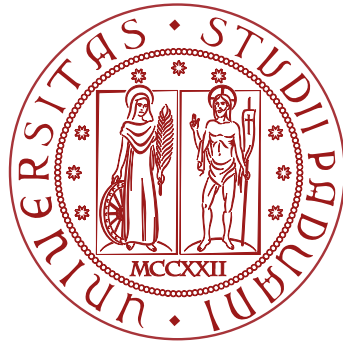


UNIVERSITÀ DEGLI STUDI DI PADOVA

Department of Civil, Environmental and Architectural Engineering

Master's Degree in Mathematical Engineering

Mathematical Modelling for Engineering and Science



**UNIVERSITÀ
DEGLI STUDI
DI PADOVA**

MASTER'S THESIS

Approximation of mean curvature flow by diffusion/redistancing schemes

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Contents

Introduction	2
Weak formulations and approximation schemes	2
Outline of the thesis	3
Notation and conventions	4
Chapter 1. Weak formulations of mean curvature flow	6
1.1. A general variational framework	6
1.2. Application to mean curvature flow	9
1.3. The distributional formulation	12
Chapter 2. Convergence of an abstract approximation scheme	13
2.1. An abstract diffusion operator	13
2.2. Construction and convergence of the scheme	15
Chapter 3. Applications	21
3.1. An elliptic scheme	21
3.2. A parabolic scheme	28
3.3. A convolution kernel scheme	35
Appendix A. Auxiliary results	38
A.1. Anisotropic geometry	38
A.2. Functional analysis	40
A.3. Inequalities	41
Bibliography	42

Introduction

A mean curvature flow is a motion of sets driven by mean curvature. Classically, the curvature is a representation of the state of bending of a regular surface, while the mean curvature is a measure of the average bending in different directions. These notions are related to the second-order approximation of the surface as a graph, but can be generalized to less regular cases. Indeed, curvature is also closely related to the perimeter of the surface: regions of greater mean curvature correspond to greater variations of the perimeter when the surface is perturbed. Thus, starting from a notion of perimeter, one could define a corresponding notion of mean curvature as the first variation of the perimeter. If the underlying notion of perimeter assigns different weights to differently oriented regions of the surface, we speak of anisotropic curvature. This variational perspective also allows us to give meaning to the curvature of sharp regions such as edges.

The kinematic quantity driven by the mean curvature is the normal velocity of the motion. Given a regular motion of sets, the normal velocity is the only geometrically meaningful notion of velocity. Indeed, if the points of the surface move tangentially, the surface remains fixed, resulting in no actual motion of the set. Just as curvature is closely related to perimeter, so velocity is closely related to the volume of the set: regions of greater normal velocity correspond to greater variations of the volume as the set moves. Thus, one could recover the normal velocity from the rate of change of the volume.

Mean curvature flow can also be understood as a motion driven by perimeter minimization. Every natural system that stores part of its energy in a surface and dissipates it as the surface moves is partially subject to some form of mean curvature flow. The contribution of the energy is to push the system towards its local minimization, a push balanced by a dissipative force that resists the motion. Since the mean curvature is the first variation of the perimeter, this driving force is precisely the mean curvature, while the dissipative force is proportional and opposite to the normal velocity. Their balance makes the normal velocity proportional and opposite to the mean curvature, which is the mathematical expression of mean curvature flow.

The canonical example is the two-phase system, in which the surface is the interface separating two phases free to transition into one another. If one of the two phases is a crystal, the surface energy density is a non-differentiable function of the orientation, replicating the geometry of the polytope emerging from the crystal lattice. Understanding phase transition phenomena involving a crystalline anisotropy requires understanding a crystalline mean curvature flow. In this thesis we present approximation schemes for anisotropic mean curvature flow, some of which apply to general convex anisotropies, including crystalline ones.

Weak formulations and approximation schemes

Even when it starts from a smooth compact surface, a mean curvature flow can develop singularities in finite time: a thin neck can contract faster than the rest of the surface and pinches, as proved by Angenent [6]. The classical formulation stops there, while the physical evolution does not. Since what happens through such an event is precisely what one wants to describe, a weak notion of solution is needed, one that is defined for irregular sets and that survives changes of topology. In the crystalline case there is a second, independent reason: the surface energy density is not differentiable, so that the mean curvature is not defined pointwise even for a smooth set, and a weak formulation is needed already to write the equation.

The first such formulation is due to Brakke [1], who describes the moving interface as a varifold and requires the equation of motion to hold in an integral form, as an inequality accounting for the possible sudden loss of area. Existence holds for arbitrary initial data, but the price of the inequality is a drastic loss of uniqueness: a flow that vanishes instantaneously satisfies the definition. Later constructions, in particular the elliptic regularization of Ilmanen [8], produce Brakke flows with better properties. However, this formulation is not the right model for the two-phase system, and may be better suited when the interface is a physical object itself.

A second formulation replaces the moving set by the level sets of an auxiliary function, which solves a degenerate parabolic equation understood in the viscosity sense. The foundational works are due to Evans and Spruck [5], and Chen, Giga and Goto [4]. The comparison principle for that equation gives existence and

uniqueness of the auxiliary function, hence a canonical evolution, at the cost of a phenomenon called fattening: the level set may develop a positive measure, and the geometric evolution is then not unique. This phenomenon may be interpreted as a physical instability of the model, and the fattened set may represent the union of all the possible evolutions under small perturbations of the initial condition. What matters here is the limitation of the method: the equation must be a local one and the anisotropy must be differentiable, so that the mean curvature is a function of the normal and of the second fundamental form.

A third formulation is variational, and is the closest in spirit to the first part of the present work. Since the flow is a motion driven by perimeter minimization, it is natural to discretize it in time as a sequence of minimization problems, in which the perimeter is minimized against a term penalizing the distance from the previous set. This is the minimizing movements scheme of Almgren, Taylor and Wang [7], whose limit is called a flat flow; Luckhaus and Sturzenhecker [10] proved that the limit solves the equation in a distributional sense, under the assumption that no perimeter is lost in the limit. The scheme is intrinsically variational, hence insensitive to the differentiability of the anisotropy, which is what makes it the natural candidate in the crystalline case.

For a crystalline anisotropy, the difficulty is that the mean curvature ceases to be a local quantity. The Wulff shape of a crystalline energy is a polytope [2], the boundary of an evolving set contains flat facets, and on each facet the curvature is determined by a minimization over the whole facet rather than by the geometry at a point. The two dimensional theory was developed first, where a facet is a segment and the flow reduces to a system of ordinary differential equations for the lengths of the sides [3]. In three dimensions, and for restricted classes of anisotropies or of initial sets, existence and uniqueness were obtained by several authors, in particular by Giga and Požár for bounded crystals [13]. The case of a general convex anisotropy in an arbitrary dimension was settled by Chambolle, Morini and Ponsiglione [14], who replaced the equation on the interface by a distributional inequality for the signed distance function to the evolving set, proving existence and uniqueness up to fattening for the natural mobility and for arbitrary closed initial sets; the extension to arbitrary convex mobilities is proved in [15]. This is the formulation adopted here, and it is stated as Definition 1.8. The discrete analogous

Alongside these formulations there are the approximation schemes. A scheme discretizes the evolution in time, replacing the motion on a time step h by one application of an operator, and the interest is twofold: on the one hand a scheme is an algorithm, and it is how such flows are actually computed; on the other hand the convergence of a scheme is often the way existence is proved, as is the case for the minimizing movements described above. Besides the variational ones, the best known scheme is the thresholding algorithm of Merriman, Bence and Osher [9], in which the characteristic function of the set is evolved by the heat equation for a time h and the new set is the region where the result exceeds one half. Closer to this thesis, Chambolle [11] proposed a scheme in which the object evolved is not the set but its signed distance function: one solves a single convex problem with the distance function as datum, and the new set is a sublevel set of the solution, which is then redistanced before the next step. Chambolle, De Gennaro and Morini [16] proposed a spatially-discrete version of such a scheme. Schemes of this kind, which we call diffusion/redistancing schemes, are the subject of this thesis.

Outline of the thesis

The schemes considered in this thesis are of the type introduced in [11]: the object that is evolved is not the set but its signed distance function, one step consists in applying an operator $\mathcal{T}_h$ to it, and the new set is the region where the result is nonpositive, whose distance function is then taken as the datum of the next step. For a scheme of this form the whole construction is determined by the single operator $\mathcal{T}_h$, and the question from which this thesis starts is which properties of that operator guarantee that the evolutions it generates converge to a superflow in the sense of [14]. We do not answer it for one operator at a time, but by isolating a list of properties: every family satisfying them converges, and a concrete scheme has then only to be checked against the list.

Summary of contributions. The main contribution of this thesis is an abstract approximation scheme, together with the identification of the properties that make it converge. The scheme is a family $\{\mathcal{T}_h\}_{h>0}$ of operators acting on Lipschitz functions, each of which is monotone, maps Lipschitz functions into Lipschitz functions, commutes with the addition of constants, is odd and is translation invariant, and which is required to satisfy three further properties: a comparison with the Wulff shape,

$$\mathcal{T}_h(\phi^\circ) \leq \sqrt{(\phi^\circ)^2 + c_0 h},$$

a stability property, asking that the incremental quotient be bounded as a distribution by

$$\left| \left\langle \frac{\mathcal{T}_h g - g}{h}, \varphi \right\rangle \right| \leq C_0 \int \phi(\nabla \varphi)$$

for every g with $\phi(\nabla g) \leq 1$, and a consistency property, asking that along sets E_n converging to E in the Kuratowski sense the same incremental quotient satisfy

$$\liminf_{n \rightarrow \infty} \left\langle \frac{\mathcal{T}_{h_n} d_{E_n} - d_{E_n}}{h_n}, \varphi \right\rangle \geq - \int \phi'(\nabla d_E; \nabla \varphi)$$

for every test function supported outside E . The result is that the evolutions generated by such a family converge, up to a subsequence, to a superflow with the prescribed initial datum. The three properties have distinct roles: the first is a comparison with the shrinking Wulff shape and is what produces the estimates on the discrete evolutions, hence the compactness; the second and the third are the two halves of a single statement, namely that the incremental quotient is the divergence of a vector field which lies in the subdifferential $\partial\phi(\nabla d)$, the one bounding that field and the other identifying it.

The second contribution is the verification of these properties for three families of schemes. The first is elliptic, and consists in solving at each step

$$-h \operatorname{div} (\phi^{p-1}(\nabla u)z) + u = g, \quad z \in \partial\phi(\nabla u),$$

which is the Euler equation of the sum of the p -th power of the anisotropy of the gradient and of a quadratic penalization of the distance from the datum. The second is parabolic, and evolves the datum by the anisotropic p -Laplacian for a time h . Both are defined for an arbitrary norm ϕ and for every $1 < p < \infty$, hence they cover the crystalline case, and for both the vector field is produced by the equation itself, so that the stability property comes for free. Their Wulff estimates are obtained by explicit radial barriers, and the parabolic one is deduced from the elliptic one by dilating the time variable. The third family is a convolution scheme, in which one step is the convolution of the datum with a kernel of width $\sqrt{h}$: it is completely explicit and it requires no solution of any equation, but it is limited to the anisotropies induced by a scalar product.

The first chapter is of a different nature from the other two, and we state this clearly: it is exploratory, and almost nothing in it is new. Its purpose is to place a variational formulation in a mechanical framework, by deriving the equation of motion from the principle of virtual powers, in a form abstract enough not to be tied to mean curvature flow. We do not claim originality for the results, but we believe that the framework has some potential, and that it may be developed into a systematic way of studying free boundary problems.

Organization of the material. In Chapter 1 we derive a variational formulation of anisotropic mean curvature flow from the principle of virtual powers, we state the distributional formulation that the rest of the thesis adopts, and we relate the two. In Section 1.1 we set up the general framework, made of a space of configurations, of a tangent structure of rates and of a balance of forces against arbitrary virtual rates, and we deduce the equation of motion. In Section 1.2 we specialize it to mean curvature flow, first in the isotropic and then in the anisotropic case. In Section 1.3 we state the distributional formulation and Definition 1.8 of superflow, which is the notion of solution used throughout.

In Chapter 2 we construct the abstract scheme and we prove its convergence. In Section 2.1.1 we give the axioms of a diffusion operator and their first consequences, among which the fact that such an operator does not increase the anisotropic Lipschitz constant, and the comparison with the Wulff shape. In Section 2.1.2 we define an approximating family through the three properties above and we deduce the two estimates on one step which are used later. Section 2.2.2 contains the compactness theorem for the discrete evolutions and the convergence theorem, together with a generic uniqueness result.

In Chapter 3 we present the three families of schemes. Both Sections 3.1 and 3.2 follow the same plan: a comparison principle, a proof of well-posedness and basic axioms, the Wulff shape estimate, and the verification of stability and consistency, ending with the statement that the scheme converges to a superflow. Also Section 3.3 follows a similar plan, the only difference being that it is explicit and requires no proof of well-posedness.

The auxiliary results of anisotropic geometry, of Kuratowski convergence, the representation lemma and the inequalities used throughout are collected in the Appendix, so as to keep the chapters self-contained but short.

Notation and conventions

General conventions. Throughout the thesis we fix as dimension of the space a natural number $N \geq 1$. The set of natural numbers $\mathbb{N}$ contains 0. Sequences are denoted by $\{a_n\}$, indexed by $n \in \mathbb{N}$. All implicit limiting operations of sequences are taken as $n \rightarrow \infty$ (for example, we write $a_n \rightarrow a$ or $\lim_n a_n = a$ for $\lim_{n \rightarrow \infty} a_n = a$).

All the topological notions must be intended in their sequential sense.

We abbreviate *if and only if* by *iff*, *right hand side* and *left hand side* by *RHS* and *LHS*, and *almost everywhere* by *a.e.*, the latter being used only for relations holding pointwise outside a negligible set.

Notation for sets and measures. Given $E \subseteq \mathbb{R}^N$, we denote by $\bar{E}$ its closure, by $\mathring{E}$ its interior, by E^c its complement and by 1_E its characteristic function. We write B_R for the Euclidean ball of radius R centred at the origin, $|E|$ for the Lebesgue measure of E and $\omega_N = |B_1|$ for the measure of the Euclidean unit ball. We denote by $\mathcal{H}^{N-1}$ the $(N-1)$ -dimensional Hausdorff measure, by ∂^*E the reduced boundary of a set of finite perimeter, by ν_E its outer normal and by $\text{Per}(E) = \mathcal{H}^{N-1}(\partial^*E)$ its perimeter. Given an open set Ω , $\mathcal{M}(\Omega)$ is the space of all Radon measures on Ω with finite total variation.

Given a norm ϕ on $\mathbb{R}^N$, the polar norm ϕ° , the Wulff shape W_R^ϕ , the anisotropic distance d^ϕ and dist^ϕ , the anisotropic signed distance sd_E^ϕ and the Kuratowski convergence $\xrightarrow{\mathcal{K}}$ of closed sets are all defined in Section A.1.

Notation for functional analysis. $\text{Lip}(\mathbb{R}^N)$ is the space of Lipschitz functions and $C_c^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support. $\mathcal{D}'(\mathbb{R}^N)$ is the space of distributions, and $\langle \cdot, \cdot \rangle$ denotes both the duality pairing of a Banach space with its dual and the action of a distribution on a test function. The subscript *loc* has its usual meaning. We write $\rightharpoonup$ for the weak convergence and $\overset{*}{\rightharpoonup}$ for the weak-* convergence.

Given $a, b \in \mathbb{R}$, we write $a \vee b$ and $a \wedge b$ for their maximum and their minimum, and $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$ for the positive and the negative part of a , so that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. The same notation is used pointwise for functions.

Given a convex function $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$, we denote by $\partial\Phi(\xi)$ its subdifferential at ξ and by $\Phi'(\xi; \eta)$ its one-sided directional derivative at ξ in the direction η .

Notation for geometry. Given a space-time set $\Lambda \subseteq \mathbb{R}^N \times [0, +\infty)$, we denote its *time-slices* by

$$\Lambda(t) = \{x \in \mathbb{R}^N : (x, t) \in \Lambda\}.$$

Given $A \subseteq \mathbb{R}^N$ and a function $u : A \rightarrow \mathbb{R}$, we denote by ∇u the spatial gradient and by $\text{div } u$ the spatial divergence, either classical, a.e., weak, or distributional. Given $B \subseteq A$, we denote by $\int_B u$ the integral of u with respect to the Lebesgue measure restricted to B and by $\bar{f}_B u$ its average. If $B = \mathbb{R}^N$ we simply write $\int u$.

Given $A \subseteq \mathbb{R}^{N+1}$ and a function $u : A \rightarrow \mathbb{R}$, we denote by ∇u the spatial gradient, by $\text{div } u$ the spatial divergence, and by $\dot{u}$ the temporal derivative, either classical, a.e., weak, or distributional. Given $B \subseteq A$, we denote by $\iint_B u$ the integral of u with respect to the Lebesgue measure restricted to B .

Given $v, w \in \mathbb{R}^N$, we write $v \otimes w$ for the matrix $(v_i w_j)_{ij}$, and given two matrices M and M' we write $M \cdot M' = \sum_{ij} M_{ij} M'_{ij}$. Given a set E of finite perimeter and a vector field $X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$, we denote by $\text{div}_E X$ the tangential divergence of X on ∂^*E .

Weak formulations of mean curvature flow

In the two-phase system, the interface separating the two phases carries a surface energy and evolves so as to dissipate it. Making this statement precise requires specifying what forces act on a moving interface and how they balance, and the natural setting for this is the principle of virtual powers, a variational principle rooted in continuum mechanics (see [3] for an historical treatment, or [12] for a modern inspiration of the general framework). We adopt it here as our starting point. Section 1.1 presents a general and experimental variational framework, Section 1.2 applies it to mean curvature flow and derives the equation of motion, recovering a first weak formulation as a force balance against arbitrary variations of the set. Section 1.3 introduces the distributional formulation of Chambolle, Morini and Ponsiglione, which is the notion of solution adopted in the rest of the thesis and which is well-posed for an arbitrary convex anisotropy. Even if the connection between the two formulations is far from being fully understood, Remarks 1.6 and 1.7 present briefly what is partially known.

1.1. A general variational framework

In this section we propose a general variational framework, currently useful only for describing and developing mechanical models whose space of configurations lacks a linear structure. We hope that it can represent a valid generalization of the classical Banach space framework, at least for modelling purposes. The main reason for this generalization lies in the lack of a linear structure for the space of configurations of the two-phase system, as well as a metric structure for which the evolution can be interpreted as a gradient flow.

1.1.1. Space of configurations. We start from a space of possible *configurations* M and assume that there are some characteristic *observables* which can be described as the elements of a separable Banach space H . We also assume that there exists an injective map $J : M \rightarrow H^*$ such that $\langle J(u), \varphi \rangle$ represents the action of the observable $\varphi \in H$ on the configuration $u \in M$.

We can now endow M with a metric structure through the distance $d(u_0, u_1) = \|J(u_1) - J(u_0)\|_{H^*}$, together with the strong convergence $u_n \rightarrow u$ defined by $d(u_n, u) \rightarrow 0$.

Finally, we assume that there exists a family of seminorms $\{\|\cdot\|_u\}_{u \in M}$ and a function $C : M \rightarrow [0, +\infty)$ with the following properties:

- (i) $\|\varphi\|_u \leq C(u)\|\varphi\|_H$ for all $u \in M$ and $\varphi \in H$;
- (ii) If $u_n \rightarrow u$, then

$$\|\varphi\|_u \leq \liminf_{n \rightarrow \infty} \|\varphi\|_{u_n} \quad \text{for all } \varphi \in H.$$

The seminorm relative to a configuration should be interpreted as a measure of the magnitude of an observable when interpreted as a force acting on that configuration.

DEFINITION 1.1 (Absolutely continuous curves). Given $T > 0$ and $1 \leq p \leq \infty$, we define $AC^p([0, T]; M)$ as the class of all continuous curves $\gamma : [0, T] \rightarrow M$ such that $C(\gamma) \in L^\infty(0, T)$ and

$$|\langle J(\gamma(s)) - J(\gamma(t)), \varphi \rangle| \leq \int_t^s m(\tau) \|\varphi\|_{\gamma(\tau)} d\tau$$

for some $m \in L^p([0, T]; [0, +\infty))$ and all $\varphi \in H$ and $0 \leq t \leq s \leq T$. If $p = 1$, we simply write $AC([0, T]; M)$.

These classes are particularly interesting, since we interpret them as the classes of all possible *processes* without jumps.

REMARK 1.1 (Measurability). The definition is well-posed, since $t \mapsto \|\varphi\|_{\gamma(t)}$ is LSC and so measurable. Indeed, let $t_n \rightarrow t$. Then $\gamma(t_n) \rightarrow \gamma(t)$ from continuity, and

$$\|\varphi\|_{\gamma(t)} \leq \liminf \|\varphi\|_{\gamma(t_n)}$$

from property (ii) of the family of seminorms. Moreover, by property (i), this class is contained in the class of absolutely continuous curves with respect to the metric structure.

1.1.2. Tangent structure. There is also a natural tangent structure: for every $u \in M$, we define the *ambient tangent space*

$$V_u = \left\{ v \in H^* : \sup_{\varphi \in H, \|\varphi\|_u \leq 1} |\langle v, \varphi \rangle| = \|v\|_u < +\infty \right\},$$

equipped with the norm $\|\cdot\|_u$. Then we define the *tangent cone* $T_u M$ as the set of all $v \in V_u$ such that

$$\frac{J(\gamma(h)) - J(u)}{h} \xrightarrow{*} v \quad \text{as } h \rightarrow 0^+$$

for some $\gamma \in \text{AC}([0, 1]; M)$. This should be interpreted as the set of possible *rates*, on which a dissipative force could depend. Given $v \in T_u M$, it is convenient to define the set of realizing curves $\Gamma_u(v)$, made of curves that realize the limit as above. We also define the *tangent bundle*

$$TM = \{(u, v) \in M \times H^* : v \in T_u M\}.$$

DEFINITION 1.2 (Curves of rates). Given $T > 0$, $\gamma \in \text{AC}([0, T]; M)$ and $1 \leq p \leq \infty$, we define $L^p([0, T]; T_\gamma M)$ as the class of all curves $w : [0, T] \rightarrow H^*$ such that:

- (i) $t \mapsto \langle w(t), \varphi \rangle$ is measurable for all $\varphi \in H$;
- (ii) $(\gamma(t), w(t)) \in TM$ for a.e. $t \in [0, T]$;
- (iii) $t \mapsto \|w(t)\|_{\gamma(t)} \in L^p(0, T)$.

This class is quotiented by a.e. equivalence.

We prove some propositions about the tangent structure.

PROPOSITION 1.3 (Completeness of the ambient). *For each $u \in M$, V_u is a Banach space.*

PROOF. Let $v_n \in V_u$ be a Cauchy sequence in V_u . Since $\|\cdot\|_{H^*} \leq C(u)\|\cdot\|_u$, v_n is also Cauchy in H^* , thus $v_n \rightarrow v$ in H^* . Let $\varphi \in H$ with $\|\varphi\|_u \leq 1$. Then we have

$$|\langle v, \varphi \rangle| \leq |\langle v - v_n, \varphi \rangle| + |\langle v_n, \varphi \rangle| \leq \|v - v_n\|_{H^*} \|\varphi\|_H + \|v_n\|_u$$

and so

$$|\langle v, \varphi \rangle| \leq \sup_n \|v_n\|_u < +\infty.$$

Taking the supremum over $\|\varphi\|_u \leq 1$ implies $v \in V_u$. Finally, fix $\epsilon > 0$ and let n_0 be such that $\|v_n - v_m\|_u < \epsilon$ for all $n, m > n_0$. We have

$$|\langle v_n - v, \varphi \rangle| \leq |\langle v_n - v_m, \varphi \rangle| + |\langle v_m - v, \varphi \rangle| \leq \epsilon + |\langle v_m - v, \varphi \rangle|.$$

Letting $m \rightarrow \infty$, $\epsilon \rightarrow 0$ and taking the supremum over $\|\varphi\|_u \leq 1$ implies $v_n \rightarrow v$ in V_u . $\square$

PROPOSITION 1.4 (Weak-* derivative). *Let $1 \leq p \leq +\infty$ and let $z \in \text{AC}^p([0, T]; M)$. Then there exists $\dot{z} \in L^p([0, T]; T_z M)$ such that*

$$\frac{J(z(t+h)) - J(z(t))}{h} \xrightarrow{*} \dot{z}(t) \quad \text{as } h \rightarrow 0, \quad \text{for a.e. } t \in [0, T].$$

Moreover,

$$\langle J(z(s)) - J(z(t)), \varphi \rangle = \int_t^s \langle \dot{z}(\tau), \varphi \rangle d\tau$$

for all $0 \leq t \leq s \leq T$ and $\varphi \in H$.

PROOF. Let $\{\varphi_n\}$ be a countable dense subset of H , and consider $H_0 = \text{span}\{\varphi_1, \varphi_2, \dots\}$. We have

$$|\langle J(z(s)) - J(z(t)), \varphi_n \rangle| \leq \int_t^s m(\tau) \|\varphi_n\|_{z(\tau)} d\tau \leq \|C(z)\|_\infty \|\varphi_n\|_H \int_t^s m(\tau) d\tau,$$

hence $\langle J(z), \varphi_n \rangle \in \text{AC}^p([0, T])$, so it is differentiable outside a null set $E_n \subset [0, T]$ such that $[0, T] \setminus E_n$ is made of Lebesgue points of m .

Let $t \notin E = \bigcup_{n=1}^\infty E_n$. We have

$$\left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi \right\rangle \rightarrow L\varphi \quad \text{for all } \varphi \in H_0,$$

where $L : H_0 \rightarrow \mathbb{R}$ is linear and bounded in H . Indeed, we have

$$\left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi \right\rangle \right| \leq \|C(z)\|_\infty \|\varphi\|_H \frac{1}{h} \int_t^{t+h} m(\tau) d\tau$$

and

$$|L\varphi| \leq \|C(z)\|_\infty m(t) \|\varphi\|_H,$$

since t is a Lebesgue point of m . Then L can be extended uniquely to $\dot{z}(t) \in H^*$.

Let $\varphi \in H$ and $\psi_n \in H_0$ such that $\psi_n \rightarrow \varphi$. We have

$$\begin{aligned} \left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi \right\rangle - \langle \dot{z}(t), \varphi \rangle \right| &\leq \left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi - \psi_n \right\rangle \right| \\ &\quad + \left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \psi_n \right\rangle - \langle \dot{z}(t), \psi_n \rangle \right| \\ &\quad + |\langle \dot{z}(t), \psi_n - \varphi \rangle| \end{aligned}$$

For the first term, we have

$$\left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi - \psi_n \right\rangle \right| \leq \|C(z)\|_\infty \|\varphi - \psi_n\|_H \frac{1}{h} \int_t^{t+h} m(\tau) d\tau,$$

so that

$$\limsup_{h \rightarrow 0^+} \left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi - \psi_n \right\rangle \right| \leq \|C(z)\|_\infty m(t) \|\varphi - \psi_n\|_H.$$

It follows

$$\limsup_{h \rightarrow 0^+} \left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi \right\rangle - \langle \dot{z}(t), \varphi \rangle \right| \leq \|C(z)\|_\infty m(t) \|\varphi - \psi_n\|_H + |\langle \dot{z}(t), \psi_n - \varphi \rangle|$$

and we conclude by taking the limit as $n \rightarrow \infty$ that

$$\frac{J(z(t+h)) - J(z(t))}{h} \xrightarrow{*} \dot{z}(t).$$

Now we prove that $t \mapsto \|\dot{z}(t)\|_{z(t)}$ is measurable. Each $t \mapsto \langle \dot{z}(t), \varphi \rangle$ is measurable as it is the limit of measurable functions $t \mapsto \langle \dot{z}(t), \varphi_n \rangle$. Then we have

$$\|\dot{z}(t)\|_{z(t)} \leq c \quad \text{iff} \quad |\langle \dot{z}(t), \varphi_n \rangle| \leq c \|\varphi_n\|_{z(t)} \quad \text{for all } n,$$

hence we have measurability (see also Remark 1.1). Given $\varphi \in H$ with $\|\varphi\|_{z(t)} \leq 1$, we have the bound

$$\left| \left\langle \frac{J(z(t+h)) - J(z(t))}{h}, \varphi \right\rangle \right| \leq \frac{1}{h} \int_t^{t+h} m(\tau) d\tau.$$

Since t is a Lebesgue point, we can pass to the limit as

$$|\langle \dot{z}(t), \varphi \rangle| \leq m(t).$$

Taking the supremum over $\|\varphi\|_{z(t)} \leq 1$ yields $\|\dot{z}(t)\|_{z(t)} \leq m(t)$. Finally, $\dot{z}(t) \in T_{z(t)}M$, since it is realized by the curve $h \mapsto z(t+h)$, so that $\dot{z} \in L^p([0, T]; T_z M)$.

Finally, let $\varphi \in H$. Since $\langle J(z), \varphi \rangle \in AC^p([0, T])$, we have

$$\langle J(z(s)) - J(z(t)), \varphi \rangle = \int_t^s \langle \dot{z}(\tau), \varphi \rangle d\tau$$

for all $0 \leq t \leq s \leq T$. □

DEFINITION 1.5 (Lower directional derivative). Given a function $F : M \rightarrow \mathbb{R} \cup \{+\infty\}$, $u \in M$ such that $F(u) < +\infty$ and $v \in T_u M$, we define the lower directional derivative

$$\underline{D}_v F(u) = \inf \left\{ \liminf_{h \rightarrow 0^+} \frac{F(\gamma(h)) - F(u)}{h} : \gamma \in \Gamma_u(v) \right\}$$

and the lower subdifferential

$$\partial F(u) = \{\xi \in V_u^* : \underline{D}_v F(u) \geq \langle \xi, v \rangle \text{ for all } v \in T_u M\}.$$

1.1.3. Mechanical problem. We assume that the constitutive data of the mechanical problem are a *free energy* $\Psi : M \rightarrow \mathbb{R}$ and a *dissipation potential* $\Phi : TM \rightarrow \mathbb{R}$ with the following properties:

- (i) Ψ is lower semicontinuous;
- (ii) $\Phi(u, \cdot) = \Phi_u$ is convex for all $u \in M$;
- (iii) $\Phi_u(v) \geq \Phi_u(0) = 0$ for all $(u, v) \in TM$.

Given $T > 0$, we say that a process $z \in AC([0, T]; M)$ is a *solution* iff $\Psi(z) \in AC([0, T])$ and

$$(1.1) \quad \partial \Psi(z(t)) + \partial \Phi_{z(t)}(\dot{z}(t)) \ni 0 \quad \text{for a.e. } t \in [0, T].$$

REMARK 1.2 (Energy balance). Since $\Psi(z) \in AC([0, T])$, we have

$$\Psi(z(s)) = \Psi(z(t)) + \int_t^s \frac{d}{d\tau} \Psi(z(\tau)) d\tau$$

for all $0 \leq t \leq s \leq T$. Let $\tau \in [0, T]$ be such that both $\dot{z}(\tau)$ and $\frac{d}{d\tau} \Psi(z(\tau))$ exist and (1.1) holds. Then there exists $\xi(\tau) \in \partial \Phi_{z(\tau)}(\dot{z}(\tau))$ such that $-\xi(\tau) \in \partial \Psi(z(\tau))$. By the definition of the lower subdifferential, we have

$$\langle -\xi(\tau), \dot{z}(\tau) \rangle \leq \frac{d}{d\tau} \Psi(z(\tau))$$

and, considering the reversed $h \mapsto z(t-h)$,

$$\langle -\xi(\tau), -\dot{z}(\tau) \rangle \leq -\frac{d}{d\tau} \Psi(z(\tau)),$$

so that

$$\frac{d}{d\tau} \Psi(z(\tau)) = -\langle \xi(\tau), \dot{z}(\tau) \rangle.$$

Thus we have

$$\Psi(z(s)) = \Psi(z(t)) - \int_t^s \langle \xi(\tau), \dot{z}(\tau) \rangle d\tau.$$

The second term is the *dissipation*: by the definition of the subdifferential, we have

$$\langle \xi(\tau), \dot{z}(\tau) \rangle \geq \Phi_{z(\tau)}(\dot{z}(\tau)) \geq 0,$$

making it an actual dissipation, so that the free energy does not increase.

EXAMPLE 1.1 (Flat space). Let M be a reflexive and separable Banach space and let $H = M^*$. Since $H^* \simeq M$, we can choose as $J : M \rightarrow M$ the identity. Finally, we choose as the family of seminorms the constant family $\{\|\cdot\|_M\}_{u \in M}$. Given $u, v \in M$, then

$$\frac{(u + hv) - u}{h} = v,$$

so $T_u M = M$. One can check that all the objects of this framework reduce to the classical ones of Banach spaces.

1.2. Application to mean curvature flow

1.2.1. Isotropic case. Let M be the class of all sets of finite volume and finite perimeter of $\mathbb{R}^N$, quotiented by the relation of being equal up to a symmetric difference of measure zero. We choose $H = C_0(\mathbb{R}^N)$ with the sup-norm. Since $H^* = \mathcal{M}(\mathbb{R}^N)$, we can choose $J : M \rightarrow H^*$ defined by

$$\langle J(E), \varphi \rangle = \int_E \varphi,$$

inducing the distance $d(E, F) = |E \Delta F|$. Finally, we choose $\{\|\cdot\|_{L^2(\partial^* E)}\}_{E \in M}$ as the family of seminorms. This is a good choice, since:

(i) Given $E \in M$ and $\varphi \in H$, we have

$$\|\varphi\|_{L^2(\partial^* E)} \leq \text{Per}(E)^{1/2} \|\varphi\|_\infty$$

(ii) If $E_n \rightarrow E$ and $\varphi \in H$, then

$$\|\varphi\|_{L^2(\partial^* E)}^2 = \int_{\partial^* E} \varphi^2 \leq \liminf_{n \rightarrow \infty} \int_{\partial^* E_n} \varphi^2 = \liminf_{n \rightarrow \infty} \|\varphi\|_{L^2(\partial^* E_n)}^2.$$

CLAIM 1.6 (Identification of the tangent cone). *Given $E \in M$, we have $T_E M \simeq L^2(\partial^* E)$.*

REMARK 1.3 (Realizing curves). The non-trivial part of proving this claim is constructing a realizing curve for every $V \in L^2(\partial^* E)$. What follows depends partly on this claim, which we expect to be true. Nevertheless, all the results regarding "smooth rates" do not depend on this claim. Indeed, it is easy to see that $X \cdot \nu_E \in T_E M$ for all $X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$, using the curves induced by the diffeomorphisms corresponding to each X .

We specify the following constitutive data: a free energy

$$\Psi(E) = \text{Per}(E)$$

and a dissipation potential

$$\Phi(E, V) = \frac{1}{2} \|V\|_{L^2(\partial^* E)}^2.$$

We can easily compute

$$\partial \Phi_E(V) = \{V\},$$

so that a process $z \in \text{AC}([0, T]; M)$ is a solution iff

$$(1.2) \quad -V_t \in \partial \Psi(E_t) \quad \text{for a.e. } t \in [0, T],$$

where $z(t) = E_t$ and $\dot{z}(t) = V_t$. By Remark 1.2, the total dissipation

$$\int_0^T \|V_t\|_{L^2(\partial^* E_t)}^2 dt$$

must be finite, so that $z \in \text{AC}^2([0, T]; M)$. Moreover, the energy balance holds as

$$(1.3) \quad \text{Per}(E_s) = \text{Per}(E_t) - \int_t^s \|V_\tau\|_{L^2(\partial^* E_\tau)}^2 d\tau$$

for all $0 \leq t \leq s \leq T$. From the definition of the lower subdifferential, at each t where it holds, (1.2) is equivalent to

$$(1.4) \quad -\langle V, \tilde{V} \rangle \leq \liminf_{h \rightarrow 0^+} \frac{\Psi(\gamma(h)) - \Psi(E)}{h} \quad \text{for all } \tilde{V} \in L^2(\partial^* E) \text{ and } \gamma \in \Gamma_E(\tilde{V}),$$

where $E = E_t$ and $V = V_t$. It can be easily shown that this implies

$$(1.5) \quad -\langle V, X \cdot \nu_E \rangle = \int_{\partial^* E} \text{div}_E X d\mathcal{H}^{N-1} \quad \text{for all } X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N),$$

using as curves the ones induced by the diffeomorphisms corresponding to each X and computing the first variation of Per^ϕ along them. This last equation is a pointwise in time version of the Luckhaus-Sturzenhecker formulation [10].

REMARK 1.4 (Failure of the converse). At a fixed time t , the converse does not necessarily hold. Indeed, consider the set $E = B_1(-1, 0) \cup B_1(1, 0) \subset \mathbb{R}^2$ with a rate $V = -1$. It can be verified that this satisfies (1.5), but along the curve $\gamma(h) = E + B_h$ we have

$$\Psi(\gamma(h)) = 4(1+h) \left(\pi - \cos^{-1} \left(\frac{1}{1+h} \right) \right),$$

so that $\frac{d}{dh} \Big|_{h=0^+} \Psi(\gamma(h)) = -\infty$, violating (1.4). This curve is admissible, since for every observable $\varphi \in C_0(\mathbb{R}^N)$ supported outside the origin, its rate can be computed as a sum of rates along diffeomorphisms. If instead φ is supported at the origin, we can write it as a sum of an observable supported outside the origin and an observable covering the entire set, whose rate is just the rate of the volume, which is finite.

If E is a smooth set, by integrating by parts in (1.5), we can derive the pointwise classical formulation of mean curvature flow

$$(1.6) \quad V + H = 0,$$

where $H : \partial E \rightarrow \mathbb{R}$ is the classical mean curvature of E . Moreover, this holds in the general case with the generalized mean curvature. Indeed,

$$\left| \int_{\partial^* E} \text{div}_E X d\mathcal{H}^{N-1} \right| \leq \|V\|_{L^2(\partial^* E)} \|X \cdot \nu_E\|_{L^2(\partial^* E)}.$$

Thus the curvature functional is bounded in $L^2(\partial^* E)$ on a dense set, so it can be extended continuously and represented by a function $H \in L^2(\partial^* E)$, for which the same equality holds.

EXAMPLE 1.2 (Shrinking of a ball). The fundamental example of such an evolution is the shrinking of a ball. Let $R > 0$ and $E_0 = B_R$. We start from the ansatz that the evolution preserves the shape, that is

$$E_t = B_{\rho(t)}$$

for some smooth $\rho : [0, T] \rightarrow \mathbb{R}$ and some $T > 0$. Then $V_t = \rho'(t)$ and $H_t = \frac{N-1}{\rho(t)}$, so we have

$$\begin{cases} \rho' + \frac{N-1}{\rho} = 0 \\ \rho(0) = R \end{cases}.$$

With $T < \frac{R^2}{2(N-1)}$, this ODE is solved by

$$\rho(t) = \sqrt{R^2 - 2(N-1)t}.$$

REMARK 1.5 (Localized energy balance). Since the energy balance (1.3) holds, we might ask whether a localized version of it could also hold. Indeed, for a diffeomorphic curve $h \mapsto F_h$ with $F_0 = E$ induced by a vector field $X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ and a cut-off function $\psi \in C_c^1(\mathbb{R}^N \times [0, 1])$, we have the classical formula

$$\frac{d}{dh} \Big|_{h=0^+} \int_{\partial^* F_h} \psi(\cdot, h) d\mathcal{H}^{N-1} = \int_{\partial^* F} (\psi(\cdot, 0)H + \nabla \psi(\cdot, 0) \cdot \nu_F)(X \cdot \nu_F) d\mathcal{H}^{N-1} + \int_{\partial^* F} \dot{\psi}(\cdot, 0) d\mathcal{H}^{N-1}.$$

We do not know whether our solution concept is able to provide such a formula for its evolutions, together with all the localized perimeter functionals being absolutely continuous. The target result would be

$$\int_{\partial^* E_s} \psi(\cdot, s) d\mathcal{H}^{N-1} = \int_{\partial^* E_t} \psi(\cdot, t) d\mathcal{H}^{N-1} + \int_t^s \int_{\partial^* E_\tau} (\psi(\cdot, \tau)H_\tau V_\tau + (\nabla \psi(\cdot, \tau) \cdot \nu_{E_\tau})V_\tau + \dot{\psi}(\cdot, \tau)) d\mathcal{H}^{N-1} d\tau$$

for all $0 \leq t \leq s \leq T$. Together with (1.6), this would be a stronger version of the Brakke formulation [1], with an equality instead of the inequality.

1.2.2. Anisotropic case. We use the same setting of the previous subsection, changing the free energy and the dissipation potential. Given a norm $\phi : \mathbb{R}^N \rightarrow [0, +\infty)$, we choose

$$\Psi(E) = \text{Per}^\phi(E) = \int_{\partial^* E} \phi(\nu_E) d\mathcal{H}^{N-1},$$

$$\Phi(E, V) = \frac{1}{2} \int_{\partial^* E} \frac{1}{\phi(\nu_E)} V^2 d\mathcal{H}^{N-1}.$$

PROPOSITION 1.7 (Characterization of the subdifferential). *Let $E \in M$. Then*

$$\partial \Psi(E) \subseteq \{\text{div}_E z \in L^2(\partial^* E) : z \in L^\infty(\partial^* E; \mathbb{R}^N), z \in \partial \phi(\nu_E)\},$$

where $\text{div}_E z$ is defined as the distribution acting on $X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ as

$$\langle \text{div}_E z, X \rangle = \int_{\partial^* E} \left((z \cdot \nu_E) \text{div} X - z \cdot (\nabla X)^T \nu_E \right) d\mathcal{H}^{N-1}$$

and $\text{div}_E z \in L^2(\partial^* E)$ means

$$(1.7) \quad |\langle \text{div}_E z, X \rangle| \leq C \|X \cdot \nu_E\|_{L^2(\partial^* E)}$$

for some $C > 0$ and all $X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$.

PROOF. As for (1.5), computing the first variation of Per^ϕ along the flow of X , we have that $\xi \in \partial \Psi(E)$ implies

$$(1.8) \quad \langle \xi, X \cdot \nu_E \rangle \leq \int_{\partial^* E} \left(\phi(\nu_E) \text{div} X - \min_{w \in \partial \phi(\nu_E)} w \cdot (\nabla X)^T \nu_E \right) d\mathcal{H}^{N-1} \quad \text{for all } X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N).$$

Now we define

$$K = \{z \in L^\infty(\partial^* E; \mathbb{R}^N) : z \in \partial \phi(\nu_E)\}$$

and $F : K \times C_c^1(\mathbb{R}^N; \mathbb{R}^N) \rightarrow \mathbb{R}$ as

$$F(z, X) = \langle \xi, X \cdot \nu_E \rangle - \int_{\partial^* E} \left(\phi(\nu_E) \text{div} X - z \cdot (\nabla X)^T \nu_E \right) d\mathcal{H}^{N-1}.$$

Note that K is convex and weakly-* compact. It is easy to show that (1.8) implies

$$(1.9) \quad \min_{z \in K} F(z, X) \leq 0 \quad \text{for all } X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N).$$

We want to prove that there exists a single $z \in K$ such that

$$(1.10) \quad F(z, X) = 0 \quad \text{for all } X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N).$$

Since $C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ is separable, let $\{X_n\}$ be a countable dense subset and consider $V_n = \text{span}\{X_1, \dots, X_n\}$. We claim that there exists $z_n \in K$ such that

$$F(z_n, X) = 0 \quad \text{for all } X \in V_n.$$

Indeed, assume by contradiction that

$$0 \notin \{F(z, X_1), \dots, F(z, X_n) : z \in K\},$$

a convex and compact set. By the Hahn–Banach theorem, we can strictly separate this set and the set $\{0\}$, so that there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that

$$\sum_{i=1}^n \lambda_i F(z, X_i) > 0 \quad \text{for all } z \in K,$$

contradicting (1.9) with $X = \sum_{i=1}^n \lambda_i X_i$. Since K is weakly- $*$ compact, $z_n \xrightarrow{*} z$ in K up to a subsequence. Passing to the limit, we have

$$F(z, X) = 0 \quad \text{for all } X \in \text{span}\{X_1, X_2, \dots\}$$

and we conclude by density of $\{X_n\}$, proving (1.10). We know now that there exists $z \in K$ such that

$$\langle \xi, X \cdot \nu_E \rangle = \int_{\partial^* E} \left(\phi(\nu_E) \operatorname{div} X - z \cdot (\nabla X)^T \nu_E \right) d\mathcal{H}^{N-1} \quad \text{for all } X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N).$$

We can conclude by observing that $z \in K$ implies that $\phi(\nu_E) = z \cdot \nu_E$ and that $\xi \in L^2(\partial^* E)$ implies (1.7). $\square$

1.3. The distributional formulation

The formulation obtained in the previous section is stated in terms of the evolving set and of variations of it. The formulation we adopt from now on, introduced in [14] and further developed in [15], is instead stated in terms of the signed distance function to the evolving set. Moreover, we do not need any assumptions about the perimeter or even the measurability of the set. We assume ϕ to be a norm on $\mathbb{R}^N$ and we use the notation of Section A.1.

DEFINITION 1.8 (Superflow). Let $E_0 \subseteq \mathbb{R}^N$ be closed. We say that a closed set $\Lambda \subseteq \mathbb{R}^N \times [0, +\infty)$ is a *superflow* of anisotropic mean curvature flow with initial condition E_0 if the following hold, where $d(x, t) = \operatorname{sd}_{\Lambda(t)}^\phi(x)$ is the signed distance to the slices of Λ :

- (a) $\Lambda(0) \subseteq E_0$;
- (b) $\Lambda(t) = \emptyset$ for all $t > T^*$, where $T^* = \inf\{t \geq 0 : \Lambda(t) = \emptyset\}$;
- (c) For every $x \in \mathbb{R}^N$, the function $t \mapsto \operatorname{dist}^\phi(x, \Lambda(t))$ is left-continuous;
- (d) There exists $z \in L^\infty(\mathbb{R}^N \times (0, T^*) \setminus \Lambda; \mathbb{R}^N)$ with $z \in \partial\phi(\nabla d)$ such that $\operatorname{div} z \in \mathcal{M}(\mathbb{R}^N \times (0, T^*) \setminus \Lambda)$, $(\operatorname{div} z)^+ \in L^\infty(\{d \geq \delta\})$ for all $\delta > 0$, and

$$(1.11) \quad \dot{d} \geq \operatorname{div} z \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (0, T^*) \setminus \Lambda).$$

We say that an open set $\Omega \subseteq \mathbb{R}^N \times [0, +\infty)$ is a *subflow* with initial condition U_0 (open) if Ω^c is a superflow with initial condition U_0^c . We say that Λ is a *flow* with initial condition E_0 if both it is a superflow with initial condition E_0 and Λ is a subflow with initial condition $\dot{E}_0$.

PROPOSITION 1.9 (Comparison principle). Let Λ be a superflow with initial condition E_0 and let Ω be a subflow with initial condition U_0 . If $\operatorname{dist}^\phi(E_0, U_0^c) = \Delta > 0$, then $\operatorname{dist}(\Lambda(t), \Omega(t)^c) \geq \Delta$ for all $t \geq 0$.

PROOF. See [14], Theorem 3.3. $\square$

REMARK 1.6 (Motivation from the comparison principle). This formulation is essentially based on the comparison principle, that is a special feature of this problem. Indeed, one could show that, in the case of $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$ with $\phi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, a comparison principle together with the smooth flow being a solution implies the distributional formulation. The argument uses the concept of *viscosity solutions*, that is shown to be equivalent to the distributional formulation in such a regular case (see [14], Proposition A.1), together with comparison with smooth barriers. If ϕ is not regular, we have two issues: the anisotropic distance function is not necessarily semi-concave (so Alexandrov theorem does not hold, and it is used in the proof of [14], Proposition A.1), and we have no smooth barriers. Indeed, in this case we have no concept of a smooth flow.

REMARK 1.7 (Comparison in the variational formulation). We could ask if the variational formulation derived in the previous chapter admits a comparison principle. In the isotropic case, if a result like that of Remark 1.5 holds, we have an *avoidance* principle (see [8], Lemma 10.6). Moreover, since our notion of solution is based on the absolute continuity with respect to the volume observables, the avoidance principle can be upgraded to a comparison principle (see [17], Lemma 2.14). Whether the anisotropic version of our variational formulation could imply the distributional one remains an open problem (it is open also in the isotropic case without assuming the Brakke type result of Remark 1.5).

Convergence of an abstract approximation scheme

In this chapter we construct and prove the convergence of an abstract approximation scheme for superflows in the sense of Definition 1.8. The scheme is also proved to *generically* converge to a flow in the sense of Corollary 2.17. Our goal is to generalize the scheme used in [14], isolating the abstract properties required for its convergence to a superflow. Throughout this chapter, we assume ϕ to be a norm on $\mathbb{R}^N$. To lighten the notation, in this chapter we drop the superscript ϕ and simply write $d_E = \text{sd}_E^\phi$ for the anisotropic signed distance to a closed set E , $\text{dist} = \text{dist}^\phi$ and W_R for the Wulff shape of radius R . All these objects are defined in Section A.1. We refer to the relevant part of Proposition A.5 as the *eikonal equation*.

2.1. An abstract diffusion operator

In this section we introduce an abstract diffusion operator through a set of axioms and prove its main properties. This operator is the core object of the *diffusion/redistancing* scheme, in which it acts by diffusing the signed distance from the previous set at each step, generating the new set as a sublevel set of the diffused distance.

2.1.1. Axioms and basic properties.

DEFINITION 2.1 (Diffusion operator). Let $X = \text{Lip}(\mathbb{R}^N) \cup \{\pm\infty\}$, which is the space of Lipschitz functions together with the identically $\pm\infty$ functions. We say that $\mathcal{T} : X \rightarrow X$ is a *diffusion operator* if the following hold:

- (i) If $u_1 \leq u_2$, then $\mathcal{T}u_1 \leq \mathcal{T}u_2$;
- (ii) If u is Lipschitz, then $\mathcal{T}u$ is Lipschitz;
- (iii) $\mathcal{T}(u + c) = \mathcal{T}u + c$ for all $c \in \mathbb{R}$;
- (iv) $\mathcal{T}(-u) = -\mathcal{T}u$;
- (v) $\mathcal{T}(u(\cdot + \tau)) = \mathcal{T}u(\cdot + \tau)$ for all $\tau \in \mathbb{R}^N$;
- (vi) $\mathcal{T}(\pm\infty) = \pm\infty$.

With a slight abuse of notation, we also write $\mathcal{T}$ for the induced operator on closed sets, defined by

$$\mathcal{T}(E) = \{\mathcal{T}d_E \leq 0\}.$$

Given a function $u : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that $u(\cdot, t) \in X$ for all $t \geq 0$, we write

$$\mathcal{T}u(x, t) = \mathcal{T}(u(\cdot, t))(x).$$

REMARK 2.1 (First properties). The following hold:

- (1) $\mathcal{T}(E)$ is well-defined, since d_E is Lipschitz (see Proposition A.5);
- (2) $\mathcal{T}(E) = E$ if $E \in \{\emptyset, \mathbb{R}^N\}$, since $d_\emptyset = +\infty$ and $d_{\mathbb{R}^N} = -\infty$;
- (3) $\mathcal{T}(E)$ is closed, since $\mathcal{T}d_E$ is Lipschitz if $E \notin \{\emptyset, \mathbb{R}^N\}$ and thus continuous;
- (4) $\mathcal{T}(\overline{E^c}) \supseteq \overline{\mathcal{T}(E)^c}$, since $\mathcal{T}d_{\overline{E^c}} = -\mathcal{T}d_E$ by property (iv) of $\mathcal{T}$.

We prove some simple consequences of the properties of a diffusion operator $\mathcal{T}$.

PROPOSITION 2.2 (Lipschitz contraction). *If $u \in \text{Lip}(\mathbb{R}^N)$, then it holds*

$$(2.1) \quad \|\phi(\nabla(\mathcal{T}u))\|_\infty \leq \|\phi(\nabla u)\|_\infty.$$

PROOF. Let $L = \|\phi(\nabla u)\|_\infty$ and let $\tau \in \mathbb{R}^N$. By Proposition A.4,

$$u - L\phi^\circ(\tau) \leq u(\cdot + \tau) \leq u + L\phi^\circ(\tau).$$

Applying the operator $\mathcal{T}$, by properties (i), (iii) and (v), it follows

$$\mathcal{T}u - L\phi^\circ(\tau) \leq \mathcal{T}u(\cdot + \tau) \leq \mathcal{T}u + L\phi^\circ(\tau),$$

proving (2.1) again by Proposition A.4. □

PROPOSITION 2.3 (Redistancing inequality). *It holds*

$$(2.2) \quad \mathcal{T}d_E \leq d_{\mathcal{T}(E)} \quad \text{in } \mathcal{T}(E)^c.$$

PROOF. We set $u = \mathcal{T}d_E$. Let $x \in \mathcal{T}(E)^c$ and let $y \in \mathcal{T}(E)$, so that $u(y) \leq 0$. By (2.1) and by the eikonal equation, it follows that $\phi(\nabla u) \leq 1$, giving

$$u(x) \leq u(x) - u(y) \leq \phi^\circ(x - y),$$

which is true by Proposition A.4. Taking the infimum over $y \in \mathcal{T}(E)$ yields $u(x) \leq d_{\mathcal{T}(E)}(x)$, proving (2.2). $\square$

PROPOSITION 2.4 (Comparison principle). *If $E \subset F$, then $\mathcal{T}(E) \subseteq \mathcal{T}(F)$.*

PROOF. Since $E \subset F$, we have $d_F \leq d_E$. By property (i) of $\mathcal{T}$, it follows that $\mathcal{T}d_F \leq \mathcal{T}d_E$, and so

$$\mathcal{T}(E) = \{\mathcal{T}d_E \leq 0\} \subseteq \{\mathcal{T}d_F \leq 0\} = \mathcal{T}(F). \quad \square$$

PROPOSITION 2.5 (Avoidance principle). *If $E \cap F = \emptyset$, then*

$$(2.3) \quad \text{dist}(\mathcal{T}(E), \mathcal{T}(F)) \geq \text{dist}(E, F).$$

PROOF. We set $u = \mathcal{T}d_E$ and $v = \mathcal{T}d_F$. If $E \in \{\emptyset, \mathbb{R}^N\}$ or $F \in \{\emptyset, \mathbb{R}^N\}$, then (2.3) is trivially true. Otherwise, by (A.1), we have $d_E + d_F \geq \text{dist}(E, F) = R$. Since $d_E \geq R - d_F$, by properties (i), (iii) and (iv) of $\mathcal{T}$, it follows that $u \geq R - v$, hence $u + v \geq R$. By (2.2), we have

$$d_{\mathcal{T}(E)} + d_{\mathcal{T}(F)} \geq u + v \geq R \quad \text{in } (\mathcal{T}(E) \cup \mathcal{T}(F))^c.$$

By (A.2), taking the infimum over $(\mathcal{T}(E) \cup \mathcal{T}(F))^c$ yields $\text{dist}(\mathcal{T}(E), \mathcal{T}(F)) \geq R = \text{dist}(E, F)$, proving (2.3). $\square$

2.1.2. A family of diffusion operators. Throughout, $\phi'(\xi; \eta)$ denotes the one-sided directional derivative of ϕ at ξ in the direction η . Recall that it is sublinear in η , that $|\phi'(\xi; \eta)| \leq \phi(\eta)$, and that $\partial\phi(\xi)$ is exactly the set of z such that $z \cdot \eta \leq \phi'(\xi; \eta)$ for all η .

DEFINITION 2.6 (Approximating family). Let $\{\mathcal{T}_h\}_{h>0}$ be a family of diffusion operators as in Definition 2.1. We say that it is an *approximating family* if the following hold:

(a) (*Wulff estimate*). There exists $c_0 > 0$ such that

$$(2.4) \quad \mathcal{T}_h(\phi^\circ) \leq \sqrt{(\phi^\circ)^2 + c_0 h}$$

for all $h > 0$;

(b1) (*Stability*). There exists $C_0 > 0$ such that

$$(2.5) \quad \left| \left\langle \frac{\mathcal{T}_h g - g}{h}, \varphi \right\rangle \right| \leq C_0 \int \phi(\nabla \varphi)$$

for all $h > 0$, every $g \in \text{Lip}(\mathbb{R}^N)$ with $\phi(\nabla g) \leq 1$ and every $\varphi \in C_c^\infty(\mathbb{R}^N)$;

(b2) (*Consistency*). Let $h_n \rightarrow 0$. Let $E_n, E \subseteq \mathbb{R}^N$ be closed sets with $E_n, E \notin \{\emptyset, \mathbb{R}^N\}$ and such that $E_n \xrightarrow{K} E$. Then, for every $\varphi \in C_c^\infty(E^c)$,

$$(2.6) \quad \liminf_{n \rightarrow \infty} \left\langle \frac{\mathcal{T}_{h_n} d_{E_n} - d_{E_n}}{h_n}, \varphi \right\rangle \geq - \int \phi'(\nabla d_E; \nabla \varphi).$$

We prove some consequences of the properties of an approximating family $\{\mathcal{T}_h\}$.

PROPOSITION 2.7 (Shrinking of the Wulff shape). *If $R > 0$ and $h < \frac{R^2}{c_0}$, then*

$$\mathcal{T}_h(W_R) \supseteq W_{\sqrt{R^2 - c_0 h}}.$$

PROOF. We set $u = \mathcal{T}_h(\phi^\circ)$ and $v = \mathcal{T}_h d_{W_R}$. It is easy to show that $d_{W_R} = \phi^\circ - R$. By property (iii) of $\mathcal{T} = \mathcal{T}_h$, we have $v = u - R$. Property (a) of $\{\mathcal{T}_h\}$ is rewritten as

$$u \leq \sqrt{(\phi^\circ)^2 + c_0 h}.$$

Therefore, since $R^2 - c_0 h > 0$, we have

$$\mathcal{T}_h(W_R) = \{v \leq 0\} = \{u \leq R\} \supseteq \{(\phi^\circ)^2 + c_0 h \leq R^2\} = W_{\sqrt{R^2 - c_0 h}}. \quad \square$$

PROPOSITION 2.8 (Comparison with the Wulff shape). *Let $E \subseteq \mathbb{R}^N$ be closed and let $h > 0$. Then*

$$(2.7) \quad d_{\mathcal{T}_h(E)} \geq \sqrt{d_E^2 - c_0 h} \quad \text{in } \{d_E > \sqrt{c_0 h}\}.$$

PROOF. If $E \in \{\emptyset, \mathbb{R}^N\}$, then (2.7) holds trivially, since in that case $\mathcal{T}_h(E) = E$. Assume that $E \notin \{\emptyset, \mathbb{R}^N\}$. Let $x \in \mathbb{R}^N$ be such that $d_E(x) > \sqrt{c_0 h}$. We set $R = d_E(x)$, and let $\sqrt{c_0 h} < R' < R$. Then we have $\text{dist}(E, x + W_{R'}) \geq R - R' > 0$. Hence, by Proposition 2.5 and properties (iii) and (iv) of $\mathcal{T} = \mathcal{T}_h$, it follows that

$$\text{dist}(\mathcal{T}_h(E), x + \mathcal{T}_h(W_{R'})) = \text{dist}(\mathcal{T}_h(E), \mathcal{T}_h(x + W_{R'})) \geq \text{dist}(E, x + W_{R'}) \geq R - R'.$$

Since $h < \frac{(R')^2}{c_0}$, by Proposition 2.7, we have

$$\begin{aligned} \text{dist}(\mathcal{T}_h(E), x + W_{\sqrt{(R')^2 - c_0 h}}) &\geq \text{dist}(\mathcal{T}_h(E), x + \mathcal{T}_h(W_{R'})) \geq R - R', \\ d_{\mathcal{T}_h(E)}(x) &\geq R + \sqrt{(R')^2 - c_0 h} - R'. \end{aligned}$$

Therefore, letting $R' \nearrow R$, it follows

$$d_{\mathcal{T}_h(E)}(x) \geq \sqrt{R^2 - c_0 h} = \sqrt{d_E(x)^2 - c_0 h},$$

proving (2.7). $\square$

LEMMA 2.9 (Estimates on one step). *Let $h > 0$ and let $E \notin \{\emptyset, \mathbb{R}^N\}$ be a closed set. Then*

$$(2.8) \quad \|\mathcal{T}_h d_E - d_E\|_\infty \leq \sqrt{c_0 h},$$

$$(2.9) \quad \frac{\mathcal{T}_h d_E - d_E}{h} \leq \frac{c_0}{2d_E} \quad \text{in } E^c.$$

PROOF. Let $\tau \in \mathbb{R}^N$. Since the function d_E is 1-Lipschitz with respect to ϕ° , we have

$$d_E(\tau) - \phi^\circ(\cdot - \tau) \leq d_E \leq d_E(\tau) + \phi^\circ(\cdot - \tau).$$

By properties (i), (iii), (iv) and (v) of $\mathcal{T} = \mathcal{T}_h$, and by property (a) of $\mathcal{T}_h$, it follows that

$$d_E(\tau) - \sqrt{\phi^\circ(\cdot - \tau)^2 + c_0 h} \leq \mathcal{T}_h d_E \leq d_E(\tau) + \sqrt{\phi^\circ(\cdot - \tau)^2 + c_0 h}.$$

Testing it at $x = \tau$ yields $d_E(\tau) - \sqrt{c_0 h} \leq \mathcal{T}_h d_E(\tau) \leq d_E(\tau) + \sqrt{c_0 h}$, and by the arbitrariness of τ this proves (2.8). On the other hand, if $\tau \in E$, then $d_E \leq \phi^\circ(\cdot - \tau)$. By properties (i) and (v) of $\mathcal{T} = \mathcal{T}_h$, and by property (a) of $\mathcal{T}_h$, we have

$$\mathcal{T}_h d_E \leq \sqrt{\phi^\circ(\cdot - \tau)^2 + c_0 h}.$$

Taking the infimum over $\tau \in E$ yields $\mathcal{T}_h d_E \leq \sqrt{d_E^2 + c_0 h}$. Since $d_E \geq 0$ in E^c , there we have

$$\frac{\mathcal{T}_h d_E - d_E}{h} \leq \frac{\sqrt{d_E^2 + c_0 h} - d_E}{h} = \frac{c_0}{\sqrt{d_E^2 + c_0 h} + d_E} \leq \frac{c_0}{2d_E},$$

proving (2.9). $\square$

2.2. Construction and convergence of the scheme

In this section we construct an approximation scheme built on an approximating family of diffusion operators, as defined in the previous section. The main result of this section is the proof of convergence of the scheme to a superflow, up to a subsequence. We consider an approximating family $\{\mathcal{T}_h\}$.

DEFINITION 2.10 (Discrete evolution). Let $E_0 \subseteq \mathbb{R}^N$ be closed, and let $\{E_n\} = \{\mathcal{T}_h^n(E_0)\}$. We define the *discrete evolution* starting from E_0 with time-step h as the space-time set

$$\Lambda = \left\{ (x, t) \in \mathbb{R}^N \times [0, +\infty) : x \in E_{\lfloor \frac{t}{h} \rfloor} \right\}.$$

We also define the *auxiliary function* $u : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ as $u(x, t) = \mathcal{T}_h d_{E_{\lfloor \frac{t}{h} \rfloor}}(x)$.

In the rest of this section we fix a closed set $E_0 \subseteq \mathbb{R}^N$ and consider the discrete evolutions starting from it with time-step $h > 0$. They are denoted by Λ_h , while the spatial signed distances are denoted by $d_h(x, t) = d_{\Lambda_h(t)}(x)$. Finally, we denote by u_h the corresponding auxiliary functions. We also write

$$D_h(x, t) = d_h(x, t)^+ = \text{dist}(x, \Lambda_h(t)),$$

which is the object in terms of which the compactness theorem below is stated.

LEMMA 2.11 (Global distance estimate). *Let $h, s > 0, t \geq 0$ and $x \in \Lambda_h(t)^c$ such that $t < s < t + \frac{d_h(x,t)^2}{c_0} - h$. Then*

$$(2.10) \quad d_h(x, s) \geq \sqrt{d_h(x, t)^2 - c_0(s - t + h)}.$$

PROOF. Set $n = \lfloor \frac{s}{h} \rfloor, m = \lfloor \frac{t}{h} \rfloor$. Let $x \in \Lambda_h(t)^c = E_m^c$ and set $E_k = \mathcal{T}_h^k(E_0)$, $\delta_k = d_{E_k}(x)$ for $k \in \{m, \dots, n\}$. Note that we have $\delta_m^2 = d_h(x, t)^2 > c_0(s - t + h) \geq c_0h(n - m)$. We prove by induction that

$$\delta_k > 0, \quad \delta_k^2 \geq \delta_m^2 - c_0h(k - m)$$

for all $k \in \{m, \dots, n\}$, by applying iteratively Proposition 2.8. The base case $k = n$ is trivially true, so assume that it is true for some $k \in \{m, \dots, n - 1\}$. It holds

$$\delta_k^2 \geq \delta_m^2 - c_0h(k - m) \geq c_0h(n - k) \geq c_0h.$$

Since $\delta_k > 0$, it follows $d_{E_k}(x) = \delta_k > \sqrt{c_0h}$. By (2.7), we have $\delta_{k+1} > 0$ and

$$\delta_{k+1}^2 = d_{\mathcal{T}(E_k)}(x)^2 \geq d_{E_k}(x)^2 - c_0h = \delta_k^2 - c_0h \geq \delta_m^2 - c_0h(k + 1 - m),$$

concluding the induction. Testing it with $k = n$ proves (2.10). $\square$

COROLLARY 2.12 (Positive global estimate). *Let $h, s > 0, t \geq 0$ and $x \in \mathbb{R}^N$ be such that $D_h(x, t) > 0$ and $t < s < t + \frac{D_h(x,t)^2}{c_0} - h$. Then*

$$(2.11) \quad D_h(x, s) \geq \sqrt{D_h(x, t)^2 - c_0(s - t + h)}.$$

PROOF. Since $D_h(x, t) > 0$, we have $x \in \Lambda_h(t)^c$ and $D_h(x, t) = d_h(x, t)$, so that (2.10) applies and gives $d_h(x, s) \geq \sqrt{d_h(x, t)^2 - c_0(s - t + h)}$. The result follows by $D_h(x, s) = d_h(x, s)$, which is true since $d_h(x, s) > 0$. $\square$

2.2.1. A compactness theorem. Applying Corollary A.9, we have

$$\overline{\Lambda_{h_n}} \xrightarrow{K} \Lambda$$

for some $h_n \searrow 0$ and some closed set $\Lambda \subseteq \mathbb{R}^N \times [0, +\infty)$. From now on in this section we fix such a sequence and such a limit set. We define the *extinction time* by $T^* = \inf\{t \geq 0 : \Lambda(t) = \emptyset\}$, which is well posed by the following proposition.

PROPOSITION 2.13 (Extinction property). *If $t > T^*$, then $\Lambda(t) = \emptyset$.*

PROOF. If $T^* = +\infty$ there is nothing to prove, so assume that $T^* < +\infty$. By contradiction, assume that $\Lambda(t) \neq \emptyset$. By definition of T^* , there exists $T^* \leq s < t$ such that $\Lambda(s) = \emptyset$. Let $x \in \Lambda(t)$. By property (ii) of the Kuratowski convergence, there exist $x_n \rightarrow x$ and $t_n \rightarrow t$ such that $t_n > s$ and $x_n \in \Lambda_{h_n}(t_n)$. This also means that $\Lambda_{h_n}(s) \neq \emptyset$, since if it was empty, then also $\Lambda_{h_n}(t_n)$ would be empty. Now we set $M = \sqrt{c_0(t - s + 1)}$. By contradiction, assume that $d_{h_n}(x_n, s) > M$. Then, by Lemma 2.11, we have

$$d_{h_n}(x_n, t_n) \geq \sqrt{d_{h_n}(x_n, s)^2 - c_0(t_n - s + h_n)} > 0,$$

for n large enough, contradicting $x_n \in \Lambda_{h_n}(t_n)$. We proved that $d_{h_n}(x_n, s) \leq M$, so we choose $y_n \in \Lambda_{h_n}(s)$ such that $\phi^\circ(x_n - y_n) \leq M$. Since x_n is bounded, also y_n is bounded, and we have $y_n \rightarrow y$ up to a subsequence. By property (i) of Kuratowski convergence, it follows that $y \in \Lambda(s)$, contradicting $\Lambda(s) = \emptyset$. $\square$

THEOREM 2.14 (Compactness of the discrete evolutions). *There exists a countable set $\mathcal{N} \subset (0, +\infty)$ such that, up to a subsequence,*

$$D_{h_n}(\cdot, t) \rightarrow \text{dist}(\cdot, \Lambda(t)) \quad \text{locally uniformly}$$

for all $t \in (0, +\infty) \setminus \mathcal{N}$. Moreover, for every $x \in \mathbb{R}^N$, the function $\text{dist}(x, \Lambda(\cdot))$ is left-continuous, and $\{D_{h_n}\}$ is bounded on every compact subset of $\mathbb{R}^N \times (0, T^*)$, so that

$$(2.12) \quad D_{h_n} \rightarrow \text{dist}(\cdot, \Lambda(\cdot)) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N \times (0, T^*)).$$

PROOF. The proof is divided in six steps, and is based on a selection argument of Helly type.

Step 1. The first part is a diagonalization argument. Let $(0, +\infty) \cap \mathbb{Q} = \{q_n\}$. The functions $D_{h_n}(\cdot, q_0)$ are nonnegative and 1-Lipschitz with respect to ϕ° , so that, exactly as in the proof of Corollary A.9, up to a subsequence $\{h_n^1\} \subseteq \{h_n\}$ they converge locally uniformly either to a nonnegative 1-Lipschitz function $D(\cdot, q_0)$ or to $+\infty$.

We repeat the argument for every q_k for $k \geq 1$, each time starting from $\{h_n^{k-1}\}$ and extracting a subsequence $\{h_n^k\} \subseteq \{h_n^{k-1}\}$ such that $\{D_{h_n^k}(\cdot, q_k)\}$ converges locally uniformly to either a nonnegative 1-Lipschitz function or $+\infty$. As is standard in diagonalization arguments, one can easily prove that for all $t \in (0, +\infty) \cap \mathbb{Q}$, the sequence $\{D_{h_n^n}(\cdot, t)\}$ (well-defined for n large enough) converges locally uniformly to either a nonnegative 1-Lipschitz function or $+\infty$. Thus, for every such t , we set

$$D(x, t) = \lim_{n \rightarrow \infty} D_{h_n^n}(x, t).$$

By Proposition 2.13, we have $D(x, t) < +\infty$ for all $t \in (0, T^*) \cap \mathbb{Q}$ and $D(x, t) = +\infty$ for all $t \in (T^*, +\infty) \cap \mathbb{Q}$. Now we rename $\{h_n^n\}$ as $\{h_n\}$.

Step 2. Let $x \in \mathbb{R}^N$ and $t \in (0, +\infty) \cap \mathbb{Q}$, and suppose that $D(x, t) > 0$. Then $D_{h_n}(x, t) > 0$ for n large enough, so that, by Corollary 2.12, we have

$$(2.13) \quad D_{h_n}(x, s) \geq \sqrt{D_{h_n}(x, t)^2 - c_0(s - t + h_n)}$$

for $s \in \mathbb{Q}$ such that $s > t$ and $s - t + h_n < \frac{D_{h_n}(x, t)^2}{c_0}$. Let $s > 0$ be such that $0 < s - t < \frac{D(x, t)^2}{c_0}$. Then $s > t$ and $s - t + h_n < \frac{D_{h_n}(x, t)^2}{c_0}$ for n large enough, hence we can take the limit in (2.13) and obtain

$$(2.14) \quad D(x, s) \geq \sqrt{D(x, t)^2 - c_0(s - t)}.$$

Step 3. Let $x \in \mathbb{R}^N$ and let $t > 0$. Set $\delta = \limsup_{s \in \mathbb{Q}, s \searrow t} D(x, s) \geq 0$. If $\delta = 0$, then $\liminf_{s \in \mathbb{Q}, s \searrow t} D(x, s) \geq 0 = \delta$, since $D \geq 0$. If $\delta > 0$, let $\{t_n\} \subset \mathbb{Q}$ be such that $t_n \searrow t$ and $D(x, t_n) \rightarrow \delta$, and let $s \in \mathbb{Q}$ be such that $0 < s - t < \frac{\delta^2}{c_0}$. Since $0 < s - t_n < \frac{D(x, t_n)^2}{c_0}$ for n large enough, by (2.14) we have

$$D(x, s) \geq \sqrt{D(x, t_n)^2 - c_0(s - t_n)},$$

$$D(x, s) \geq \sqrt{\delta^2 - c_0(s - t)}.$$

In both cases

$$(2.15) \quad \liminf_{s \in \mathbb{Q}, s \searrow t} D(x, s) \geq \delta = \limsup_{s \in \mathbb{Q}, s \searrow t} D(x, s).$$

Analogously we can derive

$$(2.16) \quad \liminf_{s \in \mathbb{Q}, s \nearrow t} D(x, s) \geq \delta = \limsup_{s \in \mathbb{Q}, s \nearrow t} D(x, s).$$

By (2.15) and (2.16), we know that there exist right and left limits along the rationals, and we denote such limits as $D(x, t^+)$ and $D(x, t^-)$ respectively.

Step 4. Set $R = \limsup_{n \rightarrow \infty, s \in \mathbb{Q}, s \rightarrow t} D_{h_n}(x, s) \geq 0$. If $R = 0$, then $D(x, t^+) \geq 0 = R$. If $R > 0$, let $k_n \rightarrow \infty$ and $t_n \rightarrow t$ be such that $D_{h_{k_n}}(x, t_n) \rightarrow R$, and let $\tau \in \mathbb{Q}$ be such that $0 < \tau - t < \frac{R^2}{c_0}$. Since $0 < \tau - t_n + h_{k_n} < \frac{D_{h_{k_n}}(x, t_n)^2}{c_0}$ for n large enough, by Corollary 2.12 we have

$$D_{h_{k_n}}(x, \tau) \geq \sqrt{D_{h_{k_n}}(x, t_n)^2 - c_0(\tau - t_n + h_{k_n})},$$

$$D(x, \tau) \geq \sqrt{R^2 - c_0(\tau - t)}.$$

In both cases

$$(2.17) \quad D(x, t^+) \geq R = \limsup_{n \rightarrow \infty, s \in \mathbb{Q}, s \rightarrow t} D_{h_n}(x, s).$$

Suppose now that $D(x, t^-) > 0$, the case $D(x, t^-) = 0$ being trivial. Let $\tau \in \mathbb{Q}$ be such that $D(x, \tau) > 0$ and $0 < t - \tau < \frac{D(x, \tau)^2}{c_0}$. Let $k_n \rightarrow \infty$ and $t_n \rightarrow t$ be such that $D_{h_{k_n}}(x, t_n) \rightarrow \liminf_{n \rightarrow \infty, s \in \mathbb{Q}, s \rightarrow t} D_{h_n}(x, s)$. Since $D_{h_{k_n}}(x, \tau) > 0$ and $0 < t_n - \tau + h_{k_n} < \frac{D_{h_{k_n}}(x, \tau)^2}{c_0}$ for n large enough, again by Corollary 2.12 we have

$$D_{h_{k_n}}(x, t_n) \geq \sqrt{D_{h_{k_n}}(x, \tau)^2 - c_0(t_n - \tau + h_{k_n})},$$

$$\liminf_{n \rightarrow \infty, s \in \mathbb{Q}, s \rightarrow t} D_{h_n}(x, s) \geq \sqrt{D(x, \tau)^2 - c_0(t - \tau)},$$

$$(2.18) \quad \liminf_{n \rightarrow \infty, s \in \mathbb{Q}, s \rightarrow t} D_{h_n}(x, s) \geq D(x, t^-).$$

By (2.17) and (2.18), it follows that

$$(2.19) \quad D(x, t^+) \geq \limsup_{n \rightarrow \infty} D_{h_n}(x, t) \geq \liminf_{n \rightarrow \infty} D_{h_n}(x, t) \geq D(x, t^-).$$

Step 5. Setting $\mathcal{N} = \{t > 0 : D(x, t^+) \neq D(x, t^-) \text{ for some } x \in \mathbb{R}^N\}$, we want to prove that it is countable. Firstly, we show that $t \in \mathcal{N}$ iff there exists $x \in \mathbb{Q}^N$ such that $D(x, t^+) \neq D(x, t^-)$. Assume that $D(q, t^+) = D(q, t^-)$ for all $q \in \mathbb{Q}^N$. Let $x \in \mathbb{R}^N$, $\epsilon > 0$ and $q \in \mathbb{Q}^N$ such that $\phi^\circ(x - q) < \frac{\epsilon}{2}$. Let $\{t_n\}, \{s_n\} \subset \mathbb{Q}$ be such that $t_n \searrow t$ and $s_n \nearrow t$. Since D is 1-Lipschitz with respect to ϕ° , we have

$$\begin{aligned} |D(x, t_n) - D(x, s_n)| &\leq |D(q, t_n) - D(q, s_n)| + |D(x, t_n) - D(q, t_n)| + |D(x, s_n) - D(q, s_n)| \\ &\leq |D(q, t_n) - D(q, s_n)| + 2\phi^\circ(x - q) \\ &\leq |D(q, t_n) - D(q, s_n)| + \epsilon. \end{aligned}$$

Taking the limit yields

$$|D(x, t^+) - D(x, t^-)| \leq \epsilon,$$

and by the arbitrariness of ϵ the claim follows. By (2.19), if $D(x, t^+) \neq D(x, t^-)$, then $D(x, t^+) > D(x, t^-)$. Hence we can write

$$\mathcal{N} = \bigcup_{k \in \mathbb{N}} \bigcup_{x \in \mathbb{Q}^N} \left\{ t > 0 : D(x, t^+) - D(x, t^-) > \frac{1}{k} \right\}.$$

Now we want to prove that for any fixed $k \in \mathbb{N}$ and $x \in \mathbb{Q}^N$ the set in the union is discrete. We write $f(t) = D(x, t)$ and $f(t^\pm) = D(x, t^\pm)$. By contradiction, assume that there exists $t_n \searrow t$ (the other case is analogous) such that $t_n \neq t$ and $f(t_n^+) - f(t_n^-) > \frac{1}{k}$. Let $\delta > 0$ such that $|f(s) - f(t^+)| \leq \frac{1}{4k}$ for $t < s < t + \delta$. Now take n large enough such that $t_n < t + \delta$ and $t < \tau < t_n < s < t + \delta$ such that $|f(t_n^+) - f(s)| \leq \frac{1}{4k}$ and $|f(t_n^-) - f(\tau)| \leq \frac{1}{4k}$. Then

$$|f(t_n^+) - f(t_n^-)| \leq |f(s) - f(t^+)| + |f(\tau) - f(t^+)| + |f(t_n^+) - f(s)| + |f(t_n^-) - f(\tau)| \leq \frac{1}{k}.$$

Since $\mathcal{N}$ is a countable union of discrete sets, it is countable. If $t \in (0, +\infty) \setminus \mathcal{N}$, we set $\tilde{D}(x, t) = D(x, t^+) = D(x, t^-)$. For such times, by (2.19), $\tilde{D}(x, t) = \lim_n D_{h_n}(x, t)$, so we can rename $\tilde{D}$ by D (they agree where the original D was defined). Since $D_{h_n}(\cdot, t)$ are equi-Lipschitz, the convergence is locally uniform. Proving that $D(\cdot, t) = \text{dist}(\cdot, \Lambda(t))$ is equivalent to showing that $E = \Lambda(t)$ is the Kuratowski limit E^* of $\overline{\Lambda_{h_n}(t)}$. Both this and the left-continuity of $\text{dist}(x, \Lambda(\cdot))$ can be proved exactly as in the proof of [14], Proposition 4.4.

Step 6. We prove (2.12). Let $K \subset \mathbb{R}^N \times (0, T^*)$ be compact and let $R > 0$ and $0 < a < b < T^*$ be such that $K \subseteq \overline{B_R} \times [a, b]$. We fix $s \in (b, T^*)$, so that $\Lambda(s) \neq \emptyset$ by the definition of T^* , and we choose $y \in \Lambda(s)$. By property (ii) of the Kuratowski convergence, there exist $(y_n, s_n) \in \overline{\Lambda_{h_n}}$ such that $(y_n, s_n) \rightarrow (y, s)$; in particular $\{y_n\}$ is bounded and $s_n > b$ for n large enough. Let $x \in \overline{B_R}$ and $t \in [a, b]$. Since $y_n \in \overline{\Lambda_{h_n}(s_n)}$, we have

$$D_{h_n}(x, s_n) \leq \phi^\circ(x - y_n).$$

If $D_{h_n}(x, t) = 0$ there is nothing to prove. Otherwise, either $s_n \geq t + \frac{D_{h_n}(x, t)^2}{c_0} - h_n$, and then $D_{h_n}(x, t)^2 \leq c_0(s_n - t + h_n)$, or $t < s_n < t + \frac{D_{h_n}(x, t)^2}{c_0} - h_n$, and then Corollary 2.12, applied between the times t and s_n , gives

$$D_{h_n}(x, t)^2 \leq D_{h_n}(x, s_n)^2 + c_0(s_n - t + h_n).$$

In both cases

$$D_{h_n}(x, t)^2 \leq \sup_n \phi^\circ(x - y_n)^2 + c_0(s_n - a + h_n) \leq C$$

for n large enough, with C depending only on K , on c_0 and on the sequence $\{(y_n, s_n)\}$. Hence $\{D_{h_n}\}$ is bounded on K . Since $D_{h_n} \rightarrow \text{dist}(\cdot, \Lambda(\cdot))$ for every $x \in \mathbb{R}^N$ and every $t \notin \mathcal{N}$, hence a.e. in $\mathbb{R}^N \times (0, T^*)$, the dominated convergence theorem gives (2.12). $\square$

2.2.2. Convergence up to a subsequence. From now on, in this section, we fix a subsequence h_n and a countable set $\mathcal{N}$ as in Theorem 2.14, and we write $d(x, t) = d_{\Lambda(t)}(x)$ for the signed distance to the limit slices, so that $d(\cdot, t) = \text{dist}(\cdot, \Lambda(t)) = \lim_n D_{h_n}(\cdot, t)$ outside $\Lambda(t)$ for every $t \notin \mathcal{N}$.

Throughout this subsection we write

$$S_n = \frac{\mathcal{T}_{h_n} d_{h_n} - d_{h_n}}{h_n}, \quad U = \mathbb{R}^N \times (0, T^*) \setminus \Lambda,$$

the latter being an open subset of $\mathbb{R}^N \times \mathbb{R}$, and for $t > 0$ we denote by $U(t) = \{x \in \mathbb{R}^N : (x, t) \in U\}$ its time-slices, which are open.

The following lemma is the point where the scheme is passed to the limit in time. Its statement is the space-time property that the whole construction needs, and its proof consists in slicing, in an application of Fatou's lemma, and in a single application of the representation lemma. No selection of vector fields, measurable or not, is ever performed.

LEMMA 2.15 (Space-time upgrade). *There exists $z \in L^\infty(U; \mathbb{R}^N)$ with $z \in \partial\phi(\nabla d)$ such that*

$$(2.20) \quad S_n \longrightarrow \operatorname{div} z \quad \text{in } \mathcal{D}'(U),$$

up to a subsequence.

PROOF. The proof is divided in three steps.

Step 1. Let $t \in (0, T^*)$ be outside the countable exceptional set of Theorem 2.14, so that

$$D_{h_n}(\cdot, t) \longrightarrow \operatorname{dist}(\cdot, \Lambda(t)) \quad \text{locally uniformly.}$$

Since $t < T^*$, the set $\Lambda(t)$ is nonempty, so that $\operatorname{dist}(\cdot, \Lambda(t)) = d_{\Lambda(t)} = d(\cdot, t)$ on $U(t) = \Lambda(t)^c$, and there $d(\cdot, t) > 0$ and $\phi(\nabla d(\cdot, t)) = 1$ by Proposition A.5. We claim that the hypotheses of (b2) are satisfied with $E_n = \Lambda_{h_n}(t)$ and $E = \Lambda(t)$. Indeed, let $K \subset U(t)$ be compact and let $\delta = \min_K d(\cdot, t) > 0$. For n large enough we have $d_{h_n}(\cdot, t)^+ \geq \frac{\delta}{2}$ on K , hence $d_{h_n}(\cdot, t) = d_{h_n}(\cdot, t)^+$ on K and

$$d_{h_n}(\cdot, t) \longrightarrow d(\cdot, t) \quad \text{uniformly on } K.$$

Therefore, by (b2),

$$(2.21) \quad \liminf_{n \rightarrow \infty} \int S_n(x, t) \varphi(x) dx \geq - \int \phi'(\nabla d(x, t); \nabla \varphi(x)) dx$$

for all $\varphi \in C_c^\infty(U(t))$, while by (b1),

$$(2.22) \quad \left| \int S_n(x, t) \varphi(x) dx \right| \leq C_0 \int \phi(\nabla \varphi(x)) dx.$$

Step 2. Let $\varphi \in C_c^\infty(U)$ and set

$$a_n(t) = \int S_n(x, t) \varphi(x, t) dx, \quad b(t) = C_0 \int \phi(\nabla \varphi(x, t)) dx,$$

so that $\langle S_n, \varphi \rangle = \int_0^{T^*} a_n(t) dt$. Since $\operatorname{supp} \varphi$ is a compact subset of U , for every t the function $\varphi(\cdot, t)$ belongs to $C_c^\infty(U(t))$, and b is bounded with compact support in $(0, T^*)$, hence integrable. By (2.22) we have $a_n + b \geq 0$, so Fatou's lemma applies and gives, using (2.21) for almost every $t \in (0, T^*)$,

$$(2.23) \quad \liminf_{n \rightarrow \infty} \langle S_n, \varphi \rangle \geq \int_0^{T^*} \liminf_{n \rightarrow \infty} a_n(t) dt \geq - \iint_U \phi'(\nabla d; \nabla \varphi) dx dt.$$

Integrating (2.22) in time we also get

$$(2.24) \quad |\langle S_n, \varphi \rangle| \leq C_0 \iint_U \phi(\nabla \varphi) dx dt.$$

Step 3. By Corollary A.11, we have $S_n = \operatorname{div} z_n$ for some $z_n \in L^\infty(U; \mathbb{R}^N)$ with $\phi^\circ(z_n) \leq C$. Then $z_n \xrightarrow{*} \tilde{z}$ up to a subsequence to some $\tilde{z} \in L^\infty(U; \mathbb{R}^N)$. This implies that $S_n \rightarrow \operatorname{div} \tilde{z} = S$ in $\mathcal{D}'(U)$ and

$$\langle S, \varphi \rangle \geq - \iint_U \phi'(\nabla d; \nabla \varphi) dx dt$$

for all $\varphi \in C_c^\infty(U)$, so that by Lemma A.10 applied with $\Phi = \phi$, $\xi = \nabla d$, we have $S = \operatorname{div} z$ with $z \in \partial\phi(\nabla d)$. $\square$

THEOREM 2.16 (Convergence of the abstract scheme). *Λ is a superflow with initial condition E_0 in the sense of Definition 1.8.*

PROOF. We verify the four properties of Definition 1.8.

Property (a). We must show that $\Lambda(0) \subseteq E_0$. Let $x \notin E_0$ and set $R = \operatorname{dist}(x, E_0) > 0$, which is positive because E_0 is closed. Since $\Lambda_h(0) = E_0$ for all h , we have $d_h(x, 0) = R$. Let $\epsilon > 0$ and $\rho > 0$ be such that

$$\epsilon + h_n < \frac{R^2}{c_0} \quad \text{and} \quad \rho < \sqrt{R^2 - c_0(\epsilon + h_n)}$$

for n large enough. By (2.10), for all $t \in (0, \epsilon)$, we have

$$d_{h_n}(x, t) \geq \sqrt{R^2 - c_0(t + h_n)} > \rho,$$

and since $d_{h_n}(\cdot, t)$ is 1-Lipschitz with respect to ϕ° it follows that $d_{h_n}(y, t) > 0$ whenever $\phi^\circ(y - x) < \rho$. Hence

$$(x + \{\phi^\circ < \rho\}) \times [0, \epsilon) \cap \Lambda_{h_n} = \emptyset$$

for n large enough, and by property (i) of the Kuratowski convergence $(x, 0) \notin \Lambda$, that is $x \notin \Lambda(0)$.

Property (b). This is exactly Proposition 2.13: $\Lambda(t) = \emptyset$ for all $t > T^*$.

Property (c). This is the second assertion of Theorem 2.14: for every $x \in \mathbb{R}^N$ the function $t \mapsto \text{dist}(x, \Lambda(t))$ is left-continuous.

Property (d). Let $\varphi \in C_c^\infty(\mathbb{R}^N \times (0, T^*) \setminus \Lambda)$ with $\varphi \geq 0$, and let $K = \text{supp } \varphi$ and let K' be a compact neighbourhood of K contained in the open set $\mathbb{R}^N \times (0, T^*) \setminus \Lambda$. Since $\text{dist}(\cdot, \Lambda) > 0$ on K' and $\text{dist}(\cdot, \overline{\Lambda_{h_n}}) \rightarrow \text{dist}(\cdot, \Lambda)$ locally uniformly by Proposition A.8, for n large enough $K' \cap \overline{\Lambda_{h_n}} = \emptyset$, so that on K' we have $d_{h_n} = D_{h_n} > 0$; in particular $x \notin \Lambda_{h_n}(t + h_n)$ for all $(x, t) \in K$ and n large enough, so that the redistancing inequality (2.2), applied with $E = \Lambda_{h_n}(t)$ and hence $\mathcal{T}_{h_n}(E) = \Lambda_{h_n}(t + h_n)$, gives

$$u_{h_n}(x, t) = \mathcal{T}_{h_n} d_{h_n}(x, t) \leq d_{h_n}(x, t + h_n) \quad \text{on } K.$$

Therefore

$$\frac{d_{h_n}(x, t + h_n) - d_{h_n}(x, t)}{h_n} \geq \frac{\mathcal{T}_{h_n} d_{h_n}(x, t) - d_{h_n}(x, t)}{h_n} \quad \text{on } K,$$

and multiplying by φ and integrating,

$$\iint \frac{d_{h_n}(x, t + h_n) - d_{h_n}(x, t)}{h_n} \varphi(x, t) dx dt \geq \iint \frac{\mathcal{T}_{h_n} d_{h_n}(x, t) - d_{h_n}(x, t)}{h_n} \varphi(x, t) dx dt.$$

A discrete integration by parts in time moves the increment onto the test function, so that the LHS equals

$$- \iint d_{h_n}(x, t) \frac{\varphi(x, t) - \varphi(x, t - h_n)}{h_n} dx dt \longrightarrow - \iint d\dot{\varphi} = \langle \dot{d}, \varphi \rangle,$$

where we used that $\frac{\varphi(\cdot, \cdot) - \varphi(\cdot, \cdot - h_n)}{h_n} \rightarrow \dot{\varphi}$ uniformly and that $d_{h_n} = D_{h_n} \rightarrow d$ in $L^1(K')$ by (2.12). The RHS is $\langle S_n, \varphi \rangle$, which by Lemma 2.15 converges, along the subsequence fixed there, to $\langle \text{div } z, \varphi \rangle$ for some $z \in L^\infty(U; \mathbb{R}^N)$ with $z \in \partial\phi(\nabla d)$. We conclude that

$$\langle \dot{d}, \varphi \rangle \geq \langle \text{div } z, \varphi \rangle$$

for all nonnegative $\varphi \in C_c^\infty(\mathbb{R}^N \times (0, T^*) \setminus \Lambda)$, which is (1.11). $\square$

COROLLARY 2.17 (Generic uniqueness). *Let $g : \mathbb{R}^N \rightarrow \mathbb{R}$ be bounded and uniformly continuous. Then there exists a countable set $\mathcal{N} \subset \mathbb{R}$ such that, for every $s \in \mathbb{R} \setminus \mathcal{N}$, the approximation scheme starting from $\{g \leq s\}$ converges to the unique flow in the sense of Definition 1.8.*

PROOF. We note that, for every $E_0 \subseteq \mathbb{R}^N$ closed, we can extract by the approximation scheme a closed Λ and an open Ω with $\Omega \subset \Lambda$, such that Λ is a superflow with initial condition E_0 and Ω^c is a superflow with initial condition $\overline{E_0^c}$, that is Ω is a subflow with initial condition $\dot{E}_0$. Indeed, once we produced Λ from E_0 , we can extract a further subsequence to produce Ω^c from $\overline{E_0^c}$, and the ordering is preserved by Kuratowski convergence. The rest of the proof is identical to [14], Corollary 4.6. $\square$

CHAPTER 3

Applications

In this chapter we present three concrete diffusion/redistancing schemes and we prove that each of them is a particular case of the abstract scheme of Chapter 2, so that Theorem 2.16 applies to it. The first two, an *elliptic* and a *parabolic* scheme, are built on the anisotropic p -Laplacian and work for an arbitrary norm ϕ . The third is built on the *convolution* with a kernel and is restricted to the anisotropies induced by a scalar product, but it is completely explicit. As in the previous chapter, we fix a norm ϕ . All the diffusion operators defined in this chapter are understood to be extended to satisfy property (vi) of Definition 2.1.

3.1. An elliptic scheme

Fix $1 < p < \infty$. Using the notation of the previous chapter, we define the scheme as $\mathcal{T}_h g = u$, where u is the unique solution to

$$(3.1) \quad -h\Delta_p^\phi u + u \ni g \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Here $\Delta_p^\phi u$ is the set-valued *anisotropic p -Laplacian*, that is the set of distributions of the form $\operatorname{div}(\phi^{p-1}(\nabla u)z)$ for some $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that $z \in \partial\phi(\nabla u)$. When ϕ is differentiable outside the origin, we have

$$\Delta_p^\phi u = \operatorname{div}(\phi^{p-1}(\nabla u)\nabla\phi(\nabla u)),$$

where we are denoting a singleton by the element itself with an abuse of notation.

The following results are necessary to prove that $\mathcal{T}_h$ is well-defined and satisfies the axioms (i) to (vi). Finally, we prove the Wulff shape estimate (2.4) and the stability and consistency properties.

REMARK 3.1 (Subdifferential of Φ). Setting $\Phi = \frac{1}{p}\phi^p$, it is useful to observe that $\tilde{z} \in \partial\Phi(\nabla u)$ iff $\tilde{z} = \phi^{p-1}(\nabla u)z$ and $z \in \partial\phi(\nabla u)$.

3.1.1. A comparison principle.

PROPOSITION 3.1 (Elliptic comparison principle). *Let $u_1, u_2 \in \operatorname{Lip}(\mathbb{R}^N)$ and $z_1, z_2 \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ be such that $z_1 \in \partial\phi(\nabla u_1)$ and $z_2 \in \partial\phi(\nabla u_2)$. If*

$$(3.2) \quad -h \operatorname{div}(\phi(\nabla u_1)^{p-1}z_1) + u_1 \leq -h \operatorname{div}(\phi(\nabla u_2)^{p-1}z_2) + u_2 \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

then $u_1 \leq u_2$.

PROOF. The function $\frac{1}{p}\phi^p$ is convex, since ϕ is convex and $t \mapsto t^p$ is convex and monotone. By the monotonicity of the subdifferential of convex functions, for all $x, y \in \mathbb{R}^N$ and all $\xi \in \partial\phi(x)$, $\eta \in \partial\phi(y)$,

$$(3.3) \quad (\phi^{p-1}(x)\xi - \phi^{p-1}(y)\eta) \cdot (x - y) \geq 0.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $\varphi = 1$ on B_1 , $0 \leq \varphi \leq 1$ and $\varphi = 0$ outside B_2 . For $R > 0$, we define

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right).$$

Let $\gamma > 2$ and $w_R = ((u_1 - u_2) \vee 0)^{\gamma-1} \varphi_R^\gamma$. Testing (3.2) with w_R by density,

$$(3.4) \quad h \int (\phi^{p-1}(\nabla u_1)z_1 - \phi^{p-1}(\nabla u_2)z_2) \cdot \nabla w_R + \int (u_1 - u_2)w_R \leq 0.$$

We compute

$$(3.5) \quad \nabla w_R = (\gamma - 1)((u_1 - u_2) \vee 0)^{\gamma-2} \varphi_R^\gamma \nabla(u_1 - u_2) + \gamma((u_1 - u_2) \vee 0)^{\gamma-1} \varphi_R^{\gamma-1} \nabla \varphi_R.$$

With $E = \{u_1 - u_2 > 0\}$, by (3.4) and (3.5),

$$\begin{aligned} & h(\gamma - 1) \int_E |u_1 - u_2|^{\gamma-2} \varphi_R^\gamma \left(\phi^{p-1}(\nabla u_1)_{z_1} - \phi^{p-1}(\nabla u_2)_{z_2} \right) \cdot (\nabla u_1 - \nabla u_2) \\ & + h\gamma \int_E |u_1 - u_2|^{\gamma-1} \varphi_R^{\gamma-1} \left(\phi^{p-1}(\nabla u_1)_{z_1} - \phi^{p-1}(\nabla u_2)_{z_2} \right) \cdot \nabla \varphi_R \\ & + \int_E |u_1 - u_2|^\gamma \varphi_R^\gamma \leq 0. \end{aligned}$$

By (3.3) applied to the first term,

$$h\gamma \int_E |u_1 - u_2|^{\gamma-1} \varphi_R^{\gamma-1} \left(\phi^{p-1}(\nabla u_1)_{z_1} - \phi^{p-1}(\nabla u_2)_{z_2} \right) \cdot \nabla \varphi_R + \int_E |u_1 - u_2|^\gamma \varphi_R^\gamma \leq 0.$$

Let $v = (u_1 - u_2) \vee 0$ and $C = 2h\gamma (\|\phi^{p-1}(\nabla u_1)_{z_1}\|_\infty \vee \|\phi^{p-1}(\nabla u_2)_{z_2}\|_\infty)$. By Hölder's inequality,

$$\begin{aligned} \|v\varphi_R\|_\gamma^\gamma & \leq C \left\| (v\varphi_R)^{\gamma-1} \nabla \varphi_R \right\|_1 \leq C \left\| (v\varphi_R)^{\gamma-1} \right\|_{\gamma'} \|\nabla \varphi_R\|_\gamma = C \|v\varphi_R\|_\gamma^{\gamma-1} \|\nabla \varphi_R\|_\gamma, \\ \|v\varphi_R\|_\gamma & \leq C \|\nabla \varphi_R\|_\gamma, \\ \|\nabla \varphi_R\|_\gamma^\gamma & = \frac{1}{R^\gamma} \int \left| \nabla \varphi \left(\frac{x}{R} \right) \right|^\gamma dx \leq \|\nabla \varphi\|_\infty^\gamma 2^N \omega_N R^{N-\gamma}. \end{aligned}$$

If we choose $\gamma > N$, then

$$\|v\|_\gamma = \lim_{R \rightarrow \infty} \|v\varphi_R\|_\gamma = 0.$$

Therefore $v = 0$ and $u_1 \leq u_2$, concluding the proof of the proposition. $\square$

3.1.2. Existence, uniqueness and other properties.

PROPOSITION 3.2 (Elliptic problem). *Let $1 < p < \infty$, $h > 0$ and $g \in \text{Lip}(\mathbb{R}^N)$. Then there exists a unique solution $u = \mathcal{T}g = \mathcal{T}_h g \in \text{Lip}(\mathbb{R}^N)$ to (3.1). Moreover:*

- (i) *If $g_1 \leq g_2$, then $\mathcal{T}(g_1) \leq \mathcal{T}(g_2)$;*
- (ii) *$\mathcal{T}(g + c) = \mathcal{T}(g) + c$ for all $c \in \mathbb{R}$;*
- (iii) *$\mathcal{T}(-g) = -\mathcal{T}(g)$;*
- (iv) *$\mathcal{T}(g(\cdot + \tau)) = \mathcal{T}(g)(\cdot + \tau)$ for all $\tau \in \mathbb{R}^N$;*
- (v) *$\mathcal{T}_h(g)(x) = \sqrt{h} \mathcal{T}_1(\tilde{g})(\frac{x}{\sqrt{h}})$, where $\tilde{g}(x) = \frac{1}{\sqrt{h}} g(\sqrt{h}x)$.*

In particular, $\mathcal{T}_h$ satisfies properties (i) to (v) of Definition 2.1.

PROOF. We start by noting that, once existence is proved, property (i) (and thus uniqueness) is a direct consequence of Proposition 3.1, while properties (ii) to (v) follow by simple computations. The proof is divided in five steps.

Step 1. We start by proving existence in a different class of functions, where the problem can be reformulated as a minimization problem. We define the spaces

$$\begin{aligned} V_0 &= \left\{ v \in L^2(\mathbb{R}^N) : \nabla v \in L^p(\mathbb{R}^N; \mathbb{R}^N) \right\}, \\ W_0 &= \text{Lip}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \end{aligned}$$

together with $V = V_0 + \mathbb{R}$ and $W = W_0 + \mathbb{R}$. We want to prove that if $g \in W$, then there exists a unique solution $u \in V$ to (3.1). With $g \in W$, we define the functional $F = F_g : V \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$F(v) = \frac{1}{p} \|\phi(\nabla v)\|_p^p + \frac{1}{2h} \|v - g\|_2^2.$$

If u is a minimizer of F , then u is a solution to (3.1). Indeed, set $\Phi = \frac{1}{p} \phi^p$, again a convex function. If u is a minimizer of F , then, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} 0 & \leq \frac{d}{d\epsilon} \Big|_{\epsilon=0} F(u + \epsilon\varphi) = \int \Phi'(\nabla u; \nabla \varphi) + \frac{1}{h} \int (u - g)\varphi, \\ & \frac{1}{h} \int (u - g)\varphi \geq - \int \Phi'(\nabla u; \nabla \varphi). \end{aligned}$$

By Lemma A.10, it follows that

$$\frac{1}{h}(u - g) = \text{div } \zeta \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

for some $\zeta \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that $\zeta \in \partial\Phi(\nabla u)$, but this is exactly (3.1) by Remark 3.1. Since F is proper, strictly convex, strongly lower semicontinuous and weakly coercive with the natural topology of V , it admits a unique minimizer that we call $\mathcal{T}g$.

Step 2. We prove a comparison principle for the solutions found in Step 1, that is if $g_1 \leq g_2$ with $g_1, g_2 \in W$, then $\mathcal{T}g_1 \leq \mathcal{T}g_2$. Let $u_1 = \mathcal{T}(g_1)$, $u_2 = \mathcal{T}(g_2)$ and $F_1 = F_{g_1}$, $F_2 = F_{g_2}$. We have

$$F_1(u_1) \leq F_1(u_1 \wedge u_2),$$

$$F_2(u_2) \leq F_2(u_1 \vee u_2).$$

With $E = \{u_1 > u_2\}$, we also have

$$\begin{aligned} \|\phi(\nabla u_1)\|_p^p + \|\phi(\nabla u_2)\|_p^p &= \int \phi(\nabla u_1)^p + \int \phi(\nabla u_2)^p \\ &= \int_E \phi(\nabla(u_1 \vee u_2))^p + \int_{\mathbb{R}^N \setminus E} \phi(\nabla(u_1 \wedge u_2))^p \\ &\quad + \int_E \phi(\nabla(u_1 \wedge u_2))^p + \int_{\mathbb{R}^N \setminus E} \phi(\nabla(u_1 \vee u_2))^p \\ &= \|\phi(\nabla(u_1 \wedge u_2))\|_p^p + \|\phi(\nabla(u_1 \vee u_2))\|_p^p. \end{aligned}$$

Combining them, we get

$$F_1(u_1) + F_2(u_2) \leq F_1(u_1 \wedge u_2) + F_2(u_1 \vee u_2),$$

so that

$$\int_E |u_1 - g_1|^2 + \int_E |u_2 - g_2|^2 \leq \int_E |u_2 - g_1|^2 + \int_E |u_1 - g_2|^2.$$

Let $g_1 = f_1 + c_1$ and $g_2 = f_2 + c_2$, with $f_1, f_2 \in V_0$ and $c_1, c_2 \in \mathbb{R}$. Then $f_1 - f_2 \leq c_2 - c_1 = c$. If $c < 0$, then $f_1 - f_2 \leq c < 0$ and $f_1 - f_2 \notin V_0$, so it must be $c \geq 0$. If $c = 0$, then $F_1(u_1 \wedge u_2) < +\infty$ and $F_2(u_1 \vee u_2) < +\infty$. If $c > 0$, then $u_1 - u_2 + c > c$ on E . Then, by Chebyshev's inequality,

$$|E| \leq \frac{1}{c^2} \|u_1 - u_2 + c\|_2^2 \leq \frac{\|u_1 - c_1\|_2^2 + \|u_2 - c_2\|_2^2}{c^2} < +\infty.$$

In both cases,

$$\int_E u_2 g_1 + \int_E u_1 g_2 \leq \int_E u_1 g_1 + \int_E u_2 g_2,$$

so that

$$\int_E (u_2 - u_1)(g_1 - g_2) \leq 0.$$

On the other hand, since $u_2 - u_1 < 0$ on E and $g_1 - g_2 \leq 0$, it must be

$$\int_E (u_2 - u_1)(g_1 - g_2) = 0.$$

It follows that

$$F_1(u_1) + F_2(u_2) = F_1(u_1 \wedge u_2) + F_2(u_1 \vee u_2).$$

Since the minimizer is unique, we have $u_1 = u_1 \wedge u_2$ and $u_1 \leq u_2$.

Step 3. We prove that if $g \in W_0$, then $u = \mathcal{T}g \in \text{Lip}(\mathbb{R}^N)$ and $\|\nabla u\|_\infty \leq \|\nabla g\|_\infty$. Let $\tau \in \mathbb{R}^N$ and $L = \|\nabla g\|_\infty$. We have

$$g - L|\tau| \leq g(\cdot + \tau) \leq g + L|\tau|.$$

By the comparison principle,

$$u - L|\tau| \leq \mathcal{T}(g(\cdot + \tau)) = u(\cdot + \tau) \leq u + L|\tau|,$$

and we conclude by the arbitrariness of τ .

Step 4. We claim that we can approximate every $g \in \text{Lip}(\mathbb{R}^N)$ with a sequence $g_n \in W_0$, in the sense that $g_n \rightarrow g$ locally uniformly. Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $\varphi = 1$ on B_1 , $0 \leq \varphi \leq 1$ and $\varphi = 0$ outside B_2 . We define

$$\varphi_n(x) = \varphi\left(\frac{x}{n}\right) \quad g_n = g\varphi_n.$$

We have that g_n are equi-Lipschitz, since

$$\begin{aligned}\|\nabla g_n\|_\infty &= \|\nabla(g\varphi_n)\|_\infty \\ &\leq \|g\|_{L^\infty(B_{2n})}\|\nabla\varphi_n\|_\infty + \|\nabla g\|_\infty\|\varphi_n\|_\infty \\ &\leq |g(0)|\frac{1}{n}\|\nabla\varphi\|_\infty + 2n\|\nabla g\|_\infty\frac{1}{n}\|\nabla\varphi\|_\infty + \|\nabla g\|_\infty \\ &\leq |g(0)|\|\nabla\varphi\|_\infty + 2\|\nabla g\|_\infty\|\nabla\varphi\|_\infty + \|\nabla g\|_\infty.\end{aligned}$$

We write $\|\nabla g_n\|_\infty \leq L$. Since

$$\|g_n\|_2 = \|g\varphi_n\|_2 \leq \|g\|_{L^2(B_{2n})} \leq \|g\|_{L^\infty(B_{2n})}|B_{2n}|^{1/2} < +\infty,$$

then $g_n \in W_0$. Moreover, for all $K \subset \mathbb{R}^N$ compact and n large enough,

$$\|g - g_n\|_{L^\infty(K)} = 0,$$

hence $g_n \rightarrow g$ locally uniformly. We set $u_n = \mathcal{T}g_n$, so that

$$(3.6) \quad -h \operatorname{div} \zeta_n + u_n = g_n \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

$$\zeta_n = \phi^{p-1}(\nabla u_n)z_n, \quad z_n \in \partial\phi(\nabla u_n).$$

We have $\|\nabla u_n\|_\infty \leq \|\nabla g_n\|_\infty \leq L$, so that $L' = \sup_n \|\zeta_n\|_\infty$ is finite, since $\phi^\circ(z_n) \leq 1$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$ and $\int \varphi = 1$. Testing (3.6) with φ ,

$$\left| \int u_n \varphi - \int g_n \varphi \right| \leq h \left| \int \phi^{p-1}(\nabla u_n)z_n \cdot \nabla \varphi \right| \leq hL' \|\nabla \varphi\|_1,$$

so the averages $\int u_n \varphi$ are equibounded. Since each u_n is L -Lipschitz, we get

$$|u_n(0)| \leq \left| \int u_n \varphi \right| + L \int |x| \varphi(x) dx,$$

so also $u_n(0)$ are equibounded, and u_n are locally equibounded since they are equi-Lipschitz. Then, by standard compactness theorems, there exist $u \in \operatorname{Lip}(\mathbb{R}^N)$ and $\zeta \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{aligned}u_n &\rightarrow u \quad \text{locally uniformly,} \\ \nabla u_n &\xrightarrow{*} \nabla u \quad \text{in } L^\infty(\mathbb{R}^N; \mathbb{R}^N), \\ \zeta_n &\xrightarrow{*} \zeta \quad \text{in } L^\infty(\mathbb{R}^N; \mathbb{R}^N).\end{aligned}$$

Step 5. Passing to the limit in (3.6), we get

$$(3.7) \quad -h \operatorname{div} \zeta + u = g \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

It remains to show that $\zeta \in \partial\Phi(\nabla u)$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$. By the Fenchel identity, $\Phi(\nabla u_n) + \Phi^*(\zeta_n) = \zeta_n \cdot \nabla u_n$, so by weak-* lower semicontinuity of the two convex integrals,

$$\int \Phi(\nabla u) \varphi + \int \Phi^*(\zeta) \varphi \leq \liminf_n \int (\zeta_n \cdot \nabla u_n) \varphi.$$

On the other hand, testing (3.6) with $u_n \varphi$ by density,

$$\int (\zeta_n \cdot \nabla u_n) \varphi = \frac{1}{h} \int (g_n - u_n) u_n \varphi - \int (\zeta_n \cdot \nabla \varphi) u_n.$$

Passing to the limit,

$$\begin{aligned}\lim_n \int (\zeta_n \cdot \nabla u_n) \varphi &= \frac{1}{h} \int (g - u) u \varphi - \int (\zeta \cdot \nabla \varphi) u \\ &= \int \zeta \cdot \nabla(u \varphi) - \int (\zeta \cdot \nabla \varphi) u \\ &= \int (\zeta \cdot \nabla u) \varphi,\end{aligned}$$

where in the first equality we used that $u_n \rightarrow u$ and $g_n \rightarrow g$ locally uniformly and that $\zeta_n \xrightarrow{*} \zeta$ in L^∞ , and in the second one (3.7) tested with $u \varphi$, by density. The result is

$$\int \left(\Phi(\nabla u) + \Phi^*(\zeta) - \zeta \cdot \nabla u \right) \varphi \leq 0$$

for all $\varphi \geq 0$, while the integrand is nonnegative by the Fenchel inequality. Then the integrand is 0, which means that $\zeta \in \partial\Phi(\nabla u)$, and we conclude by Remark 3.1. $\square$

3.1.3. Wulff shape estimate. The barriers used below are functions of ϕ° alone. For functions of this kind, the anisotropic p -Laplacian takes a simple form, as stated in the following lemma.

LEMMA 3.3 (Wulff-radial functions). *Let $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ be defined as*

$$z(x) = \frac{x}{\phi^\circ(x)} \quad \text{for } x \neq 0,$$

let $f \in C^2([0, +\infty))$ with $f' \geq 0$ and $f'(0) = 0$, and set $\eta = f(\phi^\circ)$, $\zeta = f'(\phi^\circ)^{p-1}z$. Then

$$(3.8) \quad z \in \partial\phi(\nabla\eta),$$

$$(3.9) \quad \zeta = \phi^{p-1}(\nabla\eta)z$$

and

$$(3.10) \quad \operatorname{div} \zeta = (p-1)f'(\phi^\circ)^{p-2}f''(\phi^\circ) + f'(\phi^\circ)^{p-1} \frac{N-1}{\phi^\circ} \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

The latter is true also in $L^1_{\text{loc}}(\mathbb{R}^N)$.

PROOF. **Step 1.** We prove that $z \in \partial\phi(\nabla\phi^\circ)$. By Proposition A.3, it is sufficient to show that $\phi^\circ(z) \leq 1$ and $z \cdot \nabla\phi^\circ = \phi(\nabla\phi^\circ)$. The first is trivially true, while the second is just $z \cdot \nabla\phi^\circ = 1$, since $\phi^\circ = \text{sd}_{\{0\}}^\phi$ and the eikonal equation holds. Since $\phi^\circ(\lambda x) = \lambda\phi^\circ(x)$ for $\lambda > 0$, we can differentiate at $\lambda = 1$. We get $\nabla\phi^\circ(x) \cdot x = \phi^\circ(x)$, and the second property follows.

Step 2. Let x be a point of differentiability of ϕ° . The chain rule gives $\nabla\eta(x) = f'(\phi^\circ(x))\nabla\phi^\circ(x)$ with $f'(\phi^\circ(x)) \geq 0$. If $f'(\phi^\circ(x)) > 0$, since $\partial\phi$ is 0-homogeneous, we have

$$\partial\phi(\nabla\eta(x)) = \partial\phi(\nabla\phi^\circ(x)) \ni z(x)$$

by Step 1. If instead $f'(\phi^\circ) = 0$, then $\nabla\eta(x) = 0$ and $\partial\phi(0) = \{\phi^\circ \leq 1\}$, which contains $z(x)$, proving (3.8). As for (3.9), we have

$$\phi^{p-1}(\nabla\eta) = \phi^{p-1}(f'(\phi^\circ)\nabla\phi^\circ) = f'(\phi^\circ)^{p-1}\phi(\nabla\phi^\circ)^{p-1} = f'(\phi^\circ)^{p-1}.$$

Step 3. We prove that (3.10) holds outside the origin. The functions ϕ° and $f'(\phi^\circ)^{p-1}$ are Lipschitz on compact subsets of $\mathbb{R}^N \setminus \{0\}$, so that there the distributional derivatives agree with the pointwise ones and the product rule applies. It holds

$$\operatorname{div} z = \frac{N}{\phi^\circ} - \frac{x \cdot \nabla\phi^\circ}{(\phi^\circ)^2} = \frac{N-1}{\phi^\circ}$$

and $\nabla(f'(\phi^\circ)^{p-1}) = (p-1)f'(\phi^\circ)^{p-2}f''(\phi^\circ)\nabla\phi^\circ$, so that

$$\begin{aligned} \operatorname{div}(\phi^{p-1}(\nabla\eta)z) &= \operatorname{div}(f'(\phi^\circ)^{p-1}z) \\ &= \nabla(f'(\phi^\circ)^{p-1}) \cdot z + f'(\phi^\circ)^{p-1} \operatorname{div} z \\ &= (p-1)f'(\phi^\circ)^{p-2}f''(\phi^\circ) + f'(\phi^\circ)^{p-1} \frac{N-1}{\phi^\circ}. \end{aligned}$$

Step 4. We prove that (3.10) holds across the origin. Since $f'(0) = 0$ and $p > 1$, the field ζ is continuous at the origin with $\zeta(0) = 0$. Testing with a cut-off $\varphi_\epsilon \in C_c^\infty(\mathbb{R}^N)$ equal to 1 on $\{\phi^\circ \leq \epsilon\}$, vanishing outside $\{\phi^\circ \leq 2\epsilon\}$ and with $|\nabla\varphi_\epsilon| \leq \frac{C}{\epsilon}$, we get

$$\left| \int \zeta \cdot \nabla\varphi_\epsilon \right| \leq \frac{C}{\epsilon} |\{\phi^\circ \leq 2\epsilon\}| \sup_{\{\phi^\circ \leq 2\epsilon\}} |\zeta| \leq C'\epsilon^{N-1} \sup_{\{\phi^\circ \leq 2\epsilon\}} |\zeta| \longrightarrow 0.$$

The proof concludes by noting that the RHS of (3.10) is in $L^1_{\text{loc}}(\mathbb{R}^N)$ under our hypotheses. $\square$

PROPOSITION 3.4 (Wulff shape estimate). *There exists $c_0 > 0$ such that the family $\{\mathcal{T}_h\}$ defined by (3.1) satisfies (2.4), that is*

$$\mathcal{T}_h\phi^\circ \leq \sqrt{(\phi^\circ)^2 + c_0 h}$$

for all $h > 0$.

PROOF. We set $u_h = \mathcal{T}_h \phi^\circ$. We want to find $c_0 > 0$ and a family of barriers $\eta_h = f_h \circ \phi^\circ$, with $f_h \in C^2([0, +\infty))$, $f_h' \geq 0$ and $f_h'(0) = 0$ as required by Lemma 3.3, such that, for every $h > 0$,

$$(3.11) \quad -h \operatorname{div}(\phi^{p-1}(\nabla \eta_h)z) + \eta_h \geq \phi^\circ \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N),$$

$$(3.12) \quad \eta_h \leq \sqrt{(\phi^\circ)^2 + c_0 h},$$

where z is the radial field of Lemma 3.3. If we do so, then we can apply Proposition 3.1, concluding by

$$u_h \leq \eta_h \leq \sqrt{(\phi^\circ)^2 + c_0 h}.$$

By (3.10), verifying (3.11) is equivalent to checking the inequality

$$-h \left((p-1)f_h'(r)^{p-2}f_h''(r) + f_h'(r)^{p-1} \frac{N-1}{r} \right) + f_h(r) \geq r$$

for all $h > 0$ and $r > 0$. In both cases below the barrier is self-similar, that is $f_h(r) = \sqrt{c_0 h} F(\frac{r}{\sqrt{c_0 h}})$ for a fixed profile F with the requirements of Lemma 3.3. This turns the inequality into

$$(3.13) \quad G(t) = (p-1)F'(t)^{p-2}F''(t) + F'(t)^{p-1} \frac{N-1}{t} \leq c_0(F(t) - t),$$

which does not depend on h , while (3.12) is equivalent to

$$(3.14) \quad F(t) \leq \sqrt{t^2 + 1}.$$

We consider two different cases.

Case 1: $p \geq 2$. With $c_0 = 2(N + p - 2)$, we set

$$F(t) = \sqrt{t^2 + 1}$$

We have

$$F'(t) = \frac{t}{\sqrt{t^2 + 1}}, \quad F''(t) = \frac{1}{\sqrt{t^2 + 1}^3},$$

so that

$$(p-1)F'(t)^{p-2}F''(t) + F'(t)^{p-1} \frac{N-1}{t} = \frac{t^{p-2}}{\sqrt{t^2 + 1}^{p-1}} \left(N-1 + \frac{p-1}{t^2 + 1} \right),$$

and (3.13) becomes

$$\frac{t^{p-2}}{\sqrt{t^2 + 1}^{p-1}} \left(N-1 + \frac{p-1}{t^2 + 1} \right) \leq c_0 (\sqrt{t^2 + 1} - t)$$

for all $t > 0$. Since $\sqrt{t^2 + 1} - t = \frac{1}{\sqrt{t^2 + 1} + t}$, this is equivalent to

$$\frac{t^{p-2}(\sqrt{t^2 + 1} + t)}{\sqrt{t^2 + 1}^{p-1}} \left(N-1 + \frac{p-1}{t^2 + 1} \right) \leq c_0,$$

which is true because $t^{p-2} \leq \sqrt{t^2 + 1}^{p-2}$ for $p \geq 2$, so that the first factor is at most $1 + \frac{t}{\sqrt{t^2 + 1}} \leq 2$, while the second one is at most $N-1 + p-1 = N+p-2$. Since (3.14) holds with equality, this concludes the proof for this case.

Case 2: $1 < p < 2$. With $p' > 2$ the Hölder conjugate of p , we set

$$F(t) = \frac{(t^{p'} + 1)^{1/p'}}{\sqrt{t^2 + 1}} + \sqrt{t^2 + 1} - 1.$$

Note first that $F' \geq 0$, as required by Lemma 3.3. Indeed, writing $A(t) = (t^{p'} + 1)^{1/p'}$ and $B(t) = \sqrt{t^2 + 1}$, so that $F = \frac{A}{B} + B - 1$, we have

$$F' = \frac{A'B - AB'}{B^2} + B' = \frac{A'}{B} + B' \left(1 - \frac{A}{B^2} \right),$$

where $A'(t) = (t^{p'} + 1)^{\frac{1}{p'} - 1} t^{p'-1} \geq 0$ and $B'(t) = \frac{t}{B(t)} \geq 0$. Moreover $A \leq B$ by Lemma A.12, since $p' \geq 2$, and $B \leq B^2$ since $B \geq 1$, so that $1 - \frac{A}{B^2} \geq 0$. Hence $F' \geq 0$, with $F'(0) = 0$. We may therefore set

$$G(t) = F'(t)t^{p-2} \left((p-1)F''(t) + \frac{N-1}{t}F'(t) \right).$$

We want to prove that the function $t \mapsto \frac{G(t)}{F(t)-t}$ is bounded. Firstly, the function is well-defined. Indeed, we have

$$F(t) > \frac{t}{\sqrt{t^2+1}} + \sqrt{t^2+1} - 1 \geq t + \left(\sqrt{t^2+1} - 1 \right) \left(1 - \frac{t}{\sqrt{1+t^2}} \right) \geq t.$$

Since it is continuous in $(0, +\infty)$, we need to verify that it has finite limits at 0 and $+\infty$. By an asymptotic expansion, one can check that

$$\lim_{t \rightarrow 0^+} \frac{G(t)}{F(t)-t} = N, \quad \lim_{t \rightarrow \infty} \frac{G(t)}{F(t)-t} = 2(N-1).$$

With $c_0 = \sup_{t>0} \frac{G(t)}{F(t)-t}$, (3.13) is verified. We are left with (3.14), which is true since

$$F = \frac{A}{B} + B - 1 \leq B.$$

This concludes the proof of the proposition. $\square$

3.1.4. Stability and consistency.

PROPOSITION 3.5 (Properties (b1) and (b2)). *The family $\{\mathcal{T}_h\}$ defined by (3.1) satisfies properties (b1) and (b2) of Definition 2.6, with $C_0 = 1$.*

PROOF. Property (b1). Let $h > 0$ and let $g \in \text{Lip}(\mathbb{R}^N)$ with $\phi(\nabla g) \leq 1$, and set $u = \mathcal{T}_h g$. By Proposition 3.2, there exists $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with $z \in \partial\phi(\nabla u)$ such that

$$\frac{u - g}{h} = \text{div } \zeta \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

where $\zeta = \phi^{p-1}(\nabla u)z$. By Proposition A.3, we have $\phi^\circ(z) \leq 1$, and by Proposition 2.2, we have $\phi(\nabla u) \leq \|\phi(\nabla g)\|_\infty \leq 1$, so that

$$(3.15) \quad \phi^\circ(\zeta) = \phi^{p-1}(\nabla u)\phi^\circ(z) \leq 1.$$

Therefore, for every $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\left| \left\langle \frac{\mathcal{T}_h g - g}{h}, \psi \right\rangle \right| = \left| \int \zeta \cdot \nabla \psi \right| \leq \int \phi^\circ(\zeta) \phi(\nabla \psi) \leq \int \phi(\nabla \psi),$$

which is (2.5) with $C_0 = 1$. This proves (b1).

Property (b2). Let now h_n, E_n, E be as in the statement of (b2), and we set $d_n = d_{E_n}$ and $d = d_E$. By the previous step applied to d_n , we obtain fields $\zeta_n = \phi^{p-1}(\nabla u_n)z_n$ with $\phi^\circ(\zeta_n) \leq 1$ and $S_n = \text{div } \zeta_n$, where $u_n = \mathcal{T}_{h_n} d_n$. Applying (2.8) gives $\|u_n - d_n\|_\infty \leq \sqrt{c_0 h_n}$, while $d_n \rightarrow d$ locally uniformly in E^c by Proposition A.8, so that

$$(3.16) \quad \|u_n - d\|_{L^\infty(K)} \longrightarrow 0 \quad \text{for all compact } K \subseteq E^c.$$

We fix $\varphi \in C_c^\infty(E^c)$ and consider the realizing subsequence of $\liminf_n \langle S_n, \varphi \rangle$. By standard compactness theorems, there exists $z \in L^\infty(E^c; \mathbb{R}^N)$ such that $\phi^\circ(z) \leq 1$ and, up to a further subsequence,

$$\begin{aligned} \zeta_n &\overset{*}{\rightharpoonup} z && \text{in } L^\infty(E^c; \mathbb{R}^N), \\ \nabla u_n &\overset{*}{\rightharpoonup} \nabla d && \text{in } L^\infty(E^c; \mathbb{R}^N). \end{aligned}$$

If we prove that $z \in \partial\phi(\nabla d)$, then we can conclude by

$$\langle S_n, \varphi \rangle \longrightarrow - \int z \cdot \nabla \varphi \geq - \int \phi'(\nabla d; \nabla \varphi).$$

Fixing $\psi \in C_c^\infty(E^c)$ such that $\psi \geq 0$, we have

$$\begin{aligned} \int (\zeta_n \cdot \nabla d) \psi &= - \int d(\zeta_n \cdot \nabla \psi) - \int S_n d \psi \\ &= \int \phi^p(\nabla u_n) \psi + \int S_n (u_n - d) \psi + \int (u_n - d)(\zeta_n \cdot \nabla \psi). \end{aligned}$$

For the first term of the RHS, we have $\int \phi^p(\nabla u_n) \psi \leq \int \psi$, together with

$$\int \psi = \int \phi^p(\nabla d) \psi \leq \liminf_{n \rightarrow \infty} \int \phi^p(\nabla u_n) \psi$$

by weak-* lower semicontinuity, so that $\int \phi^p(\nabla u_n) \psi \rightarrow \int \psi$. For the second term, we set $K = \text{supp } \psi$ and $\delta = \min_K d > 0$. Since $d_n \rightarrow d$ uniformly on K , for n large enough we have $d_n \geq \frac{\delta}{2}$ on K , so that $K \subseteq E_n^c$ and (2.9) gives

$$S_n \leq \frac{c_0}{2d_n} \leq \frac{c_0}{\delta} = M \quad \text{on } K.$$

This bounds S_n from above only. The bound from below is provided by (3.15). Indeed, $S_n^+ \leq M$ and $S_n^- \leq M - S_n$ on K , so that

$$(3.17) \quad \int |S_n| \psi \leq 2M \int \psi - \int S_n \psi = 2M \int \psi + \int \zeta_n \cdot \nabla \psi \leq 2M \int \psi + \int \phi(\nabla \psi) = C,$$

and therefore, by (3.16),

$$\left| \int S_n (u_n - d) \psi \right| \leq C \|u_n - d\|_{L^\infty(K)} \rightarrow 0.$$

Also the third term goes to zero, so we can pass to the limit as

$$\int (z \cdot \nabla d) \psi = \int \psi.$$

This means that $z \cdot \nabla d = 1$, which, together with $\phi^\circ(z) \leq 1$, implies $z \in \partial\phi(\nabla d)$. $\square$

THEOREM 3.6 (Convergence of the elliptic scheme). *Let ϕ be a norm on $\mathbb{R}^N$, let $1 < p < \infty$ and let $E_0 \subseteq \mathbb{R}^N$ be closed. Then the discrete evolutions generated by the elliptic scheme (3.1) converge, up to a subsequence and in the sense of Theorem 2.14, to a superflow with initial condition E_0 in the sense of Definition 1.8.*

PROOF. Properties (i) to (v) are proved in Proposition 3.2, property (vi) holds by extension, the estimate (2.4) is proved in Proposition 3.4, and properties (b1) and (b2) in Proposition 3.5. The conclusion is then Theorem 2.16. $\square$

3.2. A parabolic scheme

The elliptic scheme of the previous section can be understood as one backward Euler step of an evolutionary problem. In this section we consider directly the solution to such a problem, run for a time h . We work in the function spaces

$$\mathcal{V}_T = \left\{ u \in C(\mathbb{R}^N \times [0, T]) : \nabla u \in L^\infty(\mathbb{R}^N \times [0, T]; \mathbb{R}^N), \dot{u} \in L_{\text{loc}}^2(\mathbb{R}^N \times [0, T]) \right\}$$

for $T > 0$. Fixing $1 < p < \infty$, we define the scheme as $\mathcal{T}_h g = u(\cdot, h)$, where $u \in \mathcal{V}_h$ is the unique solution to

$$(3.18) \quad \begin{cases} \dot{u} \in \Delta_p^\phi u & \text{in } \mathcal{D}'(\mathbb{R}^N \times (0, h)) \\ u(\cdot, 0) = g \end{cases},$$

where the anisotropic p -Laplacian $\Delta_p^\phi u$ is understood, as in the previous section, as the set of distributions in the form $\text{div}(\phi^{p-1}(\nabla u)z)$ for some $z \in L^\infty(\mathbb{R}^N \times [0, T]; \mathbb{R}^N)$ with $z \in \partial\phi(\nabla u)$. We now prove results analogous to those of the elliptic scheme.

3.2.1. A comparison principle.

PROPOSITION 3.7 (Parabolic comparison principle). *Let $T > 0$, let $u_1, u_2 \in \mathcal{V}_T$ and let $z_1, z_2 \in L^\infty(\mathbb{R}^N \times (0, T); \mathbb{R}^N)$ be such that $z_i \in \partial\phi(\nabla u_i)$. Set $\zeta_i = \phi^{p-1}(\nabla u_i)z_i$. If*

$$(3.19) \quad \begin{cases} \dot{u}_1 - \text{div } \zeta_1 \leq \dot{u}_2 - \text{div } \zeta_2 & \text{in } \mathcal{D}'(\mathbb{R}^N \times (0, T)) \\ u_1(\cdot, 0) \leq u_2(\cdot, 0) \end{cases},$$

then $u_1 \leq u_2$.

PROOF. As in the proof of Proposition 3.1, the function $\frac{1}{p}\phi^p$ is convex and its subdifferential at ξ is $\phi^{p-1}(\xi)\partial\phi(\xi)$, so that by monotonicity

$$(3.20) \quad (\zeta_1 - \zeta_2) \cdot (\nabla u_1 - \nabla u_2) \geq 0.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $\varphi = 1$ on B_1 , $0 \leq \varphi \leq 1$ and $\varphi = 0$ outside B_2 , and set $\varphi_R(x) = \varphi(\frac{x}{R})$ for $R > 0$. Fix $\gamma > \max\{N, 2\}$, and we set

$$v = (u_1 - u_2) \vee 0, \quad w_R = v^{\gamma-1} \varphi_R^\gamma.$$

By density, (3.19) can be tested with such a function on a time interval $[0, t]$ with $0 \leq t \leq T$, giving

$$(3.21) \quad \int_0^t \int (\dot{u}_1 - \dot{u}_2) w_R + \int_0^t \int (\zeta_1 - \zeta_2) \cdot \nabla w_R \leq 0.$$

For the first term, on $\{u_1 > u_2\}$ we have $(\dot{u}_1 - \dot{u}_2)v^{\gamma-1} = \frac{1}{\gamma}(\dot{v}^\gamma)$, while both sides vanish elsewhere, so that, since $v(\cdot, 0) = 0$ by the initial condition,

$$\int_0^t \int (\dot{u}_1 - \dot{u}_2) w_R = \frac{1}{\gamma} \int v(\cdot, t)^\gamma \varphi_R^\gamma = \frac{1}{\gamma} \|v(\cdot, t) \varphi_R\|_\gamma^\gamma.$$

For the second term, we compute

$$(3.22) \quad \nabla w_R = (\gamma - 1) v^{\gamma-2} \varphi_R^\gamma \nabla(u_1 - u_2) 1_{\{u_1 > u_2\}} + \gamma v^{\gamma-1} \varphi_R^{\gamma-1} \nabla \varphi_R,$$

and by (3.20) the contribution of the first summand is nonnegative. Setting $M = \|\zeta_1\|_\infty \vee \|\zeta_2\|_\infty$, which is finite because $\phi^\circ(z_i) \leq 1$ and $\nabla u_i \in L^\infty$, we obtain from (3.21)

$$\frac{1}{\gamma} \|v(\cdot, t) \varphi_R\|_\gamma^\gamma \leq 2M\gamma \int_0^t \int v^{\gamma-1} \varphi_R^{\gamma-1} |\nabla \varphi_R|.$$

By Hölder's inequality in space, exactly as in the elliptic case,

$$\int v^{\gamma-1} \varphi_R^{\gamma-1} |\nabla \varphi_R| \leq \|v(\cdot, s) \varphi_R\|_\gamma^{\gamma-1} \|\nabla \varphi_R\|_\gamma,$$

so that, writing $F = \sup_{0 \leq s \leq T} \|v(\cdot, s) \varphi_R\|_\gamma$, which is finite since v is continuous and φ_R has compact support,

$$\frac{1}{\gamma} F^\gamma \leq 2M\gamma T F^{\gamma-1} \|\nabla \varphi_R\|_\gamma,$$

and therefore

$$F \leq 2M\gamma^2 T \|\nabla \varphi_R\|_\gamma.$$

Finally,

$$\|\nabla \varphi_R\|_\gamma^\gamma = \frac{1}{R^\gamma} \int \left| \nabla \varphi\left(\frac{x}{R}\right) \right|^\gamma dx \leq \|\nabla \varphi\|_\infty^\gamma 2^N \omega_N R^{N-\gamma},$$

which tends to 0 as $R \rightarrow \infty$ because $\gamma > N$. Letting $R \rightarrow \infty$ gives $v \equiv 0$, that is $u_1 \leq u_2$. $\square$

3.2.2. Existence, uniqueness and other properties.

PROPOSITION 3.8 (Parabolic problem). *Let $T > 0$ and $g \in \text{Lip}(\mathbb{R}^N)$. Then there exists a unique solution $u = \mathcal{S}g \in \mathcal{V}_T$ to (3.18) with $h = T$. Moreover:*

- (i) $\mathcal{S}(g + c) = \mathcal{S}g + c$ for all $c \in \mathbb{R}$;
- (ii) $\mathcal{S}(g(\cdot + \tau)) = \mathcal{S}g(\cdot + \tau, \cdot)$ for all $\tau \in \mathbb{R}^N$;
- (iii) $\mathcal{S}(-g) = -\mathcal{S}g$;
- (iv) If $g_1 \leq g_2$, then $\mathcal{S}g_1 \leq \mathcal{S}g_2$.

In particular, $\mathcal{T}_h$ satisfies properties (i) to (v) of Definition 2.1 for all $h > 0$.

PROOF. Uniqueness and the properties (i) to (iv) follow from Proposition 3.7 or simple computations exactly as in the elliptic case, so we concentrate on existence. We construct the solution by the backward Euler scheme built on the operator of Section 3.1: given $\tau = \frac{T}{K}$ with $K \in \mathbb{N}$, we set

$$u_\tau^0 = g, \quad u_\tau^{k+1} = \mathcal{T}_\tau^{\text{ell}}(u_\tau^k),$$

where $\mathcal{T}_\tau^{\text{ell}}$ is the operator of Proposition 3.2, so that

$$(3.23) \quad \frac{u_\tau^{k+1} - u_\tau^k}{\tau} = \text{div } \zeta_\tau^{k+1} \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

$$\zeta_\tau^{k+1} = \phi^{p-1}(\nabla u_\tau^{k+1}) z_\tau^{k+1}, \quad z_\tau^{k+1} \in \partial\phi(\nabla u_\tau^{k+1}).$$

We denote by $\hat{u}_\tau$ the piecewise affine interpolant in time of the u_τ^k and by $\bar{u}_\tau, \bar{\zeta}_\tau$ the piecewise constant interpolants taking the values $u_\tau^{k+1}, \zeta_\tau^{k+1}$ on $[k\tau, (k+1)\tau)$. The proof is divided in five steps.

Step 1. By Proposition 2.2 applied iteratively,

$$(3.24) \quad \left\| \phi(\nabla u_\tau^k) \right\|_\infty \leq \|\phi(\nabla g)\|_\infty = L,$$

so that the u_τ^k are equi-Lipschitz and, since $\phi^\circ(z_\tau^{k+1}) \leq 1$, the fields ζ_τ^{k+1} are bounded in $L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ by a constant C depending only on L, p and ϕ .

Step 2. We prove an a priori estimate. Let $R > 0$ and let $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi = 1$ on B_R , $0 \leq \varphi \leq 1$ and $\varphi = 0$ outside B_{2R} . Set $v_\tau^k = u_\tau^{k+1} - u_\tau^k$. Testing (3.23) with $v_\tau^k \varphi^2$ by density,

$$\frac{1}{\tau} \int (v_\tau^k)^2 \varphi^2 + \int (\zeta_\tau^{k+1} \cdot \nabla v_\tau^k) \varphi^2 + 2 \int (\zeta_\tau^{k+1} \cdot \nabla \varphi) v_\tau^k \varphi = 0.$$

Writing $\Phi = \frac{1}{p} \phi^p$, so that $\zeta_\tau^{k+1} \in \partial \Phi(\nabla u_\tau^{k+1})$, the subgradient inequality gives

$$\zeta_\tau^{k+1} \cdot \nabla v_\tau^k = \zeta_\tau^{k+1} \cdot (\nabla u_\tau^{k+1} - \nabla u_\tau^k) \geq \Phi(\nabla u_\tau^{k+1}) - \Phi(\nabla u_\tau^k),$$

while by Young's inequality

$$\left| 2 \int (\zeta_\tau^{k+1} \cdot \nabla \varphi) v_\tau^k \varphi \right| \leq \frac{1}{2\tau} \int (v_\tau^k)^2 \varphi^2 + 2\tau C^2 \|\nabla \varphi\|_2^2.$$

Combining them and summing over $k \in \{0, \dots, K-1\}$, the energy terms telescope to the two endpoints, and we get

$$\frac{1}{2\tau} \sum_{k=0}^{K-1} \int (v_\tau^k)^2 \varphi^2 \leq \int \Phi(\nabla g) \varphi^2 - \int \Phi(\nabla u_\tau^K) \varphi^2 + 2\tau C^2 \|\nabla \varphi\|_2^2.$$

Since $\Phi(\nabla g) \leq \frac{L^p}{p}$ by (3.24), we obtain

$$\frac{1}{2\tau} \sum_{k=0}^{K-1} \int (v_\tau^k)^2 \varphi^2 \leq \frac{L^p}{p} \|\varphi\|_2^2 + 2\tau C^2 \|\nabla \varphi\|_2^2.$$

Since $\hat{u}_\tau = \frac{v_\tau^k}{\tau}$ on $(k\tau, (k+1)\tau)$, the LHS equals $\frac{1}{2} \|\hat{u}_\tau \varphi\|_{L^2(\mathbb{R}^N \times (0, T))}^2$, so that

$$(3.25) \quad \|\hat{u}_\tau\|_{L^2(B_R \times (0, T))} \leq C_R,$$

with C_R independent of τ . With a similar computation, we also get

$$(3.26) \quad \max_k \|v_\tau^k\|_{L^2(B_R)} \leq (2\tau)^{1/2} C_R.$$

Step 3. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$ and $\int \varphi = 1$. Testing (3.23) with φ and summing over k ,

$$\left| \int u_\tau^K \varphi - \int g \varphi \right| \leq \tau \sum_{k=0}^{K-1} \left| \int \zeta_\tau^{k+1} \cdot \nabla \varphi \right| \leq \tau C \|\nabla \varphi\|_1,$$

so the averages $\int u_\tau^k \varphi$ are equibounded. Since each u_τ^k is L -Lipschitz with respect to ϕ° by (3.24), we get

$$\left| u_\tau^k(0) \right| \leq \left| \int u_\tau^k \varphi \right| + L \int \phi^\circ(x) \varphi(x) dx,$$

so also $u_\tau^k(0)$ are equibounded, and u_τ^k are locally equibounded since they are equi-Lipschitz.

Step 4. Let $0 \leq t_1 \leq t_2 \leq T$. By (3.25) and Hölder's inequality,

$$\|\hat{u}_\tau(\cdot, t_2) - \hat{u}_\tau(\cdot, t_1)\|_{L^2(B_R)} \leq |t_2 - t_1|^{1/2} \|\hat{u}_\tau\|_{L^2(B_R \times (0, T))} \leq C_R |t_2 - t_1|^{1/2}.$$

The function $w = \hat{u}_\tau(\cdot, t_2) - \hat{u}_\tau(\cdot, t_1)$ is Lipschitz with $\|\nabla w\|_\infty \leq 2L'$, where L' is a Euclidean Lipschitz bound, so Lemma A.13 upgrades the L^2 bound to

$$\|\hat{u}_\tau(\cdot, t_2) - \hat{u}_\tau(\cdot, t_1)\|_{L^\infty(B_{R/2})} \leq C'_R |t_2 - t_1|^{\frac{1}{N+2}}$$

whenever the RHS is small enough for (A.3) to hold, which is the only case of interest. Together with Step 1 and Step 3, the family $\{\hat{u}_\tau\}$ is therefore locally equibounded and equicontinuous in space-time. Then, by standard

compactness theorems, there exists $u \in \mathcal{V}_T$ such that, up to a subsequence,

$$\begin{aligned}\hat{u}_\tau &\rightarrow u && \text{locally uniformly,} \\ \nabla \hat{u}_\tau &\xrightarrow{*} \nabla u && \text{in } L^\infty(\mathbb{R}^N \times [0, T]; \mathbb{R}^N), \\ \hat{u}_\tau &\rightharpoonup \dot{u} && \text{in } L^2_{\text{loc}}(\mathbb{R}^N \times [0, T]),\end{aligned}$$

with $\|\phi(\nabla u(\cdot, t))\|_\infty \leq L$ by (3.24) and weak-* lower semicontinuity. Again by Lemma A.13 and (3.26), we have

$$\|\bar{u}_\tau - \hat{u}_\tau\|_{L^\infty(B_{R/2})} \leq \max_k \|v_\tau^k\|_{L^\infty(B_{R/2})} \leq \tau^{\frac{1}{N+2}} C_R''$$

whenever τ is small enough. Hence $\bar{u}_\tau - \hat{u}_\tau \rightarrow 0$ locally uniformly, so that, up to a further subsequence,

$$\begin{aligned}\bar{u}_\tau &\rightarrow u && \text{locally uniformly,} \\ \nabla \bar{u}_\tau &\xrightarrow{*} \nabla u && \text{in } L^\infty(\mathbb{R}^N \times [0, T]; \mathbb{R}^N), \\ \bar{\zeta}_\tau &\xrightarrow{*} \zeta && \text{in } L^\infty(\mathbb{R}^N \times [0, T]; \mathbb{R}^N).\end{aligned}$$

Step 5. Passing to the limit in (3.23), which is

$$(3.27) \quad \hat{u}_\tau = \operatorname{div} \bar{\zeta}_\tau \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (0, T)),$$

we get

$$(3.28) \quad \dot{u} = \operatorname{div} \zeta \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (0, T)).$$

It remains to show that $\zeta \in \partial\Phi(\nabla u)$, and we use the same argument as in Step 5 of Proposition 3.2. Let $\varphi \in C_c^\infty(\mathbb{R}^N \times (0, T))$ with $\varphi \geq 0$. By the Fenchel identity, $\Phi(\nabla \bar{u}_\tau) + \Phi^*(\bar{\zeta}_\tau) = \bar{\zeta}_\tau \cdot \nabla \bar{u}_\tau$, so by weak-* lower semicontinuity of the two convex integrals,

$$\iint \Phi(\nabla u) \varphi + \iint \Phi^*(\zeta) \varphi \leq \liminf_\tau \iint (\bar{\zeta}_\tau \cdot \nabla \bar{u}_\tau) \varphi.$$

On the other hand, testing (3.27) with $\bar{u}_\tau \varphi$ by density,

$$\iint (\bar{\zeta}_\tau \cdot \nabla \bar{u}_\tau) \varphi = - \iint \hat{u}_\tau \bar{u}_\tau \varphi - \iint (\bar{\zeta}_\tau \cdot \nabla \varphi) \bar{u}_\tau.$$

Passing to the limit,

$$\begin{aligned}\lim_\tau \iint (\bar{\zeta}_\tau \cdot \nabla \bar{u}_\tau) \varphi &= - \iint \dot{u} u \varphi - \iint (\zeta \cdot \nabla \varphi) u \\ &= - \langle \dot{u}, u \varphi \rangle + \langle \operatorname{div} \zeta, u \varphi \rangle + \iint (\zeta \cdot \nabla u) \varphi \\ &= \iint (\zeta \cdot \nabla u) \varphi.\end{aligned}$$

For the first term we used that $\hat{u}_\tau \rightharpoonup \dot{u}$ in L^2_{loc} and $\bar{u}_\tau \varphi \rightarrow u \varphi$ in L^2_{loc} . For the second term we used that $\bar{\zeta}_\tau \xrightarrow{*} \zeta$ in L^∞ and $\bar{u}_\tau \nabla \varphi \rightarrow u \nabla \varphi$ in L^1_{loc} . Finally we used (3.28) by density. The result is

$$\iint \left(\Phi(\nabla u) + \Phi^*(\zeta) - \zeta \cdot \nabla u \right) \varphi \leq 0$$

for all $\varphi \geq 0$, while the integrand is nonnegative by the Fenchel inequality. Then the integrand is 0, which means that $\zeta \in \partial\Phi(\nabla u)$, and we conclude by Remark 3.1. $\square$

PROPOSITION 3.9 (Absolute continuity). *Let $T > 0$ and $g \in \operatorname{Lip}(\mathbb{R}^N)$. Let $u \in \mathcal{V}_T$ be the unique solution to (3.18) with $h = T$, let z be a corresponding selection and let $\zeta = \phi^{p-1}(\nabla u)z$. If $0 \leq t_1 \leq t_2 \leq T$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$, then*

$$(3.29) \quad \int (u(\cdot, t_2) - u(\cdot, t_1)) \varphi = - \int_{t_1}^{t_2} \left(\int \zeta(x, t) \cdot \nabla \varphi(x) dx \right) dt.$$

PROOF. Let n be small enough and $\chi_n : [t_1, t_2] \rightarrow \mathbb{R}$ be defined as

$$\chi_n(t) = \begin{cases} \frac{t - t_1}{n} & \text{if } t \in \left[t_1, t_1 + \frac{1}{n} \right] \\ 1 & \text{if } t \in \left(t_1 + \frac{1}{n}, t_2 - \frac{1}{n} \right) \\ 1 - \frac{t_2 - t}{n} & \text{if } t \in \left[t_2 - \frac{1}{n}, t_2 \right] \end{cases}.$$

Testing (3.18) with $\psi_n(x, t) = \varphi(x)\chi_n(t)$ by density,

$$(3.30) \quad \iint \dot{u}(x, t)\varphi(x)\chi_n(t) dx dt = - \iint (\zeta(x, t) \cdot \nabla \varphi(x))\chi_n(t) dx dt.$$

For the LHS, we have

$$\begin{aligned} \iint \dot{u}(x, t)\varphi(x)\chi_n(t) dx dt &= - \iint u(x, t)\varphi(x)\dot{\chi}_n(t) dx dt \\ &= -\frac{1}{n} \int_{t_1}^{t_1+1/n} \left(\int u(x, t)\varphi(x) dx \right) dt + \frac{1}{n} \int_{t_2-1/n}^{t_2} \left(\int u(x, t)\varphi(x) dx \right) dt \\ &\longrightarrow \int (u(\cdot, t_2) - u(\cdot, t_1))\varphi, \end{aligned}$$

where the limit is justified by the continuity of u . Since $\chi_n \rightarrow 1_{[t_1, t_2]}$ in $L^1(0, T)$, passing to the limit in (3.30) proves (3.29). $\square$

3.2.3. Wulff shape estimate.

PROPOSITION 3.10 (Wulff shape estimate). *There exists $c_0 > 0$ such that the family $\{\mathcal{T}_h\}$ defined by (3.18) satisfies (2.4), that is*

$$\mathcal{T}_h\phi^\circ \leq \sqrt{(\phi^\circ)^2 + c_0 h}$$

for all $h > 0$.

PROOF. Let $h > 0$ and $\epsilon > 0$. We consider the barrier $\eta(x, t) = \mathcal{T}_{2(t+\epsilon)}^{\text{ell}}\phi^\circ(x)$ for $0 < t \leq h$, where $\mathcal{T}_t^{\text{ell}}$ is the operator of Proposition 3.2. We want to prove that $\eta \in \mathcal{V}_h$,

$$(3.31) \quad \begin{cases} \dot{\eta} - \text{div}(\phi^{p-1}(\nabla \eta)z) \geq 0 & \text{in } L_{\text{loc}}^1(\mathbb{R}^N \times [0, h]) \\ \eta(\cdot, 0) \geq \phi^\circ \end{cases}$$

and

$$(3.32) \quad \eta(\cdot, h) \leq \sqrt{(\phi^\circ)^2 + c_0(h + \epsilon)}.$$

If we do so, then by the comparison principle we get

$$\mathcal{T}_h\phi^\circ \leq \eta(\cdot, h) \leq \sqrt{(\phi^\circ)^2 + c_0(h + \epsilon)},$$

and we conclude by the arbitrariness of ϵ .

Step 1. We prove that $\eta(\cdot, 0) \geq \phi^\circ$, that is the second part of (3.31). Let $\xi \in \mathbb{R}^N$ such that $\phi(\xi) \leq 1$ and we set $\ell(x) = \xi \cdot x$. With a constant field $z_0 \in \partial\phi(\xi)$, we have $\text{div}(\phi^{p-1}(\nabla \ell)z_0) = \text{div}(\phi^{p-1}(\xi)z_0) = 0$, so $\mathcal{T}_t^{\text{ell}}\ell = \ell$. Since $\ell \leq \phi^\circ$, by comparison principle it follows that

$$\ell = \mathcal{T}_t^{\text{ell}}\ell \leq \mathcal{T}_t^{\text{ell}}\phi^\circ.$$

Taking the supremum over $\phi(\xi) \leq 1$ yields $\phi^\circ \leq \mathcal{T}_t^{\text{ell}}\phi^\circ$. Then we have $\eta(\cdot, 0) = \mathcal{T}_{2\epsilon}^{\text{ell}}\phi^\circ \geq \phi^\circ$.

Step 2. We set $u = \mathcal{T}_1^{\text{ell}}\phi^\circ$ and $s(t) = \sqrt{2(t + \epsilon)}$. From the self-similarity property (v) of Proposition 3.2,

$$\eta(x, t) = s(t)u\left(\frac{x}{s(t)}\right),$$

so that η is continuous. Since $\phi(\nabla u) \leq 1$, we compute

$$\begin{aligned} \dot{\eta}(x, t) &= \frac{1}{s(t)} \left(u\left(\frac{x}{s(t)}\right) - \frac{x}{s(t)} \cdot \nabla u\left(\frac{x}{s(t)}\right) \right) \\ &\geq \frac{1}{s(t)} \left(u\left(\frac{x}{s(t)}\right) - \phi^\circ\left(\frac{x}{s(t)}\right) \right) \end{aligned}$$

and

$$\nabla \eta(x, t) = \nabla u\left(\frac{x}{s(t)}\right).$$

We have

$$\begin{aligned} -\text{div} \tilde{\zeta} + u &= \phi^\circ & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \tilde{\zeta} &= \phi^{p-1}(\nabla u)\tilde{z}, & \tilde{z} \in \partial\phi(\nabla u). \end{aligned}$$

We set $z(x, t) = \tilde{z}\left(\frac{x}{s(t)}\right)$, so that

$$\begin{aligned} \operatorname{div}(\phi^{p-1}(\nabla \eta)z)(x, t) &= \frac{1}{s(t)} \operatorname{div} \tilde{\zeta}\left(\frac{x}{s(t)}\right) \\ &= \frac{1}{s(t)} \left(u\left(\frac{x}{s(t)}\right) - \phi^\circ\left(\frac{x}{s(t)}\right) \right). \end{aligned}$$

Combining them we prove the first part of (3.31).

Step 3. Let $\frac{c_0}{2}$ be the constant of Proposition 3.4. Then we have

$$\eta(\cdot, h) = \mathcal{T}_{2(h+\epsilon)}^{\text{ell}} \phi^\circ \leq \sqrt{(\phi^\circ)^2 + \frac{c_0}{2} 2(h+\epsilon)} = \sqrt{(\phi^\circ)^2 + c_0(h+\epsilon)},$$

proving (3.32). It remains to prove that $\eta \in \mathcal{V}_h$, and in particular that $\dot{\eta} \in L_{\text{loc}}^2(\mathbb{R}^N \times [0, h])$. This is true since

$$\begin{aligned} |\dot{\eta}(x, t)| &\leq \frac{1}{s(t)} \left(u\left(\frac{x}{s(t)}\right) + \phi^\circ\left(\frac{x}{s(t)}\right) \right) \\ &\leq \frac{1}{s(t)} \left(2\phi^\circ\left(\frac{x}{s(t)}\right) + \sqrt{\frac{c_0}{2}} \right) \\ &= \frac{\phi^\circ(x)}{t+\epsilon} + \frac{1}{\sqrt{2(t+\epsilon)}} \sqrt{\frac{c_0}{2}} \\ &\leq \frac{\phi^\circ(x)}{\epsilon} + \frac{1}{2} \sqrt{\frac{c_0}{\epsilon}}, \end{aligned}$$

which concludes the proof of the proposition. $\square$

3.2.4. Stability and consistency.

PROPOSITION 3.11 (Properties (b1) and (b2)). *The family $\{\mathcal{T}_h\}$ defined by (3.18) satisfies properties (b1) and (b2) of Definition 2.6, with $C_0 = 1$.*

PROOF. Property (b1). Let $h > 0$ and let $g \in \text{Lip}(\mathbb{R}^N)$ with $\phi(\nabla g) \leq 1$, and set $u = \mathcal{T}_h g$. Let $U(\cdot, \sigma) = \mathcal{T}_{h\sigma} g$, for $\sigma \in [0, 1]$, be the solution of (3.18) on $[0, h]$ with datum g , rescaled in time, so that $U(\cdot, 0) = g$ and $U(\cdot, 1) = u$, and let $Z = \phi^{p-1}(\nabla U)z$ with $z \in \partial\phi(\nabla U)$ be the associated field, so that

$$(3.33) \quad \frac{\dot{U}}{h} = \operatorname{div} Z \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (0, 1)),$$

where the dot denotes the derivative with respect to σ . We set

$$\tilde{\zeta} = \int_0^1 Z(\cdot, \sigma) d\sigma.$$

By Proposition 3.9 with $t_1 = 0$ and $t_2 = h$, and by the change of variables $t = h\sigma$, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\int \frac{u - g}{h} \varphi = -\frac{1}{h} \int_0^h \int Z\left(x, \frac{t}{h}\right) \cdot \nabla \varphi(x) dx dt = -\int_0^1 \int Z \cdot \nabla \varphi = -\int \tilde{\zeta} \cdot \nabla \varphi,$$

that is

$$\frac{\mathcal{T}_h g - g}{h} = \operatorname{div} \tilde{\zeta} \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

By Proposition A.3, we have $\phi^\circ(z) \leq 1$, and by Proposition 2.2, we have $\phi(\nabla U(\cdot, \sigma)) \leq \|\phi(\nabla g)\|_\infty \leq 1$ for all σ , so that

$$(3.34) \quad \phi^\circ(Z) = \phi^{p-1}(\nabla U)\phi^\circ(z) \leq 1,$$

and, by convexity of ϕ° , also $\phi^\circ(\tilde{\zeta}) \leq \int_0^1 \phi^\circ(Z(\cdot, \sigma)) d\sigma \leq 1$. Therefore, for every $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\left| \left\langle \frac{\mathcal{T}_h g - g}{h}, \psi \right\rangle \right| = \left| \int \tilde{\zeta} \cdot \nabla \psi \right| \leq \int \phi^\circ(\tilde{\zeta}) \phi(\nabla \psi) \leq \int \phi(\nabla \psi),$$

which is (2.5) with $C_0 = 1$. This proves (b1).

Property (b2). Let now h_n, E_n, E be as in the statement of (b2), and we set $d_n = d_{E_n}$ and $d = d_E$. By the previous step applied to d_n , we obtain U_n, Z_n and $\tilde{\zeta}_n$ with $\phi^\circ(\tilde{\zeta}_n) \leq 1$ and $S_n = \operatorname{div} \tilde{\zeta}_n$, where $U_n(\cdot, 0) = d_n$ and $U_n(\cdot, 1) = u_n$. Since $U_n(\cdot, \sigma) = \mathcal{T}_{h_n \sigma} d_n$, applying (2.8) with the step $h_n \sigma$ gives $\|U_n(\cdot, \sigma) - d_n\|_\infty \leq \sqrt{c_0 h_n \sigma} \leq$

$\sqrt{c_0 h_n}$, so that

$$(3.35) \quad \sup_{0 \leq \sigma \leq 1} \|U_n(\cdot, \sigma) - d\|_{L^\infty(K)} \longrightarrow 0 \quad \text{for all compact } K \subseteq E^c.$$

We fix $\varphi \in C_c^\infty(E^c)$ and consider the realizing subsequence of $\liminf_n \langle S_n, \varphi \rangle$. By standard compactness theorems, there exists $z \in L^\infty(E^c; \mathbb{R}^N)$ such that $\phi^\circ(z) \leq 1$ and, up to a further subsequence,

$$\begin{aligned} \tilde{\zeta}_n &\overset{*}{\rightharpoonup} z && \text{in } L^\infty(E^c; \mathbb{R}^N), \\ \nabla U_n &\overset{*}{\rightharpoonup} \nabla d && \text{in } L^\infty(E^c \times (0, 1); \mathbb{R}^N). \end{aligned}$$

If we prove that $z \in \partial\phi(\nabla d)$, then we can conclude by

$$\langle S_n, \varphi \rangle \longrightarrow - \int z \cdot \nabla \varphi \geq - \int \phi'(\nabla d; \nabla \varphi).$$

Fixing $\psi \in C_c^\infty(E^c)$ such that $\psi \geq 0$, at almost every inner time we test (3.33) with $d\psi$ and with $U_n\psi$ and we subtract, obtaining

$$\begin{aligned} \int (Z_n \cdot \nabla d)\psi &= - \int d(Z_n \cdot \nabla \psi) - \int \frac{\dot{U}_n}{h_n} d\psi \\ &= \int \phi^p(\nabla U_n)\psi + \int \frac{\dot{U}_n}{h_n} (U_n - d)\psi + \int (U_n - d)(Z_n \cdot \nabla \psi), \end{aligned}$$

where we used that $Z_n \cdot \nabla U_n = \phi^p(\nabla U_n)$, since $z_n \in \partial\phi(\nabla U_n)$. This is the identity of Proposition 3.5, between functions of the inner time. For the second term of the RHS, setting $w_n = U_n - d$, which has the same derivative as U_n because d does not depend on the inner time, we have

$$\frac{\dot{U}_n}{h_n} (U_n - d) = \frac{w_n \dot{w}_n}{h_n} = \frac{(\dot{w}_n^2)}{2h_n},$$

so that, by the fundamental theorem of calculus and by $U_n(\cdot, 0) = d_n$ and $U_n(\cdot, 1) = u_n$,

$$\int_0^1 \frac{\dot{U}_n}{h_n} (U_n - d) = \frac{(u_n - d)^2 - (d_n - d)^2}{2h_n} = S_n(\theta_n - d), \quad \theta_n = \frac{u_n + d_n}{2},$$

the chain rule being legitimate because $\sigma \mapsto U_n(\cdot, \sigma)$ is absolutely continuous with values in $L_{\text{loc}}^2(\mathbb{R}^N)$ by (3.25). Integrating the identity on the inner time interval we therefore obtain

$$(3.36) \quad \int (\tilde{\zeta}_n \cdot \nabla d)\psi = \int_0^1 \int \phi^p(\nabla U_n)\psi + \int S_n(\theta_n - d)\psi + \int_0^1 \int (U_n - d)(Z_n \cdot \nabla \psi).$$

For the first term of the RHS, we have $\int_0^1 \int \phi^p(\nabla U_n)\psi \leq \int \psi$, together with

$$\int \psi = \int_0^1 \int \phi^p(\nabla d)\psi \leq \liminf_{n \rightarrow \infty} \int_0^1 \int \phi^p(\nabla U_n)\psi$$

by weak-* lower semicontinuity, so that $\int_0^1 \int \phi^p(\nabla U_n)\psi \rightarrow \int \psi$. For the second term, the bound (3.17) holds verbatim here, its proof using only (2.9) and $\phi^\circ(\tilde{\zeta}_n) \leq 1$, so that, with $K = \text{supp } \psi$,

$$\left| \int S_n(\theta_n - d)\psi \right| \leq C \|\theta_n - d\|_{L^\infty(K)} \longrightarrow 0$$

by (3.35). Also the third term goes to zero, since by (3.34)

$$\left| \int_0^1 \int (U_n - d)(Z_n \cdot \nabla \psi) \right| \leq \sup_{0 \leq \sigma \leq 1} \|U_n(\cdot, \sigma) - d\|_{L^\infty(K)} \int \phi(\nabla \psi) \longrightarrow 0,$$

so we can pass to the limit as

$$\int (z \cdot \nabla d)\psi = \int \psi.$$

This means that $z \cdot \nabla d = 1$, which, together with $\phi^\circ(z) \leq 1$, implies $z \in \partial\phi(\nabla d)$. $\square$

THEOREM 3.12 (Convergence of the parabolic scheme). *Let ϕ be a norm on $\mathbb{R}^N$, let $1 < p < \infty$ and let $E_0 \subseteq \mathbb{R}^N$ be closed. Then the discrete evolutions generated by the parabolic scheme (3.18) converge, up to a subsequence and in the sense of Theorem 2.14, to a superflow with initial condition E_0 in the sense of Definition 1.8.*

PROOF. Properties (i) to (v) are proved in Proposition 3.8, property (vi) holds by extension, the estimate (2.4) in Proposition 3.10, and properties (b1) and (b2) in Proposition 3.11. The conclusion is then Theorem 2.16. $\square$

3.3. A convolution kernel scheme

The last scheme we present is the simplest and most explicit of the three. However, it is highly limited, since it applies only to the anisotropies induced by a scalar product.

DEFINITION 3.13 (Elliptic anisotropy). We say that the norm ϕ is *elliptic* if there exists a symmetric positive definite matrix A such that

$$\phi(\xi) = \sqrt{A\xi \cdot \xi}.$$

REMARK 3.2 (Polar norm). One can easily prove that the polar norm is

$$\phi^\circ(x) = \sqrt{A^{-1}x \cdot x}.$$

Throughout this section ϕ is assumed to be elliptic with matrix A .

DEFINITION 3.14 (Elliptic kernel). We say that a function $K \in L^1(\mathbb{R}^N)$ is an *elliptic kernel* with matrix A if the following hold:

- (i) $K \geq 0$;
- (ii) $\int K = 1$;
- (iii) $K(-x) = K(x)$ for all $x \in \mathbb{R}^N$;
- (iv) $\int K(y)y \, dy = 0$;
- (v) $\int K(x)x \otimes x \, dx = 2A$.

EXAMPLE 3.1 (Gaussian kernel). A first example is the anisotropic Gaussian kernel

$$K(x) = \frac{1}{(4\pi)^{N/2} \sqrt{\det A}} \exp\left(-\frac{1}{4}x \cdot A^{-1}x\right),$$

which can be checked to be an elliptic kernel with matrix A . Note that the convolution kernel scheme with the anisotropic Gaussian kernel is equivalent to the parabolic scheme in the case $p = 2$.

EXAMPLE 3.2 (Indicator kernel). Let $W = W_{\sqrt{2(N+2)}^\phi}^\phi$ be the Wulff shape of radius $2(N+2)$, which is the ellipsoid

$$W = \{x \in \mathbb{R}^N : A^{-1}x \cdot x \leq 2(N+2)\}.$$

One can check that $K = \frac{1}{|W|}1_W$ is an elliptic kernel with matrix A .

Given an elliptic kernel K with matrix A , we set $K_h = \frac{1}{h^N}K(\frac{x}{h})$ for all $h > 0$ and we define the scheme as

$$(3.37) \quad \mathcal{T}_h g = K_{\sqrt{h}} * g.$$

PROPOSITION 3.15 (Verification of the axioms). Let $h > 0$. The operator $\mathcal{T}_h$ defined by (3.37) is well-defined on X and satisfies properties (i) to (vi) of Definition 2.1.

PROOF. Let $g \in \text{Lip}(\mathbb{R}^N)$ with Lipschitz constant L . Since $|g(x-y)| \leq |g(x)| + L|y|$ and $\int K_{\sqrt{h}} = 1$, we have

$$|K_{\sqrt{h}} * g(x)| \leq \int K_{\sqrt{h}}(y)|g(x-y)| \, dy \leq |g(x)| + L \int K_{\sqrt{h}}(y)|y| \, dy < +\infty$$

and

$$|K_{\sqrt{h}} * g(x) - K_{\sqrt{h}} * g(y)| \leq \int K_{\sqrt{h}}(z)|g(x-z) - g(y-z)| \, dz \leq L|x-y|.$$

This means that $K_{\sqrt{h}} * g$ is Lipschitz, which is property (ii), and we extend $\mathcal{T}_h$ to X by setting $\mathcal{T}_h(\pm\infty) = \pm\infty$, which is property (vi). Property (i) is immediate from $K_{\sqrt{h}} \geq 0$, property (iii) from $\int K_{\sqrt{h}} = 1$, property (iv) from the linearity of the convolution, and property (v) from its translation invariance. $\square$

PROPOSITION 3.16 (Wulff shape estimate). The family $\{\mathcal{T}_h\}$ defined by (3.37) satisfies (2.4) with $c_0 = 2N$, that is

$$\mathcal{T}_h \phi^\circ \leq \sqrt{(\phi^\circ)^2 + 2Nh}.$$

PROOF. By a simple computation,

$$K_{\sqrt{h}} * \phi^\circ(x) = \frac{1}{h^{\frac{N}{2}}} \int K\left(\frac{y}{\sqrt{h}}\right) \sqrt{A^{-1}(x-y) \cdot (x-y)} \, dy = \int K(y) \sqrt{A^{-1}(x - \sqrt{h}y) \cdot (x - \sqrt{h}y)} \, dy.$$

Since $\int K = 1$, by Jensen's inequality and expanding the product,

$$K_{\sqrt{h}} * \phi^\circ(x)^2 \leq A^{-1}x \cdot x + hA^{-1} \cdot \int K(y)y \otimes y \, dy = \phi^\circ(x)^2 + 2hA^{-1} \cdot A = \phi^\circ(x)^2 + 2Nh. \quad \square$$

It remains to verify properties (b1) and (b2) of Definition 2.6.

REMARK 3.3 (Consistency on smooth functions). Let $h > 0$ and let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then

$$\frac{\mathcal{T}_h \varphi - \varphi}{h} \rightarrow \Delta^\phi \varphi \quad \text{uniformly as } h \rightarrow 0.$$

PROOF. Let $x \in \mathbb{R}^N$. We have

$$\begin{aligned} K_{\sqrt{h}} * \varphi(x) - \varphi(x) &= \frac{1}{h^{\frac{N}{2}}} \int K\left(\frac{y}{\sqrt{h}}\right) (\varphi(x - y) - \varphi(x)) dy \\ &= \int K(y) (\varphi(x - \sqrt{h}y) - \varphi(x)) dy \\ &= \int K(y) \left(-\sqrt{h} \nabla \varphi(x) \cdot y + h \int_0^1 (1-t) \nabla^2 \varphi(x - t\sqrt{h}y) \cdot (y \otimes y) dt \right) dy \\ &= h \int K(y) \int_0^1 (1-t) \nabla^2 \varphi(x - t\sqrt{h}y) \cdot (y \otimes y) dt dy. \end{aligned}$$

We also have $\Delta^\phi \varphi = \operatorname{div}(A \nabla \varphi) = A \cdot \nabla^2 \varphi$, so that

$$\Delta^\phi \varphi(x) = \int K(y) \int_0^1 (1-t) \nabla^2 \varphi(x) \cdot (y \otimes y) dt dy$$

and

$$\frac{K_{\sqrt{h}} * \varphi(x) - \varphi(x)}{h} - \Delta^\phi \varphi(x) = \int K(y) \int_0^1 (1-t) (\nabla^2 \varphi(x - t\sqrt{h}y) - \nabla^2 \varphi(x)) \cdot (y \otimes y) dt dy.$$

We have the two bounds

$$\begin{aligned} |\nabla^2 \varphi(x - t\sqrt{h}y) - \nabla^2 \varphi(x)| &\leq 2 \|\nabla^2 \varphi\|_\infty, \\ |\nabla^2 \varphi(x - t\sqrt{h}y) - \nabla^2 \varphi(x)| &\leq t\sqrt{h} \|\nabla^3 \varphi\|_\infty |y|. \end{aligned}$$

Fixing $R > 0$ and using the bounds, it follows

$$\left\| \frac{K_{\sqrt{h}} * \varphi - \varphi}{h} - \Delta^\phi \varphi \right\|_\infty \leq \sqrt{h} \frac{\|\nabla^3 \varphi\|_\infty}{6} \int_{B_R} K(y) |y|^3 dy + \|\nabla^2 \varphi\|_\infty \int_{B_R^c} K(y) |y|^2 dy.$$

The result follows by letting $h \rightarrow 0$ and from the arbitrariness of R . $\square$

PROPOSITION 3.17 (Properties (b1) and (b2)). *The family $\{\mathcal{T}_h\}$ defined by (3.37) satisfies properties (b1) and (b2) of Definition 2.6 with $C_0 = N$.*

PROOF. **Property (b1).** Let $h > 0$ and let $g \in \operatorname{Lip}(\mathbb{R}^N)$ with $\phi(\nabla g) \leq 1$. We fix $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since $g(x) - g(x - y) = \int_0^1 \nabla g(x - sy) \cdot y ds$ and $\int K_{\sqrt{h}}(y) y dy = 0$ by property (iv) of K , we have

$$\begin{aligned} \int \frac{\mathcal{T}_h g - g}{h} \varphi &= -\frac{1}{h} \iint K_{\sqrt{h}}(y) (g(x) - g(x - y)) \varphi(x) dy dx \\ &= -\frac{1}{h} \int K_{\sqrt{h}}(y) \int_0^1 \left(\int \nabla g(x - sy) \varphi(x) dx \right) \cdot y ds dy \\ &= \frac{1}{h} \int K_{\sqrt{h}}(y) \int_0^1 \left(\int g(x - sy) \nabla \varphi(x) dx \right) \cdot y ds dy \\ &= \frac{1}{h} \iint K_{\sqrt{h}}(y) \left(\int_0^1 (g(x - sy) - g(x)) ds \right) y \cdot \nabla \varphi(x) dx dy. \end{aligned}$$

Writing again the increment as an integral of the gradient, we have

$$\int_0^1 (g(x - sy) - g(x)) ds = - \int_0^1 (1 - \sigma) \nabla g(x - \sigma y) \cdot y d\sigma,$$

so that, by the change of variables $y = \sqrt{h}w$,

$$(3.38) \quad \frac{\mathcal{T}_h g - g}{h} = \operatorname{div} \tilde{\zeta}_h \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

$$\tilde{\zeta}_h(x) = \int K(w) \int_0^1 (1 - \sigma) (\nabla g(x - \sigma \sqrt{h}w) \cdot w) w d\sigma dw.$$

Since $\left| \nabla g(x - \sigma \sqrt{h}w) \cdot w \right| \leq \phi^\circ(w)$ and $|w \cdot \xi| \leq \phi^\circ(w)\phi(\xi)$ for all $\xi \in \mathbb{R}^N$, we have

$$\left| \tilde{\zeta}_h(x) \cdot \xi \right| \leq \frac{1}{2} \left(\int K(w) \phi^\circ(w)^2 dw \right) \phi(\xi) = N\phi(\xi),$$

where $\int K(w) \phi^\circ(w)^2 dw = A^{-1} \cdot \int K(w) w \otimes w dw = 2N$ by property (v) of K , that is $\phi^\circ(\tilde{\zeta}_h) \leq N$. Therefore

$$\left| \int \frac{\mathcal{T}_h g - g}{h} \varphi \right| = \left| \int \tilde{\zeta}_h \cdot \nabla \varphi \right| \leq \int \phi^\circ(\tilde{\zeta}_h) \phi(\nabla \varphi) \leq N \int \phi(\nabla \varphi),$$

which is (2.5) with $C_0 = N$.

Property (b2). Let $h_n \rightarrow 0$ and let E_n, E be as in the statement. We set $d_n = d_{E_n}$ and $d = d_E$, and we denote by $\tilde{\zeta}_n$ the field of (3.38) with $g = d_n$ and $h = h_n$. By Proposition A.8 we have $d_n \rightarrow d$ locally uniformly in E^c , so that

$$\nabla d_n \xrightarrow{*} \nabla d \quad \text{in } L^\infty(E^c; \mathbb{R}^N).$$

Fixing $\psi \in C_c^\infty(E^c; \mathbb{R}^N)$, by Fubini's theorem we have

$$\int \tilde{\zeta}_n \cdot \psi = \int K(w) \int_0^1 (1 - \sigma) \left(\int (\nabla d_n(x - \sigma \sqrt{h_n}w) \cdot w) (w \cdot \psi(x)) dx \right) d\sigma dw.$$

For every w and every σ we have $\psi(\cdot + \sigma \sqrt{h_n}w) \rightarrow \psi$ in $L^1(\mathbb{R}^N; \mathbb{R}^N)$, so that the inner integral converges to $\int (\nabla d \cdot w)(w \cdot \psi)$; moreover, since $\phi(\nabla d_n) = 1$ by the eikonal equation, it is bounded by $\phi^\circ(w)^2 \int \phi(\psi)$, and $K(\phi^\circ)^2$ is integrable by property (v) of K . By dominated convergence it follows that

$$\int \tilde{\zeta}_n \cdot \psi \rightarrow \frac{1}{2} \int \left(\left(\int K(w) w \otimes w dw \right) \nabla d \right) \cdot \psi = \int (A \nabla d) \cdot \psi,$$

that is $\tilde{\zeta}_n \xrightarrow{*} A \nabla d$ in $L^\infty(E^c; \mathbb{R}^N)$. Finally, since ϕ is elliptic we have $\nabla \phi(\xi) = \frac{A\xi}{\phi(\xi)}$ for $\xi \neq 0$, so that $\partial \phi(\xi)$ is the singleton $\left\{ \frac{A\xi}{\phi(\xi)} \right\}$ and, by the eikonal equation,

$$\phi'(\nabla d; \nabla \varphi) = A \nabla d \cdot \nabla \varphi \quad \text{in } E^c.$$

We conclude that, for every $\varphi \in C_c^\infty(E^c)$,

$$\left\langle \frac{\mathcal{T}_{h_n} d_n - d_n}{h_n}, \varphi \right\rangle = - \int \tilde{\zeta}_n \cdot \nabla \varphi \rightarrow - \int A \nabla d \cdot \nabla \varphi = - \int \phi'(\nabla d; \nabla \varphi),$$

so that (2.6) holds with equality. $\square$

THEOREM 3.18 (Convergence of the convolution scheme). *Let ϕ be an elliptic anisotropy and let $E_0 \subseteq \mathbb{R}^N$ be closed. Then the discrete evolutions generated by the convolution scheme (3.37) converge, up to a subsequence and in the sense of Theorem 2.14, to a superflow with initial condition E_0 in the sense of Definition 1.8.*

PROOF. By Propositions 3.15, 3.16 and 3.17, the family $\{\mathcal{T}_h\}$ satisfies all the hypotheses of Chapter 2, so the conclusion is Theorem 2.16. $\square$

Auxiliary results

A.1. Anisotropic geometry

In this section we present some geometrical results in the presence of an anisotropy ϕ , which is assumed to be a norm on $\mathbb{R}^N$. We assume Rademacher's theorem, which says that a Lipschitz function is a.e. differentiable.

A.1.1. Polar norm.

DEFINITION A.1 (Polar norm). We define the *polar norm* as

$$\phi^\circ(\xi) = \sup_{\phi(x) \leq 1} \xi \cdot x,$$

again a norm on $\mathbb{R}^N$, and $\phi^{\circ\circ} = \phi$. Moreover, $\xi \cdot x \leq \phi(x)\phi^\circ(\xi)$.

DEFINITION A.2 (Wulff shape). Given $R > 0$, we define the *Wulff shape* of radius R as

$$W_R^\phi = \{\phi^\circ \leq R\}.$$

PROPOSITION A.3 (Subdifferential characterization). We have $\xi \in \partial\phi(x)$ iff $\phi^\circ(\xi) \leq 1$ and $\xi \cdot x = \phi(x)$.

PROOF. If $\xi \in \partial\phi(x)$, then

$$\phi(y) \geq \phi(x) + \xi \cdot (y - x)$$

for all $y \in \mathbb{R}^N$. Testing the inequality with $y = 0$ and $y = 2x$ yields respectively $\phi(x) \leq \xi \cdot x$ and $\phi(x) \geq \xi \cdot x$, proving that $\phi(x) = \xi \cdot x$. The inequality can be rewritten as

$$\xi \cdot y \leq \phi(y)$$

and taking the supremum over $\phi(y) \leq 1$ yields $\phi^\circ(\xi) \leq 1$. Conversely, if $\phi^\circ(\xi) \leq 1$ and $\xi \cdot x = \phi(x)$, then

$$\xi \cdot y \leq \phi^\circ(\xi)\phi(y) \leq \phi(y)$$

and the subdifferential inequality is satisfied. □

PROPOSITION A.4 (Lipschitz bound characterization). Let $u \in \text{Lip}(\mathbb{R}^N)$. Then $\|\phi(\nabla u)\|_\infty \leq L$ iff

$$|u(x) - u(y)| \leq L\phi^\circ(x - y)$$

for all $x, y \in \mathbb{R}^N$.

PROOF. If $\|\phi(\nabla u)\|_\infty \leq L$, then

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_0^1 \nabla u(y + s(x - y)) \cdot (x - y) ds \right| \\ &\leq \int_0^1 \phi(\nabla u(y + s(x - y)))\phi^\circ(x - y) ds \\ &\leq L\phi^\circ(x - y), \end{aligned}$$

where we used that a Lipschitz function is absolutely continuous when restricted to a line. Conversely, let $\xi \in \mathbb{R}^N$ such that $\phi^\circ(\xi) \leq 1$. For a.e. $x \in \mathbb{R}^N$ we have

$$\nabla u(x) \cdot \xi = \lim_{h \rightarrow 0} \frac{u(x + h\xi) - u(x)}{h} \leq L\phi^\circ(\xi) \leq L.$$

Taking the supremum over $\phi^\circ(\xi) \leq 1$ yields

$$\phi(\nabla u(x)) \leq L,$$

concluding the proof. □

A.1.2. Anisotropic distance. All symbols in this subsection are understood to refer to the Euclidean norm when the superscript ϕ is not present. We denote by

$$d^\phi(x, y) = \phi^\circ(x - y)$$

the induced distance on $\mathbb{R}^N$, and for sets $E, F \subseteq \mathbb{R}^N$ we set

$$\text{dist}^\phi(E, F) = \inf_{x \in E, y \in F} d^\phi(x, y),$$

writing $\text{dist}^\phi(x, E)$ for $\text{dist}^\phi(\{x\}, E)$. The (anisotropic) *signed distance* to E is

$$\text{sd}_E^\phi(x) = \text{dist}^\phi(x, E) - \text{dist}^\phi(x, E^c),$$

so that $\text{sd}_E^\phi < 0$ in the interior of E and $\text{sd}_E^\phi > 0$ outside.

REMARK A.1 (First properties). The following hold:

- (1) $\text{sd}_\emptyset^\phi = +\infty$ and $\text{sd}_{\mathbb{R}^N}^\phi = -\infty$;
- (2) $\text{sd}_{E^c}^\phi = -\text{sd}_E^\phi$.

We prove some basic properties of the signed distance, assuming that the sets are closed (so that the anisotropic distance to the set is realized at some point) and not in $\{\emptyset, \mathbb{R}^N\}$.

PROPOSITION A.5 (Eikonal equation). sd_E^ϕ is Lipschitz and $\phi(\nabla \text{sd}_E^\phi) = 1$ a.e.

PROOF. Let $z \in E$. Since d^ϕ is a distance, we have $\text{dist}^\phi(x, E) \leq d^\phi(x, z) \leq d^\phi(x, y) + d^\phi(y, z)$. Taking the infimum over $z \in E$ yields $\text{dist}^\phi(x, E) \leq d^\phi(x, y) + \text{dist}^\phi(y, E)$, so that $\text{dist}^\phi(x, E) - \text{dist}^\phi(y, E) \leq d^\phi(x, y)$. Exchanging the role of x and y , it follows that $\text{dist}^\phi(\cdot, E)$ is Lipschitz with respect to d^ϕ , and so sd_E^ϕ as well. Since all norms on $\mathbb{R}^N$ are equivalent, sd_E^ϕ is also Lipschitz with respect to the Euclidean norm. Let $x \in \mathbb{R}^N$ be a point of differentiability of sd_E^ϕ . We have

$$\xi \cdot \nabla \text{sd}_E^\phi(x) = \lim_{h \rightarrow 0} \frac{\text{sd}_E^\phi(x + h\xi) - \text{sd}_E^\phi(x)}{h} \leq \lim_{h \rightarrow 0} \frac{d^\phi(x, x + h\xi)}{h} = \phi^\circ(\xi).$$

Taking the supremum over $\phi^\circ(\xi) \leq 1$ yields $\phi(\nabla \text{sd}_E^\phi(x)) \leq 1$. On the other hand, assume $x \in E^c$, and let $y \in E$ such that $\text{sd}_E^\phi(x) = d^\phi(x, y)$. With $\xi = x - y$, we have

$$\frac{\text{sd}_E^\phi(x) - \text{sd}_E^\phi(x - h\xi)}{h} \geq \frac{d^\phi(x, y) - d^\phi(x - h\xi, y)}{h} = \phi^\circ(\xi)$$

for $0 < h < 1$. Taking the limit as $h \rightarrow 0^+$ yields $\nabla \text{sd}_E^\phi(x) \cdot \xi \geq \phi^\circ(\xi)$, so that

$$\phi^\circ(\xi) \phi(\nabla \text{sd}_E^\phi(x)) \geq \xi \cdot \nabla \text{sd}_E^\phi(x) \geq \phi^\circ(\xi).$$

Since $\xi \neq 0$, we have $\phi(\nabla \text{sd}_E^\phi(x)) \geq 1$, and we can conclude that $\phi(\nabla \text{sd}_E^\phi(x)) = 1$. If $x \in E$, then $\phi(\nabla \text{sd}_E^\phi(x)) = \phi(-\nabla \text{sd}_{E^c}^\phi(x)) = \phi(\nabla \text{sd}_{E^c}^\phi(x)) = 1$, since $x \in (E^c)^c$. $\square$

PROPOSITION A.6 (Distance characterization). If $E \cap F = \emptyset$ and $E, F \notin \{\emptyset, \mathbb{R}^N\}$, then

$$(A.1) \quad \text{dist}^\phi(E, F) \leq \text{sd}_E^\phi + \text{sd}_F^\phi$$

and

$$(A.2) \quad \text{dist}^\phi(E, F) = \inf_{(E \cup F)^c} (\text{sd}_E^\phi + \text{sd}_F^\phi).$$

PROOF. If $x \in (E \cup F)^c$, then $\text{sd}_E^\phi(x) = d^\phi(x, a)$ and $\text{sd}_F^\phi(x) = d^\phi(x, b)$ for some $a \in E$ and $b \in F$, so that

$$\text{dist}^\phi(E, F) \leq d^\phi(a, b) \leq d^\phi(a, x) + d^\phi(x, b) = \text{sd}_E^\phi(x) + \text{sd}_F^\phi(x).$$

If $x \in E$, then $\text{sd}_E^\phi(x) = -d^\phi(x, a)$ for some $a \in \overline{E^c}$ and $\text{sd}_F^\phi(x) = d^\phi(x, b)$ for some $b \in F$. Since $x \in E$ and $b \in E^c$, there exists $a' \in \partial E$ such that $a' \in [x, b)$. It follows that

$$\text{sd}_E^\phi(x) + \text{sd}_F^\phi(x) = d^\phi(x, b) - d^\phi(x, a) = d^\phi(x, a') + d^\phi(a', b) - d^\phi(x, a) \geq d^\phi(a', b) \geq \text{dist}^\phi(E, F).$$

If $x \in F$ the argument is symmetrical. Finally, let $a \in E$ and $b \in F$ such that $\text{dist}^\phi(E, F) = d^\phi(a, b)$. Since $(a, b) \subseteq (E \cup F)^c$, the infimum of $\text{sd}_E^\phi + \text{sd}_F^\phi$ is attained on $(E \cup F)^c$. $\square$

A.1.3. Kuratowski convergence. The definitions and the results of this subsection are stated for closed subsets of $\mathbb{R}^N$, but they hold verbatim, with the same proofs, for closed subsets of any Euclidean space. In Chapter 2 they are applied both to closed subsets of $\mathbb{R}^N$ and to closed subsets of the space-time $\mathbb{R}^N \times [0, +\infty)$.

DEFINITION A.7 (Kuratowski convergence). Let $\{E_n\}$ be a sequence of closed sets of $\mathbb{R}^N$ and let $E \subseteq \mathbb{R}^N$ be a closed set. We say that $E_n \xrightarrow{K} E$ iff the following properties are satisfied up to deleting finitely many terms of the sequence:

- (i) If $x_n \in E_n$ and $x_n \rightarrow x$, then $x \in E$;
- (ii) If $x \in E$, then there exists a sequence $\{x_n\}$ such that $x_n \in E_n$ and $x_n \rightarrow x$.

PROPOSITION A.8 (Kuratowski characterization). $E_n \xrightarrow{K} E$ iff $\text{dist}^\phi(\cdot, E_n) \rightarrow \text{dist}^\phi(\cdot, E)$ locally uniformly.

PROOF. We assume that $E_n, E \neq \emptyset$.

Let $E_n \xrightarrow{K} E$. Firstly, we prove pointwise convergence. Let $x \in \mathbb{R}^N$ and let $y \in E$ such that $\phi^\circ(x - y) = \text{dist}^\phi(x, E)$. By property (ii), there exists a sequence $\{y_n\}$ such that $y_n \in E_n$ and $y_n \rightarrow y$. Then $\text{dist}^\phi(x, E_n) \leq \phi^\circ(x - y_n)$, so that

$$\limsup_n \text{dist}^\phi(x, E_n) \leq \phi^\circ(x - y) = \text{dist}^\phi(x, E).$$

Let $z_n \in E_n$ such that $\phi^\circ(x - z_n) = \text{dist}^\phi(x, E_n)$. Let $\{k_n\}$ be a sequence such that $\phi^\circ(x - z_{k_n}) \rightarrow \liminf_n \text{dist}^\phi(x, E_n)$. Then $z_{k_n} \rightarrow z$ up to a subsequence and $z \in E$ by property (i), so that

$$\liminf_n \text{dist}^\phi(x, E_n) = \phi^\circ(x - z) \geq \text{dist}^\phi(x, E).$$

Since $\text{dist}^\phi(\cdot, E_n)$ are equi-Lipschitz, pointwise convergence is upgraded to locally uniform convergence.

Conversely, let $\text{dist}^\phi(\cdot, E_n) \rightarrow \text{dist}^\phi(\cdot, E)$ locally uniformly. If $x_n \in E_n$ and $x_n \rightarrow x$, then $0 = \text{dist}^\phi(x_n, E_n) \rightarrow \text{dist}^\phi(x, E)$, so that $x \in E$ since E is closed, proving property (i). Let $x \in E$ and let $\{x_n\}$ be a sequence such that $x_n \in E_n$ and $\phi^\circ(x - x_n) = \text{dist}^\phi(x, E_n)$. Then $\phi^\circ(x - x_n) \rightarrow \text{dist}^\phi(x, E) = 0$, proving property (ii). $\square$

COROLLARY A.9 (Kuratowski compactness). Let $\{E_n\}$ be a sequence of closed sets of $\mathbb{R}^N$. Then there exists a closed set $E \subseteq \mathbb{R}^N$ such that

$$E_n \xrightarrow{K} E$$

up to a subsequence.

PROOF. If $E_n = \emptyset$ for infinitely many n , then along that subsequence $E_n \xrightarrow{K} \emptyset$, so we may assume that $E_n \neq \emptyset$ for all n . We consider the functions $u_n = \text{dist}(\cdot, E_n)$, which are equi-Lipschitz. If they are also locally equibounded, by the Ascoli-Arzelà theorem we have $u_n \rightarrow u$ locally uniformly up to a subsequence, where u is 1-Lipschitz and nonnegative. We set $E = \{u = 0\}$, which is closed. We want to prove that $u = \text{dist}(\cdot, E)$. Let $x \in \mathbb{R}^N$ and choose $y \in E$ such that $|x - y| = \text{dist}(x, E)$. We have

$$u(x) \leq u(y) + |x - y| = \text{dist}(x, E).$$

Now choose a sequence $y_n \in E_n$ such that $|x - y_n| = \text{dist}(x, E_n) = u_n(x)$. Since $u_n(x) \rightarrow u(x)$, the sequence y_n is bounded, hence $y_n \rightarrow y$ up to a subsequence. It follows that $|x - y| = u(x)$ and also $0 = u_n(y_n) \rightarrow u(y)$, so that $y \in E$. Therefore,

$$\text{dist}(x, E) \leq |x - y| = u(x),$$

proving that $u = \text{dist}(\cdot, E)$. If u_n are not locally equibounded, then there exists $x_0 \in \mathbb{R}^N$ such that $u_n(x_0) \rightarrow +\infty$ up to a subsequence. Since $u_n(x) \geq u_n(x_0) - |x - x_0|$, we have $u_n \rightarrow +\infty$ locally uniformly. In this case we set $E = \emptyset$. In both cases we can conclude by Proposition A.8. $\square$

A.2. Functional analysis

The following lemma is the only functional-analytic ingredient of the construction.

LEMMA A.10 (Representation lemma). Let $k \geq 0$, let $\Omega \subseteq \mathbb{R}^N \times \mathbb{R}^k$ be open and denote by ∇ the gradient with respect to the first N variables. Let $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be convex, let $\xi \in L^\infty(\Omega; \mathbb{R}^N)$ and let $S \in \mathcal{D}'(\Omega)$. Then the following are equivalent:

- (i) There exists $z \in L^\infty(\Omega; \mathbb{R}^N)$ such that $z \in \partial\Phi(\xi)$ and

$$S = \text{div } z \quad \text{in } \mathcal{D}'(\Omega);$$

(ii) For every $\varphi \in C_c^\infty(\Omega)$,

$$\langle S, \varphi \rangle \geq - \int_{\Omega} \Phi'(\xi; \nabla \varphi).$$

PROOF. The proof is based on an argument similar to the one used in the proof of Proposition 1.7. $\square$

COROLLARY A.11 (Bounded fields). *Let Ω and ∇ be as in Lemma A.10. Let $S \in \mathcal{D}'(\Omega)$ and let $C > 0$. Then $S = \operatorname{div} z$ for some $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\phi^\circ(z) \leq C$ iff*

$$\langle S, \varphi \rangle \geq -C \int_{\Omega} \phi(\nabla \varphi)$$

for all $\varphi \in C_c^\infty(\Omega)$.

PROOF. Since $\phi'(0; \eta) = \phi(\eta)$ and $\partial\phi(0) = \{\phi^\circ \leq 1\}$, the proof is a simple application of Lemma A.10. $\square$

A.3. Inequalities

In this section we collect two elementary inequalities, both used in Chapter 3.

LEMMA A.12 (Mean-like inequality). *Let $0 < \alpha < \beta$ and $t \geq 0$. Then*

$$(t^\alpha + 1)^{1/\alpha} \geq (t^\beta + 1)^{1/\beta}.$$

PROOF. With $u = t^\alpha$, we have

$$\begin{aligned} \frac{d}{d\alpha} \left(\frac{1}{\alpha} \log(t^\alpha + 1) \right) &= -\frac{1}{\alpha^2} \log(t^\alpha + 1) + \frac{1}{\alpha} \frac{t^\alpha \log t}{t^\alpha + 1} \\ &= -\frac{1}{\alpha^2} \log(u + 1) + \frac{1}{\alpha^2} \frac{u \log u}{u + 1} \\ &= \frac{1}{\alpha^2} \frac{1}{u + 1} (u \log u - (u + 1) \log(u + 1)). \end{aligned}$$

If $u < 1$, then $\log u < 0$ and $\log(u + 1) > 0$, so $\frac{d}{d\alpha} \left(\frac{1}{\alpha} \log(t^\alpha + 1) \right) \leq 0$. If $u \geq 1$, then $u + 1 > u > 0$ and $\log(u + 1) > \log(u) \geq 0$, so again $\frac{d}{d\alpha} \left(\frac{1}{\alpha} \log(t^\alpha + 1) \right) \leq 0$. $\square$

LEMMA A.13 (L^2/L^∞ inequality). *Let $R > 0$ and $u \in \operatorname{Lip}(B_R)$. With $L = \|\nabla u\|_{L^\infty(B_R)}$, if*

$$(A.3) \quad \|u\|_{L^2(B_R)} < L \omega_N^{1/2} \left(\frac{R}{2} \right)^{\frac{N+2}{2}},$$

then

$$\|u\|_{L^\infty(B_{R/2})} \leq 2 \omega_N^{-\frac{1}{N+2}} L^{\frac{N}{N+2}} \|u\|_{L^2(B_R)}^{\frac{2}{N+2}}.$$

PROOF. If $\|u\|_{L^2(B_R)} = 0$ then the conclusion is trivial, so assume that $\|u\|_{L^2(B_R)} > 0$. Let $x_0 \in B_{R/2}$. For all $0 < \rho < \frac{R}{2}$ we have

$$\begin{aligned} (A.4) \quad |u(x_0)| &\leq \left| u(x_0) - \oint_{B_\rho(x_0)} u \right| + \left| \oint_{B_\rho(x_0)} u \right| \\ &\leq \oint_{B_\rho(x_0)} |u - u(x_0)| + \frac{1}{|B_\rho|} |B_\rho|^{1/2} \|u\|_{L^2(B_\rho(x_0))} \\ &\leq L \oint_{B_\rho(x_0)} |x - x_0| + |B_\rho|^{-1/2} \|u\|_{L^2(B_R)} \\ &\leq L \rho + \omega_N^{-1/2} \|u\|_{L^2(B_R)} \rho^{-N/2}. \end{aligned}$$

If we choose

$$\rho = \omega_N^{-\frac{1}{N+2}} L^{-\frac{2}{N+2}} \|u\|_{L^2(B_R)}^{\frac{2}{N+2}},$$

then $\rho < \frac{R}{2}$ by (A.3) and $\rho > 0$. Plugging it in (A.4) yields the conclusion. $\square$

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