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Final dissertation

Models for Cosmic Birefringence

Thesis supervisor:
Prof. Nicola Bartolo

Candidate:
Alberto Rodegheri

Thesis co-supervisor:
Dr. Alessandro Greco

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Abstract

Cosmological effects of parity violation allow us to investigate some processes that took place in the early Universe, and to shed light on new physics beyond the Standard Model. We will analyze one of the effects responsible of this violation: the Cosmic Birefringence (CB). This phenomenon is related to the in-vacuo rotation of the polarization angle (denoted as χ) of photons coming from very distant sources, and it could be explained by the presence of Dark Matter and Dark Energy fields in the early universe, and by their interaction with CMB photons (through a Chern-Simons coupling). Since it is reasonable to think that these fields are affected by some fluctuations, they can generate anisotropies in the CB angle, making it space-dependent. In this work, after an overview about the CB models proposed up to now, we will extend the anisotropic birefringence effect to the case of a Chern-Simons coupling with a function of the Ricci scalar $f(R)$, including the contribution of the δR fluctuations and computing the perturbed rotation angle $\delta\chi$. We will find that δR, and so $\delta\chi$, are directly sourced by the scalar metric perturbations Φ and Ψ (which are the Bardeen's gravitational potential in the Poisson gauge), such that the birefringence angle power spectrum, given by $C_l^{\chi\chi}$, is strictly related to one of the two gravitational potentials.

Contents

Introduction

One of the most intriguing topics studied nowadays by both cosmologists and particle physicists is the searching for Dark Matter (DM) and Dark Energy (DE), required to recover the overall matter-energy content in the Universe. Up to now many evidences of their existence have been probed and studied, from the galactic velocity curves to the accelerated expansion of the Universe. This means that it is necessary to go beyond the Standard Model of particle physics, introducing new types of *exotic* particles; indeed the current cosmological model, called ΛCDM, contains this new physics in terms of both DM and DE components.

In this thesis we will investigate another interesting phenomenon that could be explained by the presence of a dark matter (or dark energy) field, and by its interactions with photons coming from very distant sources: it is the effect of *Cosmic Birefringence* (CB). As we will see it is generated by the presence of a Chern-Simons coupling term added to the standard electromagnetic Lagrangian density, modifying in this way the equations of motion (the Maxwell equations) in a parity-violating way. Then, from this modification it is possible to show that, in the end, the effect is a rotation of the linear polarization plane as the photons propagate in spacetime. This is the same effect that takes place in liquid crystals, as shown in the scheme in *figure* 1.

Thus, Cosmic birefringence represents a further way to investigate the primordial Universe and its components, in order to discover the mechanisms behind its dynamics and evolution at very early epochs. Indeed, besides the importance of CB as a probe of primordial DM and DE fields, it could be exploited also to shed light on other processes such as the primordial baryogenesis, or to study some modified GR models, for instance the so-called $f(R)$-theories of gravity.

Let's take a quick summary about the contents of this work. First of all, we will go through a deep overview on the proposed models and computations of the birefringence angle carried out up to now in the present literature, considering both the *isotropic* and *anisotropic* cases, relying also on the observability of the effect with current and future instruments. For both cases we will analyze the results obtained by considering the coupling of the photon with a scalar field ϕ, which in general is a pseudo-Nambu-Goldstone boson, and in practice it is thought to be an axion-like field or a generic quintessence field (in particular in the context of baryogenesis models). Moreover we will see that the birefringence phenomenon arises also replacing the scalar field with a generic function of the Ricci curvature scalar $f(R)$; in this context we will extend

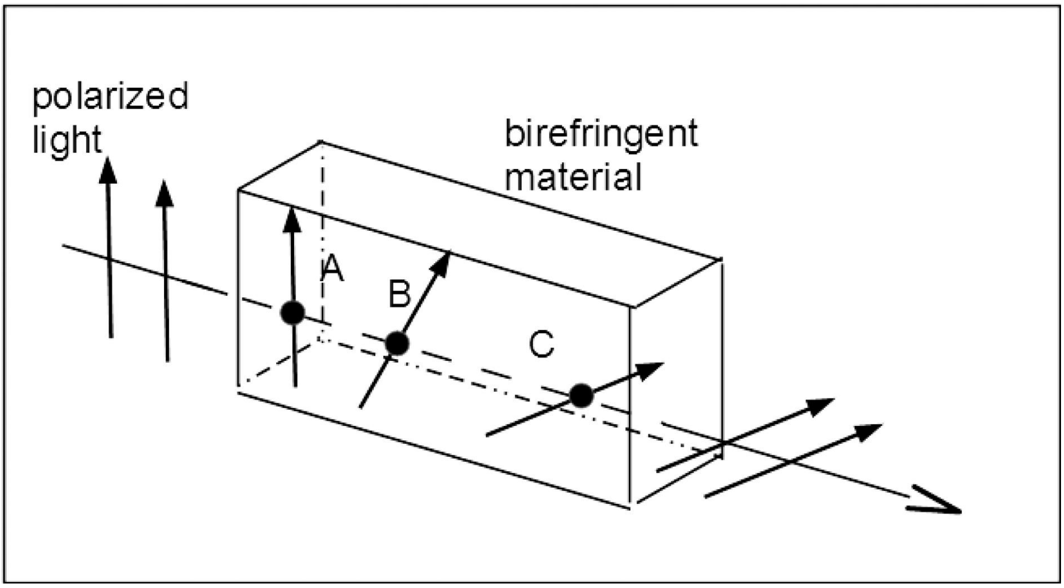

Figure 1: The effect of birefringence for a linearly polarized light that passes through a liquid crystal.

the treatment to the anisotropic case (which is not present in the literature), taking into account for the fluctuations of R, related to the perturbations of the metric. To be more precise, the thesis is organized in this way: after a short overview concerning some basic aspects about classical electrodynamics, the CMB radiation and the cosmological perturbation theory (*Chapter 1*), in *Chapter 2* we will focus on a general treatment about *isotropic* birefringence, deriving theoretically the rotation of the polarization angle in the case of a coupling with an axion-like scalar field (*Section 2.1*). Then in *Section 2.2* we will move to an overview on some observational constraints on the birefringence angle, analyzing the auto-correlation and cross-correlation power spectra, and reporting the main results obtained up to now. In *Chapter 3* we will expand the treatment to the *anisotropic* case, where the birefringence angle depends also on the direction in the sky, and not only on time; in this frame we will find out an extended expression for the polarization angle, adding the contribution of the fluctuations of the scalar field, treated by imposing both the synchronous gauge (*Section 3.1*) and the Poisson gauge (*Section 3.2*). At the end of the third chapter we will move to the analysis of the cross-spectra associated to the anisotropic birefringence angle, in order to compare the theoretical predictions with the observational data (*Section 3.3*). In *Chapter 4* we will focus on the CB effect brought by the coupling of CMB radiation with a function of the Ricci scalar $f(R)$, which replaces the scalar field ϕ in the Chern-Simons term (*Section 4.1*); then, we will see the implications of this new model in the context of baryo/leptogenesys mechanism, constraining the n_B/s ratio in dependence of the derivative of the Ricci Scalar (*Section 4.2*). Finally, in *Section 3.3*, we will extend the *f(R) birefringence* to the anisotropic case, taking into account for the perturbations of the Ricci scalar δR, directly related to the scalar perturbations of the metric. In the end, as a conclusion, we will analyze

some models for the $f(R)$ function, showing its possible physical meaning and implications.
Birefringence can be observed mainly through the analysis of CMB polarization data: indeed CMB polarization is sensitive to physics violating parity symmetry. Moreover, the rotation angle increases with the distance travelled by the photons, so it is convenient to take CMB radiation since it represents the farthest source of radiation that can be taken into account; the representation in *figure 2* shows the effect of cosmic birefringence in a very clear way.

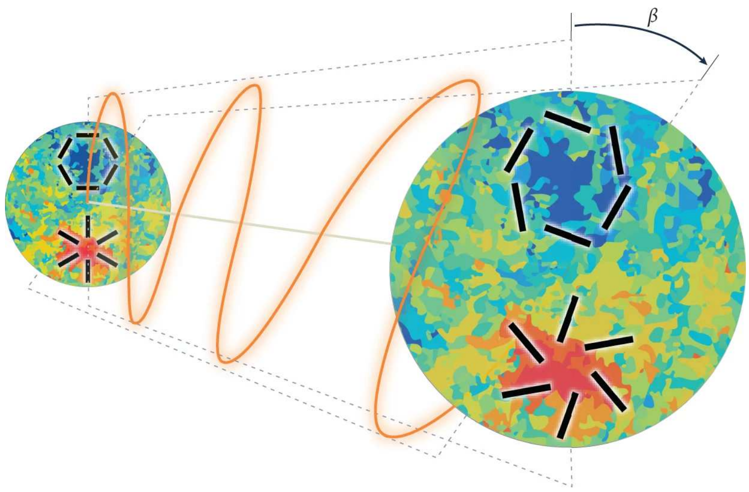

Figure 2: This picture shows the rotation of the polarization angle by cosmic birefringence. The left circle represents the surface of last scattering, while the right one is referred to the present epoch. The black lines are placed in order to display the E and B polarization modes, which are tilted by an angle β as the photons travel to reach us today. [Credit: Y.Minami]

Chapter 1

Some useful reminders

Before starting with the discussion on the Cosmic Birefringence effect, we take a quick review about classical electrodynamics (focusing mainly on the Maxwell equations) and about the polarization of the CMB radiation, paying attention in particular to the Stokes parameters formalism; in the end we will recall some relevant aspects about the theory of cosmological perturbations.

1.1 Review of classical electrodynamics

In this section we want to recall some relevant concepts and relations which will be used in the following chapters (we will adopt [**9**] and [**10**] as references). Let's start by reviewing the Maxwell equations in the context of the Lagrangian formalism; for this purpose it is convenient to define the electromagnetic field tensor (Maxwell tensor). The electric and magnetic fields written in terms of the four-potential $A^\mu = (\phi, \mathbf{A})$ read:

$$\mathbf{E} = -\frac{1}{c}\partial_0 \mathbf{A} - \nabla\phi \tag{1.1}$$

$$\mathbf{B} = \nabla \wedge \mathbf{A} \tag{1.2}$$

where $\phi \equiv A_0$. Notice that $\mathbf{E}$ is parity-odd (it transforms into - $\mathbf{E}$ under inversion of spatial coordinates $\mathbf{r} \longrightarrow -\mathbf{r}$), while $\mathbf{B}$ is parity-even. This fact is relevant for what follow since Cosmic Birefringence effect violates parity symmetry, mixing the E and B modes of the CMB polarization.
Starting from the expressions for the electric and the magnetic fields we can define a tensor, called *electromagnetic field tensor*, which encloses the information of both fields:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.3}$$

which components are: $F^{i0} = E^i$ and $F^{ij} = -\epsilon_{ijk} B^k$. Explicitly the covariant form of the Maxwell tensor can be written as:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{1.4}$$

In order to write down the Maxwell equations in terms of this tensor, we need to start from the electromagnetic Lagrangian density, which describes the dynamics of free photons:

$$\mathscr{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.5}$$

from which we can derive the equation of motion, exploiting the Euler-Lagrange equations:

$$\frac{\partial \mathscr{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = 0 \longrightarrow \frac{\partial(F_{\mu\nu}F^{\mu\nu})}{\partial(\partial_\mu A_\nu)} = 4F^{\mu\nu} \tag{1.6}$$

and $\partial \mathscr{L} / \partial A_\nu = 0$, so that the free EOM (Maxwell equation in vacuum) is given by:

$$\partial_\mu F^{\mu\nu} = 0 \tag{1.7}$$

If we want to introduce a source, in terms of a four-current $J^\mu = (c\rho, \mathbf{J})$, we need to add an interactive part to the Lagrangian density, which becomes:

$$\mathscr{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu \tag{1.8}$$

obtaining the following *coupled* EOM:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \tag{1.9}$$

This is the first set of Maxwell equations written in a covariant way; in terms of E and B vector fields they read:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{1.10}$$

$$-\frac{1}{c}\partial_0 \mathbf{E} + \nabla \wedge \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{1.11}$$

where ρ is the charge density ($\rho = J_0$).
The other pair of equations can be expressed using the *dual* of the field tensor, defined as:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \partial_\rho A_\sigma \tag{1.12}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor. The components of this dual tensor are: $\tilde{F}^{i0} = B^i$ and $\tilde{F}^{ij} = \epsilon^{ijk}E^k$. The EOM for the dual tensor is:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{1.13}$$

which can be recast in term of the electric and magnetic fields as:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.14}$$

$$\frac{1}{c}\partial_0 \mathbf{B} + \nabla \wedge \mathbf{E} = 0 \tag{1.15}$$

These are the last two Maxwell equations. To conclude this review, it is interesting to consider one particular contraction between the Maxwell tensor and its dual, i.e. :

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\mathbf{E} \cdot \mathbf{B} \tag{1.16}$$

which, evidently, changes sign under inversion of spatial coordinates; indeed, as mentioned previously, **E** is parity-odd and **B** is parity-even, so their scalar product is parity-odd, which means that it breaks parity symmetry. The term in eq.(1.16) is the one present in the Chern-Simons interaction and it is the responsible of the parity violation induced by the coupling between the photon and the axion-like field.

1.2 Basics about CMB polarization

This section is dedicated to a review about the polarization pattern in the CMB radiation, since it's the main observable exploited for constraining the birefringence angle; in particular we will focus on the analysis of the polarization in terms of the Stokes parameters. For this part we will refer mainly to [**15**] and [**16**].
Before entering in the description of the CMB polarization, it can be useful to recall some basic aspects about the Cosmic Microwave Background radiation itself. This kind of radiation was originated in the early universe during the so-called *recombination* (or *decoupling*) epoch, about 380.000 years after the Big Bang, when the temperature was low enough (~ 3000 K) that free electrons in the primordial cosmic fluid started to combine with atomic nuclei to give origin to the first atoms. This process allowed the radiation, previously trapped due to a very high scattering rate with electrons, to escape and propagate almost freely up to the present epoch; for this reason we can imagine an ideal surface from which the CMB photons have started their trip, called *surface of last scattering*: it is not possible to get direct observations and information about

what happened before that epoch. The CMB radiation spectrum observed today is a quasi-perfect black-body spectrum (described by the Planck function) emitting at a temperature of about 3 K; this surprising fact is brought about very frequent collisions between photons and free electrons, which were able to maintain a stable thermodynamic equilibrium condition.
Although the CMB radiation shows a very high degree of isotropy, i.e. it has the same intensity in whatever direction we measure it, there are some non-negligible fluctuations and departures from a perfect isotropic distribution. Indeed, one of the most relevant feature of the CMB radiation is the presence of *anisotropies* in its angular distribution, which are attributed to these fluctuations (of the order of $\Delta T/T \sim 10^{-5}$) in the temperature values point by point. The analysis of temperature anisotropies is fundamental, for example, in the study of the evolution of cosmological perturbations during the inflationary epoch, since they can be associated to the density fluctuations, related themselves to quantum fluctuations of the inflaton field, which are the seeds of the structures (galaxies and clusters) observed in the present Universe. Besides temperature anisotropies, another fundamental observable in the context of CMB radiation, is given by the polarization state of its photons, which we want to analyze in deeper way, since it is exploited to study the birefringence phenomenon. All these observables have been richly analyzed by different space missions, which allow us to acquire relevant information about the power spectra and about the CMB anisotropies. The first mission launched by NASA in 1989 was the COBE satellite, which was able to measure with a good precision the black-body CMB spectrum; then in 2001 this first mission was replaced by the WMAP satellite, which had confirmed the results obtained by COBE, but with a high precision. In the end, the last satellite launched in 2009 was the *Planck* satellite, which is still the most sensitive one in the context of CMB observations, with a very high angular resolution of about 5 arcmin. We can see an example of the CMB map from the *Planck* observatory in *figure 1.1* and the Planck power-spectra in *figure 1.5*.

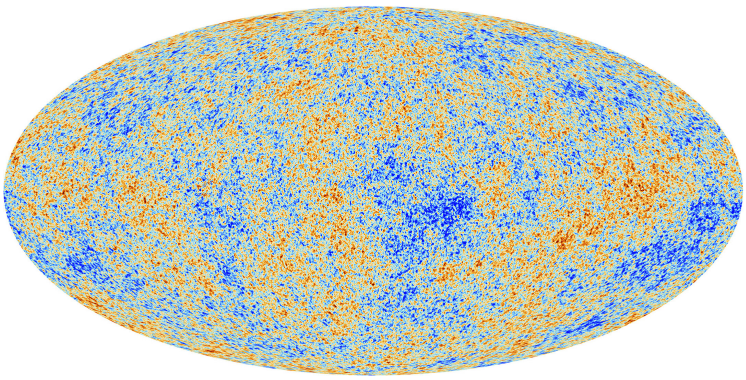

Figure 1.1: Map of the CMB as observed by the Planck satellite; the different colors are used to highlight the anisotropies in temperature (blue regions are colder, while the orange ones are hotter than the average value). Temperature fluctuations correspond to regions of slightly different density: these are the seeds of the large scale stucture of the present Universe. [Credit: ESA and the Planck Collaboration]

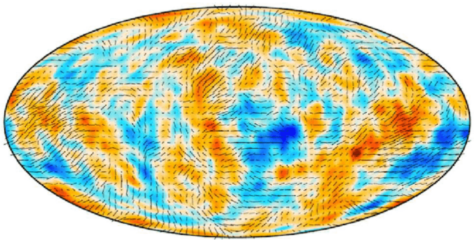

Figure 1.2: CMB polarization map: the E and B-modes are shown through a pattern of small black segments. [Credit: ESA/Planck 2018]

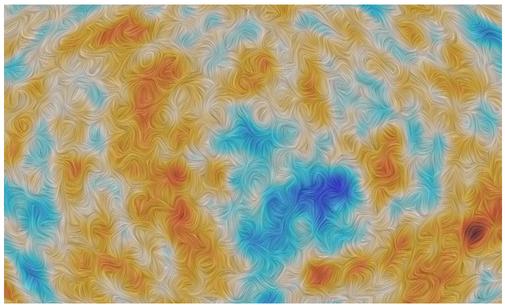

Figure 1.3: A visualization of the polarization of the Cosmic Microwave Background, or CMB, as detected by ESA's Planck satellite; the pattern seen in the texture is typical of the E-mode polarization, which is the dominant one in the CMB radiation.[Credit: ESA/Planck 2018]

CMB photons acquire a certain degree of linear polarization due to Thomson scattering of the anisotropic radiation with electrons (the anisotropy must have a non-null quadrupole moment to generate a linear polarization); moreover polarization is generated thanks to the fact that recombination has not taken place instantaneously, so the finite-thickness effect is important. It is useful to decompose the polarization pattern in E- and B-modes, which are tilted of 45° one respect to the other; they are two different eigenstates of parity symmetry, i.e. they transform differently (in an opposite way) under inversion of spatial coordinates ; through this decomposition it is possible to probe violation of parity symmetry. In *figure 1.2* and *figure 1.3* the CMB polarization pattern is shown.
In order to characterize the polarization pattern we can visualize it better through an ellipse. Let's start from a monochromatic plane wave propagating in the z direction, for which the electric field components can be written as:

$$E_x(t, \mathbf{x}) = A_x cos(z - ct + \phi_x) \tag{1.17}$$

$$E_y(t, \mathbf{x}) = A_y cos(z - ct + \phi_y) \tag{1.18}$$

where A_x and A_y are the amplitude of the electromagnetic wave and ϕ_x and ϕ_y are the phases: the polarization state depends on these four parameters. We can always redefine the starting point, such that the previous relations become:

$$E_x(t, \mathbf{x}) = A_x cos(z - ct) \tag{1.19}$$

$$E_y(t, \mathbf{x}) = A_y cos(z - ct + \beta) \tag{1.20}$$

where $\beta = \phi_x - \phi_y$ is the relative phase between the two components. Notice that, when $\beta = 0$ there is a pure linear polarization, while for $\beta = \pi/2$ it is completely circular. In general, for whatever value of β, we can identify an ellipse in the x-y plane, whose equation is:

$$\frac{E_x^2}{A_x^2} + \frac{E_y^2}{A_y^2} - \frac{2E_xE_y}{A_xA_y} cos\beta = \sin^2\beta \tag{1.21}$$

This is called *polarization ellipse*.
At this point the polarization pattern can be described with the help of the *Stokes parameters*, defined in terms of the parameters of the ellipse, in this way:

$$I = A^2 \tag{1.22}$$

$$Q = A^2 cos2\theta \tag{1.23}$$

$$U = A^2 \sin 2\theta cos\beta \tag{1.24}$$

$$V = (A^2 \sin 2\theta \sin\beta)h \tag{1.25}$$

where θ is the angle used to define the components of the amplitude $A_x = A\cos\theta$ and $A_y = A\sin\theta$, and h establishes the direction of rotation for the circular polarization (anti-clockwise for $h = 1$, clockwise for $h = -1$). From these definitions we can understand that I is simply the intensity of the wave. Then, if U and V are null, we have a pure linear polarization, since $\beta = 0$; the same happens if Q and V are zero. So the Q and U parameters describe a linear polarization with planes tilted of 45° one respect to the other. In the end, V is related to the circular polarization.
Another way to define the Stokes parameters, that will be used later, is the following:

$$I = \langle E_x^2\rangle + \langle E_y^2\rangle \tag{1.26}$$

$$Q = \langle E_x^2\rangle - \langle E_y^2\rangle \tag{1.27}$$

$$U = 2\langle E_xE_y\rangle \cos\beta \tag{1.28}$$

$$V = 2\langle E_xE_y\rangle \sin\beta \tag{1.29}$$

where the $\langle ...\rangle$ denote an *ensemble* average, equivalent to averaging over many periods of the wave. It is interesting to notice that, if we perform a rotation of an angle θ along the z direction, the linear combination of Q and U transforms as:

$$Q \pm iU \longrightarrow e^{\pm 2i\theta}(Q \pm iU) \tag{1.30}$$

This is exactly a rotation of the linear polarization plane.
A very useful tool which can be exploited in the analysis of CMB anisotropies (both in temperature and in polarization) is the formalism of the spherical harmonics, which allow us to expading any quantity or function defined on a

sphere as a linear combination of them. For instance, the temperature fluctuations can be expressed as a function of the angles θ and ϕ, defined on a sphere, as:

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{l=0}^{\inf} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi) \tag{1.31}$$

where a_{lm} are the complex coefficients of the linear combination, while $Y_{lm}(\theta,\phi)$ are the spherical harmonics functions; the indices l and m are referred respectively to the degree and the moment of each component, and they correspond to the two angular coordinates. The linear combination in eq.(1.31) is also known as the angular *multipole expansion*, being l the multipole moment. In order to better visualize the spherical harmonic decomposition, we can refer to *figure 1.4*. The expansion in eq.(1.31) can be inverted in order to get the coefficients a_{lm}:

$$a_{lm} = \int d^2\hat{n} Y_l^{*m}(\hat{n}) \frac{\Delta T}{T}(\hat{n}) \tag{1.32}$$

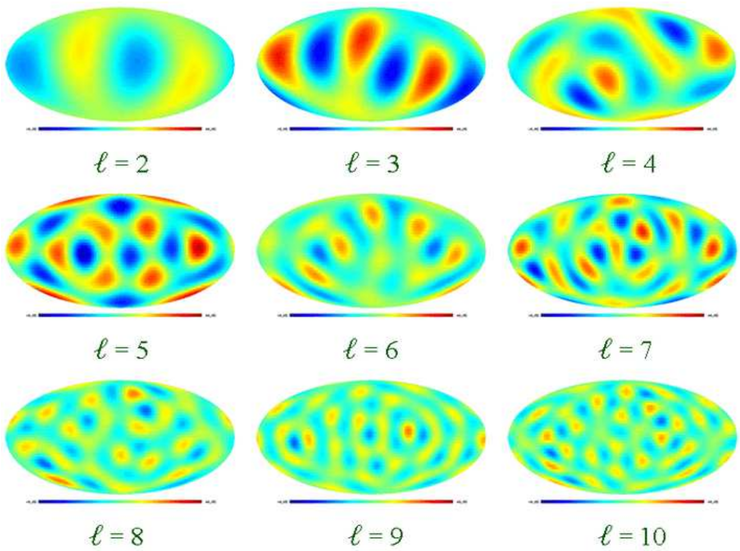

Figure 1.4: Spherical harmonic decomposition for different values of the multipole moment l. It is evident that higher multipole moments correspond to smaller angular scales, according to the relation: $\theta = \pi/l$. [Credit: NASA/WMAP/Chiang Lung-Yih]

where $\hat{n}$ is referred to the direction in which we are observing (which is identified by θ and ϕ angles), in particular: $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Then, in order to derive the CMB power spectrum we need to take into account for the variance of these coefficients, so that:

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_{TT,l} \tag{1.33}$$

where $C_{TT,l} = \langle |a_{lm}|^2 \rangle$ is the CMB power spectrum (or the auto-correlation function) of the temperature fluctuations; this definition is valid until we consider Gaussian perturbations. In the same way we can build up all the other correlations, such as EE and BB ones. In order to do this we can recall the combination defined in eq.(1.30), which can be seen as a *spin-2* field and so it can be expanded as:

$$(Q+iU)^S(\hat{n}) = \sum_{lm} a^S_{P,lm\,2}Y^m_l(\hat{n}) \tag{1.34}$$

where $a^S_{P,lm}$ are the coefficients of the linear combination and ${}_2Y^m_l(\hat{n})$ are the 2-spin spherical harmonics. The coefficients can be found by taking advantage of the orthonormality relation between the sperical harmonics.
It is convenient to redefine the expansion for $Q+iU$ in terms of the coefficients $a_{E,lm}$ and $a_{B,lm}$ instead of $a_{P,lm}$, since there is no reality condition for the last ones; moreover they are more useful since they are related to E and B polarization modes. The new coefficients can be defined as linear combination of the previous ones:

$$a_{E,lm} \equiv -(a_{P,lm} + a^*_{P,l,-m})/2 \tag{1.35}$$

$$a_{B,lm} \equiv i(a_{P,lm} - a^*_{P,l,-m})/2 \tag{1.36}$$

so that the first has parity $(-1)^l$, while the second $(-1)^{l+1}$: they behaves in an opposite way under inversion of spatial coordinates. Thus, finally, the expansion for $Q+iU$ reads:

$$(Q+iU)(\hat{n}) = \sum_{lm}(-a_{E,lm} \mp i a_{B,lm})_2Y^m_l(\hat{n}) \tag{1.37}$$

Computing the $a^{S*}_{P,lm}$ coefficients and exploiting the parity of spherical harmonics, it is possible to conclude that: $a^S_{P,lm} = a^{S*}_{P,l.-m}$, and therefore:

$$a^S_{E,lm} = -a^S_{P,lm} \qquad\qquad a^S_{B,lm} = 0 \tag{1.38}$$

which means that scalar perturbations only affect the *E-mode*; on the other hand the detection of the *B-mode* would be an indication of the existence of primordial gravitational waves (tensor perturbations).
Using the $a_{E,lm}$ and $a_{B,lm}$ coefficients we can define the following power spectra, in the same way as done in eq.(1.33) for temperature fluctuations:

$$\langle a^*_{T,lm} a_{E,l'm'}\rangle = \delta_{ll'}\delta_{mm'}C_{TE,l} \tag{1.39}$$

$$\langle a_{E,lm} a^*_{E,l'm'}\rangle = \delta_{ll'}\delta_{mm'}C_{EE,l} \tag{1.40}$$

$$\langle a^*_{B,lm} a_{B,l'm'}\rangle = \delta_{ll'}\delta_{mm'}C_{BB,l} \tag{1.41}$$

In particular the EE and TE correlations read (under the assumption of adiabatic gaussian perturbations):

$$C^S_{EE,l} = \frac{9}{64\pi}\frac{(l+2)!}{(l-2)!}\int\frac{dk}{k}\Delta^2_{\mathcal{R}}\left|\int_0^{\eta_0} d\eta S^S_P(\eta,k)\frac{j_l(kr)}{(kr)^2}\right|^2 \tag{1.42}$$

$$C^S_{TE,l} = -\frac{3}{4}\frac{(l+2)!}{(l-2)!}\int\frac{dk}{k}\Delta^2_{\mathcal{R}}\Theta_l(k)\int_0^{\eta_0} d\eta S^S_P(\eta,k)\frac{j_l(kr)}{(kr)^2} \tag{1.43}$$

where $S^S_P(\eta,k)$ is a source term, $j_l(kr)$ comes from the expansion of a plane wave in spherical harmonics and Δ^2_R is the adimensional spectrum for scalar perturbations. We can notice that we haven't mention the TB and EB correlations: these are non-zero just in case of parity-violating processes, indeed they are exploited to probe new physics, in particular the EB one, which is the direct trace of the presence of the Cosmic Birefringence effect.

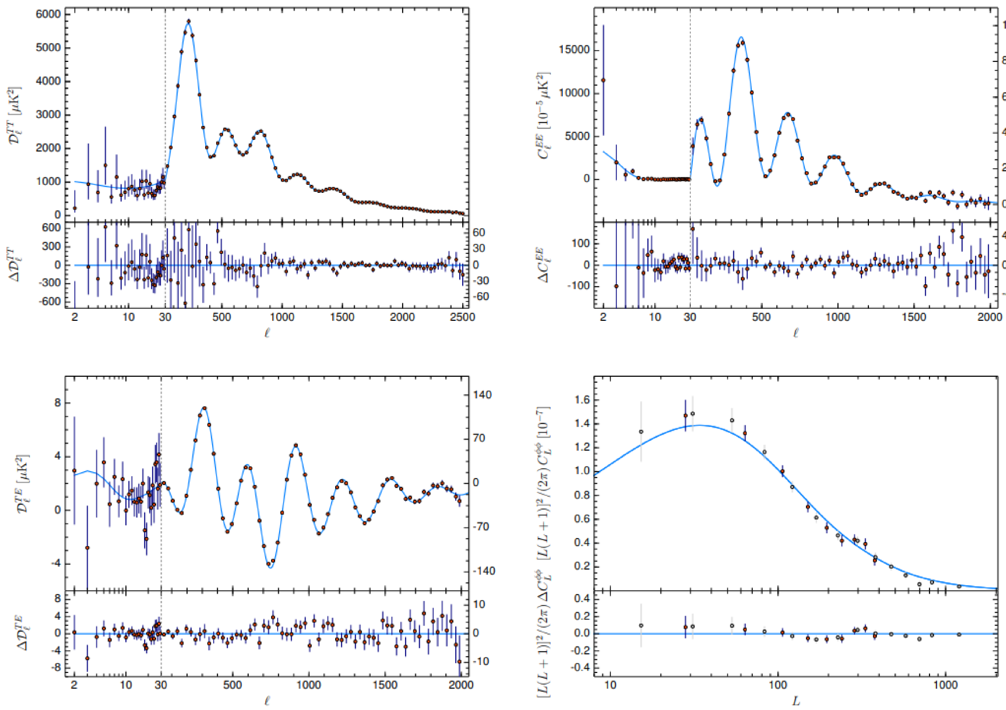

Figure 1.5: Planck 2018 CMB angular power spectra. The data (represented by the dots) are compared with the ΛCDM best fit to the Planck TT, TE, EE+lowE+lensng data (blue curves). In particular, on the left side we can see the plots for D_l^{TT} (temperature power spectrum) and D_l^{TE}, where $D_l = l(l+1)C_l/2\pi$, while on the right side the E-mode power spectrum C_l^{EE} is plotted, together with the lensing contribution (bottom panel). (Y. Akrami et al. (2020) [**31**])

1.3 Cosmological perturbation theory

In this section we want to review some relevant aspects concerning the cosmological perturbation theory, focusing in particular on the metric perturbations.

For this treatment we will follow [**32**], [**33**] and [**34**]. The main purpose of a cosmological perturbation theory is to study the properties, the origin and the evolution of primordial density fluctuations, which are the seeds of the Large Scale Structure (of galaxies and clusters) observed today in the Universe. The basic idea is to perturb an isotropic and homogeneous spacetime, described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which components read:

$$\bar{g}_{00}(t) = -1 \qquad \bar{g}_{ij}(t) = a^2(t)\delta_{ij} \tag{1.44}$$

or, in terms of the conformal time τ:

$$\bar{g}_{\mu\nu}(\tau) = a^2(\tau)\eta_{\mu\nu} \tag{1.45}$$

where $a(\tau)$ is the scale factor and $\eta_{\mu\nu}$ is the flat Minkowski metric given by $diag(-1,1,1,1)$; the relation between the cosmic time t and the conformal time τ is given by: $d\tau = dt/a(t)$. Then a generic perturbed metric can be defined as the sum of this *background* contribution, which depends only on time (since it is homogeneous and isotropic), and some fluctuations $\delta g_{\mu\nu}$, which encloses the dependence on spatial coordinates:

$$g_{\mu\nu}(\tau, \mathbf{x}) = \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau, \mathbf{x}) \tag{1.46}$$

where, in general, the perturbed part of the metric is made up of three kind of contributions: scalar, vector and tensor perturbations. We will focus on the first ones, since we are interested in the fluctuations of the Ricci scalar; however, for completeness, we mention also the vector and tensor counterparts. As we will see in *Chapter 3*, scalar field fluctuations ($\delta\phi$) are directly related to metric perturbations; indeed any fluctuation of a scalar field, especially if it dominates the matter-energy content of the Universe (for example in the case of the inflaton field during inflation), induces a perturbation of the energy-momentum tensor $T_{\mu\nu}$, which then implies a perturbation in the metric, through the Einstein field equations: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. Moreover the scalar field fluctuations and the metric perturbations are directly related by the perturbed Klein-Gordon equation (see eq.(3.15)). For these reasons it is important to consider metric perturbations in the context of cosmological perturbations.
The different components of the perturbed metric $g_{\mu\nu}$ in eq.(1.46), up to first order, read:

$$g_{00} = -a^2(\tau)\big(1 + 2\Psi(\tau, \mathbf{x})\big) \tag{1.47}$$

$$g_{0i} = a^2(\tau)\omega_i(\tau, \mathbf{x}) \tag{1.48}$$

$$g_{ij} = a^2(\tau)\big[\big(1 - 2\Phi(\tau, \mathbf{x})\big)\delta_{ij} + \chi_{ij}(\tau, \mathbf{x})\big] \tag{1.49}$$

where the scalar perturbations are given by Φ and Ψ, which are respectively the *gravitational potential* and the *lapse* perturbations; then, ω_i is the vector perturbation, and χ_{ij} is the tensor one. To be more precise, also ω_i and χ_{ij} contain some scalar contributions; it is common, indeed, to split the perturbations in their scalar, vector and tensor counterparts, on the basis of their different ways of transforming under a change of coordinates. We can do this thanks to the Helmholtz decomposition, such that they can be written as:

$$\omega_i = \partial_i \omega^{\parallel} + \omega_i^{\perp} \tag{1.50}$$

$$\chi_{ij} = \mathcal{D}_{ij}\chi^{\parallel} + \chi_{i,j}^{\perp} + \chi_{j,i}^{\perp} + \chi_{ij}^{T} \tag{1.51}$$

where $\mathcal{D}_{ij} = \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2$ is a traceless differential operator, $\chi^{\parallel}$ and $\omega^{\parallel}$ are the two scalar contributions, ω_i and χ_i are transverse vectors, and χ_{ij}^{T} is the traceless tensor part. These three different perturbation modes, at least at linear order, are completely decoupled, so that it is possible to study the related perturbed evolution equations as independent one to each other.
Since we are dealing with cosmological perturbations within General Relativity theory, we must take into account for the so-called *gauge issue*, i.e. perturbations are gauge dependent, which means that they are not invariant under a gauge transformation. This can lead to the generation of unphysical gauge modes which must be erased in order to get the correct physical solution. There are two main possible approaches in order to eliminate these spurious gauge contributions: on the one hand we can work, and do all the computations, in a specific gauge (we can choose the more convenient one on the basis of what we are searching for), while on the other hand we can rewrite the perturbed equations through some gauge-invariant quantities. In *Chapter 2* and *3* we will adopt two particular gauge choices: the synchronous gauge and the Poisson gauge, which is a GR extension of the newtonian conformal gauge (see [**38**] for a complete explanation about these two gauge choices). The first one is based on the condition $\psi = 0$; from eq.(1.47) we have: $g_{00}d\tau^2 = -a^2(\tau)d\tau^2(1+2\Psi)$, where $g_{00}d\tau^2$ is the proper time, while $a^2d\tau^2 = dt^2$ is the cosmic time. Since $\Psi = 0$ these two times are equivalent: for this reason it is called *synchronous* gauge. This choice presents a residual gauge freedom, which can be erased by imposing some suitable initial conditions. The *Poisson* gauge, on the other hand, is defined by the conditions: $\omega^{\parallel} = 0$, $\chi^{\parallel} = 0$ and $\chi_i^{\perp} = 0$. It is also called *zero-shear* gauge since the shear $\sigma = -\omega^{\parallel} + (1/2)(\chi^{\parallel})'$ is null. This two gauge choices are the most convenient to make evident the connection between the scalar field fluctuations and the scalar metric perturbations, as we will see in *Chapter 3*.
The other possible approach in order to avoid the gauge problem consists in the definition of gauge-invariant quantities: as an example, it is interesting to mention the ones proposed by Bardeen in 1980 in [**35**]. Focusing on the scalar perturbations, the perturbed metric depends on four components: Φ, Ψ, $\omega^{\parallel}$ and $\chi^{\parallel}$, from which it is possible to build two invariant scalar quantities:

$$2\Psi_A = 2\Psi + 2(\omega^{\parallel})' + 2\frac{a'}{a}\omega^{\parallel} - \left(\chi^{\parallel} + \frac{a'}{a}(\chi^{\parallel})'\right) \tag{1.52}$$

$$2\Phi_H = -2\Phi - \frac{1}{3}\nabla^2\chi^{\|} + 2\frac{a'}{a}\omega^{\|} - \frac{a'}{a}(\chi^{\|})' \tag{1.53}$$

these are the so-called Bardeen's gauge invariant gravitational potentials. It is interesting to notice that in the Poisson gauge these two quantities reduce to: $\Psi_A = \Psi$ and $\Phi_H = -\Phi$, such that we can understand their physical meaning. In the end, in order to find out the evolution equations for the cosmological perturbations, we need to perturb the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.54}$$

At first, we need to perturb the Christoffel symbols (or connection coefficients) $\Gamma^{\alpha}_{\mu\nu}$ which enter in the definition of the Riemann tensor $R_{\mu\nu\rho\sigma}$ and, consequently, of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. Let's recall the definition for the connection coefficients:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right) \tag{1.55}$$

where the *comma* denotes the partial derivative. Then, we can define the Riemann tensor as:

$$R^{\alpha}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\alpha}_{\nu\sigma} - \partial_\nu\Gamma^{\alpha}_{\mu\sigma} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{1.56}$$

and finally the expressions for the Ricci tensor, which is derived by contracting two indices in eq.(1.56), and the Ricci scalar are:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha\Gamma^{\alpha}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{1.57}$$

$$R = g^{\mu\nu}R_{\mu\nu} \tag{1.58}$$

Since the background spacetime is defined by the FLRW metric in (1.45), the non-null unperturbed Christoffel symbols are only:

$$\Gamma^0_{00} = \frac{a'}{a} \qquad \Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} \qquad \Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} \tag{1.59}$$

Thus, the components of the perturbed coefficients can be found by substituting eqs.(1.47)-(1.49) into eq.(1.55), taking into account only for scalar perturbations; in this way we get:

$$\delta\Gamma^0_{00} = \Psi' \tag{1.60}$$

$$\delta\Gamma^0_{0i} = \partial_i\Psi + \frac{a'}{a}\partial_i\omega^{\|} \tag{1.61}$$

$$\delta\Gamma^i_{00} = \frac{a'}{a}\partial^i\omega^{\parallel} + \partial^i(\omega^{\parallel})' + \partial^i\Psi \tag{1.62}$$

$$\delta\Gamma^0_{ij} = -2\frac{a'}{a}\Psi\delta_{ij} - \delta_i\delta_j\omega^{\parallel} - 2\frac{a'}{a}\Phi\delta_{ij} - \Phi'\delta_{ij} + \frac{a'}{a}D_{ij}\chi^{\parallel} + \frac{1}{2}D_{ij}(\chi^{\parallel})' \tag{1.63}$$

These relations are valid in general, whatever gauge we choose. From them it is possible to compute the perturbed Ricci tensor and Ricci scalar in order to write down the perturbed Einstein field equations. On the other hand, focusing on the right hand side of equation (1.54), we need to consider also the perturbations of the stress-energy tensor $T_{\mu\nu}$, in the case of a perfect fluid. The effect of this perturbations is the origin of an anisotropic stress tensor, which encodes the departure from an ideal fluid regime; this new contribution, decomposed in its scalar, vector and tensor contributions, can be defined as:

$$\Pi_{ij} = \mathcal{D}_{ij}\Pi^{\parallel} + \Pi^{\perp}_{i,j} + \Pi^{\perp}_{j,i} + \Pi^{T}_{ij} \tag{1.64}$$

similarly to eq.(1.51). Then we can write down the perturbed components of $T_{\mu\nu}$ as:

$$T^0_0 = \rho^{(0)}(\tau) + \delta\rho(\bar{x},\tau) \tag{1.65}$$

$$T^0_i = (\rho_0 + p_0)(v_i + \omega_i) \tag{1.66}$$

$$T^i_0 = -(\rho_0 + p_0)v^i \tag{1.67}$$

$$T^i_j = p_0\big[(1 + \Pi_L)\delta^i_j + \Pi^{i(T)}_j\big] \tag{1.68}$$

We do not want to enter too much in the details of these computations (see [**36**] for a complete treatment); in *Chapter 4* we will compute the perturbed quantities in eqs.(1.55)-(1.58) imposing a particular gauge. We just report here the final form of the perturbed Einstein equations and some interesting implications useful for our treatment; the 00-components of the perturbed equations (taking into account only for scalar perturbations of the metric) read:

$$3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Psi\right) - \nabla^2\left(\Phi + \frac{a'}{a}\sigma\right) = -4\pi G a^2\delta\rho \tag{1.69}$$

$$\Phi' + \frac{a'}{a}\Psi = -4\pi G a^2(\rho_0 + p_0)\mathcal{V} \tag{1.70}$$

where σ is the scalar part of the shear tensor, defined previously, $\delta\rho$ is the energy density perturbation and $\mathcal{V} = v^{\parallel} + \omega^{\parallel}$ is the scalar contribution to the four-velocity of the fluid. These two equations do not tell us anything about the evolution of Φ and Ψ potentials, since there are only first derivatives in time; instead, they are useful to put some constraints on the energy and momentum of the fluid, indeed $\delta\rho$ and $(\rho_0 + p_0)\mathcal{V}$ are related, respectively, to these two

quantities.
Then, the ij-components of the field equations are given by:

$$\Phi'' + 2\frac{a'}{a}\Phi' + \frac{a'}{a}\Psi' + \left[2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\Psi\right] = 4\pi G a^2 p_0 \left(\Pi_L + \frac{2}{3}\nabla^2\Pi_T\right) \quad (1.71)$$

$$\sigma' + 2\frac{a'}{a}\sigma + \Phi - \Psi = 8\pi G a^2 p_0 \Pi_T \quad (1.72)$$

where $\Pi_L = \delta P/p_0$ is the *longitudinal* part of the T_{ij} component, while $\Pi_T = \pi^{\|}/p_0$ is the traceless part. In general Π_{ij} is an anisotropic stress-energy tensor which arises from the perturbation of T_{ij}.
We can show some relevant aspects of these equations by imposing the Poisson gauge, which implies: $\Phi = -\Phi_H$ and $\Psi = \Psi_A$. Under this choice, eq.(1.72) becomes:

$$-\Phi_H - \Psi_A = 8\pi G a^2 P_0 \Pi_T \quad (1.73)$$

which tells us that, when a null anisotropic stress in considered (since we are in the case of a perfect fluid), the two Bardeen's potential assume the same value; this allow us to simplify eq.(1.71) and to write it in terms of just one of the two potentials, in this way:

$$\Phi_H'' + 3\big(1 + c_s^2\big)\frac{a'}{a}\Phi_H' + \left[2\left(\frac{a'}{a}\right)' + \big(1 + 3c_s^2\big)\left(\frac{a'}{a}\right)^2 - c_s^2\nabla^2\right]\Phi_H = 0 \quad (1.74)$$

which has the form of an evolution equation for the perturbation Φ_H with a speed equal to the speed of sound c_s; we could have found a similar equation for Ψ_A. Moreover, substituting eq.(1.70) in eq.(1.69), we can show that the Bardeen's potentials satisfy the general relativistic Poisson equations, which read:

$$\nabla^2\phi_H = -\nabla^2\psi_A = 4\pi G a^2 \epsilon_m \quad (1.75)$$

where $\epsilon_m = \delta\rho + \rho_0'(v^{\|} + \omega^{\|})$ is the gauge-invariant quantity related to the energy density of the fluid.

Chapter 2

Searching for new physics: the isotropic Cosmic Birefringence effect

Cosmic Birefringence (CB) has been studied and analyzed, both theoretically and experimentally, quite deeply in the last years. It is a fundamental observable which is generated by a parity violation in electromagnetic interaction, brought about the presence of new physics beyond the Standard Model. CB effect, indeed, can be used to get information about Dark Matter and Dark Energy components in the early universe, but it can be exploited also in the context of baryo/leptogenesys models [**1**], as we will discuss troughout this work. Moreover this effect is associated also with the detection of B-mode polarization in the CMB radiation; this mode arises in the presence of *Primordial Gravitational Waves* (PGWs) [**2**], but the CB effect is able to produce an additional contribution to it (in particular it is able to mix E and B-modes). For this reason CMB polarization can be exploited in order to investigate the birefringence angle.
At first, we will analyze the sources of the Cosmic Birefringence effect, deriving the polarization rotation angle directly from the modified Maxwell equations; we will focus on models in which the phenomenon is attributed to the coupling of a scalar field (more precisely, an axion-like field) with electromagnetic radiation, and then also on approaches which, instead, are based on the coupling with a scalar function $f(R)$, in the context of modified Einstein gravity theories. In both cases we will analyze the CB angle power-spectra in order to constrain the rotation angle from CMB observations.

2.1 Cosmic Birefringence from Chern-Simons coupling with a scalar field ϕ

The Standard Model (SM) of particle physics has been studied and confirmed by many experimental tests, in particular concerning the fundamental CPT symmetry. Nevertheless cosmological and astrophysical observations

have shown the need of exploring physics beyond the Standard Model, for instance by adding new components such as Dark Matter and Dark Energy. In this sense, a very interesting effect which can be experimentally measured and analyzed, is an unusual rotation of the photon polarization plane: this phenomenon is known under the name of *Cosmic Birefringence*, and it is the central topic treated in this work.
As we will see, since this rotation effect depends on the distance travelled by the photons, and in particular it grows evidently if the radiation covers a very long path towards us, it is reasonable to focus on photons emitted in the earliest epoch possible, i.e. at the recombination; for this reason we can get constraints on the birefringence angle mainly from the CMB polarization data. It is still a faint effect, difficult to observe and measure even with the most recent missions such as the Planck telescope, but it is very relevant in order to probe new physics beyond the SM. Indeed the CB effect is the result of the interaction of CMB photons with Dark Matter or Dark Energy fields [**1**], present at early epochs, which acts as a birefringent material (like a crystal) on the incoming radiation.
This coupling between photons from the CMB and a scalar field ϕ (let's keep it generic for the moment) modifies the Maxwell theory of electromagnetism, and the related equations of motion, and it is indeed the responsible of the parity violation: in this sense we are exploring new physics beyond the SM. Moreover, as suggested in [**5**] the addition of this new term violates also Lorentz invariance. In practice this modification is based on the addition of a *Chern-Simons* coupling term to the standard Maxwell Lagrangian density, in this way:

$$\mathscr{L} = \mathscr{L}_{EM} + \mathscr{L}_{CS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}p_\mu A_\nu \tilde{F}^{\mu\nu} \tag{2.1}$$

where p_μ is a generic four vector coupled with the EM field, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the *dual* of the strength tensor $F^{\mu\nu}$ (eq. (3)). Lorentz invariance is not fulfilled due to the fact that the vector p_μ chooses a preferred direction in spacetime, and in particular it has a non-vanishing time component p_0. As we will see in the following discussion this unidentified four-vector can be associated to a pseudoscalar field, such that $p_\mu = \partial\phi/\partial x^\mu$: in some models this ϕ is identified as an axion-like field (see for example [**2**] and [**3**]), while in other theories is treated as a generic quintessence field (see for example [**1**] and [**4**]).
Before writing down the modified Maxwell's equations of motion, it is interesting to give an idea of how much the Lorentz and parity symmetries are violated in this modified theory; this estimations are done in [**5**] by Carroll and Field, who have found an upper limit on the Chern-Simons coupling in terms of the mass m (which is given by $(p_\mu p^\mu)^{1/2}$):

$$m \leq 6 \times 10^{-26} \ GeV \tag{2.2}$$

and a more stringent bound:

$$m \leq 1.7 \times 10^{-42} h_0 \ GeV \tag{2.3}$$

This allow us to understand the smallness of this kind of effect.
At this point it's finally the time to derive the explicit expression of the birefringence angle; in order to get that relation we need to start from the EOMs of electromagnetism modified by the Chern-Simons term. We will mainly follow the treatment and computations proposed by Li and Zhang in [**4**] and in [**1**]. Let's start by analyzing the interaction term which modifies the standard electromagnetic Lagrangian density; it is defined as:

$$\mathscr{L}_{int} = \frac{c}{M} \nabla_\mu \phi J^\mu \tag{2.4}$$

It describes a derivative coupling (∇_μ is the covariant derivative) of a scalar boson ϕ to a generic fermion current J^μ; M is the typical cut-off scale of the theory which can be associated with *Grand Unification Theory* or with the Planck scale. Even though this term is CPT conserving, during the expansion of the Universe the scalar field evolves and the symmetry is spontaneously broken by a non-vanishing $\dot{\phi}$. Indeed this is the case of theories in which ϕ is identified with a dynamical Dark Energy field.
In order to clarify a little better the physical meaning of the coupling term, it is useful to make the J^μ current explicit: in [**1**] and [**4**] they have proposed that it is the left-handed part of the $B - L$ current, i.e. $J^\mu_{(B-L)_L}$. Indeed this kind of current has two very interesting implications: on one hand it helps to reproduce the baryon number asymmetry in thermal equilibrium (a fundamental aspect in models of quintessential baryo/leptogenesis), while on the other hand it shows an anomalous behaviour under the electromagnetic interaction. We can see this by considering that:

$$\nabla_\mu J^\mu_{(B-L)_L} \sim -\frac{\alpha_{em}}{3\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{2.5}$$

where $\alpha_{em} = e^2/4\pi$ is the fine structure constant. In terms of the electric and magnetic fields $F_{\mu\nu}\tilde{F}^{\mu\nu}$ can be rewritten as $-4\mathbf{B} \cdot \mathbf{E}$, which shows more clearly the violation of parity symmetry, since it changes sign under inversion of spatial coordinates. By replacing the electromagnetic tensor with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and its dual with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, and by doing simple algebraic calculations, we recover the form of the interacting Lagrangian in eq.(2.1), finding that $J^\mu = 2A_\nu \tilde{F}^{\mu\nu}$. In the end we can rewrite the coupling term in eq.(2.4) as:

$$\mathscr{L}_{int} = -\frac{2c\alpha_{em}}{3\pi M} \nabla_\mu \phi A_\nu \tilde{F}^{\mu\nu} = p_\mu A_\nu \tilde{F}^{\mu\nu} \tag{2.6}$$

where the four vector is defined as $p_\mu = (2c\alpha_{em}/3\pi M)\nabla\phi$. In this way we have found out an explicit expression for the generic four-vector p_μ introduced previously (see [**5**]).
In some models the field ϕ is recognized as the *axion* field; to understand the reason of this choice we need to refer to the so-called strong CP problem. This problem is related to the fact that the QCD Lagrangian presents an extra term, due to the QCD vacuum structure, which violates P and T symmetries,

and so it also breaks CP simmetry; however there is no evidence of CP violation in strong interactions at all. Moreover the problem is associated with the smallness of the (measured) electric dipole moment of the neutron, which requires the parameter θ to be smaller than $10^{-10} - 10^{-9}$. To explain this strange aspect, and solve the CP problem, Peccei and Quinn have proposed the introduction af a new $U(1)$ chiral symmetry (called $U(1)_{PQ}$) which is spontaneously broken, bringing θ to zero, and generating a Nambu-Goldstone axion field (for a deeper discussion on the CP problem refer to [**6**] [**7**]). In particular in [**7**] the solution to the CP problem is related to the presence of a chiral anomaly for the axial current J_5^μ which has a non-zero divergence given by:

$$\partial_\mu J_5^\mu = \frac{g^2 N}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu} \tag{2.7}$$

Notice that this current has a similar structure as the fermionic current defined in (1.5); so, in the end, it is reasonable to take an axion-like field as the scalar d.o.f. coupled with photons, which gives rise to the birefringence effect. In this sense, probing cosmological birefringence means searching for new DM particles, different from the ones described in the Standard Model.

In order to derive the expression for the birefringence angle we can exploit the Euler-Lagrange equations starting from the Lagrangian defined in (2.1): in this way we can find out the equations of motion for the photon field which interacts with a scalar field (namely the axion field). The Euler-Lagrange equation reads:

$$\frac{\partial \mathscr{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = 0 \tag{2.8}$$

and then, since the Lagrangian depends also on the scalar field, we need to consider also the other equation obtained by deriving it with respect to ϕ:

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = 0 \tag{2.9}$$

In the first equation we can notice that the *free* Lagrangian $\mathscr{L}_{EM}$ depends only on the derivative of A_μ, while the interacting part $\mathscr{L}_{CS}$ only on A_μ itself. The former can be computed as:

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\frac{\partial(F_{\mu\nu}F^{\mu\nu})}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{2}F^{\mu\nu} \tag{2.10}$$

while the latter is given by:

$$\frac{\partial \mathscr{L}}{\partial A_\nu} = \frac{\partial \mathscr{L}_{CS}}{\partial A_\nu} = p_\mu \tilde{F}^{\mu\nu} \tag{2.11}$$

Then, deriving the equation (2.10) with respect to ∂_μ and combining it with (2.11), we reach this form of the modified Maxwell EOM:

$$\nabla_\mu F^{\mu\nu} = -2p_\mu \tilde{F}^{\mu\nu} \tag{2.12}$$

On the other hand, from equation (2.9) we get the other EOM, which is not affected by the Chern-Simons term, and so it is the same as for the free photon:

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \partial_\mu (A_\nu \tilde{F}^{\mu\nu}) = 0$$

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.13}$$

The last equation can be rewritten in a more useful way:

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0 \tag{2.14}$$

which is a sort of *Bianchi identity* for electromagnetism. This can be shown more explicitly by combining together the Gauss law for the magnetic field (see [**10**] and [**11**]):

$$\partial_i B^i = 0 \longrightarrow \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \tag{2.15}$$

and the Faraday law $\epsilon_{ijk}\partial_j E^k + \partial_0 B^i = 0$ which becomes:

$$\begin{cases} \partial_2 F_{03} + \partial_3 F_{20} + \partial_0 F_{32} = 0 \\ \partial_3 F_{01} + \partial_1 F_{30} + \partial_0 F_{13} = 0 \\ \partial_1 F_{02} + \partial_2 F_{10} + \partial_0 F_{21} = 0 \end{cases} \tag{2.16}$$

Merging relations (2.15) and (2.16) we get in fact the EOM in (2.14).

For the moment, let's focus in particular on the first equation (2.10), rewriting it in terms of the four-vector A_μ instead of $F_{\mu\nu}$, in order to simplify the computations (for this treatment we will follow the work done by Li and Zhang in [**1**]):

$$\nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) = -p_\mu \epsilon^{\mu\nu\rho\sigma}(\nabla_\rho A_\sigma - \nabla_\sigma A_\rho) \tag{2.17}$$

$$\Box A^\nu - \nabla_\mu \nabla^\nu A^\mu = -p_\mu \epsilon^{\mu\nu\rho\sigma}(\nabla_\rho A_\sigma - \nabla_\sigma A_\rho) \tag{2.18}$$

where we have applied the Lorenz gauge, for which $\nabla_\mu A^\mu = 0$, to recover the equation (2.18). The term with the double derivative $(\nabla_\mu \nabla^\nu A^\mu)$ can be rewritten accounting for the commutation of covariant derivatives[1] acting on a vector field, which gives rise to a term that contains the Ricci tensor:

$$\Box A^{\nu} + R^{\nu}_{\mu} A^{\mu} = -p_{\mu} \epsilon^{\mu\nu\rho\sigma} (\nabla_{\rho} A_{\sigma} - \nabla_{\sigma} A_{\rho}) \tag{2.22}$$

In order to write down solutions of this equation we can adopt the so-called *geometric optics approximation* (or short wavelength approximation); indeed the scale of variation of the electromagnetic field is much smaller than the cosmological scale of interest; in other words this means that the wavelength of the photons is well below the typical length scale associated with the curvature of the spacetime (see [**17**]). From another point of view it means that the frequency of the photons is much larger than the mass of the axion field (this is reasonable for ultralight particles). Under this assumption, the waves can be considered as plane waves propagating in a spacetime with negligible curvature, so that the solution can be expressed (in terms of A_{μ}) as:

$$A^{\mu} = Re[(a^{\mu} + \epsilon b^{\mu} + \epsilon^2 c^{\mu} + ...)e^{iS/\epsilon}] \tag{2.23}$$

or, in terms of the strength tensor:

$$F^{\mu\nu} = (a^{\mu\nu} + \epsilon b^{\mu\nu} + \epsilon^2 c^{\mu\nu} + ...)e^{iS/\epsilon} \tag{2.24}$$

where S is a real function and S/ϵ is a phase; this means that the phase varies much faster than the amplitude of the electromagnetic wave. Furthermore we can define the wave-vector as: $k_{\mu} = \nabla_{\mu} S$.
Substituting the solution (2.23) in the equation of motion (2.18), and neglecting the terms with the Ricci tensor (since, under the chosen approximation, the curvature can be neglected), we obtain:

$$\nabla_{\mu}\nabla^{\mu}(a^{\nu} + \epsilon b^{\nu} + ...) + 2\frac{i}{\epsilon}k^{\mu}\nabla_{\mu}(a^{\nu} + \epsilon b^{\nu} + ...) + \frac{i}{\epsilon}(\nabla_{\mu}k^{\mu})(a^{\nu} + \epsilon b^{\nu} + ...) -$$

$$-\frac{1}{\epsilon^2}k_{\mu}k^{\mu}(a^{\nu} + \epsilon b^{\nu} + ...) = -p_{\mu}\epsilon^{\mu\nu\rho\sigma}[\nabla_{\rho}(a^{\sigma} + \epsilon b^{\sigma} + ...) - \nabla_{\sigma}(a^{\rho} + \epsilon b^{\rho} + ...) +$$

[1] Here we have exploited the property of the commutators applied to a four-vector:

$$[\nabla_{\mu}, \nabla_{\nu}]A^{\mu} = \nabla_{\mu}\nabla_{\nu}A^{\mu} - \nabla_{\nu}\nabla_{\mu}A^{\mu} \tag{2.19}$$

applying the Lorentz gauge and the definition of covariant derivative we get:

$$\nabla_{\mu}\nabla_{\nu}A^{\mu} + \nabla_{\mu}(\Gamma^{\mu}_{\nu\rho}A^{\rho}) - \Gamma^{\lambda}_{\mu\nu}(\nabla_{\lambda}A^{\mu}) - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\mu}_{\lambda\rho}A^{\rho} - (\mu \longleftrightarrow \nu) \tag{2.20}$$

and finally, recalling the definition of the Ricci tensor ($[\nabla_{\mu}, \nabla_{\nu}]A^{\mu} = R_{\mu\nu}A^{\mu}$):

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \nabla_{\mu}\Gamma^{\mu}_{\nu\rho} - \nabla_{\lambda}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\mu}_{\lambda\rho} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\nu}_{\lambda\rho} \tag{2.21}$$

$$+\frac{i}{\epsilon}(k_\rho(a^\sigma + \epsilon b^\sigma + ...) - k_\sigma(a^\rho + \epsilon b^\rho + ...)] \tag{2.25}$$

If we focus on terms of order $\frac{1}{\epsilon^2}$ we get the usual relation: $k_\mu k^\mu = 0$, which simply means that photons propagate along null geodesics; this relation it is not affected by the Chern-Simons modification, so it is not so interesting for us; still it is a confirmation of the consistency of the theory and the procedure that we are adopting. Differentiating the norm of k_μ, and using the previous definition in terms of S, we get:

$$0 = \nabla_\nu(k_\mu k^\mu) = \nabla_\nu(\nabla_\nu S \nabla^\nu S) = 2\nabla^\mu S \nabla_\nu \nabla_\mu S = 2k^\mu \nabla_\nu k_\mu \tag{2.26}$$

where the last factor can be rewritten as a geodesic equation:

$$\frac{dx^\mu}{d\lambda}\nabla_\nu k_\mu = 0 \tag{2.27}$$

which means that the vector k_μ is parallely transported along the light curve $x^\mu(\lambda)$; this remarks the fact that at this order $(1/\epsilon^2)$ the Chern-Simons term doesn't have any role.
On the other hand, if we focus on the $(1/\epsilon)$-order terms in equation (2.25), we can see that they are affected by the coupling; indeed we have:

$$k^\mu \nabla_\nu a^\nu + \frac{1}{2}\nabla_\mu k^\mu a^\nu = -p_\mu \epsilon^{\mu\nu\rho\sigma}(k_\rho a_\sigma - k_\sigma a_\rho) \tag{2.28}$$

where we can define $\mathcal{D} \equiv k^\mu \nabla_\mu$ and $\theta \equiv \nabla_\mu k^\mu$, so that in the end we can write down the final propagation equation for a^ν as:

$$\mathcal{D}a^\nu + \frac{1}{2}\theta a^\nu = -p_\mu \epsilon^{\mu\nu\rho\sigma}(k_\rho a_\sigma - k_\sigma a_\rho) \tag{2.29}$$

We can also recast this relation in terms of the polarization vector, instead of a^ν, taking into account for the fact that $a^\nu = A\epsilon^\nu$, where A is the scalar amplitude of the photon field:

$$\mathcal{D}\epsilon^\nu + \frac{1}{2}\theta \epsilon^\nu = -p_\mu \epsilon^{\mu\nu\rho\sigma}(k_\rho \epsilon_\sigma - k_\sigma \epsilon_\rho) \tag{2.30}$$

Without the modification due to the Chern-Simons coupling the right-hand side of the above equation would vanish (as in (2.27)), obtaining the previous situation of the parallel transport. Instead in this case the k_μ vector is not parallely tansported (equation 2.28) and also the polarization vector ϵ_ν, which means that the polarization plane rotates as the photon propagates in spacetime. Since the photons that we are considering come from the CMB radiation, they are linearly polarized due to the effect of Thomson scattering at the recombination epoch; the coupling with the scalar field doesn't change the type of polarization but only the angle of the linear polarization plane.

We can show the effect of the Chern-Simons term with a different approach, used by E. Komatsu in [**2**]. Differently from the method carried out by Li and Zhang he makes the computations taking the vector $\mathbf{A}$ instead of the four-vector A_μ, obtaining the equation of motion:

$$\mathbf{A}''_\pm + \left(k^2 \mp \frac{k\alpha\chi'}{f}\right)\mathbf{A}_\pm = 0 \tag{2.31}$$

where the $'$ denotes the time derivative. The $\pm$ indicates the two helicity states for the photon, α is a dimensionless coupling constant and χ is the scalar field in this case, while f is a *decay constant* with the dimension of an energy. In this case the constant factor is different from the one in equation (2.6) but they have the same physical meaning of a coupling factor. From this EOM we can observe two interesting aspects: at first the effect of the CS term vanishes when the field doesn't depend on time, so that it must have a background evolution in order to see some effects; then, the second consideration is that this equation leads to two different dispersion relations for the + and - helicity states, which causes a difference in the phase velocity, bringing to a rotation of the polarization plane. Indeed the birefringence angle, which quantifies this rotation, depends on the difference between the two phase velocities ω_+ and ω_-, given by:

$$\frac{\omega_\pm}{k} \simeq 1 \mp \frac{\alpha\chi'}{2kf} \tag{2.32}$$

and in the end the polarization angle (here it is denoted as β) can be computed as follows:

$$\beta = -\frac{1}{2}\int d\tau(\omega_+ - \omega_-) \tag{2.33}$$

where τ is the conformal time.

Coming back to the previous treatment, still following Li and Zhang, we can proceed by introducing the formalism of the Stokes parameters, since they are very useful to analyze the polarization of radiation. Since we are dealing with a curved spacetime, we need to generalize them in the context of general relativity. In order to do this, let's start from defining the Stokes parameters in Minkowsky spacetime, for a monochromatic wave propagating along the z axis with $E_x = a_x(t)\exp[i(\omega_0 t - \theta_x(t))]$ and $E_y = a_y(t)\exp[i(\omega_0 t - \theta_y(t))]$:

$$\begin{aligned} I &\equiv \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\ Q &\equiv \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle \\ U &\equiv \langle E_x E_y^* \rangle + \langle E_x^* E_y \rangle \\ V &\equiv i(\langle E_x E_y^* \rangle + \langle E_x^* E_y \rangle) \end{aligned} \tag{2.34}$$

where the $\langle ... \rangle$ parenthesis stand for an *ensemble* average, equivalent to averaging over many periods of the wave, and the * indicates the complex conjugate field. These relations can be generalized in GR exploiting the so-called *tetrad formalism*. A tetrad is a set of four basis vectors $e^{\mu}_{(a)}$ which allow us to move from the coordinate frame to the local inertial frame (L.I.F) in each point x of the spacetime. Indeed, in general, a vector B_μ can be rewritten in a L.I.F. in this way:

$$\bar{B}_a = e^{\mu}_{(a)} B_\mu \tag{2.35}$$

where the latin index a is the label used to indicate the component of the tetrad basis, while the greek letter μ denotes, as always, the spacetime index (from 0 to 3). In order to practically set the tetrad frame at each point it is convenient to consider the rest frame of the free fall observer, where the four-velocity acquires a very simple form: $\bar{u}^a = \delta^a_0$; moreover we assume that the observer sees the light travelling along the z direction, so that: $\bar{k}^a = \omega(\delta^a_0 + \delta^a_3)$. With these definitions we can rewrite the four-velocity and the four-momentum in the coordinate frame as:

$$u^\mu = e^{\mu}_{(a)} \delta^a_0 = e^{\mu}_{(0)} \tag{2.36}$$

$$k^\mu = e^{\mu}_{(a)} \bar{k}^a = \omega(\delta^a_0 + e^{\mu}_{(3)}) \tag{2.37}$$

then, from these two relations, we can recover the expressions for the tetrad basis vectors: $e^{\mu}_{(0)} = u^\mu$ and $e^{\mu}_{(3)} = \frac{1}{\omega}(k^\mu - \omega u^\mu)$, while the other two components $e^{\mu}_{(1)}$ and $e^{\mu}_{(2)}$ are unit spacelike vectors and they are orthogonal to k^μ. Since in relations (2.34) the components of the electric field are present , it is very useful to redefine them in the local inertial frame. In order to do so, we need to start from the definition of the electric field vector in a *generic* spacetime, as seen by an observer with four-velocity u^μ:

$$E^\mu = F^{\mu\nu} u_\nu \tag{2.38}$$

Exploiting the G.O.A. solution in (1.19) it can be rewritten as:

$$E^\mu = a^{\mu\nu} u_\nu e^{iS/\epsilon} = (k^\mu a^\nu - k^\nu a^\mu) u_\nu e^{iS/\epsilon} \tag{2.39}$$

From this generic relation we can find out the components of the electric vector in the L.I.F.:

$$\bar{E}_1 = E_x = E_\mu e^{\mu}_{(1)} \qquad \bar{E}_2 = E_y = E_\mu e^{\mu}_{(2)} \tag{2.40}$$

Since we are in the local inertial frame, the previous definitions of the Stokes parameters in a flat spacetime are applicable (eqs.(2.34)), and we can apply the above relations for the electric field in order to rewrite them in a generic curved spacetime. For this purpose we will follow the notation and the procedure proposed by A.M. Anile and R.A. Breuer in [**8**], and also in S. Kopeikin and P. Korobkov in [**9**], where they start from the polarization tensor defined in this way:

$$J_{\mu\nu\rho\sigma} = \frac{1}{2}\left\langle F_{\mu\nu}F^{*}_{\rho\sigma}\right\rangle \tag{2.41}$$

In the rest frame of the free-fall observer we can rewrite this tensor as $J_{\mu\nu} = J_{\mu\nu\rho\sigma}u^{\rho}u^{\sigma}$, and, applying the relation (2.31) it becomes:

$$J_{\mu\nu} = \left\langle E_{\mu}E^{*}_{\nu}\right\rangle \tag{2.42}$$

Notice that this polarization tensor is orthogonal both to the four-velocity and to the four-momentum. It is useful to redefine $J_{\mu\nu}$ in a most suitable form:

$$J_{\mu\nu} = \frac{1}{2}\omega^{2}L_{\mu\nu} \tag{2.43}$$

where $\omega = u_{\mu}k^{\mu}$ and $L_{\mu\nu} \equiv \left\langle a_{\mu}a^{*}_{\nu}\right\rangle$ (the same definition used in [**4**]). With the help of these relations we can write down the four Stokes parameters in the tetrad frame:

$$\begin{aligned}
I = S_0 &= J_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(1)} + e^{\mu}_{(2)}e^{\nu}_{(2)})\\
Q = S_1 &= J_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(1)} - e^{\mu}_{(2)}e^{\nu}_{(2)})\\
U = S_2 &= J_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(2)} + e^{\mu}_{(2)}e^{\nu}_{(1)})\\
V = S_3 &= iJ_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(2)} - e^{\mu}_{(2)}e^{\nu}_{(1)})
\end{aligned} \tag{2.44}$$

which indeed are consistent because if we use the relation (2.42) we recover the Stokes parameters in a flat spacetime. Then, if we insert the relation (2.43) inside these last equations for the Stokes parameters we get:

$$S_A = \frac{1}{2}\omega^{2}F_A \tag{2.45}$$

where A = 0,1,2,3; so in the end we can rewrite the *generalized* Stokes parameters as:

$$\begin{aligned}
I &= \omega^{2}L_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(1)} + e^{\mu}_{(2)}e^{\nu}_{(2)})\\
Q &= \omega^{2}L_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(1)} - e^{\mu}_{(2)}e^{\nu}_{(2)})\\
U &= \omega^{2}L_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(2)} + e^{\mu}_{(2)}e^{\nu}_{(1)})\\
V &= i\omega^{2}L_{\mu\nu}(e^{\mu}_{(1)}e^{\nu}_{(2)} - e^{\mu}_{(2)}e^{\nu}_{(1)})
\end{aligned} \tag{2.46}$$

Since the tensor $L_{\mu\nu}$ is defined trough the amplitude of the electromagnetic field as $\langle a_\mu a_\nu^* \rangle$, we can rewrite the propagation equation (2.24) in terms of it, which is very useful for showing explicitly the effect of the Chern-Simons term on the polarization plane. The equation reads:

$$L_{\mu\nu} + \theta L_{\mu\nu} = -p_\alpha k_\beta \big(\epsilon_\mu^{\alpha\beta\gamma} L_{\gamma\nu} + \epsilon_\nu^{\alpha\beta\gamma} L_{\mu\gamma} \big) \tag{2.47}$$

Notice that the Stokes parameters are not Lorentz scalars, and so we must require the tetrad frame to be not rotating, i.e. $De_a^\mu = 0$: this can be done by imposing the parallel transport for the basis vector $e_{(1)}^\mu$ and $e_{(2)}^\mu$. From (2.47) it is easy to get the propagation equation for the four Stokes parameters:

$$DF_0 + \theta F_0 = 0 \tag{2.48}$$

$$DF_1 + \theta F_1 = 2p_\mu k^\mu F_2 \tag{2.49}$$

$$DF_2 + \theta F_2 = -2p_\mu k^\mu F_1 \tag{2.50}$$

$$DF_3 + \theta F_3 = 0 \tag{2.51}$$

where F_A is defined in relation (2.45). The first equation is simply the conservation of the light flux, while the last one (related to V) tells us that the circular polarization remains null if it is zero at the beginning: since CMB photons are not circularly polarized they don't acquire this kind of polarization along their journey towards us. It is interesting to highlight that the Chern-Simons coupling enters only in the equations for Q and U, which indeed are the ones related to the linear polarization: this means that the presence of the axion field rotates the polarization plane, maintaining a linear polarization. Notice also that the two equations are mixed in F_1 and F_2, so they are not independent. In general the polarization angle is defined as $2\chi = \arctan \frac{U}{Q}$, so that:

$$\chi = \frac{1}{2} \arctan \frac{F_2}{F_1} \tag{2.52}$$

This angle still satisfies a propagation equation, obtained by dividing both sides of equation (2.50) by F_1, and assuming that χ is a small angle:

$$D\chi + p_\mu k^\mu = 0 \tag{2.53}$$

In the end, this equation can be easily integrate along the light path of the photon, by recalling the definition $k^\mu = \frac{dx^\mu}{d\lambda}$:

$$\frac{d\chi}{d\lambda} = -p_\mu k^\mu \longrightarrow \int_i^f d\chi = -\int_i^f p_\mu k^\mu d\lambda \tag{2.54}$$

and so the birefringence angle can be found with:

$$\chi_f - \chi_i = -\int_i^f p_\mu dx^\mu(\lambda) \tag{2.55}$$

where the integral is computed from the recombination epoch to today. If we want to write the explicit expression for the rotation angle, we can insert the definition of p_μ, obtaining:

$$\Delta\chi = \frac{2c\alpha_{em}}{3\pi M}\int_i^f \partial_\mu\phi dx^\mu(\lambda) = \frac{2c\alpha_{em}}{3\pi M}(\phi_f - \phi_i) \tag{2.56}$$

This relation confirms the fact that the birefringence angle depends on the difference between the value of the scalar field at the recombination and at the present epoch, and so on the distance travelled by the photons: for this reason using the CMB radiation as a probe is the best choice possible in order to ensure an obserable angle today. Moreover. from (2.56) it is evident that ϕ must evolve (at least) in time in order to have a non-zero effect on the polarization angle. In general the scalar field could also be space-dependent, not homogeneous, as we will discuss in *Chapter 3*.

2.2 Observational constraints on the isotropic birefringence angle

Up to now we have keep the discussion on the birefringence effect on a theoretical point of view, deriving the expressions for the rotation angle; now we want to find a connection between theory and observations, exploiting in particular the different power spectra related to the birefringence angle. There are a lot of articles which analyze deeply the power spectra in order to obtain a measure of the polarization angle: in this section we will refer mainly to [**22**], [**23**] and [**24**].

As suggested in the *Section 1.2*, we can visualize better the rotation of the polarization plane due to the birefringence effect by exploiting a linear combination of the Stokes parameters Q and U, i.e. $(Q \pm iU)(\hat{n})$, which is used to describe the linear polarization plane. This combination can be written through a decomposition of spin-2 spherical harmonics, since it behaves as a spin-2 field, in this way:

$$(Q \pm iU)(\hat{n}) = -\sum_{l=2}^{l_{max}}\sum_{m=-l}^{l}(E_{lm} \pm B_{lm})_{\pm 2}Y_l^m(\hat{n}) \tag{2.57}$$

where E_{lm} and B_{lm} are the spherical harmonics coefficients of the E and B polarization modes. In particular, under inversion of spatial coordinates ($\hat{n} \longrightarrow -\hat{n}$), the first ones are parity-even, i.e. they transform with $(-1)^l E_{lm}$, while the second ones are parity-odd, since they transform through $(-1)^{l+1}B_{lm}$. Then, since E_{lm} and B_{lm} are stochastic variables, it is very useful to analyze their angular power spectra, defined from the variance of the two coefficients:

$$\langle E_{lm}E^*_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}C_l^{EE} \tag{2.58}$$
$$\langle B_{lm}B^*_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}C_l^{BB} \tag{2.59}$$

so that the power spectrum can be defined through the squared amplitude of the spherical harmonic coefficients:

$$C_l^{EE} = \langle |E_{lm}|^2\rangle \tag{2.60}$$
$$C_l^{BB} = \langle |B_{lm}|^2\rangle \tag{2.61}$$

In the end, averaging over m, and assuming that the Universe is statistically isotropic (see [**2**]), we obtain:

$$C_l^{EE} = \frac{1}{2l+1}\sum_{m=-l}^{l}|E_{lm}|^2 \tag{2.62}$$

$$C_l^{BB} = \frac{1}{2l+1}\sum_{m=-l}^{l}|B_{lm}|^2 \tag{2.63}$$

Both of these auto-correlation functions are parity-even, together with the cross-correlation C_l^{TE}; this means that they are invariant (i.e. they don't change sign) under parity transformation. On the other hand, the two parity-odd CMB power spectra are:

$$C_l^{TB} = \frac{1}{2l+1}\sum_{m=-l}^{l}Re(T_{lm}B^*_{lm}) \tag{2.64}$$

$$C_l^{EB} = \frac{1}{2l+1}\sum_{m=-l}^{l}Re(E_{lm}B^*_{lm}) \tag{2.65}$$

Both of these have been used to probe the birefringence effect, but the EB spectrum is the most sensitive one; we can directly compute the birefringence angle χ from this last cross-correlation.
In order to make the rotation of the polarization plane more evident, we can still consider the combination defined in eq.(2.57) and see how it transforms under a rotation of the $\hat{n}$ vector, i.e.:

$$(Q \pm iU)^o = (Q \pm iU)e^{\pm 2i\chi} \tag{2.66}$$

where the o on the left hand side indicates the value observed today, while on the right hand side we find the value at the last scattering surface (which can be unaffected by the birefringence phenomenon); the exponential term is given by the rotation induced by the interaction of CMB photons with the axion-like field. It is more convenient to recast the previous relation in terms of E and B modes coefficients:

$$E^o_{lm} \pm iB^o_{lm} = (E_{lm} \pm B_{lm})e^{\pm 2i\chi} \tag{2.67}$$

Replacing the exponential with $\cos 2\chi \pm i \sin \chi$ we get:

$$E^o_{lm} = E_{lm} \cos 2\chi - B_{lm} \sin 2\chi \tag{2.68}$$

$$B^o_{lm} = E_{lm} \sin 2\chi + B_{lm} \cos 2\chi \tag{2.69}$$

In this way the rotated coefficients are written through the usual rotation matrix (rotation of an angle 2χ):

$$R = \begin{pmatrix} \cos 2\chi & -\sin 2\chi \\ \sin 2\chi & \cos 2\chi \end{pmatrix} \tag{2.70}$$

Adopting the rotated coefficients in eqs.(2.68) and (2.69), it's possible to re-define the rotated power spectra in (2.62), (2.63) and (2.65), in the following way:

$$C_l^{EE,o} = \cos^2(2\chi)C_l^{EE} + \sin^2(2\chi)C_l^{BB} \tag{2.71}$$

where we have considered the squared amplitude $|E^o_{lm}|^2$ as the product between E^o_{lm} and its complex conjugate $E^{o,*}_{lm}$, and then we have used the relations (2.60) and (2.61). In the same way we get the expression for the BB correlation function:

$$C_l^{BB,o} = \sin^2(2\chi)C_l^{EE} + \cos^2(2\chi)C_l^{BB} \tag{2.72}$$

At this point we can insert (2.71) and (2.72) in equation (2.65) in order to obtain the observed EB spectrum:

$$C_l^{EB,o} = \frac{1}{2}\sin 4\chi(C_l^{EE} - C_l^{BB}) + C_l^{EB}\cos 4\chi \tag{2.73}$$

Looking to this fundamental relation we can notice two relevant aspects: on one hand the intensity of the EB spectrum depends on the *asymmetry* between E and B modes, on the other hand this correlation is non-null even if the intrinsic contribution C_l^{EB} at LSS vanishes (on the right hand side of equation (1ì2.73)). In the end, we can also rewrite the EB spectrum in terms of the *observed* EE and BB spectra, starting from the rotated relations (2.71) and (2.72) and recasting them in order to get the intrinsic quantities as a function of the observed ones:

$$C_l^{BB} = \frac{\cos^2(2\chi)}{\cos^4(2\chi) - \sin^4(2\chi)} C_l^{BB,o} - \frac{\sin^2(2\chi)}{\cos^4(2\chi) - \sin^4(2\chi)} C_l^{EE,o} \tag{2.74}$$

$$C_l^{EE} = \frac{\cos^2(2\chi)}{\cos^4(2\chi) - \sin^4(2\chi)} C_l^{EE,o} - \frac{\sin^2(2\chi)}{\cos^4(2\chi) - \sin^4(2\chi)} C_l^{BB,o} \tag{2.75}$$

Inserting these relations in (2.73), we obtain:

$$C_l^{EB,o} = \frac{1}{2} \tan 4\chi (C_l^{EE,o} - C_l^{BB,o}) + \frac{C_l^{EB}}{\cos 4\chi} \tag{2.76}$$

Another time we can observe that, even in absence of an intrinsic EB correlation at the last scattering surface, the observed EB spectrum is not null, due to the coupling of photons with DM particles along their trip towards us. One possibility to explain the presence of an intrinsic contribution to the EB spectrum is related to parity-violating primordial gravitational waves.
The main aspect that we can highlight from the last equation (2.76) is the dependence of the EB correlation amplitude from the difference between EE and BB correlations: this is connected to the fact that χ mixes E and B polarization modes, such that part of the first one is transferred in the second one. From the observations about the CMB power spectra we can see that there is a large asymmetry between $C_l^{EE,o}$ and $C_l^{BB,o}$, which makes the EB spectrum a very sensitive probe of cosmic birefringence (see *figure 2.1*).

One important issue, which cannot be neglected, in measuring the birefringence angle χ, is the miscalibration of the polarization angle in the detectors. Indeed this miscalibration angle (called α) leads to the same effect as isotropic birefringence, so we need a tool to separate and differentiate the two contributions. The main problem is that from the observations we can only get estimations of the combination $\alpha + \chi$, and not two independent measurements of them; this means that α and χ are degenerate. One interesting solution to this issue has been proposed by Minami and Komatsu in [**22**], where they have exploited another source of polarized light, the Galactic foreground emission, for which the cosmic birefringence effect can be neglected since the path length of the photons is much smaller than in the case of CMB radiation, which instead is affected by both α and χ. Thus, polarization from the foreground emission is tilted only by α, while the one from CMB by $\alpha + \chi$. In this case the *extended* observed E and B-mode coefficients read:

$$\begin{aligned} E_{lm}^o = E_{lm}^{FG} \cos(2\alpha) - B_{lm}^{FG} \sin(2\alpha) + E_{lm}^{CMB} \cos(2\alpha + 2\chi) \\ -B_{lm}^{CMB} \sin(2\alpha + 2\chi) \end{aligned} \tag{2.77}$$

$$\begin{aligned} B_{lm}^o = E_{lm}^{FG} \sin(2\alpha) + B_{lm}^{FG} \cos(2\alpha) + E_{lm}^{CMB} \sin(2\alpha + 2\chi) \\ +B_{lm}^{CMB} \cos(2\alpha + 2\chi) \end{aligned} \tag{2.78}$$

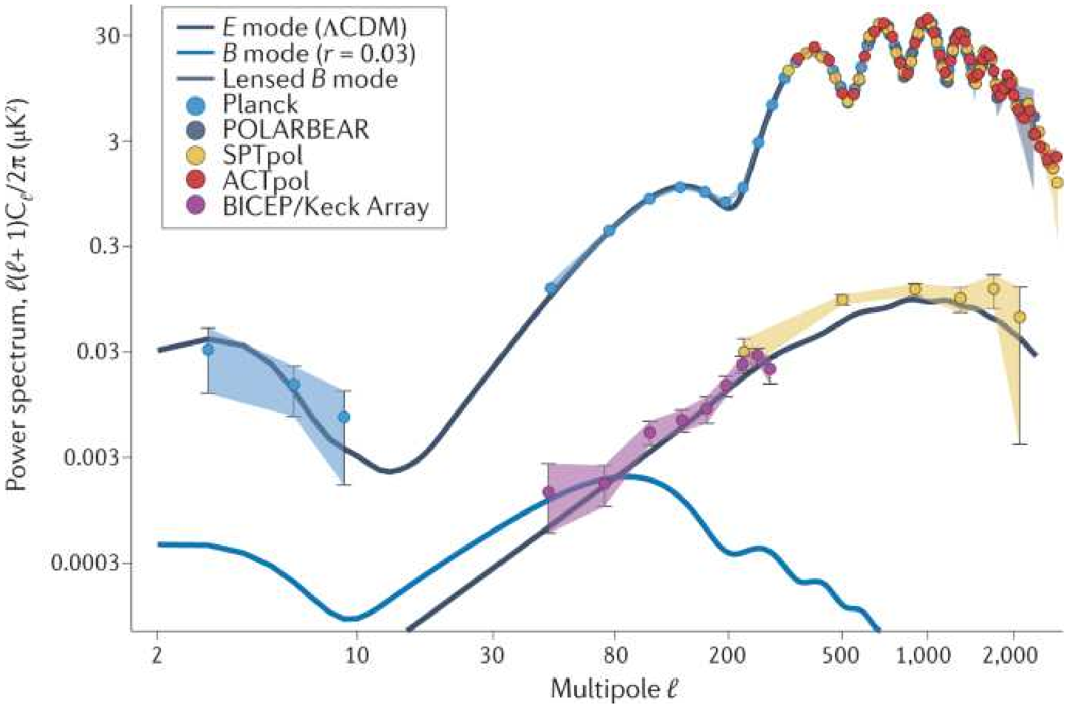

Figure 2.1: In this plot the observed CMB polarization power spectra are shown, in particular the ones related to E and B modes: C_l^{EE} and C_l^{BB}, in units of μK^2. The data are taken from the current generation of CMB experiments: *Planck*, Polarbear, South Pole Telescope (SPTpol), Atacama Cosmology telescope (ACTpol) and BICEP/Keck Array. The filled dots are the data points, averaged over multipoles in bins centered at l, while the error bars are related to the 68% confidence level. The lines represents the best-fitting ΛCDM model for the E-mode and the lensed B-mode, and the B-mode power-spectrum of the primordial GWs for the tensor-to-scalar parameter of 0.03. (Credits: Komatsu (2022)[**2**])

Then, by adopting a similar approach as the one used to get equation (2.76), from these last two relations we obtain:

$$C_l^{EB,o} = \frac{\tan(4\alpha)}{2}\left(C_i^{EE,o} - C_l^{BB,o}\right) + \frac{\sin(4\chi)}{2\cos(4\alpha)}\left(C_l^{EE,CMB} - C_l^{BB,CMB}\right) + \\ + \frac{1}{\cos(4\alpha)}C_l^{EB,FG} + \frac{cos(4\chi)}{cos(4\alpha)}C_l^{EB,CMB} \tag{2.79}$$

This allow us to determine simultaneously α and χ, as $C_l^{EE,CMB}$ and $C_l^{BB,CMB}$ are known precisely. Notice that the knowledge of $C_l^{EE,FG}$ and $C_l^{BB,FG}$ is not required, but we need only $C_l^{EB,FG}$. For the sensitivity of the current experiments, we can ignore $C_l^{EB,CMB}$, which is the intrinsic EB correlation at the last scattering surface.
The measurements about the two angles from *Planck* polarization data are shown in [**22**], [**23**] and [**24**]. In particular in [**22**] the results for χ and α from the 2018 *Planck* polarization data are reported. At first they have considered

Angles	Results (deg)
β	0.35 ± 0.14
α_{100}	-0.28 ± 0.13
α_{143}	0.07 ± 0.12
α_{217}	-0.07 ± 0.11
α_{353}	-0.09 ± 0.11

Figure 2.2: Cosmic birefringence (β angle) and miscalibration angles from the Planck 2018 polarization data with 1σ (68%) uncertainties. (Credit: Y. Minami and E. Komatsu (2020)[**22**])

the case of a perfect calibration of the detectors, such that $\alpha = 0$, finding a value for the birefringence angle of $\chi = 0.289 \pm 0.048$ deg. On the other hand, they have estimated χ and α simultaneously at four different frequencies, in order to eliminate the systematic uncertainty caused by the miscalibration of detectors. The results are shown in *Table 2.2* and in *Figure 2.3*, where it is evident that α_ν and χ are anticorrelated, since their degeneracy is broken by the foreground emission. In the end the measure for the rotation angle is: $\chi = 0.35 \pm 0.14$ deg, which excludes a zero value by 99.2%; this corresponds to a quite large statistical significance of 2.4σ. These measurements are based on the assumption of a null intrinsic EB correlation for the galactic foreground emission. However, we can see what should happen if we take into account also for this contribution to the overall EB spectrum. The dominant foreground source in *Planck* HFI channels is the dust emission, whose power spectrum can be parametrized by a further frequency-dependent angle $\gamma(\nu)$ in this way:

$$C_l^{EB,dust} = \frac{\sin(4\gamma(\nu))}{2}\left(C_l^{EE,dust} - C_l^{BB,dust}\right) \tag{2.80}$$

similarlly to the power spectrum in (2.73) for the χ angle. In the case in which γ doesn't depend on frequency, it cannot be distinguish from the birefringence angle, so that we can measure only the combination $\chi - \gamma = 0.35 \pm 0.14$ deg. Since a positive velue of γ is expected, it means that the previous results (in the case with $\gamma = 0$) give a lower bound for χ.

In ref. [**23**] the same procedure used in [**22**] is exploited, but it has been applied to the *Planck* data release 4 ($PR4$), finding a value for the birefringence angle of $\beta = 0.30 \pm 0.11$ deg. All the results for the birefringence and miscalibration angles are in agreement with the previous analysis, as can be seen in the table in *Figure 2.4*, where the measurements for the different effective sky fractions f_{sky} are reported. We can notice that the birefringence angle (here is denoted as β) decreases for smaller f_{sky}, i.e. for larger Galactic masks. This fact could be explained by the presence of an intrinsic EB correlation in the foreground emission, attributed mainly to the polarized dust emission, as suggested previously. Thus, improving the characterization of the foreground emission, in particular of its EB power spectrum, is fundamental to determine

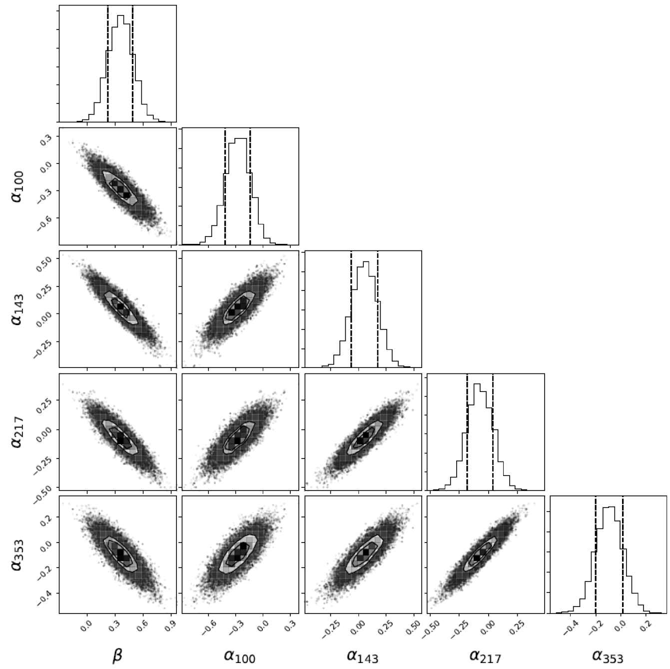

Figure 2.3: Distributions of β against the miscalibration angles. The solid contour lines in the 2D histograms show 1σ and 2σ of each area. The dashed lines of the 1D histograms show 1σ (from 16% to 84%) of each area.(Credit: Y. Minami and E. Komatsu (2020)[**22**])

which f_{sky} value gives us the most robust value for the birefringence angle.
In order to conclude this overview on the main results reached up to now on the CB angle, in ref. [**24**] Eskilt and Komatsu have exploited a joint analysis of the EB spectra from both WMAP and *Planck* (PR4 maps) missions, which cover a wide range of frequencies (from 23 Hz to 353 Hz). In this way they have measured a value of $\chi = 0.342 + 0.094/-0.091$ deg (at 68 % C.L.) for a nearly full-sky data ($f_{sky} = 0.92$), excluding zero at 99.987 %. This correspond to a statistical significance of 3.6σ, which is the largest value reached up to now. In the future experiments it is reasonable to think that this statistical significance will be overcome by full-sky missions such as LiteBIRD.
One last interesting aspect that can be mentioned about the CB observations, as reported in [**22**], is the possibility to estimate the coupling constant $g_{\phi\gamma}$ contained in the Chern-Simons term ($\mathscr{L}_{int} \propto g_{\phi\gamma}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$), so that:

$$g_{\phi\gamma}\big(\phi_0(t_0) - \phi_0(t_{rec}) + \delta\phi\big) = (1.2 \pm 0.5) \cdot 10^{-2} rad \tag{2.81}$$

This allow us to constrain and differentiate among different models for the scalar field ϕ.

f_{sky}	0.93	0.90	0.85	0.75	0.63
β	0.36 ± 0.11	0.26 ± 0.14	0.14 ± 0.17	0.10 ± 0.21	0.29 ± 0.28
$\alpha_{100\mathrm{A}}$	-0.32 ± 0.13	-0.17 ± 0.16	-0.07 ± 0.19	-0.01 ± 0.23	-0.21 ± 0.29
$\alpha_{100\mathrm{B}}$	-0.43 ± 0.13	-0.32 ± 0.16	-0.20 ± 0.19	-0.14 ± 0.22	-0.28 ± 0.29
$\alpha_{143\mathrm{A}}$	0.03 ± 0.11	0.13 ± 0.14	0.29 ± 0.18	0.40 ± 0.21	0.22 ± 0.28
$\alpha_{143\mathrm{B}}$	0.15 ± 0.11	0.25 ± 0.14	0.37 ± 0.18	0.39 ± 0.22	0.21 ± 0.28
$\alpha_{217\mathrm{A}}$	-0.06 ± 0.11	0.10 ± 0.14	0.22 ± 0.17	0.21 ± 0.21	0.02 ± 0.28
$\alpha_{217\mathrm{B}}$	-0.07 ± 0.11	0.07 ± 0.14	0.23 ± 0.17	0.23 ± 0.21	0.003 ± 0.28
$\alpha_{353\mathrm{A}}$	-0.19 ± 0.10	-0.08 ± 0.13	0.12 ± 0.17	0.03 ± 0.21	-0.09 ± 0.28
$\alpha_{353\mathrm{B}}$	-0.23 ± 0.11	-0.10 ± 0.13	0.10 ± 0.17	0.02 ± 0.21	-0.02 ± 0.29
$10^2 A_{51-130}$	$2.5^{+1.6}_{-1.4}$	$5.7^{+2.5}_{-2.4}$	$3.4^{+1.8}_{-1.7}$	$18.8^{+6.0}_{-6.1}$	$14.1^{+3.8}_{-3.7}$
$10^2 A_{131-210}$	$0.8^{+1.2}_{-0.6}$	$4.3^{+5.3}_{-3.1}$	$9.8^{+4.2}_{-4.0}$	$4.3^{+3.4}_{-2.8}$	$2.6^{+2.9}_{-1.8}$
$10^2 A_{211-510}$	$1.5^{+2.4}_{-1.1}$	$7.3^{+6.0}_{-4.7}$	$5.1^{+4.9}_{-3.4}$	$1.6^{+2.2}_{-1.2}$	$3.1^{+3.2}_{-2.1}$
$10^2 A_{511-1490}$	$6.2^{+5.7}_{-4.1}$	$4.2^{+4.2}_{-2.9}$	$6.2^{+6.7}_{-4.3}$	$4.9^{+5.3}_{-3.4}$	$5.8^{+5.3}_{-3.8}$

Figure 2.4: Cosmic birefringence and miscalibrtion angles with 1σ uncertainty, correcting for the foreground EB correlation. In the bottom part the foreground EB amplitudes A_l, in four multipole bins, are shown. (P.D.-Palazuelos et al. (2022)[**23**])

Chapter 3

The anisotropic birefringence effect

In the second chapter we have focused our analysis on the computation of the isotropic birefringence angle in different kind of models, and we have shown that there is a non null rotation of the polarization plane only if the scalar field ϕ evolves in time, i.e. it is characterized by a background evolution. However, in general, an axion-like field, which is a dynamical field, can also depend on spatial coordinates, which means that it is a function of $\hat{n}$, the direction in the sky. This spatial dependence is introduced by taking into account for fluctuations of the scalar field ϕ; indeed we can rewrite it by adopting a perturbative approach, in this way:

$$\phi(\mathbf{x}, \tau) = \phi_0(t) + \delta\phi(\mathbf{x}, \tau) \tag{3.1}$$

where $\delta\phi$ are small fluctuations with a null vacuum expectation value $\langle\delta\phi\rangle$. The relevant aspect brought about these perturbations is an additional anisotropic contribution to the birefringence angle, so that we can write also the rotation angle as the sum of its *background* isotropic value and a space-dependent small perturbation which depends on $\hat{n}$. We can recover the extended perturbed expression for the birefringence angle by inserting eq.(3.1) in (2.55) and (2.56):

$$\chi_f - \chi_i = \frac{\lambda}{2f}\int_i^f d(\phi_0 + \delta\phi) = \frac{\lambda}{2f}[\phi_0(\tau_0) - \phi_0(\tau)] + \frac{\lambda}{2f}[\delta\phi(\tau_0, \mathbf{x}) - \delta\phi(\tau, \mathbf{x})] =$$

$$= \frac{\lambda}{2f}[\phi_0(\tau_0) - \phi_0(\tau) - \delta\phi(\tau, \Delta\tau\hat{n})] \tag{3.2}$$

where we have used $\mathbf{x} = (\tau_0 - \tau)\hat{n}$, and we have replaced the coupling constant in (2.55) with $\lambda/2f$. This means that we can consider the birefringence angle as split into its isotropic background part and the anisotropic contribution, as done in perturbation theories:

$$\chi(\tau, \hat{n}) = \chi_0(\tau) + \delta\chi(\tau, \hat{n}) \tag{3.3}$$

where the fluctuations $\delta\chi$ are directly related to the fluctuations of the scalar field $\delta\phi$, i.e. :

$$\delta\chi(\tau,\hat{n}) = \frac{\lambda}{2f}\delta\phi(\tau,\Delta\tau\ \hat{n}) \tag{3.4}$$

This is the anisotropic birefringence angle. Another way to write it is the following (see [**25**]):

$$\delta\chi(\tau,\hat{n}) = -\frac{\beta}{M}\delta\phi(\mathbf{x}_{dec},\tau_{dec}) \tag{3.5}$$

where β is the dimensionless coupling constant and M is the mass scale for the effective field theory; notice that the scalar field fluctuations are referred to the decoupling epoch only, since they only give ride to a dipole contribution due to our motion with respect to the CMB frame.
Following the same reasoning adopted in *Section 2.2* for the isotropic CB angle, we can treat the rotation angle fluctuations $\delta\chi$ as a spin-2 field (two-dimensional fluctuation field), which can be expanded via a spherical harmonic decomposition as follows:

$$\delta\chi(\hat{n}) = \sum_{lm}\chi_{lm}Y_{lm}(\hat{n}) \tag{3.6}$$

where χ_{lm} are the coefficients of the decomposition. In order to find out a relation to define these coefficients, it is useful to expand $\delta\phi(\mathbf{x}_{dec},\tau_{dec})$ in Fourier space, in this way:

$$\delta\phi(\mathbf{x}_{dec},\tau_{dec}) = \int\frac{d^3k}{(2\pi)^{3/2}}\delta\phi_{\mathbf{k}}(\tau_{dec})e^{i\mathbf{k}\cdot\hat{n}\Delta\tau} \tag{3.7}$$

where $\delta\phi_{\bar{k}}$ is the Fourier transform of the $\delta\phi$ field. Comparing this relation with eqs.(3.5) and (3.6) we obtain (see [**4**] and [**39**]):

$$\chi_{lm} = -\frac{1}{2\pi^2}(-i)^l\frac{\beta}{M}\int d^3k\delta\phi_{(\bar{k})}(\tau_{dec})j_l(k\Delta\tau)Y^*_{lm}(\hat{k}) \tag{3.8}$$

Then, if the perturbation field $\delta\chi$ satisfies a Gaussian distribution, its statistical properties are fully determined by the two-point correlation function, which is defined as:

$$\langle\chi^*_{lm}\chi_{l'm'}\rangle = C_l^{\chi\chi}\delta_{ll'}\delta_{mm'} \tag{3.9}$$

where $C_l^{\chi\chi}$ is the auto-correlation power-spectrum of the CB rotation angle. In the end the angular power spectrum is given by (see [**4**] and [**25**]):

$$C_l^{\chi\chi} = 4\pi \left(\frac{\beta}{M}\right)^2 \int \frac{k^2 dk}{2\pi^2} P_{\delta\phi}(k)[j_l(k\Delta\tau)T_k(\tau_{dec})]^2 \tag{3.10}$$

where $j_l(x)$ is the spherical Bessel function, $\Delta\tau = \tau_0 - \tau_{dec}$, and $T_k(\tau_{dec})$ is the transfer function for the evolution of the perturbations up to τ_{dec}. Thus, the CB angle power spectrum is directly related to the power spectrum of $\delta\phi$ fluctuations at the decoupling time, which is defined as:

$$\langle \delta\phi^*_{k'}(\tau_{dec})\delta\phi_k(\tau_{dec})\rangle \equiv \frac{2\pi^2}{k^3} P_{\delta\phi}(k, \tau_{dec})\delta^3(\mathbf{k} - \mathbf{k}') \tag{3.11}$$

This means that, at first, we need to compute the fluctuation field $\delta\phi$ and its power spectrum, in order to get $C_l^{\chi\chi}$.
It is reasonable to ask why we are interested in the analysis of these tiny fluctuations in the rotation angle; the main reason is that anisotropic birefringence allows us to extract additional information about the scalar field, responsible of this effect, in a complementary way with respect to the isotropic case; in particular it encodes some relevant aspects on the model related to the axion field ϕ. Moreover, as suggested in [**26**], it opens the possibility to probe a wider range of masses for ϕ: indeed, larger is the mass, larger is the amplitude of the power spectra related to the anisotropic birefringence.

3.1 Evolution of scalar field fluctuations in the synchronous gauge

Since the fluctuations in the χ angle are sourced by the perturbations $\delta\phi$ of the axion-like field, we need to consider their evolution by solving the related equations of motion. As suggested in [**25**] it is more convenient to work out all the computations in the synchronous gauge, i.e. with $\delta\phi \equiv (\delta\phi)_{sync}$; however we will also move to the newtonian conformal gauge (Poisson gauge) later on. Let's quickly recall the main features of these two gauge choices. The synchronous gauge is based on the condition $\Psi = 0$, where Ψ is the so-called *lapse* perturbation, which is associated to the metric perturbation δg_{00} in the cosmological perturbation theory. Since the perturbed metric component can be defined as $g_{00} = -a^2(\tau)[1+2\Psi(\mathbf{x}, \tau)]$, and recalling the relation between the cosmic and the conformal time $dt^2 = a^2(\tau)d\tau$, then, imposing the condition $\Psi = 0$, we get the equality between the proper time ($t^2_{prop} = g_{00}d\tau^2$) and the cosmic time t; for this reason it is called *synchronous* gauge. On the other hand, the newtonian conformal gauge is very useful to express the evolution of scalar field perturbations in terms of the potentials ψ and Φ, which in this case are equal to the gauge-invariant Bardeen potentials Ψ_A and Φ_H (see *Section 1.3*).
We can start from recalling the general equation of motion for the background

field evolution (for ϕ_0):

$$\phi_0'' + 2\mathcal{H}\phi_0' + a^2\frac{\partial V(\phi_0)}{\partial\phi_0} = 0 \tag{3.12}$$

where the prime ($'$) indicates the derivative with respect to the conformal time; in this equation we have only the background field, whose evolution allow us to derive the isotropic birefringence angle in eq.(2.56). In this chapter we focus on the equation for the fluctuation field $\delta\phi$. As always we can exploit the least action principle in order to derive the EOM, starting from the following action:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4f}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}\right] \tag{3.13}$$

where the last term in the integral is the Chern-Simons coupling between the axion-like field and the photon. By varying this action with respect to ϕ (i.e. computing $\delta S/\delta\phi$) we get the following equation of motion:

$$\partial^\mu\partial_\mu\phi - \frac{\partial V}{\partial\phi} + \frac{\lambda}{4f}F_{\mu\nu}\tilde{F}^{\mu\nu} = 0 \tag{3.14}$$

where, as usual, we can rewrite $\partial^\mu\partial_\mu$ through the d'Alambertian operator in order to get a Klein-Gordon equation for ϕ, in this way:

$$\Box\phi = \frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\partial_\mu\phi\right)_{,\nu} \tag{3.15}$$

In this case we need to take into account for the metric $g^{\mu\nu}$ defined in the synchronous gauge:

$$g_{\mu\nu} = a^2(\tau)\begin{pmatrix}-1 & 0\\ 0 & \delta_{ij} + h_{ij}(\mathbf{x},\tau)\end{pmatrix} \tag{3.16}$$

from which we have: $\sqrt{-g} = a^4(\delta_{ij} + h_{ij})^{3/2}$, where h_{ij} is the perturbation of the metric in this particular gauge; usually this formalism is exploited in order to study primordial gravitational waves. Inserting the expression for $\sqrt{-g}$ inside equation (3.15) and limiting ourselves only up to *linear* perturbations (up to the linear order in h), we get this equation:

$$\frac{1}{a^2}\left[\mathcal{H}\phi' - \frac{3}{2}h_{ij}'\frac{\phi'}{\delta_{ij} + h_{ij}} - \phi'' - \frac{3}{2(\delta_{ij} + h_{ij})}\nabla h_{ij}\cdot\nabla\phi - \nabla^2\phi\right] - \frac{\partial V}{\partial\phi} = 0 \tag{3.17}$$

in which, for the moment, we have not included the Chern-Simons term.
At this point we can expand ϕ as in equation (3.1) and we can use the definition $h_{ij} = \frac{h}{3}\delta_{ij} + h_{ij}''$, where $h_{ij}'' = \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2$, in order to reach this final form of the EOM for $\delta\phi$:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + a^2\frac{\partial^2 V}{\partial\phi_0^2}\delta\phi - \nabla^2\delta\phi = -\frac{1}{2}h'\phi_0' \tag{3.18}$$

where h is the synchronous scalar metric perturbation; we can notice that, in the case of a vanishing background evolution of the scalar field ($\phi_0' = 0$), the right-hand side of equation (3.18) is null, so that the evolution of the fluctuations is uncorrelated with the perturbations of the metric.
What about the Chern-Simons term present in the action (3.13)? Since the strength tensor $F_{\mu\nu}$ contains the four vector A_μ, we need to take into account also for fluctuations in the photon field, such that we can decompose it in this way:

$$A_\mu(\tau, \mathbf{x}) = A_{0,\mu}(\tau) + \delta A_\mu(\tau, \mathbf{x}) \tag{3.19}$$

Then, using this relation inside the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we can rewrite the Chern-Simons term as:

$$\frac{\lambda}{4f}F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{i\lambda}{f}\epsilon^{ijk}\partial_0 A_{0,i}\partial_j\delta A_k \tag{3.20}$$

which should be added to the equation (3.18). However this coupling factor is negligible since, in order to preserve statistical isotropy of the Universe, the vacuum expectation value A_0 of the electromagnetic field must vanish; so the vector contribution to the EOM can be neglected (see [**26**]).
We can analyze two main possibilities which have been proposed in different works: we have models based on a *massless* scalar field (null scalar potential $V(\phi) = 0$) , while in other models ϕ is identified with the *quitessence* dark energy field. In the first case the background isotropic rotation angle is absent and the anisotropic counterpart could be quite large, so that it could be detected, for instance, by the *Planck* satellite or the CMBPol missions. On the contrary, in the second scenario, the background angle can be large, while the fluctuations are too much small to be detectable by the present experiments. Furthermore, another interesting difference between the two models, consists in the fact that in the massless case the cross-spectra $C_l^{\chi T}$ and $C_l^{\chi E}$ are null, while they are non-vanishing for the quintessence field.
Let's start from the first case, in which ϕ is a massless scalar field; the evolution equation in (3.12) is simplified in this way, since the scalar potential is null:

$$\phi_{0_k}'' + 2\mathcal{H}\phi_{0_k}' = 0 \tag{3.21}$$

On the other hand, the equation (3.18) for the evolution of fluctuations becomes:

$$\delta\phi_k'' + 2\mathcal{H}\delta\phi_k' + k^2\delta\phi_k = -\frac{1}{2}h_k'\phi_{0_k}' \tag{3.22}$$

These equations are written in Fourier space; ϕ_k is the Fourier transform of ϕ and $k^2 = -\nabla^2$, so that each component ϕ_k evolves independently from the others. It is very easy to find out the solution of equation (3.21): since it is a second order differential equation it can be rewritten as $\lambda^2 + 2\mathcal{H}\lambda = 0$ whose solutions are $\lambda = 0$ and $\lambda = -2\mathcal{H}$. This means that there are only constant solutions, i.e. $\phi'_{0_k} = 0$, which brings to a null *isotropic* birefringence angle. Thus, inserting the null derivative inside equation (3.22) we get a further simplified relation:

$$\delta\phi''_k + 2\mathcal{H}\delta\phi'_k + k^2\delta\phi_k = 0 \tag{3.23}$$

so that the evolution of the fluctuation field $\delta\phi$ doesn't depend any more on the metric perturbation h: for this reason $\delta\phi$ just corresponds to entropy perturbations. It is possible to write down the solution of the last equation as:

$$\delta\phi_k(\tau) = \delta\phi_k(\tau_{in})T_k(\tau) \tag{3.24}$$

where $T_k(\tau)$ is the transfer function which traces the evolution of the perturbations. The initial conditions can be set at the beginning of radiation dominated epoch (or at the end of inflation), where $\delta\phi'_k(\tau_{in}) = 0$ and the power spectrum of fluctuations is given by: $P_{\delta\phi}(k, \tau_{in}) = H_I^2/(4\pi^2)$, where H_I is the Hubble parameter during inflation. We can distinguish two relevant cases in which T_k assumes two different behaviours: the super-horizon regime ($k\tau_{dec} << 1$) where $T_k(\tau_{dec}) = 1$, and the sub-horizon one, where the transfer function starts to oscillate with a damped amplitude during the expansion of the Universe ([**25**] and [**28**]). *Figure 3.1* shows the two different trends of the transfer function in the two different regimes. We can find out an explicit relation for $\delta\phi_k(\tau_{in})$, at the end of inflation, in equation (3.24), assuming a de-Sitter phase, during which H is constant. Under this assumption, and taking into account for the initial conditions, the background evolution equation for the inflaton field can be written as:

$$u''_k(\tau) + (k^2 - 2a^2H^2)u_k(\tau) = 0 \tag{3.25}$$

where $u_k = \hat{\delta\phi} \cdot a$, with $\hat{\delta\phi}$ defined as the quantization of the fluctuation field ($\hat{\delta\phi} = a\delta\phi$). On sub-horizon scales, i.e. $k >> aH$, the last equation can simplified as: $u''_k(\tau) + k^2u_k(\tau) = 0$, which is an harmonic oscillator equation; so we can consider this kind of solution:

$$\delta\phi_k(\tau_{in}) = \frac{1}{a}\frac{e^{-ik\tau}}{\sqrt{2k}} \tag{3.26}$$

Thus, the field is affected by a damped oscillation. On the other hand, on super-horizon scales ($k << aH$) $\delta\phi_k$ is constant; in particular: $\delta\phi_k(\tau_{in}) =$

$H_I/\sqrt{2k^3}$, which gives a power spectrum $P_{\delta\phi} = H_I^2/2k^3$. This is the initial value for the fluctuation field whose evolution during radiation and matter dominated epochs is ruled by the transfer function given in eq.(3.24).

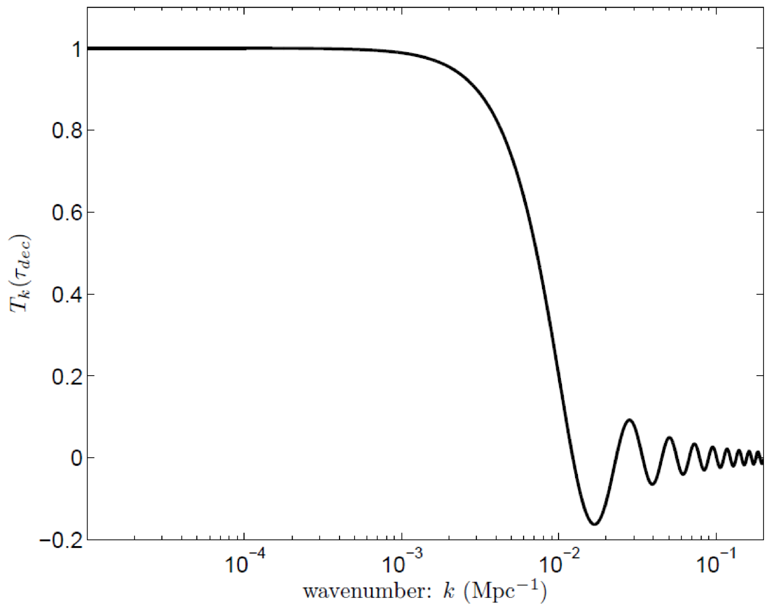

Figure 3.1: The transfer function $T_k(\tau_{dec})$ as a function of the wavenumber k; the constant and the oscillatory contributions are evident. (Credit: Zhao, Li (2014)[**25**])

At this point, we can compute the power spectrum of the rotation angle in the case of a massless scalar field, as done in eq.(3.10), taking into account for a generic transfer function: $T_k(\tau) \propto j_l(k\tau)/(k\tau)$ and inserting the power spectrum for the fluctuation field given previously. Thus, the CB angle power spectrum reads:

$$C_l^{\chi\chi} = \frac{1}{\pi}\left(\frac{\beta H_I}{M}\right)^2 \int \frac{dk}{k}[j_l(k\Delta\tau)T_k(\tau_{dec})]^2 \tag{3.27}$$

As we can see in *Figure 3.2* the amplitude of $C_l^{\chi\chi}$ decreases rapidly at higher multipoles, such that it is possible to detect it only at lower multipoles (large angular scales), let's say for $l < 100$. As seen previously, in this regime the transfer function is constant, i.e. $T_k(\tau_{dec}) \simeq 1$, so that the power spectrum is scale-invariant and it is convenient to recast it as (see [**25**] and [**27**]):

$$C_l^{\chi\chi} \simeq \frac{(\beta H_I/M)^2}{2\pi l(l+1)} \tag{3.28}$$

From this relation it is possible to compute the variance of the rotation angle, which reads:

$$\langle(\Delta\chi)^2\rangle = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l^{\chi\chi}[W_l(\theta)]^2 \simeq 332\left(\frac{\beta H_I}{M}\right)^2 deg^2 \tag{3.29}$$

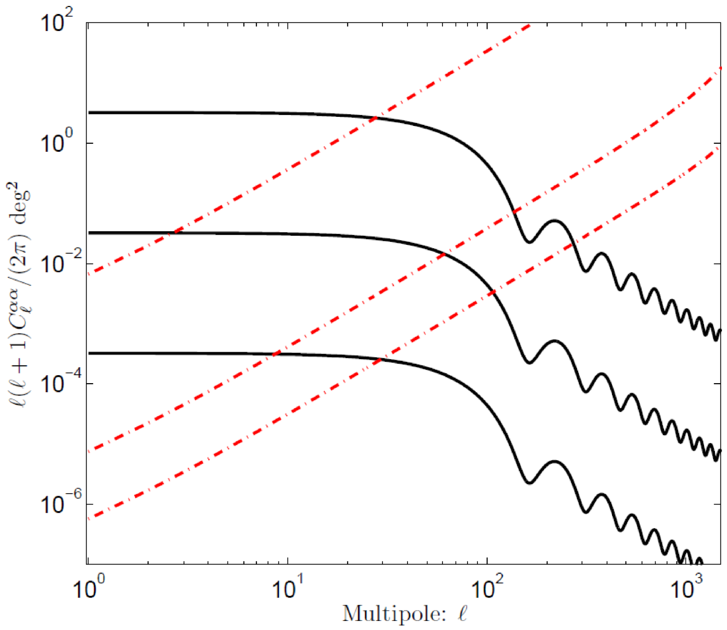

Figure 3.2: The black solid lines denote the rotation angle power spectrum for three different values of the quantity $\beta H_I/M$: the upper line is referred to a value of 0.02, the middle one to a value of. 0.02 and the lower one to 0.002. Notice that in all the cases the spectrum decreases very fast as the multipole moment l increases. The red dashed lines show the noise power spectrum: the upper one is for the Planck noise, the middle is for the CMBPol noise, and the lower one is for the case of the reference experiment. (Credit: Zhao, Li (2014)[**25**])

where $W_l(\theta)$ is a Gaussian window function of full-width half-maximum θ. The current constrain on the variance comes from AGN data an it is given by: $\langle(\Delta\chi)^2\rangle \leq 3.7$, from which a bound on $\beta H_I/M \lesssim 0.2$ can be derived. Thus, the power spectrum in eq.(3.28) can be also rewritten in this way:

$$l(l+1)C_l^{\chi\chi} = 40.56\left(\frac{\beta H_I/M}{0.2}\right)^2 deg^2 \tag{3.30}$$

Let's make some relevant remarks about $C_l^{\chi\chi}$: since it depends on the quantity $\beta H_I/M$, from observational data it is possible to constrain the coupling factor β and also the inflationary Hubble parameter H_I, which depends on the *tensor-to-scalar* ratio r in this way: $H_I = 2piM_{pl}/\sqrt{P_{\mathcal{R}}/8} \simeq 4r$ [**43**], where $M_{pl} \simeq 10^{18}\ GeV$ and $P_{\mathcal{R}} \simeq 2.1 \times 10^{-9}$ is the amplitude of the primordial scalar perturbations power spectrum. Thus, in this way we can get a bound for the r ratio, which is a fundamental cosmological parameter, and compare it with estimations from other kind of observations. Furthermore, from eq.(3.30), the constraints on the CB power spectrum can be translated into constraint of the coupling constant β, which is proportional to r, i.e. $\beta \propto 1/\sqrt{r}$.
To conclude this analysis about the massless field case, we can highlight the fact that, as can be seen in equation (3.23), where there isn't any source term on

the right hand side (as, instead, in (3.22)), the axion field fluctuations are uncorrelated with perturbations of the metric or of the matter/energy density. As a consequence all the cross-correlations are null, in particular $C_l^{\chi T} = C_l^{\chi E} = 0$: the only contribution is the auto-correlation of the rotation angle reported in eq.(3.27).

Now we can move to the case in which the scalar field ϕ is identified as a *quintessence* field (in the form of Dark Energy or Dark Matter). The main difference with respect to the massless field is a larger value of the birefringence angle. In this case we need to study the full evolution equation for $\delta\phi$ in eq.(3.18)), whose complete solution is given by a combination of the homogeneous solution (with a null right hand side) plus a particular solution of the inhomogeneous equation. The former is associated with the entropy mode, the latter is related to the adiabatic mode; the general solution is a superposition of these two modes. Moreover the homogeneous part belong to a two-parameter family specified by the initial condition on $\delta\phi$ and $\delta\phi'$ and it is not affected by the perturbations of the metric (as in the massless case). The inhomogeneous part, instead, arises as the response of the quintessence field to the fluctuations of the metric [**29**]. This kind of behaviour is similar to the one of the forced oscillator, in which the oscillations are driven by an external force, so that there are two possible modes in which the it can move, at two different frequencies. In our case the external force acting on the system is attributed to the coupling between the quintessence field and the perturbations of the metric , as shown in the right-hand side of equation (3.22), where the fluctuation h is matched with the background field ϕ_0.
Now we want to discuss deeply about this general solution, focusing on both adiabatic and entropy modes (we will follow the work done by Zhao and Li in [**25**]). For the moment we can rely on the homogeneous equation of motion, which reads:

$$\delta\phi_k'' + 2\mathcal{H}\delta\phi_k' + \left(k^2 + a^2\frac{\partial^2 V}{\partial\phi_0^2}\right)\delta\phi_k = 0 \tag{3.31}$$

We can notice that, whenever $k^2 \gg a^2 V_{\phi_0\phi_0}$, this equation coincides with the one for the massless scalar field. A typical example of this situation is the slow-rolling quintessence field, for which the equation of state reaches a value very close to -1 ($w \sim -1$). Indeed in the case we can rewrite the potential term as:

$$a^2\frac{\partial^2 V}{\partial\phi_0^2} \simeq 3a^2\eta\Omega_\phi H_0^2 \ll H_0^2 \tag{3.32}$$

where $\eta = (1/8\pi G)(V_{\phi_0\phi_0}/V)$ is the slow-rolling parameter, and Ω_ϕ is the current energy density parameter of the quintessence field. Since the condition $k^2 > H_0^2$ must hold in order to guarantee an observable cosmic birefringence effect, then we can conclude that the scalar field follows the condition $k^2 \gg a^2(\partial^2 V/\partial\phi_0^2)$, so that we come back to the massless case. Since we have already discuss about the solution for $\delta\phi$ for a massless field, let's consider a more general model for quintessence, without restricting on the case $w \sim -1$. Thus, keeping w completely general, we can express the second derivative of the potential in (3.31) as:

$$a^2\frac{\partial^2 V}{\partial \phi_0^2} = -\frac{3}{2}(1-w)\left[\frac{a''}{a} - \mathcal{H}\left(\frac{7}{2}+\frac{3}{2}w\right)\right] + \frac{1}{1+w}\left[\frac{w'^2}{4(1+w)} - \frac{w''}{2} + w'\mathcal{H}(3w+2)\right] \tag{3.33}$$

Assuming a small time evolution of ω, i.e. neglecting w' and w'' ([**25**]), and taking $1 + w = O(1)$, this expression can be simplified as:

$$a^2\frac{\partial^2 V}{\partial \phi_0^2} \simeq \frac{3\mathcal{H}^2}{4}(1-w)(7+3w) \tag{3.34}$$

Since this is proportional to τ^{-2}, the $a^2(\partial^2 V/\partial\phi_0^2) > k^2$ is fulfilled. Thus, the homogeneous equation (2.31) can be rewritten as:

$$\delta\phi_k'' + 2\mathcal{H}\delta\phi_k' + a^2\frac{\partial^2 V}{\partial\phi_0^2}\delta\phi_k = 0 \tag{3.35}$$

for which it is possible to find explicit solutions. In the radiation dominated epoch the solution is given by:

$$\delta\phi_k \propto \tau^{\frac{1}{2}(-1\pm\sqrt{1-4d})} \tag{3.36}$$

where $d \equiv \frac{3}{2}(1-w)(7+3w)$; while in the matter dominated epoch we have:

$$\delta\phi_k \propto \tau^{\frac{1}{2}(-3\pm\sqrt{9-4d})} \tag{3.37}$$

It is evident that in both cases the perturbation field decay in time for $-1 < w < 1$; in conclusion, for these kind of quintessence models, the homogeneous solutions decay to zero: this means that the homogeneous part is subdominant with respect to the inhomogeneous part.
Now we can move to the full inhomogeneous equation; to analyze the solutions in this case we just mention the approach adopted in [**29**] by Dave et al.. This method is based on a parametrization of the quintessence models in terms of the equation of state as a function of the scale factor, i.e. $w(a)$; this allow us to study the evolution of the quintessence fluctuations in models with different equations of state and different types of scalar potential. In this way, as done in eq.(3.33), it is useful to rewrite the first and second derivatives of $V(\phi)$ in terms of w and a, so that the evolution equation for $\delta\phi$ can be recast in this way:

$$\delta\phi'' + \left(2\frac{a'}{a} + \frac{w'}{1+w}\right)\delta\phi' + \left(k^2 - \frac{3}{2}(1-w)\left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\left(\frac{7+3w}{2}\right)\right] + \right.$$

$$+3w'\frac{a'}{a}\bigg)\delta\phi = -\frac{1}{2}h'\phi_0' \tag{3.38}$$

We don't want to enter in details about this treatment, since it is possible to find out a general solution in the synchronous gauge in a simpler way,i.e. moving to the Poisson gauge. For those who are interested, see [**29**] for a complete treatment on solutions to eq.(3.38).

3.2 Evolution of scalar field fluctuations in the Poisson gauge

In [**25**] and [**26**] we can find a different method in order to derive the solution for $\delta\phi$ in the synchronous gauge, starting from the Poisson gauge (or longitudinal gauge). Following this idea, let's now move from the synchronous gauge, adopted up to now, to the Poisson gauge; in this case the EOM for the fluctuations of the scalar field is a little bit more complicated; we report the main steps in order to reach the final form of this equation. We can proceed in a similar way as for the synchronous gauge, starting from the same action in (3.13) and the related EOM in (3.14), but now the form of the metric $g_{\mu\nu}$ is different:

$$g_{\mu\nu} = a^2(\tau)\begin{pmatrix} -[1+2\Psi(\tau,\mathbf{x})] & 0 \\ 0 & [1-2\Phi(\tau,\mathbf{x})]\delta_{ij} \end{pmatrix} \tag{3.39}$$

where Ψ and Φ are respectively the *shear* perturbation and the gravitational potential perturbation; in this gauge both of them are related to the two gauge-invariant Bardeen potentials, in particular: $\Psi_A = \Psi$ and $\Phi_H = -\Phi$. Using the metric in (3.39) it is possible to derive the expression for $\sqrt{-g} = a^4(1+2\Psi)^{1/2}(1-2\Phi)^{3/2}$, which must be inserted in the EOM, in this way:

$$\Box\phi = \frac{1}{a^4(1+2\Psi)^{1/2}(1-2\Phi)^{3/2}}\cdot\partial_0\bigg(g^{00}a^4(1+2\Psi)^{1/2}(1-2\Phi)^{3/2}\partial_0\phi\bigg)+$$
$$+\frac{1}{a^4(1+2\Psi)^{1/2}(1-2\Phi)^{3/2}}\cdot\partial_i\bigg(g^{ii}a^4(1+2\Psi)^{1/2}(1-2\Phi)^{3/2}\partial_i\phi\bigg) \tag{3.40}$$

where we have accounted only for diagonal components, since the metric itself is diagonal. As previously done, we need to expand the scalar field as $\phi = \phi_0(\tau) + \delta\phi(\tau,\mathbf{x})$ and to keep terms up to the linear order. In the end we get this equation:

$$-2\mathcal{H}\phi_0' - 2\mathcal{H}\delta\phi' - 2\mathcal{H}(2\Psi-4\Phi)\phi_0' + (\Psi'+3\phi')\phi_0' - \phi_0'' - (2\Psi-4\Phi)\phi_0''-$$
$$-\delta\phi'' + \nabla^2\delta\phi - a^2\frac{\partial V}{\partial\phi_0} - a^2(4\Psi-4\Phi)\frac{\partial V}{\partial\phi_0} - a^2\frac{\partial^2 V}{\partial\phi_0^2}\delta\phi = 0 \tag{3.41}$$

Taking into account for the background evolution equation in (3.12), the terms that survive are:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\frac{\partial^2 V}{\partial\phi_0^2}\delta\phi = \phi_0'(\Psi' + 3\Phi') - 2a^2\frac{\partial V}{\partial\phi_0}\Psi \tag{3.42}$$

Finally, this is the evolution equation for the fluctuation field $\delta\phi$ in the Poisson gauge. It is evident that this equation is similar to the one in the synchronous gauge except for the right hand side, where there are different kind of scalar metric perturbations. Moreover, still looking to the right hand side of eq.(3.42), we can notice that, even if there is a null background evolution for the scalar field ($\phi_0' = 0$), the term sourced by Ψ survives, unless ϕ is massless. This means that the fluctuation field $\delta\phi$ is correlated with metric perturbations even in the case of a zero time-derivative of the background field ϕ_0 (which implies a null isotropic birefringence effect).

Let's now find out solutions to equation (3.42), considering some useful approximations. Since we are dealing with the computation of the ACB angle $\delta\chi$ at the recombination epoch, we are allowed to take the perturbations Ψ and Φ as constant, so that their time derivatives Ψ' and Φ' vanish and the EOM in (3.42) can be simplified in this way:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + \left(k^2 + a^2\frac{\partial^2 V}{\partial\phi_0^2}\right)\delta\phi = -2a^2\frac{\partial V}{\partial\phi_0}\Psi \tag{3.43}$$

where $-\nabla^2$ has been replaced with k^2. Moreover, since it is reasonable to take into account for a very small mass for the axion-like field ($m_\phi \lesssim 10^{-31}eV$), we can adopt a slow-roll approximation, i.e. the second derivative of the potential can be neglected; then, since we are interested in perturbations on superhorizon scales, also the k^2 term can be cancelled out. In the end we remain with this equation:

$$\delta\phi'' + 2\mathscr{H}\delta\phi' = -2a^2\frac{\partial V}{\partial\phi_0}\Psi \tag{3.44}$$

for which we can find out a solution of this kind ([**26**] and [**27**]):

$$\delta\phi \propto a^2\tau^2\frac{\partial V}{\partial\phi_0}\Psi(\tau, k) \tag{3.45}$$

We can notice that the fluctuation field depends on the first derivative of the scalar potential, which is related to the mass of the field ϕ: this means that an higher mass corresponds to a larger amplitude of the anisotropic birefringence power spectra. Moreover, from (3.45), it is evident that, in the case of a quintessence non-massless field, the fluctuations are directly sourced by the scalar perturbation of the metric Ψ: an interesting implication of this is that scalar perturbations are able to produce a cross-correlation between anisotropic cosmic birefringence and the other CMB observables, such as the temperature and the polarization. In this way the power spectra $C_l^{\chi T}$ and $C_l^{\chi E}$ are generated, besides the auto-correlation one $C_l^{\chi\chi}$; a joint investigation of these cross-spectra allows us to extract some interesting additional informations about the axion field, such as its mass m_ϕ and the scalar potential $V(\phi)$. We can consider different shapes for the scalar potential; in the case in which ϕ is the axion field it takes this form (see [**27**]):

$$V(\phi) = m_\phi^4\left(1 - \cos\frac{\phi}{f}\right) \tag{3.46}$$

where f is the constant that appears in the Chern-Simons coupling term. On the other hand, in [**26**], the scalar field plays the role of the quintessence dark energy field and is characterized by this potential:

$$V(\phi_0) = m_\phi^2 M_{Pl}^2\left[1 - cos\left(\frac{\phi_0}{M_{Pl}}\right)\right]^2 \tag{3.47}$$

Analyzing the power-spectra of anisotropic birefringence we can also understand which is the best model that fits the observational data, and have a more clear idea on the nature of the field ϕ.

Moreover for different masses m_ϕ there are different kind of birefringence mechanisms; we can recognize different mass intervals, as done in [**26**]:

- $m_\phi \gg 10^{-27}eV$: here there is no isotropic birefringence (anisotropic birefringence only), since $\phi_0 = 0$.
- $10^{-29}eV \ll m_\phi < 10^{-27}eV$: onlu recombination, and not reionization epoch, contributed to the isotropic birefringence.
- $10^{-32}eV \ll m_\phi < 10^{-29}eV$: both recombination and reionization contribute to isotropic birfringence, with different angles.
- $m_\phi < 10^{-32}eV$: also in this case both recombination and reionization contributes, but giving the same birefringence angle.
- $m_\phi \ll 10^{-32}eV$: there is no isotropic birefringence, because the value of the background field at the recombination and at the reionization is equal to the value today (so $\phi_0(\tau_{rec}) - \phi_0(\tau_0) = 0$).

We can notice that the isotropic birefringence effect is visible only in a precise range of masses, between 10^{-32} eV and 10^{-27} eV, while the anisotropic birefringence is dominant for higher masses: in this way the anisotropic contribution allow us to probe a wider range of values for the axion field mass with respect to the isotropic case. Moreover, as suggested previously, as the mass increases the amplitude of the anisotropic birefringence power spectra increases, so that it should be easier to detect it with future experiments.

Starting from the solution for $\delta\phi$ in the newtonian conformal gauge (equation 3.45), it is possible to derive the solution in the synchronous gauge by applying this transformation:

$$\delta\phi_{syn} = \delta\phi_{con} - \alpha\dot{\phi}_0 \tag{3.48}$$

where $\alpha \simeq (2/3)\psi/\mathcal{H}$ during matter domination epoch. Then we can express the derivative of the scalar potential exploiting the *attractor solution* of a slow-rolling scalar field, such that:

$$a^2 V' \simeq -3\mathcal{H}\dot{\phi}_0 \longrightarrow V' \simeq -\frac{3\mathcal{H}}{a^2}\dot{\phi}_0 \tag{3.49}$$

with $V' = \partial V/\partial\phi_0$ and $\dot{\phi}$ denotes the time derivative. In order to rewrite the time derivative of the scalar field we can exploit this relation:

$$\dot{\phi}_0^2 = a^2\rho_\phi(1+w_\phi) \tag{3.50}$$

where w_ϕ is the equation of state parameter for the scalar field; this relation can be rewritten by considering $\rho_\phi = \Omega_\phi\rho_c$, with the critical density $\rho_c = 3H^2M_{pl}^2/8\pi$; so in the end the derivative of the potential becomes:

$$V' = -\frac{3\mathcal{H}}{a^2}\left[a^2\Omega_\phi\frac{3H^2M_{pl}^2}{8\pi}(1+w_\phi)\right]^{1/2} \tag{3.51}$$

where Ω_ϕ is the density parameter of the axion field. After some trivial mathematical passages we reach this expression for V':

$$V' = -3\frac{\mathcal{H}^2}{a^2}M_{pl}\left(\frac{3}{8\pi}\Omega_\phi(1+w_\phi)\right)^{1/2} \tag{3.52}$$

Then, inserting (3.52) into (3.45), we get another way to write the solution for the fluctuation field in the conformal gauge:

$$\delta\phi_{con} \simeq -\frac{1}{27}a^2\tau^2V'\Psi = \frac{1}{27}a^2\tau^2\left[3\frac{\mathcal{H}^2}{a^2}M_{pl}\left(\frac{3}{8\pi}\Omega_\phi(1+w_\phi)\right)^{1/2}\right]\Psi =$$
$$= \frac{4}{9}\left(\frac{3}{8\pi}\Omega_\phi(1+w_\phi)\right)^{1/2}M_{pl}\Psi \tag{3.53}$$

where we have used the relation between the Hubble constant and the conformal time: $\mathcal{H} \sim 2/\tau$.
Finally we can insert this solution in eq.(3.48) in order to directly move to the synchronous gauge, in this way:

$$\delta\phi_{syn} = \frac{4}{9}\left(\frac{3}{8\pi}\Omega_\phi(1+w_\phi)\right)^{1/2} M_{pl}\Psi - \frac{2}{3}aH\left(\frac{3}{8\pi}\Omega_\phi(1+w_\phi)\right)^{1/2} M_{pl}l\frac{\Psi}{\mathcal{H}} \quad (3.54)$$

and recalling the relation $H = \mathcal{H}/a$, we get [**27**]:

$$\delta\phi_{syn} = -\frac{2}{9}\left(\frac{3}{8\pi}\Omega_\phi(1+w_\phi)\right)^{1/2} M_{pl}\Psi \quad (3.55)$$

After this theoretical derivation of the fluctuations of the scalar field, we are ready to compute the power-spectrum for the anisotropic CB angle, exploiting the relations in eqs.(3.6)-(3.11). Differently with respect to the massless case, now the power spectrum of the quintessence field fluctuations $P_{\delta\phi}$ is directly related to the one of the perturbations of the gravitational potential P_Ψ, so that, substituting eq.(3.54) into eq.(3.11), we get:

$$\langle\delta\phi^*_{k'}(\tau_{dec})\delta\phi_k(\tau_{dec})\rangle = \frac{4}{81}\left(\frac{3\Omega_\phi(1+w_\phi)}{8\pi}\right)M^2_{pl}\langle\Psi^*_{k'}(\tau_{dec})\Psi_k(\tau_{dec})\rangle \quad (3.56)$$

and inserting it into eq.(3.10) we get the angular power spectrum:

$$C_l^{\chi\chi} = \frac{2}{27}\Omega_\phi(1+w_\phi)\left(\frac{\beta M_{pl}}{M}\right)^2 \int \frac{k^2dk}{2\pi^2}P_\Psi(k)\big[j_l(k\Delta\tau)T_k(\tau_{dec})\big] \quad (3.57)$$

which makes evident the link between the CB-angle power spectrum and the metric perturbations given by the gravitational potential Ψ. It is possible to use also the power spectrum P_Φ instead of P_Ψ, considering an ideal fluid in the perturbed Einstein equations; indeed in this case there is a null anisotropic contribution to the stress-energy tensor, which leads to the equality of the two Bardeen's potentials, i.e. $-\Psi = \Phi$, as in eq.(1.73); this means that we can consider one of the two perturbations without distinction.
Assuming adiabatic initial conditions for the perturbations (this is reasonable in the context of single-field inflationary models, which predicts the production of an adiabatic power spectrum after inflation) it is possible to redefine the two-point correlation function for the field fluctuations in terms of $P_\mathcal{R}(k)$, which is the adimensional power spectrum of the comoving curvature perturbation $\mathcal{R}$; so that, eq. (3.11) can be rewritten as:

$$\langle\delta\phi^*_{k'}(\tau_{dec})\delta\phi_k(\tau_{dec})\rangle = \frac{16\pi^5}{k^3}P_\mathcal{R}(k)\delta\phi^2(\tau_{dec})\delta^{(3)}(k-k') \quad (3.58)$$

The comoving curvature perturbation $\mathcal{R}$ is a gauge-invariant quantity. In addition, it is relevant in cosmology because, under adiabatic initial conditions, it is conserved when the perturbations move outside the horizon, i.e. remains constant on super-horizon scales (for $k \ll aH$); for this reason the initial conditions for the evolution of perturbations ($\delta\phi$, ψ and ϕ) are totally enclosed in $\mathcal{R}$. Thus, it can be useful to express P_Ψ in terms of $P_\mathcal{R}$: taking into account for large scales, i.e. small k and small l ($l \lesssim 100$), the primordial power spectrum for Ψ can be defined as:

$$P_\psi(k) = \frac{9}{25}\frac{2\pi^2}{k^3}P_\mathcal{R} \tag{3.59}$$

It is more convenient to express $C_l^{\chi\chi}$ in terms of $P_\mathcal{R}$ since this amplitude has been estimated from observations; in particular $P_\mathcal{R} \simeq 2.1\times10^{-9}$. For simplicity it is possible to rewrite the power spectrum in eq.(3.57) defining a parameter ϵ as in [**39**], which contains all the constants:

$$\epsilon = \frac{1}{100rad} \times \frac{1}{9\pi}\frac{\beta M_{pl}}{M}\left(\frac{3\Omega_\phi(\tau_{dec})(1+w_\phi(\tau_{dec}))}{8\pi}\right)^{1/2} \tag{3.60}$$

so that $C_l^{\chi\chi}$ becomes:

$$C_l^{\chi\chi} = 8\times10^4\pi\epsilon^2\int k^2dkP_\Psi(k)\left[j_l(k\Delta\tau)T_k(\tau_{dec})\right] \tag{3.61}$$

For $l \lesssim 100$ the transfer function is almost constant, i.e. $T_k(\tau_{dec}) \simeq 1$ and, exploiting eq.(3.59) the CB angle power spectrum can be recast as:

$$C_l^{\chi\chi} = \frac{7.2\times10^5\pi^3\epsilon^2P_\mathcal{R}}{25l(l+1)} \tag{3.62}$$

for which, taking the value for $P_\mathcal{R}$ previously mentioned, we get an estimation $D_l^{\chi\chi} = l(l+1)C_l^{\chi\chi}/2\pi \sim 3.0\times10^{-4}\epsilon^2\ rad^2$; this can be compared with the current observational results from ACTPol: $D_l^{\chi\chi} < 1.0\times10^{-5}\ rad^2$ at 95% C.L., allowing us to put a constraint on the parameter $\epsilon < 0.18$, so that it is possible to estimate the cosmological parameters in eq.(3.60).
Besides the auto-correlation power spectrum in (3.57), when the scalar field is a quintessence field (not massless), two additional cross-correlations of the CB angle to the CMB temperature and the E-mode polarization arise. This is due to the fact that also the temperature and polarization power spectra are sourced by the primordial power spectrum of the gravitational potential Ψ. Indeed the harmonic coefficients used to decompose the CMB temperature field and the polarization pattern can be defined in a similar way as in eq.(2.8):

$$T_{lm} = -\frac{1}{2\pi^2}(-i)^l\int d^3k\phi_{\bar{(k)}}(\tau_{dec})\Delta_{T,l}(k,\tau)Y^*_{lm}(\hat{(k)}) \tag{3.63}$$

$$E_{lm} = -\frac{1}{2\pi^2}(-i)^l\int d^3k\phi_{\bar{(k)}}(\tau_{dec})\Delta_{E,l}(k,\tau)Y^*_{lm}(\hat{(k)}) \tag{3.64}$$

from which we can write the following auto-correlation power spectra:

$$C_l^{TT} = \frac{2}{\pi} \int k^2 dk [\Delta_{T,L}(k)]^2 P_\psi(k) \tag{3.65}$$

$$C_l^{EE} = \frac{2}{\pi} \int k^2 dk [\Delta_{E,L}(k)]^2 P_\psi(k) \tag{3.66}$$

where $\Delta_{T,L}(k)$ and $\Delta_{E,L}(k)$ represent the transfer functions that quantify the contribution of a mode of wavenumber k to the related spectrum. Then, the two-point cross-correlation function is given by:

$$\langle \chi_{lm}^*(\tau) T_{l'm'}(\tau) \rangle = C_l^{\chi T} \delta_{ll'} \delta_{mm'} \tag{3.67}$$

Finally, exploiting this last relation and eqs.(2.61) and (2.65)-(2.66), we get the cross-correlation power spectra, which read [**39**]:

$$C_l^{\chi T} = 4 \times 10^2 \epsilon \int k^2 dk P_\psi(k) \Delta_{T,l}(k,\tau) j_l(k\Delta\tau) T_k(\tau_{dec}) \tag{3.68}$$

$$C_l^{\chi E} = 4 \times 10^2 \epsilon \int k^2 dk P_\psi(k) \Delta_{E,l}(k,\tau) j_l(k\Delta\tau) T_k(\tau_{dec}) \tag{3.69}$$

We can see the different behaviours of the $C_l^{\chi\chi}$, $C_l^{\chi T}$ and $C_l^{\chi E}$ in $figure$ 3.3 [**39**]. It is evident that the auto-correlation power spectrum is very similar to the one related to the massless field (see *figure 3.2*), especially at large scales, i.e. for $l \lesssim 100$, where the transfer function can be taken as a constant in both scenarios. On the other hand, the two cross-correlations show a similar oscillatory behaviour at higher multipole moments. For a complete overview on the results from *Planck* 2018 data, see [**41**] and [**42**]. It is interesting to see how the cross-correlation power spectra are affected by a rotation of the polarization plane, relating the observed spectra with the unrotated ones, in this way [**40**] :

$$C_{l,obs}^{\chi T} = C_l^{\chi T} \tag{3.70}$$

$$C_{l,obs}^{\chi E} = C_l^{\chi E} \cos 2\chi_0 - C_{l,obs}^{\chi B} \sin 2\chi_0 \tag{3.71}$$

$$C_{l,obs}^{\chi B} = C_l^{\chi E} \sin 2\chi_0 - + C_{l,obs}^{\chi B} \cos 2\chi_0 \tag{3.72}$$

where we can see that the observed power spectra are the result of the primordial ones (at decoupling epoch) after a rotation of an angle χ_0, which is the isotropic birefringence angle. Furthermore we can notice that the observed cross-correlation are non-zero if, and only if, they were non-null at early epochs; this means that they are sourced just by phenomena at the recombination epoch, not along the travel of photons towards us.

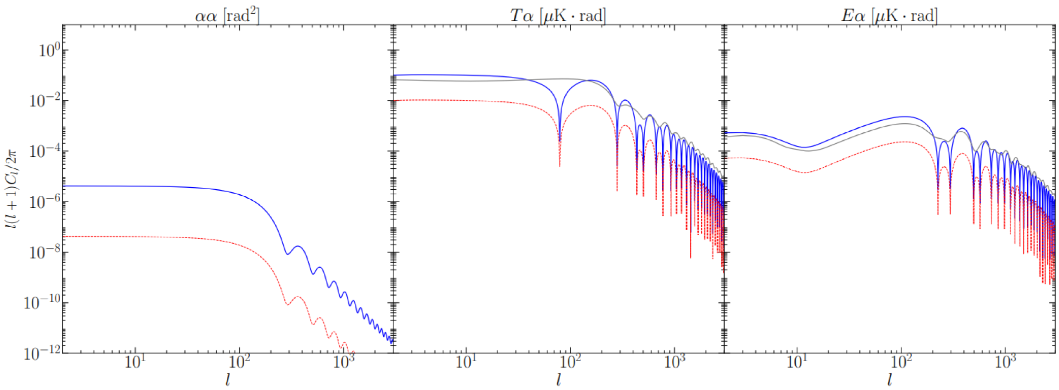

Figure 3.3: Power spectra of the anisotropic CB angle (left panel) and its cross-correlations with the CMB temperature (in the middle) and with the E-mode polarization (right panel), in axion-like field models. The blue and red curves correspond to different values of the ϵ parameter: $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively. Ar lower l the spectra are practically constant, and then decay in an oscillatory way. (Credits: H. Zhai et al. (2020)[**39**])

In conclusion, we can see how the fluctuations in the CB angle $\delta\chi$ affect the observed CMB power spectra (in particular the TE, TB, EE, BB, EB correlations); we report here the relations derived in [**4**]:

$$C_l^{TE,o} = C_l^{TE}\cos(2\Delta\chi_0)(1 - 2\langle\Delta\delta\chi^2\rangle) \tag{3.73}$$
$$C_l^{TB,o} = C_l^{TB}\sin(2\Delta\chi_0)(1 - 2\langle\Delta\delta\chi^2\rangle) \tag{3.74}$$
$$C_l^{EE,o} = \left[C_l^{EE}\cos^2(2\Delta\chi_0) + C_l^{BB}\sin^2(2\Delta\chi_0)\right](1 - 4\langle\Delta\delta\chi^2\rangle) \tag{3.75}$$
$$C_l^{BB,o} = \left[C_l^{EE}\sin^2(2\Delta\chi_0) + C_l^{BB}\cos^2(2\Delta\chi_0)\right](1 - 4\langle\Delta\delta\chi^2\rangle) \tag{3.76}$$
$$C_l^{EB,o} = \frac{1}{2}\sin(4\Delta\chi_0)\big(C_l^{EE} - C_l^{BB}\big)(1 - 4\langle\Delta\delta\chi^2\rangle) \tag{3.77}$$

These relations include also the correction for the anisotropic angle $\delta\chi$ with respect to the ones for the isotropic angle in eqs.(2.71)-(2.73). We can notice that the two power spectra C_l^{TB} and C_l^{EB}, which are sensitive to the birerin-gence effect, are proportional to $\sin(\Delta\chi_0)$ and they vanish when the isotropic angle $\Delta\chi_0$ is null, even though the anisotropic angle was different from zero. This makes evident the fact that CPT symmetry is violated only by the background field (and by its evolution in time), not by its anisotropic counterpart. On the contrary, C_l^{EE}, C_l^{BB} and C_l^{TE}, as expected, are present even in absence of an isotropic birefringence effect (when $\Delta\chi_0 = 0$), in which case they can be still affected by the correction $\Delta\delta\chi$.
It is possible to give an idea of the magnitude of this correction to the CMB power spectra. Indeed, from the auto-correlation $C_l^{\chi\chi}$ it is straightforward to estimate the variance of the fluctuations $\delta\phi$, in this way [**4**]:

$$\sum_l (2l+1)C_l^{\chi\chi} = 4\pi\langle\Delta\delta\chi^2\rangle = 4\pi\left(\frac{\beta}{M}\right)^2\langle\delta\phi^2\rangle = \frac{16c^2\alpha_{em}^2}{9\pi M^2}\langle\delta\phi^2\rangle \tag{3.78}$$

where, in the last equality we have used the same coupling factor defined in eq.(2.56). In the case of quintessence field, in the context of baryo/leptogenesis models, the variance is given by: $\langle \Delta\delta\chi^2 \rangle \sim (10^{-5}/M^2)\langle \delta\phi^2 \rangle$. Then, taking into account for the solution for $\delta\phi$ in eq.(3.53) or (3.55), and considering a tracking behaviour for the field ϕ (required in order to generate enough baryon asymmetry), we get this estimation:

$$\langle \Delta\delta\chi^2 \rangle \sim 10^{-7} \frac{M_{pl}^2}{M^2} \langle \Phi^2 \rangle \sim 10^{-17} \frac{M_{pl}^2}{M^2} \tag{3.79}$$

where these values have been considered: $w_\phi = w_m = 0$ for the tracking behaviour, $\Omega_\phi \lesssim 0,04$ [**43**] and $\langle \Phi^2 \rangle \sim 10^{-10}$ for adiabatic perturbations.

Chapter 4

Cosmic Birefringence from the coupling with an $f(R)$ scalar function

It is known that nowadays the Universe is dominated by matter particles instead of antimatter ones; so there must have been a process in a very early epoch, for which baryons were preferred over anti-baryons: this process is the so-called *baryogenesis*. Many proposed models for the this mechanism are based on the *Sakharov conditions*: the first one requires the violation of the baryon number B, the second imposes the violation of charge conjugation and of CP symmetry, and the third states that particles must be out of thermodynamical equilibrium. This last requirement is applied only in models where CPT symmetry is conserved; this means that if it is violated, then baryogenesis and leptogenesis can take place in a condition of thermal equilibrium. Thus, CPT violation could allow us to investigate this fundamental process, occurred at very early epochs. In this way the baryo/leptogenesis phenomenon can be associated with the presence of a quintessence field, or a generic scalar function $f(R)$, coupled with CMB photons, which gives rise to a parity violation in the electromagnetic Lagrangian density (in this case the entire CPT is violated), leading to the same birefringence effect studied up to now.
Following this idea, two interesting kinds of models are the ones of *spontaneous* baryogenesis and of *gravitational* baryogenesis. In the first ones CPT is violated by an additional scalar field which is a *quintessence* field ϕ (Dark Matter or Dark Energy), the same considered in the previous chapters, while in the second ones a scalar function $f(R)$ replaces the field ϕ. We will deeply analyze this second case in the following section.

4.1 Isotropic birefringence angle from a modified Chern-Simons coupling

In this section we will study the effect of the Chern-Simons coupling, seen in the first chapter, substituting the scalar field ϕ with a generic function of the

Ricci scalar $f(R)$. The relevant aspect is that, since the addition of an $f(R)$ function breaks the CPT symmetry, it leads to the same effect observed in the previous sections: the rotation of the polarization plane of CMB photons . In this way the cosmic birefringence effect can be exploited also to constrain and study the baryo/leptogenesis mechanism in the early universe, in the context of gravitational baryogenesis models.
In practice, we start from an interacting Lagrangian density as the one in eq.(2.4), but now the scalar field is replaced by a scalar function $f(R)$, which depends on the Ricci curvature scalar R (see [**12**]):

$$\mathscr{L}_{int} = c\partial_\mu f(R) J^\mu \tag{4.1}$$

where c is the coupling constant related to the strength of the interaction, and J^μ is a fermion current, which, as previously, must satisfy two requirements: it can't be orthogonal to the B-L current and it needs to be anomalous with respect to the electromagnetic interaction. So, also in this situation, it is consistent to take the $J^\mu_{(B-L)_L}$ current, which contains the parity-breaking term $F_{\mu\nu}\tilde{F}^{\mu\nu}$ as shown in eqs.(2.5) and (2.6); in the end we get an interacting Lagrangian density written in a similar way to the one for the Chern-Simons coupling:

$$\mathscr{L}_{int} = -\frac{1}{2}\sigma\partial_\mu f(R) K^\mu \tag{4.2}$$

where the current K^μ is given by $A_\nu\tilde{F}^{\mu\nu}$ (i.e. the Chern-Simons current) and σ is a coupling constant with a shape depending on the underlying fundamental theory; in [**1**] it is defined in this way:

$$\sigma \equiv -\frac{4\alpha_{em}}{3\pi} c \tag{4.3}$$

Notice, indeed, that inserting this definition in (4.2), we recover the constant coefficient present in (2.6). At this point we can start with the computations to derive the birefringence angle.
Following the same procedure as in the case of a scalar field, we can start by deriving the modified Maxwell's EOMs, exploiting another time the Euler-Lagrange equations. The only term that changes with respect to the previous case is the one containing the interacting lagrangian:

$$\frac{\partial \mathscr{L}_{int}}{\partial A_\nu} = -\frac{1}{2}\sigma\partial_\mu f(R)\tilde{F}^{\mu\nu} \tag{4.4}$$

such that the EOM in terms of the stregth tensor $F_{\mu\nu}$ is:

$$\nabla_\mu F^{\mu\nu} = \sigma\partial_\mu f(R)\tilde{F}^{\mu\nu} \tag{4.5}$$

The other equation of motion is still given by:

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 \tag{4.6}$$

which, however, doesn't contain any interesting effect produced by the Chern-Simons coupling. For this reason we will just focus on equation (4.5), recasting it in terms of the four-vector A_μ, for convenience:

$$\nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) = \frac{\sigma}{2}\partial_\mu f(R)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{4.7}$$

We can easily notice that this equation contains a gauge freedom, which can be erased by imposing, for example, the Lorenz gauge: $\nabla_\mu A^\mu = 0$. Under this choice the EOM can be rewritten as:

$$\nabla_\mu\nabla^\mu A^\nu + R^\mu_\nu A^\nu = \frac{\sigma}{2}\partial_\mu f(R)\epsilon^{\mu\nu\rho\sigma}(\nabla_\rho A_\sigma - \nabla_\sigma A_\rho) \tag{4.8}$$

in which R^μ_ν comes from the commutation of covariant derivatives reported in the *footnote* 1 in the first chapter.
We can still exploit the geometrical optics approximation, under which the solutions to the Maxwell equations can be written in the same form used in (2.23). Substituting these solutions in the EOM, and considering only the terms that scale as $1/\epsilon$ we derive the *propagation* equation:

$$k^\mu\nabla_\mu a^\nu + \frac{1}{2}\nabla_\mu k^\mu a^\nu = \frac{\sigma}{4}\partial_\mu f(R)\epsilon^{\mu\nu\rho\sigma}(k_\rho a_\sigma - k_\sigma a_\rho) \tag{4.9}$$

where $k_\mu \equiv \nabla_\mu S$. On the other hand, taking into account for terms that scale as $1/\epsilon^2$ we simply get the relation: $k_\mu k^\mu = 0$, which means that photons propagates along null geodesics, unaffected by the $f(R)$ function. Focusing instead on equation (4.9), we can multiply all the terms by a_ν and use the definition $a^\mu = A\varepsilon^\mu$ and the normalization $\varepsilon_\mu\varepsilon^\mu = 1$ in order to rewrite it as:

$$k^\mu\nabla_\mu\varepsilon^\nu = \frac{\sigma}{4}\partial_\mu f(R)\epsilon^{\mu\nu\rho\sigma}(k_\rho\varepsilon_\sigma - k_\sigma\varepsilon_\rho) \tag{4.10}$$

It is evident that in this case we have not a parallel transport of the polarization vector ε_μ, since the term $k^\mu\nabla_\mu\varepsilon^\nu$ is non-vanishing: this means that the plane of polarization rotates as the photon propagates; it's the same birefringence effect obtained for the scalar field ϕ.
In order to find out explicitly the polarization angle (birefringence angle) we can start by considering a flat FLRW spacetime, whose metric is:

$$ds^2 = a^2(\tau)(d\tau^2 - \delta_{ij}dx^i dx^j) \tag{4.11}$$

where τ is the conformal time. For the moment we take R as time-dependent only, so that $f(R)$ depends only on time and not on spatial coordinates (more generally, R could also have fluctuations, so that it becomes dependent on space; we will discuss about that case later in the next sections). Moreover, photons are assumed to propagate along the x axis, i.e. the four-momentum is $k^\mu = (k^0, k^1, 0, 0)$, with the null condition $k_\mu k^\mu = 0$. Under all these assumptions the polarization vector satisfies the following equation:

$$k^\mu \nabla_\mu \varepsilon^i = \frac{\sigma}{4} \partial_\mu f(R) \epsilon^{0ijk} (k_j \varepsilon_k - k_k \varepsilon_j) \tag{4.12}$$

Then, using the relation $\epsilon^{\mu\nu\rho\sigma} = e^{\mu\nu\rho\sigma}/\sqrt{-det|g|}$ (in FLRW $\sqrt{-det|g|} = a^4$) and the permutation property of the Levi-Civita tensor, this equation can be recast as:

$$k^\mu \nabla_\mu \varepsilon^i = \frac{\sigma}{2} \partial_\mu f(R) \frac{e^{ijk}}{a^4} g_{ij} g_{lk} k^j \varepsilon^k = \frac{\sigma}{2} \partial_\mu f(R) e^{ijk} k^j \varepsilon^k = -\frac{\sigma}{2} \partial_\mu f(R) e^{1ik} k^1 \varepsilon^k \tag{4.13}$$

where we have exploited the metric $g_{\mu\nu}$ to raise the indexes of k and ε, and we have taken the j index equal to 1, due to the assumption of propagation along x direction only; the minus sign in the last equality comes from the permutation properties of the e^{ijk} tensor.
In order to going on with the computations it is more convenient to rewrite equation (4.13) in another way, recalling that $k^\mu = dx^\mu/d\lambda$ and exploiting the definition of the covariant derivative: $\nabla_\mu \varepsilon^i = \partial_\mu \epsilon^i \Gamma^i_{\mu j} \varepsilon^j$. The Christoffel symbols can be found quickly in a FLRW metric; the only two that survives are:

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{1}{2} g^{i\lambda} (\partial g_{\lambda 0, j} + \partial g_{\lambda j, 0} - \partial g_{0j,\lambda}) = \frac{1}{2} g^{ii} (\partial g_{ij,0}) = \frac{a'}{a} \delta^i_j = \mathcal{H} \delta^i_j \tag{4.14}$$

where $\mathcal{H}$ is the Hubble constant in the conformal time (indeed the $'$ denotes the derivative with respect to the conformal time τ). Inserting all these information in equation (4.13), we derive the two *geodesic* equations for the polarization vector components ε^2 and ε^3, both orthogonal to the propagation direction determined by k^1:

$$\frac{d\varepsilon^2}{d\lambda} + \mathcal{H} k^0 \varepsilon^2 = -\frac{\sigma}{2} \partial_\mu f(R) k^1 \varepsilon^3 \tag{4.15}$$

$$\frac{d\varepsilon^3}{d\lambda} + \mathcal{H} k^0 \varepsilon^3 = \frac{\sigma}{2} \partial_\mu f(R) k^1 \varepsilon^2 \tag{4.16}$$

The different sign between the two equations is due to e^{1ik}: in the first case $i = j = 2$ and $k = 3$ are taken, while in the second one there are $i = j = 3$ and $k = 2$. Furthermore, it is interesting that these two equations are *mixed* in ε^2 and ε^3, reminding the situation seen in equations (2.47) and (2.48): this means that this rotation effect is able to inter-change the polarization components during the propagation of the photons.

Exploiting the fact that $k^0 = k^1 = d\tau/d\lambda$ and substituting it in the previous relations, we obtain:

$$\frac{d\varepsilon^2}{d\lambda}a + \frac{da}{d\lambda}\varepsilon^2 = -\frac{\sigma}{2}\partial_\mu f(R)\frac{d\tau}{d\lambda}\varepsilon^3 a \tag{4.17}$$

where we have also multiplied both sides by the scale factor a. Using the Leibniz rule for derivatives on the left hand side of the equation, and the assumption that $f(R)$ depends only on time, in the end we reach this equation for ε^2:

$$\frac{d}{df(R)}(a\varepsilon^2) = -\frac{\sigma}{2}a\varepsilon^3 \tag{4.18}$$

and, following the same reasoning, the equation for ε^3 reads:

$$\frac{d}{df(R)}(a\varepsilon^3) = \frac{\sigma}{2}a\varepsilon^2 \tag{4.19}$$

Finally at this point we can derive the expression for the birefringence angle for a photon coupled with a generic $f(R)$ scalar function by integrating the last equation:

$$\int \frac{d}{df(R)}(a\varepsilon^3) = \int \frac{\sigma}{2}a\varepsilon^2 \longrightarrow \varepsilon^3 = \frac{\sigma}{2}\varepsilon^2 f(R) + constant \tag{4.20}$$

and so:

$$\alpha = arctan\left(\frac{\varepsilon^3}{\varepsilon^2}\right) = \frac{\sigma}{2}f(R) + constant \tag{4.21}$$

For a source at redshift z (for the CMB radiation we take z_{rec}) the rotation of the polarization angle is given by:

$$\Delta\alpha = \frac{\sigma}{2}[f(R(0)) - f(R(z_{rec}))] \tag{4.22}$$

making evident that the birefringence angle depends on the difference between the value of the scalar function computed at the recombination epoch and the one computed today (at $z = 0$), along the photon trajectory. This means that there is a non-zero effect only if $f(R)$ evolves (at least) in time, as suggested previously. In most cases, the adopted convention for the sign of the rotation angle in (4.22) is the following:

$$\Delta\chi = -\Delta\alpha = \frac{}{2}[f(R(z_{rec})) - f(R(0))] \tag{4.23}$$

so that if $\Delta\chi > 0$ there is a clockwise rotation. This choice of the sign comes from the fact that a vector rotated by an angle $\Delta\alpha$ in a *fixed* coordinate frame is equivalent to a fixed vector measured in a coordinate frame which is *rotated* by the same angle (in practice in eq.(4.22) we are considering ourselves in a rotating system).

4.2 Primordial baryo/leptogenesys mechanism originated by an $f(R)$ scalar function

Up to now we have kept $f(R)$ completely generic, but there are different proposed models based on different shapes of the scalar function $f(R)$; for instance in [**1**] and in [**14**] they have adopted a logarithmic dependence on R, i.e. $f(R) = lnR$, while in other cases, such as in [**13**], a simple linear behaviour is used, with $f(R) = R/M^2$. We want to discuss a little bit further about these two kind of models, with a particular focus on the baryo/leptogenesys mechanism.
In this context it is interesting to spend a few words on the estimation of the coupling factor σ in equation (4.23), which in practice is a measure of the detectability of the birefringence effect. This estimation is related to the computation of the baryon number asymmetry, which in this kind of models (see reference [**12**]) is given by:

$$\frac{n_B}{s} \sim c\frac{\dot{f}(R)}{T} \tag{4.24}$$

where s is the entropy density and T is the temperature. It is evident, another time, that the time-derivative of the scalar function $f(R)$ must be non-vanishing. In order to ensure this, one possibility is to take a model in which $f(R) \sim lnR$, as done in reference [**1**] and [**18**], so that the interacting Lagrangian in equation (4.2) can be rewritten as:

$$\mathscr{L}_{int} = c\frac{\partial_\mu R}{R} J^\mu \tag{4.25}$$

Indeed in this case the term $\partial_\mu f(R)$ doesn't vanish during the radiation dominated epoch, and it brings to the violation of the CPT symmetry. Following the computations in [**18**], it is possible to rewrite the last equation in a more useful way, in order to give an estimation of the baryon asymmetry. At first the vector current in (4.25) can be replaced with J^μ_B (but the same calculation is valid also for a J^μ_{B-L} current), so that it gives rise to an effective chemical potential for baryons :

$$-c\frac{\dot{R}}{R}n_B \longrightarrow \mu_b = -c\frac{\dot{R}}{R} = -\mu_{\bar{b}} \tag{4.26}$$

From this, inserting the expression for n_B, the ratio of the baryon number to the entropy density can be derived:

$$\frac{n_B}{s} = -\frac{15 g_b}{4\pi^2 g_{*s}} \frac{c\dot{R}}{RT} \tag{4.27}$$

where the entropy density is given by: $s = \frac{2\pi}{45} g_{*s} T^3$ and g_{*s} is the number of degrees of freedom of the particles that contribute to the entropy of the universe. Then, the term $\dot{R}/R$ can be recovered from the Friedmann equation, and it reads:

$$\frac{\dot{R}}{R} = -3H(1-3\omega) = -4H = -6.64 g_*^{1/2} \frac{T^2}{m_{pl}} \tag{4.28}$$

where we have inserted the value 1/3 for ω, since we are in the radiation dominated epoch, and the expression for the Hubble parameter: $H = 1.66 g_*^{1/2} \frac{T^2}{m_{pl}}$. Substituting this relation into (4.27) we get the final expression for the baryon number over the entropy density:

$$\frac{n_B}{s} \simeq 2.52 c g_b g_*^{-1/2} \frac{T_D}{m_{pl}} \simeq 0.1 c \frac{T_D}{m_{pl}} \tag{4.29}$$

where $g_b \sim \mathcal{O}(1)$ and $g_* \sim \mathcal{O}(100)$; since the value of the baryon number asymmetry is of the order of 10^{-10} (and taking $c \sim \mathcal{O}(1)$) a *decoupling temperature* T_D of order $10^{-9} m_{pl} \sim 10^{10}$ GeV is required. We can get a similar result even if we take the vector current J^μ_{B-L} instead of J^μ_B in (4.25), so that we finally have the B-L number asymmetry:

$$\frac{n_{B-L}}{s} \simeq 0.1 c \frac{T_D}{m_{pl}} \tag{4.30}$$

where T_D is the decoupling temperature related to the B-L violating interactions, i.e. the temperature below which the B-L interactions freeze-out. In the Standard Model the B-L symmetry is conserved, but there are many other models in which it is violated, for example in the presence of an interacting lagrangian of this kind:

$$\mathscr{L}_{\not{L}} = \frac{2}{f} l_L l_L \phi\phi + H.c. \tag{4.31}$$

where f is the scale related to *new physics* beyond the SM, l_L is the left-handed lepton (taken as a *neutrino*) and ϕ are the Higgs doublets. The coupling in (4.31) induces an interaction rate of B-L violating processes given by: $\gamma_{\not{L}} \sim 0.04 T^3/f^2$, so that the interactions are more efficient at higher temperatures. From the requirement of an interaction rate larger than the universe expansion rate, given by H, in order to maintain a thermal equilibrium condition, we can derive a lower limit on the neutrino mass:

$$\sum_i m_i^2 = \left(0.2eV \left(\frac{10^{12} GeV}{T_D}\right)^{1/2}\right)^2 \tag{4.32}$$

where the neutrino masses are related to the decoupling temperature. The estimations of the masses come from experiments on neutrino oscillations and from cosmological tests (in particular from the analysis carried out by WMAP and SDSS, which have given $\sum_i m_i < 0.69$ eV and < 1.7 eV, respectively. Depending on the adopted neutrino mass hierarchy, the required freeze-out temperature has values in the range: $10^{10} GeV \leq T_D \leq 10^{13} GeV$. Putting these values in equation (4.30) we have a constrain on the coupling constant $c \geq 10^{-3}$, and, from equation (4.3), we can conclude that a $|\sigma| \geq 10^{-6}$ is needed to ensure a successful baryogenesis; this value for σ is effectively inside the detectable window of the future CMB experiments.

Besides the model with a logaritmic scalar function investigated up to now, there are also some models with $f(R) \sim R$, where unfortunately the time derivative $\dot{R}$ would vanish due to the fact that $R = 8\pi G(1-3\omega)\rho$ is null for ω = 1/3 (i.e. during the radiation dominated epoch). However there are some methods which allow us to reach a non-vanishing $\dot{R}$ also in the case of a linear scalar function, such as the one proposed by Davoudiasl et al. in [**13**], where they adopt a CP-violating interaction between the derivative of the Ricci scalar and the baryon current J^μ, whose action can be expressed as:

$$S = \frac{1}{M_*^2} \int d^4x \sqrt{-g} (\partial_\mu R) J^\mu \tag{4.33}$$

where M_* is the cut-off energy scale of the effective background theory; it is reasonable to take it of the same order of the Planck mass $M_P \sim 2.4 \times 10^{18}$ GeV. The vector current J^μ could be any current that lead to an asymmetry in the baryon or lepton number (as seen previously). We can notice that the action in (4.33) is very similar to the one related to the coupling between the scalar field and a vector current analyzed in the previous section; indeed this model is closely connected to the one of the *spontaneous baryogenesis*, in which the scalar field ϕ replaces the Ricci scalar R.
In this kind of approach the baryon asymmetry is ensured by taking into account for a non-null $(1-3\omega)$ term, which is possible due to interactions among massless particles which lead to running coupling constants; in particular in [**13**] they have adopted a value for $1-3\omega$ of the order of 10^{-2}-10^{-1}, and a relation for the baryon number density of this kind:

$$\frac{n_B}{s} \approx (1-3\omega) \frac{T_D^5}{M_*^2 M_P^3} \tag{4.34}$$

where T_D is still the decoupling temperature, at which the baryon number violating interactions are efficient.

In order to conclude this section, we quickly analyze another model that adopts a linear $f(R)$ function, referring in particular to the work done by Shiromizu and Koyama in [**19**], where the Randall-Sundrum model (in the brane world scenario) is investigated. Here the starting point to ensure the baryon number asymmetry is the same as in the previous approach, with the interacting action in eq.(4.33) and the same relation for the baryon number to entropy ratio, which can be rewritten also as:

$$\frac{n_B}{s} \sim \frac{\dot{R}}{M_*^2 T} \tag{4.35}$$

estimated at the decoupling temperature T_D. The novelty of this model is in the fact that the non-zero time derivative $\dot{R}$ during radiation dominated epoch is realised due to the higher order curvature corrections in the effective theory. These corrections leads to an expression for the Ricci scalar which reads:

$$R = (1-3\omega)\frac{\rho}{M_4^2} - \frac{1}{6}(1+3\omega)\frac{\rho^2}{M_5^6} \tag{4.36}$$

where the term on the right hand side comes from higher order corrections to conventional cosmology. Here the two masses M_4 and M_5 are the four and five dimensional Planck scales related by the relation $M_4^2 = lM_5^3$ and M_* is proportional to the ratio M_5^3/M_4^2. From the relation (4.36) the time-derivative of the Ricci scalar can be derived:

$$\dot{R} = \frac{8}{3}\frac{H\rho^2}{M_5^6} \sim \frac{T^{10}}{M_5^6 M_4} \tag{4.37}$$

By inserting this relation for $\dot{R}$ into equation (4.35), it can be seen that the baryon number to entropy ratio depends on the ratio between the Planck scales in four dimensional and five dimensional spacetimes, in this way:

$$\frac{n_B}{s} \sim \frac{T_D^9}{M_*^2 M_5^6 M_4} \sim \frac{M_4^3 T_D^9}{f^2 M_5^1 2} \sim \frac{1}{f^2}\left(\frac{M_5}{M_4}\right)^{3/2} \tag{4.38}$$

where f is a parameter related to the CP violation processes considered in the model, and is the coefficient that defines the proportionality among M_* and the Planck scales, i.e. $M_* = fM_5^3/M_4^2$. Then, finally, taking into account for the estimations of the different quantities appearing in the previous equation (in particular setting $M_5 > 10^8$ GeV from experiments), the baryon number is espected to be:

$$\frac{n_B}{s} \sim 10^{-10}\left(\frac{0.001}{f}\right)^2\left(\frac{10^8 GeV}{M_5}\right)^2\left(\frac{T_D}{10^{2.5}GeV}\right)^9 \tag{4.39}$$

For this scenario we should require $T_D < T_R < M_I$ (as suggested in [**13**]), where T_R is the *reheating* temperature, i.e. the temperature at which the universe becomes radiation dominated, and M_I is the inflationary scale. This last quantity depends on the type of scalar potential chosen for the inflationary model: for example, in the chaotic inflation scenario, where the potential is

$V = \frac{1}{2}m^2\phi^2$, the scale M_I (which is $\sim v^{1/4}$) is constrained to be of the order of $10^{-0.5}M_5$, which satisfies the previous condition.
Besides the case of $\omega = \frac{1}{3}$, there are other two possibilities to generate a baryon asymmetry with $\omega = 0$ (matter dominated universe) and with $\omega > \frac{1}{3}$; both of them are analyzed in [**13**], but we don't want to enter too much in details about this, since it goes outside the purpose of this work.

We conclude this section with the analysis of another model based on the addition of an $f(R)$ function in the Einstein-Hilbert action: it is the so-called *Starobinsky* model. This is one of the most relevant and consistent models proposed to explain the inflationary scenario, so that we can explore an interesting connection between the inflationary epoch and the dark matter production in the early universe. The model takes into account for an action modified by the addition of an extra term proportional to R^2, brought by a quantum correction related to the presence of an additional scalar degree of freedom, which should play the role of the *inflaton* field. There are some interesting extensions and modifications of the "classical" Starobinsky inflationary model, as the ones presented in [**20**] and [**21**]. In the first one the scalar d.o.f. is the Higgs boson, and the R^2 additional term is brought by its coupling with curvature (i.e. the graviton), while in the second one they consider an extension of the Starobinsky model by introducing a dark-sector which contains DM particles.
The modified Einstein-Hilbert action exploited in both these models is given by:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}M_P^2 R - \frac{M_P^2}{12M^2}R^2\right] \tag{4.40}$$

Very briefly, without entering in the computations, in [**20**] the DM candidate comes from an auxiliary field ξ, called *scalaron*, introduced in this action; it is shown that these additional scalar field plays the role of a dark matter particle as a thermal relic abundance from inflation. In [**21**] a similar approach is used, but they discuss the possibility that the *Dark Glueball* in the dark sector becomes DM as a consequence of the inflaton decay.

4.3 Anisotropic rotation angle induced by perturbations of the Ricci scalar

At this point we want to extend the treatment about $f(R)$ birefringence, seen in the previous sections, adding the fluctuations of the Ricci scalar inside $f(R)$. The birefringence angle in equation (4.23) can be rewritten as the sum of its isotropic counterpart (dependent on R) and its fluctuations, in this way:

$$\chi = \chi_0(\tau) + \delta\chi(\tau, \bar{x}) \tag{4.41}$$

where the isotropic part depends only on time (since it is proportional to the

Ricci scalar R), while the perturbed one depends also on the spatial coordinates, as in eq.(3.3); this last dependence is brought by the fluctuations in R, which are injected by the metric perturbations. For this reason, in order to compute the anisotropic contribution $\delta\chi$ we need, at first, to derive the expression for δR; we will proceed in the context of the first order cosmological perturbation theory, exploiting the perturbed metric defined in the newtonian conformal gauge (Poisson gauge), which components are [**32**]:

$$g_{00} = -1 - 2\Psi(t, \mathbf{x}) \tag{4.42}$$

$$g_{ij} = a^2(t)[1 - 2\Phi(t, \mathbf{x})]\delta_{ij} \tag{4.43}$$

where Φ and Ψ are the scalar perturbations of the metric already introduced in eqs.(1.46) and (1.49). Raising the indices of the two metric components we get:

$$g^{00} = -\frac{1}{1+2\Psi} \simeq -1 + 2\Psi(t, \mathbf{x}) \tag{4.44}$$

$$g^{ij} = \frac{1}{a^2(1+2\Phi)}\delta_{ij} \simeq \frac{1}{a^2}[1 - 2\Phi(t, \mathbf{x})]\delta_{ij} \tag{4.45}$$

where we have used a Taylor expansion to obtain the relations on the right hand side. Now, since the Ricci scalar is defined through the Crhristoffel symbols, making use of equations (1.55), (4.44) and (4.45) , we can perturb (at first order) their components in this way:

$$\Gamma^0_{00} = \frac{1}{2}g^{00}(\partial_0 g_{00}) = \partial_0\Psi \tag{4.46}$$

$$\Gamma^i_{00} = \frac{1}{2}g^{ij}(-\partial_i g_{00}) = \frac{1}{a^2}\partial^i\Psi \tag{4.47}$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{1}{2}g^{ij}(\partial_0 g_{ij}) = \left(\frac{\dot{a}}{a} + \dot{\Phi}\right)\delta_{ij} \tag{4.48}$$

$$\Gamma^0_{ij} = \frac{1}{2}g^{00}(-\partial_0 g_{ij}) = a^2\left(\frac{\dot{a}}{a} + 2\frac{\dot{a}}{a}(\Phi - \Psi) + \dot{\Phi}\right)\delta_{ij} \tag{4.49}$$

$$\Gamma^0_{0i} = \frac{1}{2}g^{00}(\partial_i g_{00}) = \partial_i\Psi \tag{4.50}$$

Exploiting these components and taking $\mu = \nu = 0$ in eq.(1.57) we can compute the 00 component of the Ricci tensor, whose expression is given by:

$$R_{00} = \Gamma^\alpha_{00,\alpha} - \Gamma^\alpha_{0\alpha,0} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \tag{4.51}$$

Notice that for $\alpha = 0$ we get $R_{00} = 0$, so α must be a spatial index, i.e. $\alpha = i$; thus, the different terms in eq.(4.51) are:

$$\Gamma^i_{00,i} = \frac{1}{a^2}\nabla^2\Psi \tag{4.52}$$

$$\Gamma^i_{0i,0} = \partial_0\left(\frac{\dot a}{a} + \dot\Phi\right)\delta_{ij} = 3\left(\frac{\ddot a}{a} - H^2 + \ddot\Phi\right) \tag{4.53}$$

where we have used the relation for the Hubble parameter: $H = \dot a/a$ (the *dot* indicates the derivative with respect to the *cosmic* time). Then, the products of the connection coefficients in (4.51) can be computed as:

$$\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} = \frac{\dot a}{a}\dot\Psi\delta_{ij} = 3H\dot\Psi \tag{4.54}$$

$$\Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} = \left(\frac{\dot a}{a} + \dot\Phi\right)^2\delta_{ij} = 3\left(H^2 + 2H\dot\Phi\right) \tag{4.55}$$

where, in the first equation, we have taken $\beta = 0$, while in the second one $\beta = j$, in order to keep all the terms at the linear order in the perturbations. Summing all the contributions in eqs.(4.52)-(4.55) we obtain the 00 component of the Ricci tensor:

$$R_{00} = \frac{1}{a^2}\nabla^2\Psi - 3\frac{\ddot a}{a} - 3\ddot\Phi + 3H(\dot\Psi - 2\dot\Phi) \tag{4.56}$$

At this point we need to find out the expression for the spatial component of the Ricci tensor, R_{ij}, which reads:

$$R_{ij} = \Gamma^\alpha_{ij,\alpha} - \Gamma^\alpha_{i\alpha,j} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} \tag{4.57}$$

for which it is necessary to calculate an additional connection coefficient, i.e.:

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\sigma}\left(\partial_k g_{\sigma j} + \partial_j g_{\sigma k} - \partial_\sigma g_{jk}\right) \tag{4.58}$$

Since the metric $g_{\mu\nu}$ is diagonal, we can take $\sigma = i$ only, such that:

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}\left(\frac{1-2\Phi}{a^2}\right)\left[2a^2\left(\partial_k\Phi\delta_{ij} + \partial_j\Phi\delta ik - \partial_i\Phi\delta jk\right)\right] = \\ &= \partial_k\Phi\delta_{ij} + \partial_j\Phi\delta_{ik} - \partial_i\Phi\delta_{jk}\end{aligned} \tag{4.59}$$

The non-zero terms in equation (4.57) are:

$$R_{ij} = R^{(1)}_{ij} + R^{(2)}_{ij} + R^{(3)}_{ij} = \tag{4.60}$$

$$= \left(\Gamma^0_{ij,0} - \Gamma^0_{i0,j} + \Gamma^0_{00}\Gamma^0_{ij} - \Gamma^0_{0j}\Gamma^0_{i0}\right) + \left(-\Gamma^0_{kj}\Gamma^k_{i0}\right) + \left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j}\right) \tag{4.61}$$

where we have considered $\alpha = 0$ and $\beta = 0$ for the terms in the first parenthesis,

$\alpha = 0$ and $\beta = k$ in the second one, and $\alpha = k$ in the last one. Focusing on the first part we obtain:

$$R_{ij}^{(1)} = \Bigg[2a^2H^2 + 4a^2H^2(\Phi - \Psi) + 2a\dot{a}\dot{\Phi} + \ddot{a}a - a^2H^2 + \\ +2a^2\left(\frac{\ddot{a}}{a} - H^2\right)(\Phi - \Psi) + a^2\ddot{\Phi} + a^2H\dot{\Psi}\Bigg]\delta_{ij} - \partial_i\partial_j\Psi \tag{4.62}$$

For the second term, instead, we get:

$$R_{ij}^{(2)} = a^2\big[H^2 + 2\dot{\Phi}H + 2H^2(\Phi - \Psi)\big]\delta_{ij} \tag{4.63}$$

while the third part reads:

$$R_{ij}^{(3)} = -\nabla^2\Phi - \partial_i\partial_j\Phi \tag{4.64}$$

Summing all the contributions in eqs.(4.62)-(4.64) we can finally write down the expression for the spatial component of the Ricci tensor:

$$R_{ij} = \delta_{ij}\Bigg[\big(2a^2H^2+\ddot{a}a\big)\big(1+2\Phi-2\Psi\big)+a^2H\big(6\dot{\Phi}-\dot{\Psi}\big)+a^2\ddot{\Phi}-\nabla^2\Phi\Bigg]-\partial_i\partial_j(\Phi+\Psi) \tag{4.65}$$

In the end, exploiting equation (1.58), it is possible to compute the Ricci scalar in this way:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij} \tag{4.66}$$

which, inserting all the relations derived previously, becomes:

$$R = (-1 + 2\Psi)\left[\frac{1}{a^2}\nabla^2\Psi - 3\frac{\ddot{a}}{a} - 3\ddot{\Phi} + 3H(\dot{\Psi} - 2\dot{\Phi})\right] + \frac{1 + 2\Phi}{a^2}\Bigg[3\big(2a^2H^2 + \ddot{a}a\big)\cdot \\ \cdot\big(1 + 2\Phi - 2\psi\big) + 3a^2H\big(6\dot{\Phi} - 3\dot{\Psi}\big) + 3a^2\ddot{\Phi} - 3\nabla^2\Phi - \partial_i\partial_j(\Phi + \Psi)\Bigg] \tag{4.67}$$

Taking into account only for the unperturbed terms, we can write the *zero-order* Ricci scalar as:

$$R^{(0)} = 3\frac{\ddot{a}}{a} + \frac{3}{a^2}(2a^2H^2 + \ddot{a}a) = 6\left(\frac{\ddot{a}}{a} + H^2\right) \tag{4.68}$$

This is completely consistent with the relation that we would obtain by using the unperturbed FLRW metric; indeed, in this case the Christoffel symbols are given by: $\Gamma^0_{ij} = a\dot{a}\delta_{ij}$ and $\Gamma^i_{0j} = \Gamma^i_{j0} = (\dot{a}/a)\delta_{ij}$, so that the components of the Ricci tensor are:

$$R^{(0)}_{00} = -3\frac{\ddot{a}}{a} \tag{4.69}$$

$$R^{(0)}_{ij} = 3\big(2\dot{a}^2 + \ddot{a}a\big) \tag{4.70}$$

from which the Ricci scalar is given by (considering $g^{00} = -1$ and $g^{ij} = 1/a^2$ in eq.(4.66)):

$$R^{(0)} = (-1)\left(-3\frac{\ddot{a}}{a}\right) + \frac{1}{a^2}\big(6\dot{a}^2 + 3\ddot{a}a\big) = 6\left(\frac{\ddot{a}}{a} + H^2\right) \tag{4.71}$$

which is exactly the same result obtained in equation (4.68). On the other hand, focusing on the perturbed terms in equation (4.67) we can write the expression for the fluctuations of the Ricci scalar as:

$$\delta R = -12\Psi\left(H^2 + \frac{\ddot{a}}{a}\right) - 2\frac{\nabla^2\Psi}{a^2} + 6\ddot{\Phi} - 6H(\dot{\Psi} + 4\dot{\Phi}) - 4\frac{\nabla^2\Phi}{a^2} \tag{4.72}$$

We can notice that, as expected, the unperturbed part of the Ricci scalar depends only on time, through the scale factor $a(t)$ and its derivatives, while the fluctuations depend also on spatial coordinates: this additional dependence is brought by the scalar perturbations of the metric Ψ and Φ, and by their evolution in time. This situation is similar to the one seen for the scalar field ϕ and its fluctuations (*Chapter 2*), even though in that case $\delta\phi$ was just proportional to Ψ, as can be seen in eq.(3.55).
Exploiting eq.(4.72) we can write down the explicit expression for the CB angle fluctuations, taking into account for two assumptions: Ψ and Φ can be taken as constant during matter-domination epoch, and the ∇^2 terms can be neglected since we are interested in perturbations on superhorizon scales; so the anisotropic angle can be defined as:

$$\delta\chi(\mathbf{x}, t) = \frac{\sigma}{2}\delta R(\mathbf{x}_{dec}, \tau_{dec}) \simeq \frac{\sigma}{2}\left[-12\Psi\left(H^2 + \frac{\ddot{a}}{a}\right)\right] \tag{4.73}$$

Since the birefringence angle can be written as the sum of an homogeneous background part plus some fluctuations, as in equation (4.41), and considering the *perturbed* Ricci scalar $R = R^{(0)} + \delta R$, we can recast equation (4.23) exploiting a Taylor expansion around $R^{(0)}$, keeping a generic $f(R)$ function:

$$\Delta\chi(t, \mathbf{x}) \simeq \frac{\sigma}{2}\left[f\big(R^{(0)}(t_{dec})\big) - f\big(R^{(0)}(t_0)\big) + f^{'}(R^{(0)})\delta R(t_{dec}, \mathbf{x}_{dec})\big)\right] \tag{4.74}$$

where $f'(R^{(0)})$ is the derivative of f, computed in $R^{(0)}$, with respect to R. Considering the two models for $f(R)$ previously analyzed, becomes:

$$\Delta\chi(t,\mathbf{x}) = \frac{\sigma}{2}\left[R^{(0)}(t_{dec}) - R^{(0)}(t_0) + \delta R(t_{dec},\mathbf{x}_{dec})\right] \tag{4.75}$$

for $f(R) = R$, and in the case of $f(R) = lnR$:

$$\Delta\chi(t,\mathbf{x}) = \frac{\sigma}{2}\left[ln\left(\frac{R^{(0)}(t_{dec})}{R^{(0)}(t_0)}\right) + \frac{1}{R^{(0)}}\delta R(t_{dec},\mathbf{x}_{dec})\right] \tag{4.76}$$

where the factor $1/R^{(0)}$ is given by the derivative of the logarithm. From this relations it is evident that the presence of δR fluctuations makes the birefringence angle increases, i.e. it enhances the rotation of the polarization plane with respect to the pure isotropic case.
Similarly to what we have done in the case of the scalar field, we can compute the power spectrum for the anisotropic CB angle, starting from the two-point correlation of δR fluctuations:

$$\langle \delta R(\mathbf{x},t)\delta R(\mathbf{x}',t)\rangle = 144\left(H^2 + \frac{\ddot{a}}{a}\right)^2 \langle \Psi(\mathbf{x},t)\Psi(\mathbf{x}',t)\rangle \tag{4.77}$$

Where $t = T_{dec}$. This means that the power spectrum of δR is related to the one of the metric perturbation Ψ. Thus, also in this case, $C_l^{\chi\chi}$ depends on the power spectrum P_Ψ; we can show this exploiting the fact that $\delta\chi$ can be decomposed trough spherical harmonics, as in eq.(3.6), and we can use the same expansion in the Fourier space in eq.(3.7) for δR; in this way the harmonic coefficients are given by:

$$\chi_{lm} = \frac{1}{2\pi^2}(-i)^l\frac{\sigma}{2}\int d^3k\delta R(k,t_{dec})j_l(k\Delta t)Y^*_{lm}(k) \tag{4.78}$$

where $\Delta t = t_0 - t_{dec}$. Then, given the definition of the anisotropic angle power spectrum in eqs.(3.9) and (3.10), and using the correlation in eq.(4.77), we get:

$$C_l^{\chi\chi} = 4\pi\left[144\left(H^2 + \frac{\ddot{a}}{a}\right)^2\right]\left(\frac{\sigma}{2}\right)^2\int\frac{dkk^2}{2\pi^2}P_\Psi(k)\left[j_l(k\Delta t)T_k(t_{dec})\right] \tag{4.79}$$

This result is similar to the one obtained in the case of a scalar field, but now there is a different factor outside the integral, and a different coupling constant σ, which encloses the information about the strength of the coupling between the $f(R)$ function and the CMB photons. As seen in *Chapter 2*, it is possible to relate P_Ψ to $P_\mathcal{R}$ in order to give an estimation for $C_l^{\chi\chi}$ and a bound to the constant σ (as previously done for the coupling factor β). This estimation can be done by considering the values of H and $a(t)$ at the decoupling epoch (during matter domination epoch, where $a \sim t^{-3/2}$).

In order to conclude this chapter we can make some final comments about this new scenario of cosmic birefringence sourced by an $f(R)$ function and

fluctuations of R. At first, we can notice that in this case, in order to compute δR in eq.(4.72) we have not imposed some strong approximations, as, instead, for the solution of $\delta\phi$ fluctuations. This is related to the fact that, on one hand, δR is linked to the metric perturbations (in particular Ψ and Φ in the Poisson gauge) in a *natural* way, while, on the other hand, $\delta\phi$ is related to Ψ (or Φ) through the adiabatic initial conditions. Moreover in this last scenario, there isn't any dependence of $f(R)$ or δR on the scalar potential, such that the dependence on an underline scalar field model is not present; this also imply, obviously, no dependence on the field mass.
One last aspect to be clarified is the following: for the scalar field ϕ we have found some specific physical meanings, identifying it as an axion-like field or as a general DM or DE quintessential field; what about $f(R)$? Could it be associated to a scalar field or it hides a new physical interpretation? We will address this question in the next section, making an overview on different proposed models for the $f(R)$ function's shape.

Conclusions

In this thesis we have investigated some models which are able to explain the Cosmic Birefringence effect, an observable phenomenon related to the rotation of the polarization plane of CMB photons. In general, this effect is generated by the presence of a Chern-Simons coupling term between a presudo-scalar field ϕ, or a scalar function $f(R)$, and a photon. We can conclude this work with a summary on different models based on both the presence of a scalar field and of an $f(R)$ function, in order to analyze their different implications, focusing on their observability too.
Let's start from the case of a scalar field ϕ coupled to the electromagnetic interaction. As we have seen in the second chapter this needs to be a pseudo-scalar field in order to ensure the parity invariance of the electromagnetic Lagrangian density. To be more precise, this mysterious field can be taken as a pseudo-Nambu-Goldstone boson generated by the spontaneous breaking of the $U(1)$ global Peccei-Quinn symmetry: in this case ϕ is accounted as the *QCD axion*, since it has been introduced to solve the *QCD problem*. The interesting aspect about axion-like fields is that they are able to produce a scale-invariant power spectrum of the CB angle (at least at lower multipoles): this is consistent with the theoretical predictions seen in *Chapter 3*; this allow us to make some constraints on the strength of the coupling (on the constant β or $g_{\phi\gamma}$) between the axion field and the CMB photons, as done in [**43**].
More in general ϕ could be a Dark Matter or (early) Dark Energy field: in this second case, the presence of ϕ can be exploited to solve the *Hubble tension* problem [**45**] (a disagreement between the *local* value of H and the one estimated from the CMB), and to explain the dynamics of the accelerated expansion of universe. Even in this scenario it is possible to find a bound for the coupling constant $g_{\phi\gamma}$ which is directly related to the cross-correlation power spectrum between the CB angle and the CMB temperature, i.e. $C_l^{\chi T}$. For this reason, precise measurements of the cosmic birefringence power-spectra are relevant in discriminating between different models based on an early DE field, since they produce different $C_l^{\chi\chi}$ and $C_l^{\chi T}$ spectral shapes. These kind of models are analyzed in [**45**] and [**46**], where, one the one hand, an oscillating DE field with the potential shown in eq.(3.46) is considered, while on the other hand, they take a slow-rolling scalar field, as seen in *Chapter 3*.

In the case of birefringence models based on the coupling with an $f(R)$ function, which replaces the scalar field ϕ, there are different scenarios based on different shapes of the scalar function itself, especially in the context of mod-

ified GR theories. As we have seen in *Chapter 4* the two models exploited in order to explain the baryo/leptogenesys process are the ones with $f(R) = R$ and $f(R) = \ln R$; indeed, in both cases, a non-null time derivative of the Ricci scalar can be considered, allowing the generation of the required baryon number asymmetry in the early universe.
Moreover, as suggested previously, the *f(R)* function can be found also in the context of modified theories of gravity (which are modifications of General Relativity) where an $f(R) \propto R^n$ is added to the Einstein-Hilbert action. Following this line, one interesting hypothesis is proposed by Carroll et al. in [**47**], who consider an $f(R) = R - (\mu^4/R)$. These kind of models would be able to explain the accelerated expansion in the early universe without requiring a cosmological constant Λ in Einstein field equations: this is a sort of *gravitational* alternative to Dark Energy. The same idea is exploited by Starobinsky in [**48**], where a different type of $f(R)$ function is considered.

The main result of this work is illustrated in *Chapter 4*, where in the end we have extended the treatment about the isotropic birefringence angle χ_0 generated by an *f(R)-coupling* to the anisotropic case, including the perturbations $\delta\chi$ (up to now this was done only in the scalar field case). These fluctuations depend on the perturbed Ricci scalar δR, whose complete expression in eq.(4.72) has been derived exploiting the Poisson gauge, and it is proportional to the two scalar perturbations of the metric Ψ and Φ, which in this case are equal to the gauge-invariant Bardeen's gravitational potentials. Thus, the anisotropic CB angle is directly sourced by the metric perturbations: this result is similar to the one obtained for the $\delta\phi$ fluctuations in *Chapter 3*. However, it is interesting to stress two relevant differences: on one hand, δR depends also on time and spatial derivatives of Ψ and Φ, whose evolution can be determined by solving eq.(1.74) and eq.(1.75); on the other hand, the relation for the Ricci scalar fluctuations has been computed without imposing some stringent approximations, as for obtaining the solution in eq.(3.55) for $\delta\phi$. Moreover, the scalar field fluctuations are related to the metric perturbations only thanks to the initial adiabatic conditions on ϕ, while δR is *naturally* sourced by Ψ and Φ.
Finally, through this extension to the anisotropic CB angle, it is possible to compute the auto-correlation $C_l^{\chi\chi}$ and the cross-correlation power spectra $C_l^{\chi T}$ and $C_l^{\chi E}$ even in *f(R)-birefringence* models: this allow us to make some constraints on the coupling constant σ, in order to be able to differentiate between different shapes of the scalar function, and to get much more information on the underlying physics and on the nature of $f(R)$ itself.

Future perspectives. Up to now, mainly thanks to the *Planck* observatory, consistent measurements of the isotropic CMB angle have been reached, analyzing the CMB power spectra (in particular EE, BB, TB and EB correlations). The most precise results obtained are shown in *Section 2.2* and they reach a statistical significance of 3.6σ, excluding a zero value at 99.987% C.L.. These data are mainly affected by a miscalibration of the detectors, which generates some spurious birefringence effects in the measurements. With the help of on-going and future experiments, we can expect to reach a significance $> 5\sigma$, which could lead to a convincing discovery of the birefringence effect. Among the new generation ground-based observatories we can mention the BICEP/Keck array (on-going) [**49**] and the CMB-S4 (future) [**50**], while, considering the future space-borne experiments, we can refer to the LiteBIRD satellite [**51**]. The last one, in particular, is of great interest since it can improve the current data on the CB angle power-spectra, in order to get better constraints on the coupling constant $g_{\phi\gamma}$, improving the current bounds of about one order of magnitude.

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