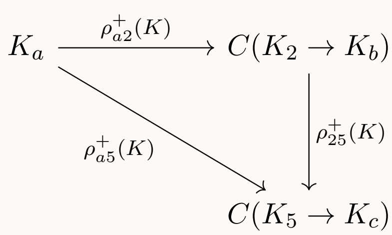
Composing with the restriction map associated with $2 \leq 5$ in X_1, we obtain

$$\rho^+_{25}(K) \circ \rho^+_{a2}(K) = \begin{pmatrix} r_{25}[1] & 0 \\ 0 & r_{bc} \end{pmatrix} \begin{pmatrix} 0 \\ r_{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ r_{bc} r_{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ r_{ac} \end{pmatrix}.$$

On the other hand, since $a \leq c \leq_- 5$, the mixed relation $a \leq_- 5$ factors through $c = f(5)$, and therefore

$$\rho^+_{a5}(K) = \rho^+_{c5}(K) \circ \rho^+_{ac}(K) = \begin{pmatrix} 0 \\ r_{ac} \end{pmatrix}.$$

This is exactly the compatibility encoded by the following diagram:



The two computations above show that the two morphisms from K_a to $C(K_5 \to K_c)$ coincide. Therefore $\rho^+_{a5}(K) = \rho^+_{25}(K) \circ \rho^+_{a2}(K)$, so the required compatibility holds. This makes visible, in the running example, the mechanism behind the general argument: a mixed relation is handled by factoring it through the intermediate vertex $f(x)$. The remaining compatibility relations are checked in exactly the same way, and are already covered by the general verification above. A completely analogous computation can be made for R^-.

Construction of the natural transformations $R^+R^- \Rightarrow [1]$ and $[1] \Rightarrow R^-R^+$:

Having defined the formulas ξ^+ and ξ^-, and having checked that they determine well-defined functors

$$R^+ \colon C(\mathcal{A})^{P^+} \to C(\mathcal{A})^{P^-}, \qquad R^- \colon C(\mathcal{A})^{P^-} \to C(\mathcal{A})^{P^+},$$

we now compare the compositions R^+R^- and R^-R^+ with the translation functor [1].
Recall that the composition of general formulas is computed pointwise. For $y \in Y$, both general formulas assign the same formula to a point ξ_y. Therefore, their compositions satisfy

$$(\xi^+\xi^-)_y = \xi_y[1], \qquad (\xi^-\xi^+)_y = \xi_y[1].$$

Let $x \in X$. At the vertex x, the local situation is the same as in the case of the chain $1 \to 2$: the composition $\xi^+\xi^-$ produces the formula ξ_{121}, while the composition $\xi^-\xi^+$ produces the formula ξ_{212}.
More precisely,

$$(\xi^+\xi^-)_x = \xi_{121} \circ \xi_{Y_x,x}, \quad (\xi^-\xi^+)_x = \xi_{212} \circ \xi_{x,Y_x},$$

where ξ_{121} and ξ_{212} are the formulas introduced in the local computation for the chain $1 \to 2$.
Let ν denote the general formula inducing the translation functor [1]. Thus, for every vertex $z \in X \sqcup Y$, the value of ν at z is the shifted formula to a point $\xi_z[1]$.
We now define morphisms of formulas

$$\epsilon^{+-} \colon \xi^+\xi^- \to \nu, \qquad \epsilon^{-+} \colon \nu \to \xi^-\xi^+,$$

by specifying their components at each vertex.
For $y \in Y$, we define both components to be the identity morphism

$$\epsilon_y^{+-} \colon \xi_y[1] \xrightarrow{1} \xi_y[1], \qquad \epsilon_y^{-+} \colon \xi_y[1] \xrightarrow{1} \xi_y[1].$$

For $x \in X$, we use the morphisms β_2 and α_1 coming from the local $1 \to 2$ case. We define

$$\epsilon_x^{+-} \colon \xi_{121} \circ \xi_{Y_x,x} \xrightarrow{\beta_2 \circ \xi_{Y_x,x}} \xi_2[1] \circ \xi_{Y_x,x} = \xi_x[1],$$

and

$$\epsilon_x^{-+} \colon \xi_x[1] = \xi_1 \circ \xi_{x,Y_x} \xrightarrow{\alpha_1 \circ \xi_{x,Y_x}} \xi_{212} \circ \xi_{x,Y_x}.$$

The equalities

$$\xi_2[1] \circ \xi_{Y_x,x} = \xi_x[1] \quad \text{and} \quad \xi_1 \circ \xi_{x,Y_x} = \xi_x[1]$$

follow from the fact that $\xi_{Y_x,x}$ and ξ_{x,Y_x} are formulas to the chain $1 \to 2$: the formula ξ_2 selects the second vertex, while ξ_1 selects the first one.

We claim that ϵ^{+-} and ϵ^{-+} are morphisms of general formulas. On the vertices of Y this is immediate since all components are identities. On the vertices of X, the claim reduces to the local commutative squares already verified in the case of the chain $1 \to 2$; composing those local morphisms with $\xi_{Y_x,x}$ and ξ_{x,Y_x} yields the required compatibility with the restriction maps.
Therefore, by functoriality of the construction of general formulas, ϵ^{+-} and ϵ^{-+} induce natural transformations

$$\epsilon^{+-} \colon R^+R^- \Rightarrow [1], \qquad \epsilon^{-+} \colon [1] \Rightarrow R^-R^+.$$

It remains to show that these natural transformations are quasi-isomorphisms. Since quasi-isomorphisms of diagrams are checked pointwise, it is enough to verify this at each vertex. For $y \in Y$, both components are the identity morphisms $\xi_y[1] \to \xi_y[1]$, hence they are quasi-isomorphisms. For $x \in X$, the components are obtained by evaluating the local morphisms β_2 and α_1 on the formulas $\xi_{Y_x,x}$ and ξ_{x,Y_x}. In the local $1 \to 2$ case, we already proved that α_1 and β_2 induce quasi-isomorphisms after evaluation on any diagram. Therefore, the same holds here.

We conclude that, for every abelian category $\mathcal{A}$, the natural transformations

$$\epsilon^{+-} \colon R^+R^- \Rightarrow [1], \qquad \epsilon^{-+} \colon [1] \Rightarrow R^-R^+$$

are pointwise quasi-isomorphisms.

Passing to derived categories, we obtain natural isomorphisms

$$\tilde{R}^+\tilde{R}^- \cong [1], \qquad \tilde{R}^-\tilde{R}^+ \cong [1].$$

In particular, $\tilde{R}^+$ and $\tilde{R}^-$ are equivalences, mutually quasi-inverse up to the shift functor. This proves that

$$D(\mathcal{A}^{P^+}) \simeq D(\mathcal{A}^{P^-}).$$

Since the construction works for every abelian category $\mathcal{A}$, the two posets P^+ and P^- are universally derived equivalent.

Example 2.2.28

First consider the vertices of Y. For every $y \in Y$, the composition R^+R^- acts on L_y by

$$R^+R^-(L)_y = R^+(R^-(L))_y = R^-(L)_y = L_y[1],$$

since R^- shifts the Y-vertices by [1] and R^+ leaves them unchanged. Therefore, the component ϵ_y^{+-} is the identity map $\epsilon_y^{+-} = id_{L_y[1]} \colon L_y[1] \to L_y[1]$, and no further verification is needed at these vertices.

The interesting case is at the vertices of X, where the iterated cone construction appears. We illustrate this explicitly at the vertex 2.

Since $Y_2 = \{b\}$, the local situation is exactly the chain $b \to 2$. Therefore, applying first R^- and then R^+, the value at the vertex 2 is obtained by iterating the cone construction:

$$R^+R^-(L)_2 = C(C(L_b \to L_2) \to L_b[1]).$$

Writing the inner cone explicitly as $C(L_b \to L_2) = L_b[1] \oplus L_2$, we may view this iterated cone as the complex

$$R^+R^-(L)_2 = L_b[2] \oplus L_2[1] \oplus L_b[1],$$

endowed with the differential coming from the iterated cone construction.

The component ϵ_2^{+-} is induced by the local morphism β_2 from the case of the chain $1 \to 2$. Under this identification, it acts by projecting onto the middle summand:

$$\epsilon_2^{+-} = (0, id_{L_2[1]}, 0) \colon L_b[2] \oplus L_2[1] \oplus L_b[1] \to L_2[1].$$

In other words, ϵ_2^{+-} collapses the two outer terms coming from the iterated cone and retains only the copy of $L_2[1]$. This is the explicit realization, at the vertex 2, of the general morphism

$$\epsilon_x^{+-} \colon \xi_{121} \circ \xi_{Y_x,x} \to \xi_x[1].$$

The same description applies at every vertex $x \in X_1$, with $b = f(2)$ replaced by $f(x)$, and dually for ϵ^{-+}.

3 Admissible cuts and reconstruction problems

In this chapter, we investigate some inverse and combinatorial questions related to the singleton case of the construction introduced in the previous chapter. More precisely, we restrict to the situation in which each subset Y_x consists of a single point, so that the data are encoded by an order-preserving map $f\colon X \to Y$. This is exactly the situation discussed in Example 2.2.3, and it gives rise to the following special case of Theorem 2.2.4:

> **Proposition 3.0.1 —** Let X and Y be finite posets, and let $f\colon X \to Y$ be an order-preserving map. Define two partial orders $\leq_f^+$ and $\leq_f^-$ on the disjoint union $X \sqcup Y$ by requiring that they restrict to the given orders on X and Y, and that, for $x \in X$ and $y \in Y$, we have
>
> $$x \leq_f^+ y \iff f(x) \leq y, \qquad y \leq_f^- x \iff y \leq f(x).$$
>
> Then the two posets $(X \sqcup Y, \leq_f^+)$ and $(X \sqcup Y, \leq_f^-)$ are universally derived equivalent.

★ **Remark.** Although Proposition 3.0.1 is formally a special case of Theorem 2.2.4, in many natural classes of posets condition (i) forces $|Y_x| \leq 1$ for every $x \in X$. Indeed, if Y is a join-semilattice [1] and $y \neq y' \in Y_x$, then $y \vee y'$ belongs to $[y, \bullet] \cap [y', \bullet]$, contradicting condition (i) of Theorem 2.2.4. Dually, if Y is a meet-semilattice, then $y \wedge y'$ belongs to $[\bullet, y] \cap [\bullet, y']$, again contradicting condition (i). Hence $|Y_x| \leq 1$ for every $x \in X$. In particular, this applies whenever Y is a lattice. The same conclusion holds whenever Y has a least element or a greatest element. Therefore, in all these cases, if the sets Y_x are non-empty, the data of Theorem 2.2.4 reduce automatically to an order-preserving map $f\colon X \to Y$.

★ **Remark.** Condition (ii) of Theorem 2.2.4 implies that, on each connected component of X, all sets Y_x have the same cardinality. Indeed, for every comparable pair $x \leq x'$ there is a bijection $\varphi_{xx'}\colon Y_x \to Y_{x'}$, and by composing these bijections and their inverses along a path in the underlying undirected graph of X, we obtain bijections between the sets attached to any two points in the same connected component. In particular, if X is connected, then all sets Y_x have the same cardinality.

★ **Remark.** The disconnected case reduces componentwise to the connected one. Indeed, if $P = P_1 \sqcup P_2$ is a disjoint union of two connected components, then the category of diagrams decomposes as

$$\mathcal{A}^P \cong \mathcal{A}^{P_1} \times \mathcal{A}^{P_2},$$

and similarly at the derived level. For this reason, when studying classification and reconstruction problems, it is natural to focus on connected posets.

We are thus led to two questions:

[1] Recall that, for two elements x, y in a poset, a **meet** is their greatest lower bound, denoted by $x \wedge y$, while a **join** is their least upper bound, denoted by $x \vee y$. A poset in which every pair of elements admits a meet is called **meet-semilattice**; if every pair admits a join, it is called a **join-semilattice**; if both exist for every pair, it is called **lattice**. For more details, see Garrett Birkhoff [5, Chapter 1, page 9].

(i) How many monotone maps $f\colon A \to B$ exist for fixed posets A and B?

(ii) Given a fixed poset P, in how many ways can P be written as $(A \sqcup B, \leq_f^+)$, so as to produce another universally derived equivalent poset $(A \sqcup B, \leq_f^-)$?

The first question is purely combinatorial. The second one is more structural, since once a decomposition $P = A \sqcup B$ is fixed, the map f is not arbitrary; indeed, we will see that, whenever such a map exists, it is determined by the mixed relations of P.

3.1 Counting monotone maps

We now formalize the notion of a cut.

A **cut** of a finite poset P is a decomposition $P = A \sqcup B$ of the underlying set of P such that A is an ideal of P; equivalently, $B = P \setminus A$ is a filter of P. Throughout this section, A and B are endowed with the partial orders induced from P.

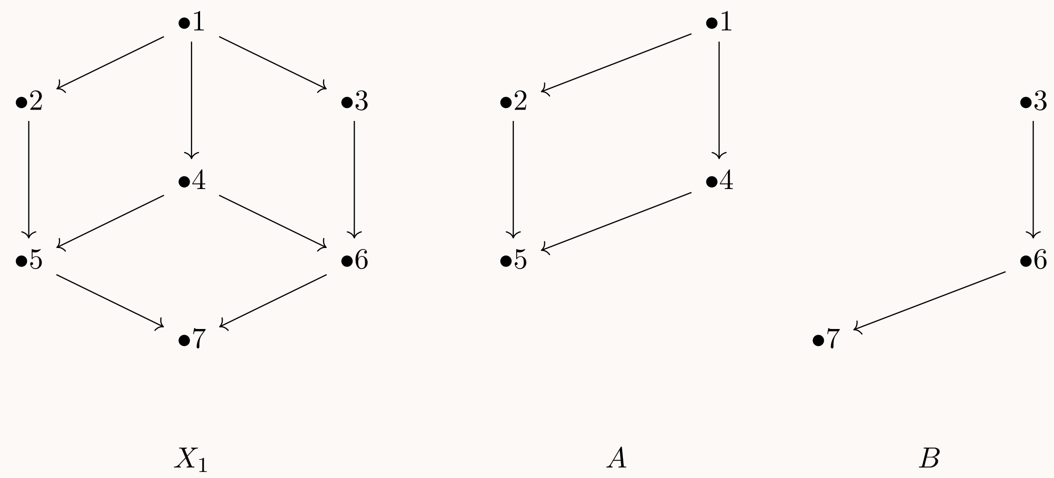
Recall that a subset $A \subseteq P$ is called an **order ideal** if, whenever $a \in A$ and $x \leq_P a$, then $x \in A$. Dually, a subset $B \subseteq P$ is called an **order filter** if, whenever $b \in B$ and $b \leq_P x$, then $x \in B$. From now on, we simply write ideals and filters.

Once a cut $P = A \sqcup B$ is fixed, the special case of the construction recalled above in which each Y_x is a singleton reduces to the choice of an order-preserving map $f\colon A \to B$.

By Proposition 3.0.1, every order-preserving map $f\colon A \to B$ determines two posets $P_f^+ := (A \sqcup B, \leq_f^+)$ and $P_f^- := (A \sqcup B, \leq_f^-)$ which are universally derived equivalent. Hence, for a fixed cut $P = (A \sqcup B)$ the number of possible choices of f is exactly the number of monotone maps from A to B, namely $|Hom_{Pos}(A, B)|$.

Example 3.1.1

As an example, we consider again the poset X_1 and decompose it into two subposets A and B, with underlying sets $A = \{1, 2, 4, 5\}$ and $B = \{3, 6, 7\}$, as shown below.



We now count all order-preserving maps $f\colon A \to B$ for the cut considered above. In the poset A we have the relations $1 \leq 2 \leq 5$ and $1 \leq 4 \leq 5$, therefore a map $f\colon A \to B$ is order-preserving if and only if $f(1) \leq f(2) \leq f(5)$ and $f(1) \leq f(4) \leq f(5)$. We count these maps by considering the value of $f(1)$.

- If $f(1) = 7$, then the values of the function on the remaining elements are forced to be $f(2) = f(4) = f(5) = 7$. So in this case there is exactly one map.

- If $f(1) = 6$, then $f(2), f(4)$ and $f(5)$ can only take values in $\{6, 7\}$, and they must satisfy $f(2) \leq f(5)$ and $f(4) \leq f(5)$.

  If $f(2) = f(4) = 6$, then $f(5)$ can be either 6 or 7, giving two maps.

  If at least one among $f(2)$ and $f(4)$ is equal to 7, then necessarily $f(5) = 7$. In this way, we have three further maps.

  Then in total, if $f(1) = 6$, we have five maps.

- Finally, if $f(1) = 3$, we distinguish three cases:

  – If $f(2) = f(4) = 3$, then $f(5)$ may be 3, 6, or 7, so we obtain three maps.

  – If neither $f(2)$ nor $f(4)$ is equal to 7, but at least one of them is equal to 6, then $f(5)$ can be either 6 or 7, and for both there are three options for the pair $(f(2), f(4))$, that are the pairs $(3, 6), (6, 3)$ and $(6, 6)$. So this contributes six maps.

  – If at least one among $f(2)$ and $f(4)$ is equal to 7, then $f(5)$ is forced to be equal to 7. The possible values of the pair $(f(2), f(4))$ are $(7, 3), (3, 7), (6, 7), (7, 6)$ and $(7, 7)$, so this gives five maps.

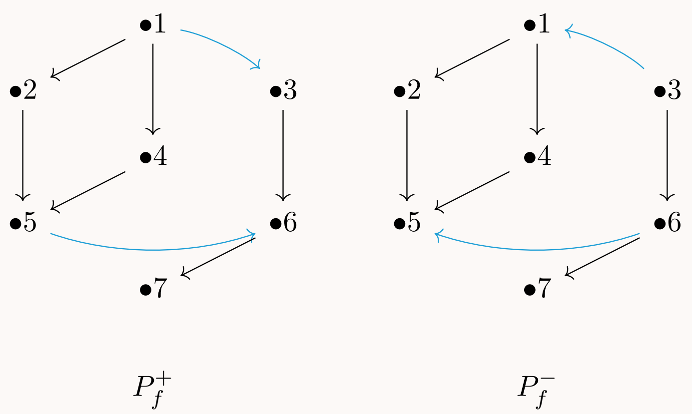
  Thus, when $f(1) = 3$, there are fourteen possible maps.

Summing up, the total number of order-preserving maps $f : A \to B$ is 20.

Example 3.1.2

Let us now choose one of the order-preserving maps from the previous example and describe explicitly the corresponding posets P_f^+ and P_f^-.

Consider the order-preserving map $f : A \to B$ defined by $f(1) = 3$ and $f(2) = f(4) = f(5) = 6$. The corresponding posets P_f^+ and P_f^- are depicted below. The blue arrows indicate the mixed relations from A to B in the first case, and from B to A in the second. The two posets are universally derived equivalent.



This example shows explicitly that, even after fixing a cut $P = A \sqcup B$, an arbitrary order-preserving map $f : A \to B$ produces a poset P_f^+ different from the original poset P. Indeed, an order-preserving map always produces a pair of universally derived equivalent posets, but in general the poset $(A \sqcup B, \leq_f^+)$ does not coincide with the original poset P. In the next section, we analyze precisely when this reconstruction occurs. This suggests an algorithmic way to generate candidates for posets universally derived equivalent to a given poset P.

3.2 Characterizing admissible cuts

In this section, we study the reconstruction problem in the special case of the construction above in which each Y_x is a singleton. Given a partition $P = A \sqcup B$, we determine when there exists an order-preserving map $f: A \to B$ such that $P = (A \sqcup B, \leq_f^+)$, and show that, whenever it exists, this map is uniquely determined by the order relations of P. This yields an algorithmic criterion for reconstruction.

Definition 3.2.1 (Admissible cut)

A cut $P = A \sqcup B$ is said to be **admissible** if there exists an order-preserving map $f: A \to B$ such that $P = (A \sqcup B, \leq_f^+)$.

Theorem 3.2.2 (Characterization of admissible cuts) — Let P be a poset, and let $P = A \sqcup B$ be a partition of the underlying set of P. We denote by $\leq_P$ the order relation on the poset P. We endow A and B with the induced orders from P. For each $a \in A$, define

$$U_B(a) := \{b \in B : a \leq_P b\}.$$

The following are equivalent:

(i) There exists an order-preserving map $f: A \to B$ such that $P = (A \sqcup B, \leq_f^+)$;

(ii) A is an ideal of P, B is a filter of P, and for every $a \in A$, the set $U_B(a)$ has a minimum.

Proof. Let P be a poset.

$(i) \Rightarrow (ii)$: Assume that there exists an order-preserving map $f: A \to B$ such that $P = (A \sqcup B, \leq_f^+)$. By construction, the orders on A and B coincide with the original ones, and the only mixed relations in $(A \sqcup B, \leq_f^+)$ are of the form $a \leq_f^+ b$ with $a \in A$ and $b \in B$. Hence no element of B can lie below an element of A, so A is an ideal and B is a filter.

Now fix $a \in A$. Since $P = (A \sqcup B, \leq_f^+)$, for every $b \in B$ we have

$$a \leq_P b \iff a \leq_f^+ b \iff f(a) \leq_P b.$$

Therefore

$$U_B(a) = \{b \in B : f(a) \leq_P b\}.$$

Since $f(a) \in B$ and $f(a) \leq_P f(a)$, we have $f(a) \in U_B(a)$. Moreover, for every $b \in U_B(a)$, by definition $f(a) \leq_P b$. Hence $f(a)$ is the minimum of $U_B(a)$.

$(i) \Leftarrow (ii)$: Assume that A is an ideal of P, B is a filter of P, and that for every $a \in A$ the set $U_B(a)$ has a minimum. Define $f: A \to B$ by

$$f(a) := \min U_B(a).$$

We first show that f is order-preserving. Let $a, a' \in A$ with $a \leq_P a'$. If $b \in U_B(a')$, then $a' \leq_P b$, hence also $a \leq_P b$. Therefore $b \in U_B(a)$, and so

$$U_B(a') \subseteq U_B(a).$$

Since both sets have a minimum, it follows that

$$f(a) = \min U_B(a) \leq_P \min U_B(a') = f(a').$$

Thus f is order-preserving.

It remains to show that $P = (A \sqcup B, \leq_f^+)$. Since A and B are endowed with the induced orders from P, the orders inside A and B already coincide in the two posets. It is therefore enough to compare the mixed relations.

Let $a \in A$ and $b \in B$. If $a \leq_P b$, then $b \in U_B(a)$, and since $f(a) = \min U_B(a)$, we get $f(a) \leq_P b$.

Conversely, assume that $f(a) \leq_P b$. Since $f(a) = \min U_B(a)$, we have $f(a) \in U_B(a)$. By definition of $U_B(a)$, this implies $a \leq_P f(a)$. Hence

$$a \leq_P f(a) \leq_P b,$$

and therefore $a \leq_P b$. Thus

$$a \leq_P b \iff f(a) \leq_P b,$$

that is exactly the condition defining the mixed relations in $\leq_f^+$. Moreover, since A is an ideal and B is a filter, there are no mixed relations of the form $b \leq_P a$ with $b \in B$ and $a \in A$; and by construction, there are no such relations in $(A \sqcup B, \leq_f^+)$. Hence

$$P = (A \sqcup B, \leq_f^+).$$

□

★ **Remark.** In particular, whenever such a map exists, it is uniquely determined by P and the decomposition $P = A \sqcup B$, and is given by

$$f(a) = \min U_B(a) \quad \text{for all} \quad a \in A.$$

> **Corollary 3.2.3** — Let P be a finite poset, and let $P = A \sqcup B$ be a partition satisfying the conditions of Theorem 3.2.2. Then it canonically produces a second poset
>
> $$Q := (A \sqcup B, \leq_f^-),$$
>
> which is universally derived equivalent to P.

Concretely, given a finite poset P and a partition $P = A \sqcup B$, the following procedure determines whether there exists an order-preserving map $f : A \to B$ such that $P = (A \sqcup B, \leq_f^+)$ and recovers it when it exists.

Algorithm 3.2.4 (Recovering f from a fixed partition.)
Input: a finite poset P together with a partition $P = A \sqcup B$.
Output: either failure, if there is no order-preserving map $f : A \to B$ such that $P = (A \sqcup B, \leq_f^+)$, or the unique such map.
Step 1: Check whether A is an ideal of P and B is a filter of P. If not, stop: no such map f exists.
Step 2: For each $a \in A$, compute set $U_B(a) = \{b \in B : a \leq_P b\}$.
Step 3: Check whether $U_B(a)$ has a minimum for every $a \in A$. If this fails for at least one a, stop: no such map f exists.
Step 4: Define $f(a) = \min U_B(a)$ for every $a \in A$.
Step 5: Return the map f.

3.3 An algorithmic search for admissible cuts

We now describe a second procedure, which starts from a finite poset P alone and detects all admissible cuts of P. It does so by running Algorithm 3.2.4 on every cut of P.

Here, by a non-degenerate cut we mean a cut $P = A \sqcup B$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Algorithm 3.3.1 (Detecting all admissible cuts of a poset.)

Input: a finite poset P.
Output: the set of all non-degenerate admissible cuts $P = A \sqcup B$, together with the corresponding uniquely determined maps $f : A \to B$.
Step 1: List all non-degenerate cuts $P = A \sqcup B$.
Step 2: For each cut $P = A \sqcup B$, apply Algorithm 3.2.4.
Step 3: Discard those cuts for which Algorithm 3.2.4 returns failure.
Step 4: For every remaining cut, record the corresponding map $f(a) = \min U_B(a)$, $a \in A$.
Step 5: Return the list of all admissible cuts, together with their associated maps f.

By Theorem 3.2.2, a cut $P = A \sqcup B$ is admissible if and only if for every $a \in A$, the set $U_B(a) = \{b \in B : a \leq_P b\}$ has a minimum, equivalently, if Algorithm 3.2.4 succeeds on the partition $P = A \sqcup B$. Therefore, the previous algorithm detects exactly all admissible cuts of P, and for each of them the corresponding map f is uniquely determined.

For each admissible cut, Corollary 3.2.3 produces canonically a second poset $Q = (A \sqcup B, \leq_f^-)$, which is universally derived equivalent to P.

Applying Algorithm 3.3.1 to the poset X_1, we obtain 6 non-trivial admissible cuts. However, some of these decompositions are related by symmetries of the poset X_1, and should therefore be regarded as equivalent. Indeed, X_1 admits a non-trivial automorphism, namely the involution exchanging 2 with 3 and 5 with 6, while fixing the remaining elements. This automorphism sends admissible decompositions to admissible decompositions, so it is natural to consider the action of $\mathrm{Aut}(X_1)$ on the set of admissible decompositions and classify them into orbits. In this way, the 6 non-trivial admissible decompositions split into 4 orbits.

A	$B = X_1 \setminus A$	f	Orbit size
$\{1,2,3\}$	$\{4,5,6,7\}$	$1 \mapsto 4,\ 2 \mapsto 5,\ 3 \mapsto 6$	1
$\{1,2,4,5\}$	$\{3,6,7\}$	$1 \mapsto 3,\ 2 \mapsto 7,\ 4 \mapsto 6,\ 5 \mapsto 7$	2
$\{1,2,3,4,5\}$	$\{6,7\}$	$1 \mapsto 6,\ 2 \mapsto 7,\ 3 \mapsto 6,\ 4 \mapsto 6,\ 5 \mapsto 7$	2
$\{1,2,3,4,5,6\}$	$\{7\}$	$a \mapsto 7$ for all $a \in A$	1

Table 3.1: Representatives of the $\mathrm{Aut}(X_1)$-orbits of non-trivial admissible decompositions of X_1.

This example suggests, more generally, that admissible decompositions lying in the same $\mathrm{Aut}(P)$-orbit should give rise to isomorphic output posets. The following lemma shows that such decompositions yield isomorphic output posets.

Lemma 3.3.2 — Let P be a finite poset, let $P = A \sqcup B$ be a decomposition, and let $f : A \to B$ be an order-preserving map. If $\sigma \in \mathrm{Aut}(P)$, set $A' = \sigma(A)$ and $B' = \sigma(B)$, and define $f' : A' \to B'$ by $f'(\sigma(a)) = \sigma(f(a))$ for all $a \in A$. Then

(i)
$$(A \sqcup B, \leq_f^+) \cong (A' \sqcup B', \leq_{f'}^+).$$

(ii)
$$(A \sqcup B, \leq_f^-) \cong (A' \sqcup B', \leq_{f'}^-).$$

Proof. We prove (i); the proof of (ii) is analogous.

Consider the map
$$\tilde{\sigma} : A \sqcup B \to A' \sqcup B'$$
defined by $\tilde{\sigma}(x) = \sigma(x)$. Since σ is a bijection and $\sigma(A) = A'$, $\sigma(B) = B'$, the map $\tilde{\sigma}$ is a bijection.

It remains to check that $\tilde{\sigma}$ preserves the order. On A and on B, this follows immediately from the fact that $\sigma \in \mathrm{Aut}(P)$. For mixed relations, let $a \in A$ and $b \in B$. Then

$$a \leq_f^+ b \iff f(a) \leq_P b.$$

Since σ is an automorphism of P, this is equivalent to $\sigma(f(a)) \leq_P \sigma(b)$. By definition of f', we have $\sigma(f(a)) = f'(\sigma(a))$, hence

$$f'(\sigma(a)) \leq_P \sigma(b),$$

which is equivalent to

$$\sigma(a) \leq_{f'}^+ \sigma(b).$$

Therefore $\tilde{\sigma}$ is an isomorphism of posets. □

Although quotienting by $\mathrm{Aut}(P)$ removes the redundancy coming from symmetries of the initial poset, it does not yet amount to counting the resulting posets up to isomorphism. Indeed, different admissible decompositions, even if not lying in the same $\mathrm{Aut}(P)$-orbit, may still produce isomorphic output posets. This suggests a natural further problem, namely to classify the resulting posets up to isomorphism.

A further source of redundancy may come from automorphisms of the pieces A and B. For instance, in the cut $A = \{1, 2, 4, 5\}$, $B = \{3, 6, 7\}$ considered above, the poset A admits a non-trivial automorphism exchanging 2 and 4. This suggests that different order-preserving maps $f : A \to B$ may still produce isomorphic output posets, even when the cut is fixed.

Motivated by the previous example, we now formulate the corresponding general statement.

Proposition 3.3.3 — Let A and B be finite posets, let $f : A \to B$ be an order-preserving map, let $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$. Define a map $g : A \to B$ by

$$g = \beta \circ f \circ \alpha^{-1}.$$

Then

$$(A \sqcup B, \leq_f^+) \cong (A \sqcup B, \leq_g^+).$$

In order to prove this proposition, we need the following lemma:

Lemma 3.3.4 — Let A and B be finite posets, let $f : A \to B$ be an order-preserving map.

(i) For every $\alpha \in \mathrm{Aut}(A)$, if $g = f \circ \alpha^{-1}$, then

$$(A \sqcup B, \leq_f^+) \cong (A \sqcup B, \leq_g^+).$$

(ii) For every $\beta \in \mathrm{Aut}(B)$, if $g = \beta \circ f$, then

$$(A \sqcup B, \leq_f^+) \cong (A \sqcup B, \leq_g^+).$$

Proof. We prove (i); the proof of (ii) is analogous. Define a bijection

$$\tilde{\alpha} : A \sqcup B \to A \sqcup B$$

by

$$\tilde{\alpha}(a) = \alpha(a) \text{ for } a \in A, \quad \tilde{\alpha}(b) = b \text{ for } b \in B.$$

We show that $\tilde{\alpha}$ is an isomorphism of posets.

On A, the map $\tilde{\alpha}$ coincides with α, which is an automorphism of A. On B, it is the identity. Hence $\tilde{\alpha}$ preserves the order inside A and inside B.

It remains to check the mixed relations. Let $a \in A$ and $b \in B$. Then

$$a \leq_f^+ b \iff f(a) \leq b.$$

Since $g = f \circ \alpha^{-1}$, we have
$$g(\alpha(a)) = f(a).$$
Therefore
$$f(a) \leq b \iff g(\alpha(a)) \leq b \iff \alpha(a) \leq_g^+ b.$$
Since $\tilde{\alpha}(a) = \alpha(a)$ and $\tilde{\alpha}(b) = b$, this shows that
$$a \leq_f^+ b \iff \tilde{\alpha}(a) \leq_g^+ \tilde{\alpha}(b).$$
Thus $\tilde{\alpha}$ preserves all order relations, and therefore it is an isomorphism. Hence
$$(A \sqcup B, \leq_f^+) \cong (A \sqcup B, \leq_g^+).$$
□

Proof of Proposition 3.3.3. By the first point of Lemma 3.3.4 applied to α, we have
$$(A \sqcup B, \leq_f^+) \cong (A \sqcup B, \leq_{f \circ \alpha^{-1}}^+).$$
By the second point of Lemma 3.3.4 applied to β, we also have
$$(A \sqcup B, \leq_{f \circ \alpha^{-1}}^+) \cong (A \sqcup B, \leq_{\beta \circ f \circ \alpha^{-1}}^+).$$
Since $g = \beta \circ f \circ \alpha^{-1}$, the claim follows. □

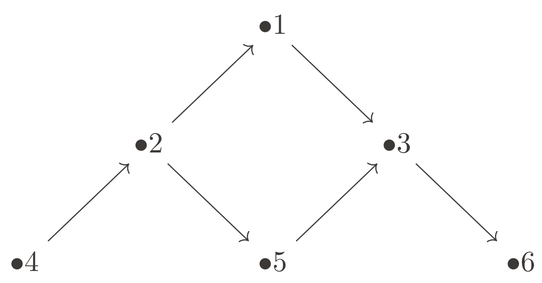
★ **Remark.** Although throughout this chapter we work with finite posets, Theorem 3.2.2 and the isomorphism results of Section 3.3 are purely order-theoretic and do not use finiteness. They therefore remain valid for arbitrary posets, whenever the required minima exist. The finiteness assumption is only needed here to stay within the framework of Proposition 3.0.1 and of the algorithmic procedures.

For the cut $A = \{1, 2, 4, 5\}$, $B = \{3, 6, 7\}$, the 20 order-preserving maps $f : A \to B$ found in Example 3.1.1 fall into 15 orbits under the action of $\mathrm{Aut}(A)$ given by $\alpha \cdot f := f \circ \alpha^{-1}$ where $\mathrm{Aut}(A)$ is generated by the involution exchanging 2 and 4. This provides a first refinement of the purely combinatorial count from Question 1.

3.4 A concrete example

We now illustrate the previous construction on a concrete finite poset. Starting from a fixed poset P, we apply Algorithm 3.3.1 to determine all admissible cuts. For each admissible cut, the corresponding map f is uniquely determined by Theorem 3.2.2, and Corollary 3.2.3 produces a new poset $Q = (A \sqcup B, \leq_f^-)$ universally derived equivalent to P.

Let P be the following finite poset:



Let us apply Algorithm 3.3.1 to the poset P.

Step 1: we list all possible cuts of P:

We do not include the degenerate cuts $A = \emptyset, B = P$ and $A = P,\ B = \emptyset$. The first one is trivial, while in the second case no map $f\colon A \to B$ can exist.

Cut	A	$B = P \setminus A$
(a)	$\{4\}$	$\{1, 2, 3, 5, 6\}$
(b)	$\{2, 4\}$	$\{1, 3, 5, 6\}$
(c)	$\{2, 4, 5\}$	$\{1, 3, 6\}$
(d)	$\{1, 2, 4\}$	$\{3, 5, 6\}$
(e)	$\{1, 2, 4, 5\}$	$\{3, 6\}$
(f)	$\{1, 2, 3, 4, 5\}$	$\{6\}$

Table 3.2: Non-degenerate cuts of the poset P.

Steps 2-5: for each cut listed above, we apply Algorithm 3.2.4 in order to determine whether it is admissible and, in the positive case, recover the corresponding map f.

Cut	$U_B(a)$	f
(a)	$U_B(4) = \{1, 2, 3, 5, 6\}$	$f(4) = 2$
(b)	$U_B(2) = U_B(4) = \{1, 3, 5, 6\}$	failure
(c)	$U_B(2) = U_B(4) = \{1, 3, 6\}, \quad U_B(5) = \{3, 6\}$	$f(2) = f(4) = 1, \quad f(5) = 3$
(d)	$U_B(1) = \{3, 6\}, \quad U_B(2) = U_B(4) = \{3, 5, 6\}$	$f(1) = 3, \quad f(2) = f(4) = 5$
(e)	$U_B(a) = \{3, 6\} \quad \forall a \in A$	$f(a) = 3 \quad \forall a \in A$
(f)	$U_B(a) = \{6\} \quad \forall a \in A$	$f(a) = 6 \quad \forall a \in A$

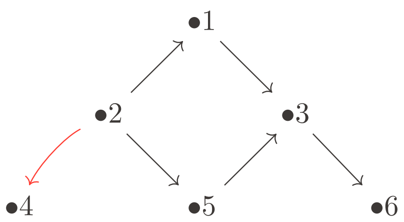
Table 3.3: Application of Algorithm 3.2.4 to the cuts listed in Table 3.2.

Thus, among six non-degenerate cuts, five are admissible and one is not.

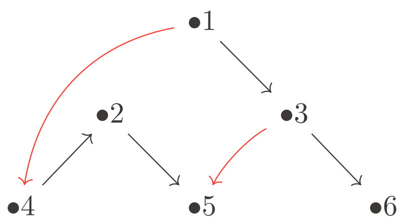
For each admissible cut listed in Table 3.2, Theorem 3.2.2 shows that the reconstructed poset $(A \sqcup B, \leq_f^+)$ coincides with the original poset P. Therefore, in each admissible case, Corollary 3.2.3 produces a new poset

$$Q = (A \sqcup B, \leq_f^-),$$

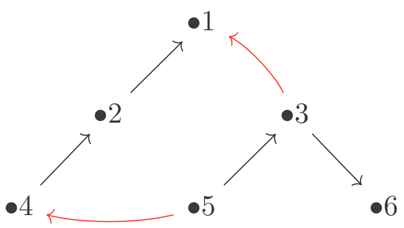
which is universally derived equivalent to P. We now draw the posets Q corresponding to the admissible cuts listed above.



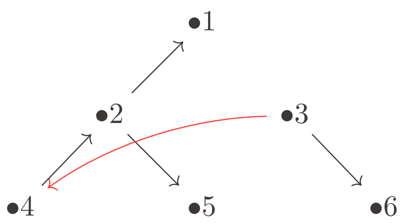
(a)



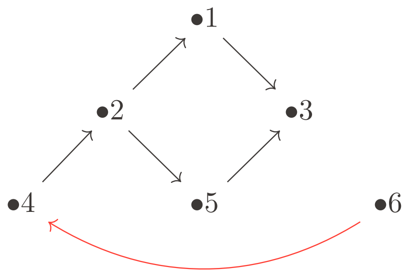
(c)



(d)



(e)



(f)

In each diagram, the red arrows represent the new mixed relations defining $\leq_f^-$, while the black arrows represent the original relations inside A and B.

By Proposition 3.0.1, each of these posets is universally derived equivalent to P. Consequently, they are universally derived equivalent to one another.

Notice that the posets Q obtained in cases (c) and (d) are isomorphic. Indeed, the automorphism $\sigma \in \mathrm{Aut}(P)$ exchanging 1 and 5 sends the cut in case (c) to the cut in case (d). Thus the two cuts lie in the same $\mathrm{Aut}(P)$-orbit, and Lemma 3.3.2 (ii) implies that the corresponding output posets are isomorphic.

4 Conclusion and further directions

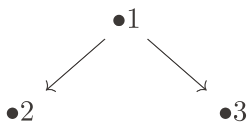
Building on the construction studied in Chapter 2, the main contribution of Chapter 3 is a change of perspective on Ladkani's construction. Theorem 2.2.4 takes two posets X and Y with suitable combinatorial data and produces two universally derived equivalent posets P^+ and P^- on the set $X \sqcup Y$. In the singleton case, we reversed this viewpoint: starting from a single poset P, we asked whether it can be decomposed as $(A \sqcup B, \leq_f^+)$ for some admissible cut and some order-preserving map $f : A \to B$. Theorem 3.2.2 gives a complete characterization: such a decomposition exists if and only if A is an ideal, B is a filter, and every element of A has a least upper bound in B; moreover, the map f is then uniquely determined. Each admissible cut produces a new poset $Q = (A \sqcup B, \leq_f^-)$ universally derived equivalent to P, and the isomorphism results of Section 3.3 show how symmetries reduce the combinatorial count.

In the singleton case, this approach transforms the abstract problem of finding universal derived equivalences into a concrete combinatorial one: enumerate the admissible cuts of a given poset. Several natural questions arise from this perspective.

4.1 On the number of admissible cuts

The number of admissible cuts of a finite poset depends on its structure in a way that does not admit a simple description in terms of the number of vertices alone. The following three examples illustrate the range of possible behaviors.

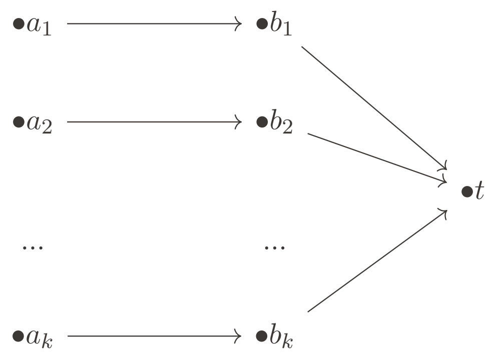
First, a connected poset may have no non-degenerate admissible cut at all. Consider the poset on three elements $\{1, 2, 3\}$ with $1 < 2$ and $1 < 3$.



There are three non-degenerate cuts: $A = \{1\}$, $B = \{2, 3\}$; $A = \{1, 2\}$, $B = \{3\}$; and $A = \{1, 3\}$, $B = \{2\}$. In the first case, $U_B(1) = \{2, 3\}$ has no minimum since 2 and 3 are incomparable. In the second case, $U_B(2) = \emptyset$. In the third case, $U_B(3) = \emptyset$. Therefore, none of the three cuts is admissible.

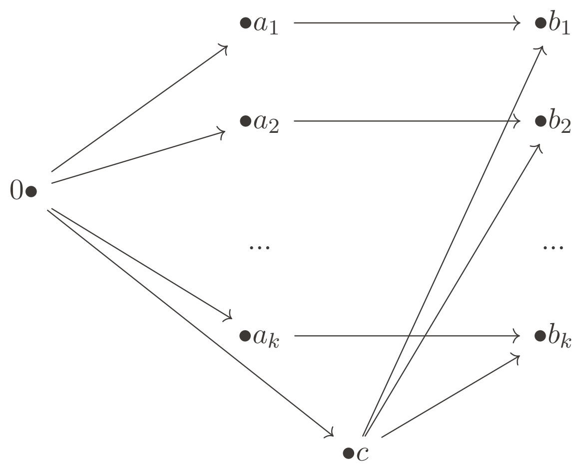
★ **Remark.** This example shows that the absence of admissible cuts does not preclude the existence of other universally derived equivalent posets. Indeed, the above poset is an orientation of the tree A_3, and is therefore universally derived equivalent to the chain on three elements by Ladkani's result that any two orientations of a tree are universally derived equivalent [3, Corollary 1.8]. Thus, the singleton construction studied in Chapter 3 does not, in general, capture all universally derived equivalent partners of a given poset.

At the other extreme, the number of admissible cuts can grow exponentially in the number of vertices. Consider the poset on $2k + 1$ elements consisting of k chains $a_i < b_i$ $(i = 1, \ldots, k)$ together with a maximum t, with $b_i < t$ for all i and no other relations.



This poset is connected and has $n = 2k + 1$ elements. A non-degenerate cut $P = A \sqcup B$ corresponds to a choice, for each chain $a_i < b_i$, of how many elements to place in A: none, only a_i, or both a_i and b_i. Since t must belong to B for A to be an ideal, this gives $3^k - 1$ non-degenerate cuts (excluding the empty-ideal). All of these cuts are admissible: if $a_i \in A$ and $b_i \notin A$, then $\min U_B(a_i) = b_i$, if both $a_i, b_i \in A$, then $\min U_B(a_i) = \min U_B(b_i) = t$.

Between these two extremes, there exist connected posets on arbitrarily many vertices with exactly one non-degenerate admissible cut. Consider the poset on $2k+2$ elements with underlying set $\{0, a_1, \ldots, a_k, b_1, \ldots, b_k, c\}$ and relations $0 < a_i < b_i$, $0 < c$, and $c < b_i$ for all $i = 1, \ldots, k$, with no other relations.



Since 0 is the minimum of the poset, it must belong to A in every non-degenerate cut, and therefore $U_B(0) = B$ must have a minimum. If the minimum of B were some b_i, then $B = \{b_i\}$ and the elements a_j with $j \neq i$ would have $U_B(a_j) = \emptyset$. If the minimum of B were some a_i, then $B = \{a_i, b_i\}$ and again $U_B(a_j) = \emptyset$ for $j \neq i$. Therefore the minimum of B must be c, which forces $B = \{c, b_1, \ldots, b_k\}$ and $A = \{0, a_1, \ldots, a_k\}$. This cut is indeed admissible: $\min U_B(0) = c$ and $\min U_B(a_i) = b_i$ for every i.

These examples show that the number of admissible cuts is not controlled by the number of vertices alone, and that no simple bound — neither linear nor polynomial — can hold in general. It would be interesting to understand which structural features of a poset govern the number of admissible cuts, and to obtain meaningful bounds within specific families.

Question 1. Which finite connected posets admit at least one non-degenerate admissible cut?

Question 2. For natural classes of posets (chains, trees, posets of Dynkin or Euclidean type), what is the typical number of admissible cuts?

4.2 Classification of outputs up to isomorphism

Different admissible cuts of a fixed poset P may produce isomorphic output posets, even when the cuts do not lie in the same $\mathrm{Aut}(P)$-orbit. Lemma 3.3.2 and Propositions 3.3.3 and 3.3.4 account for part of this redundancy, but a complete classification of the outputs up to isomorphism remains open.

Question 3. Given a finite poset P, how many pairwise non-isomorphic posets can be obtained from P by a single application of the singleton construction?

This classification problem has two distinct layers. One may first fix an admissible cut $P = A \sqcup B$ and ask when two order-preserving maps $f, g\colon A \to B$ give rise to isomorphic output posets. One may then let the cut vary and ask how the resulting isomorphism classes interact across different admissible cuts of the same poset.

Even for a fixed cut $P = A \sqcup B$, automorphisms of A and B may identify different order-preserving maps $f\colon A \to B$ yielding isomorphic outputs. It is therefore natural to ask whether the resulting equivalence classes are completely described by the action of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ on $\mathrm{Hom}_{Pos}(A, B)$, or whether genuinely different maps may still produce isomorphic outputs. More globally, one may ask whether every isomorphism between two outputs can be explained by symmetries of the input data, or whether different admissible cuts may lead to accidentally isomorphic posets.

4.3 Iteration of the construction

Because Corollary 3.2.3 produces, from an admissible cut $P = A \sqcup B$, a new poset $(A \sqcup B, \leq_f^-)$ on the same underlying set, iterating the singleton construction never changes the number of vertices. Therefore, if one starts from a finite poset P, the procedure can generate only finitely many posets up to isomorphism. In particular, the interesting issue is not finiteness itself, but rather the structure of the closure of P under iteration and the extent to which this procedure captures the whole derived-equivalence class of P.

One may think of the iterated singleton construction as generating a finite graph whose vertices are the isomorphism classes reachable from P, and whose edges correspond to a single application of the construction.

A first natural question is therefore the following.

Question 4. Given a finite poset P, what is the closure of P under iterated applications of the singleton construction? More precisely, how many isomorphism classes of posets are reachable from P by repeated applications of the algorithm?

A deeper question is whether this closure exhausts the universally derived equivalent class of P. The example from section 4.1 shows that this is not the case in general: a finite poset may admit no non-degenerate admissible cut and yet have other universally derived equivalent partners. Therefore, the iterated singleton construction is, in general, not exhaustive.

This leads to a more refined problem.

Question 5. How can one characterize the universally derived equivalent posets that lie in the closure of P under iterated applications of the singleton construction, and those that lie outside it?

In particular, this shows that being forced into the singleton case is not sufficient, by itself, to guarantee that the procedure recovers all universally derived equivalent partners of a poset.

Another natural direction concerns the stability of the singleton regime under iteration. In many interesting situations the combinatorial conditions force one to remain in the singleton case; however, as observed above, this does not imply that the singleton construction captures all universally derived equivalent posets. It is therefore natural to ask whether the singleton regime is at least stable under the procedure itself, or whether iteration can eventually lead to posets for which genuinely non-singleton instances of Ladkani's construction become possible.

Question 6. Suppose that P is a finite poset for which the singleton case is forced. Can iterating the singleton construction produce a poset Q for which the singleton case is no longer forced? Equivalently, is the class of posets for which the singleton regime is forced closed under iteration?

These questions seem particularly interesting for specific families of posets, such as chains, trees, and posets arising from representation-theoretic contexts, such as those associated with oriented Dynkin or Euclidean quivers, where additional structure may help to control both the closure under iteration and the relation with derived equivalence.

A natural continuation would also be to study the singleton construction systematically on small finite posets, with the aim of classifying, up to isomorphism, the outputs obtainable from a given P by a single application or by iteration of the construction. From a more computational perspective, it would be interesting to implement the algorithm of Chapter 3 in order to generate explicit data and examples.

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