Università degli Studi di Padova

Dipartimento di Scienze Economiche ed Aziendali "Marco Fanno"
Dipartimento di Matematica "Tullio Levi Civita"
Corso di Laurea Magistrale Specialistica in Economia e Finanza

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Tesi di Laurea

Inflation modelling with affine continuous-time models

An application to the Italian case with
an estimation of the cost of public debt

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A.A. 2019 - 2020 – DCCXCVIII

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Giulio Carlo Venturi

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Introduction

Generally speaking, inflation is an increase of prices of goods and services in an economy over a period of time. In this work, we focus on inflation at the national aggregate level on a monthly basis. We analyse Italian and European inflation indexes: we are interested in modelling the behaviour of the European inflationary process. Relying on the historical evolution over the last five years, we embrace an empirical point of view. Of course, we set up a solid theoretical framework, in which a model is developed and calibrated, but we do not follow a general equilibrium approach. Indeed, we do not specify individual preferences or utility functions. Instead, we dig in the market to retrieve information, i.e. prices which implicitly summarise the results of interactions of individuals' preferences. This is the extent to which we claim to adopt an empiricist view. We make the simple theoretical hypothesis that inflation follows a stochastic version of the Fisher equation (see Section 3.3). Then, we calibrate our model so that it matches the evidence of historical data. The scope of this work is not an analysis of the inflationary process from the historical and macroeconomic point of view, rather the application of a certain modelling framework with the aim of estimating numerical figures regarding the Italian public debt.

From a monetarist point of view, inflation is a monetary phenomenon, because of the tight link between money supply and inflation. Indeed, central banks control inflation through money supply and interest rates. When they want to stimulate inflation, they lower interest rates raising money supply. Greater availability of a good or service typically lowers its cost. Therefore, thinking of the interest rate as the cost of money, greater money supply means lower interest rates. Cheaper interest rates stimulate credit demand, hence demand for goods and services to be payed with easily available money. Finally, higher demand raises overall prices and generates inflation. For a thorough introductory explanation, please refer to (Blanchard, 2016, Chapter 23)

This chain of causal links shows that interest rates play a major role as determinants of inflation: the Fisher equation postulates that inflation is the difference between nominal and real interest rates. Interest rates are retrieved from the prices of Italian traded government securities. We adopt in this work the Jarrow-Yildirim model (Jarrow & Yildirim, 2003) to jointly model, analyse and simulate real interest rates, nominal interest rates and inflation rate. This model gained prominence because of its computational solidity and straightforwardness. Forward and inflation rates are modelled in a Gaussian setting, which substantially eases computations and ensures a smooth analytical tractability. Assuming absence of arbitrage, it is possible to derive the evolution dynamics of bond prices under the risk neutral probability. Such robustness poses a major drawback. Gaussianity entails that a positive probability is assigned to highly unlikely occurrences, such as strongly negative interest rates. It must be said, though, that this negative feature can be mitigated by suitable values of parameters, so that relevant probability distributions become almost null for undesired outcomes. The model employs a forward specification of the Hull-White model for interest rates.

We match bond theoretical and market prices to retrieve forward interest rates. This approach is conditioned to the availability of relevant bonds. Nominal bonds are typically traded in large numbers, while fewer real bonds are offered. Indeed, eleven Italian real bonds and ninety nominal ones were traded at the date of data retrieval. Denoting with n the number of bonds in the smaller set, we can retrieve $n-1$ forward rates to ensure that the system of equations we solve at each trading day has a solution. Therefore, we build a 5 rate piecewise-constant term structure, which shall capture and summarise the behaviour of rates for short, medium and long maturities. It may be argued that five rates are not enough for a proper analysis. On the contrary, the behaviour of this limited set of rates offers an interesting picture of the evolution of interest rates over time. Moreover, it is able to synthesise the evolution patterns of interest rates grouped for maturity. Jarrow and Yildirim built their analysis relying on four rates with two-year long time series. We employ five rates with five year-long time series. We grouped interest rates for maturity as follows: 0-3 years, 3-5 years, 5-10 years, 10-20 years and 20-30 years. The eight parameters of the model are calibrated to historical data between 2015 and 2020.

The analysis over the last five years highlights that Italian interest rates were unusually high between May-2018 and May-2019. We rely on the Jarrow-Yildirim model to assess whether the interest rates in this time window were excessively high. We simulate thousands of evolution paths for interest rates and inflation rate consistent with pre-May-2018 data. Simulated data shape confidence intervals, within which the observed evolution paths are likely to lie. These upper and lower benchmarks make us conclude with a good degree of statistical confidence that the evolution during the highlighted time window had not been compatible with previous patterns. Then, we look for the causes and we estimate the cost of such temporary interest rate raise. Since the monetary effects will hit Italian finances gradually over the future thirty years, we compute the value of future monetary amounts at 2020 prices with our simulated inflation rate evolution. Therefore, the cost estimation makes estensive use of the modelling framework to take into account the depleting effect of inflation.

The thesis is structured in six chapters as follows. In chapter 1 we recall the main theoretical foundations of the modelling of the term structure of interest rates. In chapter 2 we describe three common financial instruments traded in the market to counterbalance the effect of inflation. In chapter 3 we explain some of the most popular models employed in the literature to model interest and inflation rates. In chapter 4 we discuss in detail the model employed in this work, namely the Jarrow-Yildirim model, which relies on the Hull-White model for interest rate modelling. In chapter 5 we retrieve and analyse data, in order to calibrate the Jarrow-Yildirim model. In Chapter 6 we put the model at work to assess the characteristics of the evolution of interest rates. Finally, we offer an estimation of the costs incurred by Italian finances, corrected for the effect of inflation, as a result of the raise in interest rates between May-2018 and May-2019.

Chapter 1

Theoretical Preliminaries

1.1 Preliminary Definitions

Individuals and economic aggregates often need monetary funds they do not posses. The reasons of such needs are the most disparate, but they all share the necessity of suitable legal and economic instruments to temporarily move money between parties. Bonds are common vehicles to fulfil this task. The etymology of the term is traced back to the verb "to bind", in the sense that the instrument binds someone to pay back an amount to someone else. Being more formal, the issuer requires funds by issuing debt securities and binds herself to pay back the lender an amount, comprehensive of the original sum and interests. The lender proves to be the creditor, holding the bond as credit title. The issuance process and the operative framework of a debt instrument may prove out to be fairly complicated, thus requiring specific professional figures. The interest rate is the cost of a financing operation for the borrower, the profit for the lender. As for any cash flow, it is of primary relevance to be able to understand and forecast the behaviour of interest rates. Let us state the assumptions and features of our modelling framework. In this section we follow primarily (Björk, 2009, Chapter 22), further discussions can be found in (Musiela & Rutkowski, 1995, Sections 9.1 and 9.2), (Brigo & Mercurio, 2007, Chapter 1) and (Hull, 2018, Chapter 4).

The maturity of a bond is the time in which the principal value must be payed back. Broadly speaking, two categories of bonds exist: zero-coupon bonds (ZCB) and coupon bonds (CB). These securities guarantees the holder a deterministic cash flow and are, therefore, known as fixed income instruments. A **zero coupon bond** with maturity date T is a contract which guarantees the holder the principal value to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t, T)$ and corresponds to a fraction of the principal value: ZCB are priced at discount. The principal value is set to 1, by convention. A **coupon** bond with maturity date T guarantees the payment of the principal value at maturity and of coupons at fixed intermediate date. The frequency and size of coupons are contractually determined.

For modelling purposes, we assume that

- There exists a market for T-bonds for every $T > 0$.
- $p(t,t) = 1 \quad \text{for all} \quad t.$
- For each fixed t, the bond price $p(t,T)$ is differentiable with respect to the time of maturity T.

Thus, the bond price $p(t,T)$ is a stochastic function of two variables, t and T. Several interest rates can be defined in this bond market. Let us fix the following time points t, S and T, with $t < S < T$. We call **forward rate** the interest rate fixed at t for the future interval $[S,T]$. We call **spot rate** the interest rate fixed at t for the interval $[t,T]$. Spot rates are forward rates, in which $t = S$. The forward rate with simple compounding $F_s(t;S,T)$ solves the equation

$$p(t,S) = \big(1 + (T-S) \cdot F_s(t;S,T)\big) \cdot p(t,T).$$

The forward rate with continuous compounding $F_c(t;S,T)$ solves the equation

$$p(t,S) = e^{F_c(t;S,T)\cdot(T-S)} \cdot p(t,T).$$

These facts lead to the following definitions.

Definition 1.1

1. The ***simple forward rate*** *for $[S,T]$ contracted at t is defined as*

$$F_s(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)\cdot p(t,T)}. \tag{1.1}$$

2. The ***simple spot rate*** *for $[S,T]$ is defined as*

$$F_s(S,T) = -\frac{p(S,T) - 1}{(T-S)\cdot p(S,T)}. \tag{1.2}$$

3. The ***continuous forward rate*** *for $[S,T]$ contracted at t is defined as*

$$F_c(t;S,T) = -\frac{\log p(t,T) - \log p(t,S)}{(T-S)}. \tag{1.3}$$

4. The ***continuous spot rate*** *for $[S,T]$ is defined as*

$$F_c(S,T) = -\frac{\log p(S,T)}{(T-S)}. \tag{1.4}$$

5. The ***instantaneous forward rate*** *with maturity T, contracted at t for the infinitesimal interval $[T, T+dT]$, is defined as*

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}. \tag{1.5}$$

6. The ***instantaneous short rate*** *at time t for the infinitesimal interval $[t, t+dt]$ is defined as*

$$r(t) = f(t,t). \tag{1.6}$$

A further definition completes this introductory setting.

Definition 1.2
*The **money account**, or bank account, is the process defined as*

$$B_t = \exp\left\{\int_0^t r(s)ds\right\}, \tag{1.7}$$

or, equivalently, as

$$\begin{cases} dB(t) = r(t)B(t)dt, \\ B(0) = 1. \end{cases} \tag{1.8}$$

It describes the trajectory of evolution of a sum of money left in the bank account, subject to the continuous compounding of the stochastic short rate of interest.

As consequence of these definitions, the following statement can be proved.

Lemma 1.1 (ZCB Price)
Fot $t \leq s \leq T$,

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

and for $t = s$

$$p(t,T) = \exp\left\{-\int_t^T f(t,u)du\right\} \tag{1.9}$$

Proof. Using the definition of instantaneous forward rate

$$\int_s^T f(t,u)du = -\log p(t,T) + \log p(t,s) = \log \frac{p(t,s)}{p(t,T)}.$$

Hence,

$$\exp\left\{\int_s^T f(t,u)du\right\} = \frac{p(t,s)}{p(t,T)}$$

and

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\}, \quad \forall \quad t \leq s \leq T$$

□

Let us consider the following generic dynamics for the short rate, the forward rate and the bond price.

Short rate dynamics $$dr(t) = a(t)dt + b(t)dW(t), \tag{1.10}$$

Forward rate dynamics $$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{1.11}$$

Bond price dynamics $$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(T). \tag{1.12}$$

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $W^{\mathbb{P}}(t)$ process is a $\mathbb{P}$-Brownian Motion. We assume the existence of a filtration $(\mathcal{F}_t)_{t\geq 0}$, with respect to which all the stochastic processes we will present are adapted. . $W(t)$ can be d-dimensional. In such case, $v(t,T)$ and $\sigma(t,T)$ are row vectors. We shall study these dynamics in a one dimensional setting. The processes $a(t)$ and $b(t)$ are scalar adapted processes, whereas $m(t,T), v(t,T), \alpha(t,T)$ and $\sigma(t,T)$ are adapted processes parameterized by time of maturity T. For each fixed t all the objects $m(t,T)$, $v(t,T)$, $\alpha(t,T)$ and $\sigma(t,T)$ are assumed to be continuously differentiable in the T-variable. We denote the T-derivative with a T subscript.

Proposition 1.1
If $p(t,T)$, $f(t,T)$ and $r(t,T)$ satisfy respectively (1.10), (1.11) *and* (1.12), *it holds that:*

1. $$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{1.13}$$

   where α and σ are given by
   $$\begin{cases} \alpha(t,T) = v_T(t;T)\cdot v(t,T) - m_T(t,T), \\ \sigma(t,T) = -v_T(t,T) \end{cases}$$

2. $$dr(t) = a(t)dt + b(t)dW(t), \tag{1.14}$$

   where
   $$\begin{cases} a(t) = f_T(t,t) + \alpha(t,t) \\ b(t) = \sigma(t,t) \end{cases}$$

3. $$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + p(t,T)S(t,T)dW(t), \tag{1.15}$$

   where $||\cdot||$ denotes the Euclidean norm and
   $$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases}$$

Proof.

1. Given the bond price dynamics and applying the Itô formula,

   $$d\log p(t,T) = \left(m(t,T) - \frac{1}{2}v(t,T)^2\right)dt + v(t,T)dW(t).$$

   Recalling the instantaneous forward rate formula, we take the T-derivative and change the sign of $d\log p(t,T)$. This yields

   $$df(t,T) = \left(v_T(t,T)\cdot v(t,T) - m_T(t,T)\right)dt - v_T(t,T)dW(t).$$

2. Recalling that $r(t) = f(t,t)$ and integrating the corresponding forward rate dynamics, we write

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s).$$

Then, we write

$$\begin{aligned} \alpha(s,t) =& \alpha(s,s) + \int_s^t \alpha_T(s,u)du, \\ \sigma(s,t) =& \sigma(s,s) + \int_s^t \sigma_T(s,u)du, \end{aligned}$$

and insert these results into the previous equation:

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t \int_s^t \alpha_T(s,u)duds + \int_0^t \sigma(s,s)dW(s) + \int_0^t \int_s^t \sigma_T(s,u)dudW(s) \\ &= \int_0^t f_T(s,s)ds, + \int_0^t \alpha(s,s)ds + \int_0^t \sigma(s,s)dW(s), \end{aligned}$$

where $f_T(t,T)$ denotes the derivative with respect to the maturity of the forward rate. Differentiation of this expression yields the desired equation.

3. Using $Y(t,T) = -\int_t^T f(t,s)ds$ the bond price becomes $p(t,T) = e^{Y(t,T)}$. Applying the Itô formula to this expression, we obtain the dynamic

$$dp(t,T) = p(t,T)dY(t,T) + \frac{1}{2}p(t,T)d\langle Y(\cdot,T)\rangle_t,$$

where $d\langle Y(\cdot,T)\rangle_t$ is the quadrati variation of the process $(Y(t,T))_{t\in[0,T]}$. Recalling that $dY(t,T) = -d\left(\int_t^T f(t,s)ds\right)$, we can write

$$\begin{aligned} dY(t,T) &= -\frac{\partial}{\partial t}\left(\int_t^T f(t,s)ds\right)dt - \int_t^T df(t,s)ds \\ &= f(t,t)dt - \int_t^T \alpha(t,s)dtds - \int_t^T \sigma(t,s)dW(t)ds \\ &= r(t)dt + A(t,T)dt + S(t,T)dW(t), \end{aligned}$$

and

$$d\langle Y(\cdot,T)\rangle_t = ||S(t,T)||^2 dt.$$

□

We shall now focus on fixed coupon bonds. The holder of a fixed CB receives the payments of fixed coupon c_i at time T_i, for $i = 1, \ldots, T$, and the face value K at time time T. The pricing of a fixed CB stems from the fact that the stream of cash flows of a CB is a portfolio of ZCBs. More precisely, the price of a CB, $p(t)$, is given by

$$p(t,T,c,K) = K \cdot p(t,T) + \sum_{i=1}^{T} c_i \cdot p(t,T_i). \tag{1.16}$$

Definition 1.3
*The **yield to maturity**, $y(t,T)$, of a fixed CB at time t with market price $p(t)$, payments c_i at T_i for $i = 1, \ldots, T$ and face value K, is defined as the value of y which solves the equation*

$$p(t,T,c,K) = Ke^{-y(T_T - t)} + \sum_{i=1}^{T} c_i \cdot e^{-y(T_i - t)}. \tag{1.17}$$

1.2 Arbitrage Pricing

We want to model an arbitrage free family of ZCB price processes $\{p(\cdot, T); T \geq 0\}$. We follow the discussion presented in (Björk, 2009, Chapter 23). We assume the existence of one exogenously given (locally risk free) asset, the bank account, with dynamics dB(t)=r(t)B(t)dt. Aiming at pricing ZCBs, we assume that there is an arbitrage-free market for T-bonds for every T and that the price of a T-bond has the form p(t,T)=F(t,r(t),T), where F is a "smooth" function and $F(T, r; T) = 1$ for all r. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we model the short rate under the objective probability measure as

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{P}}(t). \tag{1.18}$$

This theoretical market is not complete, because we have one source of randomness, the Brownian motion $W^{\mathbb{P}}$ in equation (1.18), but no exogenously given traded asset besides the bank account. Even ZCBs cannot be replicated by the bank account, which just evolves according to the "ex-post" realisations of the short rate dynamics. We cannot determine a unique price for a specific bond, but we can price securities in terms of one benchmark bond.

We fix two maturities, S and T, and apply Itô formula to $F(t, r(t); i)$ with $i = \{S, T\}$. We shall ease the notation by writing F^i for $F(t, r(t); i)$ and F^i_x for $\frac{\partial F^i}{\partial x}$. This yields

$$dF^i = F^i \alpha_i(t)dt + F^i \sigma_i(t)dW^{\mathbb{P}}(t),$$

where

$$\begin{aligned}
\alpha_i(t) =& \frac{F^i_t + \mu F^i_r + \frac{1}{2}\sigma^2 F^i_{rr}}{F^i}, \\
\sigma_i(t) =& \frac{\sigma F^i_r}{F^i}.
\end{aligned}$$

We build a portfolio (u_S, u_T) investing a fraction of wealth u_i in the i-bond, whose value evolves according to

$$\begin{aligned}
dV &= V\left\{u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S}\right\} \\
&= V \cdot \{u_T \alpha_T(t) + u_S \alpha_S(t)\}dt + V \cdot \{u_T \sigma_T(t) + u_S \sigma_S(t)\}dW^{\mathbb{P}}(t).
\end{aligned} \tag{1.19}$$

The portfolio weights must sum to one and the diffusion term must vanish to ensure that the portfolio is locally riskless:

$$u_T + u_S = 1,$$
$$u_t\sigma_T(t) + u_S\sigma_S(t) = 0.$$

The solution of this system for u_T and u_S are inserted in the portfolio value dynamics (1.19). This yields

$$dV = V \cdot \left\{ \frac{\alpha_S(t)\sigma_T(t) - \alpha_T(t)\sigma_S(t)}{\sigma_T(t) - \sigma_S(t)} \right\} dt.$$

The money account $B(t)$, with dynamics $dB(t) = B(t)r(t)dt$, is the only risk-free asset; in order for arbitrage to be avoided, the finite-variation process dV must have a local rate of return equal to $r(t)$. Mathematically, arbitrage-free pricing requires that

$$\frac{\alpha_S(t)\sigma_T(t) - \alpha_T(t)\sigma_S(t)}{\sigma_T(t) - \sigma_S(t)} = r(t), \text{ for all } t.$$

We rewrite this expression as

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)},$$

in order to underline that this ratio is independent of the choice of maturity. In this formulation, we stress that every coefficient is time dependent.

Proposition 1.2 (Market Price of Risk)
Assume that the bond market is free of arbitrage. Then there exists a process λ such that the relation

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t) \tag{1.20}$$

holds for all t and for every maturity T.

Economically, λ is the quotient between the excess rate of return of a T-bond over the riskless rate and the local volatility of the T-bond. The process λ is the risk premium per unit of volatility and it is known as market price of risk. The market price of risk is central towards the determination of arbitrage-free prices. If we insert the formulas for α_T and σ_T in λ (from now one we will write F^T to denote the pricing function F for a generic maturity T), we obtain the term structure equation.

Proposition 1.3 (Term Structure Equation)

$$\begin{cases} F_t^T + \{\mu(s, r(s)) - \lambda(s, r(s))\sigma\}F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0, \\ F^T(T, r) = 1. \end{cases}$$

Writing r instead of $r(t)$ points out that the relation holds for every time realisation of the short rate stochastic process. Furthermore, we assume that the market price of risk is function of both time and short interest rate. Given the boundary value problem of this proposition for F^T, we can make use of the Feynman-Kač stochastic representation formula (Björk, 2009, p. 72-73) applied to the process $e^{-\int_t^u r(s)ds} F^T(u, r(u))$.

Theorem 1.1 (ZCB Price)
ZCB arbitrage-free prices are given by the formula $p(t,T) = F(t, r(t); T)$*, where*

$$F(t, r; T) = E_{t,r}^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} \times 1 \right].$$

The expectation sign $E_{t,r}^{\mathbb{Q}}$ *denotes that the expectation is taken given the following dynamics of the short rate under the risk neutral measure* $\mathbb{Q}$*, whose diffusion term is driven by the* $\mathbb{Q}$*-Brownian motion* $W^{\mathbb{Q}}(t)$.

$$\begin{cases} dr(s) = \{\mu(s, r(s)) - \lambda(s, r(s))\sigma\}ds + \sigma dW^{\mathbb{Q}}(s), \\ r(t) = r. \end{cases}$$

This discussion proves useful for the pricing of a set of contingent T-claims.

Corollary 1.1 (Contingent Claim Pricing)
Let X *be a contingent* T*-claim of the form* $X = \Phi(r(T))$*, where* Φ *is a real valued function. The arbitrage-free price* $\Pi(t; \Phi)$ *is given by*

$$\Pi(t; \Phi) = F(t, r(t)),$$

where F *solves the boundary value problem*

$$\begin{cases} F_t + \{\mu(s, r(s)) - \lambda(s, r(s))\sigma\}F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ F(T, r) = \Phi(r); \end{cases}$$

and has the stochastic representation f

$$F(t, r; T) = E_{t,r}^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} \times \Phi(r(T)) \right].$$

The expectation sign $E_{t,r}^{\mathbb{Q}}$ *denotes that the expectation is taken given the following dynamics of the short rate under the risk neutral measure* $\mathbb{Q}$

$$\begin{cases} dr(s) = \{\mu(s, r(s)) - \lambda(s, r(s))\sigma\}ds + \sigma dW^{\mathbb{Q}}(s), \\ r(t) = r. \end{cases}$$

Remark 1.1
A market is complete if every contingent claim can be replicated by a portfolio based on the underlying assets. The bond market we postulate is, then, incomplete. We cannot form a

portfolio based on the short interest rate, since r is not a traded asset. The bank account is the only exogenously given asset and it is not sufficient to hedge the randomness of the Brownian motion in the r dynamics. The value of a portfolio based on the bank account just varies according to the evolution of the short rate and it is clearly insufficient to replicate most types of derivatives. Incompleteness implies that prices are not unique. More formally, completeness ensures that the martingale measure $\mathbb{Q}$ is unique, hence the market price of risk λ is unique too. Martingale measures and market prices of risk are not unique in an incomplete market. Thus, for each measure there exists an associated market price of risk, hence a price consistent with absence of arbitrage. Prices in incomplete markets are partially determined by the aggregate supply and demand. Market forces interact and implicitly determine a specific market price of risk or equivalently a specificities martingale measure. Now we can make sense of the assumption regarding the existence of a market for every T-bond. Existence of the market implies that the model can be calibrated to market data, i.e. parameters of the theoretical price expression are calibrated to fit the observed prices on the market. Prices are determined partly by the dynamics of the short rate and partly by market forces. There are several conceivable market prices of risk, eventually one is "chosen" by agents when forming the observed price.

1.3 Change of Numeraire

A numeraire is a traded asset without dividends, which is used to normalize the asset prices. Given the prices of the $n+1$ assets in the market $S_0, S_1, \dots, S_n$ and choosing S_0 as numeraire, we can construct a normalised economy dividing each price by the chosen numerarie price:

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \dots, \frac{S_n(t)}{S_0(t)},$$

where $S_i(t)$ denotes the realisation of the i-th price process at time t. By martingale price theory (Björk, 2009, Chapter 10), we know that the market is free of arbitrage if there exists a martingale measure $\mathbb{Q}^i \sim \mathbb{P}$, associated to the numeraire S_i, such that the processes

$$\frac{S_0(t)}{S_i(t)}, \frac{S_1(t)}{S_i(t)}, \dots, \frac{S_n(t)}{S_i(t)}$$

are martingales under $\mathbb{Q}^i$. Therefore, the price of a T-claim X at time t, $\Pi(t;X)$, is arbitrage-free, when

$$\Pi(t;X) = S_i(t) E^i \left[\frac{\Pi(T;X)}{S_i(T)} \middle| \mathcal{F}_t \right], \tag{1.21}$$

where E^i denotes the expectation taken under risk neutral probability $\mathbb{Q}^i$ associated to the numeraire S_i. Choosing $B(t) = e^{\int_0^t r(s)ds}$ as numeraire,

$$\frac{\Pi(t;X)}{e^{\int_0^t r(s)ds}} = E^{\mathbb{Q}} \left[\frac{\Pi(T;X)}{e^{\int_0^T r(s)ds}} \middle| \mathcal{F}_t \right],$$

yields the risk neutral valuation formula

$$\Pi(t;X) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \cdot \Pi(T;X)\middle|\mathcal{F}_t\right].$$

It may be the case that switching from a numeraire to another one simplifies the computational complexity of the pricing task. We must find a way to move from one numeraire to another ending up with consistent prices in both cases. To this extent we follow (Björk, 2009, Chapter 26), please refer to (Brigo & Mercurio, 2007, Section 2.2-2.5) for a meticulous theoretical discussion. Suppose we want to switch from S_0 to S_1. By equation (1.21), we have

$$\Pi(0;X) = S_0(0)E^0\left[\frac{\Pi(T;X)}{S_0(T)}\right] = S_1(0)E^1\left[\frac{\Pi(T;X)}{S_1(T)}\right]. \tag{1.22}$$

Denoting by $L_0^1(T)$ the Radon-Nikodym derivative

$$L_0^1(T) = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0},$$

we can state (1.22) as

$$\Pi(0;X) = S_0(0)E^0\left[\frac{\Pi(T;X)}{S_0(T)}\right] = S_1(0)E^0\left[\frac{\Pi(T;X)}{S_1(T)} \cdot L_0^1(T)\right].$$

It is clear that, for the equivalence to hold, for every contingent claim X

$$\frac{S_0(0)}{S_0(T)} = \frac{S_1(0)}{S_1(T)}L_0^1(T).$$

Thus,

$$L_0^1(T) = \frac{S_0(0)}{S_1(0)} \cdot \frac{S_1(T)}{S_0(T)}.$$

Theorem 1.2
Assume that $\mathbb{Q}^0$ is a martingale measure for the numeraire S_0 and assume that S_1 is a positive asset price process such that $\frac{S_1(t)}{S_0(t)}$ is a $\mathbb{Q}^0$-martingale. Define $\mathbb{Q}^1$ by the likelihood process

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \cdot \frac{S_1(t)}{S_0(t)}, \quad 0 \le t \le T.$$

Then $\mathbb{Q}^1$ is a martingale measure for S_1.

Proof. The theorem states that the normalised process $\frac{\Pi(t;X)}{S_1(t)}$ is a $\mathbb{Q}^1$-martingale. We know that

$\frac{\Pi(t;X)}{S_0(t)}$ is a $\mathbb{Q}^0$-martingale. For $s \leq t$, it holds that (Björk, 2009, p. 501-502)

$$
\begin{aligned}
E^1\left[\frac{\Pi(t;X)}{S_1(t)}\middle|\mathcal{F}_s\right] &= \frac{E^0\left[L_0^1(t)\frac{\Pi(t;X)}{S_1(t)}\middle|\mathcal{F}_s\right]}{L_0^1(s)} \\
&= \frac{E^0\left[\frac{S_0(0)}{S_1(0)}\cdot\frac{S_1(t)}{S_0(t)}\cdot\frac{\Pi(t;X)}{S_1(t)}\middle|\mathcal{F}_s\right]}{L_0^1(s)} \\
&= \frac{\frac{S_0(0)}{S_1(0)}\cdot E^0\left[\frac{\Pi(t;X)}{S_0(t)}\middle|\mathcal{F}_s\right]}{L_0^1(s)} \\
&= \frac{\frac{S_0(0)}{S_1(0)}\cdot\frac{\Pi(s;X)}{S_0(s)}}{L_0^1(s)} \\
&= \frac{\Pi(s)}{S_1(s)}.
\end{aligned}
$$

□

Corollary 1.2

Assume absence of arbitrage and that numeraires evolve as $dS_i(t) = \alpha_i(t)S_i(t)dt+S_i(t)\sigma_i(t)dW^0(t)$ for $i = \{0,1\}$, where W^0 denotes the $\mathbb{Q}^0$-Brownian motion. The dynamics of the likelihood process L_0^1 are given by

$$dL_0^1(t) = L_0^1(t)\phi_0^1(t)dW^0(t),$$

where ϕ_0^1 is the Girsanov kernel[1] *for the transition from $\mathbb{Q}^0$ to $\mathbb{Q}^1$, given by*

$$\phi_0^1(t) = \sigma_1(t) - \sigma_0(t).$$

Proof. The likelihood process $L_0^1(t)$ is martingale (Björk, 2009, p.501) under $\mathbb{Q}^0$. Recall that

$$dL_0^1(t) = d\left[\frac{S_0(0)}{S_1(0)}\cdot\frac{S_1(t)}{S_0(t)}\right]$$

and that

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sigma_i(t)dW^0(t).$$

Applying Itô formula, we know that $dL_0^1(t)$ includes the first and second degree differential effects of S_0 and S_1.

$$
\begin{aligned}
dL_0^1(t) &= \frac{\partial L_0^1(t)}{\partial S_1(t)}dS_1(t) + \frac{\partial L_0^1(t)}{\partial S_0(t)}dS_0(t) + d\langle S_0, S_1\rangle(t) \\
&= \frac{S_0(0)}{S_1(0)}\left[\frac{1}{S_0(t)}[\alpha_1(t)S_1(t)dt + \sigma_1(t)S_1(t)dW^0(t)]\right] + \\
&+ \frac{S_0(0)}{S_1(0)}\left[-\frac{S_1(t)}{S_0(t)^2}[\alpha_0(t)S_0(t)dt + \sigma_0(t)S_0(t)dW^0(t)] + S_0(t)S_1(t)\sigma_0(t)\sigma_1(t)dt\right] \\
&= L_0^1(t)[\cdots]dt + L_0^1(t)[\sigma_1(t) - \sigma_0(t)]dW^0(t).
\end{aligned}
$$

[1] By the Girsanov theorem, $dW^P(t) = \phi(t)dt + dW^{\mathbb{Q}}(t)$. (Björk, 2009, Section 11.3)

Since $L_0^1(t)$ is $\mathbb{Q}^0$ martingale, the drift term must vanish. Thus,

$$dL_0^1(t) = L_0^1(t)\{\sigma_1(t) - \sigma_0(t)\}dW^0(t).$$

□

The T-forward measure $\mathbb{Q}^T$ is defined as the martingale measure for the numeraire process $p(t,T)$ and yields the price of any T-claim X as

$$\Pi(t;X) = p(t,T)E^T[\Pi(T;X)|\mathcal{F}_t].$$

Whenever r is deterministic, pricing under the measure $\mathbb{Q}$ is equivalent to pricing under $\mathbb{Q}^T$. For this choice of numerare we have that

$$L^T(T) = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{B(0)\cdot p(T,T)}{B(T)\cdot p(0,T)} = \frac{e^{-\int_0^T r(u)du}}{p(0,T)}.$$

Given that the $\mathbb{Q}$-dynamics of the T-bond price are

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)v(t,T)dW^{\mathbb{Q}}(t),$$

the L^T dynamics are given by

$$dL^T(t) = L^T(t)v(t,T)dW^{\mathbb{Q}}(t),$$

where $v(t,T)$ is the volatility parameter.

1.4 Foreign Numeraire

Broadly speaking, inflation connects the nominal and real pricing sides of any quantities. Indeed, the analysis of inflation requires both nominal and real data. For the moment, we just simplistically say that there are two regimes: the nominal and the real one. Please refer to Chapter 2 for a thorough explanation. They are separated, but closely linked.; it is important to nimbly move between the two. To this extent, it is common to read of the analogy with currency market (Brigo & Mercurio, 2007, Section 2.9). As it is possible to switch from the domestic currency to a foreign one with an exchange rate, similarly we can switch from the nominal regime to the real one with an inflation index. With this analogy we can refer to the nominal regime as the domestic market and to the real regime as the foreign regime.

Let us consider a foreign market, where an asset X^f is traded, B^f is the money account and $\mathbb{Q}^f$ is corresponding risk-neutral martingale measure. This market acts as counterpart of the domestic market with money account B and risk-neutral measure $\mathbb{Q}$. The process $\mathcal{E}$ is the spot exchange rate between the foreign and the domestic currency, meaning that one unit of foreign currency is worth $\mathcal{E}(t)$ units of domestic currency at time t. Assume that the asset is a derivative paying out X_T^f at time T. Therefore, the arbitrage-free price of X^f at time t, P_t^f, in the foreign market is

$$P_t^f = B_t^f E^{\mathbb{Q}^f}\left[\frac{X_T^f}{B_T^f}\middle|\mathcal{F}_t\right].$$

This amount equals to the following amount in the domestic currency:

$$P_t = \mathcal{E}_t B_t^f E^{\mathbb{Q}^f}\left[\frac{X_T^f}{B_T^f}\middle|\mathcal{F}_t\right].$$

From the point of view of a domestic investor the foreign price converted, $\mathcal{E}_t P_t^f$, must equal the arbitrage-free price of the converted payoff:

$$\mathcal{E}_t B_t^f E^{\mathbb{Q}^f}\left[\frac{X_T^f}{B_T^f}\middle|\mathcal{F}_t\right] = B_t E^{\mathbb{Q}}\left[\frac{X_T^f \mathcal{E}_T}{B_T}\middle|\mathcal{F}_t\right].$$

Theorem 1.3
The Radon-Nikodym derivative defining the change of measure from the foreign risk-neutral probability measure Q_f to the domestic risk-neutral probability measure $\mathbb{Q}$ is given by

$$\frac{dQ^f}{dQ} = \frac{\mathcal{E}_T B_T^f}{\mathcal{E}_0 B_T}.$$

Proof. The arbitrage condition between currencies at time 0 is

$$E^{\mathbb{Q}^f}\left[\frac{\mathcal{E}_0 X_T^f}{B_T^f}\right] = E^{\mathbb{Q}}\left[\frac{X_T^f \mathcal{E}_T}{B_T}\right],$$

whereas the rules of change of measure yields

$$E^{\mathbb{Q}^f}\left[\frac{\mathcal{E}_0 X_T^f}{B_T^f}\right] = E^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^f}{d\mathbb{Q}}\frac{\mathcal{E}_0 X_T^f}{B_T^f}\right].$$

It is straightforward to see that the two expressions coincide for every contingent claim X_T^f only for $\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{\mathcal{E}_T B_T^f}{\mathcal{E}_0 B_T}$. □

As a corollary, changing measure form $\mathbb{Q}^f$ to $\mathbb{Q}$ is equivalent to changing the numeraire from B^f to $\frac{B}{\mathcal{E}}$. Economically, this change of measure amounts to changing the numeraire from the foreign bank account to the domestic bank account translated into foreign currency through the related exchange rate. This theory obviously holds for the discount factor, $p(\cdot, T)$, too. Moving from the measure associated with the foreign discount factor to the one associate with the domestic factor is equivalent to changing the numeraire from $P^f(\cdot, T)$ to $\frac{P(\cdot,T)}{\mathcal{E}}$.

Chapter 2

Inflation-Linked Instruments

2.1 Inflation-Protected Securities

So far, we have been using nominal rates and quantities, that is we have been measuring values in terms of money. ZCBs cost a certain amount of money and guarantee a sum of money at maturity, CBs guarantee also the money of coupons during its lifetime, the bank account describes the evolution over time of a stock money. The intrinsic value of these claims depends on the value of their unit of measurement: money. Suppose $r(t) = 0$, then $dB(t) = 0$. Therefore any sum deposited at time t in the bank account, say €100, will remain the same in the future. Furthermore, suppose inflation between t and $t+1$ is 2%. €100 deposited at time t will be worth roughly €98 at $t+1$, in this setting. The amount of money remained constant over time, but its value decreased. Prices raised, therefore the same amount of money is capable of buying less goods. This simple example shows the difference between nominal and real values. In nominal terms, the deposit remained constant, but, in real terms, its value decreased. Nominal values are measured in terms of money, real values in terms of goods and services.

Generally, claims are built upon nominal quantities. For example, a ZCB pays €1 at maturity. We recall from Section 1.1 that the yield of a ZCB is y such that

$$p(t,T) = e^{-(T-t)\cdot y} \times 1.$$

This is the nominal yield, because it measures in monetary terms the profit for the lender and the cost for the borrower. Disentangling the effect of inflation, we find the real yield. Typically, nominal yields are higher than real ones, as a result of inflation. Now it is clear why we need to define nominal and real quantities. In the following, we will use r and n as subscripts to denote real and nominal quantities. The price of a real/nominal T-ZCB at time t is

$$P_k(t,T) = e^{-\int_t^T f_k(t,u)du} = E\left[e^{-\int_t^T r_k(u)du}|\mathcal{F}_t\right], \text{ for } k \in \{r,n\}$$

and, similarly, the time t value of the real/nominal bank account is

$$B_k(t) = e^{\int_0^t r_k(u)du}, \text{ for } k \in \{r, n\}.$$

We already defined nominal forward and short rates and, for the sake of completeness, we specularly define real analogues. The real/nominal instantaneous forward rate contracted at time time for the infinitesimal interval $[T, T + dT]$ are defined as

$$f_k(t,T) = -\frac{\partial \log P_k(t,T)}{\partial T}, \text{ for } k \in \{r, n\}.$$

Therefore, we denote real/nominal instantaneous short rates as

$$r_k(t) = f_k(t,t), \text{ for } k \in \{r, n\}.$$

Eventually, we denote $I(t)$ the price index value at time t. For a more thorough discussion about price indexes, please refer to Section 3.3.

Remark 2.1
$P_n(t,T)$ is the price a time t of nominal T-ZCB expressed in euros. $P_r(t,T)$ is the price at time t of a real T-ZCB expressed in price index units. This statement seems counterintuitive at first sight, let us give some clarification. Assume we have two economies, a domestic and foreign one. We live in the domestic economy and use the domestic currency. The domestic economy, and its associated currency, is the nominal economy with nominal prices. The foreign economy is the real economy with real prices. The price index $I(t)$ is the spot exchange rate to convert amounts between the two currencies at time t. This framework is known as foreign currency analogy and it is motivated by the fact the we live in a nominal-term world, whereas real values are abstract quantities. Therefore, $P_r(t,T)$ is expressed in the foreign real currency and we have to apply the exchange rate $I(t)$, in order to get the price in our nominal domestic currency.

Hence, we define the price in euros (or whatever other nominal currency in use) of a real T-ZCB as

$$P_{TIPS}(t,T) = I(t)P_r(t,T).$$

"TIPS" stands for Treasury-Inflation-Protected-Security. TIPS are debt instruments issued by governments to raise funds, guaranteeing a fixed real yield. Specifically, TIPS are issued by the US Treasury and came to denote by analogy the broad variety of sovereign inflation protected securities, notwithstanding the issuing government. We define the price at time t of a TIPS-CB in euros, issued at time $t_0 < t$, with constant coupon payments C and face value K as

$$P_{TIPS}(t) = K\frac{I(t)}{I(t_0)}P_r(t,T) + \sum_{i=1}^{T} C_i \frac{I(t)}{I(t_0)} P_r(t,T_i). \tag{2.1}$$

Coupons and face value are upwardly adjusted by the inflation factor $\frac{I(t)}{I(t_0)}$ between the issuance date t_0 and the price valuation date, to keep the real yield constant. Should $\frac{I(t)}{I(t_0)} < 1$, that is should prices deflate, the ratio would be kept constant at 1. The inflation correction factor has a lower bound floor at unity, which constitutes an embedded put option for investors. The value of this protection remains negligible, as harsh deflation is a remote occurrence.

US TIPS have been traded since 1997, albeit improved liquidity of these instruments was mildly displayed only after 2003. They are issued in terms of 5, 10, and 30 years, they pay coupons twice a year. They are indexed to the CPI-u, not seasonally adjusted. BTP€i and BTPi are Italian analogues. BTP€i are indexed to the HICP-ex.tobacco. BTPi are indexed to the Italian CPI FOInt (Famiglia di Operai ed Impiegati, escluso tabacco). They are issued in terms of 5, 10, 15 and 30 years and pay semestral coupons. Payments are modified according to the indexation coefficient $IC_{y,m,d}$ for day d of month m in year y, in order to keep the real yield constant

$$IC_{y,m,d} = \frac{RI_{y,m,d}}{BI}.$$

BI is base inflation, i.e. the reference inflation at time of issuance. $RI_{y,m,d}$ is the reference inflation for day d of month m in year y and it is determined as

$$RI_{y,m,d} = I_{y,m-3} + \frac{d-1}{\text{days in } m} \cdot (I_{y,m-2} - I_{y,m-3}).$$

The Treasury Ministry publishes monthly $CI_{y,m,d}$ values for each month.

2.2 Zero-Coupon Inflation-Indexed Swap

An inflation-indexed swap is a swap where, on each payment date, A pays to B the inflation rate of a given price index over a certain period, and B pays to A a fixed rate for the same period. The Zero-Coupon inflation-indexed swap provides a single swap at maturity. Denoted maturity as $T_M = M$ years, the fixed rate as K, the nominal amount as N, the fixed amount is

$$N[(1+K)^M - 1],$$

and the floating amount is

$$N\left[\frac{I(T_M)}{I(0)} - 1\right].$$

Section 1.2 implies that the value at time t of a T_M-ZCIIS inflation-indexed leg is

$$\mathbf{ZCISS}_{fl}(t, T_M, I(0), N) = NE^{Q_n}\left\{e^{-\int_t^{T_M} r_n(u)du}\left[\frac{I(T_M)}{I(0)} - 1\right]\middle|\mathcal{F}_t\right\}. \tag{2.2}$$

We denote by $\mathbb{Q}_n$ and $\mathbb{Q}_r$ the risk neutral measure associated to nominal and real rates, respectively. E^{Q_n} denotes the expectation taken under the nominal risk neutral measure $\mathbb{Q}_n$. As explained in (Brigo & Mercurio, 2007, Chapter 16), by the foreign currency analogy, the nominal price of a real ZCB is equal to the nominal price of a contract paying one real unit converted in nominal currency:

$$I(t)P_r(t,T) = I(t)E^{Q_r}\left[e^{-\int_t^T r_r(u)du}\middle|\mathcal{F}_t\right] = E^{Q_n}\left[e^{-\int_t^T r_n(u)du}I(T)\middle|\mathcal{F}_t\right].$$

Hence, the ZCIIS value is given by

$$\mathbf{ZCIIS}_{fl}(t, T_M, I(0), N) = N\left[\frac{I(t)}{I(0)}P_r(t, T_M) - P_n(t, T_M)\right].$$

At time $t = 0$ this simplifies into

$$\mathbf{ZCIIS}_{fl}(0, T_M, I(0), N) = N\left[P_r(0, T_M) - P_n(0, T_M)\right].$$

This formulation requires only exogenous information. The price of the fixed leg at time t=0 is

$$\begin{aligned}\mathbf{ZCIIS}_{fix}(0, T_m, I(0), N) &= NE^{Q_n}\left[e^{-\int_0^{T_M} r_n(u)du}\left((1+K)^M - 1\right)\right] \\ &= NP_n(0, T_M)\left((1+K)^M - 1\right).\end{aligned}$$

The payoff for the fixed leg payer is given by the formula

$$\begin{aligned}&\{\mathbf{ZCIIS}_{fl}(0, T_m, I(0), N) - \mathbf{ZCIIS}_{fix}(0, T_m, I(0), N)\} = \\ &= \left\{N\left[P_r(0, T_M) - P_n(0, T_M)\right] - NP_n(0, T_M)\left[(1+K)^M - 1\right]\right\}.\end{aligned}$$

Since IIS are quoted in terms of K, we can equate the nominal present value of the fixed leg to the floating leg and obtain the real discount factor. Since the contract should have null value at time $t = 0$,

$$N\left[P_r(0, T_M) - P_n(0, T_M)\right] = NP_n(0, T_M)\left[(1+K)^M - 1\right].$$

Knowing $K = K(T_M)$ from market prices and $P_n(0, T_M)$ from the current nominal ZCB curve, we can solve for $P_r(0, T_M)$:

$$P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M.$$

We can, thus, to strip real ZCB prices from the quoted prices of ZCIIS.

2.3 Year-on-Year Inflation-Indexed Swap

The year-on-year inflation-indexed swap shares the same structure of a ZCIIS, but the scheme is repeated at each time T_i. The contract fixed-leg year fraction for the interval $[T_{i-1}, T_i]$ is denoted as ϕ_i, while ψ_i is the floating-leg year fraction for the interval $[T_{i-1}, T_i]$; $T_0 = 0$ and N is the contract nominal value. The fixed leg and the floating leg respectively amount to

$$N\phi_i K, \quad \text{and} \quad N\psi_i\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right].$$

The value at time $t \leq T_i$ of the floating leg payoff at time T_i is

$$\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E^{Q_n}\left\{e^{-\int_t^{T_i} r_n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]\middle|\mathcal{F}_t\right\}. \tag{2.3}$$

We follow the explanation of (Brigo & Mercurio, 2007, Chapter 16). For $T_{i-1} \leq t \leq T_i$ the value is computed as $\mathbf{ZCIIS}_{fl}(t, T_i, I(T_{i-1}), \psi_i N)$; for $t < T_{i-1}$ we can write

$$\begin{aligned}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N) =& N\psi_i E^{Q_n}\left\{e^{-\int_t^{T_{i-1}} r_n(u)du}\cdot\right.\\ & \left. E^{Q_n}\left[e^{-\int_{T_{i-1}}^{T_i} r_n(u)du}\left(\frac{I(T_i)}{I(T_{i-1})}-1\right)\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_t\right\}.\end{aligned}$$

The inner expectation is a $\mathbf{ZCIIS}_{fl}(T_{i-1}, T_i, I(T_{i-1}), 1)$ and we use equation (2.2) to write

$$\begin{aligned}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i E^{Q_n}\left[e^{-\int_t^{T_{i-1}} r_n(u)du}[P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)]\middle|\mathcal{F}_t\right]\\ &= N\psi_i E^{Q_n}\left[e^{-\int_t^{T_{i-1}} r_n(u)du}P_r(T_{i-1}, T_i)\middle|\mathcal{F}_t\right] - N\psi_i P_n(t, T_i).\end{aligned}$$

The value at time t of the inflation-indexed leg is given by the sum of the floating payments up to T_M

$$\begin{aligned}\mathbf{YYIIS}_{fl}(t, \mathcal{T}, \Psi, N) &= \mathbf{YYIIS}_{fl}(t, T_{\iota(t)-1}, T_{\iota(t)}, \psi_{\iota(t)}, N) + \sum_{i=\iota(t)+1}^{M}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N)\\ &= \mathbf{ZCIIS}_{fl}(t, T_{\iota(t)}, I(\iota_{t-1}), \psi_{\iota(t)}N) + \sum_{i=\iota(t)+1}^{M}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N)\end{aligned} \tag{2.4}$$

where $\mathcal{T} = \{T_1, \dots, T_M\}$, $\Psi = \{\psi_1, \dots, \psi_M\}$ and $\iota(t) = min\{i : T_i \geq t\}$. The formulation reduces at time $t = 0$ to

$$\mathbf{YYIIS}_{fl}(0, \mathcal{T}, \Psi, N) = \mathbf{ZCIIS}_{fl}(0, T_1, I(0), \psi_1 N) + \sum_{i=2}^{M}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N).$$

If real rates were deterministic, pricing of YYIIS would only require information available on the market:

$$\begin{aligned}E^{Q_n}\left[e^{-\int_t^{T_{i-1}} r_n(u)du}P_r(T_{i-1}, T_i)\middle|\mathcal{F}_t\right] &= P_r(T_{i-1}, T_i)P_n(t, T_{i-1})\\ &= \frac{P_r(t, T_i)}{P_r(t, T_{i-1})}P_n(t, T_{i-1}).\end{aligned}$$

However, real rates are stochastic and the expected value is model dependent. We discuss the pricing of YYIIS with a market model. To this end, define the forward CPI, $\mathcal{I}_i(t)$, as

$$\mathcal{I}_i(t) = I(t)\frac{P_r(t, T_i)}{P_n(t, T_i)}, \tag{2.5}$$

which is a martingale under the nominal T_i-forward measure for a generic maturity T_i, $\mathbb{Q}_n^{T_i}$, with associated expectation sign $E_n^{T_i}$. We assume that $\mathcal{I}_i(t)$ evolves with log-normal dynamics

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)dW_i^I(t),$$

where $\sigma_{I,i}$ is a positive constant and W_i^I is a $\mathbb{Q}_n^{T_i}$-Brownian motion. Then, we can write the value at time $t < T_{i-1}$ of the floating leg under the T_n^i-forward measure using (2.5) as

$$\begin{aligned}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P_n(t, T_i) E_n^{T_i}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\middle|\mathcal{F}_t\right] \\ &= N\psi_i P_n(t, T_i) E_n^{T_i}\left[\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - 1\middle|\mathcal{F}_t\right] \\ &= N\psi_i P_n(t, T_i) E_n^{T_i}\left[\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1\middle|\mathcal{F}_t\right]. \end{aligned} \tag{2.6}$$

In equation (2.6) we use the fact that $\mathcal{I}_i(T_i)$ is a martingale with respect to the probability $\mathbb{Q}_n^{T_i}$, as postulated in equation (2.5) The evolution dynamics of $\mathcal{I}_{i-1}(t)$ under $\mathbb{Q}_n^{T_{i-1}}$ are $d\mathcal{I}_{i-1}(t) = \sigma_{I,i-1}\mathcal{I}_{i-1}(t)dW_{i-1}^I(t)$, symmetrically to the evolution of $\mathcal{I}_i(t)$ under $\mathbb{Q}_n^{T_i}$. However, we need to derive the dynamics under the measure $Q_n^{T_i}$. Section 1.3 and the change of numerarie theory [see (Brigo & Mercurio, 2007, Section 2.3)] yield

$$d\mathcal{I}_{i-1}(t) = \mathcal{I}_{i-1}(t)\sigma_{I,i-1}\left[-\frac{\tau_i\sigma_{n,i}F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_n(t; T_{i-1}, T_i)}\rho_{I,n,i}dt + dW_{i-1}^I(t)\right], \tag{2.7}$$

where W_{i-1}^I is a $\mathbb{Q}_n^{T_i}$-Brownian motion with $dW_{i-1}^I(t)dW_i^I(t) = \rho_{I,i}dt$ and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}_{i-1}(\cdot)$ and $F_n(\cdot; T_{i1}, T_i)$. We keep the drift of $d\mathcal{I}_{i-1}(t)$ in equation (2.7) constant at the time t value to ease calculations:

$$-\frac{\tau_i\sigma_{n,i}F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_n(t; T_{i-1}, T_i)}\rho_{I,n,i} = \bar{\mu}_t.$$

This simplification allows $\mathcal{I}_{i-1}(T_{i-1})$ to be log-normally distributed also under $Q_n^{T_i}$. Before proceeding in the evaluation of the floating leg, we recall that GBM evolution dynamics of variable X lead to the level formulation $X(T) = X(t)e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)}$. This justifies at time t

$$\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} = \frac{\mathcal{I}_i(t) \cdot e^{\left(-\frac{\sigma_{I,i}^2}{2}\right)(T_{i-1}-t) + \sigma_{I,i}(W_i^I(T_{i-1}) - W_i^I(t))}}{\mathcal{I}_{i-1}(t) \cdot e^{\left(\frac{\sigma_{I,i-1}^2}{2} - \bar{\mu}_t\right)(T_{i-1}-t) + \sigma_{I,i-1}(W_{i-1}^I(T_{i-1}) - W_{i-1}^I(t))}}. \tag{2.8}$$

Finally, we recall that the exponentials in equation (2.8) are log-normally distributed; then we take the $Q_n^{T_i}$-expectation of equation (2.8) and write

$$E_n^{T_i}\left[\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})}\middle|\mathcal{F}_t\right] = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}e^{C_i(t)},$$

where

$$C_i(T) = -\sigma_{I,i-1}\left[\sigma_{I,i-1} - \bar{\mu}_t - \rho_{I,i}\sigma_{I,i}\right](T_{i-1} - t).$$

Therefore,

$$\begin{aligned}\mathbf{YYIIS}_{fl}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P_n(t, T_i)\left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}e^{C_i(t)} - 1\right] \\ &= N\psi_i P_n(t, T_i)\left[\frac{P_n(t, T_{i-1})P_r(t, T_i)}{P_n(t, T_i)P_r(t, T_{i-1})}e^{C_i(t)} - 1\right].\end{aligned}$$

The price at time t of the inflation-indexed leg of the swap (generally, the price of the YYIIS contract) is given by the ZCIIS and YYIIS formulas, as showed by equation (2.4), resulting in

$$\begin{aligned}\mathbf{YYIIS}(t,\mathcal{T},\Psi,N) &= N\psi_{\iota(t)}P_n(t,T_{\iota(t)})\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right] \\ &+ N\sum_{i=\iota(t)+1}^{M}\psi_i P_n(t,T_i)\left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}e^{C_i(t)} - 1\right] \\ &= N\psi_{\iota(t)}\left[\frac{\mathcal{I}_{\iota(t)}(t)}{I(T_{\iota(t)-1})} - 1\right] + \\ &+ N\sum_{i=\iota(t)+1}^{M}\psi_i\left[P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C_i(t)} - P_n(t,T_i).\right] \end{aligned} \tag{2.9}$$

Valuation a time $t = 0$ reduces to

$$\begin{aligned}\mathbf{YYIIS}(0,\mathcal{T},\Psi,N) &= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{\mathcal{I}_i(0)}{\mathcal{I}_{i-1}(0)}e^{C_i(0)} - 1\right] \\ &= N\psi_1[P_r(0,T_1) - P_n(0,T_1)] + \\ &+ N\sum_{i=2}^{M}\left[P_n(0,T_{i-1})\frac{P_r(0,T_i)}{P_r(0,T_{i-1})}e^{C_i(0)} - P_n(0,T_i)\right] \\ &= N\sum_{i=1}^{M}P_n(0,T_i)\left[\frac{1+\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)}e^{C_i(0)} - 1\right], \end{aligned} \tag{2.10}$$

where the last equation follows by definition of simple nominal/real forward rate (see equation (1.1) for the general case).

This YYIIS price requires the volatilities of forward inflation indices, $\sigma_{I,i}$; the volatilities of nominal forward rates, $\sigma_{n,i}$; the instantaneous correlations between forward inflation indices and nominal forward rates, $\rho_{I,n,i}$. This setting has a weakness in the approximation we have to assume over $\bar{\mu}_t$. Such solution would be harmless if correlations $\rho_{I,n,i}$ were null. Since, generally, the correlation between inflation and nominal rates is positive, especially for longer maturities, correlations may have a significant impact on the evolution of $C_i(t)$ for long maturities.

Chapter 3

Model Survey

In this chapter we will offer a brief summary of the literature regarding interest rate and inflation modelling. Several models and techniques are presented in a fairly general context. Some of them will be selected and applied in the next chapter for modelling purposes.

3.1 Short Rate Models

The discussion of Section 1.2 proves that, given the $\mathbb{P}$-dynamics of the short rate

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{P}}(t),$$

the general term structure equation is

$$\begin{cases} F_t + \{\mu - \lambda\sigma\}F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ F(T, r) = \Phi(r). \end{cases}$$

The term $(\mu - \lambda\sigma)$ is precisely the drift term of the short rate under measure $\mathbb{Q}$. Therefore, modelling short rates under the martingale measure $\mathbb{Q}$

$$dr(t) = \mu_Q(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{Q}}(t) \tag{3.1}$$

completely determines the term structure

$$\begin{cases} F_t + \mu_Q F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ F(T, r) = \Phi(r). \end{cases}$$

We closely follow (Björk, 2009, Chapter 24) in this discussion. We assume in the following discussion, unless otherwise stated, that the drift is specified under the martingale measure, i.e. $W = W^{\mathbb{Q}}$ and $\mu = \mu_Q$.

Some models considerably ease the computational involvement required to solve the term structure equation. Affine models share this feature.

Definition 3.1 (Affine Term Structure)
Given the term structure

$$\{p(t,T); 0 \leq t \leq T,\ T > 0\} \text{ with } p(t,T) = F(t, r(t); T),$$

the model is said to posses an affine term structure when F has the form

$$F(t, r(t); T) = e^{A(t,T) - B(t,T)r}.$$

$A(t,T)$ and $B(t,T)$ are deterministic functions.

Remark 3.1
The computational simplification of some models derives from the implied distribution of the short rate. The Vasiček, Ho-Lee and Hull-White models yield normally distributed short rates.

Vasiček model (Vasiček, 1977)	$dr(t) = (\beta - \alpha r(t))dt + \sigma dW(t), \quad \alpha > 0$
Ho-Lee model (Ho & Lee, 1986)	$dr(t) = \theta(t)dt + \sigma dW(t)$
Hull-White model (Hull & White, 1990)	$dr = (\theta(t) - \alpha(t)r(t))dt + \sigma(t)dW(t), \quad \alpha(t) > 0$

Since the integral $\int_0^T r(s)ds$ can be thought of a a sum over a normally distributed variable, bond prices

$$p(0,T) = E^{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\right]$$

result form the expected value of a log-normal stochastic variable.

Given the $\mathbb{Q}$-dynamics of the short rate (3.1) and assumed that bond prices have affine representation, the term structure is given by

$$F(t, r(t); T)[A_t(t,T) - B_t(t,T)r] - F(t, r(t); T)\mu(t,r)[B(t,T)] + \\ + \frac{1}{2}\sigma^2(t,r)F(t, r(t); T)[B^2(t,T)] - rF(t, r(t); T) = 0,$$

or equivalently

$$A_t(t,T) - [1 + B_t(t,T)]r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0. \tag{3.2}$$

Proposition 3.1
Assuming μ and σ have the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}, \end{cases}$$

bond prices have affine term structure when

$$F(t, r(t); T) = e^{A(t,T) - B(t,T)r},$$

where $A(t,T)$ and $B(t,T)$ satisfy the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{3.3}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{3.4}$$

Proof. The boundary value $F(T,r;T) = 1$ implies that

$$\begin{cases} A(T,T) = 0 \\ B(T,T) = 0. \end{cases}$$

We assume that μ and σ have the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases}$$

Hence, the term structure equation transforms (3.2) into

$$A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) - \left[1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T)\right] r = 0.$$

This equation holds for every t, T and r, thus the coefficient for the r-term must equal 0:

$$B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1.$$

The solution is given by

$$A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T).$$

□

3.2 Forward Rate Models

David Heath, Robert Jarrow and Andrew Morton proposed (Heath *et al.*, 1992) a different method to analyze interest rates. This bundle of concepts and results is known as Heath-Jarrow-Morton framework. It is interesting to note the use of the term “framework” instead of the usual “model”. The HJM framework does not postulate a specific model, rather it offers a different approach: it shifts the perspective of analysis from short rates to forward rates. Still, short rate models and forward rates models are connected by the relation $r(t) = f(t,t)$. In the following discussion we follow (Björk, 2009, Chapter 25).

We assume that the forward rate $f(\cdot, T)$, for every fixed $T > 0$, evolves under the objective measure as

$$\begin{cases} df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW^{\mathbb{P}} \\ f(0,T) = f^{\star}(0,T), \end{cases} \tag{3.5}$$

where $W^{\mathbb{P}}$ is d-dimensional $\mathbb{P}$-Brownian motion and $f^{\star}(0,T)$ is the forward rate observed in the market at time $t = 0$ for maturity T. This specification offers a stochastic representation in the t-variable for each T: maturity acts as index for the theoretically infinite dynamics (one for each maturity). The initial condition ensures the perfect fit between theoretical and observed bond prices at $t = 0$ without necessity of inverting the yield curve. From equation (1.15), bond dynamics are

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + p(t,T)S(t,T)dW^{\mathbb{P}}(t),$$

where

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases}$$

We identify the risk premium of the T-bond in the quantity

$$A(t,T) + \frac{1}{2}||S(t,T)||^2$$

and state (1.20) in the multidimensional version

$$A(t,T) + \frac{1}{2}||S(t,T)||^2 = \sum_{i=1}^{d} S_i(t,T)\lambda_i(t).$$

Taking the T-derivative, we can make sense of the following theorem.

Theorem 3.1 (HJM drift condition)
Given an arbitrage-free bond market and the forward rate dynamics (3.5)*, there exist a d-dimensional process*

$$\lambda(t) = [\lambda_1(t), \ldots, \lambda_n(t)]'$$

such that for all T and for all $t \leq T$

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds - \sigma(t,T)\lambda(t). \tag{3.6}$$

If we wish to specify forward rates dynamics under the martingale measure $\mathbb{Q}$, we can write

$$\begin{cases} df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW^{\mathbb{Q}} \\ f(0,T) = f^{\star}(0,T). \end{cases}$$

Intuitively, modelling under the martingale measure requires that $\lambda = 0$. The market price of risk is the Girsanov kernel for transition from the $\mathbb{Q}$ measure to the $\mathbb{P}$ measure, but here we are working directly under $\mathbb{Q}$. Being more formal, under a martingale measure the local rate return of bond prices must equal the short rate

$$r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2 = r(t).$$

Theorem 3.2
Under the martingale measure $\mathbb{Q}$ the HJM drift condition reduces to

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds, \forall\ T \text{ and } \forall\ 0 \le t \le T \tag{3.7}$$

The HJM framework can be applied to construct a HJM model with the following steps. Define the volatility structure $\sigma(t,T)$ and obtain the drift parameter $\alpha(t,T)$ by the HJM drift condition. Then, retrieve from the market the current forward rate structure $\{f^\star(0,T); T \ge 0\}$. Compute forward rates by

$$f(t,T) = f^\star(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s).$$

Eventually, compute bond prices by

$$p(t,T) = e^{-\int_t^T f(t,s)ds}.$$

Remark 3.2
Bond prices are equivalently computed by

$$p(t,T) = e^{-\int_t^T f(t,s)ds} = E^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \times 1 \middle| \mathcal{F}_t\right].$$

The current market forward rates can be stripped from market bond prices. These are the froward rates implied by the market. We need to obtain froward rates for every maturity T, but typically we can have access only to a limited number of finite maturities, thus we may need to approximate the continuous setting with a discretised one.

3.3 Inflation Modelling

Inflation/Deflation is defined as the percentage increase/decrease in the general price level of goods and services in an economy over a period of time. The term “deflation” is rarely used, since price changes have historically been positive the great majority of times. Therefore, the term “inflation” has almost gained the meaning of price change *tout court*, notwithstanding the sign of the change. We need to be more specific regarding the price level and the period of time. There are many way to measure price level, hence there are many values of inflation at the same time. Typically, when referring to price level we refer to the value of a price index. Statistical agencies select a basket of goods and services relevant for a social category. Then, they retrieve the price of these goods and services over time. Each entry receives a weight based on the relevance of the item for the category under consideration. A price index is a weighted average of the price of the selected basket of goods and services. The resulting value is usually indexed to 100 at some fixed time, in order to ensure readability. Different baskets and categories can be taken into account. For example, there are producer price indexes (PPIs) or consumer price indexes (CPIs); among CPIs, there are urban consumer price indexes (CPI-u), seasonally adjusted consumer price indexes (CPI-sa). Price changes are measured on a monthly basis. However we shall be aware that the inflation rate for a given month is not the effective rate. There is a lag between the change in price and the detection of the change. For the US CPI-u this lag amounts to two months, meaning that the rate published on March should refer to the real inflation rate for January. This said, we can make sense of the of the formal definition of inflation:

$$\pi(t) = \frac{I(t) - I(t-1)}{I(t-1)},$$

where $\pi(t)$ is the inflation rate published on month t and $I(t)$ is the value of the price index at month t. As we pointed out, $\pi(t)$ is the price change for month $t - \ell$, where ℓ is the lag in publication of official statistics. Definitions of each concept and quantity mentioned can be found in the Eurostat Glossary. Inflation indexes are constructed with similar methodologies, but each one slightly differs from the other; for the European harmonised index of consumer prices (HICP) please refer to the Eurostat. Eurostat official explanation

Central banks manage monetary policies to ensure a stable, positive, and low inflation rate around 2-3%. This is believed to be the optimal rate for the following reasons. Unpredicted or unpredictable inflation generates economic and social tension, because agents cannot predict the value of their future monetary stakes. They will require inflation risk premium, their forecasts will often fail, credit channels and investment will get constrained. Risk and uncertainty are always dangerous for sound economic growth. Excessive inflation or hyperinflation, i.e. out of control inflation, causes price to change significantly in little time. If all parties and agents could perfectly and instantaneously adjust their positions to the new prices, then inflation would be just a numeric and innocuous shift in price values. However, this is not the case, because time, legal, economic and social constraints prevent prices from instantaneously adjust to new levels. Therefore, excessive inflation brings about again social and economic tension, as a result of wealth erosion and uncertainty regarding future monetary stakes. Deflation, i.e. negative inflation, happens primarily in contracting or stagnating economies. It can be caused by excess supply or when money supply shrinks. Low inflation occurs in sound economies, where growth happens in a systemically sustainable way. When agents and enterprises find a sound soil to grow and prosper, the growth in richness and consumption leads to a mild increase in prices. Higher demand stimulates supply and workers demand higher wages to share the economic well-being, hopefully leading to a controlled inflationary spiral. Let us stress that growth in well-being and

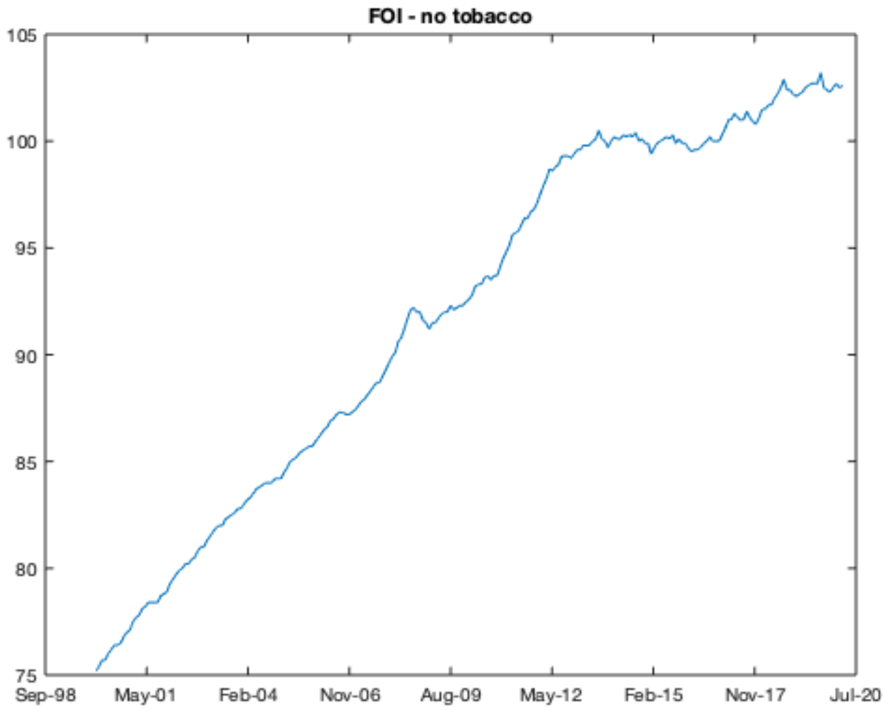


Figure 3.1: The FOInt index is an Italian CPI, which excludes tobacco from the basket under consideration. The plot shows values of the index between January, 15^{th} 2000 and March, 15^{th} 2020. The raise has been constant until May 2012, with exception of the drop due to the 2008 financial crisis. From this point onward, the evolution has been less intense and shakier. The average yearly inflation between Jan-2000 and May-2012 was 2,18%, while average yearly inflation between May-2012 and March-2020 was 0,51%.

consumption is not incompatible with socially and environmentally sustainable development. This brief analysis shows why stable, positive and low inflation rate is optimal. For a thorough analysis please refer to (Blanchard, 2016, Chapters 6,7 and 8).

Inflation plays a major in both financial and operative decision making. It may be seen as indicator of the health status of economy. It is the numerical consequence of deep economic processes and it is able to generate ample social turmoil: just think about the Italian case during 70s, 80s and 90s or, in a hyperinflation contest, to the case of Germany between the two world wars. It is imperative for financial institution to forecast its behaviour, in order to adjust and value monetary present and future monetary stakes. Moreover, governments have been issuing inflation protected securities for the last two decades roughly. It is the case of US TIPS (Treasury Inflation Protected Securities) or Italian BTP€i. These securities adjust upwards payments to contrast the nominal purchasing power reduction due to inflation and, overall, maintain a constant real rate of return.

In the following chapters we will rely on (Jarrow & Yildirim, 2003) model. Therefore, it is useful to briefly recap how Jarrow and Yildirim model inflation, the next chapter will provide more precise explanation. Inflation index evolution is modelled as a geometric brownian motion

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I dW_I^{\mathbb{Q}}(t). \tag{3.8}$$

Inflation modelling is often accompanied by interest rate modelling. Monetary policies makes clear the tight causal relation from interstate rates to inflation rates. Therefore, three factor models are usually employed. From a forward point of view , let us assume that nominal forward rate, $f_n(t,T)$, real forward rate, $f_r(t,T)$ and inflation index, $I(t)$, evolve under $\mathbb{Q}$ as

$$\begin{aligned}
df_n(t,T) &= \alpha_n(t,T)dt + \sigma_n(t,T)dW_n^{\mathbb{Q}}(t), && f_n(0,T) = f^\star(0,T)\\
df_r(t,T) &= \alpha_r(t,T)dt + \sigma_r(t,T)dW_r^{\mathbb{Q}}(t), && f_r(0,T) = f_r^\star(0,T)\\
\frac{dI(t)}{I(t)} &= \mu_I(t)dt + \sigma_I dW_I^{\mathbb{Q}}(t), && I(0) = I_0 > 0.
\end{aligned}$$

This three factor model is characterised by random drift coefficients and deterministic diffusion coefficients. Define the real and nominal money market account at time t as $B_k(t) = e^{\int_0^t r_k(s)ds}$ for $k \in \{r,n\}$; define the real and nominal T-ZCB at time t as $P_k(t,T) = e^{-\int_t^T f_k(t,s)ds}$ for $k \in \{r,n\}$. For absence of arbitrage to hold (Amin & Jarrow, 1991), the quantities

$$\frac{P_n(t,T)}{B_n(t)}, \frac{I(t)P_r(t,T)}{B_n(t)} \text{ and } \frac{I(t)B_r(t)}{B_n(t)}$$

must be $\mathbb{Q}$-martingales. We focus on the last quantity and write $\xi(t) = \frac{I(t)B_r(t)}{B_n(t)}$. Then,

$$\begin{aligned}
d\xi(t) &= -\frac{1}{B_n^2(t)}I(t)B_r(t)dB_n(t) + \frac{1}{B_n(t)}d(I(t)B_r(t))\\
&= -\xi(t)r_n(t)dt + \frac{1}{B_n(t)}(I(t))dB_r(t) + B_r(t)dI(t))\\
&= \xi(t)[r_r(t) - r_n(t) + \mu_I(t)]dt + \xi(t)\sigma_I dW_I^{\mathbb{Q}}(t).
\end{aligned} \tag{3.9}$$

This result is equivalently stated under the objective measure, recalling that $dW^{\mathbb{P}}(t) = \lambda(t)dt + dW^{\mathbb{Q}}(t)$, as

$$d\xi(t) = \xi(t)[r_r(t) - r_n(t) + \mu_I - \lambda_I(t)\sigma_I]dt + \xi(t)\sigma_I dW_I^{\mathbb{P}}(t).$$

Martingale property of $\xi(t)$ requires the drift in (3.9) to vanish, hence

$$r_n(t) = r_r(t) + \mu_I(t).$$

The index price local return follows the dynamics

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t)]dt + \sigma_I dW_I^{\mathbb{Q}}(t), \tag{3.10}$$

thus it is clear that short rate randomness makes the drift coefficient random. This result is known as Fisher equation after Irving Fisher (1867 - 1947) and estimates the relationship between nominal and real interest rates through inflation. The equation can be derived assuming an individual buys €1 of a $t+1$-bond at time t. Given the expected inflation between t and $t+1$, we have that what the investor actually gains is the real return:

$$\begin{aligned}(1+r_r(t)) &= \frac{(1+r_n(t))}{(1+\pi(t))} \\ r_n(t) &= r_r(t)+\pi(t)+r_r(t)\pi(t) \\ r_n(t) &\approx r_r(t)+\pi(t).\end{aligned}$$

In this computation we assume that $r_r(t)\cdot\pi(t)$ is negligible. The expected inflation $\pi(t)$, or better $\pi^e(t)$, is indeed equal to $\mu_I(t)$, which is the expected local rate of inflation given the $\mathbb{Q}$-dynamics of $\frac{dI(t)}{I(t)}$.

The result of equation (3.10) is achieved postulating the short rate model (Eksi & Filipović, 2014) for $X(t) = (r_n(t), r_r(t), \log I(t))'$

$$dX(t) = (B+\beta X(t))dt + \Sigma dW^{\mathbb{Q}}(t),$$

where B is a column vector $\in \mathbb{R}^3$, β and Σ are matrices $\in \mathbb{R}^{3\times 3}$ and $W^{\mathbb{Q}}$ is a 3-dimensional $\mathbb{Q}$-Brownian motion. The authors rewrite the inflation index dynamics by Itô formula

$$dI(t) = I(t)(B^{(3)} + \beta^{(3)'}X(t))dt + I(t)\Sigma^{(3)'}dW(t),$$

where numeric superscripts denotes columns of relevant vectors or matrices, and compute the dynamics of $\xi(t)$

$$d\xi(t) = \xi(t)(B^{(3)} + \beta^{(3)'}X(t) + r_r(t) - r_n(t))dt + \xi(t)\Sigma^{(3)'}dW(t).$$

By martingale property, the drift must sum to zero, hence

$$r_n(t) = B^{(3)} + \beta^{(3)'}X(t) + r_r(t).$$

Setting $B^{(3)} + \beta^{(3)'}X(t) = \mu_I(t)$ makes clear that the two approaches yield the same inflation dynamics.

The standard formulation of the Fisher equation implies that investors do not require an inflation risk premium, as if inflation dynamics were certain. However, inflation dynamics are stochastic and investors should require an inflation risk premium. Indeed, there is a tangible risk of facing a rate of inflation higher than the previously forecasted one. This point clarifies why stable and predictable inflation is desirable. The more inflation gets predictable, the more inflation risk premium reduces. The limit of this process is reached with a null risk premium for deterministic inflation. Thus, stabilising inflation reduces nominal interest rates. The previous rationale could still hold for modelling frameworks in which inflation risk is lowest. For economies with sustained inflation risk this formulation is preferable:

$$r_n(t) = r_r(t) + \pi^e(t) + \psi(t),$$

where ψ is the inflation risk premium. Accordingly, Ho, Huang and Yildirim propose (Ho *et al.*, 2014) a three-factor model to price inflation-indexed derivatives. They postulate a nominal price kernel, $M_n(t)$, and a real price kernel, $M_r(t) = \frac{M_n(t)}{I(t)}$; they incorporate unobservable economic variables in the n-dimensional latent state vector process, $x(t)$, which is mean-reverting with constant volatility. Their model employs the formulation proposed by (d'Amico *et al.*, 2018):

$$\frac{dM_n(t)}{M_n(t)} = -r_n(x(t))dt - \lambda_n(x(t))'dW(t) \tag{3.11}$$

$$d\log I(t) = \pi(x(t))dt + \sigma' dW(t) + \sigma_I' dW^I(t) \tag{3.12}$$

$$dx(t) = K(\mu - x(t))dt + \Sigma dW(t). \tag{3.13}$$

Equations (3.11) and (3.13) are, respectively, the dynamics of the nominal price kernel, M_n, and the latent state vector process, $x(t)$. For a more detailed analysis, please refer to the author's text. We shall focus on equation (3.12) for the dynamics of the logarithm of the price index. The inflation dynamics can be decomposed in a stochastic mean growth term, $\pi(x(t))$, a homogenous common shock, $\sigma' dW(t)$, and a homogenous exogenous shock, $\sigma_I' dW^I(t)$.

Chapter 4

Inflation Modelling: Models Employed

In this chapter we provide a thorough analytical discussion of the models we will employ for our simulation purposes. Specifically, we study the Hull-White model for interest rates and the Jarrow-Yildirim model for joint interest rate-inflation modelling. In the following chapters, these model will be calibrated and put in action to simulate interest rate and inflation evolution.

4.1 Hull-White Model

John Hull and Alain White proposed (Hull & White, 1990) to model the short rate as:

$$dr(t) = [\theta(t) - a(t)r(t))]dt + \sigma(t)dW^{\mathbb{Q}}(t), \quad a(t) > 0, \tag{4.1}$$

where a, σ and θ are deterministic functions of time. The model possesses an affine term structure, i.e. $P(t,T) = e^{A(t,T)-B(t,T)r(t)}$. It extends the Vasiček model, in that parameters are allowed to vary over time. We recall that the Vasiček model $dr(t) = [\theta - ar(t)]dt + \sigma dW^{\mathbb{Q}}(t)$ is mean reverting. Mean reversion characterises the behaviour by which process' realisations oscillate around a trend. This feature reasonably fits interest rate patterns of evolution, because, contrarily to stock prices, interest rates cannot rise indefinitely without significantly hindering sound economic development.

There is a simple economic rule behind mean reversion. It is known that the higher the price of a good or service, the lower the demand. Hence, high prices depress demand and the contraction forces suppliers to reduce their prices. Vice versa, low prices stimulate demand and makes profitable for suppliers to raise prices. Supply and demand continuously play a double mutual interaction: low prices and excess demand are corrected by a raise in prices, high prices and excess supply are counterbalanced by price reduction. Since we can think of interest rates as the price of the borrowing service, the same basic microeconomic reasoning holds true. The Vasiček model offers an asymptotically constant expected value and bounded variance for $r(t)$. Parameter $\frac{\theta}{a}$ describes the long term mean level, while a is the speed of reversion. For example, the drift term $(\theta - ar(t))$ is negative for $r(t) > \theta$ and thus implies a negative increment $dr(t)$, modulo the randomness of the diffusion term $\sigma dW^{\mathbb{Q}}(t)$. This example should analytically clarify what mean reversion is.

This rationale cannot be applied to stock prices, because they are typically not mean-reverting. Indeed, stock prices are usually modelled as geometric brownian motion [see, for instance, (Black & Scholes, 1973)]. Stocks are, broadly speaking, the price of a little piece of an enterprise. It entails the owner the right to take part in the firm's decisional process and profit division. To survive over years, a company shall typically grow, therefore its overall value shall increase over time. Hence, the price of a stock, as the price of a piece of the company, shall rise over time. Sound enterprises shall not exhibit mean-reverting behaviours over the long run.

The Vasiček model theoretically allows interest rates to become negative. Albeit this had been an undesirable feature up to the financial crisis, it is now common to encounter negative interest rates. For instance, Central Banks may fight economic stagnation offering negative interest rates to prevent banks from storing significant amount of money and, thus, stimulating their lending function. If the mere possibility for interest rates to become negative is not per se undesirable, an excessive hit down in the negative region is: when negative, rates are usually closest to zero. The Hull-White model extends Vasiček's rationale, allowing parameters to be time dependent. Moreover, this model guarantees a perfect fit between the theoretical and observed forward curve at time $t = 0$, i.e. $f(0,T) = f^\star(0,T)$ for all $T > 0$. This should enhance the explanatory power and precision of the model.

We shall discuss a simplified version (Hull & White, 1994), where α and σ are constant, namely

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW^{\mathbb{Q}}(t). \tag{4.2}$$

Recalling Proposition 4 in Section 3.1, we know that $A(t,T)$ and $B(t,T)$ must satisfy

$$\begin{cases} B_t(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

$$\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0. \end{cases}$$

Therefore, $A(t,T)$ and $B(t,T)$ are given by

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{4.3}$$

$$A(t,T) = \int_t^T \left[\frac{1}{2}\sigma^2 B^2(s,T) - \theta(s)B(s,T)\right] ds. \tag{4.4}$$

We require that at time $t = 0$ theoretical T-bond prices, $P(0,T)$, equal observed prices, $P^\star(0,T)$. Lemma 1 proves the biunivocal correspondence between ZCB prices and forward rates and our requirement is fulfilled mandating that the theoretical forward rate and the observed one coincide, i.e. $f(0,T) = f^\star(0,T)$. From definition of forward rate and affine models, we have that

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T} = r(t)B_T(t,T) - A_T(t,T).$$

At time $t = 0$ this becomes

$$f(0,T) = r(0)B_T(0,T) - A_T(0,T).$$

T-derivatives $B_T(t,T)$ and $A_T(t,T)$ are

$$\begin{aligned} B_T(t,T) &= e^{-a(T-t)} \\ A_T(t,T) &= \frac{\sigma^2}{2}B^2(T,T) - \theta(T)B(T,T) + \int_t^T \left[\sigma^2 B(s,T)B_T(s,T) - \theta(s)B_T(s,T)\right] ds \\ &= \frac{\sigma^2}{2a^2}\left(1 - e^{-a(T-t)}\right)^2 - \int_t^T \theta(s)B_T(s,T)ds. \end{aligned}$$

Forward rates $f(0,T)$ must satisfy

$$f(0,T) = f^\star(0,T) = e^{-aT}r(0) + \int_0^T \theta(s)B_T(s,T)ds - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2. \tag{4.5}$$

We shall look for a function $\theta(t)$, which solves equation (4.5) for all $T > 0$. To this end, we decompose $f^\star(0,T)$ into

$$f^\star(0,T) = x(T) - g(T),$$

where we define x and g as

$$\begin{cases} \dot{x} = -ax(t) + \theta(t) \\ x(0) = r(0) \end{cases}$$

$$g(t) = \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 = \frac{\sigma^2}{2}B^2(0,t).$$

It holds that

$$\begin{aligned} \theta(T) &= \dot{x}(T) + ax(T) \\ &= \frac{\partial f^\star(0,T)}{\partial T} + \dot{g}(T) + a\left(f^\star(0,T) + g(T)\right). \end{aligned} \tag{4.6}$$

Retrieved an arbitrary bond curve $\{P^\star(0,T);\ T > 0\}$, choosing θ according to (4.6) produces a term structure $\{P(0,T);\ T > 0\}$ such that $P(0,T) = P^\star(0,T)$ for all $T > 0$. We insert (4.6) into (4.4) and obtain

$$\begin{aligned}
A(t,T) &= \frac{\sigma^2}{2a^2}\int_t^T \left(1-e^{-a(T-t)}\right)^2 - \int_t^T \left[\left.\frac{\partial f^\star(0,T)}{\partial T}\right|_{T=s} + \dot{g}(s) + a\left(f^\star(0,s)+g(s)\right)\right] B(s,T)ds. \\
&= \frac{\sigma^2}{2a^2}\left\{T-t+\frac{1}{2a}-\frac{e^{-2a(t-t)}}{2a}-\frac{2}{a}+\frac{2e^{-a(T-t)}}{a}\right\}+\frac{1}{a}f^\star(0,t)(1-e^{-a(T-t)})- \\
&\quad -\int_t^T f^\star(0,s)e^{-a(T-s)}ds+\frac{1}{a}g(t)(1-e^{-a(t-t)})-\int_t^T g(s)e^{-a(T-s)}ds- \\
&\quad -\log P^\star(0,t)(1-e^{-a(T-t)})+\int_t^T a\log P^\star(0,s)e^{-a(t-s)}ds-\int_t^T g(s)ds+ \\
&\quad +\int_t^T g(s)e^{-a(T-s)}ds \\
&= f^\star(0,t)B(t,T)-\frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at})+\log\frac{P^\star(0,T)}{P^\star(0,t)}.
\end{aligned}$$

Eventually, we can characterise the affine T-bond price representation $P(t,T) = e^{A(t,T)-B(t,T)r(t)}$.

Proposition 4.1 (Hull-White Term Structure)
Given the Hull-White model with a and σ fixed. Having inverted the yield curve by choosing θ according to (4.6)*, we obtain the bond prices as*

$$P(t,T) = \frac{P^\star(0,T)}{P^\star(0,t)} e^{\left\{B(t,T)f^\star(0,t)-\frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at})-B(t,T)r(t).\right\}} \tag{4.7}$$

The short rate model can be written implicitly with the forward model

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW^{\mathbb{Q}}(t).$$

The model results from the selection of the diffusion coefficient $\sigma(t,T) = \sigma e^{-a(T-t)}$. Recalling Section 3.2 regarding the HJM framework, we know that the drift $\alpha(t,T)$ is

$$\begin{aligned}
\alpha(t,T) &= \sigma e^{-a(T-t)}\int_t^T \sigma e^{-a(s-t)}ds \\
&= \frac{\sigma^2}{a}\left[e^{-a(T-t)}-e^{-2a(T-t)}\right].
\end{aligned}$$

Proposition 4.2
The Hull-White simplified model (Hull & White, 1994) can be equivalently stated as

$$\begin{aligned}
dr(t) &= [\theta(t)-ar(t)]+\sigma dW^{\mathbb{Q}}(t) \\
df(t,T) &= \left(\sigma e^{-a(T-t)}\int_t^T \sigma e^{-a(s-t)}ds\right)dt+\sigma e^{-a(T-t)}dW^{\mathbb{Q}}(t).
\end{aligned} \tag{4.8}$$

Proof.

$$df(t,T) = \frac{\sigma^2}{a}\left(e^{-a(T-t)}-e^{-2a(T-t)}\right)dt+\sigma e^{-a(T-t)}dW^{\mathbb{Q}}(t).$$

Moving from the differential to the integral form, we have

$$f(t,T) = f^{\star}(0,T) + \frac{\sigma^2}{a}\int_0^t \left(e^{-a(T-s)} - e^{-2a(T-s)}\right) ds + \sigma \int_0^t e^{-a(T-s)} dW^{\mathbb{Q}}(s).$$

Recalling the Leibniz's rule for the differential of integral functions and matching terms with the formulation of $f(t,t) = r(t)$, we write

$$\begin{aligned} dr(t) = & \left.\frac{\partial f^{\star}(0,T)}{\partial T}\right|_{T=t} dt + \frac{\sigma^2}{a}\int_0^t \left(e^{-2a(t-s)}(2a) - e^{-a(t-s)}(a)\right) ds dt + \\ + & \ \sigma dW^{\mathbb{Q}}(t) + \sigma \int_0^t e^{-a(t-s)}(-a) dW^{\mathbb{Q}}(s) dt \\ = & \left[\left.\frac{\partial f^{\star}(0,T)}{\partial T}\right|_{T=t} - ar(t) - a\left(f^{\star}(0,t) - \frac{\sigma^2}{a}\int_0^t e^{-2a(t-s)}\right)\right] dt + \sigma dW^{\mathbb{Q}}(t). \end{aligned}$$

Finally, we identify (4.6) and write

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW^{\mathbb{Q}}(t).$$

□

4.2 Jarrow-Yildirim Model

Robert Jarrow and Yildiray Yildirim proposed (Jarrow & Yildirim, 2003) the following model to price TIPS and related derivatives. It employs the HJM framework and describes a Hull-White simplified model in forward terms. We recall assumptions and results of the modelling framework; many definitions have already been given, but, for the sake of clarity, we will repeat them in this specific notational setting.

Notation

- Nominal and real quantities are respectively denoted with subscripts r and n.
- $I(t)$ is the CPI value at time t.
- $P_n(t,T)$ is the price at time t of a nominal T-ZCB in nominal currency.
- $P_r(t,T)$ is the price at time t of a real T-ZCB in CPI units.
- $f_k(t,T)$ is the instantaneous forward rate for T, with $k \in \{r,n\}$, and $P_k(t,T) = e^{-\int_t^T f_k(t,u)du}$.
- $r_k(t) = f_k(t,t)$ is the spot rate at time t, with $k \in \{r,n\}$.
- $B_k(t) = e^{\int_0^t r_k(u)du}$ is the value of the bank account at time t, with $k \in \{r,n\}$.
- $P_{TIPS}(t,T) = I(t)P_r(t,T)$ is the price a time t in nominal currency of a real T-ZCB. By foreign currency analogy, the inflation index acts as exchange rate between the real and nominal economy.

- $CB_n(0)$ is the price at time $t = 0$ of a CB with coupon payments of C currency units and face value of K currency units, i.e.

$$CB_n(t) = K \cdot P_n(t,T) + \sum_{t=1}^{T} C \cdot P_n(t,T_i). \tag{4.9}$$

- $CB_{TIPS}(t)$ is the price at time t of TIPS-CB issued at time $t_0 < t$, with coupon payment of C units of CPI, maturity T and face value of K units of CPI, i.e.

$$CB_{TIPS}(t) = K\frac{I(t)P_r(t,T)}{I(t_0)} + \sum_{t=1}^{T} C\frac{I(t)P_r(t,T_i)}{I(t_0)}. \tag{4.10}$$

  Payments in the real currency are discounted with a real deflator and converted in nominal terms. The conversion is performed with the ratio $\frac{I(t)}{I(t_0)}$ as exchange rate. This quantity not only converts amounts between currencies, but also counteracts the depleting effect of inflation. Thus, the yield of such securities is real, in the sense that is not subject to inflation.

- Uncertainty is introduced in the model with the Brownian motions $W_n(t), W_r(t)$ and $W_I(t)$, whose correlations are $d\langle W_n, W_r\rangle(t) = \rho_{nr}dt,\ d\langle W_n, W_r\rangle(t) = \rho_{nI}dt$ and $d\langle W_r, W_I\rangle(t) = \rho_{rI}dt$.

Modelling framework

A three-factor model driven by Gaussian probability distributions is implemented. "Gaussianity" entails that a positive probability is assigned to undesired occurrences, such as strongly negative interest rates. However, such positive probability depends on the parameter values. For sufficiently small values this probability is small enough to be safely neglected. Overall, the price we have to to pay, in terms of model's reliability, is compensated by the ease in computational tractability we can enjoy.

The nominal forward rate, the real forward rate and the inflation index are the three factors, which constitutes the core of the model. Their dynamics under risk neutral measure $\mathbb{Q}$ are the following:

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n^{\mathbb{Q}}(t) \tag{4.11}$$

$$df_n(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r^{\mathbb{Q}}(t) \tag{4.12}$$

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I dW_I^{\mathbb{Q}}(t). \tag{4.13}$$

For absence of arbitrage to hold (Amin & Jarrow, 1991), the quantities

$$\eta(t) = \frac{P_n(t,T)}{B_n(t)}, \zeta(t) = \frac{I(t)P_r(t,T)}{B_n(t)} \text{ and } \xi(t) = \frac{I(t)B_r(t)}{B_n(t)}$$

must be $\mathbb{Q}$-martingales.

Proposition 4.3 (Term Structure Evolution)
The Term Structure and the price processes evolve under the risk neutral measure $\mathbb{Q}$ as follows.

$$df_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,u)dudt + \sigma_n(t,T)dW_n^{\mathbb{Q}}(t) \tag{4.14}$$

$$df_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,u)du - \rho_{r,I}\sigma_I\right]dt + \sigma_r(t,T)dW_r^{\mathbb{Q}}(t) \tag{4.15}$$

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t)]dt + \sigma_I dW_I^{\mathbb{Q}}(t) \tag{4.16}$$

$$\frac{dP_n(t,T}{P_n(t,T)} = r_n(t)dt - \int_t^T \sigma_n(u)dudW_n^{\mathbb{Q}}(t) \tag{4.17}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) + \rho_{r,I}\sigma_I\int_t^T \sigma_r(t,u)du\right]dt - \int_t^T \sigma_r(u)dudW_r^{\mathbb{Q}}(t) \tag{4.18}$$

$$\frac{dP_{TIPS}(t,T)}{P_{TIPS}P(t,T)} = r_n(t)dt + \sigma_I dW_I^{\mathbb{Q}}(t) - \int_t^T \sigma_r(t,u)dudW_r^{\mathbb{Q}}(t) \tag{4.19}$$

Proof.

1.
$$\begin{aligned} d\eta(t) &= \frac{dP_n(t,T)}{B_n(t)} - \frac{P_n(t,T)}{B_n^2(t)}dB_n(t) \\ &= \eta(t)\left\{\left[r_n(t) + A_n(t,T) + \frac{1}{2}||S_n(t,T)||^2\right]dt + S_n(t,T)dW^{\mathbb{Q}}(t) - r_n(t)dt\right\} \\ &= \eta(t)\left\{\left[A_n(t,T) + \frac{1}{2}||S_n(t,T)||^2\right]dt + S_n(t,T)dW^{\mathbb{Q}}(t)\right\} \end{aligned}$$

We made use of equation (1.15) and by martingale property write

$$A_n(t,T) = -\frac{1}{2}||S_n(t,T)||^2,$$

or, in differentiated form,

$$\alpha_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,u)du.$$

2.
$$\begin{aligned} d\zeta(t) &= \frac{d\left(I(t)P_r(t,T)\right)}{B_n(t)} - \frac{I(t)P_r(t,T)}{B_n^2(t)}dB_n(t) \\ &= \frac{I(t)dP_r(t,T) + P_r(t,T)dI(t)}{B_n(t)} + dP_r(t,T)dB_n(t) - \frac{I(t)P_r(t,T)}{B_n(t)}r_n(t)dt \\ &= \zeta(t)\left\{r_r(t) + A_r(t,T) + \frac{1}{2}||S_r(t,T)||^2 + \mu_I(t) - r_n(t) - \sigma_I S_r(t,T)\rho_{r,I}\right\}dt + \\ &+ \zeta(t)\left\{S_r(t,T)dW_r^{\mathbb{Q}}(t) + \sigma_I dW_I^{\mathbb{Q}}(t)\right\} \end{aligned}$$

Martingality requires drift to vanish, hence we set the dt-term to 0. Taking the T-derivative we can write

$$-\alpha_r(t,T) + \sigma_r(t,T)\int_t^T \sigma_r(t,u)du - \sigma_r(t,T)\sigma_I\rho_{r,I} = 0$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,s)ds - \sigma_I\rho_{r,I}\right]$$

3.
$$\begin{aligned} d\xi(t) &= -\frac{1}{B_n^2(t)}I(t)B_r(t)dB_n(t) + \frac{1}{B_n(t)}d(I(t)B_r(t)) \\ &= -\xi(t)r_n(t)dt + \frac{1}{B_n(t)}(I(t))dB_r(t) + B_r(t)dI(t)) \\ &= \xi(t)[r_r(t) - r_n(t) + \mu_I(t)]dt + \xi(t)\sigma_I dW_I^{\mathbb{Q}}(t). \end{aligned}$$

Martingale property requires the drift to vanish, hence

$$r_n(t) = r_r(t) + \mu_I(t).$$

4. Drift condition $A_n(t,T) = -\frac{1}{2}||S_n(t,T)||^2$ implies these dynamics for $P_n(t,T)$
$$\frac{dP_n(t,T}{P_n(t,T)} = r_n(t) - \int_t^T \sigma_n(u)dudW_n^{\mathbb{Q}}(t).$$

5. Drift condition $A_r(t,T) = -\frac{1}{2}||S_r(t,T)||^2 + \rho_{r,I}\sigma_I\int_t^T \sigma_r(t,u)du$ implies these dynamics for $P_r(t,T)$
$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) + \rho_{r,I}\sigma_I\int_t^T \sigma_r(t,u)du\right]dt - \int_t^T \sigma_r(u)dudW_r^{\mathbb{Q}}(t).$$

Recalling that $P_{TIPS}(t,T) = I(t)P_r(t,T)$, we write TIPS-ZCB dynamics as

6.
$$\begin{aligned} dP_{TIPS}(t,T) &= I(t)dP_r(t,T) + P_r(t,T)dI(t) + dI(t)dP_r(t,T) \\ \frac{dP_{TIPS}(t,T)}{P_{TIPS}(t,T)} &= \left[r_r(t) + \rho_{r,I}\sigma_I\int_t^T \sigma_r(t,u)du\right]dt - \int_t^T \sigma_r(u)dudW_r^{\mathbb{Q}}(t) + \\ &\quad + [r_n(t) - r_r(t)]dt + \sigma_I dW_I^{\mathbb{Q}}(t) - \sigma_I\rho_{r,I}\int_t^T \sigma_r(t,s)dsdt \\ &= r_n(t)dt + \sigma_I dW_I^{\mathbb{Q}}(t) - \int_t^T \sigma_r(t,u)dudW_r^{\mathbb{Q}}(t). \end{aligned}$$

□

As explained in Section 3.2, forward models are characterised by the choice of the volatility coefficient $\sigma(t,T)$. Jarrow and Yildirim use

$$\sigma_k(t,T) = \sigma_k e^{-a_k(t-T)} \text{ for } k \in \{r,n\}, \tag{4.20}$$

where σ_k and a_k are constants. We proved in Proposition 4.2 that the $df(t,t)$ is equal to $dr(t)$ in the simplified Hull-White model for $\sigma_n(t,T) = \sigma_n e^{-a(t-T)}$. Analogously, we show that, for $\sigma_r(t,T) = \sigma_r e^{-a_r(t-T)}$, $df_r(t,t)$ is equivalent to $dr_r(t)$ in this model.

Proposition 4.4
For $\sigma_r(t,T) = \sigma_r e^{-a_r(t,T)}$, the Hull-White simplified model (Hull & White, 1994) can be equivalently stated as

$$dr_r(t) = [\theta_r(t) - a_r r(t) - \sigma_r\sigma_I\rho_{r,I}] + \sigma_r dW_r^{\mathbb{Q}}(t)$$
$$df_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,u)du - \rho_{r,I}\sigma_I\right]dt + \sigma_r(t,T)dW_r^{\mathbb{Q}}(t)$$

Proof.

$$df_r(t,T) = \frac{\sigma_r^2}{a_r}\left(e^{-a_r(T-t)} - e^{-2a_r(T-t)}\right)dt - \sigma_r e^{-a_r(T-t)}\rho_{r,I}\sigma_I dt + \sigma_r e^{-a_r(T-t)}dW_r^{\mathbb{Q}}(t).$$

Moving from the differential to the integral form, we have

$$f_r(t,T) = f_r^\star(0,T) + \frac{\sigma_r^2}{a_r}\int_0^t \left(e^{-a_r(T-s)} - e^{-2a_r(T-s)}\right)ds -$$
$$- \frac{\sigma_r\sigma_I\rho_{r,I}}{a_r}\left(e^{-a_r(T-t)} - e^{-a_r T}\right) + \sigma_r\int_0^t e^{-a_r(T-s)}dW_r^{\mathbb{Q}}(s).$$

Recalling the Leibniz's rule for differential of integral functions and matching terms with the formulation of $f_r(t,t) = r_r(t)$, we write

$$dr(t) = \left.\frac{\partial f_r^\star(0,T)}{\partial T}\right|_{T=t} dt + \frac{\sigma_r^2}{a_r}\int_0^t \left(e^{-2a_r(t-s)}(2a_r) - e^{-a_r(t-s)}(a)\right)dsdt -$$
$$- \sigma_r\sigma_I\rho_{r,I}e^{-a_r t}dt + \sigma_r dW_r^{\mathbb{Q}}(t) + \sigma_r\int_0^t e^{-a_r(t-s)}(-a)dW_r^{\mathbb{Q}}(s)dt$$
$$= \left[\left.\frac{\partial f_r^\star(0,T)}{\partial T}\right|_{T=t} - a_r r_r(t) - a_r\left(f_r^\star(0,t) - \frac{\sigma_r^2}{a_r}\int_0^t e^{-2a_r(t-s)}\right)\right]dt -$$
$$- \sigma_r\sigma_I\rho_{r,I}dt + \sigma_r dW_r^{\mathbb{Q}}(t).$$

Finally, we identify the expression for $\theta_r(t)$ and write

$$dr_r(t) = [\theta_r(t) - a_r r_r(t) - \sigma_r\sigma_I\rho_{r,I}]dt + \sigma dW_r^{\mathbb{Q}}(t).$$

□

We see that (4.14) and (4.15) have similar formulations, but (4.15) displays a link with the dynamics of inflation. Specifically, the volatility of inflation depletes the expected change in real forward rates. The more inflation is volatile, the lower the expected evolution of real forward rates. We have already discussed the meaning of (4.16) in Section 3.3: this equation represents the Fisher equation. Equations (4.17) and (4.18) are conceptually similar to (4.14) and (4.15). The rates of change of nominal and real ZCBs evolve similarly, but real ZCBs are affected by inflation volatility proportionally to the correlation coefficient. In this case, however, higher volatility implies higher returns for real bonds. Since inflation volatility lowers real forward rates, real ZCB prices should increase, because the discounting effect diminishes. Interestingly, the expected return of TIPS depends on nominal short rates and it is clearly increasing in the volatility of inflation.

Proposition 4.5 (Hull-White Short Rates)
In the Jarrow-Yildirim model nominal and real forward rates are denoted respectively as

$$df_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,u)du + \sigma_n(t,T)dW_n^{\mathbb{Q}}(t) \tag{4.21}$$

$$df_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,u)du - \rho_{r,I}\sigma_I\right]dt + \sigma_r(t,T)dW_r^{\mathbb{Q}}(t). \tag{4.22}$$

For $\sigma_k(t,T) = \sigma_k e^{-a_k(T-t)}$, this is consistent with the Hull-White simplified formulation of nominal and real short rates

$$dr_n(t) = [\theta_n(t) - a_n r_n(t)]dt + \sigma_n dW_n^{\mathbb{Q}}(t) \tag{4.23}$$

$$dr_r(t) = [\theta_r(t) - a_r t_r(t) - \sigma_r\sigma_I\rho_{r,I}]dt + \sigma_r dW_r^{\mathbb{Q}}(t). \tag{4.24}$$

Chapter 5

Inflation Modelling: Model Calibration

In this chapter we perform the needed steps to calibrate the eight parameters of the Jarrow-Yildirim model. We retrieve market forward rates from traded bond prices and perform the calibration. The authors applied in (Jarrow & Yildirim, 2003) a procedure, which they had explained in the textbook (Jarrow, 2002, Chapter 16). The calibrated model will be used in the next chapter to simulate the evolution of interest rates and inflation.

5.1 Italian Public Debt Survey

The Ministry of Economy and Finance determines the regular issuance on the domestic market of six categories of government securities, available for both private and institutional investors.

1. **BOTs** (Buoni ordinari del Tesoro) are treasury bills with maturity of 3, 6, 12 months or less than 12 months (flexible BOTs). They are sold at discount and redeemed at par with a single payment at maturity.

2. **CTZs** (Certificato del Tesoro Zero-Coupon) are ZCB with maturity of 24 months. They are sold at discount and redeemed at par with a single payment at maturity.

3. **CCTs** (Certificato di Credito del Tesoro) are treasury certificates with maturity of 7 years. Semi-annual coupons are computed dividing by two the gross annual simple yield of the 6 month BOT issued in the last auction preceding the accruing of coupon. Then, a spread of 15 bps is added. CCTeus are CCT, whose simple gross annual yield is calculated by adding a contractually fixed spread to the the 6 month Euribor.

4. **BTPs** (Buoni del Tesoro Poliennali) are treasury bonds with maturity of 3, 5, 7, 10, 15, 20, 30 and 50 years. They pay fixed semi-annual coupons and are redeemed, at maturity, with a single payment.

5. **BTP€is** are TIPS indexed to the Euro-zone HICP-ex.tobacco index. They have maturity of 5, 10, 15 and 30 years. They are redeemed with a single payment at maturity. Coupons and principal are adjusted on the basis of the Indexation Coefficient for the relevant date and cannot be less than their nominal value.

6. **BTPItalia** are TIPS indexed to the Italian FOI-no.tobacco index. They have maturity of 4, 6 and 8 years. They share similar indexation rules with BTP€i.

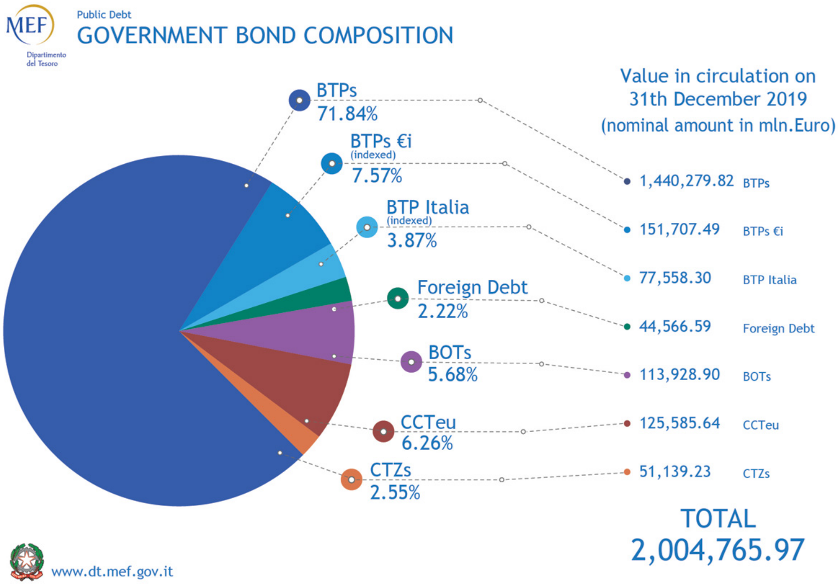


Figure 5.1: The graph shows the composition of outstanding government securities as of December, 31st 2019. Source: Ministero dell'Economica e delle Finanze - Dipartimento del Tesoro.

5.2 Data Description

Our analysis employs daily prices of BTPs and BTP€is, thus covering the majority of the total debt exposure. Securities' prices and characteristics were retrieved on April 14th, 2020 on Datastream-Eikon for a 5-year window, spanning from April 13th, 2015 to April 13th, 2020. The time window consists of 1306 daily observations; 91 BTPs and 11 BTP€is were traded on the day of retrieval. We move the starting date to October 14th, 2015 in order to include one more precious BTP€i in the relatively small sample. Thus, our window reduces to 1174 daily observations. Many securities were too "young" on that date, because they had been issued after the starting day of our time window. Other were too "old", because they will have matured in a year from the end day of our window. We removed from the dataset both types, because the former lacks of observations, the latter may entail distortionary pricing due to taxation issues near maturity. We do not apply any outlier-detecting procedure, because classic methods remove few tens of BTP€i observations, but almost three hundred BTP observations. This would mean loosing a peculiar feature of the BTP price behaviour. Hence, we prefer to keep the dataset as retrieved from data providers.

5.2.1 BTP€i

Traded BTP€is

ISIN	Issuance	Maturity	Coupon
IT0004604671	28-Apr-2010	15-Sep-2021	2,10%
IT0005188120	25-May-2016	15-May-2022	0,1%
IT0005329344	28-Mar-2018	15-May-2023	0,1%
IT0004243512	27-Jun-2007	15-Sep-2023	2,60%
IT0005004426	19-Mar-2014	15-Sep-2024	2,35%
IT0004735152	15-Jun-2011	15-Sep-2026	3,1%
IT0005246134	14-Mar-2017	15-May-2028	1,30%
IT0005387052	09-Oct-2019	15-May-2030	0,4%
IT0005138828	14-Oct-2015	15-Sep-2032	1,25%
IT0003745541	27-Oct-2004	15-Sep-2035	2,35%
IT0004545890	28-Oct-2009	15-Sep-2041	2,55%

Table 5.1: BTP€is traded on April, 14^{th} 2020, in bold the securities employed in the analysis. Source: Datastream-Eikon

BTP€is are treasury inflation-protected securities[1], whose monetary flows after issuance are indexed to Euro-zone inflation. BTP€is have been offered to the public with the auction mechanism since 2004. These securities are linked to the HICP-ex.tobacco. As explained in Section 2.1, the Ministry of Economy and Finance publishes on a monthly basis daily values for the Indexation Coefficient $IC_{y,m,d}$ for day d of month m in year y:

$$IC_{y,m,d} = \frac{RI_{y,m,d}}{BI}.$$

BI is base inflation, i.e. the reference inflation at time of issuance. $RI_{y,m,d}$ is the reference

[1] http://www.dt.mef.gov.it/en/debito_pubblico/titoli_di_stato/quali_sono_titoli/btpei/index.html

inflation for day d of month m in year y and it is determined as

$$RI_{y,m,d} = I_{m-3} + \frac{d-1}{\text{days in } m} \cdot (I_{m-2} - I_{m-3}).$$

The variable amount of the semi-annual coupons is calculated by multiplying half the real annual coupon rate by the nominal principal amount recalculated as at the coupon's payment date. The recalculated nominal principal amount is the subscribed nominal principal amount multiplied by the Indexation Coefficient at the coupon's payment date. The principal to be redeemed at maturity is calculated by multiplying the subscribed nominal principal amount by the Indexation Coefficient, calculated for the maturity date. If the value of the Indexation Coefficient for the maturity date is less than one, the amount of the principal redeemed shall be the nominal value of the bonds. Consequently, if during the bonds' term there is a reduction in prices, the amount redeemed at maturity shall be in any event equal to the bonds' nominal value (100). Currently BTP€is have maturities of 5, 10, 15 and 30 years.

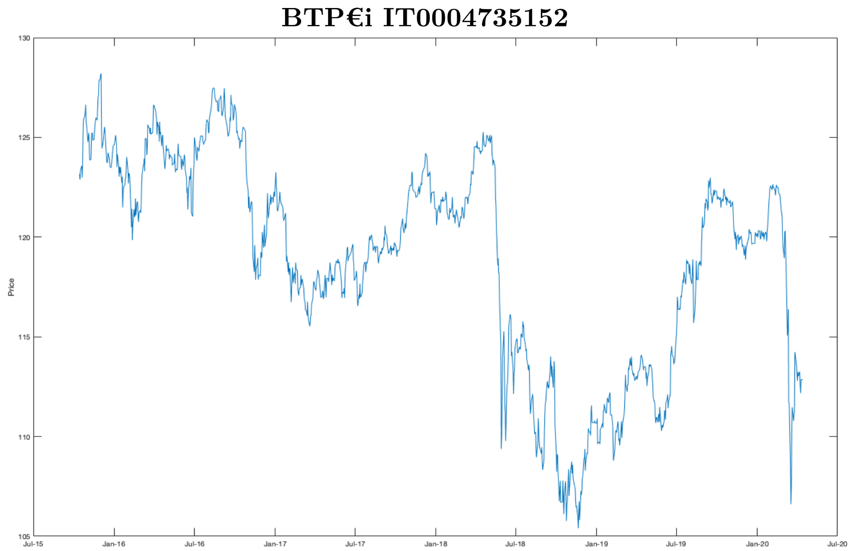


Figure 5.2: Daily Price of BTP€i IT0004735152, issued on 15-Jun-2011 and maturing on 15-Sep-2026, coupon rate 3,1%. Clean prices are employed in the analysis, but here, with the investor point of view, dirty daily prices are plotted, i.e. inclusive of the accrued interest. The time window spans between 14-Oct-2015 and 13-Apr-2020. The plot shows several harsh falls in the price of the security, highlighting some kind of market tension to be further analysed. This behaviour is representative for the other five Italian TIPSs.

5.2.2 BTP

BTPs are issued with maturities of 3, 5, 7, 10, 15, 20, 30 and 50 years. Actually, the Treasury issues 5½ year, 10½ year, 15½ year and 31 year BTPs, so that when the series of reopenings ends, residual maturity is near 5, 10, 15, 30 years[2]. BTPs are offered through a marginal auction with discretional determination of price and quantity issued. Coupons are paid twice a year. BTPs are redeemed at maturity at par with a single payment. BTPs account for almost 72% of the total Italian public debt exposure. They are among the most liquid and traded Italian government securities. 91 BTPs were traded on April 14th, 2020. Our selection rules reduced the dataset to 31 securities, with a full 5-year price time series and maturity at least one year after retrieval date.

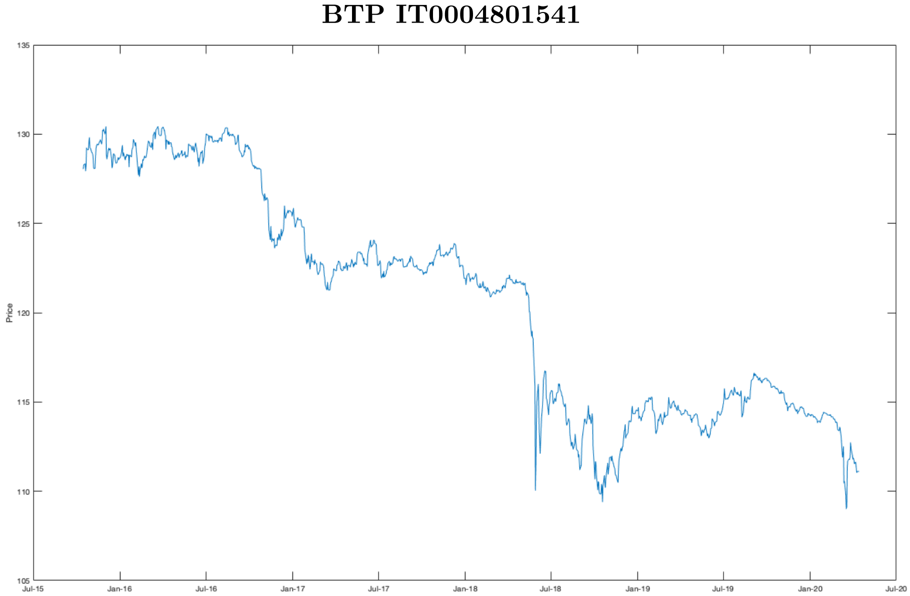


Figure 5.3: Daily dirty Price of BTP IT0004801541, issued on 01-Mar-2012 and maturing on 01-Sep-2022, coupon rate 5,5%. The time window spans between 14-Oct-2015 and 13-Apr-2020. The plot shows again several harsh falls in the price of the security. This behaviour is representative of the other thirty Italian BTPs.

5.2.3 HICP-ex.tobacco

Consumer price inflation in the Euro-area is measured by the Harmonised Index of Consumer Prices (HICP)[3]. It measures the change over time in the prices of consumer goods and services

[2] http://www.dt.mef.gov.it/en/debito_pubblico/titoli_di_stato/quali_sono_titoli/btp/index.html
[3] https://www.ecb.europa.eu/stats/macroeconomic_and_sectoral/hicp/html/index.en.html

acquired, used or paid for by euro area households. The term “harmonised” denotes the fact that all the countries in the European Union follow the same methodology. This ensures that the data for one country can be compared with the data for another. The main task of the ECB is to maintain price stability, defined as an annual HICP inflation rate of below 2% over the medium term. The HICP is also used in assessing whether a country is ready to join the euro area. The HICP is compiled by Eurostat and the national statistical institutes in accordance with harmonised statistical methods. HICP are produced and published using a common index reference period (2015 = 100). Indexes are neither calendar nor seasonally adjusted. Data are released monthly.

Almost all consumer goods and services purchased by means of monetary transactions come within the scope of the HICP. The technical name for these expenditures is “household final monetary consumption expenditure”. This includes everyday items, durable goods and services. The only significant area of consumption currently not covered is expenditure on housing by homeowners. HICP-ex.tobacco does not take into account prices relating to tobacco and its derivatives. Broadly speaking, HICP is a Laspeyres price index, i.e. a fixed-basket price index which defines a basket of goods and services in the base period that is priced in each subsequent period. These goods and services are weighted according to their share in overall consumption in the base period. However, the HICP is not a strict fixed-basket index. It measures the development of prices over time for fixed “consumption segments”, that is sets of consumer expenditures that serve a common purpose. Although these consumption segments are fixed, the specific products that are included in particular segments may change over time. In other words, certain items may exit the basket and new ones may enter as they become relevant to household consumption expenditure.

5.3 Stripping Real ZCB Prices

The price of a TIPS with face value K and coupon C is given by

$$P_{TIPS}(t) = IC_{d,m,y} \cdot K \cdot P_r(t,T) + \sum_{i=1}^{T} IC_{d,m,y} \cdot C \cdot P_r(t,T_i). \tag{5.1}$$

Face value K is redeemed at maturity. Given a coupon rate of $c\%$ and provided that coupons are payed m times per year, coupons are computed as $\frac{c}{m} \cdot K$. Treasury securities’ coupons are typically paid twice a year, hence the coupons are computed as $\frac{c}{2} \cdot K$. Coupon payment dates are stored in the vector $\mathbf{T_i} = \{t_1, \ldots, t_T\}$ for security i, maturity dates in the vector $\mathbf{T} = \{T_1, \ldots, T_6\}$. At valuation date $d/m/y$, coupons and face value are multiplied by the indexation coefficient $IC_{d,m,y}$, so that the depleting effect of inflation from issuance up to valuation date is corrected. Finally, we note that real ZCBs, $P_r(t,T)$, are employed as price deflators to discount future cash flows.

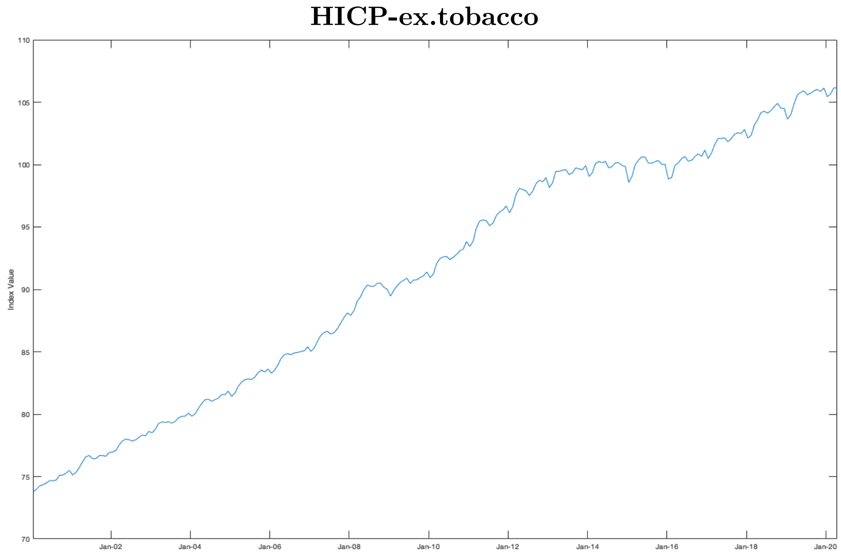


Figure 5.4: Plot of HICP-ex.tobacco monthly evolution between January, 15$^{\text{th}}$ 2000 and April, 15$^{\text{th}}$ 2020. Some degree of seasonality is shown. Source: Eurostat.

Our dataset of six BTP€is spans over $n = 1174$ daily observations. For each security at each valuation date, we computed the relevant indexation coefficient $IC_{d,m,y}$. Then, we computed coupon payment dates for each TIPS. For each security, we computed the distance in fraction of years between each day of valuation and coupon payment dates. This yielded a three dimensional matrix of n rows, number of coupons to be payed columns, and six pages[4]. The same procedure is applied to compute the distance between valuation date and maturity, yielding a $n \times 6$ matrix.

Discounting future payments requires the computation of real ZCBs. Longer lasting maturities in our dataset guarantee more than fifty future payments, but many more real ZCBs are required because payment dates of other securities rarely match. This theoretically requires the computation of many real bond prices $P_r(t, t_i)$, from date of valuation t to each payment date in $\mathbf{T_i}$ and $\mathbf{T}$. We recall that real T-ZCBs are computed as

$$P_r(t, T) = e^{-\int_t^T f_r(t,u)du}. \tag{5.2}$$

In Section 1.1 we assumed that there exist a T-bond for each $T > 0$. This assumption generates an infinite set of T-ZCB, which allows us to compute the continuous forward term structure between t and T. In theory, we could retrieve some ZCB prices by solving (5.1) for $P_r(t, T_i)$, given the market price $P_{TIPS}(t)$. We would be still irremediably far from computing a continuous

[4]We refer to the the third dimension as “page” with the MATLAB jargon. This dimension indexes the $m \times n$ matrixes, therefore we can equivalently say that we have i $m \times n$ matrixes or just a $m \times n \times i$ three dimensional matrix.

curve. We overcome this impasse with a simple, but robust assumption (Jarrow, 2002, pp. 304-306). We assume that the forward rate curve is piecewise constant between fixed intervals, thus we parametrise the entire curve with a finite numbers of points. The price of a real ZCB reduces to

$$P_r(t,T) = e^{-\int_t^T f_r(t,k)du} = e^{-f_r(t,k)\cdot\Delta},$$

where $f_r(t,k)$ is the constant forward rate for the (*a priori* fixed) time interval in which $\Delta = T - t$ is contained. Since we have six securities, we fix five rates $f_r(t,k_i)$ for five time intervals: 0-3 years, 3-5 years, 5-10 years, 10-20 years, 20-max(Δ) years. For instance, suppose $\Delta_1 = 6$ years and $\Delta_2 = 7$ years, then they would be both associated with forward rate $f_r(t,k_3)$.

We determine the forward rates for each of the five maturities on a day by day basis, in order to study the evolution of this discretised forward rate curve. At each of the n days, we look for the five rates $f_r(t,k_i)$ which minimise the sum of the squared difference between the market price of each bond, $P_j^\star(t)$, and its theoretical price:

$$\min_{\mathbf{f(t,k)}} \sum_{j=1}^{6} \left\{ P_j^\star(t) - IC_{d,m,y} \left[\sum_{i=1}^{T_j} C_j \cdot e^{-f_r(t,k_{t_i})\Delta_{j,t_i}} + K \cdot e^{-f_r(t,k_{T_J})\cdot\Delta_{j,T_j}} \right] \right\}^2. \tag{5.3}$$

This minimisation is repeated n times, yielding n vectors of five real forward rates for the five time intervals.

5.4 Stripping Nominal ZCB Prices

The price of a nominal CB with face value K and coupon C is given by

$$P(t) = K \cdot P_n(t,T) + \sum_{i=1}^{T} C \cdot P_n(t,T_i).$$

Face value K is redeemed at maturity. Coupons are computed as $\frac{c}{2} \cdot K$. Coupon payment dates are stored in the vector $\mathbf{T_i} = \{t_1, \dots, t_T\}$ for security i, maturity dates in the vector $\mathbf{T} = \{T_1, \dots, T_{31}\}$. We note that nominal ZCBs, $P_n(t,T)$, are employed as price deflators to discount future cash flows.

Our dataset of 31 BTPs spans over $n = 1174$ daily observations. For each security, we computed the coupon payment schedule and the distance in fraction of years between each day of valuation and coupon payment dates. This yielded a three dimensional matrix of n rows, number-of-coupons-to-be-payed columns, and 31 pages. The same procedure is applied to compute the distance between valuation date and maturity, yielding a $n \times 31$ matrix.

Even if we could theoretically construct a 30-parameter nominal forward curve, we stick to the previously explained five parameters scheme, in order to ensure comparability between real and nominal rates. We fix five parameters $f_n(t,k_i)$ for the same five time intervals: 0-3 years,

Time-Series of Real Forward Rates
(October 14, 2015 - April 13, 2020)

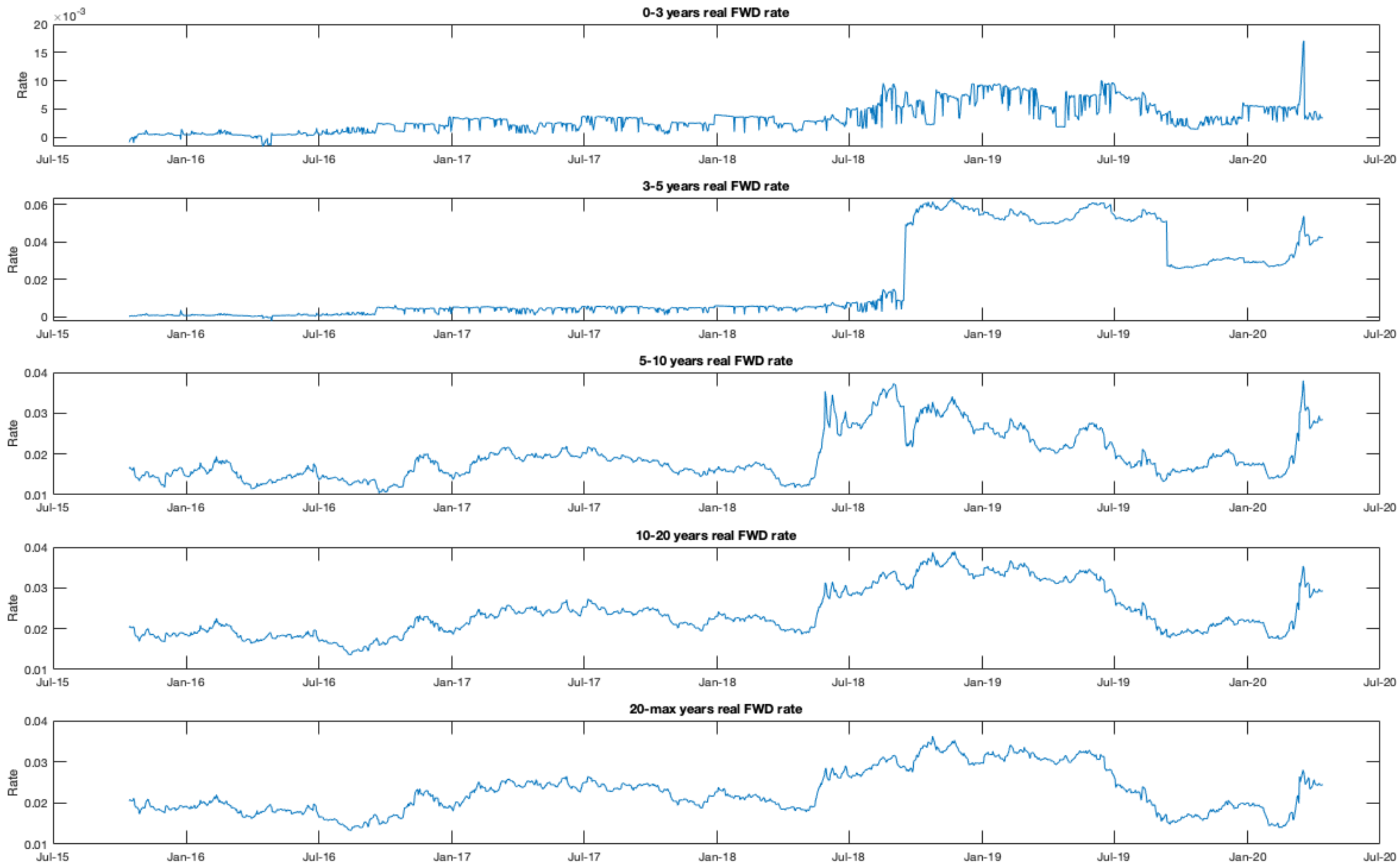


Figure 5.5: Real forward rates are quite smooth up to May, 2018. From this point on, all rates significantly increase and 0-3 year rates become more volatile. This evidence was suggested by the visual inspection of Figure 5.2 and deserves further analysis.

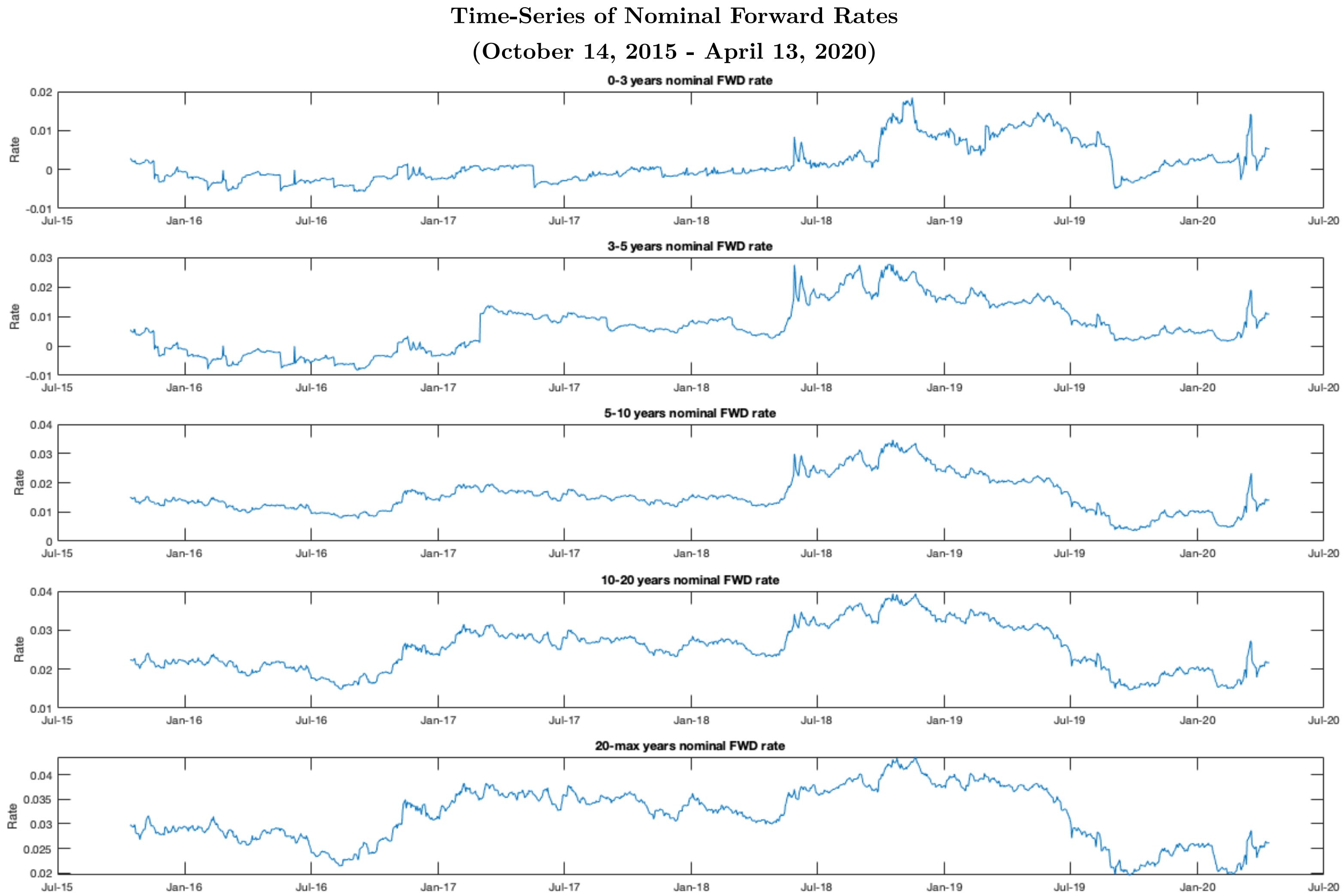
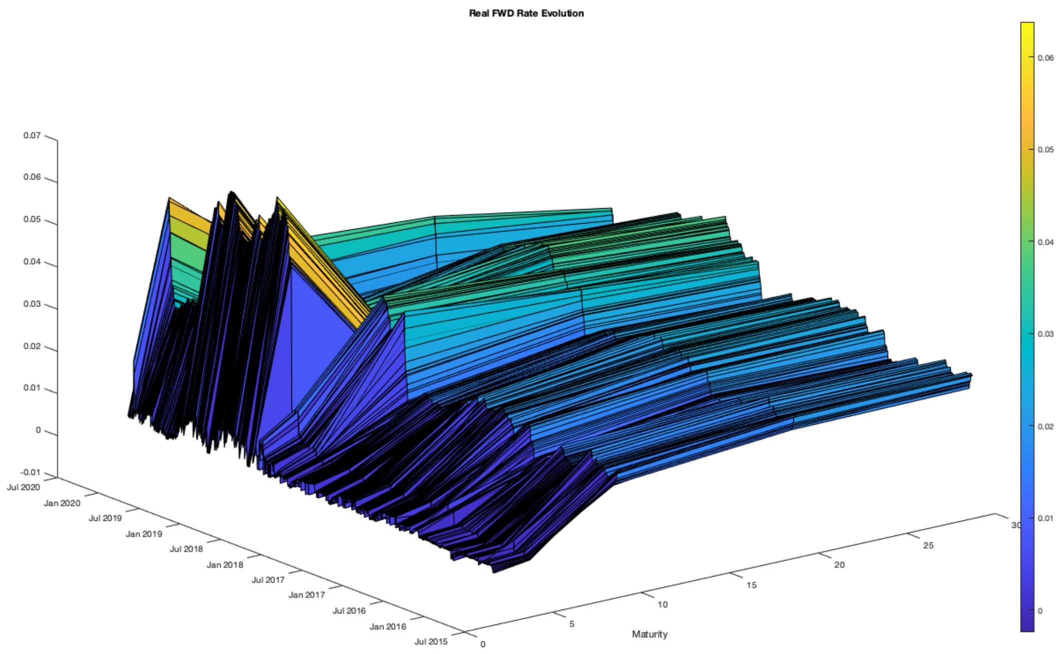
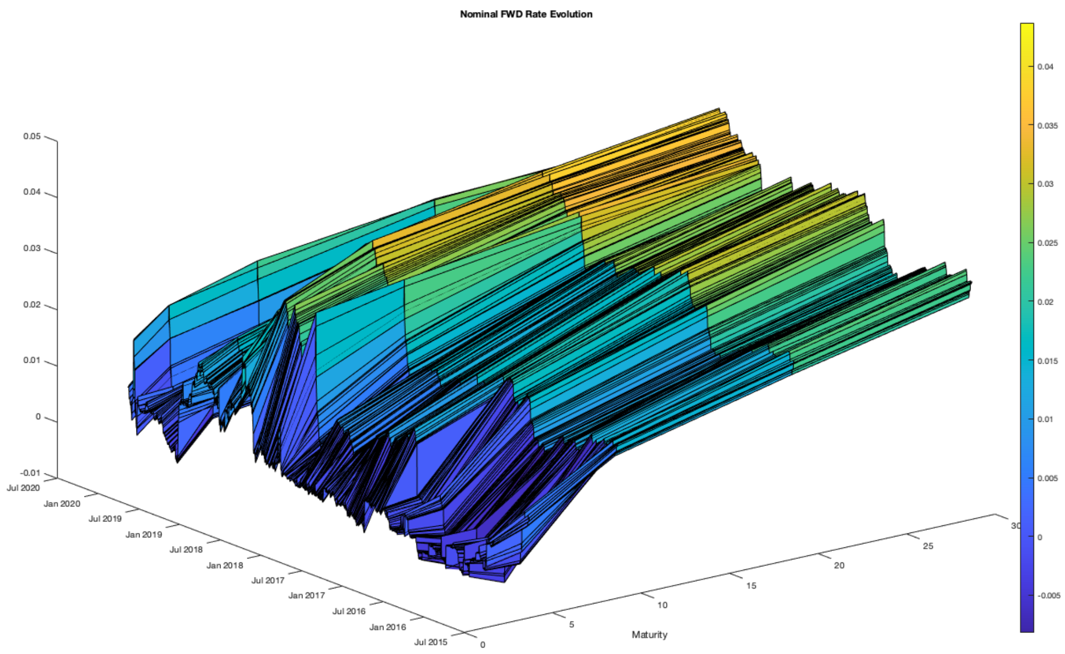


Figure 5.6: Nominal forward rates confirm the evidence offered by real rates. The overall behaviour is less spiky, however several peaks can still be clearly recognised.



(a) Real Forward Rates Evolution



(b) Nominal Forward Rates Evolution

Figure 5.7: There dimensional plot of Forward Rates Evolution between 14-Oct-2015 and 13-Apr-2020. We note again the sharp increase in shorter maturity real rates. Nominal rates clearly visualise the term structure, which is increasing in maturity.

3-5 years, 5-10 years, 10-20 years, 20-max(Δ) years. At each of the n days, we look for the five parameters $f_n(t, k_i)$ which minimise the sum of the squared difference between the market price of each bond, $P_j^\star(t)$, and its theoretical price:

$$\min_{\mathbf{f(t,k)}} \sum_{j=1}^{31} \left\{ P_j^\star(t) - \left[\sum_{i=1}^{T_j} C_j \cdot e^{-f_n(t,k_{t_i})\Delta_{j,t_i}} + K \cdot e^{-f_n(t,k_{T_J})\cdot\Delta_{j,T_j}} \right] \right\}^2 . \tag{5.4}$$

This minimisation is repeated n times, yielding n vectors of five nominal forward rates for the five time intervals.

5.5 Parameter Calibration

Having stripped forward rates from coupon bond prices, we compute ZCB prices on a daily basis for our five maturities. Specifically, we know that

$$P(t, t + \Delta_T) = e^{-\int_t^{t+\Delta_T} f(t,t+\Delta_T)du} = e^{-f(t,t+\Delta_T)\cdot\Delta_T}.$$

From our assumption regarding maturity intervals, we know that $\Delta_T = (T - t) \in \{3, 5, 10, 20, 30\}$. We stress again that $f(t, t+\Delta_T)$ is the piecewise constant forward rate associated to the maturity Δ_T. We apply this reasoning both to real and nominal froward rates, in order to retrieve both real and nominal ZCB daily prices. For the sake of clarity, we give a brief example with real forward rates retrieved for October 14th, 2015.

Real ZCB price on October 14th, 2015

$\Delta_T = (T - t)$	3	5	10	20	20
$f(t, t + \Delta_T)$	-0,087551%	0,033173%	1,64%	2,03%	2,05%
$P_r(t, t + \Delta_T)$	1,0026	0,9983	0,8483	0,6663	0,5401

Relying on the assumption of normality (see Figure 5.8) and recalling equation (4.18), we know that real bond returns evolve according to this normal distribution under the risk neutral probability measure $\mathbb{Q}$

$$\frac{\Delta P_r(t,T)}{P_r(t,T)} - \left[r_r(t) + \rho_{r,I}\sigma_I \int_t^T \sigma_r(t,u)du \right] \Delta t \sim \mathcal{N}\left[0, \left(\int_t^T \sigma_r(t,u)du \right)^2 \Delta t \right] . \tag{5.5}$$

Our analysis runs on a daily basis and $\Delta t = 1/365 \approx 0,00274$. The expected bond return $\left[r_r(t) + \rho_{r,I}\sigma_I \int_t^T \sigma_r(t,u)du \right] \Delta t$ may result to be small enough to be neglected[5]. This approximation simplifies our computational tasks, because we can employ the normal distribution of

[5]The expected return can be computed for a 3-year tenure with the calibrated parameters as $[0,0227 + 0,0084 \cdot 0,0042 \cdot 0,0086 \cdot 3] \cdot 0,00274 = 6,22 \cdot 10^{-5}$. The variance is similarly small: $(0,0086 \cdot 3)^2 \cdot 0,00274 = 1,8 \cdot 10^{-6}$. However, the standard deviation is much bigger: $\sqrt{1,8 \cdot 10^{-6}} = 1,35 \cdot 10^{-3}$. This is why we neglect the drift, but keep the variance.

(5.5) also for the bond returns of (4.18). Since we are aiming at estimating the volatility term in (5.5) and we know that diffusion components are invariant with respect to equivalent changes of probability measure, this calibration procedure is valid both under the real-world measure $\mathbb{P}$ and under the risk-neutral measure $\mathbb{Q}$. Recalling that $\sigma_r(t,T) = \sigma_r e^{-a_r \Delta_T}$, we can write the variance of the daily real bond returns as

$$\text{var}\left(\frac{\Delta P_r(t,T)}{P_r(t,T)}\right) = \frac{\sigma_r^2 (e^{-a_r \Delta_T} - 1)^2 \Delta t}{a_r^2}. \tag{5.6}$$

Let us clarify a possible notational issue. Δ is the difference operator, for instance $\Delta P_r(t,T) = P_r(t+,1,T+1) - P_r(t,T)$. $\Delta_T = (T-t)$ is the maturity time interval, which can take five values. $\Delta t = (t+1) - t$ is equal to one day or approximately 0,00274 365-day years.

We retrieve an estimation of the left side of (5.6) computing the sample variance of real ZCB price returns for each of the five maturities. Then, we run a non-linear regression of the sample variances on Δ_T through the function in the right side of (5.6). Estimations of parameters are $\hat{\sigma}_r = 0,01794$ and $\hat{a}_r = 0,03612$. The same procedure is applied to nominal forward rates, starting from equation (4.17). Estimations of nominal parameters are $\hat{\sigma}_n = 0,01509$ and $\hat{a}_n = 0,0258$. Estimates are fairly similar for nominal and real rates.

For the time window of this analysis 55 monthly observations of HICP-ex.tobacco are available. We focus on the left side of equation (4.16) to estimate σ_I, relying on $n_I = 54$ monthly rates of change. The standard deviation of the monthly rate of change of the inflation index, σ_I, is estimated as

$$\hat{\sigma}_I = \sqrt{\text{var}\left(\frac{\Delta I(t)}{I(t)}\right)}.$$

This estimation yields $\hat{\sigma}_I = 0,0042$, a value almost four times smaller than nominal rate volatility. This confirms that inflation had been fairly stable over the last years. Since $\frac{\Delta I(t)}{I(t)}$ is reasonably approximated with a normal distribution [see. Figure 5.8, number 6], we can compute 95% confidence intervals

$$\sqrt{\frac{(n_I - 1)\hat{\sigma}_I^2}{\chi^2_{53;0,025}}} \leq \sigma_I \leq \sqrt{\frac{(n_I - 1)\hat{\sigma}_I^2}{\chi^2_{53:0,975}}}.$$

We calibrate the three correlation parameters as:

$$\rho_{r,I} = \text{cor}\left(\Delta r_r(t), \frac{\Delta I(t)}{I(t)}\right) \quad \rho_{n,I} = \text{cor}\left(\Delta r_n(t), \frac{\Delta I(t)}{I(t)}\right) \quad \rho_{r,n} = \text{cor}\left(\Delta r_r(t), \Delta r_n(t)\right) \tag{5.7}$$

We consider the forward rate for maturities between 0 and 3 years as proxy for the short rate. Since inflation data are published monthly, we match the inflation rate changes' dates in the first two expressions of equation (5.7) with the corresponding real or nominal rate. Correlations between real rates and inflation changes is almost null. This proves that real return evolution is distinct from inflation patterns. There is positive, albeit small, correlation between nominal rates and inflation changes. Nominal and real rates are more heavily correlated.

Parameter Estimates

Parameter	Estimate	95% Confidence Interval
$\hat{a}_r$	0,0361	-0,0733; 0,1456
$\hat{\sigma}_r$	0,0179	-0,0043; 0,0402
$\hat{a}_n$	0,0258	0,006624; 0,04498
$\hat{\sigma}_n$	0,01509	0,0115; 0,01869
$\hat{\sigma}_I$	0,0042	0,0035; 0,0052
$\hat{\rho}_{r,I}$	0,00084	–
$\hat{\rho}_{n,I}$	0,0571	–
$\hat{\rho}_{r,n}$	0,1464	–

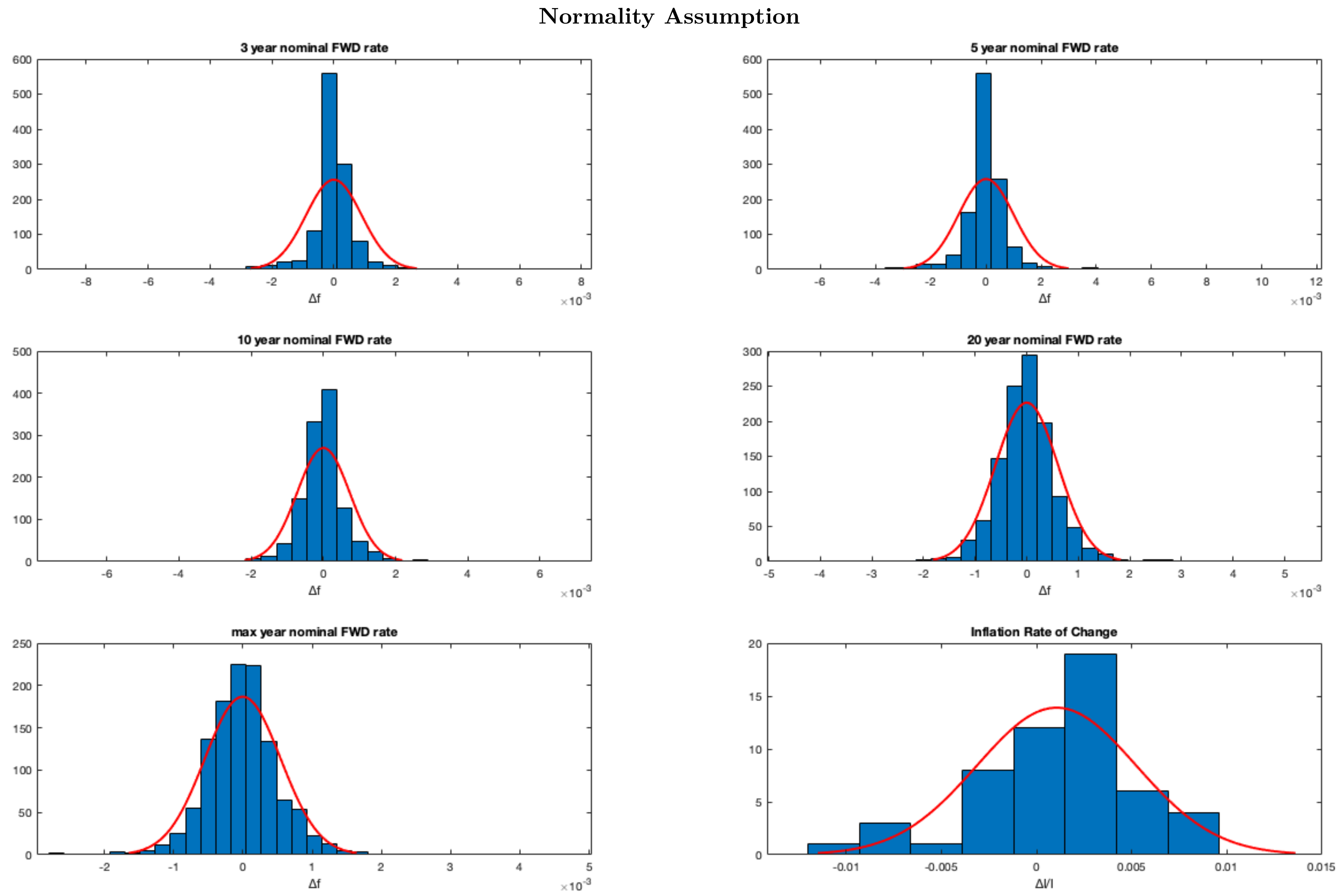


Figure 5.8: These plots offer proof of the quality of the normality assumption. The first five plots depict the frequency distribution of $\Delta f_n = f_n(t+1,\cdot) - f_n(t,\cdot)$ with a normal distribution superimposed. We plot nominal rates, real deltas behave similarly. The last plot depicts the monthly rate of change of the inflation index.

Chapter 6

Public Debt Cost Estimation

We shall now analyse in deeper detail the abrupt collapse of bond prices and the consequent rise in interest rates, which we emphasised during the first visual inspection of data. Roughly on May 2018 Italian real bond prices fell heavily and remained depressed for some time, before recovering almost as fast (see Figure 5.2]). On the other hand, nominal bond prices displayed a negative trend over the entire time window of our study. The price reduction seems to be driven by environmental persistent negative shocks: prices fell abruptly and stabilised around lower values. Two major reductions occurred for nominal bonds, the most severe of which again around May 2018 without any sign of recovering (see Figure 5.3). Both nominal and real bonds exhibit a further reduction around March, as a clear consequence of the outbreak of COVID-19. We shall not comment this highly unexpected and unusual occurrence, which is disrupting global economy with new and harmful threats. We shall focus on the contraction suffered after May 2018, whose magnitude and persistence suggest a sharp structural break rather than a momentary misperception. Rating agencies, institutions and investors may of course momentarily misunderstand or overstate the political and economic situation of a country, thus depressing the prices of its sovereign debt securities. However, this is clearly not the case, because the reduction in prices is too deep and persistent to be explained by a simplistic and unlikely bias in overall valuations.

ISTAT monthly reports on Italian economy described in mild terms the flattening of the GDP curve and the progressive deterioration of the business confidence climate (see Figure 6.2). Reports highlighted the stagnation of the business cycle at a global level, with contractions also in the US and Germany and increases in oil prices. The economic climate had of course been better, but the tortuous gestation of the Italian executive was definitely not desirable. After elections on March 4$^{\text{th}}$, 2018 Italy waited until June 5$^{\text{th}}$, 2018 for the executive to be officially formed and acknowledged. Internecine conflicts and political struggles hindered the stability of a government which was born not to last: the first executive led by Giuseppe Conte ceased on

August 20th, 2019. Political instability made impossible the survival of the 65th Italian executive, let alone the feasibility of any major and much begged reform. Investors did not condone this dangerous blend of present, past and future institutional weaknesses: the spread between 10-year Italian and German bond yield jumped from 113,30 bps on April 25th, 2018 to 289,35 bps on May 30th, 2018 and remained consistently high for more than a year.

The word "spread" in an economic context sounds treacherous to many Italians. In reality, it is as harmful as a number can be. Simply put, it is the differential between the yield of 10-year Italian BTP and a 10-year German Bund. Germany is taken as benchmark for economic comparison in the Euro area for its, at least alleged but often effective, economic solidity. As we said in the introductory chapter, the more lenders perceive borrowers as risky (in the sense that repayment becomes less and less certain), the higher the interest rate they require. Taking the German case as benchmark, the higher the spread, the higher the interest rates asked to Italy and hence the overall cost of debt. Therefore, the spread is a proxy of the Italian economic and financial reputation. The higher the spread, the higher the interest rate asked, hence the lower the confidence of investors in the Italian case. Spread is measured in basis points: 1 bp = 0,01%.

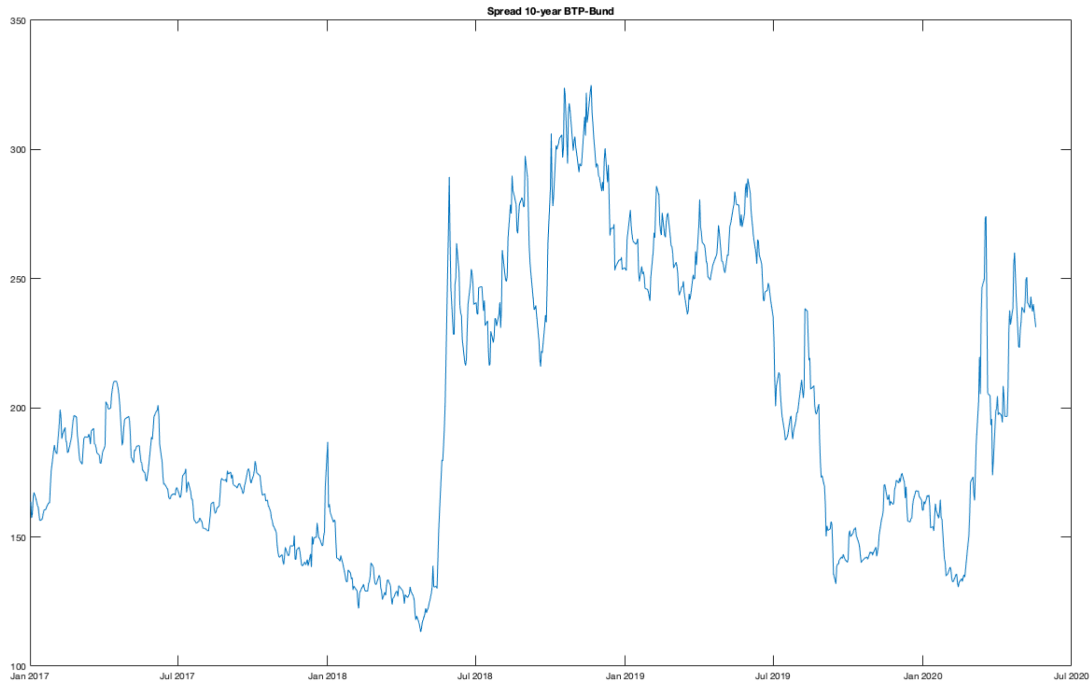


Figure 6.1: 10-year BTP-Bund Spread from 02-Jan-2017 to 18-May-2020. Source: Il Sole 24 Ore Database.

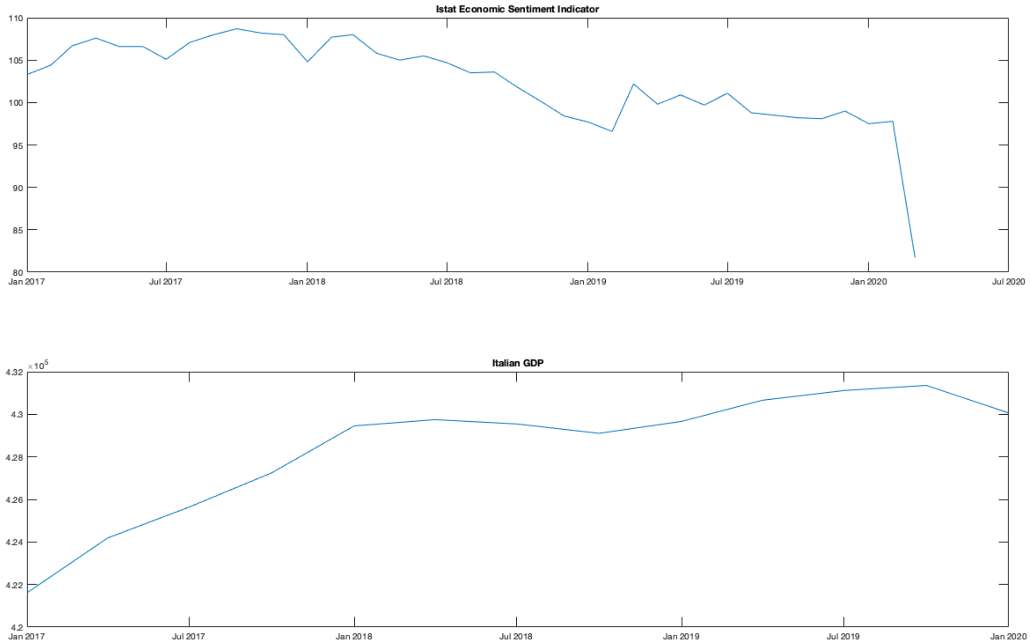


Figure 6.2: The ISTAT economic sentiment indicator is an indicator of the business confidence climate disseminated by ISTAT. Both IESI and GDP plot emphasise that for almost a year, from January 2018 to January 2019, Italian economy stagnated and the overall confidence of enterprises in the future constantly decreased. Source: ISTAT.

6.1 Forward Rate Simulation

The previous introductory summary conveys the idea that after May 2018 economic conditions changed severely. However, this may not have been the case. If we think in terms of stochastic processes, the blue lines in the plots are realisations of the underlying processes. The evolutions of stochastic quantities are modelled as $dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$. Realisations are theoretically infinitely many and are driven by a deterministic component, the drift term, and a stochastic component, the diffusion term. It may be that the parameters of the relevant stochastic processes remained constant over our entire sample. Thus, the spiky evolutions we described earlier may just be realisations driven by extreme values of the diffusion term. Specifically, the stochastic component, $dW(t)$, is modelled with a normal distribution, i.e. $W(t) \sim \mathcal{N}(0, t)$, which can take values on the entire real line. Hence, the realisation we observed may in theory be just an extreme realisation of the underlying process, unlikely yet possible. In order to evaluate the degree of unlikeliness of such realisations, we shall compute some confidence intervals for the stochastic process evolutions. As explained in Section 5.5, we calibrate again our eight parameters with observations up to May 1^{st}, 2018, thus we reduce our nominal and real forward rate sample to 665 daily observations.

Parameter Estimates on Reduced Sample

Parameter	**Estimate**	**95% Confidence Interval**
$\hat{a}_r$	-0,0029	-0,06267; 0,0569
$\hat{\sigma}_r$	0,0086	0,001157; 0,016
$\hat{a}_n$	-0,0031	-0,02196; 0,01575
$\hat{\sigma}_n$	-0,0089	-0,01147; -0,00643
$\hat{\sigma}_I$	0,0045	0,0036; 0,0060
$\hat{\rho}_{r,I}$	-0,0325	–
$\hat{\rho}_{n,I}$	0,0648	–
$\hat{\rho}_{r,n}$	0,0619	–

Table 6.1: Estimates of the first four parameters become more precise with reduced confidence intervals. $\hat{\sigma}_n, \hat{\rho}_{r,I}$ and $\hat{\rho}_{n,I}$ do not change much; on the contrary, $\hat{\rho}_{r,n}$ more than halved. Overall, confidence intervals for the reduced calibration shrink with respect to the full sample calibration. However, the intervals still remain quite wide for $\hat{a}_r$ and $\hat{\sigma}_r$. Longer time span would probably help in reducing the variability of parameters' estimates.

Once we have obtained parameter estimates, we can estimate the volatility function for both nominal and real forward rate evolutions. We recall that the Jarrow-Yildirim model employs the volatility function $\sigma_k(t,T) = \sigma_k e^{-a_k \cdot (T-t)}$, with $k = \{n, r\}$. In our setting $(T - t) = \Delta_T$ and $\Delta_T = \{3, 5, 10, 20, 30\}$. Therefore, we can compute the ten volatility functions for each Δ_T and for both nominal and real rates. For example, given estimates $\hat{\sigma}_n = -0,0089$ and $\hat{a}_n = -0,0031$, $\sigma_n(\Delta_{T,1}) = -0,0089 \cdot e^{0,0031 \cdot 3} = -0,0090$. Recalling equations (4.14) and (4.15) and knowing that volatility functions are constant for each maturity, we can compute the dynamics of nominal and real forward rates:

$$\begin{aligned} df_n(\Delta_{T,i}) &= \sigma_n(t,T) \int_t^T \sigma_n(t,u) du dt + \sigma_n(t,T) dW_n^{\mathbb{Q}}(t) \\ &= \left(\sigma_n(\Delta_{T,i})^2 \cdot \Delta_{T,i}\right) dt + \sigma_n(\Delta_{T,i}) dW_n^{\mathbb{Q}}(t) \end{aligned} \tag{6.1}$$

$$\begin{aligned} df_r(\Delta_{T,i}) &= \sigma_r(t,T) \left[\int_t^T \sigma_r(t,u) du - \rho_{r,I}\sigma_I \right] dt + \sigma_r(t,T) dW_r^{\mathbb{Q}}(t) \\ &= \sigma_r(\Delta_{T,i}) \left[\sigma_r(\Delta_{T,i}) \cdot \Delta_{T,i} - \rho_{r,I}\sigma_I\right] dt + \sigma_r(\Delta_{T,i}) dW_r^{\mathbb{Q}}(t). \end{aligned} \tag{6.2}$$

For example, $df_n(\Delta_{T,1}) = ((-0,0090)^2 \cdot 3)dt - 0,0090 \cdot dW_n^{\mathbb{Q}}(t)$. However, we must simulate trajectories under the real-world measure $\mathbb{P}$. By the Girsanov theorem, we know that $dW^{\mathbb{Q}}(t) = dW^P(t) - \phi(t)dt$, where $\phi(t)$ is the market price of risk. Rather than working with a time dependent market price of risk, we model the Girsanov kernel as constant across maturities, i.e. ϕ_k with $k \in \{n, r\}$. Therefore, we associate to each class of forward rate a market price of risk and we describe the following real-world dynamics:

$$df_n(\Delta_{T,i}) = \sigma_n(\Delta_{T,i})\left[\sigma_n(\Delta_{T,i})\cdot\Delta_{T,i} - \phi_n\right]dt + \sigma_n(\Delta_{T,i})dW_n^P(t) \tag{6.3}$$

$$df_r(\Delta_{T,i}) = \sigma_r(\Delta_{T,i})\left[\sigma_r(\Delta_{T,i})\cdot\Delta_{T,i} - \rho_{r,I}\sigma_I - \phi_r\right]dt + \sigma_r(\Delta_{T,i})dW_r^P(t). \tag{6.4}$$

We showed that it is reasonable to assume that $\Delta f_k(\Delta_{T,i}) \sim \mathcal{N}$ (see Figure 5.8).

$$E\left[\frac{\Delta f_n(\Delta_{T,i})}{\Delta t}\right] = \sigma_n(\Delta_{T,i})\left[\sigma_n(\Delta_{T,i})\cdot\Delta_{T,i} - \phi_n\right] \tag{6.5}$$

$$E\left[\frac{\Delta f_r(\Delta_{T,i})}{\Delta t}\right] = \sigma_r(\Delta_{T,i})\left[\sigma_r(\Delta_{T,i})\cdot\Delta_{T,i} - \rho_{r,I}\sigma_I - \phi_r\right]. \tag{6.6}$$

We fit a normal distribution to each differenced time series in the left side of (6.5) and (6.6), and retrieved two 5×1 mean vectors, $\mu_{\mathbf{n}}$ and $\mu_{\mathbf{r}}$. To this extent, we employed the reduced time series of the first 665 daily observations, in order to exclude the possibly anomalous time window. Market prices of risk are computed as the parameters ϕ_k, minimising the difference between the left side and right side of (6.5) and (6.6) over 5 equations respectively.

Market Price of Risk on Reduced Sample

	Estimate	**95% Confidence Interval**
$\hat{\phi}_n$	-0,2108	-0,2773; -0,1143
$\hat{\phi}_r$	0,1131	-0,3309; 0,5570

Table 6.2: Confidence intervals are wide, especially for $\hat{\phi}_r$. Typically, the estimation of market prices of risk proves to be a difficult task, because long time series are needed for a precise calibration. Several techniques may be used to retrieve market prices of risk, such as regression over time series or Kalman filters. We chose to specify constant market prices of risk, but a more complex functional form may be specified. Overall, contributions to the relevant literature are still looking for a univocal consensus.

Now we can make sense of equations (6.3) and (6.4) and write

$$df_n(\Delta_{T,i}) = \sigma_n(\Delta_{T,i})\left[\sigma_n(\Delta_{T,i})\cdot\Delta_{T,i} + 0,2108\right]dt + \sigma_n(\Delta_{T,i})dW_n^P(t)$$
$$df_r(\Delta_{T,i}) = \sigma_r(\Delta_{T,i})\left[\sigma_r(\Delta_{T,i})\cdot\Delta_{T,i} - \rho_{r,I}\sigma_I - 0,1131\right]dt + \sigma_r(\Delta_{T,i})dW_r^P(t).$$

With these evolution dynamics we simulate froward rates for 510 days from May 1$^{\text{st}}$, 2018 to April 13$^{\text{th}}$, 2020. The red lines in Figure 6.4 are the simulations of one trajectory for each maturity. We see that the model reasonably mimics the behaviour or forward rates up to May 2018. However, one simulation is not informative of the true depth of the infinite trajectory bundle. Since simulating trajectories is a fast procedure, we run 100 000 simulations for each of the ten rates, obtaining ten 510 $\times$ 100000 matrixes of simulated values. For each rate we build a quantile matrix, where we store the 0.005, 0.5 and 0.995 quantiles for each simulation day over the 100 000 simulations. The dashed black lines depicts these three quantiles, offering empiric confidence intervals. When the blue lines of realised trajectories exceed the dashed black lines, it is statistically unreasonable to assert that underlying parameters of the relevant stochastic processes remained constant. Indeed, realisations exceeding the confidence interval may happen but they are extremely unlikely: with 99% of statistical confidence we can say that such observations entail a sharp change in evolution patterns. In our case, observations exceed the upper confidence interval bound: thus they can still be traced back to parameters calibrated over the first sub-sample, but they would happen 0,5% of the times over 100 000 repetitions. The present situation teaches us that imponderable scenarios may realise, but in this case we may confidently discard the possibility that no major environmental changes occurred after May 2018.

We perform the same statistical analysis in analytical terms. Instead of computing confidence intervals over simulation results, we retrieve intervals from the implied normal distributions of forward rates. Indeed,

$$
\begin{aligned}
f_n(\Delta_{T,i}, t) &= f_n(\Delta_{T,i}, 0) + \int_0^t \sigma_n(\Delta_{T,i}) \left[\sigma_n(\Delta_{T,i}) \cdot \Delta_{T,i} - \phi_n\right] du + \\
&\quad + \int_0^t \sigma_n(\Delta_{T,i}) dW_n^P(u) \\
f_r(\Delta_{T,i}, t) &= f_r(\Delta_{T,i}, 0) + \int_0^t \sigma_r(\Delta_{T,i}) \left[\sigma_r(\Delta_{T,i}) \cdot \Delta_{T,i} - \rho_{r,I}\sigma_I - \phi_r\right] du + \\
&\quad + \int_0^t \sigma_r(\Delta_{T,i}) dW_r^P(u)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_n(\Delta_{T,i}, t) &\sim \mathcal{N}\left[f_n(\Delta_{T,i}, 0) + \sigma_n(\Delta_{T,i}) \left[\sigma_n(\Delta_{T,i}) \cdot \Delta_{T,i} - \phi_n\right] \cdot t; \sigma_n^2(\Delta_{T,i}) \cdot t\right] \\
f_r(\Delta_{T,i}, t) &\sim \mathcal{N}\left[f_r(\Delta_{T,i}, 0) + \sigma_r(\Delta_{T,i}) \left[\sigma_r(\Delta_{T,i}) \cdot \Delta_{T,i} - \rho_{r,I}\sigma_I - \phi_r\right] \cdot t; \sigma_r^2(\Delta_{T,i}) \cdot t\right].
\end{aligned}
$$

Therefore, at each of the 510 simulation days we compute the 0.005, 0.5 and 0.995 quantiles of the distribution of nominal and real forward rates. We build again a 510×3 quantile matrix, whose values are retrieved analytically rather than empirically. The analytical and empirical confidence intervals almost match, because, by the law of large numbers, the parameters of the normal distributions implied by the empirical simulations asymptotically converge to the analytical values.

Our simulations offer a clear evidence. Between May 2018 and January 2019, medium term rates cannot be considered representative and long term are very unlikely to have been representative. We refer here to "representativeness" as the extent to which realisations of the relevant stochastic processes can be considered after May 2018 coherent with pre-May 2018 evolution. We observe that 3-5 year and 5-10 year both nominal and real rates severely peaks outside the 99.5% confidence interval. This implies that it is statistically unreasonable to believe that the realised trajectories are compatible with previous evolution: something has changed in the financial, economic and political environment. 10-20 year and 20-30 year rates are similarly problematic because their realisations are closest to the confidence bound and often exceed it. Even if we cannot assert with 99% confidence that such trajectories are extraneous, it is much more likely that they are rather than they are not. Lowering the statistical confidence interval to 95%, we would reject the hypothesis of coherence between the two temporal samples. 0-3 year rates can still be traced back to the previous evolution. However, we emphasise that the volatility of these short-maturity rates greatly increased. Especially for nominal 0-3 year rates some peaks get closest to the upper confidence bound, but this evidence shall remain a clue rather than a proof of some environmental mutation.

What triggers short-maturity rates' volatility and boosts longer-maturity rates? We believe precarious political stability, disputable credibility and poor long-term vision. Indeed, in the short run instability just makes the financial environment more and more volatile and uncertain. The uncertainty, then, propagates to medium-term rates. In the medium run the instability transforms into lack of systemic reforms and overall performance and competitiveness deterioration. Uncertainty temporally propagates with investors' beliefs and forecasts. Just as it is unlikely that an athlete who has been training and performing poorly recently will win competitions in the near future, it is similarly unlikely that a country, whose executive is not at present in the position to properly guide it, will experience a the future sustainable growth. Investors' forecasts rapidly incorporate this simple principle in the medium term rate. Over the long run, short and medium term imbalances and instabilities shall be hopefully resolved. Long run rates do increase as outlooks deteriorate, but they do not explode as medium rates do. This is an instance of mean reversion due to political and economic cycle. Passing instability can last for several months. It causes turmoil in the short run, it wastes opportunities of growth in the medium terms, but shall be resolved and corrected in the long run to avoid much more unpleasant consequences.

6.2 Inflation Simulation

The simulation and analysis of forward rates proved that the evolution between May 2018 and May 2019 had entailed a shift in underlying parameters. We want to estimate the cost of this abrupt change in environmental conditions. To this aim, we need to simulate the future evolution of the inflation index, so that we can estimate the overall cost net of the inflation depleting effect. It is known that the government itself stimulated during the 70s and 80s the outbreak of inflation, so that nominal stocks of debt rapidly lost value in real terms and became easier to be repaid. This is instance is known as inflation tax, because the government, stimulating the increase of inflation, in practice raises an implicit tax over debt holders. Such contribution can be computed as the difference between the real value of the debt at issuance and its real value at

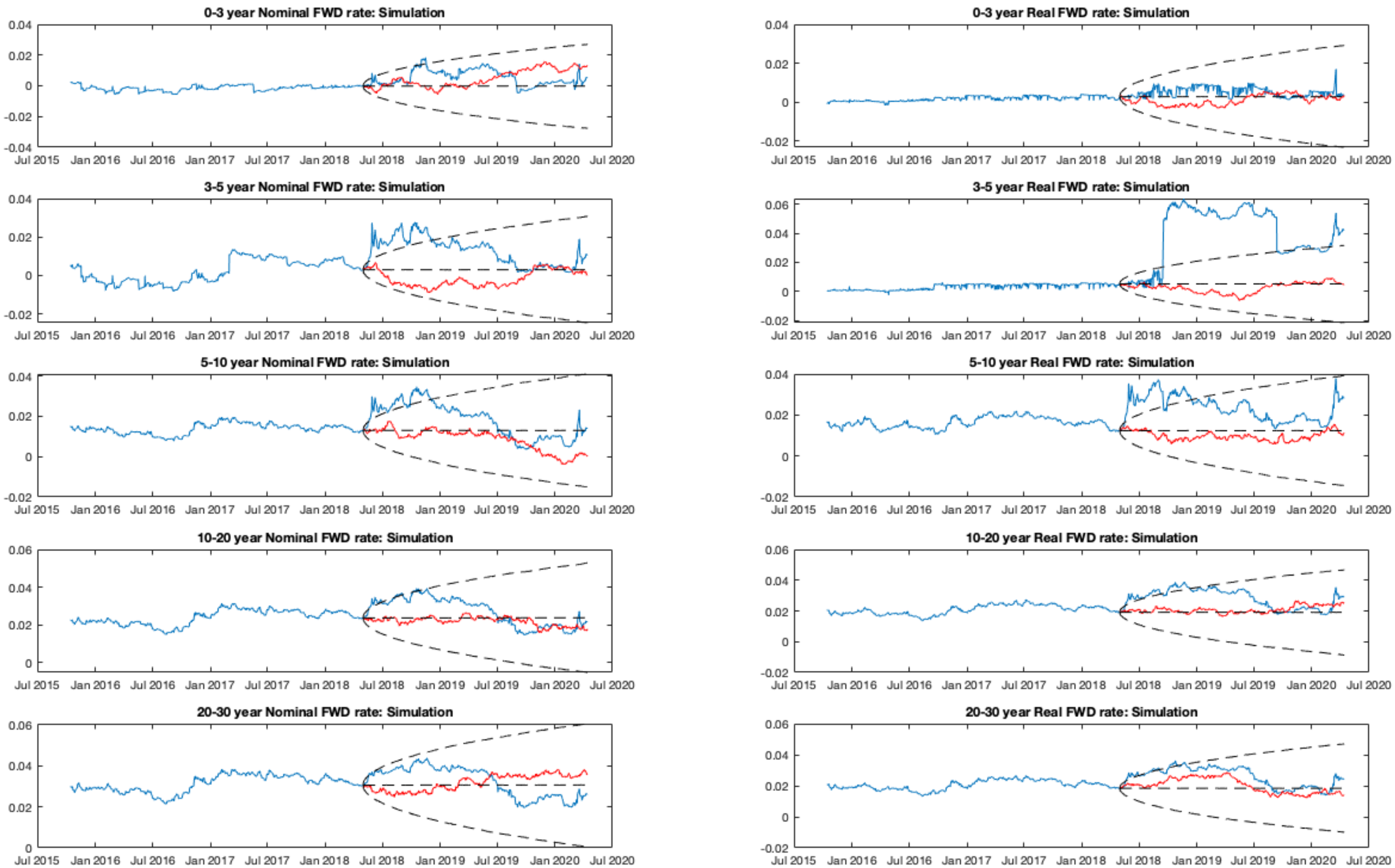


Figure 6.3: Nominal rates on the left, real rates on the right. Blu lines are rates stripped from CB prices. Each red line a simulated trajectories with parameters from the 14-Oct-2015 – 1-May-2018 sample. Dashed lines are the 0.005, 0.5 and 0.995 quantiles, computed over 100 000 simulations.

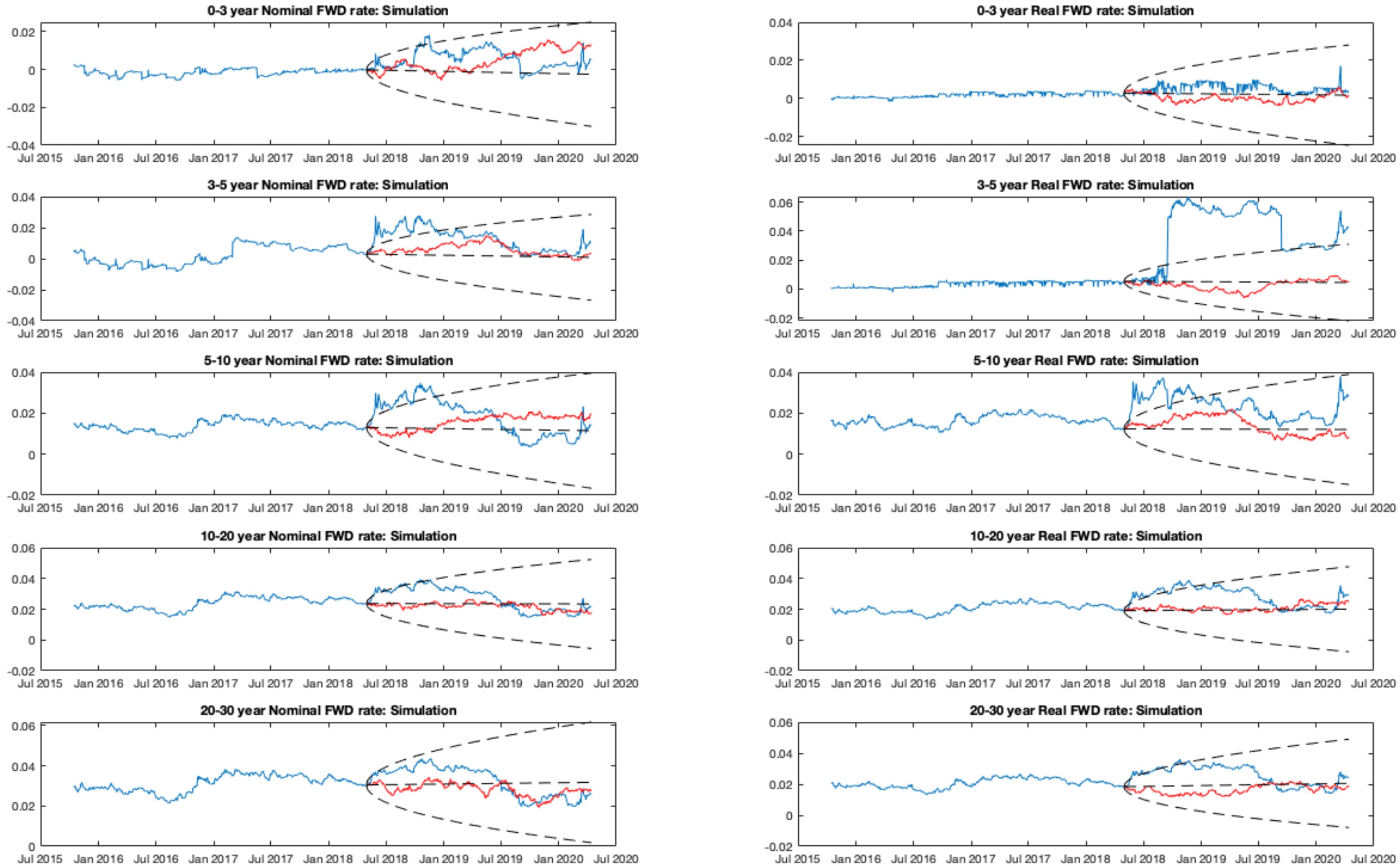


Figure 6.4: Nominal rates on the left, real rates on the right. Blu lines are rates stripped from CB prices. Each red line a simulated trajectories with parameters from the 14-Oct-2015 – 1-May-2018 sample. Dashed lines are the 0.005, 0.5 and 0.995 quantiles, analytically computed by the implied normal distribution.

payment and maturity dates. It is clear that the occurrence we are analysing does not match the traits of inflation tax, because of the very nature of the inflationary process. It is indeed strictly supervised and guided at a centralised level by both European and national Central Banks. Even though a mild instance of depletion in the real value of debts occurs, it happens to the smallest degree. From the point of view of a creditor whose only assets are credits toward the government, null inflation or even deflation may desirable. However, it is clear that such condition is very unpleasant and harmful for the economy and the nation at aggregate level. Even for creditors mild inflation is possibly beneficial, because the loss in the real value of their credits may be offset by the overall growth of the economy, implied by such an inflation rate. Therefore, the simulation of inflation is required to compute a precise estimate of the cost, rather than substantiating any instance of inflation taxing. Indeed, among observed securities some will be fully repaid as far as in 2050. For such bonds even low inflation naturally entails a substantial loss of real value over thirty years. This pattern is clear and known to both issuer and subscribers.

Recalling equation (4.16), Jarrow and Yildirim model the evolution of inflation rate at each time t with the geometric Brownian motion

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t)]dt + \sigma_I dW_I^{\mathbb{Q}}(t), \tag{6.7}$$

whose drift is the difference between nominal and real short rates. We are interested in the real-world dynamics for our simulation purposes. Recalling again the Girsanov theorem, such dynamics are given by

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t) - \sigma_I \phi(t)]dt + \sigma_I dW_I^P(t). \tag{6.8}$$

In the following we will work on a monthly basis to match the inflation data timing: $dt = 1/12 \approx 0,0833$. We selected twelve bonds, whose coupons are anomalous compared to previous issuances for homogenous maturities. We mean by "anomalous" issuance that the coupon rate is significantly different (higher in our case) from the coupon rate of the previous reference issuance. For example, the 10-year BTP issued on March 2019 with annual coupon rate of 3% is anomalous compared to the 10-year BTP issued on February 2018 with annual coupon rate of 2%. We will address in detail this issue in the next section, now we just say that the furthest coupon will be due on January 2050. Therefore, we need to simulate 358 monthly observations starting from March 2020. We start by simulating both nominal and real short rates for 358 months, using the procedure explained in the previous section and employing parameters retrieved from the calibration on the reduced sample. Since the analysis proved that the evolution between May 2018 and May 2019 is anomalous, we do not want our simulation to be upwardly biased or made moire noisy than necessary. Our simulation environment tries to mimic dynamics of quiet and predictable market conditions, knowing that financial turmoil often occurs, but may be difficult to predict and model in size. Again, we stress that simulation of rates shifts from a daily basis to a monthly one. For each rate we run 1000 simulations and retrieve the relevant time series as the mean of the simulations. Then, we compute the monthly quantity $\Delta_{rate} = r_n(t) - r_r(t)$ and retrieve the mean of the underlying normal distribution, $\mu_{\Delta_{rate}}$. We compute also the mean of the implied normal distribution of $\frac{\Delta I(t)}{I(t)\Delta t}$, μ_{Δ_I}, from our 33 monthly observations spanning from September 2015 to May 2018. These quantities are used to retrieve the constant market price of risk, ϕ. Indeed:

$$E\left[\frac{\Delta I(t)}{I(t)\Delta t}\right] = E[r_n(t) - r_r(t) - \sigma_I \phi]$$

$$\mu_{\Delta_I} = \mu_{\Delta_{rate}} - \sigma_I \phi.$$

Finally, the market price of risk is computed as

$$\phi = \frac{\mu_{\Delta_{rate}} - \mu_{\Delta_I}}{\sigma_I} = -2,6848,$$

and we can use the real-world dynamics to simulate inflation:

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t) + 2,6848\sigma_I]dt + \sigma_I dW_I^P(t).$$

6.3 Cost Estimation

We notice that the increase in forward rates is matched by a raise in coupon rates regardless of maturity. It seems straightforward to assume that forward rates, as quantitative indicators of market sentiment and forecasts, substantially determine the size of coupon rates. Indeed, when interest rates are low, the market is expressing confidence and optimism with respect to the future performance of a country. Forecasts of sound growth and political stability make investors optimistic regarding a country's outlook, whose risk of default lowers. Investors are willing to acknowledge higher quotations for securities they believe are safer. We know that higher prices mean lower yields and imply lower interest rates. Therefore, the more positive market sentiment with respect to a country's future evolution is, the lower implied interest rates will be. As this chain of causal links shows, interest rates are proxies of market sentiment. When rates are low, there is no need for the issuer to offer higher compensation to lenders in the form of higher coupon rates. The utility function of an issuer is certainly decreasing in coupon rates, therefore issuers are willing to acknowledge higher coupon rates only when it is necessary for their security to be subscribed. Figure 6.6 shows that, after the peak of the 2012 financial turmoil, a new wave of tension hit the market between the end of 2016 and the first half of 2019, albeit with lower severity. This second wave is characterised by a first increase at the beginning of 2017, but the major jump occurred between May 2018 and May 2019, which is precisely the time window in which forward rates increased well over the confidence intervals of our simulations.

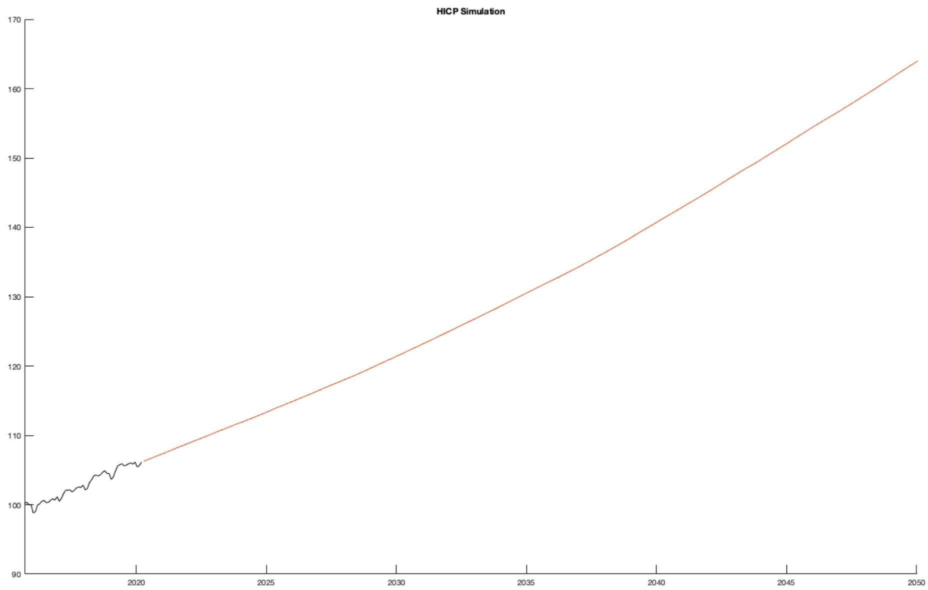


Figure 6.5: In black HICP from Sep-2015 to May-2018, in red the mean of 1000 future HICP simulations. We omit confidence intervals, because they are closest to the mean trajectory and make it undistinguishable. The simulation implies an average annual inflation rate of 1,46%, which is a credible forecast value nearest to the 2% target of the European Central Bank. We acknowledge that this simulation runs far in the future, relying on a limited number of observations for calibration purposes. However, the inflationary process has been overall stable for the last two decades and this simulation seems convincing.

We want to estimate the cost incurred by the Italian finances as a result of the raise of coupon rates. To this aim, we selected BTP issued between May 2018 and May 2019, whose coupon rates were higher than the ones of the previous issuance for homogenous maturity. These selection criteria identifies 12 bonds, covering every standard maturity between 3 and 30 years. We conduct this analysis only for BTPs, because none of the BTP€is matched our selection criteria. We paired each selected bond with its reference security, that is the first bond with the same maturity issued before May 2018. Then, we computed the differential between the two coupon rates and multiplied such value by the issuance amount of the bond under consideration. We shall clarify the procedure with an example regarding 10-year BTPs. Of the fourteen securities available for this maturity, we selected two of them issued respectively on August 2018 and March 2019 with annual coupon rate of 2,8% and 3%. We chose as reference bond the first bond issued before May 2018, namely the 10-year BTP issued on February 2018 with annual coupon rate of 2%. Therefore, the annual coupon differential is respectively 0,8% and 1%. These bonds were issued for €20,750 billions and €21,899 billions. Therefore, every six months roughly

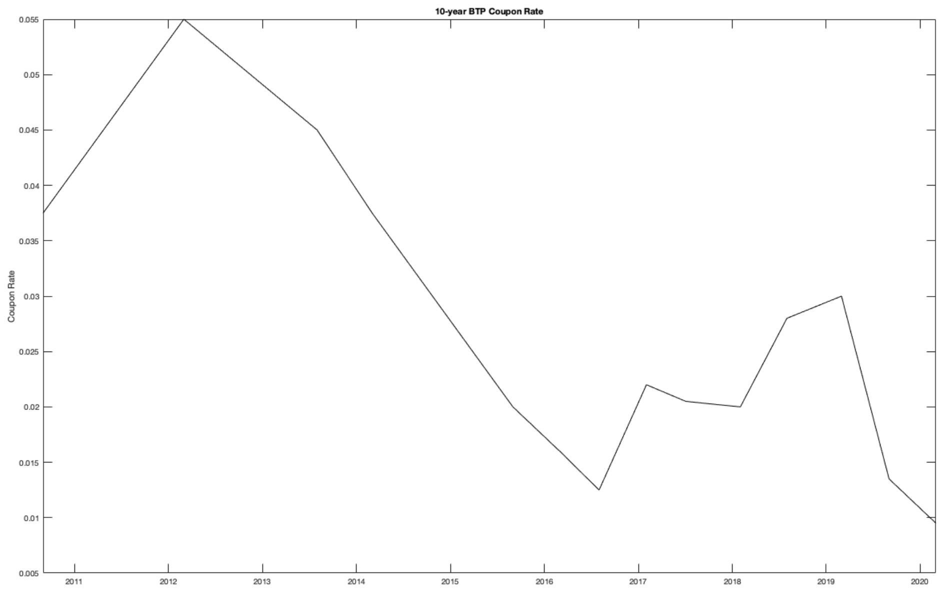


Figure 6.6: Coupon rates of the fourteen 10-year BTPs traded on 14-Apr-2020. We notice that the coupon rate peaked in 2012 during the second wave of financial crisis, before starting to decrease steadily as market conditions became calmer. The rate increased again after 2017 in three major steps up to the first half 2019. Finally, the rate drop to minimum levels. This pattern is representative of the evolution of coupon rates for securities with different maturity.

€$83 \cdot 10^6$+€$109{,}5 \cdot 10^6$=€$192, 5 \cdot 10^6$ are spent in excess over reference interest due payments. This is the cost of the positive differential between the two securities and the reference bond. If the coupon rate remained constant at 2%, semiannual coupon payments for the two BTPs would amount to €$426, 5 \cdot 10^6$, but the increase in coupon rates brought the figure to almost €$620 \cdot 10^6$. We perform these computations for each of the twelve bonds selected and obtained a stream of cash flows to be paid in "excess". Now we must stress that such payments will be due at different times in the future. For example, the 30-year bond selected will have its last coupon paid on 22-Jan-2050. Hence, conjecturing a constant annual inflation rate of 2% for 30 years, this coupon payment will be worth in real term roughly 55% of its nominal value. For this reason, we have to discount the impact of inflation on the nominal value of coupon payments. To this extent, we employed the inflation historical and simulated data, as previously obtained (see Figure 6.5). We computed the coupon payment dates for each selected bond, then we matched the dates with the due differential amounts and finally we divided such amount by a revaluation coefficient, $RC_{CD,i}$. Namely, we took as reference the Inflation Index value for March 2020 and compute the coefficient as the ratio between the value of the index at coupon payment date and the reference value: $RC_{CD,i} = \frac{I_{CD,i}}{I_{\text{March 2020}}}$.

We report the cost estimation at March 2020 prices, performed as explained with different set of parameters and values of σ_I. Parameters calibrated on the reduced sample yield slightly smaller inflation estimations than parameters calibrated on full sample do. The bigger σ_I, the greater the depleting impact of inflation. Overall, estimates remain rather constant: a value of σ_I three time as big as the original reduces the cost estimation by 0,024% and 0,022% for the reduced and full simulation respectively.

Cost Estimation Comparison

	σ_I	$\frac{\sigma_I}{2}$	$2\sigma_I$	$3\sigma_I$
Reduced Calibration	13 142 519 880	13 143 314 253	13 140 933 863	13 139 349 240
Full Calibration	13 107 026 028	13 107 757 200	13 105 561 331	13 104 099 390

This analysis shows that over the next thirty years 14 billions will be due in nominal terms or 13,1 billions at March 2020 prices. We call this amount "excess interest spending", because it represents what must be paid in addition to the already heaviest burden of interest payments. The public debt amount to almost 135% of the Italian GDP and required €63,984 billions to cover interest due on 2019[1]. These figures show the depth and harmfulness of the debt problem. Huge amounts of taxpayer contributions must be diverted from investments and welfare subsidies to the mere and impellent repayment of debt obligations. We estimated the cost of the raise in coupon rates, which we believe was due to the political struggle Italy faced during the first executive led by Giuseppe Conte. As figure 6.6 shows, this political impasse, in connection with a stagnating economic conjuncture, had a negative, albeit restrained, effect. Yet, it will cost Italian taxpayers more than thirteen billions at March 2020 prices over years. We kept the estimate conservative computing differential coupons with respect to the very previous debt issuance rather than lowest-coupon issuances. We can only imagine the excess costs previously incurred, because of poor economic and political management. As Figure 6.7 shows, the average weighted cost of interest payments hit historical minimums over the last five years, thus bringing down the negative impact of interest rate raise on a much smaller scale. Interest rates are kept at lowest level, often negative, by the expansive monetary policies of the European Central Bank, started with the Quantitative Easing. Such policies will cease sooner or later and the cost of indebtedness will jump again to much higher levels. We saw what impact even the smallest variation in interest and coupon rates may have, when managing pool of securities worth tens of billions. The resources wasted over the previous two decades must have been unacceptably large.

The main cause of the recent increase in forward rates was political instability. Even though

[1] Documento di Economia e Finanza 2019, Doc. LVII, n.2, page 42

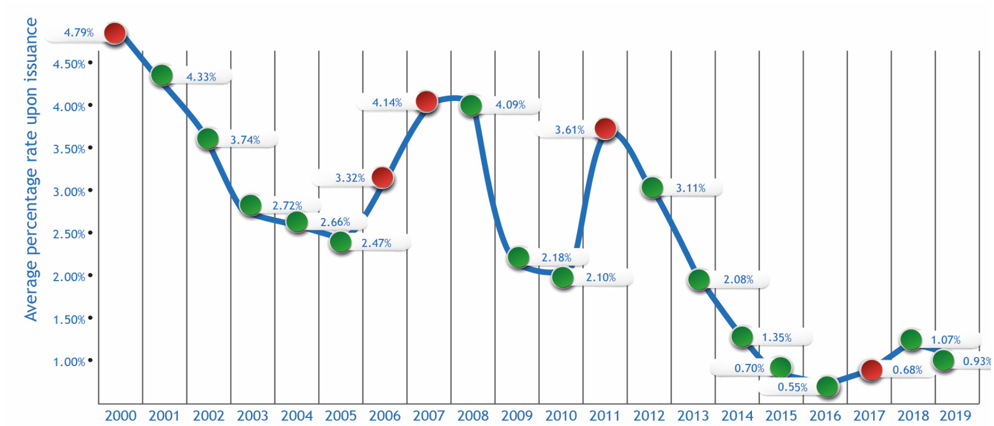


Figure 6.7: Trend of the average interest rate of government bonds computed on the basis of the gross yields at issuance for the securities issued during a single year. Source: Ministero Economia e Finanza.

the global economy had not been experiencing a fully-expansive phase for almost a decade, no major crisis or financial turmoil was registered from 2016 onwards. The ISTAT economic trend section[2] gathers indicators to asses the evolution of Italian economy. Many indicators remained stable or mildly improved from 2017 onwards. Therefore, Italy was experiencing a period of modest or null growth, which cannot explain the abrupt jump of interest rates. As the previous analysis shows, this shift is not compatible with the evolution observed from 2015 onwards. Something at environmental level deeply changed during 2018. As economic factors are discarded, we found a major political break after the elections held on March 4th, 2018. The political struggle that followed is perfectly compatibile with the inversion of tendency showed in financial data. An executive with a most difficult gestation, politically weak and thus focused more on political manoeuvres than on reforms is a suitable blend to explain the abrupt increase in volatility and value of interest rates.

On February 12th 2019, Belgian politician Guy Verhofstadt declared during his speech at the Euro-Parliament «I wonder how much longer you [Italian Prime Minister Giuseppe Conte] will be the puppet moved by Salvini and Di Maio. [...] This beautiful country [Italy] has become, from a convinced defender of Europe, a laggard in the Union». Such pronouncement lacks any instance of elegance, but cynically depicts how the Italian political and economical situation have been perceived over the last twenty years outside Italian borders. If Italy must play the harmful game of indebtedness, it must adhere as much as possible to its rules to limit unpleasant scars. This means convincing and proving the market that Italy has the willingness, the human resources and the technological means, as it does, to excel at global level and ensure regular fulfilment

[2] `https://www.istat.it/en/economic-trends`

of its obligations. It is only with virtuous and rigorous policies, possibly uncomfortable in the present but effective in the future, that the oppressive debt burden may become lighter and lighter. Belgium reduced its Debt/GDP ratio from 131% to 100% over the last twenty years and was able to free substantial resources to boost its development. This case study shall serve as stimulus to believe that such a goal can and must be achieved. The political impasse we highlighted had minor consequences, compared to the deep drops experienced after the financial crisis. The waste of resources we estimated was large, but the waste must have been much bigger during previous turmoils. Our analysis shall give an example of the dynamics and of the size of stakes at play. It is in the interest of every taxpayer to know what she will be paying for. Recognising and understanding the problem is the first step to demand and seek proper executives, able and willing to overcome it. The most precious legacy generations can leave to their heirs is the possibility to run as fast as possible toward sustainable growth and development: present indebtedness is seriously slowing down this run.

Cost Estimation

Maturity	Issue	Annual Coupon Diff.	Amount Issued	Semi-Annual Excess Payment	Num. of Payments
3 years	*16-Oct-2017*	*0,20%*	17 240 391 682	–	–
	15-Oct-2018	2,10%	17 240 391 682	181 024 113	6
	15-Mar-2019	0,80&	18 444 975 357	73 779 901	6
	26-Nov-2018	1,25%	2 338 989 883	14 618 687	8
5 years	*01-Mar-2018*	*0,95%*	18 522 221 261	–	–
	03-Sep-2018	1,50%	19 225 706 843	144 192 801	10
	01-Apr-2019	0,80%	18 722 299 079	74 889 196	10
7 years	*15-Mar-2018*	*1,45%*	16 987 079 028	–	–
	17-Sep-2018	1,05%	17 864 118 498	93 786 622	14
	15-Apr-2019	0,65%	18 141 585 966	58 960 154	14
10 years	*01-Feb-2018*	*2,00%*	*24365717003*	–	–
	01-Aug-2018	0,80%	20 750 499 329	83 001 997	20
	01-Mar-2019	1,00%	21 899 723 315	109 498 617	20
15 years	*25-Jan-2017*	*2,45%*	*17190380874*	–	–
	22-Jan-2019	0,90%	14 698 888 936	66 145 000	32
20 years	*17-Jan-2018*	*2,95%*	*16172830833*	–	–
	19-Jun-2019	0,70%	10 505 274 307	36 768 460	41
30 years	*14-Jun-2017*	*3,45%*	*17705950888*	–	–
	13-Feb-2019	0,40%	14 420 189 356	28 840 379	60
Total Nominal Cost				14 012 664 356	
Total Cost at March 2020 Prices				13 142 519 880	

In italics Reference Bonds for given maturity. For Reference Bonds the third column is the annual coupon rate.

Conclusions

We divided the thesis in three macro-sections. They all aim at the modelling of inflation, but each deals with it from a different point of view. Indeed, the path we followed required three major stages: laying the theoretical foundations, choosing the modelling framework, applying the chosen framework to the Italian case.

The first macro-section incorporates the first three chapters. It constitutes a theoretical summary of the concepts and techniques needed for proper modelling of inflation. Chapter one recapitulates the foundations of interest rate term structure modelling. Chapter two presents in detail three common financial instruments, traded in the market to counteract the effect of inflation. Chapter three synthesises some prominent contributions to the literature regarding interest rate and inflation rate modelling. The second macro-section is the smallest in size, but still is most important. In chapter four we thoroughly describe the Hull-White and Jarrow-Yildirim models from an analytical point view. This discussion is central, because we chose to build our modelling framework around these two models following the example of (Jarrow & Yildirim, 2003). The third macro-section applies the theoretical background we presented to the Italian case. In chapter five the modelling framework is calibrated to real-world Italian data. In chapter six we make use of the calibrated model to draw evidences, conclusions and estimates regarding the cost of the Italian public debt though the analysis of interest rates.

The main contribution of this thesis lies in the willingness to study the Italian case. Indeed, the Jarrow-Yildirim model has gained substantial prominence for the last two decades. Its robustness and straightforwardness more than compensate the drawbacks more recent works try to amend. However, contributions to the literature typically perform the application of theoretical findings to the American case, partly for a superior data availability. Indeed, data retrieval for Italian government securities proved to be slightly more laborious, but the effort was well worth. The application to Italian data disclosed surprisingly interesting evidences. More specifically, this analysis shows that investors are very sensitive to political and economic outlooks. In chapter five we retrieved interest rates implied by prices of Italian bonds and we noted that rates rose and became more volatile between May-2018 and May-2019. This increase proved to be quite sharp and abrupt, compared to the previous smooth evolution. The second wave of the financial crisis hit the markets around 2012 and put serious pressure on European countries with higher debt burdens. Figure 6.7 shows that Italy suffered a higher cost of public debt during this period of financial tension. Then, the situation gradually stabilised and yields of government securities, as precise indicators of the health of country, decreased and remained quite stable. We must point out that interest rates have been kept depressed since the European Central Bank started the Quantitative Easing program in 2015. ECB has been buying massive amounts of public debt to inject liquidity in the system and keep it stable. In this environmental setting the abrupt increase seemed quite amiss and we decided to investigate its characteristics and causes.

Relying on the Jarrow-Yildirim model, we calibrated our framework on observations between Apr-2015 and May-2018. Our goal was to determine if the evolution in the highlighted time window could have been compatibile with the previous smooth evolution. Calibration on this reduced dataset of observations, which we took as benchmark for stability around lower values, generated simulated evolutions, which should share on average the same stable behaviour. With solid statistical confidence, we rejected the hypothesis that evolution between May-2018 and May-2019 had been compatibile with the previous realisations. Something changed at a core level and the change persisted for roughly a year.

We looked for the possible causes of such change in two directions: economic and institutional. The analysis of aggregate indicators for the Italian economy showed that no major economic turmoil occurred: the economic situation did not worsen as the raise in rates suggested. Indicators remained overall rather stable, with the exception of business confidence indicators, which decreased (see Figure 6.2). Economic sentiment indicators acted as link between the economic and institutional side. Indeed, the reduction in business confidence could not have been driven by a stable economic outlook, but rather by a tumultuous institutional situation. After elections on March 2018, the Italian executive faced a very tortuous gestation until June 2018. The subsequent evolution proved to be as much controversial and this executive survived until September 2019. This time window matches the period in which interest rates rose. It matches also the way in which interest rates increased: short term rates became more volatile, medium term rates jumped and long term rates mildly increased. Political impasses, indeed, make the present uncertain, hinder medium-term growth, but shall be resolved and absorbed in the long run.

Having pointed out some possible causes of this shift in interest rates, we gave a monetary estimate of the costs it generated for Italian finances. Interest rates we retrieved are implied by market prices of securities. They are not the pure cost of the debt, but signal the market sentiment regarding a country. When interest rates persist at higher levels, the market signals that issuances of a given country are perceived as more risky. At the next issuance round, issuers are forced to offer higher compensation to subscribers. This is what happened in the Italian case. Issuances between May-2018 and May-2019 systematically showed coupon rates higher than those of the previous issuances. We estimated the cost of this institutional crisis in a differential way. We computed the differential in coupon rates between securities issued in the highlighted time window and those issued just before the outbreak of the impasse. These differential were multiplied by the nominal amount of each issuance to generate cash flows at each coupon payment date. We call these streams of payments excess spending, because it represents what will be paid over next years in addition to the cost generated had coupon rates remained constant at the values observed before May-2018. Such cash flows will be due in the future, some as far as in 2050. Therefore we discounted each payment to March 2020 prices with a simulated inflation evolution, in order to take into account the depleting effect of inflation. This procedure yielded an excess spending of roughly 13 billion euros at March 2020 prices.

The evidences offered by the analysis are interesting and plausible, but may be revised by an updated model. The Jarrow-Yildirim model is robust and functional, but more complex frameworks could offer superior precision of estimates. One possible extension in this direction is a deeper analysis of drivers of changes in yields and term structures. A possibility is the introduction of a latent state vector process, which drives the evolution of the term structure. This approach builds on contributions such as (Duffie & Kan, 1996), (Dai & Singleton, 2000) and (Duffee, 2002), and it was recently implemented in (d'Amico *et al.*, 2018) and (Ho *et al.*, 2014). A similar pattern of analysis acknowledges the presence latent factors, but tries to precisely identify them. (Dewachter & Lyrio, 2006) and (Rudebusch & Wu, 2008) implement such rationale, building on (Kozicki & Tinsley, 2001). (Modena, 2011) offers a review of the relevant literature for this approach. A different direction of improvement concerns market prices of risk. The link between the risk neutral and the real-world probability is a central quantity, whose estimation proves to be difficult. Indeed, our estimates are plausible, but call for improvement. For example, (Stanton, 1997) uses regression analysis on historical yield data and (Dempster *et al.*, 2011) use the Kalman filter. We estimated constant market prices of risk, but time dependence and specific functional assumptions could tune the estimation.

The Italian public debt issue has come again to prominence during the last decade. The huge debt burden accumulated over years has become a serious threat to economic and political stability of a country, whose growth rate has been stagnating after the financial crisis. Due interest payments divert billions from welfare subsidies and systemic investments. The size of this problem has been made even bigger by institutional crisis, resulting in poor management of the country and lack of forward-looking reforms. The inability to put in place a consistent shift to a virtuous path hinders Italian credibility among investors and raises the cost of indebtedness. This analysis shows the order of magnitude of the debt problems: even the smallest increase in interest rates results in hundreds of millions to be payed. The crisis we highlighted was a minor one, which occurred in a context of generally low interest rates: still the monetary consequences will amount to thirteen billions of euros. We only want to offer an example of the direction in which taxpayers and electors shall evaluate electoral programmes and policy proposals. It should help people realise the seriousness of the Italian debt problem: awareness is the first step toward its solution.

Appendix A

Matlab Codes

Datasets and Codes are available at
https://www.dropbox.com/sh/66koikku9bbvojr/AACPUOWYXsXc2_iphhIXVBY6a?dl=0

A.1 Real ZCB Price Stripping

The following code is used to retrieve the data needed for real ZCB price stripping from 6 BTP€is. We first retrieve our data set of securities and inflation index. Secondly, we prepare the relative information regarding ISINs, maturities, issue dates, indexation coefficients and coupons. Then, we compute the distance of each valuation day from each coupon and maturity date. We also compute dirty prices for explanatory purposes. These preliminary data are stored in a `.mat` file, which is passed to the minimisation code.

```
%HICP inflation data
[Inflazione]=xlsread('Inflation.xls','HICP');
HICP=Inflazione(:,2);
DateINF=Inflazione(:,1); % select original data in Excel format
DateINF = datenum('30-Dec-1899') + DateINF; % convert data in Matlab format
name={'HICP'};
HICP=array2timetable(HICP,'RowTimes',datetime(datestr(DateINF(:,1))),'VariableNames',
    name);
%€BTPi traded and selection of 6/11 used €BTPi
[Prezzo,text]=xlsread('ITtraded.xlsx','€BTPi');
ISIN_BTPi=text(1,2:end);
Coupon_BTPi=Prezzo(1,2:end)/100;
Maturity_BTPi=datetime(datestr(Prezzo(2,2:end)+datenum('30-Dec-1899')));
Issue_BTPi=datetime(datestr(Prezzo(3,2:end)+datenum('30-Dec-1899')));
Time=Prezzo(4:end,1); % select original data in Excel format
Time = datenum('30-Dec-1899') + Time; % convert data in Matlab format
yi=[Time Prezzo(4:end,2:end)];
```

```
Titoli_BTPi=array2timetable(yi(:,2:end),'RowTimes',datetime(datestr(yi(:,1))),'
    VariableNames',ISIN_BTPi);
Titoli_BTPi=Titoli_BTPi(find(datenum(Titoli_BTPi.Time)==datenum(datetime('14-Oct-2015')
    )):end,:);
BTPi=rmmissing(Titoli_BTPi,2);
clear Prezzo Time text yi;
BTPi=BTPi(:,2:end);
N=BTPi.Properties.VariableNames;
%Coupon Payment
for i=1:size(BTPi,2)
Coupon(i,1)=Coupon_BTPi(find(strcmp(ISIN_BTPi,N(i))))*100;
end
%Maturity Date
for i=1:size(BTPi,2)
Maturity(i,1)=Maturity_BTPi(find(strcmp(ISIN_BTPi,N(i))));
end
%Issue Date
for i=1:size(BTPi,2)
Issue(i,1)=Issue_BTPi(find(strcmp(ISIN_BTPi,N(i))));
end
%Indexation Coefficient
A=HICP.Variables;
for i=1:size(BTPi,2)
for j=1:size(BTPi,1)
[yy,mm,dd]=ymd(Issue(i));
IB(i,1)=A(find(month(HICP.Time)==mm & year(HICP.Time)==yy),1);
[y,m,d]=ymd(BTPi.Time(j));
R(j,i)=(d-1)/eomday(y,m);
IE2(j,i)=A(find(month(HICP.Time)==m & year(HICP.Time)==y)-2,1);
IE3(j,i)=A(find(month(HICP.Time)==m & year(HICP.Time)==y)-3,1);
IR(j,i)=IE3(j,i)+R(j,i)*(IE2(j,i)-IE3(j,i));
CR(j,i)=IR(j,i)/IB(i,1);
end
end
%Lower Bound IC = 1
for i=1:size(CR,1)
for z=1:size(CR,2)
if CR(i,z)>=1
    CR(i,z)=CR(i,z);
else CR(i,z)=1;
end
end
end
%Coupon Date
for i=1:size(BTPi,2)
for j=1:(year(Maturity(i))-year(Issue(i)))*2
CD(j,1,i)=datetime(Issue(i)) + calmonths(6*j);
if datenum(Maturity(i))-datenum(CD(j,1,i))<0
```

```
    CD(j,1,i)=NaT;
end
if year(CD(j,1,i))==year(Maturity(i)) && month(Maturity(i))-month(CD(j,1,i))<6
    break
end
end
end
%Distance day from Coupon payment
for i=1:size(BTPi,2)
for z=1:size(BTPi,1)
for j=1:size(CD,1)
Dist_C(z,j,i)=yearfrac(BTPi.Time(z),CD(j,1,i));
if Dist_C(z,j,i)<0 || isnan(Dist_C(z,j,i))==1
    Dist_C(z,j,i)=0;
end
end
end
end
%Distance day from Maturity payment
for i=1:size(BTPi,2)
for z=1:size(BTPi,1)
Dist_M(z,i)=yearfrac(BTPi.Time(z),Maturity(i));
end
end
%Accrued Interest
for i=1:size(BTPi,2)
for z=1:size(BTPi,1)
AccruInterest(z,i) = acrubond(Issue(i),BTPi.Time(z),CD(1,1,i),100*CR(z,i),Coupon(i)
    /100);
end
end
AccruInterest=array2timetable(AccruInterest,'RowTimes',BTPi.Time,'VariableNames',BTPi.
    Properties.VariableNames);
BTPi_dirt=BTPi.Variables+AccruInterest.Variables;
BTPi_dirt=array2timetable(BTPi_dirt,'RowTimes',BTPi.Time,'VariableNames',BTPi.
    Properties.VariableNames);
save RealStripIT.mat
```

We write the custom function `RealStripfunIT`, which computes theoretical bond prices from equation (5.1) given five forward rates. Then, the function computes the sum of the squared differences between market and theoretical prices.

```
function g = RealStripfunIT(f)
load RealStripIT.mat Dist_C Dist_M CR BTPi Coupon
load nday.mat
%Coupon Present Value
for i=1:size(BTPi,2)
for z=nday:nday
```

```
for j=1:size(Dist_C,2)
VA(j,i)= (CR(z,i)*(Dist_C(z,j,i)>0)*(Coupon(i)/2)*exp(-f(1)*Dist_C(z,j,i)*(Dist_C(z,j,i
    )>0 & Dist_C(z,j,i)<=3)-(f(2))*Dist_C(z,j,i)*(Dist_C(z,j,i)>3 & Dist_C(z,j,i)<=5)
    -(f(3))*Dist_C(z,j,i)*(Dist_C(z,j,i)>5 & Dist_C(z,j,i)<=10)-(f(4))*Dist_C(z,j,i)*(
    Dist_C(z,j,i)>10 & Dist_C(z,j,i)<=20)-f(5))*Dist_C(z,j,i)*(Dist_C(z,j,i)>20 &
    Dist_C(z,j,i)<=30)));
end
end
end
%Face Value Present Value
for i=1:size(BTPi,2)
for z=nday:nday
VA(size(Dist_C,2)+1,i)=  (CR(z,i)*100*exp(-f(1)*Dist_M(z,i)*(Dist_M(z,i)>0 & Dist_M(z,i
    )<=3)-(f(2))*Dist_M(z,i)*(Dist_M(z,i)>3 & Dist_M(z,i)<=5)-(f(3))*Dist_M(z,i)*(
    Dist_M(z,i)>5 & Dist_M(z,i)<=10)-(f(4))*Dist_M(z,i)*(Dist_M(z,i)>10 & Dist_M(z,i)
    <=20)-(f(5))*Dist_M(z,i)*(Dist_M(z,i)>20 & Dist_M(z,i)<=30)));
end
end
%Theoretical Price
for i=1:size(BTPi,2)
Pteorico(i)=sum(VA(:,i));
end
BTP_ifull=BTPi.Variables;
%Difference bewteen market and theoretical price
for s=1:size(BTPi,2)
Scarto(1,s)=(BTP_ifull(nday,s)-Pteorico(s))^2;
end
g=sum(Scarto);
end
```

Finally, we employ the function `lsqnonlin` in order to find the five rates, which minimise the output of `RealStripfunIT`. The minimisation is repeated for each valuation date, so that we can build a timetable to evaluate the evolution of real forward rates.

```
% RealFWDIT
x0=[0 0 0 0 0];
options = optimoptions(@lsqnonlin,'Algorithm','levenberg-marquardt');
load RealStripIT.mat
for i=1:size(BTPi,1)
nday=i;
save nday.mat nday
RealFWDIT(i,:)=lsqnonlin(@RealStripfunIT,x0,[],[],options);
end
name={'0-3yrs','3-5yrs','5-10yrs','10-20yrs','20-30yrs'};
RealFWDIT=array2timetable(RealFWDIT,'RowTimes',BTPi.Time,'VariableName',name);
save FWD.mat RealFWDIT -append
```

A.2 Nominal ZCB Price Stripping

The following code is used to retrieve the data needed for nominal ZCB price stripping from 31 BTPs. We first retrieve our data set of securities and select suitable bonds. Secondly, we prepare the relative information regarding ISINs, maturities, issue dates and coupons. Then, we compute the distance of each valuation day from each coupon and maturity date. We also compute dirty prices for explanatory purposes. These preliminary data are stored in a `.mat` file, which is passed to the minimisation code.

```
%BTP traded
[Prezzo,text]=xlsread('ITtraded.xlsx','BTP');
ISIN_BTP=text(1,2:end);
Coupon_BTP=Prezzo(2,2:end)/100;
Maturity_BTP=datetime(datestr(Prezzo(3,2:end)+datenum('30-Dec-1899')));
Issue_BTP=datetime(datestr(Prezzo(4,2:end)+datenum('30-Dec-1899')));
Time=Prezzo(5:end,1); % select original data in Excel format
Time = datenum('30-Dec-1899') + Time; % convert data in Matlab format
y=[Time Prezzo(5:end,2:end)];
Titoli_BTP=array2timetable(y(:,2:end),'RowTimes',datetime(datestr(y(:,1))),'
    VariableNames',ISIN_BTP);
Titoli_BTP=Titoli_BTP(find(datenum(Titoli_BTP.Time)==datenum(datetime('14-Oct-2015'))):
    end,:);
BTP=rmmissing(Titoli_BTP,2);
clear Prezzo Time text y;
N=BTP.Properties.VariableNames;
%Maturity Date
for i=1:size(BTP,2)
Maturityf(i,1)=Maturity_BTP(find(strcmp(ISIN_BTP,N(i))));
end
Maturity=Maturityf(year(Maturityf)-2020>1);
%ISIN
A=find(year(Maturityf)-2020>1);
for i=1:size(A,1)
ISIN(i)=N(A(i));
end
%On the run Bond
Titoli_BTPfullf=BTP.Variables;
for i=1:size(BTP,2)
if year(Maturityf(i))-2020<=1
    Titoli_BTPfullf(:,i)=NaN;
end
end
BTP=rmmissing(Titoli_BTPfullf,2);
BTP=array2timetable(BTP,'RowTimes',Titoli_BTP.Time,'VariableNames',ISIN);
BTP=BTP(1:1174,:);
%Coupon
N=BTP.Properties.VariableNames;
```

```
for i=1:size(N,2)
   Coupon(i,1)=Coupon_BTP(find(strcmp(ISIN_BTP,N(i))))*100;
end
%Issue Date
for i=1:size(N,2)
   Issue(i,1)=Issue_BTP(find(strcmp(ISIN_BTP,N(i))));
end
%Coupon Date
for i=1:size(BTP,2)
for j=1:(year(Maturity(i))-year(Issue(i)))*2
CD(j,1,i)=datetime(Issue(i)) + calmonths(6*j);
if datenum(Maturity(i))-datenum(CD(j,1,i))<0
    CD(j,1,i)=NaT;
end
if year(CD(j,1,i))==year(Maturity(i)) && month(Maturity(i))-month(CD(j,1,i))<6
    break
end
end
end
%Distance day from Coupon payment
for i=1:size(BTP,2)
for z=1:size(BTP,1)
for j=1:size(CD,1)
Dist_C(z,j,i)=yearfrac(BTP.Time(z),CD(j,1,i));
if Dist_C(z,j,i)<0 || isnan(Dist_C(z,j,i))==1
    Dist_C(z,j,i)=0;
end
end
end
end
%Distance day from Maturity payment
for i=1:size(BTP,2)
for z=1:size(BTP,1)
Dist_M(z,i)=yearfrac(BTP.Time(z),Maturity(i));
end
end
%Accrued Interest
for i=1:size(BTP,2)
for z=1:size(BTP,1)
AccruInterest(z,i) = acrubond(Issue(i),BTP.Time(z),CD(1,1,i),100,Coupon(i)/100);
end
end
AccruInterest=array2timetable(AccruInterest,'RowTimes',BTP.Time,'VariableNames',BTP.
    Properties.VariableNames);
AccruInterest=AccruInterest(1:1174,:);
BTP_dirt=BTP.Variables+AccruInterest.Variables;
BTP_dirt=array2timetable(BTP_dirt,'RowTimes',BTP.Time,'VariableNames',BTP.Properties.
    VariableNames);
```

```
save NominalStripIT.mat
```

We write the custom function `NominalStripfunIT`, which computes theoretical bond prices from equation (5.1) given five forward rates. Then, the function computes the sum of the squared differences between market and theoretical prices.

```
function g = NominalStripfunIT(f)
load NominalStripIT.mat Dist_C Dist_M BTP Coupon
load nday.mat
%Coupon Presen Value
for i=1:size(BTP,2)
for z=nday:nday
for j=1:size(Dist_C,2)
VA(j,i)= ((Dist_C(z,j,i)>0)*(Coupon(i)/2)*exp(-f(1)*Dist_C(z,j,i)*(Dist_C(z,j,i)>0 & Dist_C(z,j,i)<=3)-(f(2))*Dist_C(z,j,i)*(Dist_C(z,j,i)>3 & Dist_C(z,j,i)<=5)-(f(3))*Dist_C(z,j,i)*(Dist_C(z,j,i)>5 & Dist_C(z,j,i)<=10)-(f(4))*Dist_C(z,j,i)*(Dist_C(z,j,i)>10 & Dist_C(z,j,i)<=20)-(f(5))*Dist_C(z,j,i)*(Dist_C(z,j,i)>20 & Dist_C(z,j,i)<=max(max(Dist_M)))));
end
end
end
%Face Value Present Value
for i=1:size(BTP,2)
for z=nday:nday
VA(size(Dist_C,2)+1,i)=  (100*exp(-f(1)*Dist_M(z,i)*(Dist_M(z,i)>0 & Dist_M(z,i)<=3)-(f(2))*Dist_M(z,i)*(Dist_M(z,i)>3 & Dist_M(z,i)<=5)-(f(3))*Dist_M(z,i)*(Dist_M(z,i)>5 & Dist_M(z,i)<=10)-(f(4))*Dist_M(z,i)*(Dist_M(z,i)>10 & Dist_M(z,i)<=20)-(f(5))*Dist_M(z,i)*(Dist_M(z,i)>20 & Dist_M(z,i)<=max(max(Dist_M)))));
end
end
%Theoretical Price
for i=1:size(BTP,2)
Pteorico(i)=sum(VA(:,i));
end
BTP_full=BTP.Variables;
%Difference bewteen market and theoretical price
for s=1:size(BTP,2)
Scarto(1,s)=(BTP_full(nday,s)-Pteorico(s))^2;
end
g=sum(Scarto);
end
```

Finally, we employ the function `lsqnonlin` in order to find the five rates, which minimise the output of `NominalStripfunIT`. The minimisation is repeated for each valuation date, so that we can build a timetable to evaluate the evolution of nominal forward rates.

```
% NominalFWDIT
x0=[0 0 0 0 0];
options = optimoptions(@lsqnonlin,'Algorithm','levenberg-marquardt');
```

```
load NominalStripIT.mat
for i=1:size(BTP,1)
nday=i;
save nday.mat nday
NominalFWDIT(i,:)=lsqnonlin(@NominalStripfunIT,x0,[],[],options);
end
name={'0-3yrs','3-5yrs','5-10yrs','10-20yrs','20-30yrs'};
RealFWDIT=array2timetable(NominalFWDIT,'RowTimes',BTP.Time,'VariableName',name);
save FWD.mat NominalFWDIT -append
```

A.3 Calibration

Once we obtained forward rates, we compute ZCB prices for the five maturities. Then, we compute the day-after-day differences of ZCB prices and forward rates. Finally, we fit data as explained in Section 5.5, in order to retrieve the eight parameters. We also offer visual proof of the goodness of the normality assumption.

```
%Real Calibration
load FWDrate.mat
MaturityZCB=[3 5 10 20 30];
FWDrate=RealFWDIT.Variables;
Time=RealFWDIT.Time;
%ZCB Price
for i=1:size(RealFWDIT,2)
for j=1:size(RealFWDIT,1)
ZCBprice(j,i)=exp(-FWDrate(j,i)*MaturityZCB(i));
end
end
%Delta FWD
for i=1:size(RealFWDIT,2)
DeltaFWDrate(:,i)=diff(FWDrate(:,i));
end
%Sigma_r a_r
ZCBchange=diff(ZCBprice)./ZCBprice(1:end-1,:);
VarZCBchange=var(ZCBchange);
[xData, yData] = prepareCurveData( MaturityZCB, VarZCBchange );
ft = fittype( '(sigma^2*(exp(-a*x)-1)^2*(1/365))/(a^2)', 'independent', 'x', 'dependent
    ', 'y' );
opts = fitoptions( 'Method', 'NonlinearLeastSquares' );
opts.Algorithm = 'Levenberg-Marquardt';
opts.Display = 'Off';
opts.Robust = 'Bisquare';
opts.StartPoint = [0.653485019911401 0.957506835434298];
[fitresult, gof] = fit( xData, yData, ft, opts );
Parameter=coeffvalues(fitresult);
ConfInt_R=confint(fitresult);
```

```
SigmaR=Parameter(2);
aR=Parameter(1);
%Sigma_I
load RealStripIT.mat HICP BTPi
[y,m,d]=ymd(Time(1));
HICP=HICP(find(month(HICP.Time)==m & year(HICP.Time)==y):end,:);
HICPTime=HICP.Time;
HICP_=HICP.Variables;
HICP_=HICP_(1:end-1);
DeltaHICP=HICP.Variables;
DeltaHICP=diff(DeltaHICP)./HICP_;
VarI=var(DeltaHICP);
SigmaI=sqrt(VarI);
alpha_up=chi2inv(0.975,size(DeltaHICP,1)-1);
alpha_low=chi2inv(0.025,size(DeltaHICP,1)-1);
ConIntI_up=sqrt(((size(DeltaHICP,1)-1)*VarI)/alpha_up);
ConIntI_low=sqrt(((size(DeltaHICP,1)-1)*VarI)/alpha_low);
%Nominal Calibration
load NominalStripIT.mat Dist_M
MaturityZCB=[3 5 10 20 max(max(Dist_M))];
FWDrate=NominalFWDIT.Variables;
Time=NominalFWDIT.Time;
%ZCB Price
for i=1:size(RealFWDIT,2)
for j=1:size(RealFWDIT,1)
ZCBprice(j,i)=exp(-FWDrate(j,i)*MaturityZCB(i));
end
end
%Delta FWD
for i=1:size(RealFWDIT,2)
DeltaFWDrate(:,i)=diff(FWDrate(:,i));
end
%Sigma_n a_n
ZCBchange=diff(ZCBprice)./ZCBprice(1:end-1,:);
VarZCBchange=var(ZCBchange);
[xData, yData] = prepareCurveData( MaturityZCB, VarZCBchange );
ft = fittype( '(sigma^2*(exp(-a*x)-1)^2*(1/365))/(a^2)', 'independent', 'x', 'dependent
    ', 'y' );
opts = fitoptions( 'Method', 'NonlinearLeastSquares' );
opts.Algorithm = 'Levenberg-Marquardt';
opts.Display = 'Off';
opts.Robust = 'Bisquare';
opts.StartPoint = [0.217942564867815 0.957506835434298];
[fitresult, gof] = fit( xData, yData, ft, opts );
Parameter=coeffvalues(fitresult);
ConfInt_N=confint(fitresult);
SigmaN=Parameter(2);
aN=Parameter(1);
```

```
%Normality Plot
figure;
subplot(3,2,1)
histfit(DeltaFWDrate(:,1));
title('3 year nominal FWD rate');
xlabel(' f')
subplot(3,2,2)
histfit(DeltaFWDrate(:,2));
title('5 year nominal FWD rate');
xlabel(' f')
subplot(3,2,3)
histfit(DeltaFWDrate(:,3));
title('10 year nominal FWD rate');
xlabel(' f')
subplot(3,2,4)
histfit(DeltaFWDrate(:,4));
title('20 year nominal FWD rate');
xlabel(' f')
subplot(3,2,5)
histfit(DeltaFWDrate(:,5));
title('max year nominal FWD rate');
xlabel(' f')
subplot(3,2,6)
histfit(DeltaHICP(:,1));
title('Inflation Rate of Change');
xlabel(' I/I')
%Correlation r-I
load FWDrate.mat
FWDr=RealFWDIT.Variables;
Time=RealFWDIT.Time;
Deltar=diff(FWDr(:,1));
load RealStripIT.mat HICP BTPi
[y,m,d]=ymd(Time(1));
HICP=HICP(find(month(HICP.Time)==m & year(HICP.Time)==y):end-1,:);
HICPTime=HICP.Time;
HICP_=HICP.Variables;
HICP_=HICP_(1:end-1);
DeltaHICP=HICP.Variables;
DeltaHICP=diff(DeltaHICP)./HICP_;
DeltaHICP=timetable(DeltaHICP,'RowTimes',HICPTime(2:end,:));
DeltaHICPv=DeltaHICP.Variables;
DeltaHICPt=DeltaHICP.Time;
Deltar=timetable(Deltar,'RowTimes',Time(2:end,:));
FWDrd=Deltar.Variables;
Timerd=Deltar.Time;
for j=1:size(DeltaHICP,1)
    if find(Timerd==DeltaHICPt(j))>0
       Deltarr(j,1)=FWDrd(find(Timerd==DeltaHICPt(j)),1);
```

```
    elseif find(Timerd==DeltaHICPt(j))==0
        Deltarr(j,1)=FWDrd(find(datenum(Timerd)==datenum((DeltaHICPt(j)+1))),1);
    else find(Timerd==DeltaHICPt(j)+1)==0
        Deltarr(j,1)=FWDrd(find(Timerd==DeltaHICPt(j)+2),1);
    end
end
Rho_rI=corr(Deltarr,DeltaHICPv);
%Correlation n-I
FWDn=NominalFWDIT.Variables;
Time=NominalFWDIT.Time;
Deltan=diff(FWDn(:,1));
Deltan=timetable(Deltan,'RowTimes',Time(2:end,:));
FWDnd=Deltan.Variables;
Timend=Deltan.Time;
for j=1:size(DeltaHICP,1)
    if find(Timend==DeltaHICPt(j))>0;
       Deltann(j,1)=FWDrd(find(Timend==DeltaHICPt(j)),1);
    elseif find(Timend==DeltaHICPt(j))==0;
        Deltann(j,1)=FWDnd(find(datenum(Timend)==datenum((DeltaHICPt(j)+1))),1);
    else find(Timend==DeltaHICPt(j)+1)==0;
        Deltann(j,1)=FWDnd(find(Timend==DeltaHICPt(j)+2),1);
    end
end
Rho_nI=corr(Deltann,DeltaHICPv);
%Correlation r-n
Rho_rn=corr(FWDrd,FWDnd);
```

A.4 Calibration on Reduced Sample

We perform the same steps of the previous calibration on a reduced sample: we employ daily observation up to 01-May-2018. The retrieved parameters are stored in a .mat file, which is passed to the simulation stage.

```
%Real Calibration Reduced
load FWDrate.mat
MaturityZCB=[3 5 10 20 30];
Time=RealFWDIT.Time;
FWDrate=RealFWDIT.Variables;
%Reduced Sample
FWDrate=FWDrate(1:find(Time==datetime('01-May-2018')),:);
Time=Time(1:size(FWDrate,1));
%ZCB Price
for i=1:size(FWDrate,2)
for j=1:size(FWDrate,1)
ZCBprice(j,i)=exp(-FWDrate(j,i)*MaturityZCB(i));
end
```

```
end
%Delta FWD
for i=1:size(FWDrate,2)
DeltaFWDrate(:,i)=diff(FWDrate(:,i));
end
%Sigma_r a_r
ZCBchange=diff(ZCBprice)./ZCBprice(1:end-1,:);
VarZCBchange=var(ZCBchange);
[xData, yData] = prepareCurveData( MaturityZCB, VarZCBchange );
ft = fittype( '(sigma^2*(exp(-a*x)-1)^2*(1/365))/(a^2)', 'independent', 'x', 'dependent
    ', 'y' );
opts = fitoptions( 'Method', 'NonlinearLeastSquares' );
opts.Algorithm = 'Levenberg-Marquardt';
opts.Display = 'Off';
opts.Robust = 'Bisquare';
opts.StartPoint = [0.587669305850536 0.957506835434298];
[fitresult, gof] = fit( xData, yData, ft, opts );
Parameter=coeffvalues(fitresult);
ConfInt_R=confint(fitresult);
SigmaR=Parameter(2);
aR=Parameter(1);
%Sigma_I
load RealStripIT.mat HICP
[y,m,d]=ymd(Time(1));
HICP=HICP(find(month(HICP.Time)==m & year(HICP.Time)==y):find(month(HICP.Time)==5 &
    year(HICP.Time)==2018),:);
HICPTime=HICP.Time;
HICP_=HICP.Variables;
HICP_=HICP_(1:end-1);
DeltaHICP=HICP.Variables;
DeltaHICP=diff(DeltaHICP)./HICP_;
VarI=var(DeltaHICP);
SigmaI=sqrt(VarI);
alpha_up=chi2inv(0.975,size(DeltaHICP,1)-1);
alpha_low=chi2inv(0.025,size(DeltaHICP,1)-1);
ConIntI_up=sqrt(((size(DeltaHICP,1)-1)*VarI)/alpha_up);
ConIntI_low=sqrt(((size(DeltaHICP,1)-1)*VarI)/alpha_low);
%Nominal Calibration
load NominalStripIT.mat Dist_M
MaturityZCB=[3 5 10 20 max(max(Dist_M))];
Time=NominalFWDIT.Time;
FWDrate=NominalFWDIT.Variables;
%Reduced Sample
FWDrate=FWDrate(1:find(Time==datetime('01-May-2018')),:);
Time=Time(1:size(FWDrate,1));
%ZCB Price
for i=1:size(FWDrate,2)
for j=1:size(FWDrate,1)
```

```
ZCBprice(j,i)=exp(-FWDrate(j,i)*MaturityZCB(i));
end
end
%Delta FWD
for i=1:size(FWDrate,2)
DeltaFWDrate(:,i)=diff(FWDrate(:,i));
end
%Sigma_n a_n
ZCBchange=diff(ZCBprice)./ZCBprice(1:end-1,:);
VarZCBchange=var(ZCBchange);
[xData, yData] = prepareCurveData( MaturityZCB, VarZCBchange );
ft = fittype( '(sigma^2*(exp(-a*x)-1)^2*(1/365))/(a^2)', 'independent', 'x', 'dependent
    ', 'y' );
opts = fitoptions( 'Method', 'NonlinearLeastSquares' );
opts.Algorithm = 'Levenberg-Marquardt';
opts.Display = 'Off';
opts.Robust = 'Bisquare';
opts.StartPoint = [0.970592781760616 0.957166948242946];
[fitresult, gof] = fit( xData, yData, ft, opts );
Parameter=coeffvalues(fitresult);
ConfInt_N=confint(fitresult);
SigmaN=Parameter(2);
aN=Parameter(1);
%Correlation r-I
Time=RealFWDIT.Time;
FWDr=RealFWDIT.Variables;
FWDr=FWDr(1:find(Time==datetime('01-May-2018')),:);
Time=Time(1:size(FWDrate,1));
Deltar=diff(FWDr(:,1));
DeltaHICP=timetable(DeltaHICP(1:end-1),'RowTimes',HICPTime(2:end-1,:));
DeltaHICPv=DeltaHICP.Variables;
DeltaHICPt=DeltaHICP.Time;
Deltar=timetable(Deltar,'RowTimes',Time(2:end,:));
FWDrd=Deltar.Variables;
Timerd=Deltar.Time;
for j=1:size(DeltaHICP,1)
if find(Timerd==DeltaHICPt(j))>0;
Deltarr(j,1)=FWDrd(find(Timerd==DeltaHICPt(j)),1);
elseif find(Timerd==DeltaHICPt(j))==0;
Deltarr(j,1)=FWDrd(find(datenum(Timerd)==datenum((DeltaHICPt(j)+1))),1);
else find(Timerd==DeltaHICPt(j)+1)==0;
Deltarr(j,1)=FWDrd(find(Timerd==DeltaHICPt(j)+2),1);
end
end
Rho_rI=corr(Deltarr,DeltaHICPv);
%Correlation n-I
FWDn=NominalFWDIT.Variables;
FWDn=FWDn(1:find(Time==datetime('01-May-2018')),:);
```

```
Deltan=diff(FWDn(:,1));
Deltan=timetable(Deltan,'RowTimes',Time(2:end,:));
FWDnd=Deltan.Variables;
Timend=Deltan.Time;
for j=1:size(DeltaHICP,1)
if find(Timend==DeltaHICPt(j))>0;
Deltann(j,1)=FWDrd(find(Timend==DeltaHICPt(j)),1);
elseif find(Timend==DeltaHICPt(j))==0;
Deltann(j,1)=FWDnd(find(datenum(Timend)==datenum((DeltaHICPt(j)+1))),1);
else find(Timend==DeltaHICPt(j)+1)==0;
Deltann(j,1)=FWDnd(find(Timend==DeltaHICPt(j)+2),1);
end
end
Rho_nI=corr(Deltann,DeltaHICPv);
%Correlation r-n
Rho_rn=corr(FWDrd,FWDnd);
save Parameters.mat aR aN SigmaR SigmaN SigmaI Rho_nI Rho_rI Rho_rn
```

A.5 Simulation

We simulate forward rate evolutions with parameters obtained after the reduced calibration. Firstly, we compute the volatility functions. Secondly, we describe a stochastic differential equation for each rate. We run simulations of these SDE objects, in order to retrieve a single trajectory and the quantile confidence intervals. The procedure is repeated for each of the ten rates. Finally, we plot 10 subplots, in each of which we find the observed evolution, the simulated trajectory and the quantiles.

```
clear;
clc;
%%Nominal Simulation 1/5/18-13/4/20
load Parameters.mat%Reduced Parameters
load NominalStripIT.mat BTP Dist_M
load FWDrate.mat
MaturityZCBn=[3 5 10 20 max(max(Dist_M))];
Time=BTP.Time;
Ref=find(Time==datetime('01-May-2018'));
FWDn=NominalFWDIT.Variables;
nPeriods=size(Time,1)-find(Time==datetime('01-May-2018'));
dt=1/365;
%Volatility Function
for j=1:size(MaturityZCBn,2)
VolFunN(j,1)=SigmaN*exp(-aN*MaturityZCBn(j));
end
rng('default')
%Market Price Risk
```

```
DeltaFWDn=diff(FWDn(1:Ref,:),1)/dt;
Distr03n=fitdist(DeltaFWDn(:,1),'Normal');
ExpValn(1,1)=Distr03n.mu;
Distr35n=fitdist(DeltaFWDn(:,2),'Normal');
ExpValn(2,1)=Distr35n.mu;
Distr510n=fitdist(DeltaFWDn(:,3),'Normal');
ExpValn(3,1)=Distr510n.mu;
Distr1020n=fitdist(DeltaFWDn(:,4),'Normal');
ExpValn(4,1)=Distr1020n.mu;
Distr2030n=fitdist(DeltaFWDn(:,5),'Normal');
ExpValn(5,1)=Distr2030n.mu;
ExpValn=ExpValn./VolFunN;
Driftn=diag(MaturityZCBn.*VolFunN);
[xData, yData] = prepareCurveData( Driftn, ExpValn );
ft = fittype( 'x-phi', 'independent', 'x', 'dependent', 'y' );
opts = fitoptions( 'Method', 'NonlinearLeastSquares' );
opts.Algorithm = 'Levenberg-Marquardt';
opts.Display = 'Off';
opts.Robust = 'Bisquare';
opts.StartPoint = 0.63235924622541;
[fitresult, gof] = fit( xData, yData, ft, opts );
MktPriceRskn=fitresult.phi;
%SDE 0-3 years
drift03n=drift(0,VolFunN(1)^2*MaturityZCBn(1)-VolFunN(1)*MktPriceRskn);
diffusion03n=diffusion(0,VolFunN(1));
FWD03n=sdeddo(drift03n,diffusion03n,'StartState',FWDn(find(Time==datetime('01-May-2018'
    )),1));
FWD03nSim=simulate(FWD03n,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD03nTraj=simulate(FWD03n,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD03nSim=squeeze(FWD03nSim);
Time03nsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD03nSim,1)
ConfInt03n(j,:)=quantile(FWD03nSim(j,:),[0.005 0.5 0.995]);
end
%SDE 3-5 years
drift35n=drift(0,VolFunN(2)^2*MaturityZCBn(2)-VolFunN(2)*MktPriceRskn);
diffusion35n=diffusion(0,VolFunN(2));
FWD35n=sdeddo(drift35n,diffusion35n,'StartState',FWDn(find(Time==datetime('01-May-2018'
    )),2));
FWD35nSim=simulate(FWD35n,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD35nTraj=simulate(FWD35n,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD35nSim=squeeze(FWD35nSim);
Time35nsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD35nSim,1)
ConfInt35n(j,:)=quantile(FWD35nSim(j,:),[0.005 0.5 0.995]);
end
%SDE 5-10 years
drift510n=drift(0,VolFunN(3)^2*MaturityZCBn(3)-VolFunN(3)*MktPriceRskn);
```

```
diffusion510n=diffusion(0,VolFunN(3));
FWD510n=sdeddo(drift510n,diffusion510n,'StartState',FWDn(find(Time==datetime('01-May
    -2018')),3));
FWD510nSim=simulate(FWD510n,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD510nTraj=simulate(FWD510n,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD510nSim=squeeze(FWD510nSim);
Time510nsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD510nSim,1)
ConfInt510n(j,:)=quantile(FWD510nSim(j,:),[0.005 0.5 0.995]);
end
%SDE 10-20 years
drift1020n=drift(0,VolFunN(4)^2*MaturityZCBn(4)-VolFunN(4)*MktPriceRskn);
diffusion1020n=diffusion(0,VolFunN(4));
FWD1020n=sdeddo(drift1020n,diffusion1020n,'StartState',FWDn(find(Time==datetime('01-May
    -2018')),4));
FWD1020nSim=simulate(FWD1020n,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD1020nTraj=simulate(FWD1020n,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD1020nSim=squeeze(FWD1020nSim);
Time1020nsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD1020nSim,1)
ConfInt1020n(j,:)=quantile(FWD1020nSim(j,:),[0.005 0.5 0.995]);
end
%SDE 20-30 years
drift2030n=drift(0,VolFunN(5)^2*MaturityZCBn(5)-VolFunN(5)*MktPriceRskn);
diffusion2030n=diffusion(0,VolFunN(5));
FWD2030n=sdeddo(drift2030n,diffusion2030n,'StartState',FWDn(find(Time==datetime('01-May
    -2018')),5));
FWD2030nSim=simulate(FWD2030n,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD2030nTraj=simulate(FWD2030n,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD2030nSim=squeeze(FWD2030nSim);
Time2030nsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD2030nSim,1)
ConfInt2030n(j,:)=quantile(FWD2030nSim(j,:),[0.005 0.5 0.995]);
end
%%Real Simulation 1/5/18-13/4/20
load RealStripIT.mat BTPi Dist_M
MaturityZCBr=[3 5 10 20 30];
Time=BTPi.Time;
FWDr=RealFWDIT.Variables;
nPeriods=size(Time,1)-find(Time==datetime('01-May-2018'));
dt=1/365;
%Volatility Function
for j=1:size(MaturityZCBr,2)
VolFunR(j,1)=SigmaR*exp(-aR*MaturityZCBr(j));
end
rng('default')
%Market Price of Risk
DeltaFWDr=diff(FWDr(1:Ref,:),1)/dt;
```

```
Distr03r=fitdist(DeltaFWDr(:,1),'Normal');
ExpValr(1,1)=Distr03r.mu;
Distr35r=fitdist(DeltaFWDr(:,2),'Normal');
ExpValr(2,1)=Distr35r.mu;
Distr510r=fitdist(DeltaFWDr(:,3),'Normal');
ExpValr(3,1)=Distr510r.mu;
Distr1020r=fitdist(DeltaFWDr(:,4),'Normal');
ExpValr(4,1)=Distr1020r.mu;
Distr2030r=fitdist(DeltaFWDr(:,5),'Normal');
ExpValr(5,1)=Distr2030r.mu;
ExpValr=ExpValr./SigmaR;
Driftr=diag(MaturityZCBr.*VolFunR-(Rho_rI*SigmaI));
[xData, yData] = prepareCurveData( Driftr, ExpValr );
ft = fittype( 'x-phi', 'independent', 'x', 'dependent', 'y' );
opts = fitoptions( 'Method', 'NonlinearLeastSquares' );
opts.Algorithm = 'Levenberg-Marquardt';
opts.Display = 'Off';
opts.Robust = 'Bisquare';
opts.StartPoint = 0.905791937075619;
[fitresult, gof] = fit( xData, yData, ft, opts );
MktPriceRskr=fitresult.phi;
%SDE 0-3 years
drift03r=drift(0,VolFunR(1)^2*MaturityZCBr(1)-VolFunR(1)*Rho_rI*SigmaI-VolFunR(1)*
    MktPriceRskr);
diffusion03r=diffusion(0,VolFunR(1));
FWD03r=sdeddo(drift03r,diffusion03r,'StartState',FWDr(find(Time==datetime('01-May-2018'
    )),1));
FWD03rSim=simulate(FWD03r,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD03rTraj=simulate(FWD03r,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD03rSim=squeeze(FWD03rSim);
Time03rsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD03rSim,1)
ConfInt03r(j,:)=quantile(FWD03rSim(j,:),[0.005 0.5 0.995]);
end
%SDE 3-5 years
drift35r=drift(0,(VolFunR(2)^2*MaturityZCBr(2)-VolFunR(2)*Rho_rI*SigmaI-VolFunR(2)*
    MktPriceRskr));
diffusion35r=diffusion(0,VolFunR(2));
FWD35r=sdeddo(drift35r,diffusion35r,'StartState',FWDr(find(Time==datetime('01-May-2018'
    )),2));
FWD35rSim=simulate(FWD35r,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD35rTraj=simulate(FWD35r,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD35rSim=squeeze(FWD35rSim);
Time35rsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD35rSim,1)
ConfInt35r(j,:)=quantile(FWD35rSim(j,:),[0.005 0.5 0.995]);
end
%SDE 5-10 years
```

```
drift510r=drift(0,(VolFunR(3)^2*MaturityZCBr(3)-VolFunR(3)*Rho_rI*SigmaI-VolFunR(3)*
    MktPriceRskr));
diffusion510r=diffusion(0,VolFunR(3));
FWD510r=sdeddo(drift510r,diffusion510r,'StartState',FWDr(find(Time==datetime('01-May
    -2018')),3));
FWD510rSim=simulate(FWD510r,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD510rTraj=simulate(FWD510r,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD510rSim=squeeze(FWD510rSim);
Time510rsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD510rSim,1)
ConfInt510r(j,:)=quantile(FWD510rSim(j,:),[0.005 0.5 0.995]);
end
%SDE 10-20 years
drift1020r=drift(0,(VolFunR(4)^2*MaturityZCBr(4)-VolFunR(4)*Rho_rI*SigmaI-VolFunR(4)*
    MktPriceRskr));
diffusion1020r=diffusion(0,VolFunR(4));
FWD1020r=sdeddo(drift1020r,diffusion1020r,'StartState',FWDr(find(Time==datetime('01-May
    -2018')),4));
FWD1020rSim=simulate(FWD1020r,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD1020rTraj=simulate(FWD1020r,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD1020rSim=squeeze(FWD1020rSim);
Time1020rsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD1020rSim,1)
ConfInt1020r(j,:)=quantile(FWD1020rSim(j,:),[0.005 0.5 0.995]);
end
%SDE 20-30 years
drift2030r=drift(0,(VolFunR(5)^2*MaturityZCBr(5)-VolFunR(4)*Rho_rI*SigmaI-VolFunR(5)*
    MktPriceRskr));
diffusion2030r=diffusion(0,VolFunR(5));
FWD2030r=sdeddo(drift2030r,diffusion2030r,'StartState',FWDr(find(Time==datetime('01-May
    -2018')),5));
FWD2030rSim=simulate(FWD2030r,nPeriods,'DeltaTime',dt,'nTrials',100000);
FWD2030rTraj=simulate(FWD2030r,nPeriods,'DeltaTime',dt,'nTrials',1);
FWD1020rSim=squeeze(FWD2030rSim);
Time2030rsim=Time(find(Time==datetime('01-May-2018')):end,1);
for j=1:size(FWD2030rSim,1)
ConfInt2030r(j,:)=quantile(FWD2030rSim(j,:),[0.005 0.5 0.995]);
end

%Confidence Interval Normal Distribution
%SDE 0-3 years
ConfInt03nn(1,1:3)=FWDn(find(Time==datetime('01-May-2018')),1);
for j=1:nPeriods
mu03(j,1)=FWDn(find(Time==datetime('01-May-2018')),1)+(VolFunN(1)^2*MaturityZCBn(1)-
    VolFunN(1)*MktPriceRskn)*j*dt;
sigma03(j,1)=sqrt(VolFunN(1)^2*j*dt);
ConfInt03nn(j+1,1)=norminv(0.005,mu03(j,1),sigma03(j,1));
ConfInt03nn(j+1,2)=norminv(0.5,mu03(j,1),sigma03(j,1));
```

```
ConfInt03nn(j+1,3)=norminv(0.995,mu03(j,1),sigma03(j,1));
end
%SDE 3-5 years
ConfInt35nn(1,1:3)=FWDn(find(Time==datetime('01-May-2018')),2);
for j=1:nPeriods
mu35(j,1)=FWDn(find(Time==datetime('01-May-2018')),2)+(VolFunN(2)^2*MaturityZCBn(2)-
    VolFunN(2)*MktPriceRskn)*j*dt;
sigma35(j,1)=sqrt(VolFunN(2)^2*j*dt);
ConfInt35nn(j+1,1)=norminv(0.005,mu35(j,1),sigma35(j,1));
ConfInt35nn(j+1,2)=norminv(0.5,mu35(j,1),sigma35(j,1));
ConfInt35nn(j+1,3)=norminv(0.995,mu35(j,1),sigma35(j,1));
end
%SDE 5-10 years
ConfInt510nn(1,1:3)=FWDn(find(Time==datetime('01-May-2018')),3);
for j=1:nPeriods
mu510(j,1)=FWDn(find(Time==datetime('01-May-2018')),3)+(VolFunN(3)^2*MaturityZCBn(3)-
    VolFunN(3)*MktPriceRskn)*j*dt;
sigma510(j,1)=sqrt(VolFunN(3)^2*j*dt);
ConfInt510nn(j+1,1)=norminv(0.005,mu510(j,1),sigma510(j,1));
ConfInt510nn(j+1,2)=norminv(0.5,mu510(j,1),sigma510(j,1));
ConfInt510nn(j+1,3)=norminv(0.995,mu510(j,1),sigma510(j,1));
end
%SDE 10-20 years
ConfInt1020nn(1,1:3)=FWDn(find(Time==datetime('01-May-2018')),4);
for j=1:nPeriods
mu1020(j,1)=FWDn(find(Time==datetime('01-May-2018')),4)+(VolFunN(4)^2*MaturityZCBn(4)-
    VolFunN(4)*MktPriceRskn)*j*dt;
sigma1020(j,1)=sqrt(VolFunN(4)^2*j*dt);
ConfInt1020nn(j+1,1)=norminv(0.005,mu1020(j,1),sigma1020(j,1));
ConfInt1020nn(j+1,2)=norminv(0.5,mu1020(j,1),sigma1020(j,1));
ConfInt1020nn(j+1,3)=norminv(0.995,mu1020(j,1),sigma1020(j,1));
end
%SDE 20-30 years
ConfInt2030nn(1,1:3)=FWDn(find(Time==datetime('01-May-2018')),5);
for j=1:nPeriods
mu2030(j,1)=FWDn(find(Time==datetime('01-May-2018')),5)+(VolFunN(5)^2*MaturityZCBn(5)-
    VolFunN(5)*MktPriceRskn)*j*dt;
sigma2030(j,1)=sqrt(VolFunN(5)^2*j*dt);
ConfInt2030nn(j+1,1)=norminv(0.005,mu2030(j,1),sigma2030(j,1));
ConfInt2030nn(j+1,2)=norminv(0.5,mu2030(j,1),sigma2030(j,1));
ConfInt2030nn(j+1,3)=norminv(0.995,mu2030(j,1),sigma2030(j,1));
end
%SDE 0-3 years
ConfInt03rn(1,1:3)=FWDr(find(Time==datetime('01-May-2018')),1);
for j=1:nPeriods
mu03(j,1)=FWDr(find(Time==datetime('01-May-2018')),1)+(VolFunR(1)^2*MaturityZCBr(1)-
    VolFunR(1)*MktPriceRskr)*j*dt;
sigma03(j,1)=sqrt(VolFunR(1)^2*j*dt);
```

```
ConfInt03rn(j+1,1)=norminv(0.005,mu03(j,1),sigma03(j,1));
ConfInt03rn(j+1,2)=norminv(0.5,mu03(j,1),sigma03(j,1));
ConfInt03rn(j+1,3)=norminv(0.995,mu03(j,1),sigma03(j,1));
end
%SDE 3-5 years
ConfInt35rn(1,1:3)=FWDr(find(Time==datetime('01-May-2018')),2);
for j=1:nPeriods
mu35(j,1)=FWDr(find(Time==datetime('01-May-2018')),2)+(VolFunR(2)^2*MaturityZCBr(2)-
    VolFunR(2)*MktPriceRskr)*j*dt;
sigma35(j,1)=sqrt(VolFunR(2)^2*j*dt);
ConfInt35rn(j+1,1)=norminv(0.005,mu35(j,1),sigma35(j,1));
ConfInt35rn(j+1,2)=norminv(0.5,mu35(j,1),sigma35(j,1));
ConfInt35rn(j+1,3)=norminv(0.995,mu35(j,1),sigma35(j,1));
end
%SDE 5-10 years
ConfInt510rn(1,1:3)=FWDr(find(Time==datetime('01-May-2018')),3);
for j=1:nPeriods
mu510(j,1)=FWDr(find(Time==datetime('01-May-2018')),3)+(VolFunR(3)^2*MaturityZCBr(3)-
    VolFunR(3)*MktPriceRskr)*j*dt;
sigma510(j,1)=sqrt(VolFunR(3)^2*j*dt);
ConfInt510rn(j+1,1)=norminv(0.005,mu510(j,1),sigma510(j,1));
ConfInt510rn(j+1,2)=norminv(0.5,mu510(j,1),sigma510(j,1));
ConfInt510rn(j+1,3)=norminv(0.995,mu510(j,1),sigma510(j,1));
end
%SDE 10-20 years
ConfInt1020rn(1,1:3)=FWDr(find(Time==datetime('01-May-2018')),4);
for j=1:nPeriods
mu1020(j,1)=FWDr(find(Time==datetime('01-May-2018')),4)+(VolFunR(4)^2*MaturityZCBr(4)-
    VolFunR(4)*MktPriceRskr)*j*dt;
sigma1020(j,1)=sqrt(VolFunR(4)^2*j*dt);
ConfInt1020rn(j+1,1)=norminv(0.005,mu1020(j,1),sigma1020(j,1));
ConfInt1020rn(j+1,2)=norminv(0.5,mu1020(j,1),sigma1020(j,1));
ConfInt1020rn(j+1,3)=norminv(0.995,mu1020(j,1),sigma1020(j,1));
end
%SDE 20-30 years
ConfInt2030rn(1,1:3)=FWDr(find(Time==datetime('01-May-2018')),5);
for j=1:nPeriods
mu2030(j,1)=FWDr(find(Time==datetime('01-May-2018')),5)+(VolFunR(5)^2*MaturityZCBr(5)-
    VolFunR(5)*MktPriceRskr)*j*dt;
sigma2030(j,1)=sqrt(VolFunR(5)^2*j*dt);
ConfInt2030rn(j+1,1)=norminv(0.005,mu2030(j,1),sigma2030(j,1));
ConfInt2030rn(j+1,2)=norminv(0.5,mu2030(j,1),sigma2030(j,1));
ConfInt2030rn(j+1,3)=norminv(0.995,mu2030(j,1),sigma2030(j,1));
end
%%Plots
figure;
subplot(5,2,1)
plot(Time,FWDn(:,1));
```

```
hold on
plot(Time03nsim,FWD03nTraj,'red');
plot(Time03nsim,ConfInt03n(:,1),'k--');
plot(Time03nsim,ConfInt03n(:,2),'k--');
plot(Time03nsim,ConfInt03n(:,3),'k--');
title('0-3 year Nominal FWD rate: Simulation');
hold off
subplot(5,2,3)
plot(Time,FWDn(:,2));
hold on
plot(Time35nsim,FWD35nTraj,'red');
plot(Time35nsim,ConfInt35n(:,1),'k--');
plot(Time35nsim,ConfInt35n(:,2),'k--');
plot(Time35nsim,ConfInt35n(:,3),'k--');
title('3-5 year Nominal FWD rate: Simulation');
hold off
subplot(5,2,5)
plot(Time,FWDn(:,3));
hold on
plot(Time510nsim,FWD510nTraj,'red');
plot(Time510nsim,ConfInt510n(:,1),'k--');
plot(Time510nsim,ConfInt510n(:,2),'k--');
plot(Time510nsim,ConfInt510n(:,3),'k--');
title('5-10 year Nominal FWD rate: Simulation');
hold off
subplot(5,2,7)
plot(Time,FWDn(:,4));
hold on
plot(Time1020nsim,FWD1020nTraj,'red');
plot(Time1020nsim,ConfInt1020n(:,1),'k--');
plot(Time1020nsim,ConfInt1020n(:,2),'k--');
plot(Time1020nsim,ConfInt1020n(:,3),'k--');
title('10-20 year Nominal FWD rate: Simulation');
subplot(5,2,9)
plot(Time,FWDn(:,5));
hold on
plot(Time2030nsim,FWD2030nTraj,'red');
plot(Time2030nsim,ConfInt2030n(:,1),'k--');
plot(Time2030nsim,ConfInt2030n(:,2),'k--');
plot(Time2030nsim,ConfInt2030n(:,3),'k--');
title('20-30 year Nominal FWD rate: Simulation');
subplot(5,2,2)
plot(Time,FWDr(:,1));
hold on
plot(Time03rsim,FWD03rTraj,'red');
plot(Time03rsim,ConfInt03r(:,1),'k--');
plot(Time03rsim,ConfInt03r(:,2),'k--');
plot(Time03rsim,ConfInt03r(:,3),'k--');
```

```
title('0-3 year Real FWD rate: Simulation');
hold off
subplot(5,2,4)
plot(Time,FWDr(:,2));
hold on
plot(Time35rsim,FWD35rTraj,'red');
plot(Time35rsim,ConfInt35r(:,1),'k--');
plot(Time35rsim,ConfInt35r(:,2),'k--');
plot(Time35rsim,ConfInt35r(:,3),'k--');
title('3-5 year Real FWD rate: Simulation');
hold off
subplot(5,2,6)
plot(Time,FWDr(:,3));
hold on
plot(Time510rsim,FWD510rTraj,'red');
plot(Time510rsim,ConfInt510r(:,1),'k--');
plot(Time510rsim,ConfInt510r(:,2),'k--');
plot(Time510rsim,ConfInt510r(:,3),'k--');
title('5-10 year Real FWD rate: Simulation');
hold off
subplot(5,2,8)
plot(Time,FWDr(:,4));
hold on
plot(Time1020rsim,FWD1020rTraj,'red');
plot(Time1020rsim,ConfInt1020r(:,1),'k--');
plot(Time1020rsim,ConfInt1020r(:,2),'k--');
plot(Time1020rsim,ConfInt1020r(:,3),'k--');
title('10-20 year Real FWD rate: Simulation');
hold off
subplot(5,2,10)
plot(Time,FWDr(:,5));
hold on
plot(Time2030rsim,FWD2030rTraj,'red');
plot(Time2030rsim,ConfInt2030r(:,1),'k--');
plot(Time2030rsim,ConfInt2030r(:,2),'k--');
plot(Time2030rsim,ConfInt2030r(:,3),'k--');
title('20-30 year Real FWD rate: Simulation');
hold off
%%Plots Confidence Interval Normal Distribution
figure;
subplot(5,2,1)
plot(Time,FWDn(:,1));
hold on
plot(Time03nsim,FWD03nTraj,'red');
plot(Time03nsim,ConfInt03nn(:,1),'k--');
plot(Time03nsim,ConfInt03nn(:,2),'k--');
plot(Time03nsim,ConfInt03nn(:,3),'k--');
title('0-3 year Nominal FWD rate: Simulation');
```

```
hold off
subplot(5,2,3)
plot(Time,FWDn(:,2));
hold on
plot(Time35nsim,FWD35nTraj,'red');
plot(Time35nsim,ConfInt35nn(:,1),'k--');
plot(Time35nsim,ConfInt35nn(:,2),'k--');
plot(Time35nsim,ConfInt35nn(:,3),'k--');
title('3-5 year Nominal FWD rate: Simulation');
hold off
subplot(5,2,5)
plot(Time,FWDn(:,3));
hold on
plot(Time510nsim,FWD510nTraj,'red');
plot(Time510nsim,ConfInt510nn(:,1),'k--');
plot(Time510nsim,ConfInt510nn(:,2),'k--');
plot(Time510nsim,ConfInt510nn(:,3),'k--');
title('5-10 year Nominal FWD rate: Simulation');
hold off
subplot(5,2,7)
plot(Time,FWDn(:,4));
hold on
plot(Time1020nsim,FWD1020nTraj,'red');
plot(Time1020nsim,ConfInt1020nn(:,1),'k--');
plot(Time1020nsim,ConfInt1020nn(:,2),'k--');
plot(Time1020nsim,ConfInt1020nn(:,3),'k--');
title('10-20 year Nominal FWD rate: Simulation');
subplot(5,2,9)
plot(Time,FWDn(:,5));
hold on
plot(Time2030nsim,FWD2030nTraj,'red');
plot(Time2030nsim,ConfInt2030nn(:,1),'k--');
plot(Time2030nsim,ConfInt2030nn(:,2),'k--');
plot(Time2030nsim,ConfInt2030nn(:,3),'k--');
title('20-30 year Nominal FWD rate: Simulation');
subplot(5,2,2)
plot(Time,FWDr(:,1));
hold on
plot(Time03rsim,FWD03rTraj,'red');
plot(Time03rsim,ConfInt03rn(:,1),'k--');
plot(Time03rsim,ConfInt03rn(:,2),'k--');
plot(Time03rsim,ConfInt03rn(:,3),'k--');
title('0-3 year Real FWD rate: Simulation');
hold off
subplot(5,2,4)
plot(Time,FWDr(:,2));
hold on
plot(Time35rsim,FWD35rTraj,'red');
```

```
plot(Time35rsim,ConfInt35rn(:,1),'k--');
plot(Time35rsim,ConfInt35rn(:,2),'k--');
plot(Time35rsim,ConfInt35rn(:,3),'k--');
title('3-5 year Real FWD rate: Simulation');
hold off
subplot(5,2,6)
plot(Time,FWDr(:,3));
hold on
plot(Time510rsim,FWD510rTraj,'red');
plot(Time510rsim,ConfInt510rn(:,1),'k--');
plot(Time510rsim,ConfInt510rn(:,2),'k--');
plot(Time510rsim,ConfInt510rn(:,3),'k--');
title('5-10 year Real FWD rate: Simulation');
hold off
subplot(5,2,8)
plot(Time,FWDr(:,4));
hold on
plot(Time1020rsim,FWD1020rTraj,'red');
plot(Time1020rsim,ConfInt1020rn(:,1),'k--');
plot(Time1020rsim,ConfInt1020rn(:,2),'k--');
plot(Time1020rsim,ConfInt1020rn(:,3),'k--');
title('10-20 year Real FWD rate: Simulation');
hold off
subplot(5,2,10)
plot(Time,FWDr(:,5));
hold on
plot(Time2030rsim,FWD2030rTraj,'red');
plot(Time2030rsim,ConfInt2030rn(:,1),'k--');
plot(Time2030rsim,ConfInt2030rn(:,2),'k--');
plot(Time2030rsim,ConfInt2030rn(:,3),'k--');
title('20-30 year Real FWD rate: Simulation');
hold off
save Parameters.mat MktPriceRskn MktPriceRskr -append
```

A.6 Inflation Simulation and Cost Estimation

We select bonds in the May 2018-May 2019 time window and their reference bonds. We compute the coupon rate differentials and the coupon payment dates to obtain the steam of excess cash flows due. Then, we simulate short nominal and real rates. We employ simulated rates to simulate the evolution of inflation on monthly basis over the next 30 years. Finally, we multiply each cash flow by its revaluation coefficient, in order to obtain a figure of the "excess" cost incurred at March 2020 prices.

```
load FWDrate.mat
%BTP traded
```

```
[Prezzo,text]=xlsread('ITtraded.xlsx','BTP');
ISIN_BTP=text(1,2:end);
Coupon_BTP=Prezzo(2,2:end)/100;
Maturity_BTP=datetime(datestr(Prezzo(3,2:end)+datenum('30-Dec-1899')));
Issue_BTP=datetime(datestr(Prezzo(4,2:end)+datenum('30-Dec-1899')));
Time=Prezzo(5:end,1); % select original data in Excel format
Time = datenum('30-Dec-1899') + Time; % convert data in Matlab format
y=[Time Prezzo(5:end,2:end)];
Titoli_BTP=array2timetable(y(:,2:end),'RowTimes',datetime(datestr(y(:,1))),'VariableNames',ISIN_BTP);
Titoli_BTP=Titoli_BTP(find(datenum(Titoli_BTP.Time)==datenum(datetime('14-Oct-2015'))):end,:);
AmountIssued=Prezzo(1,2:end)';
BTP=[datenum(Issue_BTP) datenum(Maturity_BTP) yearfrac(Issue_BTP,Maturity_BTP) Coupon_BTP' AmountIssued];
BTP(:,3)=round(BTP(:,3));
BTP=sortrows(BTP,[3,1]);
Issue=datetime(datestr(BTP(:,1)));
Maturity=datetime(datestr(BTP(:,2)));
for i=1:size(BTP,1)
for j=1:(year(Maturity(i))-year(Issue(i)))*2
CD(j,1,i)=Issue(i) + calmonths(6*j);
if datenum(Maturity(i))-datenum(CD(j,1,i))<0
    CD(j,1,i)=NaT;
end
if year(CD(j,1,i))==year(Maturity(i)) && month(Maturity(i))-month(CD(j,1,i))<6
    break
end
end
end
BTP_2016=BTP(find(Issue>datetime('01-Jan-2016')),:);
CD_2016=CD(:,:,find(Issue>datetime('01-Jan-2016')));
Time=datetime(datestr(BTP_2016(:,1)));
BTP_2016=timetable(BTP_2016(:,3:end),'RowTimes',Time);
BTP_2016=BTP_2016(1:end-1,:);
CD_2016=CD_2016(:,:,1:end-1);
Var=BTP_2016.Variables;
Coupon=Var(:,2);
Amount=Var(:,3);
%Inflation Data
[Inflazione]=xlsread('Inflation.xls','HICP');
HICP=Inflazione(:,2);
DateINF=Inflazione(:,1); % select original data in Excel format
DateINF = datenum('30-Dec-1899') + DateINF; % convert data in Matlab format
name={'HICP'};
HICP=array2timetable(HICP,'RowTimes',datetime(datestr(DateINF(:,1))),...
    'VariableNames',name);
HICP=HICP(find(HICP.Time==datetime('15-Sep-2015')):end-1,:);
```

```
HICPVar=HICP.Variables;
HICPTime=datenum(HICP.Time);
%%Inflation Simulation
load Parameters.mat %Reduced Calibration
MaturityZCB=[3];
%Nominal Rates
FWDn=NominalFWDIT.Variables;
Time=NominalFWDIT.Time;
nPeriods=months(datenum(max(NominalFWDIT.Time)),datenum(max(max(CD_2016))));
dt=1/12;
%Volatility Function
for j=1:size(MaturityZCB,2)
VolFunN(j,1)=SigmaN*exp(-aN*MaturityZCB(j));
end
rng('default')
%SDE Nominal Short Rate
drift03n=drift(0,VolFunN(1)^2*MaturityZCB(1)-VolFunN(1)*MktPriceRskn);
diffusion03n=diffusion(0,VolFunN(1));
FWD03n=sdeddo(drift03n,diffusion03n,'StartState',FWDn(end,1));
FWD03nSim=simulate(FWD03n,nPeriods,'DeltaTime',dt,'nTrials',1000);
FWD03nSim=squeeze(FWD03nSim);
FWD03nSim=mean(FWD03nSim,2);
%Real Rates
FWDr=RealFWDIT.Variables;
%Volatility Function
for j=1:size(MaturityZCB,2)
VolFunR(j,1)=SigmaR*exp(-aR*MaturityZCB(j));
end
rng('default')
%SDE 0-3 years
drift03r=drift(0,VolFunR(1)^2*MaturityZCB(1)-VolFunR(1)*Rho_rI*SigmaI-VolFunR(1)*MktPriceRskr);
diffusion03r=diffusion(0,VolFunR(1));
FWD03r=sdeddo(drift03r,diffusion03r,'StartState',FWDr(end,1));
FWD03rSim=simulate(FWD03r,nPeriods,'DeltaTime',dt,'nTrials',1000);
FWD03rSim=squeeze(FWD03rSim);
FWD03rSim=mean(FWD03rSim,2);
%Delta Fisher
DeltaRates=FWD03nSim-FWD03rSim;
for j=1:size(DeltaRates,1)
DeltaDates(j,1)=datetime('15-Mar-2020')+calmonths(j);
end
DeltaRates=timetable(DeltaRates,'RowTimes',DeltaDates);
DeltaRatesVar=DeltaRates.Variables;
%Inflation Market Price of Risk
HICPreduced=HICPVar(1:find(HICP.Time==datetime('15-May-2018')));
DeltaInf=(diff(HICPreduced)./HICPreduced(1:end-1));
DeltaInf=DeltaInf./dt;
```

```
DistrInf=fitdist(DeltaInf,'Normal');
EVinflation=DistrInf.mu;
DistrDeltaRates=fitdist(DeltaRatesVar,'Normal');
EVdeltarates=DistrDeltaRates.mu;
MktPriceRskI=(EVdeltarates-EVinflation)./SigmaI;
%Inflation Simulation
InflationSim(1,1)=HICPVar(end,1);
for j=1:size(DeltaRates,1)
InflationSDE=gbm(DeltaRatesVar(j,1)-(MktPriceRskI*SigmaI),SigmaI,'StartState',
    InflationSim(j,1));
InflationSimSDE=simulate(InflationSDE,1,'DeltaTime',dt,'nTrials',1000);
InflationSimSDE=squeeze(InflationSimSDE);
InflationConfInt(j,:)=quantile(InflationSimSDE(2,:),[0.025 0.975]);
InflationSim(j+1,1)=mean(InflationSimSDE(2,:));
end
Inflationfull=[HICPVar; InflationSim(2:end)];
InflationTime=[HICP.Time; DeltaDates];
InflationFull=timetable(Inflationfull,'RowTimes', InflationTime);
figure;
hold on
plot(HICP.Time,HICPVar,'k')
plot(DeltaDates,InflationSim(2:end))
title('HICP Simulation')
%Coupon Delta net of Inflation
Ref=Inflationfull(find(month(InflationFull.Time)==3 & year(InflationFull.Time)==2020));
Diff3(:,1)=repelem(((Coupon(4)-Coupon(2))/2)*Amount(4),size(rmmissing(CD_2016(:,:,4))
    ,1))';
for j=1:size(Diff3,1)
Diff3(j,1)=Diff3(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,4)) & year(InflationFull.Time)==year(CD_2016(j,:,4)))))/Ref);
end
Diff3(:,2)=repelem(((Coupon(5)-Coupon(2))/2)*Amount(5),size(rmmissing(CD_2016(:,:,5))
    ,1))';
for j=1:size(Diff3,1)
Diff3(j,2)=Diff3(j,2)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,5)) & year(InflationFull.Time)==year(CD_2016(j,:,5)))))/Ref);
end
Diff3(7:8,:)=zeros;
Diff3(:,3)=repelem(((Coupon(8)-Coupon(2))/2)*Amount(8),size(rmmissing(CD_2016(:,:,8))
    ,1))';
for j=1:size(Diff3,1)
Diff3(j,3)=Diff3(j,3)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,8)) & year(InflationFull.Time)==year(CD_2016(j,:,8)))))/Ref);
end
%Coupon Delta 5yrs
Diff5(:,1)=repelem(((Coupon(14)-Coupon(13))/2)*Amount(14),size(rmmissing(CD_2016
    (:,:,14)),1))';
for j=1:size(Diff5,1)
```

```
Diff5(j,1)=Diff5(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,14)) & year(InflationFull.Time)==year(CD_2016(j,:,14))))/Ref);
end
Diff5(:,2)=repelem(((Coupon(15)-Coupon(13))/2)*Amount(15),size(rmmissing(CD_2016
    (:,:,15)),1))';
for j=1:size(Diff5,1)
Diff5(j,2)=Diff5(j,2)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,15)) & year(InflationFull.Time)==year(CD_2016(j,:,15))))/Ref);
end
%Coupon Delta 7yrs
Diff7(:,1)=repelem(((Coupon(24)-Coupon(23))/2)*Amount(24),size(rmmissing(CD_2016
    (:,:,24)),1))';
for j=1:size(Diff7,1)
Diff7(j,1)=Diff7(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,24)) & year(InflationFull.Time)==year(CD_2016(j,:,24))))/Ref);
end
Diff7(:,2)=repelem(((Coupon(25)-Coupon(23))/2)*Amount(25),size(rmmissing(CD_2016
    (:,:,25)),1))';
for j=1:size(Diff7,1)
Diff7(j,2)=Diff7(j,2)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,25)) & year(InflationFull.Time)==year(CD_2016(j,:,25))))/Ref);
end
%Coupon Delta 10yrs
Diff10(:,1)=repelem(((Coupon(36)-Coupon(35))/2)*Amount(36),size(rmmissing(CD_2016
    (:,:,36)),1))';
for j=1:size(Diff10,1)
Diff10(j,1)=Diff10(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,36)) & year(InflationFull.Time)==year(CD_2016(j,:,36))))/Ref);
end
Diff10(:,2)=repelem(((Coupon(37)-Coupon(35))/2)*Amount(37),size(rmmissing(CD_2016
    (:,:,37)),1))';
for j=1:size(Diff10,1)
Diff10(j,2)=Diff10(j,2)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,37)) & year(InflationFull.Time)==year(CD_2016(j,:,37))))/Ref);
end
%Coupon Delta 15yrs
Diff15(:,:)=repelem(((Coupon(40)-Coupon(42))/2)*Amount(40),size(rmmissing(CD_2016
    (:,:,40)),1))';
for j=1:size(Diff15,1)
Diff15(j,1)=Diff15(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,40)) & year(InflationFull.Time)==year(CD_2016(j,:,40))))/Ref);
end
%Coupon Delta 20yrs
Diff20(:,:)=repelem(((Coupon(45)-Coupon(44))/2)*Amount(45),size(rmmissing(CD_2016
    (:,:,45)),1))';
for j=1:size(Diff20,1)
Diff20(j,1)=Diff20(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,45)) & year(InflationFull.Time)==year(CD_2016(j,:,45))))/Ref);
```

```
end
%Coupon Delta 30yrs
Diff30(:,:)=repelem(((Coupon(48)-Coupon(47))/2)*Amount(48),size(rmmissing(CD_2016
    (:,:,48)),1))';
for j=1:size(Diff30,1)
Diff30(j,1)=Diff30(j,1)/(Inflationfull(find(month(InflationFull.Time)==month(CD_2016(j
    ,:,48)) & year(InflationFull.Time)==year(CD_2016(j,:,48)))))/Ref);
end
sum(sum(Diff3))+sum(sum(Diff5))+sum(sum(Diff7))+sum(sum(Diff10))+sum(Diff15)+sum(Diff20
    )+sum(Diff30)
```

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