UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Worldsheet scattering for string mirror models

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Academic Year 2022/2023

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Introduction

String theory is one of the major theoretical frameworks of modern physics. It attempts to unify the matter content and all the interactions, including gravity, in a unique quantum mechanical theory, by invoking the fact that the constituent building blocks of the universe are not point particles but rather extended objects called strings. This theory was first introduced as an attempt to explain the strong interaction. This proposal originates from an insight by Veneziano [1] and was subsequently developed by Nambu and Susskind [2]–[4]. However, the discovery of Quantum Chromo Dynamics (QCD) which was able to describe the theory of strong interaction and the subsequent development of the Standard Model of particle physics set string theory aside. It came back into play when Scherk and Schwarz [5] suggested that the massless spin-2 excitation in the string spectrum might be the graviton. In fact, every other attempt to quantise Einstein's theory of general relativity in the usual way and incorporate it into the Standard Model failed creating a non-renormalizable quantum theory. This suggestion prompts the development of the so-called bosonic string theory. Since in this framework the different particles are given by the different oscillation modes of the strings, in order to take into account also the fermionic particles present in nature, it was necessary to introduce also fermionic modes in the string spectrum. This was achieved through a series of works (e.g [6]–[9]) that introduced supersymmetry and led to the emergence of superstring theory.

On the other hand, another major step forward in theoretical physics is the holographic conjecture, introduced by 't Hooft [10]. It states that in order to have a consistent theory of quantum gravity, at the Planckian scale the degrees of freedom of our world can be seen as defined on a two-dimensional lattice at the boundary, evolving with time.
This conjecture, despite being a general feature of quantum gravity theories, has been mostly studied in the context of string theory [11], where a concrete example of it was provided through the AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence [12]. It states that a string theory defined on a background containing an AdS_{n+1} as a factor is dual to a n-dimensional conformal field theory, i.e. a QFT with conformal symmetry, defined on the boundary of the Anti-de Sitter space.

The major example of this duality is the one involving the maximally supersymmetric $\mathcal{N}=4$ Super Yang-Mills theory (SYM) in four dimensions and type IIB superstring theory defined on $AdS_5 \times S^5$. It is worth noting that, since $\mathcal{N}=4$ SYM is a cousin of QCD, this correspondence explains why at the beginning string theory was introduced as a theory of strong interaction.
As in the previous example, AdS/CFT in many cases relates a string theory to a gauge theory, for this reason it is also known as gauge-string correspondence. In particular, the duality relates the gauge coupling g_{YM} and the number of colors N_c, i.e. the rank of the gauge group of the theory, to the string tension T and the string coupling g_s in the following way

$$\lambda \propto T^2\,, \quad \frac{1}{N_c} \propto \frac{g_s}{T^2}\,, \tag{1}$$

where $\lambda = g_{YM}^2 N_c$ is the 't Hooft coupling.
An important regime in which to study the duality is the *planar limit* (also known as 't Hooft limit) [13], where $N_c \longrightarrow \infty$ and $\lambda = g_{YM}^2 N_c$ is kept fixed. In this limit, the only surviving Feynman diagrams in the perturbative expansion around $\lambda = 0$ in the gauge theory are the planar diagrams, namely those which can be drawn on a plane. On the other hand, in the string theory this corresponds to considering the free string ($g_s \longrightarrow 0$). This is one of the most useful aspects of the correspondence;

in fact, it allows to study a strongly coupled gauge theory ($\lambda \longrightarrow \infty$) by means of a weakly coupled string theory ($g_s \longrightarrow 0$ and $T \longrightarrow \infty$), which allows for perturbative computations.

In the 't Hooft limit AdS/CFT relates the string energy levels as a function of T to the planar scaling dimensions for local operators as a function of λ.
Strings freely propagating in a curved background are described by means of a non-linear sigma model (NLSM) action. In general, there is no method for solving the spectrum in the coupling regions where perturbative theory is not defined for either the string or the gauge theory. Nevertheless, in the case of the $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ duality, several results coming from both the gauge and the string sides suggest the presence of a phenomenon called *integrability* which allows to exactly solve the theories in the planar limit, for arbitrary values of the couplings.

Integrability was first introduced for classical Hamiltonian dynamical systems. Physicists realised that there were some non-trivial Hamiltonian systems in which the solutions could be explicitly written down. One of these was the well-known Kepler problem, which describes the motion of the planets around the Sun. Liouville realised that this feature of some systems is related to the presence of n independent conserved quantities in involution, where n is the number of degrees of freedom of the system. If the set of conserved quantities is not spoilt under the quantisation procedure, then the corresponding quantum model is said to be *quantum integrable.* A first step in the study of quantum integrability was taken by Bethe [14], when he was able to exactly solve the energy spectrum of the Heisenberg XXX spin chain, by means of a set of equations known as *Bethe ansatz equations* (BAE). Furthermore, studying the thermodynamic properties of a one-dimensional bosonic gas with a delta function interaction, Yang and Yang [15] introduced the *thermodynamic Bethe ansatz* (TBA) by which it is possible to find the free energy of a thermodynamic quantum integrable system.
In field theories, where there is an infinite number of degrees of freedom, integrability requires the presence of an infinite set of conserved charges. At the classical level, for a two-dimensional field theory, this is equivalent to finding a *Lax connection* by which the equations of motion can be written as the zero curvature condition of that connection. On the other hand, in two-dimensional quantum field theories integrability is strictly connected with the structure of the scattering processes. Specifically, in each scattering the number of particles in the initial state is equal to the number of particles in the final state, the set of the incoming momenta is equal to the set of the outgoing ones, and the S-matrix is factorised into the product of two-body S-matrices. Finally, in order to make this latter condition well-defined, the two-body S matrix must obey the *Yang-Baxter* equation.

Regarding the $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ duality, the first signs of integrability appeared on the gauge theory side, where some spin chain structures similar to the Heisenberg Hamiltonian were found [16]–[19]. Subsequently, on the string theory side, it was found that the NLSM is classically integrable [20] and, in addition, some hints suggested that the integrable structure is not spoilt in the quantum theory [21], [22].
Therefore, both theories were considered as quantum integrable theories and a complete (at all-loop) S-matrix was proposed for both the gauge theory [23]–[25] and the NLSM [26].

These results led to the research of integrability in less supersymmetric cases of the Maldacena duality, such as the AdS$_3$/CFT$_2$ correspondence. The case we shall consider is the $AdS_3 \times S^3 \times T^4$ background. This NLSM has been proven to be classically integrable [27]. Moreover, unlike the AdS_5 case, this superstring background can be supported by both Ramond-Ramond (RR) and Neveu- Schwarz-Neveu-Schwarz (NSNS) fluxes. It is worth noting that even in the mixed-flux case, the classical theory remains integrable [28]. Eventually, assuming that integrability persists at the quantum level, a complete S-matrix has been proposed for the $AdS_3 \times S^3 \times T^4$ background [29]–[31]. Knowing the S-matrix, the string energy spectrum can be computed by means of the asymptotic Bethe equations[1]:

$$e^{ip_j l} \prod_{k \neq j}^{M} S^{i_k i_j}_{i_j i_k}(p_j, p_k) = 1 \,, \qquad j = 1, ..., M, \tag{2}$$

[1] In particular, these are the Bethe equations when the S-matrix is diagonal in the flavour space. In the case of non diagonal scattering, it can be diagonalised by means of the nested algebraic Bethe ansatz.

where l is the length of the system, p_j, $j = 1, ..., M$ are the momenta of the M particles in the asymptotic state and the index i labels the flavour of the particles. In fact, once the momenta are known, the energy can be calculated using the dispersion relations, namely

$$E = \sum_{k=1}^{M} \omega_{i_k}(p_k)\,. \tag{3}$$

More precisely, these equations do not take into account the finite-size corrections due to the wrapping effects of the string [32]. These corrections are exponentially suppressed in l and can be perturbatively added to the spectrum found by the asymptotic BAE following the Lüscher approach [33], [34]. On the other hand, since in the thermodynamic limit $l \longrightarrow \infty$, the TBA approach gives the correct free energy of the model.
This fact, suggested to introduce a double Wick rotation on the worldsheet coordinates

$$\tau \longrightarrow -i\tilde{\sigma}\,, \quad \sigma \longrightarrow i\tilde{\tau}\,. \tag{4}$$

It was pointed out in [35] that the ground state energy (GSE) of the finite-size theory can be found from the free energy of the finite temperature theory after the double Wick rotation, which can be computed by means of the TBA. In the case of a relativistic theory, the double Wick rotation acts trivially and the theory remains unchanged, while in the non-relativistic case, like the gauge-fixed NLSM, it produces a new theory, the so-called *mirror theory*.
It is worth pointing out that in the mixed-flux $AdS^3 \times S^3 \times T^4$ background, the double Wick rotation leads to a nonunitary theory and unitarity is recovered only in the pure RR flux case.

The aim of this work is to study the perturbative aspects of both the $AdS_3 \times S^3 \times T^4$ mixed-flux worldsheet theory and its mirror theory and to study how the results found in one model can be mapped into the other. The motivation lies in the fact that in order to find the ground state energy using the TBA approach on the mirror theory, this has to be an integrable theory and in a 1+1-dimensional QFT, scattering processes provide information on whether the theory is integrable or not.
In particular, the thesis is structured as follows. In chapter 1 we introduce the concept of integrability both in classical and in quantum mechanics. Eventually, we discuss the QFT case introducing the factorised scattering theory. In chapter 2, we briefly discuss the main characteristics of the bosonic string that are useful in our discussion. In particular, we introduce the Polyakov action in a flat Minkowski space and point out its gauge symmetry and how it can be fixed. Finally, we generalise the treatment to the non-linear sigma model on curved backgrounds. In chapter 3 we specialise the NLSM to the mixed-flux $AdS_3 \times S^3 \times T^4$ background. We work in the first-order formalism and fix the light-cone gauge. Then, we find the quadratic and quartic worldsheet Hamiltonian by perturbatively solving the Virasoro constraints in the large string tension expansion and we compute the two-body tree-level S matrix. In chapter 4 after having introduced the Bethe equations and the thermodynamic Bethe ansatz, we discuss how the mirror theory can be used to find the GSE of the finite-size NLSM. Then, we find the quadratic and quartic mirror Lagrangian for the $AdS^3 \times S^3 \times T^4$ gauge-fixed background and we quantise this theory. Eventually, we compute the the two-body tree level S matrix comparing the results with the ones found in the previous chapter.
In chapter 5 we investigate the behaviour of the worldsheet mirror theory under some production processes. These are important because they provide information on the integrability of the theory. In particular, we consider six-point processes and finally we discuss how their amplitudes can be mapped into the NLSM.

Chapter 1

Integrability

Integrability is a powerful tool for both classical and quantum theories that allows to exactly solve a system exploiting its symmetries. Intuitively, the system has enough mutually independent symmetries (conserved charges) to constrain all its degrees of freedom. This property was first studied in the context of classical Hamiltonian systems and then it was implemented in quantum mechanics and in quantum field theory. As mentioned in the introduction, we will deal with a model which is integrable at the classical level and that is also supposed to preserve this structure after the quantisation. For this reason, it is worth beginning our discussion by introducing the main concepts and features about integrability.

1.1 Classical integrability

Let us start by introducing the concept of integrability in the context of classical Hamiltonian systems. In this section we refer to [36], [37], [37].

Let $H(p_i(t), q_i(t))$ be the Hamiltonian of an $n-$dimensional classical system, where q_i, $i = 1, ..., n$ are the coordinates and p_i are the canonically conjugated momenta. The dynamic of the system is given by the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \, , \tag{1.1}$$

which are, according to the Hamilton second variational principle, the critical points of the Hamiltonian functional

$$S[q,p] := \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}_i - H(q_i(t), p_i(t)) \right) \, , \tag{1.2}$$

where $\cdot = \frac{d}{dt}$.

The solutions of the Hamilton equations are curves defined in the space $\Gamma \subseteq \mathbb{R}^{2n}$, known as *phase space*.

Given two generic function $F(q_i, p_i)$ and $G(q_i, p_i)$ defined on the phase space, it is possible to define a bilinear antisymmetric operator, known as *Poisson bracket*, given by

$$\{F, G\} := \sum_i \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) \, . \tag{1.3}$$

In general, if two functions have zero Poisson bracket, we say that they are in *involution*.

If a system has a symmetry, i.e. a transformation that leaves the action invariant, due to the Noether theorem it has a corresponding conserved charge, namely a function of the phase space which remains constant along the solutions of the Hamilton equations (1.1).

In other words, let F be a Noether charge, since its total time derivative has to be zero along the solutions, using (1.3) and (1.1), it follows that its Poisson bracket with the Hamiltonian vanishes. i.e.

$$\frac{dF}{dt} = \{H, F\} = 0 \, . \tag{1.4}$$

Let us note that the knowledge of a conserved charge reduces the dynamic to a $2n-1$ subsurface in Γ given by the implicit equation $F = const.$
Following this intuition it is possible to give a definition of integrability :

Definition 1.1.1. The system defined by the Hamiltonian $H(q_i, p_i)$ is said to be *Liouville integrable* if it admits n independent conserved charges in involution, i.e. if there exists a set of n functions F_i, $i = 1, ..., n$ such that

1. $\{F_i, H\} = 0 \; \forall i = 1, ..., n$;
2. $\sum_{i=1}^{n} c_i \nabla F_i = 0 \Rightarrow c_1 = \cdots = c_n = 0$
3. $\{F_i, F_j\} = 0 \; \forall i, j = 1, ..., n$;

The first condition just impose that the functions are conserved quantities, while the second one imposes that, given the constants $a_1, \ldots, a_n$, the set defined by the implicit equations

$$M_a := \{(q_i, p_i) \in \Gamma \;,\; F_1 = a_1, ..., F_n = a_n\} \,, \tag{1.5}$$

is a well-defined n-dimensional manifold embedded in the phase space. Note that we could expect that, in order to completely constrain the motion on a curve in the phase space, we should have required $2n-1$ conserved charges, in such a way that the manifold M_a becomes a one dimensional curve. However, it turns out that, due to the Hamiltonian structure, n is the maximum possible number of independent conserved quantities in involution and they are sufficient to completely solve the system. In fact the Liouville-Arnol'd theorem establishes that a Liouville integrable Hamiltonian system can be solved by quadratures, namely by solving integrals. Moreover, if M_a is, connected and compact it follows that

1. M_n is diffeomorphic to the n-dimensional torus
$$\mathbb{T}^n = S^1 \times \ldots \times S^1 \,; \tag{1.6}$$
2. There exists a canonical transformation, i.e. a transformation which preserves the structure of the Hamilton equations, to angle-action variables $(q_i, p_i) \to (\phi_i, I_i)$. The action variables are defined by
$$I_i = \frac{1}{2\pi} \int_{C_i} \sum_{j=1}^{n} p_j dq_j \,, \tag{1.7}$$
where $C_1, ...C_n$ are independent cycles on M_a that cannot be deformed into each other. Furthermore, in these variables both the Hamiltonian and the conserved quantities become a function only of the action variables, namely $H = H(I_i)$ and $F_j = F_j(I_i)$;

Therefore, in a Liouville integrable system, the phase space is foliated in tori and the Hamilton equations in the angle-action variables become

$$\dot{\phi}_i = \frac{\partial H}{\partial I_i} \quad , \quad \dot{I} = \frac{\partial H}{\partial \phi_i} = 0 \,. \tag{1.8}$$

Thus, using the fact that the Hamiltonian depends only on the action variables and these are constant along the motion, the solutions are

$$I_i(t) = I_i(0) \quad , \quad \phi(t) = \phi(0) + \omega_i \, t \,, \tag{1.9}$$

where

$$\omega_i = \frac{\partial H}{\partial I_i} \,. \tag{1.10}$$

This approach allows to explicitly define and understand the intuition behind integrability. However, it does not provide any method of building the conserved charges of a system. For this reason, let us introduce a modern approach, known as *Lax formalism* that allows to explicitly write down all the conserved quantities of a model once its equations of motion are recast in a special matrix form called Lax representation.

1.1.1 Lax formalism

Let us introduce two $N \times N$ matrices L and M, whose entries take values in a phase space and which satisfy the equation

$$\dot{L} = [M, L]\,, \tag{1.11}$$

where $[\cdot,\cdot]$ denotes the usual matrix commutator. If this matrix equation is equivalent to an Hamiltonian problem, namely there exists an Hamiltonian defined in the same phase space such that (1.1) are solved iff (1.11) are solved, then L and M are called *Lax pair*, and the Hamiltonian system is said to have a *Lax representation*.
The Lax equation (1.11) is solved by

$$L(t) = g(t)L(0)g^{-1}(t)\,, \tag{1.12}$$

where the matrix $g(t)$ is given by solving the equation

$$M(t) = \dot{g}(t)g^{-1}(t)\,. \tag{1.13}$$

According to the solution (1.12), the L matrix at the time t is given just by a similarity transformation of the matrix at the initial time. Therefore, the eigenvalues and the trace are preserved along the Hamiltonian flux. In general, given this representation we can define the functions

$$Q_j = \mathrm{Tr}\, L^j\,. \tag{1.14}$$

These are conserved charges of the Hamiltonian system. Indeed

$$\begin{aligned} \dot{Q}_j &= \mathrm{Tr}\left(\dot{L}L^{j-1}\right) + \mathrm{Tr}\left(L\dot{L}L^{j-2}\right) + ... + \mathrm{Tr}\left(L^{j-1}\dot{L}\right) \\ &= j\,\mathrm{Tr}\left(\dot{L}L^{j-1}\right) = j\,\mathrm{Tr}\left([M,L]L^{j-1}\right) = j\,\mathrm{Tr}\left(\left[M,L^j\right]\right) = 0\,. \end{aligned} \tag{1.15}$$

In this way, the Lax representation allows to build a tower of conserved charges of a dynamical system. However, at this stage we do not have any information about the Poisson structure and if this charges are in involution. In order to impose the involution condition in the Lax representation, let us briefly introduce a tensorial notation that will be useful also when we will discuss the quantum mechanical case.
Let E_{ij} be the canonical base of the vector space of the $N \times N$ matrices. We define

$$L_1 := L \otimes 1 = \sum_{ij} L_{ij}\left(E_{ij} \otimes 1\right) \quad , \quad L_2 := 1 \otimes L = \sum_{ij} L_{ij}\left(1 \otimes E_{ij}\right)\,, \tag{1.16}$$

where L_{ij} are the components of the L matrix. Clearly, by generalising this notation we have $L_3 = 1 \otimes 1 \otimes L$ and

$$T_{12} = \sum_{ijkl} T_{ijkl}\left(E_{ij} \otimes E_{kl}\right) \quad \text{and} \quad T_{21} = \sum_{ijkl} T_{ijkl}\left(E_{kl} \otimes E_{ij}\right)\,, \tag{1.17}$$

where T is a generic tensor.
It can be shown [36] that the involution condition of the charges (1.14) corresponds to the existence of a matrix r such that

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]\,, \tag{1.18}$$

where the Poisson bracket is given by

$$\{L_1, L_2\} = \sum_{ijkl} \{L_{ij}, L_{kl}\}\left(E_{ij} \otimes E_{kl}\right)\,. \tag{1.19}$$

Furthermore, a well-known property of Poisson brackets is that they respect the Jacobi identity

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0\,. \tag{1.20}$$

Using this property on L_1, L_2 and L_3 together with the condition (1.19), we find that the $r-$matrix is constrained by the relation

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cycl. perm} = 0\,, \tag{1.21}$$

where cycl. perm denotes all the cyclic permutations of the indices 1,2 and 3. It is worth considering the case in which the r-matrix is independent of the phase space points. In this case, the Poisson brackets involving r vanish and (1.21) is solved by imposing

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0\,. \tag{1.22}$$

Imposing the condition $r_{12} = -r_{21}$ this equation is known as *classical Yang-Baxter* equation. We will come back to this expression when dealing with quantum integrable systems, in which a quantum version of this equation plays an important role.
In general, one can go further and introduce a complex parameter z, known as spectral parameter, such that the Lax matrices become a function of this parameter: $L = L(z)$ and $M = M(z)$. This allows to generalise the Lax formalism and include more integrable systems. Further discussion of this aspect can be found, for example, in Chapter 1 of [37].

So far, we have considered matrices with a finite number of entries. This corresponds to considering integrable systems with finite degrees of freedom. In fact, an $N \times N$ L matrix can give rise to maximum N independent conserved quantities, since, by construction, the Q_n defined in (1.14) are all functions of the eigenvalues of L, which in the best case are N independent functions. For this reason, in order to discuss integrability in classical field theory, where there are infinite degrees of freedom, and then we need an infinite number of independent charges in involution, we have to generalise this formalism.
Let us consider a $(1+1)$-dimensional field theory. Let τ be the time-like coordinate and σ be the space-like coordinate, we assume that the equation of motion can be cast in the form

$$\begin{aligned} (\partial_\sigma - \mathcal{L}_\sigma(\tau, \sigma, x))\, \Psi(\tau, \sigma, x) = 0\,, \\ (\partial_\tau - \mathcal{L}_\tau(\tau, \sigma, x))\, \Psi(\tau, \sigma, x) = 0\,, \end{aligned} \tag{1.23}$$

where Ψ, $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ are matrices depending on the fields of the system and on the complex parameter x. In particular, Ψ is referred to as the wavefunction, while $\mathcal{L}_\tau$ and $\mathcal{L}_\sigma$ are the two components of a connection known as the *Lax connection.* In fact, the two equations (1.23) can be thought of as the parallel transport condition of the wavefunction under the covariant derivative defined as $D_\mu = (\partial_\mu - \mathcal{L}_\mu)$, where $\mu = \tau, \sigma$. Given the expression of the connection, the solution of the linear system (1.23) is

$$\Psi(\tau, \sigma, x) = \overleftarrow{\exp} \left(\int_\gamma \mathcal{L}_\tau d\tau + \mathcal{L}_\sigma d\sigma \right) \Psi(\tau_0, \sigma_0, x)\,, \tag{1.24}$$

where $\overleftarrow{\exp}$ denotes the path-ordering symbol and γ is a generic curve with initial point (τ_0, σ_0) and final point (τ, σ). Clearly, the expression of $\Psi(\tau_0, \sigma_0, x)$ is given by the boundary conditions of the fields and then the solution is the parallel transport of the wave function along the curve γ using the Lax connection.
Imposing the so-called compatibility condition $\partial_\sigma \partial_\tau \Psi = \partial_\tau \sigma_\tau \Psi$, (1.23) are equivalent to

$$\partial_\tau \mathcal{L}_\sigma - \partial_\sigma \mathcal{L}_\tau - [\mathcal{L}_\tau, \mathcal{L}_\sigma] = 0\,. \tag{1.25}$$

This is the zero curvature condition. In fact, this equation imposes that the curvature tensor of the Lax connection is equal to zero, i.e. $F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\tau \mathcal{L}_\sigma - \partial_\sigma \mathcal{L}_\tau - [\mathcal{L}_\tau, \mathcal{L}_\sigma] = 0$. Furthermore, since the curvature is null, the expression (1.24) is well defined and does not depend on the chosen curve, but only on the initial and final point.
Solving non-linear Hamilton equations by means of the linear problem (1.23) is referred to as the inverse scattering method. For further discussion and examples, we refer to Chapter 13 of [36]. Finally, explicit expressions of the fields can be found by means of the linear Gel'fand-Levitan-Marchenko equation [38] [39].

Instead, let us focus on the conserved charges. We introduce the *monodromy matrix*, as the path-ordered exponential along the path $\sigma \in [-r/2, r/2]$ at fixed time

$$T(\tau, x) = \overleftarrow{\exp}\left(\int_{-r/2}^{r/2} \mathcal{L}_\sigma(\tau, \sigma, x)\, d\sigma\right). \tag{1.26}$$

This operator act just by parallel transporting a quantity along the whole period r.
By deriving with respect to time this expression one can find

$$\begin{aligned} \partial_\tau T &= \int_{-r/2}^{r/2} d\sigma\, e^{\int_\sigma^{r/2} \mathcal{L}_\sigma\, d\sigma}\, \partial_\tau \mathcal{L}_\sigma\, e^{\int_{-r/2}^{\sigma} \mathcal{L}_\sigma\, d\sigma} \overset{(1.25)}{=} \int_{-r/2}^{r/2} d\sigma\, e^{\int_\sigma^{r/2} \mathcal{L}_\sigma\, d\sigma} \left(\partial_\sigma \mathcal{L}_\tau + [\mathcal{L}_\tau, \mathcal{L}_\sigma]\right) e^{\int_{-r/2}^{\sigma} \mathcal{L}_\sigma\, d\sigma} \\ &= \int_{-r/2}^{r/2} d\sigma\, \partial_\sigma \left(e^{\int_\sigma^{r/2} \mathcal{L}_\sigma\, d\sigma} \mathcal{L}_\tau e^{\int_{-r/2}^{\sigma} \mathcal{L}_\sigma\, d\sigma}\right) = \mathcal{L}_\tau(\tau, r/2, x) T(\tau, x) - T(\tau, x) \mathcal{L}_\tau(\tau, -r/2, x). \end{aligned} \tag{1.27}$$

And imposing periodic boundary conditions on the fields, such that $\mathcal{L}_\mu(\tau, r/2, x) = \mathcal{L}_\mu(\tau, -r/2, x)$

$$\partial_\tau T = [\mathcal{L}_\tau(\tau, -r/2, x), T(\tau, x)]. \tag{1.28}$$

This is the analogous of the Lax equation (1.11) that we found in finite degrees of freedom, where now the L matrix is substituted by the monodromy matrix. Therefore, the function

$$\mathcal{T}(x) := \operatorname{Tr} T(x), \tag{1.29}$$

called *transfer matrix* is constant along the solutions and then its eigenvalues are conserved charges. Expanding the eigenvalues (or alternatively, the quantities $\mathcal{T}_j := \operatorname{Tr} T^j$) in x we obtain an infinite set of conserved charges. In conclusion, if the Hamilton equations of a two-dimensional classical field theory can be cast in the form (1.25), automatically it admits an infinite tower of conserved quantities and the monodromy matrix contains the information about all of them. Clearly, in order to impose that these charges are in involution, one still has to assume the existence of an r-matrix which satisfies a generalisation of the constraint (1.18).

1.2 Quantum integrability

In his paper [14], while solving the Heisenberg spin chain model for ferromagnetism, Bethe introduced a powerful tool, called *coordinate Bethe ansatz*, by which it is possible to exactly solve some quantum systems. This technique, as we will extensively see in Chapter 4, consists of making a specific ansatz on the expression of the wavefunction. Then, imposing the periodic boundary conditions, this gives rise to the so-called Bethe equations, which, when solved, return the complete spectrum of the system. As in the classical case, these systems are related to the presence of several conserved charges, which constrain the structure of the model. Moreover, following the same reasoning as in the Lax formalism in classical integrability, it is possible to find a quantum version of the L operator, of the monodromy matrix and of the Yang-Baxter equations, that allows to explicitly build the conserved quantities of the system. This approach, in contrast to the coordinate Bethe ansatz, is known as *Algebraic Bethe ansaz*, since it allows to construct an integrable model and to find its Bethe equations starting from an underlying algebraic structure.
In this section, we sketch some of the main results of this approach in quantum mechanical systems, referring to [40], [41], [37], and [42].
Let us assume that the quantisation of a classical integrable system does not spoil its integrable structure. Therefore, let $\{Q_j\}$ be the set of conserved charges of the system, after the quantisation this becomes a set of quantum operators $\{\mathbb{Q}_j\}$ such that each of them remains constant under the Heisenberg evolution, i.e.

$$[\hat{H}, \mathbb{Q}_i] = 0, \tag{1.30}$$

where $\hat{H}$ is the Hamiltonian operator. Furthermore, the involution condition is obtained by replacing the Poisson bracket with the commutator

$$[\mathbb{Q}_i, \mathbb{Q}_j] = 0\,, \qquad \forall i, j\,. \tag{1.31}$$

This condition means that it is possible to simultaneously diagonalise all the charges.
In analogy to the classical case, let us introduce the monodromy matrix $T(\lambda)$ depending on a complex parameter λ. This is a matrix defined on an auxiliary vector field $\mathcal{A}$ and whose entries are operators acting on the Hilbert space $\mathcal{H}$ on which our quantum system is defined. From $T(\lambda)$ let us define the transfer matrix $\mathcal{T}$ as

$$\mathcal{T}(\lambda) := \mathrm{Tr}_{\mathcal{A}}\, T(\lambda)\,. \tag{1.32}$$

We are tracing over the auxiliary space, therefore $\mathcal{T}(\lambda)$ is an operator in $\mathcal{H}$. We can expand $\mathcal{T}(\lambda)$ in λ around a point λ_0 :

$$\mathcal{T}(\lambda) = \sum_k \frac{1}{k!}(\lambda - \lambda_0)^k\, I_k\,, \tag{1.33}$$

where $\{I_k\}$ is a set of operators in $\mathcal{H}$, that can be either finite or infinite. Furthermore, imposing the condition

$$[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0, \quad \forall \lambda, \mu\,, \tag{1.34}$$

one can find that all the operators are in involution, i.e.

$$[I_k, I_j] = 0, \quad \forall k, j\,. \tag{1.35}$$

Indeed,

$$0 = \frac{d^k}{d\lambda^k}\frac{d^j}{d\mu^j}[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = \frac{d^k}{d\lambda^k}\frac{d^j}{d\mu^j}[\mathcal{T}(\lambda), \mathcal{T}(\mu)]\Big|_{\substack{\lambda=\lambda_0\\ \mu=\mu_0}} \overset{(1.33)}{=} [I_k, I_j]\,. \tag{1.36}$$

If one of the I_k is the Hamiltonian of some quantum system, the expansion of the transfer matrix provides a set of conserved quantities in involution for that model, which can be identified with the remaining I_k operators. However, it is worth stressing that nothing guarantees that the charges are independent of each other, and furthermore, in general they can be not self-adjoint, non-local *et cetera.* To have local operators, it is important to expand $\mathcal{T}$ around the correct point λ_0, and in some cases one needs to expand not the transfer matrix itself but rather one of its functions. For instance, to recover the Heisenberg model, the correct way is to expand the logarithm of $\mathcal{T}$, in such a way that

$$\mathcal{T}(\lambda) = \exp\left(\sum_k \frac{1}{k!}(\lambda - \lambda_0)^k\, I_k\right)\,. \tag{1.37}$$

Let us consider operators acting on the space $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{H}$. We define

$$T_1(\lambda) := T(\lambda) \otimes 1_2 \quad , \quad T_2(\lambda) := 1_1 \otimes T_2(\lambda)\,. \tag{1.38}$$

This means that T_1 and T_2 act trivially on $\mathcal{A}_2$ and $\mathcal{A}_1$ respectively. Using this notation and the relation (1.32), we can write the lhs of (1.34) as follows:

$$[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = [\mathrm{Tr}_{\mathcal{A}}\, T(\lambda), \mathrm{Tr}_{\mathcal{A}}\, T(\mu)] = [\mathrm{Tr}_{\mathcal{A}_1}\, T_1(\lambda), \mathrm{Tr}_{\mathcal{A}_2}\, T_2(\mu)] = \mathrm{Tr}_{\mathcal{A}_1\otimes\mathcal{A}_2}\, [T_1(\lambda), T_2(\mu)]\,. \tag{1.39}$$

Therefore, the involution condition of the operators on the Hilbert space becomes the matrix relation

$$\mathrm{Tr}_{\mathcal{A}_1\otimes\mathcal{A}_2}\, T_1(\lambda)T_2(\mu) = \mathrm{Tr}_{\mathcal{A}_1\otimes\mathcal{A}_2}\, T_2(\mu)T_1(\lambda)\,. \tag{1.40}$$

Two matrices related by a similarity transformation have the same trace. Hence, let us assume the existence of an invertible matrix R, such that

$$R_{12}(\lambda, \mu)T_1(\lambda)T_2(\mu)R_{12}^{-1}(\lambda, \mu) = T_2(\mu)T_1(\lambda)\,. \tag{1.41}$$

The subscripts denote that it acts non-trivially on both the spaces. In fact, since it is a transformation in the space where we take the trace, R_{12} is defined on $\mathcal{A}_1 \otimes \mathcal{A}_2$. The relation (1.41) can be recast in the form

$$R_{12}(\lambda,\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda,\mu)\,. \tag{1.42}$$

This is known as $RTT-TTR$ relation or *intertwining* relation and it guarantees that (1.40) is satisfied. The effect of R is to permute monodromy matrices that act on different copies of the auxiliary space. For this reason, the R-matrix cannot assume any value, but it has to satisfy some compatibility conditions, which come from the fact that when there are more matrices (i.e. $T_1(\lambda_1),...,T_n(\lambda_n)$) defined on several copies of the space $\mathcal{A}$, there are different ways of going from an initial configuration to a final one. In fact, the expression of the R-matrix must be such that all possible permutations give the same result. In particular, in the case of three monodromy matrices, we have two possibilities of going from $T_1T_2T_3$ to $T_3T_2T_1$:

$$\begin{aligned}
&T_1(\lambda_1)T_2(\lambda_2)T_3(\lambda_3) = \left(R_{12}(\lambda_1,\lambda_2)\right)^{-1} T_2(\lambda_2)T_1(\lambda_1)T_3(\lambda_3)R_{12}(\lambda_1,\lambda_2)\\
&= \left(R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2)\right)^{-1} T_2(\lambda_2)T_3(\lambda_3)T_1(\lambda_1)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2)\\
&= \left(R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_2,\lambda_3)\right)^{-1} T_3(\lambda_3)T_2(\lambda_2)T_1(\lambda_1)R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2)
\end{aligned} \tag{1.43}$$

And

$$\begin{aligned}
&T_1(\lambda_1)T_2(\lambda_2)T_3(\lambda_3) = \left(R_{23}(\lambda_2,\lambda_3)\right)^{-1} T_1(\lambda_1)T_3(\lambda_3)T_2(\lambda_2)R_{23}(\lambda_2,\lambda_3)\\
&= \left(R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)\right)^{-1} T_3(\lambda_3)T_1(\lambda_1)T_2(\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)\\
&= \left(R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)\right)^{-1} T_3(\lambda_3)T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)
\end{aligned} \tag{1.44}$$

Where we have used the fact that R_{12} and T_3 commute since R_{12} acts trivially on $\mathcal{A}_3$ and so on. Thus, by matching (1.43) and (1.44) we obtain

$$R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3) = R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2)\,. \tag{1.45}$$

This is the quantum Yang-Baxter equation. Once this relation is fulfilled, also all the other higher-order compatibility conditions are respected. In addition to this equation, the R-matrix has another constraint. In fact, by exchanging $1 \leftrightarrow 2$ and $\lambda \leftrightarrow \mu$ in (1.42) we find

$$R_{21}(\mu,\lambda)T_2(\mu)T_1(\lambda) = T_1(\lambda)T_2(\mu)R_{21}(\mu,\lambda) \Longrightarrow \left(R_{21}(\mu,\lambda)\right)^{-1} T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)\left(R_{21}(\mu,\lambda)\right)^{-1}\,. \tag{1.46}$$

And comparing this expression with the original equation (1.42) it follows

$$R_{12}(\lambda,\mu)R_{21}(\mu,\lambda) = \mathbb{I}\,. \tag{1.47}$$

To be precise, (1.46) implies that $R_{21}(\mu,\lambda) = f(\lambda,\mu)\,R_{12}^{-1}(\lambda,\mu)$, where $f(\lambda,\mu)$ is a generic function. However, without loss of generality, we can set $f(\lambda,\mu) = 1$.

Note that the Yang-Baxter equation and (1.47) are algebraic equations defined in the vector space $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ without any physical connection. However, by plugging a generic solution of these into (1.42), one can find a monodromy matrix T and then construct an integrable quantum system. This R-matrix is the quantum version of the r-matrix that we have seen in the classical case and, as the name suggests, the quantum Yang-Baxter equation (1.45) is the quantum version of the classical equation (1.22). To check this fact, let us expand the matrix in powers of $\hbar$

$$R = \mathbb{I} + \hbar r + \hbar^2 A + o(\hbar^2)\,. \tag{1.48}$$

Substituting this expression in (1.45) we found

$$\begin{aligned}
&3\cdot\mathbb{I} + \hbar\Big(r_{12}(\lambda_1,\lambda_2) + r_{13}(\lambda_1,\lambda_3) + r_{23}(\lambda_2,\lambda_3)\Big) + \hbar^2\Big(r_{12}(\lambda_1,\lambda_2)r_{13}(\lambda_1,\lambda_3) + r_{12}(\lambda_1,\lambda_2)r_{23}(\lambda_2,\lambda_3)\\
&+ r_{13}(\lambda_1,\lambda_3)r_{23}(\lambda_2,\lambda_3) + A_{12}(\lambda_1,\lambda_2) + A_{13}(\lambda_1,\lambda_3) + A_{23}(\lambda_2,\lambda_3)\Big) + o(\hbar^2) = 3\cdot\mathbb{I} + \hbar\Big(r_{23}(\lambda_2,\lambda_3)\\
&+ r_{13}(\lambda_1,\lambda_3) + r_{12}(\lambda_1,\lambda_2)\Big) + \hbar^2\Big(r_{23}(\lambda_2,\lambda_3)r_{13}(\lambda_1,\lambda_3) + r_{23}(\lambda_2,\lambda_3)r_{12}(\lambda_1,\lambda_2) + r_{13}(\lambda_1,\lambda_3)r_{12}(\lambda_1,\lambda_2)\\
&+ A_{23}(\lambda_2,\lambda_3) + A_{13}(\lambda_1,\lambda_3) + A_{12}(\lambda_1,\lambda_2)\Big) + o(\hbar^2)
\end{aligned}$$

The order $\hbar^0$ and $\hbar^1$ are trivially equal, while imposing the equality on the order $\hbar^2$ one can find

$$r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23} = r_{23}r_{13} + r_{23}r_{12} + r_{13}r_{12} \Longrightarrow [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0\,, \quad (1.49)$$

where for simplicity, we have omitted the parameters dependency and we have used the property $r_{23} = -r_{32}$ to pass from the first to the second equation. Note that this property in the quantum theory comes from expanding the relation (1.47), while in the classical theory it was set by hand. Therefore, we found that at the first order in $\hbar$ the R matrix fulfils the classical Yang-Baxter equation (1.22) and we can conclude that it is given by the quantisation of the classical r-matrix.

Let us now give an example of how it is possible to construct a quantum integrable system starting from this algebraic structure. In particular, we build the Heisenberg XXX spin-chain model.
Let us specialise the discussion in the case in which the auxiliary vector space is $\mathcal{A} = \mathbb{C}^2$. The monodromy matrix becomes

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (1.50)$$

where A, B, C and D are Hilbert space operators. Thus, the transfer matrix is

$$\mathcal{T}(\lambda) = \mathrm{Tr}\, T(\lambda) = A(\lambda) + D(\lambda)\,. \quad (1.51)$$

Let us find some simple solutions of the Yang-Baxter equation together with (1.47). The first trivial solution is given by the identity matrix. Another trivial solution is the permutation matrix $\mathbb{P}$. This is defined by the action $\mathbb{P}\,(\mathcal{A}_1 \otimes \mathcal{A}_2) = \mathcal{A}_2 \otimes \mathcal{A}_1$ and in $\mathbb{C}^2 \otimes \mathbb{C}^2$ it reads

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.52)$$

The first non trivial solution is given by the combination of these two in the following way

$$R(\lambda, \mu) = R(\lambda - \mu) = (\lambda - \mu)\mathbb{I} + c\mathbb{P}\,, \quad (1.53)$$

and putting $c = i$

$$R(\lambda - \mu) = \begin{pmatrix} \lambda - \mu + i & 0 & 0 & 0 \\ 0 & \lambda - \mu & i & 0 \\ 0 & i & \lambda - \mu & 0 \\ 0 & 0 & 0 & \lambda - \mu + i \end{pmatrix}. \quad (1.54)$$

Clearly, any other matrix given by the multiplication of this one by a generic function $\mathcal{F}(\lambda, \mu)$ is a solution as well. In fact, the Yang-Baxter equation is solved up to an overall scalar factor. Plugging this expression of the R-matrix into the RTT-TTR relation (1.42) imposes some relations between the operators that compose the monodromy matrix. Developing the operator algebra set by R it is possible to find the Bethe equations. All the details about this discussion can be found in the references at the beginning of this section.
Let us focus instead on the construction of the Hilbert space and the Hamiltonian. The Heisenberg model is composed by N spin-1/2 particles in a lattice. The Hilbert space of each particle is $\mathcal{H} = \mathbb{C}^2$. First, let us define the quantum L-operator as

$$L_{a,j}(\lambda) := R_{aj}\left(\lambda - \frac{i}{2}\right) = \left(\lambda - \frac{i}{2}\right)\mathbb{I}_{aj} + i\mathbb{P}_{aj} = \begin{pmatrix} \lambda + \frac{i\sigma_j^z}{2} & i\sigma_j^- \\ i\sigma_j^+ & \lambda - \frac{i\sigma_j^z}{2} \end{pmatrix}, \quad (1.55)$$

where σ^x, σ^y and σ^z are the Pauli matrices and $\sigma^\pm = \sigma^x \pm i\sigma^y$. This is the quantum Lax operator, and it has to be interpreted as acting on $\mathcal{A} \otimes \mathcal{H}_j$, where $\mathcal{H}_j$ is the Hilbert space of the j-th particle. On the other hand, a labels the copy of the auxiliary space in which the operator acts. The last expression, where we have a 2×2 matrix, whose entries are operators acting on $\mathcal{H}_j$, makes this interpretation manifest. Since R solves the Yang -Baxter equation, if we set $T_a(\lambda) = L_{a,j}(\lambda)$, this trivially solves the RTT-TTR relation, where $T_1(\lambda) = R_{1j}(\lambda - i/2$ and $T_2(\mu) = R_{2j}(\mu - i/2)$, because it becomes equal to the Yang-Baxter. However, the quantum model obtained by this choice contains only one particle. Therefore, in order to recover the complete Hilbert space of the XXX spin-chain model with N particles, let us write the monodromy matrix as

$$T_a(\lambda) = L_{a,N}(\lambda)L_{a,N-1}(\lambda)...L_{a,1}(\lambda)\,. \tag{1.56}$$

Since each single L-operator satisfies the RTT-TTR condition and matrices acting on different spaces commute, also this monodromy matrix solves the RTT-TTR relation(1.42). Indeed,

$$\begin{aligned}
R_{12}T_1T_2 &= R_{12}L_{1,N}L_{1,N-1}\cdots L_{1,1}L_{2,N}L_{2,N-1}\cdots L_{2,1}\\
&= R_{12}L_{1,N}L_{2,N}L_{1,N-1}L_{2,N-1}\cdots L_{1,1}L_{2,1}\\
&= R_{12}L_{1,N}L_{2,N}R_{12}^{-1}R_{12}L_{1,N-1}L_{2,N-1}R_{12}^{-1}R_{12}\cdots R_{12}^{-1}R_{12}L_{1,1}L_{2,1}R_{12}^{-1}R_{12}\\
&= L_{2,N}L_{1,N}L_{2,N-1}L_{1,N-1}\cdots L_{2,1}L_{1,1}R_{12}\\
&= L_{2,N}L_{2,N-1}\cdots L_{2,1}L_{1,N}L_{1,N-1}\cdots L_{1,1}R_{12} = T_2T_1R_{12}\,.
\end{aligned}$$

Note that in general one can change the definition of the Lax operator with $L_{a,j} = R_{aj}(\lambda - w)$, $\forall w \in \mathbb{C}$. This only changes the point around which the logarithm of the transfer matrix will be expanded.
Now that we have the monodromy matrix, we can find the transfer matrix and, according to the expansion (1.37), we can construct a set of operators in involution in the following way:

$$I_k = \frac{d}{d\lambda^k}\log\mathcal{T}(\lambda)\Big|_{\lambda=i/2}\,, \tag{1.57}$$

where we are expanding around the point $\lambda = i/2$, which, as we will see, in the correct point that allows to recover the Heisenberg Hamiltonian. In particular, after some algebra, one can find

$$iI_1 = i\frac{d}{d\lambda}\log\mathcal{T}(\lambda)\Big|_{\lambda=i/2} = i\frac{d\mathcal{T}}{d\lambda}\mathcal{T}^{-1}(\lambda)\Big|_{\lambda=i/2} = \sum_{j=1}^{N}\mathbb{P}_{jj+1}\,. \tag{1.58}$$

On the other hand, the Hamiltonian of the XXX spin-chain is

$$\begin{aligned}
H &= -J\sum_{j=1}^{N}\left(\vec{S}_{j+1}\cdot\vec{S}_j - \frac{1}{4}\right) = -\frac{J}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z - 1\right)\\
&= -\frac{J}{2}\sum_{j=1}^{N}(\mathbb{P}_{jj+1} - 1) = -J\left(\frac{i}{2}I_1 - \frac{N}{2}\right)\,.
\end{aligned} \tag{1.59}$$

Therefore, the Hamiltonian commutes with all the other operators of the expansion, which can be identified with the set of independent conserved quantities in involution $\{\mathbb{Q}_k\}$ of the spin-chain.

1.3 Factorised scattering theory in two-dimensional QFTs

Let us finally discuss the integrability for (1+1)-dimensional QFTs that supports scattering processes. This is the case of the non-linear sigma model and the corresponding mirror theory that we will discuss in the next chapters. It turns out that infinitely many independent conserved charges in involution have a non-trivial physical effect on the scattering processes of the theory. Furthermore, as we shall see, in these theories, the R-matrix is related to the two-body S-matrix.
As pointed out in [43], integrability in field theory is associated with the presence of infinitely many conserved charges that transform according to higher and higher representations of the Lorentz group. This is the reason why we consider only (1+1)-theories. In fact, due to the Coleman-Mandula theorem [44], in dimensions $d > 2$, under some reasonable physical hypothesis, a theory possessing conserved charges of higher rank under the Lorentz group would necessarily have a trivial S matrix. Following

[43] and [45], let us give a physical intuition about this argument. Let Q_s be a s-th rank conserved tensor. Its action on a one-particle state in momentum space is

$$e^{icQ_s} |p\rangle = e^{icp^s} |p\rangle \ , \tag{1.60}$$

where p is the momentum of the particle. Furthermore, if we suppose that Q_s is an integral of a local current, its action on a multiparticle state will be the sum of the actions on the individual states, namely

$$e^{icQ_s} |p_1, \cdots, p_N\rangle = e^{ic(p_1^s + \cdots + p_N^s)} |p_1, \cdots, p_N\rangle \ . \tag{1.61}$$

For simplicity here we are considering the case in which all the particles have the same flavour. We will generalise to different flavours in the next subsection.
Let us consider a wavepacket localised both in coordinate and momentum space:

$$\psi(x) \propto \int_{-\infty}^{+\infty} dp \, e^{-a^2(p-p_0)^2} e^{ip(x-x_0)} \ . \tag{1.62}$$

This is a gaussian-like wavepacket localised around the value p_0 in momentum space. This means that in position space it is localised around the value for witch the phase is stationary for $p = p_0$, namely x_0. Let us now apply the operator e^{icQ_s}. According to (1.60), its action on the wavepacket gives

$$\tilde{\psi}(x) = \int_{-\infty}^{+\infty} dp \, e^{-a^2(p-p_0)^2} e^{ip(x-x_0)} e^{icp^s} \ . \tag{1.63}$$

In momentum space it is again localised around p_0. On the other hand, now the stationary condition in position space is $x - x_0 + scp_0^{s-1} = 0$. Therefore, after the action of e^{icQ_s} the particle is localised around the point $x_0 - csp_0^{s-1}$. In general, given N particles described by localised wavepackets in position $x_1, ..., x_N$ and with momenta $p_1, ..., p_N$, Q_s generates the shifts

$$x_i \longrightarrow x_i - csp_i^{s-1} \ , \qquad \forall i = 1, ..., N \ . \tag{1.64}$$

For $s = 1$ this is the standard momentum action, which translates the entire system by the same amount. On the other hand, for $s > 1$, each particle is translated by different amounts depending on its momentum.
If Q_s is a conserved charge of the model, the scattering matrix $\mathbb{S}$ commutes with it, and then

$$\langle f|\mathbb{S}|i\rangle = \langle f|e^{icQ_s} \mathbb{S} e^{-icQ_s}|i\rangle \ , \tag{1.65}$$

where $|f\rangle$ and $|i\rangle$ are respectively the final and the initial states of a scattering process. Since Q_s shifts particles in different ways, this means that the elements of the S matrix do not change if the positions of the particles are reshuffled. In three spatial dimensions, it is always possible to change the relative positions of the particles in such a way that they no longer collide with each other. This is a physical explanation of the fact that in $d > 2$, the presence of an higher-rank conserved charge, necessarily implies that the S-matrix is trivial. On the other hand, in one spatial dimension, there can be non-trivial scattering processes despite the presence of one or more high-rank charges. In fact, particles are constrained on a line, and therefore any reshuffling cannot change their fate of colliding.

In one spatial dimension, the presence of a high-rank charge does not necessarily imply that the scattering processes are trivial. However, it still imposes strong constraints on the collisions. Let us consider a $3 \longrightarrow 3$ process. In Figure 1.1, the possible scattering configurations are shown depending on the initial relative positions of the particles. In (a) and (c) the scattering is decomposed into a sequence of $2 \longrightarrow 2$ processes and the S-matrix factorised in the product of the corresponding two-body S-matrices. On the other hand, the situation (b), in which the three collisions occur simultaneously, cannot be reduced to just the product of scattering between two particles. This is what usually happens in multi-particle collisions, where non-trivial terms are added to the interactions of each individual pair. However, if the S-matrix commutes with Q_s, the three configurations can be moved one into the other and their S-matrices must be equal. Therefore, also (b) can be seen as the product of $2 \longrightarrow 2$

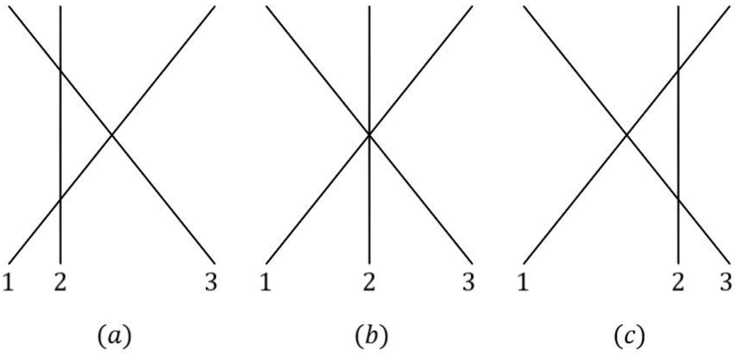


Figure 1.1: Possible configurations of a $3 \longrightarrow 3$ scattering process. The configurations (a) and (c) can be reduced to the product of three two body processes. For instance the (a) process can be see as the sequence $(12)-(13)-(23)$. On the other hand, in general (without the presence of an higher-rank charge), (b) can not be reduced to two-body scattering.

scattering events. This means that all the information is encoded into the two-body S-matrix and all the other processes are obtained by the product of these.
Finally, by imposing that (a) and (c) give the same contribution one obtain

$$\mathbb{S}(p_1,p_2)\mathbb{S}(p_1,p_3)\mathbb{S}(p_2,p_3) = \mathbb{S}(p_2,p_3)\mathbb{S}(p_1,p_3)\mathbb{S}(p_1,p_2)\,, \tag{1.66}$$

This is again the Yang-Baxter equation. In this context, as pointed out above, it arises imposing that $3 \longrightarrow 3$ processes do not depend on the order in which the different two-body scattering events occur. On the other hand, in the previous section, we found that this equation arises from the consistency condition on the permutations of an algebraic structure. We will come back soon to this expression, showing that also in this case it is related to the permutation relations of an algebra, known as the Zamolodchikov-Faddeev (ZF) algebra.

Before discussing this algebra, let us first consider the other physical conditions that arise from the integrability of a $(1+1)$-dimensional QFT. In fact, even though in $1+1$ dimensions the Coleman-Mandula theorem is no longer valid and the scattering processes are not necessary trivial, the presence of an infinite set of conserved quantities with different ranks severely constrains the structure of the S-matrix. Let us discuss how.
Let $\{\mathbb{Q}_k\}$, $k=1,\ldots,\infty$ be a set of independent conserved charges in involution. As already pointed out, since they commute to each other, there exists a basis that simultaneously diagonalises all of them. Thus,

$$\mathbb{Q}_k\,|p\rangle = \mathbf{Q}_k(p)\,|p\rangle\ , \qquad \forall k\,. \tag{1.67}$$

Let us consider a $n \longrightarrow m$ scattering event and write the asymptotic initial and final states as

$$|p_1,\cdots,p_n\rangle^{(in)}_{\alpha_1,\cdots,\alpha_n}\ , \qquad |\tilde{p}_1,\cdots,\tilde{p}_m\rangle^{(out)}_{\beta_1,\cdots,\beta_m}\ , \tag{1.68}$$

where α_j and β_k label the flavours of the particles. The asymptotic initial state is prepared at the time $\tau=-\infty$ when all particles are infinitely separated. During the evolution, the particles scatter each other until they become again infinitely separated in the final state at $\tau=+\infty$. The conservation of the charges implies that the eigenvalues of each of them are equal in the initial and in the final state, i.e.

$$\sum_{i=1}^{n}\mathbf{Q}_k(p_i;\alpha_i) = \sum_{j=1}^{m}\mathbf{Q}_k(\tilde{p}_j;\beta_j)\,, \qquad \forall k\,. \tag{1.69}$$

Thus, there are infinite many constraints. The only solution is given when $n=m$ and the set of initial momenta $\{p_i\}$ is equal to the set of final momenta $\{\tilde{p}_j\}$. Therefore, in two-dimensional integrable QFTs collision processes are such that the number of particles in the initial state is always equal to the number of particles in the final state as well as their set of momenta. In other words, scattering is reduced to the exchange of momenta (at least when the two particles have the same mass and dispersion relation) between the particles involved.

1.3.1 Zamolodchikov-Faddeev algebra

In the previous section, we saw that the two-body S matrix of a two-dimensional integrable QFT obeys the Yang-Baxter equation. We also pointed out that, as seen in the quantum mechanical case, this equation is the consequence of a consistency relation of the permutations in some algebra. Furthermore, imposing the conservation of infinitely many charges, we found that scattering processes correspond to the exchange (permutation) of the momenta of the particles in the flavour space. Therefore, by combining these two results, we expect that the Yang-Baxter equation for the S-matrix comes from the permutation relations in the algebra defined by the creation and annihilation operators of the asymptotic states. This is known as the Zamolodchikov-Faddeev (ZF) algebra and was first introduced in [46] and [47]. Let us discuss the ZF algebra following [48] and [49].

Let $A^\dagger_\alpha(p)$ and $A^\alpha(p)$ be the creation and annihilation operators. $A^\dagger_\alpha(p)$ acts on the vacuum state of the theory $|\Omega\rangle$ by creating a one-particle state of momentum p and flavour labelled by the index α. On the other hand, $A^\alpha(p)$ is the hermitian conjugate of the creation operator and annihilates the vacuum.

Therefore, the asymptotic *in* and *out* states can be written in the following way

$$\begin{aligned} |p_1,\dots,p_n\rangle^{(in)}_{\alpha_1,\dots,\alpha_n} &= A^\dagger_{\alpha_1}(p_1)\dots A^\dagger_{\alpha_n}(p_n)\,|\Omega\rangle \ , \\ |p_1,\dots,p_n\rangle^{(out)}_{\beta_1,\dots,\beta_n} &= (-1)^{\sum_{k<l}\epsilon_{\alpha_k}\epsilon_{\beta_l}} A^\dagger_{\beta_n}(p_n)\dots A^\dagger_{\beta_1}(p_1)\,|\Omega\rangle \ , \end{aligned} \tag{1.70}$$

where $p_1 > p_2 > \dots > p_n$. Let x_j be the position in space of the particle with momentum p_j. Clearly, to properly define the position, one should smear the momenta and consider wavepackets. In the *in* state the particles are sufficiently separated that they do not feel any interaction between each other and are located as follows : $x_1 < x_2 < \dots < x_n$. This allows the particles to become closer and closer during the motion, and there is a sequence of $\frac{1}{2}n(n-1)$ collisions. After sufficient time, the interactions become again negligible and the system reaches the *out* state in which $x_1 > x_2 > \dots > x_n$. In (1.70) the order of the operators reflects the order of the particles in the position space. If in the initial state the particles are not positioned in the order discussed earlier, the process can be seen as a sequence of independent processes with that order, and the S-matrix is just the product of the single matrices. For this reason, it is sufficient to discuss *in* states with the configuration (1.70). On the other hand, the final state can only have the configuration discussed above. In fact, any other configurations cannot be the final state because further collisions will necessarily occur in a finite amount of time.

The parameters ϵ_α in (1.70) are defined in such a way that $\epsilon_\alpha = 0$ when α labels a boson and $\epsilon_\alpha = 1$ when α labels a fermion. In fact, in the free interaction limit, the ZF operators become the usual *in* and *out* free creation and annihilation operators, and the term $(-1)^{\sum_{k<l}\epsilon_{\alpha_k}\epsilon_{\alpha_l}}$ takes into account the passage from $a^{\dagger\, out}_{\alpha_1}(p_1)\dots a^{\dagger\, out}_{\alpha_n}(p_n) \to a^{\dagger\, out}_{\alpha_n}(p_n)\dots a^{\dagger\, out}_{\alpha_1}(p_1)$.

The S-matrix interpolates between the *in* and *out* states

$$|p_1,\dots p_n\rangle^{(in)}_{\alpha_1,\dots,\alpha_n} = \mathbb{S}^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}(p_1,\dots,p_n)\,|p_1,\dots p_n\rangle^{(out)}_{\beta_1,\dots,\beta_n} \ , \tag{1.71}$$

where we have used the fact that the number of particles and the set of momenta are the same before and after the scattering. In particular, since the S-matrix is factorised in the product of two-body processes, let us consider

$$\begin{aligned} &|p_1,p_2\rangle^{(in)}_{\alpha_1,\alpha_2} = \mathbb{S}^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p_1,p_2)\,|p_1,p_2\rangle^{(out)}_{\alpha_3,\alpha_4} \\ &\Longrightarrow A^\dagger_{\alpha_1}(p_1)A^\dagger_{\alpha_2}(p_2)\,|\Omega\rangle = \mathbb{S}^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p_1,p_2)(-1)^{\epsilon_{\alpha_3}\epsilon_{\alpha_4}}A^\dagger_{\alpha_4}(p_2)A^\dagger_{\alpha_3}(p_1)\,|\Omega\rangle \ . \end{aligned} \tag{1.72}$$

Therefore, defining the matrix

$$R^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p_1,p_2) := \mathbb{S}^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p_1,p_2)(-1)^{\epsilon_{\alpha_3}\epsilon_{\alpha_4}} \ , \tag{1.73}$$

we find the commutation relation between the ZF creation operators

$$A^\dagger_{\alpha_1}(p_1)A^\dagger_{\alpha_2}(p_2) = A^\dagger_{\alpha_4}(p_2)A^\dagger_{\alpha_3}(p_1)R^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p_1,p_2)\,. \tag{1.74}$$

Let us simplify the notation of the flavour indices. We have already introduced the canonical base of the vector space of $N\times N$ matrices $E_i^{\ j}$. This can be decomposed into the tensor product of two canonical bases of a N-dimensional vector space $E_i^{\ j} = E_i \otimes E^j$. Let N be the number of flavours present in the theory, using these vectors, we can introduce the notation

$$\mathbf{A}^\dagger = A^\dagger_\alpha E^\alpha \,, \quad \mathbf{A} = A^\alpha E_\alpha \,, \tag{1.75}$$

and

$$R(p_1, p_2) = R^{\alpha_3;\alpha_4}_{\alpha_1,\alpha_2} E_{\alpha_3}{}^{\alpha_1} \otimes E_{\alpha_4}{}^{\alpha_2} \,, \tag{1.76}$$

where we are summing over repeated indices.
Using this notations, the complete ZF algebra can be written as

$$\mathbf{A}^\dagger_1 \mathbf{A}^\dagger_2 = \mathbf{A}^\dagger_2 \mathbf{A}^\dagger_1 R_{12} \,, \quad \mathbf{A}_1 \mathbf{A}_2 = R_{12} \mathbf{A}_2 \mathbf{A}_1 \,, \quad \mathbf{A}_1 \mathbf{A}^\dagger_2 = \mathbf{A}^\dagger_2 R_{21} \mathbf{A}_1 + \delta(p_1 - p_2) \mathbb{I} \tag{1.77}$$

The subscripts as usual denote the vector spaces on which each operator acts non trivially.
This is a deformation of the free oscillator algebra. In fact, now it is possible to explicitly see that in the free limit ($\mathbb{S} = \mathbb{I}$) these become the usual canonical commutation relations for bosons and fermions. Applying the commutation relation of $\mathbf{A}^\dagger$ (1.77) twice, where the second time we exchange the indices $1 \leftrightarrow 2$, we find

$$\mathbf{A}^\dagger_1 \mathbf{A}^\dagger_2 = \mathbf{A}^\dagger_2 \mathbf{A}^\dagger_1 R_{12} = \mathbf{A}^\dagger_1 \mathbf{A}^\dagger_2 R_{21} R_{12} \,, \tag{1.78}$$

and we obtain the analogous of (1.47), which in this case is known as *unitarity condition*

$$R_{12}(p_1, p_2) R_{21}(p_2, p_1) = \mathbb{I} \,. \tag{1.79}$$

On the other hand, considering the $3 \longrightarrow 3$ scattering, this can be obtained by permuting the creation operators in two different ways

$$\mathbf{A}^\dagger_1 \mathbf{A}^\dagger_2 \mathbf{A}^\dagger_3 = \mathbf{A}^\dagger_3 \mathbf{A}^\dagger_2 \mathbf{A}^\dagger_1 R_{12} R_{13} R_{23} \,, \quad \mathbf{A}^\dagger_1 \mathbf{A}^\dagger_2 \mathbf{A}^\dagger_3 = \mathbf{A}^\dagger_3 \mathbf{A}^\dagger_2 \mathbf{A}^\dagger_1 R_{23} R_{13} R_{12} \,, \tag{1.80}$$

and imposing that the two right-hand sides are equal we obtain the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \,. \tag{1.81}$$

This is a generalisation of the result obtained in (1.66), where now the theory contain more than one type of particle, including fermions.

In conclusion, in two-dimensional integrable QFTs, scattering processes are characterised by very specific properties, namely

- The number of particles in the initial and final state are always the same. Therefore, any production process is forbidden ;
- The set of initial and final momenta are equal ;
- The two-body S-matrix obeys the Yang-Baxter equation (1.81) ;

Furthermore, the knowledge of the two-body S-matrix is sufficient to reconstruct the complete matrix. In the next chapters we will study these properties for the worldsheet scattering in the $AdS_3 \times S^3 \times T^4$ non-linear sigma model and its mirror model in perturbation theory.

Chapter 2

Bosonic String

Before analysing the non-linear sigma model (NLSM) on $AdS_3 \times S^3 \times T^4$ let us start by discussing the free bosonic string propagating in a flat D-dimensional space-time. We will then generalise to curved backgrounds. In this chapter, we will follow [50], [51] and [52].

2.1 Point particle

First of all, let us start by considering a point particle freely moving in a D-dimensional Minkowski space-time.
Let $\eta_{\mu\nu}$ be the Minkowski metric defined as

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \dots, 1)\,, \tag{2.1}$$

and X^μ be the coordinates, with $\mu = 0, 1, \dots, D-1$.
The well-known covariant equations of motion are

$$\frac{dP^\mu}{d\tau} = 0\,, \tag{2.2}$$

where P^μ is the four-momentum and τ the proper time. In particular, in the massive case these becomes

$$m\frac{d^2X^\mu}{d\tau^2} = 0\,, \tag{2.3}$$

where m the mass of the particle. To be more generic, we can parameterise the worldline of the particle as $X^\mu = X^\mu(\lambda)$. As a consequence of this parameterisation, the proper time can be rewritten as

$$d\tau = \sqrt{-ds^2} = \sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}d\lambda\,, \tag{2.4}$$

where $\dot{X}^\mu = \frac{dX^\mu}{d\lambda}$.
Therefore (2.3) becomes

$$\frac{d}{d\lambda}\left(\frac{m\dot{X}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\right) = 0\,. \tag{2.5}$$

Let $X^\mu(\lambda_i)$ and $X^\mu(\lambda_f)$ be the initial and the final points of the trajectory; it can be seen that the action that gives the EOM (2.5) is

$$S = -m\int_{\lambda_i}^{\lambda_f} d\lambda\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}\,. \tag{2.6}$$

From the physical point of view, the extremisation of this functional selects the trajectory which maximises the proper time.

Nevertheless, this action is not defined for massless particles. Therefore, in order to generalise this expression, let us introduce a metric $h_{\lambda\lambda}$ on the worldline whose line element reads

$$ds^2 = h_{\lambda\lambda}(\lambda)d\lambda^2 = -e^2(\lambda)d\lambda^2 \,. \tag{2.7}$$

We have also introduced the field $e(\lambda)$, which is a 1-form and will be useful for rewriting the action in a more manageable way. Clearly, being a one-form, under the change of coordinates $\tilde{\lambda} = \tilde{\lambda}(\lambda)$ it transforms as follows

$$\tilde{e}(\tilde{\lambda}) = \left(e\frac{d\lambda}{d\tilde{\lambda}}\right)(\tilde{\lambda}) \,. \tag{2.8}$$

Now, using this metric, it is possible to define the action

$$S = -\frac{1}{2}\int_{\lambda_i}^{\lambda_f} d\lambda\sqrt{-h_{\lambda\lambda}}(h_{\lambda\lambda}^{-1}\dot{X}^\mu\dot{X}_\mu + m^2) \,. \tag{2.9}$$

This action, as we expected since the physics cannot depend on the parameter chosen to describe the the worldline, is invariant under reparameterisation. As a consequence, this theory has a gauge symmetry which needs to be fixed by a proper gauge choice. In other worlds, because of the reparametrisation invariance, not all the D degrees of freedom are physical, as can be seen by imposing, for instance, the gauge choice $\lambda = t$, such that

$$X^0(\lambda) = t \,, \tag{2.10}$$

where t is the time coordinate of the Minkowski space.
This action, as we will see later, is very similar to the Polyakov action for the bosonic string; therefore, we will come back to this expression and to the corresponding gauge symmetry soon. For now, let us rewrite the action in a more convenient way using the field $e(\lambda)$

$$S = \frac{1}{2}\int_{\lambda_i}^{\lambda_f} d\lambda(e^{-1}\dot{X}^\mu\dot{X}_\mu - m^2 e) \,. \tag{2.11}$$

Note that, by exploiting the equation of motion, in the massive case the additional degree of freedom $e(\lambda)$ is completely fixed in terms of X^μ. In fact,

$$e(\lambda) = \frac{\sqrt{-\dot{X}^\mu\dot{X}_\mu}}{m} \,. \tag{2.12}$$

Replacing this expression in the action (2.11) one can recover the expression (2.6), showing that the two actions are equivalent for $m \neq 0$. Furthermore, let us consider the conjugate momentum in the massive case and check that we obtain the correct result

$$P^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = e^{-1}\dot{X}^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\eta_{\rho\nu}\dot{X}^\rho\dot{X}^\nu}} \,. \tag{2.13}$$

Now that we are sure that the action correctly describes massive particles, we can consider the case $m = 0$. In this case, the EOM for $e(\lambda)$ reads

$$\dot{X}^\mu\dot{X}_\mu = 0 \,, \tag{2.14}$$

which is the null trajectories condition. Now $e(\lambda)$ is not fixed, but it can assume any value.
Finally, the EOM for X^μ are given by

$$\frac{d}{d\lambda}\left(e^{-1}\dot{X}^\mu\right) = \frac{dP^\mu}{d\lambda} = 0 \,, \tag{2.15}$$

showing that (2.11) gives the correct description of a freely moving point particle in a D dimensional Minkowski space-time. This action can be generalised to a curved space-time by replacing $\eta_{\mu\nu}$ with a generic Lorentzian metric $G_{\mu\nu}$.

2.2 Nambu-Goto Action

We have seen how to describe the motion of a free point particle in space-time. It is described by a one-dimensional curve called worldline. If, instead, we have a string, during its motion it sweeps a two-dimensional surface, known as *worldsheet*.
The worldsheet can be parameterised by two parameters, called *worldsheet coordinates* which are usually labelled by σ which is the space-like coordinate and τ which is the time-like coordinate, while the Minkowski space (or the generalised curved background) is referred to as the *target spacetime*. Let X^μ be the coordinates of the target space-time, then the string motion is described by the map

$$X^\mu = X^\mu(\sigma, \tau)\,. \tag{2.16}$$

In the following we will use the compact notation $\sigma^\alpha = (\sigma, \tau), \alpha = 0, 1$.
Let $-r \leq \sigma \leq r$ and $\tau \in \mathbb{R}$; there are two types of strings, namely:

- Open strings: the two ends do not coincide and therefore they have the topology of an interval ;
- Closed strings: the two ends coincide and therefore they have the topology of a circle, i.e. $X^\mu(\sigma+2r, \tau) = X^\mu(\sigma, \tau)$. The worldsheet swept during the motion is a cylinder of circumference $2r$.

Note that σ can be seen as an angle variable. Therefore, it can be rescaled to $0 \leq \sigma \leq \pi$ in the open case and to $0 \leq \sigma \leq 2\pi$ in the closed case. Thus, the periodicity condition can be rewritten as $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$. In the following, we will focus on closed strings.

We want to find an action whose EOM describes a free string propagating in a D-dimensional Minkowski space. This should be invariant under worldsheet reparameterizations, because these do not change the physics of the system. In order to find this action, let us recall the point particle case. In this case we found that the action is proportional to the integral of the length of the worldline (i.e. $\sqrt{-ds^2}$). Therefore, generalising this result, the action should be proportional to the integral of the area of the worldsheet.
In order to find the area of the worldsheet, let us use the pull-back to induce the metric of the target space (in this case the Minkowski metric) on the worldsheet

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}\,, \tag{2.17}$$

where $\gamma_{\alpha\beta}$ is the induced worldsheet metric.
Thus, we can write the so-called Nambu-Goto action

$$S = -T\int dA = -T\int d^2\sigma\sqrt{-\det\left(\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}\right)}\,, \tag{2.18}$$

where T is the string tension. This action, as expected, is invariant under

- Poincare' transformation of the target spacetime.
- Reparameterisation of the worldsheet.

The difference between these two symmetries is that the first one is global while the second one is local and is the manifestation of a gauge redundancy of the degrees of freedom.
We can rewrite the Nambu-Goto action by using the explicit expression

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'\cdot X' \end{pmatrix}\,, \tag{2.19}$$

where $\cdot$ denotes the scalar Minkowski product, $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ and $X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma}$.
Therefore, the action becomes

$$S = -T\int d^2\sigma\sqrt{-(\dot{X})^2(X')^2 + (\dot{X}\cdot X')^2}\,, \tag{2.20}$$

where it has been used the notation $\dot{X}\cdot\dot{X} = (\dot{X})^2$.

2.2.1 String tension

As we mentioned above, the parameter T is called string tension. In the following we show that, in fact, it has the physical meaning of energy per unit of length.
Let us move to the gauge $X^0 = \tau$. In this way we are left with $D-1$ d.o.f, namely $\vec{X} = (X^1, X^2, ..., X^{D-1})$
In order to find the potential energy we put the kinetic term equal to zero , i.e. $\frac{d\vec{X}}{d\tau} = 0$.
Then, the action becomes

$$S = -T \int d\sigma d\tau \sqrt{\left(\frac{d\vec{X}}{d\sigma}\right)^2} \,. \tag{2.21}$$

Integrating in σ we obtain

$$S = -\int d\tau V = -T \int d\tau \int d\sigma \frac{d\vec{X}}{d\sigma} = -T \int d\tau L \,, \tag{2.22}$$

and therefore

$$V = TL \,, \tag{2.23}$$

where V is the potential energy and L the length. Thus, the string tension is exactly the energy of the string per unit of length. As a consequence of this fact, the minimum energy is obtained when the string length approaches zero, and it becomes a point particle. Nevertheless, this situation is avoided due to quantum effects.

2.2.2 Equations of motion

In this section we find the EOM of the bosonic string. First of all, let us define the conjugate momenta

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-(\dot{X})^2 (X')^2 + (\dot{X} \cdot X')^2}} \,; \tag{2.24}$$

then, the equations of motion read

$$\frac{dP_\mu}{d\tau} + \frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial X^{\mu\prime}}\right) = 0 \,, \tag{2.25}$$

where

$$\frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} == -T \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-(\dot{X})^2 (X')^2 + (\dot{X} \cdot X')^2}} \,. \tag{2.26}$$

These equations can be recast in a more compact form. In fact, by exploiting the formula of the variation of the determinant

$$\delta \sqrt{-\det \gamma} = \frac{1}{2} \sqrt{-\det \gamma}\, \gamma^{\alpha\beta} \delta \gamma_{\alpha\beta} \,, \tag{2.27}$$

and computing the variation of the matrix with respect to X^μ_α, where $X^\mu_0 = \dot{X}^\mu$ and $X^\mu_1 = X^{\mu\prime}$

$$\delta\gamma = \frac{\delta\gamma}{\delta X^\mu_\alpha} \delta X^\mu_\alpha = (\partial_\beta X_\mu)(\partial_\alpha \delta X^\mu) \,, \tag{2.28}$$

we can finally rewrite the EOM in the following way

$$\partial_\alpha (\sqrt{-\det \gamma} \gamma^{\alpha\beta} \partial_\beta X^\mu) = 0 \,. \tag{2.29}$$

Note that the Nambu-Goto action, because of the presence of the square root is difficult to quantise. For this reason, it is useful to search for an equivalent Lagrangian (a Lagrangian that gives rise to the same EOM (2.29)), which does not contain any square root.

2.3 Polyakov Action

An equivalent action can be found by introducing a dynamical worldsheet metric $h_{\alpha\beta}$ and generalising the point particle case (2.9). In this way we can write the so called Polyakov action,

$$S = -\frac{T}{2}\int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma \sqrt{-h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,, \tag{2.30}$$

where $\sqrt{-h} = \sqrt{-\det h}$. Now the worldsheet metric is not induced by the target one, but it is a dinamical field by itself, which evolves according to its EOM.

The variation with respect to the worldsheet metric gives the stress-energy tensor

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu\,, \tag{2.31}$$

and the equation of motions read

$$\partial_\alpha(\sqrt{-\det h}h^{\alpha\beta}\partial_\beta X^\mu) = 0\,, \tag{2.32}$$

$$T_{\alpha\beta} = 0\,. \tag{2.33}$$

The first equation (2.32) is equal to the one obtained by the Nambu-Goto action (2.29), with the only difference that instead of having the induced metric $\gamma_{\alpha\beta}$, we have $h_{\alpha\beta}$. Solving the second equation (2.33) we find the expression of the metric

$$h_{\alpha\beta} = 2(h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu)^{-1}\partial_\alpha X^\nu \partial_\beta X_\nu = f(\sigma,\tau)\gamma_{\alpha\beta}\,, \tag{2.34}$$

where $f(\sigma,\tau) = 2(h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu)^{-1}$. Therefore, $h_{\alpha\beta}$ is given by $\gamma_{\alpha\beta}$ times a function of the worldsheet coordinates. When this expression is plugged back into the equations of motion of X^μ (2.32), we obtain

$$0 = \partial_\alpha\left(\sqrt{-\det h}h^{\alpha\beta}\partial_\beta X^\mu\right) = \partial_\alpha\left(f(\sigma,\tau)\sqrt{-\det\gamma}f^{-1}(\sigma,\tau)\gamma^{\alpha\beta}\partial_\beta X^\mu\right) = \partial_\alpha\left(\sqrt{-\det\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu\right)\,. \tag{2.35}$$

Hence we are left with the same equation obtained by the the Nambu-Goto action and thus it has been proved that the two actions are equivalent.

2.3.1 Symmetries

As the Nambu-Goto action, the Polyakov action is invariant under global Poincaré transformations

$$X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu + b^\mu\,, \tag{2.36}$$

and has the same gauge symmetry due to the worldsheet reparametrisation

$$\sigma^\alpha \longrightarrow \tilde{\sigma}^\alpha \quad , \quad h_{\alpha\beta} \longrightarrow \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}h_{\gamma\delta}\,. \tag{2.37}$$

In addition, there is another symmetry, the so-called *Weyl invariance*

$$h_{\alpha\beta} \longrightarrow \Omega(\sigma,\tau)h_{\alpha\beta}\,. \tag{2.38}$$

Note, that this symmetry is the reason why we have been able to recover the Nambu-Goto equation of motion from the Polyakov action. Furthermore, let us note that this invariance holds only in two dimensions.

2.3.2 Conformal Gauge

As seen above, the Polyakov action has two local symmetries, namely the change of worldsheet coordinate and the Weyl invariance. The latter is a local rescaling of the metric that preserves the angles. Using these two invariances, one can fix the form of the metric.

In particular, note that the metric $h_{\alpha\beta}$ is a 2×2 symmetric matrix, therefore it has 3 degrees of freedom. Moreover, the reparametrisation invariance gives two gauge conditions. Hence we are left with only one degree of freedom and the metric can be rewritten in the following form

$$h_{\alpha\beta} = e^{\phi}\eta_{\mu\nu} \,, \tag{2.39}$$

where $\phi = \phi(\sigma, \tau)$ is a generic function of the woordsheet coordinate and $\eta_{\mu\nu}$ is the 2-dimensional Lorentz matrix.
Finally, by exploiting the Weyl invariance, the overall factor can be set to one and we are left with

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.40}$$

This is the *conformal gauge* and shows that it is always possible to set the worldsheet metric as the flat one.
Nevertheless, it is worth pointing out tht after this choice, there is still a residual gauge.

In order to show the presence of this residual gauge, let us consider an infinitesimal worldsheet coordinate transformation

$$\sigma^{\alpha} \to \sigma^{\alpha} - \xi^{\alpha}(\sigma, \tau) \,; \tag{2.41}$$

the corresponding infinitesimal transformations induced on the fields and the metric are

$$\begin{aligned} &\delta X^{\mu} = \xi^{\alpha}\partial_{\alpha}X^{\mu}, \\ &\delta h_{\alpha\beta} = \xi^{\gamma}\partial_{\gamma}h_{\alpha\beta} + h_{\alpha\gamma}\partial_{\beta}\xi^{\gamma} + h_{\beta\gamma}\partial_{\alpha}\xi^{\gamma}, \\ &\delta(\sqrt{-h}) = \partial_{\alpha}(\xi^{\alpha}\sqrt{-h}) \,. \end{aligned} \tag{2.42}$$

On the other hand, an infinitesimal Weyl transformation can be written as

$$\delta h_{\alpha\beta} = \Omega(\sigma, \tau)h_{\alpha\beta}. \tag{2.43}$$

Combining both the transformations there is the residual gauge condition

$$\xi^{\gamma}\partial_{\gamma}h_{\alpha\beta} + h_{\alpha\gamma}\partial_{\beta}\xi^{\gamma} + h_{\beta\gamma}\partial_{\alpha}\xi^{\gamma} = \Omega h_{\alpha\beta} \,, \tag{2.44}$$

and considering the conformal gauge choice ($h_{\alpha\beta} = \eta_{\alpha\beta}$) in (2.44) we find

$$\begin{aligned} &\partial_{\sigma}\xi^{\tau} - \partial_{\tau}\xi^{\sigma} = 0 \,, \\ &2\partial_{\tau}\xi^{\tau} = \Omega \,, \\ &2\partial_{\sigma}\xi^{\sigma} = \Omega \,. \end{aligned} \tag{2.45}$$

Finally, subtracting the last two equations we obtain

$$\begin{aligned} &\partial_{\sigma}\xi^{\tau} - \partial_{\tau}\xi^{\sigma} = 0 \,, \\ &\partial_{\sigma}\xi^{\sigma} - \partial_{\tau}\xi^{\tau} = 0 \,. \end{aligned} \tag{2.46}$$

Now it is useful to introduce the lightcone coordinates on the worldsheet

$$\sigma^{\pm} = \tau \pm \sigma \,. \tag{2.47}$$

In these coordinates the residual gauge conditions (2.46) become

$$\partial_{+}\xi^{-} = \partial_{-}\xi^{+} = 0 \,. \tag{2.48}$$

Therefore, after fixing the conformal gauge we still have the freedom of performing a generic coordinate redefinition of the type

$$\sigma^+ \longrightarrow \tilde{\sigma}^+(\sigma^+) \quad , \quad \sigma^- \longrightarrow \tilde{\sigma}^-(\sigma^-)\,, \tag{2.49}$$

which leave the conformal choice invariant. Looking at this condition from another point of view, given the flat metric in lightcone coordinates

$$ds^2 = -d\sigma^+ d\sigma^-\,, \tag{2.50}$$

a transformation of the type (2.49) leaves the metric invariant except for an overall factor which can be reabsorbed by a Weyl transformation.
Clearly, this residual gauge condition needs to be fixed. We will discuss how to fix it soon. For now, let us make a final comment that will be useful in order to make the gauge choice. If we write the change of coordinate (2.49) in the form

$$\begin{aligned} \tau \longrightarrow \tilde{\tau} &= \frac{1}{2}(\tilde{\sigma}^+(\tau+\sigma) + \tilde{\sigma}^-(\tau-\sigma))\,, \\ \sigma \longrightarrow \tilde{\sigma} &= \frac{1}{2}(\tilde{\sigma}^+(\tau+\sigma) - \tilde{\sigma}^-(\tau-\sigma))\,, \end{aligned} \tag{2.51}$$

it is clear that both $\tilde{\tau}$ and $\tilde{\sigma}$ are solutions of the free wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)\tilde{\tau} = 0\,, \quad \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)\tilde{\sigma} = 0\,. \tag{2.52}$$

This means that as residual gauge freedom we can choose any solution of the free wave equation to be one of the worldsheet coordinates.

2.3.3 Equations of motion and constraints

In the conformal gauge the action becomes

$$S = -\frac{T}{2}\int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma \partial_\alpha X \cdot \partial^\alpha X\,, \tag{2.53}$$

and the equations of motion read

$$\Box X^\mu = 0\,. \tag{2.54}$$

In this way, the theory seems to be equivalent to free bosonic particles. However, we still need to consider the equations of motion of the metric, which provide a constraint for the field X^μ.

In particular, the stress-energy tensor in the conformal gauge becomes

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X \cdot \partial_\beta X - \frac{1}{4}\eta_{\alpha\beta}\partial_\gamma X \cdot \partial^\gamma X\,. \tag{2.55}$$

Therefore, the condition $T_{\alpha\beta} = 0$ gives

$$\begin{aligned} &\dot{X} \cdot X' = 0\,, \\ &\frac{1}{2}(\dot{X}^2 + {X'}^2) = 0\,. \end{aligned} \tag{2.56}$$

These are the *Virasoro constrains* in the flat target spacetime.
In order to solve the equations of motion and the constraints, it is useful to move to the light-cone coordinates (2.47) on the worldsheet. Changing the coordinates, the action (2.53) reads

$$S = T\int d\sigma^+ d\sigma^- \partial_+ X^\mu \partial_- X_\mu\,, \tag{2.57}$$

and the equation of motion (2.54) becomes

$$\partial_+\partial_- X^\mu = 0\,, \tag{2.58}$$

where $\partial_\pm = \partial/\partial\sigma^\pm$. This equation is solved by a generic function of only σ^+ or a generic function of only σ^-. Note that $\sigma^+ = const$ describes a particle moving on the string in the negative verse (i.e. to the left), while $\sigma^- = const$ describes a particle moving on the string in the positive verse (i.e. to the right). Therefore, the general solution can be written in terms of a right-moving and a left-moving mode

$$X^\mu(\sigma^+,\sigma^-) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)\,. \tag{2.59}$$

In addition, in the close string case there is the periodicity condition

$$X^\mu(\sigma+2\pi,\tau) = X^\mu(\sigma,\tau)\,. \tag{2.60}$$

The periodicity allows to write the solutions in the Fourier base in the following way

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^+ + i\sqrt{\frac{1}{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^- + i\sqrt{\frac{1}{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}\,, \end{aligned} \tag{2.61}$$

where x^μ and p^μ are respectively the position and the momentum of the string center of mass. Note that the single expressions X_L and X_R do not respect the periodicity condition because of the linear terms in σ^+ and σ^-, respectively. However, when they are summed, the periodicity is recovered. Furthermore, the reality of the field X^μ implies

$$\tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^* \quad , \quad \alpha_n^\mu = (\alpha_{-n}^\mu)^*\,. \tag{2.62}$$

The constraints coming from the equation of motions of the metric $T_{\alpha\beta} = 0$ in the light-cone coordinates read

$$\begin{aligned} (\partial_+ X)^2 &= 0\,, \\ (\partial_- X)^2 &= 0\,. \end{aligned} \tag{2.63}$$

Let us evaluate them using the Fourier expansion (2.61). First we compute

$$\begin{aligned} \partial_+ X^\mu &= \frac{1}{4\pi T}p^\mu + \sqrt{\frac{1}{4\pi T}}\sum_{n\neq 0}\tilde{\alpha}_n^\mu e^{-in\sigma^+}\,, \\ \partial_- X^\mu &= \frac{1}{4\pi T}p^\mu + \sqrt{\frac{1}{4\pi T}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^-}\,, \end{aligned} \tag{2.64}$$

and defining

$$\alpha_0^\mu := \sqrt{\frac{1}{4\pi T}}\,p^\mu \quad , \quad \tilde{\alpha}_0^\mu := \sqrt{\frac{1}{4\pi T}}\,p^\mu\,, \tag{2.65}$$

as the zero mode coefficients, the constraints are given by

$$\begin{aligned} (\partial_+ X)^2 &= \frac{1}{4\pi T}\sum_{\substack{n\in\mathbb{Z}\\ m\in\mathbb{Z}}}\tilde{\alpha}_n\cdot\tilde{\alpha}_m e^{-i(m+n)\sigma^+} = 0\,, \\ (\partial_- X)^2 &= \frac{1}{4\pi T}\sum_{\substack{n\in\mathbb{Z}\\ m\in\mathbb{Z}}}\alpha_n\cdot\alpha_m e^{-i(m+n)\sigma^-} = 0\,. \end{aligned} \tag{2.66}$$

These expressions can be recast in the more compact form

$$\begin{aligned} (\partial_+ X)^2 &= \frac{1}{2\pi T}\sum_{n\in\mathbb{Z}}\tilde{L}_n e^{-in\sigma^+} = 0\,, \\ (\partial_- X)^2 &= \frac{1}{2\pi T}\sum_{n\in\mathbb{Z}}L_n e^{-in\sigma^-} = 0\,, \end{aligned} \tag{2.67}$$

where

$$\tilde{L}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m \quad , \quad L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m \, , \tag{2.68}$$

and finally, these are solved by

$$\tilde{L}_n = L_n = 0 \, , \quad \forall n \in \mathbb{Z} \, . \tag{2.69}$$

Let us recall that we have defined the zero oscillator modes (2.65) in such a way that they are equal to each other $\alpha_n^0 = \tilde{\alpha}_n^0$ and are proportional to the momentum of the string center of mass. Therefore, using the mass-shell condition, they give the effective mass of the string:

$$\begin{aligned} m^2 = -p_\mu p^\mu &= -(4\pi T)\alpha_0\cdot\alpha_0 \\ &= -(4\pi T)\tilde{\alpha}_0\cdot\tilde{\alpha}_0 \, . \end{aligned} \tag{2.70}$$

Finally, exploiting the Virasoro constraints (2.69) we obtain

$$m^2 = 8\pi T\sum_{n>0}\tilde{\alpha}_n\cdot\tilde{\alpha}_{-n} = 8\pi T\sum_{n>0}\alpha_n\cdot\alpha_{-n} \tag{2.71}$$

This relation between the $\tilde{\alpha}_n$ and α_n modes is known as *level matching.*

2.4 Quantisation

There are two main ways of quantising this theory:

- The covariant quantisation. The theory is quantised in terms of the target coordinates X^μ, that are enforced to respect the usual canonical commutation relations (CCR), namely

  $$[X(^\mu\sigma,\tau), P^\nu(\sigma',\tau)] = i\delta(\sigma-\sigma')\eta^{\mu\nu} \, ,$$

  $$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = 0 \, .$$

  The physical states of the theory are then obtained by restricting the Fock space to the space that respect the Virasoro constraint conditions. This approach is manifestly covariant and involves the presence of ghost states.

- The light-cone quantisation. In this approach, the residual gauge freedom is fixed. This allows to solve the Virasoro constraints before quantising the theory. In order to proceed with the gauge fixing, a useful choice is to introduce the lightcone coordinates also on the target spacetime:

  $$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \, . \tag{2.72}$$

  The equations of motion of these fields are the usual free wave equations

  $$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)X^+ = 0 \quad , \quad \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)X^- = 0 \, . \tag{2.73}$$

  On the other hand, we have seen that one of the worldsheet coordinates can be fixed to be a general solution of the free wave equation. Therefore, a natural choice is to identify one of the lightcone coordinates on the target manifold with one of the worldsheet coordinates. This choice leads to the so-called light-cone gauge.

  $$X^+ = x^+ + p^+\tau \, , \tag{2.74}$$

  where x^+ and p^+ are constants and can be shown to be the position and momentum of the string center of mass along the X^+ direction.
  Finally, the theory is quantised using only the physical degrees of freedom. In this case the theory is no longer manifestly covariant, but it does not contain ghost states.

2.5 String spectrum

Clearly, the two approaches shown in the previous section are equivalent. Let us summarise the main results:

- Enforcing the Lorentz symmetry when quantising the theory, the dimension of the target space-time has to be D = 26.
- The spectrum contains a tachyon particle, i.e. a particle with negative mass square. More precisely, since the mass square of a quantum field is the second derivative of the potential, it means that we are expanding the quantum oscillations of this field around a maximum of the potential.
- The spectrum contains also three massless fields, namely

$$G_{\mu\nu} \quad , \quad B_{\mu\nu} \quad , \quad \Phi \, ,$$

  where

  1. $G_{\mu\nu}$ is a symmetric traceless tensor, which corresponds to a spin-2 massless particle, i.e. the graviton. Therefore, $G_{\mu\nu}$ is the background metric that is perturbed by the presence of the string ;
  2. $B_{\mu\nu}$ is an anti-symmetric tensor, i.e. a 2-form, also known as Kalb-Ramond field ;
  3. Φ is a scalar field called dilaton.

2.6 Non-linear sigma model

So far, we have considered only strings propagating in a flat spacetime. Now, let us consider a string propagating on a curved background, with a generic metric $G_{\mu\nu}$. The Polyakov action can be generalised in the following way

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu} \, . \tag{2.75}$$

This is known as *non-linear sigma model* (NLSM). The reason is due to the fact that this action was first introduced in the context of $\beta -$ decay by Gell-Mann, dealing with a field called σ.
As in the Polyakov action, this action is invariant under worldsheet coordinate redefinition and Weyl transformations. Furthermore, it is also invariant under redefinitions of the coordinates of the target space

$$X^\mu \longrightarrow \tilde{X}^\mu \quad , \quad G_{\mu\nu} \longrightarrow \tilde{G}_{\mu\nu} = \frac{\partial X^\rho}{\partial\tilde{X}^\mu}\frac{\partial X^\sigma}{\partial\tilde{X}^\nu}G_{\rho\sigma} \, . \tag{2.76}$$

However, in general, this is not a symmetry, but a field redefinition. This will become a symmetry only if the metric $G_{\mu\nu}$ remains invariant under the redefinition, and hence if the diffeomorphism is an isometry of the background space.

We have seen how strings couple with the background metric $G_{\mu\nu}$ (2.75) . Nevertheless, we have seen that there are two other fields emerging from the string spectrum, namely the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton Φ, with which the string can couple.
Let us start considering the coupling with $B_{\mu\nu}$. This can be written in the form

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}(h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu} + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu) \, , \tag{2.77}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol with convention $\epsilon^{\tau\sigma} = 1$.

In order to justify this expression, let us recall the contribution to the action given by the interaction between a charged relativistic point particle and an electromagnetic field

$$S_{int} = \int d\tau A_\mu \dot{X}^\mu \, . \tag{2.78}$$

The electromagnetic field is written in terms of the gauge potential one-form $A_\mu dx^\mu$. The pull-back of A_μ on the worldline of the particle gives exactly the expression (2.78).
Therefore, by generalising this argument, the coupling between the string and the 2-form $B_{\mu\nu}$ is given by the pull-back of the field on the worldsheet, giving exactly the expression

$$S = \int d^2\sigma \sqrt{-h}\epsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \,. \tag{2.79}$$

Finally, let us consider the coupling with the dilaton field Φ. This contributes to the action with the additional term

$$\frac{1}{2\pi T}\sqrt{-h}\Phi R^{(2)} \,, \tag{2.80}$$

where $R^{(2)}$ is the worldsheet Ricci curvature. Note that this term breaks the Weyl invariance. However, it can be proved that it is recovered by considering loop contributions. In the next sections, we will ignore the dilaton contribution and we will focus only on the background metric and on the Kalb-Ramond field.

2.7 Superstring theory

For the sake of completeness, let us make a final comment before passing to the $AdS_3 \times S^3 \times T^4$ non-linear sigma model. So far, we have considered only bosonic fields in the action and then the string spectrum is exclusively composed by bosons. In order to implement fermionic modes on the worldsheet one has to introduce the so-called superstring theory, which, as the name suggests, is a supersymmetric theory. Superstring theory has some differences with respect to the bosonic string theory, namely: the tachyon mode disappears from the spectrum, the critical dimension becomes D = 10 and in addition to the massless bosonic fields $G_{\mu\nu}$, $B_{\mu\nu}$ (often referred to as the Neveu-Schwarz-Neveu-Schwarz (NS-NS) two-form in this context) and Φ, there are further bosonic and fermionic modes in the spectrum. The peculiar aspect is that, contrary to the pure bosonic case, in which the string action is unique, the way in which the additional fermionic modes are implemented gives rise to different classes of superstring theories. In what follows, we will refer to the so-called Type IIB theory. Since we will use it in the NLSM action, it is worth pointing out that in this theory additional massless bosonic excitations, known as Ramond-Ramond (RR) 3-form, appear in the spectrum.

Chapter 3

$AdS_3 \times S^3 \times T^4$ non-linear sigma model

In this chapter we will consider a free bosonic string propagating in $AdS_3 \times S^3 \times T^4$. This NLSM has been shown to be classically integrable [27], [28]. Since the fields of the theory are functions of the worldsheet coordinates σ and τ, this is a $(1+1)-$ dimensional quantum field theory. Therefore, one can expect that if the symmetries and the integrability structure survive also at the quantum level, it should obey the factorised scattering constraints. In particular, the initial and the final sets of momenta must be equal, there is no production in the final states, and the $2 \longrightarrow 2$ S-matrix respects the Yang-Baxter equations. Furthermore, symmetries constrain the structure of the S-matrix in such a way that it is possible to find its complete (to all loop) expression. The complete non-perturbative worldsheet S-matrix for the $AdS_3 \times S^3 \times T^4$ background has been derived for both the vanishing [29], [30] and the non-vanishing [31] NS-NS flux case up to some pre-factors, known as dressing factors. In particular, so far there is a proposal for these factors for the pure-RR [53] and the pure-NSNS [54] case, but not for the generic theory.
In this chapter, we will focus on the perturbative aspects of the model. We will fix the residual gauge by exploiting the lightcone gauge choice, and then we will solve the Virasoro constraints perturbatively in large string tension expansion. Ultimately, we will compute and discuss the two-body tree-level worldsheet S matrix.

3.1 Action

Let us consider a bosonic string freely propagating in $AdS_3 \times S^3 \times T^4$. The action is described by

$$S = -\frac{T}{2}\int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma (\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu)\,, \tag{3.1}$$

where $G_{\mu\nu}$ is the target spacetime metric, $B_{\mu\nu}$ the NS-NS B-field, $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$ and $h^{\alpha\beta}$ is the worldsheet metric. As seen in the previous chapter, this action is invariant under reparameterisations of the worldsheet and Weyl transformations. These are manifestations of the fact that not all the degrees of freedom that we are considering are physical. As in the flat Minkowski spacetime we can fix the conformal gauge, i.e.

$$h^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However, unlike the flat case, discussed in the previous chapter, now the conformal gauge does not give rise to free bosons, but the field dependent metric $G_{\mu\nu}$ contributes with non trivial terms. In fact the equations of motion in conformal gauge read

$$\begin{aligned} &\partial_\alpha \partial^\alpha X^\mu + \Gamma^\mu{}_{\nu\rho}\partial_\beta X^\nu \partial^\beta X^\rho - \frac{1}{2}H^\mu{}_{\sigma\tau}\partial_\gamma X^\sigma \partial_\delta X^\tau \epsilon^{\gamma\delta} = 0\,, \\ &T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma X^\mu \partial^\gamma X^\nu G_{\mu\nu} = 0\,, \end{aligned} \tag{3.2}$$

where ${\Gamma^\mu}_{\nu\rho}$ are the Christoffel symbols and H is a 3-form defined by $H = dB$. Note that, indeed, because of the non-trivial metric structure, the equations are different from the free bosons case that we found in flat background. These can be seen as the string generalisation of the point particle geodetic equations in curved spacetime.
However, given an infinitesimal change of coordinate $\sigma^\alpha \to \sigma^\alpha + \xi^\alpha(\sigma,\tau)$ and an infinitesimal Weyl transformation $\delta h^{\alpha\beta} = \Omega(\sigma,\tau)h^{\alpha\beta}$, we know that there is still a residual gauge freedom, given by the condition

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Omega \eta^{\alpha\beta}\,, \tag{3.3}$$

We will fix this redundancy by using the lightcone gauge.

Before fixing the gauge, let us specialise the action to the background we are considering.

3.1.1 Metric

The line element reads

$$ds^2 = ds^2_{AdS_3} + ds^2_{S^3} + ds^2_{T^4}\,, \tag{3.4}$$

where

1. AdS_3 is the 3-dimensional Anti-de Sitter space. In general, the n-dimensional Anti-de Sitter space is a maximally symmetric Lorentzian space with negative curvature. Therefore, it has $n(n+2)/2$ isometries and the Riemann tensor is constrainted to be

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,,$$

where $g_{\mu\nu}$ is the metric and R the scalar curvature. As a consequence, the Ricci tensor reads

$$R_{\mu\nu} = \frac{R}{d}g_{\mu\nu}\,.$$

Since the curvature is negative ($R < 0$), from the expression of the Ricci tensor it follows that the Anti-de Sitter space is a solution of the Einstein equation in vacuum and with negative cosmological constant $\Lambda = 2n/(n-2)R$.
Given the flat space $\mathbb{R}^{n-1,2}$ with metric $diag(-1,-1,+1,+1,...,+1)$ and line element

$$ds^2 = -dx_1^2 - dx_2^2 + \sum_{i=3}^{n+1} dx_i^2\,, \tag{3.5}$$

the embedding of AdS_n in this space is defined by the constraint

$$-x_1^2 - x_2^2 + \sum_{i=3}^{n+1} x_i^2 = -1\,. \tag{3.6}$$

Note that we are considering an unitary radius. On AdS_n we can use the coordinate system $(t, r, \hat{x}_i)$ given by

$$x_1 = \sin t \cosh r\ ,\ x_2 = \cos t \cosh r\ ,\ x_i = \hat{x}_i \sinh r\,, \tag{3.7}$$

where $0 \le t \le 2\pi$, $r \ge 0$ and $\hat{x}_i$ is a coordinate system on S^{n-2}. In these coordinates, the line element induced by (3.5) on AdS_n is $ds^2_{AdS_n} = -\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\Omega^2_{n-2}$, where $d\Omega^2_{n-2}$ is the line element on S^{n-2}.
Performing the change of coordinate $\rho = \sinh r$ and specialising this result to AdS_3, the metric can be rewritten as follows

$$ds^2_{AdS_3} = -(1+\rho^2)\,dt^2 + \frac{1}{1+\rho^2}\,d\rho^2 + \rho^2\,d\psi^2\,, \tag{3.8}$$

where t can be extended to the whole real axis, i.e. $t \in \mathbb{R}$, $\rho \ge 0$ and $0 \le \psi \le 2\pi$ is the angle on S^1.

Finally, let us introduce another coordinate system, which is the one we will use in our computations

$$\sqrt{1+\rho^2} = \frac{1+\frac{z_1^2+z_2^2}{4}}{1-\frac{z_1^2+z_2^2}{4}} \quad , \quad \rho e^{i\psi} = \frac{z_1+iz_2}{1-\frac{z_1^2+z_2^2}{4}} \, , \tag{3.9}$$

where $z_1, z_2 \in \mathbb{R}$.
The line element becomes

$$ds^2_{AdS_3} = -\Big(\frac{4+z_1^2+z_2^2}{4-z_1^2-z_2^2}\Big)^2 dt^2 + \Big(\frac{4}{4-z_1^2-z_2^2}\Big)^2 (dz_1^2+dz_2^2) \, , \tag{3.10}$$

2. S^3 is the 3-dimensional sphere. Given the flat Euclidean space $\mathbb{R}^4$ with line element

$$ds^2 = \sum_{i=1}^{4} dx_i^2 \, , \tag{3.11}$$

the embedding of S^3 in this space is defined by the constraint

$$\sum_{i=1}^{4} x_i^2 = 1 \, . \tag{3.12}$$

We can use the coordinate system (ω, θ, ϕ) given by

$$x_1 = \sin\omega\cos\theta \ , \ x_2 = \cos\omega\cos\theta \ , \ x_3 = \sin\theta\sin\phi \ , \ x_4 = \sin\theta\cos\phi \, . \tag{3.13}$$

In these coordinates, the line element induced by (3.11) on S^3 is $ds^2_{S^3} = \cos^2\theta\, d\omega^2 + d\theta^2 + \sin^2\theta\, d\phi^2$.
Performing the change of coordinate $r = \sin\theta$, the metric can be rewritten in isometric coordinates:

$$ds^2_{S^3} = (1-r^2)\, d\omega^2 + \frac{1}{1-r^2}\, dr^2 + r^2\, d\phi^2 \, , \tag{3.14}$$

where $0 \le r \le 1$ and $0 \le \phi \le 2\pi$.
Finally, let us introduce the coordinate system which we will use in our computations

$$\sqrt{1-r^2} = \frac{1-\frac{y_1^2+y_2^2}{4}}{1+\frac{y_1^2+y_2^2}{4}} \quad , \quad re^{i\phi} = \frac{y_1+iy_2}{1+\frac{y_1^2+y_2^2}{4}} \, , \tag{3.15}$$

where $y_1, y_2 \in \mathbb{R}$.
The line element becomes

$$ds^2_{S^3} = \Big(\frac{4-y_1^2-y_2^2}{4+y_1^2+y_2^2}\Big)^2 d\omega^2 + \Big(\frac{4}{4+y_1^2+y_2^2}\Big)^2 (dy_1^2+dy_2^2) \, , \tag{3.16}$$

3. T^4 is the 4-dimensional torus with line element

$$ds^2_{T^4} = dx^i dx^i \, , \tag{3.17}$$

where $x^i \in \mathbb{R}$, $i = 5, 6, 7, 8$.

Now, we can write the explicit expression of the metric $G_{\mu\nu}$ and of the B-field that appear in the action (3.1). Note that we have discussed the presence and the physical meaning of the B-field in the NLSM action; however, we have not seen an explicit expression for it as a function of the background. The expression is derived by considering the supersymmetric generalisation of the bosonic string and putting the fermions to zero. The complete expression for the line element is

$$ds^2 = -G_{tt}dt^2 + G_{\omega\omega}d\omega^2 + G_{zz}(dz_1^2+dz_2^2) + G_{yy}(dy_1^2+dy_2^2) + \sum_{j=5}^{8} dx_j dx_j \, , \tag{3.18}$$

where

$$
\begin{aligned}
G_{tt} &= \left(\frac{4+z_1^2+z_2^2}{4-z_1^2-z_2^2}\right)^2 \quad , \quad G_{\omega\omega} = \left(\frac{4-y_1^2-y_2^2}{4+y_1^2+y_2^2}\right)^2 , \\
G_{zz} &= \left(\frac{4}{4-z_1^2-z_2^2}\right)^2 \quad , \quad G_{yy} = \left(\frac{4}{4+y_1^2+y_2^2}\right)^2 ,
\end{aligned}
\tag{3.19}
$$

and the NS-NS B-field is given by [31]

$$
B = \frac{q}{\left(1-\frac{z_1^2+z_2^2}{4}\right)^2}\left[z_1 dz_2 \wedge dt + z_2 dt \wedge dz_1\right] + \frac{q}{\left(1+\frac{y_3^2+y_4^2}{4}\right)^2}\left[y_3 dy_4 \wedge d\omega + y_4 d\omega \wedge dy_3\right] , \tag{3.20}
$$

where $0 \leq q \leq 1$. In particular, this formula describes both the pure RR flux model ($q = 0$) and the pure NS-NS flux model ($q = 1$), and in the other cases it supports the mixed RR-NSNS flux [55].
As discussed in the previous chapter (3.1) is invariant under reparameterisation of the target spacetime. This is a field redefinition; however, if the transformation is an isometry and leaves also the B-field invariant, then it becomes a symmetry of the action. In this case, there are the two symmetries

$$
t \longrightarrow t + \text{const} \quad , \quad \omega \longrightarrow \omega + \text{const} , \tag{3.21}
$$

which means that the action is invariant under t and ω translations. These two symmetries, because of the Noether theorem, imply the presence of the corresponding conserved charges

$$
E = -\int_{-r}^{r} d\sigma \, P_t , \qquad J = \int_{-r}^{r} d\sigma \, P_\omega , \tag{3.22}
$$

where P_t and P_ω are the conjugate momenta of t and ω, respectively. These can be interpreted as the energy and the angular momentum in the ω-direction of the string.

3.2 First-order action

In order to fix the lightcone gauge, it is useful to write the action in the first-order form. First, let us introduce the conjugate momenta

$$
P_\mu = \frac{\delta \mathcal{L}}{\delta \dot{X}^\mu} = -T\Big(\gamma^{\tau\beta} G_{\mu\nu} + \epsilon^{\tau\beta} B_{\mu\nu}\Big)\partial_\beta X^\nu . \tag{3.23}
$$

Using the expression of the momenta, the action can be rewritten in the first-order formalism

$$
S = \int_{-r}^{r} d\tau d\sigma \Big(P_\mu \dot{X}^\mu + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2 \Big) , \tag{3.24}
$$

where

$$
\begin{aligned}
C_1 &= P_\mu \acute{X}^\mu , \\
C_2 &= G^{\mu\nu} P_\mu P_\nu + T^2 G_{\mu\nu} \acute{X}^\mu \acute{X}^\nu + 2T G^{\mu\nu} B_{\nu\kappa} P_\mu \acute{X}^\kappa + T^2 G^{\mu\nu} B_{\mu\kappa} B_{\nu\lambda} \acute{X}^\kappa \acute{X}^\lambda .
\end{aligned}
\tag{3.25}
$$

As seen in the previous chapter, the equation of motion of the metric give the Virasoro constraints, which in this form are given by

$$
C_1 = C_2 = 0 . \tag{3.26}
$$

3.3 Uniform lightcone gauge

Let us introduce the lightcone coordinate and momenta on the target manifold

$$
\begin{aligned}
t &= x^+ - a x^- \quad , \quad \omega = x^+ + (1-a) x^- , \\
P_t &= (1-a) p_+ - p_- \quad , \quad P_\omega = a p_+ + p_- ,
\end{aligned}
\tag{3.27}
$$

with inverse transformations

$$x^+ = a\omega + (1-a)t \quad , \quad x^- = \omega - t \, , \\ p_+ = P_\omega + P_t \quad , \quad p_- = (1-a)P_\omega - aP_t \tag{3.28}$$

where $a \in \mathbb{R}$ is a parameter. Therefore, more precisely, this is a one-parameter family of coordinate transformations. The lightcone momenta are defined by the usual expression $p_\pm = \partial\mathcal{L}/\partial\dot{x}^\pm$.
Let us write the metric (3.18) isolating the ω and the t terms in the compact form

$$ds^2 = -G_{tt}dt^2 + G_{\omega\omega}d\omega^2 + G_{jj}dX^j dX^j \, , \tag{3.29}$$

where X^j are the remaining transversal coordinates. As we will see, these are the physical degrees of freedom because, after the gauge fixing, they are the only remaining fields.
Using this notation and the lightcone coordinates, the first-order action (3.24) becomes

$$S = \int_{-r}^{r} d\tau d\sigma \Big(p_+ \dot{x}^+ + p_- \dot{x}^- + P_j \dot{X}^j + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2 \Big) \, . \tag{3.30}$$

Now it is possible to fix the uniform lightcone gauge

$$x^+ = \tau \, , \qquad p_- = 1 \, . \tag{3.31}$$

This gauge is called uniform because the light-cone momenta p_- is constant along the string.
Exploiting the gauge condition and the Virasoro constraints $C_1 = C_2 = 0$, the action can be written in the following way

$$S = \int_{-r}^{r} d\tau d\sigma (P_j \dot{X}^j + p_+) \, . \tag{3.32}$$

With the gauge condition we have fixed p_- and x^+. Now we can exploit the two Virasoro constraints to fix the value of x^- and p_+ in terms of the transversal coordinates.
In particular, the first constraint can be used to find the expression of

$$\acute{x}^- = -P_j \acute{X}^j \, , \tag{3.33}$$

while for the second constraint we find

$$G^{\mu\nu} P_\mu P_\nu + T^2 G_{\mu\nu} \acute{X}^\mu \acute{X}^\nu + 2TG^{\mu\nu} B_{\nu\kappa} P_\mu \acute{X}^\kappa + T^2 G^{\mu\nu} B_{\mu\kappa} B_{\nu\lambda} \acute{X}^\kappa \acute{X}^\lambda = 0 \, . \tag{3.34}$$

Note that, since the metric does not depend on x^+ and x^-, this expression only depends on $p_\pm$, $\acute{x}^\pm$, X^j, $\acute{X}^j$ and P_j. However, using the gauge condition and the first Virasoro constraint we have

$$p_- = 1 \quad , \quad \acute{x}^+ = 0 \quad , \quad \acute{x}^- = -P_j \acute{X}^j \, . \tag{3.35}$$

Therefore, (3.34) is a second degree equation in p_+ as a function of X^j, $\acute{X}^j$ and P_j.

Let us make a couple of additional comments. First of all, let us note that in the expression (3.32), the Lagrangian density is written as the Legendre transform of $-p_+$. Therefore, it follows that

$$\mathcal{H} = -p_+(X^j, \acute{X}^j, P_j) \, , \tag{3.36}$$

where $\mathcal{H}$ is the Hamiltonian density. Hence, the lightcone Hamiltonian turns out

$$H(X^j, \acute{X}^j, P_j) = -\int_{-r}^{r} d\sigma p_+(X^j, \acute{X}^j, P_j) = E - J \, , \tag{3.37}$$

and it depends on the energy and the angular mometnum of the string.

Furthermore, let P_- be the total lightcone momentum, the gauge fixing also fixes the length of the string in terms of P_-. In fact

$$P_- = \int_{-r}^{r} d\sigma p_- = \int_{-r}^{r} d\sigma = 2r = aE + (1-a)J \, , \tag{3.38}$$

and thus

$$r = \frac{P_-}{2}\,. \tag{3.39}$$

The length depends also on the energy and angular momentum of the string. After the gauge fixing, we are dealing with a two-dimensional quantum field theory defined on a cylinder of length given by the total lightcone momentum. As we will see, the finiteness of the string spatial dimension will play an important role in the introduction of the mirrom model. On the other hand, let us note that, if the lightcone total momentum goes to infinity, the worldsheet becomes an infinite plane. This is known as *decompactification limit.*

Before fixing the lightcone gauge, the theory was invariant under worldsheet Lorentz transformations. In fact, by imposing the conformal gauge to the NLSM action (3.1) one can find

$$S = -\frac{T}{2}\int_{-r}^{r} d\tau d\sigma (\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu)\,. \tag{3.40}$$

This action is explicitly invariant under worldsheet proper Lorentz transformations. However, the gauge condition (3.31) breaks the Lorentz invariance. Therefore, after the gauge fixing, we are left with a non-relativistic theory. We will come back to this aspect when we consider the mirror transformation.

Moreover, since we are dealing with closed strings, we have to impose the periodic condition $X^j(r) = X^j(-r)$. In general, for an angular direction, this relation becomes

$$\omega(r) = \omega(-r) + 2\pi m\,, \qquad m \in \mathbb{Z}\,, \tag{3.41}$$

because of the presence of topologically non-trivial configurations given by the wrapping of the string around the angular direction. However, we will consider only configurations with zero winding number m. Therefore, the periodic condition for the lightcone coordinate $x^- = \omega - t$ reads

$$\Delta x^- = \int_{-r}^{r} d\sigma \acute{x}^- = 0\,. \tag{3.42}$$

Using the first Virasoro constraint (3.33), one can find

$$\Delta x^- = -\int_{-r}^{r} d\sigma P_j \acute{X}^j = 0\,. \tag{3.43}$$

Note that this expression can be written in terms of the worldsheet momentum. In fact, the gauge-fixed action is invariant under σ translations. This means that, due to the Noether theorem, there exists a conserved quantity given by

$$p_{ws} = -\int_{-r}^{r} d\sigma \frac{\partial \mathcal{L}}{\partial \dot{X}^j}\acute{X}^j = -\int_{-r}^{r} d\sigma P_j \acute{X}^j\,, \tag{3.44}$$

which is the worldsheet momentum.
From these two equalities follows that the periodicity condition i.e. the level-matching condition reads

$$\Delta x^- = p_{ws} = 0\,. \tag{3.45}$$

Therefore the physical states have zero worldsheet momentum.

Ultimately, we expect the gauge-fixed action to be invariant under time reversal. Note that before fixing the gauge, the NLSM (3.1) is not invariant neither under $\tau \longrightarrow -\tau$ nor under $\sigma \longrightarrow -\sigma$ due to the B-field term that mixes the space and time derivatives with the presence of the $\epsilon^{\alpha\beta}$ tensor. However, after imposing the lightcone gauge, the conditions $x^+ = \tau$ and $\partial\mathcal{L}/\partial\dot{x}^- = 1$ imply that under time reversal

$$x^+ \longrightarrow -x^+ \quad , \quad x^- \longrightarrow -x^-\,. \tag{3.46}$$

Therefore, time reversal in the gauge-fixed action sends

$$t \longrightarrow -t \quad , \quad \omega \longrightarrow -\omega\,. \tag{3.47}$$

Since the metric is diagonal and does not depend nether on t nor on ω, the first term in the action remains invariant under time reversal. On the other hand, the second term is given by

$$-\frac{T}{2}\int_{-r}^{r} d\tau d\sigma \epsilon^{\alpha\beta}(B_{z_1 t}\partial_\alpha z_1 \partial_\beta t + B_{z_2 t}\partial_\alpha z_2 \partial_\beta t + B_{y_1\omega}\partial_\alpha y_1 \partial_\beta \omega + B_{y_2\omega}\partial_\alpha y_2 \partial_\beta \omega)\,. \tag{3.48}$$

In fact, as we can see from the explicit expression of the B-field (3.20), these are the only non vanishing components. Therefore, this term always contains one and only one τ-derivative and one and only one t or ω field, namely two -1 sources under time reversal. For this reason we expect the lightcone Hamiltonian to be invariant under time-reversal. Clearly, the same argument does not hold for parity transformation which remains broken.

3.4 Large string tension expansion

In order to compute the worldsheet S matrix, let us consider the decompactification limit

$$P_- \longrightarrow \infty\,, \quad T \text{ fixed}, \tag{3.49}$$

in which the worldsheet cylinder becomes an infinite plane. In this limit it is possible to define the asymptotic *in* and *out* states and then the S matrix is well defined. Since $H = E - J$ has to remain finite and $P_- = \alpha E + (1-\alpha)J \longrightarrow \infty$, it turns out that also E and J go to infinity.

In order to proceed with the quantisation of the asymptotic worldsheet theory, we need to find the lightcone Hamiltonian. We have seen that this corresponds to the opposite of $p_+(X^j, \acute{X}^j, P_j)$. On the other hand, p_+ can be found by solving the second Virasoro constraints $C_2 = 0$, which is a second degree equation. This means that the expression of the Hamiltonian can be analytically computed. However, this results to be a complicated non-linear expression in the physical fields.

For this reason, let us perturbatively solve the equation in the large string tension expansion, (i.e. expanding around $1/T = 0$).

Let us note that in the second Virasoro constraint (3.34), the string tension always multiplies a $\partial/\partial\sigma$ term. Therefore in the lightcone Hamiltonian we can perform the rescaling

$$\sigma \to T\sigma\,. \tag{3.50}$$

In this way

$$\frac{\partial}{\partial_\sigma} \to \frac{1}{T}\frac{\partial}{\partial_\sigma} \quad , \quad d\sigma \to T d\sigma\,, \tag{3.51}$$

and the dependence of the lightcone momentum p_+ on the string tension T disappears:

$$H(X^j, \acute{X}^j, P_j) = -T\int d\sigma p_+(X^j, \acute{X}^j, P_j)|_{T=1}\,. \tag{3.52}$$

Performing the field redefinition

$$X^j \to \frac{X^j}{\sqrt{T}} \quad , \quad P_j \to \frac{P_j}{\sqrt{T}}\,, \tag{3.53}$$

the Hamiltonian can be expanded in powers of $1/T$

$$\begin{aligned} H(X^j, \acute{X}^j, P_j) &= T\int d\sigma\left(\frac{1}{T}\mathcal{H}^{(2)} + \frac{1}{T^2}\mathcal{H}^{(4)} + \frac{1}{T^3}\mathcal{H}^{(6)} + ...\right) \\ &= \int d\sigma\left(\mathcal{H}^{(2)} + \frac{1}{T}\mathcal{H}^{(4)} + \frac{1}{T^2}\mathcal{H}^{(6)} + ...\right), \end{aligned} \tag{3.54}$$

where $\mathcal{H}^{(2)}$ is the quadratic Hamiltonian, $\mathcal{H}^{(4)}$ the quartic and so on.

We expect the expansion to have only even terms. In fact, the gauge-fixed action remains unchanged under transformations of the type

$$X^j \longrightarrow -X^j \quad , \quad P_j \longrightarrow -P_j\,. \tag{3.55}$$

In order to see this property, let us note that using the gauge condition and the first Virasoro constraint one can find that under this transformations, t and ω do not change. Moreover, the metric (3.18) is quadratic in the physical fields, and then its expression also remains equal. This shows that the first term of the action does not change. On the other hand, the second term contains the B-field (3.20) that transforms as follows

$$B_{\mu\nu} \longrightarrow -B_{\mu\nu} \,. \tag{3.56}$$

However, the only non vanishing component are the one in which it multiplies a physical field (that changes the sign) and a non physical field (t or ω, that does not change the sign). Therefore, this second term is also invariant.
It is interesting to note that, rescaling the expression of the worldsheet momenta we obtain

$$p_{ws} = -\frac{1}{T}\int d\sigma P_j \acute{X}^j \,. \tag{3.57}$$

This means that from the physical point of view, in the large string tension expansion we are considering states with small worldsheet momenta.
Ultimately, let us stress that the procedure of going in the decompactification limit and then expanding the Hamiltonian in the large tension expansion, is different from the so-called Berenstein-Maldacena-Nastase (BMN) limit [56], in which

$$P_- \longrightarrow \infty \quad , \quad T \longrightarrow \infty \,, \tag{3.58}$$

and P_-/T is kept fixed. Also in this limit, it is possible to expand the Hamiltonian in powers of $1/T$, however, now this is also an expansion in $1/P_-$, giving finite-size corrections to the Hamiltonian. In what follows, we will use the first approach because it allows to deal with asymptotic states and to define the S matrix.

3.5 Perturbative lightcone Hamiltonian

Let us perturbatively solve the second Virasoro constraint.
First, it is convenient to introduce the complex fields and momenta

$$\begin{aligned}
z &= \frac{1}{\sqrt{2}}(z_1 + i z_2) \quad , \quad p_z = \frac{1}{\sqrt{2}}(p_{z_1} + i p_{z_2}) \,, \\
y &= \frac{1}{\sqrt{2}}(y_1 + i y_2) \quad , \quad p_y = \frac{1}{\sqrt{2}}(p_{y_1} + i p_{y_2}) \,, \\
u &= \frac{1}{\sqrt{2}}(x_5 + i x_6) \quad , \quad p_u = \frac{1}{\sqrt{2}}(p_5 + i p_6) \,, \\
v &= \frac{1}{\sqrt{2}}(x_7 + i x_8) \quad , \quad p_v = \frac{1}{\sqrt{2}}(p_7 + i p_8) \,,
\end{aligned} \tag{3.59}$$

where it has been used the notation $p_{z_1} = \partial\mathcal{L}/\partial\dot{z}_1$, $p_5 = \partial\mathcal{L}/\partial\dot{x}_5$ and so on. We will denote the corresponding complex conjugate fields and momenta by using the notation $(z)^* = \bar{z}$ and $(p)^* = \bar{p}$ and so on for all the fields.
Note that in this notation $\bar{p}_z$ is the conjugate momentum of z while p_z is the conjugate momentum of $\bar{z}$, indeed

$$\frac{\partial\mathcal{L}}{\partial\dot{z}} = \frac{\partial\dot{z}_1}{\partial\dot{z}}\frac{\partial\mathcal{L}}{\partial\dot{z}_1} + \frac{\partial\dot{z}_2}{\partial\dot{z}}\frac{\partial\mathcal{L}}{\partial\dot{z}_2} = \frac{p_{z_1}}{\sqrt{2}} - i\frac{p_{z_2}}{\sqrt{2}} = \bar{p}_z \,. \tag{3.60}$$

The same goes for the other fields.
Using these coordinates and the lightcone coordinates for t and ω the metric takes the form

$$\begin{aligned} ds^2 = & \Big[-\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + \Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2\Big]dx_+^2 \\ & + \Big[-a^2\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + (1-a)^2\Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2\Big]dx_-^2 \\ & + \Big[2a\Big(\frac{2+|z|^2}{2-|z|^2}\Big)^2 + 2(1-a)\Big(\frac{2-|y|^2}{2+|y|^2}\Big)^2\Big]dx_+dx_- \\ & + \frac{8}{(2-|z|^2)^2}dzd\bar{z} + \frac{8}{(2+|y|^2)^2}dyd\bar{y} + 2dud\bar{u} + 2dvd\bar{v}\,, \end{aligned} \tag{3.61}$$

and the B-field becomes

$$\begin{aligned} B = & \frac{4iq}{(2-|z|^2)^2}\left[\bar{z}dx_+\wedge dz - zdx_+\wedge d\bar{z} - a\bar{z}dx_-\wedge dz + azdx_-\wedge d\bar{z}\right] \\ & + \frac{8iq}{(2+|y|^2)^2}\left[\bar{y}dx_+\wedge dy - ydx_+\wedge d\bar{y} + (1-a)\bar{y}dx_-\wedge dy - (1-a)ydx_-\wedge d\bar{y}\right]\,, \end{aligned} \tag{3.62}$$

where $|z| = z\bar{z}$. Inserting these expressions in the second Virasoro constraint and solving perturbatively at the first order in $1/T$ one can find the quadratic lightcone Hamiltonian density

$$\mathcal{H}^{(2)} = -p_+^{(2)} = p_z\bar{p}_z + p_y\bar{p}_y + p_u\bar{p}_u + p_v\bar{p}_v + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2 + |z|^2 + |y|^2 + iq(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}})\,. \tag{3.63}$$

This is the free Hamiltonian of the theory. Setting the parameter $q = 0$, which correspond to the pure RR-flux, the theory describes two complex massive and two complex massless Klein-Gordon fields. However, the q-dependent term gives a non trivial additional contribution to the dispersion relations and also to the interaction vertices as we will see soon.
The equations of motion of the free Hamiltonian are

$$\begin{aligned} \dot{z} &= p_z \quad , \quad \dot{u} = p_u\,, \\ \dot{p}_z &= -z + z'' - 2iqz' \quad , \quad \dot{p}_u = u''\,. \end{aligned} \tag{3.64}$$

Deriving the first row of (3.64) with respect to τ and inserting the result in the second row gives

$$\ddot{z} - z'' + z + 2iqz' = 0 \quad , \quad \ddot{u} - u'' = 0\,. \tag{3.65}$$

The equations of motion of y and v are equal to those of z and u respectively.

3.5.1 Quantisation

In order to canonically quantise the theory let us impose

$$\begin{aligned} &[z(\sigma,\tau), \bar{p}_z(\sigma',\tau)] = [z(\sigma,\tau), \dot{\bar{z}}(\sigma',\tau)] = i\delta(\sigma-\sigma') \quad , \quad [z(\sigma,\tau), z(\sigma',\tau)] = [\dot{z}(\sigma,\tau), \dot{z}(\sigma',\tau)] = 0\,, \\ &[u(\sigma,\tau), \bar{p}_u(\sigma',\tau)] = [u(\sigma,\tau), \dot{\bar{u}}(\sigma',\tau)] = i\delta(\sigma-\sigma') \quad , \quad [u(\sigma,\tau), u(\sigma',\tau)] = [\dot{u}(\sigma,\tau), \dot{u}(\sigma',\tau)] = 0\,, \end{aligned} \tag{3.66}$$

and the same conditions holds for y and v.
Despite the presence of an additional term due to the B-field, the solutions can still be written as plane waves with a modification of the dispersion relations.
In particular, the Anti-de Sitter and the sphere directions give rise to two complex scalar massive particles, whose solutions are

$$\begin{aligned} z(\sigma,\tau) &= \int \frac{dp}{\sqrt{(2\pi)}}\left[\frac{e^{-i(\omega(p)\tau - p\sigma)}}{\sqrt{2\omega(p)}}a^z(p) + \frac{e^{i(\bar{\omega}(p)\tau - p\sigma)}}{\sqrt{2\bar{\omega}(p)}}a^\dagger_{\bar{z}}(p)\right]\,, \\ y(\sigma,\tau) &= \int \frac{dp}{\sqrt{(2\pi)}}\left[\frac{e^{-i(\omega(p)\tau - p\sigma)}}{\sqrt{2\omega(p)}}a^y(p) + \frac{e^{i(\bar{\omega}(p)\tau - p\sigma)}}{\sqrt{2\bar{\omega}(p)}}a^\dagger_{\bar{y}}(p)\right]\,, \end{aligned} \tag{3.67}$$

where $a_z^\dagger$, $a_{\bar{z}}^\dagger$ ($a_y^\dagger$, $a_{\bar{y}}^\dagger$) and a^z, $a^{\bar{z}}$ (a^y, $a^{\bar{y}}$) are the usual creation and annihilation operators, that satisfy the canonical commutation relations (CCR)

$$\begin{aligned}
&[a^z(p), a_z^\dagger(p')] = [a^{\bar{z}}(p), a_{\bar{z}}^\dagger(p')] = \delta(p-p')\,, \\
&[a^z(p), a^z(p')] = [a_z^\dagger(p), a_z^\dagger(p')] = [a^{\bar{z}}(p), a^{\bar{z}}(p')] = [a_{\bar{z}}^\dagger(p), a_{\bar{z}}^\dagger(p')] = 0\,,
\end{aligned} \tag{3.68}$$

and the dispersion relations are

$$\omega(p) = \sqrt{1-2qp+p^2} \quad , \quad \bar{\omega}(p) = \sqrt{1+2qp+p^2}\,. \tag{3.69}$$

On the other hand the torus directions give rise to two complex scalar massless particle whose solutions are

$$\begin{aligned}
u(\sigma,\tau) &= \int \frac{dp}{\sqrt{(2\pi)2|p|}}\left[e^{-i(|p|\tau-p\sigma)}a^u(p) + e^{i(|p|\tau-p\sigma)}a_{\bar{u}}^\dagger(p)\right], \\
v(\sigma,\tau) &= \int \frac{dp}{\sqrt{(2\pi)2|p|}}\left[e^{-i(|p|\tau-p\sigma)}a^v(p) + e^{i(|p|\tau-p\sigma)}a_{\bar{v}}^\dagger(p)\right],
\end{aligned} \tag{3.70}$$

where

$$\begin{aligned}
&[a^u(p), a_u^\dagger(p')] = [a^{\bar{u}}(p), a_{\bar{u}}^\dagger(p')] = \delta(p-p')\,, \\
&[a^u(p), a^u(p')] = [a_u^\dagger(p), a_u^\dagger(p')] = [a^{\bar{u}}(p), a^{\bar{u}}(p')] = [a_{\bar{u}}^\dagger(p), a_{\bar{u}}^\dagger(p')] = 0\,,
\end{aligned} \tag{3.71}$$

3.5.2 Quartic Hamiltonian

By solving the second Virasoro constraint up to the second order in $1/T$ one can find the quartic Hamiltonian

$$\begin{aligned}
\mathcal{H}^{(4)} &= 2z\bar{z}\acute{z}\acute{\bar{z}} - 2y\bar{y}\acute{y}\acute{\bar{y}} - y\bar{y}\acute{z}\acute{\bar{z}} + z\bar{z}\acute{y}\acute{\bar{y}} - y\bar{y}p_z\bar{p}_z + z\bar{z}p_y\bar{p}_y + (|z|^2-|y|^2)(p_u\bar{p}_u + p_v\bar{p}_v + |\acute{u}|^2 + |\acute{v}|^2) \\
&+ \frac{iq}{2}(|z|^2-|y|^2)(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}}) \\
&- \frac{iq}{2}(p_z\acute{\bar{z}} + \bar{p}_z\acute{z} + p_y\acute{\bar{y}} + \bar{p}_y\acute{y} + p_u\acute{\bar{u}} + \bar{p}_u\acute{u} + p_v\acute{\bar{v}} + \bar{p}_v\acute{v})(\bar{z}p_z - z\bar{p}_z - \bar{y}p_y + y\bar{p}_y) \\
&+ \frac{iq}{2}\left(p_z\bar{p}_z + p_y\bar{p}_y + p_u\bar{p}_u + p_v\bar{p}_v + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2\right)(\bar{z}\acute{z} - z\acute{\bar{z}} - \bar{y}\acute{y} + y\acute{\bar{y}}) \\
&+ \frac{(2a-1)}{2}\left((p_z\bar{p}_z + p_y\bar{p}_y + p_u\bar{p}_u + p_v\bar{p}_v + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2)^2\right) \\
&+ \frac{(2a-1)}{2}\left(-(|z|^2+|y|^2)^2 - (p_z\acute{\bar{z}} + \bar{p}_z\acute{z} + p_y\acute{\bar{y}} + \bar{p}_y\acute{y} + p_u\acute{\bar{u}} + \bar{p}_u\acute{u} + p_v\acute{\bar{v}} + \bar{p}_v\acute{v})^2\right) \\
&+ \frac{(2a-1)iq}{2}\left((\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}})\left(p_z\bar{p}_z + p_y\bar{p}_y + p_u\bar{p}_u + p_v\bar{p}_v + |\acute{z}|^2 + |\acute{y}|^2 + |\acute{u}|^2 + |\acute{v}|^2 - (|z|^2+|y|^2)\right)\right) \\
&- \frac{(2a-1)iq}{2}\left((p_z\acute{\bar{z}} + \bar{p}_z\acute{z} + p_y\acute{\bar{y}} + \bar{p}_y\acute{y} + p_u\acute{\bar{u}} + \bar{p}_u\acute{u} + p_v\acute{\bar{v}} + \bar{p}_v\acute{v})(\bar{z}p_z - zp_{\bar{z}} + \bar{y}p_y - yp_{\bar{y}})\right).
\end{aligned} \tag{3.72}$$

This Hamiltonian gives the four-legs vertices interactions. Note that the this Hamiltonian has some symmetries, namely

- It is invariant under the $U(1)$ symmetries given by

$$X_i \longrightarrow e^{iq_i\phi}X_i \quad \text{and} \quad \bar{X}_i \longrightarrow e^{-iq_i\phi}\bar{X}_i\,,$$

  with $X_i = z, y, u, v$ and $\bar{X}_i = \bar{z}, \bar{y}, \bar{u}, \bar{v}$, which constraints the possible scattering processes, e.g. it prevents processes such as $zz \longrightarrow yy$.

- It is invariant under the exchange of massless fields

$$u \longleftrightarrow v\,, \tag{3.73}$$

while, under the exchanging of the massive fields the a-independent part changes its sign and the a-dependent part remains equal, i.e.

$$z \longleftrightarrow y \Longrightarrow \begin{cases} \mathcal{H}(a=1/2) \longrightarrow \mathcal{H}(a=1/2) \\ (\mathcal{H} - \mathcal{H}(a=1/2)) \longrightarrow -(\mathcal{H} - \mathcal{H}(a=1/2)) \end{cases} \tag{3.74}$$

- It is invariant under σ and τ translations

$$\sigma \longrightarrow \sigma + \text{const} \quad , \quad \tau \longrightarrow \tau + \text{const}\,. \tag{3.75}$$

- As already pointed out in section 3.3 it is invariant under time-reversal

$$\tau \longrightarrow -\tau \tag{3.76}$$

- As pointed out in section 3.3 it is not invariant under parity transformations. This, as we mentioned above is due to the terms proportional to the B-field, which change their sign under parity. This means that if together with a parity transformation we also change the sign of the B-field , (i.e. we change the sign of the overall q factor), the Hamiltonian remains invariant. Explicitly

$$\sigma \longrightarrow -\sigma \quad \text{and} \quad q \longrightarrow -q\,. \tag{3.77}$$

3.6 Worldsheet S matrix

Now that we have the quartic Hamiltonian, which contains the first non-trivial interaction terms, it is possible to compute the two body tree-level S matrix. According to the integrability structure, knowing these processes corresponds to knowing the complete S matrix because every $n \longrightarrow n$ process can be decomposed into $2 \longrightarrow 2$ processes. In general, one can go further in the perturbative computation of the Hamiltonian and including also the fermionic modes it is possible to compute the loop corrections to the S matrix. Note that even if we are considering processes with only bosons in the initial and final states, at the quantum level one has to consider also the fermionic modes because they start to enter in the loops. However, this is not a simple task, and in addition, this procedure does not consider the non-perturbative contributions because we are dealing with a perturbative expansion in both $\hbar$ and the inverse of the string tension $1/T$. On the other hand, as mentioned above, exploiting the symmetries of the model, it is possible to fix the complete S matrix up to some prefactors called dressing factors. This approach is known as bootstrap approach. In any case, it is worth pointing out that perturbative computations are useful in order to verify that the S matrix respects the necessary conditions for integrability (at least at the considered perturbative level) and to check the perturbative expansion of the bootstrapped S matrix.

Therefore, let us compute the tree-level $2 \longrightarrow 2$ S matrix. Using the Hamiltonian approach, the S matrix operator can be written as follows

$$\mathbb{S} = \mathcal{T} \exp\left(-i \int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma\, \mathcal{H}^I(\sigma,\tau)\right), \tag{3.78}$$

where $\mathcal{T}$ is the time-ordered product and the Hamiltonian has been split in $\mathcal{H} = \mathcal{H}^f + \mathcal{H}^I$, where $\mathcal{H}^f$ and $\mathcal{H}^I$ are the free (quadratic) and the interactive Hamiltonian respectively.
By expanding the S matrix we obtain

$$\mathbb{S} = \mathbb{I} - i \int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma\, \mathcal{H}^I(\sigma,\tau) + \dots\,, \tag{3.79}$$

and a generic $2 \longrightarrow 2$ tree-level matrix element $\mathbb{S}_{ij}^{kl}$, where i and j are the two particle species in the initial state and k and l are the two particle species in the final state, is given by

$$\mathbb{S}_{ij}^{kl} = \langle kl|\mathbb{I}|ij\rangle - i \int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma\, \langle kl|\, \mathcal{H}^I(\sigma,\tau)\, |ij\rangle\,. \tag{3.80}$$

Let us note that, since there are four particles in total (two incoming and two outgoing), the only non vanishing tree-level matrix elements are obtained using the quartic Hamiltonian.
Therefore, defining the $\mathbb{T}$ matrix as

$$\mathbb{S} = \mathbb{I} + i\mathbb{T}\,, \tag{3.81}$$

its tree-level components are given by

$$\mathbb{T}^{kl}_{ij} = -\frac{1}{T}\int_{-\infty}^{+\infty} d\tau \int_{-r}^{r} d\sigma\, \langle kl|\, \mathcal{H}^{(4)}(\sigma,\tau)\, |ij\rangle\ . \tag{3.82}$$

The asymptotic initial and final states are given by the usual expressions

$$a^\dagger_z(p)a^\dagger_y(q)\,|0\rangle = |z(p)y(q)\rangle \qquad , \qquad \langle 0|\, a^z(p)a^y(q) = \langle z(p)y(q)|\ . \tag{3.83}$$

Working in interaction picture, the field operators inside the expression (3.82) can be replaced by the free plane wave expressions (3.67) and (3.70).
We will not go through the computation's details, because we will see similar calculations when we deal with the mirror theory.
Let us summarise the matrix elements, rescaling for convenience $\mathbb{T}$ to $\mathbb{T} \to T\cdot\mathbb{T}$.

- Massive-massive

$$\begin{aligned}
\mathbb{T}\,|z_\pm(p_1)z_\pm(p_2)\rangle &= \left(-\frac{(p_1+p_2)(\omega_1^\pm p_2+\omega_2^\pm p_1)}{2(p_1-p_2)} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\pm p_1)\right)|z_\pm(p_1)z_\pm(p_2)\rangle\\
\mathbb{T}\,|z_\pm(p_1)y_\pm(p_2)\rangle &= \left(\frac{(\omega_1^\pm p_2+\omega_2^\pm p_1)}{2} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\pm p_1)\right)|z_\pm(p_1)y_\pm(p_2)\rangle\\
\mathbb{T}\,|z_\pm(p_1)z_\mp(p_2)\rangle &= \left(-\frac{(p_1-p_2)(\omega_1^\pm p_2+\omega_2^\mp p_1)}{2(p_1+p_2)} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\mp p_1)\right)|z_\pm(p_1)z_\mp(p_2)\rangle\\
\mathbb{T}\,|z_\pm(p_1)y_\mp(p_2)\rangle &= \left(\frac{(\omega_1^\pm p_2+\omega_2^\mp p_1)}{2} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\mp p_1)\right)|z_\pm(p_1)y_\mp(p_2)\rangle
\end{aligned} \tag{3.84}$$

- Massive-massless

$$\mathbb{T}\,|z_\pm(p_1)U(p_2)\rangle = \left(\frac{(\omega_1^\pm p_2+|p_2|p_1)}{2} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-|p_2|p_1)\right)|z_\pm(p_1)U(p_2)\rangle \tag{3.85}$$

- Massless-massless

$$\mathbb{T}\,|U(p_1)V(p_2)\rangle = \left(a-\frac{1}{2}\right)(|p_1|p_2-|p_2|p_1)\,|U(p_1)V(p_2)\rangle\ . \tag{3.86}$$

It has been used the notation $z_+ = z$, $z_- = \bar{z}$, $\omega^+ = \omega$, $\omega^- = \bar{\omega}$ and $U, V = u, \bar{u}, v, \bar{v}$ are generic massless particles. Furthermore, let E_1 and E_2 be the energies of the two incoming particles; these matrix elements are written in the kinematic configuration $v_1 > v_2$, where $v_1 = \frac{\partial E_1}{\partial p_1}$ and $v_2 = \frac{\partial E_2}{\partial p_2}$ are the velocities of the two particles. In fact, as we will see when we perform the explicit calculations for the mirror theory, at a certain point one has to exploit the property of the composition of the Dirac delta with a function, which gives an absolute value (e.g. in the massive-massive case it gives the contribution $1/|\omega_2^\pm(p_1\mp q)-\omega_1^\pm(p_2\mp q)|$). For this reason, the choice of the kinematic condition $v_1 > v_2$ (or $v_1 < v_2$) is necessary to remove the modulus in the denominators. The remaining matrix elements can be computed considering the symmetries discussed in the previous section.
From the structure of the $\mathbb{T}$ matrix, as we expect since the classical theory is integrable, we can check that the two-body S matrix respects the Yang-Baxter equation at tree-level. In particular, since we are considering the expansion at first order in $1/T$, $\mathbb{T}$ must satisfy the classical Yang-Baxter equation:

$$\begin{aligned}
&[\mathbb{T}_{12},\mathbb{T}_{13}] + [\mathbb{T}_{12},\mathbb{T}_{23}] + [\mathbb{T}_{13},\mathbb{T}_{23}] = 0\\
&\Longrightarrow \mathbb{T}^{\alpha l}_{ik}\mathbb{T}^{nm}_{\alpha j} + \mathbb{T}^{\alpha l}_{jk}\mathbb{T}^{nm}_{i\alpha} + \mathbb{T}^{m\alpha}_{jk}\mathbb{T}^{nl}_{i\alpha} = \mathbb{T}^{n\alpha}_{ik}\mathbb{T}^{ml}_{j\alpha} + \mathbb{T}^{n\alpha}_{ij}\mathbb{T}^{ml}_{\alpha k} + \mathbb{T}^{\alpha m}_{ij}\mathbb{T}^{nl}_{\alpha k}\,,
\end{aligned} \tag{3.87}$$

where we are summing over the Greek indices.
In order to see this last point, note that the only non-vanishing matrix elements are the diagonal elements. Therefore $\mathbb{T}$ has the following structure

$$\mathbb{T}_{ij}^{kl} = \delta_i^k \delta_j^l \mathbb{T}_{ij} \,, \tag{3.88}$$

and the classical Yang-Baxter equation is satisfied

$$\begin{aligned}\mathbb{T}_{ik}^{\alpha l}\mathbb{T}_{\alpha j}^{nm} + \mathbb{T}_{jk}^{\alpha l}\mathbb{T}_{i\alpha}^{nm} + \mathbb{T}_{jk}^{m\alpha}\mathbb{T}_{i\alpha}^{nl} &= \delta_j^m\,\delta_k^l\,\delta_i^n\,\left(\mathbb{T}_{ik}\mathbb{T}_{ij} + \mathbb{T}_{jk}\mathbb{T}_{ij} + \mathbb{T}_{jk}\mathbb{T}_{ik}\right)\\ &= \mathbb{T}_{ik}^{n\alpha}\mathbb{T}_{j\alpha}^{ml} + \mathbb{T}_{ij}^{n\alpha}\mathbb{T}_{\alpha k}^{ml} + \mathbb{T}_{ij}^{\alpha m}\mathbb{T}_{\alpha k}^{nl}\,.\end{aligned} \tag{3.89}$$

It is worth noting that the matrix elements depend on the gauge parameter a. At first sight this may seem strange, because physical observables should not depend on the gauge choice. The reason is due to the fact that here we are considering asymptotic states with generic values of the momenta; however, as pointed out above, the physical states have to respect the level matching condition. Furthermore, following the discussion in [57] it is worth checking that the energy spectrum of the string does not depend on the gauge parameter. Given an asymptotic state with momenta $p_1, \ldots, p_M$, the total worldsheet momentum (3.44) and Hamiltonian (3.37) are given by

$$p_{ws} = \sum_{k=1}^{M} p_k \,, \quad H = \sum_{k=1}^{M} \omega_{i_k}(p_k) \,, \tag{3.90}$$

where $\omega_i(p)$ is the energy of a particle of flavour labelled by i, computed using the non-perturbative dispersion relations. As we shall see in the next chapter, the momenta and consequently the energy spectrum are constrained by the Bethe equations, which, after diagonalising the S matrix can be written as

$$e^{ip_j l} \prod_{k\neq j}^{M} S_{i_j i_k}^{i_k i_j}(p_j, p_k; a) = 1 \,, \tag{3.91}$$

where $S_{i_j i_k}^{i_k i_j}$ are the component of the two-body complete (at all-loop) S matrix in the flavour space and $l = 2r$ is the length of the string.
The dependence of the S matrix from the gauge parameter a can be factorised as follows [58]

$$S_{i_j i_k}^{i_k i_j}(p_j, p_k; a) = e^{ia\left(p_k \omega_{i_j}(p_j) - p_j \omega_{i_k}(p_k)\right)} S_{i_j i_k}^{i_k i_j}(p_j, p_k) \tag{3.92}$$

On the other hand, combining (3.37) and (3.38) one can find

$$l = 2r = aE + (1-a)J = aH + J \,, \tag{3.93}$$

and inserting these expressions in the Bethe equations we obtain

$$\begin{aligned} e^{ip_j(J+aH)} \prod_{k\neq j}^{M} e^{ia\left(p_k \omega_{i_j}(p_j) - p_j \omega_{i_k}(p_k)\right)} S_{i_j i_k}^{i_k i_j}(p_j, p_k) &= e^{ip_j(J+aH)} e^{-iap_j H} \prod_{k\neq j}^{M} S_{i_j i_k}^{i_k i_j}(p_j, p_k) \\ &= e^{ip_j J} \prod_{k\neq j}^{M} S_{i_j i_k}^{i_k i_j}(p_j, p_k) = 1 \,, \end{aligned} \tag{3.94}$$

where we have used the level matching condition $p_{ws} = 0$. Therefore, the Bethe equations and then the energy spectrum do not depend on the gauge choice as expected.

Finally, let us discuss parity and time reversal from the point of view of the S matrix.
Time reversal sends

$$\omega \longrightarrow -\omega \quad , \quad p = \sqrt{\omega^2 - 1 + q^2} + q \longrightarrow p \,. \tag{3.95}$$

Hence, the matrix elements gain a minus sign. However, time reversal also changes the sign of the term inside the modulus, giving another overall minus sign. Therefore, the matrix elements do not change. On the other hand parity transformation sends

$$p \longrightarrow -p \quad , \quad \omega(p) = \sqrt{1-2qp+p^2} \longrightarrow \sqrt{1+2qp+p^2} = \bar{\omega}\,, \tag{3.96}$$

changing the structure of the matrix. Adding to parity the transformation $q \longrightarrow -q$, we have

$$p \longrightarrow -p \quad , \quad \omega(p) = \sqrt{1-2qp+p^2} \longrightarrow \sqrt{1-2qp+p^2} = \omega\,, \tag{3.97}$$

and the S matrix does not change.

3.7 Lagrangian

Our purpose is to discuss the mirror model obtained by this NLSM. As we shall see, in order to pass to the mirror theory, it is convenient to move to the Lagrangian description. For simplicity, we will consider only the field z, however, the same reasoning holds for all the other fields.
The Hamilton equation for z and $\bar{z}$ are

$$\dot{z} = \frac{\delta\mathcal{H}}{\delta\bar{p}_z} = p_z + O(3) \quad , \quad \dot{\bar{z}} = \frac{\delta\mathcal{H}}{\delta p_z} = \bar{p}_z + \bar{O}(3)\,,$$

where $O(3)$ are the higher order contributions (of order grater and equal to three) in the fields and their momenta. Note that there are no order two contributions because $\mathcal{H}^{(3)} = 0$. These equations can be solved perturbatively. In particular, at the first order we have $p_z = \dot{z}$. The second order, as mentioned above, is zero. The third order is obtained by considering the contribution given by $\delta\mathcal{H}^{(4)}/\delta\bar{p}_z$ and substituting p_z with its first order solution, that is, $\dot{z}$. Proceeding in this way, it is possible to find the expression of p_z as a function of the fields and their derivatives at any order. The general solution can be written as follows

$$p_z = \dot{z} + O(3) \quad , \quad \bar{p}_z = \dot{\bar{z}} + \bar{O}(3) \tag{3.98}$$

Clearly now $O(3)$ is a different function than before. However, we keep the same notation because they are still contributions of order grater and equal to three.
Now, performing the Legiandre transform

$$\begin{aligned}\mathcal{L} &= p_z\dot{\bar{z}} + \bar{p}_z\dot{z} - \mathcal{H}\\ &= \dot{\bar{z}}(\dot{z}+O(3)) + \dot{z}(\dot{\bar{z}}+\bar{O}(3)) - (\dot{z}+O(3))(\dot{\bar{z}}+\bar{O}(3)) - z\bar{z} - \acute{z}\acute{\bar{z}} - iq(\bar{z}\acute{z} - z\acute{\bar{z}})\\ &- \mathcal{H}^{(4)}(p_z = \dot{z}, \bar{p}_z = \dot{\bar{z}}) + O(6)\\ &= \dot{z}\dot{\bar{z}} - z\bar{z} - \acute{z}\acute{\bar{z}} - iq(\bar{z}\acute{z} - z\acute{\bar{z}}) - \mathcal{H}^{(4)}(p_z = \dot{z}, \bar{p}_z = \dot{\bar{z}}) + O(6)\,,\end{aligned} \tag{3.99}$$

where $O(6)$ contains interaction terms of order grater or equal than six. Then, restoring all the fields we find that the quadratic Lagrangian is

$$\mathcal{L}^{(2)} = \dot{z}\dot{\bar{z}} + \dot{y}\dot{\bar{y}} - z\bar{z} - y\bar{y} - \acute{z}\acute{\bar{z}} - \acute{y}\acute{\bar{y}} + \dot{u}\dot{\bar{u}} - \acute{u}\acute{\bar{u}} + \dot{v}\dot{\bar{v}} - \acute{v}\acute{\bar{v}} - iq(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}})\,. \tag{3.100}$$

On the other hand, it is worth noting that the quartic Lagrangian is given by

$$\mathcal{L}^{(4)}(x, \acute{x}, \dot{x}) = -\mathcal{H}^{(4)}(x, \acute{x}, p_x = \dot{x})\,, \tag{3.101}$$

where x is a compact notation which indicates all the fields. Let us finally point out that this result is no longer valid at higher orders. In fact, in general, there are non-trivial additional terms when substituting the higher order terms of p in $\mathcal{H}$.

3.7.1 Static gauge and T-duality

We have computed the worldsheet Lagrangian of the gauge-fixed theory starting from the Hamiltonian and performing the usual Legendre transform. Alternatively, one can directly compute the gauge-fixed Lagrangian without passing from the Hamiltonian. In order to proceed in this direction, it is necessary to introduce the so-called *T-duality* in string theory.
Let us consider a general bosonic string theory propagating in a background containing at least one circular dimension of radious R and angle ϕ. The string can wrap an integer number of times around this dimension. If we consider for simplicity the background $\mathbb{R}^{1,24} \times S^1$ (note that the total spacetime dimension is 26 because in this example we are dealing with a pure bosonic theory), the action for the angular part in given by [50]

$$S[\phi] = \frac{R^2}{4\pi\alpha'} \int d^2\sigma \partial_\alpha \phi \partial^\alpha \phi \,, \tag{3.102}$$

where α' is given in terms of the string tension by $\alpha' = 2\pi T$ and the partition function is

$$Z = \int D\phi \, e^{iS[\phi]} \,. \tag{3.103}$$

The action (3.102) is invariant under the shift of the angular field $\phi \longrightarrow \phi + \lambda$. We can gauge this symmetry by replacing the derivative with the covariant derivative defined as

$$\partial_\alpha \phi \longrightarrow D_\alpha \phi = \partial_\alpha \phi + A_\alpha \,, \tag{3.104}$$

where we have introduced a new field A_α, which transforms as follows

$$A_\alpha \longrightarrow A_\alpha - \partial_\alpha \lambda \tag{3.105}$$

After gauging the shift, in order to preserve the theory, one has to add the additional term $\tilde{\phi}\, \epsilon^{\alpha\beta} \partial_\alpha A_\beta$ to the Lagrangian. Thus, the action results

$$S[\phi, \tilde{\phi}, A_\alpha] = \frac{R^2}{4\pi\alpha'} \int d^2\sigma D_\alpha \phi D^\alpha \phi + \frac{1}{2\pi} \int d^2\sigma \tilde{\phi}\, \epsilon^{\alpha\beta} \partial_\alpha A_\beta \,, \tag{3.106}$$

where $\tilde{\phi}$ is a Lagrange multiplier and its equation of motion reads $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = 0$. This means that A_α is a pure gauge and can be chosen to be $A_\alpha = 0$, restoring the expression of the starting action. Therefore, the partition function

$$Z = \int D\phi D A_\alpha D\tilde{\phi}\, e^{iS[\phi, \tilde{\phi}, A_\alpha]} \,, \tag{3.107}$$

describes the same quantum theory as (3.103). Then, we can integrate out the field ϕ by fixing the gauge condition $\phi = 0$ with a delta function contribution in the partition function. The result is given by

$$Z = \int DA_\alpha D\tilde{\phi} \exp\left(\frac{iR^2}{4\pi\alpha'} \int d^2\sigma A_\alpha A^\alpha + \frac{i}{2\pi} \int d^2\sigma \tilde{\phi}\, \epsilon^{\alpha\beta} \partial_\alpha A_\beta \right) .$$

Finally, integrating out A_α one can obtain

$$Z = \int D\tilde{\phi} \exp\left(-\frac{\alpha'}{4\pi R^2} \int d^2\sigma \partial_\alpha \tilde{\phi} \partial^\alpha \tilde{\phi} \right) . \tag{3.108}$$

This new action describes the same quantum theory of the initial one (3.102), because they have the same partition function. The relation between the two theories is known as *T-duality*. From a physical point of view, note that the duality sends $R \longrightarrow \alpha'/R$. This means that in string theory, due to the finiteness of the particles, it is not possible to distinguish between a circular dimension of radious R and a circular dimension of radious α'/R.

Taking this reasoning in particular to the NLSM (3.1), let us denote the Lagrangian by $L(\partial_\alpha x^+, \partial_\alpha x^-, X^j)$. As we pointed out, this is invariant under the shift $x^- \longrightarrow x^- + \lambda$ because it is an isometry of the background metric. Therefore, it is possible to perform a T-duality in the x^- direction. By gauging the symmetry we obtain

$$L(\partial_\alpha x^+, \partial_\alpha x^-, X^j) \longrightarrow L(\partial_\alpha x^+, \partial_\alpha x^- + A_\alpha, X^j) + \tilde{x}^- \epsilon^{\alpha\beta}\partial_\alpha A_\beta\,,$$

where $\tilde{x}^-$ is the Lagrangian multiplier. The equation of motion for A_τ gives

$$\partial_\sigma \tilde{x}^- = \frac{\partial \mathcal{L}}{\partial \dot{x}^-} = p_-\,. \tag{3.109}$$

Fixing x^- and integrating out A_α gives the T-dual Lagrangian. Furthermore, the equation (3.109) shows that fixing the lightcone gauge in the first-order Lagrangian (3.24) is equivalent to fixing the static gauge

$$x^+ = \tau \quad , \quad \tilde{x}^- = \sigma \tag{3.110}$$

in the dual Lagrangian. This gauge condition does not depend on the conjugate momenta; therefore, it can be directly applied to the Lagrangian, without moving to the first-order formalism. This is an alternative way to find the Lagrangian of the gauge-fixed theory. Examples of this way of computing the gauge-fixed Lagrangian can be found in [59].

3.8 Alternative parameterisation

Let us make a final comment on the parameterisation chosen for the Anti-de Sitter (3.9) and the sphere(3.15). There is another useful parameterisation often chosen in the letterature (e.g. in [60]). Starting from the metric written in the form

$$ds^2 = \frac{1}{1+\rho^2}\,d\rho^2 - (1+\rho^2)\,dt^2 + \rho^2\,d\psi^2 + \frac{1}{1-r^2}\,dr^2 + (1-r^2)\,d\omega^2 + r^2\,d\phi^2\,, \tag{3.111}$$

it is possible to rewrite it using the stereographic coordinates defined as follows

$$\begin{aligned} \rho &= \sqrt{(X^1)^2 + (X^2)^2} \quad , \quad \psi = -\arctan\left(\frac{X^2}{X^1}\right), \\ r &= \sqrt{(X^3)^2 + (X^4)^2} \quad , \quad \phi = \arctan\left(\frac{X^4}{X^3}\right). \end{aligned} \tag{3.112}$$

In these coordinates the metric becomes

$$\begin{aligned} ds^2 =& \frac{1+(X^2)^2}{1+\rho^2}(dX^1)^2 + \frac{1+(X^1)^2}{1+\rho^2}(dX^2)^2 - \frac{2X^1X^2}{1+\rho^2}dX^1dX^2 - (1+\rho^2)dt^2 + \\ & \frac{1-(X^4)^2}{1-r^2}(dX^3)^2 + \frac{1-(X^3)^2}{1-r^2} + \frac{2X^3X^4}{1-r^2}dX^3dX^4 + (1-r^2)d\omega^2\,. \end{aligned} \tag{3.113}$$

Using these fields, it is possible to proceed in the same way as the previous parameterisation. In particular, one can define again the lightcone coordinates (3.27) for t and ω and fix the lightcone gauge (3.31). By solving perturbatively the two Virasoro constraints $C_1 = C_2 = 0$ in large string tension expansion and introducing the complex fields

$$\begin{aligned} X^1 &= \frac{Z-\bar{Z}}{i\sqrt{2}}\,, \qquad X^2 = \frac{Z+\bar{Z}}{-\sqrt{2}}\,, \qquad X^3 = \frac{Y+\bar{Y}}{-\sqrt{2}}\,, \qquad X^4 = \frac{Y-\bar{Y}}{i\sqrt{2}}\,, \\ P_1 &= \frac{P_Z-\bar{P}_Z}{i\sqrt{2}}\,, \qquad P_2 = \frac{P_Z+\bar{P}_Z}{-\sqrt{2}}\,, \qquad P_3 = \frac{P_Y+\bar{P}_Y}{-\sqrt{2}}\,, \qquad P_4 = \frac{P_Y-\bar{P}_Y}{i\sqrt{2}}\,, \end{aligned} \tag{3.114}$$

where P_j is the conjugate momentum of X^j, one can find the quadratic and quartic Hamiltonian

$$\begin{aligned} \mathcal{H}^{(2+4)} =& Z\bar{Z} + \acute{Z}\acute{\bar{Z}} + Y\bar{Y} + \acute{Y}\acute{\bar{Y}} + P_Z\bar{P}_Z + P_Y\bar{P}_Y + \frac{1}{2}\Big\{ -2P_Z\bar{P}_Z Y\bar{Y} + 2P_Y\bar{P}_Y Z\bar{Z} + 2\acute{Y}\acute{\bar{Y}} Z\bar{Z} \\ & - 2\acute{Z}\acute{\bar{Z}} Y\bar{Y} + 4P_Z\bar{P}_Z Z\bar{Z} - 4P_Y\bar{P}_Y Y\bar{Y} + 2Y\bar{Y}Y\bar{Y} - 2Z\bar{Z}Z\bar{Z} + Z^2(\bar{P}_Z^2 - \acute{\bar{Z}}^2) + \bar{Z}^2(P_Z^2 - \acute{Z}^2) \\ & + Y^2(\acute{\bar{Y}}^2 - \bar{P}_Y^2) + \bar{Y}^2(\acute{Y}^2 - P_Y^2) \Big\}\,. \end{aligned} \tag{3.115}$$

For simplicity we have considered the pure RR-flux case i.e. $q = 0$ and we have fixed the gauge choice $a = 1/2$. The quadratic Hamiltonian is the same as we found with the previous parameterisation. However, the quartic Hamiltonian seems to be different from the previous one. Note that we are dealing with the same gauge-fixed theory. In fact, the NLSM, as mentioned above, is invariant under field redefinitions (i.e. change of coordinates of the target manifold). Furthermore, in the two parameterisations t and ω are defined in the same way, and for this reason the gauge fixing conditions $x^+ = \tau$ and $p_- = 1$ are the same in both parameterisations. This means that the two Hamiltonians describe the same theory, but the fields are named in a different way. Due to the S matrix equivalent theorem we expect that if we compute the S matrix using (3.115) we should find the same result as before. In order to check this fact, let us remember that in interaction picture the fields respect the free equation of motions that in the $q = 0$ case are

$$\ddot{Z} - Z'' + Z = 0 \quad , \quad P_Z = \dot{Z} \,. \tag{3.116}$$

Exploiting these equations one can find the following equalities

$$\begin{aligned} &4P_Y\bar{P}_Y Y\bar{Y} = 4\dot{Y}\dot{\bar{Y}}Y\bar{Y} = 4Y'\bar{Y}'Y\bar{Y} + 2Y\bar{Y}Y\bar{Y} - Y^2(\dot{\bar{Y}}^2 - \acute{\bar{Y}}^2) - \bar{Y}^2(\dot{Y}^2 - \acute{Y}^2) + \text{total derivative}\,, \\ &4P_Z\bar{P}_Z Z\bar{Z} = 4\dot{Z}\dot{\bar{Z}}Z\bar{Z} = 4Z'\bar{Z}'Z\bar{Z} + 2Z\bar{Z}Z\bar{Z} - Z^2(\dot{\bar{Z}}^2 - \acute{\bar{Z}}^2) - \bar{Z}^2(\dot{Z}^2 - \acute{Z}^2) + \text{total derivative}\,. \end{aligned} \tag{3.117}$$

Therefore, up to a total derivative the Hamiltonian can be recast in the simpler form

$$\mathcal{H}^{(2+4)} = Z\bar{Z} + \acute{Z}\acute{\bar{Z}} + Y\bar{Y} + \acute{Y}\acute{\bar{Y}} + P_Z\bar{P}_Z + P_Y\bar{P}_Y - 2\acute{Z}\acute{\bar{Z}}Y\bar{Y} + 2\acute{Y}\acute{\bar{Y}}Z\bar{Z} + 2\acute{Z}\acute{\bar{Z}}Z\bar{Z} - 2\acute{Y}\acute{\bar{Y}}Y\bar{Y}\,, \tag{3.118}$$

that is exactly the same expression found for the previous parameterisation.
Finally, one can find that the field redefinitions that transform one description to the other one are the following

$$\begin{aligned} X^1 &= \frac{4z_1}{4 - z_1^2 - z_2^2} \quad , \quad X^2 = -\frac{4z_2}{4 - z_1^2 - z_2^2}\,, \\ X^3 &= \frac{4y_1}{4 + y_1^2 + y_2^2} \quad , \quad X^4 = \frac{4y_2}{4 + y_1^2 + y_2^2} \end{aligned} \tag{3.119}$$

and the complex fields are related by

$$Z = \frac{2i}{(2 - |z|^2)}\bar{z} \quad , \quad Y = -\frac{2}{(2 + |y|^2)}\bar{y}\,. \tag{3.120}$$

Chapter 4

Mirror Theory

In the previous chapter we discussed the NLSM on $AdS_3 \times S^3 \times T^4$. As we have seen, this is a $(1+1)$-dimensional quantum field theory defined on an infinite cylinder of circumference $2r = P_-$, fixed after the gauge-fixing. An important task is to compute the spectrum of the model and in particular the ground-state energy (GSE). For a quantum integrable theory, the energy spectrum is given by solving the Bethe equations. However, due to some finite-size effects, this approach does not directly work for the gauge-fixed NLSM. On the other hand, for a quantum integrable theory defined on an infinite line, the thermodynamic Bethe ansatz (TBA) equations allow to find exactly the free energy at finite temperature. This result can be used to find the GSE of the finite-size NLSM. In fact, by performing a double Wick rotation on a theory defined on a circle of length L and at temperature T, one obtains the so-called *mirror theory* defined on a circle of length $\tilde{L} = 1/T$ and at temperature $\tilde{T} = 1/L$. It has been shown [35] that the free energy of the mirror model computed for $\tilde{L} \longrightarrow \infty$ at finite $\tilde{T}$ is related to the ground state energy at $T \longrightarrow 0$ and finite L of the starting theory; therefore, this can be found using the TBA approach.
In this chapter we will discuss the thermodynamic Bethe ansatz and how it can be implemented in order to find the ground state energy of a field theory like the gauge-fixed NLSM. Furthermore, we will consider the mirror $AdS_3 \times S^3 \times T^4$ theory. Since this is necessary to construct the energy spectrum of the corresponding gauge-fixed NLSM, it is worth studying the behaviour of this theory especially with respect to the integrability. In particular, in this chapter we compute the tree-level S matrix of the mirror theory, and we will go further with other perturbative investigations in the next chapter.

4.1 Thermodynamic Bethe Ansatz

The Bethe ansatz equations (BAE) were first discovered by Bethe in [14], while solving the Heisenberg model, and together with their thermodynamic version, the thermodynamic Bethe ansatz equations, are powerful tools that allow to find the spectrum of quantum integrable systems. We will discuss these tools considering two quantum mechanical models, namely the Lieb-Liniger model, which is the first model in which the TBA approach has been introduced [15] and the XXX spin chain. In this section, we will mainly follow [41], [61] and [62].

4.1.1 Lieb-Liniger model

Let us consider a gas of N bosonic particles interacting via a delta function potential, defined on a circle of length L. Following [63] the Hamiltonian of the model is

$$H = \sum_{j=1}^{N} -\partial_{x_j}^2 + 2c \sum_{j_1 < j_2} \delta(x_{j_1} - x_{j_2}) \,, \tag{4.1}$$

where the interaction is considered repulsive, i.e. $c > 0$.
The Schrödinger equation reads

$$\left(-\sum_{j=1}^{N} \partial_{x_j}^2 + 2c \sum_{j_1<j_2} \delta(x_{j_1} - x_{j_2}) \right) \Psi(x_1, x_2, ..., x_N) = E\Psi(x_1, x_2, ..., x_N) \,. \tag{4.2}$$

Because of the delta function, this is equivalent to free particles

$$-\sum_{j=1}^{N} \partial_{x_j}^2 \Psi(x_1, x_2, ..., x_N) = E\Psi(x_1, x_2, ..., x_N) \,, \tag{4.3}$$

with the boundary conditions

$$(\partial_{x_j} - \partial_{x_k})\Psi|_{x_j=x_k+0^+} - (\partial_{x_j} - \partial_{x_k})\Psi|_{x_j=x_k+0^-} = 2c\Psi \,. \tag{4.4}$$

Therefore, due to the contact term, the wave functions are continuous but not derivable where two particles exchange their positions. Let us now restrict the problem to the region

$$D_1 = x_1 < x_2 < ... < x_N \,. \tag{4.5}$$

We are dealing with identical bosonic particles, therefore it is possible to find the solutions in D_1 and then recover the whole domain by symmetrysing the wave function. Restricting the domain to this region and exploiting the Bose symmetrisation in the LHS of (4.4), the boundary conditions become

$$(\partial_{x_{j+1}} - \partial_{x_j} - c)|_{x_{j+1}=x_j} \Psi(x_1, x_2, ..., x_N) = 0 \,. \tag{4.6}$$

Let us introduce the so-called *Bethe ansatz* for the wave function

$$\Psi(x_1, x_2, ..., x_N) = \sum_{P\in\pi_N} A_p \, e^{i\sum_{j=1}^{N} k_{P_j} x_j} \,, \tag{4.7}$$

where $\{k\}_j$ is a set of parameters which need to be determined by imposing that the wave function solves the Schrödinger equation (4.2) together with the boundary conditions of the problem and π_N is the set of all the permutations of N elements. Inserting this ansatz in (4.6) one can find that the wave function in the region D_1 is given by

$$\Psi_{D1}(x_1, x_2, ..., x_N) = \sum_{P\in\pi_N} (-1)^{[P]} e^{i\sum_{j=1}^{N} k_{P_j} x_j + \frac{i}{2}\sum_{N\geq j_1>j_2\geq 1} \phi(k_{P_{j_1}} - k_{P_{j_2}})} \,, \tag{4.8}$$

where $[P]$ is the parity of the permutation P and ϕ is given by

$$e^{i\phi(k_i-k_j)} = \frac{c + i(k_i - k_j)}{c - i(k_i - k_j)} \,. \tag{4.9}$$

Moreover, the expression of the total energy and momentum are respectively

$$E = \sum_{j=1}^{N} k_j^2 \quad , \quad P = \sum_{j=1}^{N} k_j \,. \tag{4.10}$$

At this point it is possible to find the complete wave function defined in the whole domain by symmetrising (4.8). Note that exchanging two particles gives an overall minus sign to the wave function because of the term $(-1)^{[P]}$. Furthermore, according to (4.9), $\phi(k_i - k_j)$ is antisymmetric under the exchange of k_i and k_j. Therefore, the wave function takes the form

$$\Psi(x_1, x_2, ..., x_N) = \prod_{N\geq j_1>j_2\geq 1} sgn(x_{j_1} - x_{j_2}) \sum_{P\in\pi_N} (-1)^{[P]} e^{i\sum_{j=1}^{N} k_{P_j} x_j + \frac{i}{2}\sum_{N\geq j_1>j_2\geq 1} sgn(x_{j_1}-x_{j_2})\phi(k_{P_{j_1}} - k_{P_{j_2}})} \,. \tag{4.11}$$

Note that the wave functions are antisymmetric under the exchange of the momenta and then they vanish if at least two of them are equals.

Bethe equations

The parameters k_i are not free, but they are fixed by the periodicity conditions. In fact, the theory is defined on a circle of length L and the wave functions must obey the relations

$$\Psi(x_1, x_2, ..., x_{j-1}, x_j + L, x_{j+1}, ..., x_N) = \Psi(x_1, x_2, ..., x_{j-1}, x_j, x_{j+1}, ..., x_N)\,. \tag{4.12}$$

Imposing these relations one can find that the set $\{k_j\}$ must satisfy

$$e^{ik_jL} = (-1)^{N-1} e^{-i\sum_{i\neq j}^{N} \phi(k_j-k_i)} = \prod_{i\neq j}^{N} \frac{k_j - k_i + ic}{k_j - k_i - ic}\,. \tag{4.13}$$

These are the Bethe equations for the Bose gas, first found by Lieb and Liniger in [63].
Let us consider for simplicity the case $N = 2$, the wave function in $D_1 = x_1 < x_2$ is

$$\Psi_{D_1}(x_1, x_2) = e^{ix_1k_1 + ik_2x_2 - \frac{i}{2}\phi(k_1-k_2)} - e^{ix_2k_1 + ik_2x_1 + \frac{i}{2}\phi(k_1-k_2)}\,. \tag{4.14}$$

This is composed by two contributions, namely an incoming and an outgoing state and the S matrix which connects these two states turns out[1]

$$S(k_1, k_2) = S(k_1 - k_2) = -e^{i\phi(k_1-k_2)} = \frac{k_1 - k_2 - ic}{k_1 - k_2 + ic}\,. \tag{4.15}$$

Therefore, by generalising to N particle the Bethe equations can be written as

$$e^{ik_jL} \prod_{i\neq j}^{N} S(k_j - k_i) = 1\,. \tag{4.16}$$

It is worth noting that like the factorised scattering case in QFT, the $n \longrightarrow n$ scattering is factorised into the product of two-body processes. Furthermore, this is an important writing because we will find again this expression when we discuss field theories.
Let us take the logarithm of the Bethe equations (4.13), we obtain

$$k_j + \frac{1}{L}\sum_{k} \phi(k_j - k_k) = \frac{2\pi}{L} J_j\,, \tag{4.17}$$

where

$$J_j \in \begin{cases} \mathbb{Z} + \frac{1}{2}, & \text{for } N \text{ even} \\ \mathbb{Z}, & \text{for } N \text{ odd} \end{cases}\,. \tag{4.18}$$

Note that now the sum includes also $i = j$. This is possible because as mentioned above, ϕ is antisymmetric in the exchange of the momenta, and then it goes to zero when the two momenta are equal.
On the other hand, taking the logarithm of the expression of the S matrix as a function of ϕ (4.15) gives

$$-i \log S(k_j - k_k) = \pi + \phi(k_j - k_k)\,. \tag{4.19}$$

Hence, the logarithm form of the Bethe equations can be written as follows

$$k_j - \frac{i}{L}\sum_{k} \log S(k_j - k_k) = \frac{2\pi}{L} I_j\,, \tag{4.20}$$

where $I_j = J_j + N/2$ and then $I_j \in \mathbb{Z} + 1/2$. Alternatively, one can directly take the logarithm on (4.16)

$$k_j - \frac{i}{L}\sum_{k\neq j} \log S(k_j - k_k) = \frac{2\pi}{L} \tilde{I}_j\,, \tag{4.21}$$

where now the sum is over all the indices except i and $\tilde{I}_j \in \mathbb{Z}$.
Let us now discuss some useful properties of the solutions of the Bethe equations (4.17)

[1] Now that we know the expression of the S matrix we can write the wavefunction in an alternative form. Multiplying by $e^{\frac{i}{2}\phi(k_1-k_2)}$ we obtain $\psi_{D_1}(x_1, x_2) = e^{ix_1k_1 + ix_2k_2} + S(k_1 - k_2)e^{ix_2k_1 + ix_1k_2}$. This is the same expression that we will find when we discuss the QFT Bethe ansatz.

- In the case at hand ($c > 0$) all the $\{k_j\}$ are real.
- For every set of $\{I_j\}$ such that $I_i \neq I_j$, $\forall\, i \neq j$ there is a unique set of $\{k_j\}$ such that $k_i \neq k_j$, $\forall\, i \neq j$ that is a solution of the corresponding Bethe equations. In fact, following [15] we can introduce the Yang-Yang functional defined as follows

$$B(\{k\}) = \frac{1}{2} L \sum_{j=1}^{N} k_j^2 + \frac{1}{2} \sum_{j,l} \Phi(k_j - k_l) - 2\pi \sum_{j=1}^{N} I_j k_j \,, \tag{4.22}$$

where

$$\Phi(k) = \int_0^k dk' \phi(k') \,. \tag{4.23}$$

Note that $\Phi(k)$ is an even function. The extremum conditions of the B functional are exactly the Bethe equations (4.17). Furthermore, the Hessian

$$\partial_i \partial_j B = \delta_{ij} \left(L + \sum_{l=1}^{N} \frac{2c}{(k_i - k_l)^2 + c^2)} \right) - \frac{2c}{(k_i - k_j)^2 + c^2)} \,, \tag{4.24}$$

is definite positive, and hence the minimum is a global minimum, showing that the solutions are unique. This means that $\{I_j\}$, also referred to as the quantum numbers, completely describe the set of eigenfunctions.
- $I_j > I_i \Rightarrow k_j > k_i$. In fact, the RHS of (4.17) is obviously monotonic increasing in the quantum numbers. On the other hand, the LHS is monotonic increasing in the momenta, since

$$\phi(k)' = \frac{2c}{c^2 + k^2} > 0 \,. \tag{4.25}$$

Thermodynamic

Following the discussion of Yang and Yang [15], let us analyze the thermodynamic of the Bose gas with a delta function interaction going in the thermodynamic limit

$$L \longrightarrow \infty \quad , \quad N \longrightarrow \infty \,, \tag{4.26}$$

where the density N/L is kept fixed. Note that, according to the BAE, the difference between two consecutive momenta $k_{j+1} - k_j$ depends on the momenta themselves, but is always of the order $1/L$. This means that in the thermodynamic limit we are no longer dealing with a lattice discrete theory, but we are dealing with a theory defined on the whole real line.
First, let us rewrite the BAE in the following way

$$k_j + \frac{1}{iL} \sum_{l} \log S(k_j - k_l) = h(k_j) \,, \tag{4.27}$$

where $h(k_j) = 2\pi I_j / L$. This is possible because, as mentioned above, the map between the momenta and the quantum numbers is a one-to-one map. Given the set of all possible quantum numbers $\{I_j\}$, in a given configuration, some of them will be occupied by a particle and some of them will be empty. The latter are called *holes*. It is useful to define the particle and hole densities ρ and ρ_h in such a way that

$$\begin{aligned} L\, \rho\, dk &= \#\text{ of particles in } dk \\ L\, \rho_h\, dk &= \#\text{ of holes in } dk \,. \end{aligned} \tag{4.28}$$

The sum of these two densities gives the total density of the quantum numbers and therefore as a function of $h = 2\pi I / L$ it turns out

$$\rho(h) + \rho_h(h) = \rho_t(h) = \frac{1}{2\pi} \,. \tag{4.29}$$

Thus, as a function of the k's

$$\rho(k) + \rho_h(k) = \rho_t(k) = \frac{1}{2\pi}\frac{dh}{dk}\,. \tag{4.30}$$

Taking the thermodynamic limit of (4.27) the sum becomes an integral weighted by the particle densities and the momenta become continuous variables $k_j \to p$

$$p + \frac{1}{i}\int_{-\infty}^{+\infty} dp' \,\log S(p-p')\rho(p') = h(p)\,. \tag{4.31}$$

Deriving with respect to p one obtains

$$1 + \frac{1}{i}\int_{-\infty}^{+\infty} dp' \,\frac{d}{dp}\log S(p-p')\rho(p') = \frac{dh}{dp}(p) = 2\pi(\rho(p)+\rho_h(p))\,. \tag{4.32}$$

This result can be recast in the compact form

$$\rho(p) + \rho_h(p) = \frac{1}{2\pi} + \mathcal{C} * \rho(p)\,, \tag{4.33}$$

where

$$\mathcal{C}(p) = \frac{1}{2\pi i}\frac{d}{dp}\log S(p) = \frac{1}{\pi}\frac{c}{c^2+p^2}\,, \tag{4.34}$$

is known as Cauchy kernel and the symbol $*$ defines the convolution product

$$(f * g)(p) = \int_{-\infty}^{+\infty} dp' \, f(p-p')g(p')\,. \tag{4.35}$$

The energy and the particle density are given by

$$e = \frac{E}{L} = \int_{-\infty}^{+\infty} dp\, p^2 \rho(p) \quad , \quad n = \frac{N}{L} = \int_{-\infty}^{+\infty} dp\, \rho(p)\,. \tag{4.36}$$

In the thermodynamic limit the particle and hole densities full characterise the Bethe states and they are constrained by the equation (4.33). For example, if we want to find the ground state of the model, according to the expression of the energy (4.36), the particle density has to be different from zero only in a symmetric interval around zero, namely $-p_f \leq p \leq p_f$, where p_f is called the Fermi momentum in analogy with the Fermi gas. On the other hand, the hole density has to be zero in this interval and different from zero outside the interval. In this way, by restricting (4.33) in the region $-p_f \leq p \leq p_f$ and by fixing the total density n, the particle density in the ground state is obtained by solving the two equations

$$\rho(p) = \frac{1}{2\pi} + \int_{-p_f}^{+p_f} dp' \, \mathcal{C}(p-p')\rho(p') \quad , \quad n = \int_{-p_f}^{+p_f} dp' \, \rho(p')\,. \tag{4.37}$$

Let us point out that a general state given by ρ and ρ_h has an entropy. In fact, there are different configurations that lead to a specific state due to the fact that the density functions tell us how many particles (holes) there are in an interval, but they do not give any information about which specific slots (quantum numbers) are occupied in that interval. In particular, before performing the thermodynamic limit, the number of available states in an interval Δk_j is $L\rho_t(k_j)\Delta k_j$, while the number of occupied and unoccupied states are $L\rho(k_j)\Delta k_j$ and $L\rho_h(k_j)\Delta k_j$, respectively. Finally, the number of all possible configurations is the product of the number of the configurations in each single interval

$$\# = \prod_j \frac{(L\rho_t(k_j)\Delta k_j)!}{(L\rho(k_j)\Delta k_j)!(L\rho_h(k_j)\Delta k_j)!}\,. \tag{4.38}$$

The entropy is the logarithm of the number of configurations. Thus, taking the logarithm of (4.38) one can find the expression of the entropy. In particular, in the thermodynamic limit, using the Stirling's approximation i.e. $\log n! = n\log n - n + 1/2\log(2\pi n) + O(1/n)$ the expression of the entropy becomes

$$\mathcal{S} = L\int_{-\infty}^{+\infty} dp\,[(\rho(p)+\rho_h(p))\log(\rho(p)+\rho_h(p)) - \rho(p)\log\rho(p) - \rho_h(p)\log\rho_h(p)]\,. \tag{4.39}$$

Now that we have the expression of the entropy (4.39), the energy and the number of particles (4.36) it is possible to write the thermodynamic partition function in the grand canonical ensemble

$$Z_{gc} = \sum_n \langle\psi_n| \, e^{-(H-\mu N)/T} \, |\psi_n\rangle = \int D[\rho, \rho_h] \, e^{-G[\rho,\rho_h]/T} \,, \tag{4.40}$$

where μ is the chemical potential, $|\psi_n\rangle$ is a base of eigenstates and G is given by

$$\begin{aligned} G[\rho, \rho_h] &= E[\rho, \rho_h] - T\mathcal{S}[\rho, \rho_h] - \mu N[\rho, \rho_h] \\ &= L \int_{-\infty}^{+\infty} dp \, [\rho(p)(p^2 - \mu - T \log(\rho(p) + \rho_h(p) + T \log \rho(p)) \\ &+ \rho_h(p)(T \log \rho_h(p) - T \log(\rho(p) + \rho_h(p)))] \,, \end{aligned} \tag{4.41}$$

It is worth noting that the integral in the partition function is restricted to the particle and hole densities which respect the Bethe equations (4.33).
In the thermodynamic limit, the overall L in the G expression goes to infinity, and then it is possible to evaluate the partition function using the saddle-point approximation.
The variation is

$$\delta G = L \int_{-\infty}^{+\infty} [\delta\rho(p^2 - \mu - T \log(1 + \rho_h/\rho)) - \delta\rho_h(T \log(1 + \rho/\rho_h))] \,. \tag{4.42}$$

However, as mentioned above ρ and ρ_h are not two independent fields. They must obey

$$\delta\rho(p) + \delta\rho_h(p) = \mathcal{C} * \delta\rho(p) \,. \tag{4.43}$$

Inserting (4.43) in (4.42), and defining

$$\frac{\rho_h}{\rho} := e^{\epsilon/T} \,, \tag{4.44}$$

the extremum condition $\delta G = 0$ is given by the equation

$$\epsilon(p) = p^2 - \mu - \mathcal{C} * T \log(1 + e^{-\epsilon(p)/T}) \,. \tag{4.45}$$

This is the equilibrium condition of the system and is also known as the *thermodynamic Bethe ansatz equation.*
Inserting the solutions of the extremum condition in the partition function one can find

$$Z_{gc} = e^{-\mathcal{G}/T} \,, \tag{4.46}$$

where $\mathcal{G}$ is the grand canonical free energy and is given by

$$\mathcal{G} = -\frac{LT}{2\pi} \int_{-\infty}^{+\infty} dp \, \log(1 + e^{-\epsilon/T}) \,. \tag{4.47}$$

This shows that once the TBA are solved all the thermodynamic is solved, because the free energy and then the partition function are known. For instance it is possible to compute the pressure of the gas

$$P = -\left(\frac{\partial \mathcal{G}}{\partial L}\right) = \frac{T}{2\pi} \int_{-\infty}^{+\infty} dp \, \log(1 + e^{-\epsilon/T}) \,. \tag{4.48}$$

4.1.2 XXX Spin chain

For the sake of completeness, let us briefly discuss the Heisenberg model, first introduced in [64], dealing with ferromagnetism. This is not only the first model ever solved by the Bethe ansatz, but it also allows us to generalise the previous discussion to models containing bound states.
The Hamiltonian is

$$H = -J \sum_{j=1}^{N} \left(\vec{S}_{j+1} \cdot \vec{S}_j - \frac{1}{4}\right) \,. \tag{4.49}$$

This describes a one dimensional lattice of N spins $\vec{S}_j$, interacting with the nearest-neighbor. If the coupling constant is positive, i.e. $J > 0$ the energy is minimised when all the spins are aligned along the same direction and, therefore, it describes a ferromagnet, while if $J < 0$ the energy is minimised when the neighbour spins are anti-parallel, describing an antiferromagnet. We are considering spin-1/2 particles, hence the total Hilbert space is

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathbb{C}^2 \,. \tag{4.50}$$

Furthermore, let us consider a closed chain with the condition $\vec{S}_{N+1} = \vec{S}_1$.
The Hamiltonian commutes with the projection of the total spin along any axis, in fact, being

$$\vec{S}_{tot} = \sum_{i=1}^{N} \vec{S}_i \ , \tag{4.51}$$

it holds

$$[H, S^z_{tot}] = [H, S^y_{tot}] = [H, S^x_{tot}] = 0 \,. \tag{4.52}$$

This means that it can be chosen a common base of eigenstates for both H and one of the projections of the total spin, for instance S^z_{tot}.
Let $|\uparrow\rangle_j$ and $|\downarrow\rangle_j$ be, respectively, the up and down spin states on the j-th lattice site, defined in such a way that $S^z_j |\uparrow\rangle_j = 1/2 |\uparrow\rangle_j$ and $S^z_j |\downarrow\rangle_j = -1/2 |\downarrow\rangle_j$. We can define the ferromagnetic wave function with all spins up by

$$|0\rangle = \bigotimes_{i=1}^{N} |\uparrow\rangle_i \ . \tag{4.53}$$

This state can be used to create all the other eigenstates of the S^z_{tot} operator. In fact, given the ladder operators,

$$S^{\pm}_j = S^x_j \pm i S^y_j \,, \tag{4.54}$$

which act on the up and down states in the following way

$$S^+ |\downarrow\rangle = |\uparrow\rangle \ , \ S^+ |\uparrow\rangle = 0 \, , \ S^- |\downarrow\rangle = 0 \, , \ S^- |\uparrow\rangle = |\downarrow\rangle \ , \tag{4.55}$$

one can define a generic state with M downspins as

$$\Psi_M = \sum_{1 \leq j_1 < j_2 < \ldots < j_M \leq N} \psi(j_1, j_2, ..., j_M) \, |j_1, j_2, ..., j_M\rangle \ , \tag{4.56}$$

where

$$|j_1, j_2, ..., j_M\rangle = S^-_{j_1}, ..., S^-_{j_M} |0\rangle \ . \tag{4.57}$$

The total magnetisation of such a state is $(N/2 - M)$.
The Bethe ansatz for the spin chain is

$$\psi(j_1, j_2, ..., j_M) = \sum_{P \in \pi_N} A_p \, e^{i \sum_{i=1}^{N} k_{P_i} j_i} \ , \tag{4.58}$$

and solving the Schrödinger equation the wave functions turn out

$$\Psi(j_1, j_2, ..., j_M) = \prod_{M \geq a > b \geq 1} sgn(j_a - j_b) \sum_{P \in \pi_M} (-1)^{[P]} e^{i \sum_{a=1}^{M} k_{P_a} j_a + \frac{i}{2} \sum_{M \geq a > b \geq 1} sgn(j_a - j_b) \phi(k_{P_a}, k_{P_b})} \,, \tag{4.59}$$

where

$$\phi(k_a, k_b) = i \log \left(\frac{\cos \frac{k_a + k_b}{2} - e^{\frac{i}{2}(k_a - k_b)}}{\cos \frac{k_a + k_b}{2} - e^{-\frac{i}{2}(k_a - k_b)}} \right) \ . \tag{4.60}$$

Defining the rapidities λ in such a way that

$$k(\lambda) = i \log \left(\frac{\frac{1}{2} + i\lambda}{\frac{1}{2} - i\lambda} \right) + \pi \,, \tag{4.61}$$

the phase $\phi(k_a, k_b)$ can be written as

$$\phi(k_a, k_b) = \theta(\lambda_a - \lambda_b) = i \log \left(\frac{i + \lambda_a - \lambda_b}{i - \lambda_a + \lambda_b} \right) \,, \tag{4.62}$$

and the energy of the system is given by

$$E = J \sum_{a=1}^{M} \frac{2}{4\lambda_a^2 + 1} \,. \tag{4.63}$$

Bethe equations

In the same way as the Bose gas, imposing the periodicity condition on the lattice wave functions

$$\psi(j_1, j_2, ..., j_{a-1}, j_a + N, j_{a+1}, ..., j_M) = \psi(j_1, j_2, ..., j_{a-1}, j_a, j_{a+1}, ..., j_N) \,, \tag{4.64}$$

one obtains the Bethe equations

$$e^{ik_a N} = (-1)^{M-1} e^{-i \sum_{b \neq a}^{M} \phi(k_a, k_b)} \,, \tag{4.65}$$

that in terms of the rapidities (4.61) can be written as

$$\left[\frac{\lambda_a + \frac{i}{2}}{\lambda_a - \frac{i}{2}} \right]^N = \prod_{b \neq a}^{M} \frac{\lambda_a - \lambda_b + i}{\lambda_a - \lambda_b - i} \,. \tag{4.66}$$

Following the same argument of the Bose gas case, also in this case the S matrix is factorised in $2 \longrightarrow 2$ scatterings, which one is given by

$$S(\lambda_a - \lambda_b) = -e^{i\phi(k(\lambda_a), k(\lambda_b))} = \frac{\lambda_a - \lambda_b - i}{\lambda_a - \lambda_b + i} \,, \tag{4.67}$$

and the Bethe equations can be rewritten as

$$e^{ik_a N} \prod_{b \neq a}^{M} S(\lambda_a - \lambda_b) = 1 \quad \text{or} \quad e^{ik_a N} \prod_{b=1}^{M} S(\lambda_a - \lambda_b) = -1 \,. \tag{4.68}$$

Where in order to pass from the first to the second expression it has been used the fact that $S(\lambda_a - \lambda_a) = -1$.

String configurations

In the Heisenberg model, the momenta are not constrained to be real as in the repulsive Bose gas, but they can assume complex values. Let us consider a complex momentum $k_a = k + i\eta$ with a positive imaginary part. In the limit $N \longrightarrow \infty$ the LHS of the Bethe equation goes

$$e^{ik_a N} = e^{ikN - \eta N} \longrightarrow 0 \,. \tag{4.69}$$

Therefore, also the RHS of (4.66) has to vanish once we take the limit. This means that there must exist an other rapidity λ_b such that in this limit goes as

$$\lambda_b = \lambda_a + i \,. \tag{4.70}$$

Note that imposing that the RHS of the Bethe equation goes to zero is equivalent to requiring that the S matrix has a pole. In particular, the condition (4.70) provides a pole for the matrix $S(\lambda_a - \lambda_b)$. Now, let us consider the Bethe equation for λ_b. If k_b has a positive imaginary part, the LHS goes to zero. On the other hand, the condition (4.70) does not make vanishing the RHS of this equation (from another point of view, this condition does not provide a pole for the matrix $S(\lambda_b - \lambda_a)$). Hence, there must exist another rapidity λ_c which satisfies the condition

$$\lambda_c = \lambda_b + i \Rightarrow \lambda_c = \lambda_a + 2i \,. \tag{4.71}$$

Conversely, if we consider a momentum with negative imaginary part, we have to impose that the S matrix vanishes when $N \longrightarrow \infty$. However, we can take these solutions into account by using the fact that the Bethe equations are invariant under complex conjugation. Following this procedure, it is possible to build states with an arbitrary large number of rapidities that are separated from each other by a multiple of i and which are symmetrically distributed around the real axis. These are called *string configurations*. The l-th set of rapidities of a string composed of n down-spins can be written in the compact form

$$\lambda_l^{n,a} = \lambda_l^n + i/2(n+1-2a)\,,\ a = 1, 2, ..., n\,, \tag{4.72}$$

where $\lambda_n^l \in \mathbb{R}$ is the centre of mass of the string. These strings have length n and we will call them n-string.

It is worth pointing out that despite the fact that the repulsive Bose gas does not have string solutions because all the momenta are real, the attractive gas instead shows this type of configurations. This is intuitive from a physical point of view because, as we will see, string solutions correspond to bound states, and we expect such states in a model with an attractive potential.

We have shown that in the limit $N \longrightarrow \infty$ the Bethe solution can be written as strings of arbitrary length. Let M_n be the number of n-strings; then the total number of down spins is given by

$$M = \sum_{n=1}^{\infty} n M_n \,. \tag{4.73}$$

It is possible to show that the string configurations are bound states. In fact, let us consider, for simplicity, the case $M = 2$; the Bethe wavefunction is

$$\psi(j_a, j_b) = e^{ij_a k_a + ij_b k_b - \frac{i}{2}\phi(k_a,k_b)} - e^{ij_b k_a + ij_a k_b + \frac{i}{2}\phi(k_a,k_b)} \,, \tag{4.74}$$

where in the limit $N \longrightarrow \infty$ we have the string configuration $\lambda_a = \lambda - i/2$ and $\lambda_b = \lambda + i/2$. The phase shift becomes

$$\phi(k_a, k_b) = i \log\left(\frac{i + \lambda - \lambda - i}{i - \lambda + \lambda + i}\right) \longrightarrow -i \cdot \infty \,. \tag{4.75}$$

Therefore, the the second term of the wave function dominates

$$\psi(j_a, j_b) \approx -const \cdot e^{ik(j_a+j_b) - \eta(j_a - j_b)} \,, \tag{4.76}$$

where, since according to (4.61) $k_a = k_b^*$, it has been used the notation $k_a = k + i\eta$ and $k_b = k - i\eta$. The wave function is then composed by a term which describes the dynamic of the centre of mass of the two spins and a term that is exponentially suppressed in the distance between the two spins, representing the bounding between them.

Inserting the strings configurations in the Bethe equation (4.68) one obtain

$$e^{ik_a N} \prod_{n=1}^{\infty} \prod_{l=1}^{M_n} \prod_{b=1}^{n} S(\lambda_a - \lambda_l^{n,b}) = -1 \,, \tag{4.77}$$

where, the product is constrained by the total downspin configuration (4.73).

By taking the product of (4.77) over all rapidities within the string to which k_a belongs, the above set of equations can be simplified as follows

$$\left(-\frac{\lambda_j^m + i\frac{m}{2}}{\lambda_j^m - i\frac{m}{2}}\right)^N \prod_{n=1}^{\infty} \prod_{l=1}^{M_n} \prod_{a=1}^{m} \prod_{b=1}^{n} S(\lambda_j^{m,a} - \lambda_l^{n,b}) = (-1)^m \,. \tag{4.78}$$

The first term, written using the momenta instead of the rapidities is

$$\left(-\frac{\lambda_j^m + i\frac{m}{2}}{\lambda_j^m - i\frac{m}{2}}\right)^N = e^{iN\sum_{a=1}^m k(\lambda_j^{m,a})} = e^{ik_j^m N}\,, \tag{4.79}$$

where k_j^m is the total momentum of the string. Thus,the Bethe equations for strings is

$$e^{ik_j^m N}\prod_{n=1}^{\infty}\prod_{l=1}^{M_n}\prod_{a=1}^{m}\prod_{b=1}^{n} S(\lambda_j^{m,a} - \lambda_l^{n,b}) = (-1)^m\,. \tag{4.80}$$

In the limit $N \longrightarrow \infty$ the strings configurations are exactly solutions of the Bethe equations. It is worth noting that this limit is different from the thermodynamic limit in which also the number of downspins M goes to infinity and their ratio N/M remains fixed. Indeed, in the thermodynamic limit, there are solutions that deviates from the string configurations. However, the *string hypothesis*, first made by Bethe, assumes that the thermodynamic of the system (i.e. the free energy) is completely described in terms of only strings solutions.

Thermodynamic

The thermodynamic of the spin chain can be studied following the same procedure as for the Bose gas and also including the string configurations.
First, let us take the logarithm of the Bethe equation (4.80)

$$k_j^m - \frac{1}{N}\sum_{n=1}^{\infty}\sum_{l=1}^{M_n}\Theta_{mn}(\lambda_j^m - \lambda_l^n) = \frac{2\pi}{N}I_j^m\,, \tag{4.81}$$

where

$$\Theta_{mn}(\lambda_j^m - \lambda_l^n) = i\sum_{a=1}^{m}\sum_{b=1}^{n}\log S(\lambda_j^{m,a} - \lambda_l^{n,b})\,. \tag{4.82}$$

Note that, as one can check performing the explicit computation, the sum over all the spins along a string gives a function $\theta_{mn}(\lambda_j^m - \lambda_l^n)$ which depends only on the centre of mass of the strings, as we have already seen in the LHS of (4.78). As in the Bose gas case, let us take the thermodynamic limit. First, we introduce the particle and hole densities ρ_n and ρ_n^h. In particular, in this case ρ_n is the density of n-strings present in the configuration, while ρ_n^h is the density of n-string holes. Deriving (4.81) with respect to the rapidity and following the same procedure as before, we obtain

$$a_n(\lambda) - \sum_{m=1}^{\infty} K_{nm} * \rho_m(\lambda) = \rho_n(\lambda) + \rho_n^h(\lambda)\,, \tag{4.83}$$

where

$$a_n(\lambda) = \frac{1}{2\pi}\frac{d}{d\lambda}k_j^n(\lambda) = \frac{1}{2\pi}\frac{n}{\lambda^2 + n^2/4} \tag{4.84}$$

and

$$K_{nm} = \frac{1}{2\pi}\frac{d}{d\lambda}\Theta_{nm}(\lambda)\,. \tag{4.85}$$

Note that for convenience in (4.83) we have exchanged the labels n and m with respect to the previous notation. Furthermore, since we consider the thermodynamic limit, the centre of mass of the string λ_j^m is replaced by the continuous variable λ. In order to compute the free energy, we need the expression of the entropy, the energy, and the number of excitations (in this case downspins).
Following the same discussion of the Lieb-Liniger model, the entropy results

$$\mathcal{S} = N\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d\lambda\,[(\rho_n(\lambda) + \rho_n^h(\lambda))\log(\rho_n(\lambda) + \rho_n^h(\lambda)) - \rho_n(\lambda)\log\rho_n(\lambda) - \rho_n^h(\lambda)\log\rho_n^h(\lambda)]\,. \tag{4.86}$$

The energy is given by

$$e=\frac{E}{N}=\frac{J}{N}\sum_{n=1}^{\infty}\sum_{l=1}^{M_n}\sum_{a=1}^{n}\frac{2}{4(\lambda_l^{n,a})^2+1}=\frac{J}{N}\pi\sum_{n=1}^{\infty}\sum_{l=1}^{M_n}a_n(\lambda_l^n)=-J\pi\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}d\lambda\, a_n(\lambda)\rho_n(\lambda)\,. \quad (4.87)$$

Finally the number of excitations is

$$M=\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}d\lambda\, n\rho_n(\lambda)\,. \quad (4.88)$$

Now it is possible to write the the grand canonical free energy. In the thermodynamic limit, as the Bose gas, the overall system size factor N allows the partition function to be evaluated using the saddle-point method. Pointing out that, according to the Bethe equation (4.83), the variation of the particles and of the holes densities are connected by

$$\sum_{m=1}^{\infty}K_{nm}*\delta\rho_m+\delta\rho_n+\delta\rho_n^h=0\,, \quad (4.89)$$

then the extremum condition $\delta G=0$ gives the Thermodynamic Bethe ansatz equations for the XXX spin chain model

$$\log Y_n=\frac{J\pi}{T}a_n+\sum_{m=1}^{\infty}K_{nm}*\log\left(1+\frac{1}{Y}_m\right)\,. \quad (4.90)$$

Where in the equation above we defined

$$Y_n(\lambda):=\frac{\rho_n^h}{\rho_n}(\lambda) \quad (4.91)$$

and

$$E_n:=J\pi a_n(\lambda) \quad (4.92)$$

is the energy per unit of system size of an n-string centered around the rapidity λ. Note that the result (4.90) is the generalisation of the one found for the Lieb-Liniger model. In fact, in that case there were only one-dimensional strings, the system size was L and the energy per unit of L was $p^2-\mu$.

Furthermore, note that, plugging the expression of $\delta\rho_n^h$ as a function of $\delta\rho_n$ inside the expression of the variation of the free energy, in order to isolate the variation $\delta\rho_n$, the convolution passes from the terms K_{mn} and $\delta\rho_m$ to K_{mn} and $T\log(1+\rho_n/\rho_n^h)$. The correct way to do that would be using the convolution from the right, namely $\log(1+\rho_n/\rho_n^h)\tilde{*}K_{nm}$, defined as follows

$$f\tilde{*}g=\int_{-\infty}^{+\infty}d\lambda'\, f(\lambda')g(\lambda'-\lambda)\,. \quad (4.93)$$

However, both in the Bose gas and in the spin chain case the kernels are even functions and therefore (4.93) is equivalent to the standard convolution.
Finally, using the equilibrium condition, the grand canonical free energy is

$$\mathcal{G}=-TN\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}d\lambda\, a_n(\lambda)\log\left(1+\frac{1}{Y_n}(\lambda)\right)\,. \quad (4.94)$$

Therefore, solving the TBA equations (4.90) iteratively it is possible to find the Y_n functions. By plugging them into the free energy, the thermodynamic of the system is solved. This approach can be generalised to all models supporting the string hypothesis.

4.2 Bethe-Yang equations in field theories

In the previous section we have discussed the TBA for quantum mechanical models; however, we are interested in generalising that description to QFT, in the context of factorised scattering theory. Following [49], let us consider the asymptotic state

$$|\Psi(p_1,...,p_M)\rangle = \sum_{\pi\in S_M} \chi(p_{\pi_1},...,p_{\pi_M})\,|p_{\pi_1},...,p_{\pi_M}\rangle\ , \tag{4.95}$$

Note that these states are the analogous of the Bethe ansatz states that we have considered in the spin-chain case. In fact, these act as eigenstates of the Hamiltonian since knowing the asymptotic momenta of the particles and their dispersion relations allows to find the energy of the state.
Furthermore, let us emphasise that we are considering asymptotic states. This is due to the fact that in QFT we deal with an asymptotic S matrix, computed when particles are infinitely distant from each other in such a way that they do not interact. This means that in string theory, this treatment is valid in the decompactification limit, in which the worldsheet becomes an infinite plane and the asymptotic states can be defined.
In a two dimensional integrable theory, the asymptotic states can be written using the ZF algebra operators

$$|p_1,...,p_M\rangle = \int_{\sigma_1\ll\ldots\ll\sigma_M} d\sigma_1\ldots d\sigma_M\, e^{i(p_1\sigma_1+\cdots+p_M\sigma_M)}A^\dagger(\sigma_1)\ldots A^\dagger(\sigma_M)\,|0\rangle\ , \tag{4.96}$$

where we are considering the region $\sigma_1 \ll \ldots \ll \sigma_M$. This is a configuration with a well-defined total momenta and it is known as *magnon* or *spin wave*. The multimagnon state (4.95) is composed by different magnons and we can relate them by recalling the ZF algebra (1.74)

$$A_i^\dagger(p_1)A_j^\dagger(p_2) = A_l^\dagger(p_2)A_k^\dagger(p_1)R_{ij}^{kl}(p_1,p_2)\,, \tag{4.97}$$

where for bosons $R_{ij}^{kl} = S_{ij}^{kl}$ is the S matrix. For simplicity in this discussion we will consider only one type of excitantion and we will drop the indices.
In this way, considering for instance two particles, we can write

$$\begin{aligned}|\Psi(p_1,p_2)\rangle &= \chi(p_1,p_2)\,|p_1,p_2\rangle + \chi(p_2,p_1)\,|p_2,p_1\rangle \\ &= \chi(p_1,p_2)S(p_1,p_2)\,|p_2,p_1\rangle + \chi(p_2,p_1)S(p_2,p_1)\,|p_1,p_2\rangle\ ,\end{aligned} \tag{4.98}$$

from which follows

$$\chi(p_2,p_1) = S(p_1,p_2)\chi(p_1,p_2)\,. \tag{4.99}$$

Therefore, the magnons with two contiguous indices permuted are related by

$$\chi(p_1,...,p_{j+1},p_j,...,p_M) = S(p_j,p_{j+1})\chi(p_1,...,p_j,p_{j+1},...,p_M)\,. \tag{4.100}$$

In factorised scattering theory the S matrix is decomposed in the product of two-to-two scattering processes. Thus, the multimagnon state can be written in the following way

$$|\Psi(p_1,...,p_M)\rangle = \sum_{\pi\in S_M}\prod_{(j,k)\in\pi} S(p_j,p_k)\,|p_{\pi_1},...,p_{\pi_M}\rangle\ , \tag{4.101}$$

where it has been set $\chi(p_1,...,p_M) = 1$.
Let us stress that this has the same expression as the Bethe wavefunction found in both the Lieb-Liniber and the Heisenberg models; i.e. it is a sum of free waves weighted by the relative S matrices.
So far, as mentioned above, we have considered the decompactification limit of the string. In order to try to recover the finite-size theory, one can mimic what we did in the quantum mechanic systems and impose the periodic conditions. Let us again consider for simplicity the two-particle case. Let $l = 2r$

be the length of the string and consider the shift $\sigma_1 \longrightarrow \sigma_1 + l$. In position space, the periodicity condition reads [2]

$$|\Psi(\sigma_1, \sigma_2)\rangle = |\Psi(\sigma_2, \sigma_1 + l)\rangle \,, \tag{4.102}$$

where

$$|\Psi(\sigma_1, \sigma_2)\rangle = \int dp_1\, dp_2\, e^{-ip_1\sigma_1 - ip_2\sigma_2} \left(|p_1, p_2\rangle + S(p_1, p_2)\,|p_2, p_1\rangle\right) . \tag{4.103}$$

The condition (4.102) tells that the physics of the system does not change if we consider the particle with label 1 as the leftmost one or the rightmost one because the worldsheet space is a circle. Using (4.103) and (4.102) the explicit condition is

$$\int dp_1\, dp_2\, e^{-ip_1\sigma_1 - ip_2\sigma_2} \left(|p_1, p_2\rangle + S(p_1, p_2)\,|p_2, p_1\rangle\right) = \int dp_2\, dp_1\, e^{-ip_2\sigma_2 - ip_1(\sigma_1 + l)} \left(|p_2, p_1\rangle + S(p_2, p_1)\,|p_1, p_2\rangle\right) \,, \tag{4.104}$$

that is equivalent to

$$\int dp_1\, dp_2\, e^{-ip_1\sigma_1 - ip_2\sigma_2}\, |\Psi(p_1, p_2)\rangle = \int dp_1\, dp_2\, e^{-ip_1\sigma_1 - ip_2\sigma_2} \left(e^{-ip_1 l} S^{-1}(p_1, p_2)\, |\Psi(p_1, p_2)\rangle\right) . \tag{4.105}$$

Generalising this reasoning to M particles, the periodicity condition $\sigma_j \longrightarrow \sigma_j + l$ is

$$|\Psi(p_1, ..., p_M)\rangle = \left(e^{ip_j l} \prod_{k\neq j}^{M} S(p_j, p_k)\right) |\Psi(p_1, ..., p_M)\rangle \,, \tag{4.106}$$

which leads to the Bethe equations

$$e^{ip_j l} \prod_{k\neq j}^{M} S(p_j, p_k) = 1 \,. \tag{4.107}$$

This result is intuitive because moving a particle from the σ_j position to $\sigma_j + l$ is achieved by scattering it with all the other particles in between and this returns the product of the S matrices that we find in the Bethe equations.
In principle these equations allow us to find the momenta and then the spectrum of the worldsheet Hamiltonian. However, let us recall that we are dealing with an asymptotic S matrix defined in the decompactification limit and therefore this does not account for wrapping configurations [32]. These configurations give some corrections to the Bethe equations which are exponentially suppressed in l. This means that, while the BAE return only the asymptotic spectrum of the theory and are not able to provide a finite-size description, the TBA equations, in which $l \longrightarrow \infty$ provide the correct thermodynamics of the system. As we shall see in the next chapter, these can be used to find the ground-state energy of the string.

4.3 Mirror transformation

In the previous section, we pointed out that the BAE are not able to describe the spectrum of a finite-size worldsheet string due to wrapping effects. However, as first discussed in [35] for relativistic theories and then generalised in [65] for non-relativistic theories, it is possible to use the TBA to find the ground state energy.
Let us consider a theory in which both the time and space coordinates are periodic, i.e. $\tau \in S^1$ and $\sigma \in S^1$. This means that this theory is defined on a torus. Let R and L be respectively the length of the time and space circumference, we can write the Hamiltonian

$$H = \int_0^L d\sigma\, \mathcal{H}(p, x, x') \,. \tag{4.108}$$

[2]In particular in this expression the first spatial variable labels the leftmost particle, while the second spatial variable labels the rightmost particle. This is the reason why after the l-shift the two arguments σ_1 and σ_2 are exchanged in parentheses.

Since both time and space are periodic, the Euclidean quantum partition function Z is equal to the thermodynamic partition function Z_{TH} in which the temperature in defined by $T = 1/R$. Indeed, given the Hilbert space $\mathcal{H}$ of the theory, the thermodynamic partition function is

$$Z_{TH} := Tr_{\mathcal{H}} \left(e^{-HR}\right) = \int dy \, \langle y| \, e^{-HR} \, |y\rangle \tag{4.109}$$

where we are tracing over the coordinate basis $|y\rangle$. Using the well-known expression of the time evolution kernel (for instance, it can be found in Chapter 9 of [66])

$$\langle x_f(\tau')|x_i(\tau)\rangle = \langle x_f(\tau)|e^{-H(\tau'-\tau)}|x_i(\tau)\rangle = \int_{\substack{x(\tau)=x_i \\ x(\tau')=x_f}} Dx\, Dp \, \exp\left(i \int_{\tau}^{\tau'} d\tilde{\tau} \, \dot{x}(\tilde{\tau}) p(\tilde{\tau}) - H(p(\tilde{\tau}), x(\tilde{\tau})) \right) , \tag{4.110}$$

and exploiting the periodicity condition $y(0) = y(R) = y$ we can rewrite the thermodynamic partition function as

$$\begin{aligned} Z_{TH} &= \int dy \, \langle y| \, e^{-HR} \, |y\rangle = \int dy \, \langle y(R)|y(0)\rangle = \int dy \int_{\substack{x(0)=y \\ x(R)=y}} DpDx \, \exp\left(\int_0^R d\tau \int_0^L d\sigma \, (ip\dot{x} - \mathcal{H}) \right) \\ &= \int DpDx \, e^{\int_0^R d\tau \int_0^L d\sigma \, (ip\dot{x} - \mathcal{H})} = Z(R, L) \, , \end{aligned} \tag{4.111}$$

where the last integral is computed on all the possible paths that start at $\tau = 0$ and end at $\tau = R$ at the same spatial point. This is exactly the definition of the path integral for a quantum theory defined on a torus. The exponential $\int_0^R d\tau \int_0^L d\sigma \, (ip\dot{x} - \mathcal{H})$ is the Euclidean action written in the first-order formalism defined by a Wick rotation $\tau \longrightarrow -i\tau$ on the Minkowski action. Note that sending $R \longrightarrow \infty$, ($T \longrightarrow 0$) we recover the theory defined on an infinite cylinder. On the other hand, by sending $L \longrightarrow \infty$ we end up with a theory defined on an infinite line at finite temperature, which can be studied by means of the TBA equations.

Starting from the thermodynamic definition, another useful way of writing the partition function is

$$Z(R, L) = Tr_{\mathcal{H}} \left(e^{-HR}\right) = \sum_n \langle \psi_n| \, e^{-HR} \, |\psi_n\rangle = \sum_n e^{-E_n R} \, , \tag{4.112}$$

where $|\psi_n\rangle$ is a complete set of Hamiltonian eigenstates.

Let us now introduce a new theory, known as *mirror theory*, defined by a double Wick rotation

$$\tau \longrightarrow -i\tau \quad , \quad \sigma \longrightarrow i\sigma \tag{4.113}$$

on the original theory and let us now consider σ as the time variable and τ as the space one. In the next sections, in order to make this last point more explicit, we will denote a mirror transformation by $\tau \longrightarrow -i\sigma$ and $\sigma \longrightarrow i\tau$. However, for now, in this proof it is simpler to keep the same notation.

The mirror Hamiltonian results

$$\tilde{H} = \int_0^R d\tau \tilde{\mathcal{H}}(\tilde{p}, x, \dot{x}) \, . \tag{4.114}$$

If the starting theory is relativistic, then the two theories are the same. In fact, a Lorentz-invariant Lagrangian is composed by terms with all the Lorentz indices contracted, like $\eta^{ab}\partial_a\phi\partial_b\phi$, where η^{ab} is the Minkowski metric and ϕ a scalar field. A mirror transformation changes the overall sign of the metric ($d\tau^2 - d\sigma^2 \longrightarrow -d\tau^2 + d\sigma^2$), but this is recovered by exchanging the meanings of time and space. On the other hand, a non-relativistic theory, as the gauge-fixed worldsheet theory, gives rise to a different theory under a mirror transformation.

Given the Hamiltonian, we can define the mirror partition function

$$\tilde{Z}(R, L) = \int D\tilde{p}Dx \, e^{\int_0^R d\tau \int_0^L d\sigma \, (i\tilde{p}x' - \tilde{\mathcal{H}})} \, , \tag{4.115}$$

which in thermodynamic notation reads

$$\tilde{Z}(R,L) = \sum_n \langle\tilde{\psi}_n| \, e^{-\tilde{H}L} \, |\tilde{\psi}_n\rangle = \sum_n e^{-\tilde{E}_n L} \,. \tag{4.116}$$

Let $\mathcal{L}(x,\dot{x},x')$ be the Lagrangian density of the starting theory. The mirror Lagrangian will be $\tilde{\mathcal{L}} = \mathcal{L}(x, i\dot{x}, -ix')$. Performing the Euclidean rotation in both theories, we obtain

$$\mathcal{L}(x,\dot{x},x') \xrightarrow{\tau \longrightarrow -i\tau} \mathcal{L}(x,i\dot{x},x') \quad , \quad \mathcal{L}(x,i\dot{x},-ix') \xrightarrow{\sigma \longrightarrow -i\sigma} \mathcal{L}(x,i\dot{x},x') \,, \tag{4.117}$$

which are the same Lagrangian. This means that the two Hamiltonian in the Euclidean are the Legendre transforms of the same function but one with respect to time and the other with respect to space, namely $\mathcal{H} = i\dot{x}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \mathcal{L}$ and $\tilde{\mathcal{H}} = i\acute{x}\frac{\partial\mathcal{L}}{\partial\acute{x}} - \mathcal{L}$.
Therefore, integrating out the momenta in the Euclidean partition functions we find

$$\tilde{Z}(R,L) = Z(R,L) \,. \tag{4.118}$$

To be precise, the integration over the momenta, in addition to the Lagrangian, gives also non-trivial measure factors. We will come back to these corrections in the next chapter in section 5.1, for the moment we will suppose that they do not spoil the relation between the two partition functions.
We have found that a model defined on a circle of length L at temperature $T = 1/R$ is equivalent to its mirror model defined on a circle of length R and at temperature $1/L$. If we send $R \longrightarrow \infty$ we end up with the partition function of the string theory defined on an infinite cylinder. In particular, we have

$$Z(R,L) = \sum_n \langle\psi_n| \, e^{-HR} \, |\psi_n\rangle = \sum_n e^{-E_n R} \approx e^{-E(L)R} \,, \tag{4.119}$$

where $E(L)$ is the ground state energy of the string. On the other hand, applying the limit in the mirror theory, we end up with a theory defined on an infinite line at finite temperature $T = 1/L$. Therefore, the partition function is the exponential of the free energy

$$\tilde{Z}(R,L) = e^{-RLf(L)} \,, \tag{4.120}$$

and as we know from the previous sections this can be computed by meas of the TBA. In particular, here f is the free energy for unit of the system size. Comparing the two partition functions it follows

$$E(L) = Lf(L) \,. \tag{4.121}$$

Hence, the ground state energy depends on the free energy of the mirror model and can be exactly computed using the thermodynamic Bethe ansatz.

4.4 Mirror $AdS_3 \times S^3 \times T^4$ model

Let us come back to the $AdS_3 \times S^4 \times T^4$ NLSM. According to what we have found in the previous section, it is possible to define the corresponding mirror theory and use it to find the ground state energy. This has been done in the pure RR background case [67]. In order for the TBA equations to be used, the mirror theory has to be integrable. As mentioned above, the string and mirror theory are related by a double Wick rotation. Intuitively, we can expect that the map between the S matrices is given by the following analytic continuation

$$\omega \longrightarrow \tilde{\omega} = -ip \quad , \quad p \longrightarrow \tilde{p} = i\omega \,. \tag{4.122}$$

However, the relation between the Wick rotation and the analytic continuation of the correlation functions, which has been studied in the context of axiomatic QFT by relating the Wightman construction in Minkowski space to the Osterwalder–Schrader construction in Euclidean space, is subtle and in this case needs to be investigated.
Therefore, let us discuss the relation between the two theories at the perturbative level. First, we find

the mirror two-body S matrix at the tree-level and compare it with the result obtained in the NLSM (3.84)-(3.85)-(3.86).
By sending $\tau \longrightarrow \tilde{\tau} = -i\sigma$ and $\sigma \longrightarrow \tilde{\sigma} = i\tau$ in the gauge-fixed NLSM Lagrangian, we can find the mirror one, which up to the fourth order is

$$\begin{aligned}
\mathcal{L}^{(2)+(4)}_{mirror} =& \dot{z}\dot{\bar{z}} + \dot{y}\dot{\bar{y}} - z\bar{z} - y\bar{y} - \acute{z}\acute{\bar{z}} - \acute{y}\acute{\bar{y}} + \dot{u}\dot{\bar{u}} - \acute{u}\acute{\bar{u}} + \dot{v}\dot{\bar{v}} - \acute{v}\acute{\bar{v}} + q(z\dot{\bar{z}} - \bar{z}\dot{z} + y\dot{\bar{y}} - \bar{y}\dot{y}) \\
& - \frac{1}{2}\Big(- 4z\bar{z}\dot{z}\dot{\bar{z}} + 4y\bar{y}\dot{y}\dot{\bar{y}} + 2y\bar{y}\dot{z}\dot{\bar{z}} + 2y\bar{y}\acute{z}\acute{\bar{z}} - 2z\bar{z}\dot{y}\dot{\bar{y}} - 2z\bar{z}\acute{y}\acute{\bar{y}} \\
& + 2(y\bar{y} - z\bar{z})(\dot{u}\dot{\bar{u}} + \dot{v}\dot{\bar{v}} + \acute{u}\acute{\bar{u}} + \acute{v}\acute{\bar{v}}) \\
& + q(z\dot{\bar{z}} - \bar{z}\dot{z} + y\dot{\bar{y}} - \bar{y}\dot{y})(y\bar{y} - z\bar{z}) \\
& + q(z\dot{\bar{z}} - \bar{z}\dot{z} - y\dot{\bar{y}} + \bar{y}\dot{y})(\acute{z}\acute{\bar{z}} + \dot{z}\dot{\bar{z}} + \acute{y}\acute{\bar{y}} + \dot{y}\dot{\bar{y}} + \dot{u}\dot{\bar{u}} + \acute{u}\acute{\bar{u}} + \dot{v}\dot{\bar{v}} + \acute{v}\acute{\bar{v}}) \\
& + q(\acute{z}\bar{z} - z\acute{\bar{z}} - \acute{y}\bar{y} + y\acute{\bar{y}})(\acute{z}\dot{\bar{z}} + \acute{\bar{z}}\dot{z} + \acute{y}\dot{\bar{y}} + \acute{\bar{y}}\dot{y} + \dot{u}\acute{\bar{u}} + \acute{u}\dot{\bar{u}} + \dot{v}\acute{\bar{v}} + \acute{v}\dot{\bar{v}})\Big) \\
& - \frac{2a-1}{2}(\acute{z}\acute{\bar{z}} + \acute{y}\acute{\bar{y}} + \acute{u}\acute{\bar{u}} + \acute{v}\acute{\bar{v}} + \dot{z}\dot{\bar{z}} + \dot{y}\dot{\bar{y}} + \dot{u}\dot{\bar{u}} + \dot{v}\dot{\bar{v}})^2 \\
& + \frac{2a-1}{2}((z\bar{z} + y\bar{y})^2 + (\dot{z}\acute{\bar{z}} + \dot{\bar{z}}\acute{z} + \dot{y}\acute{\bar{y}} + \dot{\bar{y}}\acute{y} + \dot{u}\acute{\bar{u}} + \dot{\bar{u}}\acute{u} + \dot{v}\acute{\bar{v}} + \dot{\bar{v}}\acute{v})^2) \\
& + \frac{q}{2}(2a-1)(\bar{z}\dot{z} - z\dot{\bar{z}} + \bar{y}\dot{y} - y\dot{\bar{y}})(\acute{z}\acute{\bar{z}} + \acute{y}\acute{\bar{y}} + \acute{u}\acute{\bar{u}} + \acute{v}\acute{\bar{v}} + \dot{z}\dot{\bar{z}} + \dot{y}\dot{\bar{y}} + \dot{u}\dot{\bar{u}} + \dot{v}\dot{\bar{v}} + z\bar{z} + y\bar{y}) \\
& - \frac{q}{2}(2a-1)(\dot{z}\acute{\bar{z}} + \dot{\bar{z}}\acute{z} + \dot{y}\acute{\bar{y}} + \dot{\bar{y}}\acute{y} + \dot{u}\acute{\bar{u}} + \dot{\bar{u}}\acute{u} + \dot{v}\acute{\bar{v}} + \dot{\bar{v}}\acute{v})(\bar{z}\acute{z} - z\acute{\bar{z}} + \bar{y}\acute{y} - y\acute{\bar{y}})\,,
\end{aligned} \tag{4.123}$$

where $\dot{\phi} = \frac{d\phi}{d\tilde{\tau}}$ and $\acute{\phi} = \frac{d\phi}{d\tilde{\sigma}}$.
This Lagrangian has the same symmetries discussed for the string theory in the previous chapter, with the obvious modification due to the mirror transformation :

- it is invariant under parity
$$\tilde{\sigma} \longrightarrow -\tilde{\sigma}\,. \tag{4.124}$$
- it is not invariant under time-reversal. This is due to the terms proportional to the parameter q, which change their sign under time-reversal. Therefore, the Lagrangian is invariant under the transformation
$$\tilde{\tau} \longrightarrow -\tilde{\tau} \quad \text{and} \quad q \longrightarrow -q\,. \tag{4.125}$$

4.4.1 Quantisation

Recall that in the string theory, z and $\bar{z}$ were one the complex conjugate of the other. However, when passing to the path-integral description, one usually integrates on both the fields without imposing any condition. Therefore, after the Wick rotation of the path integral, there is nothing to ensure that the two fields remain one the conjugate of the other. In fact, as we will see, this is no longer valid.
The free equations of motion now are

$$\ddot{z} - z'' + z = -2q\dot{z} \qquad \ddot{\bar{z}} - \bar{z}'' + \bar{z} = 2q\dot{\bar{z}} \qquad \ddot{u} - u'' = 0\,. \tag{4.126}$$

As in the previous case, the equations for y and v are exactly the same of these two, for this reason we focus only on z and u. Note that taking the complex conjugation of the first equation, for q real, we do not find the second one.
The conjugate momenta are

$$p_z = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{z}} = \dot{\bar{z}} - q\bar{z} \quad , \quad p_{\bar{z}} = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{\bar{z}}} = \dot{z} + qz \quad , \quad p_u = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{u}} = \dot{\bar{u}}\,. \tag{4.127}$$

Imposing the canonical quantisation conditions

$$\begin{aligned}
&[z(\sigma,\tau), \dot{\bar{z}} - q\bar{z}(\sigma',\tau)] = [\bar{z}(\sigma,\tau), \dot{z} + qz(\sigma',\tau)] = i\delta(\sigma - \sigma') \quad , \quad [z(\sigma,\tau), z(\sigma',\tau)] = [\bar{z}(\sigma,\tau), \bar{z}(\sigma',\tau)] = 0\,, \\
&[u(\sigma,\tau), \dot{\bar{u}}(\sigma',\tau)] = i\delta(\sigma - \sigma') \quad , \quad [u(\sigma,\tau), u(\sigma',\tau)] = [\dot{u}(\sigma,\tau), \dot{u}(\sigma',\tau)] = 0\,,
\end{aligned} \tag{4.128}$$

the solutions are

$$z(\sigma,\tau) = \int \frac{dp}{\sqrt{(2\pi)2e}} \left(e^{-i(\omega\tau - p\sigma)} a^z(p) + e^{i(\bar{\omega}\tau - p\sigma)} a^\dagger_{\bar{z}}(p) \right) , \tag{4.129}$$

$$\bar{z}(\sigma,\tau) = \int \frac{dp}{\sqrt{(2\pi)2e}} \left(e^{-i(\bar{\omega}\tau - p\sigma)} a^{\bar{z}}(p) + e^{i(\omega\tau - p\sigma)} a^\dagger_{z}(p) \right) , \tag{4.130}$$

$$u(\sigma,\tau) = \int \frac{dp}{\sqrt{(2\pi)2|p|}} \left(e^{-i(|p|\tau - p\sigma)} a^u(p) + e^{i(|p|\tau - p\sigma)} a^\dagger_{\bar{u}}(p) \right) , \tag{4.131}$$

$$\bar{u}(\sigma,\tau) = \int \frac{dp}{\sqrt{(2\pi)2|p|}} \left(e^{-i(|p|\tau - p\sigma)} a^{\bar{u}}(p) + e^{i(|p|\tau - p\sigma)} a^\dagger_{u}(p) \right) , \tag{4.132}$$

where

$$\omega = \sqrt{p^2 + 1 - q^2} - iq \quad , \quad \bar{\omega} = \sqrt{p^2 + 1 - q^2} + iq \quad , \quad e = \sqrt{p^2 + 1 - q^2} \, , \tag{4.133}$$

and the creation and annihilation operators satisfy the usual commutation relations

$$\begin{aligned} &[a^z(p), a^\dagger_z(p')] = [a^{\bar{z}}(p), a^\dagger_{\bar{z}}(p')] = \delta(p - p') \, , \\ &[a^z(p), a^z(p')] = [a^\dagger_z(p), a^\dagger_z(p')] = [a^{\bar{z}}(p), a^{\bar{z}}(p')] = [a^\dagger_{\bar{z}}(p), a^\dagger_{\bar{z}}(p')] = 0 \, . \end{aligned} \tag{4.134}$$

Alternatively, we could find the dispersion relations directly from the Lagrangian using the path-integral and computing $\langle \bar{z} z \rangle$. In addition, also the S matrix can be computed in the path-integral approach, exploiting the LSZ formula, and therefore it may seem useless to have found the explicit expressions of the field operators. However, in order to compare the mirror results with the string ones, we have to know the fields normalisation that enter in the S matrix formula and these are explicitly found in the plane-wave expressions.
As mentioned above, from the explicit expression of the fields one can see that $(z)^* \neq \bar{z}$. For this reason, it is not obvious that the two fields should have the same creation and annihilation operators. In fact, in principle, from the free equation of motion, without imposing any relation between the two fields, we can have

$$\bar{z}(\sigma,\tau) = \int \frac{dp}{\sqrt{(2\pi)2e}} \left(e^{-i(\bar{\omega}\tau - p\sigma)} b(p) + e^{i(\omega\tau - p\sigma)} b^\dagger(p) \right) .$$

However, from the expression of the propagator $\langle \bar{z} z \rangle$ that we know from the path-integral, together with the commutation relations (4.128), $\bar{z}$ is fixed to be (4.130). This is also intuitive from a physical point of view because the mirror transformation cannot change the number of particles in the spectrum. Ultimately, these are the correct expressions of the fields in the interaction picture since they reproduce the same expression of the propagator that one can find by using the quadratic Lagrangian in the path integral. Indeed, restricting the Lagrangin only to z and $\bar{z}$ we find

$$\begin{aligned} \langle z\bar{z} \rangle &:= \frac{1}{Z} \int DzD\bar{z}\, z\bar{z} \exp\left(iS[z,\bar{z}] + i \int d^2\sigma J z + \bar{J}\bar{z} \right)\Bigg|_{J=\bar{J}=0} = (-i)^2 \frac{\delta}{\delta \bar{J}} \frac{\delta}{\delta J} \log Z[J,\bar{J}] \Bigg|_{J=\bar{J}=0} \\ &= -\frac{1}{Z} \frac{\delta}{\delta \bar{J}} \frac{\delta}{\delta J} \int Dz\, D\bar{z} \exp\left(i \int d^2p\, z(-p)(p_0^2 - 1 - p_1^2 + 2iqp_0)\bar{z}(p) + J(p)z(-p) + \bar{J}(-p)\bar{z}(p) \right)\Bigg|_{J=\bar{J}=0} \\ &= -\frac{\delta}{\delta \bar{J}} \frac{\delta}{\delta J} \int Dz\, D\bar{z} \exp\left(-i \int d^2p\, J(p) \left(\frac{1}{p_0^2 - 1 - p_1^2 + 2iqp_0} \right) \bar{J}(-p) \right)\Bigg|_{J=\bar{J}=0} \\ &= \frac{i}{p_0^2 - 1 - p_1^2 + 2iqp_0} \, . \end{aligned} \tag{4.135}$$

Where $Z[J,\bar{J}]$ is the partition function with the source terms J and $\bar{J}$. Now that we have pointed out the relation between z and $\bar{z}$, it is possible to note that the mirror Lagrangian is not hermitian for $q \in (0,1)$ and then the theory is nonunitary. In order to recover the unitarity, one should continue q to the imaginary axis; however, the mirror transformation does not act on its value and in that case it would be the string theory that would be nonunitary. The only case in which both the NLSM and the mirror theory are unitary is the pure RR flux case ($q = 0$). This aspect is not yet clear from a physical point of view, but it is still possible to perform perturbative calculations considering q a generic parameter.

4.4.2 Worldsheet S matrix

At this point, in order to check the relation between the string and the mirror theory, it is possible to compute the two-body worldsheet S-matrix at the tree-level.
Given the generic $2 \longrightarrow 2$ scattering

$$a(p_1)b(p_2) \longrightarrow c(p_3)d(p_4)\,,$$

the $\mathbb{T}_{ab\longrightarrow cd}$ matrix element reads

$$\mathbb{T} = -i \times (2\pi)^2 \times \delta(p_1+p_2-p_3-p_4)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \times (\text{fields normalisation}) \times \mathcal{M}_{ab\longrightarrow cd}\,, \tag{4.136}$$

where $\mathcal{M}$ is the Feynman amplitude and can be computed from the Lagrangian exploiting the LSZ formula. According to the plane-wave expression the fields normalisation are

$$\frac{1}{\sqrt{(2\pi)^2 2e}} \quad \text{and} \quad \frac{1}{\sqrt{(2\pi)^2 2|p|}} \tag{4.137}$$

for the massive and massless fields respectively.
Let us show a couple of example of calculations before giving the complete expression of the tree-level S matrix.

1. Massive-massive scattering between z and $\bar{z}$. It is interesting to consider this type of scattering processes because, contrary to the scattering between two equal particles (e.g. two z), which are trivially elastic due to the $U(1)$ charge conservation, in principle they can admit different final states, i.e.

$$z(p_1)\bar{z}(p_2) \longrightarrow y(p_3)\bar{y}(p_4)\ ,\ z(p_1)\bar{z}(p_2) \longrightarrow u(k_3)\bar{u}(k_4)\ ,\ z(p_1)\bar{z}(p_2) \longrightarrow v(k_3)\bar{v}(k_4)\ ,$$

$$z(p_1)\bar{z}(p_2) \longrightarrow z(p_3)\bar{z}(p_4)\,.$$

Let us start with the first process. Using the Feynman rules, we find

$$\begin{aligned}\mathbb{T}_{z\bar{z}\to y\bar{y}} =& \frac{1}{4\sqrt{e_1e_2e_3e_4}}((-1+\omega_1\omega_2)(1+\omega_3\omega_4)+p_1p_2+\omega_3\omega_4p_1p_2-\omega_4(\omega_2p_1+\omega_1p_2)p_3 \\ &-\omega_3(\omega_2p_1+\omega_1p_2)p_4+(-1+\omega_1\omega_2+p_1p_2)p_3p_4+2a(1+\omega_1p_2(\omega_4p_3+\omega_3p_4) \\ &-\omega_1\omega_2(\omega_3\omega_4+p_3p_4)+p_1(-\omega_3\omega_4p_2+\omega_2\omega_4p_3+\omega_2\omega_3p_4-p_2p_3p_4))+\omega_3(-1+\omega_1\omega_2+p_1p_2)q \\ &-\omega_4(-1+\omega_1\omega_2+p_1p_2)q-(\omega_2p_1+\omega_1p_2)(p_3-p_4)q+a(\omega_3-\omega_4-\omega_3p_1p_2+\omega_4p_1p_2 \\ &+\omega_4p_1p_3-\omega_4p_2p_3+\omega_3(p_1-p_2)p_4+\omega_2(-1+\omega_3\omega_4+p_1p_3-p_1p_4+p_3p_4)-\omega_1(-1 \\ &+\omega_2(\omega_3-\omega_4)+\omega_3\omega_4-p_2p_3+(p_2+p_3)p_4))q)\cdot\delta(p_1+p_2-p_3-p_4)\delta(\omega_1+\bar{\omega}_2-\omega_3-\bar{\omega}_4)\,,\end{aligned}$$

which vanishes for both the solutions of the delta functions, that are $p_1 = p_3$, $p_2 = p_4$ and $p_1 = p_4$, $p_2 = p_3$.
The same is true for $\mathbb{T}_{z\bar{z}\longrightarrow u\bar{u}}$ (and for $\mathbb{T}_{z\bar{z}\longrightarrow v\bar{v}}$, that has the same expression)

$$\begin{aligned}\mathbb{T}_{z\bar{z}\longrightarrow u\bar{u}} =& \frac{1}{4\sqrt{e_1e_2}\sqrt{|k3|}\sqrt{|k4|})}(-k3(k4(1+(-1+2a)\omega_1\bar{\omega}_2+(-1+2a)p_1p_2+a\omega_1q-a\bar{\omega}_2q) \\ &+(\bar{\omega}_2(p_1-2ap_1)+\omega_1(p_2-2ap_2)+a(-p_1+p_2)q)|k4|)+|k3|(k4((-1+2a)\bar{\omega}_2p_1 \\ &+(-1+2a)\omega_1p_2+a(p_1-p_2)q)+(-1+p_1(p_2-2ap_2)+a\bar{\omega}_2q \\ &+\omega_1(\bar{\omega}_2-2a\bar{\omega}_2-aq))|k4|))\cdot\delta(p_1+p_2-k_3-k_4)\delta(\omega_1+\bar{\omega}_2-|k_3|-|k_4|)\,.\end{aligned}$$

In fact the energy-momentum conservation constrains k_3 and k_4 to have different signs since there are no solutions with both the momenta either positives or negatives. This is reasonable from the physical point of view because these would correspond to configurations in which the two massless particles go in the same direction and with the same speed (i.e. the speed of light) and for this reason they cannot scatter each other. Finally, it can be seen that, considering one of the two momenta positive and the other negative, the matrix element automatically vanishes

without even inserting the explicit solution.
The fact that these three matrix elements vanish shows that also without the constraint of the $U(1)$ charges conservation the scattering are elastic. In fact, as we will see from the complete expression of the matrix, this property at tree-level holds for every process.
Furthermore, not only the scattering is elastic, but when performing the calculation of the process $z(p_1)\bar{z}(p_2) \longrightarrow z(p_3)\bar{z}(p_4)$ it can be seen that the momentum of each particle remains the same after the scattering. In fact the matrix element is given by

$$
\begin{aligned}
\mathbb{T}_{z\bar{z}\longrightarrow z\bar{z}} =& \frac{1}{2\sqrt{e_1e_2e_3e_4}}(-1+\bar{\omega}_2\bar{\omega}_4-\omega_3\bar{\omega}_4-\bar{\omega}_2\omega_3p_1p_4+p_1p_2p_3p_4+\omega_1(\omega_3+\bar{\omega}_2(-1+\omega_3\bar{\omega}_4) \\
&-\bar{\omega}_4p_2p_3-q)+\bar{\omega}_2q-\omega_3q+\bar{\omega}_4q+a(2+2\bar{\omega}_2\omega_3p_1p_4-2p_1p_2p_3p_4-\bar{\omega}_2q+\omega_3q \\
&-\bar{\omega}_4q+\bar{\omega}_2\omega_3\bar{\omega}_4q-\bar{\omega}_4p_2p_3q-\bar{\omega}_2p_1p_4q+\omega_3p_1p_4q+\omega_1(2\bar{\omega}_4p_2p_3+q+\bar{\omega}_2\bar{\omega}_4q \\
&-\omega_3\bar{\omega}_4q+p_2p_3q-\bar{\omega}_2\omega_3(2\bar{\omega}_4+q))))\delta(p_1+p_2-p_3-p_4)\delta(\omega_1+\bar{\omega}_2-\omega_3-\bar{\omega}_4)\,,
\end{aligned}
$$

and evaluated for $p_1 = p_4$ and $p_2 = p_3$ it vanishes. Therefore, using the dispersion relations (4.133) and the property of the delta function

$$
\delta(p_1+p_2-p_3-p_4)\delta(\omega_1+\omega_2-\omega_3-\omega_4) = \frac{e_1e_2}{|e_2p_1-e_1p_2|}(\delta(p_1-p_3)\delta(p_2-p_4)+\delta(p_1-p_4)\delta(p_2-p_3)) \tag{4.138}
$$

the final result is

$$
\mathbb{T}\,|z(p_1)\bar{z}(p_2)\rangle = \left(\frac{(\omega_1-\bar{\omega}_2)(\bar{\omega}_2p_1+\omega_1p_2)}{2(\omega_1+\bar{\omega}_2)}-\frac{1}{2}(-1+2a)(\bar{\omega}_2p_1-\omega_1p_2)\right)|z(p_1)\bar{z}(p_2)\rangle\ , \tag{4.139}
$$

where it has been set $p_1 > p_2$ in order to remove the modulus at denominator.

2. Massless-massless scattering between two u. It is worth considering this scattering because there are some interesting features that we will return to when we discuss production processes. For simplicity, now that we know that all processes are elastic at the tree-level, we consider the scattering

$$
u(k_1)u(k_2) \longrightarrow u(k_3)u(k_4)\,,
$$

that has only one channel. The matrix element is given by

$$
\mathbb{T}_{uu\longrightarrow uu} = \frac{-(-1+2a)(-k_1k_2+|k_1||k_2|)(-k_3k_4+|k_3||k_4|)}{2\sqrt{|k_1|}\sqrt{|k_2|}\sqrt{|k_3|}\sqrt{|k_4|}}\delta(k_1+k_2-k_3-k_4)\delta(|k_1|+|k_2|-|k_3|-|k_4|)\,.
$$

Let us note that it vanishes if k_1 and k_2 or k_3 and k_4 have the same sign. Physically, this means that the massless particles cannot scatter if they go in the same direction as we expect. Using the property of the delta function

$$
\delta(k_1+k_2-k_3-k_4)\delta(|k_1|+|k_2|-|k_3|-|k_4|) = \frac{|k_1||k_2|}{||k_2|k_1-|k_1|k_2|}(\delta(k_1-k_3)\delta(k_2-k_4)+\delta(k_1-k_4)\delta(k_2-k_3))
$$

we find

$$
\mathbb{T}\,|u(k_1)u(k_2)\rangle = (2a-1)\,k_1k_2\,|u(k_1)u(k_2)\rangle\ . \tag{4.140}
$$

for both $(k_1 > 0 \wedge k_2 < 0) \vee (k_1 < 0 \wedge k_2 > 0)$. The amplitude is proportional to $(2a-1)$ and then it vanishes in the gauge $a = 1/2$.

Let us summarize the 2 body tree-level S matrix:

- Massive-massive

$$
\begin{aligned}
\mathbb{T}\,|z_\pm(p_1)z_\pm(p_2)\rangle &= \left(\frac{(\omega_1^\pm+\omega_2^\pm)(\omega_1^\pm p_2+\omega_2^\pm p_1)}{2(\omega_1^\pm-\omega_2^\pm)} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\pm p_1)\right)|z_\pm(p_1)z_\pm(p_2)\rangle \\
\mathbb{T}\,|z_\pm(p_1)y_\pm(p_2)\rangle &= \left(-\frac{(\omega_1^\pm p_2+\omega_2^\pm p_1)}{2} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\pm p_1)\right)|z_\pm(p_1)y_\pm(p_2)\rangle \\
\mathbb{T}\,|z_\pm(p_1)z_\mp(p_2)\rangle &= \left(\frac{(\omega_1^\pm-\omega_2^\mp)(\omega_1^\pm p_2+\omega_2^\mp p_1)}{2(\omega_1^\pm+\omega_2^\pm)} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\mp p_1)\right)|z_\pm(p_1)z_\mp(p_2)\rangle \\
\mathbb{T}\,|z_\pm(p_1)y_\mp(p_2)\rangle &= \left(-\frac{(\omega_1^\pm p_2+\omega_2^\mp p_1)}{2} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-\omega_2^\mp p_1)\right)|z_\pm(p_1)y_\mp(p_2)\rangle
\end{aligned}
\tag{4.141}
$$

- Massive-massless

$$
\mathbb{T}\,|z_\pm(p_1)U(p_2)\rangle = \left(-\frac{(\omega_1^\pm p_2+|p_2|p_1)}{2} + \left(a-\frac{1}{2}\right)(\omega_1^\pm p_2-|p_2|p_1)\right)|z_\pm(p_1)U(p_2)\rangle \tag{4.142}
$$

- Massless-massless

$$
\mathbb{T}\,|U(p_1)V(p_2)\rangle = \left(a-\frac{1}{2}\right)(|p_1|p_2-|p_2|p_1)\,|U(p_1)V(p_2)\rangle\ . \tag{4.143}
$$

As in the string case, it has been used the notation $z_+=z$, $z_-=\bar{z}$, $\omega^+=\omega$, $\omega^-=\bar{\omega}$ and $U,V=u,\bar{u},v,\bar{v}$ are generic massless particles and all remaining matrix elements can be found by exploiting the symmetries of the Lagrangian. Furthermore, also in this case the matrix elements are written in the kinematic configuration $v_1>v_2$. Note that all sectors of the $\mathbb{T}$ matrix have the same structure. In fact, given two particles X and Y, with energy E_1 and E_2 and momenta p_1 and p_2 respectively, the structure of all the matrix elements is the following:

$$
\mathbb{T}_{XY\longrightarrow XY} = \begin{cases} \pm\frac{(E_1+E_2)(E_1p_2+E_2p_1)}{2(E_1-E_2)} & \text{if X and Y belong to the same species} \\ \pm\frac{E_1p_2+E_2p_1}{2} & \text{if X and Y do not belong to the same species}\,, \end{cases} \tag{4.144}
$$

where we have neglected the term proportional to $(a-1/2)$ that is manifestly the same in all the processes. Note that this formula is also valid in the massless-massless scatterings, in which $E_1p_2+E_2p_1=0$.
Another interesting aspect is that, as we wanted, the matrix has the same structure as that of the NLSM. In particular, it can be written in the following way

$$
\mathbb{T}_{ij}^{kl} = \delta_i^k\delta_j^l\,\mathbb{T}_{ij}\ . \tag{4.145}
$$

For this reason, at the tree-level it respects all the integrability requirements, including the classic Yang-Baxter equation.

Furthermore, note that the mirror matrix can be obtained from the NLSM one by applying the transformation (4.122).
To be more precise, by applying this transformation directly to the expressions (3.84)-(3.85)-(3.86) one finds the expressions (4.141)-(4.142)-(4.143) with an overall minus sign. This is due to the fact that it maps the configuration $v_1>v_2$ to the configuration $v_2>v_1$. In fact, as discussed above, the kinematic choice allows us to remove the absolute values in the denominators. For example, in the mirror massive-massive case, the Dirac delta gives the term $1/|(\omega_2^\pm\pm iq)p_1-(\omega_1^\pm\pm iq)p_2|$. On the other hand, let us recall that in the NLSM in these processes we had the term $1/|\omega_2^\pm(p_1\mp q)-\omega_1^\pm(p_2\mp q)|$. The transformation (4.122) sends $\omega_2^\pm(p_1\mp q)-\omega_1^\pm(p_2\mp q)>0$ to $(\omega_2^\pm\pm iq)p_1-(\omega_1^\pm\pm iq)p_2<0$. For this reason, it maps to each other opposite kinematic configurations of the S matrix, explaining the overall minus sign.

In order to get rid of this change of sectors, we can change the definition of mirror transformation in the following way:

$$p \longrightarrow \tilde{p} = -i\omega \quad , \quad \omega \longrightarrow \tilde{\omega} = -ip \, . \tag{4.146}$$

In fact, owing to the time-reversal invariance of the $AdS_3 \times S^3 \times T^4$ NLSM (or alternatively from the mirror point of view to the parity invariance) this transformation is equivalent to the previous one; however, in this case, it preserves the kinematics.
Finally we can do the same specular discussion of parity and time reversal made in the case of the string matrix.
Under parity transformation, momenta and energy transform as follows

$$p \longrightarrow -p \quad \omega \longrightarrow \sqrt{(-p)^2 + 1 - q^2} + iq = \omega \, ,$$

and $v_1 > v_2$ goes to $v_1 < v_2$, leaving the matrix invariant. If instead we consider time reversal, we have

$$\omega \longrightarrow -\omega \quad p^2 = (\omega - iq)^2 - 1 + q^2 \longrightarrow (-\omega - iq)^2 - 1 + q^2 \neq p^2 \, ,$$

and the matrix elements change. Adding to time-reversal the transformation $q \longrightarrow -q$, we have

$$\omega \longrightarrow -\omega \quad , \quad p^2 = (\omega - iq)^2 - 1 + q^2 \longrightarrow (-\omega + iq)^2 - 1 + q^2 = p^2 \, , \tag{4.147}$$

and again $v_1 > v_2$ goes to $v_1 < v_2$. Therefore, the S matrix does not change.

Chapter 5

Production processes

One of the main properties of a quantum integrable field theory is the fact that in every process, the number of particles in the initial state is equal to the number of particles in the final state and therefore there is no particle production. This property is very peculiar if we consider four-dimensional QFTs such as the Standard Model in which in general initial and final states can have a very different number of particles. However, if a theory has an infinite number of conserved charges, these constrain the S matrix in such a way that particle production is no longer allowed. Moreover, scattering processes with the same number of particles in the initial and final states $m \to m$ are obtained by the product of two-body processes, and hence, in general, they do not vanish. Therefore, it seems that these processes are not obtained by crossing the corresponding production processes. As we shell see, this is not exactly true. In fact, we will see that only the configurations that give a pole to the propagator contribute to the amplitudes of the $m \to m$ scatterings . Furthermore, in these cases, the propagator degenerates to a delta function. The values of the momenta that give a pole to the propagator are such that they are physical values only in the $m \to m$ processes, while they are unphysical in the production cases. Thus, the two amplitudes are still obtained by crossing; however, when we insert the physical values into a production process, the delta function, which is the only contributing term, vanishes.

In this chapter, we investigate the behaviour of the mirror model with respect to the production processes. Furthermore, we want to understand the relation between the amplitudes in the gauge-fixed NLSM and in the mirror theory. In order to proceed, we will consider production processes involving six external legs. These processes have two contributions: the first one is given by the quartic Lagrangian and consists of Feynman diagrams with two vertices and one propagator in between, while the second one is given by the sixth-order Lagrangian vertex. As we expect, according to the integrability hypothesis, the sum of the two contributions vanishes, giving a zero probability of production.

5.1 Hamiltonian vs Lagrangian

Before going through the computations of the amplitudes of the production processes, we pause for a moment to make a reflection. Both the string NLSM and the mirror model are theories with derivative interactions; i.e. their interactive Lagrangians contain derivative terms. For this reason, we are not dealing with a theory with the standard relation $\mathcal{L}_{int} = -\mathcal{H}_{int}$, because the derivative terms give non-trivial contributions to the Legendre transform. However, note that as seen in section 3.7, if we stop at the fourth order, the relation between the Hamiltonian and the Lagrangian is still the trivial one. This is the reason why we have not discussed this problem before when we have computed the two-body processes in which only the quartic Lagrangian contributes. Nevertheless, in the production computations, we need to use the sixth-order Lagrangian. For this reason, it is worth understanding what is the connection between the canonical quantisation with the Hamiltonian and the path-integral quantisation with the Lagrangian in the presence of derivative interactions.

In order to discuss this problem, let us consider a simple toy model in which the relation between the interaction Lagrangian and Hamiltonian is not the trivial one :

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda\phi^2\partial_\mu\phi\partial^\mu\phi\,. \tag{5.1}$$

The conjugate momenta is

$$\pi = \dot{\phi} - 2\lambda\phi^2\dot{\phi} \Longrightarrow \dot{\phi} = \frac{\pi}{1-2\lambda\phi^2}\,.$$

Therefore, the corresponding Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi^2}{1-2\lambda\phi^2}\right) + \frac{1}{2}(\nabla\phi)^2 - \lambda\phi^2(\nabla\phi)^2 = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 - \lambda\phi^2(\nabla\phi)^2 + \frac{1}{2}\pi^2(2\lambda\phi^2 + (2\lambda\phi^2)^2 + ...)\,. \tag{5.2}$$

Note that, passing from the Lagrangian to the Hamiltonian, an infinite series of interaction terms appears.
The Hamiltonian density can be divided into the free part and the interactive part, namely

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 \quad , \quad \mathcal{H}_{int} = -\lambda\phi^2(\nabla\phi)^2 + \frac{1}{2}\pi^2(2\lambda\phi^2 + (2\lambda\phi^2)^2 + ...)\,.$$

The free theory is described by the usual Klein-Gordon Hamiltonian. Therefore, the solution are the well-known plane waves, and the ETCR reads $[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$, with $\pi = \dot{\phi}$. Now, let us consider a $3 \longrightarrow 3$ process at the tree-level. Using the Lagrangian there is only one diagram that contributes to the amplitude, given by two vertices $\lambda\phi^2\partial_\mu\phi\partial^\mu\phi$, connected by a propagator. On the other hand, using the Hamiltonian approach, in addition to this diagram, there is also another contribution, given by the vertex $2\lambda^2\pi^2\phi^4$.
Computing the first diagram with the Hamiltonian, the tree-level S matrix turns out

$$\mathbb{S} = \langle f|\,\frac{(-i)^2}{2}\int d^4x d^4y T((\lambda\phi^2\partial_\mu\phi\partial^\mu\phi)(x)(\lambda\phi^2\partial_\mu\phi\partial^\mu\phi)(y))\,|i\rangle\;, \tag{5.3}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states and T is the time-ordered product. Using the Wick's theorem, the time-ordered product hits all the possible pairs, giving Feynman propagators. Among these, let us focus on the one that contains two time derivatives:

$$\mathbb{S} = \ldots + \frac{(-i)^2}{2}4\,\langle f|\int d^4x d^4y(\lambda\phi^2\dot{\phi})(x)(\lambda\phi^2\dot{\phi})(y))\,|i\rangle\,\langle 0|\,T(\dot{\phi}(x)\dot{\phi}(y))\,|0\rangle\;. \tag{5.4}$$

The overall factor 4 occurs because both the terms have two derivatives and therefore they can be contracted in 4 different ways.
This term gives rise to a contribution that is not present in the path integral. In fact, when computing amplitudes with the path integral, the space-time derivatives always act externally on the propagators, while in the term (5.4) the derivatives are inside the propagator. Therefore, the non-commutability of the time derivatives with the time-ordered product gives rise to additional terms, which are not present in the path integral approach.

$$\begin{aligned}
&\frac{d}{dt_1}\frac{d}{dt_2}\,\langle 0|\,T(\phi(\vec{x},t_1)\phi(\vec{y},t_2))\,|0\rangle = \frac{d}{dt_1}\frac{d}{dt_2}\,\langle 0|\,\theta(t_1-t_2)\phi(\vec{x},t_1)\phi(\vec{y},t_2) + \theta(t_2-t_1)\phi(\vec{y},t_2)\phi(\vec{x},t_1)\,|0\rangle \\
=&\frac{d}{dt_1}\,\langle 0|-\delta(t_1-t_2)[\phi(\vec{x},t_1)\phi(\vec{y},t_2)] + \theta(t_1-t_2)\phi(\vec{x},t_1)\dot{\phi}(\vec{y},t_2) + \theta(t_2-t_1)\dot{\phi}(\vec{y},t_2)\phi(\vec{x},t_1)\,|0\rangle \\
&= \langle 0|-\frac{d}{dt_1}(\delta(t_1-t_2)[\phi(\vec{x},t_1)\phi(\vec{y},t_1)]) + \delta(t_1-t_2)[\phi(\vec{x},t_1),\dot{\phi}(\vec{y},t_2)]\,|0\rangle + \langle 0|\,T(\dot{\phi}(x)\dot{\phi}(y))\,|0\rangle\;.
\end{aligned}$$

And using $[\phi(\vec{x},t),\phi(\vec{y},t)] = 0$ and $[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$, it gives

$$\langle 0|\,T(\dot{\phi}(x)\dot{\phi}(y))\,|0\rangle = -i\delta^{(4)}(x-y) + \frac{d}{dt_1}\frac{d}{dt_2}\,\langle 0|\,T(\phi(\vec{x},t_1)\phi(\vec{y},t_2))\,|0\rangle\;. \tag{5.5}$$

Inserting this expression in (5.4) one can find

$$\begin{aligned}\mathbb{S} &= \text{path integral} - 2(-i)^2 \int d^4x d^4y \, \langle f| \, (\lambda\phi^2\dot{\phi})(x)(\lambda\phi^2\dot{\phi})(y) \, |i\rangle \, i\delta^{(4)}(x-y) \\ &= \text{path integral} + 2i\lambda^2 \int d^4x \, \langle f| \, \dot{\phi}^2\phi^4 \, |i\rangle \; .\end{aligned} \tag{5.6}$$

This new term is exactly the same term that arises from the second diagram with an opposite sign. Therefore, it seems that at least at the tree-level, despite the fact that there are a different set of diagrams, the total amplitude computed with the canonical approach and the path integral is exactly the same. As we will see soon, this result can be generalised.
Now let us consider what is the behaviour of the two methods with respect to loop diagrams, considering the one-loop correction to the propagator. The only diagram that contributes in both the Hamiltonian and the path integral approach is the one given by the vertex $\lambda\phi^2\partial_\mu\phi\partial^\mu\phi$ with two legs contracted.
Therefore, following the same logic as before, the difference between the two approaches is

$$\begin{aligned}\mathbb{S} &= \text{path integral} + (-i)(-i) \, \langle\phi| \int d^4x \lambda\phi^2\delta^{(4)}(x-x) \, |\phi\rangle \\ &= \text{path integral} - \delta(0) \, \langle\phi| \int d^4x \lambda\phi^2 \, |\phi\rangle \; .\end{aligned} \tag{5.7}$$

This is a contact term and this time there is no term in the Hamiltonian that can cancel out this additional contribution. The reason is that the Lagrangian is incomplete. In fact, following the path-integral proof, before passing to the Lagrangian, there is also the integral over the momenta

$$Z = \int DpDq e^{\frac{i}{\hbar}\int dt(p\dot{q}-H(p,q))} \, . \tag{5.8}$$

Here, for simplicity, we consider the quantum mechanical case with momentum p and position q. Then it can be easily generalised to the QFT case.
It is important to stress that at the exponent there is exactly the action $S(p,q)$ as a function of p and q; however, since we are integrating over all the possible q and p, these two variables are not constrained by the canonical relation $\dot{q} = \frac{\delta H}{\delta p}$. In order to obtain the usual formula for the partition function, we have to integrate out the momenta. To do that, let us expand the action around its stationary point $\bar{p}$ such that $\dot{q} = \frac{\delta H}{\delta p}(q,\bar{p})$

$$S(p,q) = S(q,\bar{p}) + \frac{1}{2}\frac{\delta^2 S}{\delta p^2}(q,\bar{p})(p-\bar{p})^2 + \frac{1}{3!}\frac{\delta^3 S}{\delta p^3}(q,\bar{p})(p-\bar{p})^3 + ... \tag{5.9}$$

Now $S(q,\bar{p})$ is exactly the action written in terms of the Lagrangian $S(q,\bar{p}) = \int dt\mathcal{L}(q,\dot{q})$. Therefore, Z becomes

$$Z = \int Dq e^{\frac{i}{\hbar}W} \, , \tag{5.10}$$

where W is the Wilsonian action defined as

$$\begin{aligned}e^{\frac{i}{\hbar}W} &= \int Dp e^{\frac{i}{\hbar}\int dt(p\dot{q}-H(p,q))} \\ &= e^{\frac{i}{\hbar}S} \int Dp e^{\frac{i}{\hbar}(\frac{1}{2}\frac{\delta^2 S}{\delta p^2}(q,\bar{p})(p-\bar{p})^2 + \frac{1}{3!}\frac{\delta^3 S}{\delta p^3}(q,\bar{p})(p-\bar{p})^3 + ...)} \, .\end{aligned} \tag{5.11}$$

Therefore

$$W = S - i\hbar \ln\left(\int Dp e^{\frac{i}{\hbar}(\frac{1}{2}\frac{\delta^2 S}{\delta p^2}(q,\bar{p})(p-\bar{p})^2 + \frac{1}{3!}\frac{\delta^3 S}{\delta p^3}(q,\bar{p})(p-\bar{p})^3 + ...)}\right) . \tag{5.12}$$

Note that at the tree level $W = S$. This result can also be seen from the effective field theory (EFT) perspective. In fact, the momenta can be considered just as additional fields that are integrated out from the spectrum of the theory. A well-known result of EFT is that when integrating out a field, at the tree level the exponent in the path integral becomes just the exponent itself computed at the solution of the classical equation of motion of the field that is integrated out. In our case, the classical EOM of p imposes the canonical relation, and then the exponent becomes the action. However, as can be seen from (5.12) there are quantum corrections. Generally, most theories (such as the Standard Model) are at most quadratic in p , containing terms such that $\frac{\delta^2 S}{\delta p^2} = const$, which are absorbed in the measure and do not provide any quantum corrections to $\mathcal{L}$. This is not the case for neither the toy model we are considering nor the NLSM. In fact, in the case of the toy model,

$$\begin{aligned} W &= S - i\hbar \ln \int D\pi e^{\frac{i}{\hbar}\frac{\delta^2 S}{\delta \pi^2}(\phi,\bar{\pi})(\pi-\bar{\pi})^2} \\ &= S + \frac{i\hbar}{2} \ln\det\left(\frac{\delta^2 S}{\delta\pi^2}\right) + \text{const} \\ &= S + \frac{i\hbar}{2} Tr\ln\left(\frac{\delta^2 S}{\delta\pi^2}\right) + \text{const} = S + \frac{i\hbar}{2}\int d^4x \,\langle x|\ln\left(\frac{-2}{1-2\lambda\phi^2}\right)|x\rangle + \text{const} \\ &= S - \frac{i\hbar}{2}\int d^4p\,\langle x|p\rangle\,\langle p|x\rangle \int d^4x \ln(1-2\lambda\phi^2(x)) + \text{const} \\ &= S - \frac{i\hbar}{2}\int \frac{d^4p}{(2\pi)^4}\int d^4x \ln(1-2\lambda\phi^2(x)) + \text{const} \\ &= S - \frac{i\hbar}{2}\delta(0)\int d^4x \ln(1-2\lambda\phi^2(x)) + \text{const}\,. \end{aligned} \tag{5.13}$$

Therefore, the Lagrangian density becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda\phi^2\partial_\mu\phi\partial^\mu\phi - \frac{i\hbar}{2}\delta(0)\ln(1-2\lambda\phi^2) \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda\phi^2\partial_\mu\phi\partial^\mu\phi + i\hbar\delta(0)\lambda\phi^2 + ... \end{aligned} \tag{5.14}$$

The last term, obtained by expanding the logarithm to the first order, is exactly the additional term that appears in the Hamiltonian computation. In this way, we are able to recover the result obtained with the canonical quantisation.
We have seen that, in general, if the theory has higher derivative terms, the Lagrangian in the path-integral receives some corrections, which correspond to corrections to diagrams with one leg contracted to itself. This is the reason why in section 4.3, as pointed out in [65], we stressed the fact that when integrating out the momenta in the Euclidean partition functions, additional non-trivial terms arise. Finally, we have shown that for our tree-level production computations we can still safely relay on the Lagrangian, without adding any correction. However, we expect that at the loop-level these start to appear and modify the Feynman rules.

5.2 Production and scattering

As pointed out in the introduction of this chapter, in factorised scattering theory the $m \longrightarrow m$ scatterings do not seem to be obtained by the crossing of the corresponding production processes, which, as we know, must have vanishing amplitudes. The reason is due to the structure of the propagator.
Let us consider, for simplicity, the standard relativistic propagator

$$\Delta(p) = \lim_{\epsilon\to 0} \frac{i}{E^2 - p^2 - m^2 + i\epsilon}\,. \tag{5.15}$$

The NLSM is not a relativistic theory and the propagator has a different structure, however, following the same logic, this discussion can be generalised to any type of propagator.

Performing the limit we find

$$
\begin{aligned}
\Delta(p) &= \lim_{\epsilon \to 0} \frac{i}{E^2 - p^2 - m^2 + i\epsilon} \\
&= \lim_{\epsilon \to 0} i \cdot \left(Re\left(\frac{1}{E^2 - p^2 - m^2 + i\epsilon} \right) + i\, Im \left(\frac{1}{E^2 - p^2 - m^2 + i\epsilon} \right) \right) \\
&= \lim_{\epsilon \to 0} i \cdot \left(\frac{1}{E^2 - p^2 - m^2} - \frac{i\epsilon}{(E^2 - p^2 - m^2)^2 + \epsilon^2} \right) \\
&= \frac{i}{E^2 - p^2 - m^2} + \pi \delta(E^2 - p^2 - m^2)\,,
\end{aligned}
\tag{5.16}
$$

where, in order to have a non-vanishing real part, the first term is considered outside the pole (i.e. $E^2 - p^2 - m^2 \neq 0$). Therefore, the first term corresponds to the propagator when the particle is off-shell, while the second term is the expression of the propagator when the particle is on-shell. Clearly, this discussion does not depend on the dispersion relation of the particles, and then this holds in the same way for the NLSM. From a more formal point of view, this relation can be seen as the well-known distribution relation

$$
\lim_{\epsilon \to 0} \frac{1}{x + i\epsilon} = P\left(\frac{1}{x}\right) - i\pi\delta(x)\,, \tag{5.17}
$$

where P denotes the Cauchy principal value defined in such a way that

$$
\langle P\left(\frac{1}{x}\right) | f(x) \rangle := \lim_{\epsilon \to 0} \int_{|x| > \epsilon} dx\, \frac{f(x)}{x}\,. \tag{5.18}
$$

Therefore, we expect (and as we shall see, this is exactly what happens) that in scattering processes the propagators have a pole only in the factorised configurations, namely in the configurations in which the number of the incoming particles is equal to the number of the outgoing particles and the set of the initial and final momenta are the same. In particular, we will see that the propagator pole occurs when a particle in the final state has the same momentum as a particle in the initial state. This gives a delta function contribution, which, together with the delta functions coming from the conservation of the energy-momentum, constrains the configuration to the factorised one. On the other hand, all the other configurations, i.e. production processes or in general processes in which the initial and final sets of momenta are different, do not provide any pole to the propagator, and then the computation can be done in a straightforward way by using the principal value. Hence, removing $i\epsilon$ from the propagator, as we will do, automatically corresponds to considering only this kind of process, which we expect to vanish.

5.3 Light-cone momenta

Before analysing the production processes, let us introduce a useful notation. In a two-dimensional theory, the momentum has only two components, namely (p^0, p^1). Furthermore, the dispersion relation constrains the possible values that p^0 and p^1 can take. Therefore, the energy and the momentum of each particle depend only on one parameter. A useful way to make this feature manifest, without imposing every time the dispersion relations is to introduce the so-called light-cone momenta, defined as

$$
(p, \bar{p}) = (p^0 + p^1, p^0 - p^1)\,. \tag{5.19}
$$

Let us first discuss the case of relativistic theories, then we will move to the string NLSM and mirror model. In these variables, the relativistic dispersion relation $(p^0)^2 - (p^1)^2 = m^2$ becomes $p\bar{p} = m^2$. Hence, imposing the dispersion relation, we can rewrite the light-cone momenta introducing a real variable a such that

$$
(p, \bar{p}) = (ma, ma^{-1})\,. \tag{5.20}
$$

In this way all the information about the energy and momentum of a particle is encoded in the parameter a.

The inverse relations are

$$p^0 = \frac{1}{2}m\left(a + \frac{1}{a}\right) \quad , \quad p^1 = \frac{1}{2}m\left(a - \frac{1}{a}\right). \tag{5.21}$$

Note that particles with $a > 0$ have positive energy and then travel forwards in time, while particles with $a < 0$ have negative energy and travel backwards in time. This means that crossing is obtained by sending $a \longrightarrow -a$.

Now, let us consider a scattering process $a + b \longrightarrow c + d$, where all the particles have the same mass. The energy momentum conservation gives the two equations $p^0_a + p^0_b = p^0_c + p^0_d$ and $p^1_a + p^1_b = p^1_c + p^1_d$, which in light-cone coordinates becomes

$$a + b - c - d = 0 \quad , \quad \frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d} = 0. \tag{5.22}$$

We now have all the ingredients to move on to the mirror theory, however, for the sake of completeness, let us discuss a little further some interesting features of the relativistic case.

Another useful notation is obtained by introducing the parameter $\theta = \log a$, known as *rapidity*. In this way we obtain

$$(p, \bar{p}) = (me^{\theta}, me^{-\theta}), \tag{5.23}$$

and thus using (5.21)

$$p^0 = m\cosh\theta \quad , \quad p^1 = m\sinh\theta. \tag{5.24}$$

According to what discussed above, a real value of the rapidity describes a forward particle. Furthermore, crossing is given by the shift

$$\theta \longrightarrow \theta + i\pi. \tag{5.25}$$

Let us discuss how a Lorentz transformation acts in these coordinates. In $1+1$ dimensions, a Lorentz matrix is a matrix Λ that respects the equality $\Lambda^T\eta\Lambda = \eta$, where $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Therefore, given a generic matrix $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, enforcing the Lorentz condition one can find

$$\Lambda^T\eta\Lambda = \eta \Rightarrow \begin{cases} a^2 - c^2 = 1 \\ ab - cd = 0 \\ d^2 - b^2 = 1 \end{cases} \Rightarrow \Lambda = \begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \cosh\omega \end{pmatrix},$$

where ω is a real parameter. Note that this choice corresponds to the proper orthochronous subgroup. Other choices can be made by changing the relative signs of the entries in a coherent manner.

Therefore, applying a Lorentz transformation to the vector (p^0, p^1), given by (5.24) we find that the effect is to shift the rapidities, i.e. $\theta \longrightarrow \theta + \omega$. This means that the two-body S matrix of a $1+1$ relativistic theory has to be a function only of the difference of the rapidities $\theta_{12} = \theta_1 - \theta_2$ to preserve Lorentz invariance. Further discussions can be found in [45].

Let us come back to the non-relativistic mirror model. Let us rewrite the dispersion relations of the mirror massive particle z and y (4.133) in the following way:

$$(\omega + iq)^2 - p^2 = 1 - q^2. \tag{5.26}$$

This form is equal to the relativistic expression where the energy is replaced by $(\omega - iq)$ and the mass square is replaced by $1 - q^2$. Therefore, as in the relativistic case, we can introduce the parameter a, such that

$$(\omega + iq) + p = \sqrt{1 - q^2}a \quad , \quad (\omega + iq) - p = \sqrt{1 - q^2}a^{-1}, \tag{5.27}$$

and the inverse map is

$$\omega = \frac{1}{2}\sqrt{1 - q^2}\left(a + \frac{1}{a}\right) - iq \quad , \quad p = \frac{1}{2}\sqrt{1 - q^2}\left(a - \frac{1}{a}\right). \tag{5.28}$$

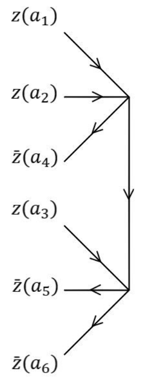


Figure 5.1: Feynman diagram of the process $z(a_1)z(a_2)z(a_3)\bar{z}(a_4)\bar{z}(a_5)\bar{z}(a_6) \longrightarrow 0$ given by the quartic Lagrangian. The arrows describe the direction of the $U(1)$ currents. There are 9 diagrams of this type, given by the z and $\bar{z}$ permutations of this one.

On the other hand, following the same procedure for $\bar{z}$ and $\bar{y}$ we find

$$\bar{\omega} = \frac{1}{2}\sqrt{1-q^2}\left(a+\frac{1}{a}\right)+iq \quad , \quad p = \frac{1}{2}\sqrt{1-q^2}\left(a-\frac{1}{a}\right) . \tag{5.29}$$

Now that we have a manageable tool to deal with the momenta without having the square roots that arise from the dispersion relations, we can start discussing the production processes.

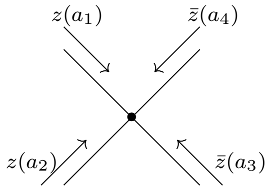
5.4 Massive production processes

Let us start considering the production processes. In particular, we consider scattering with six external legs, which at the tree-level receive a contribution by the quartic and the sixth-order Lagrangian. If the theory is integrable, we expect that the two contributions cancel each other out, as shown in [45] in the case of the sine-Gordon model.
Let us start by considering a massive production process, namely, a production process in which all the particles involved are massive.
In particular, we consider the scattering

$$z(a_1)z(a_2)z(a_3)\bar{z}(a_4)\bar{z}(a_5)\bar{z}(a_6) \longrightarrow 0 . \tag{5.30}$$

Let us stress that we are dealing with a non-relativistic theory, that in the case of the mirror model is not even unitary. Hence, the crossing symmetry property becomes more subtle. In [68] is shown how to implement crossing symmetry in the gauge-fixed worldsheet $AdS_5 \times S^5$ NLSM. It is important that these types of relations hold also for the gauge-fixed string because they allow to constrain the values of the dressing factors [22] of the S matrix in the bootstrap approach. Anyway, for our purposes, from the plane-wave expression of the fields (4.129), we note that z either annihilates a particle with energy ω and momentum p, or creates an anti-particle with energy $\bar{\omega}$ and momentum p. Therefore, in perturbative computations, a scattering with $\bar{z}$ in the final state is obtained by a process with z in the initial state by sending $\omega \longrightarrow -\bar{\omega}$ and $p \longrightarrow -p$ and *vice versa*. Let us stress that this map is well-defined since it also sends the dispersion relations into each other. According to the light-cone expressions (5.28) and (5.29), crossing is obtained by the map $a \longrightarrow -a$ and since in our computations we do not make any assumptions on the value of the a_i, this means that showing that the process (5.30) vanishes corresponds to showing that all the other processes obtained by crossing also vanish.

Only for this process, for simplicity we will consider the gauge $a = 1/2$. The Feynman diagrams are shown in Figure 5.1 and the Feynman rules for the vertices are :



$$
\begin{aligned}
&= i\left(-2\omega_1\bar{\omega}_3 + \frac{iq}{2}\left((\bar{\omega}_3 - \omega_1)(1 + p_2p_4 + \omega_2\bar{\omega}_4) + (p_1 - p_3)(\omega_2p_4 + \bar{\omega}_4p_2)\right)\right) \\
&+ \text{permutations} \\
&= \frac{i}{16a_1a_2a_3a_4}\Big(8a_2a_4(2ia_1q + (1+a_1^2)\sqrt{1-q^2})(2ia_3q - (1+a_3^2)\sqrt{1-q^2}) + iq \\
&(-2(a_1 - a_3)(1 + a_1a_3)(1 + a_2a_4)(-1 + q^2)(-i(a_2 - a_4)q - \sqrt{1-q^2} + a_2a_4\sqrt{1-q^2}) \\
&- (4ia_1a_3q + (1+a_1^2)a_3\sqrt{1-q^2} - a_1(1+a_3^2)\sqrt{1-q^2})(4a_2a_4 - (-1+a_2^2)(-1+a_4^2) \\
&(-1+q^2) - (2ia_2q + (1+a_2^2)\sqrt{1-q^2})(2ia_4q - (1+a_4^2)\sqrt{1-q^2})))\Big) + \text{permutations}\,.
\end{aligned}
$$

While, according to (4.135), the propagator is

$$
\Delta(x-y) = i\int \frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p_0^2 - 1 - p_1^2 + 2iqp_0 + i\epsilon}\,. \tag{5.31}
$$

It can be interpreted either as a particle propagating from x to y, or as an anti-particle propagating from y to x. If we compute the propagator for the configuration shown in Figure 5.1, in which the momenta a_1, a_2 and a_4 enter in the vertex, in momentum space it becomes

$$
\Delta = \frac{i}{1-q^2}\,\frac{a_1a_2a_4}{(a_1+a_2)(a_1+a_4)(a_2+a_4) + i\epsilon\frac{a_1a_2a_4}{1-q^2}}\,. \tag{5.32}
$$

And the same is true for all the other diagrams. As pointed out above, the propagator has a pole when there are two particles such that $a_i = -a_j$. These configurations give the factorised $3 \longrightarrow 3$ processes in which all the momenta in the initial state are the same of the final state. Therefore, dealing with production processes, we can safely remove $i\epsilon$, considering the principal value.

The amplitude is a function of the light-cone momenta of the six particles, however, using the energy-momentum conservation

$$
\begin{aligned}
&a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0\,, \\
&\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} = 0\,,
\end{aligned} \tag{5.33}
$$

we can write, for instance, a_4 and a_5 as a function of the momenta of the other particles. Thus, the amplitude depends only on four momenta, namely $\mathcal{M} = \mathcal{M}(a_1, a_2, a_3, a_4)$. Note that when the propagator has a pole, as we expect, there are three constraints and then the independent momenta are three.

In order to simplify the computation, we note that, since $z(a_1)$, $z(a_2)$ and $z(a_3)$ are identical particles, the amplitude must be symmetric under the exchange of a_1, a_2 and a_3. Therefore, we can introduce the variables

$$
\begin{gathered}
s_1 = a_1 + a_2 + a_3 \quad , \quad s_2 = a_1a_2 + a_2a_3 + a_3a_1 \quad , \quad s_3 = a_1a_2a_3 \quad , \\
a_4 = a_4\,.
\end{gathered} \tag{5.34}
$$

This new variables are manifestly symmetric under the exchange of a_1, a_2 and a_3 and allow to simplify the expression of the amplitude. We performed the calculation using *Mathematica*. In order to make the calculation of many terms manageable, we expanded each diagram in powers of q and $Q = \sqrt{1-q^2}$ in the following way:

$$
\mathcal{M}_i = \sum_{m,n} q^n Q^m \mathcal{M}^i_{mn}\,, \tag{5.35}
$$

where $\mathcal{M}_i$ is the amplitude of the $i-$th diagram, and we treated each element of the expansion separately. At the end, after all the simplifications, we recovered the complete expression by summing all the coefficients of the expansion. The final amplitude computed using the fourth Lagrangian in

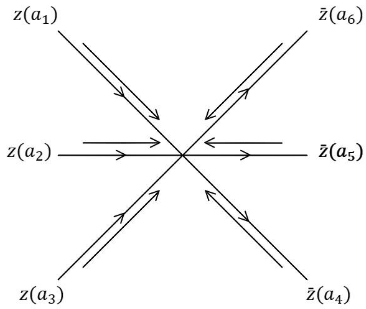


Figure 5.2: Six-legs vertex diagram of the scattering $z(a_1)z(a_2)z(a_3)\bar{z}(a_4)\bar{z}(a_5)\bar{z}(a_6) \longrightarrow 0$. The arrows above the legs describe the direction of the momenta, while the arrows along the legs describe the direction of the $U(1)$ currents.

the variables (5.34) is :

$$
\begin{aligned}
i\mathcal{M}_4 = {} & \frac{i\left(1-q^2\right)}{8a_4^2\left(a_4+s_1\right)s_3^2\left(a_4s_2+s_3\right)}\Bigg(-iqs_2\Big(2iq\left(q^2-1\right)s_2^2+\Big(4\left(1-5q^2\right)\sqrt{1-q^2}+3iq\left(q^2-1\right)s_1\Big) \\
& s_3s_2+i\left(q^2-1\right)s_3\left(21qs_3+2s_1\left(q+2i\sqrt{1-q^2}s_3\right)\right)\Big)a_4^6-2iqs_1s_2\Big(2iq\left(q^2-1\right)s_2^2+(4\left(1-5q^2\right) \\
& \sqrt{1-q^2}+3iq\left(q^2-1\right)s_1)s_3s_2+i\left(q^2-1\right)s_3\left(21qs_3+2s_1\left(q+2i\sqrt{1-q^2}s_3\right)\right)\Big)a_4^5-s_2\Big(\Big(\Big(-5s_1^2 \\
& +21s_2+81\Big)s_3^2+3s_1\left(s_2\left(-s_1^2+s_2+38\right)+7\right)s_3+s_2\left(2s_2\left(s_2+8\right)-s_1^2\left(s_2-3\right)\right)\Big)q^4+4i\sqrt{1-q^2} \\
& \left(-\left(\left(s_2-5\right)s_3s_1^2\right)+\left(-s_1^2+s_2+24\right)s_3^2s_1+s_2\left(4s_2+1\right)s_1+s_2\left(5s_2+24\right)s_3\right)q^3-\Big(\Big(11s_1^2+21s_2 \\
& +279\Big)s_3^2+s_1\left(s_2\left(-3s_1^2+3s_2+142\right)+21\right)s_3+s_2\left(2s_2\left(s_2+16\right)-3s_1^2\left(s_2-1\right)\right)\Big)q^2-4i\sqrt{1-q^2}\Big(s_2 \\
& +s_1s_3\Big)\left(-s_3s_1^2+\left(2s_2+1\right)s_1+\left(s_2+42\right)s_3\right)q-2\left(s_1^2-8\right)s_2^2+2\left(8s_1^2-9\right)s_3^2+52s_1s_2s_3\Big)a_4^4-\Big(s_1s_2 \\
& +s_3\Big)\Big(\left(\left(16s_1^2+21s_2+81\right)s_3^2+s_1\left(2s_1^2+3s_2\left(s_2+38\right)+21\right)s_3+s_2\left(\left(s_2+3\right)s_1^2+2s_2\left(s_2+8\right)\right)\right)q^4 \\
& +4i\sqrt{1-q^2}\left(\left(4s_2+5\right)s_3s_1^2+\left(s_2+24\right)s_3^2s_1+s_2\left(4s_2+1\right)s_1+s_2\left(5s_2+24\right)s_3\right)q^3-((32s_1^2+21s_2 \\
& +279)s_3^2+s_1\left(2s_1^2+s_2\left(3s_2+142\right)+21\right)s_3+s_2\left(2s_2\left(s_2+16\right)-s_1^2\left(s_2-3\right)\right))q^2-4i\sqrt{1-q^2}(s_2 \\
& +s_1s_3)\left(s_1\left(2s_2+1\right)+\left(s_2+42\right)s_3\right)q-2\left(s_1^2-8\right)s_2^2+2\left(8s_1^2-9\right)s_3^2+52s_1s_2s_3\Big)a_4^3+s_1\Big(\Big(3s_1s_2^3 \\
& +\left(\left(s_2-3\right)s_1^2+5s_2\right)s_3s_2-\left(16s_1^2+21s_2+81\right)s_3^3-s_1\left(2s_1^2+3s_2\left(s_2+38\right)+21\right)s_3^2\Big)q^4+4i\sqrt{1-q^2} \\
& \left(s_2^3+s_1\left(s_2-1\right)s_3s_2-s_1\left(s_2+24\right)s_3^3-\left(\left(4s_2+5\right)s_1^2+s_2\left(5s_2+24\right)\right)s_3^2\right)q^3-\Big(3s_1s_2^3+(3s_1^2\left(s_2-1\right) \\
& -11s_2)s_3s_2-\left(32s_1^2+21s_2+279\right)s_3^3-s_1\left(2s_1^2+s_2\left(3s_2+142\right)+21\right)s_3^2\Big)q^2-4i\sqrt{1-q^2}\left(s_2+s_1s_3\right) \\
& \left(s_2^2-s_3\left(2s_1+s_3\right)s_2-s_3\left(s_1+42s_3\right)\right)q+2s_3\left(\left(s_1^2-8\right)s_2^2-26s_1s_3s_2+\left(9-8s_1^2\right)s_3^2\right)\Big)a_4^2-2iqs_1s_2s_3 \\
& \left(4\sqrt{1-q^2}\left(\left(1-5q^2\right)s_1s_3-\left(q^2-1\right)s_2\right)-iq\left(1-q^2\right)\left(3s_1s_2+2\left(s_1^2+s_2\right)s_3+21s_3\right)\right)a_4-iqs_1s_3^2 \\
& \left(4\sqrt{1-q^2}\left(\left(1-5q^2\right)s_1s_3-\left(q^2-1\right)s_2\right)-iq\left(1-q^2\right)\left(3s_1s_2+2\left(s_1^2+s_2\right)s_3+21s_3\right)\right)\Bigg).
\end{aligned}
\tag{5.36}
$$

Now, let us consider the six-leg vertex. In order to find its amplitude, we need the expression of the sixth-order Lagrangian. We can find it in the usual way by solving the Virasoro constraints $C_1 = C_2 = 0$ at the second order in $1/T$ and performing the Legendre transform to pass from the Hamiltonian to the Lagrangian. Finally, we perform the double Wick rotation to pass from the string to the mirror theory.

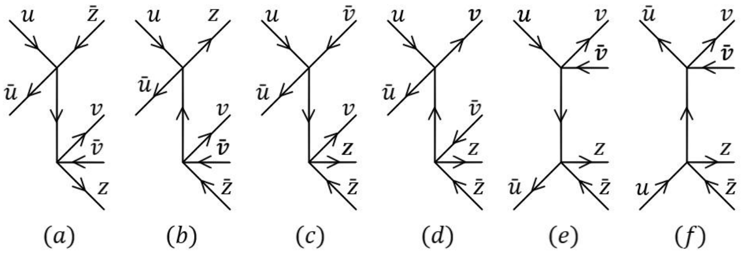


Figure 5.3: Feynman diagrams of the scattering $u(k_1)\bar{u}(k_2) \longrightarrow v(k_3)\bar{v}(k_4)z(a_1)\bar{z}(a_2)$ given by the quartic Lagrangian. The arrows describe the direction of the $U(1)$ currents. Diagrams (a) and (b) have a massive z propagator in between, (c) and (d) have a massless v propagator, while (e) and (f) have a massless u propagator.

In the gauge $a = 1/2$ we find

$$\begin{aligned} \mathcal{L}^{(6)} &= \frac{1}{4}|z|^4 \left(|z|^2(9|\dot{z}|^2 - |\acute{z}|^2) - (\dot{z}^2 - \acute{z}^2)(\bar{z}^2 - \acute{\bar{z}}^2)\right) \\ &- \frac{q}{4}|z|^2 \left(|z|^2(\dot{z}\bar{z} - z\dot{\bar{z}}) + 2(\acute{z}^2\bar{z}\dot{\bar{z}} - \acute{\bar{z}}^2 z\dot{z}) + 6|\dot{z}|^2(z\dot{\bar{z}} - \bar{z}\dot{z})\right) \\ &- \frac{q^2}{4}|\dot{z}|^2 \left(-z^2(\dot{\bar{z}}^2 - \acute{\bar{z}}^2) - \bar{z}^2(\dot{z}^2 - \acute{z}^2) + 2|z|^2(|\dot{z}|^2 - |\acute{z}|^2)\right) . \end{aligned} \tag{5.37}$$

The Feynman diagrams are shown in Figure 5.2 and the amplitude given by this Lagrangian is exactly the opposite of (5.36).
Thus,

$$i\mathcal{M}_{tot} = i\mathcal{M}_4 + i\mathcal{M}_6 = 0 . \tag{5.38}$$

Therefore, we have shown that the production processes of the type (5.30) have vanishing amplitudes at tree-level.

5.5 Massive-massless production processes

Now let us consider production processes in which there are both massless and massive particles.
In particular, we consider the scattering

$$u(k_1)\bar{u}(k_2) \longrightarrow v(k_3)\bar{v}(k_4)z(a_1)\bar{z}(a_2) . \tag{5.39}$$

In this case we use a mixed notation, i.e. the light-cone momenta for the massive particles and the standard momenta notation for the massless ones. Clearly, following the same argument, also in this case, showing that this process has zero amplitude corresponds to showing that all the processes obtained by the crossing of this one have vanishing amplitude.
The propagators of the particles u and v are the usual propagators of a massless relativistic particle, that in momentum space reads

$$\Delta = \frac{i}{p_0^2 - p_1^2 + i\epsilon} . \tag{5.40}$$

In this case there are six Feynman diagrams, which are shown in Figure 5.3. Note that, as in the previous case, the poles of the propagators occur in the configuration in which the process becomes a $3 \longrightarrow 3$ scattering with the same initial and final sets of momenta. In fact, let us consider, for instance, the (a) diagram in Figure 5.3. From the structure of the vertex and the expression of the massive propagator (5.31), we conclude that it gains a pole when $k_1 = -k_2$ and $|k_1| = -|k_2|$, as expected. Furthermore, now the diagrams with a massless propagator have a pole also when all the momenta of the massless particles in the vertex have the same sign. However, as pointed out in the previous chapter when we computed the two-body S matrix, a vertex with four massless vanishes when the incoming/outgoing particles go in the same direction. Thus, the contribution of these configurations to the delta function part of the propagator is zero, and we are left with only the factorised configurations. For this reason, once again we can safely remove the $i\epsilon$ prescription from the propagator when dealing

with production processes.
The energy momentum conservation in this mixed notation reads

$$\begin{aligned} k_1+|k_1|+k_2+|k_2| &= k_3+|k_3|+k_4+|k_4|+\sqrt{1-q^2}(a_1+a_2)\,, \\ |k_1|-k_1+|k_2|-k_2 &= |k_3|-k_3+|k_4|-k_4+\sqrt{1-q^2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\,, \end{aligned} \tag{5.41}$$

and we solve it in a_1 and a_2, writing the amplitude as a function of the massless momenta.
In order to proceed with the computation of the amplitude, due to the modulus coming from the massless dispersion relation, it is useful to separate the cases according to the sign of the momenta:

- $k_1>0,\ k_2>0$ and $k_3>0,\ k_4>0$: There are no solutions to the energy-momentum constraint with this configuration.
- $k_1>0,\ k_2>0$ and $k_3<0,\ k_4<0$:

$$\begin{aligned} i\mathcal{M}_4 = &\frac{8ik_1k_2k_3k_4}{(k_1+k_2)(k_3+k_4)}\Bigg(i\left(2a^2-3a+1\right)(k_3+k_4)\,q\Big((k_1+k_2)(k_3+k_4)\left(k_1(k_3+k_4)+k_2(k_3+k_4)\right. \\ &\left.+q^2-1\right)\Big)^{1/2}+k_1\Big(-i\left(2a^2-3a+1\right)q\Big((k_1+k_2)(k_3+k_4)\left(k_1(k_3+k_4)+k_2(k_3+k_4)+q^2-1\right)\Big)^{1/2} \\ &+2k_3\Big(4(1-2a)^2k_2k_4+(a-1)\left(a\left(2q^2-1\right)-q^2\right)\Big)-2(a-1)k_4\left(-2aq^2+a+q^2\right)+4(1-2a)^2k_2k_3^2 \\ &+4(1-2a)^2k_2k_4^2\Big)-(a-1)k_2\Big(i(2a-1)q\Big((k_1+k_2)(k_3+k_4)\left(k_1(k_3+k_4)+k_2(k_3+k_4)+q^2-1\right)\Big)^{1/2} \\ &+2k_3\left(-2aq^2+a+q^2\right)+2k_4\left(-2aq^2+a+q^2\right)\Big)+2(1-2a)^2k_1^2\,(k_3+k_4)^2+2(1-2a)^2k_2^2\,(k_3+k_4)^2\Bigg)\,. \end{aligned} \tag{5.42}$$

- $k_1>0,\ k_2>0$ and $k_3>0,\ k_4<0$:

$$\begin{aligned} i\mathcal{M}_4 = &-\frac{4i(2a-1)k_1k_2k_3k_4}{k_1+k_2-k_3}\Bigg(2iaq\Big((k_1+k_2-k_3)\,k_4\left(k_1k_4+k_2k_4-k_3k_4+q^2-1\right)\Big)^{1/2}-2ak_4\left(q^2-1\right) \\ &-k_1q^2-k_2q^2+k_3q^2-2iq\Big((k_1+k_2-k_3)\,k_4\left(k_1k_4+k_2k_4-k_3k_4+q^2-1\right)\Big)^{1/2}+k_4\left(q^2-1\right) \\ &+k_1+k_2-k_3\Bigg)\,. \end{aligned} \tag{5.43}$$

- $k_1>0,\ k_2>0$ and $k_3<0,\ k_4>0$ is obtained from (5.43) by exchanging $k_3\longleftrightarrow k_4$.
- $k_1>0,\ k_2<0$ and $k_3>0,\ k_4<0$:

$$i\mathcal{M}_4 = 0\,. \tag{5.44}$$

Even though, in order to remove the absolute values, we have set the sign of the momenta, to recover the crossing configurations we allow them to assume any value. In fact, taking for example $k_1<0$ in (5.42), the amplitude corresponds to the situation in which in the final state there is a massless particle $\bar{u}$ with positive momenta.
All the other cases are obtained from these. In particular, parity transformations, i.e. $k_i\longrightarrow -k_i\,,\ i=1,2,3,4$, map between sectors with opposite signs of the momenta and time reversal maps between sectors in which the signs of k_1 and k_3 and the signs of k_2 and k_4 are exchanged. Furthermore, recalling that the mirror model is not invariant under time reversal, in order to find the correct expression, we also have to send $q\longrightarrow -q$. Let us explicitly see how this map works. First, by crossing the massive particles, we obtain the scattering $u(k_1)\bar{u}(k_2)z(a_1)\bar{z}(a_2)\longrightarrow v(k_3)\bar{v}(k_4)$. The amplitude is still the same because once the energy-momentum conservation has been exploited, it does not depend on the momenta of the massive particles. At this point, we note that the Lagrangian

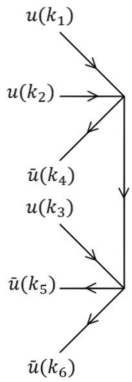


Figure 5.4: Feynman diagrams of the scattering $u(k_1)u(k_2)u(k_3)\bar{u}(k_4)\bar{u}(k_5)\bar{u}(k_6) \longrightarrow 0$ given by the quartic Lagrangian. The arrows describe the direction of the $U(1)$ currents. There are 9 diagrams of this type, given by the u and $\bar{u}$ permutations of this one.

is invariant under the exchange of u and v and their respective antiparticles. Therefore, by relabelling the momenta in such a way that $k_1 \leftrightarrow k_3$ and $k_2 \leftrightarrow k_4$, this amplitude is equal to that of the process $v(k_3)\bar{v}(k_4)z(a_1)\bar{z}(a_2) \longrightarrow u(k_1)\bar{u}(k_2)$. Finally, we can use the generalised time reversal to recover the process (5.39). This means, for example, that we can find the amplitude in the case in which $k_1 > 0$, $k_2 < 0$ and $k_3 > 0$, $k_4 > 0$ just by exchanging k_1 and k_3 together with k_2 and k_4 and sending $q \longrightarrow -q$ in the expression (5.43). Using a combination of this transformation and parity, one can recover all the sectors.

Ultimately, solving the Virasoro constraints, we find the six-point vertex of this process. The pieces of the sixth-order Lagrangian responsible for this vertex are written in the Appendix A. As in the previous case, we find that this vertex cancels out exactly $i\mathcal{M}_4$ and then the total amplitude is zero.

5.6 Massless production processes

Finally, let us consider a production process involving only massless particles. We consider the scattering

$$u(k_1)u(k_2)u(k_3)\bar{u}(k_4)\bar{u}(k_5)\bar{u}(k_6) \longrightarrow 0\,. \tag{5.45}$$

The same considerations that we did for crossing and the propagator pole structure in the previous section are also valid here.
The Feynman diagrams are shown in Figure 5.4 and the energy-momentum conservation reads

$$\begin{aligned} &k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 0\,, \\ &|k_1| + |k_2| + |k_3| + |k_4| + |k_5| + |k_6| = 0\,. \end{aligned} \tag{5.46}$$

Also in this case, in order to remove the absolute values, it is convenient to split the discussion in different cases according to the signs of the momenta.

- $k_1 > 0$ and $k_2 > 0$:
  - $k_3 > 0$ and $k_4 > 0$:
  $$i\mathcal{M}_4 = 0 \tag{5.47}$$
  - $k_3 < 0$ and $k_4 > 0$:
  $$i\mathcal{M}_4 = 0 \tag{5.48}$$
  - $k_3 < 0$ and $k_4 < 0$:
  $$i\mathcal{M}_4 = -64i(1-2a)^2 k_1 k_2 (k_1 + k_2)\, k_3 k_4 (k_3 + k_4)\ . \tag{5.49}$$

All the other cases can be found from these ones using crossing, parity, or time-reversal transformations, as shown in the previous section.
On the other hand the sixth-order Lagrangian is

$$\mathcal{L}^{(6)} = \frac{1}{2}(1-2a)^2(\dot{u}^2 - \acute{u}^2)(\dot{u}\dot{\bar{u}} - \acute{u}\acute{\bar{u}})(\dot{\bar{u}}^2 - \acute{\bar{u}}), \tag{5.50}$$

and the amplitude of the six-point vertex once again cancels out $i\mathcal{M}_4$.
In conclusion, we have seen that in the mirror theory, the production processes with six external legs at the tree-level have vanishing amplitude, according to the integrability hypothesis.

5.7 NLSM production processes

Due to the relation discussed in the previous chapter between the S matrices at the tree level of the two theories, we expect that also these results can be mapped between the two theories.
Let us start by finding the map between an amplitude computed in the mirror theory and an amplitude computed in the NLSM. Remaining at the tree-level, a Feynman diagram is composed by vertices and propagators. The map between the two theories is given by[1] $\partial_{\tilde{\tau}} \longrightarrow i\partial_\sigma$ and $\partial_{\tilde{\sigma}} \longrightarrow i\partial_\tau$, where the tilde notation denotes the mirror theory, while the non-tilde notation denotes the NLSM. According to the plane wave expressions, this means that the map between the vertices is

$$\tilde{P} \longrightarrow -iE \quad , \quad \tilde{E} \longrightarrow -iP\,, \tag{5.51}$$

where P and E denote respectively the momentum and the energy of any massive or massless particle of the theory.
On the other hand, (5.51) also maps between the dispersion relations of the two theories, and since these provide the pole of the propagators, this map also provides the relation between the propagators. In fact, taking for example the massive case, we have

$$\frac{i}{(\tilde{\omega}+iq)^2 - \tilde{p}^2 - 1 + q^2} \xrightarrow{(5.51)} \frac{i}{\omega^2 - (p-q)^2 - 1 + q^2}\,. \tag{5.52}$$

Finally, as we request since we are dealing with the same scattering process in the two theories, the energy-momentum constraints are left invariant by (5.51). It only gives an overall factor $\pm i$. Therefore, since each element that composes the tree-level amplitudes transforms coherently, this transformation maps the amplitudes of the two theories into each other.
Let us stress that this argument does not guarantee by itself that this is also the map between the tree-level S matrices. In fact, as seen before, in the calculation of the S matrix the normalisation factors (4.137) also appear. These explicitly break this map for generic values of the momenta. However, as we have found in the previous chapter, when they are combined with the factors coming from the delta functions in the allowed kinematic configurations, this map is restored also for the S matrix.

We have seen the expressions of the light-cone momenta for massive particles in the case of the mirror theory. Following the same procedure, the corresponding expressions in the NLSM are

$$\omega = \frac{1}{2}\sqrt{1-q^2}\left(a + \frac{1}{a}\right) \quad , \quad p = \frac{1}{2}\sqrt{1-q^2}\left(a - \frac{1}{a}\right) + q\,, \tag{5.53}$$

for z and y, and

$$\bar{\omega} = \frac{1}{2}\sqrt{1-q^2}\left(a + \frac{1}{a}\right) \quad , \quad p = \frac{1}{2}\sqrt{1-q^2}\left(a - \frac{1}{a}\right) - q\,, \tag{5.54}$$

for $\bar{z}$ and $\bar{y}$.
Hence, according to the expressions (5.28) and (5.29), in this parameterisation, the map (5.51) becomes

$$\tilde{a} \longrightarrow -ia\,. \tag{5.55}$$

[1] Note that we have seen in the previous chapter that the double Wick rotation is given by $\tau \to -i\tilde{\sigma}$ and $\sigma \to i\tilde{\tau}$. However, by exploiting the time-reversal invariance of the NLSM or equivalently the parity invariance of the mirror theory, this rotation is equivalent to $\tau \to i\tilde{\sigma}$ and $\sigma \to i\tilde{\tau}$. From this it follows that the derivatives transform in the manner written in the text.

Alternatively, using the equivalent mirror transformation (4.122) the map reads $\tilde{a} \longrightarrow \frac{i}{a}$.
Clearly, these two maps are equivalent. In fact, in light-cone momenta a parity and a time reversal transformation are given by

$$\text{parity:} \quad a \longrightarrow \frac{1}{a} \quad , \quad \text{time reversal:} \quad a \longrightarrow -\frac{1}{a}\,, \tag{5.56}$$

and exploiting the fact that the mirror theory is invariant under parity, while the NLSM is invariant under time reversal it follows that the two descriptions are the same.
On the other hand, for the massless particles, given the momentum $\tilde{k}$ in the mirror model and k in the NLSM, the relation (5.51) is just

$$\tilde{k} \longrightarrow -i|k| \quad , \quad |\tilde{k}| \longrightarrow -ik\,, \tag{5.57}$$

where the standard dispersion relation of a massless particle has been exploited. Let us note that, according to this expression, the analytic continuation of $\tilde{k} > 0$ to $k > 0$ is given by $\tilde{k} \longrightarrow -ik$, while the analytic continuation of $\tilde{k} < 0$ to $k < 0$ is given by $\tilde{k} \longrightarrow ik$. Therefore, written in a compact notation it becomes

$$\tilde{k} \longrightarrow \pm ik\,, \tag{5.58}$$

depending on the sign of the momenta. Clearly, we can equivalently choose to map the two theories in the opposite sectors just by changing the signs.
We have found that, given an amplitude in the mirror model $\mathcal{M}(k_i, a_i)$, we can find the corresponding amplitude in the NLSM by the analytic continuations (5.55) + (5.58). It is worth recalling that when we were computing the amplitudes, we stressed the fact that, in order to also take into account the crossing processes, the light-cone momenta were free to take any real values. Now, we can go further and consider the momenta in the complex plane. In fact, all the expressions and computations we did are well defined for any complex number. In other worlds, we did not assume in any passage that the light-cone momenta were real. In this way, we have an expression defined for any complex value of the momenta, which, when restricted to the NLSM physical region, describes the amplitude of the scattering in the NLSM theory, while, when restricted to the mirror physical region, describes the amplitude of the mirror scattering. Finally, since we have found that the expressions of the amplitudes vanish for any complex value, we conclude that six-point production processes also vanish in the NLSM, according to its classical integrability.
It is worth pointing out that for a generic theory this map does not guarantee that the amplitudes in the two theories have the same structure. In fact, has mentioned at the beginning of this section the propagator is split into a principal value plus a delta function term. In general, this map can mix the two terms when passing from one theory to the other. However, the simple pole structure of an integrable theory, which consists of setting the initial momenta equal to the finale ones, is invariant under this map, and then the structure is preserved. Non integrable theories may have a more generic pole structure which is not preserved under this map.

Conclusions

In this thesis we have studied some perturbative aspects of both the mixed-flux $AdS_3 \times S^3 \times T^4$ gauge-fixed non-linear sigma model and its corresponding mirror theory in order to check the integrability hypothesis and to investigate their relation.
First of all, we have quantised the bosonic gauge-fixed NLSM. The worldsheet theory is composed of four complex massive modes and four complex massless modes, the latter coming from the torus. Then, we found the tree-level two-body S matrix involving both massive and massless particles. As follows from its structure (3.84)-(3.85)-(3.86) it preserves the set of momenta and obeys the classical Yang-Baxter equation, according to the classical integrability of the theory.
Performing the double Wick rotation on the NLSM Lagrangian we end up with a non-unitary mirror theory. In fact, as can be seen from the dispersion relations (4.133), the massive particles have complex energies for the values of the parameter q for which the NLSM is defined, namely $q \in (0, 1)$. More generally, they are complex for $q \in \mathbb{R}$, while they become real when q is analytically continued on the imaginary axis. On the other hand, in the latter case, the NLSM becomes non-unitary. Since q interpolates between the pure RR flux model ($q = 0$) and the pure NS-NS flux model ($q = 1$), the only case in which both theories are unitary is the pure RR case ($q = 0$).
Thereafter, we have quantised the mirror theory leaving q as a free parameter on the whole complex plane and we have computed the two-body tree-level S matrix (4.141)-(4.142)-(4.143). This is related to the NLSM one by the map $p \to i\omega$ and $\omega \to ip$ and respects all the integrability conditions.

Another requirement for integrability is the absence of particle production processes. To investigate this aspect, we have considered scattering processes with six external legs. At the tree-level these receive a contribution from the quartic and from the sixth-order Lagrangian. The latter has been computed in the NLSM by solving the Virasoro constraints at second order in $1/T$ and has been mapped to the mirror theory using the usual double Wick rotation. Since these theories have derivative interactions, which means that they have derivative terms in the potential, the relation $\mathcal{L}_{int} = -\mathcal{H}_{int}$ is no longer valid at order higher than the quartic Lagrangian. As we expect, the Hamiltonian and the Lagrangian approaches still give the same result in the computation of the Feynman diagrams once we consider additional terms coming from the non-commutability between the time derivatives and the time-ordered product in the Hamiltonian. Furthermore, we expect that integrating out the momenta in the path-integral formula (5.8), the Lagrangian receives some corrections because of the derivative interaction terms. However, we have shown that these corrections start to appear at the quantum level, and thus, at the tree-level we can safely rely on the Feynman rules drawn from the Lagrangian, without considering any corrections.
In order to compute the six-point amplitudes in the production configurations, we have removed the $i\epsilon$ prescription from the propagator, considering the virtual particle propagating in between the two vertices always off-shell. In fact, the propagator can be split into a Cauchy principal value term, which is the off-shell contribution, and a delta function term, which is the on-shell contribution. We have shown that the latter is responsible for the factorisation of the $m \to m$ processes. Therefore, it does not contribute to the production amplitudes and can be neglected in our discussion. This corresponds to the physical intuition of what factorised scattering is. In fact, as seen in the first chapter, in integrable QFTs, a $m \to m$ process is decomposed into a series of two-to-two processes. The on-shell propagator splits the Feynman diagram into two vertices, where the propagator becomes a physical incoming particle in one of them and a physical outgoing particle in the other one, manifestly showing

the factorisation of the scattering.
Finally, with all the ingredients at hand, we have computed some production processes with six external legs involving both massive and massless modes at the tree-level in the mirror theory. We have shown that all of them have vanishing amplitudes, according to the integrability hypothesis.
Moreover, the mirror amplitudes can be mapped to the NLSM by the analytic continuation (5.55) + (5.58). Therefore, since we found that the amplitudes vanish for any complex value of the light-cone momenta and since in these processes this map acts coherently on the propagators in the sense that it maps the principal values to the principal values and the delta functions to the delta functions; the absence of production in the mirror theory is mapped to the absence of production in the NLSM and *vice versa*. Let us stress that all these results and the tree-level integrability structure are still valid for any value of the parameter q in the complex plane, without assuming $q \in (0, 1)$, that is the interval in which the NLSM is defined .
From these results, further investigations can be carried out. First, the tree-level matrix can be used to check the complete non-perturbative S-matrix by expanding it in $T >> 1$. In particular, this is known up to four overall factors, called dressing factors, and the perturbative results can be used to give a perturbative evaluation of these factors. Furthermore, one can investigate the integrability structure at the one- or higher-loop levels. Since in this case fermionic particles start to appear in the loops, the two-body S matrix is no longer diagonal; however, it is still supposed to respect the Yang-Baxter equations. Some steps in this direction can be found e.g. in [69], [70].
Finally, since the mirror theory is necessary to compute the ground state energy of the finite-size string, further investigations need to be carried out about its non-unitarity. In particular, in order to have a well-defined description, it must return a real energy spectrum for the string. So far, the TBA approach has been carried out in the special cases $q = 0$ [67], [71], [72] and $q = 1$ [73].

Appendix A

Sixth-order Lagrangian

Here we write the explicit expression of the pieces of the sixth-order Lagrangian which contribute to the $u(k_1)\bar{u}(k_2) \longrightarrow v(k_3)\bar{v}(k_4)z(a_1)\bar{z}(a_2)$ scattering:

$$
\begin{aligned}
\mathcal{L}^{(6)} = \frac{1}{2}i\Bigg(& -\dot{u}\dot{v}(-2(1-2a)aq\bar{z}(-3i\dot{z}\dot{\bar{u}}\dot{\bar{v}} + i\acute{z}\acute{\bar{u}}\dot{\bar{v}} + i\acute{z}\dot{\bar{u}}\acute{\bar{v}} + i\dot{z}\acute{\bar{u}}\acute{\bar{v}}) + 2(1-2a)aqz(-3i\dot{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} + i\acute{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} \\
& + i\dot{\bar{z}}\acute{\bar{u}}\acute{\bar{v}}) - iz\bar{z}(-((-1+4(1-2a)-(1-2a)^2)\dot{\bar{u}}\dot{\bar{v}}) + i\acute{\bar{u}}(i\acute{\bar{v}} + i(1-2a)^2\acute{\bar{v}})) + 2i(1-2a)^2(-\acute{z}\dot{\bar{z}}(\acute{\bar{u}}\dot{\bar{v}} + \dot{\bar{u}}\acute{\bar{v}}) \\
& - \acute{z}\acute{\bar{z}}(-\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}}) + i\dot{z}(-3i\dot{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} + i\acute{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\dot{\bar{z}}\acute{\bar{u}}\acute{\bar{v}}))) - \acute{u}\acute{v}(iz\bar{z}((1+(1-2a)^2)\dot{\bar{u}}\dot{\bar{v}} - (1+4(1-2a) \\
& + (1-2a)^2)\acute{\bar{u}}\acute{\bar{v}}) - 2(1-2a)aq\bar{z}(-i\acute{z}(\acute{\bar{u}}\dot{\bar{v}} + \dot{\bar{u}}\acute{\bar{v}}) - i\dot{z}(-\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}})) - 2(1-2a)aqz(i\acute{\bar{z}}(\acute{\bar{u}}\dot{\bar{v}} + \dot{\bar{u}}\acute{\bar{v}}) + i\dot{\bar{z}}(-\dot{\bar{u}}\dot{\bar{v}} \\
& - \acute{\bar{u}}\acute{\bar{v}})) + 2i(1-2a)^2(\dot{z}\acute{\bar{z}}(\acute{\bar{u}}\dot{\bar{v}} + \dot{\bar{u}}\acute{\bar{v}}) + \dot{z}\dot{\bar{z}}(-\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}}) + i\acute{z}(-i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + 3i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}}))) + \acute{u}\dot{v}(iz\bar{z}(\acute{\bar{u}}\dot{\bar{v}} \\
& - \dot{\bar{u}}\acute{\bar{v}}) + (1-2a)q(-i\dot{z}\bar{z}(\acute{\bar{u}}\dot{\bar{v}} - \dot{\bar{u}}\acute{\bar{v}}) + i\acute{z}\bar{z}(\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}}) + z(-i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} + i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}})) - (1-2a)^2 \\
& (-i(z\bar{z}(\acute{\bar{u}}\dot{\bar{v}} - \dot{\bar{u}}\acute{\bar{v}}) + 2i\acute{z}(-i\dot{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} - i\acute{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} + i\acute{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\dot{\bar{z}}\acute{\bar{u}}\acute{\bar{v}}) - 2i\dot{z}(i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} + i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} - i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}})) + q(-i\dot{z}\bar{z}(\acute{\bar{u}}\dot{\bar{v}} \\
& - \dot{\bar{u}}\acute{\bar{v}}) + i\acute{z}\bar{z}(\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}}) + z(-i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} + i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}})))) + \dot{u}\acute{v}(iz\bar{z}(-\acute{\bar{u}}\dot{\bar{v}} + \dot{\bar{u}}\acute{\bar{v}}) + (1-2a)q(-i\dot{z}\bar{z}(-\acute{\bar{u}}\dot{\bar{v}} \\
& + \dot{\bar{u}}\acute{\bar{v}}) + i\acute{z}\bar{z}(\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}}) + z(-i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} + i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}})) - (1-2a)^2(i(z\bar{z}(\acute{\bar{u}}\dot{\bar{v}} - \dot{\bar{u}}\acute{\bar{v}}) - 2i\acute{z}(-i\dot{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} \\
& - i\acute{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\dot{\bar{z}}\acute{\bar{u}}\acute{\bar{v}}) + 2i\dot{z}(i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} + i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} - i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} - i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}})) + q(-i\dot{z}\bar{z}(-\acute{\bar{u}}\dot{\bar{v}} + \dot{\bar{u}}\acute{\bar{v}}) + i\acute{z}\bar{z}(\dot{\bar{u}}\dot{\bar{v}} - \acute{\bar{u}}\acute{\bar{v}}) + z(-i\acute{\bar{z}}\dot{\bar{u}}\dot{\bar{v}} \\
& - i\dot{\bar{z}}\acute{\bar{u}}\dot{\bar{v}} + i\dot{\bar{z}}\dot{\bar{u}}\acute{\bar{v}} + i\acute{\bar{z}}\acute{\bar{u}}\acute{\bar{v}})))) \Bigg).
\end{aligned}
\tag{A.1}
$$

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