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Cosmological Implications of Mimetic Gravity

Relatori:

Prof. Sabino Matarrese

Prof. Nicola Bartolo

Laureando:

Niccolò Bertazzo

Contents

Chapter 1

Introduction

The aim of this work is to present a relatively new concept about modifying gravity, dubbed *Mimetic Gravity*, arising in the more general framework of Scalar-Tensor-Vector theories of gravity. Since the first pubblication [1], several authors [2, 3, 4] started to expand the new idea looking at it from different perspectives and each time finding new features. In this thesis I offer a review of some articles, presenting the ideas behind mimetic gravity and then discussing some aspects. The main topic of this thesis fits well into the so called Horndeski theory of gravity, one of the most general type of scalar-tensor theory with second-order equations of motions. This framework is usually used for describing gravitation with some additional degrees of freedom and one of its most attracting features is that it can accomodate a wide range of *classic* ideas about General Relativity and its extensions. For examples Horndeski theory can describe

- GR with a minimally coupled scalar field, the most basic extension with a scalar degree of freedom. Non minimal couplings (as well as derivative couplings) also can be accommodated;
- Brans-Dicke theory in which the gravitational coupling to be a function of space-

time coordinates is allowed;

- $f(R)$ gravity, theory for which the Ricci scalar enters the action through a general function $f(R)$, (as we will see there exists essentially three possibles versions of $f(R)$ theories);
- covariant Galileon theory, in which the field equations of motions are invariant under galilean-type 4D transformations (in a flat spacetime);
- Gauss-Bonnet coupling theory in which the action contains scalars build up from higher rank tensors than the only Ricci tensor $R_{\mu\nu}$;
- Inflationary theory from modified GR;
- Dark Matter and Dark Energy modeling, the main topic of this work.

Horndenski theory essentially tells us how to build a general Lagrangian function of the field ϕ, its kinetic term $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ plus some geometric scalars coming from ordinary differential geometry. The standard way to proceed is to perturbe a choosen metric and then look at the cosmological evolution of the perturbations under the influence of gravity and other added scalar degrees of freedom. A common step is to choose a particular gauge parameterizing the metric and then perturbing it, restricting the attention to a particular hypersurface on spacetime, a constant t hypersurface Σ_t. In the unitary gauge $\delta\phi = 0$ the constant time slices can be identified as the constant ϕ surfaces. On each slice the only relevant perturbations are those of the metric, the scalar degree of freedom is "eaten" by the metric. At the end one wants to calculate the equations of motions of the perturbed quantity, and also in expanding the action up to second order.

On the surface Σ_t, an induced metric $h_{\mu\nu}$ is defined along with some scalars derived from its first and second derivative in the usual manner. In all generality one allows the Lagrangian to be a functional of several scalar quantities related to the geometry of the hypersurface. Once the second-order Lagrangian is calculated, shifting to the Hamiltonian point of view allows to study the so called *ghost* and *Laplacian* instabilities under which one has a good definition of the energy of the system.

The main topic of this thesis is about a specific conformal extension of General Relativity following from imposing a functional dependence of the metric on an auxiliary metric and a scalar field subject to a constraint. Calculations show that Mimetic Gravity can describe for example Dark Matter alongside with other different cosmological features as early and late time acceleration.

The work is organized as follows

- in chapter II a brief introduction of the concordance model of Cosmology is given as well as a brief recall of General Relativity,
- in chapter III we will discuss some well know examples of modified theories of gravity,
- the Mimetic Gravity model is described in chapter IV,
- chapter V is devoted to a brief recall of the theory of cosmological perturbations,
- in chapter VI conditions for second order equations and absence of Ostrogradski and Laplacian instabilities are derived and analyzed,
- chapter VII is about Horndeski theory and Disformal transformations,
- the last chapter is devoted to some conclusions.

Chapter 2

Modern Cosmology

2.1 Standard Model of Cosmology

Modern cosmology relies on the concordance ΛCDM model in addition with some inflationary mechanism in the early universe. Recent results of the Planck mission confirm our ΛCDM model in which the matter and radiation fractions Ω_m and Ω_r, the Dark Energy (DE) fraction Ω_Λ and the Dark Matter (DM) fraction Ω_{dm} add up to the total budget of the universe. The firsts two term are well know, Ω_m is given by ordinary matter clustered as galaxies, stars and planets, Ω_r essentially comes from the cosmic background radiation CMB with mean temperature of $T \simeq 2.7$K and fluctuation of order $\Delta T/T \simeq 10^{-5}$. The total matter contribute is [5] $\Omega_m + \Omega_{DM} = 0.3089 \pm 0.0062$, while the Dark Energy amount to $\Omega_{DE} = 0.6911 \pm 0.0062$. The radiation fraction Ω_r essentially comes from CMB and its contribute is very small compared to the others. The most enigmatic contributions to the budget are the last two: DE, essentially is telling us that now our universe is in an accelerated expansion epoch because of something similar to a cosmological constant Λ, while DM is telling us that there exists some other type of matter besides the ordinary baryons, that clusters and interacts with ordinary matter only through gravity.

Actually the Standard Model (SM) of particles physic cannot offer a solution in term of a candidate for such a type of dark components. The only one offered by the SM would be the neutrino, chargeless and weakly interacting, but it cannot account for the entire Dark Matter budget for at least two reasons: first it is relativistic and so a neutrino dominated universe would result in a top-down formation instead of a bottom-up as observed (first stars, then galaxies and at the end clusters of galaxies) and second it has a too small mass: in the SM there is no solutions for DM and DE.

ΛCDM and more in general the standard Hot Big Bang model alone cannot resolve some problems of the early universe. An attempt at an explanation is given by the Inflationary Model. Developed in the early eighties, it introduces the *inflaton*, a scalar field that drives an exponential expansion. Before the introduction of IM there was the following open problems:

- the *horizon problem* that can be cast in the following question: *Why is the universe isotropic and homogeneus as stated by the cosmological principle?* We have already said that the universe within small fluctuations have about the same temperature every where. Moreover even regions of the universe never been in causal contact. Inflation provides a solution because the comoving Hubble radius $r_H \propto 1/aH$ decreases when the scale factor exponentially grows $a \sim e^{Ht}$ (which is when inflation occours), while it starts to increase at the end of the accelerated period, (here H is the Hubble parameter). Thus, regions (scales) of the universe that were in causal contact (thermalized) in the past, can reenter now in our Hubble radius with the same temperature as the whole universe. In some sense all the properties of the universe produced during inflation get frozen outside the comoving Hubble radius until the scale cross again r_H today. Homogeneity and isotropy of the universe at the largest scales we can see today fix the amount of exponential growth or e-folds we need to solve the horizon problem at about $N_e \sim 60 - 70$.

- The *flatness problem* or *why is the universe so flat today?* According to the first Firedmann equation
$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$
where $H = \dot{a}/a$ is the Hubble parameter, ρ is the energy density and k the spatial curvature ($k = 0$ corresponds to flat). The amount of spatial curvature of the Universe depends on the density of matter/energy. The latter equation can be recast as follows
$$(\Omega^{-1} - 1)\rho a^2 = -\frac{3kc^2}{8\pi G},$$
where the right hand side is a constant value and $\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G}{3H^2}\rho$. The critical energy density ρ_c corresponds to a condition for which the Universe is flat ($\Omega = 1$). In order to compensate the decrease of ρa^2 of a factor of 10^{60} keeping the right hand side of the equation constant, the quantity $(\Omega^{-1} - 1)$ must have been increased of the same amount. The problem is that today we observe a universe which is completely consistent with a flat universe finding [5] $\Omega = 1.0023^{+0.0056}_{-0.0054}$ and so $\Omega - 1$ must have been less than 10^{-60} at the Planck era. Given that the initial energy density of the universe could take any value, a *fine tuning* seems to have taken place in order to set exactly $\rho \simeq \rho_c$ at the beginning. Inflation succeeds in the solution of this problem because during an inflationary expansion, the scale factor growing as $a \propto e^{Ht}$ suppresses the curvature term kc^2/a^2.

- The last issue, is the so called *problem of relics or monopoles.* In particular the fact that today we do not observe any of these *topological defects*. These exotic entities are extraordinary massive and may be the result of some mechanism of spontaneous symmetry breaking in the early universe. An inflationary mechanism can diluite these defects almost to zero due to the exponential growth of the scale factor.

A model for inflation is to allow the existence of a scalar field ϕ, the *inflaton*, with equation of state

$$w = p/\rho = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}.$$

If the potential $V(\phi)$ is set to match the so call *slow-roll conditions*

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \qquad \ddot{\phi} \ll \frac{\partial V}{\partial \phi},$$

then, $w \simeq -1$ and one is looking to a quasi-*de Sitter* solution $a(t) \simeq e^{Ht}$. This can happen for example if the potential $V(\phi)$ is sufficiently flat. Over the years, different types of potential were studied each one proposed with different motivations. As quoted on [6] the Planck full mission temperature and polarization data are consistent with the spatially flat base ΛCDM model, whose perturbations are Gaussian and adiabatic with a spectrum described by a simple power law, as predicted by the simplest inflationary models.

On the other hand, ΛCDM with the addition of inflation tell us that there exists a dark sector without giving any explanation about DM and DE. In order to have some insight into Dark Energy and Dark Matter, essentially there are two ways of reasoning: adding some scalar fields or instead try to modify Einstein Gravity.

2.2 The Action of General Relativity

The mathematical background of General Relativity, (GR), is given by Riemmanian Geometry that provides the concept of metric $g_{\mu\nu}(x^\alpha)$ from which one can build the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1}$$

The geometric object $g_{\mu\nu}$, a rank-two symmetric tensor, is the dynamical tensor field that propagates the gravitational interaction.

Taking into account that in general a manifold $\mathcal{M}$ is not flat, the common way to define the derivative concept on $\mathcal{M}$ is to appeal to the so called Christoffel connection 1-form

$$\Gamma = \Gamma^{\alpha}_{\beta} = \Gamma^{\alpha}_{\mu\beta} dx^{\mu}, \tag{2.2}$$

a non-tensorial object defining parallel transport of vectors between points on the manifold. From the latter it follows that the definition of covariant derivative acting on a vector is given by

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\alpha} V^{\alpha}. \tag{2.3}$$

Such a connection is called Levi-Civita connection if it covariantly conserves the metric

$$\nabla_{\alpha} g_{\mu\nu} = 0, \tag{2.4}$$

while it is called torsion-free if

$$\Gamma^{\alpha}_{[\mu\nu]} = 0. \tag{2.5}$$

The first relation completely determines the connection as a function of first derivatives of the metric, while the second implies the symmetry of the connection with respect to its two lower indices. Equation (2.4) fixes the form of the connection as a function of the metric and its first derivative as

$$2\Gamma^{\alpha}_{\mu\beta} = g^{\alpha\tau}(\partial_{\mu} g_{\tau\beta} + \partial_{\beta} g_{\mu\tau} - \partial_{\tau} g_{\mu\beta}). \tag{2.6}$$

Starting from a 1-form there are two natural way to build a 2-form, namely the exterior derivative of a 1-form and the product of two 1-forms. In this way the curvature 2-form is defined

$$\mathscr{R} = \mathscr{R}^{\alpha}_{\beta} = d\Gamma^{\alpha}_{\beta} + (\Gamma^2)^{\alpha}_{\beta} = (\partial_{\mu}\Gamma^{\alpha}_{\nu\beta} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\nu\beta}) dx^{\mu} dx^{\nu} = \frac{1}{2} R^{\alpha}_{\beta\mu\nu} dx^{\mu} dx^{\nu}, \tag{2.7}$$

where $R^{\alpha}_{\beta\mu\nu}$ is the Riemann tensor whose components are given by antisymmetrizing the wedge product indices. The Ricci tensor is defined as $R_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu}$ while the Ricci scalar $R = g^{\beta\nu}R_{\beta\nu}$ is the trace of the latter.

Under a generic coordinate transformation $x \mapsto x'(x)$ differents objects trasforms differentely: a scalar remains unchanged

$$\phi'(x') = \phi(x), \tag{2.8}$$

a contravariant vector transforms like

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(x), \tag{2.9}$$

instead a covariant vector

$$V'_{\mu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}(x), \tag{2.10}$$

a mixed tensor trasforms like

$$T'^{\mu\nu}_{\sigma}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial x'^{\sigma}} T^{\alpha\beta}_{\gamma}(x) \tag{2.11}$$

and finally a tensor density of weight W transforms as

$$t'^{\mu\nu} = \left|\frac{\partial x'}{\partial x}\right|^{W} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} t^{\alpha\beta}(x) \tag{2.12}$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the transformation $x \mapsto x'$. It is easy to see that the determinant of the metric $g = \det g_{\mu\nu}$ transform as $g \mapsto g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$ and so it is a tensor density of weight $W = -2$. On the other hand the volume element of integration transforms as $d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x$, hence the measure

$$d^4x\sqrt{-g} \tag{2.13}$$

is invariant.

Using the invariant volume element and the Ricci scalar, the gravity action can be written as

$$S_{EH} = \frac{1}{2}\int d^4x\sqrt{-g}R, \tag{2.14}$$

the Einstein-Hilbert action. Taking its variation and imposing $\delta S_{EH} = 0$, one obtains the Einstein Field equations in vacuum

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{2.15}$$

In fact, taking the variation we get

$$\delta S_{EH} = \frac{1}{2}\int d^4x\delta(\sqrt{-g})R + \sqrt{-g}\delta R \tag{2.16}$$

and taking into account the following definition

$$g = \det g_{\mu\nu} = e^{\mathrm{tr}(\log g_{\mu\nu})} \qquad \Rightarrow \qquad \delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu}, \tag{2.17}$$

the variation of the action reads

$$\delta S_{EH} = \frac{1}{2}\int d^4x\sqrt{-g}\Big(-\frac{1}{2}Rg_{\mu\nu}\delta g^{\mu\nu} + \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\Big). \tag{2.18}$$

The term proportional to $\delta R_{\mu\nu}$ vanish upon integration by virtue of Gauss' theorem since $\delta\Gamma \to 0$ at the boundary[1], while collecting the terms proportional to $\delta g^{\mu\nu}$ one recovers equation (2.15). In presence of matter one has to include the matter action $S_M = \int d^4x\sqrt{-g}\mathscr{L}_M$ alongside the Einstein-Hilbert action S_{EH}. Using the definition of the stress-energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}, \tag{2.19}$$

one finds that Einstein field equations read

$$G_{\mu\nu} - T_{\mu\nu} = 0, \tag{2.20}$$

[1]This is true if one postulates also that $\delta\partial_\alpha g^{\mu\nu} \to 0$ besides $\delta g^{\mu\nu} \to 0$ at the boundary.

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, build with at most second derivatives of the metric.

As mentioned, the quantity $\delta R_{\mu\nu}$ vanishes upon integration due to Gauss' theorem. In fact it can be written in term of a total derivative. Using the definition of the Ricci tensor $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ we see from (2.7) that

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\nu \delta\Gamma^\alpha_{\mu\alpha} + \delta\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\nu}\delta\Gamma^\beta_{\alpha\beta} - \delta\Gamma^\beta_{\nu\alpha}\Gamma^\alpha_{\beta\mu} - \Gamma^\beta_{\nu\alpha}\delta\Gamma^\alpha_{\beta\mu} \tag{2.21}$$

and recalling the definition of the covariant derivative this last equation can be written as

$$\delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha} \tag{2.22}$$

often called *Palatini identitiy*. Taking into account that the metric is covariantely conserved, equation (2.21) can be put in the following form

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}[\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha})]. \tag{2.23}$$

Using the definition of the Christoffel symbols it is easy to show that

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{2}g^{\alpha\tau}\partial_\beta g_{\alpha\tau} = \partial_\beta(\ln\sqrt{-g}) = \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g}, \tag{2.24}$$

then the covariant four-divergence of a four-vector can be written as

$$\nabla_\alpha V^\alpha = \partial_\alpha V^\alpha + \Gamma^\alpha_{\alpha\beta}V^\beta = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}V^\alpha) \tag{2.25}$$

and so

$$\int d^4x\,\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} d^4x\,\partial_\alpha(\sqrt{-g}V^\alpha) = \int_{\partial\mathcal{M}} d\Sigma_\alpha(\sqrt{-g}V^\alpha) \tag{2.26}$$

where

$$V^\alpha = g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha}, \tag{2.27}$$

while $d\Sigma_\alpha$ is the infinitesimal element of a three-dimensional hypersurface. With this result we can conclude that the contribution to the variation of the action due the variation of the Ricci tensor can be put in the form of a total four divergence and so it vanishes at the boundary $\partial\mathcal{M}$ if one imposes $\delta\Gamma \to 0$ on $\partial\mathcal{M}$.

Chapter 3

Modified Gravity

There are basically two approaches for the construction of models for the dark components. In order to describe DM, a possible approach is based on *modified matter models* in which the energy-momentum tensor $T_{\mu\nu}$ on the r.h.s. of the Einstein equations contains an exotic matter source. The second approach, historically used in order to describe DE, is based on *modified gravity models* in which the l.h.s. of the Einstein equations is modified. It is however important to realize that within General Relativity this division is mostly a practical way to classify the variety of dark energy models but, in general, does not carry a fundamental meaning. One can write down Einstein's equations in the standard form $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ by absorbing in $T_{\mu\nu}$ all the gravity modifications that one conventionally puts on the l.h.s. This is not always true when dealing with action with higher-order derivatives terms. As we will see in the next chapter, Mimetic Gravity is one of the few models that can accomodate both DM and DE.

3.1 Dark Energy as a modified form of matter

3.1.1 Quintessence

Historically, quintessence was thought as a canonical scalar field ϕ with a potential $V(\phi)$ responsible for the late-time cosmic acceleration. Unlike the cosmological constant, the equation of state of quintessence dynamically changes with time and the cosmological evolution can be easily understood by a dynamical system approach. In these models it is important the existence of the so called *tracker fields* solutions that correspond to attractor-like solutions in which the field energy density tracks the background fluid density for a wide range of initial conditions. We use the term “quintessence” to denote a canonical scalar field ϕ with a potential $V(\phi)$ that interacts with all the other components only through standard gravity. Following [7], the quintessence model is therefore described by the action

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{2\kappa^2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big] + S_m, \tag{3.1}$$

where S_m is the matter action and $\kappa^2 = 8\pi G$; $\kappa \equiv 1$ in most of what follows. One finds, as in the case of inflation, that the equation of state reads

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}, \tag{3.2}$$

while the Klein-Gordon field equation for ϕ can be written as

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial_\phi} = 0. \tag{3.3}$$

During radiation- or matter-dominated epochs, the energy density ρ_M of the fluid dominates over that of quintessence, i.e. $\rho_M \gg \rho_\phi$. We require that ρ_ϕ tracks ρ_M so that the dark energy density emerges at late times. Whether this tracking behavior occurs or not depends on the form of the potential $V(\phi)$. If the potential is steep so that the condition $\dot{\phi}^2 \gg V(\phi)$ is always satisfied, the field equation of state is given by $w_\phi \sim 1$.

In this case the energy density of the field evolves as $\rho_\phi \propto a^{-6}$, which decreases much faster than the background fluid density. From Einstein equations one sees that in order to realize the late-time cosmic acceleration the condition $w_\phi < -1/3$ must hold, this translates into the condition $\dot{\phi}^2 < V(\phi)$. Hence the scalar potential needs to be shallow enough for the field to evolve slowly along the potential.

This situation is similar to that in inflationary cosmology and it is convenient to introduce the slow-roll parameters

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \eta = \frac{V_{,\phi\phi}}{\kappa^2 V}. \tag{3.4}$$

It is easy to see that if the conditions $\epsilon \ll 1$ and $|\eta| < 1$ hold, then $\dot{\phi}^2 \ll V(\phi)$ and $|\ddot{\phi}| \gg 3H\dot{\phi}$. Defining $\xi = |\ddot{\phi}|/3H\dot{\phi}$, the deviation of w_ϕ from -1, when $\xi \gg 1$, can be written [7] in terms of ϵ

$$1 + w_\phi = \frac{V_{,\phi}^2}{9H^2(\xi+1)^2\rho_\phi} \sim \frac{2}{3}\epsilon \tag{3.5}$$

neglecting the matter fluid in Einstein equations, (i.e. $3H^2 \sim \kappa^2 V(\phi)$).

So far many quintessence potentials have been proposed. Roughly speaking they have been classified into *freezing* models and *thawing* models. In the former case the field was rolling along the potential in the past, but the movement gradually slows down after the system enters the phase of cosmic acceleration. The representative potentials that belong to each class are

freezing model $V(\phi) = M^{4+n}\phi^{-n}$, $n > 0$ and $V(\phi) = M^{4+n}\phi^{-n}e^{\alpha\phi^2/m_{pl}^2}$,

thawing model $V(\phi) = V_0 M^{4-n}\phi^n$, $n > 0$ and $V(\phi) = M^4\cos(\phi/f)$.

The first of the two potentials do not posses a minimum and so the field rolls down the potential toward infinity while the second potential has a minimum (in which $w_\phi = -1$) and eventually the field gets trapped in it. The second class describes a field with mass

m_ϕ that has been frozen by Hubble friction $H\dot{\phi}$ until recently when H drops below m_ϕ and begins to evolve.

3.1.2 k-essence

Quintessence is based on scalar field models using a canonical field with a slowly varying potential. It is known however that scalar fields with non-canonical kinetic terms often appear in the context of inflation [8]. The same idea applied to DE led to classes of modified matter models such as k-essence among the others. The action for such models is in general given by

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{2\kappa^2} + P(\phi, X)\Big] + S_m, \tag{3.6}$$

where P is a function of the field and its kinetic energy $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The central point is that cosmic acceleration can be realized by the kinetic energy X of the field ϕ. k-essence models are based on the assumption that

$$P = K(\phi)X + L(\phi)X^2, \tag{3.7}$$

in which the kinetic part allows a functional dependence by ϕ other than that of X. These models are usually motivated by low-energy effective string theory [9]. It can be shown that the equation of state of k-essence is

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{p}{2Xp_{,X} - p} \tag{3.8}$$

and, as long as the condition $|2Xp_{,X}| \ll |p|$ is satisfied, w_ϕ can be arbitrarily close to -1.

Another example is that of *Phantom* or *ghost* condensate models that are described by a non canonical kinetic term with the opposite sign of the canonical one, $K(\phi) = -1$ and using $L(\phi) = M^{-4}$. However the phantom field is plagued by severe ultra-violet quantum instabilities because its energy density is not bounded from below. The equation of state in this case reads

$$w_\phi = \frac{1 - X/M^4}{1 - 3X/M^4} \tag{3.9}$$

which gives $-1 < w_\phi < -1/3$ for $1/2 < X/M^4 < 2/3$. Another example of such class of models is given by the *dilatonic* ghost condensate model in which $L(\phi) = e^{\kappa\lambda\phi}/M^4$ that arises as a dilatonic higher-order correction to the tree-level string action [7].

In k-essence it can happen that the linear kinetic energy in X has a negative sign. Such a field, called phantom or ghost scalar field, suffers from a quantum instability problem unless higher-order terms in X or ϕ are taken into account in the Lagrangian density. In the (dilatonic) ghost condensate scenario it is possible to avoid this quantum instability by the presence of the term X^2. Stability conditions of k-essence can be found by considering small fluctuations $\delta\phi$ of the field ϕ about a background value ϕ_0 solution in the FLRW spacetime. The expansion of the Lagrangian up to second order allows one to write the perturbed Hamiltonian density that in this case reads [7]

$$\delta\mathcal{H} = (p_{,X} + 2p_{,XX})(\dot{\delta\phi})^2/2 + p_{,X}(\nabla\delta\phi)^2/2 - p_{,\phi\phi}(\delta\phi)^2/2. \tag{3.10}$$

The positive definiteness of the Hamiltonian is guaranteed if the following conditions holds

$$\mathcal{E}_1 = p_{,X} + 2p_{,XX} \geq 0, \qquad \mathcal{E}_2 = p_{,X} \geq 0, \qquad \mathcal{E}_3 = -p_{,\phi\phi} \geq 0. \tag{3.11}$$

These two conditions prevent an instability related to the presence of negative energy ghost states. If these conditions are violated, the vacuum is unstable under a catastrophic production of ghosts. The production rate from the vacuum is proportional to the phase space integral on all possible final states. Since only a UV cut-off can prevent the creation of modes of arbitrarily high energies, this is essentially a UV instability. The phantom model with the Lagrangian density $P(\phi, X) = -X - V(\phi)$ violates both the first two conditions, which means that its vacuum is unstable. Taking into account higher-order terms such as X^2 in $P(\phi, X)$, it is possible to avoid the quantum instability mentioned above. Let us consider the dilatonic ghost condensate model with $K(\phi) = -1$ and $L(\phi) = e^{\kappa\lambda\phi}/M^4$. It can be shows that the quantum instability is ensured for $e^{\kappa\lambda\phi}/M^4 \geq 1/2$ and in this case $w_\phi \geq -1$. The instability prevented by the last condition in (3.11) is of the tachyonic type and generally much less dramatic (infra-red (IR) type) as long as the two first conditions are satisfied.

3.2 Dark components as a modification of gravity

3.2.1 f(R) theories of Gravity

Another class of modifications of the Einstein theory of gravitation results from the so called $f(R)$ theories in which the Einstein-Hilbert action generalizes to a function of the Ricci scalar

$$\int d^4x\sqrt{-g}R + S_m \qquad \rightarrow \qquad \int d^4x\sqrt{-g}f(R) + S_m,$$

in which one thinks $f(R)$ as a power expansion in R as $f(R) = \sum_k \alpha_k R^k$. Essentially there exist three types of $f(R)$ gravity:

metric $f(R)$ gravity theory with an action depending on the metric through the function $f(R)$;

Palatini $f(R)$ gravity , extension of the latter in which one promotes the connection Γ to a dynamical field;

metric-affine $f(R)$ gravity , the most general case in which one allows also the matter action S_m to be a function of the new Γ field.

Having $f(R)$ in place of R implies a modification of the field equations in which the possibility of describing accelerated expansion emerges. Variation of the action with respect to $g_{\mu\nu}$ gives the following equations of motion

$$F(R)R_{\mu\nu}(g) - 1/2f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = T_{\mu\nu}, \tag{3.12}$$

where

$$F(R) = \frac{\partial f}{\partial R} \equiv f_R$$

and $T_{\mu\nu}$ is the matter energy momentum tensor. The trace of equation 3.12 is

$$3\Box F(R) + F(R)R - 2f(R) = T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3p \tag{3.13}$$

where ρ and p are the energy density and pressure of the matter field. If, on the other hand, one thinks - as in the Palatini formalism - the connections Γ as independent fields, the following field equations hold

$$F(R)R_{\mu\nu}(\Gamma) - 1/2f(R)g_{\mu\nu} = T_{\mu\nu} \tag{3.14}$$

and

$$R_{\mu\nu}(g) - 1/2g_{\mu\nu}R(g) = T_{\mu\nu}/F - g_{\mu\nu}(FR(T) - f)/2F + (\nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F)/F+$$

$$- 3(\partial_\mu F\partial_\nu F - 1/2g^{\mu\nu}(\nabla F)^2)/2F^2. \tag{3.15}$$

when the action is varied with respect to the metric and the independent connection respectively. The trace of (3.14) is

$$F(R)R - 2f(R) = T. \tag{3.16}$$

In General Relativity $f(R) = R - 2\Lambda$ and $F(R) = 1$, so that the term $\Box F(R)$ in (3.13) vanishes. In this case both the metric and the Palatini formalisms give the relation $R = -2T = (\rho - 3p)$, which means that the Ricci scalar R is directly determined by the matter (the trace T). In modified gravity models where $F(R)$ is a function of R, the term $\Box F(R)$ in general does not vanish. This means that, in the metric formalism, there is a propagating scalar degree of freedom, $\phi = F(R)$. The trace of equation (3.13) governs the dynamics of the scalar field ϕ (dubbed "scalaron"). In the Palatini formalism the kinetic term $\Box F(R)$ is not present in equation (3.16), which means that the scalar-field degree of freedom does not propagate freely.

3.2.2 Brans–Dicke theory and Scalar-Tensor theories

As we have seen in the last section, most models of dark energy rely on scalar fields. Scalar fields have a long history in cosmology, starting from *Brans–Dicke* theory in which gravity is mediated by a scalar field in addition to the metric tensor field. Brans–Dicke theory was an attempt to revive *Mach*'s principle (according to which inertia arises when a body is accelerated with respect to the global mass distribution in the Universe) by linking the gravitational constant to a cosmic field. At the same time, Brans–Dicke theory incorporated Dirac's suggestion that the gravitational constant G varies in time. Brans–Dicke theory is just a particular example of scalar-tensor theories. These are probably the simplest example of modified gravity models and as such one of the most studied alternatives to General Relativity.

The action for scalar-tensor theories in presence of matter field is given by

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}f(\phi, R) - \frac{1}{2}\zeta(\phi)(\nabla\phi)^2\Big] + S_m[g_{\mu\nu}, \Psi_m] \tag{3.17}$$

where f is a general function of the scalar field ϕ and the Ricci scalar R, ζ is a function of ϕ, and S_m is the matter Lagrangian that depends on the metric $g_{\mu\nu}$ and matter fields Ψ_m. The latter action includes a wide variety of theories such as $f(R)$ gravity, Brans–Dicke theory, and dilaton gravity. $f(R)$ gravity corresponds to the choice $f(\phi, R) = f(R)$ and $\zeta = 0$. The action of Brans–Dicke theory is written with $f = \phi R$ and $\zeta = \omega_{BD}/\phi$, where ω_{BD} is called "Brans–Dicke parameter". One can generalize Brans–Dicke theory by adding the field potential $U(\phi)$ to the original action, i.e. $f = \phi R - 2U(\phi)$ and $\zeta =$

ω_{BD}/ϕ. The dilaton gravity arising from low-energy effective string theory corresponds to $f = 2e^{-\phi}R - 2U(\phi)$ and $\zeta(\phi) = -2e^{-\phi}$, where we have introduced the dilaton potential $U(\phi)$. The action (3.17) can be transformed to the Einstein frame under a conformal transformation with the choice

$$\Omega^2 = F \equiv \frac{\partial f}{\partial R}. \tag{3.18}$$

It have been shown that $f(R)$ theory in the Palatini formalism corresponds to the generalized Brans–Dicke theory with $\omega_{BD} = -3/2$.

3.2.3 Gauss-Bonnet gravity

The $f(R)$ and scalar-tensor theories add to the gravitational tensor field a new degree of freedom, a scalar field. However this certainly does not exhaust the range of possible modifications of gravity. One possibility is to add vector fields. Another one is to add to the Einstein Lagrangian general functions of the Ricci and Riemann tensors, e.g., $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \dots)$. However these Lagrangians are generally plagued by the existence of ghosts, i.e. the existence of negative energy states. Even besides the quantum problems, this generally implies classical instabilities either at the background or at the perturbed level. There is however a way to modify gravity with a combination of Ricci and Riemann tensors that keeps the equations at second-order in the metric and does not necessarily give rise to instabilities, namely a Gauss–Bonnet (GB) term coupled to scalar field(s). The GB term is a topological invariant quantity. It is the unique invariant for which second derivative occurs linearly in the equations of motion, thereby ensuring the uniqueness of solutions. Moreover, it is worth noticing that the GB term naturally arises as a correction to the tree-level action of low-energy effective string theory [9, 10]. A formulation of the model is based on the following action

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - f(\phi)R^2_{GB}\Big] + S_m[g_{\mu\nu}, \Psi_m] \tag{3.19}$$

where

$$R^2_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \tag{3.20}$$

is the Gauss-Bonnet term. The action corresponds to the Einstein frame action in which the scalar field ϕ does not have a direct coupling to the Ricci scalar R.

3.3 Dynamical system approach

A dynamical system which plays an important role in cosmology belongs to the class of so called autonomous systems. For simplicity we shall study the system of two first-order differential equations, but the analysis can be extended to a system of any number of equations. Let us consider the following coupled differential equations for two variables $x(t)$ and $y(t)$

$$\dot{x} = f(x, y, t), \qquad \dot{y} = g(x, y, t), \tag{3.21}$$

where f and g are the functions in terms of x, y and t. The latter system is said to be autonomous if f and g do not contain explicit time-dependent terms. The dynamics of the autonomous systems can be analyzed in the following way. A point (x_c, y_c) is said to be a fixed point or a critical point of the autonomous system if $(f, g)(x = x_c) = 0$. A critical point (x_c, y_c) is called an attractor when it satisfies the condition

$$(x(t), y(t)) \to (x_c, y_c) \qquad \text{for} \qquad t \to \infty. \tag{3.22}$$

We can find whether the system approaches one of the critical points or not by studying the stability around the fixed points. Let us consider small perturbations δx and δy around the critical point (x_c, y_c), i.e.,

$$x = x_c + \delta x, \qquad y = y_c + \delta y. \tag{3.23}$$

Then substituting into Eqs. (3.21) leads to the first-order differential equations

$$\frac{d}{dN}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \mathcal{J}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}, \tag{3.24}$$

where $N = \ln a$ is the number of e-folding which is convenient to use for the dynamics. The Jacobian $\mathcal{J}$ evaluated at the critical point gives information about the stability of the critical point itself. The matrix possesses two eigenvalues μ_1 and μ_2. The general solution for the evolution of linear perturbations can be written as

$$\delta x = C_1 e^{\mu_1 N} + C_2 e^{\mu_2 N}, \qquad \delta y = C_3 e^{\mu_1 N} + C_4 e^{\mu_2 N} \tag{3.25}$$

where $C_1,\ C_2,\ C_3,\ C_4$ are integration constants. Thus the stability around the fixed points depends upon the nature of the eigenvalues. One generally uses the following classification

Stable node: $\mu_1 < 0$ and $\mu_2 < 0$

Unstable node: $\mu_1 > 0$ and $\mu_2 > 0$

Saddle point: $\mu_1 < 0$ and $\mu_2 > 0$ or $\mu_1 > 0$ and $\mu_2 < 0$

Stable spiral: the determinant of the matrix is negative and the real parts of the eigenvalue μ_i are negative.

A critical point is an attractor in the first and in the last cases but not in the second two cases. In the following two subsections two examples of dynamical system approach are shown in the case of quintessence and in the case of a dilatonic ghost condensate models.

3.3.1 Quintessence

For the quintessence model let's define

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \qquad x_2 = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \tag{3.26}$$

then $\Omega_m = \frac{\kappa^2 \rho_m}{3H^2}$ can be espressed as

$$\Omega_m = 1 - x_1^2 - x_2^2. \tag{3.27}$$

We also define the energy fraction of dark energy

$$\Omega_\phi \equiv \frac{\kappa^2 \rho_\phi}{3H^2} = x_1^2 + x_2^2 \tag{3.28}$$

which satisfies the relation $\Omega_m + \Omega_\phi = 1$. Deriving the Einstein equations in this model, leads to the following equation

$$\dot{H}/H^2 = -3x_1^2 - 3/2(1 + w_m)(1 - x_1^2 - x_2^2). \tag{3.29}$$

In this case the effective state equation reads $w_{eff} = w_m + (1 - w_m)x_1^2 - (1 + w_m)x_2^2$. The equation of state of the quintessence field reads $w_\phi = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$. It can be show that the autonomous dynamical system associated with the described quintessence model reads

$$\begin{aligned} \frac{dx_1}{dN} &= -3x_1 + \sqrt{6}\lambda x_2^2/2 + 3x_1/2[(1 - w_m)x_2^2 + (1 + w_m)(1 - x_2^2)] \\ \frac{dx_2}{dN} &= -\sqrt{6}\lambda x_1 x_2 + 3x_2/2[(1 - w_m)x_1^2 + (1 + w_m)(1 - x_2^2)], \end{aligned} \tag{3.30}$$

where $\lambda = -V_{,\phi}/\kappa V$ characterizes the slope of the field potential and obeys the following equation

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x_1, \tag{3.31}$$

where $\Gamma = VV_{,\phi\phi}/V_{,\phi}^2$.

If λ is constant, the integration of equation (3.31) yields an exponential potential

$$V(\phi) = V_0 e^{-\kappa\lambda\phi},$$

that corresponds to $\Gamma = 1$. In this case the autonomous equations (3.30) are closed. The cosmological dynamics can be well understood by studying fixed points of the system. If Γ is constant but λ is not, we have to solve equations (3.30) and (3.31). For the power-law potential, $V(\phi) = M^{4+n}\phi^{-n}$ $(n > 0, \phi > 0)$, we have that $\Gamma = (n+1)/n > 1$ and $x_1 > 0$, in which case the quantity $\lambda(> 0)$ decreases. Of course, for general field potentials, Γ is not necessarily constant. In such cases we need to obtain the field ϕ as a function of N together with the use of the relation $\kappa\sqrt{V} = \sqrt{3}Hx_2$. Then the evolution of the variable $\lambda = \lambda(\phi)$ is known accordingly. We can derive fixed points of the system by setting $dx_1/dN = dx_2/dN = 0$. The fixed points are in general the solution of the dynamical system and give a first qualitative description of the phase space. As we discuss below they can be classified according to their stability properties. If there are no singularities or strange attractors, the trajectories with respect to $x_1(N)$ and $x_2(N)$, in general to be obtained numerically, run from unstable fixed points to stable points, coasting along "saddle" points. When λ is constant they are found to be the five points:

- the matter dominated critical point $a = (0, 0)$ corresponding to

$$\Omega_m = 1\,, \Omega_\phi = 0\,, w_{eff} = w_m\,, w_\phi \qquad \text{undefined},$$

- points $b_1 = (1, 0)$ and $b_2 = (-1, 0)$ in which $\Omega_\phi = 1$, $w_{eff} = w_\phi = 1$, for them the kinetic energy of quintessence is dominant in which case ρ_ϕ decreases rapidly $\rho_\phi \propto a^{-6}$ relative to the background density,

- the scalar field dominated critical point $c = (\lambda/\sqrt{6}, [1 - \lambda^2/6]^{1/2})$, where $\Omega_\phi = 1$, $w_{eff} = w_\phi = -1 + \lambda^2/3$ existing if $\lambda^2 < 6$, the cosmic acceleration is realized if $w_{eff} < -1/3$, i.e. $\lambda^2 < 2$. the limit $\lambda \to 0$, $(V(\phi) \to V_0)$, corresponds to the equation of state of a cosmological constant $w_{eff} = w_\phi = -1$.

- the last critical point $d = (\sqrt{3/2}(1 + w_m)/\lambda, [3(1 - w_m^2)/2\lambda^2]^{1/2})$ is the so-called tracker solution for which the ratio Ω_ϕ/Ω_m is a non-zero constant and $\Omega_\phi = 3(1 + w_m)/\lambda^2$; this scaling solution exist when $\lambda^2 > 3(1 + w_m)$ following from $\Omega_\phi < 1$.

Since $w_\phi = w_m$ for scaling solutions, it is not possible to realize cosmic acceleration unless the matter fluid has the unusual state equation $w_m < -1/3$..

If the determinant of the Jacobian vanishes, the system becomes effectively one-dimensional around the fixed point. This classification can be extended to more dimensions: a fixed point is stable if all the real parts of the eigenvalues are negative, unstable if they are all positive, and a saddle when there are negative and positive real parts. If an eigenvalue vanishes then the stability can be established expanding to higher orders. In the realistic case in which the equation of state of the fluid is in the region $0 \leq w_m < 1$, the eigenvalues and the nature of the above fixed points are those in table (3.1)

a	b_1	b_2
$-\frac{3}{2}(1-w_m)$	$3-\frac{\sqrt{6}}{2}\lambda$	$3+\frac{\sqrt{6}}{2}\lambda$
$\frac{3}{2}(1+w_m)$	$3(1-w_m)$	$3(1-w_m)$
saddle	unstable for $\lambda < \sqrt{6}$	unstable for $\lambda > -\sqrt{6}$
saddle for $\lambda > \sqrt{6}$	saddle for $\lambda < -\sqrt{6}$	saddle for $3(1+w_m) < \lambda^2 < 6$

c	d
$\frac{1}{2}(\lambda^2-6)$	$-\frac{3(1-w_m)}{4}(1+\sqrt{\mathcal{I}})$
$\lambda^2-3(1+w_m)$	$-\frac{3(1-w_m)}{4}(1-\sqrt{\mathcal{I}})$
stable for $\lambda^2 < 3(1+w_m)$	saddle for $\lambda^2 < 3(1+w_m)$
stable for $3(1+w_m) < \lambda^2 < 6$	stable for $3(1+w_m) < \lambda^2 < \eta$
	stable spiral if $\lambda^2 > \eta$

Table 3.1: Eigenvalues of critical points for a quintessence model. $\eta = \frac{24(1+w_m)^2}{7+9w_m}$ and $\mathcal{I} = 1 - \sqrt{\frac{8(1+w_m)[\lambda^2-3(1+w_m)]}{\lambda^2(1-w_m)}}$.

The radiation ($w = 1/3$) and matter ($w = 0$) dominated epochs can be realized either by the point (a) or (d). When $\lambda^2 > 3(1 + w_m)$ the solutions approach the stable scaling fixed point (d) instead of the point (a). In this case, however, the solutions do not exit from the scaling era ($\Omega_\phi = constant$) to connect to the accelerated epoch. In order to give rise to tracking behavior in which Ω_ϕ evolves to catch up with Ω_m, we require that the slope of the potential gradually decreases. This can be realized by the field potential in which λ gets smaller with time (such as $V(\phi) = M^{4+n}\phi^{-n}$). The point ($c$) is the only fixed point giving rise to a stable accelerated attractor for $\lambda^2 < 2$. When $\lambda^2 < 2$, a physically meaningful solution (d) does not exist because $\Omega_\phi > 1$ for both radiation and matter fluids. In this case the radiation-and matter-dominated epochs are realized by the point (a). Note that when λ is close to 0 the solution starting from the point (a) and approaching the point (c) is not much different from the cosmological constant scenario. Nevertheless, since the equation of state of the attractor is given by $w_\phi = -1 + \lambda^2/3$, we can still find a difference from $w_\phi = -1$.

In figure (3.1) a plot of the trajectories of solutions in the (x_1, x_2) plane for $\lambda = 1$ and $w_m = 0$. Since $\Omega_m \geq 0$ in equation (3.28), the allowed region corresponds to $0 \leq x_2 \leq \sqrt{1 - x_1^2}$. The kinetic-energy-dominated points (b_1) and (b_2) are unstable in this case. Since the matter point (a) is a saddle, the solutions starting from $x_2 \ll 1$ temporarily approach this fixed point. The trajectories finally approach the accelerated fixed point (c), because this is stable for $\lambda^2 < 3$.

3.3.2 k-essence example: dilatonic ghost field condensate

Let us consider the cosmological dynamics of the dilatonic ghost condensate model with

$$P = -X + e^{\kappa\lambda\phi} X^2/M^4$$

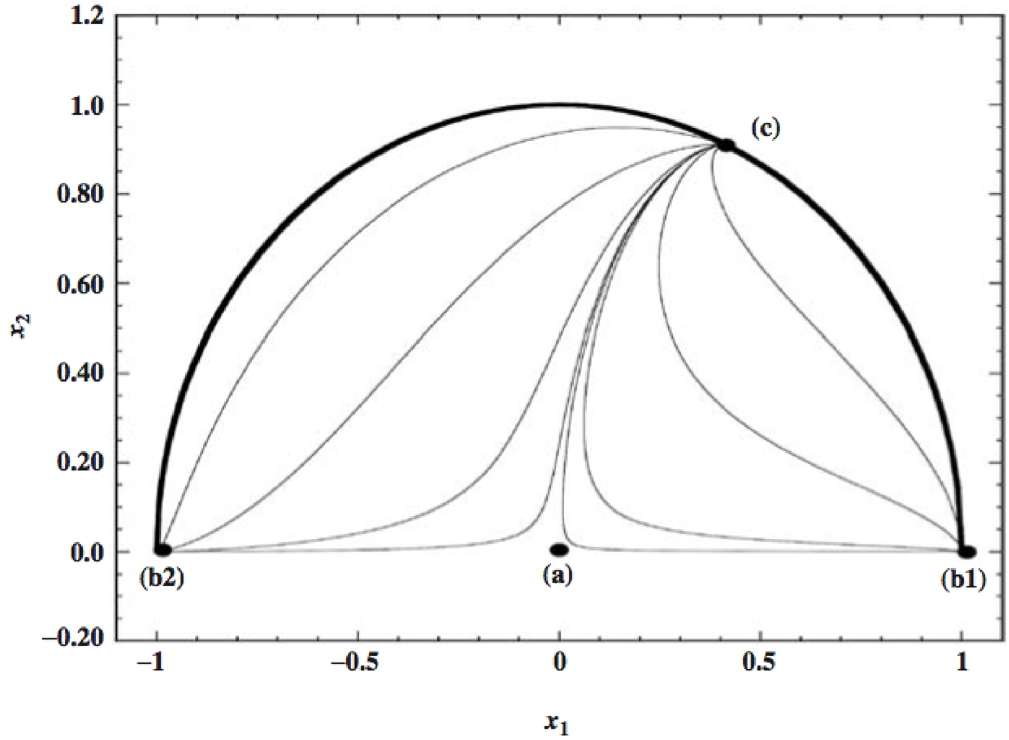

Figure 3.1: The trajectories of solutions for the exponential potential $V = V_0 e^{-\kappa\lambda\phi}$ with model parameters $\lambda = 1$ and $w_m = 0$. The attractor is the accelerated point c, the matter point a is a saddle whereas b_1 and b_2 are unstable nodes. The thick curve is the border of the allowed region characterized by $x_2 = \sqrt{1 - x_1^2}$.

in the flat FLRW background. As a matter fluid we take into account both non-relativistic matter (energy density ρ_m) and radiation (energy density ρ_r). As before, fields equations can be written in terms of several quantities related to the physical ones, then we hve the following definitions

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \qquad x_2 = \frac{\dot{\phi}^2 e^{\kappa\lambda\phi}}{2M^4}, \qquad x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}. \tag{3.32}$$

The autonomous system of equations in the case of k-essence is

$$\frac{dx_1}{dN} = -x_1\frac{6(2x_2-1)+3\sqrt{6}\lambda x_1 x_2}{6(2x_2-1)} + \frac{x_1}{2}(3-3x_1^2-3x_1^2x_2+x_3^2), \tag{3.33}$$

$$\frac{dx_2}{dN} = x_2\frac{3x_2(4-\sqrt{6}x_1)-\sqrt{6}(\sqrt{6}-\lambda x_1)}{(1-6x_2)}, \tag{3.34}$$

$$\frac{dx_3}{dN} = \frac{x_3}{2}(-1-3x_1^2+3x_1^2x_2+x_3^2), \tag{3.35}$$

together with

$$w_{eff} = -1-\frac{2\dot{H}}{3H^2} = -x_1^2+x_1^2x_2+x_3^2/3,$$

$$w_\phi = p_\phi/\rho_\phi = \frac{1-x_2}{1-3x_2},$$

$$\Omega_\phi = -x_1^2+3x_1^2x_2,$$

$$\Omega_r = x_3^2$$

and

$$\Omega_m = 1-\Omega_r-\Omega_\phi.$$

Quantum stability can be achieved if $x_2 \geq 1/2$. There are essentially three critical points: the radiation point $r=(0,1/2,1)$ with $w_{eff}=1/3$, $w_\phi=-1$, $\Omega_r=1$ and $\Omega_\phi=\Omega_m=0$, the matter point $m=(0,1/2,0)$ with $w_{eff}=0$, $w_\phi=-1$ $\Omega_\phi=\Omega_r=0$ and $\Omega_m=1$ and then the accelerated critical point

$$a = (-\sqrt{6}\lambda f_-(\lambda)/4, 1/2+f_+(\lambda)/16, 0)$$

where

$$f_{\pm}(\lambda) = 1 \pm \sqrt{1 + 16/(3\lambda^2)}.$$

The point a satisfies $w_{eff} = w_{\phi} = \frac{-8+\lambda^2 f_+}{8+\lambda^2 f_+}$, $\Omega_{\phi} = 1$ and $\Omega_r = \Omega_m = 0$. Cosmic acceleration occurs for $-1 \leq w_{eff} < -1/3$ which translates into the condition $0 \leq \lambda < \sqrt{6}/3$. There is another critical point but it lies on the quantum instability region corresponding to a phantom equation of state $w_{\phi} < -1$. In figure (3.2) a plot of the cosmological evolution of the dilatonic ghost condensate model with $\lambda = 0.2$ is given. The initial conditions at the radiation era are chosen to be close to the radiation point r with $x_2 > 1/2$. Finally we recall that the sound speed of the dilatonic ghost condensate model is smaller than the speed of light and it is given by

$$c_s^2 = \frac{2x_2 - 1}{6x_2 - 1} \tag{3.36}$$

and the condition for the existence of the late-time accelerated point gives

$$0 \leq c_s^2 < 1/3$$

and thus this model does not violate causality.

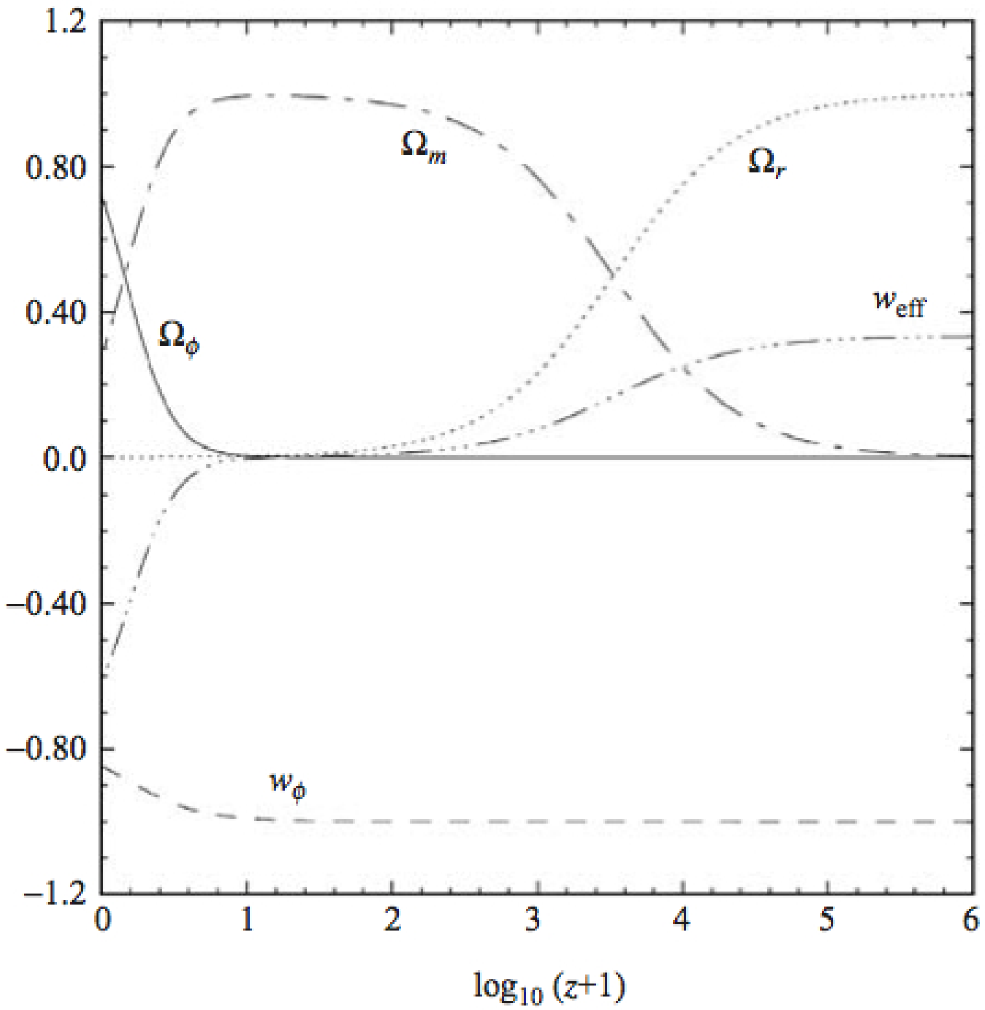

Figure 3.2: Evolution of Ω_m, Ω_r, Ω_ϕ, w_{eff} and w_ϕ for the dilatonic ghost condensate model with $\lambda = 0.2$ versus redshift z.

Chapter 4

Mimetic Gravity

4.1 Mimetic Gravity

It was recently shown [1, 2], that allowing the physical metric $\bar{g}_{\mu\nu}$ to be a function of an auxiliary metric $g_{\mu\nu}$ and of a scalar field ϕ via the relation

$$\bar{g}_{\mu\nu} = (g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)g_{\mu\nu} = Pg_{\mu\nu} \tag{4.1}$$

it is possible to describe a wide variety of gravitational phenomena. Such a theory is clearly Weyl invariant, because a rescaling

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu} \tag{4.2}$$

would preserve the physical metric $\bar{g}_{\mu\nu}$ that is a function of the auxiliary metric and its inverse.

Taking the variation of the Einstein-Hilbert action in the presence of matter

$$S_{EH}\big(\bar{g}_{\mu\nu}(g_{\mu\nu})\big) + S_M \tag{4.3}$$

with respect to the metric defined by (4.1), we find

$$G^{\mu\nu} - T_M^{\mu\nu} - (G - T_M)\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi = 0, \tag{4.4}$$

where $G - T_M = \mathrm{tr}(G^{\mu\nu} - T_M^{\mu\nu})$. This result can be obtained after noting that

$$\delta\bar{g}_{\mu\nu} = \delta P g_{\mu\nu} + P\delta g_{\mu\nu} = (\delta g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2g^{\alpha\beta}\partial_\alpha\delta\phi\partial_\beta\phi)g_{\mu\nu} + P\delta g_{\mu\nu} \tag{4.5}$$

and after restoring the relation between the physical and the auxiliary metrics, in fact from (4.1) follows that

$$\bar{g}^{\mu\nu} = P^{-1}g^{\mu\nu}.$$

In this case, in contrast to standard GR, even when matter is absent $T_M \equiv 0$, one find a contribute to the right hand side of the Einstein field equations, given by

$$G\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi. \tag{4.6}$$

This term, as we will see, can be identified with the energy-momentum tensor of some kind of fluid. Moreover, the relation between the physical metric and the auxiliary imply the existence of the constraint

$$P = g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = P\bar{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \tag{4.7}$$

or

$$\bar{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = 1, \tag{4.8}$$

which tells us that the relatives Einstein equations (4.4) are traceless. It is important to stress that the scalar field of the mimetic model it is a different entity respect to others scalar fields introduced by other existing tensor-scalar theories. In fact, due to the conformal symmetry, the scalar degree of freedom in (4.1) is equivalent to the scaling factor up to an integrating constant, and thus it is not a new dynamical degree of freedom [1]. The existence of the constraint (4.8), as suggested in [2], encourages to employ the constraint (4.8) as Lagrange multipliers inside the usual Einstein-Hilbert action.

4.1.1 Action and equations of motion of Mimetic Matter

Employing the constraint in the action and generalizing it by adding also a potential $V(\phi)$, results in the following action

$$S_\lambda = \int d^4x\sqrt{-g}\Big(\frac{R}{2} - \lambda(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - 1) + V(\phi) + \mathscr{L}_M\Big). \tag{4.9}$$

Taking variation of the latter with respect to the Lagrange multiplier leads to the constraint (4.8). Using the constraint in the calculation of the variation of S_λ with respect to the metric brings the Einstein type equations in the following form

$$G_{\mu\nu} = T^M_{\mu\nu} + 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi) \equiv T^M_{\mu\nu} + T^{\text{mimetic}}_{\mu\nu}. \tag{4.10}$$

The equation of motion of ϕ follows if instead we vary the action with respect to ϕ. After an integration by parts the quantity

$$\int d^4x\sqrt{-g}g^{\alpha\beta}\Big(2\lambda\partial_\alpha\delta\phi\partial_\beta\phi - \partial_\phi V\delta\phi\Big), \tag{4.11}$$

gives

$$\nabla^\beta(2\lambda\partial_\beta\phi) = -\partial_\phi V(\phi), \tag{4.12}$$

where 2λ is fixed by the trace of equation (4.10)

$$2\lambda = G - T - 4V. \tag{4.13}$$

Comparing $T^{\text{mimetic}}_{\mu\nu}$ in (4.10) with the energy-momentum tensor of a perfect fluid

$$T^{\text{p.fluid}}_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{4.14}$$

we can conclude that we are in the presence of a perfect fluid with energy density and pressure density given by

$$\rho = G - T - 3V, \qquad p = -V, \tag{4.15}$$

while the normalized four velocity is $u^\mu = \partial^\mu\phi$, the normalization condition being the constraint.

4.1.2 Cosmological Solutions

Equation (4.8) completely determines the form of the field ϕ and for a FLRW metric

$$\phi = \pm t + const. \equiv t \tag{4.16}$$

clearly is a solution. With this result, the equation of motion (4.12) of ϕ simplifies a lot. In fact for a flat isotropic and homogeneous universe described by the diagonal metric

$$ds^2 = dt^2 - a^3(t)\delta_{ij}dx^i dx^j, \tag{4.17}$$

the only contribution to it is given by

$$\frac{1}{\sqrt{-g}}\frac{d}{dt}\Big(\sqrt{-g}(\rho - V)\dot{\phi}\Big) = -\dot{V}. \tag{4.18}$$

Integration of the latter equation gives

$$\rho = \frac{3}{a^3}\int da\, a^2 V + \frac{const.}{a^3}, \tag{4.19}$$

that is the energy density as a function of the potential. The contribute of the integration constant reproduce the typical *dust*-type contribution given by this *mimetic fluid*.

In the case that the metric is (4.17) and the stress-energy tensor is that of this mimetic fluid, then the only relevant parts of the Ricci tensor are only along the diagonal, i.e.

$$R^0_0 = 3\frac{\ddot{a}}{a}, \qquad R^i_j = \Big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\Big)\delta^i_j \tag{4.20}$$

while the Ricci scalar is

$$R = 6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big). \tag{4.21}$$

Differentiating the Hubble parameter with respect to time one has

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 \tag{4.22}$$

and so the time-time component of the Einstein equations become

$$H^2 = \frac{1}{3}\rho \tag{4.23}$$

while the space-space component reads

$$2\dot{H} + 3H^2 = V. \tag{4.24}$$

At this point, solving these two last equations in terms of the scale factor a for a given potential, and taking into account that

$$w = p/\rho = -1 - \frac{2\dot{H}}{3H^2} \tag{4.25}$$

is the state equation of this perfect fluid, it is possible to mimic a wide range of cosmological behaviors when an appropriate potential is chosen. Equation (4.25) must be understood as a function of H once the scale factor a is calculated for a given V. If we choose the potential as a power law of time $V = \alpha t^n$, then setting $y = a^{3/2}$, the space-space Einstein equation (4.24) became the differential equation

$$\ddot{y} - \frac{3}{4}V(t)y = 0, \tag{4.26}$$

whose solutions can be obtained for example using the Frobenius method, that consists in the substitution of the ansatz $y = t^s \sum_{k=0}^{\infty} a_k t^k$. In doing this one finds an algebrical relation between the terms of our original differential equation and the a_k coefficients. Inserting the ansatz gives

$$\sum_{k=0}^{\infty} t^{k+s} a_k \left[(k+s)(k+s-1)t^{-2} - \frac{3\alpha}{4}t^n\right] = 0. \tag{4.27}$$

For example, setting $k = 0$ the case $n = -2$ results in the equation

$$t^{-2}a_0[s(s-1) - \frac{3\alpha}{4}] = 0, \tag{4.28}$$

which is solved for $t^{-2}a_0 \neq 0$ and $\alpha \geq -1/3$ ($p \leq 1/3$) by

$$s_\pm = \frac{1 \pm \sqrt{1+3\alpha}}{2}. \tag{4.29}$$

With this value a solution in the case $p \leq 1/3$ is of the form

$$y(t) = c_1 t^{s_+} + c_2 t^{s_-} = c_1 t^{s_+}(1 + ct^{s_- - s_+}) = c_1 t^{s_+}(1 + ct^{-2\sqrt{1+3\alpha}}), \tag{4.30}$$

where c_1 and c_2 are constant of integration and $c = c_2/c_1 \neq 0$. Given that $a = y^{2/3}$, $\rho = 3H^2$ and $p = -\alpha/t^2$ one finds

$$w = p/\rho = -3\alpha\Big(1 + \sqrt{1+3\alpha}\frac{1 - ct^{-2\sqrt{1+3\alpha}}}{1 + ct^{-2\sqrt{1+3\alpha}}}\Big)^{-2}, \tag{4.31}$$

which in the limit of large and small t approaches a constant. For $\alpha = -1/3$ we have $w = 1$ a ultra-hard equation of state $p = \rho$ while the case $\alpha = -1/4$ corresponds to a ultra relativistic fluid with $p = \frac{1}{3}\rho$ for $t \to \infty$ and $p = 3\rho$ for $t \to 0$.

If, in the other case $\alpha < -1/3$, then the solutions is of the form

$$y(t) = k_1 t^{1/2} \cos\big(\frac{1}{2}\sqrt{|1+3\alpha|\ln t + k_2}\big) \tag{4.32}$$

with two constants of integration. This solution shows that for large negative α, i.e. large positive pressure p, we are describing an oscillating flat universe with singularities and amplitude of oscillations growing in time. The general case of a power-law potential $V = \alpha t^n$ with $n \neq -2$ can be solved in term of the modified Bessel functions of first kind [2]

$$y(t) = t^{1/2} Z_{\frac{1}{n+2}}\Big(\frac{\sqrt{-3\alpha}}{n+2} t^{\frac{n+2}{2}}\Big). \tag{4.33}$$

If $n < -2$, that is, the potential decay faster than $1\phi^2$, the asymptotic at large t is $y \propto t$ and, correspondingly, the scale factor in the leading order behaves as in dust dominated universe, $a \propto t^{2/3}$. For $n > -2$

$$y \propto t^{-n/4} \exp\Big(\pm i\frac{\sqrt{-3\alpha}}{n+2} t^{\frac{n+2}{2}}\Big) \tag{4.34}$$

as $t \to \infty$. Here the behavoir of the scale factor drastically depends on the sign of α. For negative α (positive pressure p), the mimetic matter leads to an oscillating universe with singularities. The case of positive α corresponds to accelerated, inflationary universe. In particular, for $n = 0$ one finds an exponential expansion corresponding to the cosmological constant, while $n = 2$ leads to inflationary expansion with scale factor

$$a \propto t^{-1/3} \exp\left(\sqrt{\frac{\alpha}{12}} t^2\right). \tag{4.35}$$

4.1.3 Mimetic Matter as Quintessence

It is possible to consider the behavior of mimetic matter in the case when the universe is dominated by some other matter with constant equation of state $p = w\rho$ and where the potential si given by $V(\phi) = \alpha/t^2$. In this case the scale factor is $\alpha \propto t^2 3(1+w)$ and if $\phi = t$ then the energy density of mimetic matter given by (4.19) decays as

$$\rho_{\text{mimetic}} = -\frac{\alpha}{wt^2} \tag{4.36}$$

if one set to zero the constant of integration in (4.19). Because $p_{\text{mimetic}} = -\alpha/t^2$, the mimetic matter imitates the equation of state of the dominant matter [2]. However, since the total energy density is equal to

$$\rho = 3H^2 = \frac{4}{3(1+w)^2 t^2} \tag{4.37}$$

this mimetic matter can be subdominant only if $\alpha/w \ll 1$. The more general solution for subdominant mimetic matter, $\phi = t + t_0$, first corresponds to a cosmological constant for $t < t_0$ and only at $t > t_0$ starts to behave similar to a dominant matter.

4.1.4 Mimetic Matter as an inflaton

One can easily construct the inflationary solutions using the mimetic matter. In fact, one can take any scale factor $a(t) = y^{2/3}$ and using equation (4.26) find the potential

$$V(\phi \equiv t) = \frac{4\ddot{y}}{3y} \tag{4.38}$$

for the theory where this scale factor wil be a solution of the corresponding equation. For example, the potential

$$V(\phi) = \frac{\alpha\phi^2}{e^\phi + 1} \tag{4.39}$$

with positive α describes inflation with graceful exit to matter dominating universe. In fact, the scale factor grows as

$$a \propto \exp\Big(- \frac{\alpha}{12} t^2 \Big) \tag{4.40}$$

at large negative $\phi = t$ and is proportional to $t^{2/3}$ for positive t. Playing with potentials one can easily get any "wishful" behavior for the scale factor during inflation and after it. Thus we see that the mimetic matter can easily provide us with the inflaton. The questions is then: *how one can generate the radiations and baryons we observe?* This can be done either via gravitational particle production at the end of inflation, or via direct coupling of other fields to ϕ.

4.2 Modified Mimetic action

A further generalization for the action would be to add a higher derivative term for the scalar field to the mimetic action S_λ and setting $\mathscr{L}_M \equiv 0$ with no loss of generality. The new term can be of the form

$$S_\gamma = \int d^4x \sqrt{-g} \frac{1}{2} \gamma (\Box\phi)^2. \tag{4.41}$$

Taking variations of this gives as always two terms: the first when varied with respect to the metric, arising from the determinant, and the second when acting on the higher derivative term. These new terms results in an alteration of the stress-energy tensor already found for this mimetic matter. The variations read

$$\delta S_\gamma = \int d^4x \sqrt{-g} \big(- \frac{\gamma}{4} g_{\mu\nu} \chi^2 - \gamma \partial_\mu \chi \partial_\nu \phi \big) \delta g^{\mu\nu}, \tag{4.42}$$

where the last line follows after an integration by parts and after defining $\chi = \Box\phi$. The new contribution $Y_{\mu\nu} \equiv \partial_\mu\chi\partial_\nu\phi$ to the stress-energy tensor can be decomposed into the sum of three irreducible pieces

$$Y_{\mu\nu} = \Big(Y_{[\mu\nu]}\Big) + \Big(\frac{1}{n}\delta_{\mu\nu}\delta^{\alpha\beta}Y_{\alpha\beta}\Big) + \Big(Y_{(\mu\nu)} - \frac{1}{n}\delta_{\mu\nu}\delta^{\alpha\beta}Y_{\alpha\beta}\Big) \tag{4.43}$$

and so the stress-energy tensor for this modified action is

$$T^\mu_\nu = \Big(V + \gamma\big(\partial_\sigma\phi\partial^\sigma\chi + \frac{1}{2}\chi^2\big)\Big)\delta^\mu_\nu + 2\lambda\partial_\nu\phi\partial^\mu\phi - \gamma(\partial_\nu\phi\partial^\mu\chi + \partial_\nu\chi\partial^\mu\phi). \tag{4.44}$$

Recalling that for a given metric the covariant d'Alambertian of a scalar field corresponds to

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi), \tag{4.45}$$

we see that a scalar field ϕ solution of the constraint, i.e. $\phi = t + \mathit{const.}$, in a flat background produces a value of χ equal to

$$\chi = \ddot{\phi} + 3H\dot{\phi} = 3H \tag{4.46}$$

and so, in a flat Friedman universe, ϕ and χ are functions only of time. The Einstein equations in this case read

$$H^2 = \frac{1}{3}V + \gamma\Big(\frac{3}{2}H^2 - \dot{H}\Big) + \frac{2}{3}\lambda \tag{4.47}$$

and

$$2\dot{H} + 3H^2 = V + \frac{3}{2}\gamma(2\dot{H} + 3H^2) \tag{4.48}$$

from the time-time and space-space respectively. Therefore in place of equation (4.24) we find

$$2\dot{H} + 3H^2 = \frac{2}{2-3\gamma}V \tag{4.49}$$

different from equation (4.24) by the overall normalization of the potential V proportional to the speed of sound c_s defined [2] as

$$c_s^2 = \frac{\gamma}{2-3\gamma}. \tag{4.50}$$

It is clear that this non-vanishing speed of sound is an added feature of the standard mimetic model with potential $V(\phi)$ which instead predicts $c_s = 0$, as we saw before in (4.24). The high derivative term was analyzed for the first time by [2] in order to implement mimetic gravity in the context of inflation and in particular in the quantization of the inflaton. As pointed out by the authors, this modification can lead to a suppression of gravitational waves from inflation, seemingly without any non-Gaussianity.

Chapter 5

Cosmological perturbations

5.1 Cosmological perturbations

The present chapter is devoted to consider what consequences a small perturbation of the scalar field $\phi = \phi_0 + \delta\phi$ can have in relation of small perturbations around, for example, a conformally flat background metric for which ϕ_0 is a solution of the flat field equations. One foundamental result is that at the end we can be able to write some gauge-invariant functions.

The total metric splits into its conformally flat background term ${}^0g_{\alpha\beta}$, of the form

$$ {}^0g_{\alpha\beta}dx^\alpha dx^\beta = a^2(\eta)\big(d\eta^2 - \delta_{ij}dx^i dx^j\big), \tag{5.1} $$

plus a small perturbation $|\delta g_{\alpha\beta}| \ll |{}^0g_{\alpha\beta}|$ that in general can be of scalar, vectorial or tensorial type. Einstein equations involve rank-two symmetric tensors, so the number of degrees of freedom $n(n+1)/2$ is ten, the same as the independent components of the metric. In order to perturb every component of a symmetric metric we have to use ten degrees of freedom of various type. Therefore calling the scalar perturbations (ϕ_p, B, ψ, E), the vector perturbations (S_i, F_i) and the symmetric tensor h_{ij}, they add up to a total of

$$4 + 2 \cdot 3 + 6 = 16 \qquad \text{(degrees of freedom)}.$$

The correct amount of d.o.f. is achieved imposing the divergence free conditions $\partial_i S^i = 0$, $\partial_i F^i = 0$ and $\partial_i h^i_j = 0$ along with the traceless condition $h^i_i = 0$. A total of 6 equations that reduce the number of degrees of freedom to the desired result. Scalar perturbations are induced by energy density inhomogeneities, while vector perturbations are related to the rotational motion of the fluid. The former are the most relevant while the latter are quickly decaying and so not important to a first approximation. On the other hand, tensor perturbations (traceless and transverse) describe gravitational waves. The most general form of the perturbed metric is

$$\delta g_{\alpha\beta} dx^\alpha dx^\beta = a^2[2\phi_p d\eta^2 + 2(B_{,i} + S_i) d\eta dx^i + (2\psi\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij}) dx^i dx^j]. \quad (5.2)$$

In General Relativity there exists a freedom in the choice of the coordinate system realized by diffeomorphisms. Following [11] for a small displacement of coordinates

$$x \mapsto x'(x) = x + \xi$$

the transformation law for the metric is given by

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \simeq \left(\delta^\alpha_\mu - \xi^\alpha_{,\mu}\right)\left(\delta^\beta_\nu - \xi^\beta_{,\nu}\right)({}^0g_{\alpha\beta} + \delta g_{\alpha\beta}) \quad (5.3)$$

$$\simeq {}^0g_{\mu\nu}(x) + \delta g_{\mu\nu} - {}^0g_{\mu\lambda}\xi^\lambda_{,\nu} - {}^0g_{\nu\lambda}\xi^\lambda_{,\mu} + \text{higher order terms}.$$

The metric expressed in the new coordinate system also splits into background and perturbation part as

$$g'_{\mu\nu}(x') = {}^0g_{\mu\nu}(x') + \delta g'_{\mu\nu}, \quad (5.4)$$

while

$${}^0g_{\mu\nu}(x) ={}^0 g_{\mu\nu}(x' - \xi) \simeq {}^0g_{\mu\nu}(x') - {}^0g_{\mu\nu,\lambda}(x)\xi^\lambda + \text{high order terms}, \quad (5.5)$$

so at linear order we find

$$\delta g_{\mu\nu} \mapsto \delta g'_{\mu\nu} = \delta g_{\mu\nu} - {}^0g_{\mu\nu,\lambda}(x)\xi^\lambda - {}^0g_{\mu\lambda}\xi^\lambda_{,\nu} - {}^0g_{\nu\lambda}\xi^\lambda_{,\mu}. \tag{5.6}$$

The same recipe applied to vectors and scalars allows one to write

$$\delta u_\mu \mapsto \delta u'_\mu = \delta u_\mu - {}^0u_{\mu,\lambda}\xi^\lambda - {}^0u_\lambda\xi^\lambda_{,\mu} \tag{5.7}$$

and

$$\delta q \mapsto \delta q' = \delta q - {}^0q_{,\lambda}\xi^\lambda \tag{5.8}$$

where ${}^0u_\lambda$ and ${}^0q$ are the background values. In order to find how the scalar functions ϕ, B, ψ and E transform one must apply (5.6) taking into account that each scalar enters in a different component of the transformation law of the metric. If the traslation vector ξ^μ in $x^\mu \mapsto x^\mu + \xi^\mu$ is of the form

$$\xi^\mu = (\xi^0 = \delta t, \xi^i), \tag{5.9}$$

taking into account that from equation (5.2) one has

$$\delta g_{00} = 2a^2\phi_p, \tag{5.10}$$

then the transformation of the scalar part of the metric is

$$\begin{aligned} \delta g_{00} \mapsto \delta g'_{00} &= \delta g_{00} - {}^0g_{00,\lambda}\xi^\lambda - 2\,{}^0g_{0\lambda}\dot{\xi}^\lambda \\ &= \delta g_{00} - {}^0\dot{g}_{00}\xi^0 - 2\,{}^0g_{00}\dot{\xi}^0 \\ &= \delta g_{00} - 2a\dot{a}\delta t - 2a^2\dot{\delta t} = \delta g_{00} - 2a\tfrac{d}{dt}(a\delta t). \end{aligned} \tag{5.11}$$

because the background metric is diagonal, conformally flat and independent of the space coordinate. This last equation tells us how the scalar perturbation ϕ transforms, in fact

$$2a^2\phi_p \mapsto 2a^2\phi_p - 2a\frac{d}{dt}(a\delta t) \tag{5.12}$$

or

$$\phi_p \mapsto \phi'_p = \phi_p - \frac{1}{a}\frac{d}{dt}(a\delta t). \tag{5.13}$$

Now looking at the transformation law of the vector part of the metric, and splitting the space part of the infinitesimal translation ξ^i as

$$\xi^i = \xi^i_\perp + \zeta^{,i} \tag{5.14}$$

where $\xi^i_\perp$ is divergence free and ζ is a scalar, one finds

$$\delta g_{0i} \mapsto \delta g'_{0i} = \delta g_{0i} - {}^0g_{0i,\lambda}\xi^\lambda - {}^0g_{0\lambda}\xi^\lambda_{,i} - {}^0g_{i\lambda}\xi^\lambda_{,0}, \tag{5.15}$$

that with our diagonal background metric become

$$\begin{aligned} \delta g_{0i} \mapsto \delta g'_{0i} &= \delta g_{0i} - {}^0g_{00}\xi^0_{,i} - {}^0g_{ij}\dot{\xi}^j \\ &= \delta g_{0i} - a^2\xi^0_{,i} + a^2\dot{\xi}^i \\ &= \delta g_{0i} + a^2\big(\dot{\xi}^i_\perp + (\dot{\zeta} - \xi^0)_{,i}\big). \end{aligned} \tag{5.16}$$

After an integration by parts, from

$$\delta g_{0i}\Big|_{\text{scalar}} = a^2 B_{,i} \tag{5.17}$$

one finds

$$B \mapsto B' = B + \dot{\zeta} - \delta t, \tag{5.18}$$

because $\partial_i\dot{\xi}^i_\perp = 0$. The transformation laws of the two last scalars ψ and E follow using the same reasoning and turn out to be

$$\psi \mapsto \psi' = \psi + \mathcal{H}\delta t, \tag{5.19}$$

and

$$E \mapsto E' = E + \zeta, \tag{5.20}$$

where $\mathcal{H} = da/ad\eta$.These four relations are functions only of the two parameters δt and ζ. It is easy to see that if one chooses two of them as

$$E = -\zeta \qquad \text{and} \qquad B = \delta t - \dot{\zeta} = \delta t + \dot{E}, \tag{5.21}$$

then $E' = 0$ and $B' = 0$ while the other two become

$$\Phi = \phi_p - \frac{1}{a}\big(a(B - \dot{E})\big)^{\bullet} \tag{5.22}$$

and

$$\Psi = \psi - \mathcal{H}(B - \dot{E}), \tag{5.23}$$

the two Bardeen potentials. The latter are two gauge-invariant quantities. If both are zero in a coordinate system then they will be zero in any coordinate system and in this case the perturbations are called fictitious because these are the result of the particular coordinate system chosen. In the framework of cosmological perturbation different gauges exists and for example in the case of scalar perturbations, the Longitudinal (conformal-Newtonian) gauge is defined by the condition $\zeta = -E_l$ and $\delta t - \dot{E}_l = B_l = 0$. In this frame the metric takes the form

$$ds^2 = a^2[(1 + 2\phi_l)d\eta^2 - (1 - 2\psi_l)\delta_{ij})dx^i dx^j], \tag{5.24}$$

where only the two potentials $\Phi = \phi_p \equiv \phi_l$ and $\Psi = \psi \equiv \psi_l$ appear. Moreover, in the longitudinal gauge these two functions are gauge invariant. The latter metric simplifies if one has a stress-energy tensor with diagonal space part, in fact in that case the two functions ϕ_l and ψ_l are just the same function.

5.2 Cosmological perturbations of Mimetic Gravity

Perturbations of the metric induce a perturbation of the Einstein tensor $G_{\mu\nu}$ that can be expanded as,

$$G^\mu_\nu = {}^0G^\mu_\nu + \delta G^\mu_\nu + \cdots, \tag{5.25}$$

where δG^μ_ν is linear in metric perturbations and from this, a linearized version of Einstein equations reads

$$\delta G^\mu_\nu = \delta T^\mu_\nu, \tag{5.26}$$

where δT^{μ}_{ν} is the linear part of the perturbed stress-energy tensor. The calculation in the flat background gives

$$^{0}G^{0}_{0} = \frac{3\mathcal{H}^2}{a^2}, \qquad ^{0}G^{0}_{i} = 0 \qquad \text{and} \qquad ^{0}G^{i}_{j} = \frac{1}{a^2}\left(2\dot{\mathcal{H}} + 3\mathcal{H}^2\right)\delta^{i}_{j} \tag{5.27}$$

where $\dot{\mathcal{H}} = d\mathcal{H}/d\eta$. Consider a longitudinal Newtonian gauge in the presence of a diagonal stress-energy tensor, then the gauge invariant quantities Φ and Ψ can be identified and the metric in this case is

$$ds^2 = (1+2\Phi)dt^2 - (1-2\Phi)a^2\delta_{ij}dx^i dx^j. \tag{5.28}$$

Giving a small perturbation to the scalar field solution of the mimetic model of the last Chapter, using the constraint we see that one can choose the flat background solution as $\phi_0 \equiv t$ and so $\phi = t + \delta\phi$. Using the constraint (4.8) we find that

$$g^{00}(\dot{\phi})^2 = 1 \qquad \text{or} \qquad (1+2\Phi)^{-1}(1+\dot{\delta\phi})^2 = 1, \tag{5.29}$$

so we have

$$\Phi = \dot{\delta\phi}. \tag{5.30}$$

Thus, time derivative of the small perturbation of the scalar degree of freedom of the metric in the mimetic model can be identified with the gauge-invariant newtonian potential Φ in the longitudinal gauge.

An evolution equation for this quantity can be extracted from the linearized time-space component of the Einstein equations. Because the metric is diagonal the only contribution to the Einstein tensor G_{0i} is given by

$$R_{0i} = 2\partial_i(\dot{\Phi} + H\Phi), \tag{5.31}$$

while from the perturbation of a perfect fluid stress energy tensor we find that the time-space component can be written at linear order as

$$\delta T^{0}_{i} = (\rho + p)\partial_i\delta\phi. \tag{5.32}$$

Recalling that for mimetic dark matter the following relations hold,

$$2\dot{H} + 3H^2 = V = -p, \qquad \rho = 3H^2 \qquad \Rightarrow \qquad \rho + p = -2\dot{H}, \tag{5.33}$$

we see that combining (5.31) with (5.32) one get

$$\partial_i\left(\dot{\Phi} + H\Phi\right) = \frac{1}{2}(\rho + p)\partial_i\delta\phi \tag{5.34}$$

or

$$\ddot{\delta\phi} + H\dot{\delta\phi} + \dot{H}\delta\phi = 0 \tag{5.35}$$

because $\Phi = \dot{\delta\phi}$. Solution of this differential equation is of the form

$$\delta\phi = const. \times \frac{1}{a}\int a dt \tag{5.36}$$

and so the newtonian gravitational potential is given by

$$\Phi = \dot{\delta\phi} = const. \times \left(1 - \frac{H}{a}\int a dt\right). \tag{5.37}$$

The above solution is valid for every perturbation irrespective of its wavelength and, as pointed out in [2] it is the same one would obtain for the long wavelength solution when negleting the spatial derivative term for a hydrodynamical fluid. The mentioned spatial derivative term is usually multiplied by the speed of sound and in this sense we see that perturbations behave as dust with vanishing speed of sound even for mimetic dark matter with nonvanishing pressure. This turns out to be a problem if one wants to use mimetic matter, for example as an inflationary mechanism, because quantum fluctuation cannot be defined in the usual way. There are two ways to solve this problem: adding one more scalar degree of freedom making the theory not very plausible because such a theory can "explain" nearly everything and predict nothing, or slightly modify the action, for example by adding a higher derivative term for the scalar degree of freedom ϕ of the metric.

Recalling the result of the previous Chapter about the modification of the Mimetic action with the term $\propto (\Box\phi)^2$, the symmetric stress-energy tensor was quoted to be

$$T^\mu_\nu = \Big(V + \gamma\big(\partial_\sigma\phi\partial^\sigma\chi + \frac{1}{2}\chi^2\big)\Big)\delta^\mu_\nu + 2\lambda\partial_\nu\phi\partial^\mu\phi - \gamma(\partial_\nu\phi\partial^\mu\chi + \partial_\nu\chi\partial^\mu\phi). \tag{5.38}$$

In order to write the scalar field equations at linear order in perturbations, we need the linear order perturbation of the Einstein tensor - for a metric written in a particular gauge - and the perturbed energy-momentum tensor. From equation (5.38) one can find that, at linear order in perturbations, the energy-momentum tensor is

$$\delta T^0_i = 2\lambda\partial_i(t+\delta\phi)\frac{d}{dt}(t+\delta\phi) - \gamma(\partial_i(t+\delta\phi)\frac{d}{dt}(\chi+\delta\chi) + \partial_i(\chi+\delta\chi)(t+\delta\phi)) \tag{5.39}$$

or

$$\delta T^0_i = 2\lambda\partial_i\delta\phi - 3\gamma\dot{H}\partial_i\delta\phi - \gamma\partial_i\delta\chi, \tag{5.40}$$

using $\chi = 3H$. The quantity $\delta\chi = \delta(\Box\phi)$ must be evaluated at linear order when the metric that defines the operator

$$\Box\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \tag{5.41}$$

is given by the longitudinal Newtonian gauge. Taking into account that $-g = a^6(1 + 2\Phi)(1-2\Phi)^3$ and $g^{00} \simeq (1-2\Phi)$ while $g^{ij} \simeq -\frac{1}{a^2}(1+2\Phi)\delta^{ij}$, after some calculations one finds

$$\delta\chi = -3\ddot{\delta\phi} - 3H\dot{\delta\phi} - \frac{\Delta}{a^2}\delta\phi. \tag{5.42}$$

Using

$$\lambda = (3\gamma - 1)\dot{H} \tag{5.43}$$

which follows from (4.47) and (4.48), we see that equation (5.35), when the kinetic term χ is taken into account, is modified as

$$\ddot{\delta\phi} + H\dot{\delta\phi} - \frac{c_s^2}{a^2}\Delta\delta\phi + \dot{H}\delta\phi = 0. \tag{5.44}$$

The presence of the kinetic term drastically changes the equation that $\delta\phi$ must solve, in fact now a second-order spatial derivative term appears and the speed of sound is given by

$$c_s^2 = \frac{\gamma}{2 - 3\gamma}. \tag{5.45}$$

In the following chapter we will find that in order to avoid Laplacian instabilities, i.e. a wrong sign in front of the Laplacian Δ in (5.44), one must require $c_s^2 > 0$ that in this case is satisfied when $0 < \gamma < 2/3$.

Chapter 6

Effective Field Theories

6.1 Second order equations of motion

Each theories of interactions follows from an action built with a Lagrangian. Looking at the different interactions in nature, at first sight it seems that each theory is built in a way to have second order equations of motion. There is a reason for this. Higher order derivative theories suffer the plague of the appearance of ghost modes, i.e. states with negative energy. These states lead to a catastrophic production of normal and ghost fields out from the vacuum invalidating the theory itself. This problem, as first pointed out by *Ostrogradski*, arises because, in the case of higher derivative order theories, the Hamiltonian of the system is linear in one of its conjugate canonical momenta and then not bounded from below. Considering at the classical level, in one dimension, a Lagrangian function of N time derivatives

$$L = L(q, \dot{q}, \ddot{q}, \ldots, q^{(N)}), \tag{6.1}$$

it is easy to show that in this case the *Euler-Lagrange* equation of motion (EOM) reads

$$\sum_{k=0}^{N} (-1)^k \frac{d^k}{dt^k}\Big(\frac{\partial L}{\partial q^{(k)}}\Big) = 0. \tag{6.2}$$

Cleary, if the Lagrangian is non-degenerate, $\det \frac{\partial^2 L}{\partial q^{(N)} \partial q^{(N)}} \neq 0$, then (6.2) gives rise to $2N$-order EOM and in order to write a solution one needs $2N$ initial conditions $(q, \dot{q}, \ddot{q}, \ldots, q^{(2N-1)})_0$. The number of initial conditions is the same as the dimension of the phase-space of canonical conjugate coordinates

$$Q = (Q_1 = q, Q_2 = \dot{q}, Q_3 = \ddot{q}, \ldots, Q_N = q^{(N-1)}) \tag{6.3}$$

and

$$P = (P_N = \frac{\partial L}{\partial \dot{Q}_N}, P_{N-1} = \frac{\partial L}{\partial \dot{Q}_{N-1}} - \frac{d}{dt} P_N, \ldots). \tag{6.4}$$

The Hamiltonian H is the *Legendre* transform of L, defined in the usual manner as

$$H = \sum_i P_i \dot{Q}_i - L \tag{6.5}$$

where one has to understand the latter as a function of Qs and Ps, once the relations between each P_i and $\dot{Q}_i$ have been inverted. Consider for simplicity the case $N = 2$, then the Lagrangian is $L(q, \dot{q}, \ddot{q})$ and the EOM reads

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\Big) = 0, \tag{6.6}$$

which require the four initial conditions $(q, \dot{q}, \ddot{q}, q^{(3)})_0$ if the Lagrangian is not degenerate. The canonical variables are

$$Q_1 = q, Q_2 = \dot{q}, P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, P_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{6.7}$$

It is clear that $\ddot{q}$ can be inverted as a function of Q_1, Q_2 and P_2 only, and so P_1 appears only in the first term of the Hamiltonian meaning that there is a linear dependence on P_1 and so a non-bounded energy from below. This instability on its own is not a bad thing. It becomes bad when interactions with other degrees of freedom whose Hamiltonians are bounded from below are introduced. The presence of these negative energy states means that there exists a vast phase space where the Hamiltonian is negative, hence the modes will begin to populate them by entropic arguments alone while, by conservation of energy, creating an equally large number of positive energy modes in the interacting d.o.f.. This is the onset of the instability. Note that, while this is a classical instability, in quantum theory, negative energy modes are particularly sick - attempts to canonically quantize them will either lead to negative norm (and hence undefined) states or negative energy states (and hence runaway particle production). Since negative norm states are often called "ghosts" in quantum theory, higher derivative theories are often called "ghost-like".

As pointed out by *Dirac*, one might try to eliminate the instability by imposing constraints, i.e. one selectively restricts the trajectories of the d.o.f. such that the Hamiltonian becomes bounded from below. Those constraints for example can follows from the fact that not all the relations between conjugate momenta can be inverted to give $\dot{q}_i$ in terms of p_i. Then the theory has primary constraints $\Phi(q_i, p_i) = 0$ solely by virtue of the form of the Lagrangian. As Dirac noted [12], in such a case a theory described by a Hamiltonian $H(q_i, p_i)$ could just as well be described by a Hamiltonian $H_{\text{total}} = H + u_i \Phi_i$ for arbitrary functions u_i. The implementation of constraints into the theory requires the introduction of auxiliary variables and hence the enlargement of the total phase space. As a consequence, one may hope to change the orbits of the trajectories of the theory to a degree which is sufficient to cure it from the instability. Using a fourth order theory example [13], one can imagine a modification

$$S = \int dt \Big(\ddot{q}^2/2 - \alpha(q) + \lambda f(q, \dot{q}, \ddot{q}) \Big) \tag{6.8}$$

where $\alpha(q)$ is a potential and λ is an auxiliary field which enforces the constraint $f(q, \dot{q}, \ddot{q}) = 0$. We emphasize that the latter action is a different physical theory from the case in which no constraint is present, as long as the constraint cannot be gauged away. The question is: can f be cleverly chosen in such a way that this theory, despite being a higher derivative theory, is free of the linear instability? If these constraints are first class, or second class and give rise to a secondary class constraints, then they will remove the spurious degrees of freedom associated with the higher derivatives appearing in the action. Second class constraints are "physical" in the sense that the solutions to the equations of motion are different with or without the constraint. On the other hand, first class constraints are those associated with some gauge freedom in the theory, i.e. the solutions of the equations of motion contain some arbitrary functions of time and hence describe physically equivalent systems [14]. As shown in [13], one can "gauge fix" such theories - these so-called "gauge fixing" functions appear as new (primary) constraints in the theory, and once introduced the original first class constraint and the new gauge fixing constraint both become second class constraints. The most general second order time derivative Lagrangian with one auxiliary field λ is given by a Lagrangian $L(q, \dot{q}, \ddot{q}, \lambda)$. Calling the generalizated coordinates $Q_1 = q$, $Q_2 = \dot{q}$, $Q_3 = \lambda$, the consequent conjugate momenta are $P_1 = \partial L/\partial \dot{q} - \frac{d}{dt}\partial L/\partial \ddot{q}$, $P_2 = \partial L/\partial \ddot{q}$ and since the Lagrangian does not depend on $\dot{lambda}$ we have $P_3 = \partial L/\partial \dot{\lambda} = 0$. The primary constraint, from now calledΦ_1, has the following functional form $P_3 = 0$. The assumption of non-degeneracy $\det \partial^2 L/\partial \ddot{q}^2 \neq 0$ allows us to use the definition of P_2 to invert the relation and writing $\ddot{q} = h(Q_1, Q_2, Q_3, P_2)$. Then the total Hamiltonian becomes

$$H_T = P_1 Q_1 + P_2 h(Q_1, Q_2, Q_3, P_2) - L(Q_1, Q_2, Q_3, h) + u_1 \Phi_1, \tag{6.9}$$

where $\Phi_1 \equiv P_3$ is the primary constraint while u_1 is the Lagrange multiplier that enforce the condition $P_3 = 0$. Since $P_3 = 0$, consistency implies that its equation of motion $\dot{P}_3 = [P_3, H_T]$ must also vanish (on constraint) - this leads to a series of consistency relations which allow us to find further constraints called secondary constraints. In this case, there exists one further secondary constraint as expected (the conservation under time evolution of the primary constraint Φ_1), which is

$$\Phi_2 : [\Phi_1, H_T]_{\text{poisson bracket}} = -P_2 \frac{\partial h}{\partial Q_3} + \frac{\partial h}{\partial Q_3}\left[\frac{\partial L}{\partial \ddot{q}}\right]_{\ddot{q}=h} + \frac{\partial L}{\partial \lambda}\Big|_{\lambda=Q_3} = \frac{\partial L}{\partial \lambda}\Big|_{\lambda=Q_3} \simeq 0 \tag{6.10}$$

Here we introduce the weak equality symbol $\simeq$ for the constraint equations. The constraint equation is written as $\Phi_2 \simeq 0$, which means Φ_2 is numerically restricted to be zero but does not identically vanish throughout phase space. I.e. Φ_2 only vanishes on the hypersurface where all the constraints are satisfied.

6.2 Effective Field Theory methods

Effective Field Theory methods in the framework of cosmological perturbations [17] rely on the variation of an action expanded up to second order about its geometrical and physical variables when the unitary gauge is employed. The former variables are scalars of various type accounting for the geometry of the hypersurface at constant time Σ_t, while the latter in general can be several scalars degrees of freedom associated to gravity, for example the scalar degree of freedom ϕ of the mimetic model. Once the second order variation of the action is calculated, it is possible to impose conditions under which the model would be free of ghosts arising from a wrong sign in front of the Laplacian or from higher derivative action giving equations of motion of order higher than two, avoiding in this latter case the so called *Ostrogradski* instabilities.

Strictly speaking Effective Field Theory methods are usually employed in particle physics [15] as well as in the context of inflation [16]. Those methods are used in order to describe the nature of interactions at some particular energy scale ignoring what happen at other scales. This mathematical framework automatically limits the role which smaller distance scales can play in the description of larger objects. In the context of inflation when one has to deal with the inflaton, a scalar field, and one see that the scalar mode can be eaten by the metric by going to unitary gauge. This is analogous to what happens in a spontaneously broken gauge theory where a *Goldstone* mode, which transforms non-linearly under the gauge symmetry, can be eaten by the gauge boson (unitary gauge) to give a massive spin one particle. The usual way to study a single field inflationary model is to start from a Lagrangian for a scalar field ϕ and solve the equation of motion for ϕ together with the Friedmann equations for the FLRW metric. One is usually interested in an inflating solution, i.e. an accelerated expansion with a slowly varying Hubble parameter, with the scalar following an homogeneous time-dependent solution $\phi_0(t)$. At this point one studies perturbations around this background solution to work out the predictions for the various cosmological observables. The theory of perturbations around the time evolving solution is quite different from the theory of ϕ one started with: while ϕ is a scalar under all diffeomorphisms, the perturbation $\delta\phi$ is a scalar only under spatial diffeomorphisms while it transforms non-linearly with respect to time diffeomorphisms:

$$t \mapsto t + \xi^0(x,t), \qquad \delta\phi \mapsto \delta\phi + \dot{\phi}_0(t)\xi^0.$$

In particular one can choose a gauge $\phi(t,x) = \phi_0(t)$ where there are no inflaton perturbations, but all degrees of freedom are in the metric. The scalar variable $\delta\phi$ has been eaten by the graviton, which has now three degrees of freedom: the scalar mode and the two tensor helicities.

This quasi de Sitter background has a privileged spatial slicing, given by a physical clock which allows to smoothly connect to a decelerated hot Big Bang evolution. The slicing is usually realized by a time evolving scalar $\phi(t)$. To describe perturbations around this solution one can choose a gauge where the privileged slicing coincides with surfaces of constant t, i.e. the unitary gauge $\delta\phi(x,t) = 0$. In this gauge there are no explicit scalar perturbations, but only metric fluctuations. As time diffeomorphisms have been fixed and are not a gauge symmetry anymore, the graviton now describes three degrees of freedom: the scalar perturbation it is said to has been "eaten" by the metric.

As pointed out in [16], in the context of inflation, starting from a scenario of inflation with a scalar field with minimal kinetic term and slow-roll potential, one parameterize our ignorance about all the possible high energy effects in terms of the leading invariant operators. Experiments will put bounds on the various operators, for example with measurements of the non-Gaussianity of perturbations and studying the deviation from the consistency relation for the gravitational wave tilt. In some sense this is similar to what one does in particle physics, where one puts constraints on the size of the operators that describe deviations from the Standard Model and thus encode the effect of new physics. This is the standard definition of EFT in particle physics and inflationary models.

6.2.1 The Geometry of the hypersurface at constant time

Following [17] at constant time we choose the hypersurface Σ_t and so the induced metric is given by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{6.11}$$

where $n_\mu = -N t_{,\mu} = (-N, \vec{0})$ is a vector orthogonal to Σ_t. The metric $g_{\mu\nu}$ can be parametrized as

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{6.12}$$

where $n^\mu = g^{\mu\nu} n_\nu = (1/N, N^i/N)$ while N and N^i are the lapse function and the shift vector respectively. The lapse N is the change in proper time as one moves off the spatial surface and the shift N^i is the displacement in identification of the spatial coordinates between two adjacent slices. Clearly $n_\mu n^\mu = -1$ and so $n^\mu h_{\mu\nu} = 0$. The extrinsic curvature, the acceleration of the hypersurface, is defined by

$$K_{\mu\nu} = h^{\lambda}_{\mu} n_{\nu,\lambda} = (g^{\lambda}_{\mu} + n^{\lambda} n_{\mu}) n_{\nu,\lambda} = n_{\nu,\mu} + n_{\mu} a_{\nu}, \tag{6.13}$$

a_ν being the normal acceleration of the vector n_μ; in addition, the relation $n^\mu K_{\mu\nu} = 0$ ensures that the extrinsic curvature is a quantity on Σ_t. The extrinsic Ricci tensor $\mathcal{R}_{\mu\nu} \equiv {}^{(3)}R_{\mu\nu}$ associated to the spatial part of the induced metric $h_{\mu\nu}$ define the internal geometry of Σ_t. The Ricci scalar $\mathcal{R} = \mathcal{R}^{\mu}_{\mu}$, is related to the four dimensional Ricci scalar by the decomposition

$$R = \mathcal{R} - K_{\mu\nu} K^{\mu\nu} - K^2 + 2(K n^{\mu} - a^{\mu})_{,\mu}. \tag{6.14}$$

Alongside with the lapse function N, several geometric scalars such as

$$N, \qquad K = K^{\mu}_{\nu}, \qquad \mathcal{S} = K_{\mu\nu} K^{\mu\nu}, \qquad \mathcal{R} = \mathcal{R}^{\mu}_{\mu}, \qquad \mathcal{Z} = \mathcal{R}_{\mu\mu} \mathcal{R}^{\mu\nu}, \qquad \mathcal{U} = \mathcal{R}_{\mu\nu} K^{\mu\nu} \tag{6.15}$$

can be defined and the action of general gravitational theories that depends on these scalars, that encodes the geometry of the hypersurface at constant time Σ_t, is given by

$$S = \int d^4x \sqrt{-g} L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \mathcal{U}; t). \tag{6.16}$$

In what follows, $L_N, L_K, \ldots$ represent partial derivatives of the lagrangian with respect to $N, K, \ldots$. The latter action does not contain any scalar related to the shift vector N^i while ϕ the scalar degree of freedom of the metric (see below) and its kinetic term X, depending on N and t, enters the equations of motion through L_N and L_{NN}. Let us consider four scalar perturbations

$$(A, \psi, \zeta, E)$$

about a FLRW background with scale factor $a(t)$ described by the perturbed metric

$$ds^2 = -e^{2A} dt^2 + 2\nabla^{(h)}_i \psi dx^i dt + a^2 \Big(e^{2\zeta} \delta_{ij} + (\nabla^{(h)}_i \nabla^{(h)}_j - \frac{\delta_{ij}}{3} \Delta^{(h)}) E \Big) dx^i dx^j \tag{6.17}$$

where $\nabla_i^{(h)}$ is the covariant derivative build with the metric h_{ij}. The scalar perturbation ζ is the so called curvature perturbation. Under the perturbation realized by

$$\xi^\alpha = (\delta t, \delta^{ij}\partial_j \delta x) \tag{6.18}$$

the scalar fields ϕ of modified gravitational theories with a single scalar degree of freedom associated to the metric (as for example in mimetic gravity) and the scalar perturbation E transforms, according to (5.8) and (5.6), as

$$\delta\phi \mapsto \delta\phi - \dot{\phi}\delta t, \qquad E \mapsto E - \delta x, \tag{6.19}$$

and so the unitary gauge $\delta\phi = 0$ fixes the time slicing δt while the choice $E = 0$ fixes the spatial threading δx and allows us to concentrate on the scalar perturbation ζ of the metric. Thus the hypersurface at constant time Σ_t coincide with the constant ϕ hypersurface. As already stated, in this unitary gauge, the scalar degree of freedom associated with ϕ is eaten by the metric and so the Lagrangian in (6.16) does not have an explicit dependence on ϕ for a flat background. For a FLRW background metric $\bar{h}_{\mu\nu}$, the three dimensional geometric quantities already defined are given by

$$\bar{K}_{\mu\nu} = H\bar{h}_{\mu\nu}, \qquad \bar{K} = 3H, \qquad \mathcal{S} = 3H^2, \qquad \bar{\mathcal{R}}_{\mu\nu} = 0, \qquad \bar{\mathcal{R}} = \bar{\mathcal{Z}} = \bar{\mathcal{U}} = 0 \tag{6.20}$$

where $H = \dot{a}/a$, while the perturbations of these can be written as

$$\delta K^\mu_\nu = K^\mu_\nu - Hh^\mu_\nu, \qquad \delta K = K - 3H, \qquad \delta S = S - 3H^2. \tag{6.21}$$

Using the definition of S and the first and second relations of the latter equations, one sees that the perturbation of S can be rewritten as

$$\delta S = 2H\delta K + \delta K^\mu_\nu K^\nu_\mu, \tag{6.22}$$

while given that the quantities $\mathcal{R}$ and $\mathcal{Z}$ vanish in the background, they appear only as a perturbation that can be written as

$$\delta\mathcal{R} = \delta_1\mathcal{R} + \delta_2\mathcal{R}, \qquad \delta\mathcal{Z} = \delta\mathcal{R}^\mu_\nu \mathcal{R}^\nu_\mu. \tag{6.23}$$

From the last definition in (6.15), using (6.21) one finds that

$$\mathcal{U} = \mathcal{R}_{\mu\nu}K^{\mu\nu} = H\mathcal{R} + \mathcal{R}^{\mu}_{\nu}\delta K^{\nu}_{\mu}, \tag{6.24}$$

clearly the second term is a second order quantity.

Expanding the Lagrangian up to quadratic order in the perturbations one finds

$$\begin{aligned} L = \bar{L} + L_N\delta N + L_K\delta K + L_{\mathcal{S}}\delta\mathcal{S} + L_{\mathcal{R}}\delta\mathcal{R} + L_{\mathcal{Z}}\delta\mathcal{Z} + L_{\mathcal{U}}\delta\mathcal{U} + \\ +\tfrac{1}{2}\left(\delta N\tfrac{\partial}{\partial N} + \delta K\tfrac{\partial}{\partial K} + \delta S\tfrac{\partial}{\partial S} + \delta\mathcal{R}\tfrac{\partial}{\partial\mathcal{R}} + \delta\mathcal{Z}\tfrac{\partial}{\partial\mathcal{Z}} + \delta\mathcal{U}\tfrac{\partial}{\partial\mathcal{U}}\right)^2 L. \end{aligned} \tag{6.25}$$

Defining

$$\mathcal{F} = L_K + 2HL_S \tag{6.26}$$

one finds that the third and the fourth terms of the latter Lagrangian become

$$L_K\delta K + L_{\mathcal{S}}\delta\mathcal{S} = \mathcal{F}(K - 3H) + L_{\mathcal{S}}\delta K^{\mu}_{\nu}K^{\nu}_{\mu} \simeq -\dot{\mathcal{F}} - 3H\mathcal{F} + \dot{\mathcal{F}}\delta N + L_{\mathcal{S}}\delta K^{\mu}_{\nu}K^{\nu}_{\mu} - \dot{\mathcal{F}}\delta N^2, \tag{6.27}$$

where the last equality comes after an integration by parts of $\mathcal{F}K = \mathcal{F}n^{\mu}_{,\mu}$ and an expansion of $\frac{1}{N} = \frac{1}{1+\delta N}$ up to second order. On the other hand it can be shown that the first order contribution of the last term of (6.25) is equal to[1]

$$L_{\mathcal{U}}\delta\mathcal{U} = \frac{1}{2}\left(\dot{L}_{\mathcal{U}} + 3HL_{\mathcal{U}}\right)\delta_1\mathcal{R}. \tag{6.28}$$

Defining

$$\mathcal{E} = L_{\mathcal{R}} + \frac{1}{2}\dot{L}_{\mathcal{U}} + \frac{3}{2}HL_{\mathcal{U}} \tag{6.29}$$

up to first order one finds

$$L_0 = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F}, \qquad L_1 = (\dot{\mathcal{F}} + L_N)\delta N + \mathcal{E}\delta_1\mathcal{R}. \tag{6.30}$$

Remembering that $\mathcal{L} = \sqrt{-g}L$, and so

$$\delta\mathcal{L} = \sqrt{h}L\delta N + NL\delta\sqrt{h} + N\sqrt{h}\delta L, \qquad h = \det h_{ij}, \tag{6.31}$$

[1] Using the relation $2\alpha(t)\mathcal{U} = \alpha(t)\mathcal{R}K + \frac{1}{N}\dot{\alpha}(t)\mathcal{R}$, where $\alpha(t)$ is an arbitrary function of time.

neglecting second-order corrections one has

$$\mathcal{L}_0 = a^3 L_0 = a^3\Big(\bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F}\Big), \qquad \mathcal{L}_1 = a^3(\bar{L} + L_N - 3H\mathcal{F})\delta N + N L_0 \delta\sqrt{h} + a^3\mathcal{E}\delta_1\mathcal{R}. \tag{6.32}$$

Variations with respect to N and $\sqrt{h}$ of $\mathcal{L}_1$ gives the following equations of motion

$$\bar{L} + L_N - 3H\mathcal{F} = 0, \qquad L_0 = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} = 0; \tag{6.33}$$

the difference of these equations gives

$$L_N + \dot{\mathcal{F}} = 0. \tag{6.34}$$

Two of the last three equations are sufficient to determine the cosmological dynamics on the flat FLRW background.

6.2.2 Expansion of the action up to second order

From equations (6.21) one finds the following second-order variation of the variables

$$\delta\mathcal{S}^2 = 4H^2\delta K^2, \qquad \delta K\delta\mathcal{S} = 2H\delta K^2, \qquad \delta\mathcal{S}\delta N = 2H\delta K\delta N. \tag{6.35}$$

Furthermore, the second-order expansion of $\mathcal{U}$

$$\delta\mathcal{U}\Big|_{2^\circ ord.} = \frac{1}{2}\Big(L_{\mathcal{U}}\delta K - \dot{L}_{\mathcal{U}}\delta N\Big)\delta_1\mathcal{R} + \frac{1}{2}\Big(\dot{L}_{\mathcal{U}} + 3HL_{\mathcal{U}}\Big)\delta_2\mathcal{R}, \tag{6.36}$$

allows one to write the expansion of the lagrangian (6.25) up-to-second-order as

$$\begin{aligned} L = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_N)\delta N + \mathcal{E}\delta_1\mathcal{R} + \\ +\Big(\frac{1}{2}L_{NN} - \dot{\mathcal{F}}\Big)\delta N^2 + \frac{1}{2}\mathcal{A}\delta K^2 + \mathcal{B}\delta K\delta N + \mathcal{C}\delta K\delta_1\mathcal{R} + \mathcal{D}\delta N\delta_1\mathcal{R} + \\ +\mathcal{E}\delta_2\mathcal{R} + \frac{1}{2}\mathcal{G}\delta_1\mathcal{R}^2 + L_{\mathcal{S}}\delta K_{\mu\nu}\delta K^{\mu\nu} + L_{\mathcal{Z}}\delta\mathcal{R}_{\mu\nu}\delta\mathcal{R}^{\mu\nu}, \end{aligned} \tag{6.37}$$

when the following definitions are taken into account

$$\mathcal{A} = L_{KK} + 4HL_{\mathcal{S}K} + 4H^2 L_{SS}, \qquad \mathcal{B} = L_{KN} + 2HL_{\mathcal{S}N}$$

$$\mathcal{C} = L_{K\mathcal{R}} + 2HL_{\mathcal{S}\mathcal{R}} + \frac{1}{2}L_{\mathcal{U}} + HL_{K\mathcal{U}} + 2H^2L_{\mathcal{S}\mathcal{U}}$$
$$\mathcal{D} = L_{N\mathcal{R}} + \frac{1}{2}\dot{L}_{\mathcal{U}} + HL_{N\mathcal{U}}, \qquad \mathcal{G} = L_{\mathcal{R}\mathcal{R}} + 2HL_{\mathcal{R}\mathcal{U}} + H^2L_{\mathcal{U}\mathcal{U}}. \tag{6.38}$$

Then, expansion of the Lagrangian density up-to-second order is

$$\mathcal{L}_2 = \delta\sqrt{h}\Big[(\dot{\mathcal{F}} + L_N) + \mathcal{E}\delta_1\mathcal{R}\Big] + \tag{6.39}$$
$$+a^3\Big[\Big(L_N + \frac{1}{2}L_{NN}\Big)\delta N^2 + \mathcal{E}\delta_2\mathcal{R} + \frac{1}{2}\mathcal{A}\delta K^2 + \mathcal{B}\delta K\delta N + \mathcal{C}\delta K\delta_1\mathcal{R} +$$
$$+(\mathcal{D}+\mathcal{E})\delta N\delta_1\mathcal{R} + \frac{1}{2}\mathcal{G}\delta_1\mathcal{R}^2 + L_{\mathcal{S}}\delta K_{\mu\nu}\delta K^{\mu\nu} + L_{\mathcal{Z}}\delta\mathcal{R}_{\mu\nu}\delta\mathcal{R}^{\mu\nu}\Big].$$

From the gauge choice (6.17) with $E = 0$, the three dimensional induced metric is $h_{ij} = a^2(t)e^{2\zeta}\delta_{ij}$ and so the perturbations of the determinant and of the extrinsic three dimensional Ricci tensor and scalar can be expressed as

$$\delta\sqrt{h} = 3a^3\zeta, \qquad \delta\mathcal{R}_{ij} = -(\delta_{ij}\partial^2\zeta + \partial_i\partial_j\zeta),$$
$$\delta_1\mathcal{R} = -4\frac{\partial^2\zeta}{a^2}, \qquad \delta_2\mathcal{R} = -\frac{2}{a^2}[(\partial\zeta)^2 - 4\zeta\partial^2\zeta], \tag{6.40}$$

while it can be shown that the extrinsic curvature can be written as

$$K_{ij} = \frac{1}{N}\Big(\dot{h}_{ij} - \nabla_i^{(h)}N_j - \nabla_j^{(h)}N_i\Big) \tag{6.41}$$

and so for the perturbed metric (6.17) the first-order extrinsic curvature reads

$$\delta K^i_j = \Big(\dot{\zeta} - H\delta N\Big)\delta^i_j - \frac{1}{2a^2}\delta^{ik}(\partial_k N_i + \partial_i N_k). \tag{6.42}$$

Since the shift vector is related to the metric perturbation ψ via $N_i = \partial_i\psi$ the trace of K_{ij} can be expressed as

$$\delta K = 3\Big(\dot{\zeta} - H\delta N\Big) - \frac{1}{a^2}\partial^2\psi, \tag{6.43}$$

exhibiting the dependence of δK on δN. On using (6.40), (6.42) and (6.43), the second order Lagrangian density (6.39), up to boundary terms and using the background equation $L_N + \dot{\mathcal{F}} = 0$, reduces to

$$\mathcal{L}_2 = a^3\Big\{\frac{1}{2}(2L_N + L_{NN} + 9\mathcal{A}H^2 - 6\mathcal{B}H + 6L_{\mathcal{S}}H^2)\delta N^2 +$$

$$+\Big[(\mathcal{B}-3\mathcal{A}H-2L_{\mathcal{S}}H)\Big(3\dot{\zeta}-\frac{\partial^2\psi}{a^2}\Big)+4(3H\mathcal{C}-\mathcal{D}-\mathcal{E})\frac{\partial^2\zeta}{a^2}\Big]\delta N+$$
$$-(3\mathcal{A}+2L_{\mathcal{S}})\dot{\zeta}\frac{\partial^2\psi}{a^2}-12\mathcal{C}\dot{\zeta}\frac{\partial^2\zeta}{a^2}+\Big(\frac{9}{2}\mathcal{A}+3L_{\mathcal{S}}\Big)\dot{\zeta}^2+2\mathcal{E}\frac{(\partial\zeta)^2}{a^2}+$$
$$+\frac{1}{2}(\mathcal{A}+2L_{\mathcal{S}})\frac{(\partial^2\psi)^2}{a^2}+4\mathcal{C}\frac{(\partial^2\psi)(\partial^2\zeta)}{a^2}+2(4\mathcal{G}+3L_{\mathcal{Z}})\frac{(\partial^2\zeta)^2}{a^2}\Big\}. \tag{6.44}$$

Defining the following quantity

$$\mathcal{W}=\mathcal{B}-3\mathcal{A}H-2L_{\mathcal{S}}H, \tag{6.45}$$

then the variation of (6.44) with respect to δN and $\partial^2\psi$ leads to the following Hamiltonian and momentum constraints, respectively:

$$\big[2L_N+L_{NN}-6H\mathcal{W}-3H^2(3\mathcal{A}+2L_{\mathcal{S}})\big]\delta N+$$
$$-\mathcal{W}\frac{\partial^2\psi}{a^2}+3\mathcal{W}\dot{\zeta}+4(3H\mathcal{C}-\mathcal{D}-\mathcal{E})\frac{\partial^2\zeta}{a^2}=0 \tag{6.46}$$
$$\mathcal{W}\delta N-(\mathcal{A}+2L_{\mathcal{S}})\frac{(\partial^2\psi)^2}{a^2}+(3\mathcal{A}+2L_{\mathcal{S}})\dot{\zeta}-4\mathcal{C}\frac{(\partial^2\zeta)}{a^2}=0. \tag{6.47}$$

These two constraints give δN and $\partial^2\psi/a^2$ in terms of $\dot{\zeta}$ and $\partial^2\zeta/a^2$. An important point is that if one imposes the vanishing of the coefficients of the last three terms of (6.44)

$$\mathcal{A}+2L_{\mathcal{S}}=0,\qquad \mathcal{C}=0,\qquad 4\mathcal{G}+3L_{\mathcal{Z}}=0, \tag{6.48}$$

then the consequent equations of motion will be at most of second order.

Inverting the equations of motion as a function of $\dot{\zeta}$ and $\partial^2\zeta/a^2$ and inserting the results into the second order Lagrangian density (6.44) will result in the fact that the latter can be decomposed in the following way

$$\mathcal{L}_2=c_1(t)\dot{\zeta}^2+c_2(t)\dot{\zeta}\partial^2\zeta+c_3(t)(\partial\zeta)^2=c_1(t)\dot{\zeta}^2+\big(\frac{1}{2}\dot{c}_2(t)+c_3(t)\big)(\partial\zeta)^2, \tag{6.49}$$

where the last equality results after an integration by parts. Finally, the second order Lagrangian can be recast in a more easy to read form

$$\mathcal{L}_2=a^3\mathcal{Q}_s\Big[\dot{\zeta}^2-\frac{c_s^2}{a^2}(\partial\zeta)^2\Big] \tag{6.50}$$

where

$$\mathcal{Q}_s \equiv \frac{2L_{\mathcal{S}}(3\mathcal{B}^2 + 4L_{\mathcal{S}}(2L_N + L_{NN}))}{\mathcal{W}^2} \tag{6.51}$$

while the speed of the scalar perturbation ζ is defined as

$$c_s^2 \equiv \frac{2}{Q_s}\Big(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E}\Big), \tag{6.52}$$

with

$$\mathcal{M} \equiv \frac{4L_{\mathcal{S}}(\mathcal{D}+\mathcal{E})}{\mathcal{W}} = 4L_{\mathcal{S}}\frac{L_{\mathcal{R}} + L_{N\mathcal{R}} + HL_{N\mathcal{U}} + \frac{3}{2}HL_{\mathcal{U}}}{L_{KN} + 2HL_{\mathcal{S}N} + 4HL_{\mathcal{S}}}. \tag{6.53}$$

6.2.3 Ghost and Laplacian instability for scalar and tensor perturbations

Variation of the action $S_2 = \int d^4x\mathcal{L}_2$, built with the lagrangian density (6.50), with respect to the curvature perturbation ζ leads to the following equation of motion

$$\frac{d}{dt}\Big(a^3\mathcal{Q}_s\dot{\zeta}\Big) - a\mathcal{Q}_s c_s^2\partial^2\zeta = 0 \tag{6.54}$$

and we see that in order to avoid ghost and Laplacian instability the following conditions must hold

$$\mathcal{Q}_s > 0, \qquad c_s^2 > 0. \tag{6.55}$$

If instead one looks at tensors perturbations, a similar expansion of the Lagrangian up to second order can be made. Including a trace/divergence-free tensor mode γ_{ij} and parametrizing the three dimensional metric h_{ij} as

$$h_{ij} = a^2 e^{2\zeta}(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma_{lj}), \tag{6.56}$$

a similar treatment would end up with an up-to-second-order tensorial action

$$S_2^{(h)} = \sum_{\lambda=+,\times}\int d^4x a^3\mathcal{Q}_t\Big[\dot{h}_\lambda^2 - \frac{c_t^2}{a^2}(\partial h_\lambda)^2\Big] \tag{6.57}$$

where the sum is over (each) polarization ($\gamma_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij}$). Again variation of the action with respecto to the tensorial perturbations gives rise to the equations of motion

$$\frac{d}{dt}\left(a^3 \mathcal{Q}_t \dot{h}_\lambda\right) - a\mathcal{Q}_t c_t^2 \partial^2 h_\lambda = 0 \tag{6.58}$$

where

$$\mathcal{Q}_t = \frac{L_\mathcal{S}}{2} \tag{6.59}$$

while the speed of the tensorial perturbation is defined as

$$c_t^2 = \frac{\mathcal{E}}{L_\mathcal{S}}. \tag{6.60}$$

The conditions for the avoidance of ghost and Laplacian instabilities in the tensorial case are simpler and reads

$$L_\mathcal{S} > 0, \qquad \mathcal{E} > 0. \tag{6.61}$$

6.3 Hamiltonian analysis of Mimetic Gravity

The key point of Mimetic Gravity is a parametrization of the physical metric $\bar{g}_{\mu\nu}$ in term of an auxiliary metric $g_{\mu\nu}$ and a scalar field as

$$\bar{g}_{\mu\nu} = (-g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)g_{\mu\nu} \equiv \Phi^2 g_{\mu\nu}, \tag{6.62}$$

where Φ is related with the term P in equation (4.1) of the mimetic gravity model by $\Phi^2 = -P$. The General Relativity action

$$S[g_{\mu\nu}, \phi] = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}R\big(\bar{g}_{\mu\nu}(g_{\mu\nu}, \phi)\big) \tag{6.63}$$

after an integration by parts, can be rewritten as

$$S[g_{\mu\nu}, \phi] = \frac{1}{2}\int d^4x\sqrt{-g}\big[\Phi^2 R(g_{\mu\nu}) + 6g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi\big] \tag{6.64}$$

using the relation

$$R(\bar{g}_{\mu\nu}) = \frac{1}{\Phi^2}\Big(R(g_{\mu\nu}) - 6\frac{g^{\mu\nu}\nabla_\mu\nabla\nu\Phi}{\Phi}\Big) \tag{6.65}$$

where ∇_μ is the covariant derivative defined using $g_{\mu\nu}$. As already mentioned, the action (6.64) is invariant with respect to a Weyl rescaling $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$, and containing second-order derivatives of the field ϕ, an Hamiltonian analysis is needed in order to exclude the presence of ghosts. Besides the fields $g_{\mu\nu}$ and ϕ, it is customary to introduce an auxiliary field λ playing the role of Lagrange multiplier that enforces the constraint $\Phi^2 = -g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ and hopefully reducing the action to be first order in derivatives of ϕ, i.e. the action becomes

$$S[g_{\mu\nu}, \phi] = \frac{1}{2}\int d^4x\sqrt{-g}\big[\Phi^2 R(g_{\mu\nu}) + 6g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \lambda(\Phi^2 + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\big]. \tag{6.66}$$

Variation of the latter with respect to the *field* λ gives back the constraint and moreover, the field Φ will be treated as independent togheter with the field ϕ. In order to perform the Hamiltonian analysis, following [18], we employ the ADM formalism using a $3+1$ decomposition of the metric $g_{\mu\nu}$ as

$$g_{00} = -N^2 + h_i^j N^i N_j, \qquad g_{0i} = N_i, \qquad g_{ij} = h_{ij} \tag{6.67}$$

where N and N_i are the lapse function and shift vector respectively. The inverse metric components are

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = \frac{N^i}{N^2}, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \tag{6.68}$$

where the metric h_{ij} refers to the *Cauchy* surface Σ_t and $h_{ij}h^{jk} = \delta_i^k$. The four-dimensional scalar curvature - the Ricci scalar - is related to the extrinsic geometry via the relation

$$R(g_{\mu\nu}) = K_{ij}\mathscr{G}^{ijkl}K_{kl} + R + \frac{2}{\sqrt{-g}}\partial_\mu(\sqrt{-g}n^\mu K) - \frac{2}{\sqrt{h}N}\partial_i(\sqrt{h}h^{ij}\partial_j N) \tag{6.69}$$

where the extrinsic curvature K_{ij} and the *de Witt* metric $\mathscr{G}^{ijkl}$ are defined as

$$K_{ij} = \frac{1}{2N}\Big(\frac{\partial h_{ij}}{\partial t} - D_i N_j - D_j N_i\Big), \qquad D_i \equiv \nabla_i^{(h_{jk})} \tag{6.70}$$

and

$$\mathscr{G}^{ijkl} = \frac{1}{2}(h^{ik}h^{jl} + h^{il}h^{jk}) - h^{ij}h^{kl}, \tag{6.71}$$

$D_i \equiv \nabla_i^{(h_{jk})}$ being the spatial part of the covariant derivative built using the metric of the hypersurface Σ_t. The future pointing vector n^μ normal to Σ_t has component

$$n^0 = \sqrt{-g^{00}} = \frac{1}{N}, \qquad N^i = -\frac{g^{0i}}{\sqrt{-g^{00}}} = -\frac{N^i}{N}. \tag{6.72}$$

With the previous results, it can be shown that, ignoring boundary terms, and using

$$\nabla_n \Phi = \frac{1}{N}(\partial_t \Phi - N^i \partial_i \Phi), \tag{6.73}$$

the original action can be rewritten as

$$\begin{aligned} S[N, N^i, h_{ij}, \Phi, \lambda, \phi] = \frac{1}{2}\int dtd\mathbf{x}\sqrt{h}N[K_{ij}\mathscr{G}^{ijkl}K_{kl}\Phi^2 + R\Phi^2 - 4K\Phi\nabla_n\Phi+ \\ -\frac{2}{\sqrt{h}}\partial_i(\sqrt{h}h^{ij}\partial_j\Phi^2) - 6(\nabla_n\Phi)^2 + 6h^{ij}\partial_i\Phi\partial_j\Phi+ \\ - \lambda\Phi^2 + \lambda(\nabla_n\phi)^2 - \lambda h^{ij}\partial_i\Phi\partial_j\Phi]. \end{aligned} \tag{6.74}$$

From this action the following conjugate momenta of h_{ij}, Φ, λ and ϕ can be extracted

$$\pi^{ij} = \frac{1}{2}\sqrt{g}\mathscr{G}^{ijkl}K_{kl}\Phi^2 - \sqrt{h}h^{ij}\Phi\partial_n\Phi, \tag{6.75}$$

$$p_\Phi = -2K\Phi\sqrt{h} - 6\sqrt{h}\nabla_n\Phi, \tag{6.76}$$

$$p_\lambda \simeq 0, \tag{6.77}$$

and

$$p_\phi = \sqrt{h}\lambda\nabla_n\phi. \tag{6.78}$$

From these, the primary constraint $\mathcal{D}$ can be obtained as combination of the previous conjugate momenta as

$$\mathcal{D} = p_\Phi \Phi - 2\pi^{ij} h_{ij} \simeq 0. \tag{6.79}$$

A Legendre transformation of the Lagrangian gives the Hamiltonian of the model that can be written as a sum of constraints that vanish for any physical configuration on the constraint surface on the phase space. The Hamiltonian reads

$$H = \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i + v_\mathcal{D}\mathcal{D} + v_N\pi_N + v^i\pi_i + v_\lambda p_\lambda) \tag{6.80}$$

where

$$\begin{aligned}\mathcal{H}_T = {} & \frac{2}{\sqrt{h}\Phi^2}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{2}\sqrt{h}R\Phi^2 + \frac{1}{2\sqrt{h}\lambda}p_\phi^2 + \partial_i(\sqrt{h}h^{ij}\partial_j\Phi^2) \\ & - 3\sqrt{h}h^{ij}\partial_i\Phi\partial_j\Phi + \frac{1}{2}\sqrt{h}\lambda(\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi)\end{aligned} \tag{6.81}$$

and

$$\mathcal{H}_i = p_\Phi\partial_i\Phi + p_\phi\partial_i\phi - 2h_{ij}D_k\pi^{jk}. \tag{6.82}$$

These results hold up to boundary terms and express the Hamiltonian of local degrees of freedom rather than the global gravitational energy. On the other hand, a complete Hamiltonian must contain also these boundary terms defining the total energy conserved in time. According to [18], the total energy is conserved in time, and according to the positive energy theorem of general relativity the total energy is positive, except for flat Minkowski spacetime, which has zero energy. The field Φ is not dynamical, since it is a gauge degree of freedom associated with the conformal symmetry. The total gravitational energy is independent of the chosen gauge for the conformal symmetry. Fixing the gauge of the conformal symmetry one obtains a minimally coupled scalar field theory. It must be required that the energy density of the scalar field is positive on the initial Cauchy surface, at time $t = 0$ for example, since only those initial configurations are physically meaningful. Then the energy conditions of the positive energy theorem of general relativity are satisfied at the inital time $t = 0$ and the total gravitational energy is positive. Since the total energy is conserved, it remains positive.

The preservation of primary constraints $\pi_N \simeq 0$, $\pi_i \simeq 0$ implies the secondary constraints

$$\mathcal{H}_T \simeq 0, \qquad \mathcal{H}_i \simeq 0. \tag{6.83}$$

The next step is to require the preservation of primary constraints under time evolution and for further analysis it is common to introduce two smeared quantities

$$\mathbf{T}_T(N) = \int d^3\mathbf{x} N \mathcal{H}_T, \qquad \mathbf{T}_S(N^i) = \int d^3\mathbf{x}(N^i\mathcal{H}_i + p_\lambda \partial_i \lambda). \tag{6.84}$$

The preservation of p_λ implies

$$\mathcal{C}_\lambda \equiv \frac{1}{N}\partial_t p_\lambda = \frac{1}{N}\{p_\lambda, H\} = \frac{1}{2\sqrt{h}\lambda^2}p_\phi^2 - \frac{1}{2}\sqrt{h}(\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi) \simeq 0. \tag{6.85}$$

In order to impose the preservation of the constraint $\mathcal{D} \simeq 0$ let us consider the following linear combination with $p_\lambda \simeq 0$

$$\tilde{\mathcal{D}} = \mathcal{D} + 2p_\lambda\lambda \tag{6.86}$$

which has the following non-zero Poisson brackets:

$$\begin{aligned}
\{\tilde{\mathcal{D}}(\mathbf{x}), h_{ij}(\mathbf{y})\} &= 2h_{ij}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \\
\{\tilde{\mathcal{D}}(\mathbf{x}), \pi_{ij}(\mathbf{y})\} &= -2\pi_{ij}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \\
\{\tilde{\mathcal{D}}(\mathbf{x}), \Phi(\mathbf{y})\} &= -\Phi(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \\
\{\tilde{\mathcal{D}}(\mathbf{x}), p_\Phi(\mathbf{y})\} &= p_\Phi(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \\
\{\tilde{\mathcal{D}}(\mathbf{x}), \lambda(\mathbf{y})\} &= -2\lambda(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \\
\{\tilde{\mathcal{D}}(\mathbf{x}), p_\lambda(\mathbf{y})\} &= 2p_\lambda(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{6.87}$$

It can be shown that $\tilde{\mathcal{D}}$ is conserved without imposing any additional constraint, in fact one finds

$$\partial_t\tilde{\mathcal{D}} = \{\tilde{\mathcal{D}}, \mathcal{H}_T\} = -N\mathcal{H}_T + \partial_i(N^i\tilde{\mathcal{D}}) + 2v_\lambda p_\lambda \simeq 0. \tag{6.88}$$

Finally the Poisson brackets between $\mathcal{H}_T$ and $\mathcal{H}_i$ in their smeared form read

$$\{\tilde{\mathbf{T}}_T(N), \mathbf{T}_T(M)\} = \mathbf{T}_S((N\partial_i M - M\partial_i N)h^{ij}) - \int d^3\mathbf{x}(\partial_i MN - N\partial_i M)h^{ij}\frac{\partial_j\Phi}{\Phi}\mathcal{D}, \tag{6.89}$$

vanishing on the surface $\mathcal{H}_i \simeq 0$ $\mathcal{D} \simeq 0$. Further one finds

$$\{\tilde{\mathbf{T}}_S(N^i), \mathbf{T}_S(M^i)\} = \mathbf{T}_S(N^i\partial_i M^j - M^i\partial_i N^j), \tag{6.90}$$

and lastly

$$\{\tilde{\mathbf{T}}_S(N^i), \mathbf{T}_T(M)\} = \mathbf{T}_T(N^i\partial_i M). \tag{6.91}$$

Hence the Hamiltonian and momentum constraints (6.83) are preserved under time evolution. According to *Dirac* formula, the number of extra degrees of freedom following from the presented argument amounts to

$$\#\,\text{of canonical variables}/2 - \#\,\text{of primary constraints} - \#\,\text{of secondary constraints}/2 \tag{6.92}$$

and so comparing this model with standard General Relativity one finds that an extra degree of freedom is present. If one sets the secondary constraints $C_\lambda \simeq 0$ to vanish strongly, then solving the latter with respect to λ allows one to find

$$\lambda = \pm\frac{p_\phi}{\sqrt{h}(\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi)}. \tag{6.93}$$

Setting $p_\lambda = 0$ leads to the disappearing of the conjugate variables λ and p_λ from the Hamiltonian that can be rewritten as

$$\mathcal{H}_T = \frac{2}{\sqrt{h}\Phi^2}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{2}\sqrt{h}R\Phi^2 + \partial_i(\sqrt{h}h^{ij}\partial_j\Phi^2) +$$

$$- 3\sqrt{h}h^{ij}\partial_i\Phi\partial_j\Phi \pm p_\phi\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi}. \tag{6.94}$$

The Hamiltonian dependence on p_ϕ is linear, and as we have already seen, this can lead to ghost intabilities.

Choosing for convention the Hamiltonian with positive sign in front of p_ϕ leads to an interpretation of the latter as proportional to the energy density of the *mimetic dust* on the surface Σ_t, i.e. p_ϕ can be view as the rest mass of the mimetic fluid per coordinate volume $d^3\mathbf{x}$ as measured by the *Eulerian* observers with four-velocity n^μ. Since p_ϕ has the physical meaning of density of rest mass, we require that p_ϕ is initially nonnegative everywhere, i.e. $p_\phi \geq 0$ everywhere on the initial *Cauchy* surface Σ_0 at time $t = 0$. On the other hand, the physical meaning of ϕ is that its gradient $\partial_\mu\phi$ represents the direction of the rest mass current of the mimetic dust in spacetime. The equation of motion of ϕ reads

$$\partial_t\phi = \{\phi, H\} = N\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi} + N^i\partial_i\phi, \tag{6.95}$$

and the square of the latter tells us that the rest mass current of the mimetic fluid is a timelike vector, $\Phi^2 = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Equation (6.95) is rather unusual, in fact the evolution of ϕ is not driven by its canonical conjugate momentum p_ϕ; this kind of systems where the evolution of a coordinate does not depend on canonical momenta have been studied in the past in the context of *'t Hooft*'s deterministic quantum mechanics [19]. The evolution of ϕ in the gauge where $N = const. \geq 0$, $N^i = 0$ and $\Phi = const. \geq 0$ is monotonic and always increasing and the rate of growth experiences an increase from the minimal value $\partial_t\phi = N\Phi$ when spatial inhomogeneities in ϕ are present. It is the spatial gradient of ϕ the relevant quantity and not the local value of the field on Σ_t.

The other relevant equation of motion belongs to p_ϕ and it can be shown to be

$$\partial_t p_\phi = N p_\phi \partial_i\Big(\frac{h^{ij}\partial_j\phi}{\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi}}\Big) + \partial_i\Big(\frac{N h^{ij}\partial_i p_\phi\partial_j\phi}{\sqrt{\Phi^2 + h^{ij}\partial_i\phi\partial_j\phi}}\Big) \tag{6.96}$$

when the aforementioned gauge has been employed. Physically, the latter represents the continuity equation for the mimetic rest mass current ensuring the conservation of the total rest mass on the surface Σ_t under time evolution. This equation shows that there exists a ground state where $p_\phi = 0$, and if there exists a region of space in which $p_\phi = 0$ and $\partial_i p_\phi = 0$, then $\partial_t p_\phi = 0$. Considering now a situation in which $p_\phi > 0$ somewhere in space, the question is: can p_ϕ evolve to the negative side of the phase space where $p_\phi < 0$? Inside a region of space in which h_{ij} and $\partial_i \phi$ are nearly constant, only the second term of equation (6.96) really drives the evolution of p_ϕ, because the first one becomes negligible. Considering now the case in which the two gradients $\partial_i \phi$ and $\partial_j p_\phi$ are contradirectional to each other, then $h^{ij}\partial_i p_\phi \partial_j \phi < 0$. The crucial point is that no matter how small p_ϕ is, it would eventually evolve towards zero crossing then the $p_\phi = 0$ surface at some later time. This discussion shows that under certain circumstances, the energy density of the mimetic dust can become negative, and consequently the system can become unstable. Also for the mirror image of this system, the one that follows choosing the minus sign on the initial Hamiltonian, one is forced to conclude that the system can still become unstable for some given kind of initial configurations.

6.4 Another example of Hamiltonian analysis of Mimetic Gravity

In the last section, following [18] the mimetic constraint

$$\Phi^2 = -g^{\alpha\beta}\partial_\alpha \phi \partial_\beta \phi, \tag{6.97}$$

was treated as an independent field and then the Hamiltonian analysis was presented. There exists another way to perform the Hamiltonian analysis without the assumption that Φ is also an independent field. As shown by [4] in order to get the canonical formalism it is possible to start directly from the original mimetic action with the constraint as Lagrange multiplier

$$S = -\int d^4x\sqrt{-g}\Big(\frac{1}{2}R + \frac{1}{2}\lambda(1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) + V(\phi)\Big). \tag{6.98}$$

This action can be rewritten in a $3+1$ dimensional form and its ϕ-dependent part reads

$$S_\phi = -\int d^4x\frac{1}{2}N\sqrt{h}\lambda\Big(1 - g^{00}\dot{\phi}^2 - 2g^{0i}\dot{\phi}\partial_i\phi + h^{ij}\partial_i\phi\partial_j\phi + \\ - \frac{N^iN^j}{N^2}\partial_i\phi\partial_j\phi\Big) + N\sqrt{h}V(\phi), \tag{6.99}$$

where N and N^i are the lapse and shift functions and $g^{00} = \frac{1}{N^2}$, $g^{0i} = \frac{-N^i}{N^2}$ and $g^{ij} = -h^{ij} + \frac{N^iN^j}{N^2}$. There are two conjugate momenta, respectively

$$p_\lambda = \frac{\partial L}{\partial\dot{\lambda}} = 0 \tag{6.100}$$

and

$$p = \frac{\partial L}{\partial\dot{\phi}} = N\sqrt{h}\lambda(g^{00}\dot{\phi} + g^{0i}\partial_i\phi). \tag{6.101}$$

Equation (6.100) is a primary constraint that implies a secondary constraint by demanding its time constancy

$$0 = \dot{p}_\lambda = \{p_\lambda, H\} = \frac{\delta H}{\delta\lambda}. \tag{6.102}$$

Equation (6.101) can be inverted giving $\dot{\phi}$ in function of its conjugate momenta, $\dot{\phi} = \dot{\phi}(p)$, thus allowing to perform a Legendre transform and writing the Hamiltonian as

$$H = \frac{Np^2}{2\sqrt{h}\lambda} + \frac{1}{2}N\sqrt{h}\lambda(1 + h^{ij}\partial_i\phi\partial_j\phi) + pN^i\partial_i\phi + N\sqrt{h}V(\phi). \tag{6.103}$$

The dependence of the latter Hamiltonian with respect to the Lagrange multiplier λ can be excluded by solving equation (6.102) in terms of λ and it can be shown that

$$\lambda = \frac{p}{\sqrt{h}(1 + h^{ij}\partial_i\phi\partial_j\phi)}. \tag{6.104}$$

The total action can be cast in the following form [20]

$$S = S_g + S_\phi = \int d^4x \Big(L_{ADM} + p\dot{\phi} - Np\sqrt{1 + h^{ij}\partial_i\phi\partial_j\phi} - N^i p\partial_i\phi + \\ - N\sqrt{h}V(\phi), \tag{6.105}$$

where

$$L_{ADM} = \dot{h}^{ij}\pi_{ij} - NR^0 - N^i R_i, \tag{6.106}$$

$$\pi^{ij} = \sqrt{-h}(\Gamma^0_{kl} - h_{kl}\Gamma^0_{mn}h^{mn})h^{ik}h^{jl}, \tag{6.107}$$

and the intrinsic curvature is given by

$$R^0 \equiv -\sqrt{h}\Big[\,{}^3R + h^{-1}\Big(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\Big)\Big], \tag{6.108}$$

$$R_i \equiv -2h_{ik}\pi^{kj}_{|j}, \tag{6.109}$$

where $|_j$ indicates the covariant derivative given by the metric h_{ij} while $\pi = \pi^i_i$. The quantity ${}^3R$ is understood as three-dimensional. After the previous definitions the total action can then be written as

$$S = \int d^4x \Big[\dot{h}^{ij}\pi_{ij} + p\dot{\phi} - N\Big(R^0 + p\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1}\Big) - N^i(R_i + p\partial_i\phi) - N\sqrt{h}V(\phi)\Big]. \tag{6.110}$$

Then the equations of motion are found by varying with respect to the variables h_{ij}, π_{ij}. Variation with respect to π_{ij} gives the six equations

$$\dot{h}^{ij} = \{h^{ij}, H\} = \frac{2N}{\sqrt{h}}\Big(\pi^{ij} - \frac{1}{2}h^{ij}\pi\Big) + N^{i|j} + N^{j|i} \tag{6.111}$$

independent of the scalar field ϕ, since the action S_ϕ is independent of π_{ij}. On the other hand, variation with respect to h_{ij} gives

$$\dot{\pi}_{ij} = \{\pi_{ij}, H\} = -N\sqrt{h}\Big({}^3R_{ij} - \frac{1}{2}h_{ij}\,{}^3R\Big) + \frac{1}{2\sqrt{h}}Nh_{ij}\Big(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\Big) + \\ -\frac{2}{\sqrt{h}}N\Big(\pi_{im}\pi^m_j - \frac{1}{2}\pi\pi_{ij}\Big) + \sqrt{h}(N_{|ij} - h_{ij}N^{|m}_{|m}) + (\pi_{ij}N^m)_{|m} +$$

$$- N_i^{|m}\pi_{mj} - N_j^{|m}\pi_{mi} + \frac{Np\partial_i\phi\partial_j\phi}{2\sqrt{h^{kl}\partial_k\phi\partial_l\phi + 1}} - \frac{1}{2}N\sqrt{h}V(\phi)h_{ij}, \tag{6.112}$$

that differ from Einstein's gravity by the presence of two terms function of ϕ.

Variation of the action with respect to N and N^i yields four constraint equations

$$R^0 + p\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1} + \sqrt{h}V(\phi) = H_{grav} + H_\phi = 0$$

$$R_i + p\partial_i\phi = H_{i\,grav}H_{i\,\phi} = 0. \tag{6.113}$$

Differently from standard General Relativity in the case of mimetic gravity there are two more phase space variables and consequently two more equations of motion, namely those of ϕ and p, respectively

$$\dot{\phi} - N\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1} - N^i\partial_i\phi = 0, \tag{6.114}$$

$$\dot{p} - \partial_k\Big(\frac{Nph^{kl}\partial_l\phi}{\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1}} + N^k p\Big) + N\sqrt{h}\frac{dV(\phi)}{d\phi} = 0. \tag{6.115}$$

The crucial point here is that these two last equations do not add new information as these are nothing else that the constraint equation $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$ and the Bianchi identity $\nabla_\mu T_i^\mu = 0$ that follow after calculating the stress-energy tensor from the action S_ϕ. Summarizing, the equations of motion of mimetic gravity are those of Einstein's gravity plus two more equations that can be reinterpreted as the conservation of the energy-momentum tensor and the constraint equation. However, the equations of motion obtained varying with respect to h^{ij}, N and N^i are those of pure Einstein's gravity [20] but including extra terms as a function of the scalar field ϕ. As in the last section, with the help of smeared functions in order to have well-defined algebraic relationships, it is possible to define a Dirac algebra showing its closure.

As was shown in the last two sections, the presence of the Lagrange multipliers in the action has strong impact on the form of the resulting equations of motions. Then it was natural to ask the question how the presence of Lagrange multipliers modifies Hamiltonian structure of given theory. Moreover, one would like to see whether the Hamiltonian of these systems is again given as a linear combination of constraints and whether these constraints are the first class and their Poisson algebra respects the basic principles of geometrodynamics [21]. It turns out that Hamiltonian structure of given theory is very interesting. It was shown that the presence of the first scalar field that plays the role of the Lagrange multiplier implies an existence of the second class constraints. Then after their solving we find the Hamiltonian equations of motions for the second scalar field that are autonomous in the sense that the time evolution of the scalar field does not depend on its conjugate momenta. The final result is that the resulting theory is a fully constrained system with the algebra of constraints that has the same form as in General Relativity.

Chapter 7

Horndeski theory and Disformal transformations

7.1 The invariance of the Horndeski action under Disformal transformations

Concepts of mimetic gravity can be analyzed within the most general framework of Horndeski theories built from an action that gives rise to second-order equations of motion. This framework is a general description of how a scalar degree of freedom ϕ fits into a theory of gravity and in particular how ϕ and its first and second derivatives along with the kinetic term $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ enter into the action. The Lagrangian density is fully described by the four functions ($\mathcal{K}(\phi, X)$, $G_i(\phi, X)$) and read

$$\mathcal{L}_{\text{Hor}} = \sum_{i=2}^{5} \mathcal{L}_i \tag{7.1}$$

where

$$
\begin{aligned}
\mathcal{L}_2 &= \mathcal{K}(\phi, X) \\
\mathcal{L}_3 &= G_3(\phi, X)\Box\phi \\
\mathcal{L}_4 &= G_4(\phi, X)R - G_{4X}(\phi, X)[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2] \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\partial^\nu\phi + \tfrac{G_{5X}(\phi,X)}{6}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\nu\partial_\mu\phi)^2 + 2(\nabla_\mu\partial_\nu\phi)^2].
\end{aligned}
\tag{7.2}
$$

It can be shown that for the Horndeski action, conditions (6.48) of the last chapter are satisfied and so $\mathcal{L}_{\text{Hor}}$ gives rise to second-order equations of motion.

In chapter IV we saw that Mimetic Matter emerges if we map the metric $g_{\mu\nu}$ to a well defined function of a scalar field ϕ via

$$
g_{\mu\nu} \mapsto \bar{g}_{\mu\nu} = (g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)g_{\mu\nu} = P(\phi)g_{\mu\nu}. \tag{7.3}
$$

There exists a more general class of transformations [22, 23], dubbed *disformal* transformations, realized by

$$
g_{\mu\nu} \mapsto \bar{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi, \qquad X = \bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2. \tag{7.4}
$$

In order to be a physical transformation, the following three conditions have to be satisfied:

Lorentzian signature: considering a frame in which $\partial_\mu\phi = (\dot{\phi}, \vec{0})$, then the lorentzian signature is guarantee if

$$
\bar{g}_{00} = A(\phi, X)g_{00} + B(\phi, X)\dot{\phi}^2 < 0
$$

or

$$
A(\phi, X) + 2B(\phi, X)X > 0. \tag{7.5}
$$

Causal behaviour: the sign of B can alter the light-cone introducing superluminal or a-causal effects, but the requirement that physical particles obey $ds^2 < 0$ will ensure the absence of such problematic situations.

Invertibility: the conditions by which the inverse and the volume element are never singular. Searching for an inverse metric of the form

$$\bar{g}^{\mu\nu} = \frac{1}{A} g^{\mu\nu} + x \partial^\mu \phi \partial^\nu \phi, \tag{7.6}$$

contracting one index of the latter with one index of the disformal transformed metric (7.4) and requiring $\bar{\delta}^\mu_\nu = \delta^\mu_\nu$, one finds $x = -\frac{B}{A}\frac{\omega}{2X}$ or

$$\bar{g}^{\mu\nu} = \frac{1}{A} g^{\mu\nu} - \frac{B(\phi, X)}{A(\phi, X)} \frac{\omega}{2X} \partial^\mu \phi \partial^\nu \phi, \qquad \omega = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{7.7}$$

and $\sqrt{-\bar{g}} = A^2(1 + 2XB/A)^{1/2}\sqrt{-g}$.

As already pointed out, the Horndeski action gives rise to second-order equations of motion and this is because of a fine cancellation between higher derivative terms from the non-minimally coupled part of the Lagrangian and those produced from derivative counterterms. This happens because of the antisymmetric structure of $\mathcal{L}_4 = G_4(\phi, X)R - G_{4X}(\phi, X)[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2]$ that can be rewritten as

$$\mathcal{L}_4 = (g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta})[G_4(\phi, X) R_{\mu\nu\alpha\beta} - G_{4X}(\phi, X) \nabla_\mu \partial_\nu \phi \nabla_\alpha \partial_\beta \phi], \tag{7.8}$$

whit clearly an antisymmetric structure.

A conformal transformation of the type $g_{\mu\nu} \mapsto A(X) g_{\mu\nu}$ will spoil the main feature of Horndeski theory, namely second-order equations of motion. In fact the conformal transformations will produce on $\nabla_\mu \partial_\nu \phi$ a contribution that inserted into (7.8) gives rise to a symmetric term responsible for altering the Horndeski action antisymmetry. Thus we are forced to conclude that in order to guarantee the peculiarity of $\mathcal{L}_{\text{Hor}}$ we have to restrict the attention on trasformations of the type

$$g_{\mu\nu} \mapsto \bar{g}_{\mu\nu} = A(\phi) g_{\mu\nu} + B(\phi) \partial_\mu \phi \partial_\nu \phi, \tag{7.9}$$

where the dependence of the two functions A and B on X is dropped [23]. It can be shown that the effect of the latter class of transformations on the Horndeski Lagrangian is only a rescaling of the four functions ($\mathcal{K}$ and G_is) while no modification to the antisymmetry properties are introduced by (7.9). Moreover, $\mathcal{L}_{\rm Hor}$ also admits the field redefinition $\phi \mapsto s(\phi)\phi$ symmetry.

The invariance of the action under disformal transformations means that (7.9) is a symmetry of $\mathcal{L}_{\rm Hor}$ and all functions are defined modulo a conformal and a disformal transformation.

7.1.1 Special cases

Setting $A \neq 0$ and $B = 0$ will produce a purely conformal transformation that alters in a non trivial way two of the four functions, while it only rescales G_4 and X and leaves unchanged G_5, respectively

$$\bar{\mathcal{K}}(\phi, X) = A^2\mathcal{K}(\phi, X_C) + f(X, A, A', A'', G_3, G_4, G_5) \tag{7.10}$$

$$\bar{G}_3(\phi, X) = AG_3(\phi, X_C) - g(X, A, A', A'', G_4, G_5) \tag{7.11}$$

and

$$\bar{G}_4(\phi, X) = AG_4(\phi, X_C), \qquad X_C = \frac{X}{A} \tag{7.12}$$

$$\bar{G}_5(\phi, X) = G_5(\phi, X_C). \tag{7.13}$$

The form of the two functions f and g are rather complicated and the key point is that if one starts with zero $\mathcal{K}$ or G_3, they would appear after a purely conformal transformation. In fact $\mathcal{L}_{j<i}$ receives contributions from all the $\mathcal{L}_i$s, while $\mathcal{L}_5$ cannot be generated in this way. Given that, a purely conformal transformation cannot eliminate non minimal couplings (NMC) for any choice of the conformal factor $A(\phi)$.

On the other hand, purely disformal transformations are achieved setting $A = 1$ and $B \neq 0$ and the corresponding transformations on the two NMC functions G_4, G_5 reads

$$\bar{G}_4(\phi, X) = (1 + 2XB)^{1/2}G_4(\phi, X_D) + m(X, B, B', G_5) - \frac{\partial H_R}{\partial \phi}(\phi, X)X \tag{7.14}$$

and

$$\bar{G}_5(\phi, X) = \frac{G_5(\phi, X_D)}{(1+2BX)^{1/2}} + H_R(\phi, X), \tag{7.15}$$

where

$$X_D = \frac{X}{1+2BX},$$
$$H_R = B \int dX \frac{G_5(\phi, X_D)}{(1+2BX)^{3/2}}.$$

The resulting transformation appears to be richer than in the purely conformal case and the form of $m(X, B, B', G_5)$

$$m(X, B, B', G_5) = \frac{G_5(\phi, X_D)B'(\phi)X^2}{(1+2BX)^{3/2}}$$

forces one to conclude that even in this case NMC terms cannot be generically eliminated with a purely disformal transformation. Imposing that the NMC terms disappear from the action, i.e. $\bar{G}_4 = 1$ and $\bar{G}_5 = 0$, leads for G_5 to the relation

$$\int dX \frac{G_{5X}(\phi, X_D)}{(1+2BX)^{1/2}} = 0 \tag{7.16}$$

satisfied for example if

$$G_5 = G_5(\phi) \tag{7.17}$$

and if G_4 is

$$G_4(\phi, X) = (1-2BX)^{1/2} - \frac{\partial G_5}{\partial \phi} X. \tag{7.18}$$

It can be concluded that the NMC part of the action

$$S_{NMC} = \int d^4x\sqrt{-g}[G_4(\phi, X)R - G_{4X}[(\Box\phi)^2 - (\nabla_\mu \partial\nu\phi)^2] + G_5(\phi)G_{\mu\nu}\nabla^\mu\partial^\nu\phi], \tag{7.19}$$

where G_4 is given by (7.18), is the only one that admits a disformal map able to eliminate all the NMC terms in the context of Horndeski theory. The more general transformation with $A = A(\phi)$ would simply result in a conformal rescaling of G_4.

7.1.2 Disformal Frames

The analysis of modified theories of gravity as for example $f(R)$, Horndeski and Brans-Dicke among the others, can be exploited in two different frames, dubbed the *Jordan frame* and the *Einstein frame* that correspond to somewhat opposite situations:

Einstein frame in which the gravitational dynamics is described by an Einstein-Hilbert action and matter field are coupled to gravity via some functions of the scalar field and its derivatives.

Jordan frame in which the gravitational sector Lagrangian includes a NMC scalar field.

Given the following action

$$S = \int d^4x \Big[G(\phi)R - \frac{f(\phi)}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\Big] + S_m[e^{2\alpha(\phi)}g, \psi], \tag{7.20}$$

in which the gravitational coupling have been promoted to a function of a scalar field $G(\phi)$, while $f(\phi)$, $V(\phi)$ and $\alpha(\phi)$ are general functions of their argument, the difference between the Jordan and the Einstein frame can be appreciated fixing two out of four of the latter functions. The Einstein frame is defined by the choice

$$G(\phi) = 1, \qquad f(\phi) = 1 \tag{7.21}$$

while the Jordan frame by

$$G(\phi) = \phi, \qquad \alpha(\phi) = 0. \tag{7.22}$$

Recalling the total action for which is possible to eliminate all the NMC terms with a disformal transformation and completing that form including an action for matter fields leads to

$$S = \int d^4x\sqrt{-g}[G(\phi,X)R - G_X[(\Box\phi)^2 - (\nabla_\mu\partial_\nu\phi)^2] + \mathcal{K}(\phi,X) + G_3(\phi,X)\Box\phi] + S_m[\bar{g},\psi], \tag{7.23}$$

where the function $G(\phi, X)$ is parametrized by the two functions $C(\phi)$ and $D(\phi)$ as

$$G(\phi, X) = C(\phi)^2 \Big(1 - 2\frac{D(\phi)}{C(\phi)}X\Big)^{1/2}$$

and $S_m[\bar{g}_{\mu\nu}]$ is the total matter action defined in terms of the physical metric

$$\bar{g}_{\mu\nu} = e^{\alpha(\phi)} g_{\mu\nu} + \beta(\phi)\partial_\mu\phi\partial_\nu\phi. \tag{7.24}$$

The above definitions leave us with six free functions, four from the previous argument about Horndeski theories and two from the physical metric definitions. Choosing appropriately $A(\phi) = e^{\alpha(\phi)}$ and $B(\phi) = \beta(\phi)$, a disformal transformations allows to fix two out of the four functions $C(\phi)$, $D(\phi)$, $\alpha(\phi)$, $\beta(\phi)$ and in this way it is possible to select a particular frame in which different features of the theory can emerge. For example, as well as the Jordan and the Einstein frames discussed above, as pointed out by [23], there exist also others two frames, namely the *Galileon* frame and the *Disformal* frame which can be seen as a sort of intermediates states between the former. The name Galileon is given because the conformal part enters the matter Lagrangian explicitly and the field couples directly to gravity as a DBI Galileion, [25]. The Jordan frame is given setting $\alpha = 1$ and $\beta = 0$, the Einstein frame correspond to the choice $C(\phi) = 1$ and $D(\phi) = 0$, on the other hand the Galileon frame is given by $C(\phi) = 1$ and $\beta(\phi) = 0$ while the Disformal frame is choosen setting $D(\phi) = 0$ and $\alpha(\phi) = 1$. In conclusion, the equivalence of the frames allows to claim the equivalence of many apparentely unrelated models given that one can move from one to another through appropriately chosen disformal transformations and field redefinitions.

7.2 Disformal Transformation Method

Recalling the form of the inverse metric

$$\bar{g}^{\mu\nu} = \frac{1}{A} g^{\mu\nu} - \frac{B(\phi, X)}{A(\phi, X)} \frac{\omega}{2X} \partial^\mu\phi\partial^\nu\phi, \qquad \omega = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

contracting the latter with $\partial_\mu\phi\partial_\nu\phi$ allows one to find

$$\omega = \frac{2XA}{1-2XB} = \frac{Ag^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{1-Bg^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} \tag{7.25}$$

where clearly $1 - B\partial_\mu\phi\partial_\nu\phi \neq 0$. Using (7.25), as in [3], it is possible to define a function G of ω and ϕ of the form

$$G(\phi,\omega) \equiv \omega\frac{1-B(\phi,\omega)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{A(\phi,\omega)} = \bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{7.26}$$

For fixed ϕ, if

$$\frac{dG(\phi,\omega)}{d\omega}\Big|_{\omega=\omega_*} \neq 0, \tag{7.27}$$

the inverse function theorem ensure that the inverse function G^{-1} exists near ω_*, and it is possible to write $\omega = G^{-1}(\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)$. On the other hand $\frac{dG(\phi,\omega)}{d\omega}\Big|_{\omega=\omega_*} = 0$ implies the non existence of G^{-1}. The latter is solved for example [3, 24] by

$$G(\phi,\omega) = \frac{1}{b(\phi)}, \tag{7.28}$$

and in this exceptional case of non invertibility, the relation between $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ cannot be inverted. Moreover from (7.25) one has

$$B(\phi,\omega) = -\frac{A(\phi,\omega)}{\omega} + b(\phi). \tag{7.29}$$

Then the disformal transformation can be written as

$$\bar{g}_{\mu\nu} = A(\phi,\omega)g_{\mu\nu} + \Big(b(\phi) - \frac{A(\phi,\omega)}{\omega}\Big)\partial_\mu\phi\partial_\nu\phi, \tag{7.30}$$

and so, it is possible to claim, as the authors of [24] does, that mimetic gravity - for which $A = A(\phi,\omega)$ and $b(\phi) = 1$ - emerges as particular case of a non invertible disformal transformation of the phisical metric $\bar{g}_{\mu\nu}$ in term of an auxiliary metric $g_{\mu\nu}$ and a scalar field ϕ.

7.2.1 Equations of motion

Given the relation between the physical metric and the auxiliary metric, it is possible to calculate the generalized field equations taking variations of the total action, [3]. If the latter is

$$S = \int d^4x\sqrt{-g}\mathcal{L}[\bar{g}_{\mu\nu}, \partial\bar{g}_{\mu\nu}, \phi, \partial\phi] + S_m[\bar{g}_{\mu\nu}, \psi_m] \tag{7.31}$$

where ψ_m are matter fields, then the variation reads

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}(E^{\mu\nu} + T^{\mu\nu})\delta\bar{g}_{\mu\nu} + \int d^4x\Omega_\phi\delta\phi + \int d^4x\Omega_m\delta\psi_m, \tag{7.32}$$

where

$$\Omega_\phi = \frac{\delta(\sqrt{-\bar{g}}\mathcal{L})}{\delta\phi}, \tag{7.33}$$

$$E^{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}}\frac{\delta(\sqrt{-\bar{g}}\mathcal{L})}{\delta\bar{g}_{\mu\nu}}, \tag{7.34}$$

and

$$T^{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}}\frac{\delta(\sqrt{-\bar{g}}\mathcal{L}_m)}{\delta\bar{g}_{\mu\nu}}. \tag{7.35}$$

On the other hand, taking variations of the metric $\bar{g}_{\mu\nu}$, disformally related to $g_{\mu\nu}$, the result would be

$$\begin{aligned}\delta\bar{g}_{\mu\nu} = A\delta g_{\mu\nu} - \Big(g_{\mu\nu}\frac{\partial A}{\partial\omega} + \partial_\mu\phi\partial_\nu\phi\frac{\partial B}{\partial\omega}\Big)[(g^{\alpha\rho}\partial_\alpha\phi)(g^{\beta\sigma}\partial_\beta\phi)\delta g_{\rho\sigma} - 2g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\delta\phi] + \\ + \Big(g_{\mu\nu}\frac{\partial A}{\partial\phi} + \partial_\mu\phi\partial_\nu\phi\frac{\partial B}{\partial\phi}\Big)\delta\phi + B(\partial_\mu\phi\partial_\nu\delta\phi + \partial_\nu\phi\partial_\mu\delta\phi).\end{aligned} \tag{7.36}$$

Inserting the last relation in (7.32) then the generalized Einstein equations of motion $\frac{\delta S}{\delta\bar{g}^{\mu\nu}} = 0$ read

$$A(E^{\mu\nu} + T^{\mu\nu}) = \Big(\alpha_1\frac{\partial A}{\partial\omega} + \alpha_2\frac{\partial B}{\partial\omega}\Big)g^{\mu\rho}\partial_\rho\phi g^{\nu\sigma}\partial_\sigma\phi \tag{7.37}$$

and the generalized Klein-Gordon equation $\frac{\delta S}{\delta\phi} = 0$

$$\frac{1}{\sqrt{-g}}\partial_\rho\Big\{\sqrt{-g}\partial\sigma\phi\Big[B(E^{\rho\sigma}+T^{\rho\sigma})+\Big(\alpha_1\frac{\partial A}{\partial\omega}+\alpha_2\frac{\partial B}{\partial\omega}\Big)g^{\rho\sigma}\Big]\Big\}-\frac{\Omega_\phi}{\sqrt{-g}} = \frac{1}{2}\Big(\alpha_1\frac{\partial A}{\partial\phi}+\alpha_2\frac{\partial B}{\partial\phi}\Big), \tag{7.38}$$

where the following quantities have been defined

$$\alpha_1 \equiv (E^{\rho\sigma} + T^{\rho\sigma})g_{\rho\sigma}, \qquad \alpha_2 \equiv (E^{\rho\sigma} + T^{\rho\sigma})\partial_\rho\phi\partial_\sigma\phi. \tag{7.39}$$

Contracting the metric equation of motion (7.37) with $g_{\mu\nu}$ and with $\partial_\mu\phi\partial_\nu\phi$ gives the following two-dimensional linear system

$$\alpha_1\Big(A - \omega\frac{\partial A}{\partial\omega}\Big) - \alpha_2\omega\frac{\partial B}{\partial\omega} = 0, \qquad \alpha_1\omega^2\frac{\partial A}{\partial\omega} - \alpha_2\Big(A - \omega^2\frac{\partial B}{\partial\omega}\Big) = 0. \tag{7.40}$$

One may write the latter system of equations in matrix form as

$$M\begin{pmatrix}\alpha_1\\ \\ \alpha_2\end{pmatrix} = 0, \qquad M = \begin{pmatrix} A - \omega\frac{\partial A}{\partial\omega} & -\omega\frac{\partial B}{\partial\omega} \\ \\ \omega^2\frac{\partial A}{\partial\omega} & -A + \omega^2\frac{\partial B}{\partial\omega}\end{pmatrix} \tag{7.41}$$

and the determinant of the system is

$$\det M = \omega^2 A\frac{\partial}{\partial\omega}\Big(B + \frac{A}{\omega}\Big). \tag{7.42}$$

In the generic case, when the determinant is not vanishing, the only solution is $\alpha_1 = \alpha_2 = 0$ and the two field equations reads

$$E^{\mu\nu} + T^{\mu\nu} = 0, \qquad \Omega_\phi = 0. \tag{7.43}$$

It is possible to conclude that in the general case, it does not matter with respect to what metric one takes variations, using the metric $g_{\mu\nu}$ or its disformally related $\bar{g}_{\mu\nu}$ one always recovers General Relativity: $G_{\mu\nu} = -E^{\mu\nu} = T_{\mu\nu}$. As pointed out by [24], this fact corresponds to a generalizations of standard *veiled* General Relativity where the disformed metric reduces to a conformal metric $\bar{g}_{\mu\nu} = P(\phi)g_{\mu\nu}$.

7.3 Mimetic Gravity

In the case of vanishing determinant, one finds that

$$B(\phi,\omega) = -\frac{A(\phi,\omega)}{\omega} + b(\phi), \tag{7.44}$$

with a non-zero constant of integration $b(\phi)$. The latter relation between the coefficients of the disformal map, is the same condition that follows from the non-invertibility of the disformal map itself. Equations of motion in this case are

$$E^{\mu\nu} + T^{\mu\nu} = \frac{\alpha_1}{\omega}(g^{\mu\alpha}\partial_\alpha\phi)(g^{\nu\beta}\partial_\beta\phi) \tag{7.45}$$

and

$$\partial_\alpha(\sqrt{-\bar{g}}b\alpha_1 g^{\alpha\beta}\partial_\beta\phi) - \Omega_\phi = \frac{\sqrt{-\bar{g}}}{2}\alpha_1\omega\frac{db}{d\phi}. \tag{7.46}$$

Taking into account that the inverse metric in this case is found to be

$$\bar{g}^{\mu\nu} = \frac{1}{A}\Big(g^{\mu\nu} + \frac{A-\omega B}{Ab\omega^2}(g^{\mu\alpha}\partial_\alpha\phi)(g^{\nu\beta}\partial_\beta\phi)\Big), \tag{7.47}$$

with the latter inverse and the starting metric $\bar{g}_{\mu\nu}$ it is possible to find, using contractions with $\partial_\mu\phi$, $g^{\mu\alpha}\partial_\alpha\phi = b\omega\partial^\mu\phi$, $\alpha_1 = (E+T)/(b\omega)$ and

$$b(\phi)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1. \tag{7.48}$$

Inserting these results into the equations of motion leads to the already known set of equations

$$E_{\mu\nu} + T_{\mu\nu} = (E+T)b\partial\mu\phi\partial_\nu\phi, \qquad \nabla_\alpha[(E+T)b\partial^\alpha\phi] - \frac{\Omega_\phi}{\sqrt{-g}} = \frac{1}{2}(E+T)\frac{d\ln b}{d\phi}. \tag{7.49}$$

These equations correspond to the case of Mimetic Matter already analyzed. It is clear that the realizing transformations dubbed "mimetic disformal transormation", drastically change the set of equations one finds varying with respect to the metric $g_{\mu\nu}$ in place of the disformally related $\bar{g}_{\mu\nu}$.

Chapter 8

Conclusions

Within this thesis work I presented a (relatively) new way of modifying General Relativity emerged in recent years, see for example [1], [2]. Since the first Einstein's formulation of gravitation, possible modifications of the theory started to get appeal within the scientific comunity. Einstein itself was among the first who started to modify the theory of GR in order to explain his static version of the Universe. Today people start to think to modifications of the same theory in order to accomodate observational evidences for the existence of dark components. Several aspects of these components, as we saw in this work, can be *mimicked* once one accepts the idea to add a scalar degree of freedom to the metric. Provided that one consider the possibility that the physical metric can have a scalar degree of freedoom ϕ, considering also a potential $V(\phi)$ for this scalar allows the description of Dark Matter and Dark Energy. As discussed, the mimetic model offer a wide range of applications: besides the dark sector, it provide us also a possible description of an inflationary mechanism. It is important to stress that, differently from the inflationary paradigm or to other tensor-scalar theories, mimetic matter is a modification of GR that offer a solution without appealing to the existence of a new propagating field. The scalar field of mimetic matter it is thought to be a new degree of freedom of the metric, and in his first appearance and easier formulation [1, 2] it is not a propagating field.

A symmetric metric, in general, possesses ten independent components so, allowing a relation between the physical metric $\bar{g}_{\mu\nu}$ and a combination of a scalar field ϕ and an auxiliary metric $g_{\mu\nu}$, generally reduces to a map from ten elements to eleven. Let me recall that the disformal map is given by

$$\bar{g}_{\mu\nu} = A(\phi,\omega)g_{\mu\nu} + B(\phi,\omega)\partial_\mu\phi\partial_\nu\phi, \qquad \omega = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

We saw that two cases exists. They are related to the possibility that the quantity

$$\det M = \omega^2 A \frac{\partial}{\partial\omega}\Big(B + \frac{A}{\omega}\Big),$$

associated with the sets of field equations following from the action, vanishes or not. When $\det M \neq 0$ then this version of the theory reduces to standard General Relativity, showing that the theory itself is disformally invariant. On the other hand, when the determinant vanishes, i.e. when the relation

$$B(\phi,\omega) = -\frac{A(\phi,\omega)}{\omega} + b(\phi),$$

holds, then we are in the presence of a modified version of General Relativity with a mimetic dust fluid component.

The Hamiltonian view provides a further insight at the heart of this new idea. In fact, once the Hamiltonian is written, conditions for positive energy definiteness can be written in order to avoid the presence of UV ghosts in the theory. What emerged is that the Hamiltonian related to Mimetic Gravity turns out to be linear with respect to one of its conjugate momenta, namely that of ϕ. Time evolution in the phase space is given by the Hamiltonian flow, and it is clear that orbits live on a subspace of the total constant energy surface given by the set of primary constraints emerging from the formalism.

The idea of a scalar degree of freedom of the metric fit well into a general theory called *Horndeski* theory. This theory describe how a scalar degree of freedom can be accomodated into a theory of gravitation. As we saw, the Horndeski action is generally invariant under disformal transformations and gives rise to second-order equations of motion.

The different representations of a theory, written in terms of disformally related metrics, are often referred to as being written in different 'frames'. In some cases these transformations can be used to remove non-minimal coupling between the scalar field and the Ricci scalar at the level of the action, leaving only a canonical Einstein-Hilbert term. This particular frame, if it exists, is referred to as the Einstein frame. In other words, the formulation in the Einstein frame represents one conformal gauge of the mimetic theory, as usual there exist alternative gauges.

In conclusion, despite the potential problem of the presence of ghosts discussed above, the original theory of mimetic dark matter could be useful for astrophysical and cosmological modeling, provided that one considers only those initial configurations that do not cross the *critical line* under time evolution leading to negative energy states. All the ideas presented in this work must be tested not only in their ability of describe the dark sector of cosmology, the key feature of Mimetic Gravity, but they must also agree with Solar System measurements. As we have seen, when one consider the action of the Mimetic Model, the only modification with respect to Einstein GR it is reduced to the appearance of a perfect-fluid that can mimic the observed behavior of Dark Matter and Dark Energy. It is possible to claim that no modifications are given to gravity at scales of the Solar Sistem.

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