UNIVERSITÀ DEGLI STUDI DI PADOVA

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Tesi di Laurea

Impact of the Sommerfeld corrections on the dark matter relic density

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Contents

Introduction

The origin and the composition of the dark matter is one of the most urgent problems in fundamental Physics. Although since Fritz Zwicky ('30s) and Vera Rubin's ('70s) discoveries we have gathered a lot of evidence of the existence of abundant massive (thus interacting by gravity) but invisible (so that we exclude electromagnetic and strong interactions) matter that governs the behaviour of objects with dimensions as of galaxies, we are convinced that there is no place for such a peculiar matter inside the Standard Model, that is all the matter of which we know the composition.

Among several plausible candidates, weakly interacting massive particles (WIMPs), produced in the early universe in a quantity set by thermal freeze-out from the primordial thermic bath in equilibrium, provide an appealing and motivating scenario. In this framework we develop the formalism to account for an effect known as Sommerfeld enhancement, that is significant when attractive forces mediated by interactions with small coupling constants become strong at small velocities.

This thesis work will begin with a general presentation of the formalism and scenario describing the evolution of the Universe, defining the notation and the basic concepts that lead us to the physical exploration of the early Universe. Then, we develop the formalism for thermal models, applying it to freeze-out of the WIMPs, both in an instantaneous and naive way and in a more accurate one by solving the Boltzmann equation with the proper considerations. Afterwards, a general theory of non-relativistic quantum scattering is developed, with a particular regard for the formalism of partial waves, in which the role of the angular momentum is essential. Finally, we characterize the effect of the Sommerfeld enhancement, as a parameter that modifies the cross section in the particle-antiparticle annihilation for WIMPs, and quantify how it affects the relic density of dark matter that we measure today.

Chapter 1

The Standard Cosmological Model

In this *Chapter* we will discuss the fundamental aspects of modern Cosmology in order to reach a basic knowledge of the concepts and the tools useful for the following *Chapter*s.

1.1 The large-scale structure of the Universe

The aim of this section is to indicate the spatial scale in which the results that will be discussed in the following *Paragraph*s can be considered true. In fact, we are interested to describe the universe as a smooth continuum, where homogeneity and isotropy hold. This request is enforced by observations in several zones of the electromagnetic spectrum surveying larger regions of the Universe, on a scale of hundreds of *megaparsecs* (*Mpc*).

We spend just few words talking about this measure unit, since it will appear in most of the results of this *Chapter*. A *parsec* equals approximately 3.261 light years, that is 1 $pc = 3.086 \cdot 10^{16} \ m$. So, while parsecs are very suitable to describe distances between stars close to each other, one megaparsec is the typical separation between neighbouring galaxies. When we survey galaxies on the scale of hundreds of megaparsecs, a large-scale structure becomes evident: in fact at this scale galaxies are grouped in superclusters joined together by filaments, and between these structures lie voids as large as tens of *Mpc*.

Once we get to this scale, there is no evidence of larger structures and the Universe seems to repeat itself indefinitely, as revealed by extremely large galaxy surveys like the 2dF galaxy redshift survey and the Sloan Digital Sky Survey (see Figure 1.1). Moreover Penzias and Wilson accidentally discovered in 1965 that we are merged in microwave radiation, called Cosmic Microwave Background (CMB), with a black-body spectrum of a temperature measured $2.725 \pm 0.001 \ K$ by FIRAS (Far InfraRed Absolute Spectrometer) experiment on board of COBE (COsmic Background Explorer) satellite in 1992. Furthermore, the radiation appears uniform seen from all directions, with variations small as one part in a hundred thousand, as measured by the *Planck* satellite in 2013 (see Figure 1.2).

These results are in agreement with our first assumption of using homogeneity and isotropy on a sufficiently large scale. Therefore, this will be presented as a matter of fact in the following *Paragraph*s.
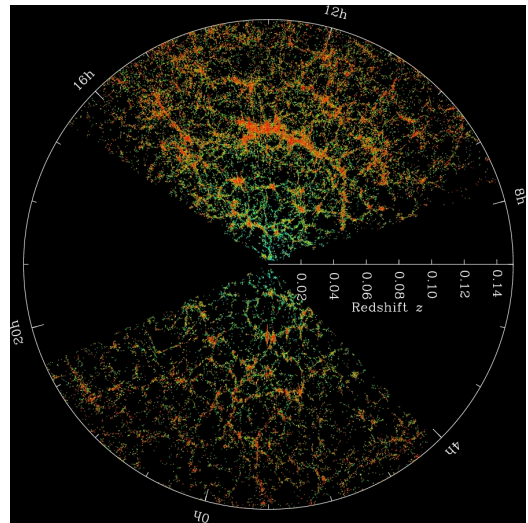

Figure 1.1: Sloan Digital Sky Survey image of filaments and voids between them. Picture from `sdss.org`.
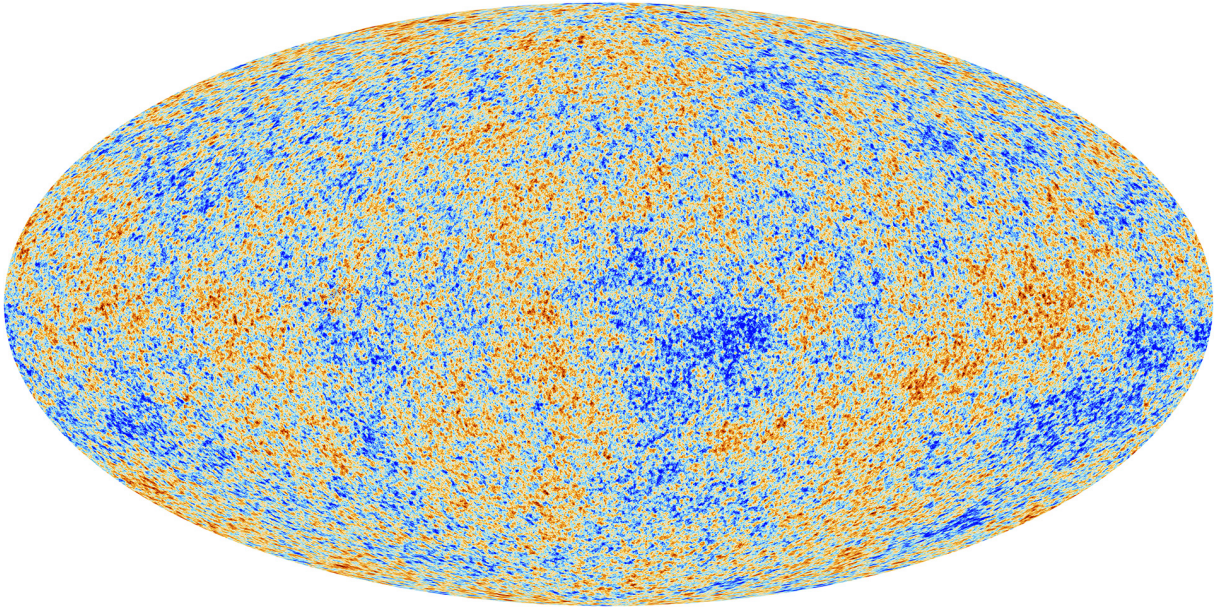
Figure 1.2: Planck satellite measurements of intensity variations in the Cosmic Microwave Background. Picture from `esa.int`.

1.2 Hubble's law and comoving coordinates

In 1912 Vesto Slipher and afterwards (and more systematically) Edwin Hubble used the shift of observed wavelengths coming from distant galaxies compared to the ones we expect (for example observing galaxies nearby) to measure the relative velocity from us and each galaxy. From surveying 1355 galaxies Edwin Hubble published his well-known law:

$$\vec{v} = H_0 \vec{r} \tag{1.1}$$

where $\vec{r}$ is the relative position of the galaxy with respect to us and H_0 is known as the Hubble's constant, that assumes a positive value. Notice that the linearity of Hubble's law perfectly joins our requests of homogeneity and isotropy. Furthermore, we have assumed homogeneity and isotropy only of space, so it is not necessary for the Hubble's constant to be constant in time: its value H_0 is the nowadays value.

Being the Hubble's constant positive, what one observes is the expansion of the Universe and the fact that the velocity of moving away galaxies increases with their distance from us. Given these facts, it is convenient to adopt different coordinates $\vec{x}$, named comoving coordinates, instead of $\vec{r}$, the actual separation between objects, such that at every time $\vec{r} = a(t)\,\vec{x}$ and $\vec{x}$ remains constant. In fact, a is known as the *scale factor* of the Universe. It is important to notice that the homogeneity and isotropy principles permit this change of coordinates, since in a very euristic way, we are assuming that while objects are moving away each other, the "space grid" is stretched along with them such that galaxies assume the same position in this grid. It becomes now trivial that

$$H(t) = \frac{\dot{a}}{a}$$

where $H(t)$ is the Hubble's constant at a time t.

1.3   Friedmann equations

We present here the Friedmann equations, commenting in detail most of their fundamental aspects. The first one, although a full comprehension of it requires knowledge in general relativity, can be seen as the conservation of energy in the Universe. In fact, the mechanical energy of a particle of mass m at the boundary of a sphere of radius r, which gravitationally interacts with the energy density ρ in it, is:

$$U = \frac{1}{2}m\dot{r}^2 - \frac{4}{3}\pi G \frac{m\rho\, r^3}{r}$$

that after having changed to comoving coordinates $r = a\,x$ and imposed $k = -\dfrac{2U}{mc^2 x^2}$ becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k\,c^2}{a^2} \tag{1.2}$$

We need to spend more words about the value k, known as the curvature. Firstlt, it is a constant in the common sense, that is independent from space and time, since it can be shown that $k \propto -U$, where U is the total energy in the Universe, and we are convinced that energy is always conserved. Moreover, the sign of k, which is linked to the sign of U, is fundamental to distinguish the possible behaviours of the Universe in a complete analogy with the Kepler's problem:

- If $k > 0$, then the Universe is in a bonded state, meaning that although it is now expanding, at a finite time there will be a turning point when gravity will dominate and the Universe will collapse. This kind of Universe is called *close* Universe and the geometry describing it is spherical.

- On the other hand, if $k < 0$, gravity is not strong enough to stop expansion (always one may think to the analogy with the Kepler's problem) and the Universe will expand forever. In the following *Paragraph* we will show that in this case for very large t, we can have $a(t) \propto t$, so expansion becomes linear in time. This Universe is called *open* and is described by hyperbolic geometry.

- The case $k = 0$ corresponds to the Kepler's problem in the particular case we are at the escape velocity. Therefore, although $a(t)$ continues to grow with time, $\dot{a}$ tends to 0 for large t. This case is called *flat* Universe and is described by Euclidean geometry.

Since in order to solve 1.2 we need to know ρ, we have to derive a second law, considering the Universe as a homogeneous fluid with a time-dependent density ρ. From the first principle of thermodynamics:

$$dE + pdV = TdS$$

Assuming that on the large-scale we can apply $E = mc^2$ and assume every transformation to be adiabatic, so that $TdS = 0$, we have now:

$$dE = 4\pi a^2 \rho c^2 da + \frac{4\pi}{3}a^3 c^2\, d\rho$$

$$dV = 4\pi a^2\, da$$

Considering the change of E and V in the time dt and putting all these results together we obtain the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{1.3}$$

The pressure p assumes here an unusual meaning: in fact, there are no forces associated to this pressure, since in a homogeneous Universe there cannot be any gradient in the pressure. Therefore, the role of p is merely to express the work done by the Universe to raise potential energy as the Universe expands. In order to solve equations 1.2 and 1.3 combined, one needs to know the behaviour of p. In the following *Paragraph* we will present two possible solutions assuming $p = p(\rho)$, known as equation of state.

Matter and radiation

We define matter as any kind of pressureless material, independently from its composition. Therefore, $p(\rho) \equiv 0$, so that (3) can be rewritten as:

$$\frac{1}{a^3}\frac{d}{dt}\left(\rho a^3\right) = 0$$

that implies $\rho_{mat} \propto a^{-3}$, which is obvious since volume increases with a^3, and one may assume that matter inside this volume remains constant.

On the other hand, radiation is defined by the equation of state $p = \frac{1}{3}\rho c^2$, so that (3) becomes

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0$$

which is solved by $\rho_{rad} \propto a^{-4}$. This result is also consistent with what we expect because the number density of photons scales with a^{-3}, as it is a density and we expect radiation to be homogeneous, but also the energy of each photon scales with a^{-1} since $E = hc/\lambda \propto a^{-1}$.

1.4 Observational parameters

Since $H(t) = \dfrac{\dot{a}}{a}$, (2) can assume the form:

$$\rho = \frac{3H^2}{8\pi G} + \frac{3kc^2}{8\pi G\, a^2} \tag{1.4}$$

Therefore, if ρ assumes the particular value $\rho_c(t) = 3H^2/(8\pi G)$, called *critical density*, k is obliged to have null value (for all times!). We see that in natural units $(8\pi G)^{-\frac{1}{2}}$ is an energy, and we define this value as the reduced Planck mass $M_{Pl} \approx 2.4 \cdot 10^{18}\, GeV$.

Once we have measured H_0, the value of H at the present time, we may easily obtain ρ_c. The measured value of H_0 is $100\, h\, km/(Mpc \cdot s)$, where there is tension between the measurements of h (see Table 1.1), but we consider that its value should be 0.70 with a few percents of uncertainty. This gives $\rho_c = 1.88\, h^2 \cdot 10^{-26}\, kg\, m^{-3}$: although it seems a very small density, doing the maths it equals the mass of an average galaxy in a Mpc^3, that is the typical separation between neighbouring galaxies, as seen in *Chapter* 1. Therefore, if we introduce the parameter $\Omega = \rho/\rho_c$ (that is a function of time), observations oblige us to consider that $\Omega \approx 1$ and, since $k \approx 0$, we can say that the Universe is approximately flat.

We are now interested in determining the value Ω_0 of Ω, considering just the contributions of

Value of h	Experiment
0.72(8)	HSTKP
0.75(3)	SH0ES
0.673(12)	*Planck*

Table 1.1: Values of h from different measurements. The numbers in brackets indicate the error on the least significant digit. HSTKP = Hubble Space Telescope Key Project. SH0ES = Supernovae and H_0 for the Equation of State.

matter, with the above definition.

- Of course, stars in the galaxies give a non-negligible contribution to the total density. However, one observes $\Omega_{stars} \approx 0.005 - 0.01$, that is thus a little percentage of the total density.

- Another contribution comes from hot gases that surround superclusters, so hot that they emit radiation in the X-ray spectrum. From the theory of nucleosynthesis it can be derived that the *baryonic* density (i.e., protons and neutrons) is such that $0.021 \leq \Omega_B\, h^2 \leq 0.025$.

- There is a lot of evidence of the existence of another type of matter, that is thought to surround each galaxy with a spherical halo, called *dark matter*. The value of its density can be calculated knowing that galaxies and hot gases in superclusters do not provide enough

gravitational attraction to maintain all this structure in a bonded state. Since *Chandra* X-ray satellite measures

$$\frac{\Omega_B}{\Omega_0} \approx 0.161$$

and from further accurate confirmations from WMAP and *Planck* satellite, we estimate $\Omega_0 \, h^2 = 0.1428 \pm 0.0011$.

Therefore, by subtraction, we can compute

$$\Omega_{DM,0} \, h^2 = 0.1198 \pm 0.0012$$

We conclude this *Paragraph* by affirming that the total amount of energy of the Universe (or energy budget) Ω_{tot} must be very close to 1 (thus, the density equals the critical density). Without giving further details, that are not useful for our scopes, we are convinced of this fact, since both in a matter and in a radiation dominated Universe, the function $|\Omega_{tot}(t) - 1|$ is increasing with t. Therefore, any deviation from 1 at the beginning of the Universe, could give a Universe totally different from the one we observe.

1.5 Cosmological Statistical Mechanics

In the primordial plasma each particle species i is described by its phase-space distribution $f_i(\mathbf{x}, \mathbf{p}, t)$. Imposing the cosmological principle, f_i may only depend by $p = |\mathbf{p}|$ and t, since the Universe is expanding and cooling down.

We can therefore define the number density n_i, the energy density ρ_i and the pressure p_i of the species i as follows:

$$n_i = g_i \int \frac{d^3 p}{(2\pi)^3} f_i(p, t) \tag{1.5}$$

$$\rho_i = g_i \int \frac{d^3 p}{(2\pi)^3} f_i(p, t) E_i(p)$$

$$p_i = g_i \int \frac{d^3 p}{(2\pi)^3} f_i(p, t) \frac{p^2}{3 \, E_i(p)},$$

where g_i are the internal degrees of freedom and $E_i = \sqrt{p_i^2 + m_i^2}$.

Defining with S the total entropy in a comoving volume and with s the entropy density, we have $S = s \, a^3$. From the Euler relation (we apply it since we have homogeneity and isotropy), we have

$$s = \frac{\rho + p}{T} \tag{1.6}$$

which can be computed once we have computed the energy density ρ and the pressure p as described above.

A very useful adimensional quantity is the comoving number density defined as

$$Y_i = \frac{n_i}{s}$$

because it is a constant variable since the values $Ni = n_i \, a^3$ and $S = s \, a^3$ are constant by homogeneity, if we are away from a number-changing process, such as the thermal equilibrium.

In the case of non-relativistic particle i, the energy density of the species i is $\rho_i = m_i n_i$. So, being Y_i constant away from equilibrium, we can compute:

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_c} = \frac{m_i n_{i,0}}{\rho_c} = \frac{m_i Y_i s_0}{\rho_c}. \tag{1.7}$$

Chapter 2

Freeze-out of WIMPs

The Weakly Interacting Massive Particles are, amongst all the dark matter hypothetical candidates, the most popular ones, because they can solve also the hierarchy problem, another unsolved problem in the Standard Model. Broadly speaking, we consider WIMPs to satisfy this typical values for their mass and cross section to Standard Model (SM) final state:

$$1\,GeV \leq m_{WIMP} \leq 1\,TeV$$

$$\sigma_{WIMP} \simeq 1\,pb$$

We develop in the following *Paragraph* some estimates of the dark matter relic density, from the moment it decoupled from the primordial plasma.

2.1 Thermal processes

We define thermal models those in which dark matter was in thermal equilibrium in the early universe at very high temperatures, and we impose that the departure from thermal equilibrium is the process setting the relic density that we observe nowadays.
Of course, since the Universe is expanding, there should have been interactions between particles in order to maintain the thermal equilibrium. However, gravitational interactions are not sufficiently strong to achieve equilibrium, so we add some interactions between the dark matter particle χ and the Standard Model particle ψ to give the annihilation process

$$\chi\,\overline{\chi} \longrightarrow \psi\,\overline{\psi}$$

with the rate $\Gamma_{ann} = n_\chi \langle \sigma\,v_{rel} \rangle$, where $\langle \sigma\,v_{rel} \rangle$ is the thermally averaged over all the possible states of the product between the cross section and the Moeller velocity: for brevity, we will improperly call this quantity *cross section* for the rest of the *Chapter* and we will define better this concept later. We evaluate the ratio Γ_{ann}/H between the annihilation rate and the Hubble's constant, because it is an interesting value that quantifies whether the interactions were sufficient to contrast the expansion. In particular, if the ratio is lower than 1, then the equilibrium could not be achieved, while if it was grater than 1, we could have equilibrium.
From equation 1.5 we can compute in the non-relativistic case ($T << m_\chi$)

$$n_\chi^{eq} = g_\chi \left(\frac{m_\chi\,T}{2\pi} \right)^{3/2} e^{-\frac{m_\chi}{T}}. \tag{2.1}$$

Now, since if dark matter particles were nowadays in thermal equilibrium, their number density would be approximately zero due to the exponential factor, we are convinced that there existed a specific temperature, namely *freeze-out temperature* (T_{FO}), at which dark matter departed from equilibrium. We estimate T_{FO} solving the equation:

$$\Gamma_{ann}(T_{FO}) = n_\chi^{eq}(T_{FO})\langle \sigma\,v \rangle_{T=T_{FO}} = H(T_{FO}). \tag{2.2}$$

The solutions in which $T_{FO} > m_\chi$, in which particles decouple when they are relativistic, is what characterizes *hot relics*, while the opposite case is known as *cold relics*, and will be the subject of the following *Paragraph*.

It is very important to underline the difference in the behaviour of Y_χ in the two situations $T > T_{FO}$ and $T < T_{FO}$. In the first case, dark matter is in thermal equilibrium in the primordial bath and therefore $Y_\chi(T)$ is computed with n_χ and s at the equilibrium; in the second one, dark matter is decoupled from primordial bath and therefore its evolution is governed by the fact that the number of dark matter particles and the entropy per comoving volume remain constant, and so does Y_χ.

2.2 Instantaneous freeze-out

It can be shown from the Friedmann equations that in a radiation-dominated Universe, where $a(t) \propto t^{1/2}$, we have:

$$H(t) = \frac{1}{2t} = \frac{\pi\sqrt{g_*(T)}}{3\sqrt{10}}\frac{T^2}{M_{Pl}}$$

where g_* are the total degrees of freedom of the Standard Model. This equation is a straightforward consequence of the two possible descriptions of the density ρ of the particles in thermal equilibrium: from Statistical Mechanics $\rho = \frac{\pi^2}{30}g_*(T)T^4$, and from Cosmology $\rho = 3H(t)^2 M_{Pl}^2$.

We substitute the above result in equation 2.2 and rewrite it with the new dimensionless parameter $x_{FO} = m_\chi/T_{FO}$. We obtain the equation

$$\frac{g_\chi m_\chi^3}{(2\pi)^{3/2}} x_{FO}^{-3/2} e^{-x_{FO}} \langle\sigma v\rangle \approx \frac{\pi\sqrt{g_*(T)}}{3\sqrt{10}}\frac{T_{FO}^2}{M_{Pl}} \tag{2.3}$$

Taking the logarithm of each side, we can write:

$$x_{FO}(x_{FO}) \approx \frac{1}{2}ln(x_{FO}) + ln\left(\frac{3\sqrt{5}}{2\pi^{5/2}}\right) + ln\left(\frac{g_\chi}{g_*^{1/2}(x_{FO})}\right) + ln(m_\chi M_{Pl}\langle\sigma v\rangle) \tag{2.4}$$

We choose as typical values $m_\chi \approx 100\,GeV$ and $\langle\sigma v\rangle \approx 1\,pb \approx 2.6 \cdot 10^{-9}\,GeV^{-2}$. Therefore, we have $x_{FO} \approx 25$, where the biggest contribution is the one of the last logarithm, where the Planck mass dominates, and the other logarithms give much smaller corrections to the final value. Note that we have found $m_\chi \approx 25T_{FO}$, which is quite accurate to infer that the relics are cold. To be more precise, we developed a code that computes x_{FO} from equation 2.4 by looking where the function $x_{FO} - x_{FO}(x_{FO})$ is null. First of all, we suppose that our dark matter particle has two internal degrees of freedom: $g_\chi = 2$. We have a x_{FO} dependence both in the logarithm and in the function g_*: for $x_{FO} \approx 25$, we have $T_{FO} \approx 4\,GeV$, that gives $g_*(4\,GeV) \approx 84$, from which a contribution to equation 2.4 of the order

$$ln\left(\frac{g_\chi}{g_*^{1/2}(x_{FO})}\right) \approx -1.6.$$

Therefore, we implemented our code being confident that $20 < x_{FO} < 50$, to which corresponds $82.50 < g_*(T) < 85.60$, and we will verify this condition *a posteriori*. To account for the changing values of $g_*(T)$, we extrapolated a straight line that for $x = 20$ gives $g_*(T) = 82.50$ and for $x = 50$ gives $g_*(T) = 85.60$.

We obtained with a precision at the second decimal digit $x_{FO} = 23.65$, that is completely coherent with our expectations and assumptions.

In order to compute the nowadays relic density for this process, it is convenient to compute the comoving number density at the freeze-out, since we have seen that it is conserved. Therefore, we assume that the freeze-out is instantaneous.

$$Y_\chi(T_{FO}) = \frac{n_\chi(T_{FO})}{s(T_{FO})} \approx \frac{H(T_{FO})/\langle\sigma v\rangle}{\frac{2\pi^2}{45}g_{*s}(T_{FO})T_{FO}^3}$$

In this expression the entropy has been computed from the Euler relation at equation (6), and g_{*s} are the degrees of freedom that contribute to the entropy. From equation (1.7), we can finally write:

$$\Omega_\chi = 2\frac{m_\chi\, Y_\chi(T_{FO})s_0}{\rho_c} = \frac{3\sqrt{5}}{\sqrt{2\pi}}\frac{g_*^{1/2}(T_{FO})}{g_{*s}(T_{FO})}\frac{x(T_{FO})\,s_0}{\rho_c\, M_{Pl}\langle\sigma\,v\rangle}.$$

Being $s_0 = 2891.2\,cm^{-3}$ the nowadays entropy density, $\rho_c = 1.05\cdot 10^{-5}\,h^2\,GeV\,cm^{-3}$ and $\langle\sigma v\rangle = 2.6\cdot 10^{-9}\,GeV^{-2}$, we obtain the final value at the second decimal digit $\Omega_\chi\,h^2 = 0.15$, which is the same order of magnitude of the measured $\Omega_{DM,0}\,h^2 = 0.1198 \pm 0.0012$ discussed in *Paragraph* 1.4. The 2 in the previous equation accounts also for the contribution of the dark matter antiparticle.

2.3 Boltzmann equation

We present here a way to find Y_χ for every possible value of x and, in particular, not only at the freeze-out. This formalism permits us to avoid the approximation of instantaneous freeze-out and to visualize also the process of decoupling from thermal equilibrium. Therefore, we suppose that our dark matter particle χ makes a binary scattering in the form:

$$\chi\,\overline{\chi} \longrightarrow \phi\,\overline{\phi}$$

and the thermally averaged cross section:

$$\langle\sigma\,v_{rel}\rangle_{\chi\overline{\chi}\longrightarrow\phi\overline{\phi}} = \frac{\int d^3p_{\chi_1}d^3p_{\chi_2}f_\chi^{eq}(|\mathbf{p}_{\chi_1}|)\,f_\chi^{eq}(|\mathbf{p}_{\chi_2}|)\,\sigma(\mathbf{p}_{\chi_1},\mathbf{p}_{\chi_2})_{\chi\overline{\chi}\longrightarrow\phi\overline{\phi}}\,v_{rel}}{\int d^3p_{\chi_1}d^3p_{\chi_2}f_\chi^{eq}(|\mathbf{p}_{\chi_1}|)f_\chi^{eq}(|\mathbf{p}_{\chi_2}|)}$$

where v_{rel} is the Moeller velocity between the initial state particles: in the non-relativistic case, which we are interested to, v_{rel} can be considered just as the relative velocity between the colliding particles.

We therefore arrive to the Boltzmann equation for the number density of χ:

$$\frac{d\,n_\chi}{dt} + 3H(t)n_\chi = -\langle\sigma\,v_{rel}\rangle\left(n_\chi^2 - n_\chi^{eq2}\right) \tag{2.5}$$

where n_χ^{eq} is the number density for the particles in thermal equilibrium.
We can consider the derivative of the comoving number density and obtain:

$$\frac{d\,Y_\chi}{dt} = \frac{d}{dt}\left(\frac{n_\chi\,a^3}{s\,a^3}\right) = \frac{1}{s\,a^3}\left(\frac{d\,n_\chi}{dt}a^3 + 3n_\chi\,a^2\dot{a}\right) = \frac{1}{s}\left(\frac{d\,n_\chi}{dt} + 3Hn_\chi\right)$$

that gives

$$\frac{d\,Y_\chi}{dt} = -\langle\sigma\,v_{rel}\rangle\,s\left(Y_\chi^2 - Y_\chi^{eq2}\right) \tag{2.6}$$

We are now interested to compute $Y_\chi(x)$: see Appendix A for details. The final form of the differential equation that we are interested to solve is

$$\frac{d\,Y_\chi}{dx} = -\frac{\lambda}{x^2}\frac{g_{*s}(x)}{\sqrt{g_*(x)}}\left(1 - \frac{x}{3}\frac{g'_{*s}(x)}{g_{*s}(x)}\right)(Y_\chi^2 - Y_\chi^{eq\,2}(x)) \tag{2.7}$$

with $\lambda = \dfrac{2\pi\sqrt{10}}{15}\,m_\chi\,M_{Pl}\,\langle\sigma\,v_{rel}\rangle$ and $g'_{*s}(x) = \dfrac{d\,g_{*s}(x)}{dx}$.

Solutions to the Boltzmann Equation

The Boltzmann equation is a particular Riccati equation, therefore there is no closed analytic solution to it, but we can compute it via numerical solution.
In order to make a first estimate, we consider a s-wave process (this concept will be fully defined in the next *Chapter*), from which we expect

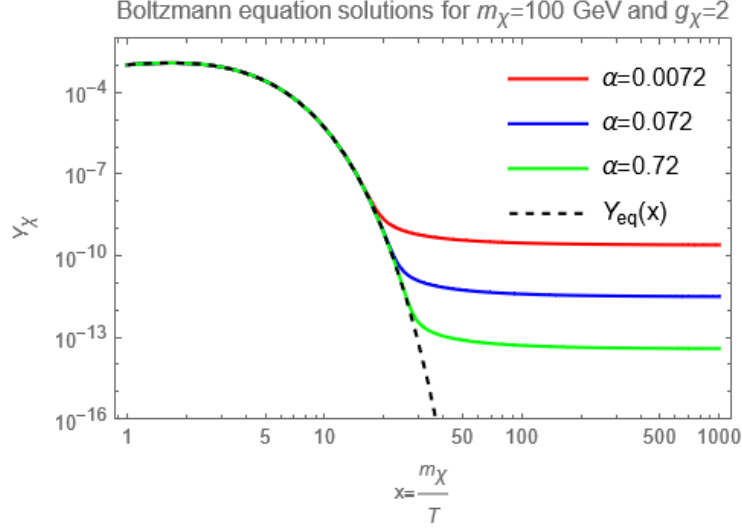$$\langle\sigma\,v_{rel}\rangle = \frac{\alpha_{DM}^2}{32\pi\,m_\chi^2}$$

Figure 2.1: Simulation of the Boltzmann equation for different values of the coupling constants. The dashed black line represents the equilibrium behaviour.

with α_{DM} a coupling parameter that we suppose to give a cross section of the order of magnitude of $1\,pb$, so $\alpha_{DM} \simeq 0.072$.

We develop our simulations for three coupling constants $\alpha_1 = 0.1\alpha_{DM}$, $\alpha_2 = \alpha_{DM}$ and $\alpha_3 = 10\alpha_{DM}$. We assume that $Y_\chi(x_{in}) = Y_\chi^{eq}(x_{in})$, with x_{in} the starting point of the simulation, chosen to be $x_{in} = 1$, because we want to limit ourselves in the non-relativistic case. To be consistent with the previous calculation, we choose $m_\chi = 100\,GeV$ and $g_\chi = 2$. The functions g_* and g_{*s} are computed as proposed in [SSH]. In Figure 2.1 we can see the graphs obtained: as expected the higher the coupling contant, the later the dark matter particles decouple from the primordial bath, that is at lower temperatures.

Now, we can compute the relic density today $Y_{\chi,0}$: because of computational reasons, we could not reach the value $x_0 \approx 10^{15}$, which is approximately the value of x today. On the other hand, is is obvious that Y_χ becomes asymptotically constant $(Y_\chi'(x) \propto x^{-2})$, thus we choose the value $Y_\chi(x = 10^6)$ as $Y_{\chi,0}$. In Table 2.1 are shown the results obtained from $\Omega_{\chi,0} = 2m_\chi Y_{\chi,0}\,s_0/\rho_c$ computed for different coupling constants: note that no one reproduces the observed result $\Omega_{DM,0}\,h^2 = 0.1198 \pm 0.0012$, but this fact does not worry: the simulation is strongly dependent from our assumptions concerning m_χ and α_i. Therefore, the mere knowledge of $\Omega_{DM,0}$ does not explicitly return the values of these two parameters.

α_i	$\Omega_{\chi,0}\,h^2$
0.1	13.6
1	0.174
10	0.00208

Table 2.1: Relic density of dark matter assuming different coupling constants α_i, expressed in $\alpha_{DM}\,units$.

Chapter 3

Non-relativistic Quantum Scattering

In this *Chapter* we will develop the formalism necessary for the discussion of the subjects concerning the next *Chapter*: in particular, we will discuss the theory of elastic scattering by a central potential described by the partial waves formalism, as fully defined in the following *Paragraphs*.

3.1   Definition of the problem

We are interested in describing collisions between particles, that, in principle, can occur between an arbitrary number of both incident particles and outgoing ones. However, we define the *non-relativistic scattering* the collisions between particles (1), called incident, and (2), called target, in which the final state is composed only by the same (1) and (2). Therefore, the formalism that we will develop will not take into consideration the phenomena of creation and annihilation, that are not of interest for our scopes.

In order to obtain simpler results, we need to impose more conditions, that are anyway the same ones we assume that are valid in the next *Chapter*.

1. We do not take the spin of both particles (1) and (2) into consideration: this is just an approximation and does not imply that the theory can be applied only to spinless particles.

2. Particles (1) and (2) are considered point-like. Otherwise, we should take into consideration the evolution of the more elementary particles composing (1) and (2).

3. We ignore multiple scattering processes: this condition is also called *thin target condition*.

4. We neglect entangled final states between (1) and (2).

5. We assume that the interactions can occur only between incident beam and the target, and are described by a potential energy depending only on the relative position between (1) and (2) $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Therefore, we can study the interaction in the center-of-mass reference frame adopting the reduced mass μ defined by $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$.

6. We assume a short-ranged potential, that is one going to zero more rapidly than the coulombian one for r going to infinite.

3.2   Cross section and stationary states

We suppose that a detector, subtending an infinitesimal solid angle $d\Omega$, is located far away from the scattering region (i.e., where the potential is effective) in a direction defined by the angles θ and ϕ, respectively the polar angle from the incident beam direction $\hat{z}$ and the azimutal angle in the orthogonal plane. Being, F_i the incident flux defined as the number of incident particles per unit time and surface, the number dn of particles (1) scattered in the origin and revealed by the detector described above per unit time is

$$dn = F_i \, \sigma(\theta, \phi) \, d\Omega \tag{3.1}$$

where $\sigma(\theta, \phi)$ is called differential cross section and has the dimensions of a surface, measures in *barn* (b) such that $1\,b = 10^{-24}\,cm^2$.

The cross section is defined as the integral over the solid angle of the differential cross section

$$\sigma = \int \sigma(\theta, \phi)\, d\Omega$$

and represents the effective surface of the target as seen from the incident particles.

In order to describe the time evolution of the particle state $|\psi\rangle$ in the potential $V(\mathbf{r})$, we need to solve the Schrödinger's eigenvalues equation

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{r})\right]\varphi(\mathbf{r}) = E\,\varphi(\mathbf{r}) \tag{3.2}$$

Defining $E = \dfrac{\hbar^2 k^2}{2\mu}$ and $V(\mathbf{r}) = \dfrac{\hbar^2}{2\mu}U(\mathbf{r})$ enables us to simplify the above equation in the form

$$\left[\nabla^2 + k^2 - U(\mathbf{r})\right]\varphi(\mathbf{r}) = 0 \tag{3.3}$$

Note that we have assumed that the energy E is a positive number, but this is not a limitation, because we are interested only to non bonded states. Moreover, the spectrum of the Hamiltonian is continuous: we expect then unnormalisable eigenstates, called *stationary scattering states* and written as $v_k(\mathbf{r})$.

The asymptotic form of the $v_k(\mathbf{r})$ can be found considering that for large values of $r = |\mathbf{r}|$, $U(\mathbf{r}) \approx 0$, thus the Schrödinger equation becomes the one of a free particle and the asymptotic form of the eigenstates can be seen as the superposition of a z-axis incoming plane wave $v_k^{in}(\mathbf{r}) = e^{ikz}$ and an outgoing scattered wave $v_k^{out}(\mathbf{r}) = f_k(\theta, \phi)\dfrac{e^{ikr}}{r}$, giving

$$v_k = e^{ikz} + f_k(\theta, \phi)\frac{e^{ikr}}{r} \tag{3.4}$$

where $f_k(\theta, \phi)$ is the only term depending on the potential $V(\mathbf{r})$ and is called *scattering amplitude*. Note that, while v_k^{in} is a proper solution to the free particle Hamiltonian, v_k^{out} is a solution only for large r values, as can be shown by explicit calculation.

Now, if we want to compute the differential cross section as a function of the quantities defined so far, we can introduce the probability current

$$\mathbf{J}(\mathbf{r}) = \frac{1}{\mu}Re\left[\varphi^*(\mathbf{r})\frac{\hbar}{i}\nabla\varphi(\mathbf{r})\right]$$

Therefore, substituting v_k^{in} as φ in the above equation, the incident flux is

$$F_i = C|\mathbf{J}_i| = C\frac{\hbar\,k}{\mu}$$

where C is a constant having no physical meaning that will be reabsorbed. On the other hand, substituting v_k^{out} and proceeding with the calculation in spherical coordinates, the scattered flux is

$$dn = C\mathbf{J}(\mathbf{r}) \cdot d\mathbf{S} = C\,J_r(\mathbf{r})\,r^2 d\Omega + O(\frac{1}{r}) \approx C\frac{\hbar\,k}{\mu}|f_k(\theta, \phi)|^2 d\Omega$$

Finally, combining these results with equation (3.1), we obtain

$$\sigma(\theta, \phi) = |f_k(\theta, \phi)|^2.$$

3.3   Partial waves formalism

If we now consider a central potential $V(r)$, we can find solutions to (3.2) that also eigenfunctions of $\mathbf{L}^2$ and L_z, that we will call $|\varphi_{k,l,m}\rangle$. It is well-known that $\mathbf{L}^2 |\varphi_{k,l,m}\rangle = \hbar^2 l(l+1) |\varphi_{k,l,m}\rangle$ and $L_z |\varphi_{k,l,m}\rangle = \hbar m |\varphi_{k,l,m}\rangle$. Is is also well-known that in the case of central potential $|\varphi_{k,l,m}\rangle$ can be rewriiten as

$$\varphi_{k,l,m} = \frac{1}{r} u_{k,l}(r) Y_l^m(\theta, \phi)$$

with $Y_l^m(\theta, \phi)$ the spherical harmonic of numbers l and m, and $u_{k,l}(r)$ solves

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 - U(r) \right] u_{k,l}(r) = 0 \tag{3.5}$$

with the condition $u_{k,l}(0) = 0$ (no transmitted wave at the origin: we imagine that the mathematical fact that $r \geq 0$ by definition can be represented as an infinite potential well placed in the region $r < 0$).

The particular case of $V(r) = 0$ has well-known solutions called *free spherical waves* indicated as $\left| \varphi_{k,l,m}^0 \right\rangle$, such as

$$\varphi_{k,l,m}^0(r, \theta, \phi) = \sqrt{\frac{2k^2}{\pi}} \, j_l(kr) \, Y_l^m(\theta, \phi)$$

The functions $j_l(\rho)$ are the Bessel functions, defined as

$$j_l(\rho) = (-1)^l \rho^l \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^l \frac{\sin \rho}{\rho}$$

It can be shown that $j_l(kr) \sim \frac{1}{kr} \sin\left(kr - l\frac{\pi}{2} \right)$ when $r \to +\infty$.

Now, for large values of r, the equation (3.5) reduces to

$$\left[\frac{d^2}{dr^2} + k^2 \right] u_{k,l} \simeq 0$$

giving $u_{k,l}(r) \sim A\,e^{ikr} + B\,e^{-ikr}$. Since there is no absorbed wave at the origin, the amplitude of the incident wave e^{-ikr} ($|B|$) must be the same as the outgoing e^{ikr} ($|A|$). This leads to $u_{k,l}(r) \sim C \left(e^{ikr} e^{i\phi_A} + e^{-ikr} e^{i\phi_B} \right)$, that can be rewritten as

$$u_{k,l}(r) \sim C \sin\left(kr - l\frac{\pi}{2} + \delta_l \right)$$

underlying the *phase shift* δ_l, that can be well determined when the equation (3.5) has been solved and the limit $r \to +\infty$ has been taken.
This leads to

$$\varphi_{k,l,m}(\mathbf{r}) \sim C \frac{\sin\left(kr - l\frac{\pi}{2} + \delta_l \right)}{r} Y_l^m(\theta, \phi)$$

that can be rewritten, redefining the constant C as $e^{i\delta_l}/k$, so that the asymptotic normalization overlaps with the one of a free spherical wave. We obtain

$$\tilde{\varphi}_{k,l,m} = -Y_l^m(\theta, \phi) \frac{e^{-ikr} e^{il\frac{\pi}{2}} - e^{ikr} e^{-il\frac{\pi}{2}} e^{2i\delta_l}}{2ikr}. \tag{3.6}$$

Now, if we want to write the stationary scattering states v_k in the basis of the $|\tilde{\varphi}_{k,l,m}\rangle$ that have well-defined angular momentum, firstly, we consider the well-known expansion:

$$e^{ikz} = \sum_{l=0}^{+\infty} i^l \sqrt{4\pi(2l+1)} \, j_l(kr) Y_l^0(\theta)$$

In the case of central potential we admit only scattering states that are ϕ-independent, thus we can write $v_k \sim \sum_{l=0}^{+\infty} c_l \, \tilde{\varphi}_{k,l,0}$ asymptotically. Moreover, the only difference between the asymptotic form of the partial wave $|\tilde{\varphi}_{k,l,0}\rangle$ and the free spherical wave $\left|\varphi_{k,l,0}^0\right\rangle$ is the phase term $e^{2i\delta_l}$. Hence, we guess that $c_l = i^l \sqrt{4\pi(2l+1)}$ exactly like in the case of zero potential. In particular we suppose that

$$v_k = \frac{1}{kr} \sum_l i^l \sqrt{4\pi(2l+1)} e^{i\delta_l} u_{k,l}(r) Y_l^0(\theta) \tag{3.7}$$

We show that this guess is accurate: considering that

$$e^{2i\delta_l} = 1 + 2ie^{\delta_l} \sin \delta_l$$

we can write

$$\sum_{l=0}^{+\infty} i^l \sqrt{4\pi(2l+1)} \tilde{\varphi}_{k,l,0} = \sum_{l=0}^{+\infty} i^l \sqrt{4\pi(2l+1)} \, Y_l^0(\theta) \, \frac{1}{2ikr} \left(e^{ikr} e^{-il\frac{\pi}{2}} e^{2i\delta_l} - e^{-ikr} e^{il\frac{\pi}{2}} \right) =$$

$$= \sum_{l=0}^{+\infty} i^l \sqrt{4\pi(2l+1)} \, Y_l^0(\theta) \left[\frac{e^{ikr} e^{-il\frac{\pi}{2}} - e^{-ikr} e^{il\frac{\pi}{2}}}{2ikr} + \frac{1}{k} e^{-il\frac{\pi}{2}} e^{i\delta_l} \sin \delta_l \frac{e^{ikr}}{r} \right] \sim$$

$$\sim \sum_{l=0}^{+\infty} i^l \sqrt{4\pi(2l+1)} \, Y_l^0(\theta) \left(j_l(kr) + \frac{1}{k} e^{-il\frac{\pi}{2}} e^{i\delta_l} \sin \delta_l \frac{e^{ikr}}{r} \right) \sim$$

$$\sim e^{ikz} + f_k(\theta) \frac{e^{ikr}}{r}.$$

Where we have redefined $f_k(\theta) = \frac{1}{k} \sum_{l=0}^{+\infty} \sqrt{4\pi(2l+1)} \, e^{i\delta_l} \sin \delta_l \, Y_l^0(\theta)$. Hence, we have proved that our assumption on the c_l reproduces the exact asymptotic behaviour of the stationary scattering states v_k.

Finally, we can calculate the cross section σ as

$$\sigma = \int d\Omega \, \sigma(\theta) = \int d\Omega \, |f_k(\theta)|^2 = \frac{4\pi}{k^2} \sum_{l,l'=0}^{+\infty} \sqrt{(2l+1)(2l'+1)} \, e^{i(\delta_l - \delta_{l'})} \sin \delta_l \, \sin \delta_{l'} \times$$

$$\times \int d\Omega \, Y_{l'}^{0*}(\theta) \, Y_l^0(\theta) = \frac{4\pi}{k^2} \sum_{l,l'=0}^{+\infty} \sqrt{(2l+1)(2l'+1)} \, e^{i(\delta_l - \delta_{l'})} \sin \delta_l \, \sin \delta_{l'} \, \delta_{ll'}$$

because the Y_l^m form an orthonormal basis in the solid angle space. This leads to

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{+\infty} (2l+1) \sin^2 \delta_l. \tag{3.8}$$

This formalism is very useful for studying processes with well-defined angular momentum, for which the computation is easier. The terminology used is s-wave for $l = 0$, p-wave for $l = 1$, then d, f, g and so on. Clearly, also processes with a superposition of different angular momenta are described by the formalism, provided that the number of different values of l taken into account is sufficiently small that the sum in the above equation is easy to compute. On the other hand, if the number of different l is large or unknown, other formalisms are more helpful, such as the *Born Approximation*.

3.4 s, p and d waves in the dark matter annihilation processes

We reconsider now the process of dark matter annihilation $\chi\,\overline{\chi} \longrightarrow \psi\,\overline{\psi}$ described in *Chapter* 2. We have solved the Boltzmann equation describing the process in the case of null angular momentum, that is an s-wave: in this case, we have seen that the thermally averaged cross section assumes a constant value depending only by the two parameters α_{DM} and m_χ

$$\langle \sigma\, v \rangle = \frac{\alpha_{DM}^2}{32\pi\, m_\chi^2} \equiv a_0$$

Now, it is a fact, that we do not justify here, but frequent in literature, that the cross section can be expanded as

$$\langle \sigma\, v \rangle = \langle a_0 + a_1\, v^2 + a_2\, v^4 + \ldots \rangle = \langle \sum_{l=0}^{+\infty} a_l v^{2l} \rangle$$

where v is the Moeller velocity (in this *Chapter* we use this easier notation). The choice of the label l is not arbitrary, since the term $a_l v^{2l}$ is the contribution to the cross section of the wave with angular momentum number l. Therefore, in order to fine the thermally averaged cross section for the wave corresponding to the angular momentum l, we need to find the thermal average of v^{2l} using the Maxwell-Boltzmann distribution (we are in the non-relativistic case)

$$f(\mathbf{v}_i) = \left(\frac{m}{2\pi\, T} \right)^{\frac{3}{2}} exp \left(-\frac{m\mathbf{v}_i^2}{2T} \right)$$

We can now proceed with the calculation of

$$\langle v^{2l} \rangle = \int d^3\mathbf{v}_1 d^3\mathbf{v}_2 \left(\frac{x}{2\pi} \right)^3 e^{-\frac{x}{2}(\mathbf{v}_1^2 + \mathbf{v}_2^2)} |\mathbf{v}_1 - \mathbf{v}_2|^{2l}$$

where $x = \dfrac{m}{T}$. The solution to the integral gives (see Appendix A for details)

$$\langle v^{2l} \rangle = 2^{2l+1} \frac{\Gamma(l + \frac{3}{2})}{\sqrt{\pi}} \frac{1}{x^l}$$

Firstly, the result is consistent, because a direct calculation gives for $l = 0$ that $\langle \sigma\, v \rangle = a_0$ as we wanted. Now, a direct calculation gives

$$\langle \sigma\, v \rangle = \begin{cases} a_0 & l = 0 \\ a_1 \frac{6}{x} & l = 1 \\ a_2 \frac{60}{x^2} & l = 2 \end{cases} \tag{3.9}$$

As expected from the expansion of $\langle \sigma\, v \rangle$, where, being $v \ll 1$, the terms with higher values of l tens to be smaller than than the ones with smaller value of l, the thermally averaged cross section is smaller for higher x, that is for low temperatures, as imposed by the dependence $\propto x^{-l}$. Hence, we suppose that the s-wave processes decouple later than p-ones, that do it later than d-ones and so on. As a consequence, we expect that the higher the value of l in the process, the higher is the relic density we have today.

Now, we want to apply these results in the numerical solution of the Boltzmann equation, for which we use the same code, making sure to consider the proper x-dependence of the thermally averaged cross section. For our scopes we assume that all the a_l are equal to a_0. Since, as we can convince ourselves by looking at Table 2.1, the most suitable cross section is the one corresponding to $\langle \sigma\, v \rangle \simeq 1\, pb$, we set the coupling constant in a_0 to be α_{DM}. The graph obtained is exposed in Figure 3.1 and shows that the solutions have the exact behaviour we have described just above. Again, we computed the relic density today for the processes with different angular momentum and reported them in the following Table. Also in this case we chose as nowadays relic density value, the one corresponding to $x = 10^6$.

l	$\Omega_{\chi,0}\, h^2$
0	0.171
1	1.11
2	3.12

Table 3.1: Relic density of dark matter for processes with well-defined angular momentum.
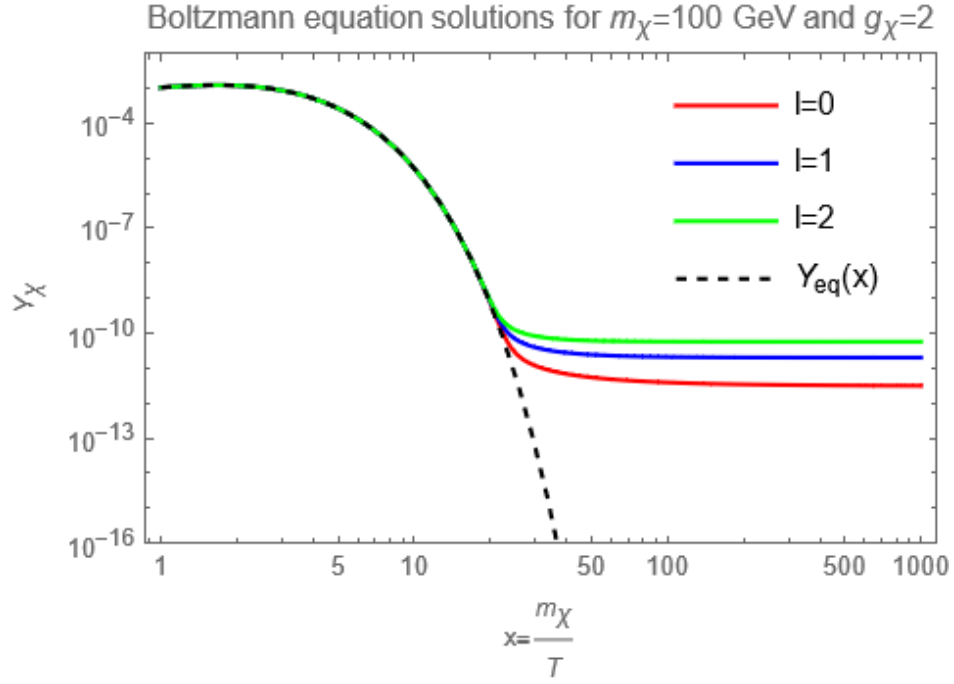
Figure 3.1: Simulation of the Boltzmann equation for different values of the angular momentum number l. The dashed black line represents the equilibrium behaviour.

Chapter 4

Sommerfeld Enhancement

The Sommerfeld enhancement is an elementary effect that occurs in nonrelativistic quantum mechanics. We can introduce the main idea, considering a nonrelativistic particle moving in a properly defined frame. As discussed in the previous *Chapter*s, we are interested in the processes of annihilation and their characterization in the context of the theory of scattering. So, our particle undergoes an interaction $H_{ann} = U_{ann}\delta^3(\mathbf{r})$ localized in the origin and responsible for the annihilation of the particle and the creation of others. Imagining that the particle is moving along the z-axis, without any other potential than H_{ann} its wavefunction is

$$\psi_k^{(0)}(\mathbf{r}) = e^{ikz}$$

Trivially, the rate for the annihilation process, or, equivalently, the cross section, is proportional to the square modulus of the wavefunction in the origin: $\sigma^{(0)} \propto |\psi_k^{(0)}(0)|^2$.

Now, we suppose that there is a central potential $V(r)$ in the space, that characterizes the interactions between particles before their annihilation. Although it is always possible to treat $V(r)$ perturbatively, for small asymptotic velocities the effect of the potential on the wavefunction is significantly distorting. The determination of the new wavefunction ψ_k involves of course the solution of the Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r) \right]\psi_k = \frac{\hbar^2 k^2}{2\mu}\,\psi_k$$

However, the rate of annihilation in the origin is analogously proportional to the probability for the perturbed particle to be in the origin, that is expressed by $|\psi_k(0)|^2$. Therefore, the new cross section will be

$$\sigma = \sigma^{(0)}\frac{|\psi_k(0)|^2}{|\psi_k^{(0)}(0)|^2} \equiv \sigma^{(0)}S_k$$

and the quantity S_k is called Sommerfeld enhancement.

Now, we have shown in *Chapter* 3 that under the proper considerations we can expand ψ_k as seen in Equation 3.7:

$$\psi_k(r) = \frac{1}{kr}\sum_l i^l\sqrt{4\pi(2l+1)}e^{i\delta_l}u_{k,l}(r)Y_l^0(\theta) \tag{4.1}$$

because of the rotational invariance around the z-axis. Moreover, if the potential energy $V(r)$ blows up to infinity slower than $1/r^2$ as $r \longrightarrow 0$, we can ignore its term in the 1-dimensional Schrödinger equation for $u_{k,l}$ (see Equation 3.5) in the neighbourhoods of the origin. This gives $u_{k,l}(r) \simeq rj_{k,l}(r)$, and since $j_{k,l}(r) \propto r^l$ as $r \longrightarrow 0$, we have that in Equation 4.1 all the terms different than $l = 0$ can be ignored at the origin, giving

$$\psi_k(0) = \frac{1}{k}\sqrt{4\pi}e^{i\delta_0}R_{k,0}(0)\,Y_0^0(\theta)$$

with $R_{k,l}(r) = \dfrac{u_{k,l}(r)}{r}$ by definition. This results in

$$S_k = \left|\frac{R_{k,0}(0)}{k}\right|^2 \tag{4.2}$$

The fact that the Sommerfeld enhancement depends only by the $l = 0$ process can be understood by a classical argument: in fact, if the particle has a non-zero angular momentum, classically it cannot travel through the origin by conservation of angular momentum. This classical property is found also in the quantum mechanics formalism, since we have found that the probability of transition through the origin is 0. Therefore, the impossibility for the particle to reach the annihilation point placed in the origin cannot give the enhancement of the cross section for $l \neq 0$.

4.1 Coulomb potential

We are interested to develop a model concerning the interaction between dark matter particles also before their annihilation interaction described in *Chapter* 2. In this thesis work we analyze the case of the Coulomb potential $V(r) = \alpha/r$, with α the constant that quantifies the intensity of the potential and whose sign determines if the potential is attractive ($\alpha < 0$) or repulsive otherwise. Calling $\chi_k(r) = rR_{k,0}(r)$ and remembering we are interested only in the $l = 0$ case, we can write the Schrödinger equation for our scope (in natural units) as

$$-\frac{1}{2\mu}\frac{d^2\chi_k}{dr^2} + \frac{\alpha}{r}\chi_k = E\chi_k$$

Now, calling v the asymptotic relative velocity, we have that $k^2 = 2\mu E = \mu^2 v^2$. Moreover, we adopt a new variable $\rho = kr$ and call $\eta = \mu\alpha/k = \alpha/v$. Putting all together we can rewrite the above equation in a more compact form as

$$\chi_k'' + (1 - \frac{2\eta}{\rho})\chi_k = 0 \tag{4.3}$$

The solution to this differential equation is exact and involves hypergeometric functions (see [ABST]). In particular, the general solution is

$$\chi_k = AF_0(\eta, \rho) + BG_0(\eta, \rho)$$

where $F_l(\eta, \rho)$ is the regular Coulomb wavefunction associated to the angular momentum number l and $G_l(\eta, \rho)$ the irregular logarithmic one. The condition $\chi_k(0) = 0$, ensures $B = 0$, that gives

$$\chi_k(\rho) = C_0(\eta)\rho e^{-i\rho} M(1 - i\eta, 2, 2i\rho)$$

with $M(a, b, z)$ is the Kummer's function and $C_0(\eta)$ such that $C_0^2(\eta) = 2\pi\eta(e^{2\pi\eta} - 1)^{-1}$ (see Appendix A for mathematical definitions of these functions). Then,

$$R_{k,0}(r) = kC_0(\eta)e^{-ikr} M(1 - i\eta, 2, 2ikr)$$

Finally, from Equation 4.2, being $M(a, b, 0) = 1$, we get the final result

$$S_k = \left|\frac{R_{k,0}(0)}{k}\right|^2 = C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

Now, we can analyze the interesting limits of S_k with respect to the values of η.
In the limit $\eta \longrightarrow 0$, that corresponds to $v >> \alpha$, we have that $S_k \longrightarrow 1$. This means that in the case of high asymptotic velocity, the potential has no effect. This is corroborated from Equation 4.3, since for $\eta = 0$, it is the equation of a free particle. Therefore, in this case the cross section σ trivially remains unaltered.
In the limit $\eta \longrightarrow +\infty$ we have a strong repulsion with respect to the asymptotic velocity. This results, as expected, in $S_k \longrightarrow 0$, since the potential throws the particle away from the origin, reducing the possibility to have the annihilation.
Finally, in the case of strong attraction, that is $\eta \longrightarrow -\infty$, the Sommerfeld Enhancement blows up to infinity as $S_k \simeq 2\pi\eta = e\pi\alpha/v$, that is the behaviour frequently cited in literature, which affirms that the Sommerfeld Enhancement is inversely proportional to the velocity at small velocities, resulting in the enhancement of the cross section, as we will analyze in the following *Paragraph*.

4.2 Impact on the dark matter relic density

We want to put all together in order to study the effect of the Sommerfeld enhancement, caused by the Coulomb potential, on the dark matter relic density. For the sake of simplicity we consider an s-wave annihilation process, such that the non-enhanced cross section $\langle \sigma_{ann} v \rangle$ assumes a constant value, as seen in *Chapter* 1 and 2. Now, in order to solve the Boltzmann equation, we have to previously thermally average the Sommerfeld enhancement

$$\langle S_k \rangle = \left\langle \frac{\frac{2\pi\alpha}{v}}{e^{\frac{2\pi\alpha}{v}} - 1} \right\rangle$$

As discussed in the detail in Appendix A, the calculation of $\langle S_k \rangle$ involves the integral

$$S(x) = \int_0^{+\infty} dv \, 2\pi\alpha v \, e^{-\frac{x}{4}v^2} \frac{1}{e^{\frac{2\pi\alpha}{v}} - 1}$$

that is finite, but impossible to determine analytically. Therefore, we compute it in a numerical way, assuming $|\alpha| = 1/137$ the usual coupling constant of electromagnetic force. We plot the obtained values for $S(x)$ in the following Figure.
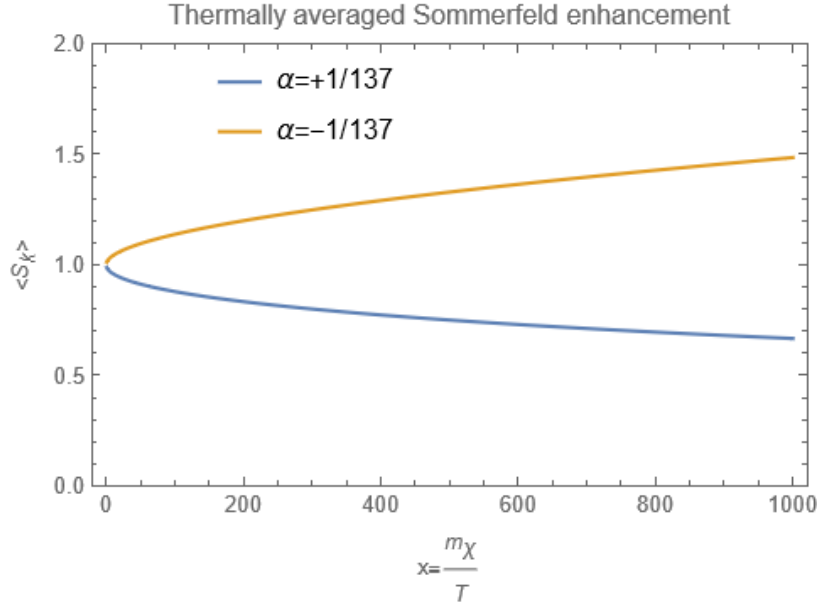

Figure 4.1: Numerical solution for thermally averaged Sommerfeld enhancement in the cases of attractive and repulsive potential as a function of x.

The thermal averaged cross section in the Boltzmann Equation 2.7 needs thus to be modified as

$$\langle \sigma v \rangle = \sigma_0 \langle S_k \rangle = \frac{\alpha_{DM}^2}{32\pi \, m_\chi^2} \left(\frac{x^3}{4\pi} \right)^{\frac{1}{2}} S(x)$$

Since we are interested in the annihilation $\chi\overline{\chi} \longrightarrow \psi\overline{\psi}$ between dark matter particles and antiparticles, we can state that the Coulomb potential needs to be attractive with $\alpha = -1/137$. In this case, we have seen that S_k is always greater than 1, causing an higher rate of annihilation or, in other words, the enhancement of the cross section. As we discussed in *Chapter* 2, a higher cross section means longer coupling with the thermal bath during the cooling down of the Universe. Therefore, the decoupling takes place later, giving a lower dark matter density, than in the $\alpha = 0$ (no interaction before annihilation) case. However, for the sake of completeness, we solved the Boltzmann equation with Sommerfeld enhancement in both the attractive and the repulsive case, to visualize the effect of each case and, in particular, to verify that the repulsive case reproduces
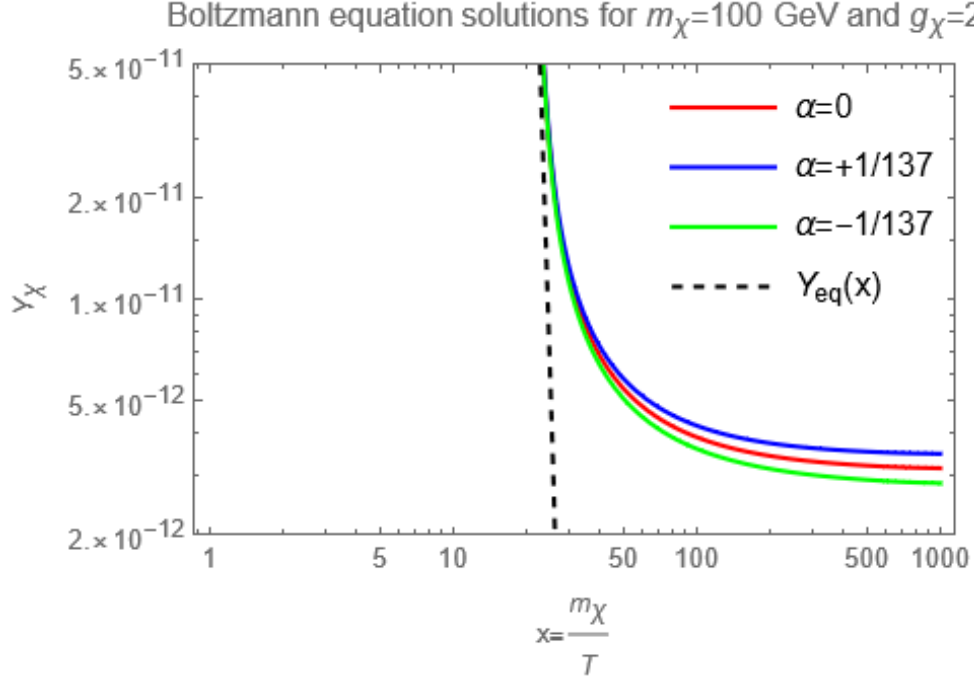
Figure 4.2: Effect of the Sommerfeld enhancement by attractive and repulsive Coulomb potentials on the dark matter comoving number density.

the higher dark matter density.

Since, as we have seen that the cross section that more likely reproduces that density we observe today is $\sigma_0 \simeq 1\, pb$, we adopt this value also in this computation. The results obtained, resumed in Table 4.1, completely agree with our suppositions, causing a significant deviation of the order of the 10% both in the repulsive and attractive case, that cause, always as described before, respectively an increase and a reduction in the dark matter relic density.

$\alpha\ (\alpha_{fs})$	$\Omega_{\chi,0}\, h^2$
0	0.171
-1	0.155
+1	0.189

Table 4.1: Relic density of dark matter for an s-wave with Sommerfeld corrections. The coupling constant is in units of the fine structure constant $\alpha_{fs} \simeq 1/137$.

Chapter 5

Conclusions

In this thesis work we have at first developed a non rigorous formalism to create a model of the evolution of the Universe. In particular, through the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k\,c^2}{a^2}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

we could achieve some important calculations of the number densities, energy densities and temperatures in our expanding flat Universe.

The second main idea was the fact that in the primordial Universe, that is for smaller scale factors a, collisions between particles were so frequent that originally all of them were in a thermal bath. Of course, while the Universe was expanding and cooling down, collisions needed stronger interactions to keep the thermal equilibrium. However, the process in which several particles decoupled from the mentioned thermal bath, due to the fact that their rate of interaction was not strong enough to counteract the expansion of the Universe, is the one setting the relic density of there particles. In fact, Statistical Mechanics impose that if the particles remain coupled to the primordial bath, their number density is exponentially reduces as temperature decreases, while after the decoupling particles dilute as the Universe expands, but do not vanish. The formalism we adopted to quantify the process of the decoupling for our dark matter candidate, the WIMP, was the one of the Boltzmann equation, that in its complete form is:

$$\frac{d\,Y_\chi}{dx} = -\frac{\lambda}{x^2}\frac{g_{*s}(x)}{\sqrt{g_*(x)}}\left(1 - \frac{x}{3}\frac{g'_{*s}(x)}{g_{*s}(x)}\right)(Y_\chi^2 - Y_\chi^{eq\,2}(x))$$

with $\lambda = 2\pi\sqrt{10}/15 \cdot m_\chi\,M_{Pl}\,\langle\sigma\,v_{rel}\rangle$. We chose the mass of our WIMP χ to be the reasonable $m_\chi = 100\,GeV$ and computed the Boltzmann equation for the cross sections (that is, rates of collisions) 0.01, 1 and $100\,pb$ to verify that the stronger the interaction was, the later the particle decoupled and therefore, the smaller is their relic density.

Then, since we were interested to account for the impact of interaction before the annihilation (consequent to the collision $\chi\overline{\chi}$) between dark matter particle and antiparticle, we exposed the fundamental ideas of non relativistic theory of scattering in Quantum Mechanics, in particular through the partial waves formalism, that is particularly useful for interaction potentials with spherical symmetry and permits to compute the cross section σ in relation with the angular momenta l as

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{+\infty}(2l+1)\,\sin^2\delta_l.$$

Then, after having introduced the formalism of partial waves, we focused on the behaviour of well-defined angular momentum processes and computed that in the case of angular momentum l, the thermally averaged cross section is

$$\langle\sigma\,v\rangle = \sigma_0\,2^{2l+1}\frac{\Gamma(l+\frac{3}{2})}{\sqrt{\pi}}\frac{1}{x^l}.$$

It is clear that the $l \neq 0$ term causes a reduction in the s-wavecross section σ_0. Therefore, as exposed previously, we observe an earlier decoupling from the thermal bath as the angular momentum increases.

Finally, we were able to compute the Sommerfeld enhancement S_k as the quantity that modifies the annihilation cross section σ_0 as the cross section σ in the case of interaction before annihilation:

$$\sigma = \sigma_0 \, S_k$$

We found, from the formalism of scattering, that

$$S_k = \left| \frac{R_{k,0}(0)}{k} \right|^2$$

So, we were able to compute $R_{k,0}$ once chosen the Coulomb potential $V(r) = \frac{\alpha}{r}$, and being the asymptotic relative velocity v and $\eta = \frac{\alpha}{v}$ we obtained

$$S_k = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

Applying the thermally averaged $\langle S_k \rangle$ to σ_0 in the case of s-wave annihilation, we obtained, as expected that in the case of attractive interaction, the cross section is increased and this leads to a lower relic density. In Figure 5.1 and in Table 5.1 are summed up the most relevant results. As we can see our assumptions on WIMPs were correct at least as order of magnitude. In fact the result obtained with σ_0 and the Sommerfeld correction caused by attractive Coulomb potential $(\Omega_{\chi,0} h^2)$ is the same order of magnitude as the observed dark matter density in the Universe

$$\Omega_{DM,0} h^2 = 0.1198 \pm 0.0012.$$

$$\Omega_{\chi,0} \, h^2 = 0.155.$$

case	$\Omega_{\chi,0} \, h^2$
0	0.171
1	1.11
S	0.155

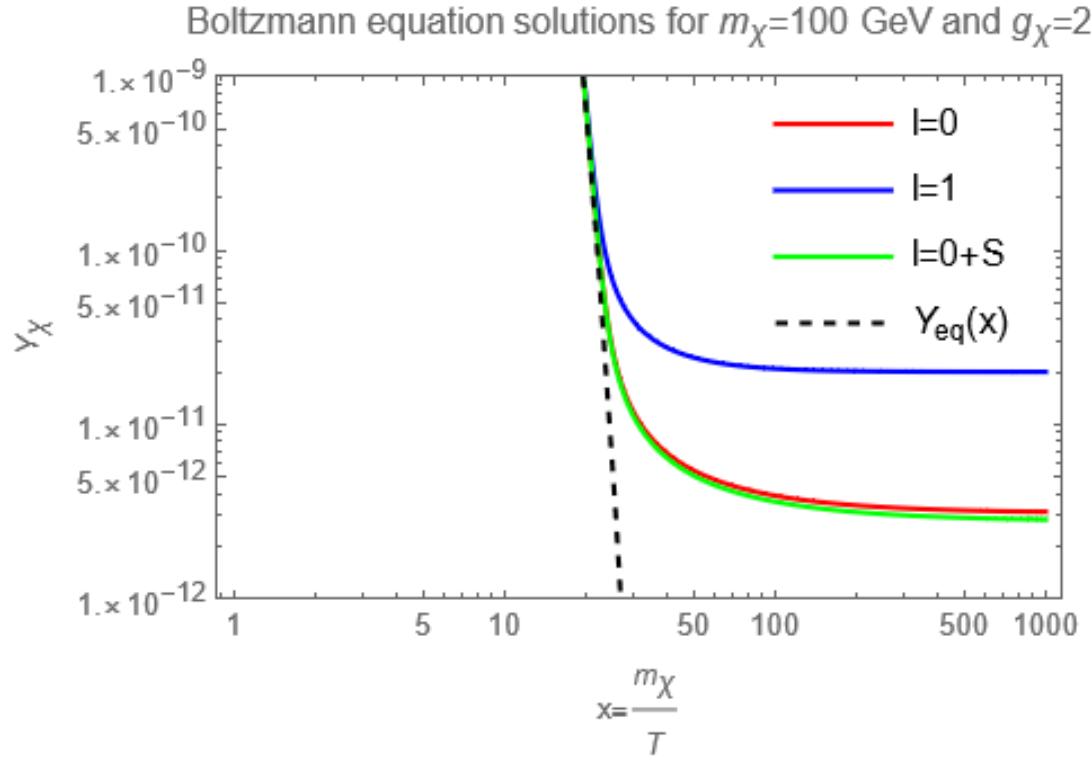Table 5.1: Relic density of dark matter for the processes described in Figure 5.1.

Figure 5.1: Final graph in which we resume the possible effects. In red, the s-wave without Sommerfeld corrections. In blue, the p-wave. In green, s-wave with Sommerfeld correction.

Appendix A

Boltzmann equation: computational details

Being the entropy in a comoving volume $S = \dfrac{2\pi^2}{45} g_{*s}(T) T^3 a^3 = const$ because of homogeneity, we have (with $x = m_\chi/T$)

$$0 = \frac{d}{dt}\left[g_{*s}(x) \left(\frac{a}{x}\right)^3 \right]$$

$$= \left(\frac{1}{g_{*s}(x)} \frac{d\, g_{*s}(x)}{dx} - \frac{3}{x} \right) \frac{d\, x}{dt} + 3\frac{\dot{a}}{a}$$

Therefore, (calling $g'_{*s}(x) = \dfrac{d\, g_{*s}(x)}{dx}$)

$$\frac{d\, x}{dt} = Hx \left(1 - \frac{x}{3}\frac{g'_{*s}(x)}{g_{*s}(x)} \right)^{-1}.$$

Now, since $\dfrac{d\, Y_\chi}{dt} = \dfrac{d\, Y_\chi}{dx}\dfrac{dx}{dt} = \dfrac{d\, Y_\chi}{dx} Hx \left(1 - \dfrac{x}{3}\dfrac{g'_{*s}(x)}{g_{*s}(x)} \right)^{-1}$, we can combine this result with Equation 2.6, obtaining

$$\frac{d\, Y_\chi}{dx} = -\langle \sigma\, v_{rel} \rangle \frac{s}{Hx} \left(1 - \frac{x}{3}\frac{g'_{*s}(x)}{g_{*s}(x)} \right) (Y_\chi^2 - Y_\chi^{eq\,2}(x))$$

Finally, being $s = \dfrac{2\pi^2}{45} g_{*s}(x) \left(\dfrac{m_\chi}{x}\right)^3$ and $H = \dfrac{\pi \sqrt{g_*(x)}}{3\sqrt{10}} \dfrac{m_\chi^2}{M_{Pl}\, x^2}$, we obtain the final form of the Boltzmann equation

$$\frac{d\, Y_\chi}{dx} = -\frac{\lambda}{x^2} \frac{g_{*s}(x)}{\sqrt{g_*(x)}} \left(1 - \frac{x}{3}\frac{g'_{*s}(x)}{g_{*s}(x)} \right) (Y_\chi^2 - Y_\chi^{eq\,2}(x))$$

with $\lambda = \dfrac{2\pi\sqrt{10}}{15} m_\chi\, M_{Pl}\, \langle \sigma\, v_{rel} \rangle$.

We are now interested to compute thermal averages of generic functions $f(v)$, where v depends only on the relative speed. To achieve this, it is convenient to change variables with the ones of the center of mass frame $\mathbf{V} = \dfrac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$ the velocity of the center of mass of a particle and antiparticle system and the relative velocity $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$. Consistently with the previous definitions, we will

call $V = |\mathbf{V}|$ $v = |\mathbf{v}|$ the moduli of the velocities. This leads to

$$\langle f(v) \rangle = \left(\frac{x}{2\pi}\right)^3 \int d^3\mathbf{V} d^3\mathbf{v}\, e^{-\frac{x}{2}[(\mathbf{V}+\frac{\mathbf{v}}{2})^2 + (\mathbf{V}-\frac{\mathbf{v}}{2})^2]} f(v)$$

$$= \left(\frac{x}{2\pi}\right)^3 \int d^3\mathbf{V}\, e^{-xV^2} \int d^3\mathbf{v}\, f(v)\, e^{-\frac{x}{4}v^2}$$

$$= \left(\frac{x}{2\pi}\right)^3 \cdot \left(\int dV e^{-xV^2}\right)^3 \cdot 4\pi \int_0^{+\infty} dv\, v^2 f(v)\, e^{-\frac{x}{4}v^2}$$

$$= \left(\frac{x}{2\pi}\right)^3 \cdot \left(\frac{\pi}{x}\right)^{\frac{3}{2}} \cdot 4\pi \int_0^{+\infty} dv\, v^2 f(v)\, e^{-\frac{x}{4}v^2}$$

Therefore,

$$\langle v^{2l} \rangle = \left(\frac{x}{2\pi}\right)^3 \cdot \left(\frac{\pi}{x}\right)^{\frac{3}{2}} \cdot 4\pi \int_0^{+\infty} dv\, v^{2l+2}\, e^{-\frac{x}{4}v^2}$$

$$= \left(\frac{x}{2\pi}\right)^3 \cdot \left(\frac{\pi}{x}\right)^{\frac{3}{2}} \cdot 4\pi \frac{2^{2l+2}}{x^{l+\frac{3}{2}}} \int_0^{+\infty} t^{l+\frac{1}{2}}\, e^{-t}\, dt$$

$$= 2^{2l+1} \frac{\Gamma(l+\frac{3}{2})}{\sqrt{\pi}} \frac{1}{x^l}.$$

Appendix B

Hypergeometric functions

The general solution of the equation

$$\frac{d^2\chi}{d\rho^2} + \left(1 - \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right)\chi = 0$$

is $\chi = A F_l(\eta, \rho) + B G_l(\eta, \rho)$, where $F_l(\eta, \rho)$ is the regular Coulomb wave function

$$F_l(\eta, \rho) = C_l(\eta)\rho^{l+1} e^{-i\rho} M(l+1-i\eta, 2l+2, 2i\rho)$$

and $G_l(\eta, \rho)$, that is useless to define for our scopes.
Then,

$$C_l(\eta) = 2^l e^{-\frac{\pi\eta}{2}} \frac{|\Gamma(l+1+i\eta)|}{\Gamma(2l+2)}$$

and remarkably

$$C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}.$$

Finally, we define the Kummer's function

$$M(a, b, z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

with $(q)_k = q(q+1)\ldots(q+k-1)$ and $(q)_0 = 1$ for each q.

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