UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Towards multi-center AdS black holes

Supervisor

Prof. Gianguido Dall'Agata

Candidate

Mario Giacobbo

Accademic Year 2021/2022

Abstract

In the last decades interest in black holes with an anti-de Sitter asymptotic behaviour has been renewed, mainly thanks to the possibilities opened by the gauge/gravity correspondence. Progress has been made in reproducing the Bekenstein–Hawking entropy through the counting of degenerate microstates by making use of the dual conformal field theories. Despite this, we still lack a proper understanding of these solutions from the gravity side, mainly with respect to rotating solutions and multi-center configurations. This work is motivated by recent developments: the finding of a first order description for rotating BPS black holes in gauged supergravity and of stable bound states from probe analysis in an AdS black hole background. The aim of this work is to recover the first order description of rotating BPS black holes in gauged supergravity in a duality invariant formulation and to work out possible developments with the ambition of proving the existence of multi-center configurations. The flow equations are formulated in an explicit way by making use of the symplectic invariance of the vector sector of the theory, starting from an appropriate ansatz for the metric and vector fields. We provide an analysis of the solutions in a simplified setting and this gives hints as to which is the role of the various quantities entering these equations. Lastly we show that, in this simplified case, a superpotential drives the flow of the scalar fields and warp factors.

Contents

Introduction

Black holes in General Relativity

General relativity is the classical and relativistic theory describing the gravitational interaction as a geometric property of spacetime. Black holes are peculiar solutions of the Einstein equations of general relativity which are characterised by region of space, enclosed by the so called event horizon, where the gravitational attraction becomes so strong that no ingoing particles can ever come back. A solution with these characteristics was found by Schwarzschild as early as 1916 [1], even though it was not interpreted as a black hole at first. In this solution the event horizon presents itself as a coordinate singularity which can be removed with an appropriate extension of the coordinates [2], although, even with such an extension, the metric inside of the horizon is still affected by singularities. The fact that the Einstein equations can have solutions that evolve from a smooth asymptotic configuration to a central singularity is a consequence of their non-linearity [3]. It is now known that black holes are ubiquitous in our universe and can form as the final step in the evolution of stars, provided that these are massive enough to completely collapse under their own gravitational attraction. The first astrophysical object identified as a black hole was Cygnus X-1 [4, 5], while recent and remarkable observations are the first black hole merger through its gravitational wave's signature by the LIGO–Virgo collaboration [6] and the first direct image of a black hole shadow by the EHT collaboration [7]. The cosmic censorship hypothesis states that all singularities must be "dressed" by an horizon that protects the outer region from causally interacting with the inner one and, as of now, observations are consistent with this hypothesis.

From the theoretical point of view, black holes are among the most elementary solutions of General Relativity and can be thought as elementary particles that are completely specified by their mass, charges and angular momentum. This can be explained by the fact that, once a black holes is formed, the gravitational field around its horizon sweeps away all the details of the original star and we are left with a small set of parameters. On the other hand, in a series of famous papers [8, 9] published in the 70's, it was shown that black holes must emit particles with a black body spectrum whose temperature is related to the surface gravity at the horizon. With this in mind, the simplicity of black hole solutions could also be attributed to the fact that they can be interpreted as thermodynamical systems, once a semi-classical approach is used, and as such are described by a small number of parameters like their energy, temperature and possibly their charges and angular momentum. The insightful results obtained with a semi-classical approach, however, also reveal some inconsistencies with respect to the classical description of black holes provided by general relativity. First of all, according to general relativity, the emitted radiation must not carry any information about the matter that has previously fallen inside the horizon, which leads to an information loss paradox. As a matter of facts, it was shown that one cannot use black hole solutions of general relativity in a system with quantum particles without breaking the unitarity of the evolution of states [10]. Another important result of the semi-classical approach, and the one we are mainly concerned with in this work, comes from Bekenstein and Hawking [11–14], who showed that black holes as thermal states must posses an entropy that is proportional to the area of the horizon A_H

$$S_{BH} = \frac{A_H}{4} ,$$

known as the Bekenstein–Hawking entropy. This entropy must then depend only on the mass, the charges and the angular momentum of the solution. A fundamental question arises from this observation, is it possible to give a statistical interpretation of the Bekenstein–Hawking entropy as a counting of degenerate microstates? The answer in the context of picture of general relativity comes in the form of a set of theorems, known as no-hair theorems [15, 16], which states that only one geometry corresponds to a given macrostate[1]. This leads to a contradiction with the predictions of the Bekenstein–Hawking entropy based on the area of the horizon. We can grasp the magnitude of the problem by considering, for instance, the black hole at the center of our galaxy (Sgr A*) that has an estimated entropy of $S = 5 \cdot 10^{66} J/K$ [17] based on it radius. This entropy corresponds to around $exp(3.6 \cdot 10^{89})$ degenerate microstate geometries, while general relativity is only able to predict one of these geometries. It is clear that a consistent statistical interpretation of the Bekenstein–Hawking entropy cannot be achieved in a classical theory of gravity.

Black holes in quantum gravity and supergravity

The microstate geometries contributing to the Boltzmann counting of the entropy must be the states of the underlying fundamental theory and as such must have a quantum counterpart. Their construction and counting is one of the most important requests that candidate theories of quantum gravity must satisfy. String Theory is, today, the only one of these candidates to be able to successfully reproduce the Bekenstein–Hawking entropy as a counting of microstates, even if only in some special cases. Examples have been built where the fundamental degrees of freedom, that are strings and branes, give rise to effective solutions in the low energy and four-dimensional limit that correspond to what we interpret as a black hole in general relativity. With this setup, it was found that different configurations of string and branes can produce the same effective solution and as such are a good fit for the role of degenerate microstates. In this case the counting of these degenerate configurations could be used to reproduce the entropy. As a matter of facts, this program was carried out successfully for asymptotically flat, extremal and near-extremal black holes that preserve some amount of supersymmetry, starting from the first groundbreaking result

[1] The no-hair theorems are valid for asymptotically flat black holes or other specific cases.

of Strominger and Vafa [18]. Supersymmetry, in this context, protects the perturbative results from quantum corrections, which means that we can count the number of degenerate microstates in the weak coupling limit without loss of generality and this result remains the same in the strongly coupled regime, where gravity becomes effectively dynamical. In these cases, the effective solutions correspond to black hole solutions of supergravity. One, then, can gain important insights about the entropy and the quantum structure of black holes without using a full theory of quantum gravity.

In this work we will be concerned mainly with charged extremal black holes in $\mathcal{N}=2$ supergravity, as these solutions present many interesting properties that facilitate the analysis of their entropy and have a well defined derivation from string theory, with an effective theory under control. These solutions can be seen as a generalisation of the Reissner–Nordström extremal solutions, that are characterised by the fact that their mass and charges satisfy a BPS bound $M=\sqrt{p^2+q^2}$. They have vanishing temperature and non-vanishing entropy, hence are thermodynamically stable, leading to a regular near horizon geometry. We will pay special attention to the role of scalar fields, whose presence in black hole solutions of supergravity is unavoidable. The geometry of the solution will depend on the scalars and this could be dangerous for a consistent statistical interpretation of the entropy: the area of the horizon depends, in principle, on the initial values of the scalars, which are continuous parameters. This is cured by the fact that the horizon of asymptotically flat extremal solutions in supergravity presents what is known as attractor behaviour: the horizon is found at the critical point of an appropriate effective black hole potential whose minimisation stabilises the scalar fields at values that depend solely on the charges [19–23].
An important point is that the vector sector of $\mathcal{N}=2$ supergravity presents an extension of the electromagnetic duality known as U-duality. Duality transformations do not change the equations of motion of the theory and the metric while they do affect the charges of the black hole. This means that we can use a charged "seed" solution to build a class of different solutions that fall in the same orbit by making use of duality transformations, while keeping the geometry unchanged. All of the solutions built by making use of U-duality are said to be in the same orbit and share the same entropy. With this in mind, we are going to dedicate the first two chapters to a brief review of $\mathcal{N}=2$ supergravity, both in the gauged and ungauged cases, and to black holes solutions of supergravity in the ungauged case.

Multi-center configurations

Extremal charged (Reissner–Nordström) solutions of a Maxwell–Einstein theory have the fascinating feature that they can be superposed and combined into solutions that describe several objects at equilibrium. The existence of static multi-center black hole configurations can be explained by the presence of a cancellation of the gravitational attraction and the electromagnetic repulsion between the centers. This feature is also present for extremal black holes in supergravity, where, however, the fact that a superposition principle is applicable to solutions of highly non-linear equations is not trivial. The main point is that the set of equations describing extremal BPS (and non-BPS) black holes in supergravity can be reduced to first order thanks to supersymmetry. These equations are known as flow equations of the solution. The reduction of the problem to first order allows us to take superpositions of multiple single-center solutions and construct to multi-center configurations [24]. In the BPS case these bound states of black holes provided us with prime candidate "microstate geometries" for BPS extremal black holes, since they descend from 5-dimensional smooth horizonless solutions that have the same charges and mass as a black hole [25–27]. In the final section of chapter 2 we will take a closer look at asymptotically flat multi-center solutions in supergravity.

Anti-de Sitter black holes

Up until now we have been mainly concerned with asymptotically flat black holes. In the latter half of this thesis, however, we are going to focus on the case of black hole solutions with an AdS_4 asymptotic behaviour. Black holes of this kind are solutions of gauged supergravity, where the gauging procedure has the effect of introducing a scalar potential which plays the role of a cosmological constant at spatial infinity. Supersymmetry, then, forces the cosmological constant to negative values. Although these solutions are not realised in our universe there is a strong theoretical interest in them, sparked by the introduction of the gauge/gravity (AdS/CFT) correspondence [28]. The AdS/CFT correspondence states that a string theory in a d-dimensional Anti-de Sitter spacetime is dual to a $(d-1)$-dimensional conformal field theory (CFT) built on the boundary of AdS_d. A particularly powerful application is that duality links the weakly coupled regime of the CFT to to the strongly coupled one of the gravitational theory and vice versa. In principle, then, one can gain insights onto the quantum structure of AdS black holes by making use of a dual conformal field theory in the accessible perturbative regime. On the other side, using the weakly coupled limit of the gravity theory, which is gauged supergravity, one can find many interesting applications in condensed matter systems and strongly coupled field theories. Another fascinating aspect of the CFT/AdS correspondence is that one can use the conformal field theory on the boundary to provide a non-perturbative definition of quantum gravity in the special case in which the gravitational field is asymptotically anti-de Sitter.

Thanks to the possibilities opened by the gauge/gravity correspondence, it has been possible to achieve successful microstate counting for AdS black holes in the corresponding dual CFT. Despite this, we still lack a proper understanding of these microstates in the effective supergravity theory. As we will see in chapter 3, solutions for charged, static and supersymmetric AdS_4 black holes with spherical horizons in supergravity are known [29–31]. These solutions are described by first order flow equations, provided that the effective black hole potential has appropriate modifications with respect to the ungauged case. An attractor mechanism is still present, with some key differences with respect to the flat case. The main difference is that the values of the scalars at the boundary are stabilised by the presence of the cosmological constant. This means that the AdS_4 vacuum at spatial infinity requires the scalars to have fixed initial values, starting from which the scalars flow towards the attractor point where the horizon is found. This means that the S^2 radius at the horizon, and thus the entropy, is completely determined by the charges and gaugings. Microstate counting for these static configurations in the simple case of the STU model, which has an uplifting to M-theory, has been recently performed [32, 33]. We will focus on static asymptotically AdS_4 black hole solutions in $\mathcal{N}=2$ supergravity in chapter 3. The situation with respect to non-static AdS solutions is less clear. Supersymmetric rotating solutions that are $1/4$-BPS, with compact horizons and a consistent static limit have been recently found by Hristov, Katmadas and Toldo [34], where, however, heavy assumptions on the form of the metric and the sections have been made in order to find an explicit expression for the entropy. In the latter part of this work we will mainly focus on these solutions.

Since a first order description of rotating, single center solutions in AdS seems to be possible, one could hope to be able to find the corresponding multi-center configurations. These would be useful in order to build many classes of microstate geometries for AdS black holes, similarly to what has been done in the asymptotically flat case. Proving the existence of stationary multi-center AdS configurations, however, has been a long-standing challenge. The main difficulty can be heuristically understood by the fact that the presence of a negative cosmological constant acts as an effective attractive force between the centers and has to be accounted for in the cancellation of the gravitational attraction and "electromagnetic" repulsion between the centers. This new contribution has implications for the masses and charges of the centers needed in order to build stable configurations. Nevertheless, recent results from probe analysis in [35] and [36] revealed that stable bound states for probes in an AdS_4 black hole background are possible and could imply the existence of the corresponding stationary multi-center configurations. We will give further details on the state of the art of the search for multi-center AdS black holes at the end of chapter 3.

Entropy functional for rotating AdS black holes

Another interesting open problem that motivates the study of AdS_4 rotating black holes in supergravity $\mathcal{N}=2$ is the explanation of the entropy functional reported in (1.2) of [37]. This functional is derived by making use of the dual field theory and some educated guesses based on simple models, for spherical black objects in $AdS_4\times S^7$ and $AdS_5\times S^5$ which can be reduced to AdS_4 rotating black holes solutions of $\mathcal{N}=2$ gauged supergravity. The main obstacle in finding a proper expression for the entropy of this kind of black holes is the lack of knowledge regarding the attractor mechanism for rotating AdS black holes. This entropy functional is given by

$$\mathcal{I}(p^\Lambda,\chi^\Lambda,\omega)=\frac{\pi}{4G_N}\left(\sum_{\sigma=1,2}\mathcal{B}(X^\Lambda_{(\sigma)},\omega_{(\sigma)})-2i\chi^\Lambda q_\Lambda-2\omega\mathcal{J}\right),$$

where the gravitational block $\mathcal{B}$ is the one reported in (1.3) of the same work:

$$\mathcal{B}(X^\Lambda,\omega)-\frac{\mathcal{F}(X)}{\omega}\,.$$

The q_Λ are the electric charges and the χ^Λ are the electric potentials, $\mathcal{J}$ is the angular momentum and ω is the conjugated potential. This functional gives us the entropy through

$$S_{BH}(p^\Lambda, q_\Lambda, \mathcal{J}) = \mathcal{I}(p^\Lambda, \chi^\Lambda, \omega)|_{crit.} ,$$

once is extremized with respect to χ^Λ and ω and is subject to an appropriate gauge-fixing constraint. In order to do so the $X^\Lambda_{(\sigma)}$ and $\omega_{(\sigma)}$ need to be expressed in terms of χ^Λ, q_Λ and ω. Hosseini, Hristov and Zaffaroni showed that this can be done in two different ways, depending on the considered black hole solution and special Kähler model. The two different ways of "gluing" the gravitational block differ by a sign, as reported in equations (1.4) and (1.5) of [37]. This sign choice was not present in the case of static black holes and needs to be explained by an analysis of the attractor mechanism for rotating AdS black holes.

Original content of the thesis

The main problem addressed by this work is the construction of rotating BPS solutions in $\mathcal{N} = 2$ supergravity that are characterised by first order equations. This is motivated by the recent developments with regards to BPS rotating black holes in [34] and multi-center configurations of AdS_4 black holes coming from [35, 36]. We are going to work with an ansatz for the metric and the vector fields that takes into account a space-like Killing vector related to the rotation of the vector fields. The starting point of our analysis is provided by the equations, found by Meessen and Ortín in [38], that describe all $d = 4$ stationary solutions of gauged $\mathcal{N} = 2$ supergravity and contain both first and second order equations. In order to achieve a first order description, in the static case [30], one makes use of the fact that the electric potentials only appear in the action through their first derivatives to remove their contribution in favour of the charges. We will show that such a procedure is also possible in the case at hand and that the Meessen and Ortín equations can be reduced to first order. Although such a reduction is possible, the resulting equations depend on both the radial and the angular variables in such a way that the two cannot be immediately separated.

In order to gain some initial insights we make use of a couple of simplifying assumptions on the form of the vector fields and the sections. With these in place, we find a separation of variables in the dependencies of the various quantities which noticeably simplifies the equations. It becomes clear at this point that these simplified solutions do not reproduce an AdS_4 vacuum at spatial infinity unless we remove the non-static contribution, leading us back to the static case. Despite this issue, we proceed in their analysis, hoping to find some indication on the procedure to follow in the general case. We find, in particular, that these solutions are determined by first order flow equations for the warp factors, the scalar fields and the phase α associated to the spinor projector used in the Killing spinor equations. These equations reduce correctly to the ones of [30] in the static limit. Then, we show that these equations can be rewritten by making use of a superpotential $\mathcal{W}_0$ that generalises the one in [30] and reduces to it when we remove the non-static contribution. The superpotential drives the flow of the warp factors and the scalar fields. We lack however a gradient flow expression for the flow equations. Lastly we look at the near–horizon limit of our simplified solutions. We find that the squared radius ${R_S}^2$ of the S^2 part of the near-horizon metric is found as the solution of a second order equation, leading to the necessity of choosing a sign which is reminiscent of the similar property of the entropy functional of Hosseini, Hristov and Zaffaroni. A commented summary of the results is given at the end of this work.

Chapter 1

$\mathcal{N} = 2$ supergravity

This chapter provides an introduction to $\mathcal{N} = 2$, $D = 4$ supergravity theories. It does not aim to be a comprehensive exposition of the topic but rather focuses on the main features of supergravity that will be important in the following chapters. In particular we will focus on $\mathcal{N} = 2$ supergravity theories without hypermultiplets. In these theories the scalar fields are part of vector multiplets together with fermions and vector fields and, in order for the theory to be supersymmetric, the action has to be invariant under transformations that mix the fields in the same multiplet. As we will see, this has consequences on the couplings of the theory and on the geometry of the scalar manifold. One particularly important consequence is that the geometry of the scalar manifold and the electro-magnetic duality of the vector sector are tied together by supersymmetry. This interplay has the effect of further constraining the geometry of the scalar manifold, producing many important relations between the fundamental quantities of the theory. These relations make $\mathcal{N} = 2$ supergravity a fairly tractable theory from the mathematical point of view, without it being so much constrained so we lose too many classes of solutions. General introductions to supersymmetry and supergravity can be found in [39–43].

1.1 Supergravity field content

The gravity multiplet of $\mathcal{N} = 2$ supergravity is

$$\left\{g_{\mu\nu}\,,\psi^{(1)}_{\mu}\,,\psi^{(2)}_{\mu}\,,A^{0}_{\mu}\right\}\,, \tag{1.1}$$

composed of the spin 2 graviton $g_{\mu\nu}$, two gravitinos $\psi^{(i)}_{\mu}$ and a vector A^0_μ, which is usually referred to as graviphoton. This multiplet can be coupled to matter in vector multiplets and hypermultiplets. Vector multiplets are

$$\left\{z,\lambda^{(1)},\lambda^{(2)},A_{\mu}\right\}, \tag{1.2}$$

where we find a vector field A_μ, two gauginos $\lambda^{(i)}$ and a complex scalar z. Hypermultiplets are composed of 4 real scalars (hyperscalars) q^u, with U$= 1,\dots,4$, and 2 fermions (hyperinos) $\zeta^{(i)}_\alpha$, with $\alpha = 1,2$. The geometric structure of the scalar manifold can be factorised as

$$\mathcal{M}_{scal} = \mathcal{M}_{vec} \times \mathcal{M}_{hyper} \;, \tag{1.3}$$

where:

- $\mathcal{M}_{scal}$ is the manifold parameterized by the complex scalars in the vector multiplets; supersymmetry constrains the geometry of this manifold to be a special Kähler manifold. Depending if we ask for global or local supersymmetry, we will have either global or local special Kähler geometry.
- The scalars in hypermultiplets parameterize $\mathcal{M}_{hyper}$, which turns out to be an hyper-Kähler manifold in supersymmetry or a quaternionic-Kähler manifold in supergravity. We will mostly neglect the contribution from the hypermultiplets, as these will not play a significant role in the models we are going to use.

1.2 Kähler geometry and the gauge kinetic matrix in $\mathcal{N}=2$ supersymmetry

We can already gain some insights on the structure of $\mathcal{N}=2$ supergravity from looking at global $\mathcal{N}=2$ supersymmetric theories. Let us consider the case of a global $\mathcal{N}=2$ supersymmetric theory with n vector multiplets, the lagrangian has the following kinetic terms [43]

$$\begin{aligned}\mathscr{L}^{\mathcal{N}=2}_{kin} = & -G_{a\bar{b}}\left[\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \overline{\lambda}^{a(1)}_L \not{D}\lambda^{\bar{b}(1)}_R + h.c.\right] \\ & -\frac{1}{4}Re(f_{ab})\left[F^a_{\mu\nu}F^{b\,\mu\nu} + 8\overline{\lambda}^{a(2)}\not{\partial}\lambda^{b(2)}\right] \\ & +\frac{1}{4}Im(f_{ab})\left[F^a_{\mu\nu}\tilde{F}^{b\,\mu\nu} - 4i\partial_\mu\left(\overline{\lambda}^{a(2)}\gamma_5\gamma^\mu\lambda^{b(2)}\right)\right] .\end{aligned} \tag{1.4}$$

The index $a=1,\dots,n$ labels different multiplets, we use barred indices to label the conjugate scalar fields and right handed spinors. The matrix $f_{ab}(z)$ is required to be symmetric and have holomorphic dependence on the scalars. The covariant derivative on $\lambda^{(1)}$ will be defined after introducing the geometry of the scalar manifold. Global $\mathcal{N}=1$ supersymmetry constrains the geometry of the scalar manifold to what is known as Kähler geometry:

Definition 1: A Kähler manifold $\mathcal{M}$, parameterized by coordinates z^a, is a complex n-dimensional manifold endowed with an hermitian metric $G_{a\bar{b}}(z,\bar{z})$ and a closed fundamental form

$$J = iG_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}} \;, \tag{1.5}$$

called Kähler form.

Closure of the Kähler form J implies the local existence of a real function $\mathcal{K}(z,\bar{z})$, known as the Kähler potential, such that

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K} \qquad \text{and} \qquad J = i\partial_a \partial_{\bar{b}} \mathcal{K} dz^a \wedge d\bar{z}^{\bar{b}} \,. \tag{1.6}$$

This means that one can derive the whole geometry of the scalar manifold from the Kähler potential $\mathcal{K}(z,\bar{z})$. Notice that the Kähler form J and the metric do not change under Kähler transformations

$$\mathcal{K}(z,\bar{z}) \to \mathcal{K}(z,\bar{z}) + h(z) + h^*(\bar{z}) \,, \tag{1.7}$$

where $h(z)$ is a holomorphic function. The covariant derivative $D_\mu \lambda^a$ is

$$D_\mu \lambda^a \equiv \partial_\mu \lambda^a + \Gamma^a{}_{bc} \partial_\mu z^b \lambda^c \,, \tag{1.8}$$

where the connection is given by

$$\Gamma^a{}_{bc} \equiv \frac{1}{2} G^{a\bar{d}} \left(\partial_b G_{c\bar{d}} + \partial_c G_{b\bar{d}} - \partial_{\bar{d}} G_{bc} \right) = G^{a\bar{d}} \partial_b G_{c\bar{d}} \,. \tag{1.9}$$

The $\mathrm{U}(2)_R$ R-symmetry of $\mathcal{N} = 2$ supersymmetry contains, in particular, a discrete transformation that exchanges the gauginos as

$$\lambda^{a(1)} \to \lambda^{a(2)} \qquad \text{and} \qquad \lambda^{a(2)} \to -\lambda^{a(1)} \,. \tag{1.10}$$

The lagrangian (1.4) must be invariant under this transformation. In particular, expanding the covariant derivative on $\lambda^{(1)}$ and $\partial_\mu Im(f_{ab}) = (\partial_\mu z^c \partial_c + \partial_\mu \bar{z}^{\bar{c}} \partial_{\bar{c}}) Im(f_{ab})$, we need to request:

$$2Re(f_{ab}) = G_{a\bar{b}} = G_{b\bar{a}} \,, \tag{1.11}$$

$$2i\partial_c Im(f_{ab}) = \Gamma_{ca\bar{b}} = \partial_c G_{a\bar{b}} = \Gamma_{cb\bar{a}} \,. \tag{1.12}$$

Notice that, since f_{ab} is holomorphic, the second equation can be found as a consequence of the first.

It is convenient to introduce a complex matrix $\mathcal{N}_{ab} \equiv -i\overline{f}_{ab} = \mathcal{R}_{ab} + i\mathcal{I}_{ab}$, known as gauge kinetic matrix. Notice that, in global $\mathcal{N} = 2$ supersymmetry, its imaginary part is related to the scalar metric $G_{a\bar{b}} = -2\mathcal{I}_{a\bar{b}}$. The bosonic sector of the lagrangian (1.4) can then be written as

$$\mathscr{L}^{\mathcal{N}=2}_{vec} = -G_{a\bar{b}} \partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \frac{1}{4} \mathcal{I}_{ab} F^a_{\mu\nu} F^{b\,\mu\nu} - \frac{1}{4} \mathcal{R}_{ab} F^a_{\mu\nu} \tilde{F}^{b\,\mu\nu} \,. \tag{1.13}$$

In the next section we will show that theories with a vector sector in this form present an invariance of the equations of motion and Bianchi identities under a particular group of transformations, known as U-duality of the theory.

1.3 Symplectic invariance of the vector sector

In this section we will look into electromagnetic U-duality in theories with multiple abelian vector fields. This topic was introduced by Gaillard and Zumino [44] as a generalisation of the electro-magnetic duality of source free electromagnetism. This duality consist in an invariance of the equations of motion under transformation in a so-called U-duality group. This duality will be present in $\mathcal{N}=2$ supersymmetric theories and is one of the crucial ingredients in the construction of special Kähler geometry.

We are going to consider a class of theories with n_V abelian vector fields A^Λ and n_S scalar fields z^a. Even if we do not specify n_S and n_V, the two must satisfy:

$$\begin{aligned} n_V &= n_S && \text{in global } \mathcal{N}=2 \text{ supersymmetry,} \\ n_V &= n_S + 1 && \text{in } \mathcal{N}=2 \text{ supergravity,} \end{aligned}$$

because of the introduction of the graviphoton in the second case. The vectors can be coupled to the scalars through the complex, symmetric and scalar dependent gauge kinetic matrix $\mathcal{N}$ (also known as the period matrix). Since we are going to use the results of the following discussion in the context of supergravity, it is convenient to work in curved spacetime. The lagrangian is

$$e^{-1}\mathscr{L} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu} - \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu} + e^{-1}\mathscr{L}_{rest}\,, \tag{1.14}$$

where $e \equiv \sqrt{-g}$; the matrices $\mathcal{R}$ and $\mathcal{I}$ are the real and imaginary parts of the period matrix, in particular $\mathcal{I}$ describes the gauge kinetic couplings and must be negative definite, $\mathcal{R}$ can be seen as a generalisation of the θ angle term; $\mathscr{L}_{rest}$ is a generic lagrangian[1] that depends on the scalars but not on the vectors, this means in particular that the scalars are taken to be neutral under the U(n_V) gauge group. The abelian field strengths are given by

$$F^\Lambda_{\mu\nu} = 2\partial_{[\mu}A^\Lambda_{\nu]} \qquad \text{and} \qquad \tilde{F}^\Lambda_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\Lambda\,\rho\sigma}\,. \tag{1.15}$$

The dual field strengths are constructed using the the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$:

Definition 2: The Levi-Civita tensor is a 4-component tensor whose components in the veirbein basis correspond to the Levi-Civita symbol

$$\varepsilon_{mnrs} \equiv \epsilon_{mnrs} \qquad \text{with } \varepsilon_{mnrs} = e^\mu_m e^\nu_n e^\rho_r e^\sigma_s \varepsilon_{\mu\nu\rho\sigma}\,, \tag{1.16}$$

[1] In a theory with multiple fields, $\mathscr{L}_{rest}$ can depend on any of these fields (and their first derivatives) apart from the vector fields. Furthermore, we could introduce another coupling between these fields and the vector in the form $\sim \mathcal{O}_{\mu\nu\,\Lambda}F^{\Lambda\,\mu\nu}$, where $\mathcal{O}$ is a tensor combination of these fields and their first derivatives. For a detailed discussion see [43].

where the Levi-Civita symbol is

$$\epsilon_{mnrs} = \begin{cases} 1 & \text{for } m=0, n=1\,, r=2\,, s=3 \text{ and event permutations,} \\ -1 & \text{for odd permutations,} \\ 0 & \text{if there are any repeated indices.} \end{cases} \tag{1.17}$$

Hodge duality in 4-dimensional curved spacetime acts on a 2-form as

$$\star A = \star\left(\frac{1}{2}A_{\mu\nu}dx^{\mu}\wedge dx^{\nu}\right) = \frac{1}{2}\underbrace{\left(\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}A^{\rho\sigma}\right)}_{\tilde{A}_{\mu\nu}}dx^{\mu}\wedge dx^{\nu}\,, \tag{1.18}$$

which means in particular that $\tilde{A}_{\mu\nu}$ depends on the metric $g_{\mu\nu}$. The volume form is consequentially defined as

$$\sqrt{-g}d^4x \equiv \star(1) = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho}\wedge dx^{\Sigma}\,. \tag{1.19}$$

The equations of motion and Bianchi identities of the theory in consideration can be written as

$$dF^{\Lambda} = 0 \qquad \text{and} \qquad dG_{\Lambda} = 0\,, \tag{1.20}$$

where we introduced dual field strengths $G_{\Lambda\mu\nu}$, such that

$$G_{\Lambda\,\mu\nu} \equiv \varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathscr{L}}{\partial F^{\Lambda}_{\rho\sigma}} \qquad \Rightarrow \qquad \begin{cases} G_{\Lambda} = \mathcal{I}_{\Lambda\Sigma}\star F^{\Sigma} + \mathcal{R}_{\Lambda\Sigma}F^{\Sigma} \\ \star G_{\Lambda} = \mathcal{I}_{\Lambda\Sigma}F^{\Sigma} - \mathcal{R}_{\Lambda\Sigma}\star F^{\Sigma} \end{cases} . \tag{1.21}$$

Since both F^{Λ} and G_{Λ} are closed forms, we can introduce the dual vector fields as 1-forms A_{Λ}, such that $G_{\Lambda} = dA_{\Lambda}$. In this case we are able to find a dual description where we use $A_{\Lambda\mu}$ in place of the starting fields A^{Λ}_{μ}, then the Bianchi identities would be $dG_{\Lambda} = 0$ and the equations of motion would be $dF^{\Lambda} = 0$. The theory is said to have a duality invariance since these dual descriptions have the same equations. Notice that the equations of motions and Bianchi identities are actually invariant under more general transformations

$$\mathcal{F} = \begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix} \to \mathcal{F}' = S\mathcal{F}\,, \tag{1.22}$$

where S could be, in principle, any constant $\mathrm{GL}(2n_V, \mathbb{R})$ matrix. However, in order for the identification of G_{Λ} as the dual field strengths to be valid in any frame, we need to impose

$$G_{\mu\nu\,\Lambda}{}' = \varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathscr{L}'}{\partial F^{\Lambda}_{\rho\sigma}{}'}\,. \tag{1.23}$$

This condition will restrict the actual U-duality group. Let us work, for the sake of simplicity, with an infinitesimal transformation

$$\mathcal{F}' = \left[1 + \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]\mathcal{F} \qquad \Rightarrow \qquad \begin{cases} \delta F^{\Lambda} = A^{\Lambda}{}_{\Sigma} F^{\Sigma} + B^{\Lambda\Sigma} G_{\Sigma} \\ \delta G_{\Lambda} = C_{\Lambda\Sigma} F^{\Sigma} + D_{\Lambda}{}^{\Sigma} G_{\Sigma} \end{cases} . \tag{1.24}$$

In order to satisfy (1.23), we need to ask that the matrices $\mathcal{I}$ and $\mathcal{R}$ also transform under U-duality and that

$$\begin{aligned} \delta \star G_{\Lambda} =& C_{\Lambda\Sigma} \star F^{\Sigma} + D_{\Lambda}^{\Sigma} \star G_{\Sigma} \\ =& \delta\mathcal{R}_{\Lambda\Sigma} \star F^{\Sigma} - \delta\mathcal{I}_{\Lambda\Sigma} F^{\Sigma} + \mathcal{R}_{\Lambda\Sigma} \delta \star F^{\Sigma} - \mathcal{I}_{\Lambda\Sigma} \delta F^{\Sigma} , \end{aligned} \tag{1.25}$$

i.e. the relations between G_{Λ} and F^{Λ} in (1.21) are preserved. We can now remove any contribution from G_{Λ} in favour of F^{Λ} from the previous equation. The resulting equation is satisfied for any configuration of the fields provided that the matrices $\mathcal{R}$ and $\mathcal{I}$ transform as

$$\delta\mathcal{R} = C + D\mathcal{R} - \mathcal{R}A + \mathcal{I}B\mathcal{I} - \mathcal{R}B\mathcal{R} , \tag{1.26a}$$

$$\delta\mathcal{I} = D\mathcal{I} - \mathcal{I}A - \mathcal{R}B\mathcal{I} - \mathcal{I}B\mathcal{R} . \tag{1.26b}$$

Since we must have $\delta\mathcal{R}^T = \delta\mathcal{R}$ and $\delta\mathcal{I}^T = \delta\mathcal{I}$ in order to preserve the symmetry of the gauge kinetic couplings, we can easily derive the following conditions on the transformation (1.24)

$$C = C^T , \quad B = B^T , \quad \text{and} \quad A^T = -D \quad \Rightarrow \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(2n_V, \mathbb{R}) , \tag{1.27}$$

which means that the U-duality group is the symplectic group $\mathrm{Sp}(2n_V, \mathbb{R})$, i.e. the matrix S must satisfy

$$S^T \Omega S = \Omega \qquad \text{with} \quad \Omega = \begin{pmatrix} \mathbb{0} & -\mathbb{1}_n \\ \mathbb{1}_n & \mathbb{0} \end{pmatrix} . \tag{1.28}$$

We will call symplectic vector any object that transform under duality transformations S in the same way as the field strength $\mathcal{F}$. Let X and Y be symplectic vectors, we introduce an inner product

$$\langle X, Y \rangle \equiv X^T \Omega Y , \tag{1.29}$$

which we will refer to as symplectic product as it is manifestly invariant under symplectic rotations. This property will be useful as we will often construct objects that need to be symplectic invariants as symplectic products.

Finally, notice that U-duality is not necessarily a symmetry of the theory, since duality invariance of $d\mathcal{F} = 0$ does not imply invariance of the lagrangian. As a matter of fact, under an infinitesimal transformation we have

$$\delta\mathscr{L} = \frac{1}{4} F C \tilde{F} + \frac{1}{4} G D \tilde{G} , \tag{1.30}$$

where the first term is a total derivative, while the second is not. This means that symplectic transformations are a symmetry of the action (1.14) only in the case in which $B = 0$. In order to fully appreciate the effect of symplectic transformations, it is convenient to introduce (anti) self-dual field strengths

$$\mathcal{F}^{\pm} = \begin{pmatrix} F^{\Lambda\pm} \\ G^{\pm}_{\Lambda} \end{pmatrix} \equiv \frac{1}{2}\left(\mathcal{F} \mp i\star\mathcal{F}\right) \qquad \text{such that } \star\mathcal{F}^{\pm} = \pm i\mathcal{F}^{\pm} \,. \tag{1.31}$$

The components of the (anti) self-dual field strengths are related one another by the period matrix $\mathcal{N} = \mathcal{R} + i\mathcal{I}$, as

$$G^{+}_{\Lambda} = \mathcal{N}_{\Lambda\Sigma}F^{\Sigma+} \qquad \text{and} \qquad G^{-}_{\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma}F^{\Sigma-} \,, \tag{1.32}$$

which allows us to rewrite the action as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}Im\left(\mathcal{N}_{\Lambda\Sigma}F^{\Lambda+}_{\mu\nu}F^{\mu\nu\,\Sigma+}\right) + \mathscr{L}_{rest}\right] . \tag{1.33}$$

A symplectic transformation $S \in \mathrm{Sp}(2n_V, \mathbb{R})$

$$S = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} , \tag{1.34}$$

acts on the self-dual field strengths as

$$F^{+\,\prime} = \left(\hat{A} + \hat{B}\mathcal{N}\right)F^{+} \qquad \text{and} \qquad G^{+\,\prime} = \left(\hat{C} + \hat{D}\mathcal{N}\right)F^{+} \,, \tag{1.35}$$

while the period matrix must transform as

$$\mathcal{N}' = \left(\hat{C} + \hat{D}\mathcal{N}\right)\left(\hat{A} + \hat{B}\mathcal{N}\right)^{-1} . \tag{1.36}$$

These transformations are compatible with the relation between self-dual field strengths in (1.32). Transformations with $\hat{B} \neq 0$ correspond to non-perturbative duality transformations since they involve an inversion of $\mathcal{N}$, hence exchange weak and strong couplings. In the quantum theory these non-perturbative transformations need to be restricted to $\mathrm{Sp}(2n_V, \mathbb{Z})$ in order to not ruin the path integral formulation of the theory.

There is one last important result from [44] that we need to mention and it regards the coupling of the vector sector with matter. The stress energy tensor for a theory like the one considered in this discussion, where we could also add rather general interactions, is invariant under duality transformations. This means that the Einstein equations are invariant under U-duality transformations, which allows us to "rotate" one particular solution of our theory into another without modifying the metric. For charged black hole solutions the magnetic and electric charges form a symplectic vector $\mathcal{Q} = (p^{\Lambda}; q_{\Lambda})$, which means that we can rotate solutions in the same orbit with a U-duality transformation and this will not modify the metric. One can take advantage of this result and identify a simple "seed" solution, for example with only electric charges, and then use appropriate U-duality transformations to obtain new configurations with different charges. Solutions in the same orbit share the same metric and hence crucial geometric quantities, like the horizon area. When considering black hole solutions, we will try for the most part to work with symplectic covariant objects, i.e. without fixing a particular symplectic frame.

1.4 Special Kähler Geometry

We have seen that an $\mathcal{N}=2$ supersymmetric theory has a U-duality of the equations of the vector sector under $\mathrm{Sp}(2n_V,\mathbb{R})$ transformations of the field strengths $\mathcal{F}$. These transformations act on the period matrix $\mathcal{N}$, which is related to the Kähler metric by $\mathcal{N}=2$ supersymmetry. This, however, appears to be a problem since we want the metric to be invariant under symplectic transformations. On the other side the geometry of scalar manifold is invariant under general reparameterizations of the scalars. These reparameterizations cannot act on symplectic vectors, since these only admit linear transformations. In this section we will see how the geometry of the scalar manifold needs to be constrained in order to solve these tensions in the case of local supersymmetry. One can find in depth reviews of special Kähler geometry, both in the global and local cases, in [45–48].

Let us consider a supergravity theory with n vector multiplets, then:

- We have $n+1$ vector fields, this means that the symplectic group will be $\mathrm{Sp}(2n+2,\mathbb{R})$. We will use the indices $\Lambda=0,1,\ldots,n$ to label these vector fields.
- The manifold $\mathcal{M}_{vec}$ spanned by the scalar fields belonging to the vector multiplets, on the other side, is still parameterized by n complex scalars z^a. We will use the indices $a=1,\ldots,n$ to label these scalars and barred indices to label their complex conjugates.
- Local supersymmetry transformations require the scalar manifold to have a Kähler–Hodge structure, i.e. the spinors must behave non-trivially under Kähler transformations.

The bosonic sector of the lagrangian is

$$e^{-1}\mathscr{L}_{bos}=\frac{R}{2}-G_{a\bar{b}}\partial_\mu z^a\partial^\mu\bar{z}^{\bar{b}}+\frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\,\mu\nu}-\frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu}\,, \tag{1.37}$$

where we add the Ricci scalar contribution in order to be able to find the Einstein equations as equations of motions. Now $\mathcal{R}$ and $\mathcal{I}$ are $(n+1)\times(n+1)$ matrices. We ask that the symplectic transformations act only on symplectic vectors without touching the scalars and spinors. On the other side, the scalars and spinors should transform under general reparameterizations of the scalar manifold as

$$z^a\to\tilde{z}^a(z)\qquad\text{and}\qquad\lambda^{a(i)}\to\tilde{\lambda}^{a(i)}=\frac{\partial\tilde{z}^a}{\partial z^b}\lambda^{b(i)}\,, \tag{1.38}$$

while symplectic vectors are invariant.

The coupling with gravity, as in $\mathcal{N}=1$ supergravity, constrains the scalar manifold to be Kähler-Hodge manifold. This comes from the fact that, in order for the local supersymmetry transformations to be satisfied and the lagrangian to be Kähler invariant, the fermions need to transform under Kähler transformations as

$$\mathcal{K}\to\mathcal{K}+h+\bar{h}\qquad\Rightarrow\qquad\chi\to exp\big(-iq_\chi Im(h)\gamma_5\big)\chi\,, \tag{1.39}$$

where q_χ is the Kähler "charge" of the fermion χ. This means in particular that fermions are sections in a principal U(1)-bundle, with a connection

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \frac{i}{2}\left(\partial_\mu \bar{z}^{\bar{a}} \partial_{\bar{a}} \mathcal{K} - \partial_\mu z^a \partial_a \mathcal{K}\right) dx^\mu \ , \tag{1.40}$$

called composite Kähler connection, such that the Kähler form is $J = d\mathcal{A}$. The action is then covariantized by introducing Kähler covariant derivatives on spinors

$$\mathcal{D}_\mu \chi \equiv D_\mu \chi + i q_\chi \mathcal{A}_\mu \gamma_5 \chi \ , \tag{1.41}$$

where $D_\mu \chi$ is the previous covariant derivative on χ, that include the spin connection ω. In particular we find that the gravitinos and the spinor parameters must transform as [43]

$$\psi_\mu^{(i)} \to exp\left[-\frac{i}{2} Im(h)\gamma_5\right] \psi_\mu^{(i)} \qquad \Rightarrow \qquad \mathcal{D}_{[\mu} \psi_{\nu]}^{(i)} = D_{[\mu} \psi_{\nu]}^{(i)} + \frac{i}{2} \mathcal{A}_{[\mu} \gamma_5 \psi_{\nu]}^{(i)} \ , \tag{1.42a}$$

$$\varepsilon^{(i)} \to exp\left[-\frac{i}{2} Im(h)\gamma_5\right] \varepsilon^{(i)} \qquad \Rightarrow \qquad \mathcal{D}_\mu \varepsilon^{(i)} = D_\mu \varepsilon^{(i)} + \frac{i}{2} \mathcal{A}_\mu \gamma_5 \varepsilon^{(i)} \ . \tag{1.42b}$$

We can now look at supersymmetric transformations of the left handed gauginos along $\varepsilon^{(1)}$, as these are simpler than the most general transformations but already give us all the information we need. We have

$$\delta_{\varepsilon^{(1)}} \lambda_L^{a(1)} = \frac{1}{2} \partial\!\!\!/ z^a \varepsilon_L^{(1)} \ , \tag{1.43}$$

$$\delta_{\varepsilon^{(1)}} \lambda_L^{a(2)} = -\frac{1}{4} G^{a\bar{b}} \overline{f_b^\Lambda} \mathcal{I}_{\Lambda\Sigma} F_{\mu\nu}^{\Sigma -} \gamma^{\mu\nu} \varepsilon_L^{(1)} \ , \tag{1.44}$$

where the f_a^Λ are complex functions of the scalar fields. From these transformations we can find that:

- From the first we see that the left handed gauginos $\lambda^{a(1)}$ must transform under scalar reparameterizations as holomorphic tangent vectors. Because of the R-symmetry the $\lambda^{a(2)}$ must transform in the same way, hence

$$\lambda^{a(i)} \to (\partial \tilde{z}^a / \partial z^b) \lambda^{b(i)} \ . \tag{1.45}$$

- In order for the second equation to be consistent under scalar reparameterizations we need to ask that the f_a^Λ transform as holomorphic covectors

$$f_a^\Lambda \to (\partial z^b / \partial \tilde{z}^a) f_b^\Lambda \ . \tag{1.46}$$

- The anti-self dual field strength $F^{\Lambda -}$ and the matrix $\mathcal{I}$ transform under symplectic transformations, while $\lambda^{a(2)}$ must be invariant. In order to keep this property we need to ask that the left side of (1.44) is invariant. This is realised if the f_a^Λ are the upper components of a symplectic vector

$$\mathcal{U}_a(z,\bar{z}) \equiv \begin{pmatrix} f_a^\Lambda \\ h_{\Lambda a} \end{pmatrix} \ , \tag{1.47}$$

such that $h_{\Lambda a} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_a^\Lambda$. If this is the case we have

$$\overline{f_b^\Lambda} \mathcal{I}_{\Lambda\Sigma} F_{\mu\nu}^{\Sigma -} = -\frac{1}{2i} \langle \mathcal{F}_{\mu\nu}^-, \overline{\mathcal{U}_b} \rangle \ , \tag{1.48}$$

which is manifestly invariant under symplectic transformations.

- Under Kähler transformations, using (1.42b) in (1.43), we find that $\lambda^{a(1)}$ has a Kähler charge $q = -1/2$. This must also hold for $\lambda^{a(2)}$ since the R-symmetry mixes the two. Equation (1.44) seems to tell us otherwise, since $\delta\lambda_L^{a(2)}$ is proportional to $\varepsilon_L^{(1)}$, which has opposite Kähler charge. This is solved by asking that some of the other objects in the left hand side of this equation also transform non trivially under Kähler transformations. Since the inverse metric is Kähler invariant and non trivial transformation of the period matrix $\mathcal{N}$ or the field strengths would lead to a non Kähler invariant gauge sector, the only option left is to admit that the symplectic vector $\mathcal{U}_a$ transforms as

$$\mathcal{U}_a \to exp\big[-iIm(h)\big]\mathcal{U}_a \ . \tag{1.49}$$

These simple observations lead us directly to the heart of local special Kähler geometry. Let us introduce the symplectic section

$$\mathcal{V}(z,\bar{z}) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} \ , \tag{1.50}$$

such that the $\mathcal{U}_a$ are obtained from its Kähler covariant derivative

$$\mathcal{U}_a = \mathcal{D}_a \mathcal{V} = \left(\partial_a + \frac{1}{2}\partial_a \mathcal{K}\right)\mathcal{V} \ , \tag{1.51}$$

then the section $\mathcal{V}$ transforms as $\mathcal{V} \to exp\big[-iIm(h)\big]\mathcal{V}$ under Kähler transformations. $\mathcal{V}$ cannot be a holomorphic section since its transformation depends on both $h(z)$ and $\bar{h}(\bar{z})$. We can, however, introduce another section

$$\mathscr{V} = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \equiv e^{-\frac{1}{2}\mathcal{K}(z,\bar{z})}\mathcal{V}(z,\bar{z}) \ , \tag{1.52}$$

in such a way that this transforms as $\mathscr{V} \to e^{-h}\mathscr{V}$ under Kähler transformations. We ask $\mathscr{V}$ to be holomorphic and this is equivalent to asking $\mathcal{V}$ to be covariantly holomorphic

$$\mathcal{U}_{\bar{a}} = \mathcal{D}_{\bar{a}}\mathcal{V} = \left(\partial_{\bar{a}} - \frac{1}{2}\partial_{\bar{a}}\mathcal{K}\right)\mathcal{V} = 0 \ . \tag{1.53}$$

Notice that, in order for the sections $\mathcal{V}, \mathscr{V}, \dots$ to be invariant under reparametrizations of the scalar manifold, their components must be simple scalar functions of the scalar fields. This means that the sections live in a complex $(2n+2)$-dimensional vector bundle over the scalar manifold. Since this bundle is endowed with a structure group $\mathrm{Sp}(2n+2,\mathbb{R})$, we will call it symplectic bundle $\mathcal{SV}$.

In order for the geometry of the scalar manifold to be invariant under symplectic transformations, the Kähler potential should be expressed as a symplectic product between sections. This expression should also satisfy the correct behaviour for the Kähler potential under Kähler transformation. A consistent choice is provided by asking the Kähler potential to satisfy

$$\mathcal{K} = -log\big(i\langle\overline{\mathscr{V}},\mathscr{V}\rangle\big) \qquad \Longleftrightarrow \qquad \langle\mathcal{V},\overline{\mathcal{V}}\rangle = i \,. \tag{1.54}$$

The Kähler-Hodge structure of the scalar manifold means that now the holomorphic section $\mathscr{V}$ is not only a section in $\mathcal{SV}$, but also in a topologically non-trivial holomorphic line bundle $\mathcal{L}$, whose $\mathrm{U}(1)$ connection is given by the Kähler composite connection $\mathscr{A}$. Sections of $\mathcal{SV}\times\mathcal{L}$, on different patches of the manifold, can differ by a constant $\mathrm{Sp}(2n_V+2,\mathbb{R})$ matrix and a holomorphic factor $e^{-h(z)}$.

For $n>1$, one has to to impose the following condition

$$\langle\mathcal{D}_a\mathscr{V},\mathcal{D}_b\mathscr{V}\rangle = 0 \qquad \Leftrightarrow \qquad \langle\mathcal{U}_a,\mathcal{U}_b\rangle = 0 \,, \tag{1.55}$$

in order to ensure the existence and uniqueness of the period matrix. With this condition one can show the existence of a prepotential $F(X)$, such that the lower components of $\mathscr{V}$ are $F_\Lambda = \partial_\Lambda F$, in a particular symplectic frame[2] [47]. In this symplectic frame we can write the period matrix as [43]

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F_{\Lambda\Sigma}} + 2i\frac{Im(F_{\Lambda\Omega})Im(F_{\Sigma\Gamma})X^\Omega X^\Gamma}{Im(F_{\Xi\Phi})X^\Xi X^\Phi} \,, \tag{1.56}$$

where $F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$.

Definition 3: An *affine (or local) special Kähler manifold* is an n-dimensional Kähler-Hodge manifold of restricted type, equipped with a tensor bundle $\mathscr{H}$

$$\mathscr{H} = \mathcal{L}\times\mathcal{SV} \,, \tag{1.57}$$

where $\mathcal{SV}$ is a flat, holomorphic, $(2n+2)$-dimensional vector bundle with a symplectic structure group, $\mathcal{L}$ is a holomorphic line bundle. On each patch U_A of a good cover of the manifold, a section of $\mathscr{H}$ is

$$\mathscr{V}_A = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \,, \qquad \Lambda = 1,\ldots,n \,, \tag{1.58}$$

such that the transition function between two local trivializations of $\mathscr{H}$ on patches U_A and U_B is

$$\mathscr{V}_A = e^{-h_{AB}}S_{AB}\mathscr{V}_B \,, \tag{1.59}$$

[2] for $n_V = 1$ the condition (1.55) is empty, in this case counter-examples where there is no prepotential in any symplectic frame have been constructed, see [49].

where S_{AB} is a constant $\mathrm{Sp}(2n+2,\mathbb{R})$ matrix and $h_{AB}(z)$ is a holomorphic function. The Kähler Kähler potential is

$$\mathcal{K} = -log\big(i\langle\mathscr{V},\overline{\mathscr{V}}\rangle\big) \ , \tag{1.60}$$

where the $\langle\cdot,\cdot\rangle$ is a symplectic and hermitian inner product on $\mathcal{H}$ such that

$$\langle\mathcal{D}_a\mathscr{V},\mathcal{D}_b\mathscr{V}\rangle = 0 \qquad\qquad \text{with } \mathcal{D}_a\mathscr{V} = \big(\partial_a + \partial_a\mathcal{K}\big)\mathscr{V} \ . \tag{1.61}$$

If $n>1$, the last equation ensures the existence of a holomorphic prepotential F in some symplectic frame and the symmetry of the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$.

Special Kähler geometry identities and constrains

Local special Kähler geometry is rich of constrains and relations between the various objects that we introduced. These are an important mathematical tool which make $\mathcal{N}=2$ supergravity theories fairly tractable and as such we will be use them extensively in later chapters. In addition to the constraints that we already introduced in (1.53), (1.54) and (1.55), we have [46]

$$\langle\mathcal{V},\mathcal{U}_a\rangle = 0 = \langle\overline{\mathcal{V}},\mathcal{U}_a\rangle \ , \tag{1.62a}$$

$$\langle\mathcal{U}_a,\overline{\mathcal{U}_b}\rangle = iG_{a\bar{b}} \ , \tag{1.62b}$$

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma \ . \tag{1.62c}$$

One can show that, using these constrains, the matrix $(\overline{f}^{\Lambda}_{\bar{a}}\,,\,\mathcal{L}^\Lambda)$ is invertible, which allows us to express the period matrix as

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix}\overline{h}_{\Lambda\bar{a}}\\ \mathcal{M}_\Lambda\end{pmatrix}\cdot\begin{pmatrix}\overline{f}^{\Sigma}_{\bar{a}}\\ \mathcal{L}^\Sigma\end{pmatrix}^{-1} \ , \tag{1.63}$$

this expression ensures the symmetry of $\mathcal{N}$ and the fact that its imaginary part is invertible and negative definite. From (1.62c) and the previous relations we can derive some particularly useful identities

$$\mathcal{I}_{\Lambda\Sigma}\mathcal{L}^\Lambda\mathcal{L}^\Sigma = -\frac{1}{2}\ , \quad \mathcal{I}_{\Lambda\Sigma}f^\Lambda_a\overline{f}^\Sigma_{\bar{b}} = -\frac{1}{2}G_{a\bar{b}}\ , \quad f^\Lambda_a G^{a\bar{b}}\overline{f}^\Sigma_{\bar{b}} = -\frac{1}{2}\mathcal{I}^{-1\,\Lambda\Sigma} - \overline{\mathcal{L}}^\Lambda\mathcal{L}^\Sigma \ . \tag{1.64}$$

1.5 Ungauged $\mathcal{N}=2$ bosonic truncation

We will now provide the explicit lagrangian and supersymmetric transformations for the model that we will use in chapter 3 when working out asymptotically flat black hole solutions. This model is obtained as a bosonic truncation of a $\mathcal{N}=2$ supergravity theory without hypermultiplets. This means, in particular, that we set the expectation values of the spinors and the hyperscalars to zero. With these requests the lagrangian reads

$$e^{-1}\mathscr{L}_{bos} = \frac{R}{2} - G_{a\bar{b}}\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\,\mu\nu} - \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu} \,. \tag{1.65}$$

Let ε be the doublet of Majorana spinors that parameterize the local supersymmetry transformation, then the theory needs to be invariant under

$$\delta_\varepsilon z^a = 0 \qquad \text{and} \qquad \delta_\varepsilon A^{\Lambda}_{\mu} = 0 \,, \tag{1.66a}$$

since these would only be proportional to the expectation values of the fermions, and under

$$\delta_\varepsilon \psi_{\mu A} = \left[\partial_\mu - \frac{1}{4}(\omega_\mu)^{mn}\gamma_{mn} + \frac{i}{2}\mathcal{A}_\mu\right]\varepsilon_A + \langle \mathcal{V}, \mathcal{F}^-_{\mu\nu}\rangle \gamma^\nu \epsilon_{AB}\varepsilon^B \,, \tag{1.66b}$$

$$\delta_\varepsilon \lambda^{aA} = -\,i\partial\!\!\!/ z^a \varepsilon^A + \frac{i}{2}G^{a\bar{b}}\langle \overline{\mathcal{U}_b}, \mathcal{F}^-_{\mu\nu}\rangle \gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B \,. \tag{1.66c}$$

Since the ε spinors carries 8 independent parameters, the $\mathcal{N}=2$ action is invariant under the transformations generated by 8 real supercharges. A generic solution of the theory does not need, however, to be invariant under all of the 8 transformations. For instance, let us admit to have a solution such that

$$\delta_{\varepsilon_{sol}}\psi_{\mu A} = 0 = \delta_{\varepsilon_{sol}}\lambda^{aA} \,, \tag{1.67}$$

for a particular doublet of spinors ε_{sol}, when computed along the solutions. Let $N \leq 8$ be the (even) number of independent components of ε_{sol}, then the solutions is said to preserve N supersymmetries. Solutions that preserve some of the supersymmetries are called BPS solutions and, as we already mentioned, are a bridge with the UV theory, since they are protected from quantum corrections. The equations in (1.67) are called Killing spinor equations (or BPS equations) of the solution.

1.6 Gauged $\mathcal{N}=2$ supergravity

It is possible to introduce gauge symmetries in $\mathcal{N}=2$ supergravity, provided that the group $\mathcal{G}$ that we choose to gauge is a subgroup of the isometries of $\mathcal{M}_{scal} = \mathcal{M}_{vec}\times\mathcal{M}_{hyper}$. At first we are going to focus on the gauging of isometries of $\mathcal{M}_{vec}$ as this will allow us to introduce the basic concepts of the gauging procedure. In this case one gets a bosonic sector modified by charges for the scalars and a scalar potential. On the other hand, instead of gauging an isometry, one could also promote a subgroup of the $\mathrm{U}(2)_R$ R-symmetry to a local symmetry. This case, known as Fayet–Iliopoulos (F.I.) gauging, will be the one we are going to work with in later chapters. One particular consequence of F.I. gauging is that the gravitinos and gauginos gain charges, while the bosons remain neutral, this means that, in the bosonic truncation, the whole $\mathrm{Sp}(2n+2,\mathbb{R})$ group of U-duality is preserved. At the end of this section we are going to consider the case of $\mathrm{U}(1)$ Fayet–Iliopoulos gauged supergravity without hypermultiplets. As we will see, the main difference between this model and the ungauged case is the introduction of a scalar potential. This last point is of particular importance for the content of this thesis, as it will allow for solutions that reproduce an anti-de Sitter vacuum at spatial infinity. For additional references regarding gauged supergravity see for instance [48, 50–52].

1.6.1 Gauging of special Kähler manifold

Let us focus on the special Kähler manifold $\mathcal{M}_{vec}$, parameterized by the n_V complex scalars z^a from the vector multiplets. Let $\mathcal{G}$ be the global group of the isometries of $\mathcal{M}_{vec}$. An infinitesimal transformation of $\mathcal{G}$ acts on the scalars as

$$\delta_{\mathcal{G}} z^a = \alpha^\Lambda k^a_\Lambda \,, \tag{1.68}$$

where the α^Λ are infinitesimal constants and the $k^a_\Lambda(z,\bar{z})$ are Killing vectors. The Killing vectors span the algebra of $\mathcal{G}$

$$\left[k_\Lambda\,,\,k_\Sigma\right] = f^\Gamma{}_{\Lambda\Sigma} k_\Gamma \,, \qquad \text{with} \qquad k_\Lambda = k^a_\Lambda \partial_a + \overline{k^a}_\Lambda \partial_{\bar{a}} \,, \tag{1.69}$$

where $f^\Gamma{}_{\Lambda\Sigma}$ are the structure constants of $\mathcal{G}$. The Killing vectors have to be holomorphic in order for transformations of this type to preserve the Kähler structure of the manifold. The Killing equations are satisfied once we ask that the Kähler potential transforms at most as

$$\delta_{\mathcal{G}} \mathcal{K} \equiv k^a_\Lambda \partial_a \mathcal{K} + \overline{k^a}_\Lambda \partial_{\bar{a}} \mathcal{K} = \alpha^\Lambda \left(r_\Lambda(z) + \bar{r}_\Lambda(\bar{z}) \right) . \tag{1.70}$$

The isometry has a lifting to the symplectic vector bundle, such that the sections also transform under $\mathcal{G}$. The lifting acts on the sections as

$$\delta_{\mathcal{G}} \mathcal{V} = \alpha^\Lambda \left[T_\Lambda \mathcal{V} + r_\Lambda \mathcal{V} \right] , \tag{1.71}$$

where the T_Λ are infinitesimal, symplectic and block diagonal matrices that coincide, in each block, to the adjoint representation of $\mathfrak{g}$

$$T_\Lambda = \begin{pmatrix} a_\Lambda & 0 \\ 0 & -a_\Lambda^T \end{pmatrix} , \tag{1.72}$$

where $(a_\Lambda)^\Gamma{}_\Sigma = -f^\Gamma{}_{\Lambda\Sigma}$, while $r_\Lambda(z)$ is the same function that appears in the Kähler transformation. Closure of the gauge algebra tells us that [38, appendix B]

$$k^a_\Lambda \partial_a r_\Sigma - k^a_\Sigma \partial_a r_\Lambda = f^\Gamma{}_{\Lambda\Sigma} r_\Sigma \,. \tag{1.73}$$

We can now gauge the group $\mathcal{G}$. This procedure involves the following substitutions:

$$\partial_\mu z^a \quad \to \quad \nabla_\mu z^a = \partial_\mu z^a + g k^a_\Lambda A^\Lambda_\mu \,, \tag{1.74}$$

$$F^\Lambda_{\mu\nu} = 2\partial_{[\mu} A^\Lambda_{\nu]} \quad \to \quad F^\Lambda_{\mu\nu} = 2\partial_{[\mu} A^\Lambda_{\nu]} + g f^\Lambda{}_{\Sigma\Gamma} A^\Sigma_\mu A^\Gamma_\nu \,, \tag{1.75}$$

where g is the coupling constant of the gauge group. We will need new terms to be added to the lagrangian in order to keep it invariant under local transformations of this kind. In particular, the period matrix $\mathcal{N}$ also transforms non trivially under gauge transformations and needs to be accounted for. The resulting lagrangian must still preserve supersymmetry. We will not delve into details regarding the final structure of the resulting gauged theory since we will not use it, one can find the full lagrangian in [48, section 8]

We showed that, thanks to the Kähler structure of the scalar manifold, the metric can be expressed in terms of derivatives of the Kähler potential. In the same way, we can find real scalar functions $\mathcal{P}_\Lambda(z,\bar{z})$, called momentum maps, such that their derivatives are related to the Killing vectors

$$k^a_\Lambda = -iG^{a\bar{b}}\partial_{\bar{b}}\mathcal{P}_\Lambda \qquad\qquad \overline{k^a}_\Lambda = iG^{b\bar{a}}\partial_b\mathcal{P}_\Lambda\,. \tag{1.76}$$

Using the holomorphicity of the Killing vectors and inserting the Kähler potential in place of the metric, we find

$$\mathcal{P}_\Lambda = ik^a_\Lambda\partial_a\mathcal{K} - ir_\Lambda = -i\overline{k^a}_\Lambda\partial_{\bar{a}}\mathcal{K} + i\bar{r}_\Lambda\,. \tag{1.77}$$

Finally, using the momentum maps in (1.73) we find the so-called equivariance condition

$$2k^a_{[\Lambda}G_{a\bar{b}}\overline{k^b}_{\Sigma]} = if^\Gamma{}_{\Lambda\Sigma}\mathcal{P}_\Gamma\,. \tag{1.78}$$

1.6.2 Abelian F.I. gauged supergravity without hypermultiplets

We are going to focus on the bosonic truncation of a $\mathrm{U}(1)$ F.I. gauged $\mathcal{N}=2$ supergravity theory without hypermultiplets. This is the model that will be used in later chapters of this work.

It is possible to have non-zero quaternionic momentum maps[3] $\mathcal{P}^x_\Lambda$ even in absence of hypermultiplets, provided that we gauge a subgroup of the R-symmetry group. If we gauge a subgroup $\mathrm{U}(1)$ of $\mathrm{U}(2)_R$ we will have

$$\mathcal{P}^x_\Lambda = \xi^x_\Lambda\,, \qquad\qquad \epsilon^{xyz}\xi^y_\Lambda\xi^z_\Lambda = 0\,. \tag{1.79}$$

The second equation comes from an equivariance condition for the $\mathcal{P}^x_\Lambda$ momentum maps, similar to (1.73). In this case the constants ξ^x_Λ are called Fayet–Iliopoulos parameters[4]. Let us, then, admit to have $p \leq n_V + 1$ gauge fields A^ρ_μ and non vanishing constant Fayet–Iliopoulos parameters $\xi^x_\rho = (0, \xi_\rho, 0)$, such that $A_\mu = \xi_\rho A^\rho_\mu$ gauges an $\mathrm{U}(1)$ subgroup of $SU(2)_R$. Once we put all the expectation values of the fermions to zero, the lagrangian reads

$$e^{-1}\mathscr{L}_{bos} = \frac{R}{2} - G_{a\bar{b}}\partial_\mu z^a\partial^\mu\overline{z^a} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu} - \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu} - V_g\,, \tag{1.80}$$

[3]The quaternionic momentum maps are the momentum maps coming from the gauging of isometries of the hyperscalar manifold. They form a triplet, hence the index x is to be taken from 1 to 3. Further details regarding this can be found in [48, 53]

[4]We mention that, with this procedure, it is also possible to gauge the $\mathrm{SU}(2)$ subgroup of the R-symmetry, which leads to non abelian F.I. gauged supergravity [54, 55].

where the field strengths are abelian, the scalars are neutral (the only charged fields are the gravitinos and gauginos). The scalar potential V_g is generated by the non vanishing F.I. constants and can be written as

$$V_g = g^2 \xi_\Lambda \xi_\Sigma \left(G^{a\bar{b}} f_a^\Lambda \overline{f}_b^\Sigma - 3 \overline{L}^\Lambda L^\Sigma \right) . \tag{1.81}$$

Notice that the scalar potential, which should be invariant under symplectic transformation, depends on the components of the sections $\mathcal{V}$ and $\mathcal{U}_a$. In order to obtain a manifestly invariant expression we can introduce the gauging superpotential

$$\mathcal{L} \equiv g \xi_\Lambda L^\Lambda = g_\Lambda L^\Lambda = \langle \mathcal{G}, \mathcal{V} \rangle , \tag{1.82}$$

such that the scalar potential turns out to be

$$V_g = |\mathcal{D}\mathcal{L}|^2 - 3|\mathcal{L}|^2 , \qquad \text{where } \mathcal{D}_a \mathcal{L} = \partial_a \mathcal{L} + \frac{1}{2} \partial_a \mathcal{K} \mathcal{L} = \langle \mathcal{G}, \mathcal{U}_a \rangle , \tag{1.83}$$

and this is invariant if the F.I. constants are the lower component of a symplectic vector

$$\mathcal{G} \equiv \begin{pmatrix} 0 \\ g\xi_\Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ g_\Lambda \end{pmatrix} . \tag{1.84}$$

In this model the supersymmetric transformations for the fermions gain additional terms due to the gauging, in particular [56, appendix A]

$$\delta_\varepsilon \psi_{\mu A} = \mathcal{D}_\mu \varepsilon_A + \langle \mathcal{V}, \mathcal{F}^-_{\mu\nu} \rangle \gamma^\nu \epsilon_{AB} \varepsilon^B - \frac{i}{2} \mathcal{L} \delta_{AB} \gamma^\nu \eta_{\mu\nu} \varepsilon^B , \tag{1.85}$$

$$\delta_\varepsilon \lambda^{aA} = - i \partial\!\!\!/ z^a \varepsilon^A + \frac{i}{2} G^{a\bar{b}} \langle \overline{\mathcal{U}}_b, \mathcal{F}^-_{\mu\nu} \rangle \gamma^{\mu\nu} \epsilon^{AB} \varepsilon_B + G^{a\bar{b}} \overline{\mathcal{D}_b \mathcal{L}} \delta^{AB} \varepsilon_B , \tag{1.86}$$

where the covariant derivative on ε is

$$\mathcal{D}_\mu \varepsilon_A = \partial_\mu \varepsilon_A - \frac{1}{4} (\omega_\mu)^{mn} \gamma_{mn} \varepsilon_A + \frac{i}{2} \mathcal{A}_\mu \varepsilon_A + g_\Lambda A_\mu^\Lambda \delta_{AB} \epsilon^{BC} \varepsilon_C . \tag{1.87}$$

The fact that in (1.84) the Fayet–Iliopoulos gaugings are gathered in a symplectic vector with only lower components suggests that we are choosing a preferred symplectic frame. One could, in principle, restore symplectic covariance by introducing magnetic gaugings g^Λ, such that

$$\mathcal{G} = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix} . \tag{1.88}$$

As shown in [57, 58], the extension to magnetic gaugings needs the introduction of tensor fields, hence one needs to add vector–tensor multiplets to the theory. The supersymmetry transformations and scalar potential contribution for such a theory were worked out in [59]. For what we are concerned, when dealing with gauged supergravity, we are going to ask for vanishing expectation values of the tensors.

In the following chapters we are going to work with charged black hole solutions, where the black hole charges are given by

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \frac{1}{Vol(\Sigma)} \int_\Sigma \mathcal{F} \,, \tag{1.89}$$

for a closed surface Σ that envelopes the black hole. In a theory with multiple non-local charges we need to impose a Dirac quantization condition like

$$\langle \mathcal{G}, \mathcal{Q} \rangle = n \in \mathbb{Z} \,, \tag{1.90}$$

which will have important applications in the case of black holes solutions in gauged supergravity.

Chapter 2

Asymptotically flat black holes

In this chapter we will first consider static, asymptotically flat, spherically symmetric and charged black holes solutions of $\mathcal{N}=2$, $d=4$ ungauged supergravity. These can be seen as generalisations of the Reissner–Nördstrom solutions in General Relativity. As we will see, the main difference we encounter when considering black holes in supergravity comes from the presence of scalar fields in our theory. This means that the metric could depend on the values of the scalars at the boundary, which are continuous parameters that could ruin the statistical interpretation of entropy. We will show how, for extremal solutions, independently that they preserve supersymmetry or not, there is an attractor mechanism at work, which requires the scalar fields to lose all information of their initial condition during the flow towards the horizon. This ensures that the entropy solely depends on discrete parameters.

We will also look at multi-center configurations for extremal black holes, both for the BPS case and the less understood non-BPS case. The physics of supersymmetric multi-center black holes is quite rich and has led to many further developments in string theory. Among other results, it has led to a deeper understanding of the quantization of spacetime [60] and the identification of new candidate microstate geometries for extremal BPS black holes [27, 61, 62].

2.1 Black holes in supergravity

We will now work out the main features of static, spherically symmetric, charged and asymptotically flat black holes in supergravity. In order to do so we use the $\mathcal{N}=2$ supergravity model from (1.65), where no charged particles nor scalar potential appear

$$e^{-1}\mathscr{L}=\frac{R}{2}-G_{a\bar{b}}\partial_\mu z^a\partial^\mu\bar{z}^{\bar{b}}+\frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu}-\frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu}\,. \tag{2.1}$$

We will use a general ansatz for the metric that reproduces static and spherically symmetric solutions

$$ds^2=-e^{2U}dt+e^{-2U(r)}\left[dr^2+f(r)^2d\Omega^2\right]\,, \tag{2.2}$$

where (r,θ,ϕ) are spherical coordinates. We can further constrain our ansatz by fixing $f(r)$ [63, Section 4.1] to be

$$f(r)^2 = r^2 e^{2U(r)} = (r-r_+)(r-r_-) \, , \tag{2.3}$$

where $r_\pm$ are the two roots of e^{2U} where we expect to find coordinate singularities. These are the radial values of the horizons, which allow us to introduce an extremality parameter $c \equiv r_+ - r_-$. We can then introduce a new radial coordinate ρ such that

$$\rho(r) \equiv -\int_r^\infty \frac{1}{f(s)^2} ds \, , \tag{2.4}$$

which means that the metric (2.2) can be rewritten as

$$ds^2 = -e^{2U(\rho)}dt^2 + e^{-2U(\rho)}\left[\frac{c^4}{sinh^4(c\rho)}dr^2 + \frac{c^2}{sinh^2(c\rho)}d\Omega^2\right] . \tag{2.5}$$

The extremality parameter that appears in this expression allows us to encompass both extremal and non-extremal solutions. Notice that in the extremal case we only have one horizon, hence the ρ coordinate takes the simple form

$$\rho = \frac{1}{r - r_H} \, . \tag{2.6}$$

Following the discussion in section 1.1, we introduce n_V magnetic and electric charges for our black hole, which are gathered in a symplectic vector

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \frac{1}{4\pi}\int_{S^2} \mathcal{F} \, , \tag{2.7}$$

where S^2 is a 2-sphere at spatial infinity. We can also provide ansatze for the vectors A^Λ and their duals, that are consistent with the charges and the isometries of the metric:

$$A^\Lambda = \chi^\Lambda(\rho)dt - p^\Lambda cos(\theta)d\phi \qquad | \quad F^\Lambda = dA^\Lambda \, , \tag{2.8a}$$

$$A_\Lambda = \psi_\Lambda(\rho)dt - q_\Lambda cos(\theta)d\phi \qquad | \quad G_\Lambda = dA_\Lambda \, . \tag{2.8b}$$

Here χ^Λ and ψ_Λ can be seen, respectively, as the electric and magnetic potentials. In the original action we only find A^Λ, however we will show that χ^Λ can be integrated out, leaving us with a scalar potential that depends only on the charges. As a matter of fact, one can already make use of the duality relation

$$G_\Lambda = \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma + \mathcal{R}_{\Lambda\Sigma}F^\Sigma \, , \tag{2.9}$$

which, using the explicit expressions from (2.8a) and (2.8b), implies

$$\dot{\chi}^\Lambda = e^{2U}(\mathcal{I}^{-1})^{\Lambda\Sigma}\left(q_\Sigma - \mathcal{R}_{\Sigma\Omega}p^\Omega\right) , \tag{2.10}$$

in order to see that the electric potential can be removed in favour of a combination of the charges and the matrices $\mathcal{I}$ and $\mathcal{R}$.

The original action (2.1) can be reduced to a one dimensional effective theory thanks to the isometries that we imposed on the solutions. The gravity and scalar sector of the action are reduced to

$$S_{grav,scal} = -N\int d\rho\left[\dot{U}^2 - c^2 + G_{a\bar{b}}(z,\bar{z})\dot{z}^a\dot{\bar{z}}^{\bar{b}}\right] , \tag{2.11}$$

where N is a constant factor coming from the integration in (t,θ,ϕ). The dimensional reduction of the vector sector gives

$$S_{vec} = \frac{N}{2}\int d\rho\left[\mathcal{I}_{\Lambda\Sigma}\left(e^{2U}p^\Lambda p^\Sigma - e^{-2U}\dot{\chi}^\Lambda\dot{\chi}^\Sigma\right) - \mathcal{R}_{\Lambda\Sigma}\left(p^\Lambda\dot{\chi}^\Sigma + p^\Sigma\dot{\chi}^\Lambda\right)\right] , \tag{2.12}$$

where, as anticipated, the electric potentials only appear through their first derivatives. In order for the χ^Λ's equations of motion to reproduce (2.10) we need to add a total derivative

$$S'_{vec} = S_{vec} + N\int d\rho\, q_\Lambda\dot{\chi}^\Lambda , \tag{2.13}$$

such that the boundary term appearing in the variation of the action in χ^Λ is zero along the solutions. The on-shell action for the vector sector can be, finally, written as

$$S'_{vec} = \frac{N}{2}\int d\rho\, e^{2U}\mathcal{Q}^T\mathcal{M}\mathcal{Q} = -N\int d\rho\, e^{2U}V_{BH}(z,\bar{z}) , \tag{2.14}$$

where $\mathcal{M}$ is a real, symmetric and scalar dependent symplectic matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1}\end{pmatrix} \qquad | \quad \mathcal{M}\Omega\mathcal{M} = \Omega . \tag{2.15}$$

The resulting 1-dimensional effective action is [23]

$$S_{1-dim} = -N\int d\rho\left[\dot{U}^2 - c^2 + G_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} + e^{2U}V_{BH}\right] , \tag{2.16}$$

from which we find the following equations of motion for the scalars and the warp factor

$$\ddot{U} = e^{2U}V_{BH} , \tag{2.17a}$$

$$\ddot{z}^a + \Gamma^a_{bc}\dot{z}^a\dot{z}^b = e^{2U}G^{a\bar{b}}\partial_{\bar{b}}V_{BH} . \tag{2.17b}$$

These equations, however, do not contain all the information of the original theory. From the Einstein equations of the 4 dimensional theory one finds the previous equations and a constraint

$$\dot{U}^2 + G_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} = c^2 + e^{2U}V_{BH} , \tag{2.18}$$

where no second order derivatives appear. This constraint has to be added to the equations of motion from the reduced theory in order to achieve full equivalence with the original theory. As we will see, the presence of the constraint is essential for the reduction of the equation of motion to first order.

The expression for the scalar potential V_{BH} is a remarkable consequence of special geometry. One can introduce the central charge $\mathcal{Z}$ of the $\mathcal{N}=2$ SUSY algebra

$$\mathcal{Z} \equiv \langle \mathcal{Q}, \mathcal{V} \rangle \,, \tag{2.19}$$

which satisfies the following relation [46]

$$|\mathcal{Z}|^2 + |\mathcal{D}\mathcal{Z}|^2 = \mathcal{Z}\overline{\mathcal{Z}} + G^{a\bar{b}} \mathcal{D}_a \mathcal{Z} \overline{\mathcal{D}_b \mathcal{Z}} = -\frac{1}{2} \mathcal{Q}^T \mathcal{M} \mathcal{Q} = V_{BH} \,, \tag{2.20}$$

where $\mathcal{D}_a \mathcal{Z} \equiv \langle \mathcal{Q}, \mathcal{U}_a \rangle$. This means that the potential is invariant under symplectic rotations of the charges, then equation (2.17a) tells us that the metric is, as expected, invariant under U-duality transformations.

2.2 The attractor mechanism

The equations of motion that we found in the previous discussion

$$\ddot{U} = e^{2U} V_{BH} \qquad \text{and} \qquad \ddot{z}^a + \Gamma^a_{bc} \dot{z}^a \dot{z}^b = e^{2U} G^{a\bar{b}} \partial_{\bar{b}} V_{BH} \tag{2.21}$$

are coupled differential equations, meaning that the metric will depend on the behaviour of the scalar fields and vice versa. The horizon area, in particular, will depend on the the initial values z^a_∞ of the scalars. This is obviously a problem since these are continuous parameters that would ruin the statistical interpretation of entropy, which should depend only on discrete quantities. Extremal black holes, however, have the special property that the scalars lose all information regarding their initial values during their flow towards the horizon. This result is known as attractor mechanism and implies that, for extremal black holes, the entropy does only depend on the quantized charges. We will now look at the details of this mechanism.

In order to understand the special role of extremal black holes we have to first look at the near horizon behaviour of the metric (2.5). Notice that, as we approach the outer horizon, we have $\rho \to -\infty$. Then, assuming that the horizon area is finite, the metric should have a near horizon limit in the form $AdS_2 \times S^2$, where the factor in front of the angular part is the square of the radius r_H of the horizon. In our case we can set

$$e^{-2U} \frac{c^2}{sinh^2(c\rho)} \underset{\rho\to-\infty}{\longrightarrow} r_H^2 \,, \tag{2.22}$$

then one can introduce a proper radial coordinate $\omega(r)$ by asking that

$$e^{-2U} \frac{c^4}{sinh^4(c\rho)} d\rho^2 \underset{\rho\to-\infty}{\longrightarrow} r_H^2 d\omega^2 \,. \tag{2.23}$$

In the non-extremal case this means that $\omega = 2e^{c\rho}$, the horizon sits at $\omega_H = 0$ and an observer at $\omega_O > 0$ measures a finite proper distance from the horizon

$$L_H = \int_{\omega_H}^{\omega_O} r_H d\omega < \infty \,. \tag{2.24}$$

On the other side, in the extremal case we find $\omega = -log(-\rho)$, the horizon sits at $\omega_H = -\infty$ and any observer measures an infinite proper distance from the horizon. This result has particular consequences for the behaviour of the scalars in extremal black holes.
In order to have regular solutions one has to ask that [23]

$$G_{a\bar{b}} g^{\mu\nu} \partial_\mu z^a \partial_\nu \bar{z}^{\bar{b}} < \infty \qquad\qquad \text{for } r \to r_H \,. \tag{2.25}$$

This means that, using the proper radial coordinate, in the near horizon limit the derivatives of the scalars are finite constants

$$\frac{dz^a}{d\omega} \underset{\omega\to\omega_H}{\longrightarrow} \text{const.} \qquad\qquad \Rightarrow \qquad\qquad z^a(\omega) \propto \omega \,. \tag{2.26}$$

On the other side we do not want to scalars to blow up at the horizon, hence:

- in the extremal case the only possibility is that $dz^a/d\omega = 0$, such that $z^a(\omega) \simeq 0$ near the horizon.
- in the non-extremal case the scalars do not have time to blow up since they flow for a finite proper distance, hence we do not need a similar condition.

Extremal black holes have, then, a precise behaviour in the near horizon limit, as vanishing derivatives for the scalars mean that

$$\ddot{z}^a + \Gamma^a_{bc} \dot{z}^a \dot{z}^b = e^{2U} G^{a\bar{b}} \partial_{\bar{b}} V_{BH} \qquad\qquad \Rightarrow \qquad\qquad \partial_a V_{BH}|_{hor} = 0 \,, \tag{2.27}$$

i.e. the horizon is a critical point for the potential. This means that the scalars, which flow for an infinite proper distance, must always reach the critical value z_c at the horizon, independently of their values at spatial infinity. Let V_c be the critical value of the potential, the warp factor behaves in the near horizon limit as

$$\ddot{U} \simeq e^{2U} V_c \qquad\qquad \Rightarrow \qquad\qquad U(\rho) \simeq -log\left(\sqrt{V_c}\rho\right) \,, \tag{2.28}$$

which allows us to factorise the near-horizon metric as

$$ds^2 = \frac{1}{\rho^2}\left(-\frac{dt^2}{V_c} + V_c d\rho^2\right) + V_c d\Omega^2 \,, \tag{2.29}$$

which corresponds to an $AdS_2 \times S^2$ metric. The horizon radius is $r_H = \sqrt{V_c}$ and the entropy follows from the Bekenstein–Hawking formula. Furthermore, since the critical values of the scalars depend only on the black hole charges, the entropy will actually be dependent only on quantized quantities

$$S_{BH} = \pi V_c = \pi V_{BH}\left(p, q, z_c(p,q)\right) \,. \tag{2.30}$$

On the other side, for non-extremal black holes we have two horizons at radii that depend on the initial values of the scalars and on the scalar charges [64]. There could be cases where the scalar potential has multiple basins of attraction for different critical points for a given choice of charges. In this situations the entropy depends on a discrete index that labels the different basins, see for example [65].

2.3 Flow equations

Solutions that preserve some amount of supersymmetry, i.e. BPS solutions, admit a first order description, meaning that the equations for the scalars and warp factors can actually be reduced to first order differential equations. This is not surprising, since supersymmetric states must be solutions of the supersymmetry equations, namely the Killing spinor equations (1.67), which are first order equations. It can be shown, however, that a first order formalism is also possible for extremal non-BPS solutions [66]. We will now look at how the first order flow equations are produced in both the BPS and non-BPS cases.

In order to grasp the basic idea behind the reduction to first order equations, let us consider a 1-dimensional toy-model with n real scalar fields φ^A, an action

$$S = \int dx \left(\tilde{G}_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta + V(\varphi) \right) , \tag{2.31}$$

and an hamiltonian constraint in the form

$$H = \tilde{G}_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta - V(\varphi) = 0 , \tag{2.32}$$

where $\tilde{G}$ is an symmetric[1] metric on the scalar manifold. Notice that the action (2.16) and the constraint (2.18) obtained in section 2.1, in the extremal case, are in these forms once we allow for complex scalar fields. Using the hamiltonian constraint we can rewrite the action as

$$S = \int dx \left[\tilde{G}_{\alpha\beta} (\dot{\varphi}^\alpha + n^\alpha \sqrt{V})(\dot{\varphi}^\beta + n^\beta \sqrt{V}) - 2\tilde{G}_{\alpha\beta} n^\alpha \dot{\varphi}^\beta \sqrt{V} \right] , \tag{2.33}$$

where n^α is a unit vector in moduli space. The second term in (2.33) is a total derivative provided that we can find a function $\mathcal{W}(\varphi)$, known as superpotential, that satisfies

$$\sqrt{V} n^\alpha = \nabla^\alpha \mathcal{W} = \tilde{G}^{\alpha\beta} \partial_\beta \mathcal{W} \qquad \Rightarrow \qquad V = |\nabla \mathcal{W}|^2 , \tag{2.34}$$

since, when this is the case, we have

$$-2\tilde{G}_{\alpha\beta} n^\alpha \dot{\varphi}^\beta \sqrt{V} = -2\dot{\varphi}^\alpha \partial_\alpha \mathcal{W} = -2 \frac{d}{dx} \mathcal{W} . \tag{2.35}$$

[1] When we will work with complex scalar fields we will ask the metric $\tilde{G}$ to be hermitian.

The resulting action is made up of a sum of squares of first order expressions and a boundary term

$$S = \int dx \left[\tilde{G}_{\alpha\beta}(\dot{\varphi}^{\alpha} + \nabla^{\alpha}\mathcal{W})(\dot{\varphi}^{\beta} + \nabla^{\beta}\mathcal{W}) - 2\frac{d}{dx}\mathcal{W} \right] , \tag{2.36}$$

which is referred to as BPS squared form. The equations of motion correspond to setting to zero the squared expressions, which give us the actual flow equations in a gradient flow form

$$\dot{\varphi}^{\alpha} = -\nabla^{\alpha}\mathcal{W} . \tag{2.37}$$

For the extremal solutions of the model in section 2.1, identifying the scalar fields as $\varphi^A = \{U, z, \bar{z}\}$, we have to set

$$\tilde{G}_{UU} = 1 \qquad\qquad \tilde{G}_{a\bar{b}} = \frac{1}{2}G_{a\bar{b}} , \tag{2.38}$$

in order to have the correct identification with the toy model. The complete potential is $V = e^{2U}V_{BH}$, which suggests to introduce a fake superpotential $W = e^{-U}\mathcal{W}$, such that W only depends on the scalar fields. The fake superpotential has to satisfy

$$V = e^{2U}V_{BH} = \tilde{G}^{\alpha\beta}\partial_{\alpha}\mathcal{W}\partial_{\beta}\mathcal{W} = e^{2U}W^2 + 4e^{4U}G^{a\bar{b}}\partial_a W\partial_{\bar{b}}W . \tag{2.39}$$

Using the fake superpotential we can rewrite the action as a sum of squares

$$\begin{aligned} S = -N\int d\rho \Big[&(\dot{U} + e^U W)^2 + G_{a\bar{b}}(\dot{z}^a + 2e^U G^{a\bar{c}}\partial_{\bar{c}}W)(\dot{\bar{z}}^{\bar{b}} + 2e^U G^{d\bar{b}}\partial_d W) \\ &- 2\frac{d}{d\rho}(e^U W) \Big] , \end{aligned} \tag{2.40}$$

such that the flow equations are

$$\dot{U} = -\tilde{G}^{UU}\partial_U(e^U W) = -e^U W , \tag{2.41a}$$

$$\dot{z}^A = -2G^{a\bar{b}}\partial_{\bar{b}}(e^U W) = -2e^U G^{a\bar{b}}\partial_{\bar{b}}W . \tag{2.41b}$$

It is clear now that, by comparison with the asymptotic gravitational potential, the mass of the black hole is given by

$$M = \left(\frac{dU}{d\rho}\right)_{\rho\to 0} = W_{\infty} , \tag{2.42}$$

which is related to the boundary term of (2.40). The problem resides now in if we can find a superpotential $\mathcal{W}$ such that the above conditions are satisfied.

2.3.1 Supersymmetric attractors

The geometric properties of the $\mathcal{N} = 2$ special Kähler manifold allows us to write the potential V_{BH} as

$$V_{BH} = |\mathcal{Z}|^2 + |\mathcal{D}\mathcal{Z}|^2 = |\mathcal{Z}|^2 + 4G^{a\bar{b}}\partial_a|\mathcal{Z}|\partial_{\bar{b}}|\mathcal{Z}| \,, \tag{2.43}$$

where we used the fact that $\mathcal{D}_a\overline{\mathcal{Z}} = 0$. By comparing this expression with (2.41b), we immediately find that the fake superpotential can be identified with the modulus of the central charge, up to a sign. The flow equation for the warp factor is $d(e^{-U})/d\rho = \mp|\mathcal{Z}|$, and the sign can be fixed by looking at the boundary and horizon behaviours of e^{-U}. In order to have asymptotically flat solutions we need to impose that $e^U \to 1$ at the boundary. On the other side, in order for the horizon to have finite area we need to ask that the angular part of the metric (2.22), with $c = 0$, is finite in the near horizon limit, which means that $e^{-U} \propto |\rho|$ at the horizon. The only acceptable sign is $W = |\mathcal{Z}|$, from which we get that the superpotential is $\mathcal{W} = e^U|\mathcal{Z}|$, the flow equations are

$$\dot{U} = -\, e^U|\mathcal{Z}| \,, \tag{2.44a}$$

$$\dot{z}^A = -\, 2e^U G^{a\bar{b}}\partial_{\bar{b}}|\mathcal{Z}| \,, \tag{2.44b}$$

and the ADM mass of the solution will be $M = |\mathcal{Z}|_\infty$. This means that these solutions are at the threshold of the BPS bound $M \geq |\mathcal{Z}|$, hence they must preserve some amount of supersymmetry of the original theory. As a matter of fact, let us use the Killing spinor equations (1.67) and impose on the Killing spinor ε_A

$$\varepsilon_A(\rho) = e^{g(\rho)}\chi_A \qquad \chi_A = \text{constant} \,, \tag{2.45}$$

that ensures that the solution has the right symmetries, and the projection relation between the components of χ_A

$$\gamma_0\chi_A = i\frac{\mathcal{Z}}{|\mathcal{Z}|}\epsilon_{AB}\chi^B = ie^{i\alpha}\epsilon_{AB}\chi^B \,, \tag{2.46}$$

where $\alpha(\rho)$ is the phase of $\mathcal{Z}$. With these conditions, the solutions of the Killing spinor equations are $1/2$-BPS, since (2.46) halves the number of independent components of ε_A. The resulting first order equations, once we impose our ansatz for the metric, are

$$\dot{U} = -\, e^U|\mathcal{Z}| \,, \tag{2.47a}$$

$$\dot{z}^a = -\, 2e^{-i\alpha}e^U G^{a\bar{b}}\overline{\mathcal{D}_b\mathcal{Z}} \,. \tag{2.47b}$$

These are very similar to the the flow equations (2.44a) and (2.44b), where we set $W = |\mathcal{Z}|$; full equivalence with them is achieved by imposing a first order equation on the phase

$$\dot{\alpha} + \mathcal{A}_\rho = 0 \,, \tag{2.48}$$

where $\mathcal{A}_\rho$ is the radial component of the composite Kähler connection. It will be important later to keep in mind that α in not an independent degree of freedom of the theory and hence equations like (2.48) need to be identically satisfied once all the flow equations are. All in all, the attractor mechanism for BPS solutions is fully determined by the modulus of the central charge, as:

- The horizon is a critical point of the potential that is also a critical point of $|\mathcal{Z}|$. Special Kähler geometry further contrains this critical point to be a minimum, since

$$\partial_a \partial_{\bar{b}} |\mathcal{Z}| = G_{a\bar{b}} |\mathcal{Z}| > 0 \,. \tag{2.49}$$

- Let $\mathcal{Z}_c$ be the value of the central charge at the critical point, then the near-horizon metric is

$$ds^2_{NH} = -\frac{\rho^2}{|\mathcal{Z}_c|^2} dt^2 + \frac{|\mathcal{Z}_c|^2}{\rho^2} d\rho^2 + |\mathcal{Z}_c|^2 d\Omega^2 \,, \tag{2.50}$$

which means that the radius of the horizon is $r_H = |\mathcal{Z}_c|^2$ and the entropy is

$$S_{BH} = \pi |\mathcal{Z}_c|^2 \,. \tag{2.51}$$

This makes it clear the knowledge of the superpotential is of great interest in order to find the dependency of entropy on the charges.

2.3.2 Non supersymmetric attractors

For a given choice of charges there could be extremal solutions where a critical point of the potential V_{BH} is not a critical point of $|\mathcal{Z}|$. These correspond to non-BPS solutions of the Killing spinor equations, i.e. solutions that do not preserve any of the original supersymmetries. In order for this situation to arise we have to ask that the horizon is at a critical point of a fake superpotential $W \neq |\mathcal{Z}|$. The geometry at the horizon is, then, determined by the critical value W_c of W and the metric has a near horizon behaviour

$$ds^2_{NH} = -\frac{\rho^2}{W_c^2} dt^2 + \frac{W_c^2}{\rho^2} d\rho^2 + W_c^2 d\Omega^2 \,. \tag{2.52}$$

In this case the entropy turns out to be $S_{BH} = \pi W_c^2$. Notice that not all possible choices of charges will lead to the existence of a fake superpotential $W \neq |\mathcal{Z}|$ that satisfies the requirements. Furthermore, even if techniques have been developed [67–69], finding a superpotential is often a challenging task. As a last point, we mention that in the non-BPS case the potential at the critical point can have flat directions, then these flat direction extend to the whole potential and the solution will not depend at all on these scalar fields. This is reflected by the fact that the fake superpotential is independent on these scalar fields [67, 68, 70].

2.4 Multi-center solutions

It is clearly possible, in the context of Newtonian gravity and Maxwell electromagnetism, to build configurations of charged particles at equilibrium by tuning their masses and charges, thus balancing the gravitational attraction and electric repulsion. In an Einstein–Maxwell theory, however, the Einstein equations are non-linear and we have no reason to believe that, a priori, a superposition principle should be applicable to their solutions. Yet, composite objects made up of massive and charged constituents have been known for a long time in general relativity, see [71–73] for static configurations and [74] for stationary configurations. Hertle and Hawking showed that there are static configurations that describe a system of multi-center extremal Reissner–Nordström black holes. It turns out that these configurations satisfy rather simple first order equations and this explains why we can take superpositions of their elementary components, despite the non-linearity of the Einstein–Maxwell equations.

Asymptotically flat, static or rotating[2] black hole solutions of supergravity obey first order equations, independently that they conserve some amount of the original supersymmetries. The reduction of the equations to first order suggests, then, that multi-center configurations could be possible in both the BPS and non-BPS cases. The study of multi-center black holes in supergravity revealed a very rich physical landscape and has led to many developments in our understanding of black hole physics and string theory. The best understood case is the supersymmetric one, for which we have the general stationary 4-dimensional solution found by Denef [24]. Among other important results, we remark that some of the BPS multi-center configurations descend from 5-dimensional smooth horizonless solutions [25, 26], that have the same charges and mass of the 4-dimensional black hole and hence are prime candidate microstates of extremal BPS black holes. Furthermore, in the vicinity of these 5-dimensional solutions one can find an infinite dimensional family of smooth horizonless solutions, parameterized by arbitrary functions of one variable [27], whose quantization may yield an entropy with the same parametric dependence on the charges and mass as the black holes.

The case of non-BPS multi-center configurations is more involved as many particular solutions have been found but a general solution is still missing. Progress in this case has come from the discovery of single center almost-BPS solutions, where supersymmetry is weakly broken [75], from these solutions one can build multi-center configurations that have non-trivial constrains on the positions of the charges. Satisfying these constrain, however, is challenging already in the two centers case. Over the last decades many particular non-BPS multi-center solutions have been found, see for example [76, 77].

2.4.1 String theory origin

In order to construct multi-center configurations in $\mathcal{N}=2$, $d=4$ supergravity we will consider a setup which makes contact with a particular compactification of 11-dimensional string theory. These solutions, however, can be extended to more general situations. We will follow the derivation in [78], where one can find a much more detailed discussion, and consider an 11-dimensional type IIB string theory, compactified on an inner space $CY_6 \times S^1$. The 6-dimensional Calaby–Yau manifold is taken to be $CY_6 \simeq (T^2)^3$, such that this kind of compactification leads to an $\mathcal{N}=2$ supergravity theory in 4 dimensions, with a scalar

[2] For example, an analysis of asymptotically flat, rotating, BPS black hole solutions of $\mathcal{N}=2$ supergravity can be found in [24].

manifold described by a STU model and no hypermultiplets. In the low dimensional theory there will be 6 real charges (p^I, q_I) and two geometric charges (p^0, q_0). Let us call $(t, \vec{x})$ the coordinates of 4-dimensional spacetime, ψ the coordinate on the circle S^1 and $(y_{I,1}, y_{I,2})$ the two coordinates on the I-th torus of CY_6. The index I will be taken to run from 1 to 3. We use the 11-dimensional stationary metric

$$ds_{11}^2 = -Z^{-2}(dt+k)^2 + Zds_4^2 + \sum_{I=1}^{3} \frac{Z}{Z_I} ds_I^2 \ , \tag{2.53}$$

where k is a one-form, ds_4^2 is a metric on a Ricci-flat 4-dimensional eucledean space parameterized by $(\psi, \vec{x})$, ds_I^2 is the metric on the I-th torus. In order for no hypermultiplets to appear in the low dimensional theory we fix the volume of the internal manifold to be 1. The warp factors $\{Z, Z_I\}$ only depend on the coordinates $(\psi, \vec{x})$ and are related by

$$Z = \left(Z_1 Z_2 Z_3\right)^{1/3} . \tag{2.54}$$

By providing the floating brane ansatz for the 3-form potential

$$A^{(3)} = \sum_I A_I^{(3)} \wedge dT_I = \sum_I \left(\frac{-dt+k}{Z_I} + a_I \right) \wedge dT_I \ , \tag{2.55}$$

where dT_I is the volume form of the I-th torus, the equations of motion and Bianchi identities of the 11-dimensional theory will reduce to almost linear equations in the coordinates $(\psi, \vec{x})$ [25]. In order to arrive to the 4-dimensional theory, let us set the ds_4^2 metric to be that of a Gibbons–Hawking space with a U(1) isometry along ψ

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V ds_3^2 \ , \tag{2.56}$$

where V and A only depend on the spatial coordinates $\vec{x}$ and satisfy

$$\star_3 dA = \pm V \ . \tag{2.57}$$

The sign in (2.57) specifies the orientation of the space and leads to different types of solutions: the plus sign will lead to supersymmetric ones, while the minus sign to non-supersymmetric ones. The a_I and k one-forms decompose accordingly as

$$a_I = P_I(d\psi + A) + w^I \ , \qquad k = \mu(d\psi + A) + \omega \ , \tag{2.58}$$

where ω and w^I are 1-forms on the 3 dimensional space.

Once we perform the integration on the whole internal manifold we are left with the metric

$$ds_{4d}^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \ , \tag{2.59}$$

for the 4-dimensional spacetime. We also find a 4-dimensional lagrangian whose bosonic sector is in the familiar form (1.65). We have 3 vector multiplets, meaning that now the indices can be identified as $\Lambda = \{0, I\}$ and $a = I$. Notice that the metric (2.59) allows for a non-static contribution ω, which is to be expected since the vector fields for a configuration of multiple static charges carry an intrinsic angular momentum[3]. The scalars, the warp factor U and the vectors have a constrained form with respect to the various quantities descending from the 11-dimensional theory

$$e^{-2U} = \sqrt{Z_1 Z_2 Z_3 V - \mu^2 V^2}\,, \tag{2.60}$$

$$z^I = \frac{(V Z_I P_I - V\mu) - ie^{-2U}}{V Z_I}\,, \tag{2.61}$$

and $A^\Lambda = w^\Lambda + \chi^\Lambda(dt+\omega)$, where

$$A^0 = -A + e^{4U}\mu V^2(dt+\omega)\,, \tag{2.62}$$

$$A^I = w^I - \frac{e^{4U}V}{Z_I}(Z_1 Z_2 Z_3 - \mu V P_I Z_I)(dt+\omega)\,. \tag{2.63}$$

The w^Λ will be related to the magnetic charges and the χ^Λ to the electric potentials. Duality invariance of the vector sector allows to define the closed dual field strength $G_\Lambda = dA_\Lambda$. The metric and the gauge kinetic couplings are fully determined in terms of the scalar by the STU model structure, see [78, appendix A] for details.

Once we know a solution we can generate new solutions in the same orbit by making use of U-duality transformations. The U-duality group, in the case of the STU model, is $SU(1,1)^3 \in Sp(n,\mathbb{R})$ and its action is described by three 2×2 matrices

$$M_I = \begin{pmatrix} a_I & b_I \\ c_I & d_I \end{pmatrix} \qquad \text{such that} \qquad Det(M_I) = 1\,. \tag{2.64}$$

These transformations act on the scalars as

$$z^I \longrightarrow \frac{a_I z^I + b_I}{c_I z^I + d_I} \qquad \text{not summed over I}\,, \tag{2.65}$$

and on symplectic vectors as

$$a'_{abc} = (M_1)_a^{\ d}(M_2)_b^{\ e}(M_1)_c^{\ f}\, a_{def}\,, \tag{2.66}$$

where we identify the components of a symplectic vector $\mathcal{V} = (V^\Lambda; V_\Lambda)$ as

$$\begin{array}{llll} V^0 = a_{222} & V^1 = a_{211} & V^2 = a_{121} & V^3 = a_{112} \\ V_0 = -a_{111} & V_1 = a_{122} & V_2 = a_{212} & V_3 = a_{221}\,. \end{array} \tag{2.67}$$

2.4.2 The BPS case

The BPS case is obtained by asking for the plus sign in the orientation of the Gibbons–Hawking space in equations (2.57). This leads to a set of almost linear equations in the

[3] we will expand on the role and physical meaning of the angular momentum later on in this section.

spatial coordinates $\vec{x}$

$$
\begin{aligned}
d\star_3 dZ_I =& \frac{|\epsilon_{IJK}|}{2} d\star_3 d(VP_JP_K) \,, && (2.68a)\\
\star_3 dw^I =& -d(VP_I) \,, && (2.68b)\\
\star_3 d\omega =& Vd\mu - \mu dV - VZ_I dP_I \,, && (2.68c)
\end{aligned}
$$

which can be solved, in general, in terms of 8 harmonic functions $\{H^\Lambda, H_\Lambda\}$. This means, in turn, that the scalars, the warp factors and the vector fields can be all expressed in terms of these harmonic functions. We can gather these functions in a symplectic vector $\mathcal{H}$, such that

$$
\mathcal{H} = \begin{pmatrix} H^\Lambda \\ H_\Lambda \end{pmatrix} \qquad \Rightarrow \qquad \star_3 d\mathcal{A} = \star_3 d \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix} = d\mathcal{H} \,. \tag{2.69}
$$

The equations for the warp factor and the scalars are then fully determined by the harmonic functions as:

$$
\begin{aligned}
e^{-2U} =& \sqrt{I_4} \,, && (2.70a)\\
z^I =& \left(H^I + i\frac{\partial\sqrt{I_4}}{\partial H_I}\right)\left(H^0 + i\frac{\partial\sqrt{I_4}}{\partial H_0}\right)^{-1} \,, && (2.70b)
\end{aligned}
$$

where $I_4 = I_4(H^\Lambda, H_\Lambda)$ is the quartic invariant of the STU model.

Notice that, since the harmonic functions form a symplectic vector, then they simply mix one with the other under U-duality transformations in $\mathrm{SU}(1,1)^3$. This means that U-duality does not change the ansatz we have given for the 11-dimensional metric and the 3-form potential. This means that we are encompassing all possible BPS solutions, even though the details of the specific solution depend on the choice of the harmonic functions.

Following the original derivation of the multi-center BPS solution by Denef [24], we can find an explicit expression for the harmonic functions in terms of the charges and positions of the centers. In order to do so, we can first look at the static single center solution that we described in this chapter and use the flow equations (2.47a), (2.47b) and the phase equation (2.48) to show that

$$
\mathcal{Q} = -\frac{d}{d\rho}\left[e^{-U} Im(e^{-i\alpha}\mathcal{V})\right] \qquad \Rightarrow \qquad d\star_3 d\left[e^{-U} Im(e^{-i\alpha}\mathcal{V})\right] = 0 \,, \tag{2.71}
$$

from which we can set the symplectic vector of harmonic functions to be $\mathcal{H} \equiv e^{-U} Im(e^{-i\alpha}\mathcal{V})$. Integration of the first one gives us an equivalent expression for $\mathcal{H}$

$$
\mathcal{H} = -\frac{\mathcal{Q}}{r - r_H} + 2Im(e^{-i\alpha}\mathcal{V})|_{r=\infty} \,. \tag{2.72}
$$

This expression can be easily generalised to a configuration with N static centers by asking that

$$\mathcal{H} = -\sum_{n=1}^{N} \frac{\mathcal{Q}_n}{|\vec{x}-\vec{x}_n|} + 2Im(e^{-i\alpha}\mathcal{V})|_{r=\infty} \,, \tag{2.73}$$

where $\mathcal{Q}_n$ is the vector of magnetic and electric charges for the n-th center and $\vec{x}_n$ is its position. Let us then introduce central charges for the centers as $\mathcal{Z}_n = \langle \mathcal{Q}_n, \mathcal{V}\rangle$ and "inverse radial distances" from them as $\rho_n = |\vec{x}-\vec{x}_n|^{-1}$, these allow us to introduce a 1-form

$$\zeta \equiv -\langle d\mathcal{H}, \mathcal{V}\rangle = \sum_n \mathcal{Z}_n d\rho_n \,, \tag{2.74}$$

such that taking symplectic products of $\mathcal{H}$ with $\mathcal{V}$ and $\mathcal{U}_a$ gives the flow equations for the general solution

$$d\alpha + \mathcal{A} = e^{U} Im(e^{-i\alpha}\zeta) = -\frac{1}{2}e^{2U}\langle d\mathcal{H}, \mathcal{H}\rangle \,, \tag{2.75a}$$

$$dU = e^{-U} Re(e^{-i\alpha}\zeta) \,, \tag{2.75b}$$

$$dz^a = -\, e^{i\alpha} G^{a\bar{b}} e^{U} \overline{D_b}\zeta \,, \tag{2.75c}$$

and the non-static contribution ω obeys

$$\star_3 d\omega = \langle d\mathcal{H}, \mathcal{H}\rangle \,. \tag{2.75d}$$

Equation (2.75d) implies, as we already mentioned, that one has to introduce a non-static contribution to the metric in order to find multi-center solutions. The presence of this contribution, however, must not be interpreted as a rotation of the centers but as the angular momentum contained in the vector fields produced by a static distribution of charges. As a matter of fact, in classical electrodynamics one can have a distribution of non-local charges at equilibrium, for example the monopole-electron system, that has an intrinsic angular momentum even if the particles are at rest. With this in mind, we can interpret the ω contribution as being is related to the intrinsic angular momentum $\vec{J}$ associated to the vector fields, such that asymptotically

$$\omega_i = 2\epsilon_{ijk}\frac{J^j x^k}{r^3} + O\left(\frac{1}{r^3}\right) \qquad \Rightarrow \qquad \vec{J} = \frac{1}{2}\sum_{n<m}\langle Q_n, Q_m\rangle \frac{\vec{x}_m - \vec{x}_n}{|\vec{x}_m - \vec{x}_n|} \,. \tag{2.76}$$

Consistency of these solutions implies the following constraint on the charges of the centers and their positions:

$$\sum_{m=1}^{n} \frac{\langle \mathcal{Q}_n, \mathcal{Q}_m\rangle}{|\vec{x}_n - \vec{x}_m|} = 2Im\big(e^{-i\alpha}\mathcal{Z}_n\big)|_{r=\infty} \,, \tag{2.77}$$

whose zeros give the positions in moduli space related to marginal stability of the solutions.

2.4.3 Non-BPS case

The non-BPS case is obtained by using the minus sign in equation (2.57), which leads to a mild supersymmetry breaking and gives what are known as almost-BPS configurations [75]. The floating brane ansatz gives us almost linear equations in the $\vec{x}$ coordinates

$$d\star_3 dZ_I = \frac{|\epsilon_{IJK}|}{2} V d\star_3 d(P_J P_K) \, , \tag{2.78a}$$

$$\star_3 dw^I = P_I dV - V dP_I \, , \tag{2.78b}$$

$$\star_3 d\omega = d(\mu V) - V Z_I dP_I \, . \tag{2.78c}$$

From these equations we can still find harmonic functions to associate to the P_I, however this has the consequence that the Z_I cannot be harmonic. The single center case is special, in that we set $P_I = 0$ and then V and Z_I are harmonic. On the other side, the multi-center case is rather complicated: we only have four of the eighth harmonic functions that where found in the BPS case. We do not report the full solution, however the details can be found in [76, 78].

The solution found in [76] can be used as a seed to generate new solutions by making use of U-duality. In this case, however, U-duality transformations change the form of the 5-dimensional metric obtained from compactification of the Calabi-Yau manifold [78]. This means that our ansatz does not encompass all possible solutions, hence non supersymmetric configurations are far richer that the supersymmetric ones. Any attempt to construct a general multi-center non-BPS solution, similar to the Denef solution in the BPS case, have to face much harder challenges. Furthermore, an important property of the solution in [76] is that, in order for it to be regular, we need to introduce non-trivial constraints on the positions of the centers already in the 2-centers case, which are quite difficult to satisfy. Some interesting configurations have been found, we cite as an example the case of a line of rotating black holes [77].

As a side note, we mention that progress towards a general solutions encompassing all multi-center non-BPS solutions have been made, in particular in the 4-dimensional case, where one can make use of the timelike dimensional reduction [79], that allows us to relate black holes, regardless of supersymmetry, to geodesics on the scalar manifold. Then, provided that the scalar manifold has sufficient symmetries, one can generate solutions, even multi-center ones, by employing group theoretical methods [80]. With this method one can find the general non-BPS multi-center solution [81], at the expense of the result being expressed in a less explicit way. On the other side, we still lack a general non-BPS solution using the explicit superpotential approach, which generalises the method used by Denef in the BPS case, even if some progress has been made [82, 83].

Chapter 3

Anti-de Sitter black holes

For a long time, black holes with an anti-de Sitter asymptotic behaviour have been neglected in favour of the much more studied Minkowski ones, mainly because they do not seem to be of particular relevance for the description of observable objects in our universe. The situation drastically changed in the last decades, mainly thanks to the introduction of the gauge/gravity correspondence. As a matter of fact, we can exploit the dual conformal field theory built on the AdS boundary of these solutions to provide a microscopic interpretation of the degenerate geometries that contribute to the black hole entropy [32]. These gravitational systems also provide non trivial asymptotically AdS backgrounds that have rich holographic structure and have many applications in the context of field theory and condensed matter systems at strong coupling.

The zoology of AdS black holes is much broader than the one of asymptotically flat ones, as the horizons can be compact Riemann surfaces of any genus [84] or non-compact surfaces, corresponding for example to black brane solutions. For black holes with a compact horizon, the Bekenstein–Hawking formula, which relates the entropy to the area of the horizon, is still valid. The known solutions fall into a general rotating solution with electric, magnetic and NUT charges [85]. In particular, purely electric black holes admit both extremal and thermal, rotating and static solutions [86–89], while if we ask for supersymmetric solutions we only have rotating configurations with constant scalars [84, 90]. In the magnetic case we can have BPS solutions only for static black holes with non constant scalars [29, 30, 91], while non-BPS and non-extremal solutions must have vanishing angular momentum [92–95]. Recently, a 1/4-BPS rotating solution has been proposed [34], this solution has either compact or non-compact horizon and in the static limit it reduces to the solution of [30]. We will focus on rotating AdS black holes in the next chapter.

In this chapter we are going to investigate supersymmetric static black hole solutions in $\mathcal{N}=2$, $d=4$, $\mathrm{U}(1)$ Fayet–Iliopoulos gauged supergravity. As we will see, we can still find an attractor mechanism, although the situation is radically different from the asymptotically flat case. The content of this chapter is based on the results of [30], which generalises, using a full symplectic covariant description, a previous work [29] where the supersymmetric

AdS black holes with non constant scalars where first described.

3.1 Setup and dimensional reduction

We will look for dyonic black holes solutions in $\mathcal{N}=2$ gauged supergravity in 4 dimensions. We use the U(1) Fayet–Iliopoulos gauged theory with bosonic lagrangian

$$e^{-1}\mathscr{L}=\frac{R}{2}-G_{a\bar{b}}\partial_\mu z^a\partial^\mu\bar{z}^{\bar{b}}+\frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu}-\frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\mu\nu}-V_g \tag{3.1}$$

and set the expectation values for the fermions to zero, as we have done in the previous case. The potential V_g, introduced because of the gauging, takes the symplectic invariant form

$$V_g=|\mathcal{D}\mathcal{L}|^2-3|\mathcal{L}|^2\ , \tag{3.2}$$

with $\mathcal{L}\equiv\langle\mathcal{G},\mathcal{V}\rangle$ known as gauging superpotential. We let both electric and magnetic gaugings be part of the symplectic vector $\mathcal{G}=(g^\Lambda;g_\Lambda)$, such that can work out a fully symplectic covariant description. One can recover the case where we only have electric gaugings by moving to the appropriate symplectic frame.

Since we are looking for static, spherically symmetric and charged black holes we can employ the following ansatz for the metric:

$$ds^2=-e^{2U(r)}dr^2+e^{-2U(r)}\left(dr^2+e^{2V(r)}d\Omega^2\right)\ , \tag{3.3}$$

where (r,θ,ϕ) are spherical coordinates, $U(r)$ and $V(r)$ are warp factors that depends only on the radial coordinate. In accordance with this ansatz for the metric, we ask the scalar fields to only depend on the radial coordinate, then the scalar and gravity sectors of the lagrangian can be reduced to an effective 1-dimensional action. In the same way, we introduce an appropriate ansatz for the vector fields

$$A^\Lambda=\chi^\Lambda(r)dt-p^\Lambda cos(\theta)d\phi \qquad\qquad A_\Lambda=\psi_\Lambda(r)dt-q_\Lambda cos(\theta)d\phi\ , \tag{3.4}$$

such that the integration of the field strengths $\mathcal{F}=(F^\Lambda;G_\Lambda)$ on a closed surface provides us with the electric and magnetic charges

$$\mathcal{Q}=\frac{1}{4\pi}\int_{S^2}\mathcal{F}\ . \tag{3.5}$$

The electric potentials χ^Λ only appear in the action through their first derivative in r and can be integrated out using their equations of motion. These are in accordance with the results of the duality relation (1.21), that, for our metric and vector fields ansatze, requires

$$\dot{\chi}^\Lambda=e^{2(U-V)}(\mathcal{I}^{-1})^{\Lambda\Sigma}\left(q_\Sigma-\mathcal{R}_{\Sigma\Omega}p^\Omega\right)\ . \tag{3.6}$$

As in the asymptotically flat case we need to add to the action a total derivative "$q_\Lambda \dot{\chi}^\Lambda$" in order to match the equations of motion for χ^Λ and the request (3.6). All in all, the dimensional reduction of the vector sector amounts to the introduction of a scalar potential

$$V_{BH} = -\frac{1}{2}\mathcal{Q}^T \mathcal{M} \mathcal{Q} = |\mathcal{Z}|^2 + |\mathcal{D}\mathcal{Z}|^2 \tag{3.7}$$

that only depends on the charges. These reductions allow us to find a one dimensional effective action

$$S_{1d} = -N \int dr \left[e^{2V}\left(\dot{U}^2 - \dot{V}^2 + G_{a\bar{b}}\dot{z}^a \dot{\bar{z}}^{\bar{b}} + e^{-2U}V_g + e^{2U-4V}V_{BH} \right) - 1 \right], \tag{3.8}$$

where we removed the Einstein–Gibbons boundary term and the integration in time and in the angular variables factors out. The equations of motion coming from the effective action (3.8) govern the dynamics of the scalar fields and the two warp factors. They are:

$$\ddot{z}^a + \Gamma^a{}_{bc}\dot{z}^a \dot{z}^b + 2\dot{V}\dot{z}^a = e^{2U} G^{a\bar{b}} \partial_{\bar{b}}\left(V_g + e^{-4U}V_{BH} \right), \tag{3.9a}$$

$$\ddot{U} + 2\dot{U}\dot{V} = e^{2(U-2V)}V_{BH} - e^{-2U}V_g , \tag{3.9b}$$

$$\ddot{V} + \dot{V}^2 + \dot{U}^2 + G_{a\bar{b}}\dot{z}^a \dot{\bar{z}}^{\bar{b}} = e^{2(U-2V)}V_{BH} - e^{-2U}V_g . \tag{3.9c}$$

Full equivalence between these equations of motion and the Einstein equations of the original 4-dimensional theory is ensured by the introduction of an hamiltonian constraint

$$\dot{U}^2 - \dot{V}^2 + G_{a\bar{b}}\dot{z}^a \dot{\bar{z}}^{\bar{b}} = e^{-2U}V_g + e^{2(U-2V)}V_{BH} - e^{-2V} . \tag{3.10}$$

The appearance of this hamiltonian constraint suggests that these solutions admit a first order description. As we will see in the next sections, this is the case for BPS solutions, where we can find first order flow equations that satisfy both the equations of motion and the hamiltonian constraint.

3.2 The BPS rewriting and flow equations

In order to show that these solutions are characterised by a set of first order flow equations, one has to achieve a rewriting of the action (3.8) as a sum of squares of first order expressions up to boundary terms. One can, equivalently, derive these equations from the Killing spinor equations and, in the case of supersymmetric solutions, one needs to prove that the Killing spinors used in the Killing spinor equations do indeed preserve some amount of the original supersymmetries. It was shown in [30] that, following the same procedure used in the ungauged case [24], the action admits the following rewriting

$$\begin{aligned} S_{1d} = -N \int dr \Bigg\{ & -\frac{1}{2}e^{2U-2V}\mathcal{E}^T \mathcal{M}\mathcal{E} - e^{2V}\left[(\dot{\alpha} + \mathcal{A}_r) + 2e^{-U} Re(e^{-i\alpha}\mathcal{L}) \right]^2 \\ & - e^{2V}\left[\dot{V} - 2e^{-U} Im(e^{-i\alpha}\mathcal{L}) \right]^2 - 1 - \langle \mathcal{G}, \mathcal{Q} \rangle \\ & - 2\frac{d}{dr}\left[e^{2V-U} Im(e^{-i\alpha}\mathcal{L}) + e^{U} Re(e^{-i\alpha}\mathcal{Z}) \right] \Bigg\} , \end{aligned} \tag{3.11}$$

where $\alpha(r)$ is a real phase. Notice that, at this point, the phase α can be freely introduced thanks to the properties of symplectic sections. We will see that α has a deeper meaning when considering the KSE derivation of the flow equations. $\mathcal{A}_r = Im(\dot{z}^a \partial_a \mathcal{K})$ is the radial component of the composite Kähler connection. The quantity $\mathcal{E}$ is a symplectic vector made up of multiple contributions

$$\begin{aligned}\mathcal{E} =& 2e^{2V} \frac{d}{dr}\big(e^{-U} Im(e^{-i\alpha}\mathcal{V})\big) + 4e^{2V-U}(\dot{\alpha} + \mathcal{A}_r) Re(e^{-i\alpha}\mathcal{V}) \\ &+ e^{2V-2U}\mathcal{M}^{-1}\Omega\mathcal{G} + \mathcal{Q}\,. \end{aligned} \tag{3.12}$$

In order to show the equivalence of the two actions it is necessary to use a plethora of relations that follow from the basic identities of special Kähler geometry reported at the end of section 1.3. The main relation, that is extensively used in deriving the other ones, is

$$\frac{1}{2}\big(\mathcal{M} - i\Omega\big) = \Omega\overline{\mathcal{V}}\mathcal{V}^T\Omega + \Omega\mathcal{U}_a G^{a\bar{b}} \mathcal{U}_{\bar{b}}^T\Omega\,, \tag{3.13}$$

from which one finds that there are the following relations between the actions of Ω and $\mathcal{M}$ on the sections:

$$\mathcal{M}\mathcal{V} = i\Omega\mathcal{V} \qquad \text{and} \qquad \mathcal{M}\mathcal{U}_a = -i\Omega\mathcal{U}_a\,. \tag{3.14}$$

Other useful relations can be found in [30]. Using (3.13), one can also show that the potential V_g can be rewritten as

$$V_g = |\mathcal{D}\mathcal{L}|^2 - 3|\mathcal{L}^2| = -\frac{1}{2}\mathcal{G}^T\mathcal{M}\mathcal{G} - 4|\mathcal{L}^2|\,. \tag{3.15}$$

Notice that the rewriting (3.11) is a sum of squares only if we impose a constraint on the charges and gaugings

$$\langle \mathcal{G}, \mathcal{Q}\rangle = -1\,, \tag{3.16}$$

which is consistent with the request of the Dirac quantization condition (1.90). The equations of motion are all first order and follow directly from the action (3.11)

$$\mathcal{E} = 0\,, \tag{3.17a}$$

$$\dot{V} = 2e^{-U} Im(e^{-i\alpha}\mathcal{L})\,, \tag{3.17b}$$

$$\dot{\alpha} + \mathcal{A}_r = -2e^{-U} Re(e^{-i\alpha}\mathcal{L})\,, \tag{3.17c}$$

where the equations for the scalar fields and the U warp factor are contained in the first equation and can be extracted by taking appropriate projections of $\mathcal{E}$ along the sections $\mathcal{V}$ and $\mathcal{U}_a$. The projections along the real and imaginary parts of the section $\mathcal{V}$ provide

$$\dot{U} = e^{-U} Im(e^{-i\alpha}\mathcal{L}) - e^{U-2V} Re(e^{-i\alpha}\mathcal{Z})\,, \tag{3.18a}$$

$$\dot{\alpha} + \mathcal{A}_r = -e^{-U} Re(e^{-i\alpha}\mathcal{L}) - e^{U-2V} Im(e^{-i\alpha}\mathcal{Z})\,, \tag{3.18b}$$

while the projection along $\mathcal{U}_a$ provides

$$\dot{z}^a = -e^{i\alpha} G^{a\bar{b}} \left(e^{U-2V} \overline{\mathcal{D}_b \mathcal{Z}} + i e^{-U} \overline{\mathcal{D}_b \mathcal{L}} \right) . \tag{3.18c}$$

Projections along the charges and the gaugings give already known identities, once the other first order equations are used. Let us now focus on the fact that, in addition to the two expected equations for the scalars and the remaining warp factor, we have found one further equation for the phase α. Compatibility of the two different equations for the phase introduces a constraint

$$e^{2(U-V)} Im(e^{-i\alpha} \mathcal{Z}) = Re(e^{-i\alpha} \mathcal{L}) . \tag{3.19}$$

The presence of such a constraint is important as it confirms that α is not an independent degree of freedom of the theory. As a matter of fact, using the flow equations (3.17b, 3.18a, 3.18c) and the constraint (3.19), one can show that α is the phase of a complex combination of the warp factors, the central charge and the gauging superpotential

$$e^{2i\alpha} = \frac{\mathcal{Z} + i e^{2(V-U)} \mathcal{L}}{|\mathcal{Z} + i e^{2(V-U)} \mathcal{L}|} . \tag{3.20}$$

This expression is remarkable, since:

- using this identification and the flow equations for the scalars and warp factors, the phase equation (3.18b) is trivially satisfied.
- the phase α gets identified with the phase of a complex quantity that in ungauged case reduces to the central charge.

This last point suggests that the solutions are supersymmetric, which we can check from the Killing spinor equations. The flow equations follow from the Killing spinor equations obtained by setting to zero the variations of the fermions in the gauged case, i.e. the variations in (1.85, 1.86), along a Killing spinor $\tilde{\varepsilon}(r)$ that satisfies

$$\gamma^0 \tilde{\varepsilon}_A = i e^{i\alpha} \epsilon_{AB} \tilde{\varepsilon}^B \qquad \text{and} \qquad \gamma^1 \tilde{\varepsilon}_A = e^{i\alpha} \delta_{AB} \tilde{\varepsilon}^B , \tag{3.21}$$

where $\alpha(r)$ will correspond to the previously introduced phase. Once we impose the metric ansatz, the vector fields ansatz and the requirement that the scalar fields only depend on the radial coordinate, we find that the Killing spinor must be in the form $\varepsilon_A = e^{f(r)} \chi_A$, for a constant spinor χ_A. Then:

- we find the flow equations for the U warp factor from the time component of the variation of the gravitinos ;
- we find the phase equation (3.18b) from the radial component of the variation of the gravitinos;
- the angular components of the variation of the gravitinos give the flow equation for the V warp factor, the constraints (3.19) and the Dirac quantization constraint $\langle \mathcal{G}, \mathcal{Q} \rangle = -1$;

- we find the flow equations for the scalars from the variations of the gauginos.

Since we impose two independent projections, the resulting solutions are $1/4$-BPS, meaning that they conserve 2 of the original 8 supersymmetries. A detailed derivation of the flow equations from the KSE can be found in [30, appendix A].

3.3 Superpotential and gradient flow equations

It is remarkable that these solutions are supersymmetric and that the phase α, appearing in the projectors on the Killing spinor, corresponds to the phase of a complex quantity that reduces to the central charge in the ungauged case. This suggests that the flow equations for the degrees of freedom $\{U,V,z,\bar{z}\}$ can be expressed in a simple and suggestive way, inspired by the discussion in section 2.3. As a matter of fact, we can find a superpotential $\mathcal{W}$ that allows us to write the flow equations as

$$\dot{\varphi}^A = -\tilde{G}^{\alpha\beta}\partial_\beta \mathcal{W} \,, \tag{3.22}$$

for an appropriate choice of $\tilde{G}_{\alpha\beta}$, where $\varphi = \{U,V,z,\bar{z}\}$. The superpotential can be expressed as $\mathcal{W} = e^U W$, for a "fake superpotential" W that replaces the modulus of the central charge in the flat BPS case. Expression (3.20) suggests that we identify

$$\mathcal{W} = e^U |\mathcal{Z} - ie^{2(V-U)}\mathcal{L}| \,. \tag{3.23}$$

It is important to stress that the phase α, at this point, needs to be treated as a dependent degree of freedom and the derivatives of the superpotential $\mathcal{W}$ need to take this into account. In order to do so it is convenient to use the following expression for $\mathcal{W}$:

$$\mathcal{W} = \sqrt{e^{2U}|\mathcal{Z}|^2 + e^{4V-2U}|\mathcal{L}|^2 + 2e^{2V} Im(\mathcal{L}\overline{\mathcal{Z}})} \,, \tag{3.24}$$

where the phase α does not appear. With this, the derivatives of $\mathcal{W}$ in the warp factors are

$$\partial_U \mathcal{W} = \mathcal{W} - 2e^{2V-U} Im(e^{-i\alpha}\mathcal{L}) \qquad \text{and} \qquad \partial_V \mathcal{W} = 2e^{2V-U} Im(e^{-i\alpha}\mathcal{L}) \,, \tag{3.25a}$$

while the derivatives in the scalar fields can be obtained from the expression (3.23), by making use of the definition of the phase in (3.20), as

$$\partial_{\bar{a}} \mathcal{W} = \frac{1}{2} e^{i\alpha} \left(e^U \overline{\mathcal{D}_b \mathcal{Z}} + ie^{2V-U} \overline{\mathcal{D}_b \mathcal{L}} \right) \,. \tag{3.25b}$$

Notice that, by making use of the constraint (3.19), the superpotential can also be written as

$$\mathcal{W} = e^U Re(e^{-i\alpha}\mathcal{Z}) + e^{2V-U} Im(e^{-i\alpha}\mathcal{L}) \,, \tag{3.26}$$

which means, in particular, that

$$\partial_U \mathcal{W} = e^U Re(e^{-i\alpha}\mathcal{Z}) - e^{2V-U} Im(e^{-i\alpha}\mathcal{L}) \,. \tag{3.27}$$

We can then provide the following non-trivial coefficients for the metric $\tilde{G}_{\alpha\beta}$

$$\tilde{G}_{UU} = -\tilde{G}_{VV} = e^{2V} \qquad \text{and} \qquad \tilde{G}_{a\bar{b}} = \frac{1}{2}e^{2V}G_{a\bar{b}} \,, \tag{3.28}$$

such that the flow equations can be written in a gradient flow form:

$$\dot{U} = -e^{-2V}\partial_U\mathcal{W} = -e^{U-2V}Re(e^{-i\alpha}\mathcal{Z}) + e^{-U}Im(e^{-i\alpha}\mathcal{L}) \,, \tag{3.29a}$$

$$\dot{V} = -e^{-2V}\partial_V\mathcal{W} = 2e^{-U}Im(e^{-i\alpha}\mathcal{L}) \,, \tag{3.29b}$$

$$\dot{z}^a = -e^{-2V}G^{a\bar{b}}\partial_{\bar{b}}\mathcal{W} = -e^{-i\alpha}G^{a\bar{b}}\left(e^{U-2V}\overline{\mathcal{D}_b\mathcal{Z}} + ie^{-U}\overline{\mathcal{D}_b\mathcal{L}}\right) . \tag{3.29c}$$

The equation for the phase (3.17c) is identically satisfied once we have these three equations and we impose the constraint (3.19). The prepotential $\mathcal{W}$ satisfies the condition (2.34), that in this case is

$$|\nabla\mathcal{W}|^2 = \tilde{G}^{\alpha\beta}\partial_\alpha\mathcal{W}\partial_\beta\mathcal{W} = V_{tot} = e^{2(U-V)}V_{BH} + e^{2(V-U)}V_g - 1 \,, \tag{3.30}$$

and it allows us to write the action in the BPS squared form

$$S_{1D} = -N\int dr\left[|\dot{\varphi} - \nabla\mathcal{W}|^2 - 2\frac{d}{dr}(\mathcal{W})\right] . \tag{3.31}$$

We have found a first order description of supersymmetric solutions, where the flow equations can be written in a gradient flow form and hence are driven by a superpotential $\mathcal{W}$, that mimics the situation in ungauged case. This provides a powerful tool to compute explicit solutions and to study their properties at the horizon. The presence of a negative cosmological constant forbids, however, a direct relation between the mass of the black hole and the value of superpotential at spatial infinity.

3.4 The gauged attractor

We will now work out the properties of the attractor mechanism for supersymmetric black holes in $\mathrm{U}(1)$ F.I. gauged supergravity. First of all, we need to expand on the fact that the attractor mechanism in the gauged case is fundamentally different from the ungauged one. It still fixes the values of the scalar fields at the horizon in terms of algebraic equations on the charges and on the symplectic sections, however the attractor point cannot be reached from any initial condition. This happens because the presence of a cosmological constant stabilises the initial values of the scalars, i.e. the scalars at the boundary need to minimise the potential V_g such that its expectation value reproduces the cosmological constant. This means that we have an asymptotic constraint in terms of the gaugings $\mathcal{G}$ that fixes the possible initial values of the scalars. With this in mind, before proceeding with the analysis of the near horizon behaviour, we introduce a different parameterization of the warp factors where, in place of V, we use $A = V - U$. The metric takes the form

$$ds^2 = -e^{2U}dt^2 + e^{-2U}dr^2 + e^{2A}d\Omega^2 \,, \tag{3.32}$$

while the superpotential takes the form $\mathcal{W} = e^U|\mathcal{Z} - ie^{2A}\mathcal{L}|$. It is important to notice that, with such a reparameterization of the warp factors, we need to replace the derivatives (3.25) with

$$\partial_U \mathcal{W} = \mathcal{W}\ , \qquad \partial_A \mathcal{W} = 2e^{2A-U} Im(e^{-i\alpha}\mathcal{L}) \tag{3.33}$$

and the derivatives in the scalar with

$$\partial_{\bar{a}} \mathcal{W} = \frac{1}{2} e^{i\alpha} \left(e^U \overline{\mathcal{D}_b \mathcal{Z}} + ie^{2A+U} \overline{\mathcal{D}_b \mathcal{L}} \right) . \tag{3.34}$$

The flow equations are now given by

$$\dot{U} = -e^{-2(U+A)} \left(\partial_U \mathcal{W} - \partial_A \mathcal{W} \right) = -e^{-2(U+A)} \left(\mathcal{W} - \partial_A \mathcal{W} \right) , \tag{3.35a}$$

$$\dot{A} = e^{-2(U+A)} \partial_U \mathcal{W} = e^{-2(U+A)} \mathcal{W} , \tag{3.35b}$$

$$\dot{z}^a = -2e^{-2(U+A)} G^{a\bar{b}} \partial_{\bar{b}} \mathcal{W} . \tag{3.35c}$$

Let us admit to be working with a spherical horizon, which means that the metric approaches an $AdS_2 \times S^2$ form in the near horizon limit

$$ds^2_{NH} = -\frac{r^2}{R_A^2} dr^2 + \frac{R_A^2}{r^2} dr^2 + R_S^2 d\Omega^2 , \tag{3.36}$$

where R_S is the radius of the two dimensional S^2 space, and R_A is the radius of the two dimensional AdS_2 spacetime. As we have already seen in the asymptotically flat case, we also need to impose a regularity condition on the scalars, i.e. their derivatives must vanish at the horizon. Furthermore, the warp factors should have the following near-horizon behaviour:

$$e^U \to \frac{r}{R_A} \qquad \text{and} \qquad e^A \to R_S . \tag{3.37}$$

These requests lead us to the following attractor conditions:

$$\dot{z}^a \to 0 \qquad \Rightarrow \qquad \partial_a |\mathcal{Z} - ie^{2A}\mathcal{L}| = 0 , \tag{3.38a}$$

$$\dot{A} \to 0 \qquad \Rightarrow \qquad \mathcal{W} = 0 . \tag{3.38b}$$

Notice the equation (3.38a) leads to

$$\mathcal{D}_a \mathcal{Z} - ie^{2A} \mathcal{D}_a \mathcal{L} = 0 , \tag{3.39}$$

which can be used to show that at the attractor point the sections, the charges and the gaugings must satisfy the attractor equation

$$\mathcal{Q} + e^{2A} \Omega \mathcal{M} \mathcal{G} = -2Im(\overline{\mathcal{Z}} \mathcal{V}) + 2e^{2A} Re(\overline{\mathcal{L}} \mathcal{V}) , \tag{3.40}$$

which can be derived from special Kähler geometry identities as well as from the equation $\mathscr{E} = 0$, once we insert the other flow equations and the horizon condition (3.39). This condition specifies the values of the scalars at the horizon in terms of the charges and the gaugings once we have the corresponding value of A. In order to do so we make use of the second condition, namely that $\mathcal{W}=0$ at the horizon, and imposing $e^A \in \mathbb{R}$, then

$$e^{2A} = R_S^2 = -i\frac{\mathcal{Z}}{\mathcal{L}} \in \mathbb{R}\,. \tag{3.41}$$

Notice that the above ratio of the central charge and gauging superpotential is real only if the phases $\phi_{\mathcal{Z}}$ for the central charge and $\phi_{\mathcal{L}}$ for the gauging superpotential differ by $\pi/2$. From the definition of α we find that, using the relation between $\phi_{\mathcal{Z}}$ and $\phi_{\mathcal{L}}$, we must have

$$e^{2i\alpha} = e^{2i\phi_{\mathcal{Z}}} \qquad \Rightarrow \qquad \alpha = \phi_{\mathcal{Z}} + k\pi\,, \quad k \in \mathbb{Z}\,. \tag{3.42}$$

Finally, inserting the near-horizon limits of the warp factors in the flow equations, we find that

$$e^{-i\alpha}\mathcal{Z} = -\frac{R_S^2}{2R_A} < 0\,, \tag{3.43}$$

that is only possible if at the horizon $\alpha = \phi_{\mathcal{Z}} + \pi$. The attractor equations in (3.40) and (3.41) are not all independent, since they provide $2n_V + 4$ conditions and we need to fix the values of $2n_V$ scalars and the A warp factor. As a matter of fact, one can find identities by contracting the (3.40) equations with $\mathcal{V}$ and $\mathcal{U}_a$. On the other side, contracting the same equation with the charges and gauging provides the following equations

$$e^{-2A} = 2\big(|\mathcal{D}\mathcal{L}|^2 - |\mathcal{L}|^2\big) \qquad\qquad e^{2A} = 2\big(|\mathcal{D}\mathcal{Z}|^2 - |\mathcal{Z}|^2\big)\,, \tag{3.44}$$

that are, interestingly, related to the second quadratic invariant [46]

$$I_2(\mathcal{Z}) = -\mathcal{Q}^T\mathcal{M}(F)\mathcal{Q} = |\mathcal{Z}|^2 - |\mathcal{D}\mathcal{Z}|^2\,, \tag{3.45}$$

where $\mathcal{M}(F)$ is similar to the matrix in (2.15), where we use the real and imaginary parts of the matrix $F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$ in place of the ones of the period matrix $\mathcal{N}_{\Lambda\Sigma}$. From the first equation in (3.44), starting from an AdS$_4$ vacuum with $|\mathcal{D}\mathcal{L}| = 0$ and keeping the scalars constant, one finds the contradictory result $e^{-2A} = -2|\mathcal{L}|^2 < 0$. This means that supersymmetric solutions with an AdS asymptotic are not possible for constant scalars and explains the results of the early analysis of AdS black holes [84, 96, 97].

3.5 The open search for multi-center AdS black holes

Although the study of asymptotically flat composite systems of black holes has been really successful and has played a crucial role in the understanding of the quantum structure of the gravitational interaction, the same cannot be said with regards to multi-center configurations in asymptotically anti-de Sitter space. As a matter of facts, a definite proof of the existence of AdS stationary multi-center configurations remains elusive. Heuristically, the main difficulty in building stable configurations lies in the fact that the negative cosmological constant provides an effective attractive force between the centers that plays a role in the equilibrium between the gravitational attraction and electromagnetic repulsion. As a consequence of this fact, stable configurations in an asymptotically AdS spacetime should be more complex than the one in the flat case and, up until recently, it was believed that stationary solutions were not possible.

Despite these problems, the search for AdS multi-center has not stopped and recent results are very promising. Indications that such bound states indeed exist came from the construction of hovering black holes on top of black branes [98, 99], AdS black Saturns [100] and probe multi-center black holes [35, 36]. These last results are of particular interest since they open up the possibility of the existence of the full solution, once the backreaction is taken into account. The analysis performed in [35] uses a simplified version of 4-dimensional F.I. gauged $\mathcal{N}=2$ supergravity, where only two gauge vectors and non-minimally coupled scalar are present, such that the gauging potential is analogous to the one used in this chapter. Non-extremal charged black holes in AdS_4 are used as a background solutions. Numerical analysis shows that small charged probes are subject to an effective potential - that accounts for the contributions from gravity and the two $\mathrm{U}(1)$ interactions - with stable and metastable minima. These minima represent stationary bound states and are only present when the background black hole and the probe have mutually non-local charges (i.e. in a symplectic frame where the background is only electrically charged the probe must be only magnetically charged). Furthermore, the holographic duals of these bound states represent structural glasses. We stress that the analysis in [35] does not take into account the embedding of the supergravity theory in string theory, which would introduce many additional features. From this point of view, recent developments were made in [36], where stable bound states in the probe approximation have been found when working with a supergravity theory derived by the reduction of M-theory on a Sasaki–Einstein manifold.

We mention that, on the other side, much more progress has been made in the search for time-dependent multi-center solutions. Asymptotically de Sitter multi-center solutions of a cosmological Einstein–Maxwell theory have been long known [101, 102]. These solutions can be converted to supersymmetric multi-center configurations in $\mathcal{N}=2$, $d=5$ supergravity, provided that we use an euclideanized metric or imaginary couplings [102]. The euclidean time-dependent solutions have been generalised to $\mathcal{N}=2$ supergravity with arbitrary vector couplings [103] and to a general n-dimensional FLRW background [104]. These last kind of configurations are, however, affected by big bang/big crunch singularities which become real once we consider dynamical black holes and this ruins the possibility of an AdS/CFT interpretation.

Chapter 4

Towards multi-center AdS black holes

The promising results regarding the existence of stable bound states in the probe approximation for anti-de Sitter black hole [35, 36] renewed the motivation for the search of full AdS multi-center black holes. The main point is that if some of these bound states survive when taking into account the backreaction then we would have multi-center configurations. The hope is that, at least for solutions that preserve some amount of supersymmetry, one could be able to build them in a way analogous to the one followed by Denef in the asymptotically flat case [24]. Multi-center configurations require, in general, rotation of the vector fields involved and hence the metric describing such configurations needs to be at least stationary. Construction of multi-center solutions requires, then, knowledge of single-center supersymmetric rotating solutions, in particular regarding the first order flow equations and the BPS square rewriting of the action. The construction of explicit BPS black holes in gauged supergravity with non-zero angular momentum is, however, non trivial. As a matter of fact, the known solutions found in the literature are affected by some problems: 1/4-BPS electrically charged solutions of [105] do not have a consistent static limit and the 1/4-BPS magnetically charged solutions of [84, 91] must have a non compact horizon. Non-extremal solutions are also known [85, 87–89] and display similar problems. Recent and promising developments in the description of non-static BPS black holes in anti-de Sitter come from the proposed 1/4-BPS dyonic rotating solutions in [34]. These solutions have the correct behaviour in the static limit and admit compact horizons. The proposed first order equations, however, do not make use of the symplectic invariance of the vector sector in order to remove the contributions from the potentials in favour of the charges. Furthermore, the proposed solutions are derived using important assumptions on the form of the warp factors and the sections.

In this chapter we are going to investigate stationary supersymmetric black hole solutions of $\mathrm{U}(1)$ F.I. gauged supergravity in a simplified setting, where we ask for an additional space-like Killing vector. This additional symmetry is introduced with the idea that the angular momentum of the vector fields should be directed along our Killing vector. What we are concerned with is if these solutions admit a consistent first order reduction of the equations of motion. If these conditions were to be satisfied by our solutions then we would have a solid basis upon which to construct multi-center generalisations.

The chapter is organised as follows. In the first section we provide the ansatze for the metric and vector fields and justify them based on known single-center AdS solutions. We will then briefly report the results of [38], where a useful characterisation of the equations governing time-like BPS solutions of gauged supergravity is provided. The equations governing our solutions will be found as a particular application of the ones by Meessen and Ortín to our ansatz. Among these equations, however, we find second order ones. Inspired by the procedure used in the asymptotically flat case and in the anti-de Sitter static case, we will employ symplectic invariance of the vector sector to reduce the equations to first order. We will then look at the main consequences of these equations in a simplified setting.

4.1 Supersymmetric stationary solutions with one spatial isometry

We will work with the same $\mathcal{N}=2$, U(1) Fayet–Iliopoulos gauged supergravity theory as the one considered in the previous chapter. We ask, however, the solutions to have both a time-like and a space-like Killing vectors. In order to do so we need to specify ansatze for both the metric and vector fields that are in accordance with these symmetries. All the known rotating black hole solutions in AdS$_4$ have a metric that can be brought in the form [85]

$$ds^2 = -f(dt+\omega\, dz)^2 + f^{-1}\left[v\left(\frac{dq^2}{Q}+\frac{dp^2}{P}\right)+QP\,dz^2\right], \tag{4.1}$$

where ω, v, f are functions of the coordinates (q,p), while Q and P are polynomials in q and p respectively. Metrics of this kind have an additional spatial isometry generated by a Killing vector along the z direction. With this in mind, the ansatz we are going to use for the metric is

$$ds^2 = -e^{2A}(dt+\hat{\omega})^2 + e^{2B}(dx_1^2+dx_2^2) + e^{2C}dx_3^2\,. \tag{4.2}$$

The warp factors A, B and C can depend only the coordinates (x_1,x_2). Notice that we introduced a non-static contribution through the 1-form $\hat{\omega} = \omega(x_1,x_2)\,dx_3$. The ansatz for the vector fields that we are going to use is

$$A^\Lambda = \chi^\Lambda(x_1,x_2)dt + \psi^\Lambda(x_1,x_2)dx_3\,, \tag{4.3}$$

where χ^Λ are the electric potentials and ψ_Λ are related to the magnetic charges. A similar ansatz applies to the dual vector fields

$$A_\Lambda = \phi_\Lambda(x_1,x_2)dt + \eta_\Lambda(x_1,x_2)dx_3\,. \tag{4.4}$$

Let us remark that the proposed forms for the metric and the vector fields are not the most general ones allowed by the two symmetries.

4.1.1 Meessen–Ortín equations for BPS timelike solutions

In order to obtain BPS solutions that fall into our ansatz for the metric and vector fields, one should, in principle, solve the Killing spinor equations and the equations of motion of the full theory. We will, however, make use of the results of [38], where Meessen and Ortín characterised all stationary supersymmetric solutions of gauged $\mathcal{N}=2$, $d=4$ supergravity. We are going to summarise the result in the restricted case of Fayet–Iliopoulos U(1) gauging and no hypermultiplets, which is the one we are interested in. We cite [106] as an useful reference for this kind of reduction, where some corrections to the original paper by Meessen and Ortín are also provided.

Let us recall the bosonic truncation of the U(1) F.I. gauged supergravity action, which will be our starting point,

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{2}-G_{a\bar{b}}\partial_\mu z^a\partial^\mu\bar{z}^{\bar{b}}+\frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu}-\frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\mu\nu}-V_g\right), \tag{4.5}$$

where the scalar potential takes the form

$$V_g=-\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\left(\overline{\mathcal{L}}^\Lambda\mathcal{L}^\Sigma+\frac{1}{8}(\mathcal{I}^{-1})^{\Lambda\Sigma}\right) \tag{4.6}$$

thanks to the introduction of the quaternionic momentum maps $\mathcal{P}^x_\Lambda$. These are related to the gauging of an U(1) subgroup of the R-symmetry $U(2)_R$. Meessen and Ortín showed that, by making use of the Killing spinor equations, supersymmetric timelike solutions can be obtained as solutions of a reduced number of first and second order equations. This provides a systematic method to find BPS solutions. The equations only depend on the spatial coordinates and involve a number of time-independent building blocks constructed from the following bilinears of the Killing spinors ε_A:

$$S=\frac{1}{2}\epsilon^{AB}\overline{\varepsilon}_A\varepsilon_B\,,\qquad V_\mu=i\,\overline{\varepsilon}^A\gamma_\mu\varepsilon_A\,,\qquad\text{and}\qquad V^x_\mu=i\,(\sigma^x)_A{}^B\overline{\varepsilon}^A\gamma_\mu\varepsilon_B\,. \tag{4.7}$$

It will be particularly useful to rescale the sections $\mathcal{V}$ with the scalar bilinear S as $\mathcal{V}/S\equiv\mathcal{R}+i\mathcal{I}$, where the real and imaginary parts of $\mathcal{V}/S$ are

$$\mathcal{R}=\begin{pmatrix}\mathcal{R}^\Lambda\\ \mathcal{R}_\Lambda\end{pmatrix}\qquad\text{and}\qquad\mathcal{I}=\begin{pmatrix}\mathcal{I}^\Lambda\\ \mathcal{I}_\Lambda\end{pmatrix}. \tag{4.8}$$

Using the special geometry identity $\langle\mathcal{V},\overline{\mathcal{V}}\rangle=i$, Meessen and Ortín find that the modulus of the scalar bilinear is fixed by the symplectic product:

$$\frac{1}{2|S|^2}=\langle\mathcal{R},\mathcal{I}\rangle\,. \tag{4.9}$$

At this point we introduce a time-independent phase α for the scalar bilinear S as $S=e^{i\alpha}|S|$. This phase has a particular physical interpretation: it appears in the projectors that constrain the number of independent components of the Killing spinor, see [38, section

5]. This is completely analogous to how the phase α, later identified with the phase of the superpotential $\mathcal{W}$, appears in the projector conditions in (3.21) for the static case. It is important to keep in mind that the phase α is not an independent degree of freedom of the theory. With the introduction of the phase we find that the real and imaginary parts of the rescaled sections $\mathcal{V}$ can be written as

$$\mathcal{R} = |S|^{-1} Re(e^{-i\alpha}\mathcal{V}) \,, \qquad \mathcal{I} = |S|^{-1} Im(e^{-i\alpha}\mathcal{V}) \,. \tag{4.10}$$

The last two bilinears in (4.7) can be seen as components of 1-forms:

$$V \equiv V_\mu \, dx^\mu \qquad \text{and} \qquad V^x \equiv V^x_\mu \, dx^\mu \,. \tag{4.11}$$

The triplet of 1-forms V^x is the dreibein basis of an auxiliary 3-dimensional space such that the 4-dimensional metric can be written as

$$ds^2 = -2|S|^2(dt+\hat{\omega})^2 + \frac{1}{2|S|^2}\delta_{xy}V^xV^y \,. \tag{4.12}$$

In order to avoid confusion, we use underlined indices to label the components in the dreibein basis, i.e. $x = \{\underline{1},\underline{2},\underline{3}\}$. The quaternionic momentum maps $\mathcal{P}^x_\Lambda$ form a triplet in this auxiliary space. The 1-form V, on the other side, must be in the form $V = 2\sqrt{2}|S|^2(dt+\hat{\omega})$ once the metric is fixed to (4.12) and it constrains the vector fields[1] to be

$$A^\Lambda = -\mathcal{R}^\Lambda V + \tilde{A}^\Lambda = -2\sqrt{2}|S|^2\mathcal{R}^\Lambda(dt+\hat{\omega}) + \tilde{A}^\Lambda \,, \tag{4.13}$$

where $\tilde{A}^\Lambda$ is a 1-form in the auxiliary 3-dimensional space.

With this setup, the degrees of freedom that still need to be fixed are just V^x, $\hat{\omega}$, $\tilde{A}^\Lambda$ and $\mathcal{I}$, since, once we fix the special Kähler geometry, for example by providing an explicit prepotential, we can always find the sections $\mathcal{R}$ in terms of the sections $\mathcal{I}$ using to the stabilization equations. Meessen and Ortín show that these quantities are governed by a reduced set of first and second order equations such that, once these are satisfied, also the equations of motion and the Killing spinor equations are. In particular, the V^x, $\hat{\omega}$ and $\tilde{A}^\Lambda$ 1-forms are governed by first order equations. These equations can be conveniently written using the components in the dreibein basis as

$$dV^x = \left(\frac{1}{2}\epsilon^{xyz}\tilde{A}^\Lambda\mathcal{P}^y_\Lambda - \frac{1}{\sqrt{2}}\mathcal{I}^\Lambda\mathcal{P}^y_\Lambda V^y\delta^x_z\right)\wedge V^z \,, \tag{4.14a}$$

$$(d\hat{\omega})_{xy} = 2\epsilon_{xyz}\left(\langle\mathcal{I},\partial_z\mathcal{I}\rangle - \frac{1}{2\sqrt{2}|S|^2}\mathcal{R}^\Lambda\mathcal{P}^z_\Lambda\right) \,, \tag{4.14b}$$

$$(d\tilde{A}^\Lambda)_{xy} = -\sqrt{2}\epsilon_{xyz}\left(\partial_z\mathcal{I}^\Lambda + \mathfrak{B}^\Lambda_z\right) \,, \tag{4.14c}$$

where we introduced the combination

$$\mathfrak{B}^\Lambda_x \equiv \sqrt{2}\mathcal{P}^x_\Sigma\left(\mathcal{R}^\Lambda\mathcal{R}^\Sigma + \frac{1}{8|S|^2}(\mathcal{I}^{-1})^{\Lambda\Sigma}\right) \,. \tag{4.15}$$

[1] Notice that, using the normalisation of the field strengths such the vector contribution to the action is the one in (4.5), the vector fields in our notation must be double the ones used in [38].

On the other side, the sections $\mathcal{I}$ are fixed by second order equations. These can be conveniently expressed by introducing covariant derivative $\tilde{\nabla}_x$ in the auxiliary 3-dimensional space, such that for a function $f(\vec{x})$, one has covariant laplacian

$$d(\star_3 df) \equiv (\tilde{\nabla}^2 f) V^{\underline{1}} \wedge V^{\underline{2}} \wedge V^{\underline{3}} \,, \tag{4.16}$$

and, for a 1-form $\nu = \nu_x V^x$, one has covariant gradient

$$d(\star_3 \nu) \equiv (\tilde{\nabla}_x \nu_x) V^{\underline{1}} \wedge V^{\underline{2}} \wedge V^{\underline{3}} \,. \tag{4.17}$$

With these definitions, the second order equations for the components of $\mathcal{I}$ are

$$\tilde{\nabla}^2 \mathcal{I}^\Lambda + \tilde{\nabla}_x \mathfrak{B}^\Lambda_x = 0 \,, \tag{4.18a}$$

$$\tilde{\nabla}^2 \mathcal{I}_\Lambda + \tilde{\nabla}_x \mathfrak{B}_{\Lambda x} = \frac{1}{4\sqrt{2}} \epsilon_{xyz} (d\hat{\omega})_{xy} \mathcal{P}^z_\Lambda \,, \tag{4.18b}$$

where we introduce the combination

$$\mathfrak{B}_{\Lambda x} \equiv \sqrt{2} \mathcal{P}^x_\Sigma \left(\mathcal{R}_\Lambda \mathcal{R}^\Sigma + \frac{1}{8|S|^2} (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^\Sigma \right) . \tag{4.19}$$

The presence of second order equations seems to be problematic as one would expect solutions of a supersymmetric theory to be characterised by only first order equations. Notice, however, that the first one of these second order equations is the integrability condition of equation (4.14c) and hence it is automatically satisfied once (4.14c) is. On the other side, the second equation has no immediate reduction to first order.

4.1.2 The Meessen–Ortín equations in our ansatz

We will now specialise the equations of Meessen and Ortín for BPS solutions to our ansatze for the metric and vector fields, i.e. the ones in (4.2-4.3). In order to do so, we gauge the U(1) subgroup of the R-symmetry along the σ^2 generator[2], which means that we can set the momentum maps to be

$$\mathcal{P}^x_\Lambda = \gamma\, g_\Lambda \delta^{x\underline{2}} \,, \tag{4.20}$$

where γ is a constant factor that will be fixed later in the discussion. Since we are working with only electric gaugings, this means that we are in a special symplectic frame where $\mathcal{G} = (0, g_\Lambda)$. Notice that, by comparing the expressions for the potential in equation (3.15) and the one in (4.6), once we impose that the momentum maps are the ones in (4.20) then the constant factor γ is fixed to either ± 2.

[2]The gauging along the σ^1 direction is completely analogous to the one along σ^2, it just consists in the exchange of the x^1 and x^2 coordinates. On the other hand, the gauging along the σ^3 direction leads to trivial equations because of the presence of the Killing vector along this direction.

The direct comparison of our ansatz for the metric and the one used by Meessen and Ortín, which are

$$\begin{aligned} ds^2 &= -\, e^{2A}(dt+\omega dx_3)^2 + e^{2B}(dx_1^2+dx_2^2) + e^{2C}dx_3^2 \\ &= -\, 2|S|^2(dt+\hat{\omega})^2 + \frac{1}{2|S|^2}\delta_{xy}V^xV^y \ , \end{aligned} \tag{4.21}$$

shows that the matching of the two implies the following identifications for the dreibein and the $\hat{\omega}$ 1-form:

$$V^{\underline{1}} = e^{A+B}dx_1 \ , \quad V^{\underline{2}} = e^{A+B}dx_2 \ , \quad V^{\underline{3}} = e^{A+C}dx_3 \quad \text{and} \quad \hat{\omega} = \omega dx_3 \ . \tag{4.22}$$

Furthermore, the modulus of the scalar bilinear is related to the warp factor $A(x_1,x_2)$ by

$$|S|^2 = \frac{e^{2A}}{2} \qquad \Rightarrow \qquad S = \frac{e^{i\alpha}}{\sqrt{2}}e^A \ . \tag{4.23}$$

This allows us to rewrite the sections $\mathcal{R}$ and $\mathcal{I}$ in terms of the warp factor A, the sections $\mathcal{V}$ and the phase as

$$\mathcal{R} = \sqrt{2}e^{-A}Re(e^{-i\alpha}\mathcal{V}) \qquad \text{and} \qquad \mathcal{I} = \sqrt{2}e^{-A}Im(e^{-i\alpha}\mathcal{V}) \ . \tag{4.24}$$

On the other side, the matching of the vector fields in our ansatz and the expression from Meessen and Ortín, which are

$$A^\Lambda = \chi^\Lambda dt + \psi^\Lambda dx_3 = -2\sqrt{2}|S|^2\mathcal{R}^\Lambda(dt+\hat{\omega}) + \tilde{A}^\Lambda \ , \tag{4.25}$$

gives us the following identifications for the electric potential and "magnetic charges":

$$\chi^\Lambda = -2\sqrt{2}|S|^2\mathcal{R}^\Lambda = -2e^A Re(e^{-i\alpha}\mathcal{L}^\Lambda) \qquad \text{and} \qquad \tilde{\psi}^\Lambda dx_3 = \tilde{A}^\Lambda \ , \tag{4.26}$$

where $\tilde{\psi}^\Lambda \equiv \psi^\Lambda - \omega\chi^\Lambda$.

Now that all of the identifications are in place, we will look at the first order equations for V^x, $\hat{\omega}$ and $\tilde{A}^\Lambda$ and their consequences with regards to the dependencies on (x_1,x_2) of the various objects. First of all, we will look at the dreibein equations, which give us equations for the warp factors. These take a simple form once we introduce the reparameterization of the warp factors:

$$A = A \ , \qquad E = A+B \qquad \text{and} \qquad F = C-B \ , \tag{4.27}$$

such that the metric takes the form

$$ds^2 = -e^{2A}(dt+\omega\, dx_3) + e^{-2A}\left[e^{2E}(dx_1^2+dx_2^2) + e^{2(E+F)}dx_3^2\right] . \tag{4.28}$$

The dreibein basis is now identified as

$$V^{\underline{1}} = e^E dx_1 \ , \qquad V^{\underline{2}} = e^E dx_2 \qquad \text{and} \qquad V^{\underline{3}} = e^{E+F}dx_3 \tag{4.29}$$

and the equations (4.14a) tell us that

$$\partial_1 E = 0\,, \qquad\qquad \partial_2 E = -\frac{\gamma}{\sqrt{2}} e^{E} g_\Lambda \mathcal{I}^\Lambda\,, \tag{4.30}$$

$$\partial_1 F = \frac{\gamma}{2} e^{-F} g_\Lambda \tilde{\psi}^\Lambda\,, \qquad\qquad \partial_2 F = 0\,, \tag{4.31}$$

from which we find that the dependencies of the warp factors on the coordinates are $E = E(x_2)$ and $F = F(x_1)$. Let us notice that, since the warp factors must be invariant under symplectic transformations, then the non-trivial equations in (4.89) and (4.31) can be written as

$$\partial_2 e^{-E} = \frac{\gamma}{\sqrt{2}} \langle \mathcal{G}, \mathcal{I}^\Lambda \rangle \qquad \text{and} \qquad \partial_1 e^{F} = \frac{\gamma}{2} \langle \mathcal{G}, \tilde{\Psi} \rangle\,, \tag{4.32}$$

where we introduced the symplectic vector $\tilde{\Psi}$ such that

$$\mathcal{A} = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix} = \Phi dt + \Psi dt \qquad \text{with} \qquad \Psi = \tilde{\Psi} + \omega \Phi\,. \tag{4.33}$$

This expression is justified by the fact that our ansatz for the vector fields must also apply to the dual fields.
Equation (4.14b) for the 1-form $\hat{\omega}$ has non trivial components $\underline{13}$ and $\underline{23}$, these give us the following equations for the derivatives of $\omega(x_1, x_2)$:

$$\partial_1 \omega = -\,2 e^{F+E} \left[\langle \mathcal{I}, \partial_2 \mathcal{I} \rangle - \frac{\gamma}{\sqrt{2}} e^{E-2A} g_\Lambda \mathcal{R}^\Lambda \right]\,, \tag{4.34a}$$

$$\partial_2 \omega = 2 e^{F+E} \langle \mathcal{I}, \partial_1 \mathcal{I} \rangle\,. \tag{4.34b}$$

Analogously the $\underline{13}$ and $\underline{23}$ components of the equation for the 1-form $\tilde{A}^\Lambda$ in (4.14b) fix the derivatives of $\tilde{\psi}^\Lambda$ as

$$\partial_1 \tilde{\psi}^\Lambda = \sqrt{2} e^{E+F} \left(\partial_2 \mathcal{I}^\Lambda + e^{E} \mathfrak{B}^\Lambda_{\underline{2}} \right)\,, \tag{4.35a}$$

$$\partial_2 \tilde{\psi}^\Lambda = -\,\sqrt{2} e^{E+F} \partial_1 \mathcal{I}^\Lambda\,, \tag{4.35b}$$

while the $\underline{12}$ component is trivial. In addition to this set of equations, we can use the special Kähler identities

$$\langle \mathcal{V}, \partial_i \mathcal{V} \rangle = 0 \qquad \text{and} \qquad \langle \mathcal{V}, \partial_i \overline{\mathcal{V}} \rangle = \mathcal{A}_i\,, \tag{4.36}$$

where $\mathcal{A}_i = Im(\partial_i z^a \partial_a \mathcal{K})$ are the components of the composite Kähler connection, to express the derivatives of the phase α as

$$e^{2A} \langle \mathcal{I}, \partial_i \mathcal{I} \rangle = \partial_i \alpha + \mathcal{A}_i\,. \tag{4.37}$$

Lastly, we can look at the second order equations. The second order equations (4.18a-4.18b) can written in a covariant form as

$$\tilde{\nabla}^2 \mathcal{I} + \tilde{\nabla}_x \mathfrak{B}_x = \frac{1}{4\sqrt{2}} \epsilon_{xyz} (d\tilde{\omega})_{xy} \mathcal{P}^z\,, \tag{4.38}$$

where we introduced a triplet of symplectic vectors $\mathcal{P}^x \equiv (0\,,\mathcal{P}^x_\Lambda)$ and gathered the $\mathfrak{B}_{\underline{2}\Lambda}$ and $\mathfrak{B}^\Lambda_{\underline{2}}$ combinations in a symplectic vector

$$\mathfrak{B}_{\underline{2}} \equiv \begin{pmatrix} \mathfrak{B}^\Lambda_{\underline{2}} \\ \mathfrak{B}_{\underline{2}\Lambda} \end{pmatrix} = \sqrt{2}\gamma\left(\mathcal{R}\,\langle\mathcal{G},\mathcal{R}\rangle - \frac{e^{-2A}}{4}\Omega\mathcal{M}\mathcal{G}\right)\,. \tag{4.39}$$

By making use of the explicit expressions for the covariant laplacian and gradient in our ansatz, we find that equation (4.38) is

$$\partial_2\left(e^E\partial_2\mathcal{I} + e^{2E}\mathfrak{B}_{\underline{2}}\right) + e^{E-F}\partial_1\left(e^F\partial_1\mathcal{I}\right) = -\frac{\gamma}{2\sqrt{2}}e^{E-F}\partial_1\omega\,\mathcal{G}\,. \tag{4.40}$$

4.1.3 Separation of variables

By making use of the Meessen–Ortín equations, we have found the set of equations that determines the supersymmetric solutions in our ansatz. These equations have important consequences on the structure and dependencies of the various quantities at play. We will now focus on the equations for the warp factors

$$\partial_1 E = 0\,, \qquad\qquad \partial_1 F = 0\,, \tag{4.41a}$$

$$\partial_2 e^{-E} = \frac{\gamma}{\sqrt{2}}\langle\mathcal{G},\mathcal{I}\rangle\,, \qquad\qquad \partial_1 e^F = \frac{\gamma}{2}\langle\mathcal{G},\tilde{\Psi}\rangle\,. \tag{4.41b}$$

An immediate and important result from the application of the Meessen–Ortín equations to our ansatze is that equations in (4.41) imply a separation of the dependence on the variables (x_1,x_2) for the warp factors, such that $E = E(x_2)$ and $F = F(x_1)$. Consistency of the equations (4.41) with the separation of variables means that

$$\partial_1\partial_2\,e^{-E} = 0 \qquad \Rightarrow \qquad \partial_1\langle\mathcal{G},\mathcal{I}\rangle = 0\,, \tag{4.42a}$$

$$\partial_2\partial_1\,e^{F} = 0 \qquad \Rightarrow \qquad \partial_2\langle\mathcal{G},\tilde{\Psi}\rangle = 0\,, \tag{4.42b}$$

from which we have a factorisation in the structure of $\mathcal{I}$ and $\tilde{\Psi}$, such that

$$\mathcal{I}(x_1,x_2) = \mathcal{I}_0(x_2) + \mathcal{G}\,\Delta\mathcal{I}(x_1,x_2) \quad \text{and} \quad \tilde{\Psi}(x_1,x_2) = \tilde{\Psi}_0(x_1) + \mathcal{G}\,\Delta\tilde{\Psi}(x_1,x_2)\,. \tag{4.43}$$

4.2 Static limit

Before proceeding with the analysis of the equations that describe BPS solutions in our ansatz, let us work out their static limit. This is useful in order to fix the value of the constant γ and in order to gain some insights as to which is the physical role of the various quantities at play. The static supersymmetric solutions considered in [30] should be reproduced by our solutions when the non-static contribution vanish. In order to do so we need to impose that ω is a constant, such that it can be reabsorbed in the time coordinate by a redefinition $t+\omega dx_3 \rightarrow t$. With this, our metric takes the form

$$ds^2 = -e^{2A}dt^2 + e^{-2A}\left[e^{2E}dx_2^2 + e^{2E}\left(dx_1^2 + e^{2F}dx_3^2\right)\right] . \tag{4.44}$$

The coordinates $\{r,\theta,\phi\}$ in the metric (2.5) of [30] can be reproduced by asking that:

$$dr = e^E dx_2 \, , \qquad d\theta = dx_1 \, , \qquad d\phi = dx_3 \, , \tag{4.45}$$

while the warp factors[3] are identified as

$$A = U(r) \, , \qquad E = V(r) \qquad \text{and} \qquad e^F = -sin(\theta) \, . \tag{4.46}$$

Since ω must be a constant, its derivatives vanish and the equations (4.34) tell us that:

$$0 = -2e^{F+E}\left[\langle \mathcal{I}, \partial_2 \mathcal{I}\rangle - \frac{\gamma}{\sqrt{2}}e^{E-2A}g_\Lambda \mathcal{R}^\Lambda\right] \qquad \text{and} \qquad 0 = 2e^{F+E}\langle \mathcal{I}, \partial_1 \mathcal{I}\rangle \, . \tag{4.47}$$

In order to satisfy the second one it is sufficient to ask that $\partial_1 \mathcal{I} = 0$, which has important consequences on the dependencies of the sections, the phase and the A warp factor. As a matter of facts, by asking $\partial_1 \mathcal{I} = 0$ we find

$$\partial_1 \alpha + \mathcal{A}_1 = 0 \qquad \text{and} \qquad \partial_1\left(e^{-A}Im(e^{-i\alpha}\mathcal{V})\right) = 0 \, , \tag{4.48}$$

that are satisfied if we ask that the scalar fields, the phase and the A warp factor only depend on x_2. This is consistent with the fact that these quantities only depend on the radial coordinate in the static case. Let us notice that, since the scalar fields do not depend on x_1, the period matrix $\mathcal{N}_{\Lambda\Sigma}$ will not depend on it either. Lastly, the sections $\mathcal{R}$, obtained from $\mathcal{I}$ through the stabilisation equations, will also only depend on x_2.
Let us notice that in the static case there is no distinction between $\tilde{\psi}^\Lambda$ and ψ^Λ. The equations for $\tilde{\psi}^\Lambda$ (4.35) are now

$$e^{-F}\partial_1 \tilde{\psi}^\Lambda = \sqrt{2}e^E\left(\partial_2 \mathcal{I}^\Lambda + e^E \mathfrak{B}_{\underline{2}}{}^\Lambda\right) \qquad \text{and} \qquad \partial_2 \tilde{\psi}^\Lambda = 0 \, , \tag{4.49}$$

the second one tells us that $\tilde{\psi}^\Lambda$ only depends on x_1, the first one is satisfied by imposing

$$e^{-F}\partial_1 \tilde{\psi}^\Lambda = \sqrt{2}c^\Lambda \, , \tag{4.50}$$

for a set of constants c^Λ, since the left hand side only depends on x_1 while the right hand side only depends on x_2. By employing the symplectic invariance of the vector sector we can show that similar equations are also present for the η_Λ component of the dual fields A_Λ:

$$e^{-F}\partial_1 \tilde{\eta}_\Lambda = \sqrt{2}e^E\left(\partial_2 \mathcal{I}_\Lambda + e^E \mathfrak{B}_{\underline{2}\,\Lambda}\right) \qquad \text{and} \qquad \partial_2 \tilde{\eta}_\Lambda = 0 \, . \tag{4.51}$$

[3]We call the $\psi(r)$ warp factor in the paper by Dall'Agata and Gnecchi as $V(r)$, as we have done in the previous chapter. This is done in order to avoid confusion with the component of the vector fields ψ^Λ.

This means that we can introduce constants c_Λ such that

$$e^{-F}\partial_1\tilde{\eta}_\Lambda = \sqrt{2}c_\Lambda\ . \tag{4.52}$$

The second order equations, written as an equation for symplectic vectors, in the static case take the simpler form

$$0 = \partial_2\left(e^E\partial_2\mathcal{I} + e^{2E}\mathfrak{B}_{\underline{2}}^{\Lambda}\right)\ , \tag{4.53}$$

that are trivially satisfied by introducing a symplectic vector of constants $\mathcal{C}$ such that

$$\mathcal{C} = e^E\partial_2\mathcal{I} + e^{2E}\mathfrak{B}_{\underline{2}}^{\Lambda} \tag{4.54}$$

and, in order for this to be consistent with the first two equations in (4.49) and (4.51), we need to ask that $\mathcal{C} = (c^\Lambda\,;c_\Lambda)$. At this point, by making use of the coordinates $\{r,\theta,\phi\}$, the warp factors $\{U,V\}$ and fixing[4] $e^F = -sin(\theta)$, our set of equations takes the form

$$V' = -\gamma e^{-U}Im(e^{-i\alpha}\mathcal{L})\ , \tag{4.55a}$$

$$\alpha' + \mathcal{A}_r = \gamma e^{-U}Re(e^{-i\alpha}\mathcal{L})\ , \tag{4.55b}$$

$$\frac{d}{d\theta}\tilde{\Psi} = -\sqrt{2}\mathcal{C}sin(\theta) \tag{4.55c}$$

$$\frac{d}{d\theta}e^F = \frac{\gamma}{2}\langle\mathcal{G},\tilde{\Psi}\rangle \tag{4.55d}$$

and

$$\sqrt{2}e^{2V}\left(e^{-U}Im(e^{-i\alpha}\mathcal{V})\right)' + \sqrt{2}\gamma e^{2V-2U}\left(\sqrt{2}Re(e^{-i\alpha}\mathcal{L})Re(e^{-i\alpha}\mathcal{V}) - \frac{1}{4}\Omega\mathcal{M}\mathcal{G}\right) - \mathcal{C} = 0\ . \tag{4.55e}$$

We can integrate equation (4.55c) for $\tilde{\Psi}$ and find

$$\tilde{\Psi} = \sqrt{2}\mathcal{C}\,cos(\theta)\ , \tag{4.55f}$$

which, once compared with the ansatz for the vector fields used in the static case, tells us that the constants $\mathcal{C}$ are related to the charges by $\mathcal{Q} = -\sqrt{2}\mathcal{C}$. Since we fixed $e^F = -sin\theta$, the fourth equation is now an algebraic constraint that tells us that

$$\frac{d}{d\theta}e^F = cos(\theta) = \frac{\gamma}{2}\langle\mathcal{G},\mathcal{Q}\rangle cos(\theta) \qquad \Rightarrow \qquad \langle\mathcal{G},\mathcal{Q}\rangle = \frac{2}{\gamma} = \pm 1\ , \tag{4.55g}$$

depending on the value of γ. Lastly, using the equation for the phase (4.55g) and the charges $\mathcal{Q}$ in place of $\mathcal{C}$, equation (4.55e) gives us the vanishing combination:

$$\mathscr{E} = 2e^{2V}\left(e^{-A}Im(e^{-i\alpha}\mathcal{V})\right)' + 4e^{2V-U}(\alpha' + \mathcal{A}_r)Re(e^{-i\alpha}\mathcal{V}) - \frac{\gamma}{2}e^{2(V-U)}\Omega\mathcal{M}\mathcal{G} + \mathcal{Q}\ . \tag{4.55h}$$

It is clear now that the all of the equations and constraints describing BPS static black holes are reproduced once we ask that $\gamma = -2$.

[4] We introduced a minus sign in $e^F = -sin(\theta)$ in order to obtain the correct relation between c^Λ and the magnetic charges.

4.3 Reduction to first order

Since the value of γ does not depend on the presence of the non-static contribution, from now on we will set $\gamma = -2$ in all of our equations. The complete set of equations that describes supersymmetric solutions in our ansatz is then provided by

$$\partial_1 E = 0\,, \qquad \partial_1 F = 0\,, \tag{4.56a}$$
$$\partial_2 e^{-E} = -\sqrt{2}\langle \mathcal{G}, \mathcal{I}\rangle\,, \qquad \partial_1 e^{F} = -\langle \mathcal{G}, \tilde{\Psi}\rangle\,, \tag{4.56b}$$
$$\partial_1 \omega = -2e^{F+E}\left[\langle \mathcal{I}, \partial_2 \mathcal{I}\rangle + \sqrt{2}e^{E-2A} g_\Lambda \mathcal{R}^\Lambda\right]\,, \qquad \partial_2 \omega = 2e^{F+E}\langle \mathcal{I}, \partial_1 \mathcal{I}\rangle\,, \tag{4.56c}$$
$$\partial_1 \tilde{\psi}^\Lambda = \sqrt{2}e^{E+F}\left(\partial_2 \mathcal{I}^\Lambda + e^{E}\mathfrak{B}_{\underline{2}}^{\Lambda}\right)\,, \qquad \partial_2 \tilde{\psi}^\Lambda = -\sqrt{2}e^{E+F}\partial_1 \mathcal{I}^\Lambda\,, \tag{4.56d}$$

coupled with the phase equations

$$\partial_1 \alpha + \mathcal{A}_1 = e^{2A}\langle \mathcal{I}, \partial_1 \mathcal{I}\rangle\,, \qquad \partial_2 \alpha + \mathcal{A}_2 = e^{2A}\langle \mathcal{I}, \partial_2 \mathcal{I}\rangle\,, \tag{4.56e}$$

and the algebraic relations

$$\chi^\Lambda = -2e^{A} Re(e^{-i\alpha}\mathcal{V})\,, \qquad e^{2A} = \langle \mathcal{R}, \mathcal{I}\rangle\,. \tag{4.56f}$$

Lastly, the second order equations have been gathered into

$$\partial_2\left(e^{E}\partial_2 \mathcal{I} + e^{2E}\mathfrak{B}_{\underline{2}}\right) + e^{E-F}\partial_1\left(e^{F}\partial_1 \mathcal{I}\right) = \frac{1}{\sqrt{2}}e^{E-F}\partial_1 \omega\, \mathcal{G}\,, \tag{4.56g}$$

where

$$\mathfrak{B}_{\underline{2}} = -2\sqrt{2}\left(\mathcal{R}\langle \mathcal{G}, \mathcal{R}\rangle - \frac{e^{-2A}}{4}\Omega \mathcal{M}\mathcal{G}\right)\,. \tag{4.56h}$$

We are now ready to look at how this set of equations can be reduced to first order. For static AdS_4 black holes, reduction to first order of the supersymmetric solutions was possible thanks to the fact that the potentials can be integrated out in favour of the electric and magnetic charges [30]. This could be done since the potentials only appeared in the action through their first derivatives. The idea that we are going to work on in the following is that a similar substitution of the charges in place of the potentials should also be possible, although with some complications, in the case at hand. The equations derived in the previous section are all expressed in terms of the components χ^Λ and ψ^Λ of the vector fields A^Λ, which are related to the electric potentials and the magnetic charges. We have, however, introduced the dual vector fields A_Λ that contain information regarding the electric charges and magnetic potentials. Using the duality relations between the field strengths F^Λ and the dual field strengths G_Λ we should, in principle, be able to rewrite our equations in terms of only the electric and magnetic charges.

4.3.1 Symplectic invariance in the vector sector

As we have shown in section 1.3, the vector sector of a $\mathcal{N}=2$ supergravity theory is invariant under symplectic transformations. This means that we can introduce dual vector fields A_Λ from which we can build dual field strengths

$$G_\Lambda = dA_\Lambda \qquad \text{such that} \qquad \mathcal{F} = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \Omega \mathcal{M} \star_4 \mathcal{F} \,. \tag{4.57}$$

In our case, the ansatz for the vector fields can be extended to the duals, meaning that we can impose

$$A_\Lambda = \phi_\Lambda dt + \eta_\Lambda dx_3 = \phi_\Lambda (dt + \omega dx_3) + \tilde{\eta}_\Lambda dx_3 \,, \tag{4.58}$$

where ϕ_Λ are the magnetic potentials and η_Λ are related to the electric charges. The invariance condition (4.57) produces the following relations between the differentials of the components of A_Λ and A^Λ:

$$d\phi_\Lambda = \mathcal{R}_{\Lambda\Sigma} d\chi^\Sigma + e^{2A-E-F} \mathcal{I}_{\Lambda\Sigma} \star_2 \left(d\psi^\Sigma - \omega d\chi^\Sigma \right) \,, \tag{4.59}$$

$$d\eta_\Lambda = \mathcal{R}_{\Lambda\Sigma} d\psi^\Sigma + e^{2A-E-F} \mathcal{I}_{\Lambda\Sigma} \star_2 \left[d\psi^\Sigma - (\omega^2 - e^{2(E+F-2A)}) d\chi^\Sigma \right] \,. \tag{4.60}$$

From equation (4.59), by inserting the factorisation $\psi^\Lambda = \tilde{\psi}^\Lambda + \omega \chi^\Lambda$, we find

$$d\phi_\Lambda = \mathcal{R}_{\Lambda\Sigma} d\chi^\Sigma + e^{2A-E-F} \mathcal{I}_{\Lambda\Sigma} \star_2 \left(d\tilde{\psi}^\Sigma + \chi^\Sigma d\omega \right) \tag{4.61}$$

and from this we can extract equations for the derivatives of the potential ϕ_Λ. Let us notice that, since in the static case the x_1 coordinate is related to the angular coordinate θ, while x_2 is related to the radial one r, it seems natural to impose that Hodge duality in the (x_1, x_2) space is given by

$$\star_2 dx_1 = -dx_2 \qquad \text{and} \qquad \star_2 dx_2 = dx_1 \,. \tag{4.62}$$

By making use of these rules for $\star_2$, we find that the equations for the derivatives of ϕ_Λ from (4.61) are:

$$\partial_1 \phi_\Lambda = \mathcal{R}_{\Lambda\Sigma} \partial_1 \chi^\Sigma + e^{2A-E-F} \mathcal{I}_{\Lambda\Sigma} \left(\partial_2 \tilde{\psi}^\Sigma + \chi^\Sigma \partial_2 \omega \right) \,, \tag{4.63a}$$

$$\partial_2 \phi_\Lambda = \mathcal{R}_{\Lambda\Sigma} \partial_2 \chi^\Sigma - e^{2A-E-F} \mathcal{I}_{\Lambda\Sigma} \left(\partial_1 \tilde{\psi}^\Sigma + \chi^\Sigma \partial_1 \omega \right) \,. \tag{4.63b}$$

On the other side, rewriting (4.60) using $\tilde{\eta}_\Lambda$ and making use of (4.61) to replace the contributions from the derivatives of ϕ_Λ, we find

$$d\tilde{\eta}_\Lambda = -\phi_\Lambda d\omega + \mathcal{R}_{\Lambda\Sigma} (d\tilde{\psi}^\Sigma + \chi^\Sigma d\omega) + e^{E+F-2A} \mathcal{I}_{\Lambda\Sigma} \star_2 d\chi^\Sigma \,, \tag{4.64}$$

from which we find that the derivatives of $\tilde{\eta}_\Lambda$ satisfy

$$\partial_1 \tilde{\eta}_\Lambda = -\phi_\Lambda \partial_1 \omega + \mathcal{R}_{\Lambda\Sigma} (\partial_1 \tilde{\psi}^\Sigma + \chi^\Sigma \partial_1 \omega) + e^{E+F-2A} \mathcal{I}_{\Lambda\Sigma} \partial_2 \chi^\Sigma \,, \tag{4.65a}$$

$$\partial_2 \tilde{\eta}_\Lambda = -\phi_\Lambda \partial_2 \omega + \mathcal{R}_{\Lambda\Sigma} (\partial_2 \tilde{\psi}^\Sigma + \chi^\Sigma \partial_2 \omega) - e^{E+F-2A} \mathcal{I}_{\Lambda\Sigma} \partial_1 \chi^\Sigma \,. \tag{4.65b}$$

At this point we can manipulate these derivatives by making use of the first order equations, the expression for χ^Λ in (4.56f), the equations for the derivatives of the phase in (4.56e) and the fact that the upper and lower components of the sections $\mathcal{V}$ and $\mathcal{U}_a$ are related by

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma \qquad \text{and} \qquad h_{a\,\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_a^\Sigma \,. \tag{4.66}$$

The overall results is that the derivatives of ϕ_Λ can be written as

$$\partial_1\phi_\Lambda = \partial_1\left[-2e^A Re(e^{-i\alpha}\mathcal{M}_\Lambda)\right], \qquad \partial_2\phi_\Lambda = \partial_2\left[-2e^A Re(e^{-i\alpha}\mathcal{M}_\Lambda)\right] + e^E g_\Lambda\,, \tag{4.67}$$

which mean that the potential ϕ_Λ is in the form

$$\phi_\Lambda = -2e^A Re(e^{-i\alpha}\mathcal{M}_\Lambda) + f g_\Lambda \tag{4.68}$$

where $f = f(x_2)$ is a new warp factor that satisfies $\partial_2 f = e^E$. On the other side, the derivatives of $\tilde{\eta}_\Lambda$ are:

$$\begin{aligned}
\partial_1\tilde{\eta}_\Lambda =& \sqrt{2}e^{E+F}\left(\partial_2\mathcal{I}_\Lambda + e^E\mathfrak{B}_{\underline{2}\,\Lambda}\right) - (\partial_1\omega)\,f\,g_\Lambda\,, &(4.69)\\
\partial_2\tilde{\eta}_\Lambda =& -\sqrt{2}e^{E+F}\partial_1\mathcal{I}_\Lambda - 2e^{E+F}\langle\mathcal{I},\partial_1\mathcal{I}\rangle\,f\,g_\Lambda\,. &(4.70)
\end{aligned}$$

4.3.2 First order equations

We are now ready to show that the BPS solutions in our ansatz are described by a set of equations that are all of first order. The consequences of duality invariance in the vector sector, analysed in the previous section, can be resumed in two equations for the derivatives of the symplectic vector $\tilde{\Psi} = (\tilde{\psi}^\Lambda\,;\,\tilde{\eta}_\Lambda)$

$$\begin{aligned}
e^{-F}\partial_2\tilde{\Psi} &= -\sqrt{2}e^E\partial_1\mathcal{I} - e^{-F}\partial_2\omega f\mathcal{G}\,, &(4.71)\\
e^{-F}\partial_1\tilde{\Psi} &= \sqrt{2}e^E\left(\partial_2\mathcal{I} + e^E\mathfrak{B}_{\underline{2}}\right) - e^{-F}\partial_1\omega f\mathcal{G}\,, &(4.72)
\end{aligned}$$

which also gather the equations in (4.56d), and the algebraic equation for the potentials

$$\Phi = -2e^A Re(e^{-i\alpha}\mathcal{V}) + f\,\mathcal{G}\,. \tag{4.73}$$

At this point, let us notice now that the mixed second derivatives of $\tilde{\Psi}$ calculated from both equations (4.71) and (4.72) must be the same, this means that

$$\begin{aligned}
\partial_1\partial_2\tilde{\Psi} &= -\sqrt{2}e^E\partial_1(e^F\partial_1\mathcal{I}) - \partial_1\partial_2\omega f\mathcal{G}\\
&= \sqrt{2}e^F\partial_2\left(e^E\partial_2\mathcal{I} + e^{2E}\mathfrak{B}_{\underline{2}}\right) - \partial_2\left(f\partial_1\omega\right)\mathcal{G}\,, &(4.74)
\end{aligned}$$

from which follows that

$$\partial_2\left(e^E\partial_2\mathcal{I} + e^{2E}\mathfrak{B}_{\underline{2}}\right) + e^{E-F}\partial_1\left(e^F\partial_1\mathcal{I}\right) = \frac{1}{\sqrt{2}}e^{E-F}\partial_1\omega\mathcal{G}\,. \tag{4.75}$$

This equation is exactly the second order equation that we found by starting from the Meessen–Ortín equations. This means that our supersymmetric solutions are completely determined by first order equations. Before listing them all, let us notice that the equations for the derivatives of ω and $\tilde{\Psi}$

$$\partial_1\omega = -2e^{F+E}\left[\langle\mathcal{I},\partial_2\mathcal{I}\rangle + \sqrt{2}e^{E-2A}\langle\mathcal{G},\mathcal{R}\rangle\right] , \qquad \partial_2\omega = 2e^{F+E}\langle\mathcal{I},\partial_1\mathcal{I}\rangle , \tag{4.76a}$$

$$e^{-F}\partial_2\tilde{\Psi} = -\sqrt{2}e^{E}\partial_1\mathcal{I} - e^{-F}\partial_2\omega f\mathcal{G} , \tag{4.76b}$$

$$e^{-F}\partial_1\tilde{\Psi} = \sqrt{2}e^{E}\left(\partial_2\mathcal{I} + e^{E}\mathfrak{B}_{\underline{2}}\right) - e^{-F}\partial_1\omega f\mathcal{G} , \tag{4.76c}$$

can be written as equations for differential forms $d\omega$ and $d\tilde{\Psi}$ if we introduce the 1-form

$$\mathfrak{B} = \mathfrak{B}_2 dx_2 = \mathfrak{B}_{\underline{2}}V^{\underline{2}} \qquad \Rightarrow \qquad \mathfrak{B}_2 = e^{E}\mathfrak{B}_{\underline{2}} , \tag{4.77}$$

such that, by using the definition of $\mathfrak{B}_{\underline{2}}$ in (4.56h), we have

$$\langle\mathcal{I},\mathfrak{B}\rangle = \frac{3\sqrt{2}}{2}e^{E-2A}\langle\mathcal{G},\mathcal{R}\rangle dx_2 . \tag{4.78}$$

Using this result, we are able to find that the two equations in (4.76b) can be written as

$$e^{-F}\star_2 d\omega = 2e^{E}\langle\mathcal{I},d\mathcal{I} + \frac{2}{3}\mathfrak{B}\rangle , \tag{4.79}$$

while equations (4.76b-4.76c) can be written as

$$e^{-F}\star_2\left(d\tilde{\Psi} + f\mathcal{G}d\omega\right) = -\sqrt{2}e^{E}\left(d\mathcal{I} + \mathfrak{B}\right) , \tag{4.80}$$

where we can remove the contribution from $\star_2 d\omega$ by employing equation (4.79). The complete set of differential and algebraic equations describing our solution is then:

$$\partial_2 e^{-E} = -\sqrt{2}\langle\mathcal{G},\mathcal{I}\rangle , \tag{4.81a}$$

$$\partial_1 e^{F} = -\langle\mathcal{G},\tilde{\Psi}\rangle , \tag{4.81b}$$

$$e^{-F}\star_2 d\omega = 2e^{E}\langle\mathcal{I},d\mathcal{I} + \frac{2}{3}\mathfrak{B}\rangle , \tag{4.81c}$$

$$e^{-F}\star_2 d\tilde{\Psi} = -2e^{E}f\mathcal{G}\langle\mathcal{I},d\mathcal{I} + \frac{2}{3}\mathfrak{B}\rangle - \sqrt{2}e^{E}\left(d\mathcal{I} + \mathfrak{B}\right) , \tag{4.81d}$$

in addition to which we also have an equation for the new "warp factor" $f(x_2)$

$$\partial_2 f = e^{E} , \tag{4.81e}$$

the phase equations

$$\partial_i\alpha + \mathcal{A}_i = e^{2A}\langle\mathcal{I},\partial_i\mathcal{I}\rangle \qquad \text{with } i = 1,2 \tag{4.81f}$$

and the algebraic relations

$$\Phi = -2e^{A}Re(e^{-i\alpha}\mathcal{V}) + f\mathcal{G} , \qquad e^{2A} = \langle\mathcal{R},\mathcal{I}\rangle . \tag{4.81g}$$

Although we have not proved it, these should correspond to the ones proposed in (2.45) of [34], once the appropriate identifications are in place. This reduction gives us hope that this kind of solutions could admit a description in term of a superpotential, a rewriting of the action in a BPS squared form and possibly a generalisation to multi-center configurations. What is still missing is an identification of the charges $\mathcal{Q}$ and the angular momentum $\mathcal{J}$ of the solution. We will comment on this at the end of this work.

4.4 The solutions in a simplified case

The procedure that needs to be followed in the general case to achieve explicit expressions for the flow equations in terms of the charges and angular momentum and, hopefully, find a superpotential that drives the flow is still elusive. In order to gain some insights, for the rest of this chapter we are going to employ simplifying assumption on the structure of $\mathcal{I}$ and $\tilde{\Psi}$, that are

$$\Delta\mathcal{I} = 0 \qquad \Rightarrow \qquad \mathcal{I} = \mathcal{I}(x_2)\,, \tag{4.82a}$$
$$\Delta\tilde{\Psi} = 0 \qquad \Rightarrow \qquad \tilde{\Psi} = \tilde{\Psi}(x_1)\,. \tag{4.82b}$$

These assumptions are inspired by the fact that this is the simplest way of satisfying the requests of (4.42). As we will see, this will lead to solutions with interesting properties. The first consequence is that the dependence on x_1 and x_2 of the various quantities is separated in such a way that the dependence on the angular variable x_1 can be completely solved and we are left with a simple radial flow in x_2. We will find that the first order equations are driven by a real superpotential that generalises the one in the static case, which suggests that an attractor mechanism is at work, but no gradient flow form can be found for all the equations. However, despite these nice properties, we will see that the solutions resulting from our simplifying assumptions can be asymptotically AdS_4 only in the case in which the non-static contribution disappears. Nonetheless the following analysis could be useful in order to understand the physical role played by the coordinates and the various quantities at play in the general case.

4.4.1 Separation of variables

The assumptions (4.82) have important consequences on the dependencies on x_1 and x_2 of the various quantities entering our equations. Similarly to the static case, we find that the only way of satisfying

$$\partial_1\mathcal{I} = 0 \qquad \text{and} \qquad \partial_1\alpha + \mathcal{A}_1 = 0 \tag{4.83}$$

is to ask for the scalar fields, the phase and the A warp factor to not depend on x_1. Notice that this means that the period matrix $\mathcal{N}_{\Lambda\Sigma}$ is independent of x_1. Then the sections $\mathcal{R}$ should also only depend on x_2 because of the stabilisation equations. On the other side, we find that the non-static contribution ω must only depend on x_1, since equation (4.56c) implies that $\partial_2\omega = 0$. The overall result of the separation of variables produced by our assumptions is here summarised:

$$E = E(x_2)\,, \qquad A = A(x_2)\,, \qquad F = F(x_1) \qquad \text{and} \qquad \omega = \omega(x_1) \tag{4.84a}$$

for the warp factors and the non-static contribution,

$$z^a = z^a(x_2) \tag{4.84b}$$

for the scalar fields, while

$$\Phi = \Phi(x_2) \qquad \text{and} \qquad \tilde{\Psi} = \tilde{\Psi}(x_1) \tag{4.84c}$$

for the components of the vector fields $\mathcal{A}$. This separation of variables simplifies noticeably the set of supersymmetric equations. The warp factors E and F still satisfy (4.81b). The equations for $\partial_2\omega$ and $\partial_2\tilde{\Psi}$ are trivial. On the other side, the equation for $\partial_1\omega$ is

$$e^{-F}\partial_1\omega = -2e^E\left[\langle\mathcal{I},\partial_2\mathcal{I}\rangle + \sqrt{2}e^{E-2A}\langle\mathcal{G},\mathcal{R}\rangle\right] , \tag{4.85}$$

where the left hand side only depends on x_1, while the right hand side only depends on x_2. This means that the only possible solution is to introduce a constant $\mathcal{J}$, which should be related to the angular momentum of the solution, such that

$$e^{-F}\partial_1\omega = \mathcal{J} . \tag{4.86}$$

The other remaining non-trivial first order equation is the one for $\partial_1\tilde{\Psi}$, which takes the form

$$e^{-F}\partial_1\tilde{\Psi} = \sqrt{2}e^E\left(\partial_2\mathcal{I} + e^E\mathfrak{B}_{\underline{2}}\right) - f\,\mathcal{J}\mathcal{G} , \tag{4.87}$$

where, again, the two sides depend on different coordinates. We can, then, solve this by introducing a symplectic vector of constants $\mathcal{C}$, such that

$$e^{-F}\partial_1\tilde{\Psi} = \sqrt{2}\mathcal{C} . \tag{4.88}$$

By looking back at the static limit, the constants $\mathcal{C}$ should be related to the charges $\mathcal{Q}$ of the solution. Notice that, thanks to the introduction of the constant $\mathcal{J}$, we can recast the the non-trivial equation for the phase as

$$\partial_2\alpha + \mathcal{A}_2 = -\frac{e^{2A-E}}{2}\mathcal{J} - 2e^{E-A}Re(e^{-i\alpha}\mathcal{L}) . \tag{4.89}$$

On the other side, thanks to the introduction of the constants $\mathcal{C}$, by taking another derivative in x_1 of equation for the warp factor F we find

$$e^{-F}\partial_1{}^2 e^F = -\sqrt{2}\langle\mathcal{G},\mathcal{C}\rangle \tag{4.90}$$

and this means that the x_1 dependence is completely solvable, since $\langle\mathcal{G},\mathcal{C}\rangle$ is a constant. As a matter of fact, in order to keep the correct static limit, we will ask that $\sqrt{2}\mathcal{C} = -\mathcal{Q}$ and then, depending on the sign of $\langle\mathcal{G},\mathcal{Q}\rangle = n$, we will have

$$e^F = \begin{cases} c_1 sin(\sqrt{-n}x_1) + c_2 cos(\sqrt{-n}x_1) & \text{if } n < 0 , \\ c_1 e^{\sqrt{n}x_1} + c_2 e^{-\sqrt{n}x_1} & \text{if } n > 0 , \\ c_1 x_2 + c_2 & \text{if } n = 0 . \end{cases} \tag{4.91}$$

The first two possibilities describe, respectively, the case of spherical and hyperbolic horizons. The last one is only admitted in the $\mathcal{G} = 0$ case, i.e. in the ungauged case, where we have a discontinuity of our solutions. With this in mind, we will only focus on the x_2 dependence from now on.

4.4.2 First order flow equations

From the discussion in the previous section we have found that, thanks to our simplifying assumption, the set of first order equations is composed of equations for the warp factors

$$\partial_2 E = 2e^{E-A} Im(e^{-i\alpha}\mathcal{L}) \,, \qquad \partial_2 f(x_2) = e^{E} \,, \tag{4.92a}$$

equations for the non-static contribution ω and the $\tilde{\Psi}$ components of the vector fields

$$\partial_1 \omega = e^{F}\mathcal{J} \,, \qquad \partial_1 \tilde{\Psi} = -e^{F}\mathcal{Q} \,, \tag{4.92b}$$

an equation for the phase

$$\partial_2 \alpha + \mathcal{A}_2 = -2e^{E-A} Re(e^{-i\alpha}\mathcal{L}) - \frac{1}{2}e^{2A-E}\mathcal{J} \tag{4.92c}$$

and the vanishing combination

$$\begin{aligned}\mathscr{E} =& 2e^{E}\partial_2(e^{-A} Im(e^{-i\alpha}\mathcal{V}) + 4e^{E-A}\big(\partial_2\alpha + \mathcal{A}_2 + \frac{1}{2}e^{2A-E}\mathcal{J}\big)Re(e^{-i\alpha}\mathcal{V}) \\ &+ e^{2(E-A)}\Omega\mathcal{M}\mathcal{G} + \mathcal{Q} - f\,\mathcal{J}\mathcal{G} = 0 \,. \end{aligned}\tag{4.92d}$$

As in the static case, we expect to be able to extract further information from $\mathscr{E} = 0$ by asking that its projections on various symplectic vectors vanish. The projections on the real and imaginary parts of the section $\mathcal{V}$ give us the flow equations for the warp factor A and another equation for the phase α. Let us introduce the shorter notation:

$$\tilde{\mathcal{Q}} \equiv \mathcal{Q} - f\,\mathcal{J}\mathcal{G} \,, \qquad \tilde{\mathcal{Z}} = \mathcal{Z} - f\,\mathcal{J}\mathcal{L} \,, \tag{4.93}$$

then the two equations that come from $\langle \mathscr{E}, Re(e^{-i\alpha}\mathcal{V}\rangle = 0$ and $\langle \mathscr{E}, Im(e^{-i\alpha}\mathcal{V}\rangle = 0$ are

$$\partial_2 A = e^{E-A} Im(e^{-i\alpha}\mathcal{L}) - e^{A-E} Re(e^{-i\alpha}\tilde{\mathcal{Z}}) \,, \tag{4.94a}$$

$$\partial_2 \alpha + \mathcal{A}_2 = -e^{2A-E}\mathcal{J} - e^{E-A} Re(e^{-i\alpha}\mathcal{L}) - e^{A-E} Im(e^{-i\alpha}\tilde{\mathcal{Z}}) \,. \tag{4.94b}$$

On the other side, by taking the projection of $\mathscr{E}$ on $\mathcal{U}_a$ we find the equations for the scalar fields

$$\partial_2 z^a = -e^{i\alpha} G^{a\bar{b}}\big(e^{A-E}\,\overline{\mathcal{D}_b\tilde{\mathcal{Z}}} + ie^{E-A}\overline{\mathcal{D}_b\mathcal{L}}\big) \,. \tag{4.94c}$$

The other possible projections of $\mathscr{E}$ provide us with some already known identities:

$$0 = \langle \mathcal{G}, \mathscr{E}\rangle \qquad \Rightarrow \qquad -\frac{1}{2}\mathcal{G}^T\mathcal{M}\mathcal{G} = |\mathcal{L}|^2 + |\mathcal{D}\mathcal{L}|^2 \,, \tag{4.95a}$$

$$0 = \tilde{\mathcal{Q}}^T\mathcal{M}\mathscr{E} \qquad \Rightarrow \qquad -\frac{1}{2}\tilde{\mathcal{Q}}^T\mathcal{M}\tilde{\mathcal{Q}} = |\tilde{\mathcal{Z}}|^2 + |\mathcal{D}\tilde{\mathcal{Z}}|^2 \,. \tag{4.95b}$$

As in the static case, we have obtained two different equations for the phase α. In order for (4.92c) and (4.94b) to be compatible we need to impose the constraint

$$\frac{e^{2A-E}}{2}\mathcal{J} + e^{A-E} Im(e^{-i\alpha}\tilde{\mathcal{Z}}) = e^{E-A} Re(e^{-i\alpha}\mathcal{L}) \,. \tag{4.96}$$

The presence of such a constraint is not a surprise, as the phase is not an independent degree of freedom in our theory. Notice that equations (4.94a, 4.94b, 4.94c), together with the constraint (4.96), all correctly reduce to the ones of the static case once we set $\mathcal{J} = 0$.

We can see that our simplified solutions cannot be asymptotically AdS_4, mainly because the non-static contribution ω does not depend on the radial coordinate x_2, which means that it does not disappear at spatial infinity. This make it clear that no AdS_4 vacuum can be reproduced at spatial infinity by this kind of solutions.

4.5 Attractor mechanism

We introduced the concept of the superpotential and its role in the attractor mechanism in section 2.2. We are interested in looking for a superpotential which drives the flow of the scalars and the warp factors because it would give us a powerful tool to work out the attractor mechanism, the near–horizon behaviour of the solution and hence the entropy of the black hole. We are now going to show that the flow of our simplified solutions is driven by a real superpotential $|\mathcal{W}_0|$, which generalises the one found in the static case [30]. Despite this encouraging result, we have not been able to find a rewriting of the flow equations in a gradient flow form.

4.5.1 A complex quasi-superpotential

In order to completely factor out the dependence on x_1, it will be convenient to work in the case of spherical horizon, where we normalise $\langle \mathcal{G}, \mathcal{Q} \rangle = -1$, such that

$$e^F = -sin(x_1) \; . \tag{4.97}$$

We will also introduce the coordinates $\{r, \theta, \phi\}$, inspired by the ones used in the reduction to the static case, as

$$dr = e^E dx_2 \; , \qquad \theta = x_1 \qquad \text{and} \qquad \phi = x_3 \; , \tag{4.98}$$

and a reparameterization of the warp factors $\{U, A_{new}\}$ as

$$U = A_{old} \qquad \text{and} \qquad A_{new} = E - A_{old} \; , \tag{4.99}$$

from now on we will label A_{new} as A. In this way we have the same notation as the one in section 2.3 of [30]. Notice that with this choice of coordinates in place we have

$$' \equiv \frac{d}{dr} = e^{-E} \frac{d}{dx_2} \qquad \text{and} \qquad f' = 1 \quad \Rightarrow \quad f = r - r_0 \; , \tag{4.100}$$

for a constant r_0. With these choices made, the set of flow equations is

$$U' = e^{-U} Im(e^{-i\alpha}\mathcal{L}) - e^{-U-2A} Re(e^{-i\alpha}\tilde{\mathcal{Z}}) \,, \tag{4.101a}$$

$$A' = e^{-U} Im(e^{-i\alpha}\mathcal{L}) + e^{-U-2A} Re(e^{-i\alpha}\tilde{\mathcal{Z}}) \,, \tag{4.101b}$$

$$z^{a\,\prime} = - e^{i\alpha} G^{a\bar{b}} \left(e^{-U-2A}\overline{\mathcal{D}_b\tilde{\mathcal{Z}}} + ie^{-U}\overline{\mathcal{D}_b\mathcal{L}} \right) \,, \tag{4.101c}$$

$$\begin{aligned}\alpha' + \mathcal{A}_r &= - 2e^{-U} Re(e^{-i\alpha}\mathcal{L}) - \frac{1}{2}e^{-2A}\mathcal{J} \\ &= - e^{-U} Re(e^{-i\alpha}\mathcal{L}) - e^{-U-2A} Im(e^{-i\alpha}\tilde{\mathcal{Z}}) - e^{-2A}\mathcal{J}\end{aligned} \tag{4.101d}$$

and we add $f(r)$ as an independent warp factors that obeys a first order flow equation

$$f' = 1 \,. \tag{4.101e}$$

With this notation, the constraint for the phase takes the form

$$\frac{1}{2}e^{U-A}\mathcal{J} + e^{-A} Im(e^{-i\alpha}\tilde{\mathcal{Z}}) = e^{A} Re(e^{-i\alpha}\mathcal{L}) \,. \tag{4.102}$$

Let us now introduce a complex function

$$\mathcal{W} \equiv e^{U}\left(\tilde{\mathcal{Z}} - ie^{2A}\mathcal{L}\right) \,, \tag{4.103}$$

that depends on the warp factors $\{U, A, f\}$, the scalar fields and the phase α, which we treat as an independent degrees of freedom for the time being. This combination is a direct generalisation of the superpotential found in the static case. Let us notice that, by making use of the constraint, we have

$$Re(e^{-i\alpha}\mathcal{W}) = e^{U} Re(e^{-i\alpha}\tilde{\mathcal{Z}}) + e^{2A+U} Im(e^{-i\alpha}\mathcal{L}) \,, \tag{4.104}$$

and, since we now have to treat $\{A, U, f, z, \bar{z}, \alpha\}$ as independent degrees of freedom, we can introduce derivatives in the warp factors as

$$\partial_A Re(e^{-i\alpha}\mathcal{W}) = 2e^{2A} Im(e^{-i\alpha}\mathcal{L}) \quad \text{and} \quad \partial_A Im(e^{-i\alpha}\mathcal{W}) = -2e^{2A} Re(e^{-i\alpha}\mathcal{L}) \,, \tag{4.105}$$

derivatives in the scalar fields as

$$\partial_a\mathcal{W} = e^{U}\left(\partial_a\tilde{\mathcal{Z}} - ie^{2A}\partial_A\mathcal{L}\right) \,. \tag{4.106}$$

Using these derivatives we can write the flow equation in (4.101) as

$$U' = - e^{-2(A+U)}\left[Re(e^{-i\alpha}\mathcal{W}) - \partial_A Re(e^{-i\alpha}\mathcal{W})\right] \,, \tag{4.107a}$$

$$A' = e^{-2(A+U)} Re(e^{-i\alpha}\mathcal{W}) \,, \tag{4.107b}$$

$$f' = 1 \,, \tag{4.107c}$$

$$z^{a\,\prime} = - G^{a\bar{b}} e^{-2(A+U)} e^{i\alpha}\overline{\mathcal{D}_b\mathcal{W}} \,, \tag{4.107d}$$

$$\alpha' + \mathcal{A}_r = - e^{2(A+U)}\left[Im(e^{-i\alpha}\mathcal{W}) - \partial_A Im(e^{-i\alpha}\mathcal{W}) + e^{2U}\mathcal{J}\right] \,, \tag{4.107e}$$

and the constraint is written as

$$\mathcal{J} = -2e^{-2U} Im(e^{-i\alpha}\mathcal{W}) \,. \tag{4.108}$$

As we have not provided a definition of the phase α in terms of the independent degrees of freedom, we have not yet reached the description in terms of a superpotential that we are looking for.

4.5.2 A real superpotential

We are now ready to introduce a new complex combination, derived from the previous one, as

$$\mathcal{W}_0 \equiv \mathcal{W} + \frac{i}{2} e^{2U+i\alpha} \mathcal{J} \ . \tag{4.109}$$

Let us impose that the phase of $\mathcal{W}_0$ corresponds with α, which means that we can set

$$\mathcal{W}_0 = e^{i\alpha} |\mathcal{W}_0| \ , \tag{4.110}$$

it follows from this request that

$$\mathcal{W} = e^{i\alpha}\left(|\mathcal{W}_0| - \frac{i}{2} e^{2U} \mathcal{J}\right) \qquad \Rightarrow \qquad |\mathcal{W}_0| = \sqrt{|\mathcal{W}|^2 - \frac{1}{4} e^{4U} \mathcal{J}^2} \ . \tag{4.111}$$

With this request, the real and imaginary parts of $e^{-i\alpha}\mathcal{W}$ that appear in equations (4.107), are

$$Im(e^{-i\alpha}\mathcal{W}) = -\frac{1}{2} e^{2U} \mathcal{J} \ , \tag{4.112a}$$

$$Re(e^{-i\alpha}\mathcal{W}) = Re(e^{-i\alpha}\mathcal{W}_0) = |\mathcal{W}_0| \ . \tag{4.112b}$$

Notice that the constraint (4.108) is identically satisfied because of (4.112a). Since the constraint is now implicit, we can take

$$e^{i\alpha} \equiv \frac{|\mathcal{W}_0| + \frac{i}{2} e^{2U} \mathcal{J}}{\mathcal{W}} \tag{4.113}$$

to be the definition of α as a dependent degree of freedom and hence, from now on, we will have to treat the phase as a function $\alpha(U, A, f, z, \bar{z})$. With this in mind, we want to rewrite the flow equations for the warp factors and the scalar fields in terms of the real superpotential $|\mathcal{W}_0|$. We have to remark, however, that this cannot be done by simply substituting $|\mathcal{W}_0|$ in place of $Re(e^{-i\alpha}\mathcal{W})$ in the equations, because now the derivatives of $|\mathcal{W}_0|$ have to take into account that α is a dependent degree of freedom. This is clear once we use (4.111), which gives us an expression for $|\mathcal{W}_0|$ where the phase does not appear, to compute the following derivatives:

$$\partial_A |\mathcal{W}_0| = 2e^{2A+U} Im(e^{-i\alpha}\mathcal{L}) + \frac{e^{2A+3U}}{|\mathcal{W}_0|} \mathcal{J} Re(e^{-i\alpha}\mathcal{L}) \ , \tag{4.114a}$$

$$\partial_U |\mathcal{W}_0| = \frac{1}{|\mathcal{W}_0|} \left(|\mathcal{W}_0|^2 - \frac{e^{4U}}{4} \mathcal{J}^2 \right) \ , \tag{4.114b}$$

$$\partial_f |\mathcal{W}_0| = -e^U \mathcal{J} Re(e^{-i\alpha}\mathcal{L}) + \frac{e^{3U}}{2|\mathcal{W}_0|} \mathcal{J}^2 Im(e^{-i\alpha}\mathcal{L}) \ . \tag{4.114c}$$

We can use these to find the coefficients of the metric $\tilde{G}$ need to reproduce

$$\begin{aligned} U' &= -\tilde{G}^{UA} \partial_A |\mathcal{W}_0| - \tilde{G}^{UU} \partial_U |\mathcal{W}_0| - \tilde{G}^{Uf} \partial_f |\mathcal{W}_0| \\ &= e^{-U} Im(e^{-i\alpha}\mathcal{L}) - e^{-U-2A} Re(e^{-i\alpha}\tilde{\mathcal{Z}}) \ , \end{aligned} \tag{4.115}$$

which turn out to be

$$\tilde{G}^{UA} \equiv -\frac{e^{-2(A+U)}|\mathcal{W}_0|^2}{|\mathcal{W}_0|^2+\frac{1}{4}e^{4U}\mathcal{J}^2}\,, \quad \tilde{G}^{UU} \equiv \frac{e^{-2(A+U)}|\mathcal{W}_0|^2}{|\mathcal{W}_0|^2-\frac{1}{4}e^{4U}\mathcal{J}^2}\,, \quad \tilde{G}^{Uf} \equiv -\frac{|\mathcal{W}_0|}{|\mathcal{W}_0|^2+\frac{1}{4}e^{4U}\mathcal{J}^2}\,. \tag{4.116}$$

For the A warp factor we still have the simple equation

$$A' = e^{-2(A+U)}|\mathcal{W}_0|\,, \tag{4.117}$$

which is unmodified since no derivatives of $|\mathcal{W}_0|$ appear. In order to express the equations for the scalar fields in terms of $|\mathcal{W}_0|$ we have to first calculate

$$\begin{aligned}\partial_{\bar{a}}e^{i\alpha} =& \partial_{\bar{a}}\left(\frac{\mathcal{W}}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\right) = \frac{\partial_{\bar{a}}\mathcal{W}}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}} - \frac{\mathcal{W}}{(|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J})^2}\partial_{\bar{a}}|\mathcal{W}_0| \\ =& e^{i\alpha}\left(\frac{1}{2}\partial_{\bar{a}}\mathcal{K} - \frac{\partial_{\bar{a}}|\mathcal{W}_0|}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\right)\,,\end{aligned} \tag{4.118}$$

where we used the fact that $\mathcal{D}_{\bar{a}}\mathcal{W} = \partial_{\bar{a}}\mathcal{W} - \frac{1}{2}\partial_{\bar{a}}\mathcal{K}\mathcal{W} = 0$. From this we can take the conjugate equation and find

$$i\partial_a\alpha + \frac{1}{2}\partial_a K = \frac{\partial_a|\mathcal{W}_0|}{|\mathcal{W}_0|+\frac{i}{2}e^{2U}\mathcal{J}}\,. \tag{4.119}$$

It is now trivial to use this to show that

$$e^{i\alpha}\overline{\mathcal{D}_a\mathcal{W}} = e^{i\alpha}\left(\partial_b\mathcal{W} + \frac{1}{2}\partial_a K\mathcal{W}\right)^* = \frac{2|\mathcal{W}_0|}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\partial_{\bar{a}}|\mathcal{W}_0|\,, \tag{4.120}$$

which allows us to write the equations for the scalar fields as

$$z^{a\,\prime} = -\tilde{G}^{a\bar{b}}\partial_{\bar{b}}|\mathcal{W}_0| = -G^{a\bar{b}}e^{-2(A+U)}\frac{2|\mathcal{W}_0|}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\partial_{\bar{b}}|\mathcal{W}_0|\,, \tag{4.121}$$

which, as expected, is still driven by the real function $|\mathcal{W}_0|$.

4.5.3 The phase equations

The remaining point is to show that the equation for the phase (4.101d) is an identity once all of the other flow equations are in place. This is needed because now we cannot treat the phase as independent, but it must be viewed as a function $\alpha = \alpha(A,U,f,z,\bar{z})$. This means that its derivative in r is given by the chain rule

$$\alpha' = z^{a\,\prime}\partial_a\alpha + \bar{z}^{\bar{a}\,\prime}\partial_{\bar{a}}\alpha + A'\partial_A\alpha + U'\partial_U\alpha + f'\partial_f\alpha\,. \tag{4.122}$$

In order to evaluate it, let us first consider the contribution from the scalar fields, we know from (4.119) that

$$z^{a\,\prime}\partial_a\alpha = z^{a\,\prime}\left[\frac{i}{2}\partial_a K - i\frac{\partial_a|\mathcal{W}_0|}{|\mathcal{W}_0|+\frac{i}{2}e^{2U}\mathcal{J}}\right] = \frac{i}{2}\partial_a K z^{a\,\prime} + i\frac{2G^{a\bar{b}}\partial_a|\mathcal{W}_0|\partial_{\bar{b}}|\mathcal{W}_0|}{|\mathcal{W}_0|^2-\frac{1}{4}e^{4U}\mathcal{J}^2}\ , \quad (4.123)$$

where in the second step we used the equation for the scalar fields. This means that the first two contributions to α' are

$$z^{a\,\prime}\partial_a\alpha + \bar{z}^{\bar{a}\,\prime}\partial_{\bar{a}}\alpha = -\mathcal{A}_r\ . \quad (4.124)$$

The three remaining contributions can be computed by taking derivatives of (4.113):

$$\partial_A\alpha = -\,ie^{-i\alpha}\partial_A\left(\frac{\mathcal{W}}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\right) = -\frac{1}{|\mathcal{W}_0|}\left[2e^{2A+U}Re(e^{-i\alpha}\mathcal{L})\right]\ , \quad (4.125a)$$

$$\partial_U\alpha = -\,ie^{-i\alpha}\partial_U\left(\frac{\mathcal{W}}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\right) = \frac{1}{|\mathcal{W}_0|}\left(\frac{1}{2}e^{2U}\mathcal{J}\right)\ , \quad (4.125b)$$

$$\partial_f\alpha = -\,ie^{-i\alpha}\partial_f\left(\frac{\mathcal{W}}{|\mathcal{W}_0|-\frac{i}{2}e^{2U}\mathcal{J}}\right) = -\frac{1}{|\mathcal{W}_0|}e^{U}\mathcal{J}Im(e^{-i\alpha}\mathcal{L})\ , \quad (4.125c)$$

where we made use of the derivatives of $|\mathcal{W}_0|$ reported in (4.114) At this point we insert these results and the explicit flow equations for A, U and f in (4.122) and we find that

$$\alpha' + \mathcal{A}_r = A'\partial_A\alpha + U'\partial_U\alpha + f'\partial_f\alpha = -2e^{-U}Re(e^{-i\alpha}\mathcal{L}) - \frac{1}{2}e^{-2A}\mathcal{J}\ , \quad (4.126)$$

which corresponds with the equation for the phase (4.101d). This means that α can be consistently treated as a dependent degree of freedom.

4.5.4 Summary of attractor behaviour and "near-horizon" limit

Summarising the results, we have found that in our simplified solutions the flow of the warp factors and the scalar fields from the boundary to the horizon is driven by a superpotential $\mathcal{W}_0$, given by

$$\mathcal{W}_0 = e^U\left(\mathcal{Z} - f\mathcal{J}\mathcal{L} - ie^{2A}\mathcal{L}\right) + \frac{i}{2}e^{2U+i\alpha}\mathcal{J}\ , \quad (4.127)$$

provided that we fix the phase α by asking that

$$\mathcal{W}_0 = e^{i\alpha}|\mathcal{W}_0|\ , \quad (4.128)$$

which means that the phase is given by

$$e^{i\alpha} \equiv \frac{e^U\left(\mathcal{Z} - f\mathcal{J}\mathcal{L} - ie^{2A}\mathcal{L}\right)}{|\mathcal{W}_0| - \frac{i}{2}e^{2U}\mathcal{J}}\ . \quad (4.129)$$

The constraint for the phase is recovered from the fact that

$$|\mathcal{W}_0| \in \mathbb{R} \qquad \Rightarrow \qquad Im(e^{-i\alpha}\mathcal{W}_0) = 0\,. \tag{4.130}$$

The real function $|\mathcal{W}_0|$ is given by

$$|\mathcal{W}_0| = \sqrt{|e^U\left(\mathcal{Z} - f\mathcal{J}\mathcal{L} - ie^{2A}\mathcal{L}\right)|^2 - \frac{1}{4}e^{4U}\mathcal{J}^2} = e^U Re(e^{-i\alpha}\tilde{\mathcal{Z}}) + e^{U+2A}Im(e^{-i\alpha}\mathcal{L})\,, \tag{4.131}$$

which drives the flow equations for the warp factors and the scalar fields as:

$$A' = e^{-2(A+U)}|\mathcal{W}_0|\,, \tag{4.132a}$$
$$U' = -\tilde{G}^{UA}\partial_A|\mathcal{W}_0| - \tilde{G}^{UU}\partial_U|\mathcal{W}_0| - \tilde{G}^{Uf}\partial_f|\mathcal{W}_0|\,, \tag{4.132b}$$
$$z^{a\,\prime} = -\tilde{G}^{a\bar{b}}\partial_{\bar{b}}|\mathcal{W}_0|\,, \tag{4.132c}$$
$$f' = 1\,, \tag{4.132d}$$

where the hermitian metric $\tilde{G}$ of our moduli space has coefficients:

$$\tilde{G}^{UA} \equiv -\frac{e^{-2(A+U)}|\mathcal{W}_0|^2}{|\mathcal{W}_0|^2 + \frac{1}{4}e^{4U}\mathcal{J}^2}\,, \qquad \tilde{G}^{UU} \equiv \frac{e^{-2(A+U)}|\mathcal{W}_0|^2}{|\mathcal{W}_0|^2 - \frac{1}{4}e^{4U}\mathcal{J}^2}\,, \tag{4.133a}$$
$$\tilde{G}^{Uf} \equiv -\frac{1}{|\mathcal{W}_0|^2 + \frac{1}{4}e^{4U}\mathcal{J}^2}\,, \qquad \tilde{G}^{a\bar{b}} \equiv \frac{2e^{-2(A+U)}|\mathcal{W}_0|}{|\mathcal{W}_0| - \frac{i}{2}e^{2U}\mathcal{J}}G^{a\bar{b}}\,. \tag{4.133b}$$

Notice that in the static limit $\mathcal{J} \to 0$ the superpotential $|\mathcal{W}_0|$ reduces to the one in equation (2.47) of [30] and the flow equations reduce to the ones in (2.48) of [30]. With these, the equation for the phase can be recovered by making use of (4.129). Despite these nice properties we have not found a consistent choice of the metric $\tilde{G}$ that allows us to write all of the flow equations in a gradient flow form. We will comment on these results in the last chapter of this work.

Despite the fact that our simplified solutions are not asymptotically AdS_4, we can still work out the near–horizon limit and the attractor conditions. Regularity of the solutions still requires the scalar fields to approach the horizon with vanishing derivatives, i.e. we need to ask that

$$z^{a\,\prime} = 0 \qquad \Rightarrow \qquad \partial_a|\mathcal{W}_0| = 0 \tag{4.134}$$

at the horizon. The other request is that the angular sector of our metric should approach an S^2 form. This means that the contribution e^{2A} should reduce to R_S^2, which is the radius squared of the S^2 part of the near-horizon metric, which in turn means that

$$A' = 0 \qquad \Rightarrow \qquad |\mathcal{W}_0| = 0 \tag{4.135}$$

at the horizon. The two horizon conditions can, then, be written as

$$\begin{cases} |\mathcal{W}_0|^2 = 0 \\ \partial_a |\mathcal{W}_0| = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} |\mathcal{W}|^2 = \frac{e^{4U}}{4}\mathcal{J}^2 \\ \mathcal{D}_a\mathcal{W} = 0\,. \end{cases} \tag{4.136}$$

The condition $\mathcal{D}_a\mathcal{W} = 0$ leads to

$$\mathcal{Q} - f\mathcal{J}\mathcal{G} + e^{2A}\Omega\mathcal{M}\mathcal{G} = -2Im(\overline{\mathcal{Z}} - 2f\mathcal{J}\overline{\mathcal{L}}\mathcal{V}) + 2e^{2A}Re(\overline{\mathcal{L}}\mathcal{V})\,, \tag{4.137}$$

which is a direct generalisation of the similar condition for the static case that we reported in (3.40). It is obtained by either making use of special Kähler identities or by making use of the flow equations in the vanishing combination $\mathscr{E}$.
The condition $|\mathcal{W}_0|^2 = 0$ lead to a second order equation for R_S^2

$$|\mathcal{L}|^2(R_S^2)^2 + 2Im(\overline{\mathcal{Z}}\mathcal{L})R_S^2 + |\mathcal{Z} - f\mathcal{J}\mathcal{L}|^2 - \frac{e^{2U}}{4}\mathcal{J}^2 = 0\,. \tag{4.138}$$

This means that we have two possible values for R_S^2, given by

$$R_S^2 = \frac{1}{|\mathcal{L}|^2}\left[-Im(\overline{\mathcal{Z}}\mathcal{L}) \pm \sqrt{Im(\overline{\mathcal{Z}}\mathcal{L})^2 - |\mathcal{L}|^2\left(|\mathcal{Z} - f\mathcal{J}\mathcal{L}|^2 - \frac{e^{2U}}{4}\mathcal{J}^2\right)}\right]\,. \tag{4.139}$$

In the static case this reduces to

$$R_S^2 = \begin{cases} +i\overline{\mathcal{Z}\mathcal{L}} \\ -i\overline{\mathcal{Z}}\mathcal{L} \end{cases}\,, \tag{4.140}$$

which, once we ask R_S^2 to be real, correspond both to the result found in (3.41).

Summary and Outlook

The main goal of this thesis was to analyse BPS, charged and rotating black hole solutions of U(1) Fayet–Iliopoulos gauged $\mathcal{N}=2$ supergravity in $d=4$ dimensions. The overall objectives of this kind of analysis are to find:

- a first order reduction of the equations of motion that would be consistent with the already known results of [31];
- a rewriting of the flow equations in terms of a superpotential, which indicates the existence of an attractor mechanism;
- a BPS rewriting of the action, analogously to the one found for the static case in [30].

Although this program was quite ambitious and not all of the initial objectives were reached, we were still able to provide some, hopefully useful, results with regards to the second and third objectives in the case in which some simplifying assumptions are in place.

We used as a starting point an ansatz for that metric that is stationary and has an extra spatial isometry related to the rotation of the vector fields. The equations describing BPS solutions found in [38] have been specialised to our ansatz, resulting in a set of first and second order equations for the sections, the vector fields and the warp factors. We have been able, by making use of the symplectic invariance of the vector sector, to reduce to first order the Meessen and Ortín equations. The resulting set of first order equations is the one reported in (4.81). It is remarkable that the reduction to first order has the effect of introducing a new warp factor f, which is governed by a first order equation. This warp factor should obviously disappear in the static limit as it is not found in the solutions of [30]. Although we have not proved it explicitly, the equations in (4.81) should be equivalent to the equations in section 5 of [34]. Our formulation, however, differs from the one provided by Hristov, Katmadas and Toldo by the fact that we removed to contribution of the electric and magnetic potentials in favour of the charges. This formulations is the first step in the procedure to find the characteristics of the attractor mechanism, based on the similar steps followed in the asymptotically flat and static AdS_4 cases. We still lack, however, a clear relation between the components Ψ of the vector fields and the actual charges $\mathcal{Q}$ of the solution as well as a relation between the non-static contribution ω and the angular momentum of the solution $\mathcal{J}$.

In order to, hopefully, provide some insights on the role of the various quantities at play, we focused on a simplified case obtained by making the assumptions reported in (4.82). With these assumptions the equations (4.81) reduced to the flow equations in (4.92) and (4.94). Among these we found two different equations for α', from which we derived the constraint (4.96). Both the flow equations and the constraint can be seen as a generalisations of the ones in the static case [30], in the sense that they reduce to them once we send to zero the parameter $\mathcal{J}$ which controls the flow of the non-static contribution ω. The equation for ω does, however, make it clear that no AdS$_4$ vacuum can be reproduced at spatial infinity by this kind of solutions, as our metric turns out to be

$$ds^2 = -e^{2U}\left(dr + \mathcal{J} cos\theta\, d\phi\right)^2 + e^{-2U}\left(dr^2 + e^{2A} d\Omega^2\right) ,$$

when considering the closed horizon case with $\langle \mathcal{G}, \mathcal{Q} \rangle = -1$. This is to be attributed to our simplifying assumptions, from which it turns out that ω does not depend on the radial coordinate and hence cannot vanish at spatial infinity. It seems plausible that in the full solution this could be cured by an appropriate dependence of ω on the radial coordinate. Despite this issue, we persevered in the analysis of this simplified case and showed that there exist a superpotential $\mathcal{W}_0(z,\bar{z},U,A,f)$, reported in (4.109), such that:

- once we impose that $\mathcal{W}_0 = e^{i\alpha}|\mathcal{W}_0|$ the constraint (4.96) is identically satisfied. Equation (4.129) can then be taken to be a definition of the phase as a dependent degrees of freedom.
- the flow equations for the warp factors and scalar fields can be all written in terms of the real combination $|\mathcal{W}_0|$ and its derivatives in the scalar fields and warp factors.
- the phase equation is identically satisfied once we ask that $\alpha(z,\bar{z},U,A,f)$ is a dependent degree of freedom of the theory and the other flow equations are satisfied.
- both the superpotential and the flow equations reduce to the ones of the static case, found in section 2 of [30].

The flow of the scalar fields and warp factors is, then, clearly driven by the real superpotential $|\mathcal{W}_0|$. Despite this, we did not achieve a full rewriting of the equations in a gradient flow form

$$\varphi^{\alpha\,\prime} = -\nabla^{\alpha}|\mathcal{W}_0| ,$$

which was only possible for the equations for the scalar fields and the U warp factor. The cause of this difficulty could be attributed to the fact that we are working in a simplified setting, where we integrated out the non-static contribution ω. It could be that the superpotential for the full solution has a dependency of ω which contributes to the gradient flow equations. Another point upon which we need to comment is that it may be that the newly introduced warp factor f is not a good choice and should be replaced with a different combination of the warp factors.

Our simplified solutions are not a true black holes as they do not describe a flow from AdS_4 to $AdS_2 \times S^2$. Despite this, we still tried to look at the the near-horizon limit of the angular sector and used our rewriting of the flow equations in terms of the superpotential. We are interested in the behaviour of the superpotential in the near–horizon because, as already seen in the asymptotically flat case and in the static AdS_4 case, its value at the horizon is closely related to the entropy of the black hole. We ask that the angular part of our metric approaches S^2 in the near-horizon and find the two conditions reported in (4.136). These lead to two horizon conditions which seem to generalise the ones found in the static case [30], which are (4.137) and (4.138). The latter condition gives us that the possible values for R_S^2, from which we would find the entropy of the black hole. It is remarkable that these are found as solution of a second order equation, which means that we have to make a choice of sign. This is reminiscent of the properties of the entropy functional proposed by Hosseini, Hristov and Zaffaroni in [37].

The analysis developed in this work shows that rotating black hole solutions have, at least in the simplified case considered, promising properties. The direction upon which to expand these results in the future is to work out the explicit flow equations in the general case, starting from the equations in (4.81). The main obstacle to this kind of development is the difficulty of finding the relation between the components $\tilde{\Psi}$ of the vector fields and the charges $\mathcal{Q}$ of the black hole and between the contribution ω and the angular momentum $\mathcal{J}$. If this is addressed then we would be able to find explicit first order equations describing the flow of the scalars and the warp factors in two dimensions. At this point a further question arises, the radial and angular flow can be factorised? The hope is that this could be done, for example, by making use of a first integral along the flow, possibly related to the angular momentum, which would allow us to decouple the flow in the radial direction from the one in the angular one. In this case we would recover a simple radial flow that could lead to an attractor mechanism. One could, then, use this to explain and generalise the result for the entropy functional of Hosseini, Hristov and Zaffaroni.

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