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Fourier transform restriction and fractal sets of $\mathbb{R}$

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Introduction

This thesis, written under the supervision of Professor Paolo Ciatti of the University of Padua, is intended as a study of some of the several works that have been published in the second half of the 20th century regarding the problem of Fourier restriction to a set of measure 0 in $\mathbb{R}^d$. In the following pages we will assume the reader to be familiar with the concept and properties of Fourier transforms, L^p (and Sobolev $W^{k,p}$) spaces, as well as with the concept of measure and with the basic notions on hyperplanes and curvature. In the last chapter it will also be fairly necessary to be acquainted with Hausdorff dimension and measure and with fractal sets.

The problem originally comes from the idea that the Fourier transform of f is only continuous and defined everywhere in $\mathbb{R}^d$ if f is L^1. As it is known from classical Fourier theory, the Fourier transform operator can be extended to a surjective L^2 isometry. However, this comes at the cost of losing the ability to be able to ascertain the value of the resulting function on any given point (or set of measure 0), since L^2 functions are only defined almost everywhere. It is also possible, by interpolation on L^p spaces using the Riesz-Thorin interpolation theorem, to define the Fourier transform $\mathcal{F}$ on L^p spaces for $p \in [1, 2]$, where the transform would then belong to the $L^{p'}$ space such that p and p' are conjugate exponents (which means they satisfy $\frac{1}{p} + \frac{1}{p'} = 1$). While functions in $L^{p'}$ are not by any means defined everywhere, we will see how an exception can be made for p close enough to 1 if the set over which we take the restriction satisfies certain specific properties.

Stein and Tomas (see [11] and [13]) were among the first to study this concept. Stein managed to prove that the Fourier transform of a function in L^p with $p < \frac{2d}{d+1}$ is continuous, and as such well defined in every point, as long as the function is radial. For further results, while continuity itself cannot be easily obtained, Stein and Tomas managed to prove that the restriction of the Fourier transform of an L^p function is well defined on hypersurfaces in the sense of L^2 functions, for appropriate values of p, as long as we take a hypersurface with non-vanishing Gauss curvature. Specifically, the more principal curvatures of M are equal to 0 for the hypersurface and the more it is necessary for the f to be close to L^1 for the restriction of its Fourier transform to exist in the L^q sense on M.

Later on, in the early 2000s, Mockenhaupt (in [8]) managed to develop an extension to this theory for subsets of $\mathbb{R}$, where a curvature assumption cannot be formulated. He managed to find that the existence of Fourier transform restrictions descends from an intrinsic characteristic of sets of any dimension, which depends on the order of decay of Fourier transforms of measures over that set. Such a property has been called *Fourier dimension* of the set. As we will see in Section 2, this translates into the existence of a curvature for hyperplanes in dimension $d \geq 2$, but for $d = 1$ constructing sets with nonzero Fourier dimension is more complicated, and one has to look for unevenly generated fractal constructions, such as those by Salem [9]. This paved the road to a whole new theory for which new works and results are still being published nowadays, such as those by Bak and Seeger [1], Łaba and Pramanik [16] and very recently Fraser, Hambrook and Ryou [5].

The sections of this thesis are organized as it follows. In Section 1 we will introduce the concept of oscillatory integral, which is necessary to formulate some of the estimates we will work with later on. In Section 2 we will then start working on the geometrical side of the problem, analyzing the properties of sets with non-vanishing Gauss curvature to produce some new estimates that depend only on the geometry of the set. In Section 3 we will state the main result from Stein's book on Fourier restriction transform, and after formulating the problem we will state and prove all of the intermediate steps still necessary to solve it, and finish by discussing the optimality of the hypothesis given by Stein. Finally in Section 4 we will work on the problem in dimension $d = 1$ and prove a result from Mockenhaupt and compare it with what had been attained by Stein; also at the end we will present a construction that can be used to attain the existence of a Fourier restriction on the real line.

1 Estimates on oscillatory integrals

We will now begin by studying the concept of *oscillatory integrals* as described by Stein in [11].

Definition 1.1. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a *phase* function of class at least C^1, $\psi : \mathbb{R} \to \mathbb{C}$ be an *amplitude* function at least of class C^1 then for all $\lambda \in \mathbb{R}$ we define an *oscillatory integral* as one of the following kind:

$$I(\lambda) = \int_{\mathbb{R}} e^{i\lambda\Phi(x)}\psi(x)dx$$

We want to study the decay of integrals of this kind as $|\lambda| \to \infty$. The first thing we can see is that as long as $\nabla\Phi(x) \geq c > 0$ such an integral decays to 0 faster than λ^{-N} for all $N \in \mathbb{N}$, provided the phase and amplitude functions are smooth (it suffices for them to be of class C^N).

Proposition 1.2 (Stationary phase)**.** *Assume Φ, $\psi \in C^\infty$ and* $\operatorname{supp}\psi$ *compact. If $|\nabla\Phi(x)| \geq c > 0$ for all $\lambda > 0$ then for all $N \geq 0$ we have:*

$$|I(\lambda)| \leq c_N \lambda^{-N}$$

for all $\lambda > 0$ and some constant c_N depending only on N.

Proof. We will work with the following vector field:

$$L \coloneqq \frac{1}{i\lambda}\sum_{k=1}^{d} a_k \frac{\partial}{\partial x_k} = \frac{1}{i\lambda}(a \cdot \nabla) \tag{1.1}$$

with $a = (a_1, \cdots, a_d) = \frac{\nabla\Phi}{|\nabla\Phi|^2}$, which is bounded on $\operatorname{supp}\psi$ by hypothesis. We can also see how the same holds for any of the partial derivatives of order N of a, for any fixed N, as they are smooth functions on a compact set: all of the functions that appear in them are of class C^∞ and the denominators are non-zero.

Now we consider the transpose operator L^t which must satisfy:

$$\int L(F)g dx = \int \frac{1}{i\lambda}\sum_{k=1}^{d}\left(a_k \frac{\partial F}{\partial x_k} g\right) dx = -\int \frac{1}{i\lambda}\sum_{k=1}^{d}\left(F\frac{\partial(a_k g)}{\partial x_k}\right) dx = \int F L^t(g) dx$$

for any given F with values in $\mathbb{R}^d$ and g with values in $\mathbb{R}$ and compact support, both of class at least C^1. We have used the compact support of g to integrate by parts. Hence:

$$L^t(g) = -\frac{1}{i\lambda}\sum_{k=1}^{d}\frac{\partial(a_k g)}{\partial x_k}. \tag{1.2}$$

Now we can see that:

$$L(e^{i\lambda\Phi}) = \frac{1}{i\lambda}(a \cdot i\lambda\nabla\Phi)e^{i\lambda\Phi} = e^{i\lambda\Phi}$$

and by reiterating, for all N positive integers it holds:

$$L^N(e^{i\lambda\Phi}) = e^{i\lambda\Phi}. \tag{1.3}$$

Combining (1.2) and (1.3), integrating by parts again, and since $\operatorname{supp}\psi$ is compact, we get:

$$I(\lambda) = \int_{\mathbb{R}^d} L^N(e^{i\lambda\Phi})\psi dx = \int_{\mathbb{R}^d} e^{i\lambda\Phi}(L^t)^N(\psi)dx$$

hence:

$$|I(\lambda)| \leq \int_{\operatorname{supp}\psi} \left|(L^t)^N(\psi)\right| dx.$$

Now $\left|(L^t)^N(\psi)\right|$ is of the form $\left|\lambda^{-N}\right| P_N$ where P_N is a polynomial in the derivatives up to order N of ψ and a which are all bounded by some constant c_N. Hence by multiplying the constant by the finite measure of $\operatorname{supp}\psi$ we have proven our thesis. □

So we have proven that when studying the order of decay of oscillatory integrals it really suffices to look at the points x for which $\nabla\Phi(x) = 0$. We will now make estimates for *non-degenerate critical points*, for which $\det\{\nabla^2\Phi\} \neq 0$ on the support of ψ, where $\nabla^2\Phi$ is the *Hessian matrix* of Φ.

Proposition 1.3. *Assume Φ, $\psi \in C^\infty$ and $\operatorname{supp}\psi$ compact. If we have that $\det\{\nabla^2\Phi(x)\} \neq 0$ for all $x \in \operatorname{supp}\psi$ then $I(\lambda) = O(\lambda^{-d/2})$ as $\lambda \to \infty$.*

Proof. Let us assume $\operatorname{supp}\psi \subseteq B_\varepsilon$ for some B_ε ball of radius $\varepsilon > 0$ as small as we would like. We can prove our thesis for such ψ and then extend it to functions whose compact support has greater size summing over smaller domains through a partition of unity.

We have:

$$|I(\lambda)|^2 = \overline{I(\lambda)}I(\lambda) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\lambda[\Phi(y)-\Phi(x)]}\psi(y)\overline{\psi(x)}dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\lambda[\Phi(x+u)-\Phi(x)]}\psi(x,u)dxdu \tag{1.4}$$

where we have set $y = x + u$ and defined $\psi(x,u) = \psi(x+u)\overline{\psi(x)}$, which is a smooth function with compact support. Specifically we have $\operatorname{supp}\psi(x,u) \subset \{|u| \leq 2\varepsilon\}$.

We now call $J_\lambda(u) = \int_{\mathbb{R}^d} e^{i\lambda[\Phi(x+u)-\Phi(x)]}\psi(x,u)dx$ and we claim

$$|J_\lambda(u)| \leq c_N(\lambda|u|)^{-N} \tag{1.5}$$

for some constant c_N and for each $N \geq 0$. If (1.5) holds, then we would have, by combining the estimates for $N = 1$ and $N = d+1$:

$$|J_\lambda(u)| \leq c\frac{1}{(1+\lambda|u|)^{d+1}},$$

and by combining that with (1.4) we get:

$$|I(\lambda)|^2 = \int_{\mathbb{R}^d} J_\lambda(u)du \leq c\int_{\mathbb{R}^d} \frac{1}{(1+\lambda|u|)^{d+1}}du = c\lambda^{-d}\int_{\mathbb{R}^d}\frac{1}{(1+|t|)^{d+1}}dt = c'\lambda^{-d}$$

hence $|I(\lambda)| = O(\lambda^{-d/2})$ and we have our thesis.

We are now left to prove (1.5). To do so we will follow the same ideas as Proposition 1.2 and define $L = \frac{1}{i\lambda}(a\cdot\nabla)$ with $L^t(f) = -\frac{1}{i\lambda}\nabla\cdot(af)$:

$$a = \frac{\nabla_x(\Phi(x+u)-\Phi(x))}{|\nabla_x(\Phi(x+u)-\Phi(x))|} = \frac{b}{|b|}$$

and $b = \nabla_x(\Phi(x+u) - \Phi(x))$. Now $|b| \le C|u|$ for some C depending only on the second order derivatives of Φ on $\operatorname{supp}\psi$, which are bounded since ψ is smooth. On the other hand, since we know that $\nabla^2\Phi(x)$ is invertible, we have that $\left|\nabla^2\Phi(x) \cdot u\right| \ge c|u|$ for some constant $c > 0$ and by using Taylor's theorem, we get:

$$|\nabla_x(\Phi(x+u) - \Phi(x))| \ge \left|\nabla^2\Phi(x) \cdot u + O(|u|^2)\right| \ge \left|c|u| - \left|O(|u|^2)\right|\right| \ge c|u|$$

where we have used $|u| \le 2\varepsilon$, by fixing ε small enough. So we have:

$$c|u| \le |b| \le C|u| \tag{1.6}$$

for some appropriate constants. Also $|\partial_x^\alpha b| \le c_\alpha |u|$ for some appropriate constant c_α, since all the derivatives of Φ on the compact support of ψ are bounded. Hence, by combining this result with (1.6), we get $|\partial_x^\alpha a| \le c|u|^{-1}$ ($\partial_x^\alpha a$ can be written in the form $P_\alpha/|b|^{|\alpha|+2}$ where $P(\alpha)$ is a polynomial of homogeneous degree $|\alpha|+1$ in b and its derivatives, and is as such bounded by $c'_\alpha |u|^{|\alpha|+1}$, while $|b|^{|\alpha|+2} \ge c|u|^{\alpha+2}$ by (1.6)). So by using this last result we get:

$$\left|L^t(\psi(x,u))\right| \le |\lambda|^{-1} |\nabla \cdot (a\psi(x,u))| \le c|\lambda u|^{-1}$$

and by iterating:

$$\left|(L^t(\psi(x,u)))^N\right| \le c_N |\lambda u|^{-N}.$$

But we can also see that, by using our results in Proposition 1.2:

$$|J_\lambda(u)| = \left|\int_{\mathbb{R}^d} L^N \left(e^{i\lambda[\Phi(x+u)-\Phi(x)]}\right) \psi(x,u)dx\right| \le \int_{\mathbb{R}^d} \left|(L^t)^N(\psi(x,u))\right| dx \le c_N(|\lambda u|)^- N$$

which proves our claim. □

We will now make a few remarks on this result. First of all it is evident from the proof that it is not necessary to require Φ and ψ to be smooth, but it suffices for them to be of class C^{d+2} and C^{d+1} respectively since c_N only depends on $\|\Phi\|_{C^{d+2}}$, on $\|\psi\|_{C^{d+1}}$, on $\inf\left\{\left|\det\{\nabla^2\Phi\}\right|\right\}$ and on $\operatorname{diam}(\operatorname{supp}\psi)$. It is also possible to prove a weaker estimate in case we have degenerate critical points in the domain for which $\det\{\nabla^2\Phi(x)\} = 0$.

Corollary 1.4. *Assume Φ, $\psi \in C^\infty$ and $\operatorname{supp}\psi$ compact and let $0 < m \le d$ be an integer. If we have that $\operatorname{rk}\nabla^2\Phi(x) \ge m$ for all $x \in \operatorname{supp}\psi$ then $I(\lambda) = O(\lambda^{-m/2})$ as $\lambda \to \infty$.*

The proof of this proposition is omitted, but it descends directly from the same line of reasoning used to prove Proposition 1.3, after diagonalizing the Hessian matrix and working separately on each coordinate.

The estimate proven in Proposition 1.3 is extremely useful when dealing with oscillatory integrals. In particular, in the next chapter, we will see how it can be applied to measures restricted to manifolds to study the decay of their Fourier transforms.

Before doing that, we would actually like to show some similar but more specific results for $d = 1$ which will allow us to prove a classic estimate on Bessel functions that will be helpful to us later on. We will start for some estimates for Φ' without any critical points.

Proposition 1.5. *Let $d = 1$ and assume Φ, ψ of class C^∞ and* $\operatorname{supp}\psi$ *compact. Let Φ' be monotone and such that $|\Phi'(x)| \geq c > 0$ for all $\lambda > 0$ and let $\psi(x) = \chi_{[a,b]}(x)$ be the indicator function of some interval $[a, b]$. Let $I_1(\lambda) = \int_a^b e^{i\lambda\Phi(x)}dx$ be the associated oscillatory integral for such ψ, then for all $\lambda > 0$ we have:*

$$|I_1(\lambda)| \leq 3(\lambda\mu)^{-1}$$

Proof. Since Φ is continuous we can assume without any loss of generality that $\Phi' \geq \mu > 0$ on $[a, b]$ (the other case is analogous by taking complex conjugates). Let's define as before $L = \frac{1}{i\lambda\Phi'(x)}\frac{d}{dx}$, $L^t(f) = -\frac{1}{i\lambda}\frac{d}{dx}(f/\Phi')$, we have, integrating by parts:

$$I_1(\lambda) = \int_a^b L\left(e^{i\lambda\Phi(x)}\right)dx = \int_a^b e^{i\lambda\Phi(x)}L^t(1)dx + \left[e^{i\lambda\Phi}\frac{1}{i\lambda\Phi'}\right]_a^b \tag{1.7}$$

Let us now make some estimates on the terms on the right hand side of the above equation.

On the one hand we have that

$$\left|\left[e^{i\lambda\Phi}\frac{1}{i\lambda\Phi'}\right]_a^b\right| \leq \frac{1}{\lambda}\left|\frac{1}{\Phi'(b)}\right| + \frac{1}{\lambda}\left|\frac{1}{\Phi'(a)}\right| \leq 2(\lambda\mu)^{-1}.$$

Meanwhile, since Φ' is monotone continuous, $\frac{d}{dx}\left(\frac{1}{\Phi'}\right)$ does not change sign on $[a, b]$, so the other term can be estimated as it follows:

$$\begin{aligned}\left|\int_a^b e^{i\lambda\Phi(x)}L^t(1)dx\right| \leq \int_a^b \left|L^t(1)\right|dx &= \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\left(\frac{1}{\Phi'}\right)\right|dx \\ &= \frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\left(\frac{1}{\Phi'}\right)dx\right| \\ &= \left|\frac{1}{\Phi'(b)} - \frac{1}{\Phi'(a)}\right| \leq \frac{1}{\lambda\mu}\end{aligned}$$

By combining these estimates with (1.7) we get:

$$|I_1(\lambda)| \leq \left|\int_a^b e^{i\lambda\Phi(x)}L^t(1)dx\right| + \left|\left[e^{i\lambda\Phi}\frac{1}{i\lambda\Phi'}\right]_a^b\right| \leq 3(\lambda\mu)^{-1}$$

which gives us the estimate we wanted. □

We will now look at a generalization of this result, for a generic function ψ on $[a, b]$ rather that the indicator function of the interval.

Corollary 1.6. *Let Φ and μ be as in Proposition 1.5, ψ of class C^1 on $[a, b]$, then there exists some constant c_ψ depending only on the choice of ψ such that:*

$$\left|\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx\right| \leq c_\psi(\lambda\mu)^{-1}$$

Proof. Let $J(x) = \int_a^x e^{i\lambda\Phi(u)}du$. By Proposition 1.5 we have that $|J(x)| \leq 3(\lambda\mu)^{-1}$. Since $J(a) = 0$, by integrating by parts we get:

$$\left|\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx\right| \leq \left|\int_a^b J(x)\psi'(x)dx\right| + |J(b)\psi(b)| \leq 3(\lambda\mu)^{-1}\left(\int_a^b |\psi'(x)|dx + |\psi(b)|\right)$$

which gives us the desired estimate. □

Now we will turn to the following estimate of van der Corput for cases in which Φ has non-degenerate critical points.

Proposition 1.7. *Let Φ be of class C^2 on $[a,b]$, $|\Phi(x)''| \geq \mu$ in $[a,b]$, then, if $I_1(\lambda)$ is defined as in Proposition 1.5, we get, for all $\lambda > 0$:*

$$|I_1(\lambda)| \leq 8(\lambda\mu)^{-1/2}.$$

Proof. Since Φ' is continuous we can assume without any loss of generality that $\Phi'' \geq \mu > 0$ on $[a,b]$ (the other case is analogous by taking complex conjugates). Hence Φ' is increasing and Φ has at most one critical point x_0 for which $\Phi'(x_0) = 0$. Let us split $[a,b] = [a, x_0 - \delta] \cup [x_0 - \delta, x_0 + \delta] \cup [x_0 + \delta, b]$

- On the intervals $[a, x_0 - \delta]$ and $[x_0 + \delta, b]$ we have $\Phi'(x) \geq \delta\mu$. In fact, we have that $\Phi'(x_0) = 0$ and $\Phi'' \geq \mu$, thus necessarily $\Phi'(x_0 - \delta) \geq \delta\mu$ and $\Phi'(x_0 + \delta) \geq \delta\mu$ and by monotonicity we can extend to the full intervals. So by Proposition 1.5 we have:

$$\left|\int_a^{x_0-\delta} e^{i\lambda\Phi(x)}dx\right| + \left|\int_{x_0+\delta}^b e^{i\lambda\Phi(x)}dx\right| \leq 6(\lambda\delta\mu)^{-1} \tag{1.8}$$

- On $[x_0 - \delta, x_0 + \delta]$ we have a contribution of at most the length of the interval, which is 2δ:

$$\left|\int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\Phi(x)}dx\right| \leq \int_{x_0-\delta}^{x_0+\delta} \left|e^{i\lambda\Phi(x)}\right|dx = 2\delta \tag{1.9}$$

Finally by combining (1.8) and (1.9) with $\delta = (\lambda\mu)^{-1/2}$ we get:

$$|I_1(\lambda)| \leq \left|\int_a^{x_0-\delta} e^{i\lambda\Phi(x)}dx\right| + \left|\int_{x_0+\delta}^b e^{i\lambda\Phi(x)}dx\right| + \left|\int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\Phi(x)}dx\right| \leq 6(\lambda\mu)^{-1/2} + 2(\lambda\mu)^{-1/2}$$

which gives us the estimate we required. □

We can extend this result to more generic ψ functions on intervals as well.

Corollary 1.8. *Let Φ and μ be as in Proposition 1.7, ψ of class C^1 on $[a,b]$, then there exists some constant c_ψ depending only on the choice of ψ such that:*

$$\left|\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx\right| \leq c_\psi(\lambda\mu)^{-1/2}$$

Proof. Let $J(x) = \int_a^x e^{i\lambda\Phi(u)}du$. By Proposition 1.7 we have that $|J(x)| \leq 8(\lambda\mu)^{-1/2}$. Since $J(a) = 0$, by integrating by parts we get:

$$\left|\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx\right| \leq \left|\int_a^b J(x)\psi'(x)dx\right| + |J(b)\psi(b)| \leq 8(\lambda\mu)^{-1/2}\left(\int_a^b |\psi'(x)|dx + |\psi(b)|\right)$$

which gives us the desired estimate. □

We will now apply the above results to Bessel functions to find a useful estimate which we will later employ to study the decay of Fourier transforms of radial functions. A proper introduction to Bessel functions and their properties can be found in Chapter 6 of [10]. It can be proven that we can write a Bessel function of first kind as:

$$J_m(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin x}e^{-imx}dx.$$

Proposition 1.9. *Let $J_m(r)$ be as above, then it holds that $J_m(r) = O(r^{-1/2})$ as $r \to \infty$.*

Proof. We will apply the previous propositions with $\lambda = r$, $\Phi(x) = \sin x$ and $\psi(x) = \frac{1}{2\pi}e^{-imx}$. We split the interval $[0, 2\pi]$ in two parts:

- $\left[\frac{1}{4}\pi, \frac{3}{4}\pi\right] \cup \left[\frac{5}{4}\pi, \frac{7}{4}\pi\right]$ over which $|\Phi''(x)| = |\sin x| \geq \frac{1}{\sqrt{2}}$. Over each one of these intervals we can thus apply Corollary 1.8 with $\mu = \frac{1}{\sqrt{2}}$, obtaining a decay of order $O(r^{-1/2})$.
- $\left[0, \frac{1}{4}\pi\right] \cup \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right] \cup \left[\frac{7}{4}\pi, 2\pi\right]$ over which $|\Phi'(x)| = |\cos x| \geq \frac{1}{\sqrt{2}}$. Over each one of these intervals we can thus apply Corollary 1.6 with $\mu = \frac{1}{\sqrt{2}}$, obtaining a decay of order $O(r^{-1})$.

By combining the above results we get overall that $J_m(r) = O(r^{-1/2})$ as $r \to \infty$, which is what we needed. □

2 Decay of the Fourier transform of carried measures on hypersurfaces with nonzero curvature

In this chapter we will recall some notions and properties of hypersurfaces, then we will define surface carried measures and show how their Fourier transform decays at infinity as long as the surface has some degree of curvature.

Let M be a smooth hypersurface in $\mathbb{R}^d$. For any $x_0 \in M$ we know that there exists some ball $\tilde{B}$ centered in x_0 in which M is the graph of a function, specifically such that there exists a set of coordinates $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and some φ of class $C^\infty(\mathbb{R}^{d-1}, \mathbb{R})$ such that:

$$M \cap \tilde{B} = \left\{(x', x_d) \in \tilde{B} : x_d = \varphi(x')\right\} \tag{2.1}$$

we may also ask, via a further change of coordinates, for φ to be such that $\varphi(0) = 0$ (and $x_0 = 0$) and also $\nabla_{x'}\varphi(x')|_{x'=0} = 0$.

We then take $\rho_0(x) = \varphi(x') - x_d$ which is a *defining function* of M in $\tilde{B}$ (a function such that $\rho_0(x) = 0$ if and only if $x \in M \cap \tilde{B}$). If we take $\rho(x) = \frac{\rho_0(x)}{\nabla \rho_0(x)}$, such that $|\nabla \rho(x)| = 1$ on M, then we can then write the *curvature form* or *second fundamental form* of M at $x \in M$ as $\sum\limits_{1 \le j,k \le d} \xi_k \xi_j \frac{\partial^2 \rho}{\partial x_k \partial x_j}(x)$, uniquely up to an isometry.

On the other hand we have a representation of the curvature form as a $(d-1) \times (d-1)$ matrix as well. In fact since $\nabla_{x'}\varphi(x')|_{x'=0} = 0$ then we have:

$$\varphi(x') = \frac{1}{2} \sum_{1 \le j,k \le d-1} a_{kj} x_k x_j + O\left(|x'|^3\right)$$

and the curvature form is represented by the matrix $\left\{\frac{\partial^2 \varphi}{\partial x_k \partial x_j}\right\}_{k,j} = \{a_{kj}\}_{k,j}$. We can, up to one more isometry, diagonalize the curvature form matrix obtaining:

$$\varphi(x') = \frac{1}{2} \sum_{1 \le j \le d-1} \lambda_j x_j^2 + O\left(|x'|^3\right). \tag{2.2}$$

Definition 2.1. Let M, x_0, φ be defined as above, then the coefficients λ_j for $j = 1, \cdots, d-1$ in (2.2) are called the *principal curvatures* of M at x_0. Also, $\prod_j \lambda_j = \det \nabla^2 \varphi$ is the *total curvature* or *Gauss curvature* of M at x_0.

Before moving on, it is necessary to discuss a bit about Fourier transforms of measures. Let $d\sigma$ be a measure, then its Fourier transform is defined by the following integral:

$$\widehat{d\sigma}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\sigma(x)$$

which is well defined as long as $d\sigma$ is nice enough; in our case we will always consider $d\sigma$ to be a surface measure, finite and compactly supported, and as such $\widehat{d\sigma}(\xi)$ will be of class C^∞. Furthermore it can be proven that most of the theory for Fourier transforms of functions also holds for measures. For example the properties on convolution can be extended to measures as well:

$$\widehat{f * d\sigma}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} \int_{\mathbb{R}^d} f(x-y)d\sigma(y)dx = \iint e^{-2\pi i z\cdot\xi} f(z)e^{-2\pi i y\cdot\xi} d\sigma(y)dz = \hat{f}(\xi)\widehat{d\sigma}(\xi).$$

For a more in depth study of Fourier transforms of measures one could look at [11] among others.

Let us now take the induced Lebesgue measure on M, $d\sigma$, in such a way that for all f functions on M with compact support it holds:

$$\int_M f d\sigma = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{M_\varepsilon} F dx \tag{2.3}$$

where $M_\varepsilon = \{x : d(x, M) < \varepsilon\}$, $\varepsilon > 0$ and F is a continuous extension of f to a neighborhood of M. In our coordinate system we have:

$$\int_M f d\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x'))(1 + |\nabla_{x'}\varphi|^2)^{\frac{1}{2}} dx'$$

Now let us introduce the concept of *surface carried measure* on M.

Definition 2.2. A measure $d\mu$ is a *surface-carried measure on M with smooth density* if it has the form $d\mu = \psi d\sigma$ for some function ψ of class C^∞ and compact support.

Let $d\mu$ be a surface-carried measure on M, then, as seen before, we can write $\widehat{d\mu} = \int_M e^{-2\pi i x\cdot\xi} d\mu$, which is bounded since the measure μ is finite.

We will now see how the previous section estimates can be applied in this setting.

Theorem 2.3. *Let M be a smooth hypersurface in $\mathbb{R}^d$ and μ surface-carried measure on M with smooth density ψ. Assume M to have non-vanishing Gauss curvature at each point of* $\operatorname{supp} d\mu$, *then we have:*

$$\left|\widehat{d\mu}(\xi)\right| = O\left(|\xi|^{-(d-1)/2}\right) \quad as \quad |\xi| \to \infty \tag{2.4}$$

Proof. We begin by writing ψ as a finite sum of ψ_j with support centered in a small ball via a partition of unity of sorts. As such we can assume $\operatorname{supp}\psi \subset \tilde{B}$ as small as we want, and specifically we would like for it to satisfy (2.1) and have $M \cap \tilde{B} = \left\{(x', x_d) \in \tilde{B} : x_d = \varphi(x')\right\}$ for some φ smooth and some coordinate system obtained via an isometry from our initial coordinates. We now keep working in the new coordinate system, since isometries only multiply the Fourier transform by a factor of absolute value 1 and as such the estimate remains unchanged.

In the new coordinate system, by using $x_d = \varphi(x')$ and $d\mu = \psi d\sigma$, we can write:

$$\widehat{d\mu}(\xi) = \int_M e^{-2\pi i x'\cdot\xi' + \varphi(x')\xi_d} \psi(x', \varphi(x'))d\sigma = \int_M e^{-2\pi i x'\cdot\xi' + \varphi(x')\xi_d} \tilde{\psi}(x')dx' \tag{2.5}$$

with $\tilde{\psi}(x') = \psi(x', \varphi(x')) \left(1 + |1 + \nabla_{x'}\varphi|^2\right)^{\frac{1}{2}}$ which is a smooth function since both φ and ψ are, and its support is compact since $\operatorname{supp}\psi$ is compact as well. Hence we can just consider x' to belong to a compact set in the integral. We now split our space in the following regions:

- We first study the estimate on the cone $|\xi_d| \geq c|\xi'|$ for some $c > 0$ small to be fixed. We will only study the half cone with $\xi_d > 0$ since the other case is analogous. Let us choose $i\lambda\Phi(x') = -2\pi i(x' \cdot \xi' + \varphi(x')\xi_d)$, with $\lambda = 2\pi\xi_d$, $\Phi(x') = -\varphi(x') - \frac{x'\xi'}{\xi_d}$, hence $\nabla^2_{x'}\Phi = -\nabla^2_{x'}\varphi$. Now, since by hypothesis the Gauss curvature is non-zero, we have $\det \nabla^2_{x'}\Phi = (-1)^{d-1} \det \nabla^2_{x'}\varphi \neq 0$. Also for all $N \in \mathbb{N}$ we have that $\|\Phi\|_{C^N}$ is bounded uniformly in ξ on the cone (in fact, the derivatives of order 2 and above depend only on the derivatives of φ, which are bounded on a compact set, while $|\Phi| \leq |\varphi| + c|x'|$ and $|\nabla\Phi| \leq |\nabla\varphi| + c$ which are once again bounded on a compact set, independently of ξ). By applying Proposition 1.3 in dimension $d-1$ we get for all ξ in the cone:

$$\left|\widehat{d\mu}(\xi)\right| \leq \left|\int_M e^{i\lambda\Phi(x')}\tilde{\psi}(x')dx'\right| = O\left(\lambda^{-\frac{d-1}{2}}\right) = O\left(\xi_d^{-\frac{d-1}{2}}\right) = O\left(|\xi|^{-\frac{d-1}{2}}\right) \tag{2.6}$$

since $\xi_d \geq c|\xi'|$. We notice that after applying the proposition we could change to a uniform bound in ξ because the constants involved only depend on variables which are uniformly bounded in ξ.

- Now we proceed to study the estimate on the complementary region $|\xi_d| < c|\xi'|$. We choose $\lambda = 2\pi|\xi'|$ and $\Phi(x') = -\varphi(x')\frac{\xi_d}{|\xi'|} - \frac{x'\cdot\xi'}{|\xi'|}$. As before we can see that $\|\Phi\|_{C^N}$ is bounded uniformly in ξ. Also:

$$|\nabla_{x'}\Phi(x')| \geq \left|\left|\nabla_{x'}\left(\frac{x'\cdot\xi'}{|\xi'|}\right)\right| - \left|\nabla_{x'}\varphi(x')\frac{\xi_d}{|\xi'|}\right|\right| \geq |1 - c|\nabla_{x'}\varphi(x')|| \geq \frac{1}{2}$$

as long as we choose c small enough that $c|\nabla_{x'}\varphi(x')| \leq \frac{1}{2}$. Hence by Proposition 1.2 we have for all $N \geq 0$:

$$\left|\widehat{d\mu}(\xi)\right| \leq \left|\int_M e^{i\lambda\Phi(x')}\tilde{\psi}(x')dx'\right| = O\left(\lambda^{-N}\right) = O\left(\xi'^{-N}\right) = O\left(|\xi|^{-N}\right) \tag{2.7}$$

where once again in the last two equalities we have used respectively the fact that bound is uniform in ξ and $|\xi_d| < c|\xi'|$.

So by using (2.6) and (2.7) for $N \geq \frac{d-1}{2}$ we get that (2.5) holds on all $\mathbb{R}^d$ as $|\xi| \to \infty$. □

We have now proven that any surface carried measure on a hypersurface with nonzero Gauss curvature has Fourier transform vanishing at infinity with a decay of order $O\left(|\xi|^{-\frac{d-1}{2}}\right)$. It comes naturally to ask oneself what happens if the Gaussian curvature is 0. It can be proven that, as long as the principal curvatures are not all 0, a weaker estimate holds for surface carried measures, in a similar fashion to Corollary 1.4.

Corollary 2.4. *Let M, μ and ψ be as in Theorem 2.3, but assume M has only at least m non-vanishing principal curvatures at each point of* $\operatorname{supp} d\mu$*. Then we have:*

$$\left|\widehat{d\mu}(\xi)\right| = O\left(|\xi|^{-m/2}\right) \quad as \quad |\xi| \to \infty.$$

The proof is exactly the same as the one from Theorem 2.3, except for the estimate in the cone, where instead of utilizing Proposition 1.3, we apply Corollary 1.4.

We will now look at some application of the above estimates.

Aside from being used for the Fourier transform restriction (as we will see in Section 3), the estimates above can also be used to bound other functionals. An example from [11] would be the averaging operator. We will just give an intuitive example: on $d = 3$ we can define the averaging operator on the sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ as $A(f)(x) = \frac{1}{4\pi}\int_{S^2} f(x-y)d\sigma(y) = \frac{1}{4\pi}(f * d\sigma)$, where σ is the Lebesgue induced measure on the sphere. Assuming $f \in L^2(\mathbb{R}^3)$ we can see that $A(f) \in W^{1,2}(\mathbb{R}^3)$.

Now we can see that, when computing $\widehat{d\sigma}(\xi)$, we can make use of ξ being along an axis of S^2 for all ξ, and as such:

$$\widehat{d\sigma}(\xi) = \int_{S^2} e^{-2\pi i x\xi} d\sigma(x) = 2\pi \int_0^\pi e^{-2\pi i|\xi|\cos\varphi} \sin\varphi d\varphi = 2\pi \left[\frac{e^{-2\pi i|\xi|\cos\varphi}}{2\pi i|\xi|}\right]_0^\pi = 2\frac{\sin(2\pi|\xi|)}{|\xi|}. \tag{2.8}$$

Hence we can write $\left|\widehat{d\sigma}(\xi)\right| \leq (1+|\xi|)^{-1}$ If we combine this result and Plancherel's Theorem, we are now able to estimate the derivatives of the average operator:

$$\left\|\partial_{x_j} A(f)\right\|_{L^2}^2 = \left\|\widehat{\partial_{x_j} A(f)}\right\|_{L^2}^2 = C\int \left|\xi_j \hat{f}(\xi)\widehat{d\sigma}(\xi)\right|^2 d\xi \leq C\int \frac{|\xi|^2}{1+|\xi|^2}\hat{f}^2(\xi)d\xi \leq C\|f\|_{L^2}^2$$

which means that the average operator smoothens the function f, operating between L^2 and $W^{1,2}$. More in general we can extend the above theory to any dimension d and any surface M given a surface-carried measure μ.

Definition 2.5. Let μ be a surface-carried measure on some hypersurface M in $\mathbb{R}^d$ for $d > 1$. We define the *averaging operator* for μ on M as

$$A(f)(x) := \int_M f(x-y)d\mu(y). \tag{2.9}$$

And similar results as before hold in higher dimension, as we can see from the statement of the following theorem.

Theorem 2.6. *Let M be a hypersurface in $\mathbb{R}^d$ and $d\mu$ surface-carried measure for M of the form $d\mu = \psi d\sigma$, with $d\sigma$ induced by the Lebesgue measure on M as seen in (2.3). Assume the Gauss curvature of M to be non vanishing at each point $x \in M$ of the support of $d\mu$. Let A be the averaging operator for μ on M defined as in (2.9). Then the following statements hold:*

(a) $A : L^2(\mathbb{R}^d) \to W^{k,2}(\mathbb{R}^d)$ is well defined for $k = \frac{d-1}{2}$.

(b) A extends to a bounded linear transformation from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ with $p = \frac{d+1}{d}$, $q = d+1$.

Proof of (a). By Theorem 2.3 we have $\left|\widehat{d\mu}(\xi)\right| = O\left(|\xi|^{-\frac{d-1}{2}}\right)$, and, since $d\mu$ has compact support and as such its Fourier transform is continuous, that means we can write $\left|\widehat{d\mu}(\xi)\right| \leq C(1+|\xi|)^{-\frac{d-1}{2}}$. As such, by combining this with Plancherel's Theorem and the formula for the Fourier transform of derivative we get:

$$\|A(f)\|_{W^{\frac{d-1}{2},2}}^2 \leq C\left\|(1+|\xi|)^{\frac{d-1}{2}}\widehat{A(f)}\right\|_{L^2}^2 = C\left\|(1+|\xi|)^{\frac{d-1}{2}}\hat{f}(\xi)\widehat{d\mu}(\xi)\right\|_{L^2}^2 \leq C\|f\|_{L^2}^2$$

which gives us the required estimate for (a). □

We will not prove (b). However the proof descends directly from the propositions in the next section (see Stein [11]).

3 Fourier transform restriction on hypersurfaces

We now turn to the actual main topic of our thesis, studying the theory of Fourier restriction over different sets of $\mathbb{R}^d$. The concept itself, as anticipated in Section 1, is fairly straightforward. We do know that for $f \in L^1$ we have $\hat{f}$ continuous and as such well defined on any set of $\mathbb{R}^d$. While for L^2 functions this doesn't hold, since $\hat{f}$ is L^2 and defined only almost everywhere, once we start questioning whether restricting the Fourier transform of an L^p function with $1 < p < 2$ to a set of zero measure in $\mathbb{R}^d$ the answer becomes the following: it depends on p and the set itself. As a base we only know that $\hat{f} \in L^{p'}$ with p' conjugate exponent of p, and as such is normally defined only almost everywhere. However, when the set considered is a hypersurface, we get that a restriction is feasible, but it strictly depends on the principal curvatures of the surface. The more principal curvatures of a hypersurface are zero, the weaker our restriction becomes. On the other hand, if the Gauss curvature is non-vanishing at each point, we get the maximum range of L^p spaces for which our restriction holds. This was proven by both Stein and Tomas in [11] and [13].

A first more specific result was given by Stein for the case in which f is a radial function. In this case we can utilize the theory on Bessel functions we have proven in Proposition 1.9 as well as the formula for the Fourier transform of a radial function.

Let $f(x) = f_0(|x|)$ then we can write $\hat{f}(\xi) = F(|\xi|)$ with:

$$F(\rho) = 2\pi\rho^{-\frac{d}{2}+1} \int_0^\infty J_{\frac{d}{2}-1}(2\pi\rho r) f_0(r) r^{\frac{d}{2}}\, dr. \tag{3.1}$$

At which point it becomes quite feasible to use the estimates on Bessel functions seen in Section 2 to prove the following.

Proposition 3.1. *Let $d \geq 2$, $f \in L^p(\mathbb{R}^d)$ radial then $\hat{f}$ is continuous for $\xi \neq 0$ whenever $1 \leq p < \frac{2d}{d+1}$.*

Notice how $\frac{2d}{d+1} < 2$ and $\frac{2d}{d+1} \to 2$ for $d \to \infty$.

Proof. We can assume $f \equiv 0$ on the unit ball B, since otherwise if $f \in L^p(B)$ then it is also L^1 and its Fourier transform is continuous.

Let $\hat{f}(\xi) = F(|\xi|)$ as above, we have that by Proposition 1.9 it holds that $J_{\frac{d}{2}-1}(2\pi\rho r) = O(r^{-1/2})$ and as such by (3.1):

$$|F(\rho)| \leq c \int_0^\infty |f_0(r)| r^{\frac{d}{2}-\frac{1}{2}}\, dr$$

Now let $q = p'$ be the conjugate exponent of p, then by splitting $r^{\frac{d-1}{2}} = r^{\frac{d-1}{p}} r^{\frac{d-1}{q}} r^{-\frac{d-1}{2}}$ and by Hölder inequality we get:

$$|F(\rho)| \leq c \left(\int_1^\infty |f_0(r)|^p r^{d-1} dr \right)^{\frac{1}{p}} \left(\int_1^\infty r^{d-1-q\left(\frac{d-1}{2}\right)} dr \right)^{\frac{1}{q}} = c\|f\|_{L^p(\mathbb{R}^d)} \left(\int_1^\infty r^{d-1-q\frac{d-1}{2}}\, dr \right)^{\frac{1}{q}},$$

where the integral on the right hand side is uniformly bounded as long as $\left(\frac{d-1}{2}\right) < -1$ which means $q > \frac{2d}{d-1}$ and hence $p < \frac{2d}{d+1}$. Hence F is in L^∞ and in L^q and as such continuous in ρ for such values of p. $\square$

More in general we will turn to the following problem. Let M be a hypersurface and $d\mu = \psi d\sigma$ a surface-carried measure.

Definition 3.2. We say that the (L^p, L^q) *restriction holds for* M for $1 < p < 2$, $1 \leq q \leq \infty$, if for some $c > 0$ it holds:

$$\left(\int_M \left| \hat{f}(\xi) \right|^q d\mu(\xi) \right)^{\frac{1}{q}} \leq c \|f\|_{L^p(\mathbb{R}^d)} \tag{3.2}$$

Now, of course, it is easy to see that this holds for the trivial case $p = 1$ and $q = \infty$, since the Fourier transform would be continuous. However, how we will see in the rest of this section, Stein managed to prove (3.2) to hold for $1 \leq p \leq \frac{2d+2}{d+3}$ and $q \leq \frac{d-1}{d+1} p'$ with p' conjugate exponent to p. At the end of this section we will also look at an example from Knapp that will prove the optimality of this exponent range.

To prove the theorem mentioned above, Stein first states the result for some specific exponents and then later proceeds by interpolation.

Theorem 3.3. *Let $M \subset \mathbb{R}^d$ be a smooth hypersurface with nonzero Gauss curvature on* $\operatorname{supp} d\mu$ *for some $d\mu = \psi d\sigma$ surface-carried measure. Then it holds:*

$$\left(\int_M \left| \hat{f}(\xi) \right|^q d\mu(\xi) \right)^{\frac{1}{q}} \leq c \|f\|_{L^p(\mathbb{R}^d)}$$

for $q = 2$ and $p = \frac{2d+2}{d+3}$.

Before proving the above result, it will be necessary to state and prove some partial results. The first one descends directly from the three-lines lemma, which we will state below for simplicity, but we will not give a proof of (See Chapter 2, Lemma 2.2 in Stein).

Lemma 3.4 (Three-lines)**.** *Let $\Phi(z)$ be a holomorphic function on the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re}\{z\} < 1\}$ that is also continuous and bounded on the closure of S. If*

$$M_0 = \sup_{y \in \mathbb{R}} |\Phi(iy)| \quad \textit{and} \quad M_1 = \sup_{y \in \mathbb{R}} |\Phi(1 + iy)|$$

then $\sup_{y \in \mathbb{R}} |\Phi(t + iy)| \leq M_0^{1-t} M_1^t$.

Now we will use this to prove the first proposition: an interpolation result on a carefully constructed class of linear operators.

Proposition 3.5. *Let $a, b \in \mathbb{R}$, and for all $s \in S = \{s \in \mathbb{C} : a \leq \operatorname{Re}\{s\} \leq b\}$ let us take a class of linear operators $T_s : X \to L^1_{loc}(\mathbb{R}^d)$ where X is the space of simple functions on $\mathbb{R}^d$. Assume that for any pair of simple functions $f, g \in X$ the functional $\Phi_0(s) = \int_{\mathbb{R}^d} T_s(f) g dx$ is continuous and bounded on S and analytic in $S \setminus \partial S$. Assume also that for some $p_0, p_1, q_0, q_1 \in [1, +\infty]$ we have the following estimates:*

$$\sup_{t \in \mathbb{R}} \|T_{a+it}(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \textit{and} \quad \sup_{t \in \mathbb{R}} \|T_{b+it}(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}};$$

then it holds $\|T_c(f)\|_{L^q} \leq M \|f\|_{L^p}$ for all $c \in [a, b]$ where, if we write $c = (1-\theta)a + \theta b$ for some $\theta \in [0, 1]$, then $p = \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1} \right)^{-1}$ and $q = \left(\frac{1-\theta}{q_0} + \frac{\theta}{q_1} \right)^{-1}$.

Proof. Let us take $c \in [a,b]$ and fix θ, p, q as above. Let $f = \sum_k a_k \chi_{E_k}$ and $g = \sum_j b_j \chi_{F_j}$ be simple functions, without loss of generality we may assume $\|f\|_{L^p} = 1$ (otherwise proven the estimate for $f/\|f\|_{L^p}$ we may multiply by the L^p norm and the statement would still hold by linearity). We may also assume $\|g\|_{L^{q'}} = 1$ where q' is the conjugate exponent to q.

Let $s = (1-z)a + bz$, or in other terms $z = \frac{s-a}{b-a}$, then the strip considered becomes $S = \{0 \leq \operatorname{Re}\{z\} \leq 1\}$. We will call $\tilde{T}_z = T_{(1-z)a+bz}$.

Let $f_z = |f|^{\gamma(z)} \frac{f}{|f|}$ and $g_z = |g|^{\delta(z)} \frac{g}{|g|}$ be their rescaled versions, where $\gamma(z) = p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)$ and $\delta(z) = q'\left(\frac{1-z}{q_0'} + \frac{z}{q_1'}\right)$, where q_0' and q_1' are the conjugate exponents of q_0 and q_1 respectively. We notice that f_z and g_z are simple as well, specifically of the form $f_z = \sum_k |a_k|^{\gamma(z)} \frac{a_k}{|a_k|} \chi_{E_k}$ and $g_z = \sum_j |b_j|^{\delta(z)} \frac{b_j}{|b_j|} \chi_{F_j}$, where the sums are finite and the sets E_k are pairwise disjoint, as well as the sets F_j.

We may also see that for $\operatorname{Re}\{z\} = 0$ we get $\|f_z\|_{L^{p_0}} = \|f\|_{L^p} = 1$ and $\|g_z\|_{L^{q_0'}} = \|g\|_{L^{q'}} = 1$ and for $\operatorname{Re}\{z\} = 1$ we get $\|f_z\|_{L^{p_1}} = \|f\|_{L^p} = 1$ and $\|g_z\|_{L^{q_1'}} = \|g\|_{L^{q'}} = 1$.

Let us now set:

$$\Phi(z) = \int_{\mathbb{R}^d} \tilde{T}_z(f_z) g_z = \sum_{k,j} |a_k|^{\gamma(z)} |b_j|^{\delta(z)} \frac{a_k}{|a_k|} \frac{b_j}{|b_j|} \int_{\mathbb{R}^d} T_{(1-z)a+bz}(\chi_{E_k}) \chi_{F_j} dx,$$

which is continuous and bounded on S and analytic in $S \setminus \partial S$ since we have by hypothesis that $\sum_{k,j} \int_{\mathbb{R}^d} T_{(1-z)a+bz}(\chi_{E_k}) \chi_{F_j} dx = \Phi_0((1-z)a + bz) = \Phi_0(s)$ satisfies the same conditions. By combining Hölder's inequality with the bounds in our hypothesis we get for all $t \in \mathbb{R}$:

$$|\Phi(it)| \leq \left\| \tilde{T}_{it}(f_{it}) \right\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}} \leq M_0 \|f_{it}\|_{p_0} \|g_{it}\|_{L^{q_0'}} = M_0;$$

$$|\Phi(1+it)| \leq \left\| \tilde{T}_{1+it}(f_{1+it}) \right\|_{L^{q_1}} \|g_{1+it}\|_{L^{q_1'}} \leq M_1 \|f_{1+it}\|_{p_1} \|g_{1+it}\|_{L^{q_1'}} = M_1.$$

Therefore by Lemma 3.4 we get $\sup_{\operatorname{Re}\{z\}=r} |\Phi(z)| \leq M_0^{1-r} M_1^r$, independently from the choice of g. Hence, by using the definition of L^q norm as operator norm and $\theta = \frac{c-a}{b-a}$, we get:

$$\|T_c(f)\|_{L^q} = \sup_{\|g\|_{L^{q'}}=1} \left| \int_{\mathbb{R}^d} T_c(f) g \right| \leq \sup_{\|g\|_{L^{q'}}=1} |\Phi(\theta)| \leq M_0^{1-\theta} M_1^{\theta} = M$$

which is the estimate we needed. □

The second partial result we need is a proposition that descends from the estimates of Section 2. Before stating and proving this result we will need to prove another preliminary lemma.

Lemma 3.6. *Let $F \in C^\infty(\mathbb{R})$ be a function with compact support and $N \in \mathbb{N}$ an integer. Let us set:*

$$I_s(\rho) := s(s+1)\cdots(s+N) \int_0^\infty u^{s-1} F(u) e^{-2\pi i u \rho} du$$

defined for all $\rho \in \mathbb{R}$ and $s \in \mathbb{C}$ with $\operatorname{Re}\{s\} > 0$. Then the following hold:

(a) I_s has analytic continuation in the half plane $\mathrm{Re}\{s\} > -N-1$*;*

(b) $|I_s(\rho)| \leq c_s \|F\|_{C^{N+1}}(1+|\rho|)^{-\mathrm{Re}\{s\}}$ *when* $-N-1 < \mathrm{Re}\{s\} \leq 1$ *for come constant c_s which depends only on* $\operatorname{supp} F$ *and s, with at most polynomial growth in* $\mathrm{Im}(s)$*;*

(c) $I_0(\rho) = N!F(0)$.

Proof. The compact support of F allows us to easily integrate by parts, which we will use several times. We may notice straight away that $s(s+1)\cdots(s+N)u^{s-1} = \left(\frac{d}{du}\right)^{N+1} u^{s+N}$ and thus, integrating $N+1$ times by parts, we get:

$$I_s(\rho) = (-1)^{N+1} \int_0^\infty u^{s+N} \left(\frac{d}{du}\right)^{N+1} \left(F(u)e^{-2\pi i u\rho}\right) du.$$

Now, since $\left(\frac{d}{du}\right)^{N+1}\left(F(u)e^{-2\pi i u\rho}\right)$ is bounded with compact support in u, and u^{s+N} is integrable on compact sets as long as $\mathrm{Re}\{s\} > -N-1$, $I_s(\rho)$ admits analytic continuation in this half plane, and (a) is proven.

To prove (b) we split ρ into two different cases.

- If $|\rho| \leq 1$ then we only require for $I_s(\rho)$ to be bounded by a constant which depends polynomially on $\mathrm{Im}(\rho)$. This is readily obtained since we would have $|I_s(\rho)| \leq Cs(s+1)\cdots(s+N)$.
- If $|\rho| \geq 1$ let us take a bump function $\eta \in C^\infty(\mathbb{R})$, with $\eta(u) = 1$ if $u \leq \frac{1}{2}$ and $\eta(u) = 0$ if $|u| \geq 1$, $\eta(u) \in [0,1]$ for all $u \in \mathbb{R}$. We can also ask for $\|\eta\|_{C^\infty}$ to be bounded by some fixed constant M. We can write:

$$I_s(\rho) = \prod_{j=0}^{N}(s+j)\left[\int_0^\infty \eta(u\rho)u^{s-1}F(u)e^{-2\pi i u\rho}du + \int_0^\infty (1-\eta(u\rho))u^{s-1}F(u)e^{-2\pi i u\rho}du\right] \tag{3.3}$$

The first integral on the right hand side of the equation (3.3) can be estimated as for the proof of (a) and, if we call $\sigma = \mathrm{Re}\{s\}$, we get:

$$\begin{aligned}\left|\prod_{j=0}^{N}(s+j)\right|\left|\int_0^\infty \eta(u\rho)u^{s-1}F(u)e^{-2\pi i u\rho}du\right| &= \\ &= \left|\int_0^\infty u^{s+N}\left(\frac{d}{du}\right)^{N+1}\left(\eta(u\rho)u^{s-1}F(u)e^{-2\pi i u\rho}du\right)\right| \\ &\leq C_s\|F\|_{C^{N+1}}(1+|\rho|)^{N+1}\left|\int_0^{\frac{1}{|\rho|}} u^{\sigma+N}du\right| \\ &\leq C_s\|F\|_{C^{N+1}}(1+|\rho|)^{N+1}|\rho|^{-\sigma-N-1} \leq C_s\|F\|_{C^{N+1}}(1+|\rho|)^{-\sigma}.\end{aligned}$$

Where we have used $\sigma + N > -1$ to integrate on $\operatorname{supp}\eta$ and the constant obtained only depends on $\mathrm{Im}(s)$ and $\|\eta\|_{C^{N+1}}$, the latter of which we can preliminarily bound uniformly.

We now turn to estimate the second integral in (3.3). Let $k > \sigma = \mathrm{Re}\{s\}$ and A be such that $\operatorname{supp} F \subset \{|u| \leq A\}$, we have:

$$\begin{aligned}
\left|\prod_{j=0}^{N}(s+j)\right|&\left|\int_0^\infty (1-\eta(u\rho))u^{s-1}F(u)e^{-2\pi iu\rho}du\right| = \\
&= C_s\left|\int_0^\infty \frac{1}{(-2\pi i\rho)^k}(1-\eta(u\rho))u^{s-1}F(u)\left(\frac{d}{du}\right)e^{-2\pi iu\rho}du\right| \\
&= C_s\rho^{-k}\left|\int_0^\infty e^{-2\pi iu\rho}\left(\frac{d}{du}\right)\left[(1-\eta(u\rho))u^{s-1}F(u)\right]du\right| \\
&\leq C_s\|F\|_{C^{N+1}}\rho^{-k}\int_{\frac{1}{2|\rho|}}^{A} u^{\sigma-k-1}du \\
&= C_s\|F\|_{C^{N+1}}\rho^{-k}\left[u^{\sigma-k}\right]_{\frac{1}{2|\rho|}}^{A} \leq C_s\|F\|_{C^{N+1}}\rho^{-\sigma},
\end{aligned}$$

where the constant is polynomial in s.

And as such we have proven (b).

To prove (c) we integrate once more by parts and get:

$$I_s(\rho) = -\prod_{j=1}^{N}(s+j)\int_0^\infty u^s\frac{d}{du}\left(F(u)e^{-2\pi iu\rho}\right)du.$$

From which:

$$I_0(\rho) = N!\left[-F(u)e^{-2\pi iu\rho}\right]_{u=0}^{\infty} = N!F(0).$$

and the last part of our thesis is proven. □

We can now prove a proposition which gives us some required properties for the Fourier transforms of some specific convolution kernels on hypersurfaces, which we will use to prove Theorem 3.3. Let us first set, as in Section 2, $M \subset \mathbb{R}^d$ as a smooth hypersurface and let $\mu = \psi d\sigma$ be a surface-carried measure with $\operatorname{supp}\mu \subset \tilde{B}$ for $\tilde{B}$ ball such that, as in (2.1) we have $M \cap \tilde{B} = \left\{(x', x_d) \in \tilde{B} : x_d = \varphi(x')\right\}$ for some $\varphi \in C^\infty(\mathbb{R}^{d-1})$.

Proposition 3.7. *Let M, $d\mu = \psi d\sigma$ and φ be as above and fix $N \geq \frac{d-1}{2}$. For all $s \in \mathbb{C}$, with $\operatorname{Re}\{s\} > 0$ let $K_s(x) = \gamma_s|x_d - \varphi(x')|_+^{s-1}\psi_0(x)$ where $\gamma_s = e^{s^2}\prod_{j=0}^{N}(s+j)$, $|f|_+ = \sup\{0, f\}$ is the positive part of f and $\psi_0(x) = \psi(x)(1 + |\nabla_{x'}\varphi(x')|^2)^{\frac{1}{2}}$, then the following hold:*

(a) $K_s \in L^1(\mathbb{R}^d)$ for $\operatorname{Re}\{s\} > 0$;

(b) $\hat{K}_s$ is analytically continuable into the half plane $-\frac{d-1}{2} \leq \operatorname{Re}\{s\}$;

(c) $\sup_{\xi\in\mathbb{R}^d}\left|\hat{K}_s(\xi)\right| \leq M$ for $-\frac{d-1}{2} \leq \operatorname{Re}\{s\} \leq 1$.

Proof. (a) descends directly from the compact support and boundedness of the functions in play, as long as the only singularity $|x_d - \varphi(x')|^{s-1}$ scales as $|x|^\alpha$ with $\alpha > -1$ as $|x| \to 0$, which holds for $\operatorname{Re}\{s\} > 0$.

Now by recalling Lemma 3.6 we can change variables with $u = x_d - \varphi(x')$ and set $F_\xi(u) = \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x'\xi' + \varphi(x')\xi_d)}\psi_0(x', u + \varphi(x'))dx'$. By Theorem 2.3 we get $|F_\xi(u)| \leq c(1+|\xi|)^{-\frac{d-1}{2}}$. Furthermore, we can observe that the same estimate also holds for the derivatives of F_ξ in u by smoothness of ψ. So we can write $\|F_\xi\|_{C^{N+1}} \leq c(1+|\xi|)^{-\frac{d-1}{2}}$. Thus we get:

$$\begin{aligned}\hat{K}_s(\xi) &= \gamma_s \int_{\mathbb{R}^d} |x_d - \varphi(x')|_+^{s-1} \psi_0(x) e^{-2\pi i(x'\xi' + \varphi(x')\xi_d)} dx \\ &= \gamma_s \int_0^\infty u^{s-1} e^{-2\pi i u\xi_d} F_\xi(u) du = e^{s^2} I_s(\xi_d)\end{aligned}$$

and finally we have by Lemma 3.6:

$$\left|\hat{K}_s(\xi)\right| = \left|e^{s^2}\right| |I_s(\xi_d)| \leq c_s \left|e^{s^2}\right| (1+|\xi_d|)^{-\operatorname{Re}\{s\}} \|F_\xi\|_{C^{N+1}} \leq c_s \left|e^{s^2}\right| (1+|\xi_d|)^{-\operatorname{Re}\{s\}} (1+|\xi|)^{-\frac{d-1}{2}}.$$

If $-\frac{d-1}{2} \leq \operatorname{Re}\{s\}$ we have that $(1+|\xi_d|)^{-\operatorname{Re}\{s\}}(1+|\xi|)^{-\frac{d-1}{2}}$ is bounded uniformly with respect to ξ, and as such the integrals above are well defined and as in Lemma 3.6 we can extend analytically $\hat{K}_s$, which gives us (b). On the other hand, by also requiring $\operatorname{Re}\{s\} \leq 1$ we get

$$\left|c_s e^{s^2}\right| \leq k \left|c_s e^{\operatorname{Re}\{s\}^2 - \operatorname{Im}(s)^2}\right| \leq c \left|c_s e^{-\operatorname{Im}(s)^2}\right|$$

which is bounded since c_s is polynomial in $\operatorname{Im}(s)$, which proves (c). □

We are now ready to proceed with the proof of Theorem 3.3.

Proof of Theorem 3.3. We will begin by defining the restriction operator $\mathcal{R} : C_c^0(\mathbb{R}^d) \to C^0(M)$ as it follows:

$$\mathcal{R}(f)(\xi) = \hat{f}(\xi)|_M = \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x) dx \bigg|_M.$$

Now we can define the dual operator $\mathcal{R}^*$ through the duality identity

$$\int_{\mathbb{R}^d} f(x)\overline{\mathcal{R}^*(F)(x)}dx =: (f, \mathcal{R}^*(F))_{\mathbb{R}^d} = (\mathcal{R}(f), F)_M := \int_M \mathcal{R}(f)(\xi)\overline{F(\xi)}d\mu(\xi);$$

and as such we get:

$$\mathcal{R}^*(F)(x) = \int_M e^{2\pi i x\xi} F(\xi) d\mu(\xi).$$

Hence:

$$\mathcal{R}^*\mathcal{R}(f)(x) = \int_M e^{2\pi i x\xi} \int_{\mathbb{R}^d} e^{-2\pi i y\xi} f(y) dy d\mu(\xi) = (f * \mathcal{F}^{-1}(d\mu))(x). \tag{3.4}$$

We shall now see that it suffices to work with the composition of the two operators to prove our thesis.

Proposition 3.8. *Let $p \geq 1$ be given, then the following statements are equivalent:*

$$\begin{aligned}&(i) \quad \|\mathcal{R}(f)\|_{L^2(M,d\mu)} \leq c\|f\|_{L^p(\mathbb{R}^d)} \\ &(ii) \quad \|\mathcal{R}^*(F)\|_{L^{p'}(\mathbb{R}^d)} \leq c\|F\|_{L^2(M,d\mu)} \\ &(iii) \; \|\mathcal{R}^*\mathcal{R}(f)\|_{L^{p'}(\mathbb{R}^d)} \leq c\|f\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

where $\|\mathcal{R}(f)\|^2_{L^2(M,d\mu)} = \int_M |\mathcal{R}(f)(\xi)|^2 d\mu(\xi)$ and p' is the conjugate exponent to p.

The proof of the proposition is fairly straightforward. We can see that (i) and (ii) are equivalent since by the general duality theorem:

$$\sup_{F\in L^2(M,d\mu)} \frac{\|\mathcal{R}^*(F)\|_{L^{p'}(\mathbb{R}^d)}}{\|F\|_{L^2(M,d\mu)}} = \sup_{f\in L^p(\mathbb{R}^d)} \frac{\|\mathcal{R}(f)\|_{L^2(M,d\mu)}}{\|f\|_{L^p(\mathbb{R}^d)}} = c.$$

On the other hand if we assume (i) (and thus (ii)) to be true we have:

$$\|\mathcal{R}^*\mathcal{R}(f)\|_{L^{p'}(\mathbb{R}^d)} \leq c\|\mathcal{R}(f)\|_{L^2(M,d\mu)} \leq c^2\|f\|_{L^p(\mathbb{R}^d)}$$

which gives us (iii). Finally assuming (iii) to be true we can use the duality statement and Hölder's inequality to get:

$$\|\mathcal{R}(f)\|^2_{L^2(M,d\mu)} = (\mathcal{R}(f),\mathcal{R}(f))_M = (\mathcal{R}^*\mathcal{R}(f),f)_M \leq \|\mathcal{R}^*\mathcal{R}(f)\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} \leq c^2\|f\|^2_{L^p(\mathbb{R}^d)}$$

which proves (i).

So to prove Theorem 3.3 it suffices to prove statement (iii) in Proposition 3.8. Now let us set a class of functionals $S_s(f) = f * k_s$ where $k_s(x) = \hat{K}_s(-x) = \mathcal{F}^{-1}(K_s)(x)$, and let us define as in Proposition 3.7 $K_s = \gamma_s|x_d - \varphi(x')|_+^{s-1}\psi_0(x)$. Now, we know that in the strip $-\frac{d-1}{2} \leq \mathrm{Re}\{s\} \leq 1$ by Proposition 3.7 there exists an analytical extension of $\hat{K}_s$ to the whole strip which is bounded. Also we have from the proof of Proposition 3.7 combined with Proposition 3.5, that $\hat{K}_0(\xi) = I_0(\xi_d) = N!F_\xi(0) = N!\widehat{d\mu}(\xi)$, hence by (3.4):

$$S_0(f)(x) = N!f * \mathcal{F}^{-1}(d\mu)(x) = N!\mathcal{R}^*\mathcal{R}(f) \tag{3.5}$$

Once again by Proposition 3.7, for $\mathrm{Re}\{s\} = 1$, since we have that

$$\sup_{t\in\mathbb{R}} \|K_{1+it}\|_{L^\infty} \leq \sup_{t\in\mathbb{R}} c\|\psi_0\|_{L^\infty} e^{-t^2} \prod_{j=1}^{N+1}(j+it) \leq M$$

we then get:

$$\|S_{1+it}(f)\|_{L^2} = \|f * k_{1+it}\|_{L^2} = \left\|\widehat{f * k_{1+it}}\right\|_{L^2} = \left\|\hat{f}\hat{k}_{1+it}\right\|_{L^2} = \left\|\hat{f}K_{1+it}\right\|_{L^2},$$

which in turn implies:

$$\sup_{t\in\mathbb{R}} \|S_{1+it}(f)\|_{L^2} \leq \sup_{t\in\mathbb{R}} \left\|\hat{f}K_{1+it}\right\|_{L^2} \leq M\|f\|_{L^2}. \tag{3.6}$$

On the other hand for $\mathrm{Re}\{s\} = -\frac{d-1}{2}$ we get, by Hölder's inequality:

$$\sup_{t\in\mathbb{R}} \left\|S_{-\frac{d-1}{2}+it}(f)\right\|_{L^\infty} = \sup_{t\in\mathbb{R}} \left\|f * \mathcal{F}^{-1}(K_{1+it})\right\|_{L^\infty} \leq M\|f\|_{L^1}. \tag{3.7}$$

We also have that by setting $\Phi_0(s) = \int_{\mathbb{R}^d} S_s(f)g\,dx$ for some f, g simple functions we can prove Φ_0 to be continuous and bounded for $-\frac{d-1}{2} \leq \mathrm{Re}\{s\} \leq 1$ and analytic in the interior, which descends directly from the fact that by Proposition 3.7 we have $\hat{K}_s$ analytic in the chosen strip and thus $k_s(\,\cdot\,) = \hat{K}_s(-\,\cdot\,)$, $S_s(f) = f * k_s$ are analytic as well as Φ_0. Boundedness comes from following the same steps as in (3.7), obtaining $\sup_{-\frac{d-1}{2}\leq\mathrm{Re}\{s\}\leq 1} |\Phi_0(s)| \leq M\|f\|_{L^1}\|g\|_{L^1}$.

As such we see that by (3.6) and (3.7) the hypothesis of Proposition 3.5 are satisfied for $a = -\frac{d-1}{2}$, $b = 1$, $p_0 = 1$, $q_0 = \infty$, $p_1 = 2$, $q_1 = 2$. We set $c = 0$, which implies $\theta = -\frac{a}{b-a} = \frac{d-1}{d+1}$ and $p = (1-\theta/2)^{-1} = \frac{2d+2}{d+3}$, which coincides with the result we were expecting, and finally $q = p'$. So by Proposition 3.5 we get:

$$\|S_0(f)\|_{L^{p'}(\mathbb{R}^d)} \leq c\|f\|_{L^p(\mathbb{R}^d)}$$

which then by (3.5) gives us:

$$\|\mathcal{R}^*\mathcal{R}(f)\|_{L^{p'}(\mathbb{R}^d)} \leq c\|S_0(f)\|_{L^{p'}(\mathbb{R}^d)} \leq c\|f\|_{L^p(\mathbb{R}^d)}$$

which then by Proposition 3.8 proves our thesis. □

Now we can obtain the more general result by interpolation.

Corollary 3.9. *Under the assumptions of Theorem 3.3, the restriction inequality (3.2) holds for all* $1 \leq p \leq \frac{2d+2}{d+3}$ *and* $q \leq \frac{d-1}{d+1}p'$.

Proof. First of all we know that the trivial case $p_0 = 1$, $q_0 = \infty$ holds. Then by the critical case $p_1 = \frac{2d+2}{d+3}$, $\tilde{q}_1 = 2$ which was proven in Theorem 3.3, considering that $\operatorname{supp}\psi = \operatorname{supp} d\mu$ is compact, proving that $\hat{f} \in L^2(M, d\mu)$ implies that $\hat{f} \in L^{q_1}(M, d\mu)$ for all $q_1 \leq 2$.

At this point given some generic p, q, we can show that a restriction exists via Riesz-Thorin interpolation theorem. We get

$$p = \left(\theta + (1-\theta)\frac{d+3}{2d+2}\right)^{-1} = \frac{2d+2}{d+3+(d-1)\theta} \leq p_1$$

and specifically $\theta = \frac{2d+2-(d+3)p}{(d-1)p}$ which implies $\theta \in [0,1]$ iff $p \in [1, p_1]$ as we wanted to prove. For q we see that $q = \frac{q_1}{1-\theta}$, hence

$$[1,2] \ni q_1 = (1-\theta)q = 2\frac{(d+1)q}{(d-1)p'}$$

which implies that $\frac{(d-1)p'}{2(d+1)} \leq q \leq \frac{(d-1)p'}{d+1}$, where the lower bound can be lifted as before, giving us the required range. □

We now would be interested in seeing what happens when the Gauss curvature of M vanishes. As before, we get a weaker result.

Corollary 3.10. *Assume that there exists some* $\delta > 0$ *such that* $\left|\widehat{d\mu}(\xi)\right| = O\left(|\xi|^{-\delta}\right)$ *as* $|\xi| \to \infty$. *Then the restriction property holds for* $p = \frac{2\delta+2}{\delta+2}$, $q = 2$.

In particular if M has m non vanishing principal curvatures, by Corollary 2.4 this holds for $p = \frac{2m+4}{m+4}$.

Proof. The proof descends directly by the proof of Proposition 3.7. In fact, we would have $\|F_\xi(u)\|_{C^{N+1}} \leq c(1+|\xi|)^{-\delta}$ and as such the half plane over which the function is continuable would be $-\delta \leq \text{Re}\{s\}$. At which point following the steps of the proof of Theorem 3.3 we would have that (3.7) would hold in such a half plane, and we would need to apply Proposition 3.5 for $a = -\delta$ instead and get $\theta = \frac{\delta}{\delta+1}$ and $p = \frac{2\delta+2}{\delta+2}$ as requested. □

We may notice that this result can also be extended as we have done in Corollary 3.9, by following exactly the same steps.

We will now show a construction by Knapp (which can be found for example in [14]) which proves that the range of exponents from Corollary 3.9 is optimal.

Let S^{d-1} be the unit sphere and $d\sigma$ the measure induced on it by the Lebesgue measure on $\mathbb{R}^d$ as per usual. We start by taking a spherical cap of height δ^2 on the unit sphere: $C_\delta = \{x \in S^{d-1} : 1 - (x \cdot e_d) \leq \delta^2\}$.

Now we notice that $|x - e_d|^2 = 2(1 - x \cdot e_d)$ which gives us $|x - e_d| \leq c\delta$ for some $c \geq 0$ for all $x \in C_\delta$. Now for $f = \chi_{C_\delta}$ indicator function of C_δ we can see that $\|f\|_{L^2}(S^{d-1}) = \sqrt{\sigma(C_\delta)} \approx \delta^{\frac{d-1}{2}}$ and we would like to bound $\left\|\widehat{fd\sigma}\right\|_{L^{p'}(\mathbb{R}^d)}$ for p' conjugate exponent of p.

We already know that supp $fd\sigma \subset \mathbb{R}$ for some rectangle R with sizes of order $c\delta^2$ along e_d and of order $c\delta$ along the other orthogonal directions. Because of that, its Fourier transform will mainly be concentrated in the dual rectangle $R^* \subset \mathbb{R}^d$, which is centered at 0 with sizes of order $c^{-1}\delta^{-2}$ for ξ_d and $c^{-1}\delta^{-1}$ for ξ_j, $j = 1, \cdots, d-1$. Hence, we will study the behavior of the Fourier transform of $fd\sigma$ on R^*, which has a volume of order δ^{-d-1} from its above definition. So let us take $|\xi_d| \leq c^{-1}\delta^{-2}$ and $|\xi_j| \leq c^{-1}\delta^{-1}$ for $j \neq d$ and for some c large enough, then we have, by changing variables from x to $x - e_d$:

$$\left|\widehat{fd\sigma}(\xi)\right| = \left|\int_{C_\delta} e^{-2\pi i x\cdot\xi} d\sigma(x)\right| = \left|\int_{C_\delta} e^{-2\pi i(x-e_d)\cdot\xi} d\sigma(x)\right| \geq \int_{C_\delta} \cos(2\pi i(x - e_d) \cdot \xi) d\sigma(x).$$

By our conditions on ξ we have that there exists c large enough such that $|(x - e_d) \cdot \xi| \leq \frac{1}{3}$ for all $x \in C_\delta$, and so we would get $\cos(2\pi i(x - e_d) \cdot \xi) \geq \frac{1}{2}$, and:

$$\left|\widehat{fd\sigma}(\xi)\right| \geq \frac{1}{2}|C_\delta| \approx \delta^{d-1}$$

and hence, since the volume of R^* has order δ^{-d-1} we have:

$$\left\|\widehat{fd\sigma}\right\|_{L^{p'}(\mathbb{R}^d)} \geq C\left(\int_{R_0} \delta^{(d-1)p'}\right)^{\frac{1}{p'}} \geq (\delta^{-d-1}\delta^{d-1}p')^{\frac{1}{p'}} = \delta^{d-1-\frac{d+1}{p'}}. \tag{3.8}$$

On the other hand by Theorem 3.3, specifically stated in dual form as in (ii) of Proposition 3.8, we have:

$$\left\|\widehat{fd\sigma}\right\|_{L^{p'}(\mathbb{R}^d)} \leq C\|f\|_{L^2(S^{d-1})} \lesssim \delta^{\frac{d-1}{2}}. \tag{3.9}$$

So by combining (3.8) and (3.9) we get:

$$\delta^{d-1-\frac{d+1}{p'}} \lesssim \delta^{\frac{d-1}{2}}$$

which holds uniformly for all $0 < \delta \leq 1$, and as such necessarily we need $p' \geq \frac{2n+2}{n-1}$ which gives us p as in Corollary 3.9.

4 Fourier restrictions for $d = 1$ and Salem constructions

In this section we will study the results of G. Mockenhaupt from [8]. We have already seen how for $d > 1$ we can effectively define a restriction of Fourier's transform on surfaces with non-zero Gauss curvature for functions in $L^p(\mathbb{R}^d)$, with $p \leq \frac{2d+2}{d+3}$. We would now like to see if a similar result can be obtained in dimension $d = 1$. Of course the notion of curvature now has no geometrical meaning and furthermore for $d = 1$ the estimate obtained before would only admit functions in $L^1(\mathbb{R})$, for which the Fourier transforms would be continuous and the existence of a restriction would be trivial.

However, it is possible to attain other results in $\mathbb{R}$ by working with sets of non-integer dimension. Of course, it is first necessary to search for an equivalent, monodimensional, idea of curvature to be able to study the existence of a restriction that can hold even in dimension $d = 1$; for that, Mockenhaupt remarks that the curvature condition has also been expressed in previous studies as an estimate on the size of arithmetic progressions contained in the set. As such we would need a fractal set of $\mathbb{R}$ without a regular repeating structure to get a proper bound on the size of arithmetic progressions, which is why we will (in the latter part of this section) look at Salem sets, fractal sets generated explicitly in a non-regular, random way. However, before that, we will study a scaling property that distinguishes these sets, derived from their *Fourier dimension.* We will then study how a set with strictly positive Fourier dimension is required on $\mathbb{R}$ to have some kind of restriction theory and lastly we will see how Salem's construction attains exactly that.

Throughout this chapter we will assume the reader to know the definitions of Hausdorff measure, Hausdorff dimension and probability measure. We will begin by giving a few more definitions:

Definition 4.1. Let us take a probability measure μ supported on a compact set $E \subset \mathbb{R}^d$, let $0 < \beta < d$ and $x \in E$. We define the *β-potential* of μ at x as:

$$I_\beta(\mu)(x) := \int \frac{d\mu(y)}{|x-y|^\beta}.$$

We define the *β-energy* of μ as:

$$\mathcal{E}_\beta(\mu) := \iint \frac{d\mu(x)d\mu(y)}{|x-y|^\beta}.$$

The β-energy of a compactly supported measure is a way of comparing the scaling of such a measure against the integration kernel $|x-y|^\beta$. Specifically if μ scales with power at most α, with $\alpha < \beta$, then its β-energy integral is finite. Now we will state a theorem from Frostman which imposes a strict correlation between the finiteness of β-energy and the Hausdorff dimension of a set (see [4]).

Theorem 4.2. *Let $E \subset \mathbb{R}^d$ be a compact set.*

(i) Let $\alpha = \dim_H E$ be the Hausdorff dimension of the set E, then there exists a a probability measure μ such that $\operatorname{supp}\mu \subseteq E$ *and for any given $x \in \mathbb{R}^d$, for all $r > 0$ we have:*

$$\mu(B_r(x)) \leq Cr^\alpha$$

and consequently $\mathcal{E}_\alpha(\mu) < \infty$*.*

(ii) Let $\mathcal{E}_\beta(\nu) < \infty$ *for some* ν *probability measure supported on* E*, then* $\dim_H E \geq \beta$*.*

In particular we will now see how this leads to the definition of the Fourier dimension of our set.

We first remark that $\mathcal{F}(|x|^{-\alpha})(\xi) = C|\xi|^{d-\alpha}$, which can be noticed by remembering that as seen in (3.1) the Fourier transform of a radial function is radial and that $\mathcal{F}(f(\,\cdot\,))(\xi\lambda^{-1}) = \int e^{-iy\frac{\xi}{\lambda}} f(y)dy = |\lambda|^d \mathcal{F}(f(\lambda\,\cdot\,))(\xi)$.

As such we can see that formally by Parseval's identity:

$$\mathcal{E}_\alpha(\mu) = c\int_{\mathbb{R}^n} \frac{|\widehat{d\mu}(y)|^2}{|y|^{d-\alpha}} \tag{4.1}$$

where however a proper proof of this statement requires approximation by convolution of the integration kernel and by taking a more regular restriction of the measure μ (see for example [7] Section 2, Lemma 3). Again, the finiteness of the energy integral gives us information on the size of $\widehat{d\mu}$, albeit this time the information provided is of a weaker form. Specifically we notice that if $\left|\widehat{d\mu}(x)\right| \leq C|x|^{-\frac{\beta}{2}}$ then $\mathcal{E}_\alpha(\mu) < \infty$ for any $\alpha < \beta$, since $\widehat{d\mu}(y)$ is continuous and as such $\frac{|\widehat{d\mu}(y)|^2}{|y|^{d-\alpha}}$ is locally integrable, hence:

$$\mathcal{E}_\alpha(\mu) = c\int_{\mathbb{R}^n \setminus B_1(0)} \frac{|\widehat{d\mu}(y)|^2}{|y|^{d-\alpha}} dy \leq \tilde{C}\int_{\mathbb{R}^n \setminus B_1(0)} |y|^{-\beta+\alpha-d} dy$$

with $-\beta+\alpha-d < -d$, which makes the rightmost term finite. Then, by Theorem 4.2, we have $\dim_H E \geq \beta$ for some E compact containing the support of μ, however this time the converse is not true. As such this scaling property of our set is indeed a weaker notion of dimension which is called *Fourier dimension.*

Definition 4.3. We define the *Fourier dimension* of a compact set $E \subset \mathbb{R}^d$ as:

$$\dim_F E := \sup\left\{\beta \geq 0 : \left|\widehat{d\mu}(x)\right| \leq C|x|^{-\frac{\beta}{2}} \text{ for some } \mu \text{ probability measure on } E\right\}.$$

From the previous remarks we get that $\dim_F E \leq \dim_H E$. If $\alpha = d-1$ and E is a compact smooth α-dimensional manifold in $\mathbb{R}^d$ with μ surface-carried measure, then the decay of order $|x|^{-\beta/2}$ is equivalent to certain assumptions on the curvature of the set. For example, we have seen that the Fourier transform of any surface-carried measure on a hypersurface with non-vanishing Gauss curvature has decay of order $|x|^{-(d-1)/2}$ (see Theorem 2.3). So let us take for example the sphere with surface measure induced by Lebesgue measure in $\mathbb{R}^d$ (see Section 2), which has nonzero curvature on each point of its surface, and we get $d-1 \leq \dim_F S^{d-1} \leq d-1 = \dim_H S^{d-1}$, hence $\dim_F S^{d-1} = d-1 = \dim_H S^{d-1}$.

However for other sets these two definitions of dimensions do not coincide. For example by taking the surface Q of a unit cube in $\mathbb{R}^d$ for $d > 1$, we can see that a uniform measure on it does not give a scaling of order $|x|^{-(d-1)/2}$ (it suffices to check that the decay of the Fourier transform along the normal to any of the faces of the cube has power 0). It is actually possible to prove that $\dim_F Q = 0 \lneq d-1 = \dim_H Q$. Another example, this time for $d = 1$, is the Cantor middle third set where $\dim_F C_{1/3} = 0 < \frac{\log 2}{\log 3} = \dim_H C_{1/3}$. Let's notice that Q only has principal curvatures equal to 0, while $C_{1/3}$ is a regular fractal set, and as such admits arithmetic progressions of increasing sizes. We will later see how we can construct a set that does not satisfy this last property in dimension $d = 1$ to get a set of nonzero Fourier dimension.

Now we can proceed to prove a restriction estimate similar to that of Theorem 3.3, which will however also work for sets of fractal dimensions.

Theorem 4.4. *Let μ be a compactly supported positive measure on $\mathbb{R}^d$ such that:*

(i) there exists some $\beta > 0$ such that for all $x \in \mathbb{R}^d$

$$\left|\widehat{d\mu}(x)\right| \leq C|x|^{-\frac{\beta}{2}}; \tag{4.2}$$

(ii) there exists some $\alpha > 0$ such that for all $x \in \mathbb{R}^d$, $r > 0$:

$$\mu(B_r(x)) \leq Cr^{\alpha}; \tag{4.3}$$

then for $1 \leq p < \frac{2(2d-2\alpha+\beta)}{4(d-\alpha)+\beta}$ we have

$$\left(\int |\hat{f}|^2 d\mu\right)^{1/2} \leq C\|f\|_{L^p(\mathbb{R}^d)}. \tag{4.4}$$

Before proving this theorem we will make a couple of remarks. First of all, we can notice how (4.4) is analogous to statement (*ii*) of (3.8) in that it is a dual form of the statement of the Fourier transform restriction. The required hypothesis (4.2) and (4.3) respectively imply that $\dim_F(\operatorname{supp}\mu) \geq \beta$ and by Theorem 4.2 that $\dim_H(\operatorname{supp}\mu) \geq \alpha$. Furthermore we can see that, in the case of the sphere (assuming μ uniform as before), we can take $\alpha = \beta = d-1$ and get $1 \leq p < \frac{2d+2}{d+3}$, which, is consistent with Theorem 3.3 and with the example from Knapp in the previous section. On the other hand, if we consider Corollary 3.10, we see how a hypersurface with only m nonzero principal curvatures would have the same scaling as a set with Fourier dimension $\beta = m$, which shows an even deeper connection between the concept of curvature and Fourier dimension. Notice however, that unlike the estimate in Theorem 3.3 we lack the endpoint in the range of admissible values of p. The endpoint case has been later proven by Bak and Seeger in [1], and very recently the range has been proven optimal in dimension $d = 1$ by Fraser, Hambrook and Ryou in [5].

The idea of the proof derives from Stein and Tomas approach on the study of the manifold problem, as seen in [13] and [12] (see also [14]).

Proof. We are only truly interested in the case in which $f \in L^p(\mathbb{R}^d)$, otherwise the statement is trivial. Let $T_f := \widehat{d\mu} * f$ for all $f \in L^p(\mathbb{R}^d)$. We can see that, by combining Parseval's identity with Hölder's inequality:

$$\|\hat{f}\|^2_{L^2(d\mu)} = \int |\hat{f}|^2 d\mu = \int \hat{\hat{f}} \cdot \mathcal{F}(\hat{f} d\mu) = \int \hat{\hat{f}} \cdot (f * \widehat{d\mu}) \leq C\|Tf\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence it would suffice to prove that the functional $T : f \mapsto Tf$ is bounded from L^p to $L^{p'}$; in fact, if that were the case, we would have:

$$\|\hat{f}\|^2_{L^2(d\mu)} \leq C\|T\|_{L^p \to L^{p'}}\|f\|_{L^p} \cdot \|f\|_{L^p}$$

from which we would get the thesis by simply taking the square root. So let's just prove $f \mapsto Tf$ is bounded as above for all $1 \leq p < \frac{2(2d-2\alpha+\beta)}{4(d-\alpha)+\beta}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}\varphi \subset \{\frac{1}{2} \leq |x| \leq 2\}$, with $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^d$ and $\varphi(x) = 1 - \varphi(x/2)$ for all x such that $1 \leq |x| \leq 2$ (to be sure it respects smoothness we can first take a bump function between the inner disk of radius $1/2$ and the outer infinite annulus of inner radius 1 and then change it in the outer annulus to have $\varphi(x) = 1 - \varphi(x/2)$). We know that $\hat{\varphi}$ is a Schwartz function, and as such it holds that for any $N > 0$ we have $\hat{\varphi}(x) \leq \frac{1}{(1+|x|)^N}$. Specifically let us take $N > d$.

For all integers $k \geq 1$ we will take $\varphi_k = \varphi(\cdot/2^k) \in C_0^\infty$. We can see that such a function has support in the spherical annulus $A_k = \{2^{k-1} \leq |x| \leq 2^{k+1}\}$ and $\sum_{k\geq 1} \varphi_k(x) = 1$ when $|x| \geq 2$. Then we take $\varphi_0 \in C_0^\infty(A_0)$ with $A_0 = \{|x| \leq 2\}$ such that $\sum_{k\geq 0} \varphi_k = 1$ on the whole $\mathbb{R}^d$. It holds that

$$\hat{\varphi}_k(x) = 2^{kd}\hat{\varphi}(2^k x) \leq 2^{kd}\frac{1}{(1+2^k|x|)^N} \tag{4.5}$$

Let $T_k f = (\varphi_k \widehat{d\mu}) * f$ for all $k \geq 0$. Then $Tf = \sum_{k\geq 0} T_k f$.

Firstly we see that we from (4.2) can get:

$$\|T_k\|_{L^1\to L^\infty} \leq \|\varphi_k \widehat{d\mu}\|_{L^\infty(\mathbb{R}^d)} \leq \|C|x|^{-\frac{\beta}{2}}\|_{L^\infty(A_k)} \leq \tilde{C} \cdot 2^{-k\frac{\beta}{2}}. \tag{4.6}$$

Secondly, by Plancharel's Theorem we get that:

$$\|T_k f\|_{L^2} \leq C\|(\hat{\varphi}_k * d\mu(x)) \cdot \hat{f}\|_{L^2} \leq C \sup_{x\in\mathbb{R}^d} |\hat{\varphi}_k * d\mu(x)| \cdot \|f\|_{L^2},$$

from which, by using (4.5):

$$\|T_k\|_{L^2\to L^2} \leq C \sup_{x\in\mathbb{R}^d} |\hat{\varphi}_k * d\mu(x)| \leq C2^{kd}\hat{\varphi}(2^k x) \leq C2^{kd} \sup_{x\in\mathbb{R}^d} \left| \int \frac{d\mu(y)}{(1+2^k|x-y|)^N} \right|.$$

Since the argument of the integral is radial, by a change of variable to polar coordinates with $r = |x-y|$ we get, by using 4.3:

$$\|T_k\|_{L^2\to L^2} \leq C2^{kd} \sup_{x\in\mathbb{R}^d} \int_0^\infty \mu(B_{r/2^k}(x))(1+r)^{-N-1}dr \leq \tilde{C}2^{k(d-\alpha)}. \tag{4.7}$$

We can now use Riesz's Interpolation Theorem to combine (4.6) and (4.7) and get:

$$\|T_k\|_{L^p\to L^{p'}} \leq C2^{k(d-\alpha)(1-\theta)-k\frac{\beta}{2}\theta}$$

with θ satisfying $\frac{1}{p'} = \frac{1-\theta}{2}$, hence $\theta = 1 - \frac{2}{p'}$ and $k(d-\alpha)(1-\theta) - k\frac{\beta}{2}\theta = k\left(\frac{2d-2\alpha}{p'} - \frac{\beta}{2} + \frac{\beta}{p'}\right)$. So finally we get:

$$\|T_k\|_{L^p\to L^{p'}} \leq C2^{k\left(\frac{2d-2\alpha+\beta}{p'} - \frac{\beta}{2}\right)},$$

and by summing over k we get:

$$\|T\|_{L^p\to L^{p'}} \leq \sum_{k\geq 0} \|T_k\|_{L^p\to L^{p'}} \leq C\sum_{k\geq 0} 2^{k\left(\frac{2d-2\alpha+\beta}{p'} - \frac{\beta}{2}\right)}$$

where the series on the right converges if $\frac{2d-2\alpha+\beta}{p'} - \frac{\beta}{2} < 0$ hence $p' > 2\frac{2d-2\alpha+\beta}{\beta}$ and thus $p < \frac{2(2d-2\alpha+\beta)}{4(d-\alpha)+\beta}$. So we have proven that for these values of p the functional T is bounded and as such (4.4) is satisfied. □

We can see that for any fixed α we have $\frac{2(2d-2\alpha+\beta)}{4(d-\alpha)+\beta} = 1 + \frac{\beta}{4(d-\alpha)+\beta}$ which is increasing in β. So naturally the ideal construction that would come to mind for a set able to optimize the above result would be by taking some set E for which $\dim_F(E)$ and $\dim_H(E)$ are close to each other such that it would be possible to have some kind of measure μ that satisfies (4.2) and (4.3) with $\alpha \leq \beta + \varepsilon$ for some small $\varepsilon > 0$.

Constructing a set with nonzero Fourier dimension in $\mathbb{R}$ is not easy. Such a set is called a *Salem set* and their constructions have been studied by Salem [9], Kaufman [6] and Buhm [2], [3].

To construct a Salem set on $\mathbb{R}$ we will begin by taking some $\xi \in (0, 1/N)$ with $N = M^M$ for some even integer $M > 2$. We then fix a sequence $0 < a_1 < a_2 < \cdots < a_N < 1$ such that they are linearly independent over rational numbers and such that $0 < a_1 < \frac{1}{N} - \xi$ and $\xi < a_k - a_{k-1} < \frac{1}{N}$ for $2 \leq k \leq N$.

Definition 4.5. Over a segment $[A, B]$ we define a *dissection* of type $(N, a_1, \cdots, a_N, \xi)$ by calling the closed intervals $[A + La_k, A + L(a_k + \xi)]$ for $k = 1, \cdots, N$ *white* and the complementary intervals *black*.

Then we take a sequence $\Xi = (\xi_k)_{k\geq 1}$ such that $\left(1 - \frac{1}{2k^2}\right)\xi \leq \xi_k \leq \xi$ for all $k \geq 1$.

Now starting with $E_0 = [0, 1]$ we perform a dissection of type $(N, a_1, \cdots, a_N, \xi_1)$ and remove the black intervals to get E_1, obtaining N intervals of length ξ_1. Then we repeat the process by taking a dissection of type $(N, a_1, \cdots, a_N, \xi_2)$ on each of the remaining intervals of E_1, and remove the black intervals once more obtaining E_2 made of N^2 intervals of length $\xi_1\xi_2$. After n steps we get a set E_n of N^n intervals of length $\prod_{i=1}^{n} \xi_i$.

Let's set $E = \bigcap_{n\geq 0} E_n$. It is a perfect set since each of the E_n is closed, and E has no isolated points by iterated construction. Furthermore it has 1-dimensional Hausdorff measure equal to 0 since $\xi < 1/N$ implies $N^n\xi^n \to 0$. In fact it has Hausdorff dimension $\alpha = -\log_\xi N$ (this can be seen since $\dim_H E \leq \alpha$ by taking the E_n as covers, and to prove the other inequality it suffices to see that the α-dimensional measure of E is nonzero, which can be done analogously to the proof for computing the dimension of Cantor sets, see also [4]).

Now for all $n \in \mathbb{N}$ let F_n be a continuous non-decreasing cumulative distribution function that increases linearly on E_n and remains otherwise constant. Specifically we need:

- $F_n(x) = 0$ for all $x \leq 0$, $F_n(1) = 1$ for all $x \geq 1$;
- F_n increases linearly by $1/N^n$ on each (white) interval of E_n;
- F_n is constant over $E_0 \setminus E_n$.

Since for a given x we have that $F_n(x)$ will eventually be constant as n increases, we can take $F = \lim_{n\to\infty} F_n$. F is itself non-decreasing and continuous, and ideally it represents a cumulative distribution function of a uniform probability measure on E. Via a result from Zygmund in [15] we can also compute:

$$\widehat{dF}(x) = P(x) \prod_{n \geq 1} P(x\xi_1 \cdots \xi_n)$$

where $P(x) = \frac{1}{N} \sum_{1 \leq k \leq N} e^{i a_k x}$. The main result we are interested in however, is the following from Salem [9].

Proposition 4.6. *Given $\alpha \in (0,1)$, $\varepsilon > 0$, there is $M > 2$ for which we can pick $\xi = N^{-\frac{1}{\alpha}}$ and a sequence Ξ as above such that:*

$$\left|\widehat{dF}(x)\right| \leq C_\varepsilon |x|^{-\frac{\alpha}{2}+\varepsilon}. \tag{4.8}$$

Which gives us (4.2) for values as close to α as we would like. On the other hand, we now have to show that F does satisfy (4.3).

Proposition 4.7. *There exists $C > 0$, depending only on M such that for all $x, y \in \mathbb{R}$:*

$$|F(x) - F(y)| \leq C|x - y|^\alpha \tag{4.9}$$

Proof. Let $x, y \in [0,1]$, without loss of generality suppose $y > x$, F is constant on the black intervals complementary to E, so we can assume $x, y \in E$ (otherwise we can see that there are points in E, at the border of the same black intervals, which we will call x' and y' respectively such that $|x' - y'| < |x - y|$ and $|F(x') - F(y')| = |F(x) - F(y)|$ and as such proving the inequality for the new points is sufficient to imply it for x and y). Let k be the smallest integer such that after k dissections there are at least two black intervals in $[x, y]$ (it exists since the length of the white intervals converges to 0 as $k \to \infty$). We have that $[x, y]$ fully contains at least a white interval at step k and:

$$y - x \geq \xi_1 \cdots \xi_k \geq \xi^k \prod_{1 \leq m \leq k} \left(1 - \frac{1}{2m^2}\right) \geq \xi^k \prod_{m \geq 1} \left(1 - \frac{1}{2m^2}\right) \geq C\xi^k$$

since the product $\prod_{m \geq 1} \left(1 - \frac{1}{2m^2}\right)$ converges to a positive value, which is independent of any of our variables.

On the other hand, after $k - 1$ dissections there is at most one black interval (a, b) contained in $[x, y]$, for which $F(b) = F(a)$, so by using $N = \xi^{-\alpha}$ we get:

$$F(y) - F(x) = F(y) - F(b) + F(a) - F(x) \leq \frac{2}{N^{k-1}} \leq C\xi^{k\alpha} \leq C|y - x|^\alpha$$

Specifically we notice that the new constant introduced only depends on N, hence on M. $\square$

So now by combining all of the previous results we get the following corollary of Theorem 4.4.

Corollary 4.8. *Let $1 \leq p < \frac{2(2-\alpha)}{4-3\alpha}$ and choose M such that (4.8) is satisfied, then there exists C depending only on M for which:*

$$\int_E |\hat{f}|^2 dF \leq C\|f\|_{L^p(\mathbb{R})}$$

Proof. We would like to apply Theorem 4.4 with $d = 1$, $\beta = \alpha - \varepsilon$. We have proven (4.3) by using (4.9). For hypothesis (4.2), it can be obtained by (4.8) for all $\beta < \alpha$ (by choosing ε small enough), and specifically we have that, since $\frac{\beta}{4d-4\alpha+\beta}$ is increasing in β, then

$$1 + \frac{\beta}{4d - 4\alpha + \beta} < \frac{2(2-\alpha)}{4-3\alpha}$$

holds for all $\beta < \alpha$, and for $\beta = 0$ we get $1 + \frac{\beta}{4d-4\alpha+\beta} = 1$. Since the function is continuous in β for $\beta \in [0, \alpha]$ we have that fixed $1 \leq p < \frac{2(2-\alpha)}{4-3\alpha}$ there exists $\beta^* < \alpha$ such that $p = 1 + \frac{\beta^*}{4d-4\alpha+\beta}$. So by (4.8) we can get E satisfying (4.2) for β^*. □

As such we have proven the existence of sets in dimension 1 that admit a restriction for the Fourier transforms of f with $f \in L^p$ and p close enough to 1. Such constructions have later been extended by Łaba and Pramanik in [16] to fractal sets of dimension greater than 1, greatly improving the range of known sets over which Fourier transforms are well defined, and investigating even deeper the connection between the curvature in a hypersurface and the idea of Fourier dimension of a set.

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