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The Kakeya problem

From Besicovitch sets to maximal estimates

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Introduction

In 1917, the Japanese mathematician Sōichi Takeya asked for the least area of a plane region inside which a unit segment, called a *needle*, can be continuously rotated through a half turn, returning to its initial position with its two ends interchanged.

While working on a question concerning Riemann integration, Besicovitch constructed a compact subset of the plane which contains a unit segment in every direction and has Lebesgue measure zero. Sets with this property are now referred to as *Takeya sets*, or *Besicovitch sets*. His construction, later simplified by Oskar Perron and known as the *Perron tree*, consists in iteratively bisecting a triangle and sliding the resulting subtriangles over one another, so as to preserve the set of directions of the segments contained in it while reducing its area.

Connecting the subtriangles of a Perron tree by corridors of small area also provides an answer to Takeya's original question: as shown in [Fal85], it is possible to rotate a needle within sets of arbitrarily small area. Hence the infimum is zero, even though it is not achieved, because any planar set in which a segment can be continuously turned must have positive measure [Tao08]. As Takeya sets can have Lebesgue measure zero, Lebesgue measure is no longer the appropriate notion of size, and is naturally replaced by the Hausdorff and Minkowski notions of dimension. The *Takeya conjecture* states that every Takeya set in $\mathbb{R}^n$ has Hausdorff dimension n , while a weaker version uses the lower Minkowski dimension in its place. Both versions were settled in the plane by Davies [Dav71], and recently proved in $\mathbb{R}^3$ by Wang and Zahl [WZ25]. In dimension $n \geq 4$ only partial lower bounds are available.

Beyond their geometric interest, Takeya sets are important for the role they play in harmonic analysis. Fefferman was the first to exploit the Besicovitch construction in this setting, showing that the characteristic function of the unit ball of $\mathbb{R}^n$, $n \geq 2$, is a Fourier multiplier on L^p only for $p = 2$ [Fef71]. The mechanism behind this and a number of other conjectures is that the object one is trying to estimate can be decomposed into elementary pieces concentrated on thin tubes; since the curvature of the problem makes these tubes point in different directions, they can be arranged in a Takeya-type configuration. Stein's restriction conjecture is one such example, as a Takeya set of Hausdorff dimension less than n would produce a counterexample to it. For this argument we refer the reader to [Mat15].

This thesis focuses on the connection between geometry and analysis provided by the *Takeya maximal function*, the operator that takes the supremum of the averages of a function over the thin tubes pointing in a given direction. The best one can hope for is a bound whose constant grows, as the width tends to zero, more slowly than any negative power of it: this is known as the *Takeya maximal conjecture*. Bounds of this type yield lower bounds for the dimension of Takeya sets; in particular, the Takeya maximal conjecture implies the Takeya conjecture. Our aim is to discuss the Takeya problem from an analytic point of view and to give a complete proof of the Takeya maximal conjecture in the plane.

In Chapter 1 we construct Besicovitch sets of measure zero, we discuss the needle problem, we recall the definitions of the Hausdorff and Minkowski dimensions and we state the two versions of the conjecture.

In Chapter 2 we introduce the Takeya maximal function, we show that L^p estimates give lower bounds for the Hausdorff dimension of Takeya sets and we prove Córdoba's L^2 estimate, which settles the planar case.

Chapter 1

Keakeya sets

We first introduce the main object of study of this thesis. We call *Keakeya set* (or, equivalently, *Besicovitch set*) a compact subset of $\mathbb{R}^n$ that contains a line segment of length 1 (also called a *needle*) in every possible direction. To be more formal, we will adopt the following definition.

Definition 1.1. We say that a compact set $K \subseteq \mathbb{R}^n$ is a *Keakeya set* if, for every direction $\omega \in \mathbb{S}^{n-1}$, there exists a position $a \in \mathbb{R}^n$ such that the line segment

$$\ell_{\omega,a} := \left\{ a + t\omega : -\frac{1}{2} \leq t \leq \frac{1}{2} \right\}$$

satisfies $\ell_{\omega,a} \subseteq K$.

In order to build some intuition let us look at three simple planar *Keakeya sets*: a disc of diameter 1, an equilateral triangle of height 1 and a deltoid¹. The last one has an interesting property: as the needle rotates inside the deltoid, its endpoints slide along the curve and the segment stays tangent to it at an interior point, as one can see in Figure 1.1.

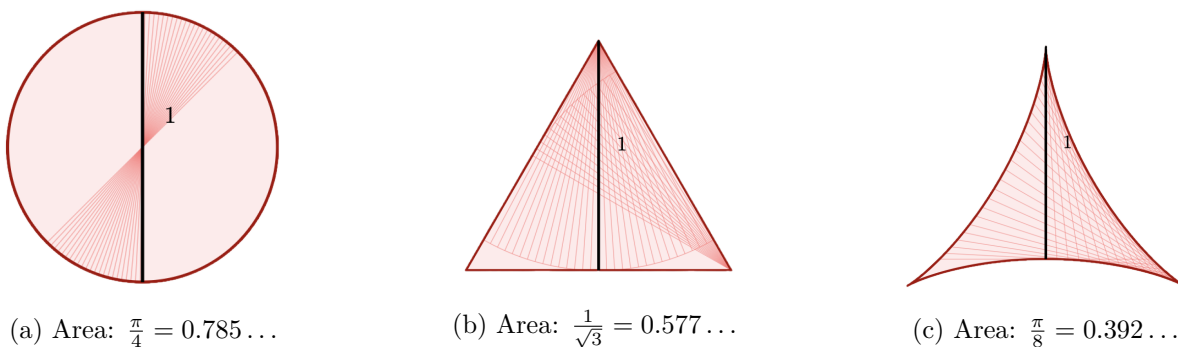


Figure 1.1: Examples of planar Keakeya sets.

Observation 1.2. The definition only asks the set to contain a segment in each direction; it does not require the segment to be moved continuously (i.e., translated and rotated). In each example in Figure 1.1, however, this stronger continuous motion is also possible. Sets with this additional property will be studied in Section 1.2.

The example we shall dwell on the most is the triangle. It allows the needle to perform a full rotation: the segment is first rotated about a vertex, sweeping out the angle at that vertex, and then translated so that the procedure can be repeated at the next vertex. Besides being the convex Keakeya set of least area, as proved by Julius Pál in [Pál21], the triangle is, when

¹ One possible characterization of this particular deltoid is as the region bounded by the curve traced by a point on a circle rolling inside a circle of three times its radius.

we focus on a single vertex, the simplest shape in which the segment can be rotated through a prescribed angle; for this reason it will serve as the basic building block for constructing different sets, with rather different properties.

If we compare the areas of the three Kakeya sets in Figure 1.1, we see they get successively smaller. In particular, the deltoid even has half the area of the disc. This means that, intuitively, inside the deltoid the segments are packed much more tightly.

This suggests the first natural question one may ask about such sets: what is the minimal Lebesgue measure of a Kakeya set? In other words, in a terminology that will prove useful in later chapters, how tightly can the needles be packed together?

1.1 Besicovitch's construction

Following the heuristic at the end of the previous section, that needles can be packed in more or less efficient ways, Besicovitch found a way to construct planar Kakeya sets of arbitrarily small area. In the literature, procedures where objects are repacked to reduce some notion of measure are referred to as the *Besicovitch compression phenomenon*.

Theorem 1.3. *For every $\varepsilon > 0$ there exists a Kakeya set $K \subseteq \mathbb{R}^2$ with $\mathcal{L}^2(K) < \varepsilon$.*

To construct such a set we shall use a procedure based on iteratively bisecting a triangle and sliding the two halves onto one another. At each step the resulting figure preserves every direction in which the original triangle contains a unit segment, while its area decreases, since parts of the triangle are made to overlap. We follow the presentation given in [Fox21]. The main idea for the construction is the following geometric argument.

Lemma 1.4. *Let T be a triangle that has a base, on line L , of length $2b$ and a height of h . Let T_1 and T_2 be triangles that result from cutting T along the median that goes to the middle of the side of T that lies on line L . Fix $\alpha \in (\frac{1}{2}, 1)$ and slide T_1 along L over T_2 by a distance of $2b(1 - \alpha)$. The resulting figure S has $(1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T)$ less area than T .*

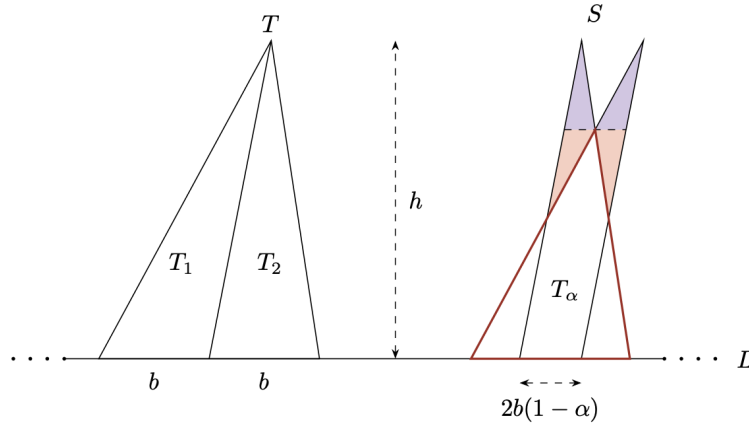


Figure 1.2

Proof. Using simple angle chasing and length measurements, we notice that S is formed by a triangle T_α similar to T scaled by a factor α , together with two additional triangles which can be cut and rearranged to form two triangles similar to T scaled by a factor $(1 - \alpha)$ (see Figure 1.2). We can therefore compute the area of S as

$$\mathcal{L}^2(S) = \alpha^2 \mathcal{L}^2(T) + 2(1 - \alpha)^2 \mathcal{L}^2(T)$$

and hence

$$\mathcal{L}^2(T) - \mathcal{L}^2(S) = (1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T). \quad \square$$

From this lemma we can prove the following theorem.

Theorem 1.5. *Let T be a triangle with base on a line L and apex $v \notin L$. For $k \in \mathbb{N}$, we can divide the base of T into 2^k equal segments, which serve as the bases of 2^k triangles $T_1, \dots, T_{2^k}$ sharing the common apex v . Then, for any $\varepsilon > 0$, there exist $\alpha \in (\frac{1}{2}, 1)$ and k such that these 2^k triangles can be slid along L so that the resulting figure S has area less than ε . Moreover, S still contains a unit segment in every direction subtended by T at v , provided that every segment joining v to the base of T has length at least 1, and, if $T \subseteq U$ for some open set U , we can also ensure that $S \subseteq U$.*

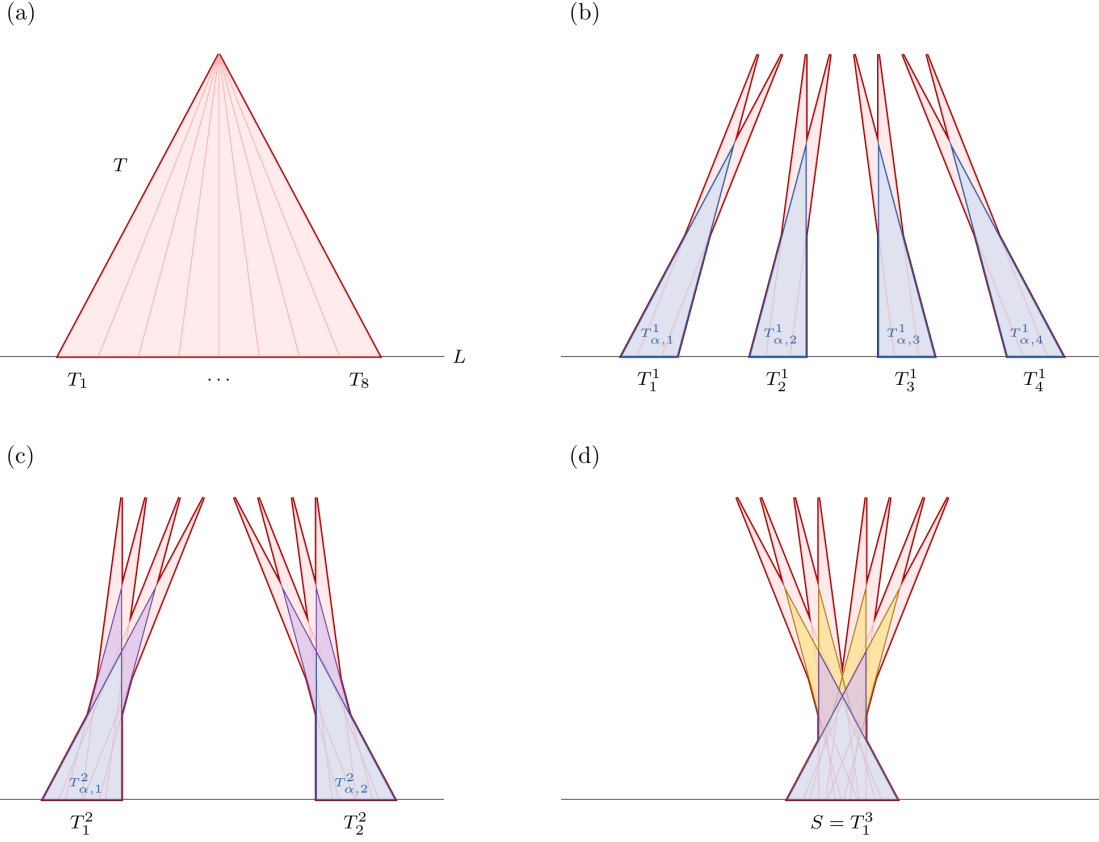


Figure 1.3

Proof. For clarity, we follow Figure 1.3 to build up the final set; there we have set $k = 3$ and $\alpha = \frac{3}{4}$. For the proof $\alpha \in (1/2, 1)$ and $k \in \mathbb{N}$ are to be determined later.

- (a) Divide the base of T into 2^k equal segments, obtaining the 2^k triangles $T_1, \dots, T_{2^k}$ as in the statement.
- (b) We first work with consecutive pairs among these triangles. For $1 \leq i \leq 2^{k-1}$, we slide T_{2i} towards T_{2i-1} along L as in Lemma 1.4 to produce a figure T_i^1 . By the same lemma, T_i^1 contains a triangle $T_{\alpha,i}^1$ which is similar to $T_{2i-1} \cup T_{2i}$ with similarity ratio α , and we have the following reduction in area:

$$\mathcal{L}^2(T_{2i-1} \cup T_{2i}) - \mathcal{L}^2(T_i^1) = (1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T_{2i-1} \cup T_{2i}).$$

- (c) We can now repeat the procedure carried out in (b), this time starting from the figures it produced: we slide T_{2i-1}^1 and T_{2i}^1 towards each other until they overlap, and we call T_i^2 the resulting figure, for $1 \leq i \leq 2^{k-2}$. Observe that the triangles $T_{\alpha,2i-1}^1$ and $T_{\alpha,2i}^1$, contained respectively in T_{2i-1}^1 and T_{2i}^1 , have their facing sides parallel and of the same length. Hence, when we apply Lemma 1.4 to superimpose T_{2i-1}^1 and T_{2i}^1 , these two smaller triangles get superimposed as well, and their union contains a triangle $T_{\alpha,i}^2$ which is similar to $T_{\alpha,2i-1}^1 \cup T_{\alpha,2i}^1$ with ratio α , while their overlap accounts for the area we save. This is what allows us to bound from below the area we save in passing from the two figures T_{2i-1}^1 and T_{2i}^1 to the single figure T_i^2 :

$$\begin{aligned} \mathcal{L}^2(T_{2i-1}^1 \cup T_{2i}^1) - \mathcal{L}^2(T_i^2) &\geq (1-\alpha)(3\alpha-1) \mathcal{L}^2(T_{\alpha,2i-1}^1 \cup T_{\alpha,2i}^1) \\ &= (1-\alpha)(3\alpha-1) \alpha^2 \mathcal{L}^2(T_{4i-3} \cup T_{4i-2} \cup T_{4i-1} \cup T_{4i}), \end{aligned}$$

the last equality being a consequence of the fact that $T_{\alpha,2i-1}^1$ and $T_{\alpha,2i}^1$ are similar, with ratio α , to $T_{4i-3} \cup T_{4i-2}$ and to $T_{4i-1} \cup T_{4i}$ respectively. Notice that in this estimate we are taking into account only the area saved by the overlapping of the triangles $T_{\alpha,i}^1$, and we are discarding any further overlapping that may occur.

- (d) Iterating this construction k times in total, we are left with a single figure, which we denote by $S := T_1^k$. At the j -th step, for $1 \leq j \leq k$, we superimpose the 2^{k-j} pairs T_{2i-1}^{j-1} , T_{2i}^{j-1} and we obtain the figures T_i^j , $1 \leq i \leq 2^{k-j}$; here it is understood that $T_i^0 = T_{\alpha,i}^0 = T_i$, so that for $j = 1$ we recover exactly the situation of (b). Each one of these superpositions gives the following lower bound for the corresponding reduction in area:

$$\begin{aligned} \mathcal{L}^2(T_{2i-1}^{j-1} \cup T_{2i}^{j-1}) - \mathcal{L}^2(T_i^j) &\geq (1-\alpha)(3\alpha-1) \mathcal{L}^2(T_{\alpha,2i-1}^{j-1} \cup T_{\alpha,2i}^{j-1}) \\ &= (1-\alpha)(3\alpha-1) \alpha^2 \mathcal{L}^2(T_{\alpha,4i-3}^{j-2} \cup \dots \cup T_{\alpha,4i}^{j-2}) \\ &= \dots \\ &= (1-\alpha)(3\alpha-1) \alpha^{2(j-1)} \mathcal{L}^2(T_{(i-1)2^{j-1}+1} \cup \dots \cup T_{i2^{j-1}}). \end{aligned}$$

The reduction is caused by the overlap of $T_{\alpha,2i-1}^{j-1}$ and $T_{\alpha,2i}^{j-1}$. When $T_{\alpha,2i-1}^{j-1}$ and $T_{\alpha,2i}^{j-1}$ overlap the superposition creates inside T_i^j the triangle $T_{\alpha,i}^j$. The intermediate equalities come from applying the similarity of ratio α , between $T_{\alpha,i}^l$ and $T_{\alpha,2i-1}^{l-1} \cup T_{\alpha,2i}^{l-1}$ $j-1$ times.

Summing over the 2^{k-j} pairs involved, we can now compute the reduction in area produced at a generic j -th step:

$$\begin{aligned} \sum_{i=1}^{2^{k-j}} \left(\mathcal{L}^2(T_{2i-1}^{j-1} \cup T_{2i}^{j-1}) - \mathcal{L}^2(T_i^j) \right) &\geq (1-\alpha)(3\alpha-1) \alpha^{2(j-1)} \mathcal{L}^2(T_1 \cup \dots \cup T_{2^k}) \\ &= (1-\alpha)(3\alpha-1) \alpha^{2(j-1)} \mathcal{L}^2(T). \end{aligned}$$

Explicitly, these are the following k inequalities:

$$\begin{aligned} \sum_{i=1}^{2^{k-1}} (\mathcal{L}^2(T_{2i-1} \cup T_{2i}) - \mathcal{L}^2(T_i^1)) &\geq (1-\alpha)(3\alpha-1) \mathcal{L}^2(T), \\ \sum_{i=1}^{2^{k-2}} (\mathcal{L}^2(T_{2i-1}^1 \cup T_{2i}^1) - \mathcal{L}^2(T_i^2)) &\geq (1-\alpha)(3\alpha-1) \alpha^2 \mathcal{L}^2(T), \\ &\vdots \\ \mathcal{L}^2(T_1^{k-1} \cup T_2^{k-1}) - \mathcal{L}^2(T_1^k) &\geq (1-\alpha)(3\alpha-1) \alpha^{2(k-1)} \mathcal{L}^2(T). \end{aligned}$$

Summing them for j from 1 to k , we obtain a lower bound for the overall reduction in area:

$$\begin{aligned} \sum_{i=1}^{2^{k-1}} \mathcal{L}^2(T_{2i-1} \cup T_{2i}) - \mathcal{L}^2(T_1^k) &\geq \sum_{j=1}^k \sum_{i=1}^{2^{k-j}} \left(\mathcal{L}^2(T_{2i-1}^{j-1} \cup T_{2i}^{j-1}) - \mathcal{L}^2(T_i^j) \right) \\ &\geq (1-\alpha)(3\alpha-1)(1+\alpha^2+\dots+\alpha^{2(k-1)}) \mathcal{L}^2(T), \end{aligned}$$

where the first inequality holds since the middle terms cancel, the sum being telescopic up to the bound $\mathcal{L}^2(T_{2i-1}^j \cup T_{2i}^j) \leq \mathcal{L}^2(T_{2i-1}^j) + \mathcal{L}^2(T_{2i}^j)$, which points in the right direction for the inequality.

Equivalently, since $T_1^k = S$ and $\sum_{i=1}^{2^{k-1}} \mathcal{L}^2(T_{2i-1} \cup T_{2i}) = \mathcal{L}^2(T)$, we have

$$\begin{aligned} \mathcal{L}^2(S) &\leq \mathcal{L}^2(T) - (1-\alpha)(3\alpha-1)(1+\alpha^2+\dots+\alpha^{2(k-1)}) \mathcal{L}^2(T) \\ &= \mathcal{L}^2(T) - \frac{(1-\alpha)(3\alpha-1)(1-\alpha^{2k})}{1-\alpha^2} \mathcal{L}^2(T) \\ &= \left(1 - \frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha} \right) \mathcal{L}^2(T). \end{aligned}$$

We now show that, with a suitable choice of α and k , the measure of S can be made as small as we like. Let $\varepsilon > 0$, set $\varepsilon' := \frac{\varepsilon}{\mathcal{L}^2(T)}$ and writing $\alpha = 1 - \delta$ with $0 < \delta < \frac{1}{2}$ we have

$$1 - \frac{3\alpha-1}{1+\alpha} = 1 - \frac{2-3\delta}{2-\delta} = \frac{2\delta}{2-\delta} \xrightarrow{\delta \rightarrow 0^+} 0,$$

so we may fix $\delta \in (0, \frac{1}{2})$, and hence $\alpha = 1 - \delta$, such that

$$1 - \frac{3\alpha-1}{1+\alpha} < \frac{\varepsilon'}{2}.$$

With α now fixed we have $0 < \alpha < 1$, so $\alpha^{2k} \rightarrow 0$ as $k \rightarrow \infty$, and we may choose $k \in \mathbb{N}$ large enough that

$$\frac{3\alpha-1}{1+\alpha} \alpha^{2k} < \frac{\varepsilon'}{2}.$$

Adding the two estimates,

$$\begin{aligned} \mathcal{L}^2(S) &\leq \left(1 - \frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha} \right) \mathcal{L}^2(T) \\ &= \left[\left(1 - \frac{3\alpha-1}{1+\alpha} \right) + \frac{3\alpha-1}{1+\alpha} \alpha^{2k} \right] \mathcal{L}^2(T) \\ &< \left(\frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} \right) \mathcal{L}^2(T) = \varepsilon' \mathcal{L}^2(T) = \varepsilon. \end{aligned}$$

This shows that, by choosing α and k appropriately, we can make $\mathcal{L}^2(S)$ arbitrarily small. As for the directions, each T_i is only translated, never rotated, so in its final position it still subtends at its apex the same set of directions as it did inside T ; since these sets together cover all the directions subtended by T , the figure S contains a unit segment in every direction subtended by T .

Suppose $U \overset{\circ}{\subset} \mathbb{R}^2$ with $T \subseteq U$. Since T is compact, U^C is closed, and the two sets are disjoint,

$$\text{dist}(T, U^C) := \inf \{ \|x - y\| : x \in T, y \in U^C \} > 0,$$

so we may fix η with $0 < \eta < \text{dist}(T, U^C)$. Observe first that if the base length of T is b , then, fixing T_1 and locating the other subtriangles relative to T_1 , no subtriangle is moved by a distance of more than b during the construction. Therefore if we split the triangle into smaller subtriangles with base length no more than η and do the process mentioned earlier on each group we end up with a figure S that has arbitrarily small area and none of the subtriangles has been moved by more than η . Since every point of S lies at distance less than $\eta < \text{dist}(T, U^C)$ from T , no point of S can belong to U^C . Thus $S \subseteq U$, as required. $\square$

Observation 1.6. Another way to see what the open-set part of the statement says is the following: for every $\varepsilon > 0$ and every $\delta > 0$ there exists a figure S with $\mathcal{L}^2(S) < \varepsilon$ and $S \subseteq T_\delta$, where $T_\delta := \{p \in \mathbb{R}^2 : \text{dist}(p, T) < \delta\}$ denotes the δ -neighbourhood of T . We could make S stick out of T by as little as we please.

This theorem shows that Kakeya sets of arbitrarily small measure exist: although the triangle has been cut several times, none of the directions associated with its apex is lost. The entire family of directions that emanates from that vertex will be preserved, whereas the measure of the new figure created can be made arbitrarily small. By rotating the construction and taking finitely many copies of it (if we start the construction with an equilateral triangle we will need 3 copies), one obtains a set which contains a unit segment in every direction. This proves Theorem 1.3.

Applying a limiting argument to the Besicovitch construction in combination with Observation 1.6 proves a stronger version of the theorem.

Theorem 1.7 (Besicovitch). *There exists a Kakeya set $K \subseteq \mathbb{R}^2$ with $\mathcal{L}^2(K) = 0$.*

Proof. Let T_1 be, for simplicity, an equilateral triangle of height 1 whose base lies on the x -axis. We choose a bounded open set U_1 such that $T_1 \subseteq U_1$ and $\mathcal{L}^2(\overline{U}_1) \leq 2\mathcal{L}^2(T_1)$. By Theorem 1.5 we know that we can divide T_1 into subtriangles and slide them to form a new figure $T_2 \subseteq U_1$ that also satisfies $\mathcal{L}^2(T_2) \leq 2^{-2}$.

We can then repeat the process, applying Theorem 1.5 to each of the subtriangles making up T_2 : we find a bounded open set U_2 such that $T_2 \subseteq U_2 \subseteq U_1$ and $\mathcal{L}^2(\overline{U}_2) \leq 2\mathcal{L}^2(T_2)$, and we find a figure $T_3 \subseteq U_2$ such that $\mathcal{L}^2(T_3) \leq 2^{-3}$, and such that it preserves all the directions of the initial triangle T_1 .

Continuing the process we get a sequence of figures $\{T_i\}_{i=1}^\infty$ with $\mathcal{L}^2(T_i) \leq 2^{-i}$ for $i \geq 2$, and a sequence of bounded open sets $\{U_i\}_{i=1}^\infty$ with

$$T_i \subseteq U_i \subseteq U_{i-1} \quad \text{and} \quad \mathcal{L}^2(\overline{U}_i) \leq 2\mathcal{L}^2(T_i) \leq 2^{-i+1}$$

for all $i \in \mathbb{N}^+$, $i \geq 2$. Let

$$K_1 = \bigcap_{i=1}^\infty \overline{U}_i.$$

Then $\mathcal{L}^2(K_1)$ has to be equal to 0, since $K_1 \subseteq \overline{U}_i$ for every i , and therefore

$$\mathcal{L}^2(K_1) \leq \mathcal{L}^2(\overline{U}_i) \leq 2^{-i+1} \quad \text{for all } i \in \mathbb{N}^+, i \geq 2.$$

We know that every T_i contains a unit segment in each direction forming an angle between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with the positive x -axis (this is true for the initial equilateral triangle T_1 , and the construction of Theorem 1.5 preserves it), and hence so does $\overline{U}_i$, since $T_i \subseteq \overline{U}_i$. What remains to prove is that K_1 has all these directions as well.

Let ρ be a direction forming an angle in $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ with the x -axis. For each $i \in \mathbb{N}^+$ there exists a unit segment $\gamma_i \subseteq \overline{U}_i$ in the direction ρ . Consider $x_i \in \gamma_i$, the endpoint of γ_i closest to the

x -axis (well defined, since ρ is not horizontal). Since the sequence $\{U_i\}$ is decreasing, every γ_i is contained in $\overline{U}_1$, which is compact; hence, up to passing to a subsequence, we can assume that $\{x_i\}$ converges to a point x . Now we consider the unit segment γ in the direction ρ whose nearest point to the x -axis is x . Since all the γ_i are unit segments in the same direction ρ , we must have that γ_i converges to γ in the Hausdorff distance².

Since $\{\overline{U}_i\}_{i=1}^\infty$ is decreasing, we have $\gamma_i \subseteq \overline{U}_j$ for all $i, j \in \mathbb{N}^+$ with $i \geq j$, and because $\overline{U}_j$ is closed, $\gamma \subseteq \overline{U}_j$ for every j . Hence $\gamma \subseteq K_1$.

Therefore K_1 is a set of measure 0 which contains a unit line segment in every direction between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. To obtain all the directions we just have to add 2 copies of K_1 , namely K_2 and K_3 , that are rotated counterclockwise by $\pi/3$ and $2\pi/3$ respectively. Then

$$K := K_1 \cup K_2 \cup K_3$$

is a finite union of null sets, so $\mathcal{L}^2(K) = 0$, and it contains a unit line segment in every direction. Figure 1.4 represents an intermediate step of the construction. $\square$

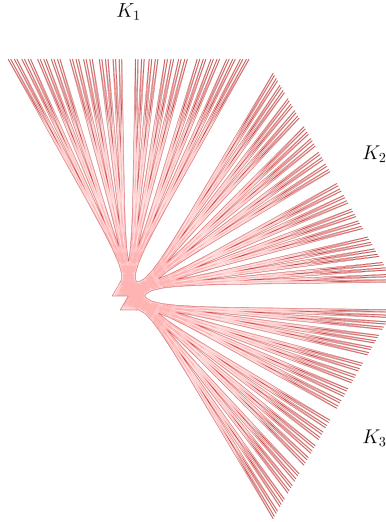


Figure 1.4

The set obtained in this way is usually referred to as the *Perron tree*, named after Oskar Perron, the German mathematician who refined the construction found by Besicovitch.

Corollary 1.8. *Given a Keakeya set $K \subseteq \mathbb{R}^2$ with $\mathcal{L}^2(K) = 0$, the product $K \times [0, 1]^{n-2}$ is a Keakeya set in $\mathbb{R}^n$ which is $\mathcal{L}^n$ -null. Hence Besicovitch's theorem also implies that for every $n \geq 2$ there exists a Keakeya set $K_n \subseteq \mathbb{R}^n$ with $\mathcal{L}^n(K_n) = 0$.*

Proof. The set $K \times [0, 1]^{n-2}$ has measure zero, since $\mathcal{L}^n(K \times [0, 1]^{n-2}) = \mathcal{L}^2(K) \cdot \mathcal{L}^{n-2}([0, 1]^{n-2}) = 0$. Let $\omega = (\omega', \omega'') \in \mathbb{S}^{n-1}$, so that $|\omega'|, |\omega''| \leq 1$, and let $b := (\frac{1}{2}, \dots, \frac{1}{2})$ be the centre of the cube. Since K is a Keakeya set there is $a \in \mathbb{R}^2$ with $\ell_{\omega'/|\omega'|, a} \subseteq K$ (if $\omega' = 0$, take any $a \in K$), and as $|\omega'| \leq 1$ the segment $\{a + t\omega' : |t| \leq \frac{1}{2}\}$ is contained in it; likewise $|\omega''| \leq 1$ gives $b + t\omega'' \in [0, 1]^{n-2}$ for $|t| \leq \frac{1}{2}$. Hence $\ell_{\omega, (a, b)} \subseteq K \times [0, 1]^{n-2}$. $\square$

² Given two non-empty compact sets $E, F \subseteq \mathbb{R}^n$, their *Hausdorff distance* is

$$d_H(E, F) := \max \left\{ \sup_{p \in E} \text{dist}(p, F), \sup_{q \in F} \text{dist}(q, E) \right\}.$$

Observation 1.9. It is worth pointing out that Besicovitch was led to these sets by a question in Riemann integration.

Question. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Riemann integrable function defined on the plane. Does there always exist a pair of orthogonal coordinate axes such that, for every $y \in \mathbb{R}$, the integral

$$F(y) = \int_{-\infty}^{+\infty} f(x, y) dx$$

exists as a Riemann integral, and such that the resulting function F of y is Riemann integrable as well?

We will show that it is possible to construct a counterexample based on the existence of a compact set $K \subseteq \mathbb{R}^2$ with $\mathcal{L}^2(K) = 0$ containing a unit segment in every direction. Set

$$D := \{p \in \mathbb{R}^2 : \text{dist}(p, (0, 0)) \in \mathbb{Q}\}, \quad f := \chi_K \cdot \chi_D = \chi_{K \cap D},$$

so that f is the characteristic function of the set of points of K lying at rational distance from the origin. We can apply Lebesgue's criterion: f is bounded, compactly supported, and its set of discontinuity points is contained in K , which has planar Lebesgue measure zero; so it is Riemann integrable on the plane.

Now we have to show that no choice of orthogonal axes works. We fix any orthogonal system we like. For every line $r \subseteq \mathbb{R}^2$ we know that both $r \cap D$ and $r \setminus D$ are dense in r . Since K is a Keakeya set, we know that it contains a unit segment ℓ parallel to the first axis, say at a height $\bar{y}$. On that segment we have $f = \chi_D$, so $x \mapsto f(x, \bar{y})$ takes both the values 0 and 1 on every subinterval of ℓ , and so the integral

$$F(\bar{y}) = \int_{-\infty}^{+\infty} f(x, \bar{y}) dx$$

will not exist in the Riemann sense.

1.2 Keakeya needle problem

Closely related to the notion of a *Keakeya set* is that of a *classical Keakeya set*. The question originally posed by the Japanese mathematician Sōichi Keakeya concerned sets of this kind. For the proofs that follow we need some formalism, so we give a precise definition of these sets.

Definition 1.10. We say that a compact set $K \subseteq \mathbb{R}^2$ is a *classical Keakeya set* if a unit line segment can be continuously rotated inside K through every direction. This means that there exist continuous functions

$$\theta : [0, 1] \rightarrow \mathbb{R}, \quad a : [0, 1] \rightarrow \mathbb{R}^2,$$

giving at each time s the direction $\omega(s) := (\cos \theta(s), \sin \theta(s)) \in \mathbb{S}^1$ and the midpoint $a(s)$ of the segment, such that the line segment

$$\ell_{\omega(s), a(s)} := \left\{ a(s) + t\omega(s) : -\frac{1}{2} \leq t \leq \frac{1}{2} \right\}$$

satisfies $\ell_{\omega(s), a(s)} \subseteq K$ for every $s \in [0, 1]$, and every direction $\omega \in \mathbb{S}^1$ is of the form $\pm \omega(s)$ for some $s \in [0, 1]$.

Observation 1.11. For clarity, and since our interest lies in the planar case, we have stated Definition 1.10 in $\mathbb{R}^2$. The notion can be extended to $\mathbb{R}^n$ in the following manner. One seeks continuous maps $\omega : [0, 1] \rightarrow \mathbb{S}^{n-1}$ and $a : [0, 1] \rightarrow \mathbb{R}^n$ which satisfy the condition that $\ell_{\omega(s), a(s)} \subseteq K$ for each value of $s \in [0, 1]$, and such that every direction of $\mathbb{S}^{n-1}$ appears as $\omega(s)$ for some s , at least up to a sign.

The *Kakeya needle problem* asks for the minimal area of such a set. After requiring the continuous rotation of the needle, the family of admissible sets becomes smaller than in the *Kakeya set* Definition 1.1, so one might expect the answer to be larger. Following the argument in [Tao08], we first show that in the plane a *classical Kakeya set* must have positive area; in particular, it cannot have measure zero as in Theorem 1.7.

Theorem 1.12. *Let $E \subset \mathbb{R}^2$ be a measurable set within which a unit line segment can be continuously rotated (through a nonzero angle). Then E has strictly positive Lebesgue measure.*

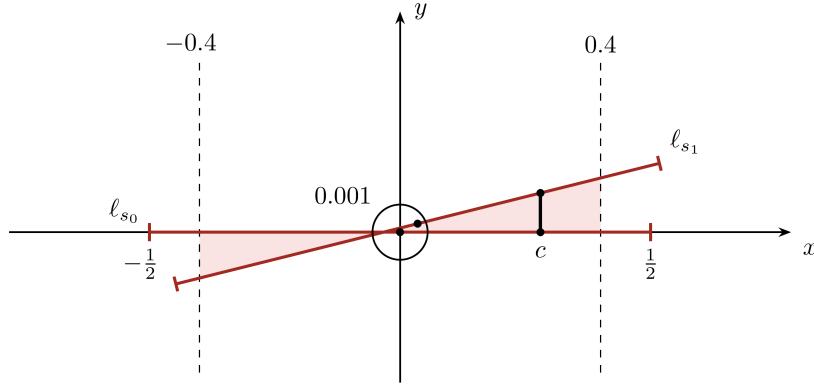


Figure 1.5

Proof. To say that a unit line segment can be continuously rotated inside E means that there exists a continuous map $s \mapsto \ell_{\omega(s), a(s)}$, defined for $s \in [0, 1]$, sending each time to a unit line segment $\ell_{\omega(s), a(s)} \subseteq E$. Each such segment is parameterised as

$$\ell_{\omega(s), a(s)} = \left\{ (x(s) + t \cos \theta(s), y(s) + t \sin \theta(s)) : -\frac{1}{2} \leq t \leq \frac{1}{2} \right\},$$

where $x, y, \theta : [0, 1] \rightarrow \mathbb{R}$ are continuous functions. Here $a(s) = (x(s), y(s))$ is the midpoint of the segment at time s , and $\theta(s)$ is the angle it makes with the horizontal axis.

On a compact set, every continuous function is uniformly continuous. In particular, there exists $\varepsilon > 0$ such that

$$|x(s) - x(s')|, |y(s) - y(s')|, |\theta(s) - \theta(s')| \leq 0.001 \quad (1)$$

whenever $s, s' \in [0, 1]$ satisfy $|s - s'| \leq \varepsilon$. Fix such an ε .

Observe that $\theta(s)$ cannot be a constant function of s , since we ask the needle to rotate even a little. We conclude that there must exist $s_0, s_1 \in [0, 1]$ with $|s_0 - s_1| \leq \varepsilon$ and $\theta(s_0) \neq \theta(s_1)$. Without loss of generality, we may assume that $s_0 < s_1$ and that

$$x(s_0) = y(s_0) = \theta(s_0) = 0,$$

after a translation and rotation of the plane (which do not affect the measure of E).

Now let c be any real number with $-0.4 \leq c \leq 0.4$. From (1) we see that, for any s with $s_0 \leq s \leq s_1$, the segment $\ell_{\omega(s), a(s)}$ intersects the vertical line $x = c$ in some point $(c, y_c(s))$, which must therefore lie in E . Indeed, since $|\theta(s)| \leq 0.001$ and $|x(s)| \leq 0.001$, the segment $\ell_{\omega(s), a(s)}$ is nearly horizontal and its x -coordinate ranges over an interval containing $[-0.4, 0.4]$, so the intersection with the line $x = c$ exists and is unique. Moreover, $y_c(s)$ is a continuous function of s . Thus, by the intermediate value theorem, the vertical segment from $(c, y_c(s_0))$ to $(c, y_c(s_1))$ lies entirely in E . Moreover $y_c(s_0) \neq y_c(s_1)$ for all but at most one value of c : since $\theta(s_0) \neq \theta(s_1)$, the two lines supporting $\ell_{\omega(s_0), a(s_0)}$ and $\ell_{\omega(s_1), a(s_1)}$ are not parallel, and hence meet in a single point.

Taking the union over all c between -0.4 and $+0.4$, we conclude that E contains the non-degenerate two-dimensional region bounded by these two non-parallel lines within the strip $|x| \leq 0.4$, and hence has strictly positive area. $\square$

Observation 1.13. Theorem 1.12 is only valid for dimension 2, since it cannot be generalised to higher dimensions. In $\mathbb{R}^n$ with $n \geq 3$ a unit segment (or even a full line) can be continuously rotated inside a set of measure zero, as proved in [Jär+11].

So being able to continuously rotate a line segment inside a set in the plane forces the set to have positive area. However, there exist *classical Kakeya sets* of arbitrarily small area. To prove this fact, we follow [Fal85]. We need the following lemma.

Lemma 1.14. *Let L_1 and L_2 be two parallel lines in $\mathbb{R}^2$ and let $\varepsilon > 0$ be given. Then there exists a set A containing L_1 and L_2 such that a unit-length segment $\gamma \subseteq L_1$ can be moved continuously onto L_2 without leaving A and such that $\mathcal{L}^2(A) < \varepsilon$.*

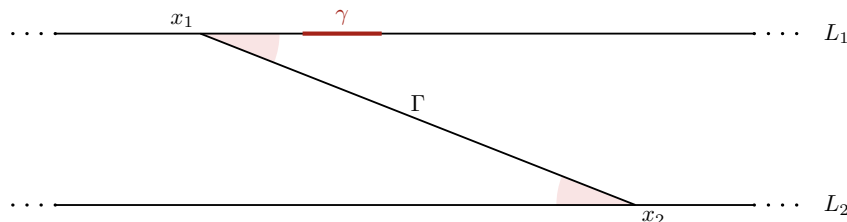


Figure 1.6

Proof. Choose $x_1 \in L_1$ and $x_2 \in L_2$ so that $|x_1 - x_2| \geq 1$, and let Γ be the segment joining x_1 to x_2 . Let $\theta \in (0, \pi/2)$ be the angle that Γ makes with the two lines, which is the same at both ends since L_1 and L_2 are parallel. Let S_1 be the sector of radius 1 and angle θ with centre at x_1 , lying between Γ and L_1 , and let S_2 be the sector of radius 1 and angle θ with centre at x_2 , lying between Γ and L_2 . Define

$$A := L_1 \cup L_2 \cup \Gamma \cup S_1 \cup S_2.$$

We can move γ in the following way: we move it along L_1 until one of its ends reaches x_1 , we rotate it about x_1 within S_1 until it lies on Γ , we move it along Γ until the other end arrives at x_2 , we rotate it about x_2 within S_2 until it lies on L_2 , and finally we move it freely along L_2 . Each of the steps in this movement is continuous and remains within A .

Since L_1 , L_2 and Γ are Lebesgue null sets, the measure of A is determined solely by the area swept out by each rotation of a unit segment, which is a sector of area $\frac{1}{2}\theta$; hence $\mathcal{L}^2(A) = \theta$.

Since $\theta \rightarrow 0$ as the distance between x_1 and x_2 tends to infinity, choosing x_1 and x_2 far enough apart gives $\mathcal{L}^2(A) < \varepsilon$. $\square$

We combine this lemma with the Besicovitch construction described in the proof of Theorem 1.5 to construct a *classical Kakeya set* of arbitrarily small measure.

Observation 1.15. We have defined *Kakeya set* as a compact set, so in the construction below we do not use the whole lines L_1 and L_2 of Lemma 1.14; we will replace each L_i by a closed subsegment.

Theorem 1.16. *For every $\varepsilon > 0$ there exists a classical Kakeya set K with $\mathcal{L}^2(K) < \varepsilon$.*

Proof. For this proof we combine the construction of Theorem 1.5 with Lemma 1.14.

Let T be an equilateral triangle of height 1 with base on the x -axis, and fix $k \in \mathbb{N}$. Following Theorem 1.5, we may split T into subtriangles $T_1, \dots, T_{2^k}$ and slide them along the x -axis, so as to obtain a figure S with

$$\mathcal{L}^2(S) < \frac{\varepsilon}{12},$$

in which all the directions of T are preserved. Notice that, because T is of height 1, any line segment connecting the apex of T_i to its base will have length at least 1; thus, within each triangle T_i , we can rotate the unit segment around the apex by the full angle that it spans. What remains to be done is to find the continuous movement from one subtriangle to another. The property which we will be using, together with Lemma 1.14, is that any two adjacent subtriangles T_i and T_{i+1} have a pair of parallel sides, due to the way they were constructed. Within T , they have a common side and the translation preserves that parallelism. (We will use the same name T_i for the subtriangles as they are moved into place.)

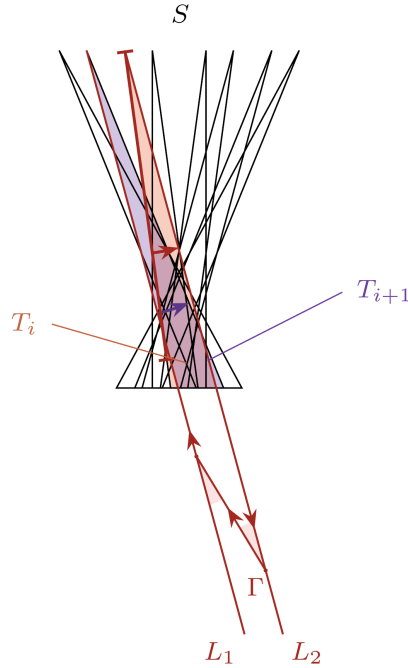


Figure 1.7

Thus, we now know, by Lemma 1.14, that for every $1 \leq i \leq 2^k - 1$ there exists a set A_i with

$$\mathcal{L}^2(A_i) < \frac{\varepsilon}{12(2^k - 1)}$$

inside which a unit segment can be moved with a continuous motion from T_i to T_{i+1} , so that the A_i adjoin the subtriangles to one another. Moreover, by Observation 1.15 we can choose the sets A_i to be compact.

We set

$$K_1 := S \cup \left(\bigcup_{i=1}^{2^k-1} A_i \right),$$

and subadditivity gives us

$$\mathcal{L}^2(K_1) < \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{\varepsilon}{6}.$$

Inside K_1 a unit segment can now be moved continuously through all the directions of T : the two extreme positions are the two non-horizontal sides of the original triangle, so the segment is rotated by $\frac{\pi}{3}$ with respect to its starting position.

To obtain a full rotation we add K_2 and K_3 , the copies of K_1 rotated by $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively, and we adjoin them by means of sets A'_1, A'_2, A'_3 with

$$\mathcal{L}^2(A'_j) < \frac{\varepsilon}{6}, \quad j = 1, 2, 3.$$

These are again given by Lemma 1.14: the final position of the segment in K_1 is parallel to its initial position in K_2 , and the same holds for the pairs K_2, K_3 and K_3, K_1 .

Hence

$$K := K_1 \cup K_2 \cup K_3 \cup A'_1 \cup A'_2 \cup A'_3$$

has

$$\mathcal{L}^2(K) < 3 \cdot \frac{\varepsilon}{6} + 3 \cdot \frac{\varepsilon}{6} = \varepsilon.$$

Following the segment through K_1, K_2 and K_3 , it rotates by $\frac{\pi}{3}$ in each of them, hence by π in total, and it comes back to its initial position with the two endpoints interchanged. In particular the segment has passed through every direction with a continuous motion, and K is a classical Keakeya set with $\mathcal{L}^2(K) < \varepsilon$. $\square$

1.3 Hausdorff and Minkowski dimension

To rigorously expand the study of these sets, in an attempt to provide a classification of sets of Lebesgue measure zero, we need to introduce a more general concept of measure. Unlike the standard Lebesgue measure, which is restricted to integer dimensions, this generalised measure allows us to compute the size of sets with fractional dimensions. We present the main definitions below; for a complete overview of the topic, we refer the reader to [Fal03].

Definition 1.17 (Hausdorff measure). Let $E \subseteq \mathbb{R}^n$, $0 \leq s < \infty$, and $0 < \delta \leq \infty$. We define

$$\mathcal{H}_\delta^s(E) := \inf \left\{ \sum_{i=1}^{\infty} \omega_s \left(\frac{\text{diam}(U_i)}{2} \right)^s : E \subseteq \bigcup_{i=1}^{\infty} U_i, \text{diam}(U_i) \leq \delta \right\}, \quad (2)$$

where

$$\omega_s := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2} + 1)}$$

and for $s \in \mathbb{N}^+$ it is the dimensional constant corresponding to the volume of the s -dimensional unit ball, and

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad \operatorname{Re}(z) > 0,$$

is Euler's Gamma function, one extension of the factorial function, satisfying $\Gamma(n+1) = n!$ for every $n \in \mathbb{N}$. We now obtain the s -dimensional Hausdorff measure on $\mathbb{R}^n$ for a set E as

$$\mathcal{H}^s(E) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E) = \sup_{\delta > 0} \mathcal{H}_\delta^s(E). \quad (3)$$

We can use this measure to define the dimension of a set in terms of the critical value s at which the measure changes from infinity to zero.

Definition 1.18 (Hausdorff dimension). The Hausdorff dimension of $E \subseteq \mathbb{R}^n$ is defined as

$$\dim_{\mathcal{H}}(E) := \inf \{s \geq 0 : \mathcal{H}^s(E) = 0\}. \quad (4)$$

Proposition 1.19. For every $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathcal{H}}(E) = \sup \{s \geq 0 : \mathcal{H}^s(E) = \infty\},$$

with the convention $\sup \emptyset := 0$.

Lemma 1.20. Let $E \subseteq \mathbb{R}^n$ and $0 \leq s < t$. If $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$.

Proof. Let $0 < \delta \leq 1$ and let $\{U_i\}$ be a cover of E with $\operatorname{diam}(U_i) \leq \delta$. So we have

$$\omega_t \left(\frac{\operatorname{diam}(U_i)}{2} \right)^t = \frac{\omega_t}{\omega_s} \left(\frac{\operatorname{diam}(U_i)}{2} \right)^{t-s} \omega_s \left(\frac{\operatorname{diam}(U_i)}{2} \right)^s \leq \frac{\omega_t}{\omega_s} \left(\frac{\delta}{2} \right)^{t-s} \omega_s \left(\frac{\operatorname{diam}(U_i)}{2} \right)^s,$$

and taking the infimum over all such covers gives

$$\mathcal{H}_\delta^t(E) \leq \frac{\omega_t}{\omega_s} \left(\frac{\delta}{2} \right)^{t-s} \mathcal{H}_\delta^s(E) \leq \frac{\omega_t}{\omega_s} \left(\frac{\delta}{2} \right)^{t-s} \mathcal{H}^s(E).$$

As $t - s > 0$ and $\mathcal{H}^s(E) < \infty$, letting $\delta \rightarrow 0$ yields $\mathcal{H}^t(E) = 0$. $\square$

Proof of Proposition 1.19. Set $A := \{s \geq 0 : \mathcal{H}^s(E) = 0\}$ and $B := \{s \geq 0 : \mathcal{H}^s(E) = \infty\}$, we have to show $\inf A = \sup B$ (with $\sup \emptyset := 0$).

Let $s \in B$ and $t \in A$. If we had $t \leq s$, then $t < s$ (the case $t = s$ being impossible) and $\mathcal{H}^t(E) = 0 < \infty$, so Lemma 1.20 would give $\mathcal{H}^s(E) = 0$, contradicting $s \in B$. Hence $s < t$ for all such s, t , and therefore $\sup B \leq \inf A$.

Suppose $\sup B < \inf A$ and pick t, s with $\sup B < t < s < \inf A$. Since $t \notin B$ we have $\mathcal{H}^t(E) < \infty$, so Lemma 1.20 gives $\mathcal{H}^s(E) = 0$, i.e. $s \in A$: this contradicts $s < \inf A$. $\square$

Furthermore, for $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define the δ -neighbourhood of E as

$$E_\delta := \{x \in \mathbb{R}^n : d(x, E) < \delta\}, \quad (5)$$

and another possible generalisation of dimension, often used, is the following.

Definition 1.21 (Minkowski dimension). The lower Minkowski dimension of a bounded set $E \subseteq \mathbb{R}^n$ is defined as

$$\underline{\dim}_{\mathcal{M}}(E) := n - \limsup_{\delta \rightarrow 0} \frac{\log \mathcal{L}^n(E_\delta)}{\log \delta}, \quad (6)$$

and the upper Minkowski dimension of E as

$$\overline{\dim}_{\mathcal{M}}(E) := n - \liminf_{\delta \rightarrow 0} \frac{\log \mathcal{L}^n(E_\delta)}{\log \delta}. \quad (7)$$

If the two values coincide, their common value is called the Minkowski dimension of E and is denoted by $\dim_{\mathcal{M}}(E)$.

An equivalent formulation of this definition of dimension, as we will prove in Theorem 1.24, can be given in terms of covers by cubes.

Definition 1.22 (Minkowski dimension via box-counting). Let $E \subseteq \mathbb{R}^n$ be any non-empty bounded set, for $\delta > 0$, let $N_\delta(E)$ be the smallest number of cubes of side-length δ which can cover E . The Minkowski dimension of E is defined as

$$\dim_{\mathcal{M}}(E) := \lim_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{\log(\frac{1}{\delta})},$$

provided the limit exists. If it does not, the lower and upper Minkowski dimensions are defined by taking the limit inferior and the limit superior, respectively:

$$\underline{\dim}_{\mathcal{M}}(E) = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{\log(\frac{1}{\delta})}, \quad (8)$$

$$\overline{\dim}_{\mathcal{M}}(E) = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{\log(\frac{1}{\delta})}. \quad (9)$$

We begin by listing some equivalent ways of defining $N_\delta(E)$ which will be useful below.

Proposition 1.23. *Let $E \subseteq \mathbb{R}^n$ be a non-empty bounded set. For $\delta > 0$, let $N_\delta(E)$ denote any of the following quantities:*

- (i) *the smallest number of closed balls of radius δ that cover E ;*
- (ii) *the smallest number of cubes of side δ that cover E ;*
- (iii) *the number of δ -mesh cubes that intersect E ;*
- (iv) *the smallest number of sets of diameter at most δ that cover E ;*
- (v) *the largest number of disjoint balls of radius δ with centres in E .*

Then the values of

$$\underline{\dim}_{\mathcal{M}}(E) = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_{\mathcal{M}}(E) = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta}$$

do not depend on which of (i)–(v) is chosen.

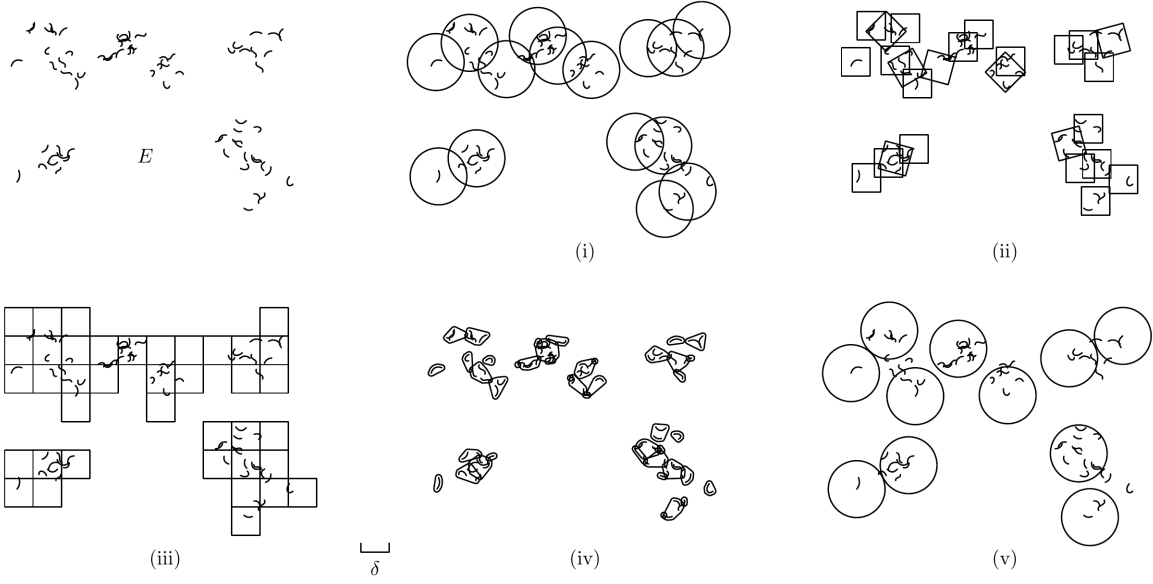


Figure 1.8

Proof. Write $N_\delta^{(i)}, \dots, N_\delta^{(v)}$ for the five quantities. Since E is non-empty and bounded, all of them are finite and each has a value of at least 1.

We start by stating a comparison principle. Suppose that N_δ and N'_δ are two of the five quantities and that there are constants $C, \lambda > 0$, independent of δ , such that

$$N_{\lambda\delta} \leq C N'_\delta$$

for all small $\delta > 0$. Taking logarithms and dividing by $-\log(\lambda\delta) = -\log \delta - \log \lambda > 0$,

$$\frac{\log N_{\lambda\delta}}{-\log(\lambda\delta)} \leq \frac{\log N'_\delta}{-\log \delta} \cdot \frac{1}{1 + \frac{\log \lambda}{\log \delta}} + \frac{\log C}{-\log \delta - \log \lambda}.$$

As $\delta \rightarrow 0$ the last summand tends to 0 and the correction factor tends to 1, while $\delta \mapsto \lambda\delta$ is a bijection of a neighbourhood of 0 onto another one, so taking $\liminf$ and $\limsup$ yields

$$\liminf_{\delta \rightarrow 0} \frac{\log N_\delta}{-\log \delta} \leq \liminf_{\delta \rightarrow 0} \frac{\log N'_\delta}{-\log \delta}, \quad \limsup_{\delta \rightarrow 0} \frac{\log N_\delta}{-\log \delta} \leq \limsup_{\delta \rightarrow 0} \frac{\log N'_\delta}{-\log \delta}.$$

We may therefore compare the five quantities, allowing for a fixed multiplicative constant and a fixed rescaling of δ , as we now do.

- (a) $N_\delta^{(ii)} \leq N_\delta^{(iv)}$: a set of diameter at most δ fits inside a cube of side δ .
- (b) $N_{\sqrt{n}\delta/2}^{(i)} \leq N_\delta^{(ii)}$: a cube of side δ lies within the closed ball of radius $\sqrt{n}\delta/2$ centred at its centre.
- (c) $N_{2\delta}^{(iv)} \leq N_\delta^{(i)}$: a closed ball of radius δ has diameter 2δ .
- (d) $N_\delta^{(iii)} \leq N_\delta^{(ii)}$: the δ -mesh cubes which intersect E are cubes of side δ and together they cover E .

- (e) $N_\delta^{(\text{iii})} \leq 2^n N_\delta^{(\text{ii})}$: an interval of length δ can meet at most two mesh intervals of the same length; consequently, a cube of side δ meets at most 2^n mesh cubes.
- (f) $N_{2\delta}^{(\text{i})} \leq N_\delta^{(\text{v})}$: let $B(x_1, \delta), \dots, B(x_m, \delta)$ be disjoint with $x_i \in E$ and $m = N_\delta^{(\text{v})}$ maximal. For $x \in E$, the ball $B(x, \delta)$ must meet some $B(x_i, \delta)$. In other words, $|x - x_i| < 2\delta$, so the closed balls $B(x_i, 2\delta)$ cover E .
- (g) $N_\delta^{(\text{v})} \leq N_{\delta/2}^{(\text{i})}$: with x_i as above, disjointness gives $|x_i - x_j| \geq 2\delta$ for $i \neq j$, so a ball of radius $\delta/2$, having diameter $\delta < 2\delta$, contains at most one x_i . Since the x_i are in E , any cover of E by balls of radius $\delta/2$ has at least m elements.

The estimates (a)–(c) form a cycle involving (i), (ii) and (iv), and therefore these three quantities have the same lower and upper limits; (d)–(e) connect (iii) to (ii), while (f)–(g) connect (v) to (i). Thus all five lower limits coincide, and so do all five upper limits. $\square$

The proposition gives us the opportunity to choose, arbitrarily, whichever definition is most convenient to adopt in a particular application; the list could also be extended further.

Theorem 1.24. *Let $E \subseteq \mathbb{R}^n$ be a non-empty bounded set. Then the quantities defined in (6) and (8) coincide, and so do those defined in (7) and (9). That is,*

$$n - \limsup_{\delta \rightarrow 0} \frac{\log \mathcal{L}^n(E_\delta)}{\log \delta} = \liminf_{\varepsilon \rightarrow 0} \frac{\log N_\varepsilon(E)}{\log(1/\varepsilon)}, \quad (10)$$

$$n - \liminf_{\delta \rightarrow 0} \frac{\log \mathcal{L}^n(E_\delta)}{\log \delta} = \limsup_{\varepsilon \rightarrow 0} \frac{\log N_\varepsilon(E)}{\log(1/\varepsilon)}. \quad (11)$$

In particular, $\dim_{\mathcal{M}}(E)$ is well defined in the sense of one definition if and only if it is in the sense of the other, and the two values agree.

Proof. If E can be covered by $N_\delta(E)$ cubes of side-length $\delta < 1$ then E_δ can be covered by the concentric cubes of side-length 3δ . Hence

$$\mathcal{L}^n(E_\delta) \leq N_\delta(E) (3\delta)^n.$$

Taking logarithms,

$$\frac{\log \mathcal{L}^n(E_\delta)}{-\log \delta} \leq \frac{\log 3^n + n \log \delta + \log N_\delta(E)}{-\log \delta},$$

so

$$\liminf_{\delta \rightarrow 0} \frac{\log \mathcal{L}^n(E_\delta)}{-\log \delta} \leq -n + \underline{\dim}_{\mathcal{M}}(E) \quad (12)$$

with a similar inequality for the upper limits. On the other hand if there are $N_\delta(E)$ disjoint balls of radius δ with centres in E , then by adding their volumes,

$$N_\delta(E) c_n \delta^n \leq \mathcal{L}^n(E_\delta).$$

Taking logarithms and letting $\delta \rightarrow 0$ gives the opposite inequality to (12). $\square$

The idea behind both definitions of dimension is essentially the same. A set should be s -dimensional if it needs about δ^{-s} pieces of size δ in order to be covered. This agrees with the examples we already know: covering a point only requires one piece, corresponding to δ^0 ; covering a unit segment requires roughly δ^{-1} intervals of length δ ; a unit square requires roughly δ^{-2} squares of side length δ ; a unit cube requires roughly δ^{-3} cubes of side length δ , and so on.

In particular there is a relationship between the Hausdorff and the Minkowski dimension which is always true.

Theorem 1.25. *Let $E \subseteq \mathbb{R}^n$ be bounded. Then*

$$\dim_{\mathcal{H}}(E) \leq \underline{\dim}_{\mathcal{M}}(E) \leq \overline{\dim}_{\mathcal{M}}(E).$$

Proof. As E is bounded, it can be covered by $N_\delta(E)$ sets of diameter at most δ , then from definition (2),

$$\mathcal{H}_\delta^s(E) \lesssim_{s,n} N_\delta(E) \delta^s,$$

where $\lesssim_{s,n}$ means inequality up to multiplicative constants $c_{s,n}$. Now let $s < \dim_{\mathcal{H}}(E)$, then $\mathcal{H}^s(E) = +\infty$ and hence $1 < \mathcal{H}_\delta^s(E)$ provided $\delta < 1$ is sufficiently small, and so $\log N_\delta(E) + s \log \delta > -\log(c_{s,n})$. Then $s \leq \liminf_{\delta \rightarrow 0} \left(\frac{\log N_\delta(E)}{-\log \delta} + \frac{\log c_{s,n}}{-\log \delta} \right) = \underline{\dim}_{\mathcal{M}} E$, the last term vanishing. Taking the supremum over $s < \dim_{\mathcal{H}}(E)$, and since $\liminf \leq \limsup$,

$$\dim_{\mathcal{H}} E \leq \underline{\dim}_{\mathcal{M}} E \leq \overline{\dim}_{\mathcal{M}} E \quad (13)$$

for every $E \subseteq \mathbb{R}^n$ bounded. $\square$

One of the main differences between these two notions of dimension is the freedom allowed in covering the set. In the Hausdorff case we may use countably many sets of *different* diameters, subject only to the constraint that none of these diameters exceed δ ; in the Minkowski case all the cubes are forced to have the *same* side-length δ , and only finitely many of them are involved.

In the Hausdorff case the covering sets may adapt to the local structure of E : small sets where E is sparse, larger ones where it is dense. The Minkowski dimension, on the other hand, remains at a fixed scale always. This can also be observed in the first definition of the Minkowski dimension where the $\mathcal{L}^n(E_\delta)$ approach is employed. In this approach, we measure the size of the set after we have increased its size by δ so as not to distinguish points closer than δ . Although Hausdorff and box dimensions are equal for many ‘reasonably regular’ sets, there are plenty of examples where this inequality is strict. We make this difference explicit through one such example.

For the proof of the example we will use the following proposition.

Proposition 1.26. *Let $E \subseteq \mathbb{R}^n$ be bounded and let $\overline{E}$ denote the closure of E . Then*

$$\underline{\dim}_{\mathcal{M}} E = \underline{\dim}_{\mathcal{M}} \overline{E} \quad \text{and} \quad \overline{\dim}_{\mathcal{M}} E = \overline{\dim}_{\mathcal{M}} \overline{E}.$$

Proof. Let $B_1, \dots, B_k$ be a finite collection of closed balls of radius δ . If the closed set $\bigcup_{i=1}^k B_i$ contains E then that closed set $\bigcup_{i=1}^k B_i$ also contains $\overline{E}$. This means that the smallest number of closed balls of radius δ that cover E is exactly the same as the smallest number of closed balls of radius δ needed to cover the larger set $\overline{E}$. The result follows. $\square$

Example. $\mathbb{Q} \cap [0, 1]$: it has Hausdorff dimension 0, since it is a countable union of points, but Minkowski dimension 1, since its closure is $[0, 1]$.

One might object that the gap above is caused only by the failure of $\mathbb{Q} \cap [0, 1]$ to be closed, since by Proposition 1.26 the box dimension is blind to the passage to the closure, while the Hausdorff dimension is not. The following compact example shows that this is not the case.

Example. The compact set $E := \{0\} \cup \{1/k : k \in \mathbb{N}^+\} \subseteq \mathbb{R}$ has $\dim_{\mathcal{H}}(E) = 0$ and $\dim_{\mathcal{M}}(E) = \frac{1}{2}$.

Proof. E is a countable union of points, so it has Hausdorff dimension 0. Now, for the Minkowski part, let $0 < \delta < \frac{1}{2}$ and let k be the integer satisfying $\frac{1}{(k-1)k} > \delta \geq \frac{1}{k(k+1)}$. If $|U| \leq \delta$, then U can cover at most one of the points $\{1, \frac{1}{2}, \dots, \frac{1}{k}\}$ since $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{(k-1)k} > \delta$. So at least k sets of diameter δ are needed to cover E , so $N_\delta(E) \geq k$ giving

$$\frac{\log N_\delta(E)}{-\log \delta} \geq \frac{\log k}{\log k(k+1)}.$$

As $\delta \rightarrow 0$ so $k \rightarrow \infty$ this gives $\underline{\dim}_{\mathcal{M}} E \geq \frac{1}{2}$. On the other hand, if $\frac{1}{2} > \delta > 0$, take k such that $\frac{1}{(k-1)k} > \delta \geq \frac{1}{k(k+1)}$. Then $k+1$ intervals of length δ can cover $[0, \frac{1}{k}]$, leaving $k-1$ points of E which can be covered by another $k-1$ intervals. So $N_\delta(E) \leq 2k$ giving

$$\frac{\log N_\delta(E)}{-\log \delta} \leq \frac{\log(2k)}{\log k(k-1)}$$

which results in

$$\overline{\dim}_{\mathcal{M}} E \leq \frac{1}{2}. \quad \square$$

1.4 Kakeya set conjecture

Theorem 1.7 has settled the question raised at the beginning of the chapter: a Kakeya set can have Lebesgue measure zero, so the needles can be packed together as tightly as one pleases. We can restate this fact in terms of δ -neighbourhoods.

Lemma 1.27. *Let $K \subseteq \mathbb{R}^n$ be compact. Then*

$$\lim_{\delta \rightarrow 0} \mathcal{L}^n(K_\delta) = \mathcal{L}^n(K).$$

In particular, if K is a Kakeya set with $\mathcal{L}^n(K) = 0$, then $\mathcal{L}^n(K_\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. The family $\{K_\delta\}_{\delta>0}$ is decreasing as $\delta \rightarrow 0$ and, K being closed, the intersection of all of them is K , that is $\bigcap_{\delta>0} K_\delta = K$. Since K is bounded we have $\mathcal{L}^n(K_1) < \infty$, so the claim follows from the fact that the Lebesgue measure is continuous from above. $\square$

Besicovitch's theorem says exactly that there are Kakeya sets such that the δ -neighbourhood of the set has a volume that approaches zero as $\delta \rightarrow 0$. What the theorem does not explain is how quickly it goes to zero. It is this speed that the concepts of dimension discussed earlier measure. Our goal is to prove that even though Kakeya sets with zero measure do exist, they are still big, in another way: they have full dimension. Heuristically, if E is covered by $N_\delta(E)$ cubes of side δ , then E_δ is essentially the union of those cubes slightly enlarged, so that

$$\mathcal{L}^n(E_\delta) \simeq_n \delta^n N_\delta(E).$$

We use the notation $A \lesssim_{p_1, \dots, p_k} B$ to mean that $A \leq CB$ for some constant $C < \infty$ depending only on the parameters $p_1, \dots, p_k$, and $A \simeq_{p_1, \dots, p_k} B$ to mean that both $A \lesssim_{p_1, \dots, p_k} B$ and $B \lesssim_{p_1, \dots, p_k} A$ hold. If E has dimension d , then $N_\delta(E) = \delta^{-d+o(1)}$ and therefore

$$\mathcal{L}^n(E_\delta) = \delta^{n-d+o(1)}.$$

And hence, the exponent governs the decay of the volume of the neighbourhood, and it coincides with the codimension of E . If a set is of full dimension n , then its volume satisfies

$\mathcal{L}^n(E_\delta) = \delta^{o(1)}$, which means that the volume tends to zero slower than any positive power of δ . Thus, saying that E has full dimension means that E can have Lebesgue measure zero and yet its δ -neighbourhoods lose their n -dimensional volume so slowly that no positive power of δ can bound their decay. The following proposition offers one of the ways to formalize this heuristic idea.

Proposition 1.28. *Let $E \subseteq \mathbb{R}^n$ be bounded and non-empty. Then $\underline{\dim}_{\mathcal{M}}(E) = n$ if and only if, for every $\varepsilon > 0$, one can find constants $c_\varepsilon > 0$ and $\delta_\varepsilon \in (0, 1)$ such that*

$$\mathcal{L}^n(E_\delta) \geq c_\varepsilon \delta^\varepsilon \quad \text{for all } 0 < \delta < \delta_\varepsilon. \quad (14)$$

Proof. We write

$$L(\delta) := \frac{\log \mathcal{L}^n(E_\delta)}{\log \delta}, \quad 0 < \delta < 1,$$

in order that $\underline{\dim}_{\mathcal{M}}(E) = n - \limsup_{\delta \rightarrow 0} L(\delta)$ by (6). The quantity $L(\delta)$ is well defined since E is non-empty and therefore the set E_δ contains a ball of radius δ , so that $\mathcal{L}^n(E_\delta) \geq \omega_n \delta^n$ and is thus positive; furthermore, since E is bounded there exists a constant $C_E < \infty$ such that $\mathcal{L}^n(E_\delta) \leq C_E$ for all $\delta < 1$.

We note that $\limsup_{\delta \rightarrow 0} L(\delta) \geq 0$, that is to say that $\underline{\dim}_{\mathcal{M}}(E) \leq n$. For indeed, $\log \mathcal{L}^n(E_\delta) \leq \log C_E$ and so $L(\delta) \geq \frac{\log C_E}{\log \delta}$, and the right-hand side tends to 0 as δ tends to 0. It follows that

$$\underline{\dim}_{\mathcal{M}}(E) = n \iff \limsup_{\delta \rightarrow 0} L(\delta) \leq 0.$$

Suppose (14) holds and let ε be a positive number. When we take the logarithm of the inequality $\mathcal{L}^n(E_\delta) \geq c_\varepsilon \delta^\varepsilon$ and divide it by $\log \delta$, which is negative, we get

$$L(\delta) \leq \frac{\log c_\varepsilon}{\log \delta} + \varepsilon \quad \text{for } 0 < \delta < \delta_\varepsilon.$$

The first term tends to 0 as δ tends to 0, so $\limsup_{\delta \rightarrow 0} L(\delta) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrarily chosen, it follows that $\limsup_{\delta \rightarrow 0} L(\delta) \leq 0$ and therefore $\underline{\dim}_{\mathcal{M}}(E) = n$.

With regard to the converse, suppose that $\underline{\dim}_{\mathcal{M}}(E) = n$, that is to say that $\limsup_{\delta \rightarrow 0} L(\delta) \leq 0$, and let ε be any positive number. By the definition of $\limsup$ there exists $\delta_\varepsilon \in (0, 1)$ such that $L(\delta) < \varepsilon$ for every δ satisfying $0 < \delta < \delta_\varepsilon$. Multiplying by $\log \delta$, which is negative, gives $\log \mathcal{L}^n(E_\delta) > \varepsilon \log \delta$, that is to say that

$$\mathcal{L}^n(E_\delta) > \delta^\varepsilon \quad \text{for all } 0 < \delta < \delta_\varepsilon,$$

which is (14) with $c_\varepsilon = 1$. □

We can now state the two conjectures, both of which are open for $n \geq 4$.

Conjecture 1.29 (Weak Kakeya conjecture). *Every Kakeya set in $\mathbb{R}^n$ has lower Minkowski dimension n .*

Conjecture 1.30 (Strong Kakeya conjecture). *Every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n .*

Theorem 1.25 shows that $\dim_{\mathcal{H}}(E) \leq \underline{\dim}_{\mathcal{M}}(E) \leq \overline{\dim}_{\mathcal{M}}(E)$ for every bounded set. So if we know that the Hausdorff dimension of a Kakeya set is n it means the Minkowski dimension must also be n . Conjecture 1.30 is therefore, at least a priori, the harder of the two. The freedom on the sizes of the covers is what makes the Hausdorff dimension difficult to bound

from below. It is this stronger statement that one usually means when speaking of the *Kakeya conjecture*.

Conjecture 1.29 and Conjecture 1.30 are trivially true for $n = 1$ (a *Kakeya set* has to contain a unit segment). They were proved in the plane by Davies [Dav71], and have recently been proved in $\mathbb{R}^3$ by Wang and Zahl [WZ25], in both the Hausdorff and the Minkowski form. For $n \geq 4$ both conjectures remain open, although lower bounds are available; for example, Wolff [Wol95] proved that every Kakeya set $K \subseteq \mathbb{R}^n$ satisfies $\dim_{\mathcal{H}}(K) \geq \frac{n+2}{2}$.

In the following chapter we shall focus on the case $n = 2$, for which we give a complete proof of Conjecture 1.30.

Chapter 2

Keakeya maximal function

Our presentation in this chapter follows [Mat15] and [Kro10]. The central idea for the rest of the thesis is to estimate the size of the neighbourhood of our set through bounds on operator norms that are meant to approximate the set. This approximation is carried out by thickening our needles, placing on them a function f which for convenience will be L^1_{loc} , and trying to estimate the norms as the tubes shrink to unit segments. These bounds will give us back the rate at which the neighbourhood of the set can decay, and therefore the dimension of the set. We now introduce the geometric notion of a δ -tube, which is fundamental for the definitions that follow. To simplify the notation, during this chapter, we will use $|\cdot|$ for the Lebesgue measure.

Definition 2.1. For $a \in \mathbb{R}^n$, $e \in \mathbb{S}^{n-1}$ and $\delta > 0$, we define the *tube* $T_e^\delta(a)$ by

$$T_e^\delta(a) := \left\{ x \in \mathbb{R}^n : |(x - a) \cdot e| < \frac{1}{2}, |(x - a)^{\perp e}| < \delta \right\},$$

where $x^{\perp e} := x - (x \cdot e)e$ is the orthogonal projection of x onto the hyperplane $e^\perp$ through the origin. Thus $T_e^\delta(a)$ is essentially the δ -neighbourhood of the unit line segment $\ell_{e,a}$ of Definition 1.1, that is, of the needle in the direction e centred at a .

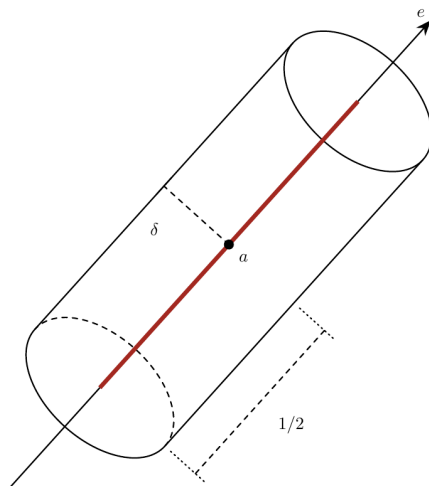


Figure 2.1

Observe that the volume of such a tube is

$$|T_e^\delta(a)| = \omega_{n-1} \delta^{n-1},$$

where ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$. In particular the volume of the tube depends neither on a nor on e . Using this geometric object we can define an operator measuring the maximal average of a function over thin tubes pointing in a given direction.

Definition 2.2. The *Keakeya maximal function* with width δ of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is the function $\mathcal{K}_\delta f : \mathbb{S}^{n-1} \rightarrow [0, \infty]$ given by

$$\mathcal{K}_\delta f(e) := \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx.$$

The following proposition follows directly from the definition above and establishes the basic boundedness properties of the Keakeya maximal operator.

Proposition 2.3. For all $0 < \delta < 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we have:

$$1. \quad \|\mathcal{K}_\delta f\|_{L^\infty(\mathbb{S}^{n-1})} \leq \|f\|_{L^\infty(\mathbb{R}^n)};$$

$$2. \quad \|\mathcal{K}_\delta f\|_{L^\infty(\mathbb{S}^{n-1})} \leq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\omega_{n-1} \delta^{n-1}}.$$

Proof. (1) Fix $e \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}^n$. Since $|f(x)| \leq \|f\|_{L^\infty(\mathbb{R}^n)}$ for almost every $x \in \mathbb{R}^n$, the average of $|f|$ over any tube is bounded by the same quantity:

$$\frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx \leq \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} \|f\|_{L^\infty(\mathbb{R}^n)} dx = \|f\|_{L^\infty(\mathbb{R}^n)}.$$

If we take the supremum over $a \in \mathbb{R}^n$, then $\mathcal{K}_\delta f(e) \leq \|f\|_{L^\infty(\mathbb{R}^n)}$ holds for every $e \in \mathbb{S}^{n-1}$.

(2) Once more fix $e \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}^n$. In this instance we merely extend the domain of integration from the tube to the whole space:

$$\frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx \leq \frac{1}{|T_e^\delta(a)|} \int_{\mathbb{R}^n} |f(x)| dx = \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\omega_{n-1} \delta^{n-1}}.$$

Taking the supremum over a we obtain

$$\mathcal{K}_\delta f(e) \leq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\omega_{n-1} \delta^{n-1}} \quad \forall e \in \mathbb{S}^{n-1}. \quad \square$$

2.1 Keakeya maximal conjecture

In the study of such operators it is natural to look for inequalities and the fewer constants are involved, the better. In this case the optimal should be

$$\|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad (15)$$

where C does not depend on δ , $p < \infty$ and $f \in L^p(\mathbb{R}^n)$. We would like (15) to hold for any $0 < \delta < 1$ and any $f \in L^p$.

To see that (15) cannot hold, we use Corollary 1.8. Let $K \subset \mathbb{R}^n$ be a Keakeya set with $\mathcal{L}^n(K) = 0$ and, for $\delta > 0$, let

$$K_\delta := \{x \in \mathbb{R}^n : \text{dist}(x, K) < \delta\}$$

be its δ -neighbourhood. We define $f_\delta := \chi_{K_\delta}$. On one hand by part 1 of Proposition 2.3, we know that

$$|\mathcal{K}_\delta f_\delta(e)| \leq \|\mathcal{K}_\delta f_\delta\|_{L^\infty(\mathbb{S}^{n-1})} \leq \|f_\delta\|_{L^\infty(\mathbb{R}^n)} = 1.$$

Moreover, K contains a unit segment in every direction, so for each $e \in \mathbb{S}^{n-1}$ there is $a \in \mathbb{R}^n$ with $T_e^\delta(a) \subset K_\delta$; hence the average of f_δ over that tube equals 1, and therefore $\mathcal{K}_\delta f_\delta(e) = 1$ for every $e \in \mathbb{S}^{n-1}$. Hence we have

$$\|\mathcal{K}_\delta f_\delta\|_{L^q(\mathbb{S}^{n-1})} = |\mathbb{S}^{n-1}|^{\frac{1}{q}} > 0.$$

On the other hand

$$\|f_\delta\|_{L^p(\mathbb{R}^n)} = |K_\delta|^{\frac{1}{p}} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0 \quad \text{for every } p < \infty.$$

Estimates of the form (15) are too strong to hold.

One possible way to weaken the inequality is to let the constant depend on the parameters involved and to allow a factor depending on δ , such as

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq C(p, n, \varepsilon) \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}, \quad (16)$$

for all $\varepsilon > 0$, $0 < \delta < 1$ and $f \in L^p(\mathbb{R}^n)$. However, this is too strong to hold for any $p < n$. To show this we simply choose $f_\delta := \chi_{B(0, \delta)}$ where $B(0, \delta)$ is the n -dimensional ball of radius δ centred at the origin. If $\delta \leq \frac{1}{2}$, then $B(0, \delta) \subset T_e^\delta(0)$ for every $e \in \mathbb{S}^{n-1}$, and no tube can contain more of $B(0, \delta)$ than this one; hence the supremum defining the maximal function is attained at $a = 0$ and

$$\mathcal{K}_\delta f_\delta(e) = \frac{1}{|T_e^\delta(0)|} \int_{T_e^\delta(0)} |f_\delta(x)| dx = \frac{|B(0, \delta)|}{|T_e^\delta(0)|} = \frac{\omega_n \delta^n}{\omega_{n-1} \delta^{n-1}} = c_n \delta,$$

with $c_n := \frac{\omega_n}{\omega_{n-1}}$. Since this value does not depend on e , we obtain

$$\|\mathcal{K}_\delta f_\delta\|_{L^p(\mathbb{S}^{n-1})} = c_n \delta |\mathbb{S}^{n-1}|^{\frac{1}{p}}.$$

On the other hand,

$$\|f_\delta\|_{L^p(\mathbb{R}^n)} = |B(0, \delta)|^{\frac{1}{p}} = \omega_n^{\frac{1}{p}} \delta^{\frac{n}{p}}.$$

Therefore, for (16) to hold, the exponents of δ must satisfy $1 \geq \frac{n}{p} - \varepsilon$, since otherwise the left-hand side would dominate as $\delta \rightarrow 0$. In the case where $p < n$, this fails for every $\varepsilon < \frac{n-p}{p}$, and therefore no estimate of the type (16) can hold in that range.

The example above says nothing about the case $p > n$. These estimates become weaker as p increases, so that the case $p = n$ already contains all the cases $p > n$. Indeed, assume (16) holds with $p = n$ and let $p > n$. Let $s := \frac{p}{n} \geq 1$, by Hölder's inequality we have

$$\frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f| \leq \left(\frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f|^s \right)^{\frac{1}{s}},$$

and taking the supremum over $a \in \mathbb{R}^n$ we obtain the pointwise bound

$$\mathcal{K}_\delta f(e) \leq (\mathcal{K}_\delta |f|^s(e))^{\frac{1}{s}} \quad \forall e \in \mathbb{S}^{n-1}.$$

Setting $g := |f|^{\frac{p}{n}}$, we have $g \in L^n(\mathbb{R}^n)$ with $\|g\|_{L^n}^n = \|f\|_{L^p}^p$, and since $\frac{p}{s} = n$,

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})}^p \leq \int_{\mathbb{S}^{n-1}} (\mathcal{K}_\delta g)^n d\sigma \leq C(n, \varepsilon)^n \delta^{-\varepsilon n} \|g\|_{L^n(\mathbb{R}^n)}^n = C(n, \varepsilon)^n \delta^{-\varepsilon n} \|f\|_{L^p(\mathbb{R}^n)}^p,$$

that is

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq C(n, \varepsilon)^{\frac{n}{p}} \delta^{-\frac{\varepsilon n}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

As $\varepsilon > 0$ is arbitrary and $\frac{n}{p} < 1$, this is exactly (16) for the exponent p , with an even smaller loss in δ .

So the best we can hope for an inequality like (16) to hold is imposing $p = n$, that is the following conjecture.

Conjecture 2.4 (Keakeya maximal conjecture).

$$\|\mathcal{K}_\delta f\|_{L^n(\mathbb{S}^{n-1})} \leq C(n, \varepsilon) \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)}, \quad (17)$$

for all $\varepsilon > 0$, $0 < \delta < 1$ and $f \in L^n(\mathbb{R}^n)$.

Observation 2.5. In studying an inequality like (17) it will suffice to assume that f is nonnegative. If (17) holds for nonnegative $f \in L^n$, then it holds for all functions in L^n .

Proof. $\|f\|_p = \| |f| \|_p$ for any $f \in L^p(\mathbb{R}^n)$ and using Definition 2.2 we see that

$$\begin{aligned} \|\mathcal{K}_\delta f\|_{L^n(\mathbb{S}^{n-1})} &= \|\mathcal{K}_\delta |f|\|_{L^n(\mathbb{S}^{n-1})} \\ &\leq C(n, \varepsilon) \delta^{-\varepsilon} \| |f| \|_{L^n(\mathbb{R}^n)} = C(n, \varepsilon) \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)}. \end{aligned}$$

If (17) holds for $f \geq 0$, then it holds for all $f \in L^n$. $\square$

2.2 Separated tubes and localisation

We now introduce some definitions and lemmas in order to obtain bounds related to the conjecture. The idea is to understand how different tubes intersect each other and to extract useful information from this. We can work with a simplified form of the Keakeya operator, one where we restrict the domain of the maximal function; we would also like to work with finitely many tubes. Since tubes whose directions are only slightly different overlap in an essentially redundant way, we restrict our attention to tubes whose directions are sufficiently separated.

Definition 2.6. We say that a set $\Omega = \{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ is a δ -separated subset of $\mathbb{S}^{n-1}$ if $|e_j - e_k| \geq \delta$ for $j \neq k$. It is *maximal* if in addition for every $e \in \mathbb{S}^{n-1}$ there is some k for which $|e - e_k| < \delta$. We say that $\{T_1, \dots, T_m\}$ are δ -separated δ -tubes if $T_k = T_{e_k}^\delta(a_k)$, $1 \leq k \leq m$, for some δ -separated subset $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ and some $a_1, \dots, a_m \in \mathbb{R}^n$.

Observation 2.7. For any δ -separated set $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$, the inequality $m \lesssim_n \delta^{1-n}$ holds. Suppose, for the sake of contradiction, that this bound does not hold. Then, for every $k \in \mathbb{N}$, there exist $\delta_k \in (0, 1]$ and a δ_k -separated set $\{e_1^k, \dots, e_{m_k}^k\}$ with $m_k > k \delta_k^{1-n}$. Define $\mathcal{C}_i^k := B(e_i^k, \delta_k/2) \cap \mathbb{S}^{n-1}$. These sets are pairwise disjoint and $|\mathcal{C}_i^k| \gtrsim_n \delta_k^{n-1}$, and therefore

$$|\mathbb{S}^{n-1}| \geq \sum_{i=1}^{m_k} |\mathcal{C}_i^k| \gtrsim_n m_k \delta_k^{n-1} > k \delta_k^{1-n} \delta_k^{n-1} = k,$$

which leads to a contradiction, since k is arbitrary. By a similar argument, it follows that $m \simeq_n \delta^{1-n}$ for a maximal δ -separated set.

We could also restrict the domain of the maximal function $\mathcal{K}_\delta(f)$, initially defined on $\mathbb{S}^{n-1}$, to the set $\mathcal{C} := \{e \in \mathbb{S}^{n-1} : |e - e_n| < \frac{1}{10}\}$ of all directions that lie within $\frac{1}{10}$ of the vertical e_n . Any spherical cap of fixed size, not depending on δ , would serve equally well.

Lemma 2.8. Let $1 \leq p < \infty$, $0 < \delta < 1$ and $A > 0$. If

$$\|\mathcal{K}_\delta f\|_{L^p(\mathcal{C})} \leq A \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n),$$

then there is a constant $N = N(n)$ such that

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq N^{1/p} A \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

Proof. Select a maximal $\frac{1}{10}$ -separated subset $\Omega \subset \mathbb{S}^{n-1}$ containing the vertical direction e_n , as defined in Definition 2.6. The maximality of Ω guarantees that the caps $\mathcal{C}_i$, each of radius $\frac{1}{10}$ and centred at points of Ω , cover $\mathbb{S}^{n-1}$. Since the sphere has finite measure, the number of such caps is a constant N depending only on n . Let $\mathcal{C}_1 = \mathcal{C}$ denote the cap centred at e_n , which corresponds to the cap $\mathcal{C}$ in the hypothesis. Since $\mathcal{K}_\delta f \geq 0$, we have

$$\begin{aligned} \|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})}^p &= \int_{\mathbb{S}^{n-1}} (\mathcal{K}_\delta f(e))^p d\sigma(e) \\ &= \int_{\bigcup_{i=1}^N \mathcal{C}_i} (\mathcal{K}_\delta f(e))^p d\sigma(e) \\ &\leq \sum_{i=1}^N \int_{\mathcal{C}_i} (\mathcal{K}_\delta f(e))^p d\sigma(e) = \sum_{i=1}^N \|\mathcal{K}_\delta f\|_{L^p(\mathcal{C}_i)}^p. \end{aligned}$$

This inequality can be obtained from subadditivity, since the caps may overlap; it is therefore enough to prove the estimate

$$\|\mathcal{K}_\delta f\|_{L^p(\mathcal{C}_i)} \leq A \|f\|_{L^p(\mathbb{R}^n)} \quad (18)$$

for each index i . Any cap can be rotated in such a way that its centre comes to rest at e_n , and thus the problem becomes one where $i = 1$, which is exactly the hypothesis in question. To see this in detail, assume (18) holds for $i = 1$ and let U be the rotation carrying the centre of $\mathcal{C}_i$ to e_n . Then, by the rotation invariance of the surface measure σ , we obtain

$$\|\mathcal{K}_\delta f\|_{L^p(\mathcal{C}_i)} = \left(\int_{\mathcal{C}_i} (\mathcal{K}_\delta f(e))^p d\sigma(e) \right)^{\frac{1}{p}} = \left(\int_{\mathcal{C}_1} (\mathcal{K}_\delta f(U^{-1}e))^p d\sigma(e) \right)^{\frac{1}{p}}. \quad (19)$$

Now write $f_U(x) := f(U^{-1}x)$. Using the identity $U(T_{U^{-1}e}^\delta(a)) = T_e^\delta(Ua)$, together with the change of variables $y = Ux$, we find

$$\begin{aligned} \mathcal{K}_\delta f(U^{-1}e) &= \sup_{a \in \mathbb{R}^n} \frac{1}{|T_{U^{-1}e}^\delta(a)|} \int_{T_{U^{-1}e}^\delta(a)} |f(x)| dx \\ &= \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(Ua)|} \int_{T_e^\delta(Ua)} |f_U(y)| dy = \mathcal{K}_\delta f_U(e). \end{aligned}$$

Substituting this into (19) yields

$$\|\mathcal{K}_\delta f\|_{L^p(\mathcal{C}_i)} = \|\mathcal{K}_\delta f_U\|_{L^p(\mathcal{C}_1)} \leq A \|f_U\|_{L^p(\mathbb{R}^n)} = A \|f\|_{L^p(\mathbb{R}^n)},$$

where in the last equality we are using the invariance of Lebesgue measure under rotations. $\square$

In certain cases this restriction simplifies geometric arguments, removing the need to consider interactions between antipodal directions. It also allows us to use the approximations $\sin \theta \sim \theta$ and $\cos \theta \sim 1 - \frac{\theta^2}{2}$ when θ is a small angle between two directions. We will use the same notation $\mathcal{K}_\delta f$ to denote the restricted maximal function.

Another useful observation is that the Kakeya maximal function is essentially constant on spherical caps of radius δ . That is:

Lemma 2.9. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $e, e' \in \mathbb{S}^{n-1}$ satisfy $|e - e'| \leq \delta$, then*

$$\mathcal{K}_\delta f(e) \leq C(n) \mathcal{K}_\delta f(e'). \quad (20)$$

Proof. Fix $a \in \mathbb{R}^n$ and let $x \in T_e^\delta(a)$. Denote by b the orthogonal projection of x onto the axis $\ell_{e,a}$ of $T_e^\delta(a)$, and by b' the orthogonal projection of b onto the line passing through a in the direction e' (see Figure 2.2). If θ denotes the angle in the interval $[0, \pi]$ between e and e' , then

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \leq 2 \sin \frac{\theta}{2} = |e - e'| \leq \delta,$$

and since $|a - b| \leq \frac{1}{2}$ we obtain

$$|b - b'| = |a - b| \sin \theta \leq \frac{\delta}{2}.$$

The point x lies at distance at most δ from b , so the triangle inequality gives

$$|x - b'| \leq |x - b| + |b - b'| \leq \frac{3}{2} \delta.$$

Moreover, $|a - b'| = |a - b| |\cos \theta| \leq \frac{1}{2}$, so that b' is in $\ell_{e',a}$. Since x was arbitrary we conclude that

$$T_e^\delta(a) \subset \mathcal{N}, \quad \mathcal{N} := \left\{ x \in \mathbb{R}^n : \text{dist}(x, \ell_{e',a}) \leq \frac{3}{2} \delta \right\},$$

That is, the tube in the direction e is contained within a fixed dilation of the tube in the direction e' with the same centre. This set can be covered by a number $C(n)$, depending only on n , of tubes $T_{e'}^\delta(y_k)$ with $y_k \in \mathbb{R}^n$. Recalling that all the tubes considered have the same volume, $\omega_{n-1} \delta^{n-1}$, we finally obtain

$$\begin{aligned} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx &\leq \frac{1}{|T_{e'}^\delta(a)|} \int_{\mathcal{N}} |f(x)| dx \\ &\leq \sum_{k=1}^{C(n)} \frac{1}{|T_{e'}^\delta(y_k)|} \int_{T_{e'}^\delta(y_k)} |f(x)| dx \leq C(n) \mathcal{K}_\delta f(e'), \end{aligned}$$

and (20) follows by taking the supremum over a on the left-hand side. $\square$

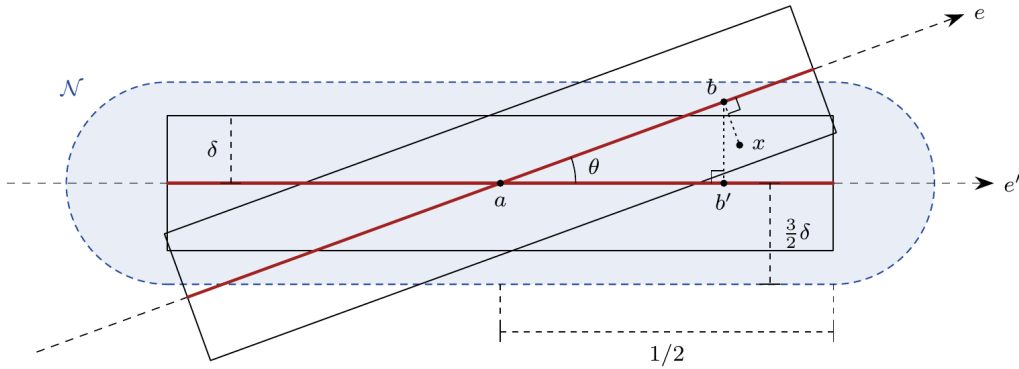


Figure 2.2

2.3 From maximal estimates to Hausdorff dimension

Our aim in this section is to show that L^p estimates for the Keakeya maximal function force lower bounds on the dimension of Keakeya sets. Suppose that, for some exponent $1 \leq p < \infty$ and some $\beta > 0$,

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{n,p,\beta} \delta^{-\beta} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 0 < \delta < 1 \text{ and } f \in L^p(\mathbb{R}^n). \quad (21)$$

For the Minkowski dimension the conclusion is almost immediate.

Theorem 2.10. *Let $1 \leq p < \infty$ and $\beta > 0$. If (21) holds for all $0 < \delta < 1$ and all $f \in L^p(\mathbb{R}^n)$, then every Keakeya set $K \subset \mathbb{R}^n$ satisfies*

$$\underline{\dim}_{\mathcal{M}} K \geq n - \beta p.$$

Proof. Let $K \subset \mathbb{R}^n$ be a Keakeya set and let K_δ denote its δ -neighbourhood. Since K contains a unit segment in every direction, K_δ contains a tube $T_e^\delta(a_e)$ for every $e \in \mathbb{S}^{n-1}$, so testing (21) on the characteristic function $f := \chi_{K_\delta}$ gives, exactly as in Section 2.1, that $\mathcal{K}_\delta f(e) = 1$ for every $e \in \mathbb{S}^{n-1}$. Consequently

$$1 \simeq_n |\mathbb{S}^{n-1}| = \|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})}^p \lesssim_{n,p,\beta} \delta^{-\beta p} \|f\|_{L^p(\mathbb{R}^n)}^p = \delta^{-\beta p} |K_\delta|,$$

that is, $|K_\delta| \gtrsim_{n,p,\beta} \delta^{\beta p}$. Hence for every $s < n - \beta p$ we have $\delta^{s-n} |K_\delta| \gtrsim_{n,p,\beta} \delta^{s-(n-\beta p)} \rightarrow \infty$ as $\delta \rightarrow 0$, and therefore

$$\underline{\dim}_{\mathcal{M}} K \geq n - \beta p. \quad \square$$

Passing to the Hausdorff dimension is more delicate. A covering of K may now consist of balls of wildly different radii, whereas (21) only sees one scale δ at a time. The usual solution, and the one we are going to use, is to divide the covering into subfamilies made up of balls which have essentially the same radius, and then to apply the maximal estimate at the scale corresponding to each of these subfamilies.

The following characterisation of Hausdorff dimension will be the one used.

Lemma 2.11. *Let $A \subset \mathbb{R}^n$ and let α be a positive number. The Hausdorff dimension of A is at least α if there exists a positive constant c such that*

$$\sum_j r_j^\alpha \geq c \quad \text{for every countable cover } A \subset \bigcup_j B(x_j, r_j) \text{ with } r_j < 1 \quad (22)$$

holds.

Proof. Given any cover of A by sets E_j of small diameter, each E_j may be replaced by a ball $B(x_j, r_j)$ centred at a point $x_j \in E_j$ with radius $r_j := \text{diam } E_j$. When the diameter is zero, a ball of radius ρ_j is selected so that the sum of the ρ_j^α taken over the degenerate indices is at most $c/2$; applying (22) to the resulting cover consisting of balls and omitting the degenerate contribution, we get

$$\sum_j (\text{diam } E_j)^\alpha \geq \sum_j r_j^\alpha - \frac{c}{2} \geq \frac{c}{2} > 0,$$

and thus the sum of the diameters raised to the power α cannot be made arbitrarily small. $\square$

The symbol σ will stand for the surface measure on $\mathbb{S}^{n-1}$. We are now ready to prove the main result of this section.

Theorem 2.12. *Let $1 \leq p < \infty$ and $\beta > 0$ be such that $n - \beta p > 0$. If*

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq C(n, p, \beta) \delta^{-\beta} \|f\|_{L^p(\mathbb{R}^n)} \quad (23)$$

holds for all $0 < \delta < 1$ and all $f \in L^p(\mathbb{R}^n)$, then every Kakeya set $K \subset \mathbb{R}^n$ satisfies

$$\dim_{\mathcal{H}} K \geq n - \beta p.$$

Proof. Let $K \subset \mathbb{R}^n$ be a Kakeya set and fix α such that $0 < \alpha < n - \beta p$. According to (22) it is enough to find a constant $c = c(n, p, \beta, \alpha) > 0$ for which

$$\sum_j r_j^\alpha \geq c \quad (24)$$

is satisfied for every covering $K \subset \bigcup_j B_j$, $B_j := B(x_j, r_j)$, by balls of radius $r_j < 1$; the theorem is then obtained by letting $\alpha \nearrow n - \beta p$. Hence we fix such a covering.

We decompose the sphere according to the size of the balls that cover the corresponding needles. For each integer $k \geq 1$ we group the balls by setting

$$J_k := \{j : 2^{-k} \leq r_j < 2^{1-k}\};$$

this is valid since each $r_j < 1$. If any of the J_k is infinite, the sum of the r_j^α is already infinite, since $\sum_j r_j^\alpha \geq \#J_k 2^{-k\alpha} = \infty$, and therefore (24) becomes trivial, so we may assume that each J_k is finite. The fact that K is a Kakeya set means that, for each direction $e \in \mathbb{S}^{n-1}$, there is a unit segment $I_e \subset K$ parallel to e , and we let a_e denote its midpoint. We define

$$S_k := \left\{ e \in \mathbb{S}^{n-1} : \mathcal{H}^1\left(I_e \cap \bigcup_{j \in J_k} B_j\right) \geq \frac{1}{2k^2} \right\},$$

the set of directions e for which the one-dimensional Hausdorff measure of the intersection of I_e with the union of the balls indexed by J_k is at least $\frac{1}{2k^2}$.

We claim that

$$\mathbb{S}^{n-1} = \bigcup_{k \geq 1} S_k. \quad (25)$$

Suppose not and choose a direction $e \in \mathbb{S}^{n-1} \setminus \bigcup_k S_k$. Then, when we sum over k , we obtain

$$\sum_{k \geq 1} \mathcal{H}^1\left(I_e \cap \bigcup_{j \in J_k} B_j\right) < \sum_{k \geq 1} \frac{1}{2k^2} = \frac{\pi^2}{12} < 1.$$

On the other hand, $I_e \subset K \subset \bigcup_j B_j = \bigcup_k \bigcup_{j \in J_k} B_j$ and by subadditivity it follows that

$$1 = \mathcal{H}^1(I_e) \leq \sum_{k \geq 1} \mathcal{H}^1\left(I_e \cap \bigcup_{j \in J_k} B_j\right).$$

This is a contradiction, and therefore (25) is established.

The decomposition just obtained is useful because on each S_k the maximal function admits a lower bound at the corresponding scale. Let us denote by $\delta_k := 2^{-k}$, and let

$$M_k := \bigcup_{j \in J_k} B(x_j, r_j), \quad F_k := \bigcup_{j \in J_k} B(x_j, 2r_j), \quad f_k := \chi_{F_k}.$$

It then follows that $f_k \in L^p(\mathbb{R}^n)$ and $\|f_k\|_{L^p}^p = |F_k|$. We assert that the estimate

$$|T_e^{\delta_k}(a_e) \cap F_k| \gtrsim_n \frac{1}{k^2} |T_e^{\delta_k}(a_e)| \quad (26)$$

holds for every direction $e \in S_k$.

We start by observing that

$$B(y, \delta_k) \subset F_k \quad \text{for every } y \in M_k. \quad (27)$$

To see this, suppose that $y \in B(x_j, r_j)$ with $j \in J_k$ and let $z \in B(y, \delta_k)$. Since $r_j \geq 2^{-k} = \delta_k$, the triangle inequality implies that

$$|x_j - z| \leq |x_j - y| + |y - z| < r_j + \delta_k \leq 2r_j,$$

so $z \in B(x_j, 2r_j) \subset F_k$.

Let us parametrize I_e as $I_e = \{a_e + te : |t| < \frac{1}{2}\}$, and let

$$A := \{t \in (-\frac{1}{2}, \frac{1}{2}) : a_e + te \in M_k\}$$

Then we know that $\mathcal{L}^1(A) = \mathcal{H}^1(I_e \cap M_k) \geq \frac{1}{2k^2}$ for $e \in S_k$. Now consider the part of the tube that lies above A , that is, the set

$$G := \{a_e + te + w : t \in A, w \perp e, |w| < \frac{\delta_k}{2}\}$$

Any point $x = a_e + te + w$ in G satisfies $(x - a_e) \cdot e = t$ with $|t| < \frac{1}{2}$ and $(x - a_e)^\perp = w$ with $|w| \leq \delta_k$, so $G \subset T_e^{\delta_k}(a_e)$. Also, $|x - (a_e + te)| \leq \delta_k/2$ with $a_e + te \in I_e \cap M_k$, so that $G \subset F_k$ by means of (27). We then apply Fubini's theorem to obtain

$$|G| = \mathcal{L}^1(A) \omega_{n-1} \left(\frac{\delta_k}{2}\right)^{n-1} = \frac{\mathcal{L}^1(A)}{2^{n-1}} \omega_{n-1} \delta_k^{n-1} \geq \frac{1}{2^n k^2} |T_e^{\delta_k}(a_e)|,$$

thus proving (26). As a result, for each $e \in S_k$ we have the lower bound

$$\mathcal{K}_{\delta_k} f_k(e) \geq \frac{1}{|T_e^{\delta_k}(a_e)|} \int_{T_e^{\delta_k}(a_e)} |f_k| = \frac{|T_e^{\delta_k}(a_e) \cap F_k|}{|T_e^{\delta_k}(a_e)|} \geq \frac{\kappa_n}{k^2}, \quad (28)$$

with $\kappa_n := 2^{-n}$. It remains to turn this pointwise bound into an estimate for the size of S_k , and then to sum over k . Since the function $\mathcal{K}_{\delta_k} f_k$ is lower semicontinuous, it is Borel measurable, and therefore the superlevel set $\{e : \mathcal{K}_{\delta_k} f_k(e) \geq \kappa_n k^{-2}\}$ is measurable and includes S_k by (28). We can thus obtain an estimate of the outer measure $\sigma^*(S_k)$ by applying Chebyshev's inequality with the hypothesis (23) for $\delta = \delta_k$, thus arriving at

$$\begin{aligned} \sigma^*(S_k) &\lesssim_{n,p} k^{2p} \int_{\mathbb{S}^{n-1}} (\mathcal{K}_{\delta_k} f_k(e))^p d\sigma(e) \\ &\leq k^{2p} C^p \delta_k^{-\beta p} \|f_k\|_{L^p(\mathbb{R}^n)}^p = k^{2p} C^p 2^{k\beta p} |F_k|. \end{aligned}$$

But $2r_j < 2^{2-k}$ for $j \in J_k$, hence we have $|F_k| \leq \#J_k \omega_n 2^{(2-k)n} = \omega_n 4^n \#J_k 2^{-kn}$, which gives

$$\begin{aligned} \sigma^*(S_k) &\lesssim_{n,p,\beta} k^{2p} 2^{k\beta p} 2^{-kn} \#J_k \\ &= \left(k^{2p} 2^{k[\alpha - (n-\beta p)]}\right) 2^{-k\alpha} \#J_k \lesssim_{n,p,\beta,\alpha} 2^{-k\alpha} \#J_k. \end{aligned}$$

This last inequality is valid since $\alpha - (n - \beta p) < 0$ and hence $k^{2p} 2^{k[\alpha - (n - \beta p)]} \rightarrow 0$ as $k \rightarrow \infty$ and it can be bounded by a constant that depends only on n, p, β and α .

We are now in a position to conclude

$$\begin{aligned} \sum_j r_j^\alpha &= \sum_{k \geq 1} \sum_{j \in J_k} r_j^\alpha \geq \sum_{k \geq 1} \#J_k 2^{-k\alpha} \gtrsim_{n,p,\beta,\alpha} \sum_{k \geq 1} \sigma^*(S_k) \\ &\geq \sigma\left(\bigcup_{k \geq 1} S_k\right) = |\mathbb{S}^{n-1}| \gtrsim_n 1, \end{aligned}$$

with a constant that does not depend on the cover, as required. $\square$

Corollary 2.13. *The Keakeya maximal conjecture 2.4 implies the Keakeya conjecture 1.30: every Keakeya set in $\mathbb{R}^n$ has Hausdorff dimension n .*

Proof. Assume (17) and fix $0 < \varepsilon < 1$ so that $n - \varepsilon n > 0$. Applying Theorem 2.12 with $p = n$ and $\beta = \varepsilon$ gives that the Hausdorff dimension of every Keakeya set K is at least $n - \varepsilon n$. Letting ε decrease to zero we obtain $\dim_{\mathcal{H}} K \geq n$, and the reverse inequality is trivial since $K \subset \mathbb{R}^n$. $\square$

2.4 Keakeya maximal conjecture in the plane

We now give a proof of the Keakeya conjecture 1.30 in $\mathbb{R}^2$. By Corollary 2.13 it suffices to establish the Keakeya maximal conjecture 2.4 in $\mathbb{R}^2$.

Clearly, by Hölder,

$$\frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f| \leq \frac{\|f\|_{L^2(\mathbb{R}^2)}}{|T_e^\delta(a)|^{1/2}} = \frac{\|f\|_{L^2(\mathbb{R}^2)}}{(\omega_1 \delta)^{1/2}}$$

holds.

We will prove the following stronger estimate.

Theorem 2.14 (Córdoba). *For all $0 < \delta < \frac{1}{5}$ and $f \in L^2(\mathbb{R}^2)$, we have the estimate*

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^1)} \lesssim \sqrt{\log \frac{1}{\delta}} \|f\|_{L^2(\mathbb{R}^2)}.$$

This result provides not only a quantitative upper bound for the norm of the maximal operator but in fact the optimal one, and resolves the Keakeya conjecture in two dimensions. The proof, however, gives little insight into the higher-dimensional cases, since it depends strongly on the L^2 structure.

The following geometric lemma is the main tool of this chapter, as it is used repeatedly to understand how different tubes intersect. Although Córdoba's estimate is stated in terms of L^2 , we formulate and prove the lemma in $\mathbb{R}^n$.

Lemma 2.15 (Córdoba). *For any pair of directions $e_k, e_l \in \mathbb{S}^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have the estimates*

$$\text{diam}(T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)) \lesssim_n \frac{\delta}{\min\{|e_k - e_l|, |e_k + e_l|\}}, \quad (29)$$

and

$$|T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)| \lesssim_n \frac{\delta^n}{\min\{|e_k - e_l|, |e_k + e_l|\}}. \quad (30)$$

Consistent with our intuition, observe that when the tubes are very close to each other in both space and direction ($|e_k - e_l| \sim \delta$), their intersection is close to a full tube, $|T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)| \lesssim_n \delta^{n-1}$. However, in the opposite situation, when the tubes are close to each other in space but orthogonal in direction ($|e_k - e_l| \sim 1$), their intersection will be a δ -box in $\mathbb{R}^n$, $|T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)| \lesssim_n \delta^n$.

Observation 2.16. The minimum appearing in Lemma 2.15 is there because a tube depends on its direction only up to sign, $T_{-e}^\delta(a) = T_e^\delta(a)$, and the relevant quantity is the smaller of the two angles determined by e_k and e_l , that is $d(e_k, e_l) = \min\{|e_k - e_l|, |e_k + e_l|\}$. This formulation was adopted so that the lemma would remain valid for all directions on $\mathbb{S}^{n-1}$. Had the analysis been limited to the directions contained in the cap $\mathcal{C}$, as allowed by Lemma 2.8, the minimum would not have been needed: for $e_k, e_l \in \mathcal{C}$ we have $|e_k - e_l| \leq \frac{1}{5}$, hence $e_k \cdot e_l > 0$ and therefore $d(e_k, e_l) = |e_k - e_l|$.

Proof. We may assume that $T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b) \neq \emptyset$, otherwise there is nothing to prove, and that $e_k \neq \pm e_l$, since in that case the right-hand sides of (29) and (30) are infinite. By Observation 2.16, replacing e_l by $-e_l$ if necessary, we may assume $e_k \cdot e_l > 0$; then $\min\{|e_k - e_l|, |e_k + e_l|\} = |e_k - e_l|$ and the angle $\theta \in (0, \frac{\pi}{2})$ between e_k and e_l satisfies $|e_k - e_l| = 2 \sin \frac{\theta}{2}$. It is also convenient to introduce two pieces of notation: for $\lambda \geq 1$ we let

$$\lambda T_e^\delta(a) := \left\{ x \in \mathbb{R}^n : |(x - a) \cdot e| \leq \frac{\lambda}{2}, |(x - a)^\perp| \leq \lambda \delta \right\}$$

be the dilation of $T_e^\delta(a)$ by the factor λ with respect to its own centre a . Furthermore, given a direction $e_j \in \mathbb{S}^{n-1}$, we write

$$x^{\perp j} := x - (x \cdot e_j) e_j$$

for the orthogonal projection of x onto the hyperplane $e_j^\perp$ through the origin.

The first thing to observe is that the two centres play no role, and that we may as well work with concentric tubes centred at the origin. Indeed, pick a point $x \in T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)$. For every $y \in T_{e_k}^\delta(a)$ the triangle inequality gives

$$|(y - x) \cdot e_k| \leq |(y - a) \cdot e_k| + |(x - a) \cdot e_k| \leq 1$$

since $x \in T_{e_k}^\delta(a)$. Similarly, since $w \mapsto w^{\perp k}$ is linear,

$$|(y - x)^{\perp k}| \leq |(y - a)^{\perp k}| + |(x - a)^{\perp k}| \leq 2\delta,$$

so that $y \in 2T_{e_k}^\delta(x)$. Applying the same argument to $T_{e_l}^\delta(b)$ gives $T_{e_l}^\delta(b) \subset 2T_{e_l}^\delta(x)$, and it follows that

$$T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b) \subset 2T_{e_k}^\delta(x) \cap 2T_{e_l}^\delta(x) = 2(T_{e_k}^\delta(x) \cap T_{e_l}^\delta(x)),$$

the last equality holding because both dilations are carried out with respect to the same point x . The diameter is unchanged under translations and, since $\text{diam}(2E) = 2 \text{diam}(E)$ and $|2E| = 2^n |E|$, it is sufficient to prove (29) and (30) in the case $a = b = 0$: the reduction costs a factor 2 in the first estimate and a factor 2^n in the second one.

Before estimating the diameter we observe the relationship between θ and $|e_k - e_l|$. Since $|e_k - e_l| = 2 \sin \frac{\theta}{2}$ and $\theta \leq \frac{\pi}{2}$, it follows that $\cos \frac{\theta}{2} \geq \frac{1}{\sqrt{2}}$ and therefore

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \geq \sqrt{2} \sin \frac{\theta}{2} = \frac{|e_k - e_l|}{\sqrt{2}}. \quad (31)$$

Let now $x \in T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)$. Being in the first tube, x lies on a unit segment parallel to e_k whose midpoint is at distance at most δ from the origin; more precisely, setting

$$t_k := x \cdot e_k, \quad m_k := x^\perp_k,$$

we have the parametrisation

$$x = m_k + t_k e_k, \quad |t_k| \leq \frac{1}{2}, \quad |m_k| \leq \delta,$$

and in the same way, being in the second tube,

$$x = m_l + t_l e_l, \quad |t_l| \leq \frac{1}{2}, \quad |m_l| \leq \delta.$$

If $t_k = 0$ or $t_l = 0$ then $x = m_k$ or $x = m_l$, so that $|x| \leq \delta$ and the bound below is immediate. We may therefore assume $t_k, t_l \neq 0$; note that in this case $m_k \neq m_l$, since $m_k = m_l$ would give $t_k e_k = t_l e_l$, which is impossible because e_k and e_l are linearly independent. Consider then the triangle with vertices m_k, m_l and x , and denote by α its angle at x and by β its angle at m_l . The sides xm_k and xm_l are parallel to e_k and e_l respectively, whence $\alpha \in \{\theta, \pi - \theta\}$ and in either case $\sin \alpha = \sin \theta > 0$; in particular the triangle is non-degenerate. The law of sines gives

$$\frac{|x - m_k|}{\sin \beta} = \frac{|m_k - m_l|}{\sin \alpha},$$

and therefore, using $\sin \beta \leq 1$, $|m_k - m_l| \leq 2\delta$ and (31),

$$|t_k| = |x - m_k| = \frac{\sin \beta}{\sin \theta} |m_k - m_l| \leq \frac{2\delta}{\sin \theta} \leq \frac{2\sqrt{2}\delta}{|e_k - e_l|}.$$

Consequently, recalling that $|e_k - e_l| \leq 2$,

$$|x| \leq |m_k| + |t_k| \leq \delta + \frac{2\sqrt{2}\delta}{|e_k - e_l|} \lesssim \frac{\delta}{|e_k - e_l|}. \quad (32)$$

The point x was arbitrary, so (32) holds at every point of the intersection and

$$\text{diam}(T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)) \leq \sup_{x \in T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)} 2|x| \lesssim \frac{\delta}{|e_k - e_l|},$$

which, after the reduction made at the beginning, proves (29).

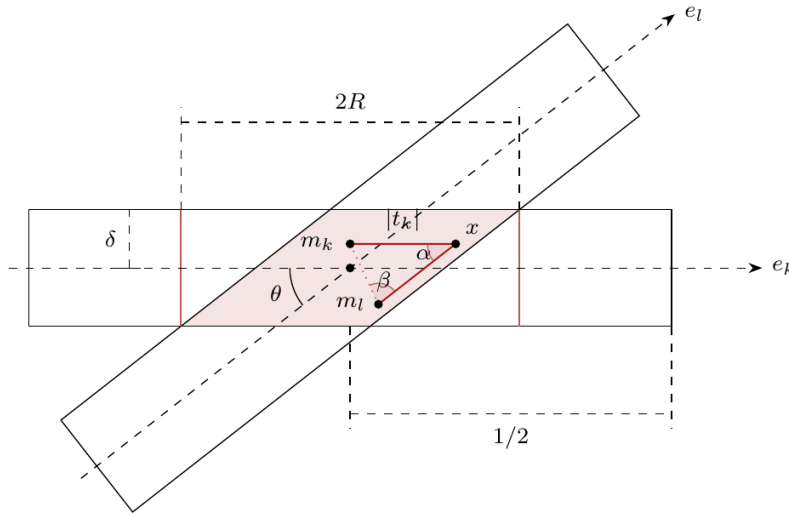


Figure 2.3

The estimate on the measure is now a consequence of the one just obtained. Let $R := \frac{C\delta}{|e_k - e_l|}$ be the bound provided by (32), where C is the absolute constant implicit there. Every $x \in T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)$ satisfies $|x^\perp| \leq \delta$, because x belongs to $T_{e_k}^\delta(0)$, and $|x \cdot e_k| \leq |x| \leq R$; hence

$$T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0) \subset \{x \in \mathbb{R}^n : |x \cdot e_k| \leq R, |x^\perp| \leq \delta\},$$

that is, the intersection is contained in a cylinder of radius δ and length $2R$, whose axis points in the direction e_k . Such a cylinder has measure $2R\omega_{n-1}\delta^{n-1}$, and therefore

$$|T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)| \leq 2R\omega_{n-1}\delta^{n-1} \lesssim_n \frac{\delta}{|e_k - e_l|} \delta^{n-1} = \frac{\delta^n}{|e_k - e_l|},$$

which gives (30) and concludes the proof. $\square$

The following proposition reduces the problem of obtaining the required estimate to one concerning families of δ -separated tubes: once suitably normalised sums of the characteristic functions of these tubes are controlled in the $L^{p'}$ norm, the desired L^p bound for $\mathcal{K}_\delta$ follows. The geometric content of the problem is thereby completely transferred to the way the tubes overlap, which is precisely what Lemma 2.15 is able to quantify.

Proposition 2.17. *Let $1 < p < \infty$, $p' = \frac{p}{p-1}$ be the dual exponent of p , $0 < \delta < 1$ and $0 < M < \infty$. Suppose that*

$$\left\| \sum_{k=1}^m t_k \chi_{T_{e_k}^\delta(a_k)} \right\|_{L^{p'}(\mathbb{R}^n)} \leq M \quad (33)$$

whenever $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ is a maximal δ -separated set, $a_1, \dots, a_m \in \mathbb{R}^n$ are arbitrary points and $t_1, \dots, t_m$ are positive numbers such that

$$\delta^{n-1} \sum_{k=1}^m t_k^{p'} \leq 1. \quad (34)$$

Then

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{n,p} M \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n). \quad (35)$$

Proof. Let $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ be a maximal δ -separated set as in Definition 2.6. By maximality the family of caps $\{\mathbb{S}^{n-1} \cap B(e_k, \delta)\}$ covers $\mathbb{S}^{n-1}$, and if $e \in \mathbb{S}^{n-1} \cap B(e_k, \delta)$ then $\mathcal{K}_\delta f(e) \leq C(n) \mathcal{K}_\delta f(e_k)$ by Lemma 2.9. Hence

$$\begin{aligned} \|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})}^p &\leq \sum_{k=1}^m \int_{\mathbb{S}^{n-1} \cap B(e_k, \delta)} (\mathcal{K}_\delta f(e))^p d\sigma(e) \\ &\leq \sum_{k=1}^m C(n)^p \int_{\mathbb{S}^{n-1} \cap B(e_k, \delta)} (\mathcal{K}_\delta f(e_k))^p d\sigma(e) \\ &= \sum_{k=1}^m C(n)^p (\mathcal{K}_\delta f(e_k))^p \sigma(\mathbb{S}^{n-1} \cap B(e_k, \delta)) \\ &\lesssim_{n,p} \sum_{k=1}^m \left(\mathcal{K}_\delta f(e_k) \delta^{\frac{n-1}{p}} \right)^p. \end{aligned}$$

By the duality of ℓ^p and $\ell^{p'}$, for any $a_k \geq 0$, $k = 1, \dots, m$,

$$\left(\sum_{k=1}^m a_k^p \right)^{\frac{1}{p}} = \max \left\{ \sum_{k=1}^m a_k b_k : b_k \geq 0 \text{ and } \sum_{k=1}^m b_k^{p'} = 1 \right\}.$$

This way,

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{n,p} \sum_{k=1}^m \mathcal{K}_\delta f(e_k) \delta^{\frac{n-1}{p}} b_k = \delta^{n-1} \sum_{k=1}^m \mathcal{K}_\delta f(e_k) t_k,$$

where we have chosen $t_k := \delta^{\frac{1-n}{p'}} b_k$, so that $\delta^{n-1} \sum_{k=1}^m t_k^{p'} = 1$ and the normalisation (34) is satisfied. Let $\varepsilon > 0$ be small. By Definition 2.2 there are points $a_k \in \mathbb{R}^n$ such that

$$\mathcal{K}_\delta f(e_k) - \varepsilon \leq \frac{1}{|T_{e_k}^\delta(a_k)|} \int_{T_{e_k}^\delta(a_k)} |f(x)| dx,$$

hence, writing $\varepsilon_\delta := \varepsilon \delta^{n-1} \sum_{k=1}^m t_k$,

$$\begin{aligned} \|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} &\lesssim_{n,p} \delta^{n-1} \sum_{k=1}^m (\mathcal{K}_\delta f(e_k) - \varepsilon) t_k + \varepsilon_\delta \\ &\leq \delta^{n-1} \sum_{k=1}^m t_k \frac{1}{|T_{e_k}^\delta(a_k)|} \int_{T_{e_k}^\delta(a_k)} |f(x)| dx + \varepsilon_\delta \\ &\lesssim_{n,p} \sum_{k=1}^m t_k \int_{T_{e_k}^\delta(a_k)} |f(x)| dx + \varepsilon_\delta \\ &= \int_{\mathbb{R}^n} \left(\sum_{k=1}^m t_k \chi_{T_{e_k}^\delta(a_k)} \right) |f(x)| dx + \varepsilon_\delta \\ &\leq \left\| \sum_{k=1}^m t_k \chi_{T_{e_k}^\delta(a_k)} \right\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} + \varepsilon_\delta \\ &\leq M \|f\|_{L^p(\mathbb{R}^n)} + \varepsilon_\delta, \end{aligned}$$

where we use Hölder in the second to last inequality. By taking $\varepsilon \rightarrow 0$ we finish the proof. $\square$

We can now proceed to the main result on $\mathbb{R}^2$.

Proof of Theorem 2.14. Recall that, by Lemma 2.8, it suffices to bound the maximal function restricted to the cap $\mathcal{C}$; in particular all the directions occurring below satisfy $|e_k - e_l| \leq \frac{1}{5}$. Proposition 2.17 and its proof remain valid with $\mathbb{S}^{n-1}$ replaced by $\mathcal{C}$ throughout, the e_k forming a maximal δ -separated subset of $\mathcal{C}$. We apply it in this form with $n = 2$ and $p = p' = 2$; the theorem follows once we prove

$$\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^2(\mathbb{R}^2)} \lesssim \sqrt{\log \frac{1}{\delta}}, \quad (36)$$

whenever $T_k := T_{e_k}^\delta(a_k)$, $1 \leq k \leq m$, are δ -separated δ -tubes associated with a maximal δ -separated set $\{e_1, \dots, e_m\} \subset \mathcal{C}$ and $t_1, \dots, t_m > 0$ satisfy the normalisation

$$\delta \sum_{k=1}^m t_k^2 \leq 1. \quad (37)$$

We first have to prove the following counting estimate: for every fixed k ,

$$\sum_{\substack{l=1 \\ l \neq k}}^m \frac{\delta}{|e_k - e_l|} \lesssim \log \frac{1}{\delta}. \quad (38)$$

Indeed, for $j \geq 1$ set

$$I_j := \{l \neq k : j\delta \leq |e_k - e_l| < (j+1)\delta\}.$$

The fact that the directions are δ -separated means that $|e_k - e_l| \geq \delta$ whenever $l \neq k$, so the families I_j , $j \geq 1$, include all the indices $l \neq k$; furthermore, I_j is empty as soon as $j\delta > \frac{1}{5}$, since all the directions are contained in the cap $\mathcal{C}$.

In order to obtain an upper bound for the number of elements in I_j we change from the chord distance to the arc distance $\vartheta(e, e') \in [0, \pi]$, the two being related by $|e - e'| = 2 \sin(\frac{\vartheta(e, e')}{2})$. That is, $\vartheta(e, e') = 2 \arcsin(\frac{|e - e'|}{2})$. The map $r \mapsto 2 \arcsin(\frac{r}{2})$ is increasing on $[0, \frac{2}{5}]$ and has derivative $(1 - \frac{r^2}{4})^{-\frac{1}{2}} \leq 2$ there, hence is Lipschitz with constant 2 on that interval; this covers the relevant range because $(j+1)\delta$ is less than or equal to $\frac{2}{5}$. If we let ϑ_j equal $2 \arcsin \frac{j\delta}{2}$, we therefore have

$$\{e \in \mathbb{S}^1 : j\delta \leq |e - e_k| < (j+1)\delta\} = \{e \in \mathbb{S}^1 : \vartheta_j \leq \vartheta(e, e_k) < \vartheta_{j+1}\},$$

and this is made up of two arcs, each having a length of $\vartheta_{j+1} - \vartheta_j \leq 2\delta$. Since $|e - e'| \leq \vartheta(e, e')$, two distinct directions of our family are at arc distance at least δ , so each of these arcs contains at most three of them and the number of elements of I_j is at most six, that is $\#I_j \leq 6$. Consequently

$$\sum_{l \neq k} \frac{\delta}{|e_k - e_l|} = \sum_{j \geq 1} \sum_{l \in I_j} \frac{\delta}{|e_k - e_l|} \leq \sum_{1 \leq j \leq \frac{1}{5\delta}} \#I_j \frac{\delta}{j\delta} \lesssim \sum_{1 \leq j \leq \frac{1}{5\delta}} \frac{1}{j} \lesssim \log \frac{1}{\delta},$$

which is (38).

We now expand the square and split the sum into its diagonal and off-diagonal parts:

$$\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^2(\mathbb{R}^2)}^2 = \sum_{k, l=1}^m t_k t_l |T_k \cap T_l| = \sum_{k=1}^m t_k^2 |T_k| + \sum_{\substack{k, l=1 \\ k \neq l}}^m t_k t_l |T_k \cap T_l|.$$

For the diagonal term, since every tube has measure $\omega_1 \delta$ and by (37), we know that

$$\sum_{k=1}^m t_k^2 |T_k| = \omega_1 \delta \sum_{k=1}^m t_k^2 \leq \omega_1 \lesssim \log \frac{1}{\delta},$$

where the last inequality holds since $\delta < \frac{1}{5}$. For the off-diagonal term we apply Lemma 2.15 in the form given by (30) with $n = 2$, that is $|T_k \cap T_l| \lesssim \frac{\delta^2}{|e_k - e_l|}$, and we distribute the factor δ symmetrically between the two indices:

$$\sum_{k \neq l} t_k t_l |T_k \cap T_l| \lesssim \sum_{k \neq l} \left(\sqrt{\delta} t_k \left(\frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} \right) \left(\sqrt{\delta} t_l \left(\frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} \right).$$

By the Cauchy-Schwarz inequality and the symmetry of the quantity $\frac{\delta}{|e_k - e_l|}$ in the two indices, the right-hand side is bounded by

$$\left(\sum_{k \neq l} \delta t_k^2 \frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} \left(\sum_{k \neq l} \delta t_l^2 \frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} = \sum_{k \neq l} \delta t_k^2 \frac{\delta}{|e_k - e_l|} = \sum_{k=1}^m \delta t_k^2 \sum_{l \neq k} \frac{\delta}{|e_k - e_l|}.$$

The inner sum is estimated by (38) and the outer one by (37), so that

$$\sum_{k \neq l} t_k t_l |T_k \cap T_l| \lesssim \log \frac{1}{\delta} \sum_{k=1}^m \delta t_k^2 \leq \log \frac{1}{\delta}.$$

If we add the two contributions together we obtain

$$\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^2(\mathbb{R}^2)}^2 \lesssim \log \frac{1}{\delta},$$

which is precisely (36). Proposition 2.17 then gives a bound for the operator, namely

$$\|\mathcal{K}_\delta f\|_{L^2(\mathcal{C})} \lesssim \sqrt{\log \frac{1}{\delta}} \|f\|_{L^2(\mathbb{R}^2)},$$

and Lemma 2.8 extends the estimate to the whole unit circle $\mathbb{S}^1$. $\square$

Observation 2.18. Since $\log \frac{1}{\delta} \leq C_\varepsilon \delta^{-\varepsilon}$ for all $0 < \delta < 1$ and every $\varepsilon > 0$, the estimate of Theorem 2.14 implies the Keakeya maximal conjecture 2.4 for $n = 2$.

Observation 2.19. The logarithmic factor in Theorem 2.14 cannot be removed. For $0 < \delta < \frac{1}{4}$ consider

$$f_\delta(x) := \frac{1}{|x|} \mathbf{1}_{\{\delta \leq |x| \leq 1\}}.$$

Passing to polar coordinates,

$$\|f_\delta\|_{L^2(\mathbb{R}^2)}^2 = \int_0^{2\pi} \int_\delta^1 \frac{1}{r^2} r \, dr \, d\vartheta = 2\pi \log \frac{1}{\delta}.$$

On the other hand, for every $e \in \mathbb{S}^1$ the tube $T_e^\delta(0)$, which is centred at the origin, gives

$$\mathcal{K}_\delta f_\delta(e) \geq \frac{1}{2\delta} \int_{T_e^\delta(0)} f_\delta \gtrsim \log \frac{1}{\delta}.$$

Therefore $\|\mathcal{K}_\delta f_\delta\|_{L^2(\mathbb{S}^1)} \gtrsim \log \frac{1}{\delta}$, whereas $\|f_\delta\|_{L^2(\mathbb{R}^2)} \simeq (\log \frac{1}{\delta})^{1/2}$, so no bound better than $\sqrt{\log \frac{1}{\delta}}$ is possible.

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