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OCCUPATION MEASURES: GAP ANALYSIS AND APPLICATIONS

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Introduction

Minimization problems are at heart of Calculus of Variations. They arise everywhere in Physics, Engineering, Biology, Modeling and other fields of study. Each problem may be formulated and approached in multiple ways, possibly depending on the aim and the features of the problem.
One fundamental and in fact strongly relevant feature for a variational problem is dimensionality. By *dimensionality* of a problem we mean the dimension of the objects and spaces involved.
One class of variational problems which enjoys particularly nice properties is the class of problems with codimension equal to 1, e.g. minimization over a space of $\mathbb{R}^1$ valued maps or minimization of area over $(n-1)$-dimensional surfaces.
In this thesis we go through two results concerning minimization problems in codimension 1. Besides being both about problems in codimension 1, these two results are structurally related, even though they are born in two quite different context.
The first one is found in [5]. This article from Robert Miller Hardt and Jon T. Pitts is a milestone in the history of Geometric Measure Theory. In it the authors consider an $(n-1)$-dimensional normal current N having compact support and rectifiable boundary and they find a way of decomposing N as the integral over $(0,1)$ of some rectifiable currents $\{R_s\}_{s\in(0,1)}$. As a corollary, they also produce an integral current T satisfying

$$\begin{cases} \partial T = \partial N \\ \mathbf{M}(T) \leq \mathbf{M}(N) \end{cases}.$$

Hardt-Pitts's Theorem guarantees that if some fixed rectifiable boundary allows for a mass-minimizing current, then the same minimum is attained by some other current which is also integral.

The second result which we will be presenting is found in [7]. While Hardt-Pitts article falls into the area of abstract Geometric Measure Theory, in this paper Milan Korda and Rodolfo Rios-Zertuche prove a decomposition result about occupation measures.
Occupation measures are a class $\mathcal{M}$ of measures in which the space[(1)] $C^1(\overline{\Omega}, \mathbb{R}^n)$ can be embedded and in which any integral functional F of the form

$$\begin{array}{rccl} F: & W^{1,\infty}(\overline{\Omega}) & \longrightarrow & \mathbb{R} \\ & y & \mapsto & \displaystyle\int_\Omega L\big(x, y(x), Dy(x)\big)\, dx + \int_{\partial\Omega} L_\partial\big(x, y(x)\big)\, d\mathcal{H}^{n-1} \end{array}$$

(where $L \in L^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and $L_\partial \in L^1\big(\partial\Omega \times \mathbb{R})\big)$ can be extended in a natural way as

$$\begin{array}{rccl} \overline{F}: & \mathcal{M} & \longrightarrow & \mathbb{R} \\ & (\mu, \mu_\partial) & \mapsto & \displaystyle\int_{\Omega\times\mathbb{R}\times\mathbb{R}^n} L\, d\mu + \int_{\partial\Omega\times\mathbb{R}} L_\partial\, d\mu_\partial \end{array}.$$

The problem of minimizing the extended functional over the class of occupation measures is linear, and thus amenable for numerical approaches (such as linear programming and convex optimization). Numerical methods are not the topic of this thesis, the interested reader is referred to [8] for fundamental definitions and [6] for a number of applications.

Our objective is to prove the decomposition result for occupation measures and use it to show that under suitable convexity assumptions the "numerically easier" relaxed problem on the space $\mathcal{M}$ has the same solution as the original problem on the space $W^{1,\infty}(\overline{\Omega})$.

We now briefly anticipate the formal definition of occupation measure and the main decomposition result.

[1] Here Ω is an open subset of $\mathbb{R}^n$ with piecewise C^1 boundary.

We say that a couple (μ, μ_∂) is a **relaxed occupation measure** (or simply **occupation measure**) on Ω if

(i) μ is a compactly supported and positive Radon measure on $\overline{\Omega} \times Y \times Z$. μ_∂ is a compactly supported and positive Radon measure on $\partial\Omega \times Y$.

(ii) $\mu(\overline{\Omega} \times Y \times Z) = \mathcal{L}^n(\Omega)$.

(iii) For any $\phi \in C^\infty(\Omega \times Y)$, the measures μ and μ_∂ satisfy

$$\int_{\overline{\Omega}\times Y\times Z} \left(\frac{\partial\phi}{\partial x}(x,y) + \frac{\partial\phi}{\partial y}\, z\right) d\mu(x,y,z) \;=\; \int_{\partial\Omega\times Y} \phi(x,y)\mathbf{n}(x)\, d\mu_\partial(x,y), \tag{1}$$

where $\mathbf{n}(x)$ is the exterior normal vector to Ω at x.

Our final result will be that for any occupation measure (μ, μ_∂) there exists a family $\{\psi_r\}_{r\in[-1,0]} \subset W^{1,\infty}(\overline{\Omega})$ such that:

- for any $\phi \in L^1(\mu)$ which is affine in z we have
$$\int_{\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n} \phi\, d\mu = \int_{[-1,0]}\int_{\Omega} \phi\big(x, \psi_r(x), D\psi_r(x)\big)\, d\mathcal{L}^n(x)\, d\mathcal{L}^1(r);$$
- for any $\phi_\partial \in L^1(\mu_\partial)$ we have
$$\int_{\partial\Omega\times\mathbb{R}} \phi_\partial\, d\mu_\partial = \int_{(-1,0)}\int_{\partial\Omega} \phi_\partial\big(x, \psi_r(x)\big)\, d\mathcal{H}^{n-1}(x)\, d\mathcal{L}^1(r).$$

For the sake of completeness we feel compelled to point out that paper [7], on which the second part of this thesis is based, has some flaws in its proofs, though in the end they do not affect the validity of the main results.
We omit the typographical mistakes[(2)] but we list here the other issues to

[2] For instance, the fundamental identity proven in [7, Lemma 2.14, page 20] lacks of a negative sign, since $S(u)$ is defined as $S(u) := \int_{\Omega\times Y\times Z}\int_y^\infty u(s)\, ds\, d\mu(x,y,z)$, while the computations at page 20 are carried out considering $S(u)$ defined as $\int_{\Omega\times Y\times Z}\int_{-\infty}^y u(s)\, ds\, d\mu(x,y,z)$. Similarly the formula which in this thesis is proven in Lemma 3.2.11(1), must have a negative sign because ρ is negative, while in the paper (e.g. at page 24) it is used as if ρ was positive.

the benefit of the interested reader who may wish to make a comparison with the original paper:

- In the proof of lemma 2.15 (and in the rest of the paper) the authors never make use of the reduced boundary, which is the actual object of interest throughout the entirety of the proof. They instead use the topological boundary. It can be proven, and it is clear at the end of the proof, that under our hypothesis the reduced boundaries and the topological boundaries which we consider coincide $\mathcal{H}^n$-almost everywhere;

- In the proof of Lemma 2.19 the authors prove that the measure m_2 is absolutely continuous with respect to the measure m_1 ($m_2 \ll m_1$, see page 103 of this thesis for the definitions), which is true, but they use it to directly deduce the existence of the maps J_r (the same maps which we define in (3.28) at page 98). This deduction is not possible without first finding the value of the limit which we prove in claim 5 at page 102.

Contents

Chapter 1

Notations and fundamental definitions

In this chapter we will be dealing with currents. Currents are one of the fundamental objects in the field of Geometric Measure Theory. We shall briefly introduce them, describe their use and recall some fundamental properties which will be at the core of our discussion.
To properly understand the formalism behind currents we shall

- define the *exterior algebra* of a vector space, its wedge product and its inner product;
- prove existence and uniqueness of the exterior algebra;
- define the space of linear and alternating maps and its wedge product;
- state the fundamental link between the exterior algebra and alternating maps;
- define the space of k-forms on an open set of $\mathbb{R}^n$;
- define what is a current, its total variation, its mass and how they extend k-surfaces and their $\mathcal{H}^k$-measure;

- define the cartesian product, define the join of two currents and state the fundamental properties of these two operations;
- define rectifiable currents and state a characterization theorem for them;
- define integral currents, which will be in a way the final objective of this chapter.

1.1 Elementary notation

Definition 1.1.1 (Open balls).
Given a metric space (X, d), an element $a \in X$ and a poritive real number r, we define The *open ball centered in a and with radius r* as

$$\mathbf{B}(a, r) := \{x \in X : d(x, a) < r\}.$$

Definition 1.1.2 (m-dimensional measure of the unit ball).
Given a non negative integer m, we'll denote by $\alpha(m)$ the measure of the unitary open ball in $\mathbb{R}^m$ with respect to the m-dimensional lebesgue measure:

$$\alpha(m) := \mathcal{L}^m(U(0, 1)).$$

Definition 1.1.3 (m-dimensional upper and lower densities of a measure at a point).
Given a metric space (X, d), a non negative integer m, and a positive measure μ on X, define the *m-dimensional upper density of μ at a* as

$$\Theta^{*m}(\mu, a) := \limsup_{r \to 0^+} \frac{1}{\alpha(m) r^m} \mu\big(B(a, r)\big)$$

and the the *m-dimensional lower density of μ at a* as

$$\Theta^m_*(\mu, a) := \liminf_{r \to 0^+} \frac{1}{\alpha(m) r^m} \mu\big(B(a, r)\big).$$

Definition 1.1.4 (Density of a measure at a point).
Given a metric space (X, d), a non negative integer m, and a positive measure

μ on X, if $\Theta^{*m}(\mu,a) = \Theta^{m}_{*}(\mu,a)$, then define the *m-dimensional density of μ at a* as

$$\Theta(\mu,a) := \Theta^{m}_{*}(\mu,a)\big(= \Theta^{*m}(\mu,a)\big).$$

Definition 1.1.5 (Tangent cone of a subset at a point)**.**
Given a normed space $(X, ||\cdot||_X)$, a subset S of X and an element $a \in X$, define the *Tangent cone of S at a* as follows:

$$\mathrm{Tan}(S,a) := \{x \in X : \forall \epsilon > 0, \exists s \in S, \exists r > 0 \text{ such that } ||s-a|| < \epsilon \text{ and } ||r(s-a)-x|| < \epsilon\}.$$

Any element of $\mathrm{Tan}(S,a)$ will be called a *tangent vector of S at a*.

Definition 1.1.6 ($(\phi,m)-$approximate tangent vectors at a)**.**
Given a normed space $(X, ||\cdot||_X)$, a positive integer m, a positive measure ϕ on X and an element $a \in X$, define the *Set of (ϕ,m)-approximate tangent vectors* as

$$\mathrm{Tan}^m(\phi,a) := \bigcap_{\substack{S\subset X \text{ s.t.} \\ \Theta^m(\phi \llcorner (X\setminus S))=0}} \mathrm{Tan}(S,a).$$

Definition 1.1.7 (m^{th} Grassmann manifold)**.**

- Given a vector space V, its m^{th} *Grassmann manifold* is the set

  $$\mathbf{G}(V,m) := \{W \leq V : \dim(W) = m\}$$

  containing all m-dimensional subspaces of V.

- The grassmann manifold $\mathbf{G}(\mathbb{R}^n, m)$ will be denoted shortly as $\mathbf{G}(n,m)$.

Definition 1.1.8 (Rectifiable sets)**.**
Let E be a sub set of a metric space X and let m be a positive integer. We will say that E is:

- *m-rectifiable* if there exists a Lipschitzian map f mapping a bounded subset of $\mathbb{R}^n$ onto E.

- *countably m-rectifiable* if E equals the union of some countable family of m-rectifiable sets.

- *countably (ϕ, m)-rectifiable* (with ϕ measure on X) if it exists a countably m rectifiable set $R \subset X$ such that $\phi(E \setminus R) = 0$.

- *(ϕ, m)-rectifiable* if it is countably (ϕ, m)-rectifiable and $\phi(E) < \infty$.

1.2 Tensor product of vector spaces

For all vector spaces considered we will take as given that the field they are defined on is $\mathbb{R}$.

Idea: the tensor product of some vector spaces $V_1, ..., V_m$ can be thought as the minimal vector space that is able to embody the properties of any multilinear map from $V_1 \times ... \times V_m$ to $\mathbb{R}$.

Definition 1.2.1 (Tensor product of vector spaces)**.**
Given m vector spaces $V_1, V_2, ..., V_m$ we call a *tensor product of* $V_1, ..., V_m$ any coupling $(\mathcal{V}, \mu)$ where $\mathcal{V}$ is a vector space and $\phi : V_1 \times V_2 \times ... \times V_m \to \mathcal{V}$ is an m-linear map such that

for any m-linear map f from $V_1 \times ... \times V_m$ to some other vector space W, $\exists! g : \mathcal{V} \to W$ linear and such that $f = g \circ \mu$.

We will denote:

- the space $\mathcal{V}$ as $V_1 \otimes V_2 \otimes ... \otimes V_m$

- the image through μ of an m-tuple $(v_1, ..., v_m)$ as $v_1 \otimes ... \otimes v_m$.

Theorem 1.2.1. *Consider m vector spaces $V_1, ..., V_m$. Their tensor product exists and is unique up to isomorphism.*

Proof.
Let F be the real vector space

$$F := \left\{ f : V_1 \times ... \times V_m \to \mathbb{R} : \, \mathrm{Card}(\{v : f(v) \neq 0\}) < \infty \right\}$$

Let, for any $(v_1, ..., v_m) \in V_1 \times ... \times V_m$, $f_{v_1,...,v_m}$ be the map

$$\begin{aligned} f_{v_1,...,v_m}: \quad V_1 \times ... \times V_m &\to \mathbb{R} \\ (w_1, ..., w_m) &\mapsto \begin{cases} 1 & \text{if } (w_1, ..., w_m) = (v_1, ..., v_m) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Let ϕ be the (injective) map

$$\begin{aligned} \phi: \quad V_1 \times ... \times V_m &\to F \\ (v_1, ..., v_m) &\mapsto f_{v_1,...,v_m} \end{aligned}$$

Let $G \leq F$ be the subspace generated by the elements of the type

$$\phi(v_1, ..., v_{i-1}, x, v_{i+1}, ..., v_m) + \phi(v_1, ..., v_{i-1}, y, v_{i+1}, ..., v_m) + \\ -\phi(v_1, ..., v_{i-1}, x + y, v_{i+1}, ..., v_m)$$

together with the elements of the type

$$\phi(v_1, ..., v_{i-1}, cx, v_{i+1}, ..., v_m) - c\phi(v_1, ..., v_{i-1}, x, v_{i+1}, ..., v_m)$$

Define then

$$V_1 \otimes ... \otimes V_m := F/G,$$

$$v_1 \otimes ... \otimes v_m := [\phi(v_1, ..., v_m)] = [f_{v_1,...,v_m}],$$

$$\text{(remembering the notation } \mu(v_1, ..., v_m) \equiv v_1 \otimes ... \otimes v_m).$$

Now

- The tensor map μ is m-linear by construction.
- $\mathrm{Im}(\mu)$ generates $V_1 \otimes ... \otimes V_m$.
- Any m-linear map $f : V_1 \times ... \times V_m \to W$ induces in a natural way a unique linear map on $\mathrm{Im}(\mu)$.

Therefore The couple $(F/G, \mu)$ defined is a tensor product of the m starting vector spaces.

It is also unique, because if we assume that there was anothe coupling (W, μ') being a tensor product, then

1. $\exists! g : F/G \to W$ linear and s.t. $\mu' = g\mu$
2. $\exists! g' : F/G \to W$ linear and s.t. $\mu = g'\mu'$
3. Putting together "1." and "2." we find that $gg'\mu' = \mu'$ and $g'g\mu = \mu$.
4. Since the maps μ and μ' must be monomorphisms (otherwise the fundamental property of tensor products is contradicted), it follows that $gg' = \mathrm{id}_W$ and $g'g = \mathrm{id}_{F/G}$, so that $g' = g^{-1}$ and g, g' are isomorphisms.

$\square$

Proposition 1.2.1 (Isomorphisms with tensor products)**.**

- *If λ is a permutation of $\{1, 2, ..., m\}$, then*

$$\begin{array}{rrcl} \Phi_\lambda : & V_1 \otimes ... \otimes V_m & \to & V_{\lambda(1)} \otimes ... \otimes V_{\lambda(m)} \\ & v_1 \otimes ... \otimes v_m & \mapsto & v_{\lambda(1)} \otimes ... \otimes v_{\lambda(m)} \\ & \text{other elements} & \mapsto & \text{by linearity} \end{array}$$

  is an isomorphism.

- *If $m < n$ then*

$$\begin{array}{rccl} \Phi_{m,n} : & \big(V_1 \otimes ... \otimes V_m\big) \otimes \big(V_{m+1} \otimes ... \otimes V_n\big) & \to & V_1 \otimes ... \otimes V_n \\ & (v_1 \otimes ... \otimes v_m) \otimes (v_{m+1} \otimes ... \otimes v_n) & \mapsto & v_1 \otimes ... \otimes v_n \\ & \text{other elements} & \mapsto & \text{by linearity} \end{array}$$

  is an isomorphism.

- *The map*

$$\begin{array}{rrcl} \Phi_{\mathbb{R}} & \mathbb{R} \otimes V & \to & V \\ & c \otimes v & \mapsto & cv \end{array}$$

  is an isomorphism.

- *Assume that V can be decomposed in the direct sum $V = P \oplus Q$. Then*

$$V \otimes W \;\cong\; (P \otimes W) \oplus (Q \otimes W).$$

As a consequence it follows that

$$\dim(V_1 \otimes ... \otimes V_m) = \prod_{i=1}^{m} \dim(V_i).$$

Proposition 1.2.2 (Naturality of the tensor product)**.**
The tensor product of vector spaces is natural, in the sense that given m morphisms of vector spaces (continuous linear maps)

$$f_1 : V_1 \to W_1$$

$$f_2 : V_2 \to W_2$$

$$\vdots$$

$$f_m : V_m \to W_m,$$

there is a unique morphism

$$\Phi : V_1 \otimes ... \otimes V_m \to W_1 \otimes ... \otimes W_m$$

such that for all $(v_1, ..., v_m) \in V_1 \times ... \times V_m$ it holds

$$\Phi(v_1 \otimes ... \otimes v_m) = f_1(v_1) \otimes ... \otimes f_m(v_m).$$

Such Φ is denoted by $f_1 \otimes ... \otimes f_m$.

Proof.
Consider the construction of the tensor product made in the proof of Theorem 1.2.1.
Then we would like

$$\begin{array}{cccc} \Phi : & V_1 \otimes ... \otimes V_m & \longrightarrow & W_1 \otimes ... \otimes W_m \\ & [f_{v_1,...,v_m}] & \mapsto & [f_{f_1(v_1),...,f_m(v_m)}] \\ & \text{other elements} & \mapsto & \text{by linearity} \end{array}$$

to be a well defined morphism of vector spaces.
To be well defined we need to show that it does not depend on the choice of

a representative for each class (uniqueness will follow as the set $\{[f_{v_1,...,v_m}] : v_i \in V_i\}$ is a generator of $V_1 \otimes ... \otimes V_m$).
Another representative is obtainen by summing to $f_{v_1,...,v_m}$ any element of the group G (defined in the proof of the theorem 1.2.1). Now by definition of G, by linearity of the maps f_i and by definition of the quotient classes in $W_1 \otimes ... \otimes W_m$ (which uses the subgroup G' of $W_1 \times ... \times W_m$ defined analogously as G) the morphism is well defined. □

1.3 Graded algebras

1.3.1 Definition of graded algebras

Idea: the concept of *graded algebra* is a completely abstract in nature. Its fundamental purpose, in our discussion, is to fix the natural set of properties which, as we will se, characterize the exterior algebra.

Definition 1.3.1 (Graded algebra)**.** We call a graded algebra any triplet $(A, \{A_n\}_{n\in\mathbb{N}}, \mu)$ where:

- A is a vector space.
- $\{A_n\}_{n\in\mathbb{N}}$ is a set of vector spaces with which A can be decomposed as $A = \bigoplus_{n=0}^{\infty} A_n$.
- $\mu : A \times A \to A$ is a bilinear map such that $\mu(A_m \times A_n) \in A_{m+n}$.

We will usually write the bilinear map μ as a product, and write $x \cdot y$ in place of $\mu(x, y)$.

Although they are not true in general, most of the graded algebras which are dealt with satisfy these three properties:

- μ is associative.

- $A_0 \cong \mathbb{R}$ and the unit element of A_0 is also a unit element for the ring A.
- Anticommutative law of the product: $\xi\cdot\eta = (-1)^{mn}\eta\cdot\xi \;\; \forall\xi \in A_m,\ \forall\eta \in A_n$.

1.3.2 Tensor algebra of a vector space V

> **Idea**: the *tensor algebra* of a vector space can be thought as the minimum vector space which is able to embody within itself the properties of any multilinear map from some power of V to another vector space W.

Definition 1.3.2 (Tensor algebra of a vector space V)**.**
We define the *tensor algebra of the vector space* V as

$$\bigotimes_* V := \bigoplus_{n=0}^{\infty} \bigotimes_n V,$$

where

$$\bigotimes_0 V = \mathbb{R}, \quad \bigotimes_1 V = V, \quad \bigotimes_2 V = V \otimes V, \quad \dots$$

and where the product is defined on the subspaces $\left(\bigotimes_m V\right) \times \left(\bigotimes_n V\right)$ as simple concatenation of tensor products, i.e.

$$(v_1 \otimes ... \otimes v_m) \cdot (w_1 \otimes ... \otimes w_n) := v_1 \otimes ... \otimes v_m \otimes w_1 \otimes ... \otimes w_n.$$

The tensor algebra has the following universal mapping property:

$$\left.\begin{array}{l} (A, \{A_n\}_{n\in\mathbb{N}}, \mu) \text{ is a graded algebra} \\ \mu \text{ is associative} \\ A \text{ has a unit element} \\ \varphi : V \to A_1 \text{ is a linear map} \end{array}\right\} \Rightarrow \begin{array}{l} \exists! \overline{\varphi} : \bigotimes_* V \to A \\ \text{unit preserving morphism of} \\ \text{algebras such that} \\ \quad \overline{\varphi}|_{\otimes_1 V} = \varphi \text{ and} \\ \quad \overline{\varphi}(\bigotimes_m V) \subset A_m \;\; \forall m \in \mathbb{N} \end{array}$$

Besides, the construction of the tensor algebra is natural, in the sense that:

$$\left.\begin{array}{l} V, V' \text{ are vector spaces} \\ f : V \to V' \text{ is a linear map} \end{array}\right\} \Rightarrow \begin{array}{l} \exists! \overline{f} : \bigotimes_* V \to \bigotimes_* V' \\ \text{extending } f \text{ and preserving the unit element} \end{array}$$

such extension $\overline{f}$ will be denoted by $\otimes_* f$.

1.3.3 Exterior algebra of a vector space

Idea: the *exterior algebra* of a vector space can be thought as the minimum vector space which is able to embody within itself the properties of any multilinear and alternating map from some power of V to another vector space W.
Notice that the exterior algebra can be defined in a way which does not depend from the space of multilinear alternating maps, which we will in fact define afterwards.

Definition 1.3.3 (The ideal $\mathcal{U}V$)**.** We define the two sided ideal $\mathcal{U}V$ generated by the set $\{x \otimes x : \; x \in V\}$ (it is indeed a subset of $\bigotimes_2 V$). In other words

$$\mathcal{U}V = \{a \otimes x \otimes x \otimes b : \; a, b \in \bigotimes_* V, \; x \in V\}.$$

Definition 1.3.4 (Exterior algebra)**.**
Given a vector space V we define its exterior algebra $\bigwedge_* V$ as the quotient

$$\bigwedge\nolimits_* V := \left(\bigotimes\nolimits_* V\right) / \mathcal{U}V.$$

We notice that the following general properties of the objects just defined:

- $\mathcal{U}V = \bigoplus_{m=2}^{\infty} \left(\bigotimes_m V \cap \mathcal{U}V\right)$.
- $\bigwedge_* V = \bigoplus_{m=0}^{\infty} \bigwedge_m V$, where $\bigwedge_m V := \bigotimes_m V \,/\, \left(\bigotimes_m V \cap \mathcal{U}V\right)$.

and that therefore the product operation induced by the product in $\otimes_* V$ (i.e. the one defined as $([v_1 \otimes ... \otimes v_m] \cdot [w_1 \otimes ... \otimes w_n] = [v_1 \otimes ... \otimes v_m \otimes w_1 \otimes ... \otimes w_n]$ on the fundamental elements and extended by linearity) makes $\bigwedge_* V$ a graded algebra.

We will denote its product operation by $\wedge$.

Observation: The product of the exterior algebra is anticommutative. In fact

$$\begin{aligned} x \wedge y + y \wedge x &= [x \otimes y + y \otimes x] \\ &= [(x+y) \otimes (x+y) - x \otimes x - y \otimes y] \\ &= [0] \end{aligned}$$

because $(x+y) \otimes (x+y) - x \otimes x - y \otimes y \in \mathcal{U}V$.

The exterior algebra has the following **universal mapping characterization**:

$$\left.\begin{array}{l} (A, \{A_n\}_{n\in\mathbb{N}}, \mu) \text{ is a graded algebra} \\ \mu \text{ is associative and anticommutative} \\ A \text{ has a unit element} \\ \varphi : V \to A_1 \text{ is a linear map} \end{array}\right\} \Rightarrow \begin{array}{l} \exists! \overline{\varphi} : \bigwedge_* V \to A \\ \text{unit preserving morphism of} \\ \text{algebras such that} \\ \quad \overline{\varphi}|_{\otimes_1 V} = \varphi \text{ and} \\ \quad \overline{\varphi}(\bigwedge_m V) \subset A_m \;\; \forall m \in \mathbb{N} \end{array}$$

Besides, the construction of the exterior algebra is natural, in the sense that:

$$\left.\begin{array}{l} V, V' \text{ are vector spaces} \\ f : V \to V' \text{ is a linear map} \end{array}\right\} \Rightarrow \begin{array}{l} \exists! \overline{f} : \bigwedge_* V \to \bigwedge_* V' \\ \text{extending } f \text{ and preserving} \\ \text{the unit element} \end{array}$$

such extension $\overline{f}$ will be denoted by $\wedge_* f$.

Moreover $\wedge_* f = \displaystyle\bigoplus_{m=0}^{\infty} \wedge_m f$, where $\wedge_m f := \wedge_* f|_{\wedge_m V}$.

1.3.4 Scalar product in the exterior algebra

Assume that a vector space V on a field K has a scalar product $\cdot$. Then the scalar product induces the natural polarity map

$$\begin{array}{rccl} \beta : & V & \longrightarrow & V^* \cong \bigwedge^1 V \\ & v & \mapsto & \beta(v) \end{array}$$

where

$$\begin{array}{rccc} \beta(v): & V & \longrightarrow & K \\ & w & \mapsto & w \cdot v \end{array}.$$

By naurality of the exterior algebras, the map β can be extended in a unique way to

$$\overline{\beta}: \bigwedge_* V \to \bigwedge^* V$$

and in particular

$$\overline{\beta} = \bigoplus_{n=0}^{\infty} \overline{\beta}|_{\wedge_n V} =: \overline{\beta}_0 + \overline{\beta}_1 + \overline{\beta}_2 + \dots .$$

Now given two simple multivectors

$$\overline{v} = u_1 \wedge \dots \wedge u_n$$

$$\overline{w} = u'_1 \wedge \dots \wedge u'_m,$$

we see that

$$\overline{g}(\overline{v}) \in \bigwedge^n (V, K) \cong \mathrm{Hom}\left(\bigwedge_n V, K\right) \cong \mathrm{Hom}^n\left(\bigwedge_* V, K\right),$$

where

$$\mathrm{Hom}^n\left(\bigwedge_* V, K\right) := \left\{\xi \in \mathrm{Hom}\left(\bigwedge_* V, K\right):\ \mathrm{Ker}(\xi) \supset \bigoplus_{m \neq n} \bigwedge_m V\right\},$$

and that more precisely it holds

$$\begin{aligned} \langle \overline{w}, \overline{g}(\overline{v})\rangle &= \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \langle u_i, u'_{\sigma(i)}\rangle \\ &= \det\left(\begin{pmatrix} - u_1^T - \\ \vdots \\ - u_n^T - \end{pmatrix} \begin{pmatrix} | & & | \\ u'_1 & \cdots & u'_n \\ | & & | \end{pmatrix} \right) \end{aligned} \tag{1.1}$$

so that an inner product $\langle \cdot, \cdot, \rangle$ can be defined on $\bigwedge_* V$ as

$$\langle v, w\rangle := \rangle v, \overline{g}(w)\rangle.$$

We see that

- $\langle \cdot, \cdot \rangle$ is symmetric
- it is bilinear wrt the sum of multivectors
- $\langle v, \lambda w \rangle = \lambda^n \langle v, w \rangle$ if $w \in \bigwedge_n V$
- $\langle v, \lambda v \rangle > 0$ for any nonzero simple multivector v.

This inner product allows to define a norm of simple multivectors as

$$|v| := \sqrt{\langle v, v \rangle}.$$

1.4 m-linear alternating maps

Idea: Multilinear alternating maps are well known to play a role in pretty much any branch of mathematics, mainly in the form of determinant of matrices. Here we present them in a form which is slightly more general than determinants, since we consider maps which are not necessarily $\mathbb{R}$ valued, but possibly take value in any real vector space W.

1.4.1 Definition

Definition 1.4.1 (m-linear alternating function, $\bigwedge^m(V, W)$)**.**
Given two vector spaces V, W, we say that f is an *m-linear alternating function* if

$$f : V^m \to W$$

is m-linear and $f(v_1, ..., v_m) = 0$ whenever $\exists i \neq j$ s.t. $v_i = v_j$.

We denote the set of m-linear alternating functions between V and W as $\bigwedge^m(V, W)$ and we will call each of its elements an *m-covector*.

Proposition 1.4.1. *The spaces* $\bigwedge^m(V, W)$ *and* $\mathrm{Hom}(\bigwedge_m(V), W)$ *are isomorphic.*

Proof. Asdf... $\square$

Notice that there is a natural linear isomorphism

$$\mathrm{Hom}(\bigwedge_m V, W) \cong \mathrm{Hom}^m(\bigwedge_* V, W)$$

where $\mathrm{Hom}^m(\bigwedge_* V, W)$ is the linear subspace of $\mathrm{Hom}(\bigwedge_* V, W)$ containing the maps which take value 0 on any element of $\bigwedge_n V$ with $n \neq m$. Also define

$$\bigwedge^*(V, W) := \bigoplus_{m=0}^{\infty} \bigwedge^m(V, W).$$

1.4.2 The diagonal map (or anticommutative product)

The **diagonal map** or **anticommutative product** in $\bigwedge_* V$ is the operation

$$\psi : \bigwedge_* V \longrightarrow \bigwedge_* V \otimes \bigwedge_* V$$

defined by:

- $\psi(1, 1) = 1$
- $\psi(v_1 \wedge ... \wedge v_m) = \prod_{i=1}^{m}(v_i \otimes 1 + 1 \otimes v_i)$, where the product rule intended in "$\prod$" is that

$$(v_i \otimes 1) \cdot (1 \otimes v_j) = v_i \otimes v_j = -(1 \otimes v_j) \cdot (v_i \otimes 1).$$

1.4.3 The wedge product of alternating maps

Definition 1.4.2 (Wedge product for alernating maps)**.** If W is an algebra (not necessarily graded) with its own product operation, then define

$$\begin{array}{rccc} \wedge : & \bigwedge^*(V, W) \times \bigwedge^*(V, W) & \to & \bigwedge^*(V, W) \\ & (\alpha, \beta) & \mapsto & \alpha \wedge \beta \end{array}$$

by defining it on the simple covectors: if $\alpha \in \bigwedge_m V$ and $\beta \in \bigwedge_n V$, then define $\alpha \wedge \beta \in \mathrm{Hom}^m(\bigwedge_* V, W)$ as the composition

$$\alpha \wedge \beta : \bigwedge_* V \xrightarrow{\psi} \bigwedge_* V \otimes \bigwedge_* V \xrightarrow{\alpha \otimes \beta} W \otimes W \xrightarrow{\nu} W$$

where ν is defined using the product of W as $\nu(s \otimes t) = s \cdot t$.

1.5 Lattices and representability by integration

Key takeaway: a lattice L is family of maps enjoying some very general hypothesis. We will not be dealing with lattices in their complete generality, but we shall spend a couple pages to appreciate their power in the context of measure theory and functional analysis. In fact, despite their generality it is possible to prove theorems of representation by integrability for functionals defined on L and taking values in $\mathbb{R}$.

Definition 1.5.1 (Lattices of functions)**.**
Let X be a set.

- We call a *lattice of functions* on X any set L of functions from X to $\mathbb{R}$ such that:
  1. $f, g \in L \;\Rightarrow\; f + g \in L$ and $\inf\{f, g\} \in L$.
  2. $0 \leq c < \infty,\; f \in L \;\Rightarrow\; cf \in L$ and $\inf\{f, c\} \in L$.
  3. $f, g \in L,\; f \leq g \;\Rightarrow\; g - f \in L$.
- Observation: $f \in L =$ lattice of functions $\Rightarrow\; f^+ = f - \inf\{f, 0\} \in L$ and $f^- = f^+ - f \in L$.
- Observation/definition: L is a lattice of functions $\Rightarrow\; L^+ := L \cap \{f \geq 0\}$ is a lattice of functions.

Theorem 1.5.1 (Representation by integration).
Let L be a lattice on X. Let $\lambda : L \to \mathbb{R}$ such that for any $f, g, h_1, h_2, h_3, ... \in L$ the following statements hold:

- $\lambda(f+g) = \lambda(f) + \lambda(g)$.
- $c \in [0, +\infty) \Rightarrow \lambda(cf) = c\lambda(f)$.
- $f \geq g \Rightarrow \lambda(f) \geq \lambda(g)$
- $h_n \uparrow f$ *as* $n \to \infty \Rightarrow \lambda(h_n) \uparrow \lambda(f)$ *as* $n \to \infty$.

Then there exists a measure ϕ on X such that

$$\lambda(f) = \int f \, d\phi \quad \forall f \in L.$$

The measure ϕ in the above theorem is not unique in general, but it has some properties. In particular the proof of the theorem constructs a ϕ which has some regularities (see **Herbert-Federer**, section 2.5).

Theorem 1.5.2.
Let L be a lattice on X. Let $\lambda : L \to \mathbb{R}$ such that for any $f, g, h_1, h_2, h_3, ... \in L$ the following statements hold:

- $\lambda(f+g) = \lambda(f) + \lambda(g)$.
- $c \in [0, +\infty) \Rightarrow \lambda(cf) = c\lambda(f)$.
- $\sup(L \cap \{g : 0 \leq g \leq f\}) < \infty$.
- $h_n \uparrow f$ *as* $n \to \infty \Rightarrow \lambda(h_n) \uparrow \lambda(f)$ *as* $n \to \infty$.

Let

$$\begin{array}{rccl} \lambda^+ : & L^+ & \longrightarrow & \mathbb{R} \\ & f & \mapsto & \sup(L \cap \{g : 0 \leq g \leq f\}) \end{array}$$

and

$$\begin{array}{rccl} \lambda^- : & L^+ & \longrightarrow & \mathbb{R} \\ & f & \mapsto & -\inf(L \cap \{g : 0 \leq g \leq f\}) \end{array}.$$

Then there exist two positive measures ψ^+ and ψ^- on X such that

$$\lambda^+(f) = \int f \, d\psi^+ \quad \text{and} \quad \lambda^-(f) = \int f \, d\psi^- \qquad \forall f \in L^+,$$

$$\lambda(f) = \int f \, d\psi^+ - \int f \, d\psi^- \quad \forall f \in L.$$

Definition 1.5.2 (Daniell integral)**.**
Let L be a lattice on X. We will call:

- a *Daniell integral* on L any function λ satisfying the hypothesis of theorem 1.5.2.
- a *monotone Daniell integral* on L any function λ satisfying the hypothesis of theorem 1.5.1

Theorem 1.5.3.

Suppose that:

1. $(E, ||\cdot||)$ *is a separable normed space.*

2. $(E^*, ||\cdot||_*)$ *is its dual.*

3. X *is a set and* L *is a lattice on* X *such that* L^+ *contains a countable subset* K *of maps such that*

$$\sum_{f\in K} f(x) \geq 1 \quad \forall x \in X.$$

4. Ω *is a vector space of functions mapping* X *into* E *such that:*

   (a) $f \in L,\ y \in E \ \Rightarrow\ f \cdot y \in \Omega.$

   (b) $\omega \in \Omega,\ \alpha \in E^* \ \Rightarrow\ \alpha \circ \omega \in L,\ ||\cdot|| \circ \omega \in L.$

   (c) $\omega \in \Omega,\ ||\cdot||\circ\omega \geq f \in L^+ \ \Rightarrow\ \exists \xi \in \Omega : ||\cdot||\circ\xi = f,\ (||\cdot||\circ\omega)\cdot\xi = f \cdot \omega$

5. $T : \Omega \to \mathbb{R}$ *is a linear map such that, for any* $f \in L^+$ *and any* $\xi_1, \xi_2, ... \in \Omega$*:*

   (a) $\lambda(f) := \sup T(\Omega \cap \{\omega :\ ||\cdot|| \circ \omega \leq f\}) \quad \forall f \in L^+,$

   (b) $||\cdot|| \circ \xi_n \downarrow 0 \ \Rightarrow\ T(\xi_n) \to 0.$

Then λ *is a monotone Daniell integral on* L^+*.*
Moreover, if ϕ *is the* L^+ *regular measure associated with* λ*, then* $\exists\, k : X \to E^*$ ϕ*-measurable such that:*

(a) $||\cdot|| \circ k$ *is* ϕ*-measurable*

(b) $T(\omega) = \int \langle \omega(x), k(x) \rangle \, d\phi(x) \quad \forall \omega \in \Omega.$

Such a function k *is* ϕ*-almost unique. Every member of* Ω *is* ϕ *measurable. For each* ϕ *measurable function* η *with values in* E*, the real valued function* $\langle \eta, k \rangle$ *is* ϕ *measurable; in case* $||\cdot|| \circ \eta$ *is* ϕ *summable, so is* $\langle \eta, k \rangle$*.*

Definition 1.5.3 (Variation integral and measure).
Under the hypothesis of theorem 1.5.3, we call λ the *variation integral* and ϕ the *variation measure* associated with T.

Definition 1.5.4 (The lattice of continuous functions with compact support).
For any locally compact Hausdorff topological space X, will denote by $\mathcal{K}(X)$ the set of continuous maps $f : X \to \mathbb{R}$ whose support is compact.

Theorem 1.5.4 (Riesz Representation Theorem).
Let X be a locally compact hausdorff space. Let $L := \mathcal{K}(X)$ be the lattice of continuous functions with compact support. If a functional $\mu : L \to \mathbb{R}$ is linear and it satisfies the property

$$\sup \mu(L \cap \{g \in L :\, 0 \leq g \leq f\} < \infty \quad \forall f \in L^+,$$

then μ is a Daniell integral.

1.5.1 Coverings, Vitali relations, approximate limits and approximate continuity

Key takeaway: Continuity is usually a property which is too strong to assume. We would like to define a weaker notion of continuity. For the sake of generality we present this topic using the notion of Vitali covering. Our use of approximate continuity will be limited to the case in which the Vitali covering is given by Borel sets and we will need these notions in order to apply Theorem 4.5.9 at page 482 of [4].

Definition 1.5.5 (Coverings and Vitali relations).
Let X be any set.

- We will call a *covering relation* any subset of the set $\{(x, S) :\, x \in S \subset X\}$.

- For any covering relation C and any $Z \subset X$, define

$$Z(C) := \{S \in \mathcal{P}(X) : \exists x \in Z \text{ s.t. } (x, S) \in C\}$$

(where $\mathcal{P}(X)$ is the power set of X).

- We will say that a covering C is *fine* at a point $x \in X$ if $\inf\{\text{diam}(S) : (x, S) \in C\} = 0$

Let now τ be a topology on X and ϕ a measure on X.

- We will call a ϕ-*Vitali relation* any covering relation V such that:

  1. $V(X)$ is a family of borel sets.
  2. V is fine at every point of X.
  3. If a covering relation $C \subset V$ is fine at each point of some set $Z \subset X$, then $C(Z)$ contains a countable disjoint subfamily that covers ϕ-almost all of Z.

Definition 1.5.6 ((ϕ, V)-density of a set A at the point x)**.**
Define the (ϕ, V)-*density of A at x* the quantity

$$(V) \lim_{S \to x} \frac{\phi(S \cap A)}{\phi(S)}.$$

Definition 1.5.7 (Approximate limit of a function between topological spaces)**.**

Let X and Y be topological spaces and consider $f : X \to Y$. We say that *$y \in Y$ is the $(\phi, V)-$approximate limit of f at x* if:

$$\forall W \subset Y \text{ neighborhood of } y, \text{ the set } X \setminus f^{-1}(W) \text{ has zero density at } x.$$

Note that if Y is a Hausdorff topological space and the approximate limit y of f at x exists, then it is unique. In that case we will write

$$y = (\phi, V)\text{ap}\lim_{z \to x} f(z).$$

Definition 1.5.8 (Approximate (ϕ, V)-continuity of a function)**.**
We say that f is (ϕ, V)-approximately continuous at $x \in \mathrm{dmn}(f)$ if

$$f(x) = (\phi, V)\mathrm{ap}\lim_{z \to x} f(z).$$

Definition 1.5.9 ((ϕ, V)-approximate upper limit)**.**
If X is a topological space and $f : X \to \overline{\mathbb{R}}$, then we define the (ϕ, V)-*approximate upper limit of* f the quantity

$$(\phi, V)\mathrm{ap}\limsup_{z \to x} f(z) := \inf \big\{ t \in \overline{\mathbb{R}} : \ \{z : f(z) > t\} \text{ has zero } (\phi, V)\text{−density at } x \big\}.$$

Definition 1.5.10 ((ϕ, V)-approximate lower limit)**.**
If X is a topological space and $f : X \to \overline{\mathbb{R}}$, then we define the (ϕ, V)-*approximate lower limit of* f the quantity

$$(\phi, V)\mathrm{ap}\liminf_{z \to x} f(z) := \sup \big\{ t \in \overline{\mathbb{R}} : \ \{z : f(z) < t\} \text{ has zero } (\phi, V)\text{−density at } x \big\}.$$

Theorem 1.5.5 (Measurability and approximate continuity)**.**
Let ϕ be a measure on a set X. Let Y be a separable metric space, and let $f : X \to Y$ be a generic function. Then:

$$f \text{ is } \phi \text{ measurable} \iff \begin{array}{l} f \text{ is } (\phi, V) - \text{approximately} \\ \text{continuous at } x \text{ for } \phi - \text{a.e. } x \in X \end{array}.$$

1.6 Currents

Idea: m-currents on $\mathbb{R}^n$ are objects which generalize the concept of m-dimensional surfaces in $\mathbb{R}^n$.
The *mass* $\mathbf{M}(T)$ of a current T is a quantity which generalizes the concept of m-dimensional Hausdorff measure for the m-dimensional surface.

Definition 1.6.1. Let Ω be an open subset of $\mathbb{R}^n$.
Then we define

- a k-form on Ω as any continuous map from Ω to $\bigwedge^k(\mathbb{R}^n)$;

- $\mathcal{E}^m(\Omega)$ as the set of smooth k forms on Ω;
  $\mathcal{D}^m(\Omega)$ as the set of smooth k forms on Ω with compact support.

- $\mathcal{E}_m(\Omega)$ as the dual space of $\mathcal{E}^m(\Omega)$;
  $\mathcal{D}_m(\Omega)$ as the dual space of $\mathcal{D}^m(\Omega)$.

We call m**-dimensional current** any element of $\mathcal{D}_m$.

To any m-dimensional orientable surface S we can associate a current $[\![S]\!]$ by integration, defining its action on an m-form ω as

$$[\![S]\!](\omega) := \int_S \langle \omega(x), \overrightarrow{S}(x) \rangle \, d\mathcal{H}^m(x),$$

where $\overrightarrow{S}(x)$ is any unitary and simple m-vector associated to the tangent space to S at x.

1.6.1 The current $\mathbf{E}^n$

Idea: the current $\mathbf{E}^n$ is the fundamental n current on $\mathbb{R}^n$ representing the classical integration with respect to the lebesgue measure and the canonical volume n-form $e_1 \wedge ... \wedge e_n$.

Definition 1.6.2 (The current associated with Lebesgue integration in $\mathbb{R}^n$)**.** Define the current $\mathbf{E}^n \in \mathcal{D}_n(\mathbb{R}^n)$ as

$$\mathbf{E}^n := \mathcal{L}^n \wedge e_1 \wedge ... \wedge e_n,$$

which, in other words, is the current acting by lebesgue integration:

$$\begin{array}{rccl} \mathbf{E}^n : & \mathcal{D}^n(\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & \phi & \mapsto & \int \langle \, e_1 \wedge ... \wedge e_n, \; \phi \, \rangle \, d\mathcal{L}^n \end{array} .$$

1.6.2 Comass of a covector and mass of a current

Idea: mass and comass of m-forms are quantities associated to m-forms. They are pretty much the equivalent of dual norms. Our purpose is to use them in the fundamental definition of mass of a current.
The mass $\mathbf{M}$ of a current generalizes the m-dimesional area of m-dimensional surfaces. In fact one can easily see, applying the definitions, that if S is an m-dimensional surface, then $\mathbf{M}([\![S]\!]) = \mathcal{H}^m(S)$.

Definition 1.6.3 (Comass of a covector).
For any covector $\xi \in \bigwedge^m(\mathbb{R}^n)$ we define its comass as [1]

$$||\xi|| := \sup\{|\langle \xi, v\rangle| : \ v \text{ is a unit, simple } m\text{-vector}\}.$$

Definition 1.6.4 (Comass of an m-form).
Define the comass of an m-form $\phi \in \bigwedge^m(\mathbb{R}^n)$ as

$$\mathbf{M}(\phi) := \sup\{||\phi(x)|| : \ x \in U\}$$

Definition 1.6.5 (Mass of an m-current).
Define the mass of an m-current T as

$$\mathbf{M}(T) := \sup\{T(\phi) : \ \phi \in \mathcal{D}^m, \, \mathbf{M}(\phi) \leq 1\}.$$

1.6.3 Pushforward of a current

Definition 1.6.6 (Pushforward of a current).
Given two open subsets U, V of some euclidian space $\mathbb{E}$, a map $f \in C^\infty(U, V)$ and a current $T \in \mathcal{D}_m(U)$ such that $f|_{\mathrm{spt}(T)}$ is proper, we define the *pushforward of T through f* as the current $f_\# T \in \mathcal{D}_m(V)$ given by

$$\begin{array}{rccc} f_\# T : & \mathcal{D}^m(V) & \longrightarrow & \mathbb{R} \\ & \phi & \mapsto & T\big(\gamma_\phi \wedge (f^\# \phi)\big) \end{array}.$$

[1]The norm of a simple m-vector $(v_1, ..., v_m)$ is simply $||(v_1, ..., v_m)|| := \sqrt{\sum_{i=1}^m ||v_i||^2}$.

where γ_ϕ is an element of $\mathcal{D}^0(V)$ such that

$$\mathrm{spt}(T) \cap f^{-1}(\mathrm{spt}(\phi)) \subset \mathrm{Int}\{x : \gamma_\phi(x) = 1\}.$$

(Of course the definition of $f_{\#}T(\phi)$ is independent on the choice of γ_ϕ).

1.6.4 Currents representable by integration

Definition 1.6.7.
We say that a current $T \in \mathcal{D}_m(\Omega)$ is **representable by integration** if there exist a map $\overrightarrow{T} : \Omega \to \bigwedge_m(\mathbb{R}^n)$ (i.e. an m-vector field on Ω) and a measure $||T||$ on Ω such that the action of T on any $m \in \mathcal{D}^m(\Omega)$ can be written as

$$T(\omega) = \int_\Omega \langle \omega(x), \overrightarrow{T}(x) \rangle \, d||T||.$$

In this case we call $\overrightarrow{S}$ the **orientation** of T and $||T||$ the **total variation measure** of T.

We state without proof two fundamental facts about currents representable by integration. For the proof the reader may see [4].

Theorem 1.6.1 (Representation by integration)**.**
An m-current T is representable by integration if and only if it has finite mass.

Proposition 1.6.1.
If a current S is representable by integration, then its orientation $\overrightarrow{S}$ is carachterized by the property

$$\langle \overrightarrow{S}(x), y \rangle = \lim_{r \to 0^+} \frac{\overrightarrow{S}(\mathbb{1}_{U(x,r)}(y))}{||S||(\mathbf{B}(x,r))} \qquad \forall y \in \bigwedge^m(\mathbb{R}^n).$$

1.6.5 Cartesian product of current

Proposition 1.6.2. *[Cartesian product of currents]*
Let $A, B \subset \mathbb{R}^n$ be two open subsets. Let

$$\begin{array}{rccc} p: & A \times B & \longrightarrow & A \\ & (a,b) & \mapsto & a \end{array}$$

and

$$\begin{array}{rccc} q: & A \times B & \longrightarrow & B \\ & (a,b) & \mapsto & b \end{array}$$

be the projections from the cartesian product onto A and B respectively. Let $i, j \in \mathbb{N}$ and let $S \in \mathcal{D}_i(A)$ and $T \in \mathcal{D}_j(B)$.
Then there exists a unique current in $\mathcal{D}_{i+j}(A \times B)$, denoted by $S \times T$, such that for any $k \in \{0, 1, ..., i+j\}$, any $\alpha \in \mathcal{D}^k(A)$ and any $\beta \in \mathcal{D}^{i+j-k}(B)$ it holds that

$$S \times T(p^{\#}\alpha \wedge q^{\#}\beta) = \begin{cases} S(\alpha)T(\beta) & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}.$$

Definition 1.6.8 (Cartesian prouct of currents)**.**
The current $T \times S$ defined in proposition 1.6.2 is said *cartesian product of S and T*.

Notation 1.6.1. Denote by P and Q the maps

$$\begin{array}{rccc} P: & \mathbb{R}^m & \longrightarrow & \mathbb{R}^m \times \mathbb{R}^n \\ & a & \mapsto & (a,0) \end{array},$$

$$\begin{array}{rccc} Q: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \times \mathbb{R}^n \\ & b & \mapsto & (0,b) \end{array}.$$

Proposition 1.6.3 (Properties of the cartesian product).
Let $S \in \mathcal{D}_i(A)$ *and* $T \in \mathcal{D}_j(B)$. *Then*

- $\operatorname{spt}(S \times T) = \operatorname{spt}(S) \times \operatorname{spt}(T)$

- *Boundaries satisfy the following equality*

$$\partial(S \times T) = \begin{cases} (\partial S) \times T + (-1)^i S \times \partial T & \text{if } i, j > 0 \\ (\partial S) \times T & \text{if } i > 0 = j \\ S \times \partial T & \text{if } j > 0 = i \end{cases}$$

- *Let*

$$\begin{array}{rccc} r: & A \times B & \longrightarrow & B \times A \\ & (a,b) & \mapsto & (b,a) \end{array}.$$

Then

$$r_{\#}(S \times T) = (-1)^{ij} T \times S.$$

- *If both* S *and* T *are representable by integration, then, defined*

$$\begin{array}{rccc} \xi: & A \times B & \longrightarrow & \bigwedge_{i+j}(\mathbb{R}^n) \\ & (a,b) & \mapsto & (\wedge_i P)\overrightarrow{S}(a) \wedge (\wedge_j Q)\overrightarrow{T}(b) \end{array},$$

it holds that

$$S \times T = \int \langle \xi, \cdot \rangle \, d(||S|| \times ||T||)$$

(therefore $S \times T$ *is representable by integration and* $||S \times T|| \leq ||S|| \times ||T||$*).*

- *If both* S *and* T *are representable by integration and at least for one among* $\overrightarrow{S}(a)$ *and* $\overrightarrow{T}(b)$ *is simple for* $(||S|| \times ||T||)$*-a.e.* $(a,b) \in A \times B$*, then*

$$||S \times T|| = ||S|| \times ||T||,$$

$$||\xi(a,b)|| = 1,$$

$$\overrightarrow{S \times T} = \xi \quad a.s..$$

1.6.6 The join of two currents

Definition 1.6.9 (Join of two currents)**.**
We define the join $S \times\!\!\!\times T$ of two currents $S \in \mathcal{D}_i(\mathbb{R}^n)$ and $T \in \mathcal{D}_j(\mathbb{R}^n)$ trhough the map

$$\begin{array}{rccc} F: & \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (x,t,y) & \mapsto & (1-t)x + ty \end{array}$$

as

$$S \times\!\!\!\times T := F_\#(S \times [\![0,1]\!] \times T).$$

Proposition 1.6.4 (Properties of the join)**.**

- *It holds that*

$$\partial(S \times\!\!\!\times T) = \begin{cases} (\partial S) \times\!\!\!\times T - (-1)^i \, S \times\!\!\!\times \partial T & \text{if } i,j > 0 \\ (\partial S) \times\!\!\!\times T - (-1)^i \, T(1) S & \text{if } i > 0 = j \\ S(1)T \; - \; S \times\!\!\!\times \partial T & \text{if } j > 0 = i \\ S(1)T - T(1)S & \text{if } i = j = 0 \end{cases}$$

- $S \times\!\!\!\times T = (-1)^{(i+1)(j+1)} T \times\!\!\!\times S$

- $A_\#(S \times\!\!\!\times T) = A_\# S \times\!\!\!\times A_\# T$ *for every affine map* A.

1.6.7 Deformation chains and the homotopy formula

Definition 1.6.10 (Proper function)**.**
We say that a function f between metric spaces X and Y is proper if $f^{-1}(K)$ is compact for every compact $K \subset Y$.

Definition 1.6.11 (Deformation chain)**.**
Given an open set $U \subset \mathbb{R}^n$, a current $T \in \mathcal{D}_m(U)$ and a proper function $h \in C^\infty([0,1] \times \operatorname{spt}(T), \mathbb{R}^m)$, we call *deformation chain of T* (associated with h) the current $h_\#([\![0,1]\!] \times T)$.

Proposition 1.6.5 (Homotopy formula).
Fix a generic $T \in \mathcal{D}_m(U)$. If $h \in C^\infty([0,1]\times \mathrm{spt}(T), \mathbb{R}^m)$ is proper and a homotopy between the functions f and g (i.e. $f(\cdot) = h(0,\cdot)$ and $g(\cdot) = h(1,\cdot)$), then the deformation chain of T associated with h satisfies the following equation:

$$g_\# T - f_\# T = \begin{cases} \partial\big(h_\#([\![0,1]\!] \times T)\big) \ + \ h_\#([\![0,1]\!] \times \partial T) & \text{if } m > 0 \\ \partial\big(h_\#([\![0,1]\!] \times T)\big) & \text{if } m = 0 \end{cases}$$

1.6.8 Polihedral and rectifiable currents

Idea: Currents, if considered in their complete generality, are very abstract objects. We could say that for our purpose, which is to study surfaces using currents, there are too many more currents than there are surfaces. Because of this, rewriting variational problems from the set of surfaces to the space of currents may lead to non-interpretable results.
We would like to describe a subclass of currents which are "more similar" to surfaces. Rectifiable currents are precisely that. They are still currents and they are still a class which is strictly wider than the set surfaces, but they are much more similar to them. They can be thought as the union of a finite amount of $(\mathcal{H}^m, m)$-rectifiable sets, possibly overlapping.
Notice that the definition is quite abstract, but that the characterization theorem below gives equivalent definitions which are much more concrete.

Definition 1.6.12 ($\mathcal{P}_{m,K}(U)$).
Given $K \subset U \subset \mathbb{R}^n$ with K compact and U open, we define $\mathcal{R}_{m,K}(U)$ as the additive subgroup of $\mathcal{D}_m(U)$ generated by all m-dimensional oriented simplexes whose convex hull is contained in K.

Definition 1.6.13 (Integral polihedral chains).
We define as $\mathcal{P}_m(U)$ the set

$$\mathcal{P}_m(U) := \bigcup_{\substack{K \subset U, \\ K \text{ compact}}} \mathcal{P}_{m,K}(U).$$

Each element of $\mathcal{P}_m(U)$ will be called *Integral polihedral chain*

Definition 1.6.14 (Polihedral chains).
Define $\mathbf{P}_m(U)$ as the vector space (on the considered field) generated by $\mathcal{P}_m(U)$ (whose elements have only integer coefficents).

Definition 1.6.15 ($\mathcal{R}_{m,K}(U)$).
Given $K \subset U \subset \mathbb{R}^n$ with K compact and U open, we define $\mathcal{R}_{m,K}(U)$ as the set of m-dimensional currents T of U with the following property: "for all $\epsilon > 0$, the exist:

- An euclidean space $\mathbb{E}$
- $C \subset Z \subset \mathbb{E}$ with C compact and Z open
- A lipschitz map $f : Z \to U$ such that $f(C) \subset K$
- An integral polihedral chain $P \in \mathcal{P}_{m,C}$

satisfying $\mathbf{M}(T - f_{\#}P) < \epsilon$."

Definition 1.6.16 (Rectifiable m-currents in U).
Define the set $\mathcal{R}_m(U)$ of rectifiable m-dimensional currents in U as

$$\mathcal{R}_m(U) := \bigcup_{\substack{K \subset U, \\ K \text{ compact}}} \mathcal{R}_{m,K}(U).$$

Theorem 1.6.2 (Equaivalent conditions for rectifiability of a current).
Given an open subset U of $\mathbb{R}^n$ and a m-current $T \in \mathcal{D}_m(U)$ with compact support, the following 5 statements are equivalent:

1. *T is rectifiable.*

2. *$T \in \mathcal{R}_{m,K}$ for every compact $K \subset U$ such that $\operatorname{spt}(T) \subset \operatorname{Int}(K)$.*

3. *$\forall \epsilon > 0$, $\exists Z \subset \mathbb{R}^m$ open, $\exists A \subset Z$ compact, $\exists f \in \operatorname{Lip}(Z, U)$ such that*
$$\mathbf{M}(T - f_{\#}\mathbf{E}^m \llcorner A) < \epsilon.$$

4. *$\exists B \subset \operatorname{spt}(T)$ which is $\mathcal{H}^m$-measurable and $(\mathcal{H}^m, m)$-rectifiable, $\exists \eta : U \to \bigwedge_m$, $\mathcal{H}^m \llcorner B$-summable m-vector field, such that*
   - *(a) $T = (\mathcal{H}^m \llcorner B) \wedge \eta$.*
   - *(b) $\eta(x)$ is simple for $\mathcal{H}^m$-a.e. $x \in B$*
   - *(c) $|\eta(x)| \in \mathbb{N} \setminus \{0\}$ for $\mathcal{H}^m$-a.e. $x \in B$*
   - *(d) $\operatorname{Tan}^m(\mathcal{H}^m \llcorner B, x)$ is associated with $\mu(x)$.*

5. *$\mathbf{M}(T) < \infty$, U is $(||T||, m)$-rectifiable and, for $||T||$-a.e. $x \in U$,*
   - *(a) $\Theta^m(||T||, x) \in \mathbb{N} \setminus \{0\}$.*
   - *(b) $\overrightarrow{T}(x)$ is simple.*
   - *(c) $\operatorname{Tan}^m(||T||, x)$ is associated with $\overrightarrow{T}(x)$.*

Moreover, if one of the above holds, then $||T|| = \mathcal{H}^m \llcorner \Theta^m(||T||, \cdot)$.

1.6.9 Integral currents

Definition 1.6.17 (Integral currents with support in K).
For a natural number n and a compact subset K of the oper set U, define the set $\mathbf{I}_{m,K}(U)$ as

$$\mathbf{I}_{m,K}(U) := \{T : T \in \mathcal{R}_{m,K}(U),\ \partial T \in \mathcal{R}_{m-1,K}(U)\}.$$

Definition 1.6.18 (Integral currents)**.**
Define the set of *integral m-currents* as the abelian additive subgroup of $\mathcal{R}_m(U)$ as the union

$$\mathbf{I}_m(U) := \bigcup_{\substack{K \subset U, \\ K \text{ compact}}} \mathbf{I}_{m,K}(U).$$

1.6.10 Boundary of a current

Idea: The boundary of a current is the notion which indeed generalizes the notion of topological boundary for surfaces.

Definition 1.6.19.
Given a current $T \in \mathcal{D}_m(\Omega)$ with $m \geq 1$, we define the **boundary** of T as the current $\partial T \in \mathcal{D}_{m-1}(\Omega)$ whose action on any $\omega \in \mathcal{D}^{m-1}$ is given by

$$\partial T(\omega) := T(d\omega).$$

1.6.11 Normal currents

Definition 1.6.20 (The set $\mathbf{I}_{m,K}$)**.**
For $K \subset U \subset \mathbb{R}^n$, with U open and K compact, and for an integer $m \geq 0$, we define

$$\mathbf{I}_{m,K}(U) := \begin{cases} \{T : \ T \in \mathcal{R}_{m,K}(U) \text{ and } \partial T \in \mathcal{R}_{m-1,K}(U)\} & \text{if } m > 0 \\ \mathcal{R}_{0,K}(U) & \text{se } m = 0 \end{cases}.$$

Definition 1.6.21 (Normal currents)**.**
Let $T \in \mathcal{D}_m(U)$.

- We say that T is **locally normal** if T is representable by integration and either ∂T is representable by integration or $m = 0$.
- We say that T is **normal** if it is locally normal and $\operatorname{spt}(T)$ is compact.
- Define
$$\mathbf{N}(T) := \begin{cases} \mathbf{M}(T) + \mathbf{M}(\partial T) & \text{if } m > 0 \\ \mathbf{M}(T) & \text{if } m = 0 \end{cases}.$$

- Denote by $\mathbf{N}_m^{loc}(U)$ the set of locally normal currents on U.
- For any compact subset K of U, denote by $\mathbf{N}_{m,K}(U)$ the set of nomal m-currents on U whose support is contained in K.
- Denote by $\mathbf{N}_m(U)$ the set of normal currents on U.

Proposition 1.6.6.
The following implications hold:

- $\mathbf{N}(T) < \infty \;\Rightarrow\; T$ *is locally normal.*
- T *is normal* $\Rightarrow\; \mathbf{N}(T) < \infty$.

Chapter 2

Hardt Pitts

2.1 First theorem: finding a suitable rectifiable current to work with

Lemma 2.1.1.
If $m \in \{1, 2, ..., n-1\}$ and B is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable subset of $\mathbb{R}^n$, then the set

$$Z := \Big\{ z \in \mathbb{S}^{n-1} : \ \mathcal{H}^m\big(\{x \in B : \ z \in \operatorname{Tan}^m(\mathcal{H}^m \llcorner B, x)\}\big) > 0 \Big\}$$

is contained in the countable union of great $(m-1)$-spheres.

Proof.
For any linear subspace P of $\mathbb{R}^n$ having dimension $j \in \{1, 2, ..., m\}$ define

$$S(P) := B \cap \Big\{ x : \ P \subset \operatorname{Tan}^m(\mathcal{H}^m \llcorner B, x) \in \mathbf{G}(n, m) \Big\}.$$

Also define

$$G_m := \mathbf{G}(n, m) \cap \{P : \ \mathcal{H}^m\big(S(P)\big) > 0\}$$

and (inductively), for any $j = m-1,\, m-2, ..., 1$,

$$G_j := \mathbf{G}(n,j) \cap \Big\{ P : \ \mathcal{H}^m\big(S(P)\big) > 0 \text{ and } P \text{ is not contained in any } Q \in \bigcup_{k=j+1}^{m} G_k \Big\}.$$

Claim 1. *The following implication holds:*

$$\left.\begin{array}{l} j \in \{1,2,...,m\} \\ P, Q \in G_j \\ P \neq Q \end{array}\right\} \Rightarrow \mathcal{H}^m(P \cap Q) = 0.$$

Proof of claim 1.
If $j = m$, then the implication is trivial, since rectifiability of B implies that

$$P = \mathrm{Tan}^m\left(\mathcal{H}^m \llcorner B, x\right) \quad \forall x \in S(P)$$

$$Q = \mathrm{Tan}^m\left(\mathcal{H}^m \llcorner B, x\right) \quad \forall x \in S(Q)$$

and so, since $P \neq Q$ and the tangent space at a point x is unique, that

$$S(P) \cap S(Q) = \varnothing.$$

Consider now the case $j \in \{1, ..., m-1\}$.
Assume by contradiction that there is a subset A^+ of $S(P) \cap S(Q)$ such that

$$\mathcal{H}^m(A^+) > 0.$$

Then the subspace $V := P + Q \leq \mathbb{R}^n$ would have dimension $\dim(V)$ strictly greater than j and it would satisfy $\mathcal{H}^m\big(S(V)\big) > 0$.
This means that there exists a subspace $V' \leq V$ such that

$$\dim(V') \in \{j+1, ..., m\}$$

$$\mathcal{H}^m\big(S(V')\big) \geq \mathcal{H}^m\big(S(V)\big) > 0$$

$$P \leq V'$$

which in other words is an element of $G_{\dim(V')}$ containing P. But this contradicts, by definition of G_j, the fact that $P \in G_j$.
We deduce that $\mathcal{H}^m\big(S(P) \cap S(Q)\big)$ must be 0.

$\square$

Claim 2. *The set G_j is countable for all $j = 1, 2, ..., m$.*

Proof of claim 2.
By definition of countable $(\mathcal{H}^m, m)$-rectifiability, B is contained in the union of a countable amount of m-rectifiable sets $\{B_j\}_{j\in\mathbb{N}}$.
By definition of m-rectifiability, each set B_j has finite $\mathcal{H}^m$ measure.

Assume by contradiction that one of the sets G_j was uncountable.
Then the uncountable G_j could be described as $G_j = \{P_\alpha\}_{\alpha\in\mathbb{A}}$, where $\mathbb{A}$ is an uncountable set of indices.
For any $\alpha \in \mathbb{A}$ define $A_\alpha := S(P_\alpha)$.
For any $\alpha \in \mathbb{A}$ define $\Gamma_\alpha := \{j \in \mathbb{N} : \mathcal{H}^m(B_j \cap A_\alpha) > 0\}$.
The set Γ_α must be non-empty for all α, otherwise $\mathcal{H}^m(A_\alpha) = 0$, which by definition of G_j is not true.
Therefore there must be at least one $\overline{k} \in \mathbb{N}$ which intersects an uncountable amount of A_α in a $\mathcal{H}^m$-non trivial set.
This means, since all sets A_α are $\mathcal{H}^m$-essentially disjoint, that $\mathcal{H}^m(B_{\overline{k}}) = \infty$.
But this contradicts the fact that $\mathcal{H}^m(B_j) < \infty \quad \forall j \in \mathbb{N}$.
We conclude that G_j must therefore be countable for all $j = 1, ..., m$, as we wished.

$\square$

Observing now that

$$Z = S^{n-1} \cap \left(\bigcup_{j=1}^{m} \bigcup_{P\in G_j} P \right) = \bigcup_{j=1}^{m} \bigcup_{P\in G_j} (S^{n-1} \cap P),$$

the statement of the lemma is immediately deduced using claim 2.

$\square$

We now state and prove the first important result in [5]. In this Theorem we consider a Normal current in codimension 1 whose support is compact and whose boundary is rectifiable and we are able to find a suitable integral current R which is linked to N by some useful properties.

Theorem 2.1.2.
If $N \in \mathbf{N}_{n-1}(\mathbb{R}^n)$ (remember that by definition this means that it has compact support) and $\partial N \in \mathcal{R}_{n-2}(\mathbb{R}^n)$, then there exists $R \in \mathbf{I}_{n-1}(\mathbb{R}^n)$ with $\partial R = \partial N$ and such that:

1. $||N - R|| = ||N|| + ||R||$.
2. $\overrightarrow{N-R}(x) + \overrightarrow{R}(x) = 0$ *for* $||R||$*-a.e.* $x \in \mathbb{R}^n$.
3. *The following implication holds:*

$$\left.\begin{array}{l} f : \mathbb{R}^n \to \mathbb{R} \text{ is } \mathcal{L}^n\text{-measurable} \\ \partial\left(\mathbf{E} \llcorner f\right) = N - R \\ \lambda := \operatorname{ap\,liminf}\, f \\ \mu := \operatorname{ap\,limsup}\, f \end{array}\right\} \Rightarrow \begin{array}{rl} 0 & < \Theta^{n-1}(||N-R||, x) \\ & = \Theta^{n-1}(||R||, x) \\ & = \mu - \lambda \in \mathbb{Z} \\ & \text{for } ||R||\text{-a.e. } x \in \mathbb{R}^n \end{array}$$

Proof.
Let $C := [\![0]\!] \times\!\!\!\times \partial N \; \in \; \mathcal{R}_{n-1}(\mathbb{R}^n)$.
Let $U \subset \mathbb{R}^n$ be a bounded open set containing $\operatorname{spt}(N) \cup \operatorname{spt}(C)$.
Let $r \in \mathbb{R}_{>0}$ be such that $\operatorname{dist}(\operatorname{spt}(\tau_{rz\,\#}C),\, U) > 0\; \forall z \in \mathbb{S}^{n-1}$.
Let, for any $z \in \mathbb{S}^{n-1}$, $h_z(t, x) := (1-t)(x + rz) + tx$ be the homotopy from τ_{rz} to the identity.
Let $C_z := h_{z\,\#}([\![0,1]\!] \times \partial N) + \tau_{rz\,\#}C$.
Let B be the set $B := \{x : \Theta^{n-1}(||C||, x) > 0\}$.
Let B_z be the set $B_z := \{x : \Theta^{n-1}(||C_z||, x) > 0\}$.

Claim 1.
For any $z \in \mathbb{S}^{n-1}$ it holds that $\partial C_z = \partial N$.

Proof of Claim 1. Using proposition 1.6.4, we deduce that

$$
\begin{aligned}
\partial C &= [\![0]\!](1)\partial N - [\![0]\!] \times\!\!\!\times \partial(\partial N)\\
&= \partial N
\end{aligned}
$$

and by the homotopy formula we have

$$
\begin{aligned}
\partial C_z &= \partial\big(h_{z\#}([\![0,1]\!] \times \partial N)\big) + \partial\big(\tau_{rz\#}C\big)\\
&= h_z(1,\cdot)_\#\partial N - h_z(0,\cdot)_\#\partial N + h_{z\#}\big([\![0,1]\!] \times \partial(\partial N)\big) + \tau_{rz\#}\partial C\\
&= \partial N - \tau_{rz\#}\partial N + 0 + \tau_{rz\#}\partial C\\
&= \partial N.
\end{aligned}
$$

□

Claim 2.
For every $z \in \mathbb{S}^{n-1}$, the current C_z is an $(n-1)$-rectifiable current.

Proof of claim 2.
The current C is rectifiable, since it is the join of two rectifiable currents. C_z is obtained from rectifiable currents through pushforwards, sums and cartesian products. □

Claim 3.
The sets B and B_z are $(\mathcal{H}^{n-1}, n-1)$-rectifiable sets and $\mathcal{H}^{n-1}$-measurable sets (for any $z \in \mathbb{S}^{n-1}$).

Proof of claim 3.
With reference to [4, 4.1.28, p. 384-385], we see that the rectifiability of C implies that the open set U is $(||C||, n-1)$-rectifiable, which by definition means that there is a countably m-rectifiable set $A \subset U$ such that $||C||(U \setminus A) = 0$ and that $||S||(U) < \infty$.

The same theorem tells that $||C|| = \mathcal{H}^{n-1} \llcorner \Theta^{n-1}(||C||, \cdot)$, so that the set $B = \{x \in U : \Theta^{n-1}(||C||, x) > 0\}$ has full $||C||$-measure (i.e. $||C||(B) = ||C||(U)$. This in particular means that the set A must contain $||C||$-almost all points of B, which by definition means that B is $(||C||, n-1)$-rectifiable.

$\mathcal{H}^{n-1}$-measurability is due to the fact that $\Theta^{n-1}(||C||,\cdot)$ is a $\mathcal{H}^{n-1}$-measurable map.

The argument that shows $(||C||, n-1)$-rectifiability and $\mathcal{H}^{n-1}$-measurability of B_z is analogous.

□

Claim 4.
For any $z \in \mathbb{S}^{n-1}$ and for $||C_z||$-a.e. $x \in U$ it holds that $z \wedge \overrightarrow{C}_z(x) = 0$ and $z \in \operatorname{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner B_z, x)$.

Proof of claim 4.
Since C_z is rectifiable, it is also representable by integration through a vector field which is simple for $||C_z||$-a.e. $x \in \mathbb{R}^n$.
By construction the current $\tau_{rz\,\#}C$ has its support disjoint from U.
The homotopy current $h_{z\,\#}([\![0,1]\!] \times \partial N)$ has a "translation component" parallel to z in all the points of its support, so that the vector z is contained in almost all the tangent spaces. Since tangent spaces of are associated with the vectori field $\overrightarrow{C_z}$, it follows by definition (of vector subspace associated with a simple m-vector) that $z \wedge \overrightarrow{C_z} = 0$ for $||C_z||$-a,e, $x \in U$.

The analogous statement on B_z follows from the fact just proven and [4, 4.1.28, p. 384-385]. □

Claim 5.
There is a function g of bounded variation such that $N - C = \partial\big(\mathbf{E}^n \llcorner g\big)$.

Proof of claim 5.
By claim 1 it holds $\partial(N - R) = \partial N - \partial R = 0$. Since we are dealing with currents in $\mathbb{R}^n$, which is simply connected, this implies that the current $N - R$ is also exact (by the version of Poincaré's lemma for currents). It is therefore the boundary ∂M of some $M \in \mathcal{D}_n(\mathbb{R}^n)$. All n-currents on $\mathbb{R}^n$

are representable by integration and therefore exists $g \in L^1_{loc}(\mathbb{R}^n)$ such that $M = \mathbf{E}^n \llcorner g$.

To see that f is BV we shall use the notation

$$\hat{dx_i} := dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n$$

and notice that, for any $(n-1)$-form $\omega = w_1(x)\hat{dx_1} + ... + w_n(x)\hat{dx_n}$, we have

$$\begin{aligned} \partial\big(\mathbf{E}^n \llcorner g\big)(w) &= \mathbf{E}^n \llcorner g\Big(\frac{\partial w_1}{\partial x_1}(x) dx_1 \wedge \hat{dx_1} + ... + \frac{\partial w_n}{\partial x_n}(x) dx_n \wedge \hat{dx_n}\Big) \\ &= \mathbf{E}^n \llcorner g\left(\Big(\sum_{i=1}^{n} (-1)^{i+1} \frac{\partial w_i}{\partial x_i}\Big) dx\right) \\ &= \int_{\mathbb{R}^n} g \operatorname{div}(\overline{w})\, dx \end{aligned}$$

where

$$\begin{array}{rccc} \overline{w}: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \mapsto & \big(w_1(x), -w_2(x), ..., (-1)^{n+1} w_n(x)\big) \end{array}.$$

Therefore

$$\begin{aligned} \sup_{\substack{\phi \in C^1_c(\mathbb{R}^n) \\ |\phi| \le 1}} \int_{\mathbb{R}^n} g \operatorname{div}(\phi)\, d\mathcal{L}^n &= \sup_{\substack{\phi \in C^\infty_c(\mathbb{R}^n) \\ |\phi| \le 1}} \int_{\mathbb{R}^n} g \operatorname{div}(\phi)\, d\mathcal{L}^n \\ &= \big\| \partial\big(\mathbf{E}^n \llcorner g\big) \big\| \\ &= \|N - R\|(\mathbb{R}^n) < \infty. \end{aligned}$$

□

Define $F := \{x : (\mathcal{L}^n)\text{ap}\liminf_{z \to x} g(z) \;<\; (\mathcal{L}^n)\text{ap}\limsup_{z \to x} g(z)\}$.

Claim 6.

The set F is a $(\mathcal{H}^{n-1}, n-1)$-rectifiable Borel set.

Proof of claim 6.

It is precisely the statement of [4, 4.5.9 (16), p.483], together with [4, 4.5.9 (2), p.482] to see that the set is in fact a Borel set.

□

Claim 7.
It exists $\overline{z} \in \mathbb{S}^{n-1}$ *such that* $\mathcal{H}^{n-1}\Big(\big(F \cup B\big) \cap B_{\overline{z}}\Big) = 0.$

Proof of claim 7. We apply the lemma above to both F and B to see that $\mathcal{H}^{n-1}$-almost all $z \in \mathbb{S}^{n-1}$ are tangent vectors for a subset of $F \cup B$ which has null $\mathcal{H}^{n-1}$-measure. Therefore there is at least one (and there are in fact infinitely many) such z. We pick one and call it $\overline{z}$.

We see that therefore:

- inside U the tangent spaces on $\mathcal{H}^{n-1}$-almost all intersection points of $F \cup B$ and B_z have dimension at most $n-2$.
- If F and B_z overlapped on a set with nonzero $\mathcal{H}^{n-1}$-measure, then that set would be $(\mathcal{H}^{n-1}, n-1)$-rectifiable and the tangent spaces of F and B_z would coincide.
  An analogous statement holds for B and $B_{\overline{z}}$.
- Outside of U there are only points of $B_{\overline{z}}$, since both F and B are contained in U.

These observations show that $\mathcal{H}^{n-1}\big((F \cup B) \cap B_z\big) = 0.$

□

We can finally define the current R that we were looking for as $R := C_{\overline{z}}$. We now prove that it has the properties we desired.

Claim 8.
The two following implications hold:

$$\left.\begin{array}{l} \mathcal{H}^{n-1}(W) < \infty \\ W \subset \mathbb{R}^n \setminus F \end{array}\right\} \Rightarrow ||N - C||(W) = 0 \tag{2.1}$$

and

$$\left.\begin{array}{l} \mathcal{H}^{n-1}(W) < \infty \\ W \subset \mathbb{R}^n \setminus (F \cup B) \end{array}\right\} \Rightarrow ||N||(W) = 0. \tag{2.2}$$

Proof of claim 8.

Proof of (2.1)**:**

Consider a $\mathcal{H}^{n-1}$-measurable set W with $\mathcal{H}^{n-1}(W) < \infty$.
Assume that $W \subset \mathbb{R}^n \setminus F$.
Then W equals the uncountable union

$$W = \bigcup_{s\in\mathbb{R}} \Big(W \cap \big\{ x : \ \operatorname{ap}\liminf_{z\to x} g(z) = s = \operatorname{ap}\limsup_{z\to x} g(z) \big\} \Big).$$

Since $\mathcal{H}^{n-1}(W) < \infty$, then only a countable amount real numbers s can satisfy

$$\mathcal{H}^{n-1}\Big(W \cap \big\{ x : \ \operatorname{ap}\liminf_{z\to x} g(z) = s = \operatorname{ap}\limsup_{z\to x} g(z) \big\} \Big) > 0.$$

This tell us, in particular, that the map $s \mapsto \mathcal{H}^{n-1}(\{x : \ \lambda(x) \le s \le \mu(x)\})$ (with $s \in \mathbb{R}$) is $\mathcal{L}^1$-a.e. null. Therefore, using [4, 4.5.9(14), p.483], we see that

$$\begin{aligned} ||N - C||(W) &= ||\partial(\mathbf{E}^n \llcorner g)||(W) \\ &= \int_{-\infty}^{+\infty} \int_{\{x:\, \lambda(x)\le s\le\mu(x)\}} \mathbb{1}_W(x) d\mathcal{H}^{n-1}(x)\, d\mathcal{L}^1(s) \\ &= 0 \end{aligned}$$

Proof of (2.2)**:**

Consider a $\mathcal{H}^{n-1}$-measurable set W with $\mathcal{H}^{n-1}(W) < \infty$.
Assume that $W \subset \mathbb{R}^n \setminus (F \cup B)$.
Notice that for any set $A \in \mathbb{R}^n \setminus B$ it holds that $||C||(A) = 0$, and that

$$||N||(A) \le ||C||(A) + ||N - C||(A) \le 0 + ||N||(A) + ||C||(A) = ||N||(A).$$

Thus $||N||(A) = ||N - C||(A)$.
It follows now from the first implication proven above that

$$||N||(W) = ||N - C||(W) = 0.$$

$\square$

Claims 7 and 8 automatically imply that

$$||N||(B_{\overline{z}}) = 0.$$

Moreover for any $A \subset \mathcal{B}_{\mathbb{R}^n}$ [1],

$$\begin{aligned} ||N - C_{\overline{z}}||(A) = & \; ||N - C_{\overline{z}}||(A \setminus B_{\overline{z}}) \, + \, ||N - C_{\overline{z}}||(A \cap B_{\overline{z}}) \\ \geq & \; ||N||(A \setminus B_{\overline{z}}) - ||C_{\overline{z}}||(A \setminus B_{\overline{z}}) + \\ & + ||C_{\overline{z}}||(A \cap B_{\overline{z}}) - ||N||(A \cap B_{\overline{z}}) \\ = & \; ||N||(A \setminus B_{\overline{z}}) \, + \, ||C_{\overline{z}}||(A \cap B_{\overline{z}}) \\ = & \; ||N||(A) \, + \, ||C_{\overline{z}}||(A). \end{aligned}$$

The inverse inequality is trivial, and therefore we have

$$||N - C_{\overline{z}}||(A) = ||N||(A) \, + \, ||C_{\overline{z}}||(A).$$

Claim 9.
For $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n \setminus (F \cup B)$ it holds that $\Theta^{n-1}(||N||, x) = 0$.

Proof of claim 9.
By definition

$$\Theta^{n-1}(||N||, x) = \lim_{r \to 0^+} \frac{||N||\big(\mathbf{B}(x, r)\big)}{\alpha(n) r^{n-1}}.$$

Assume by contraddiction that there is a set $A^+ \subset \mathbb{R}^n \setminus (F \cup B)$ of positive $\mathcal{H}^{n-1}$ measure such that

$$\Theta^{n-1}(||N||, x) > 0 \quad \forall x \in A^+.$$

Then there must be some $\varepsilon > 0$ such that

$$\mathcal{H}^{n-1}\big(\{x : \Theta^{n-1}(||N||, x) > \varepsilon\}\big) > 0.$$

[1] $\mathcal{B}_{\mathbb{R}^n} := \{\text{Borel subsets of } \mathbb{R}^n\}$

Denote by Q this positive value.
This implies, using [4, 2.10.19(3), p. 181], that

$$||N||(A^+) \geq \varepsilon Q > 0,$$

but this contradicts claim 8.
As a conclusion there can be no set outside of $F \cup B$ having positive $||N||$ measure.

□

Claim 10.
For any function f as in 3 of Theorem 2.1.2, define $E_f := \{x \in \mathbb{R}^n : \lambda(x) < \mu(x)\}$. Then

$$\mathcal{H}^{n-1}(B_{\overline{z}} \setminus E_f) = 0$$

and for any $x \in \mathbb{R}^n$ and any $r \in \mathbb{R}_{>0}$

$$\begin{aligned} &||N - C_{\overline{z}}||\big(B(x,r)\big) = \\ &\quad = ||N||\big(B(x,r) \setminus E_f\big) + ||N - C_{\overline{z}}||\big(B(x,r) \cap E_f\big). \end{aligned}$$

Proof of claim 10.
Suppose by contradiction that

$$\mathcal{H}^{n-1}(B_{\overline{z}} \setminus E_f) > 0.$$

Then:

1. Analogously as earlier in the proof of claim 9, we see that

$$||C_{\overline{z}}||(B_{\overline{z}} \setminus E_f) > 0.$$

2. In the precise same way as in the proof of the first implication of claim 8, we see that

$$||N - C||(B_{\overline{z}} \setminus E_f) = 0.$$

3. We know by claims 7 and 8 that $||N||(B_{\overline{z}}) = 0$, and so that in particular $||N||(B_{\overline{z}} \setminus E_f) = 0$.

These three observations are contradictory, since they imply that

$$0 = ||N - C||(B_{\overline{z}} \setminus E_f) = ||N||(B_{\overline{z}} \setminus E_f) \,+\, ||C_{\overline{z}}||(B_{\overline{z}} \setminus E_f) > 0.$$

This means that $\mathcal{H}^{n-1}(B_{\overline{z}} \setminus E_f)$ can not be positive, and must therefore be 0.

More in general, then, it also holds that $||C_{\overline{z}}||(\mathbb{R}^n \setminus E_f) = 0$, since

$$\begin{aligned}
&\|C_{\overline{z}}\|(\mathbb{R}^n \setminus E_f) = \\
&\quad = \int_{\mathbb{R}^n \setminus B_{\overline{z}}} \Theta^{n-1}\big(\|C_{\overline{z}}\|, x\big)\, d\mathcal{H}^{n-1}(x) \,+\, \int_{B_{\overline{z}} \setminus E_f} \Theta^{n-1}\big(\|C_{\overline{z}}\|, x\big)\, d\mathcal{H}^{n-1}(x) \\
&\quad = 0
\end{aligned}$$

Fix now some $x \in \mathbb{R}^n$ and some $r > 0$.
We see that

$$\begin{aligned}
||N - C_{\overline{z}}||\big(\mathbf{B}(x,r)\big) = \;& ||N - C_{\overline{z}}||\big(\mathbf{B}(x,r) \setminus E_f\big) + \\
& + ||N - C_{\overline{z}}||\big(\mathbf{B}(x,r) \cap E_f\big) \\
= \;& ||N||\big(\mathbf{B}(x,r) \setminus E_f\big) \,+\, ||C_{\overline{z}}||\big(\mathbf{B}(x,r) \setminus E_f\big) + \\
& + ||N - C_{\overline{z}}||\big(\mathbf{B}(x,r) \cap E_f\big) \\
= \;& ||N||\big(\mathbf{B}(x,r) \setminus E_f\big) \,+\, ||N - C_{\overline{z}}||\big(\mathbf{B}(x,r) \cap E_f\big) \\
= \;& ||N||\big(\mathbf{B}(x,r)\big) \,+\, ||C_{\overline{z}}||\big(\mathbf{B}(x,r) \cap E_f\big)
\end{aligned}$$

□

Now statement 3 of the theorem can be proven using claim 10 and [4, 4.1.28 (p.385), 4.5.9(15) (p.483)].
In fact:

- By claim 10 the set F contains $\mathcal{H}^{n-1}$-a.e. point x of $\mathbb{R}^n$ with $\Theta^{n-1}(||C_{\overline{z}}||, x) > 0$.

- Since $||N - C_{\overline{z}}|| = ||N|| + ||C_{\overline{z}}||$, then it is also true that $\Theta^{n-1}(||N - C_{\overline{z}}||, \cdot) = \Theta^{n-1}(||N||, \cdot) \,+\, \Theta^{n-1}(||C_{\overline{z}}||, \cdot)$.

- By claim 9 $||N||$ has zero $(n-1)$-density $\mathcal{H}^{n-1}$-a.e. outside of $F \cup B$, and so in particular $\mathcal{H}^{n-1}$-a.e. in $B_{\overline{z}}$ (by claim 7).
  Since $||C_{\overline{z}}||(\mathbb{R}^n \setminus B_{\overline{z}}) = 0$ and (by [4, 4.1.28 (p.384)]) $||C_{\overline{z}}|| \ll \mathcal{H}^{n-1}$, this means that $\Theta^{n-1}(||N||, \cdot) = 0 \quad ||C_{\overline{z}}||$-a.e. .
- By [4, 4.1.28 (p.384)], since $C_{\overline{z}}$ is rectifiable, $\Theta^{n-1}(||C_{\overline{z}}||, x) \in \mathbb{Z}$ for $||C_{\overline{z}}||$-a.e. $x \in \mathbb{R}^n$.
- By [4, 4.5.9(15) (p.483)], for every Borel subset W of F, it holds
$$||N - C||(W) = \int_W (\mu - \lambda)\, d\mathcal{H}^{n-1}$$

Putting these observations together we have, for every Borel subset W of F, that

$$\begin{aligned}
\Big(\mathcal{H}^{n-1} \llcorner \Theta^{n-1}(||C_{\overline{z}}||, \cdot)\Big)(W) &= ||C_{\overline{z}}||(W) \\
&= 0 \,+\, ||C_{\overline{z}}||(W) \\
&= ||N||(W) \,+\, ||C_{\overline{z}}||(W) \\
&= ||N - C_{\overline{z}}||(W) \\
&= \Big(\mathcal{H}^{n-1} \llcorner (\mu - \lambda)\Big)(W)
\end{aligned}$$

which means that

$$(\mu - \lambda)(x) = \Theta^{n-1}(||C_{\overline{z}}||, x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in F.$$

Since $||C_{\overline{z}}|| \ll \mathcal{H}^{n-1}$ and since F has full $||C_{\overline{z}}||$-measure, we deduce that in particular

$$(\mu - \lambda)(x) = \Theta^{n-1}(||C_{\overline{z}}||, x) \quad \text{for } ||C_{\overline{z}}||\text{-a.e. } x,$$

and statement 3 is proven.

Lastly we can see, using [4, p.357], that for any $y \in \bigwedge^m \mathbb{R}^n$

$$\begin{aligned}
\langle \overrightarrow{N - C_{\overline{z}}}, y \rangle &= \lim_{r \to 0^+} \frac{(N - C_{\overline{z}})(b_{x,r} y)}{||N - C_{\overline{z}}||\big(\mathbf{B}(x,r)\big)} \\
&= \lim_{r \to 0^+} \frac{N(b_{x,r} y) \,-\, C_{\overline{z}}(b_{x,r} y)}{||N||\big(\mathbf{B}(x,r)\big) \,+\, ||C_{\overline{z}}||\big(\mathbf{B}(x,r)\big)} \\
&= \lim_{r \to 0^+} \frac{-\, C_{\overline{z}}(b_{x,r} y)}{||C_{\overline{z}}||\big(\mathbf{B}(x,r)\big)} \\
&= \langle -\overrightarrow{C_{\overline{z}}}, y \rangle.
\end{aligned}$$

And this proves statement 2, concluding the proof of the theorem.

□

2.2 Main result of Hardt-Pitts: mass reducing integral current

The purpose of Theorem 2.1.2 is to provide the rectifiable current R which will allow to prove the Theorem below, which is the main result of Hardt-Pitts's paper.

Theorem 2.2.1 (Main result).
If $N \in \mathcal{D}_{n-1}(\mathbb{R}^n)$, $\mathbf{M}(N) < \infty$, $\mathrm{spt}(N)$ *is compact and* $\partial N \in \mathbf{I}_{n-2}(\mathbb{R}^n)$, *then there exists* $T \in \mathbf{I}_{n-1}(\mathbb{R}^n)$ *such that*

1. $\partial T = \partial N$;
2. $\mathbf{M}(T) \leq \mathbf{M}(N)$.

Proof.
The current N satisfies the properties of Theorem 2.1.2.
Let then R, f, λ and μ be as in Theorem 2.1.2.
By [4, 4.5.9(13), p. 483]

$$\begin{aligned} N - R &= \partial\big(\mathbf{E}^n \llcorner f\big) \\ &= \int_{-\infty}^{+\infty} \partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq s\}\big]\, d\mathcal{L}^1(s) \\ &= \int_{[0,1)} \left(\sum_{j\in\mathbb{Z}} \partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq s+j\}\big] \right) d\mathcal{L}^1(s) \end{aligned} \tag{2.3}$$

and similarly

$$||N - R|| = \int_{[0,1)} \left(\sum_{j\in\mathbb{Z}} ||\partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq s+j\}\big]|| \right) d\mathcal{L}^1(s). \tag{2.4}$$

Claim 1. *There exists* $\overline{s} \in (0,1)$ *such that*

(a) $\partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq \overline{s} + j\}\big] \in \mathcal{R}_{n-1}(\mathbb{R}^n) \quad \forall j \in \mathbb{Z}.$

(b) $\mathbf{M}(N-R) \geq \sum_{j\in\mathbb{Z}} \mathbf{M}\big(\, \partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq \overline{s} + j\}\big]\,\big)$

(c) $||R||\big(\, \lambda^{-1}(\overline{s}+j) \cup \mu^{-1}(\overline{s}+j)\big) \;=\; 0 \quad \forall j \in \mathbb{Z}.$

Proof of claim 1.
We make three observations:

1. By [4, 4.5.9(12), p.483] the current $\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s\}\big]$ is in $\mathcal{R}^{loc}_{n-1}(\mathbb{R}^n)$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. Since in our case the current $N - R$ is not just in $\mathbf{N}^{loc}_{n-1}(\mathbb{R}^n)$ (as in [4, 4.5.9(12)]), but in $\mathbf{N}_{n-1}(\mathbb{R}^n)$ (i.e. it also has compact support), the current $\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s\}\big]$ is also in $\mathcal{R}_{n-1}(\mathbb{R}^n)$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$.
   This means that the set
   $$\{s \in [0,1) :\; \partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s\}\big] \in \mathcal{R}_{n-1}(\mathbb{R}^n)\}$$
   has full (i.e. unitary) $\mathcal{L}^1$ measure.

2. The set
   $$\Big\{s \in [0,1) :\; \mathbf{M}(N-R) \geq \sum_{j\in\mathbb{Z}} \mathbf{M}\big(\, \partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq \overline{s} + j\}\big]\,\big)\Big\}$$
   has strictly positive $\mathcal{L}^1$-measure.
   Assume by contradiction that this was not the case, i.e. that
   $$\mathbf{M}(N-R) < \sum_{j\in\mathbb{Z}} \mathbf{M}\big(\partial\big[\mathbf{E}^n \llcorner \{x :\, f(x) \geq \overline{s} + j\}\big]\big) \text{ for } \mathcal{L}^1\text{-a.e. } s \in [0,1).$$

Then

$$\begin{aligned}\int_{[0,1)} \Bigg(\sum_{j\in\mathbb{Z}} ||\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s+j\}\big]||(\mathbb{R}^n)\Bigg)\, d\mathcal{L}^1(s) = \\ &= \int_{[0,1)} \left(\sum_{j\in\mathbb{Z}} \mathbf{M}(\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s+j\}\big])\right) d\mathcal{L}^1(s) \\ &> \int_{[0,1)} \mathbf{M}(N-R)\, d\mathcal{L}^1(s) \\ &= ||N-R||(\mathbb{R}^n)\end{aligned}$$

which contradicts equation (2.4).

3. Since

$$||R||(\mathbb{R}^n) < \infty \quad \text{and} \quad \mathbb{R}^n = \bigcup_{s\in[0,1)} \bigcup_{j\in\mathbb{Z}} \lambda^{-1}(s+j)$$

there can only be a countable amount of $s \in [0,1)$ for which

$$\mathcal{L}^1\left(\bigcup_{j\in\mathbb{Z}} \lambda^{-1}(s+j)\right) \;>\; 0.$$

This means that

$$\mathcal{L}^1\Big(\big\{s \in [0,1) :\; \mathcal{L}^1(\lambda^{-1}(s+j)) = 0 \;\; \forall j \in \mathbb{Z}\big\}\Big) = 1.$$

An identical statement can be analogously deduced for μ.

These three observations together show that the set of $s \in [0,1)$ which would satisfy the desired properties has strictly positive $\mathcal{L}^1$-measure, and is therefore non-empty.

$\square$

Define, for each $j \in \mathbb{Z}$, $S_j \;:=\; \partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq \overline{s} + j\}\big]$.

Claim 2. *For the selected $\overline{s}$ and $||R||$-almost every $x \in \mathbb{R}^n$ the following statements are true:*

(a) $\lambda(x) \neq \overline{s} + j$ *and* $\mu(x) \neq \overline{s} + j \quad \forall j \in \mathbb{Z}$.

(b) For all $j \in \mathbb{Z}$ the measure $||S_j||$ has density at x given by

$$\Theta(||S_j||, x) = \begin{cases} 1 & \text{if } \lambda(x) < \overline{s} + j < \mu(x) \\ 0 & \text{otherwise} \end{cases}$$

(c) $0 < \Theta^{n-1}(||R||, x) = \mu(x) - \lambda(x) \in \mathbb{Z}$.

(d) if $\lambda(x) < \overline{s} + j < \mu(x)$, *then* $-\overrightarrow{R}(x) = \overrightarrow{N - R}(x) = -\overrightarrow{S_j}$.

Proof of claim 2.
Statement *(a)* is a direct consequence of statement *(c)* from claim 1.

Statement *(b)* follows from Theorem 2.1.2 above, [4, 4.5.9(17), p.483] and [4, 4.5.6(2,3), p.478]. In fact:

Case 1 ($\lambda(x) < \overline{s} + j < \mu(x)$)**:**

- In the theorem above we proved that $\mu(x) - \lambda(x) > 0$ for $||R||$-a.e. $x \in \mathbb{R}^n$.
- By [4, 4.5.9(17), p.483] the normal exterior vector $\mathbf{n}(\{y : f(y) \geq \overline{s} + j\}, x)$ is a unit vector (instead of being 0) for $\mathcal{H}^{n-1}$-a.e. x for which $\lambda(x) < \overline{s} + j < \mu(x)$.
- Now [4, 4.5.6(2,3), p.478] can be used together with the two observations just made to deduce that $\Theta^{n-1}(||S_j||, x) = 1$ for $||R||$-a.e. $x \in \mathbb{R}^n$ for which $\lambda(x) < \overline{s} + j < \mu(x)$.

Case 2 ($\mu(x) < \overline{s} + j$)**:**

If $\lambda(x) \leq \mu(x) < \overline{s} + j$, then by definition of $\mu(x)$ it must be that

$$\lim_{\substack{S \subset \mathbb{R}^n \\ S \to x}} \frac{\mathcal{L}^n(S \cap \{y : f(y) \geq \overline{s} + j\})}{\mathcal{L}^n(S)} = 0.$$

This implies in particular, since

$$\Theta^n(\mathcal{L}^n \llcorner \{y : f(y) \geq \overline{s} + j\}, x) := \lim_{r \to 0^+} \frac{\mathcal{L}^n \llcorner \{y : f(y) \geq \overline{s} + j\}\big(\mathbf{B}(x, r)\big)}{\mathcal{L}^n\big(\mathbf{B}(x, r)\big)},$$

that

$$\Theta^n(\mathcal{L}^n \llcorner \{y : f(y) \geq \overline{s} + j\}, x) = 0.$$

This further implies, by definition of exterior normal, that

$$\mathbf{n}\big(\{y : f(y) \geq \overline{s} + j\}, x\big) = 0.$$

We have shown this inclusion:

$$\{x : \mu(x) < \overline{s} + j\} \;\subset\; \{x : \mathbf{n}\big(\{y : f(y) \geq \overline{s} + j\}, x\big) = 0\},$$

and using [4, 4.5.6(2,3), p.478] we see that $\Theta^{n-1}(||S_j||, x) = 0$ for $\mathcal{H}^{n-1}$-a.e. x having $\mu(x) < \overline{s} + j$.

Case 3 **(**$\lambda(x) > \overline{s} + j$**):**
It is analogous to case 2, only using the definition of λ instead of the definition of μ.

Statement *(c)* was proven in claim 9 at page 42, and more in general was a part of statement "*3.*" in the theorem above.

Statement *(d)* requires a couple more details:

- from *(b)*, [4, 4.5.9(17), p.483] and [4, 4.5.6(2,3), p.478] we deduce that

$$\Big\|\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq \overline{s} + j\}\big]\Big\| = \mathcal{H}^n \llcorner \{x : \lambda(x) < \overline{s} + j < \mu(x)\};$$

- The second equation in [4, 4.5.9(13), p.483] is equivalent to saying that for any $f \in L^1(||N-R||)$ we have
$$\int_{\mathbb{R}^n} f\, d\|N-R\| = \int_{\mathbb{R}} \int_{\mathbb{R}^n} f\, d\big\|\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s\}\big]\big\|\, ds,$$
so that, denoting $\mathbf{E}^n \llcorner \{x : f(x) \geq s\}$ by R_s and using also the first equation in [4, 4.5.9(13), p.483], we can write
$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}^n} \langle w(x), \overrightarrow{\partial R_s}(x)\rangle\, d\|\partial R_s\|(x)\, ds &= (N-R)(\omega)\\ &= \int_{\mathbb{R}^n} \langle w(x), \overrightarrow{N-R}(x)\rangle\, d\|N-R\|(x)\\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \langle w(x), \overrightarrow{N-R}(x)\rangle\, d\|\partial R_s\|(x)\, ds.\end{aligned}$$
This means that
$$\overrightarrow{N-R}(x) = \overrightarrow{\partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq s\}\big]}(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \text{ and } \|\partial R_s\|\text{-a.e. } x.$$

From these two observations we deduce that (up to making a different choice for $\overline{s}$ in claim 1) property *(d)* holds for $\mathcal{H}^{n-1}$-a.e. x for which $\lambda(x) < \overline{s} + j < \mu(x)$ and for all j.

□

Define the current
$$S := \sum_{j \in \mathbb{Z}} S_j.$$

Claim 3. *For $||R||$-almost every $x \in \mathbb{R}^n$, the density of the measure $||S||$ satisfies the equalities*
$$\begin{aligned}\Theta^{n-1}(||S||, x) &= \sum_{j \in \mathbb{Z}} \Theta^{n-1}(||S_j||, x)\\ &= \operatorname{Card}\big(\{j \in \mathbb{Z} :\ \lambda(x) < \overline{s} + j < \mu(x)\}\big)\\ &= \mu(x) - \lambda(x)\\ &= \Theta^{n-1}(||R||, x)\end{aligned}$$
and the orientation $\overrightarrow{S}$ satisfies
$$\overrightarrow{S}(x) = \overrightarrow{R}(x).$$

Proof of claim 3.
In claim 2 was proven that $\overrightarrow{S_j} = \overrightarrow{R}$ for all j (and so that in particular the orientation is the same for all values of j in $||R||$-a.e. point x). This allows to deduce

$$\Theta^{n-1}(||S||, x) = \sum_{j \in \mathbb{Z}} \Theta^{n-1}(||S_j||, x).$$

The other equalities easily follow from claims 1 and 2 and from Theorem 2.1.2.
The equality $\overrightarrow{S}(x) = \overrightarrow{R}(x)$ follows again from the fact that $\overrightarrow{S_j} = \overrightarrow{R}$ for all $j \in \mathbb{Z}$.

□

Claim 4. $||S|| = ||R|| + ||S - R||$.

Proof of claim 4.
By rectifiability and Theorem [4, 4.1.28, p.385]

$$||S|| = \mathcal{H}^{n-1} \llcorner \Theta(||S||, \cdot),$$

$$||R|| = \mathcal{H}^{n-1} \llcorner \Theta(||R||, \cdot),$$

$$||S - R|| = \mathcal{H}^{n-1} \llcorner \Theta(||S - R||, \cdot).$$

By claim 3

$$\Theta^{n-1}(||S||, \cdot)\mathbb{1}_{\{x:\Theta^{n-1}(||R||,x)>0\}} = \Theta^{n-1}(||R||, \cdot) \quad \mathcal{H}^{n-1}\text{-a.e.}$$

$$\Theta^{n-1}(||S||, \cdot)\mathbb{1}_{\{x:\Theta^{n-1}(||R||,x)=0\}} = \Theta^{n-1}(||S||, \cdot) - \Theta^{n-1}(||R||, \cdot) \quad \mathcal{H}^{n-1}\text{-a.e..}$$

Now for any Borel set W it holds

$$\begin{aligned}
||S||(W) = & \int_W \Theta^{n-1}(||S||,\cdot)\, d\mathcal{H}^{n-1} \\
= & \int_W \Theta^{n-1}(||S||,\cdot)\mathbb{1}_{\{x:\Theta^{n-1}(||R||,x)>0\}}\, d\mathcal{H}^{n-1}+ \\
& +\int_W \Theta^{n-1}(||S||,\cdot)\mathbb{1}_{\{x:\Theta^{n-1}(||R||,x)=0\}}\, d\mathcal{H}^{n-1} \\
= & \int_W \Theta^{n-1}(||R||,\cdot)\, d\mathcal{H}^{n-1}+ \\
& +\int_W \Big(\Theta^{n-1}(||S||,\cdot) \,-\, \Theta^{n-1}(||R||,\cdot)\Big)\, d\mathcal{H}^{n-1} \\
= & \ ||R||(W) \,+\, ||S-R||(W).
\end{aligned}$$

□

Now we know that

$$\partial R = \partial N$$

$$\partial S = \partial\left(\sum_{j\in\mathbb{Z}} S_j\right) = \sum_{j\in\mathbb{Z}} \partial S_j = 0,$$

which implies

$$\partial(R-S) = \partial N.$$

Moreover

$$\begin{aligned}
\mathbf{M}(N) \ & = ||N||(\mathbb{R}^n) \\
& = ||N-R||(\mathbb{R}^n) - ||R||(\mathbb{R}^n) \\
& = ||N-R||(\mathbb{R}^n) - ||S||(\mathbb{R}^n) + ||S-R||(\mathbb{R}^n) \\
& \geq ||S-R||(\mathbb{R}^n) \\
& = \mathbf{M}(R-S)
\end{aligned}$$

where the inequality follows from *(b)* of claim 1.
We define $T := R - S$ and the theorem is proven.

□

2.3 Hardt-Pitts decomposition and its connection to Theorem 3.2.1

In [11] Maciej Zworski pointed out one fundamental takeaway from Hardt-Pitts proof. That is the following theorem:

Theorem 2.3.1.
Let $N \in \mathbf{N}_{n-1}(\mathbb{R}^n)$.
Assume that N *has compact support and that* $\partial N \in \mathbf{I}_{n-2}(\mathbb{R}^n)$.
Then there exists a family $\{R_s\}_{s\in(0,1)}$ *such that*

(i) $R_s \in \mathcal{R}_{n-1}(\mathbb{R}^n)$ *for* $\mathcal{L}^1$*-a.e.* $s \in (0,1)$*;*

(ii) $N = \displaystyle\int_{(0,1)} R_s \, d\mathcal{L}^1(s)$*;*

(iii) $||N|| = \displaystyle\int_{(0,1)} ||R_s|| \, d\mathcal{L}^1(s)$*;*

(iv) $||\partial N|| = \displaystyle\int_{(0,1)} ||\partial R_s|| \, d\mathcal{L}^1(s)$*, and more in particular* $\partial R_s = \partial N$ *for* $\mathcal{L}^1$*-a.e.* $s \in (0,1)$.

Proof of Theorem 2.3.1.
Take R and f as in Theorem 2.1.2 and define

$$R_\omega := R + \sum_{j\in\mathbb{Z}} \partial\big[\mathbf{E}^n \llcorner \{x : f(x) \geq j + s\}\big] \quad \forall s \in (0,1).$$

We can now deduce:

- *(i)* from the proof of claim 1.
- *(ii)* from equation (2.3).
- *(iii)* from equation (2.4).
- *(iv)* using the fact that

$$\partial R_s = \partial R = \partial N \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0,1),$$

to write

$$||\partial N||(A) = \int_{(0,1)} ||\partial R||(A)\, ds = \int_{(0,1)} ||\partial R_s||(A)\, ds.$$

□

In the next Chapter, Theorem 3.2.1 will provide not only an alternative proof for the decomposition in the specific case of occupational measures, but also a procedure to compute the actual currents R_s.

We now anticipate Theorem 3.2.1 and formulate its connection with the Hardt-Pitts decomposition more concretely. To do so we need to associate a suitable current to any occupation measure.

Consider a measure μ as in Theorem 3.2.1 on $\overline{\Omega} \times Y \times Z \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and consider the maps $\{\psi_r\}_{r\in[-1,0]}$ as in Theorem 3.2.1.

Define $T_\mu, R_s \in \mathcal{D}_n\big(\Omega \times Y\big)$ as

$$T_\mu(\omega) := \int_{\Omega\times Y\times Z} \left\langle \omega(x,y), \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ z_1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ z_2 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ z_n \end{pmatrix} \right\rangle d\mu(x,y,z)$$

and

$$R_s(\omega) := \int_{\Omega} \left\langle \omega\big(x, \psi_s(x)\big), \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial \psi_s}{\partial x_1}(x) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \frac{\partial \psi_s}{\partial x_2}(x) \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \frac{\partial \psi_s}{\partial x_n}(x) \end{pmatrix} \right\rangle d\mathcal{L}^n(x).$$

Then (3.4) implies that

$$T_\mu = \int_{(-1,0)} R_s\, ds.$$

Chapter 3

Occupation measures

3.1 Variational problems and the definition of occupation measures

In this section we start by presenting a simple prototype of variational problem **(P)** on an open bounded subset Ω of $\mathbb{R}^n$. It will provide a context for some observations about $C^1(\overline{\Omega})$ maps. Such observations will show how the definition of occupation measures (made shortly after) is in fact quite natural. We shall then present the actual problem of interest, **(GP)**, and its relaxed version, **(RGP)**, made using occupation measures.

3.1.1 A tipical variational problem

Our entire discussion will be focused on techniques to solve a certain type of variational problems. A simple formulation of such type of problems is the following:

(P):

Simple problem (P):
Let $m, n \in \mathbb{N}$. We will refer to n as the **dimension** and to m as the **codimension**.
Let $\Omega \subset \mathbb{R}^n$ be open, connected and with piecewise C^1 boundary.
Let $Y := \mathbb{R}^m$ and $Z := \mathbb{R}^{mn} \cong M_{m\times n}(\mathbb{R})$.
Let $L : \Omega \times Y \times Z \to \mathbb{R}$ be a locally bounded and measurable function (which we will call **Lagrangian** of the problem).
Remember that $W^{1,\infty}(\Omega, Y) = \mathrm{Lip}(\Omega, Y)$.
Define the functional

$$\begin{array}{rccl} F: & W^{1,\infty}(\Omega, Y) & \longrightarrow & \mathbb{R} \\ & y(\cdot) & \mapsto & \displaystyle\int_\Omega L(x, y(x), Dy(x))\, dx. \end{array}$$

Find

$$\inf_{y\in W^{1,\infty}(\Omega,Y)} F(y).$$

3.1.2 Measures induced by maps in $C^1(\overline{\Omega})$

Before defining the space of occupation measures we make some observations which will make its definition quite natural.
Consider the set $C^1(\overline{\Omega})$ of real functions that are C^1 on the closure of Ω.
To each map $y \in C^1(\overline{\Omega})$ one can associate naturally the map

$$\begin{array}{rccl} \overline{y}: & \Omega \times Y \times Z & \longrightarrow & \Omega \times Y \times Z \\ & x & \mapsto & (x, y(x), Dy(x)) \end{array}$$

and therefore the Borel measure μ_y on $\Omega\times Y\times Z$ obtained by pushing forward $\mathcal{L}^n$ through $\overline{y}$:

$$\begin{array}{rccl} \mu_y = \overline{y}_\# \mathcal{L}^n: & \mathcal{B}_{\Omega\times Y\times Z} & \longrightarrow & [0, +\infty] \\ & B & \mapsto & \mathcal{L}^n\big(\overline{y}^{-1}(B)\big). \end{array}$$

The measure μ_y has the property that for any measurable $f : \Omega\times Y\times Z \to \mathbb{R}$ it holds

$$\int_{\Omega\times Y\times Z} f\, d\mu_y = \int_\Omega f(x, y(x), Dy(x))\, d\mathcal{L}^n(x).$$

Moreover for any $\phi \in C_c^\infty(\Omega \times Y)$ one may consider the map

$$\begin{array}{rccc} \phi_y : & \Omega & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \phi(x, y(x)) \end{array}$$

and notice that for all $j \in \{1, 2, ..., n\}$

$$\begin{aligned} 0 &= \int_\Omega \frac{\partial \phi_y}{\partial x_j}\, dx \\ &= \int_\Omega \left[\frac{\partial \phi}{\partial x_j}(x, y(x)) + \sum_{i=1}^m \frac{\partial \phi}{\partial y}(x, y(x)) \frac{\partial y_i}{\partial x_j}(x, y(x)) \right] dx \\ &= \int_{\Omega \times Y \times Z} \left[\frac{\partial \phi}{\partial x_j}(x, y(x)) + \sum_{i=1}^m \frac{\partial \phi}{\partial y_i}(x, y(x)) z_{ij} \right] d\mu_{y(\cdot)}(x, y, z). \end{aligned}$$

We will define occupation measures to be the ones satisfying precisely these properties, as one can see below.

3.1.3 Definition of simple occupation measures (without boundary component)

For any Radon measure on $\Omega \times Y \times Z$ define the two properties

(O1): For all $j \in \{1, ..., n\}$ and all $\phi \in C_c^\infty(\Omega \times Y)$

$$\int_{\Omega \times Y \times Z} \left[\frac{\partial \phi}{\partial x_j}(x, y(x)) + \sum_{i=1}^m \frac{\partial \phi}{\partial y_i}(x, y(x)) z_{ij} \right] d\mu(x, y, z) = 0.$$

(O2): $\displaystyle\int_{\Omega \times Y \times Z} ||z||\, d\mu(x, y, z) < \infty.$

Definition 3.1.1.
Define the set of **simple occupation measures** on $\Omega \times Y \times Z$ as

$$\mathcal{M}_0 := \Big\{ \mu : \ \mu \text{ is a Radon measure on } \Omega \times Y \times Z \\ \text{satisfying (O1) and (O2)} \Big\}.$$

We observe that

1. $\mathcal{M}_0$ is a vector space.

2. The total variation distance

$$||\mu - \nu|| := \sup\{|\mu(A) - \nu(A)| : A \in \mathcal{B}_{\Omega \times Y \times Z}\}$$

is a norm on $\mathcal{M}_0$.

3. If $y \in W^{1,\infty}(\Omega, Y)$ and $A \in \mathcal{B}_\Omega, B \in \mathcal{B}_Y, C \in \mathcal{B}_Z$, then

$$\mu_y(A \times B \times C) = \mathcal{L}^n\big(A \cap y^{-1}(B) \cap Dy^{-1}(C)\big).$$

3.1.4 Relaxing the simple variational problem using simple occupation measures

We shall now can define the relaxed version of problem **(P)**:

(RP):

> **Relaxed problem (RP):**
> Let m, n, Ω, Y, Z, L, F be as in the original problem **(P)**.
> Define the extension $\overline{F}$ of F as
>
> $$\begin{array}{rccc} \overline{F}: & \mathcal{M}_0 & \longrightarrow & \mathbb{R} \\ & \mu & \mapsto & \displaystyle\int_{\Omega \times Y \times Z} L \, d\mu \end{array}$$
>
> and find
>
> $$\inf_{\mu \in \mathcal{M}_0} \overline{F}(\mu).$$

First a general observation: $C^1(\overline{\Omega}, Y) = C^1(\Omega, Y) \cap W^{1,\infty}(\Omega, Y)$ and it is dense in $W^{1,\infty}(\Omega, Y)$.
Further immediate observations can be made about **(RP)** and **(P)**:

1. For all $y \in C^1(\overline{\Omega}, Y)$, $\mu_y \in \mathcal{M}_0$ and

$$F(y) = \overline{F}(\mu_y).$$

2. The functional F is not necessarily linear, while $\overline{F}$ is always linear.

3. The first two observations imply that

$$\inf_{y \in W^{1,\infty}(\Omega, Y)} F(y) \geq \inf_{\mu \in \mathcal{M}_0} \overline{F}(\mu).$$

3.1.5 The variational problem of our interest

The problem which we will be dealing with is a more general version of the simple problem **(P)** in which we add a boundary contribution to the functional F and we consider possible constraints either within Ω, on its boundary or on both. The formal statement of this general version is the following:

(GP):

General problem (GP):
Let m, n, Ω, Y, Z, L be as in problem **(P)**.
Let $\sigma := \mathcal{H}^{n-1} \llcorner \partial\Omega$.
Let $F, G : \Omega \times Y \times Z \to \mathbb{R}$ be two measurable maps.
Let $F_\partial, G_\partial : \partial\Omega \times Y \to \mathbb{R}$ be two $\sigma \times \mathcal{L}^m$-measurable maps.
Let $L_\partial : \partial\Omega \times Y \to \mathbb{R}$ be a bounded and σ-measurable map.
Define $\mathcal{C}$ as the set of $y \in W^{1,\infty}(\Omega, Y)$ satisfying

$$F(x, y(x), Dy(x)) = 0, \;\; G(x, y(x), Dy(x)) \leq 0 \quad \forall x \in \Omega,$$

$$F_\partial(x, y(x)) = 0, \;\; G_\partial(x, y(x)) \leq 0 \quad \forall x \in \partial\Omega.$$

Find

$$\inf_{y \in \mathcal{C}} \left(\int_\Omega L(x, y(x), Dy(x))\, dx \;+\; \int_{\partial\Omega} L_\partial(x, y(x))\, d\sigma(x) \right).$$

We shall define, for future reference,

$$\mathcal{F}(y) := \int_\Omega L(x, y(x), Dy(x))dx \;+\; \int_{\partial\Omega} L_\partial(x, y(x))\, d\sigma(x)$$

for all $y \in W^{1,\infty}(\Omega, Y)$.

3.1.6 Definition of occupation measures

We now give a formal definition of the actual occupation measures which we will be dealing with.

Definition 3.1.2 (Relaxed occupation measures).
We say that a couple (μ, μ_∂) is a **relaxed occupation measure** (or simply **occupation measure**) on Ω if

(i) μ is a compactly supported and positive Radon measure on $\overline{\Omega} \times Y \times Z$. μ_∂ is a compactly supported and positive Radon measure on $\partial\Omega \times Y$.

(ii) $\mu(\Omega \times Y \times Z) = \mathcal{L}^n(\Omega)$.

(iii) For any $\phi \in C^\infty(\Omega \times Y)$, the measures μ and μ_∂ satisfy

$$\int_{\Omega \times Y \times Z} \left(\frac{\partial \phi}{\partial x}(x, y) + \frac{\partial \phi}{\partial y} z \right) d\mu(x, y, z) \; = \; \int_{\partial\Omega \times Y} \phi(x, y)\mathbf{n}(x)\, d\mu_\partial(x, y), \tag{3.1}$$

where $\mathbf{n}(x)$ is the exterior normal vector to Ω at x.

We will denote by $\mathcal{M}$ the set of relaxed occupation measures on Ω (we will not write $\mathcal{M}(\Omega, Y, Z)$ as all the sets will be clear from the context).
In the following, when referring to the single measures in the couple (μ, μ_∂), we will refer to μ as **relaxed occupation measure** and to μ_∂ as **relaxed boundary measure**.

We make some observations:

1. The condition (3.1) is a system of n equations.

2. $\mathcal{M}$ is a vector space.

3. Every relaxed occupation measure (μ, μ_∂) satisfies

$$\int_{\Omega \times Y \times Z} ||z||\, d\mu(x, y, z) < \infty. \tag{3.2}$$

4. If $\mu \in \mathcal{M}_0$, then the couple $(\mu, 0)$ is a relaxed occupation measure.

3.1.7 Relaxed version of the general problem

We now have all the necessary tools to define a relaxed version of **(GP)** using occupation measures.

(RGP):

Relaxed general problem (RGP):
Let $m, n, \Omega, Y, Z, \sigma, L, L_\partial, F, G, F_\partial, G_\partial$ be as in problem **(GP)**.
Define $\overline{\mathcal{C}}$ as the set of $(\mu, \mu_\partial) \in \mathcal{M}$ satisfying

$$\operatorname{spt}(\mu) \subset \Big\{(x,y,z) :\ F(x,y,z) = 0 \ \text{ and } \ G(x,y,z) \le 0\Big\},$$

$$\operatorname{spt}(\mu_\partial) \subset \Big\{(x,y) :\ F_\partial(x,y) = 0 \ \text{ and } \ G_\partial(x,y) \le 0\Big\}.$$

Find

$$\inf_{(\mu,\mu_\partial)\in\overline{\mathcal{C}}} \left(\int_{\Omega\times Y\times Z} L \, d\mu \ + \int_{\partial\Omega\times Y} L_\partial(x,y)\, d\mu_\partial \right).$$

We shall define, for future reference,

$$\overline{\mathcal{F}}((\mu,\mu_\partial)) := \int_{\Omega\times Y\times Z} L \, d\mu \ + \int_{\partial\Omega\times Y} L_\partial(x,y)\, d\mu_\partial$$

for all couples $(\mu, \mu_\partial) \in \mathcal{M}$.

We notice that if $y(\cdot)$ is a map in $\mathcal{C} \cap C^1(\overline{\Omega})$, where $\mathcal{C}$ is the set described in **(GP)**, then the occupation measure $(\mu_y, 0)$ is in the set $\overline{\mathcal{C}}$ defined in **(RGP)**. From this one deduces that the solution of **(RGP)** is less or equal to the solution of **(GP)**. The viceversa is not true in general, as shown in the following counterexample.

Proposition 3.1.1.
*Problems **(GP)** and **(RGP)** are, in general, not equivalent.*

Proof.
We prove the proposition by showing a simple counterexample. Consider the particular case of problem **(GP)** in which

$$\Omega = [0,1], \quad Y = \mathbb{R}, \quad Z = \mathbb{R},$$

$$L(x,y,z) = \min\{|z-1|,\, |z+1|\}, \quad L_\partial(x,y,z) \equiv 0,$$

$$F(x,y,z) = y, \quad G(x,y,z) \equiv 0,$$

$$F_\partial(x,y) \equiv 0 \equiv G_\partial(x,y)$$

$G, F_\partial, G_\partial$ are null and therefore give no actual constraint. The constraint given by F, though, makes is so that the set $\mathcal{C}$ contains only the identically null function. This means that the solution to **(GP)** is 1, which is the minimum attained by the only feasible map $y(\cdot)$.

The relaxation of the described problem has the conditions

$$\operatorname{spt}(\mu) \subset [0,1] \times \{0\} \times \mathbb{R},$$

$$\operatorname{spt}(\mu_\partial) \subset \{0,1\} \times \mathbb{R}.$$

The defining condition of occupation measures is

$$\int_{[0,1]\times\mathbb{R}\times\mathbb{R}} \left(\frac{\partial \phi}{\partial x}(x,y) + \frac{\partial \phi}{\partial y} z \right) d\mu \; = \int_{\{0,1\}\times\mathbb{R}} \phi(x,y)\mathbf{n}(x)\, d\mu_\partial,$$

which, since $\mathbf{n}(x) = -\mathbb{1}_{\{0\}}(x) + \mathbb{1}_{\{1\}}(x)$, is the same as

$$\int_{[0,1]\times\mathbb{R}\times\mathbb{R}} \left(\frac{\partial \phi}{\partial x}(x,y) + \frac{\partial \phi}{\partial y} z \right) d\mu \; = \; - \int_{\{0\}\times\mathbb{R}} \phi(0,y)\, d\mu_\partial + \int_{\{1\}\times\mathbb{R}} \phi(1,y)\, d\mu_\partial.$$

We now simply notice that:

- $L \geq 0$, which means that

$$\inf_{(\mu,\mu_\partial)\in\overline{\mathcal{C}}} \left(\int_{\Omega\times Y\times Z} L\, d\mu \; + \int_{\partial\Omega\times Y} L_\partial(x,y)\, d\mu_\partial \right) \geq 0.$$

- The measures

$$\overline{\mu} := \mathcal{L}^1 \otimes \delta_0 \otimes \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1 \right)$$

$$\overline{\mu_\partial} = 0$$

  define a feasible variational measure for the considered problem.

- $\displaystyle \int_{\Omega\times Y\times Z} L\, d\overline{\mu} \; + \int_{\partial\Omega\times Y} L_\partial(x,y)\, d\overline{\mu_\partial} = 0.$

It is now clear that the defined occupation measure $(\overline{\mu}, \overline{\mu_\partial})$ attains the minimum of the relaxed problem, which is therefore precisely 0.

$\square$

The goal of the following section will be to prove that under suitable hypothesis the problems **(GP)** and **(RGP)** have the same solution.

3.2 Conditions for null gap in codimension 1

In this section we aim at proving that under the set of hypothesis $CV1$, $CV2$, ..., $CV6$ listed below it is possible to prove a decomposition which is of the same type as the one from Hardt-Pitts and use it to show that **(GP)** and **(RGP)** have the same solution.

3.2.1 Notation and some observations

We shall now set notation and basic assumptions for future reference:

$CV1$**:** $L : \Omega \times Y \times Z \to \mathbb{R}$ is $\mathcal{L}^n$-measurable and locally bounded,
$L_\partial : \partial\Omega \times Y \to \mathbb{R}$ is $\big(\mathcal{H}^{n-1} \llcorner \partial\Omega\big)$-measurable and locally bounded.

$CV2$**:** $F, G : \Omega \times Y \times Z \to \mathbb{R}$ is $\mathcal{L}^n$-measurable.

$CV3$**:** $F_\partial, G_\partial : \partial\Omega \times Y \to \mathbb{R}$ is $\big(\mathcal{H}^{n-1} \llcorner \partial\Omega\big)$-measurable.

$CV4$**:** L is convex in z, i.e. $L(x, y, \cdot) : Z \to \mathbb{R}$ is convex for all $(x, y) \in \Omega \times Y$.

$CV5$**:** $F^{-1}(0) \cap G^{-1}\big((-\infty, 0]\big) \cap \big(\{x\} \times \{y\} \times Z\big)$ is convex for every $(x, y) \in \Omega \times Y$.

$CV6$**:** $F^{-1}(0) \cap G^{-1}\big((-\infty, 0]\big)$ and $F_\partial^{-1}(0) \cap G_\partial^{-1}\big((-\infty, 0]\big)$ are closed.

All these hypothesis, which are rather general, are for example implied by the following set of more familiar hypothesis:

- $L, F, G, L_\partial, F_\partial, G_\partial$ are coutinuous,

- L and G are convex in z,
- either F is non-negative and convex in z or it is affine in z.

3.2.2 Main results of this section: decomposition and zero gap

We state here the three main results of this chapter and also show how the first two easily imply the third. We will then proceed to prove the first two theorems in the following sections.

Theorem 3.2.1 (Decomposition in codimension 1)**.**
*Let n, m, Ω, Y, Z be as in **(RGP)**.*
Assume that $m = \dim(Y) = 1$.
Let μ be a compactly supported, positive and finite Radon measure on $\overline{\Omega} \times Y \times Z$.
Assume that for all $\phi \in C_c^\infty(\Omega \times Y)$ the measure μ satisfies

$$\int_{\Omega\times Y\times Z} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} z \right) d\mu(x,y,z) = 0. \tag{3.3}$$

Then there exist

- *a compactly-supported, positive and finite Radon measure ν on $\mathbb{R}$,*
- *a family $\{\varphi_r\}_{r\in\mathbb{R}} \subset W^{1,\infty}(\Omega, Y) = W^{1,\infty}(\Omega)$*

such that for all $\phi \in L^1(\mu)$ which are affine in z we have

$$\int_{\Omega\times Y\times Z} \phi \, d\mu = \int_{\mathbb{R}} \int_{\Omega} \phi\big(x, \varphi(x), D\varphi(x)\big) \, dx \, d\nu. \tag{3.4}$$

Moreover the family $\{\varphi_r\}_{r\in\mathbb{R}}$ can be found such that if $r \geq r'$ then $\varphi_r(x) \leq \varphi(r')$ $\forall x \in \Omega$.

Theorem 3.2.2.
*Let $n, m, \sigma, L, F, G, L_\partial, F_\partial, G_\partial$ be as in **(RGP)**.*
Assume $m = \dim(Y) = 1$.
Assume that $L, F, G, L_\partial, F_\partial, G_\partial$ satisfy $CV1$,...,$CV6$.
*Assume that $(\mu, \mu_\partial) \in \overline{\mathcal{C}}$ ($\overline{\mathcal{C}}$ as in **(RGP)**).*

Then:

*(i) There exists $\overline{\varphi} \in W^{1,\infty}(\overline{\Omega}) \cap \mathcal{C}$ (with $\mathcal{C}$ as in **(GP)**) such that*

$$\begin{aligned}\int_\Omega L\big(x, \overline{\varphi}(x), D\overline{\varphi}(x)\big)\, dx \quad &+ \int_{\partial\Omega} L_\partial(x, \overline{\varphi}(x))\, d\sigma \ \le \\ &\le \int_{\Omega\times Y\times Z} L\, d\mu + \int_{\partial\Omega\times Y} L_\partial\, d\mu_\partial.\end{aligned} \tag{3.5}$$

(ii) If L, F and G are also continuous, then there exists a sequence $\big\{g_i : \overline{\Omega} \to Y\big\}_{i\in\mathbb{N}} \subset C^\infty(\Omega) \cap W^{1,\infty}(\overline{\Omega})$ such that

$$\begin{aligned}\lim_{i\to\infty} \left(\int_\Omega L(x, g_i(x), Dg_i(x))\, dx \quad + \int_{\partial\Omega} L_\partial(x, g_i(x))\, d\sigma \right) \ &\le \\ \le \int_{\Omega\times Y\times Z} L\, d\mu + \int_{\partial\Omega\times Y} L_\partial\, d\mu_\partial&\end{aligned} \tag{3.6}$$

and

$$\lim_{i\to\infty} F(x, g_i(x), Dg_i(x)) = 0, \ \lim_{i\to\infty} G(x, g_i(x), Dg_i(x)) \le 0 \ \forall x \in \Omega \tag{3.7}$$

$$F_\partial(x, g_i(x)) = 0, \quad G_\partial(x, g_i(x)) \le 0 \quad \forall x \in \partial\Omega. \tag{3.8}$$

Theorem 3.2.3.
*Let $n, m, \sigma, L, F, G, L_\partial, F_\partial, G_\partial$ be as in **(RGP)**.*
Assume $m = \dim(Y) = 1$.
Assume that $L, F, G, L_\partial, F_\partial, G_\partial$ satisfy $CV1$,...,$CV6$.
*Denote by M_c the solution to **(GP)** and by M_r the solution to **(RGP)**.*
Assume that $M_c < \infty$.
Then
$$M_c = M_r.$$

Proof of Theorem 3.2.3.
We prove the theorem by proving the two inequalities separately .

Proving $M_c \geq M_r$:
Consider a map $\overline{\varphi} \in W^{1,\infty}(\Omega, Y)$.
Define the functional

$$\begin{array}{llll} F_{\overline{\varphi}}: & C^0(\Omega \times Y \times Z) & \longrightarrow & \mathbb{R} \\ & \phi & \mapsto & \displaystyle\int_\Omega \phi(x, \overline{\varphi}(x), D\overline{\varphi}(x))\, dx \end{array}.$$

The functional $F_{\overline{\varphi}}$ is linear and, since $\overline{\varphi}$ is a Lipschitz function, continuous. In other words $F_{\overline{\varphi}}$ is in the dual space of $C^0(\Omega \times Y \times Z)$. By Riesz–Markov–Kakutani representation theorem there is a unique signed measure $\mu^{(\overline{\varphi})}$ such that

$$F_{\overline{\varphi}}(\phi) = \int_{\Omega \times Y \times Z} \phi\, d\mu^{(\overline{\varphi})} \quad \forall \phi \in C^0(\Omega \times Y \times Z).$$

In a completely analogous way one finds a measure $\mu_\partial^{(\overline{\varphi})}$ on $\partial\Omega \times Y$ such that

$$\int_{\partial\Omega} \phi(x, y(x)) d\sigma(x) \;=\; \int_{\partial\Omega \times Y} \phi\, d\mu_\partial^{(\overline{\varphi})} \quad \forall \phi \in C^0(\Omega \times Y).$$

The pair $(\mu^{(\overline{\varphi})}, \mu_\partial^{(\overline{\varphi})})$ is indeed in $\overline{\mathcal{C}}$ (since $\overline{\varphi}$ is in $\mathcal{C}$) and $\overline{\mathcal{F}}(\,(\mu^{(\overline{\varphi})}, \mu_\partial^{(\overline{\varphi})})\,) \leq \mathcal{F}(\overline{\varphi})$.
This shows how $M_c \geq M_r$.

Proving $M_c \leq M_r$**:**

Assume that $(\mu, \mu_\partial) \in \overline{C}$.

Then it satisfies the hypothesis of Theorem 3.2.2, and the theorem yields a $\overline{\varphi} \in \mathcal{C}$ such that $\mathcal{F}(\overline{\varphi}) \leq \overline{\mathcal{F}}((\mu, \mu_\partial))$.

This shows how $M_c \leq M_r$.

□

3.2.3 Proof of Theorem 3.2.1

We now prove a sequence of lemmas which we will use to prove the theorem. From now to the end of the proof of Theorem 3.2.1 we will assume that any considered measure μ satisfies the hypothesis of Theorem 3.2.1.

Lemma 3.2.4.
Let $\pi_\Omega : \Omega \times Y \times Z \to \Omega$ *be the projection.*
Then there exists $c \in \mathbb{R}_{>0}$ *such that* $\pi_{\Omega\#}\mu \; = \; c\mathcal{L}^n$.

Proof of Lemma 3.2.4.
By [1] there exists an orthogonal matrix whose entries are all nonzero. Taking its columns and normalizing them, one can see that there exists an orthonormal basis $\{u_1, ..., u_n\}$ whose vectors' coordinates (with respect to the canonical basis) are all nonzero.
Let $R \subset \Omega$ be a parallelepiped whose generating vectors are all multiples of the vectors $u_1, ..., u_n$.
Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ be a translation such that $\tau(R) \subset \Omega$.
Write τ as finite composition $\tau_k \circ \tau_{k-1} \circ ... \circ \tau_1$ of translations each one of which has direction parallel to one of the coordinate axis (i.e. the direction of τ_i is in $\{e_1, e_2, ..., e_n\}$ for all $i = 1, ..., k$).
Define $\tilde{\tau}_i := \tau_i \circ \tau_{i-1} \circ ... \circ \tau_1$, for every $i = 1, ..., k$.
Notice that it is possible to choose the translations $\tau_1, ..., \tau_k$ in such a way that[1] $\operatorname{co}\left(\tilde{\tau}_{i-1}(R) \cup \tilde{\tau}_i(R)\right) \subset \Omega$ for all $i = 1, ..., k$. Assume then that $\tau_1, ..., \tau_k$

[1]Given a subset A of $\mathbb{R}^n$ we denote by $\operatorname{co}(A)$ the convex hull of A and with $\overline{\operatorname{co}}(A)$ its closure.

were chosen with that property.
For each $i = 1, ..., k$ define $j(i)$ to be the index representing the direction of the translation τ_i (i.e. for all i we have $j(i) \in \{1, ..., n\}$ such that the direction of τ_i is parallel to $\mathbf{e}_{j(i)}$).
Define, for each $i \in \{1, .., k\}$, the map $\phi_i : \Omega \to \mathbb{R}$ as

$$\phi_i(x) := \int_{-\infty}^{x_{j(i)}} \Big(\mathbb{1}_{\tilde{\tau}_i(R)}(x_1, ..., x_{j(i)-1}, s, x_{j(i)+1}, ..., x_n) + \\ - \mathbb{1}_{\tilde{\tau}_{i-1}(R)}(x_1, ..., x_{j(i)-1}, s, x_{j(i)+1}, ..., x_n) \Big)\, ds.$$

Since $\tilde{\tau}_{i-1}(R)$ and $\tilde{\tau}_i(R)$ are closed parallelepiped and one is the translated of the other, $\mathrm{co}\left(\tilde{\tau}_{i-1}(R) \cup \tilde{\tau}_i(R)\right)$ is a closed set.
Notice that $\phi(x)$ is

- non-zero for every x in the interior of the set $\mathrm{co}\left(\tilde{\tau}_{i-1}(R) \cup \tilde{\tau}_i(R)\right)$;
- possibly 0 (although it is not guaranteed) on its boundary;
- equal to 0 elsewhere.

This means that $\mathrm{supp}(\phi_i) = \overline{\mathrm{co}}\left(\tilde{\tau}_{i-1}(R) \cup \tilde{\tau}_i(R)\right)$.
Define also

$$\begin{array}{rccc} \overline{\phi}_i : & \Omega \times Y & \to & \mathbb{R} \\ & (x, y, z) & \mapsto & \phi_i(x). \end{array}$$

Claim 1.
ϕ is a Lipschitz map and the partial derivative $\dfrac{\partial \phi_i}{\partial x_{j(i)}}$ *is given by*

$$\frac{\partial \phi_i}{\partial x_{j(i)}} = \mathbb{1}_{\tilde{\tau}_i(R)} - \mathbb{1}_{\tilde{\tau}_{i-1}(R)}$$

Proof of claim 1.
Lipschitz regularity follows from the choice on the vectors generating R, none of which is orthogonal to any of the vectors in the canonical basis.
The other statement is quite trivial, since $\dfrac{\partial \phi_i}{\partial x_{j(i)}}$ is

- Equal to 1 in the set $\mathrm{int}\left(\tilde{\tau}_i(R) \setminus \tilde{\tau}_{i-1}(R)\right)$.

- Equal to -1 in the set $\operatorname{int}\Big(\tilde{\tau}_{i-1}(R) \setminus \tilde{\tau}_i(R)\Big)$.
- Either $-1, 0, +1$ or not defined in the set $\partial\big(\tilde{\tau}_i(R)\big) \cup \partial\big(\tilde{\tau}_{i-1}(R)\big)$.
- Equal to 0 everywhere else.

$\square$

Now, by Lipschitz regularity, the map $\overline{\phi}$ can be approximated, in a neighborhood of $\operatorname{spt}(\mu)$ by a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega \times Y)$ of smooth functions with compact support which converge uniformly to $\overline{\phi}$.
Moreover:

- uniform convergence implies that the derivatives $\{Df_j\}_j$ converge pointwise to the derivative $D\overline{\phi}_i$ (where it exists);
- Lipschitz regularity implies that the sequence $\{f_j\}$ can be taken such that the sequence of the derivatives $\{Df_j\}$ is uniformly bounded.

These two observations show that the dominated convergence Theorem applies and that therefore

$$\int_{\Omega\times Y\times Z} \left(\frac{\partial\overline{\phi}_i}{\partial x_{j(i)}} + \frac{\partial\overline{\phi}_i}{\partial y} z_{j(i)}\right) d\mu = \lim_{m\to\infty} \int_{\Omega\times Y\times Z} \left(\frac{\partial f_m}{\partial x_{j(i)}} + \frac{\partial f_m}{\partial y} z_{j(i)}\right) d\mu = 0.$$

To conclude the proof of the lemma we simply put all these pieces together and see that

$$\begin{aligned}
\pi_{\Omega\,\#}\mu\big(\tilde{\tau}_i(R)\big) - \pi_{\Omega\,\#}\mu\big(\tilde{\tau}_{i-1}(R)\big) &= \mu(\tilde{\tau}_i(R)\times Y\times Z) - \mu(\tilde{\tau}_{i-1}(R)\times Y\times Z) \\
&= \int_{\Omega\times Y\times Z} \big(\mathbb{1}_{\tilde{\tau}_i(R)\times Y\times Z} - \mathbb{1}_{\tilde{\tau}_{i-1}(R)\times Y\times Z}\big)\, d\mu \\
&= \int_{\Omega\times Y\times Z} \left(\frac{\partial\overline{\phi}_i}{\partial x_{j(i)}} + \frac{\partial\overline{\phi}_i}{\partial y} z_{j(i)}\right) d\mu \\
&= 0.
\end{aligned}$$

The whole argument above made proves that for any parallelepiped $R \subset \Omega$ with directions parallel to $u_1, ..., u_n$ and any translation τ which keeps $\tau(R)$ in Ω, the $\pi_{\Omega\,\#}\mu$-measure of R and $\tau(R)$ is the same.

All measures with this property must be a multiple of the Lebesgue measure (cfr [10, Theorem 2.20, page 50]), and the lemma is proven.

□

For any vector field $X : \Omega \times Y \to \mathbb{R}^{n+1}$ define

$$\mu(X) := \int_{\Omega \times Y \times Z} \langle X(x,y), (z_1, ..., z_n, -1) \rangle \, d\mu(x,y,z). \tag{3.9}$$

Lemma 3.2.5.
Let $X : \Omega \times Y \to \mathbb{R}^{n+1}$ be a smooth and compactly-supported vecor field.
Assume that X vanishes on a neighborhood N of $\partial\Omega \times Y$.
Assume also that $\mathrm{div}(X) = 0$ *on Ω.*
Then

$$\mu(X) = 0.$$

Proof of Lemma 3.2.5.
Assume without loss of generality that N is an open neighborhood.
Notice that $\mathrm{spt}(\mu)$ is a compact set contained in the open set $\Omega \times Y$.
Then there is a closed (and therefore compact) neighborhood K of $\mathrm{spt}(\mu)$ which is contained in $\Omega \times Y$.
Let then $\chi \in C_c^\infty(\Omega \times Y)$ be such that it is identically 1 on K.
Define, for each $i \in \{1, ..., n\}$,

$$\tilde{X}_i(x,y) := \int_{-\infty}^{y} X_i(x,s) \, ds.$$

Then

- $\tilde{X}_i \in C^\infty(\Omega \times Y)$ and it vanishes on a neighborhood of $\partial\Omega \times Y$.
- $\tilde{X}_i \cdot \chi \in C_c^\infty(\Omega \times Y)$
- $\tilde{X}_i \cdot \chi = \tilde{X}_i \; \forall x \in K$;
- $\mu\big((\Omega \times Y) \setminus K\big) = 0$.

This means, by (3.3), that

$$\begin{aligned} 0 &= \int_{\Omega\times Y\times Z} \left(\frac{\partial(\tilde{X}_i\chi)}{\partial x_i} + \frac{\partial(\tilde{X}_i\chi)}{\partial y} z \right) d\mu \\ &= \int_{\Omega\times Y\times Z} \left(\frac{\partial \tilde{X}_i}{\partial x_i} + \frac{\partial \tilde{X}_i}{\partial y} z_i \right) d\mu \\ &= \int_{\Omega\times Y\times Z} \left(\frac{\partial \tilde{X}_i}{\partial x_i} + X_i(x,y) z_i \right) d\mu \end{aligned}$$

Now

$$\begin{aligned} \mu(X) &= \int_{\Omega\times Y\times Z} \left(-X_{n+1}(x,y) + \sum_{i=1}^n X_i(x,y) z_i \right) d\mu \\ &= -\int_{\Omega\times Y\times Z} X_{n+1}(x,y)\, d\mu \, - \sum_{i=1}^n \int_{\Omega\times Y\times Z} \frac{\partial \tilde{X}_i}{\partial x_i}(x,y)\, d\mu(x,y,z). \\ &= -\int_{\Omega\times Y\times Z} \left(X_{n+1}(x,y) + \sum_{i=1}^n \frac{\partial \tilde{X}_i}{\partial x_i}(x,y) \right) d\mu \end{aligned}$$

We can see that

- $X_{n+1}(x,y) = \displaystyle\int_{-\infty}^{y} \frac{\partial X_{n+1}}{\partial y}(x,s)\, ds.$
- Passing the derivatives inside the integral sign

$$\frac{\partial \tilde{X}_i}{\partial x_i}(x,y) = \int_{-\infty}^{y} \frac{\partial X_i}{\partial x_i}(x,s)\, ds \quad \forall i = 1, ..., n.$$

So that

$$\begin{aligned} X_{n+1}(x,y) + \sum_{i=1}^n \frac{\partial \tilde{X}_i}{\partial x_i}(x,y) &= \int_{-\infty}^{y} \left(\frac{\partial X_{n+1}}{\partial y}(x,s) + \sum_{i=1}^n \frac{\partial X_i}{\partial x_i}(x,s) \right) d\mu \\ &= \int_{-\infty}^{y} \mathrm{div}(X)(x,s)\, ds \\ &= 0. \end{aligned}$$

□

For any measurable, compactly-supported and bounded function $u : \Omega\times Y \to \mathbb{R}$, define the value

$$S(u) := \mu(X^{(u)}),$$

where

$$\begin{array}{rccl} X^{(u)}: & \Omega \times Y & \longrightarrow & \mathbb{R}^{n+1} \\ & (x,y) & \mapsto & (0,...,0,\displaystyle\int_y^{+\infty} u(x,s)\,ds) \end{array}.$$

In other words

$$S(u) = -\int_{\Omega\times Y\times Z}\int_y^{+\infty} u(x,s)\,ds\,d\mu(x,y,z). \tag{3.10}$$

Lemma 3.2.6.
Denoted by $\mathcal{K}$ the set of measurable, compactly-supported and bounded functions $u:\Omega\times Y\to\mathbb{R}$, the functional

$$\begin{array}{rccl} S: & \mathcal{K} & \longrightarrow & \mathbb{R} \\ & u & \mapsto & S(u) \end{array}$$

is representable by integration through an absolutely continuous non-positive measure.
That is to say that there exists a measurable map $\rho:\Omega\times Y\to(-\infty,0]$ such that

$$S(u) = \int_{\Omega\times Y} u(x,y)\rho(x,y)\,dx\,dy \quad \forall u\in\mathcal{K}.$$

Proof of Lemma 3.2.6.
We use Radon-Nikodym's Theorem.
First notice that the map

$$\begin{array}{rccl} \nu_S: & \mathcal{B}_{\Omega\times Y} & \longrightarrow & \mathbb{R} \\ & B & \mapsto & \begin{cases} -S(\mathbb{1}_B) & \text{if } B \text{ bounded} \\ -\displaystyle\lim_{r\to+\infty} S(\mathbb{1}_{B\cap\mathbf{B}(0,r)}) & \text{if } B \text{ unbounded} \end{cases} \end{array}$$

defines a positive measure.

Claim 2. *If $\mathcal{L}^{n+1}(A)=0$, then $S(\mathbb{1}_A)=0$.*

Proof of claim 2.
If $\mathcal{L}^{n+1}(A) = 0$, then using Tonelli-Fubini's Theorem we deduce that

$$\mathcal{L}^1\big(\{s : \mathbb{1}_A(x,s) = 1\}\big) = 0 \quad \text{for } \mathcal{L}^n\text{-a.e. } x.$$

Using Lemma 3.2.4 this means that

$$\mathcal{L}^1\Big(\big\{(s : \mathbb{1}_A(x,s) = 1\big\}\Big) = 0 \quad \text{for } \mu\text{-a.e. } (x,y,z) \in \Omega \times Y \times Z.$$

It now follows immediately from (3.10) that

$$S(\mathbb{1}_A) = 0.$$

□

The claim shows that ν_A is absolutely continuous with respect to $\mathcal{L}^{n+1}$. So by Radon-Nicodym's Theorem we deduce the existence of a measurable function $(-\rho) : \Omega \times Y \to [0, +\infty)$ such that

$$\nu_A(A) = \int_{\Omega\times Y} (-\rho)\, d\mathcal{L}^{n+1} \quad \forall A \in \mathcal{B}_{\Omega\times Y}.$$

This means that (3.10) is true for any indicator function of a measurable set. It can therefore be extended by linearity to simple functions and by convergence theorems to all measurable, bounded and compactly supported functions u, and this concludes the proof of the lemma.

□

Lemma 3.2.7.
The map ρ has the following properties:

(1) $\rho(x,\cdot) : Y \to (-\infty, 0]$ is non-increasing for $\mathcal{L}^n$-almost all $x \in \Omega$ (up to $\mathcal{L}^{n+1}$-equivalence on ρ).

(2) $(x,y) \in \Omega \times [N, +\infty) \;\Rightarrow\; \rho(x,y) = -1$.

(3) $(x,y) \in \Omega \times (-\infty, N] \;\Rightarrow\; \rho(x,y) = 0$.

(4) $\rho \in L^\infty(\Omega \times Y)$.

Proof of statement (1).

Let $R \subset \Omega \times Y$ be an $(n+1)$-dimensional cube with directions parallel to the $n+1$ axis $x_1, ..., x_n, y$.

Let $\tau_t(x,y) := (x, y+t)$ be a translation along the y direction.

Now using Lemma 3.2.6 and definitions (3.9) and (3.10) we have

$$\begin{aligned}
\int_R \rho(x, y+t)\, d\mathcal{L}^{n+1}(x,y) &= \int_{\tau_t(R)} \rho(x,y)\, d\mathcal{L}^{n+1}(x,y) \\
&= \int_{\Omega \times Y} \mathbb{1}_{\tau_t(R)}(x,y)\rho(x,y)\, d\mathcal{L}^{n+1}(x,y) \\
&= S(\mathbb{1}_{\tau_t(R)}) \\
&= \mu\Big(0, ..., 0, \int_y^{+\infty} \mathbb{1}_{\tau_t(R)}(x,s)\, ds\Big) \\
&= \mu\Big(0, ..., 0, \int_{y-t}^{+\infty} \mathbb{1}_R(x,s)\, ds\Big) \\
&= -\int_{\Omega \times Y \times Z} \int_{y-t}^{+\infty} \mathbb{1}_R(x,s)\, ds\, d\mu.
\end{aligned}$$

Since μ is a positive measure, the last expression is clearly non-increasing in t. Since this is true for all t and all R, this proves that ρ is non-increasing in the y direction.

$\square$

Proof of statements (2) and (3).

Consider now an open set $R_\Omega \subset \Omega$ and two real numbers $a < b$.

Let $R := R_\Omega \times [a, b]$.

Then

$$\int_y^{+\infty} \mathbb{1}_R(x,s)\,ds = \begin{cases} 0 & \text{if } x \notin R_\Omega \\ 0 & \text{if } x \in R_\Omega,\ y \geq b \\ b-y & \text{if } x \in R_\Omega,\ y \in (a,b) \\ b-a & \text{if } x \in R_\Omega,\ y \leq a \end{cases}$$

so that:

- if $a < b \leq -N$, then

$$\int_R \rho\, d\mathcal{L}^{n+1} = -\int_{\Omega\times Y\times Z}\int_y^{+\infty} \mathbb{1}_R(x,s)\,ds\,d\mu(x,y,z) = 0;$$

- if $N \leq a < b$, then

$$\begin{aligned}\int_R \rho\, d\mathcal{L}^{n+1} &= -\int_{\Omega\times Y\times Z}\int_y^{+\infty} \mathbb{1}_R(x,s)\,ds\,d\mu(x,y,z)\\ &= -\int_{R_\Omega\times Y\times Z}(b-a)\,d\mu\\ &= -(b-a)\mathcal{L}^n(R_\Omega).\end{aligned}$$

Using Lebesgue's Differentiation Theorem we get precisely our desired result.

□

Proof of statement (4).
Assume by contradiction that ρ is not essentially bounded.
Then let $B_j := \{(x,y) \in \Omega\times Y :\ \rho(x,y) \leq -j\}\ \forall j \in \mathbb{N}\setminus\{0\}$.
By assumption $\mathcal{L}^{n+1}(B_j) > 0\,\forall j$.
Take $N > 0$ such that $\operatorname{spt}(\mu) \subset \Omega\times(-N,N)\times Z$, as in Lemma **??**.
By the same lemma $\rho(x,\cdot)$ is non-increasing for all x and it is constant on $[N,+\infty)$. Therefore

$$\mathcal{L}^{n+1}\big(B_j \cap (\Omega\times[N,N+1])\big) > 0 \quad \forall j \in \mathbb{N}\setminus\{0\}, \tag{3.11}$$

otherwise $\exists \bar{j}, \exists A \subset \Omega$ such that

$$\mathcal{L}^n(A) = 0 \quad \text{and} \quad \rho(x,y) > -\bar{j} \quad \forall (x,y) \in \big(\Omega \setminus A\big) \times Y,$$

which would imply essential boundedness.
Take then, for all j, a set $A_j^\Omega \subset \Omega$ such that

$$\mathcal{L}^n(A_j^\Omega) > 0 \quad \text{and} \quad A_j^\Omega \times [N, N+1] \subset B_j.$$

Define $A_j := A_j^\Omega \times [N, N+1]$.
Norice that by construction

$$A_j \cap \operatorname{spt}(\mu) = \varnothing \quad \forall j. \tag{3.12}$$

Take, for all j, and open set $U_j^\Omega \subset \Omega$ such that

$$A_j^\Omega \subset U_j^\Omega \quad \text{and} \quad \mathcal{L}^n\big(U_j^\Omega \setminus A_j^\Omega\big) < \frac{\mathcal{L}^n(A_j^\Omega)}{j},$$

which exists by outer regularity of the Lebesgue measure.
Define, for all j, $U_j := U_j^\Omega \times [N, N+1]$. These open sets satisfy

$$A_j \subset U_j \quad \text{and} \quad \mathcal{L}^n\big(U_j \setminus A_j\big) < \frac{\mathcal{L}^n(A_j)}{j} \quad \forall j. \tag{3.13}$$

Define, for all j, the function $f_j := \dfrac{1}{\mathcal{L}^n(A_j} \mathbb{1}_{U_j}$.
Each function f_j trivially satisfies (since $\rho f_j \leq 0$)

$$\int_{\Omega \times Y} \rho f_j \, d\mathcal{L}^{n+1} \leq \int_{A_j} (-j) f_j \, d\mathcal{L}^{n+1} = -j. \tag{3.14}$$

Moreover, by approximation of f_j with functions in $C_c^\infty(\Omega \times Y)$, it is possible to find (for each j) a function $\phi_j \in C_c^\infty(\Omega \times Y)$ such that

$$\int_{\Omega \times Y} \rho \phi_j \, d\mathcal{L}^{n+1} \leq -\frac{j}{2} \tag{3.15}$$

$$\mathcal{L}^n(\pi_\Omega\big(\operatorname{supp}(\phi_j)\big) \leq 2\mathcal{L}^n(A_j^\Omega) \tag{3.16}$$

$$\sup_{x \in \Omega} \int_{-\infty}^{+\infty} \phi_j(x,s) \, ds \leq 2 \frac{1}{\mathcal{L}^n(A_j)} = \frac{2}{\mathcal{L}^n(A_j^\Omega} \tag{3.17}$$

Now we put together lemmas 3.2.4 and 3.2.6, identity (3.10), non-negativity of μ and ϕ_j, and the bounds just proven to get

$$
\begin{aligned}
-\frac{1}{j} &\geq \int_{\Omega\times Y} \rho\phi_j \, d\mathcal{L}^{n+1} \\
&= S(\phi_j) \\
&= \mu\Big(0, ..., 0, \int_{-\infty}^{y} \phi_j(x,s)\, ds\Big) \\
&= -\int_{\Omega\times Y\times Z} \int_{-\infty}^{y} \phi_j(x,s)\, ds\, d\mu \\
&\geq -\int_{\Omega\times Y\times Z} \int_{-\infty}^{+\infty} \phi_j(x,s)\, ds\, d\mu \\
&\geq -\int_{\Omega\times Y\times Z} \frac{2}{\mathcal{L}^n(A_j^\Omega)} \mathbb{1}_{\pi_\Omega \operatorname{supp}(\phi_j)}(x)\, d\mu(x,y,z) \\
&= -\frac{2}{\mathcal{L}^n(A_j^\Omega)} \pi_{\Omega\,\#}\mu\big(\operatorname{supp}(\phi_j)\big) \\
&= -\frac{2}{\mathcal{L}^n(A_j^\Omega)} c\, \mathcal{L}^n\big(\operatorname{supp}(\phi_j)\big) \\
&\geq -4c.
\end{aligned}
$$

Here c is independent of j, being it the constant given by Lemma 3.2.4, and we have therefore reached a contradiction.
We conclude that ρ must be $\mathcal{L}^{n+1}$-essentially bounded.

$\square$

Notation 3.2.1.
From this point on we shall

- Consider ρ to be bounded (taking a suitable representative in its class of essentially bounded maps);
- Denote the range of ρ by I (i.e. $I := \big[\inf_{\Omega\times Y} \rho(x,y),\ 0\big] = [-1, 0]$.
- Denote by ν the restriction $\mathcal{L}^1 \llcorner I$.

Lemma 3.2.8.
Let $X : \Omega \times Y \to \mathbb{R}^{n+1}$ be a smooth and compactly supported vector field. Assume that X vanishes in a neighborhood of $\partial\Omega \times Y$.
Then
$$\mu(X) = S\big(-\operatorname{div}(X)\big).$$

Proof of Lemma 3.2.8.
Let
$$\tilde{X} := X + \left(0, ..., 0, \int_y^{+\infty} \operatorname{div}(X)(x,s)\, ds\right).$$
Indeed
$$\begin{aligned}\operatorname{div}(\tilde{X}) &= \sum_{i=1}^{n} \frac{\partial X}{\partial x_i} + \frac{\partial}{\partial y}\left(X_{n+1} + \int_y^{+\infty} \operatorname{div}(X)(x,s)\, ds\right)\\ &= \operatorname{div}(X) - \operatorname{div}(X)\\ &= 0.\end{aligned}$$
So Lemma 3.2.5 gives
$$\begin{aligned}0 &= \mu(\tilde{X})\\ &= \mu(X) + \mu\left(\left(0, ..., 0, \int_y^{+\infty} \operatorname{div}(X)\right)\right)\\ &= \mu(X) + S\big(\operatorname{div}(X)\big)\\ &= \mu(X) - S\big(-\operatorname{div}(X)\big).\end{aligned}$$
□

Lemma 3.2.9.
The map ρ is in $BV(\Omega \times Y)$.

Proof of Lemma 3.2.9.
Let $X \in C_c^1(\Omega \times Y, \mathbb{R}^{n+1})$ such that $\sup_{(x,y)\in\Omega\times Y} ||X(x,y)|| \leq 1$.
We know that

- being compactly supported, X vanishes in a neighborhood of $\partial\Omega \times Y$;
- our hypothesis on μ implies
$$\int_{\Omega\times Y\times Z} ||z||\, d\mu(x,y,z) < \infty$$
(see section 3.1.6);

- an explicit representation for $\mu(\cdot), S(\cdot)$ are given by (3.9) and (3.10);

- $\mu(X), S(\operatorname{div}(X))$ and ρ are linked by lemmas 3.2.6 and 3.2.8;

- by Lemma 3.2.7 ρ is essentially bounded;

- for any non-negative real number t, the inequality$\sqrt{1+t^2} \leq 1+t$ holds.

These facts put together plus the use of Cauchy-Schwarz inequality tell us that

$$\left| \int_{\Omega\times Y} \rho \operatorname{div}(X)\, d\mathcal{L}^{n+1} \right| = |S(\operatorname{div}(X))| = |\mu(-X)| =$$
$$= \left| \int_{\Omega\times Y\times Z} \langle -X(x,y), (z,-1)\rangle\, d\mu \right| \leq \int_{\Omega\times Y\times Z} \Big(1 + ||z||\Big)\, d\mu < \infty.$$

This holds for any ϕ considered, so that

$$\sup_{\substack{\phi\in C^1_c(\Omega\times Y)\\ ||\phi||_\infty \leq 1}} \int_{\Omega\times Y} \rho \operatorname{div}(X)\, d\mathcal{L}^{n+1} \leq \int_{\Omega\times Y\times Z} \Big(1 + ||z||\Big)\, d\mu < \infty,$$

which by definition means that $\rho \in BV(\Omega\times Y)$.

□

Remark 1.
By the coarea formula (cfr. [2, Theorem 1, p.185]), $\exists A \in I$ such that

- $\mathcal{L}^1(I \setminus A) = 0$,
- For all $r \in A$, it holds

$$\sup_{\substack{\phi \in C^1_c(\rho^{-1}\big((-\infty,r)\big)\\ ||\phi||_\infty \leq 1}} \int_{\rho^{-1}\big((-\infty,r)\big)} \operatorname{div}(X)\, d\mathcal{L}^{n+1} < \infty,$$

i.e. $\rho^{-1}\big((-\infty,r)\big)$ is a set of finite perimeter (crf. [2, def 5.1]).

Using [9, prop. 12.1, p.122], we deduce the existence of a set $\{\nu_r\}_{r\in A}$ of $\mathbb{R}^{n+1}$ valued measures on $\mathbb{R}^{n+1}$ such that for all r the measure ν_r has the properties

- $\displaystyle\int_{\rho^{-1}\big((-\infty,r)\big)} \operatorname{div}(\phi)\, d\mathcal{L}^{n+1} = \int_{\mathbb{R}^{n+1}} \langle \phi, d\nu_r\rangle \quad \forall \phi \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$;
- $|\nu_r|(\mathbb{R}^{n+1}) < \infty$.

By De Giorgi's Structure Theorem (cfr. [9, Theorem 15.9, p.170]) the measures ν_r and the sets $E_r := \rho^{-1}\big((-\infty,r)\big)$ satisfy that

- $\operatorname{spt}(\nu_r) = \overline{\partial^* E_r} \subset \partial E_r \subset \mathbb{R}^{n+1}$ and $\mathcal{H}^n\big(\partial E_r \setminus \operatorname{spt}(\nu_r)\big) = 0$;
- $\nu_r = \eta_r(\cdot)\mathcal{H}^n \llcorner \partial^* E_r$ where

$$\begin{aligned} \eta_r :\ \partial^* E_r &\longrightarrow \mathbb{S}^n \\ x &\mapsto \lim_{t\to 0^+} \frac{\nu_r\big(\mathbf{B}(x,t)\big)}{|\nu_r|\big(\mathbf{B}(x,t)\big)} \end{aligned}$$

is the measure theoretic outer unit normal to E_r (cfr. [9, p.167]).

- The generalized Gauss-Green formula

$$\int_{E_r} \nabla\phi\, d\mathcal{L}^{n+1} = \int_{\partial^* E_r} \phi(x,y)\eta_r(x,y)\, d\mathcal{H}^n(x,y) \tag{3.18}$$

holds for all $\phi \in C^1_c(\Omega \times Y)$.

The generalized Gauss-Green formula is actually equivalent with the usual Gauss-Green formula

$$\int_{E_r} \operatorname{div}(X)\, d\mathcal{L}^{n+1} = \int_{\partial^* E_r} \langle X, \eta_r\rangle\, d\mathcal{H}^n \quad \forall X \in C^1_c(\Omega\times Y, \mathbb{R}^{n+1}). \tag{3.19}$$

Lemma 3.2.10.
Let $r \leq r' \leq 0$.
Let $(x, y) \in \partial^* E_r \cap \partial^* E_{r'}$.
Then

$$\eta_r(x, y) \;=\; \eta_{r'}(x, y).$$

Proof of Lemma 3.2.10.
Throughout this proof, for simplicity, each variable x or y will represent a vector in $\mathbb{R}^{n+1}$, differently from the statement of the lemma.
We recall [2, def. 5.7.2, p.198], by which for any $x \in \partial^* E_r$ we set

$$H_r(x) := \big\{ y \in \mathbb{R}^{n+1} : \langle \eta_r(x), y - x \rangle = 0 \big\},$$

$$H_r^+(x) := \big\{ y \in \mathbb{R}^{n+1} : \langle \eta_r(x), y - x \rangle \geq 0 \big\},$$

$$H_r^-(x) := \big\{ y \in \mathbb{R}^{n+1} : \langle \eta_r(x), y - x \rangle \leq 0 \big\},$$

$$E_r^{(\varepsilon)}(x) := \big\{ y \in \mathbb{R}^{n+1} : x + \varepsilon(y - x) \in E_r \big\} \quad \forall \varepsilon > 0.$$

By [2, Theorem 1, p.199]

$$\mathbb{1}_{E_r^{(\varepsilon)}(x)} \to \mathbb{1}_{H^-(x)} \;\text{ in } L^1_{\text{loc}}(\mathbb{R}^{n+1}) \tag{3.20}$$

as $\varepsilon \to 0^+$.
We know that

- $E_{r'} \subset E_r$ (obvious by definition of $E_r, E_{r'}$);
- By [2, corollary 1, p.203], for all $x \in \partial^* E_r$, the vector $\eta_r(x)$ is the exterior outer normal to E_r at x (as defined also in [4, 4.5.5, p.477]).

The inclusion $E_r \subset E_{r'}$ and the convergence (3.20) tell that

- $E_r^{(\varepsilon)}(x) \subset E_{r'}^{(\varepsilon)}(x)$ $\forall \varepsilon > 0$, so that $\mathbb{1}_{E_r^{(\varepsilon)}(x)} \leq \mathbb{1}_{E_{r'}^{(\varepsilon)}(x)}$.
- $\mathbb{1}_{E_r^{(\varepsilon)}(x)} \to \mathbb{1}_{H_r^-(x)\}}$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$.
  $\mathbb{1}_{E_{r'}^{(\varepsilon)}(x)} \to \mathbb{1}_{H_{r'}^-(x)\}}$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$.

This means that $\mathbb{1}_{H_r^-(x)} \leq \mathbb{1}_{H_{r'}^-(x)}$, and since $H_r^-(x)$ and $H_{r'}^-(x)$ are both semispaces, then they must coincide or have parallel boundaries. In both cases their defining vectors $\eta_r(x)$ and $\eta_{r'}(x)$ must coincide.

$\square$

Remark 2.

Lemma 3.2.10 tells that the two maps

$$\begin{array}{rccl} \mathcal{E}: & \Omega \times Y & \longrightarrow & \mathbb{R}^{n+1} \\ & (x,y) & \mapsto & \begin{cases} \eta_r(x,y) & \text{if } y = \varphi_r(x)\ \exists r \in I \\ 0 & \text{otherwise} \end{cases} \end{array}$$

and (in particular)

$$\begin{array}{rccl} \mathcal{E}_{n+1}: & \Omega \times Y & \longrightarrow & \mathbb{R} \\ & (x,y) & \mapsto & \begin{cases} [\eta_r(x,y)]_{n+1} & \text{if } y = \varphi_r(x)\ \exists r \in I \\ 0 & \text{otherwise} \end{cases} \end{array}$$

are well defined $\mathcal{L}^{n+1}$-a.e. .

Lemma 3.2.11.
Let $\mu_{\Omega\times Y} := \pi_{\Omega\times Y\,\#}\mu$.
Remember that $I := [-1, 0]$ *and* $\nu := \mathcal{L}^1 \llcorner I$.
Then:

(1) *If* $g \in L^1\big(\mathcal{L}^{n+1}\llcorner(\Omega\times Y)\big)$*, then the following integration formula holds:*

$$\int_I \int_{E_r} g(x, y)\, d\mathcal{L}^{n+1}(x, y)\, d\nu(r) = -\int_{\Omega\times Y} \rho g\, d\mathcal{L}^{n+1}. \tag{3.21}$$

(2) *For any Borel set* B *of* $\Omega\times Y$ *the measure* $\mu_{\Omega\times Y}$ *of* B *can be expressed as*

$$\mu_{\Omega\times Y}(B) = -\int_I \int_{\partial^* E_r} \mathbb{1}_B(x, y)[\eta_r(x)]_{n+1}\, d\mathcal{H}^n(x, y)\, d\nu(r). \tag{3.22}$$

(3) *If* $f \in L^1(\mu_{\Omega\times Y})$*, then*

$$\int_{\Omega\times Y} f\, d\mu_{\Omega\times Y} \;=\; -\int_I \int_{\partial^* E_r} f\, [\eta_r(\cdot)]_{n+1}\, d\mathcal{H}^n\, d\nu(r). \tag{3.23}$$

(4) *Define, for all* $r \in I$, $A_r^0 := \Big\{(x, y) \in \partial^* E_r : [\eta_r(x, y)]_{n+1} = 0\Big\}$.
Then

$$\mathcal{H}^n(A_r^0) = 0 \quad \textit{for } \nu\textit{-a.e. } r.$$

(5) *Define, for all* $r \in I$, $A_r^+ := \Big\{(x, y) \in \partial^* E_r : [\eta_r(x, y)]_{n+1} > 0\Big\}$.
Then

$$\mathcal{H}^n(A_r^+) = 0 \quad \textit{for } \nu\textit{-a.e. } r.$$

(6) $\forall\varepsilon > 0$, $\exists\delta_\varepsilon > 0$ *such that* $\mu_{\Omega\times Y}\Big(\mathcal{E}_{n+1}^{-1}\big((0, \delta_\varepsilon)\big)\Big) < \varepsilon$.

(7) *Define*

$$\begin{array}{rccl} J: & \Omega\times Y & \longrightarrow & \mathbb{R}\cup\{+\infty\} \\ & (x, y) & \mapsto & \begin{cases} -\dfrac{1}{\mathcal{E}(x, y)} & \textit{if } \mathcal{E}(x, y) < 0 \\ +\infty & \textit{otherwise} \end{cases} . \end{array}$$

Then

$$J \in L^1(\mu_{\Omega\times Y}).$$

(8) If $f \in L^\infty(\Omega \times Y)$, then

$$\int_I \int_{\partial^* E_r} f \, d\mathcal{H}^n \, d\nu(r) = \int_{\Omega \times Y} f J \, d\mu_{\Omega \times Y}$$

(9) If $B \subset \Omega$ is a Borel set, then the following implication holds:

$$\mathcal{L}^n(B) = 0 \Rightarrow \int_I \int_{\partial^* E_r} \mathbb{1}_B(x) \, d\mathcal{H}^n \, d\nu(r) = 0.$$

Proof of statement (1).

We prove this formula in an elementary way, using approximation with simple functions.

Without loss of generality assume $g \geq 0$ (the general case follows from the usual argument of writing a general map g as the sum $g^+ - g^-$).

For any $k \in \mathbb{N}$ define:

- $\rho^{(k)}(x, y) := \displaystyle\sum_{i=1}^{2^k} -\frac{i}{2^k} \mathbb{1}_{\rho^{-1}([-\frac{i}{2^k}, -\frac{i-1}{2^k}))}(x, y).$
- $\Phi_g(r) := \displaystyle\int_{E_r} g(x, y) \, d\mathcal{L}^{n+1}.$
- $N_k(r) := \dfrac{\lfloor 2^k r \rfloor}{2^k} \quad \forall r \in [-1, 0].$
- $\Phi_g^{(k)}(r) := \displaystyle\int_{E_{N_k(r)}} g \, d\mathcal{L}^{n+1}.$

We have the following list of elementary properties which hold for any $k \in \mathbb{N}$:

- $||\rho^{(k)} - \rho||_{L^\infty(\Omega \times Y)} \leq \dfrac{1}{2^k}.$
- $N_k(r) = -\dfrac{i}{2^k} \Leftrightarrow r \in \left[-\dfrac{i}{2^k}, -\dfrac{i-1}{2^k}\right).$
- $N_k(r) \in \left[r - \dfrac{1}{2^k}, r\right] \quad \forall r \in [-1, 0].$

- $N_k(r) \uparrow r$ as $k \to \infty$.
- $0 \le \Phi_g^{(k)} \le \Phi_g^{(k+1)} \le \Phi_g$.
- $\Phi_g^{(k)}(r) \xrightarrow{k\to\infty} \Phi_g(r) \quad \forall r$.

To conclude we have the three following facts:

(1): It holds

$$\left| \int_{\Omega\times Y} \rho g \, d\mathcal{L}^{n+1} - \int_{\Omega\times Y} \rho^{(k)} g \, d\mathcal{L}^{n+1} \right| \le \frac{1}{2^k} ||g||_{L^1} \xrightarrow{k\to\infty} 0.$$

(2): By Beppo Levi's Theorem we have

$$\int_{[-1,0]} \Phi_g^{(k)}(r) \, dr \xrightarrow{k\to\infty} \int_{[-1,0]} \Phi_g(r) \, dr.$$

(3): We have

$$\begin{aligned}
\int_{[-1,0]} \Phi_g^{(k)} &= \int_{[-1,0]} \int_{E_{N_k(r)}} g \, d\mathcal{L}^{n+1} \, dr \\
&= \int_{[-1,0]} \sum_{j=1}^{2^k} \left(\mathbb{1}_{\left[-\frac{j}{2^k}, -\frac{j-1}{2^k}\right)}(r) \int_{E_{-j/2^k}} g \, d\mathcal{L}^{n+1} \right) dr \\
&= \sum_{j=1}^{2^k} \frac{1}{2^k} \int_{\rho^{-1}\left([-1,-j/2^k)\right)} g \, d\mathcal{L}^{n+1} \\
&= \sum_{j=1}^{2^k} \frac{1}{2^k} \sum_{i=j}^{2^k-1} \int_{\rho^{-1}\left(\left[-\frac{j+1}{2^k}, -\frac{j}{2^k}\right)\right)} g \, d\mathcal{L}^{n+1} \\
&= \int_{\Omega\times Y} \left(\sum_{i=1}^{2^k-1} \frac{j}{2^k} g \mathbb{1}_{\rho^{-1}\left(\left[-\frac{j+1}{2^k}, -\frac{j}{2^k}\right)\right)} \right) d\mathcal{L}^{n+1} \\
&= \int_{\Omega\times Y} -\rho^{(k)} g \, d\mathcal{L}^{n+1} \; - \; \frac{1}{2^k} \int_{\Omega\times Y} g \, d\mathcal{L}^{n+1}
\end{aligned}$$

□

Proof of statements (2) and (3).

We can see that if $X \in C^1_c(\Omega \times Y, \mathbb{R}^{n+1})$ then

$$
\begin{aligned}
\int_I \int_{\partial^* E_r} \langle X, \eta_r \rangle \, d\mathcal{H}^n \, d\nu(r) &= \int_I \int_{E_r} \operatorname{div}(X) \, d\mathcal{L}^{n+1} \, d\nu(r) \\
&= - \int_{\Omega \times Y} \rho \operatorname{div}(X) \, d\mathcal{L}^{n+1} \\
&= -S(\operatorname{div}(X)) \\
&= \mu(X) \\
&= \int_{\Omega \times Y \times Z} \langle X(x,y), (z,-1) \rangle \, d\mu \qquad (3.24)
\end{aligned}
$$

Since the set $C^1_c(\Omega \times Y)$ is dense in $L^1(\mu_{\Omega \times Y})$, identity (3.24) holds for any map in $L^1(\Omega \times Y)$. Therefore it also holds for any $\mu_{\Omega \times Y}$-integrable map taking values in $\mathbb{R}^{n+1}$.

Let now $A \in \mathcal{B}_{\Omega \times Y}$ and define $X_A := \Big(0, ..., 0, \mathbb{1}_A\Big)$.

Then

$$
\begin{aligned}
-\mu(A \times Z) &= - \int_{\Omega \times Y \times Z} \mathbb{1}_A(x,y) \, d\mu(x,y,z) \\
&= \int_{\Omega \times Y \times Z} \langle X_A(x,y), (z,-1) \rangle \, d\mu \\
&= \int_I \int_{\partial^* E_r} \langle X_A, \eta_r \rangle \, \mathcal{H}^n \, d\nu(r) \\
&= \int_I \int_{\partial^* E_r} \mathbb{1}_A(x,y) \, [\eta_r(x,y)]_{n+1} \, d\mathcal{H}^n(x,y) \, d\nu(r)
\end{aligned}
$$

where $[\eta_r(x,y)]_{n+1}$ denotes the $(n+1)$-th component of the vector $\eta_r(x,y)$. To extend this formula to any $f \in L^1(\Omega \times Y)$, simply use the vector field $X_f := (0, ..., 0, f)$ in place of X_A above.

□

Proof of statement (4).

Let

$$A := \bigcup_{r \in I} A_r.$$

Then

$$\mu_{\Omega\times Y}(A) = -\int_I\int_{\partial^* E_r} \mathbb{1}_A(x,y)[\eta_r(x,y)]_{n+1}\,\mathcal{H}^n(x,y)\,d\nu(r) = 0.$$

To conclude we see that by definition $\langle \eta_r, \eta_r\rangle = \mathbb{1}_{\partial^* E_r}$, so that using (3.24) we can write

$$\begin{aligned}\int_I\int_{\partial^* E_r} \mathbb{1}_A(x,y)\,d\mathcal{H}^n(x,y)\,d\nu(r) &= \int_I\int_{\partial^* E_r} \langle \mathbb{1}_A\mathcal{E}, \eta_r\rangle\,d\mathcal{H}^n\,d\nu(r)\\ &= \int_{\Omega\times Y\times Z} \langle \mathbb{1}_A(x,y)\mathcal{E}(x,y), (z,-1)\rangle\,d\mu(x,y,z)\\ &= 0,\end{aligned}$$

where $\mathcal{E}$ is the map defined in Remark 2.
This concludes the proof.

□

Proof of statement (5).
By contradiction assume that there is a subset P of I having ν positive measure and whose elements r satisfy

$$\mathcal{H}^n\Big(\big\{(x,y)\in\partial^* E_r :\ [\eta_r(x,y)]_{n+1} > 0\big\}\Big) > 0.$$

For all such r define the set $B_r := \big\{(x,y)\in\partial^* E_r :\ [\eta_r(x,y)]_{n+1} > 0\big\}$.
Let $B := \bigcup_{r\in P} B_r$.
Then

$$\mu_{\Omega\times Y}(B) \;= -\int_I\int_{\partial^* E_r} \mathbb{1}_A[\eta_r(\cdot)]_{n+1}\,d\mathcal{H}^n\,d\nu(r) < 0.$$

Since $\mu_{\Omega\times Y}$ is a positive measure, this is a contradiction and the claim is proven.

□

Proof of statement (6).

This is trivial. Follows from finiteness of $\mu_{\Omega\times Y}$ and continuity from above of the measure.

$\square$

Proof of statement (7).

For all $j \in \mathbb{N}\setminus\{0\}$ and all $(x,y)\in\Omega\times Y$, define

$$g^{(j)}(x,y) := \begin{cases} -\dfrac{1}{\mathcal{E}(x,y)} & \text{if } \mathcal{E}(x,y) < -\dfrac{1}{j} \\ j & \text{otherwise} \end{cases}$$

Then we have

- $g^{(j)}$ is bounded, and more precisely $g^{(j)}(x,y)\in(0,j]\ \forall(x,y)$
- $g^{(j+1)} \le g^{(j)} \quad \forall j$;
- for all $(x,y)\in\Omega\times Y$, we have that $g^{(j)}(x,y) \xrightarrow{n\to\infty} J(x,y)$.
- since they are bounded, each $g^{(j)}$ satisfies (3.23), and therefore any bounded measurable $f:\Omega\times Y\to\mathbb{R}$ satisfies

$$\int_{\Omega\times Y} fg^{(j)}\,d\mu_{\Omega\times Y} = -\int_I\int_{\partial^* E_r} fg^{(j)}\mathcal{E}_{n+1}\,d\mathcal{H}^n\,d\nu(r).$$

Now

$$\begin{aligned}\int_{\Omega\times Y} g^{(j)}\,d\mu_{\Omega\times Y} &= \int_I\left(\int_{\{\mathcal{E}_{n+1}<-\frac{1}{j}\}\cap\partial^*E_r} 1\,d\mathcal{H}^n + \int_{\{\mathcal{E}_{n+1}\in(-\frac{1}{j},0)\}\cap\partial^*E_r} -j[\eta(\cdot)]_{n+1}\,d\mathcal{H}^n\right)d\nu(r)\\ &\le \int_I\left(\int_{\partial^*E_r} 1\,d\mathcal{H}^n\right)d\nu\\ &= ||D\rho||(\Omega\times Y)<\infty.\end{aligned}$$

Now using the Monotone Convergence Theorem, we can see that

$$\int_{\Omega\times Y} J\,d\mu_{\Omega\times Y} = \lim_{j\to\infty}\int_{\Omega\times Y} g^{(j)}\,d\mu_{\Omega\times Y} \le ||D\rho||(\Omega\times Y)<\infty$$

and the proof is concluded.

□

Proof of statement (8).
Use (7) to deduce that $fJ \in L^1(\mu_{\Omega\times Y})$ and use (3) to conclude.

□

Proof of statement (9).
Let $B \subset \Omega$ be a Borel subset with zero $\mathcal{L}^n$ measure.
Then $\mu(B \times Y \times Z) = \pi_{\Omega\,\#}\mu(B) = c\mathcal{L}^n(B) = 0$.
Then we have

$$\begin{aligned}\int_I \int_{\partial^* E_r} \mathbb{1}_B(x)\, d\mathcal{H}^n\, d\nu(r) &= \int_I \int_{\partial^* E_r} \big\langle \mathbb{1}_B(x)\mathcal{E}(x,y),\ \eta_r(x,y)\big\rangle\, d\mathcal{H}^n\, d\nu(r) \\ &= \int_{\Omega\times Y\times Z} \big\langle \mathbb{1}_B(x)\mathcal{E}(x,y),\ (z,-1)\big\rangle\, d\mu(x,y,z) \\ &= 0.\end{aligned}$$

□

In the next lemma we use the definition of measure theoretical boundary $(\partial_* E_r)$ which can be found in [2, p.208]. This set is $\mathcal{H}^n$-equivalent to the reduced boundary $\partial^* E_r$ (cfr. [2, Lemma 1, p.208]), but its definition is much more practical in this context.

Lemma 3.2.12.

Let $N > 0$ be such that $\mathrm{spt}(\mu) \subset \Omega \times [-N, N] \times Z$.
Define, for all $r \in [-1, 0]$, $E_r := \{(x, y) \in \Omega \times Y : \rho(x, y) < r$.
Define, for all $x \in \Omega$,

$$\varphi_r(x) := \begin{cases} \inf\{y : \rho(x, y) < r\} & \textit{if } \{y : \rho(x, y) < r\} \neq \varnothing \\ +\infty & \textit{otherwise} \end{cases}$$

Then:

(1) E_r is given by the union

$$E_r = \bigcup_{x \in \Omega} r_x$$

where r_x depends on ρ but must be either the open half-line $(\varphi_r(x), +\infty)$ or the close half-line $[\varphi_r(x), +\infty)$.

(2) If $y < -N$, then for all $r \in [-1, 0)$ we have

$$\Theta^{n+1}\Big(\mathcal{L}^{n+1} \llcorner E_r,\ (x, y)\Big) = 0.$$

(3) If $y > N$, then for all $r \in [-1, 0]$ we have

$$\Theta^{n+1}\Big(\mathcal{L}^{n+1} \llcorner E_r,\ (x, y)\Big) = 1.$$

(4) If $x \in \Omega$ and $y_1, y_2 \in Y$ with $y_1 < y_2$, then

$$\Theta^{n+1}\Big(\mathcal{L}^{n+1} \llcorner E_r,\ (x, y_1)\Big) \leq \Theta^{n+1}\Big(\mathcal{L}^{n+1} \llcorner E_r,\ (x, y_2)\Big).$$

(5) The following implication about the measure theoretical boundary $\partial_* E_r$ holds:

$$\left.\begin{array}{l} y_1, y_2 \in Y \\ y_1 < y_2 \\ (x, y_1) \in \partial_* E_r \\ (x, y_2) \in \partial_* E_r \end{array}\right\} \Rightarrow \{x\} \times [y_1, y_2] \subset \partial_* E_r.$$

(6) *The following double implication holds:*

$$\mathrm{Card}\Big((\{x\}\times Y)\cap\partial_* E_r\Big)>1 \Leftrightarrow \mathcal{L}^1\Big((\{x\}\times Y)\cap\partial_* E_r\Big)>0.$$

(7) *The set*

$$P_r := \Big\{x\in\Omega:\ \mathrm{Card}\Big((\{x\}\times Y)\cap\partial_* E_r\Big)>1\Big\}$$

has zero $\mathcal{L}^n$-measure.

(8) *The map*

$$\begin{array}{rccl}\psi_r: & \Omega & \longrightarrow & Y\\ & x & \mapsto & \inf\Big\{y:\ \Theta^{n+1}\Big(\mathcal{L}^{n+1}\llcorner E_r,\ (x,y)\Big)=1\Big\}\end{array}$$

is well defined for $\mathcal{L}^n$-a.e. x.

(9) *The following implication about the measure theoretical boundary $\partial_* E_r$ holds:*

$$\left.\begin{array}{l} x\in\Omega\\ \mathrm{Card}\Big((\{x\}\times Y)\cap\partial_* E_r\Big)>0\\ (x,y)\in\partial_* E_r\\ \eta_r \textit{ is well defined at } (x,y)\end{array}\right\}\Rightarrow\ [\eta_r(x,y)]_{n+1}=0.$$

(10) *Define, for all $r\in[-1,0]$, the set $A_{r,\Omega}\subset\Omega$ as*

$$A_{r,\Omega} := \Big\{x\in\Omega:\ \mathrm{Card}\Big((\{x\}\times Y)\cap\partial_* E_r\Big)>1\Big\}$$

and the set

$$A_r := \Big(A_{r,\Omega}\times Y\Big)\cap\partial_* E_r.$$

Then for ν-a.e. r, we have

$$\mathcal{H}^n(A_r)=0.$$

(11) *For ν-a.e. r, we have*

$$\mathcal{H}^n\Big(\partial_* E_r\setminus \mathrm{Gr}(\psi_r)\Big)=0.$$

(12) *Define, for all* $r \in I$, $E_r^{(\psi)} := \{(x,y) \in \Omega \times Y : y \geq \psi_r(x)\}$.
Then

$$\mathcal{L}^{n+1}\left(E_r \Delta E_r^{(\psi)}\right) = 0.$$

Proof of statement (1).
This is an immediate consequence of ρ being non-increasing along the y direction.

□

Proof of statements (2) and (3).
These two statements follow from (3) and (2) of Lemma 3.2.11.

□

Proof of statement (4).
Follows from (1) and the definition of density, through obvious inclusions.

□

Proof of statements (5), (6) and (7).
(5) is obvioius from the definition of $\partial_* E_r$ (cfr. [2]) and (4).
(6) is a direct consequence of (5).
Now (7) must be true as well, otherwise using (5) and the coarea formula we could deduce that $\mathcal{L}^{n+1}(\partial_* E_r) > 0$, which can not be true, since $\mathcal{H}^n(\partial_* E_r) < \infty$.

□

Proof of statement (7).

□

Proof of statement (8).
To be sure that ψ_r is well defined it is enough to remember (2) and (3), which actually further tell us that $\psi_r(x) \in [-N, N]$ for $\mathcal{L}^n$-a.e. x.

□

Proof of statement (9).

The Structure Theorem for sets of finite perimeter (cfr. [2, Theorem 2, p.205] and its proof) tells us that, up to excluding the points (x, y) of a set N with zero $\mathcal{H}^n$-measure, there are a locally C^1 hypersurface S and a neighborhood U of (x, y) in $\Omega \times Y$ such that

- $U \subset S$,
- $\eta_r(x, y)$ is normal to S at the point (x, y).

Since the neighborhood U must contain a segment which passes through x and which is parallel to the y direction, the normal vector $\eta_r(x, y)$ must be orthogonal to the y direction, and this is equivalent to saying that $[\eta_r(x, y)]_{n+1} = 0$.

□

Proof of statements (10) and (11).

(9) tells us that the set A_r is contained in the set A_r^0 defined in (4) of Lemma 3.2.11. Then (4) of Lemma 3.2.11 itself implies that

$$\mathcal{H}^n(A_r) = 0 \quad \text{for } \nu\text{-a.e. } r,$$

and (10) is proved.

To see that (11) is true we notice that if $\mathrm{Card}(\{x\} \times Y) = 1$, then $\{x\} \times Y = \{(x, \psi_r(x))\}$. This means that $\partial_* E_r \setminus \mathrm{Gr}(\psi_r) \subset A_r$ and the thesis follows from (10).

□

Proof of statement (12).

By (7) we have that $P_r \subset \Omega$ has zero $\mathcal{L}^n$ measure.

Moreover, because of the definition of ψ_r and because of (4), the following two implications hold:

$$\left.\begin{array}{l} x \in \Omega \setminus P_r \\ y < \psi_r(x) \end{array}\right\} \Rightarrow \Theta^{n+1}\big(\mathcal{L}^{n+1} \llcorner E_r,\ (x, y)\big) = 0$$

and

$$\left.\begin{array}{l} x \in \Omega \setminus P_r \\ y > \psi_r(x) \end{array}\right\} \Rightarrow \Theta^{n+1}\big(\mathcal{L}^{n+1} \llcorner E_r^C,\, (x,y)\big) = 0.$$

The first implication tells that $\mathcal{L}^n$-almost all points of $\big(E_r^{(\psi)}\big)^C$ have null $\mathcal{L}^{n+1} \llcorner E_r$ density.

The second one tells that $\mathcal{L}^n$-almost all points of $E_r^{(\psi)} \setminus \mathrm{Gr}(\psi_r)$ have null $\mathcal{L}^{n+1} \llcorner E_r^C$ density.

By [4, 2.10.19(1), p.181], this means that

$$\mathcal{L}^{n+1}(E_r \setminus E_r^{(\psi)}) = 0$$

and

$$\mathcal{L}^{n+1}(E_r^{(\psi)} \setminus E_r) = \mathcal{L}^{n+1}(\mathrm{Gr}(\psi_r) \setminus E_r) = 0.$$

□

Lemma 3.2.13.

(1) If $M \in M_n(\mathbb{R})$ is an $n \times n$ symmetric matrix of the form

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \\ a_n & 0 & \cdots & & 1 \end{pmatrix},$$

then

$$\det(M) = a_1 - a_2^2 - a_3^2 - \ldots - a_n^2 = a_1 - \sum_{j=2}^{n} a_j^2.$$

(2) If a matrix A is in $GL_n(\mathbb{R})$ and it is of the form

$$A = \begin{pmatrix} | & | & & | \\ \mathbf{e}_1 - \alpha_1 v & \mathbf{e}_2 - \alpha_2 v & \cdots & \mathbf{e}_n - \alpha_n v \\ | & | & & | \end{pmatrix}$$

with

$$v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n \quad \textit{and} \quad \alpha_1 \neq 0,$$

then

$$\det(A) = 1 - \alpha_1^2 - \alpha_2^2 - \ldots - \alpha_n^2.$$

(3) *For any* $B \subset H\big(x, \psi_r(x)\big)$ *we have*

$$\mathcal{H}^n(B) = \frac{1}{\big|[\eta_r\big(x, \psi_r(x)\big)]_{n+1}\big|} \mathcal{L}^n\big(\pi_\Omega(B)\big). \tag{3.25}$$

(4) *Define the map*

$$\begin{array}{rccc} \zeta_r: & \Omega & \to & \mathbb{R}^n \\ & x & \mapsto & \zeta_r(x) \end{array}$$

where

$$[\zeta_r(x)]_i := \begin{cases} 0 & \textit{if } \big(x, \psi_r(x)\big) \notin \partial^* E_r \\ -\dfrac{[\eta_r\big(x, \psi_r(x)\big)]_i}{[\eta_r\big(x, \psi_r(x)\big)]_{n+1}} & \textit{if } [\eta_r\big(x, \psi_r(x)\big)]_{n+1} \neq 0 \\ \operatorname{sgn}([\eta_r\big(x, \psi_r(x)\big)]_i)\infty & \textit{otherwise} \end{cases}$$

Then, for any x *such that* $\big(x, \psi_r(x)\big) \in \partial^* E_r$*, the map*

$$y \mapsto \big(y,\ \psi_r(x) + \langle y - x, \zeta_r(x)\rangle$$

is the map that lifts Ω *vertically onto the hyperplane* $H\big(x, \psi_r(x)\big)$.

(5) *If* $x_0 \in \Omega$, $\big(x_0, \psi_r(x_0)\big) \in \partial^* E_r$ *and* $\big[\eta_r\big(x_0, \psi_r(x_0)\big)\big]_{n+1} \neq 0$*, then*

$$\lim_{t \to 0^+} \int_{\mathbf{B}(x_0,1)} \left| \frac{\psi_r\big(x_0 + t(x - x_0)\big) - y_0}{t} - \big\langle x - x_0, \zeta_r(x_0)\big\rangle \right| d\mathcal{L}^n(x) = 0 \tag{3.26}$$

(6) *If* $x \in \Omega$, $\big(x, \psi_r(x)\big) \in \partial^* E_r$ *and* $\big[\eta_r\big(x, \psi_r(x)\big)\big]_{n+1} \neq 0$*, then*

$$\lim_{t \to 0^+} \frac{1}{t^{n+1}} \int_{\mathbf{B}(x,t)} \big|\psi_r(y) - \psi_r(x) - \langle y - x, \zeta_r(x)\rangle\big| \, d\mathcal{L}^n(y) = 0. \tag{3.27}$$

(7) *Define, for all $r \in I$, the map*

$$p^{(r)}: \begin{array}{ccc} \Omega & \longrightarrow & \Omega \times Y \\ x & \mapsto & \big(x, \psi_r(x)\big) \end{array}$$

Then, for ν-a.e. r, the measure $\mathcal{H}^n \llcorner \partial^* E_r$ is absolutely continuous with respect to the measure $p^{(r)}_{\#}\Big(\mathcal{L}^n \llcorner \pi_\Omega\big(\partial^* E_r\big)\Big)$.

In other words there exist maps $J_r : \Omega \to [0, +\infty)$ such that for any integrable $f : \partial^* E_r \to \mathbb{R}$ we have

$$\int_{\partial^* E_r} f \, d\mathcal{H}^n = \int_\Omega f\big(x, \psi_r(x)\big) J_r\big(x, \psi_r(x)\big) \, d\mathcal{L}^n(x). \tag{3.28}$$

Moreover we can say that[2]

$$J_r(x, y) = J(x, y) \quad \textit{for } \mathcal{H}^n\textit{-a.e. } (x, y) \in \partial^* E_r.$$

Proof of statement (1).

This is elementary algebra. We use Laplace formula for determinants on the first row and the rest is trivial.

□

Proof of statement (2).

We can define the real numbers

$$a_1 := \frac{1}{\alpha_1^2} - 1, \; a_2 := -\frac{\alpha_2}{\alpha_1}, \; a_3 := -\frac{\alpha_3}{\alpha_n}, \; ..., a_n := -\frac{\alpha_n}{\alpha_1}$$

and we can see, using Gauss operations between columns, that

$$\begin{aligned} \det(A) &= \alpha_1^2 \det \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \\ a_n & 0 & \cdots & & 1 \end{pmatrix} \\ &= \alpha_1^2(a_1 - a_2^2 - a_3^2 - ... - a_n^2) \\ &= 1 - \alpha_1^2 - ... - \alpha_n^2. \end{aligned}$$

[2]Here J is the map defined in Lemma 3.2.11(7)

□

Proof of statement (3).
We recall that the orthogonal projection from $\mathbb{R}^{n+1}$ onto the hyperplane $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ is the linear map with matrix

$$\begin{pmatrix} \mathbb{1}_n & \mathbf{0}_n \\ \mathbf{0}_n^T & 0 \end{pmatrix}$$

and that the $\mathcal{H}^k$ measure of the parallelogram

$$P(v_1, ..., v_k) := \left\{ \sum_{i=1}^{k} \alpha_i v_i \ : \ (\alpha_1, ..., \alpha_k) \in [0,1]^k \right\}$$

is

$$\mathcal{H}^k\Big(P(v_1, ..., v_k)\Big) = \sqrt{\det\Big(\big(\langle v_i, v_j\rangle\big)_{1\le i,j\le k}\Big)}.$$

Now the statement follows from elementary algebra and (2).

□

Proof of statement (5).
By the Blow Up Theorem (cfr. [2, page 199]), we have that

$$\mathbb{1}_{E_r^{(t)}(x_0,y_0)} \xrightarrow{t\to 0^+} \mathbb{1}_{H^-(x_0,y_0)} \quad \text{in } L^1_{loc}(\Omega \times Y),$$

which by definition means that if we fix $\mathbf{B} := \mathbf{B}(x_0, 1)\times[-N, N]$ as a bounded domain of integration, then

$$\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0 : \ ||\mathbb{1}_{\mathbf{B}\cap E_r^{(\delta)}(x_0,y_0)} - \mathbb{1}_{\mathbf{B}\cap H^-(x_0,y_0)}||_{L^1} < \varepsilon \quad \forall\delta \in (0, \delta_\varepsilon),$$

which is to say that $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ s.t.

$$\mathcal{L}^{n+1}\bigg(\mathbf{B} \cap \Big(H^-(x_0, y_0)\Delta E_r^{(\delta_\varepsilon)}(x_0, y_0)\Big)\bigg) < \varepsilon \quad \forall\delta \in (0, \delta_\varepsilon). \tag{3.29}$$

The elements of $E_r^{(\delta)}$ can be described equivalently as follows:

$$
\begin{aligned}
(x,y) \in E_r^{(\delta)}(x_0,y_0) \quad &\overset{\text{def}}{\Leftrightarrow} \quad \exists(\overline{x},\overline{y}) \in E_r : \begin{cases} \overline{x} = x_0 + r(x-x_0) \\ \overline{y} = y_0 + r(y-y_0) \end{cases} \\
&\overset{\mathcal{L}^{n+1}\text{-a.e.}}{\Leftrightarrow} \quad \exists(\overline{x},\overline{y}) \in \Omega \times Y : \begin{cases} \overline{y} \geq \psi_r(\overline{x}) \\ \overline{x} = x_0 + r(x-x_0) \\ \overline{y} = y_0 + r(y-y_0) \end{cases} \\
&\Leftrightarrow \quad y \geq y_0 + \frac{1}{r}\Big(\psi_r\big(x_0 + r(x-x_0)\big) - y_0\Big)
\end{aligned}
$$

This shows that not only E_r, but also $E_r^{(\delta)}$ is an epigraph.
Now, using the coarea formula (cfr. [2, Lemma 1, p.104], using the projection as linear map), we can rewrite (3.29) as

$$
\int_{\mathbf{B}(x_0,1)} \left| y_0 + \frac{1}{r}\Big(\psi_r\big(x_0 + r(x-x_0)\big) - y_0\Big) - \Big(y_0 + \langle x - x_0, \zeta_r(x_0)\rangle\Big) \right| \, d\mathcal{L}^n(x) < \varepsilon,
$$

which holds for all $\delta \in (0, \delta_\varepsilon)$, and this proves the statement.

□

Proof of statement (6).
This follows from (5) through change of variables.

□

Proof of statement (7).
Throughout this proof we shall consider a fixed $r_0 \in [-1, 0]$ and use, for simplicity, the notation $\psi \equiv \psi_{r_0}$, $\zeta \equiv \zeta_{r_0}$, $E \equiv E_{r_0}$, $p \equiv p^{(r_0)}$.

Claim 3.
If $\delta \in (0, \delta_\varepsilon)$ and $\alpha > 0$, then

$$
\mathcal{L}^n\Big\{ y \in \mathbf{B}(x,r) : |\psi(y) - \psi(x) - \langle y - x, \zeta(x)\rangle| \geq \alpha \Big\} \leq \frac{\varepsilon \delta^{n+1}}{\alpha}.
$$

Proof of claim 3.

This is simply a consequence of Chebychev's inequality and (6).

$\square$

Claim 4.

If $\delta \in (0, \delta_\varepsilon)$, $\big(x, \psi(x)\big) \in \partial^* E$ and $\big[\eta\big(x, \psi(x)\big)\big]_{n+1} \neq 0$, then

$$\mathcal{L}^n\bigg(\Big(\pi_\Omega\big(\mathbf{B}(x,\delta) \cap \mathrm{Gr}(\psi)\big)\Big)\Delta\Big(\pi_\Omega\big(\mathbf{B}(x,\delta) \cap H(x)\big)\Big)\bigg) \leq$$
$$\leq C(n,x)\sqrt{\varepsilon}\,\delta^n.$$

where $C(n,x) = \left(1 + \dfrac{n\alpha(n)}{\big|\big[\eta(x,\psi(x))\big]_{n+1}\big|}\right)$.

Proof of claim 4.

Define our target set

$$R := \Big(\pi_\Omega\big(\mathbf{B}(x,\delta) \cap \mathrm{Gr}(\psi)\big)\Big)\Delta\Big(\pi_\Omega\big(\mathbf{B}(x,\delta) \cap H(x)\big)\Big)$$

and, for any $\alpha \in (0, \delta/2)$, the sets

- $R_1 := \left\{ y \in \mathbf{B}(x,\delta) : \begin{cases} \big(y, \psi(x) + \langle y - x, \zeta(x)\rangle\big) \in \mathbf{B}\big((x,\psi(x)), \delta - \alpha\big) \\ \big(y, \psi(y)\big) \notin \mathbf{B}\big((x,\psi(x)), \delta\big) \end{cases} \right\}$
- $R_2 := \left\{ y \in \mathbf{B}(x,\delta) : \begin{cases} \big(y, \psi(x) + \langle y - x, \zeta(x)\rangle\big) \notin \mathbf{B}\big((x,\psi(x)), \delta - \alpha\big) \\ \big(y, \psi(y)\big) \in \mathbf{B}\big((x,\psi(x)), \delta\big) \end{cases} \right\}$
- $R_3 := \Big\{ y \in \mathbf{B}(x,\delta) : \big(y, \psi(x) + \langle y - x, \zeta(x)\rangle\big) \in \Big(\mathbf{B}\big((x,\psi(x)), \delta\big) \setminus \mathbf{B}\big((x,\psi(x)), \delta - \alpha\big)\Big) \cap H(x) \Big\}$.

About these sets we can say that

- $R_1 \cap R_2 = \varnothing$.
- By claim 3 $\mathcal{L}^n\big(R_1 \cup R_2\big) \leq \dfrac{\varepsilon\delta^{n+1}}{\alpha}$.

- Since $\mathbf{B}\big((x,\psi(x)),\delta\big)\cap H(x)$ and $\mathbf{B}\big((x,\psi(x)),\delta-\alpha\big)\cap H(x)$ are two concentric n-dimensional disks on H and since R_3 is the projection on Ω of their difference, we have

$$\begin{aligned}\mathcal{L}^n(R_3) &= \frac{1}{\big|\big[\eta(x,\psi(x))\big]_{n+1}\big|}\Big(\alpha(n)\delta^n \;-\; \alpha(n)(\delta-\alpha)^n\Big)\\ &= \frac{\alpha(n)}{\big|\big[\eta(x,\psi(x))\big]_{n+1}\big|}\Big(\delta-(\delta-\alpha)\Big)\Big(\delta^{n-1}+\delta^{n-2}(\delta-\alpha)+\\ &\quad +\delta^{n-3}(\delta-\alpha)^2+\ldots+(\delta-\alpha)^{n-1}\Big)\\ &\le \frac{n\alpha(n)}{\big|\big[\eta(x,\psi(x))\big]_{n+1}\big|}\,\alpha\,\delta^{n-1}.\end{aligned}$$

- $R\subset R_1\cup R_2\cup R_3$.

We can now obtain the statement of the claim simply taking $\alpha=\sqrt{\varepsilon}\delta$ and writing

$$\mathcal{L}^n(R)\le\mathcal{L}^n(R_1\cup R_2)+\mathcal{L}^n(R_3)\le\left(1+\frac{n\alpha(n)}{\big|\big[\eta(x,\psi(x))\big]_{n+1}\big|}\right)\sqrt{\varepsilon}\,\delta^n.$$

$\square$

Claim 5.
If $\big(x,\psi(x)\big)\in\partial^*E$ *and* $\big[\eta\big(x,\psi(x)\big)\big]_{n+1}\neq 0$, *then*

$$\lim_{r\to0^+}\frac{\mathcal{H}^n\llcorner\mathrm{Gr}(\psi)\Big(\mathbf{B}\big((x,\psi(x)),r\big)\Big)}{p_\#\mathcal{L}^n\Big(\mathbf{B}\big((x,\psi(x)),r\big)\Big)}=-\frac{1}{[\eta_r\big(x,\psi(x)\big)]_{n+1}}.$$

Proof of claim 5.
Using the notation $\mathbf{B}_r:=\mathbf{B}\big((x,\psi(x)),r\big)$, we have

$$\begin{aligned}&\frac{\mathcal{H}^n\llcorner\mathrm{Gr}(\psi)\big(\mathbf{B}_r\big)}{p_\#\mathcal{L}^n\big(\mathbf{B}_r\big)}=\\ &\quad=\frac{\mathcal{H}^n\big(\mathrm{Gr}(\psi)\cap\mathbf{B}_r\big)}{\mathcal{H}^n\big(H(x)\cap\mathbf{B}_r\big)}\,\frac{\mathcal{H}^n\big(H(x)\cap\mathbf{B}_r\big)}{\mathcal{L}^n\Big(\pi_\Omega\big(H(x)\cap\mathbf{B}_r\big)\Big)}\,\frac{\mathcal{L}^n\Big(\pi_\Omega\big(H(x)\cap\mathbf{B}_r\big)\Big)}{\mathcal{L}^n\Big(\pi_\Omega\big(\mathrm{Gr}(\psi)\cap\mathbf{B}_r\big)\Big)}\end{aligned}$$

Using [2, Corollary 1(ii), p.203], item (3), and claim 4 we have that the left hand side converges, as r goes to 0^+, to $\dfrac{1}{\left|\big[\eta(x,\psi(x))\big]_{n+1}\right|}$, so that we have finally proven that

$$\lim_{r\to 0^+} \frac{\mathcal{H}^n \llcorner \mathrm{Gr}(\psi)\big(\mathbf{B}_r\big)}{p_\# \mathcal{L}^n\big(\mathbf{B}_r\big)} = \frac{1}{\left|\big[\eta(x,\psi(x))\big]_{n+1}\right|}.$$

$\square$

Consider now the measures $m_1 := \mathcal{H}^n \llcorner \partial^* E$ and $m_2 := p_\#\Big(\mathcal{L}^n \llcorner \big(\pi_\Omega(\partial^* E)\big)\Big)$. Using [3, Theorem 1.6(b), p.8] we see that they are both Radon measures (in the sense of [2, p.4,5]).

Clearly $m_2 \ll m_1$, so that by the previous discussion and by [2, Theorem 2, p.40] we must have

$$dm_2 = |[\eta(x,\psi(x))]_{n+1}|\, dm_1.$$

Now we simply need to remember Lemma 3.2.11(4) to deduce that

$$m_1 \ll m_2$$

is true as well. We can further say, using Lemma 3.2.11(5) and again [2, Theorem 2, p.40], that

$$dm_1 = -\frac{1}{[\eta(x,\psi(x))]_{n+1}}\, dm_2.$$

$\square$

Lemma 3.2.14.
The following statements hold:

(1) If $\phi \in C^1_c(\Omega,\mathbb{R})$, then

$$\int_\Omega \psi_r \begin{pmatrix} \nabla\phi \\ 0 \end{pmatrix} d\mathcal{L}^n = \\ = -\int_\Omega \phi \mathbf{e}_{n+1}\, dx - \int_\Omega \phi(x)\eta_r\big(x,\varphi_r(x)\big) J_r\big(x,\varphi_r(x)\big)\, dx. \tag{3.30}$$

(2) For ν-a.e. r, we have[3] [4]

$$J_r(x,y) = J(x,y) \quad \forall (x,y) \in \partial^* E_r.$$

(3) ψ_r is weakly differentiable and its weak derivative is given by the map ζ_r defined in Lemma 3.2.13(4).

(4) for all $X \in C^\infty_c(\Omega \times Y, \mathbb{R}^{n+1})$

$$\int_I \int_{\{(x,y)\in\Omega\times Y: y \geq \psi_r(x)\}} \operatorname{div}(X) d\mathcal{L}^{n+1}\, d\nu = \\ = \int_I \int_\Omega \langle X(x,\psi_r(x)), (\,D\psi_r(x), -1)\rangle\, dx\, d\nu(r). \tag{3.31}$$

Proof of statement (1).
If $f \in L^1(\mathcal{H}^n \llcorner \partial^* E_r)$, since $\eta_r \in L^\infty(\Omega \times Y)$, then $f\eta_r \in L^1(\mathcal{H}^n \llcorner \partial^* E_r)$ and Lemma 3.2.13(7) applies, giving

$$\int_{\partial^* E_r} f\eta_r\, d\mathcal{H}^n = \int_\Omega f\big(x,\varphi_r(x)\big)\eta_r\big(x,\varphi_r(x)\big) J_r\big(x,\varphi_r(x)\big)\, dx. \tag{3.32}$$

[3] Here J_r is the map given by Lemma 3.2.13(7), while J is the one defined in Lemma 3.2.11

[4] This statement is actually a repetition, since the same fact was already proven in Lemma 3.2.13(7). We state and prove it again assuming only absolute continuity. This simply strengthens the proof made earlier.

Now let $N > 0$ such that $\mathrm{spt}(\mu) \subset \Omega \times [-N, N] \times Z$.
Let $\chi \in C_c^\infty(\overline{\Omega \times Y})$ such that $0 \leq \chi \leq 1$ and $\chi(x, y) = 1 \ \forall (x, y) \in \Omega \times [-N, N]$.
We recall the generalized Gauss-Green formula, stated in (3.18). It tells us that for any $F \in C_c^1(\Omega \times Y)$

$$\int_{E_r} \nabla F \, d\mathcal{L}^{n+1} \quad = \int_{\partial^* E_r} F \eta_r \, d\mathcal{H}^n.$$

Consider a function $\phi \in C_c^1(\Omega)$.
Define now $\overline{\phi}$ as

$$\begin{array}{rccc} \overline{\phi}: & \Omega \times Y & \longrightarrow & \mathbb{R} \\ & (x, y) & \mapsto & \phi(x) \end{array}.$$

We have

- $\nabla \overline{\phi}(x, y) = \big(\nabla \phi(x), 0\big)$;
- $\overline{\phi}\psi \in C_c^1(\Omega \times Y)$.

Let $\mathbf{n}_0$ denote the exterior normal vector to the set $\{(x, y) \in \Omega \times Y : \, y \geq 0\}$.
In particular

$$\begin{array}{rccc} \mathbf{n}_0: & \Omega \times Y & \longrightarrow & \mathbb{R}^{n+1} \\ & (x, y) & \mapsto & \begin{cases} -\mathbf{e}_{n+1} & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases} \end{array}.$$

Now we have

$$\begin{aligned}
\int_\Omega \psi_r \nabla\overline{\phi}\, d\mathcal{L}^n &= \int_\Omega \int_0^{\psi_r(x)} \nabla\overline{\phi}(x)\, dy\, dx \\
&= \int_\Omega \int_0^{\psi_r(x)} \nabla(\chi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \\
&= \int_{\{(x,y):x\in\Omega,\, 0\le y\le\psi_r(x)\}} \nabla(\chi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \ - \int_{\{(x,y):x\in\Omega,\, 0\ge y\ge\psi_r(x)\}} \nabla(\chi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \\
&= \int_{\{(x,y):x\in\Omega,\, y\ge 0\}} \nabla(\chi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \ - \int_{\{(x,y):x\in\Omega,\, y\ge\psi_r(x)\}} \nabla(\psi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \\
&= \int_{\{(x,y):x\in\Omega,\, y\ge 0\}} \nabla(\chi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \ - \int_{E_r} \nabla(\psi\overline{\phi})(x,y)\, d\mathcal{L}^{n+1} \\
&= \int_{\partial^*\{(x,y):x\in\Omega,\, y\ge 0\}} \chi\overline{\phi}\mathbf{n}_0\, d\mathcal{H}^n \ - \int_{\partial^* E_r} \chi\overline{\phi}\eta_r\, d\mathcal{H}^n \\
&= \int_{\Omega\times\{0\}} \overline{\phi}\mathbf{n}_0\, d\mathcal{H}^n \ - \int_{\partial^* E_r} \overline{\phi}\eta_r\, d\mathcal{H}^n \\
&= \int_\Omega \big(-\overline{\phi}\mathbf{e}_{n+1}\big)\, d\mathcal{L}^n \ - \int_\Omega \overline{\phi}\big(x,\psi_r(x)\big)\eta_r\big(x,\psi_r(x)\big)J_r\big(x,\psi_r(x)\big)\, d\mathcal{L}^n.
\end{aligned}$$

□

Proof of statement (2).

Consider (3.30) and look only at the $(n+1)$-th entry of the equation. It reads

$$\int_\Omega \phi(x)\Big(1+[\eta_r\big(x,\varphi_r(x)\big)]_{n+1}J_r\big(x,\varphi_r(x)\big)\Big)\, d\mathcal{L}^n \ = 0 \quad \forall\phi\in C_c^1(\Omega). \tag{3.33}$$

The statements follows from the density of $C_c^1(\Omega)$ in $L^1(\Omega)$.

□

Proof of statement (3).

Consider again (3.30), and this time look only at the first n entries. Applying (2) to them the statement follows.

□

Proof of statement (4).

In light of (2), equation (3.32) becomes

$$\int_{\partial^* E_r} f(x,y)\eta_r(x,y)\, d\mathcal{H}^n(x,y) = \int_{\Omega} f\big(x, \psi_r(x)\big) \begin{pmatrix} \zeta_r(x) \\ -1 \end{pmatrix} d\mathcal{L}^n(x), \quad (3.34)$$

and it holds for any $f \in L^1(\mathcal{H}^n \llcorner \partial^* E_r)$.

Now applying (3.34) to $f = X_i$ and adding over $i = 1, ..., n$ gives statement (4).

$\square$

Remark 3.

Definition of centroid and centroid-concentrated mass

Let $\mu_{\Omega\times Y} := (\pi_{\Omega\times Y})_{\#}\mu$.

Since μ is by hypothesis a finite measure, we can apply the disintegration Theorem to μ and the projection map $\pi_{\Omega\times Y}$ to obtain a family of measures μ_{xy} on Z which are defined for $\mu_{\Omega\times Y}$-a.e. $(x,y)\in\Omega\times Y$ and which are such that

$$\int_{\Omega\times Y\times Z} f\,d\mu = \int_{\Omega\times Y}\int_Z f(x,y,z)\,d\mu_{xy}(z)\,d\mu_{\Omega\times Y} \tag{3.35}$$

for any $f:\Omega\times Y\times Z\to\mathbb{R}$ measurable.

Now, by (3.2) at page 62, the quantity

$$\mathcal{Z}(x,y) := \int_Z z\,d\mu_{xy} \tag{3.36}$$

is well defined for $\mu_{\Omega\times Y}$-a.e. pair (x,y). Define $\mathcal{Z}(x,y)\in Z$ to be the **centroid** of μ at (x,y).

We also define a measure $\overline{\mu}$ on $\Omega\times Y\times Z$ whose projection $(\pi_{\Omega\times Y})_{\#}\overline{\mu}$ coincides with $\mu_{\Omega\times Y}$ and such that it is concentrated on the graph of $\mathcal{Z}(\cdot,\cdot)$:

$$\begin{array}{rccl} \overline{\mu}: & \mathcal{B}_{\Omega\times Y\times Z} & \longrightarrow & [0,+\infty) \\ & B & \mapsto & \displaystyle\int_{\Omega\times Y} \delta_{\left(x,y,\mathcal{Z}(x,y)\right)}(B)\,d\mu_{\Omega\times Y} \end{array}.$$

We call $\overline{\mu}$ the **version of μ concentrated at its centroid in the z variable**.

An equivalent definition of $\overline{\mu}$ is that for any $f:\Omega\times Y\times Z\to\mathbb{R}$ measurable it holds

$$\int_{\Omega\times Y\times Z} f\,d\overline{\mu} = \int_{\Omega\times Y} f\big(x,y,\mathcal{Z}(x,y)\big)\,d\mu_{\Omega\times Y}. \tag{3.37}$$

Proof of Theorem 3.2.1.

> **Claim 6.**
> *If $\phi \in C^\infty(\Omega \times Y \times Z)$, it is linear in z and it is compactly-supported in $\Omega \times Y$, then* (3.4) *holds.*

Proof of claim 6.
Since by hypothesis $z \mapsto \phi(x, y, z)$ is linear, there must be some $\tilde{X} \in C_c^\infty(\Omega \times Y, \mathbb{R}^n)$ such that

$$\phi(x, y, z) = \langle \tilde{X}(x, y), z \rangle.$$

Define $X \in C_c^\infty(\Omega \times Y, \mathbb{R}^{n+1})$ as $X(x, y) := \big(\tilde{X}(x, y), 0\big)$.
Now

$$\begin{aligned}
\int_{\Omega \times Y \times Z} \phi \, d\mu \quad &= \int_{\Omega \times Y \times Z} \langle X, (z, -1) \rangle \, d\mu \\
&= \mu(X) \\
&= S(-\operatorname{div}(X)) \\
&= -\int_{\Omega \times Y} \rho \operatorname{div}(X) \, d\mathcal{L}^{n+1} \\
&= \int_I \int_{E_r} \operatorname{div}(X) \, d\mathcal{L}^{n+1} \, d\nu(r) \\
&= \int_I \int_{\{(x,y): y \geq \psi_r(x)\}} \operatorname{div}(X) \, d\mathcal{L}^{n+1} \, d\nu(r) \\
&= \int_I \int_\Omega \langle X\big(x, \psi_r(x)\big), \big(D\psi_r(x), -1\big) \rangle \, dx \, d\nu(r) \\
&= \int_I \int_\Omega \phi\big(x, \psi_r(x), D\psi_r(x)\big) \, dx \, d\nu(r).
\end{aligned}$$

□

Claim 7.

If $\phi \in L^1(\mu)$ and it is linear in z, then

(1) $\exists \tilde{X} : \Omega \times Y \to Z$ *measurable and such that* $\phi(x, y, z) = \langle \tilde{X}(x, y), z \rangle$.

(2) $\langle \tilde{X}(x, y), \mathcal{Z}(x, y) \rangle \in L^1(\mu_{\Omega \times Y})$.

(3) $\displaystyle\int_{\Omega \times Y \times Z} \phi \, d\mu = \int_{\Omega \times Y \times Z} \phi \, d\overline{\mu}$.

(4) It holds

$$\begin{aligned} \int_{\Omega \times Y \times Z} \phi \, d\mu &= \int_{\Omega \times Y} \langle \tilde{X}, \mathcal{Z} \rangle \, d\mu_{\Omega \times Y} \\ &= \int_I \int_\Omega \Big\langle \tilde{X}\big(x, \psi_r(x)\big), \, \mathcal{Z}\big(x, \psi_r(x)\big) \Big\rangle \, d\mathcal{L}^n(x) \, d\nu(r). \end{aligned}$$

Proof of claim 7.

(1) is obvious by the linearity of ϕ in z.

(2) can be seen through the chain of inequalities

$$\begin{aligned} +\infty &> \int_{\Omega \times Y \times Z} \phi \, d\mu \\ &= \int_{\Omega \times Y} \int_Z |\langle \tilde{X}(x, y), z \rangle| \, d\mu_{xy} \, d\mu_{\Omega \times Y} \\ &\geq \int_{\Omega \times Y} \left| \int_Z \langle \tilde{X}(x, y), z \rangle \, d\mu_{xy} \right| \, d\mu_{\Omega \times Y} \\ &= \int_{\Omega \times Y} |\langle \tilde{X}(x, y), \mathcal{Z}(x, y) \rangle| \, d\mu_{\Omega \times Y}. \end{aligned}$$

(3) follows from (3.35), (3.36) and (3.37), since

$$\begin{aligned}
\int_{\Omega\times Y\times Z} \phi\, d\mu &= \int_{\Omega\times Y}\int_Z \langle \tilde{X}(x,y), z\rangle\, d\mu_{xy}(z)\, d\mu_{\Omega\times Y}(x,y) \\
&= \int_{\Omega\times Y}\int_Z \left(\sum_{i=1}^n \tilde{X}_i z_i\right) d\mu_{xy}(z)\, d\mu_{\Omega\times Y}(x,y) \\
&= \int_{\Omega\times Y} \sum_{i=1}^n \tilde{X}_i \mathcal{Z}_i(x,y)\, d\mu_{\Omega\times Y}(x,y) \\
&= \int_{\Omega\times Y} \langle \tilde{X}(x,y), \mathcal{Z}(x,y)\rangle\, d\mu_{\Omega\times Y} \\
&= \int_{\Omega\times Y\times Z} \phi\, d\overline{\mu}.
\end{aligned}$$

To see (4) use (2), Lemma 3.2.11(3), Lemma 3.2.13 and Lemma 3.2.14(2). In fact

$$\begin{aligned}
\int_{\Omega\times Y\times Z} \phi\, d\mu &= \int_{\Omega\times Y}\int_Z \langle \tilde{X}(x,y), z\rangle\, d\mu_{xy}\, d\mu_{\Omega\times Y} \\
&= \int_{\Omega\times Y} \langle \tilde{X}, \mathcal{Z}\rangle\, d\mu_{\Omega\times Y} \\
&= -\int_I \int_{\partial^* E_r} \langle \tilde{X}, \mathcal{Z}\rangle [\eta_r(\cdot)]_{n+1}\, d\mathcal{H}^n\, d\nu(r) \\
&= \int_I\int_\Omega \Big\langle \tilde{X}\big(x,\psi_r(x)\big), \mathcal{Z}\big(x,\psi_r(x)\big)\Big\rangle\, d\mathcal{L}^n(x)\, d\nu(r).
\end{aligned}$$

$\square$

> **Claim 8.**
> *For $\mathcal{L}^n$-a.e. $x\in\Omega$ we have that $\mathcal{Z}\big(x,\psi_r(x)\big) = D\psi_r(x)$.*

Proof of claim 8.

If $\phi \in C^\infty(\Omega\times Y\times Z)$ is compactly supported in (x,y) and linear in z, then claim 6 and claim 7(4) give

$$\begin{aligned}
\int_I\int_\Omega \langle \tilde{X}\big(x,\psi_r(x)\big), D\psi_r(x)\rangle\, dx\, d\nu(r) &= \int_I\int_\Omega \phi\big(x,\psi_r(x), D\psi_r(x)\big)\, dx\, d\nu(r) \\
&= \int_{\Omega\times Y\times Z} \phi\, d\mu \\
&= \int_I\int_\Omega \Big\langle \tilde{X}\big(x,\psi_r(x)\big),\ \mathcal{Z}\big(x,\psi_r(x)\big)\Big\rangle\, dx\, d\nu(r)
\end{aligned}$$

$\square$

Claim 9. *If $\phi \in L^1(\mu)$ and it is linear in z, then (3.4) holds for ϕ.*

Proof of claim 9.

We simply apply claim 8 to claim 7(4) and we get

$$\begin{aligned}
\int_{\Omega\times Y\times Z} \phi \, d\mu &= \int_I \int_\Omega \Big\langle \tilde{X}\big(x, \psi_r(x)\big), \, \mathcal{Z}\big(x, \psi_r(x)\big) \Big\rangle \, d\mathcal{L}^n(x)\, d\nu(r) \\
&= \int_I \int_\Omega \Big\langle \tilde{X}\big(x, \psi_r(x)\big), \, D\psi_r(x) \Big\rangle \, d\mathcal{L}^n(x)\, d\nu(r) \\
&= \int_I \int_\Omega \phi\big(x, \psi_r(x), D\psi_r(x)\big) \, d\mathcal{L}^n(x)\, d\nu(r)
\end{aligned}$$

□

Claim 10. *For ν-a.e. $r \in \mathbb{R}$, $\psi_r \in W^{1,1}(\Omega) \cap W^{1,\infty}(\Omega)$.*

Proof of claim 10.

ψ_r is clearly in $L^\infty(\Omega)$, since $\psi_r(x) \in [-N, N]$ for all $r \in (-1, 0]$, and therefore in $L^1(\Omega)$, since Ω is bounded by hypothesis.

We defined $D\psi_r(x) \equiv \zeta_r(x)$ by truncation of the vector $\eta_r\big(x, \psi_r(x)\big)$ and rescaling by a factor $-\dfrac{1}{[\eta_r\big(x, \psi_r(x)\big)]_{n+1}} = J\big(x, \psi_r(x)\big)$. This clearly implies that

$$|D\psi_r(x)| \le \big|J\big(x, \psi_r(x)\big|\,.$$

We proved in J is almost everywhere positive we have

$$\int_\Omega |J\big(x, \psi_r(x)\big)| \, dx = \int_\Omega J\big(x, \psi_r(x)\big) \, dx = \int_{\partial^* E_r} 1 \, d\mathcal{H}^n < \infty,$$

and this proves that $D\psi_r \in L^1(\Omega, \mathbb{R}^n)$.

Moreover, since $\mathcal{Z}(x, y) := \displaystyle\int_Z z \, d\mu_{xy}$ and $\{x\} \times \{y\} \times \operatorname{spt}(\mu_{xy}) \subset \operatorname{spt}(\mu)$, we see that

$$\mathcal{Z}(x, \psi_r(x)) \in K \quad \forall x \in \Omega,$$

where K is a compact subset of Z such that $\operatorname{co}\big(\operatorname{spt}(\mu)\big) \subset \Omega \times [-N, N] \times K$. This proves boundedness of $\mathcal{Z}(x, \psi_r(x))$, and therefore of $D\psi_r(x)$.

□

Claim 11.
If $\phi \in L^1(\Omega \times Y \times Z)$ is constant in z, then (3.4) holds for ϕ.

Proof of claim 11.
By construction η_r is a unitary vector, and by Lemma 3.2.11 $[\eta_r(\cdot)]_{n+1}$ is $\mathcal{H}^n$-a.e. strictly negative. Recalling the definition of ζ_r and claim 8 we see that $||\mathcal{Z}(x,y)|| > 0$ for $\mathcal{H}^n$-a.e. $(x,y) \in \partial^* E_r$.
So the following expressions are well defined $\mathcal{H}^n$-a.e. and we can say that

$$\begin{aligned}
\int_{\Omega\times Y\times Z} \phi(x,y,z)\, d\mu(x,y,z) &= \int_{\Omega\times Y\times Z} \phi(x,y)\, d\mu(x,y,z)\\
&= \int_{\Omega\times Y} \phi(x,y)\, d\mu_{\Omega\times Y}\\
&= \int_{\Omega\times Y}\int_Z \phi(x,y) \left\langle z, \frac{\mathcal{Z}(x,y)}{||\mathcal{Z}(x,y)||^2}\right\rangle d\mu_{xy}\, d\mu_{\Omega\times Y}\\
&= \int_{\Omega\times Y\times Z} \phi(x,y) \left\langle z, \frac{\mathcal{Z}(x,y)}{||\mathcal{Z}(x,y)||^2}\right\rangle d\mu(x,y,z)\\
&= \int_{\Omega\times Y\times Z} \left\langle \phi(x,y)\frac{\mathcal{Z}(x,y)}{||\mathcal{Z}(x,y)||^2},\, z\right\rangle d\mu(x,y,z)
\end{aligned}$$

Now these same computations read backwards show that

$$(x,y,z) \mapsto \left\langle \phi(x,y)\frac{\mathcal{Z}(x,y)}{||\mathcal{Z}(x,y)||^2},\, z\right\rangle$$

is a map in $L^1(\mu)$, and moreover it is clearly linear in z, so that claim 7(4) applies, and allows us to continue the chain of equalities as follows:

$$\begin{aligned}
\int_{\Omega\times Y\times Z} \phi(x,y,z)\, d\mu(x,y,z) &= \int_{\Omega\times Y\times Z} \left\langle \phi(x,y)\frac{\mathcal{Z}(x,y)}{||\mathcal{Z}(x,y)||^2},\, z\right\rangle d\mu(x,y,z)\\
= \int_I\int_\Omega &\left\langle \phi(x,\psi_r(x))\frac{\mathcal{Z}\big(x,\psi_r(x)\big)}{||\mathcal{Z}(x,\psi_r(x))||^2},\, \mathcal{Z}(x,\psi_r(x))\right\rangle d\mathcal{L}^n(x)\, d\nu(r)\\
= \int_I\int_\Omega &\phi\big(x,\psi_r(x)\big)\, d\mathcal{L}^n(x)\, d\nu(r).
\end{aligned}$$

□

To sum up we proved that $\psi_r \in W^{1,\infty}(\Omega)$ for ν-a.e. $r \in I$ and that any map in $L^1(\mu)$ which is either linear or constant in z satisfies (3.4). So the statement of the theorem follows.

□

3.2.4 Proof of theorem 3.2.2

Lemma 3.2.15.

(i) Let (μ, μ_∂) be as in Theorem 3.2.2.
Let ν, ψ_r be as in Theorem 3.2.1.
Let $L : \Omega \times Y \times Z \to \mathbb{R}$ be measurable and assume that it is convex in z.
Then

$$\int_I \int_\Omega L\big(x, \psi_r(x), D\psi_r(x)\big)\, d\mathcal{L}^n(x)\, d\nu(r) \le \int_{\Omega\times Y\times Z} L\, d\mu. \tag{3.38}$$

(ii) If, additionally, μ_∂ is as in Theorem 3.2.2, then $\psi_r|_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ is a well defined Lipschitz function and we have, for all $\phi : \partial\Omega \times Y \to \mathbb{R}$ for which the integrals below are defined, that

$$\int_{\partial\Omega\times Y} \phi\, d\mu_\partial = \int_I \int_{\partial\Omega} \phi\big(x, \psi_r(x)\big)\, d\sigma(x)\, d\nu(r),$$

where $\sigma := \mathcal{H}^{n-1} \llcorner \partial\Omega$.

Proof of statement (i).
By Jensen's inequality on the map $z \mapsto L(x, y, z)$ we have

$$L\big(x, y, \mathcal{Z}(x, y)\big) \le \int_Z L(x, y, z)\, d\mu_{xy}.$$

Integrating we find

$$\begin{aligned}
\int_{\Omega\times Y\times Z} L\, d\overline{\mu} \quad &= \int_{\Omega\times Y\times Z} L\big(x, y, \mathcal{Z}(x, y)\big)\, d\mu \\
&= \int_{\Omega\times Y} L\big(x, y, \mathcal{Z}(x, y)\big)\, d\mu_{\Omega\times Y} \\
&\le \int_{\Omega\times Y} \int_Z L(x, y, z)\, d\mu_{xy}\, d\mu_{\Omega\times Y} \\
&= \int_{\Omega\times Y\times Z} L\, d\mu. \qquad (3.39)
\end{aligned}$$

In the proof of Theorem 3.2.1 we saw that the following equivalences hold for any $\phi \in L^1(\mu)$:

$$\begin{aligned}\int_{\Omega\times Y\times Z} \phi \, d\overline{\mu} &= \int_{\Omega\times Y} \phi\big(x, y, \mathcal{Z}(x,y)\big) \, d\mu_{\Omega\times Y} \\ &= -\int_I \int_{\partial^* E_r} \phi\big(x, y, \mathcal{Z}(x,y)\big)[\eta_r(x,y)]_{n+1} \, d\mathcal{H}^n(x,y) \, d\nu(r) \\ &= \int_I \int_\Omega \phi\big(x, \psi_r(x), D\psi_r(x)\big) \, d\mathcal{L}^n(x) \, d\nu(r). \qquad (3.40)\end{aligned}$$

Now, using (3.39) and (3.40), we have

$$\int_I \int_\Omega L\big(x, \psi_r(x), D\psi_r(x)\big) \, d\mathcal{L}^n(x) \, d\nu(r) \;\leq\; \int_{\Omega\times Y\times Z} L \, d\mu.$$

□

Proof of statement (ii).
Let $\mathbf{n}_\Omega : \partial\Omega \to S^{n-1}$ be the exterior unit normal vector to Ω.
Since Ω has piecewise C^1 boundary, Gauss-Green formula holds, hence

$$\int_\Omega \operatorname{div}(X) \, d\mathcal{L}^n = \int_{\partial\Omega} \langle X, \mathbf{n}_\Omega \rangle \, d\sigma \quad \forall X \in C^1(\overline{\Omega}, \mathbb{R}^n). \qquad (3.41)$$

Taking $X = \phi\big(x, u(x)\big)\mathbf{e}_j$ for all $j = 1, ..., n$ we can see that $\forall \phi \in C^\infty(\overline{\Omega} \times Y), \forall u \in C^1(\overline{\Omega})$

$$\int_\Omega \left(\frac{\partial \phi}{\partial x}\big(x, u(x)\big) \;+\; \frac{\partial \phi}{\partial y} Du(x) \right) d\mathcal{L}^n(x) = \int_{\partial\Omega} \phi\big(x, u(x)\big) \mathbf{n}_\Omega(x) \, d\sigma(x) \qquad (3.42)$$

(this property is actually equivalent to the one before).

Claim 1.
(3.42) *also holds for a couple* (ϕ, u) *with* $\phi \in C^\infty(\Omega \times Y, \mathbb{R}^n)$ *and* $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$.

Proof of claim 1.
If $u \in W^{1,1}(\Omega)$, then (see [4, Theorem 3, p.127]) $\exists \{u_n\}_{n\in\mathbb{N}} \subset W^{1,1}(\Omega) \cap C^\infty(\overline{\Omega})$ such that

$$u_n \xrightarrow[n\to\infty]{||\cdot||_{W^{1,1}}} u.$$

Up to taking a subsequence, we can assume that the sequence $\{u_n\}_n$ converges pointwise to u and that $\{Du_n\}_n$ converges pointwise to Du.
Then, by continuity of ϕ and of its derivatives, for $\mathcal{L}^n$-a.e. x,

$$\frac{\partial\phi}{\partial x}\big(x, u_n(x)\big) + \frac{\partial\phi}{\partial y}\big(x, u_n(x)\big)Du_n(x) \to \frac{\partial\phi}{\partial x}\big(x, u(x)\big) + \frac{\partial\phi}{\partial y}\big(x, u(x)\big)Du(x).$$

If we also assume that u is essentially bounded, then the sequence $\{u_n\}_n$ can be taken uniformly bounded, meaning that

$$\exists C_0 \in [0,\infty) \text{ such that } \sup_n ||u_n||_{L^\infty} \le C_0.$$

Define

$$M := \max_{\substack{(x,y)\in\overline{\Omega}\times Y\\ |y|\le C_0}} \max\left\{\frac{\partial\phi}{\partial x}(x,y),\ \frac{\partial\phi}{\partial y}(x,y)\right\}.$$

Then, for all n and for all x, we have

$$\left|\frac{\partial\phi}{\partial x}\big(x, u_n(x)\big) + \frac{\partial\phi}{\partial y}\big(x, u_n(x)\big)Du_n(x)\right| \le M(1+|Du_n|(x).$$

By dominated convergence Theorem[(5)] this means that

$$\lim_{n\to\infty}\int_\Omega \left(\frac{\partial\phi}{\partial x}\big(x, u_n(x)\big) + \frac{\partial\phi}{\partial y}\big(x, u_n(x)\big)Du_n(x)\right) d\mathcal{L}^n(x) =$$
$$= \int_\Omega \left(\frac{\partial\phi}{\partial x}\big(x, u(x)\big) + \frac{\partial\phi}{\partial y}\big(x, u(x)\big)Du(x)\right) d\mathcal{L}^n(x).$$

[5]Actually here we use a slightly different variation of the dominated convergence Theorem, which is

$$\left.\begin{array}{l} \mu' \text{ is a positive measure}\\ g, \{g_n\}_n \text{ are in } L^1(\mu')\\ f, \{f_n\}_n \text{ are measurable}\\ |f_n| \le g_n\ \forall n\\ f_n(x)\to f(x) \text{ for } \mu'\text{-a.e. } x\\ \displaystyle\int g_n\, d\mu' \to \int g\, d\mu' \end{array}\right\} \Rightarrow \int f_n\, d\mu' \to \int f\, d\mu'.$$

We use it with $\mu' = \mathcal{L}^n \llcorner \Omega$, $g_n = M(1+|Du_n|)$ and $g = M(1+|Du|)$.

An analogous argument on the same sequence $\{u_n\}_n$ shows that

$$\lim_{n\to\infty}\int_{\partial\Omega}\phi\big(x,u_n(x)\big)\mathbf{n}_\Omega(x)\,d\sigma = \\ = \int_{\partial\Omega}\phi\big(x,u(x)\big)\mathbf{n}_\Omega(x)\,d\sigma,$$

so, since each u_n satisfies (3.42), the claim follows by passing to the limit.

$\square$

Recall that within the proof of Theorem 3.2.1 (precisely in claim 10) we showed that $\psi_r \in W^{1,\infty}(\Omega)$ (which, by boundedness of Ω, means that ψ_r is also in $W^{1,1}(\Omega)$).
Since ψ_r are Lipschitz functions[6] they are well defined and Lipschitz continuous also on $\partial\Omega$.
Now we can use (3.1), (3.4) and claim 1 (with $u = \psi_r$) to see that for any $\phi \in C^\infty(\overline{\Omega}\times Y)$ we have

$$\begin{aligned}\int_{\partial\Omega\times Y}\phi(x,y)\mathbf{n}_\Omega(x)\,d\mu_\partial(x,y) &= \int_{\overline{\Omega}\times Y\times Z}\left(\frac{\partial\phi}{\partial x}(x,y)+\frac{\partial\phi}{\partial y}(x,y)z\right)d\mu(x,y,z)\\ &= \int_I\int_\Omega\left(\frac{\partial\phi}{\partial x}\big(x,\psi_r(x)\big)+\frac{\partial\phi}{\partial y}\big(x,\psi_r(x)\big)D\psi_r(x)\right)d\mathcal{L}^n(x)\,d\nu(r)\\ &= \int_I\int_{\partial\Omega}\phi\big(x,u(x)\big)\mathbf{n}_\Omega(x)\,d\sigma(x)\,d\nu(r).\end{aligned}\tag{3.43}$$

To conclude we fix any $\phi \in C^\infty(\overline{\Omega}\times Y)$ and

- notice that $[\mathbf{n}_\Omega(\cdot)]_i$ is a function in $C^1(\partial\Omega)$, since the boundary of Ω is piecewice C^1 by hypothesis;
- apply (3.43) to $f^{(1)}, ..., f^{(n)}$, with $f^{(i)} = \phi[\mathbf{n}_\Omega(\cdot)]_i$, to obtain n vector equations of the type

$$\int_{\partial\Omega\times Y}\phi(x,y)[\mathbf{n}_\Omega(x)]_i\mathbf{n}_\Omega(x)\,d\mu_\partial(x,y) = \\ = \int_I\int_{\partial\Omega}\phi\big(x,u(x)\big)[\mathbf{n}_\Omega(x)]_i\mathbf{n}_\Omega(x)\,d\sigma(x)\,d\nu(r).$$

[6] $W^{1,\infty}(\Omega) = \mathrm{Lip}(\Omega)$.

- sum up the i-th entries to obrain

$$\int_{\partial\Omega\times Y} \phi(x,y)\langle \mathbf{n}_\Omega(x), \mathbf{n}_\Omega(x)\rangle \, d\mu_\partial(x,y) = \\ = \int_I \int_{\partial\Omega} \phi\big(x,u(x)\big)\langle \mathbf{n}_\Omega(x), \mathbf{n}_\Omega(x)\rangle \, d\sigma(x)\, d\nu(r).$$

And the lemma is proven.

□

Proof of Theorem 3.2.2.
Define

$$\mathcal{I} := F^{-1}(0) \cap G^{-1}\big((-\infty, 0]\big)$$

and

$$\mathcal{I}_\partial := F_\partial^{-1}(0) \cap G_\partial^{-1}\big((-\infty, 0]\big).$$

Claim 2.
The set $I_1 := \{r \in [-1, 0] : \psi_r$ *satisfies* $(3.5)\}$ *has strictly positive* ν *measure.*

Proof.
This is an easy consequence of $CV4$ and Lemma 3.2.15.

□

Claim 3.
For ν*-a.e.* r *we have*

$$\mathcal{L}^n\Big(\big\{x \in \Omega : \big(x, \psi_r(x), D\psi_r(x)\big) \notin \mathcal{I}\big\}\Big) = 0.$$

Denote by I_2 *the set of all such* r*.*

Proof.
We know that

- by claim 8 at page 111, for ν-a.e. r and $\mathcal{L}^n$-a.e. $x \in \Omega$ we have $D\psi_r(x) = \mathcal{Z}\big(x, \psi_r(x)\big)$;
- $\big(x, y, \mathcal{Z}(x, y)\big) \in \mathrm{co}\,\big((\{x\} \times \{y\} \times Z) \cap \mathrm{spt}(\mu)\big) \quad \forall (x, y) \in \Omega \times Y$ [7];
- by hypothesis $\mathrm{spt}(\mu) \subset \mathcal{I}$
- by hypothesis $CV5$ we have that $\mathcal{I} \cap (\{x\} \times \{y\} \times Z)$ is convex for all couples (x, y)

[7] We recall again that by $\mathrm{co}(A)$ we mean the convex hull of the set A.

From these facts it is clear that for ν-a.e. r and $\mathcal{L}^n$-a.e. $x \in \Omega$ we have

$$\big(x, \psi_r(x), D\psi_r(x)\big) \in \mathcal{I}.$$

$\square$

Claim 4.
For ν-a.e. r we have

$$\sigma\Big(\big\{x \in \partial\Omega : \big(x, \psi_r(x)\big) \notin \mathcal{I}_\partial\big\}\Big) = 0.$$

Denote by I_3 the set of all such r.

Proof.
It must be that for ν-a.e. r and σ-a.e. $x \in \partial\Omega$ we have

$$\big(x, \psi_r(x)\big) \in \mathrm{spt}(\mu_\partial),$$

otherwise $\mathbb{1}_{(\partial\Omega\times Y)\setminus \mathrm{spt}(\mu_\partial)}$ would be a measurable function contradicting Lemma 3.2.15(ii).
By hypothesis we know that $\mathrm{spt}(\mu_\partial) \subset \mathcal{I}_\partial$, so that the claim follows immediately.

$\square$

Now because of claims 2, 3 and 4, the set $I_1 \cap I_2 \cap I_3$ has strictly positive ν measure, and is therefore non-empty. Pick one $r_0 \in I_1 \cap I_2 \cap I_3$ and define $\overline{\varphi} := \psi_{r_0}$. It satisfies the seeked properties and thus *(i)* is proven.

We now proceed to prove *(ii)*.
Consider a mollifier χ, i.e. $\chi \in C^\infty\big(\mathbb{R}^n, [0, +\infty)\big)$ and it satisfies that $\mathrm{supp}(\chi) \subset \mathbf{B}_{\mathbb{R}^n}(0, 1)$ and $\displaystyle\int_{\mathbb{R}^n} \chi \, d\mathcal{L}^n = 1$.
Consider a map $h \in C^\infty(\Omega) \cap W^{1,\infty}(\overline{\Omega})$ such that $0 < h(x) < \mathrm{dist}\,\big(x, \partial\Omega\big)/2 \;\forall x \in \Omega$ and extend it to $\overline{\Omega}$ defining $h(x) := 0 \;\forall x \in \partial\Omega$.

Define

$$g_i(x) := \begin{cases} \dfrac{i^n}{h(x)^n} \displaystyle\int_{\mathbb{R}^n} \chi\left(i\frac{x-y}{h(x)}\right) \overline{\varphi}(y)\, dy & \text{if } x \in \Omega \\ \overline{\varphi}(x) & \text{if } x \in \partial\Omega \end{cases} \tag{3.44}$$

The sequence $\{g_i\}_i$ is an approximation of $\overline{\varphi}$ by convolution (with the slight modification of using the postive functions $\dfrac{h(x)}{i}$ which uniformly converge to zero in place of some positive constants ε_i converging to zero).
Since $\overline{\varphi}$ is Lipschitz continuous (and therefore absolutely continuous) we have uniform convergence (with respect to the L^∞ norm) of the sequence $\{g_i\}_i$ to $\overline{\varphi}$.
What is left to prove is that $g_i \in W^{1,\infty}(\overline{\Omega})$ $\forall i$. To prove this we denote by l the Lipschitz constant of $\overline{\varphi}$, we denote by H the Lipschitz constant of h, and we consider the following three different possibilities:

Case 1: $x_1, x_2 \in \partial\Omega$.

In this case $|g_i(x_2) - g_i(x_1)| = |\overline{\varphi}(x_2) - \overline{\varphi}(x_1)| \leq l|x_2 - x_1|$.

Case 2: $x_1, x_2 \in \Omega$.

In this case we have

$$
\begin{aligned}
&|g_i(x_1) - g_i(x_2)| = \\
&= \left| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{x_1 - y}{h(x_1)} \right) \overline{\varphi}(y)\, dy \;-\; \frac{i^n}{h(x_2)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{x_2 - y}{h(x_2)} \right) \overline{\varphi}(y)\, dy \right| \\
&= \left| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right) \overline{\varphi}(x_1 - y)\, dy \;-\; \frac{i^n}{h(x_2)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_2)} \right) \overline{\varphi}(x_2 - y)\, dy \right| \\
&\le \left| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right) \overline{\varphi}(x_1 - y)\, dy \;-\; \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right) \overline{\varphi}(x_2 - y)\, dy \right| + \\
&\;\; + \left| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right) \overline{\varphi}(x_2 - y)\, dy \;-\; \frac{i^n}{h(x_2)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_2)} \right) \overline{\varphi}(x_2 - y)\, dy \right| \\
&\le l||x_1 - x_2|| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right)\, dy + \\
&\;\; + \left| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right) \overline{\varphi}(x_2 - y)\, dy \;-\; \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{u}{h(x_1)} \right) \overline{\varphi}\left(x_2 - u \frac{h(x_2)}{h(x_1)} \right)\, du \right| \\
&\le l||x_1 - x_2|| + \\
&\;\; + \left| \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right)\, l \left\| y \left(1 - \frac{h(x_2)}{h(x_1)} \right) \right\|\, dy \right| \\
&\le l||x_1 - x_2|| \;+\; l \left(\sup_{||y|| \le \frac{h(x_1)}{i}} \left\| \frac{y}{h(x_1)} (h(x_1) - h(x_2)) \right\| \right) \frac{i^n}{h(x_1)^n} \int_{\mathbb{R}^n} \chi\left(i \frac{y}{h(x_1)} \right)\, dy \\
&\le l||x_1 - x_2|| \;+\; \frac{lH}{i} ||x_1 - x_2|| \\
&= \left(l + \frac{lH}{i} \right) ||x_1 - x_2||.
\end{aligned}
$$

Case 3: $x_1 \in \partial\Omega$, $x_2 \in \Omega$.

We have

$$
\begin{aligned}
&|g_i(x_1) - g_i(x_2)| = \\
&= \left| \overline{\varphi}(x_1) \; - \; \frac{i^n}{h(x_2)^n} \int_{\mathbb{R}^n} \chi \left(i \frac{y}{h(x_2)} \right) \overline{\varphi}(x_2 - y) \, dy \right| \\
&= \left| \frac{i^n}{h(x_2)^n} \int_{\mathbb{R}^n} \chi \left(i \frac{y}{h(x_2)} \right) \Big(\overline{\varphi}(x_1) - \overline{\varphi}(x_2 - y) \Big) \, dy \right| \\
&\leq \sup_{||y|| \leq \frac{h(x_2)}{i}} |\overline{\varphi}(x_1) - \overline{\varphi}(x_2 - y)| \\
&\leq l \sup_{||y|| \leq \frac{h(x_2)}{i}} ||x_1 - x_2 + y|| \\
&\leq l \sup_{||y|| \leq \frac{\operatorname{dist}(x_2, \partial\Omega)}{i}} ||x_1 - x_2 + y|| \\
&\leq l \sup_{||y|| \leq \frac{||x_2 - x_1||}{i}} ||x_1 - x_2 + y|| \\
&\leq l \Big(1 + \frac{1}{i} \Big) ||x_1 - x_2||.
\end{aligned}
$$

Lipschitz regularity on $\overline{\Omega}$ is proven and the proof of Theorem 3.2.2 is concluded.

$\square$

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