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Study of a nonlocal and isoperimetric variational problem arisen from a model of charged drops

Adriano Prade

Id number: 22201795

Matricola: 2055019

Supervisor:
Prof. Michael Goldman

Cosupervisor:
Prof. Annalisa Massaccesi

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Introduction

Consider a drop of a conductive liquid in the three dimensional space endowed with a positive charge $Q > 0$, in absence of gravity and at rest. Under such conditions, only two kinds of forces affect the configurations of the system by competing one against another. Surface tension is determined by intrinsic physical properties of the fluid: it acts locally on the drop's surface and it induces cohesive effects. Specifically, it prevents the drop from crumbling and it is manifested by the tendency of the liquid to shrink into the minimum surface area possible. Conversely, the positive electric charge provokes repulsive forces of Coulombic type between the particles composing the droplet. Its action is strongly nonlocal and it often dominates surface tension, deforming and breaking the shape of the liquid. Both in their stable and unstable regimes, such charged droplets significantly contribute to many applications in the experimental field, ranging from electrowetting in digital micro-fluids to optics and electronic displays.

A question which naturally arises is whether there exists or not any stable configuration (in a sense to be specified) of the system under the influence of these two forces and, in the event, how such structure may be characterized. In particular, from a mathematical standpoint, we are interested in building a model to analyse with the tools of Calculus of Variations the scenario we described above. We provide first a heuristic construction conveying the basic ideas (pioneered by Lord Rayleigh in [24]): to do so we keep sticking to the physically relevant three dimensional case just for the the time being. Calling $E \subset \mathbb{R}^3$ the set corresponding to our drop, De Giorgi's perimeter $P(E)$ is the optimal notion employable with the purpose of minimizing the surface area, as it often happens in many variational problems. In addition, the fact that it is defined locally on the drop suits perfectly the features of surface tension. On the other hand, we represent the configuration assumed by the charge with a probability measure μ supported on the set E. Its Coulombic interaction takes into account of all the possible couples of units of charge (hence it is nonlocal) and it writes as:

$$Q^2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x) d\mu(y)}{|x-y|}.$$

Since the charge is free to position itself within the drop according to its repulsive nature, it is reasonable to assume that the optimal configuration it reaches minimizes its Coulombic energy. We do not know yet whether there can be proved existence of a minimum or not, hence we just pass the previous expression to the infimum and find the Riesz interaction energy of the set E:

$$Q^2 \mathcal{I}_2(E) := Q^2 \inf_{\mu(E)=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x) d\mu(y)}{|x-y|}.$$

The general energy of the whole system is easily defined by considering both the surface tension and the repulsive interaction together:

$$\mathcal{F}_{2,Q}(E) := P(E) + Q^2 \mathcal{I}_2(E).$$

As the liquid drop is free to move in $\mathbb{R}^3$ simultaneously maintaining its volume $m > 0$, it is natural to optimize in the shape of the drop E, in an attempt to find configurations of minimal energy. Thus, the following problem is determined:

$$\min_{|E|=m} \mathcal{F}_{2,Q}(E).$$

After presenting the model for charged drops in $\mathbb{R}^3$, we are ready to examine a more general version of the problem, with dimension not necessarily $N = 3$ and whose nonlocal interaction comprehends the Coulombic one just as a particular case. For $N \geq 2$ and $\alpha \in (0, N)$, we define consequently:

$$\mathcal{I}_\alpha(E) := \inf_{\mu(E)=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x) d\mu(y)}{|x-y|^{N-\alpha}} \qquad \text{and} \qquad \mathcal{F}_{\alpha,Q}(E) := P(E) + Q^2 \mathcal{I}_\alpha(E).$$

Therefore, as the title states, the aim of the dissertation is to study the variational problem:

$$\min_{|E|=m} \mathcal{F}_{\alpha,Q}(E).$$

The constant $m > 0$ denotes a fixed volume for the set E, and we always assume without loss of generality $m = \omega_N$. Our exposition deals with multiple classical topics of Calculus of Variations: definition and well-posedness of the problem, existence and regularity of minimizers, their stability under a specific class of perturbations. More precisely, the thesis is organized as follows.

In the first chapter the variational problem is introduced in a rigorous way, outlining all the notions we need to carry out our analysis in the next ones. We begin by covering some basic concepts from potential theory, among which the two most important are the definitions of Riesz energy of a set and of potential function of a measure, presented together with their essential properties. Next, a few remarks on fractional Laplacians and Sobolev spaces are provided as well. Later on, we resume the definition of the variational problem we just sketched above, immediately stating the only significant construction of the chapter, which leads us to infer ill-posedness of the minimum problem when $\alpha > 1$. Such outcome forces us to make some delicate considerations about the class of sets which we are allowed to work over and where the functional $\mathcal{F}_{\alpha,Q}$ is well-defined, that eventually result in properly formulating the problem we want to study. Then, we conclude by defining generalized sets and measures, thanks to which we can give another different formulation of the variational problem by broadening the domain of the energy functional.

As it turns out, generalized sets play a key role when trying to prove existence of minimizers for $\mathcal{F}_{\alpha,Q}$ and the first half of the second chapter is devoted to tackling this issue. We kickoff by modifying once again the functional, initially adding a Λ-relaxation to get rid of the volume constraint and then introducing an ε-regularization for the Riesz energy $\mathcal{I}_\alpha$. These adjustments make the problem definitely less tough to handle and, as a consequence, they let us prove existence of generalized minimizers for the new functional $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, by applying a relatively standard concentration-compactness argument to a minimizing sequence. At this point, the idea is to derive some properties of such generalized minimizers, with the purpose of recovering solutions to the variational problem for $\mathcal{F}_{\alpha,Q}$ we started with. Thus, we show a first minimality property enjoyed by generalized minimizers, thanks to which we are able to deduce a few of their first good features and then existence of solutions to the problem:

$$\min_{|\tilde{E}|=\omega_N} \mathcal{F}_{\alpha,Q}(\tilde{E}).$$

Next, we are ready to focus on regularity of minimizers: our aim consists in proving $C^{1,\beta}$ regularity of their boundaries for some $\beta \in (0, \frac{1}{2})$ and to do so we are obliged to separately consider the cases $\alpha < 1$ and $\alpha = 1$. When $\alpha \in (0, 1)$ the conclusion is straightforward, as we exploit once more the first almost minimality property we found before in order to appeal to classical regularity theory for almost minimizers of the perimeter. Moreover, with the help of the Quantitative Isoperimetric Inequality, we understand that for Q small enough not only any generalized minimizer is actually a classical one, but also it converges in $C^{1,\gamma}$ with $\gamma < \beta$ to the unit ball B_1 as $Q \to 0$. Obtaining the same conclusions for $\alpha = 1$ is a bit more tricky, as, unfortunately, we are not able to argue with the same strategy. Nevertheless, with the help of a somehow similar procedure, we still manage to show that generalized minimizers of $\mathcal{F}_{1,Q}$ enjoy Reifenberg flatness, a weaker yet extremely helpful for our goals notion of regularity. Later on, a second almost minimality property for minimizers of $\mathcal{F}_{1,Q}$ is derived: very similar to the first one, their only structural difference is the presence of an $L^{\frac{2N}{N+1}}$ norm, which needs to be estimated in order to draw the same outcomes of the case $\alpha < 1$. This is probably the most convoluted result of the dissertation and it requires many tools from elliptic PDE theory. The main ingredient is Alt-Caffarelli-Friedman monotonicity formula indeed, but we stress anyway the necessity of employing Reifenberg flatness. Finally, we conclude the chapter presenting an additional non-existence result in dimension $N = 2$ and for large enough charges Q.

The third and last chapter of the exposition begins by giving the definition of nearly spherical sets, namely all the sets E with $|E| = \omega_N$ and simultaneously close to the unit ball B_1 in the $C^{1,\gamma}$ topology. Then, we prove minimality of the ball for the functional $\mathcal{F}_{\alpha,Q}$ among nearly spherical sets sufficiently close to the ball, in the slightly more general case $\alpha \in (0, 2)$. Coming back to the situation we studied before with $\alpha \in (0, 1]$, we highlight that classical minimizers of $\mathcal{F}_{\alpha,Q}$ are nearly spherical if the charge Q is small enough. Thus, the last result allows us to infer both stability of the ball under small $C^{1,\gamma}$ perturbations and its minimality for the functional $\mathcal{F}_{\alpha,Q}$. In particular, the goals of proving existence, regularity and to characterize minimizers of $\mathcal{F}_{\alpha,Q}$ are all achieved and we can wrap up the dissertation.

The main references we relied upon are the two articles [12] and [15] by Michael Goldman, Matteo Novaga and Berardo Ruffini, where the variational problem was introduced and studied first. We highlight as well the importance of Landkov's book [16], source of almost all the useful results on potential theory. Of course, many other references are employed and quoted throughout the thesis and their exhaustive list can be found in the corresponding final section of the dissertation.

Chapter 1

The variational problem

The purpose of the first chapter is to set the stage for what will follow later on in the dissertation. The variational problem we want to study is introduced, as well as some helpful tools we will employ to tackle it. In the first section we present all the basic preliminary results which allow us to handle the matter. In the second one instead the problem is defined and we provide an ill-posedness result in the case $\alpha > 1$. Finally, in the last section we discuss the right class of sets where the functional is well defined and we introduce the notions of generalized sets and measures.

1.1 Notation and preliminary results

The first section is entirely devoted to collecting all of the helpful definitions and results which we are going to apply throughout the thesis. We deal mostly with basic topics from potential theory and functional analysis, especially focusing on Riesz energy and potentials. Before beginning, please notice that the current section is far from giving an exhaustive insight into these subjects: we decided only to report what will be strictly necessary later, avoiding the majority of the proofs. For a complete account on these topics, we suggest to consult either Landkov's monograph on potential theory [16] or the book [17] by Lieb and Loss.

First of all, we set some notation. In the followings, we work in the space $\mathbb{R}^N$ with $N \geq 2$. For any measurable set $E \subset \mathbb{R}^N$ and open set $\Omega \subset \mathbb{R}^N$, we write $|E|$ and $P(E,\Omega)$ to denote respectively the Lebesgue measure of E and its relative perimeter in Ω. When $\Omega = \mathbb{R}^N$, we just write $P(E)$. For $x \in \mathbb{R}^N$ and $r > 0$, we denote by $B_r(x)$ the open ball of radius r centered in x. We drop the dependence on x when the center is $x = 0$. In particular, B_1 denotes the unit ball and we set its Lebesgue measure to be $|B_1| = \omega_N$. For $k \in [0, N]$, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Concerning measures, we write $\mathcal{M}(\Omega)$ and $\mathcal{M}^+(\Omega)$ to denote respectively the spaces of Radon measures and positive Radon measures supported on Ω, dropping the dependence on Ω in the case $\Omega = \mathbb{R}^N$. Given any set $A \subset \mathbb{R}^N$, the expression χ_A indicates its characteristic function. Finally, for any function f, we denote by f^* its symmetric decreasing rearrangement according to [17, Section 3.3].

In addition, we adopt the following notation to compare quantities: we write $A \lesssim B$ to indicate that there exists a constant $C > 0$ (typically depending on the dimension N and a parameter α) such that $A \leq CB$. In case, we will specify when C depends on other quantities and we write $A \approx B$ when $A \lesssim B \lesssim A$. Conversely, we use the notation $A \ll B$, when there exists a small universal constant $\varepsilon > 0$ (again, usually depending on N and α or possibly other quantities) such that $A \leq \varepsilon B$.

A little disclaimer before beginning: the majority of definitions and results reported here are valid more in general for signed Radon measures. Anyway, we will just need positive measures for the purposes of the dissertation, therefore we present the theory restricting ourselves to them.

Definition 1.1 (Interaction energy)**.** Let $N \geq 2$ and $\alpha \in (0, N)$. Given μ and ν positive Radon measures, we define the *interaction energy* between μ and ν as:

$$I_\alpha(\mu,\nu) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x)d\nu(y)}{|x-y|^{N-\alpha}} \in [0,+\infty].$$

When $\mu = \nu$, we simply write $I_\alpha(\mu) := I_\alpha(\mu,\mu)$. In case the measures are absolutely continuous with respect to the Lebesgue measure, so that $\mu = f\mathcal{H}^N$ and $\nu = g\mathcal{H}^N$ for some functions f and g, we denote $I_\alpha(f,g) := I_\alpha(\mu,\nu)$.

An immediate consequence of the definition is that the functional $I_\alpha(\cdot,\cdot)$ is a positive, bilinear operator on the product space $\mathcal{M}^+ \times \mathcal{M}^+$ and that $I_\alpha(\cdot)$ is a quadratic form. In particular, Cauchy-Schwarz inequality is satisfied:

$$I_\alpha(\mu,\nu) \leq I_\alpha(\mu)^{\frac{1}{2}} I_\alpha(\nu)^{\frac{1}{2}}.$$

Even though we do not need it, it can be shown that $I_\alpha(\cdot,\cdot)$ is a positive, bilinear operator also on the product space $\mathcal{M} \times \mathcal{M}$. Hence, in particular, $I_\alpha(\mu) \geq 0$ regardless of the sign of the measure μ and $I_\alpha(\mu) = 0$ if and only if $\mu = 0$. On the other hand, the next result [16, Formula 1.4.4] is true only for positive measures: we endow the space $\mathcal{M}^+$ with the weak topology given by duality with the space of continuous functions with compact support $C_c(\mathbb{R}^N)$.

Proposition 1.2. The functional I_α is lower semicontinuous with respect to weak convergence of measures, namely if $\mu_n \rightharpoonup \mu$ and $\nu_n \rightharpoonup \nu$, we have that:

$$I_\alpha(\mu,\nu) \leq \liminf_{n\to+\infty} I_\alpha(\mu_n,\nu_n).$$

Then, it is rather natural to define the Riesz potential energy of a Borel set.

Definition 1.3 (Riesz energy)**.** Let $N \geq 2$ and $\alpha \in (0, N)$, for every Borel set A we define the *Riesz potential energy* of A as:

$$\mathcal{I}_\alpha(A) := \inf\left\{ I_\alpha(\mu) : \mu \in \mathcal{M}^+(A),\, \mu(A) = 1 \right\}. \tag{1.1}$$

From the definition, we immediately infer monotonicity of $\mathcal{I}_\alpha$, namely $A \subset B$ implies $\mathcal{I}_\alpha(A) \geq \mathcal{I}_\alpha(B)$. In addition, since I_α is a quadratic form over $\mathcal{M}$, we have in particular that $Q^2 I_\alpha(\mu) = I_\alpha(Q\mu)$ for any measure μ. Passing the expression to the infimum over all measures $\mu \in \mathcal{M}^+(A)$ such that $\mu(A) = 1$ yields immediately:

$$Q^2\mathcal{I}_\alpha(A) = \inf\left\{ I_\alpha(\mu) : \mu \in \mathcal{M}^+,\, \mu(A) = Q \right\}. \tag{1.2}$$

In particular, this is true because there is a one-to-one correspondence between finite measures supported on A with total mass 1 and those with total mass Q. The next fact in slightly trickier to prove instead.

Proposition 1.4. Let $N \geq 2$ and $\alpha \in (0, N)$, for every $\lambda > 0$ there holds:

$$\mathcal{I}_\alpha(\lambda A) = \lambda^{-(N-\alpha)}\mathcal{I}_\alpha(A). \tag{1.3}$$

Proof. We define the map $T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ such that $T(x) = \lambda x$, so in this way $T(A) = \lambda A$. T is clearly a diffeomorphism, so there exists a one-to-one correspondence between probability measures supported on A and those supported on λA given by the pushforward of measures $T_\#$. In particular, for every μ

competitor for $\mathcal{I}_\alpha(A)$, the measure $\nu = T_\#\mu$ is a competitor for $\mathcal{I}_\alpha(\lambda A)$. Therefore, by simple properties of pushforward:

$$\begin{aligned}
\mathcal{I}_\alpha(\lambda A) &= \inf_{\nu(\lambda A)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\nu(x)\,d\nu(y)}{|x-y|^{N-\alpha}} = \inf_{T_\#\mu(\lambda A)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{dT_\#\mu(x)\,dT_\#\mu(y)}{|x-y|^{N-\alpha}} \\
&= \inf_{\mu(A)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\,d\mu(y)}{|T(x)-T(y)|^{N-\alpha}} = \frac{1}{\lambda^{N-\alpha}} \inf_{\mu(A)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\,d\mu(y)}{|x-y|^{N-\alpha}} \\
&= \lambda^{-(N-\alpha)}\mathcal{I}_\alpha(A).
\end{aligned}$$

□

From [16, p.131] we get the following important result.

Proposition 1.5. If A is a compact set, the infimum in 1.1 is achieved, namely there exists a probability measure $\mu \in \mathcal{M}^+(A)$ such that $\mathcal{I}_\alpha(A) = I_\alpha(\mu)$. Moreover, the minimizing measure μ is unique.

Before going on exploring other properties of Riesz energy, we need to make a small drift in the world of fractional Laplacians and of fractional Sobolev spaces. Again, we refrain from being precise and we restrict ourselves only to the essential for the purposes of the dissertation. The main sources we used are the guide [5] by Di Nezza, Palatucci and Valdinoci, and Lieb and Loss' book [17].

For the Fourier transform, we use the convention:

$$\widehat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2i\pi\xi\cdot x} u(x)\,dx.$$

In this way, exploiting [5, Proposition 3.4], we can define the homogeneous H^s semi-norm for $s \in \mathbb{R}^N$ as:

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{u}|^2 d\xi,$$

writing $[u]_{H^s}$ instead of $[u]_{H^s(\mathbb{R}^N)}$ when there is no risk of confusion. For $s \in \mathbb{R}^N$, we define the s-fractional Laplacian by its Fourier transform:

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\widehat{u},$$

so that, by Parseval identity, it immediately follows:

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u(-\Delta)^s u. \tag{1.4}$$

For $s \in (0,1)$, by [5, Proposition 3.3] there exists a constant $C(N,s) > 0$ such that we can give the explicit expression for the s-fractional Laplacian:

$$(-\Delta)^s u = C(N,s) \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,$$

where the integral is intended in the principal value sense. Applying it to 1.4, we have the alternative formula:

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dxdy. \tag{1.5}$$

In Chapter 3 we will also use fractional Sobolev spaces defined on the unit sphere ∂B_1. For these, we take 1.5 as a starting point and we define for $\varphi : \partial B_1 \longrightarrow \mathbb{R}$ and $s \in (0,1)$:

$$[\varphi]^2_{H^s(\partial B_1)} = \int_{\partial B_1\times\partial B_1} \frac{(\varphi(\sigma)-\varphi(v))^2}{|\sigma-v|^{N-1+2s}}\,d\sigma dv. \tag{1.6}$$

Moreover, the following result holds.

Proposition 1.6. Denoting $\bar\varphi = \frac{1}{P(B_1)}\int_{\partial B_1}\varphi$, we have for $0 < s < s' < 1$:

$$\int_{\partial B_1}(\varphi-\bar\varphi)^2 \lesssim [\varphi]^2_{H^s(\partial B_1)} \le 2^{2(s'-s)}[\varphi]^2_{H^{s'}(\partial B_1)} \lesssim \int_{\partial B_1}|\nabla\varphi|^2, \tag{1.7}$$

where we denote by $\nabla\varphi$ the tangential gradient when there is no risk of confusion and where the implicit constants depend on N, s and s'.

Proof. Since $s' > s$, the second inequality is easy once realized that for $\sigma, v \in \partial B_1$ we have:

$$\frac{1}{|\sigma-v|^{N-1+2s}} = \frac{|\sigma-v|^{2(s'-s)}}{|\sigma-v|^{N-1+2s'}} \le 2^{2(s'-s)}\frac{1}{|\sigma-v|^{N-1+2s'}}$$

and we can conclude by 1.6. The first inequality is provided by Cauchy-Schwarz and a computation:

$$\begin{aligned}\int_{\partial B_1}(\varphi-\bar\varphi)^2 &= \frac{1}{P^2(B_1)}\int_{\partial B_1}\left(\int_{\partial B_1}\varphi(\sigma)-\varphi(v)\,dv\right)^2 d\sigma\\ &\le \frac{1}{P^2(B_1)}\int_{\partial B_1}\left(\int_{\partial B_1}\frac{(\varphi(\sigma)-\varphi(v))^2}{|\sigma-v|^{N-1+2s}}\,dv\right)\left(\int_{\partial B_1}|\sigma-v|^{N-1+2s}\right)d\sigma\\ &\lesssim \int_{\partial B_1\times\partial B_1}\frac{(\varphi(\sigma)-\varphi(v))^2}{|\sigma-v|^{N-1+2s}}\,d\sigma dv.\end{aligned}$$

Finally, the third inequality follows from [4, Proposition 2.7 and Remark 2.8]. □

Now, we come back to Riesz energies and we start with the useful definition of α-capacity.

Definition 1.7 (α-Capacity)**.** Let $N \ge 2$ and $\alpha \in (0, N)$. Let $A \subset \mathbb{R}^N$ be a Borel set, its α-*capacity* is defined to be:

$$C_\alpha(A) := \frac{1}{\mathcal{I}_\alpha(A)}$$

From monotonicity of the Riesz energy $\mathcal{I}_\alpha$ we get monotonicity of the α-capacity C_α, specifically $A \subset B$ implies $C_\alpha(A) \le C_\alpha(B)$. Moreover, it can be shown [16, p. 141] that the α-capacity is subadditive over compact sets.

For $N \ge 3$, $\alpha = 2$ and $K \subset \mathbb{R}^N$ we have the following representation of the 2-capacity from [17]:

$$C_2(K) = \inf\left\{\int_{\mathbb{R}^N}|\nabla f|^2 : f \in C_c^\infty(\mathbb{R}^N),\ f \ge \chi_K\right\}.$$

More in general, this is true for any $\alpha \in (0, N)$ as well:

$$C_\alpha(K) = \inf\left\{[f]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)} : f \in C_c^\infty(\mathbb{R}^N),\ f \ge \chi_K\right\}.$$

Finally, considering 1.5, we get for $\alpha \in (0, 2)$:

$$C_\alpha(K) = \inf\left\{\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+\alpha}}\,dxdy : u \in C_c^\infty(\mathbb{R}^N),\ u \ge \chi_K\right\}. \tag{1.8}$$

For any particular property, we say that it holds α-quasi everywhere if it is true up to sets of zero α-capacity. The next proposition corresponds to [19, Theorem 8.9] and it clarifies the relation between the notions of capacity and Hausdorff measure.

Proposition 1.8. Let $A \subset \mathbb{R}^N$ be a Borel set. Then:

1. if $\alpha > 0$ and $\mathcal{H}^\alpha(A) < +\infty$, then $C_\alpha(A) = 0$;
2. if $\alpha > 0$ and $C_\alpha(A) = 0$, then $\mathcal{H}^t(A) = 0$ for every $t > \alpha$.

In particular, $C_\alpha(A) = 0$ for any $\alpha > 0$ implies immediately $|A| = 0$. Exploiting characterization 1.8 of the α-capacity, we get the following nice result.

Proposition 1.9. Let $N \geq 2$ and $\alpha \in (0, 2)$, for all compact sets $K \subset \mathbb{R}^N$ with $|K| > 0$ we have that $\mathcal{I}_\alpha(B_K) \geq \mathcal{I}_\alpha(K)$, where B_K denotes the ball such that $|B_K| = |K|$.

Proof. By definition of α-capacity, it is enough to prove that $C_\alpha(B_K) \leq C_\alpha(K)$ for any compact set K with strictly positive volume. Since $\alpha \in (0, 2)$, we can use the characterization given by formula 1.8. Then, by [8, Theorem A.1] we have:

$$\begin{aligned} C_\alpha(K) &= \inf_{u \in C_c^\infty(\mathbb{R}^N),\, u \geq \chi_K} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+\alpha}}\, dxdy \\ &\geq \inf_{u \in C_c^\infty(\mathbb{R}^N),\, u \geq \chi_K} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u^*(x) - u^*(y))^2}{|x-y|^{N+\alpha}}\, dxdy. \end{aligned}$$

Since the symmetric decreasing rearrangements of positive functions is order preserving [17, p. 81], we have that $u \geq \chi_K$ implies $u^* \geq \chi_K^* = \chi_{B_K}$. Thus:

$$\begin{aligned} C_\alpha(K) &\geq \inf_{u^* \in C_c^\infty(\mathbb{R}^N),\, u^* \geq \chi_{B_K}} \left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u^*(x) - u^*(y))^2}{|x-y|^{N+\alpha}}\, dxdy : u^* \text{ radially symmetric, decreasing} \right\} \\ &\geq \inf_{u^* \in C_c^\infty(\mathbb{R}^N),\, u^* \geq \chi_{B_K}} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u^*(x) - u^*(y))^2}{|x-y|^{N+\alpha}}\, dxdy. = C_\alpha(B_K). \end{aligned}$$

□

Another useful property of symmetric decreasing rearrangements is given by next inequality.

Theorem 1.10 (Riesz rearrangement inequality)**.** Let $f, g, h : \mathbb{R}^N \longrightarrow R^+$. Then:

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} f(x) g(x-y) h(y)\, dxdy \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} f^*(x) g^*(x-y) h^*(y)\, dxdy.$$

For any set A we denote by B_A the ball centered in the origin such that $|A| = |B_A|$. Observing that $\chi_A^* = \chi_{B_A}$, an easy application of Riesz rearrangement inequality is given by:

$$\int_{A \times A} \frac{dx\, dy}{|x-y|^{N-\alpha}} \leq \int_{B_A \times B_A} \frac{dx\, dy}{|x-y|^{N-\alpha}}. \tag{1.9}$$

Indeed:

$$\begin{aligned} \int_{A \times A} \frac{dx\, dy}{|x-y|^{N-\alpha}} &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \chi_A(x) \frac{1}{|x-y|^{N-\alpha}} \chi_A(y)\, dx\, dy \\ &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \chi_A^*(x) \frac{1}{|x-y|^{N-\alpha}} \chi_A^*(y)\, dx\, dy = \int_{B_A \times B_A} \frac{dx\, dy}{|x-y|^{N-\alpha}}. \end{aligned}$$

At this point, it just remains to spend a few words on the potential of a measure.

Definition 1.11 (Potential function)**.** Let $N \geq 2$ and $\alpha \in (0, N)$, given $\mu \in \mathcal{M}^+$ we define its *potential function* as:

$$u_\alpha^\mu(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\alpha}} = \mu * |x|^{-(N-\alpha)}.$$

From now on we always drop the dependence on μ and α and we refer to the function u as the potential. Here we present its main properties, taken from [16, p. 137].

Proposition 1.12. Let $K \subset \mathbb{R}^N$ be compact, μ be the minimizer for $\mathcal{I}_\alpha(K)$ and u its corresponding potential. Then, the following equation holds in distributional sense:

$$(-\Delta)^{\frac{\alpha}{2}} u = C(N, \alpha)\mu,$$

where $C(N, \alpha) > 0$. In addition, we have:

1. $u = \mathcal{I}_\alpha(K)$ α-q.e. on $\mathrm{spt}(\mu)$ and $u \geq \mathcal{I}_\alpha(K)$ α-q.e. on K;
2. if $\alpha \in (0, 2]$, then $u = \mathcal{I}_\alpha(K)$ α-q.e. on K and $u \leq \mathcal{I}_\alpha(K)$ everywhere on $\mathbb{R}^N$.

Through the thesis we will focus more on the case $\alpha \in (0, 1]$, so we will often use point (2) from the previous result. In addition, we infer:

$$I_\alpha(\mu) = \int_{\mathbb{R}^N} u\, d\mu = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{-\frac{\alpha}{2}} \mu = C(N, \alpha) \int_{\mathbb{R}^N} \mu (-\Delta)^{-\frac{\alpha}{2}} \mu = \frac{1}{C(N, \alpha)} \int_{\mathbb{R}^N} u (-\Delta)^{\frac{\alpha}{2}} u.$$

From 1.4, we deduce immediately:

$$I_\alpha(\mu) = C(N, \alpha)[\mu]^2_{H^{-\frac{\alpha}{2}}(\mathbb{R}^N)} = \frac{1}{C(N, \alpha)} [u]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)}. \tag{1.10}$$

Finally, we provide the explicit expression for the optimal measure of the ball B_1 in the case $\alpha \in]0, 2[$.

Proposition 1.13. When $\alpha \in (0, 2)$, the optimal measure μ_{B_1} for the variational problem $\mathcal{I}_\alpha(B_1)$ is absolutely continuous with respect to Lebesgue measure and it is represented by the function:

$$\mu_{B_1}(x) = \frac{C_\alpha}{(1 - |x|^2)^{\frac{\alpha}{2}}} \chi_{B_1}(x),$$

where C_α indicates the suitable renormalization constant to make it a probability measure.

1.2 Introduction of the problem and ill-posedness when $\alpha > 1$

After defining all the basic tools we will employ, we are ready to present the variational problem we would like to study in our dissertation. Be careful that the first formulation we give is exclusively heuristic and far from being precise, similarly to what we did in the introduction. Actually, problem 1.13 is not even well-posed yet, as we still need to specify for which kind of sets the functional $\mathcal{F}_{\alpha,Q}$ is well-defined and, consequently, the right class where to minimize. Before dwelling on such issues, we present right away an ill-posedness result when $\alpha > 1$, which derives from some useful considerations about the α-capacity that we will investigate further in the next section.

Let $N \geq 2$, $\alpha \in (0, N)$ and a measurable set $E \subset \mathbb{R}^N$. We consider the Riesz interaction energy $\mathcal{I}_\alpha(E)$ for the set E and from now on we use the compact notation:

$$\mathcal{I}_\alpha(E) = \inf_{\mu(E)=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}}, \tag{1.11}$$

where the infimum is implicitly taken over all probability measures supported in E. For every charge $Q > 0$, we define the functional:

$$\mathcal{F}_{\alpha,Q}(E) = P(E) + Q^2\mathcal{I}_\alpha(E)\,. \tag{1.12}$$

Fixed any mass $m > 0$, the aim of the thesis is to carry out an analysis of the variational problem:

$$\min_{|E|=m} \mathcal{F}_{\alpha,Q}(E), \tag{1.13}$$

where the precise class which we are minimizing over is yet to be specified. The following scaling argument allows us to assume $m = \omega_N$ without loss of generality. Indeed, if $|E| = m$, thanks to 1.3 we find:

$$\lambda^{N-\alpha}Q^2\mathcal{I}_\alpha(\lambda E) = Q^2\mathcal{I}_\alpha(E)$$

for any scaling factor $\lambda > 0$. Hence:

$$\begin{aligned} P(E) + Q^2\mathcal{I}_\alpha(E) &= \frac{1}{\lambda^{N-1}}P(\lambda E) + \lambda^{N-\alpha}Q^2\mathcal{I}_\alpha(\lambda E) \\ &= \frac{1}{\lambda^{N-1}}\left(P(\lambda E) + \lambda^{(N-\alpha)(N-1)}Q^2\mathcal{I}_\alpha(\lambda E)\right). \end{aligned}$$

By requiring $|\lambda E| = \omega_N$, we obtain $\lambda = (\frac{\omega_N}{m})^{\frac{1}{N}}$ and therefore it is enough to set:

$$\tilde{Q}^2 = \left(\frac{\omega_N}{m}\right)^{\frac{(N-\alpha)(N-1)}{N}} Q^2 \qquad \text{and} \qquad \tilde{E} = \lambda E.$$

In this way, we are able to study the equivalent problem with desired mass ω_N, up to a scaling factor of $(\frac{\omega_N}{m})^{N-1}$.

The first significant statement we prove about problem 1.13 is that in the case $\alpha \in (1, N)$, the functional $\mathcal{F}_{\alpha,Q}$ admits no minimizer among sets of given volume ω_N. The procedure we use in the proof consists in a geometric construction exploiting the relation between the notions of α-capacity and Hausdorff measure. The key observations are the following: first, as we will prove later on in the compact case, the Riesz energy $\mathcal{I}_\alpha$ is defined α-quasi everywhere [16, Chapter 2]. In addition, from what discussed at the beginning of [19, Chapter 8], α-capacity has a very similar behaviour to $\mathcal{H}^{N-\alpha}$. Specifically, when $\alpha > 1$, sets of positive α-capacity are not seen by the perimeter, which on the other hand operates like $\mathcal{H}^{N-1}$. Therefore, exploiting some properties of Riesz energy, we build a sequence of sets with uniformly bounded perimeter (actually converging to $P(B_1)$) and α-capacity diverging at $+\infty$. By what we just highlighted, the variation of the α-capacity is not seen by the perimeter and, by definition, the Riesz energy of our sequence of sets tends to 0. Since every compact set has positive α-capacity, we finally infer ill-posedness of problem 1.13 when $\alpha > 1$.

Theorem 1.14. For every $\alpha \in (1, N)$, there holds:

$$\inf_{|E|=\omega_N} \mathcal{F}_{\alpha,Q}(E) = P(B_1).$$

in particular, the problem does not admit minimizers.

Proof. The first inequality is straight-forward. For each Borel set $E \subset \mathbb{R}^N$ of given volume ω_N, we have that $\mathcal{I}_\alpha(E) \geq 0$; therefore, by Euclidean isoperimetric inequality [18, Theorem 14.1]:

$$P(E) + Q^2\mathcal{I}_\alpha(E) \geq P(E) \geq P(B_1),$$

We conclude by passing to the infimum over all sets E as above.

Concerning the second inequality, let $K \in \mathbb{N}$ and a number $\beta \geq 0$ to be chosen precisely later on. Consider K balls $\{B^i\}_{i=1}^K$ of radius $r_K = K^{-\beta}$ and centers $\{x_i\}_{i=1}^K$, such that $|x_i| \gg R$ for each $i \leq K$ and $|x_i - x_j| \gg R$ for each $i \neq j$, for some $R \geq r_K$. On each ball B^i, we put on an uniform charge ν^i such that $\nu^i(B^i) = \frac{1}{K}$ and we set:

$$\nu := \sum_{i=1}^{K} \nu^i.$$

Let $V_K = K(r_K)^N \omega_N$ be their total volume: we build a set E_K by taking the union of all these charged balls with a non-charged ball B^0 centered at the origin $x = 0$ of volume $\omega_N - V_K$. By our choice of α, we have that $N - \alpha < N - 1$, so we are able to choose any $\beta \in \left(\frac{1}{N-1}, \frac{1}{N-\alpha}\right)$. In this way:

- $\beta > \frac{1}{N-1}$ implies $\beta(N-1) > 1$, so that:

$$\lim_{K\to\infty} K(r_K)^{N-1} = \lim_{K\to\infty} \frac{K}{K^{\beta(N-1)}} = 0;$$

- $\beta < \frac{1}{N-\alpha}$ implies $\beta(N-\alpha) < 1$, so that:

$$\lim_{K\to\infty} \frac{1}{K} \frac{1}{(r_K)^{N-\alpha}} = \lim_{K\to\infty} \frac{K^{\beta(N-\alpha)}}{K} = 0;$$

- $\beta N > \beta(N-1)$, so that:

$$\lim_{K\to\infty} V_K = \lim_{K\to\infty} \frac{K\omega_N}{K^{\beta N}} < \lim_{K\to\infty} \frac{K\omega_N}{K^{\beta(N-1)}} = 0.$$

We are now ready to estimate $\mathcal{F}_{\alpha,Q}(E_K) = P(E_K) + Q^2\mathcal{I}_\alpha(E_K)$. Concerning the perimeter, by scaling $P(B^0) = (r_{B^0})^{N-1} P(B_1)$ and a simple computation shows that:

$$r_{B^0} = \left(\frac{\omega_N - V_K}{\omega_N}\right)^{\frac{N-1}{N}}.$$

Moreover:

$$P(\bigcup_{i=1}^{K} B^i) = \sum_{i=1}^{K} P(B^i) = C(N)K(r_K)^{N-1}.$$

On the other hand, providing an estimate for the Riesz interaction energy is more interesting. Noticing that the measure ν is a competitor for $\mathcal{I}_\alpha(E_K)$, we compute:

$$\begin{aligned}
\mathcal{I}_\alpha(E_K) &= \inf_{\mu(E_K)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} \\
&\leq \int_{E_K\times E_K} \frac{d\nu(x)\, d\nu(y)}{|x-y|^{N-\alpha}} \\
&= \int_{\bigcup_{i=1}^K B^i \times \bigcup_{j=1}^K B^j} \frac{d(\sum_{i=1}^K \nu^i)(x)\, d(\sum_{j=1}^K \nu^j)(y)}{|x-y|^{N-\alpha}} \\
&= \sum_{i=1}^{K} \int_{B^i\times B^i} \frac{d\nu^i(x)\, d\nu^i(y)}{|x-y|^{N-\alpha}} + \sum_{i\neq j} \int_{B^i\times B^j} \frac{d\nu^i(x)\, d\nu^j(y)}{|x-y|^{N-\alpha}} \\
&= \frac{1}{K^2} \sum_{i=1}^{K} \int_{B^i\times B^i} \frac{dx\, dy}{|x-y|^{N-\alpha}} + \frac{1}{K^2} \sum_{i\neq j} \int_{B^i\times B^j} \frac{dx\, dy}{|x-y|^{N-\alpha}}
\end{aligned}.$$

Regarding the first term, since $B^i = x_i + r_K B_1$ for all i, we find by changing coordinates:

$$\int_{B^i \times B^i} \frac{dx\,dy}{|x-y|^{N-\alpha}} = \frac{1}{(r_K)^{N-\alpha}} \int_{B_1 \times B_1} \frac{dx\,dy}{|x-y|^{N-\alpha}} = C(N,\alpha) \frac{1}{(r_K)^{N-\alpha}}.$$

The trick of changing coordinates does not work for the second term, due to the presence of two different indexes. Nevertheless, we can perform the following estimate by exploiting the geometry of the set E_K. For all $i, j \in \{1, \dots, K\}$ with $i \neq j$, for all $x \in B^i$ and $y \in B^j$, by triangle inequality we have that:

$$|x_i - x_j| \leq |x - x_i| + |x - y| + |y - x_j| \qquad \Longrightarrow \qquad |x - y| \geq |x_i - y_j| - |x - x_i| - |y - x_j|.$$

Now we use the fact that $|x_i - x_j| \gg R$ for each $i \neq j$ and that $|x - x_i| < R$, $|y - x_j| < R$, so:

$$|x - y| > |x_i - x_j| - 2R > \frac{1}{2}|x_i - x_j| \qquad \Longrightarrow \qquad |x - y|^{-(N-\alpha)} < |x_i - x_j|^{-(N-\alpha)}$$

Therefore, for all $i \neq j$:

$$\begin{aligned}\frac{1}{K^2} \int_{B^i \times B^j} \frac{dx\,dy}{|x-y|^{N-\alpha}} &< \frac{1}{K^2} \frac{C}{|x_i - x_j|^{N-\alpha}} \int_{B^i \times B^j} dx\,dy = \frac{C}{|x_i - x_j|^{N-\alpha}} \frac{C(N)(r_K)^{2N}}{K^2} \\ &\ll \frac{C(N)}{R^{N-\alpha} K^{2+2\beta N}}.\end{aligned}$$

Thus, putting the two estimates together we find:

$$\begin{aligned}\mathcal{I}_\alpha(E_K) &\ll \frac{1}{K^2} \sum_{i=1}^{K} C(N,\alpha) \frac{1}{(r_K)^{N-\alpha}} + \sum_{i \neq j} \frac{C(N)}{R^{N-\alpha} K^{2+2\beta N}} \\ &\leq C(N,\alpha) \left(\frac{K}{K^2} \frac{1}{(r_K)^{N-\alpha}} + \frac{1}{R^{N-\alpha}} \frac{K^2 - K}{K^{2+2\beta N}} \right) \\ &\leq C(N,\alpha) \left(\frac{1}{K} \frac{1}{(r_K)^{N-\alpha}} + \frac{1}{K^{2\beta N}} \frac{1}{R^{N-\alpha}}. \right)\end{aligned}$$

Finally, we come back to the estimate of $\mathcal{F}_{\alpha,Q}(E_K)$:

$$\begin{aligned}\inf_{|E| = \omega_N} \mathcal{F}_{\alpha,Q}(E) &\leq \mathcal{F}_{\alpha,Q}(E_K) = P(E_K) + Q^2 \mathcal{I}_\alpha(E_K) \\ &\leq \left(\frac{\omega_N - V_K}{\omega_N} \right)^{\frac{N-1}{N}} P(B_1) + C \left(K (r_K)^{N-1} + \frac{Q^2}{K} \frac{1}{(r_K)^{N-\alpha}} + \frac{1}{K^{2\beta N}} \frac{Q^2}{R^{N-\alpha}} \right).\end{aligned}$$

Since the right-hand side converges to $P(B_1 1)$ as $K \to +\infty$, we recover the required inequality.

The ill-posedness of the problem is straightforward once noticing that $\mathcal{I}_\alpha(B_1) > 0$ and thus the infimum cannot be achieved by any set. $\square$

1.3 Restriction to the interesting case $\alpha \in (0, 1]$

Once excluded the cases for which the problem is ill-posed, in this section we are ready to properly choose the class of competitors for 1.13. Afterwards, we define generalized sets and measures, consequently extending the notion of Riesz energy to them. Finally, we conclude the first chapter by stating the corresponding formulation of 1.14 for generalized sets, the starting point of our analysis in the next chapter.

At first glance, working with the class of smooth compact sets seems rather appropriate in order to minimize our functional, as both the perimeter and the Riesz interaction energy are well defined over it, as we discussed in the previous section. Awfully, we are forced to rule it out, due to its bad compactness property under many types of convergence, such as L^1_{loc} or Hausdorff convergence. On the other hand, since we are dealing with a minimum problem involving the perimeter, one could be tempted to employ sets of finite perimeter, by identifying two sets E and F agreeing up to a Lebesgue-negligible set. Unfortunately, it can be shown that the Riesz energy $\mathcal{I}_\alpha$ is well defined up to sets of 0 α-capacity, namely if $E = F$ α-quasi everywhere then $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(F)$. We prove this fact when E and F are compact sets (up to a choice of the right representative) with positive measure. If $C_\alpha(E\Delta F) = 0$, by monotonicity of α-capacity we have:

$$C_\alpha(E \setminus F) = C_\alpha(F \setminus E) = 0.$$

Recalling that $F = (F \cap E) \cup (F \setminus E)$, by subadditivity of C_α over compact sets and by monotonicity again we deduce:

$$C_\alpha(F) \leq C_\alpha(F \cap E) + C_\alpha(F \setminus E) \leq C_\alpha(E).$$

In a similar way, we have $C_\alpha(E) \leq C_\alpha(F)$ as well. Since E and F are compact sets with positive measure, $\mathcal{I}_\alpha(E), \mathcal{I}_\alpha(F) < +\infty$ by Proposition 1.9. In particular, $C_\alpha(E)$, $C_\alpha(F) \neq 0$ and:

$$\mathcal{I}_\alpha(E) = \frac{1}{C_\alpha(E)} = \frac{1}{C_\alpha(F)} = \mathcal{I}_\alpha(F).$$

We saw that $C_\alpha(K) = 0$ implies $|K| = 0$, but the converse is not true in general. Specifically, there exist sets agreeing Lebesgue almost everywhere but with different α-capacity, so we conclude that the class of sets of finite perimeter is not a feasible choice for trying to minimize our functional.

As advocated in [21], [22] for the particular case $N = 2$ and $\alpha = 1$, in our dissertation we consider the class:

$$\mathcal{S} = \left\{ E \subset \mathbb{R}^N : E \text{ is compact and } P(E) = \mathcal{H}^{N-1}(\partial E) < +\infty \right\}.$$

Identifying sets differing only a set of zero Lebesgue measure works perfectly in class $\mathcal{S}$, because in this case the Riesz interaction energy is well behaved. First of all, if $E, F \in \mathcal{S}$ with $|E\Delta F| = 0$ then $P(E) = P(F)$ by definition of perimeter. In addition, we claim that $\mathcal{H}^{N-1}(E\Delta F) = 0$ as well. Indeed, since $|E\Delta F| = 0$, we necessarily have $E^{(t)} = F^{(t)}$ for all $t \in [0, 1]$, where $E^{(t)}$ denotes the set of points of density t:

$$E^{(t)} = \left\{ x \in \mathbb{R}^N : \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}.$$

By Federer's Theorem [18, Theorem 16.2], we have that:

$$E = E^{(1)} \cup E^{(1/2)} \ \mathcal{H}^{N-1}\text{-a.e.} \qquad \text{and} \qquad F = F^{(1)} \cup F^{(1/2)} \ \mathcal{H}^{N-1}\text{-a.e.}.$$

Since $E^{(1)} = F^{(1)}$ and $E^{(1/2)} = F^{(1/2)}$, we deduce $E = F$ $\mathcal{H}^{N-1}$-almost everywhere. In particular, basic properties of Hausdorff measure imply $\mathcal{H}^{N-\alpha}(E\Delta F) = 0$ for all $\alpha \in (0, 1]$, so $C_\alpha(E\Delta F) = 0$ by Proposition 1.8. Thus, we can prove that $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(F)$ arguing as we did in the previous paragraph.

In conclusion, the variational problem we study is:

$$\min_{|E| = \omega_N, E \in \mathcal{S}} \mathcal{F}_{\alpha,Q}(E) \tag{1.14}$$

Finally, we introduce the notion of generalized sets and minimizers, which will be essential in order to prove the existence of classical minimizer for problem 1.14.

Definition 1.15 (Generalized sets, measures and energies).

- A (possibly finite) collection of sets $\tilde{E} = \{E^i\}_{i\geq 1}$ with $E^i \subset \mathbb{R}^N$ for all $i \geq 1$ is said to be a *generalized set*. We set the volume and the perimeter of a generalized set to be, respectively:

$$|\tilde{E}| = \sum_{i\geq 1} |E^i| \qquad\qquad P(\tilde{E}) = \sum_{i\geq 1} P(E^i).$$

- A (possibly finite) collection of measures $\tilde{\mu} = \{\mu^i\}_{i\geq 1}$ with $\mu^i \in \mathcal{M}^+$ for all $i \geq 1$ is said to be a *generalized measure*. The Riesz interaction energy of a generalized measure is set to be:

$$\mathcal{I}_\alpha(\tilde{\mu}) = \sum_i \mathcal{I}_\alpha(\mu^i)$$

- The *Riesz energy* of a generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$ is defined to be:

$$\mathcal{I}_\alpha(\tilde{E}) = \inf_{\tilde{\mu}} \left\{ \mathcal{I}_\alpha(\tilde{\mu}) \,:\, \sum_i \mu^i(E^i) = 1 \right\}.$$

Regarding the last definition, when minimizing over $\tilde{\mu}$, we may assume without loss of generality that μ^j is concentrated on its corresponding set E^j for all $j \geq 1$, namely μ^j is supported on E^j. Indeed, if otherwise $\mu^j = \mu^j_1 + \mu^j_2$ with μ^j_1, $\mu^j_2 \neq 0$ positive measures supported respectively on E^j and on $(E^j)^c$, by the fact that:

$$\mathcal{I}_\alpha(\mu^j_1 + \mu^j_2) \geq \mathcal{I}_\alpha(\mu^j_1) \qquad \text{and} \qquad \mu^j_1(E^j) + \sum_{i\neq j} \mu^i(E^i) = 1$$

(the energy does not increase and the condition on the weights is not affected), we can set $\mu^j = \mu^j_1$.

After introducing the last ideas, we are able to define the energy of a generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$ given a charge $Q > 0$:

$$\mathcal{F}_{\alpha,Q}(\tilde{E}) = P(\tilde{E}) + Q^2 \mathcal{I}_\alpha(\tilde{E}). \tag{1.15}$$

Moreover, we say that $\tilde{E} \in \mathcal{S}^{\mathbb{N}}$ is a (volume-constrained) generalized minimizer for the functional $\mathcal{F}_{\alpha,Q}$ if, for any collection $\tilde{F} \in \mathcal{S}^{\mathbb{N}}$ with $|\tilde{E}| = |\tilde{F}|$, we have:

$$\mathcal{F}_{\alpha,Q}(\tilde{E}) \leq \mathcal{F}_{\alpha,Q}(\tilde{F})$$

Hence, it is rather natural to introduce as well the variational problem:

$$\min_{|\tilde{E}|=\omega_N, \tilde{E}\in\mathcal{S}^{\mathbb{N}}} \mathcal{F}_{\alpha,Q}(\tilde{E}). \tag{1.16}$$

The study of this formulation will be an intermediate step in order to prove existence of minimizers for the original one 1.14 and it will constitute the starting point of our analysis in Chapter 2.

Chapter 2

Existence and regularity of minimizers

The second chapter of the exposition is devoted to proving existence and $C^{1,\beta}$ regularity of minimizers of $\mathcal{F}_{\alpha,Q}$ with volume ω_N and belonging to the class $\mathcal{S}$, in the case $\alpha \in (0,1]$. Initially, we modify the functional with a relaxation of the volume constraint and a regularization of the Riesz energy $\mathcal{I}_\alpha$, to put ourselves in a situation where is more convenient to work applying standard Calculus of Variations' techniques. In this way, we manage to get existence of generalized minimizers for the modified version of the functional $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. The continuation consists in showing regularity properties of such minimizers, with the aim of finding solutions to the initial problem as well. To do so, we separately consider the cases $\alpha \in (0,1)$ and $\alpha = 1$: the former is easier to be dealt with as we are able to rely on standard regularity theory for almost minimizers of the perimeter. Instead, when $\alpha = 1$ the situation is much more complicated and it requires the aid of tools from elliptic PDE theory. Anyhow, we finally succeed in reaching the same results valid for the case $\alpha \in (0,1)$.

Before starting our analysis, we anticipate that the second chapter is quite convoluted and it is definitely the most voluminous of the exposition. A possible strategy to lighten it could have been splitting existence and regularity of minimizers into separate parts of the dissertation. However, as we will see below, the procedures to derive them happen to be rather entangled with each other: for this reason, we decided to privilege fluidity of the general reasoning, presenting existence and regularity altogether in the same chapter.

2.1 Relaxation of the volume constraint

Since the volume constraint $|\tilde{E}| = \omega_n$ of the variational problem 1.16 results rather cumbersome to deal with, the first step of our analysis consists in getting rid of it. To do so, we consider the following relaxation of the energy functional:

$$\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) = P(\tilde{E}) + Q^2\mathcal{I}_\alpha(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right|,$$

for some $\Lambda > 0$ to be determined, together with its associated minimum problem. The next lemma is the only result presented in the section and it shows that, for some $\Lambda > 0$ large enough, the variational problem associated to the relaxed energy functional coincides with the constrained one 1.16. We do not know whether the two problems we are examining attain minimum yet, so for the moment we just consider their infima, which are always well defined.

Lemma 2.1. For every $\alpha \in (0, N)$, $Q > 0$ and every $\Lambda \gg 1 + Q^2$, we have:

$$\inf_{\tilde{E} \in \mathcal{S}^{\mathbb{N}}} \left\{ \mathcal{F}_{\alpha,Q}(\tilde{E}) \,:\, |\tilde{E}| = \omega_N \right\} = \inf_{\tilde{E} \in \mathcal{S}^{\mathbb{N}}} \mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}). \tag{2.1}$$

Moreover, for such Λ, if $\tilde{E}$ is a minimizer of the right-hand side of 2.1, then $|\tilde{E}| = \omega_N$.

Proof. The $\geq$ inequality is clear and holds true for all $\Lambda > 0$. Indeed, by taking as a competitor any $\tilde{F} \in \mathcal{S}^{\mathbb{N}}$ with volume ω_N:

$$\inf_{\tilde{E} \in \mathcal{S}^{\mathbb{N}}} P(\tilde{E}) + Q^2 \mathcal{I}_\alpha(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right| \leq P(\tilde{F}) + Q^2 \mathcal{I}_\alpha(\tilde{F}) = \mathcal{F}_{\alpha,Q,\Lambda}(\tilde{F})$$

and we conclude by passing to the infimum over all $\tilde{F}$ as above.

It remains to prove $\leq$: let $\Lambda \gg 1 + Q^2$ and assume by contradiction that there exist $\tilde{E}$ with $|\tilde{E}| \neq \omega_N$ such that:

$$\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) \leq \inf_{\tilde{E} \in \mathcal{S}^{\mathbb{N}}} \left\{ \mathcal{F}_{\alpha,Q}(\tilde{E}) \,:\, |\tilde{E}| = \omega_N \right\}.$$

Notice that in our hypothesis we must have $|\tilde{E}| \neq \omega_N$, otherwise $\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) = \mathcal{F}_{\alpha,Q}(\tilde{E})$ which is not possible. Using B_1 as a competitor, we get:

$$P(\tilde{E}) + Q^2 \mathcal{I}_\alpha(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right| \leq P(B_1) + Q^2 \mathcal{I}_\alpha(B_1) \lesssim 1 + Q^2.$$

In particular, we have $\Lambda \left| |\tilde{E}| - \omega_N \right| \lesssim 1 + Q^2$, so $\left| |\tilde{E}| - \omega_N \right| \ll 1 + Q^2$ as we assumed $\Lambda \gg 1 + Q^2$; thus, it means that there exists $t = 1 + \delta$ with $|\delta| \ll 1$ such that:

$$|t\tilde{E}| = \omega_N \qquad \Longrightarrow \qquad t^N |\tilde{E}| = \omega_N \qquad \Longrightarrow \qquad t = \left(\frac{\omega_N}{|\tilde{E}|} \right)^{\frac{1}{N}}.$$

Using the well-known Taylor expansion $(1-x)^{-\beta} = 1 + \beta x + o(x)$ for $|x| \ll 1$ and for all $\beta \in \mathbb{C}$, we get:

$$t = \left(1 - \left(1 - \frac{|\tilde{E}|}{\omega_N} \right) \right)^{-\frac{1}{N}} = 1 + \frac{1}{N} \left(1 - \frac{\tilde{E}}{\omega_N} \right) + o\left(\left(1 - \frac{\tilde{E}}{\omega_N} \right) \right) = 1 + \delta,$$

with:

$$|\delta| \lesssim \left| |\tilde{E}| - \omega_N | \right|.$$

Now, we use the set $t\tilde{E} = \{t\tilde{E^i}\}_{i \geq 1}$ (which satisfies $|t\tilde{E}| = \omega_N$) as a competitor for the constrained energy and get $\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) \leq \mathcal{F}_{\alpha,Q}(t\tilde{E})$, namely:

$$P(\tilde{E}) + Q^2 \mathcal{I}_\alpha(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right| \leq P(t\tilde{E}) + Q^2 \mathcal{I}_\alpha(t\tilde{E}) = t^{N-1} P(\tilde{E}) + t^{-(N-\alpha)} Q^2 \mathcal{I}_\alpha(\tilde{E})$$

by standard properties of perimeter and Riesz energy. This time, we use the different Taylor expansion $x^\beta = (1 + x - 1)^\beta = 1 + \beta(x-1) + o((x-1))$ for $x \simeq 1$ and for all $\beta \in \mathbb{C}$ to get

$$\Lambda \left| |\tilde{E}| - \omega_N \right| \leq (t-1) \Big((N-1) P(\tilde{E}) - (N-\alpha) Q^2 \mathcal{I}_\alpha(\tilde{E}) \Big) + o((t-1)),$$

which yield, using that $t-1=\delta$ and majorizing the term $o((t-1))$:

$$\Lambda|\delta| \lesssim \Lambda \left||\tilde{E}| - \omega_N\right| \leq \delta \left((N-1)P(\tilde{E}) - (N-\alpha)Q^2\mathcal{I}_\alpha(\tilde{E})\right). \tag{2.2}$$

If $\delta \geq 0$:

$$\Lambda\delta \lesssim \delta P(\tilde{E}) \leq \delta \mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) \lesssim \delta(1+Q^2)$$

and thus $\Lambda \lesssim 1+Q^2$, which is a contradiction with the hypothesis $\Lambda \gg 1+Q^2$. In the opposite case $\delta \leq 0$ we reach the same contradiction, since 2.2 implies this time:

$$\Lambda\delta \lesssim |\delta| \left((N-\alpha)Q^2\mathcal{I}_\alpha(\tilde{E}) - (N-1)P(\tilde{E})\right) \leq |\delta| Q^2 \mathcal{I}_\alpha(\tilde{E}) \leq |\delta|\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) \lesssim |\delta|(1+Q^2).$$

Therefore, we conclude that for all $\tilde{E} \in \mathcal{S}^{\mathbb{N}}$ such that $|\tilde{E}| \neq \omega_N$ we must necessarily have:

$$\inf_{\tilde{E}\in\mathcal{S}^{\mathbb{N}}} \left\{\mathcal{F}_{\alpha,Q}(\tilde{E}) \,:\, |\tilde{E}| = \omega_N\right\} < \mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}).$$

Passing to the infimum over all $\tilde{E} \in \mathcal{S}^{\mathbb{N}}$ yields the remaining inequality in 2.1. Instead, the second part of the statement is reached once noticed that trivially there holds $\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) = \mathcal{F}_{\alpha,Q}(\tilde{E})$ when $|\tilde{E}| = \omega_N$ instead. □

2.2 The regularized functional

As we highlighted in the first chapter, the capacitary term $\mathcal{I}_\alpha$ and hence the functional $\mathcal{F}_{\alpha,Q,\Lambda}$ are not well defined in L^1, the natural setting where to study variational problems involving the perimeter. In addition, the class $\mathcal{S}$ itself in which we are minimizing is not closed under L^1 convergence, so it is not clear how to argue directly to minimize $\mathcal{F}_{\alpha,Q,\Lambda}$. The purpose of this section is to overcome this difficulty, by introducing a regularization $\mathcal{I}_{\alpha,\varepsilon}$ of the Riesz energy $\mathcal{I}_\alpha$ together with some of its properties for both classical and generalized sets: in particular $\mathcal{I}_{\alpha,\varepsilon}$ will be well defined in L^1. In Lemma 2.4 we prove some estimates for the regularized Riesz energy, which will allow us to show existence and uniqueness of an optimal generalized measure minimizing $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$ in Lemma 2.5, similarly to what happen for $\mathcal{I}_\alpha$.

Let $\varepsilon > 0$ and μ be a positive measure, we define the regularized interaction energy of μ:

$$I_{\alpha,\varepsilon}(\mu) = I_\alpha(\mu) + \varepsilon\int_{\mathbb{R}^N}\mu^2 = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\,d\mu(y)}{|x-y|^{N-\alpha}} + \varepsilon\int_{\mathbb{R}^N}\mu^2,$$

setting $I_{\alpha,\varepsilon}(\mu) = +\infty$ if $\mu \notin L^2(\mathbb{R}^N)$. Consequently, for a measurable set $E \in \mathcal{S}$, we define its regularized Riesz interaction energy as:

$$\mathcal{I}_{\alpha,\varepsilon}(E) = \inf_{\mu(E)=1} \left\{\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\,d\mu(y)}{|x-y|^{N-\alpha}} + \varepsilon\int_{\mathbb{R}^N}\mu^2\right\}, \tag{2.3}$$

where the infimum is taken over the class of probability measures belonging to L^2 and supported in E. The fact that $\mathcal{I}_{\alpha,\varepsilon}(E)$ is well defined Lebesgue almost everywhere follows directly from its definition. Indeed, let E and F be measurable sets such that $|E\Delta F| = 0$: every measure μ such that $I_{\alpha,\varepsilon}(\mu) < +\infty$ is a positive function in $L^2(\mathbb{R}^N)$ and therefore, since the Lebesgue integral is defined up to Lebesgue negligible sets, satisfies:

$$\int_E \mu(x)\,dx = \int_F \mu(x)\,dx, \qquad \Longrightarrow \qquad \mu(E) = \mu(F).$$

Thus, in the infimum problems $\mathcal{I}_{\alpha,\varepsilon}(E)$ and $\mathcal{I}_{\alpha,\varepsilon}(F)$, both the sets of competitors and the values of the functionals coincide, hence the desired equality $\mathcal{I}_{\alpha,\varepsilon}(E) = \mathcal{I}_{\alpha,\varepsilon}(F)$.

In analogy with 1.15, it is natural to state the following definition.

Definition 2.2 (Regularized Riesz energy for generalized sets)**.**

- The *regularized energy* of a generalized measure $\tilde{\mu} = \{\mu^i\}_{i\geq 1}$ is set to be:

$$I_{\alpha,\varepsilon}(\tilde{\mu}) = \sum_i \mathcal{I}_{\alpha,\varepsilon}(\mu^i).$$

- The *regularized Riesz interaction energy* of a generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$ is defined to be:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \inf_{\tilde{\mu}} \left\{ I_{\alpha,\varepsilon}(\tilde{\mu}) \;:\; \sum_i \mu^i(E^i) = 1 \right\}.$$

An almost immediate consequence of the definition is the following characterization.

Lemma 2.3. Given a generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$, we have the equivalence:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \inf_{\{q_i\}_{i\geq 1}\subset[0,1]} \left\{ \sum_i q_i^2\, \mathcal{I}_{\alpha,\varepsilon}(E^i) \;:\; \sum_i q_i = 1 \right\}.$$

Proof. The idea is to exploit the 2-homogeneity of $I_{\alpha,\varepsilon}$, which is clear from its definition. In this way, arguing like in 1.2, we recover the relation:

$$Q^2\mathcal{I}_{\alpha,\varepsilon}(A) = \inf\{I_{\alpha,\varepsilon}(\mu) : \mu(A) = Q\} \qquad \forall\, A \subset \mathbb{R}^N \text{ Borel set.}$$

Thus, given any generalized measure $\tilde{\mu}$, we let $q_i = \mu^i(E^i)$. Using $I_{\alpha,\varepsilon}(\mu^i) \geq q_i^2\mathcal{I}_{\alpha,\varepsilon}(E^i)$, we have immediately:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \inf_{\tilde{\mu}} \left\{ \sum I_{\alpha,\varepsilon}(\mu^i) \;:\; \sum_i \mu^i(E^i) = 1 \right\} \geq \inf_{\tilde{\mu}} \left\{ \sum q_i^2\mathcal{I}_{\alpha,\varepsilon}(E^i) \;:\; \sum_i q_i = 1 \right\}.$$

Concerning the second inequality, we fix $\delta > 0$ and a sequence $\{q_i\}_{i\geq 1} \subset [0,1]$ such that $\sum_i q_i = 1$. Then, for all $i \geq 1$, we choose $\overline{\mu}^i$ such that $\overline{\mu}^i(E^i) = q_i$ and

$$I_{\alpha,\varepsilon}(\overline{\mu}^i) \leq \inf\{I_{\alpha,\varepsilon}(\mu^i) \;:\; \mu^i(E^i) = q_i\} + \frac{\delta}{2^i} = q_i^2\mathcal{I}_{\alpha,\varepsilon}(E^i) + \frac{\delta}{2^i}.$$

Hence, by definition:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \leq \sum_i I_{\alpha,\varepsilon}(\overline{\mu}^i) \leq \sum_i q_i^2\mathcal{I}_{\alpha,\varepsilon}(E^i) + \frac{\delta}{2^i} = \left(\sum_i q_i^2\mathcal{I}_{\alpha,\varepsilon}(E^i) \right) + \delta$$

Passing to the infimum over all $\{q_i\}_{i\geq 1}$ as above, we find:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \leq \inf_{\tilde{\mu}} \left\{ \sum q_i^2\mathcal{I}_{\alpha,\varepsilon}(E^i) \;:\; \sum_i q_i = 1 \right\} + \delta,$$

and the thesis follows by taking the limit as $\delta \to 0$. $\square$

Similarly to what happens for classical sets, $\mathcal{I}_{\alpha,\varepsilon}$ is well defined in L^1 over generalized sets. In other words, we can identify generalized sets agreeing up to another generalized set of measure 0, namely $\tilde{E}$ and $\tilde{F}$ such that $|E^i \Delta F^i| = 0$ for all $i \geq 1$. The next lemma provides upper and lower bounds for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$.

Lemma 2.4. Let $\tilde{E} = \{E^i\}_{i\geq 1}$ be a generalized set with $|\tilde{E}| \in (0, +\infty)$, then:

$$\frac{\varepsilon}{|\tilde{E}|} \leq \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \leq \frac{c(N,\alpha)}{|\tilde{E}|^{\frac{N-\alpha}{N}}} + \frac{\varepsilon}{|\tilde{E}|},$$

where

$$c(N,\alpha) = \frac{1}{\omega_N^{1+\frac{\alpha}{N}}} \int_{B_1\times B_1} \frac{1}{|x-y|^{N-\alpha}} dx\, dy.$$

Proof. We start with the upper bound: let $m = |\tilde{E}|$ and, for all $i \geq 1$, let B^i be a ball such that $|B^i| = |E^i|$. Choosing $\mu^i = \chi_E^i/m$ as a competitor in the definition of $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$ we find:

$$\begin{aligned}\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) &\leq \sum_i \frac{1}{m^2} \int_{E^i\times E^i} \frac{dx\, dy}{|x-y|^{N-\alpha}} + \frac{\varepsilon}{m^2}\sum_i |E^i| \\ &= \frac{1}{m^2}\sum_i \int_{E^i\times E^i} \frac{dx\, dy}{|x-y|^{N-\alpha}} + \frac{\varepsilon}{m}.\end{aligned}$$

By Riesz rearrangement inequality and its consequence 1.9, we get:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \leq \frac{1}{m^2}\sum_i \int_{B^i\times B^i} \frac{dx\, dy}{|x-y|^{N-\alpha}} + \frac{\epsilon}{m}.$$

Performing the changes of variable $\Phi_i : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ such that $\Phi_i(B_1) = B^i$

$$\Phi_i(x,y) = \left(\left(\frac{|E^i|}{\omega_N}\right)^{\frac{1}{N}} x, \left(\frac{|E^i|}{\omega_N}\right)^{\frac{1}{N}} y\right), \qquad \text{so} \qquad |\det(\mathrm{J}\Phi_i)(x,y)| = \left(\left(\frac{|E^i|}{\omega_N}\right)^{\frac{1}{N}}\right)^{2N} = \frac{|E^i|^2}{(\omega_N)^2},$$

we get for all $i \geq 1$:

$$\begin{aligned}\int_{B^i\times B^i} \frac{dx\, dy}{|x-y|^{N-\alpha}} &= \int_{B_1\times B_1} \frac{|E^i|^2}{(\omega_N)^2} \frac{dx\, dy}{|(|E^i|/\omega_N)^{\frac{1}{N}}(x-y)|^{N-\alpha}} = \frac{|E^i|^2}{(\omega_N)^2}\frac{(\omega_N)^{1-\frac{\alpha}{N}}}{|E^i|^{1-\frac{\alpha}{N}}} \int_{B_1\times B_1} \frac{dx\, dy}{|x-y|^{N-\alpha}} \\ &= c(N,\alpha)\, |E^i|^{1+\frac{\alpha}{N}}.\end{aligned}$$

Thus:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \leq \frac{c(N,\alpha)}{m^2}\sum_i |E^i|^{1+\frac{\alpha}{N}} + \frac{\varepsilon}{m} \leq \frac{c(N,\alpha)}{m^2}\left(\sum_i |E^i|\right)^{1+\frac{\alpha}{N}} + \frac{\varepsilon}{m} = \frac{c(N,\alpha)}{m^{\frac{N-\alpha}{N}}} + \frac{\varepsilon}{m}.$$

The lower bound is much shorter to obtain: for all $\tilde{\mu} = \{\mu^i\}_{i\geq 1}$ such that $\sum_i \mu^i(E^i) = 1$, we have by Cauchy-Schwarz inequality:

$$\varepsilon^{\frac{1}{2}} = \varepsilon^{\frac{1}{2}}\left(\sum_i \mu^i(E^i)\right) = \sum_i \int_{E^i} \varepsilon^{\frac{1}{2}} d\mu^i \leq \left(\sum_i |E^i|\right)^{\frac{1}{2}} \left(\varepsilon \sum_i \int_{E^i} (\mu^i)^2\right)^{\frac{1}{2}} \leq m^{\frac{1}{2}} I_{\alpha,\varepsilon}(\tilde{\mu}).$$

Passing to the infimum over all $\tilde{\mu}$ as above, we get the conclusion:

$$\frac{\varepsilon}{|\tilde{E}|} \leq \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}).$$

□

As a consequence of the previous Lemma, we can prove existence uniqueness of an optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$.

Lemma 2.5. For every $\varepsilon > 0$ and every generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$ with $|\tilde{E}| \in (0, +\infty)$ and $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) < +\infty$, there exists a unique optimal measure $\tilde{\mu} = \{\mu^i\}_{i\geq 1}$ for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$.

Proof. By Lemma 2.3, we have:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \inf\left\{\sum_i q_i^2\, \mathcal{I}_{\alpha,\varepsilon}(E^i) \,:\, \sum_i q_i = 1\right\}. \tag{2.4}$$

Hence, the existence of an optimal $\tilde{\mu}$ follows from the facts:

- (a) For every fixed set E such that $|E| + \mathcal{I}_{\alpha,\varepsilon}(E) < +\infty$, there exists a unique optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(E)$;
- (b) There exists a unique optimal distribution of charges $\{q_i\}_{i\geq 1} \subset [0, 1]$ for the problem 2.4.

(a) Let $E \subset \mathbb{R}^N$ be measurable and such that $|E| + \mathcal{I}_{\alpha,\varepsilon}(E) < +\infty$. Noticing that obviously we have $\mathcal{I}_{\alpha,\varepsilon}(E) > -\infty$, we select a minimizing sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^N)$, such that $\mu_n \geq 0$ a.e., $\int_{\mathbb{R}^N} \mu_n = 1$, $\mathrm{spt}(\mu_n) \subset E$ for all $n \in \mathbb{N}$ and:

$$\lim_{n\to+\infty} I_{\alpha,\varepsilon}(\mu_n) = \inf_{\mu(E)=1} I_{\alpha,\varepsilon}(\mu) = \mathcal{I}_{\alpha,\varepsilon}(E) < +\infty,$$

which implies, by convergence, $\sup_{n\in\mathbb{N}} I_{\alpha,\varepsilon}(\mu_n) < C$. Thus:

$$\int_{\mathbb{R}^N} \mu_n^2 = \int_E \mu_n^2 \leq \varepsilon^{-1} \sup_{n\in\mathbb{N}} I_{\alpha,\varepsilon}(\mu_n) < C\varepsilon^{-1}.$$

By Banach-Alaoglu Theorem, there exist $\mu \in L^2(\mathbb{R}^N)$ and a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ such that $\mu_{n_k} \rightharpoonup \mu$ as $k \to +\infty$. By weak lower semicontinuity of L^2 norm and of the functional I_α, we immediately deduce:

$$I_{\alpha,\varepsilon}(\mu) \leq \liminf_{k\to+\infty} I_{\alpha,\varepsilon}(\mu_{n_k}) = \mathcal{I}_{\alpha,\varepsilon}(E).$$

Therefore, we conclude that μ is the minimizer we are looking for by showing that it is an admissible competitor for the minimum problem $\mathcal{I}_{\alpha,\varepsilon}(E)$. First of all, $\{\mu_n\}_{n\in\mathbb{N}} \subset \{u \in L^2(\mathbb{R}^N) \,:\, u \geq 0 \text{ a.e.}\}$ which is closed and convex, hence weakly closed: $\mu_{n_k} \rightharpoonup \mu$ implies $\mu \geq 0$ Lebesgue almost everywhere. Now, we prove that μ is a probability measure on E: again, it easily follows by definition of weak convergence. Indeed, since $|E| < +\infty$, $\chi_E \in L^2(\mathbb{R}^N)$, so:

$$1 = \lim_{k\to+\infty} \int_E \mu_{n_k} = \lim_{k\to+\infty} \int_{\mathbb{R}^N} \chi_E\, \mu_{n_k} = \int_{\mathbb{R}^N} \chi_E\, \mu = \int_E \mu.$$

We find our minimizer by setting $\mu = 0$ on E^c without loss of generality. Uniqueness of the measure μ is easy to prove, because it suffices to show that the regularized energy functional $I_{\alpha,\varepsilon}$ is strictly convex over its domain $\mathcal{M}^+$. Since the term $\varepsilon \int_{\mathbb{R}^N} \mu^2$ is strictly convex, we just need to prove that I_α is convex. Let ρ_1, ρ_2 be positive measures and $t \in [0, 1]$, by bilinearity and Cauchy-Schwarz inequality we have:

$$\begin{aligned}
I_\alpha(t\rho_1 + (1-t)\rho_2) &= t^2 I_\alpha(\rho_1) + 2t(1-t) I_\alpha(\rho_1, \rho_2) + (1-t)^2 I_\alpha(\rho_2) \\
&\leq t^2 I_\alpha(\rho_1) + 2t(1-t)\left[\frac{1}{2} I_\alpha(\rho_1) + \frac{1}{2} I_\alpha(\rho_2)\right] + (1-t)^2 I_\alpha(\rho_2) \\
&\leq t I_\alpha(\rho_1) + (1-t) I_\alpha(\rho_2).
\end{aligned}$$

(b) Let $\tilde{E} = \{E^i\}_{i\geq 1}$ be a generalized set with $|\tilde{E}| < +\infty$, by the lower bound in the previous Lemma we get:

$$\sum_i \mathcal{I}_{\alpha,\varepsilon}^{-1}(E^i) \leq \frac{1}{\varepsilon} \sum_i |E^i| < +\infty \qquad \Longrightarrow \qquad \lim_{I\to+\infty} \sum_{i\geq I} \mathcal{I}_{\alpha,\varepsilon}^{-1}(E^i) = 0.$$

Hence, we consider a minimizing sequence for the minimum problem 2.4: $\{\{q_i^n\}_{i\geq 1}\}_{n\in\mathbb{N}}$ such that $\{q_i^n\}_{i\geq 1} \subset [0,1]$ and $\sum_i q_i^n = 1$ for all $n \in \mathbb{N}$ and see:

$$\sum_{i\geq I} q_i^n \leq \left(\sum_{i\geq I} (q_i^n)^2 \mathcal{I}_{\alpha,\varepsilon}(E^i) \right)^{\frac{1}{2}} \left(\sum_{i\geq I} \mathcal{I}_{\alpha,\varepsilon}^{-1}(E^i) \right)^{\frac{1}{2}} \longrightarrow 0 \quad \text{ as } I \to +\infty.$$

The minimizing sequence is tight and therefore, by Prohorov Theorem, there exist $\{q_i\}_{i\geq 1} \subset [0,1]$ such that $\sum_i q_i = 1$ and $\{q_i^n\}_{i\geq 1} \rightharpoonup \{q_i\}_{i\geq 1}$ in ℓ^1 as $n \to +\infty$, which turns out to be the optimal distribution of charge we are looking for by weak lower semicontinuity of the ℓ^2 norm. Uniqueness is yielded again by strict convexity of the functional, which is quadratic over ℓ^1. □

2.3 Existence of generalized minimizers for the regularized energy

After the preparation we developed in the last two sections, we can finally introduce the regularized relaxed energy functional:

$$\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}) = P(\tilde{E}) + Q^2 \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right|.$$

The current section is devoted to proving existence of generalized minimizers for the variational problem:

$$\min_{0<|\tilde{E}|<+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}), \tag{2.5}$$

thanks to a concentration-compactness technique performed in Theorem 2.7. Notice that we are not allowed to minimize over the class of sets $\mathcal{S}^{\mathbb{N}}$ with the mathematical tools we plan to work with, due to the issue with the functional $\mathcal{I}_\alpha$ we previously raised. Therefore, we minimize over the more general class of sets with finite measure which is way easier to handle for our purposes. In addition, we highlight that the kind of argument we employ in Theorem 2.7 constitutes another good motivation for introducing the regularized Riesz energy $\mathcal{I}_{\alpha,\varepsilon}$.

Anyway, before studying the problem 2.5, we need one last preparatory Lemma, stating that minimizing among classical or generalized sets gives us the same infimum energy.

Lemma 2.6. For every $\alpha \in (0,N)$, $Q > 0$, $\lambda > 0$ and $\varepsilon > 0$ we have:

$$\inf_{0<|E|<+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) = \inf_{0<|\tilde{E}|<+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}).$$

Proof. The $\geq$ inequality is trivial once noticing that every classical set is a generalized set. Therefore, it is enough to prove that for every $\delta > 0$ and for every generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$ there exists a set $E \subset \mathbb{R}^N$ with $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) \leq \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}) + \delta$. We fix $I \in \mathbb{N}$ and $R > 0$, then we let $F^i = E^i \cap B_R$ if $i \leq I$ and $F^i = \emptyset$ otherwise, and we set $\tilde{F} = \{F^i\}_{i\geq 1}$. First, we remark that, by monotone convergence, we have $\lim_{R\to+\infty} |E^i \cap B_R| = |E^i|$ for all $i \leq 1$. Therefore, since $\sum_i |E^i| \leq +\infty$, we get:

$$\lim_{I,R\to+\infty} |\tilde{F}| = \lim_{I,R\to+\infty} \sum_{i\leq I} |F^i| = \lim_{I\to+\infty} \sum_{i\leq I} \lim_{R\to+\infty} |F^i| = \lim_{I\to+\infty} \sum_{i\leq I} |E^i| = |\tilde{E}|,$$

again by monotone convergence. So, by continuity of the function $\Lambda|\cdot - \omega_N|$, we can choose I and R large enough such that:

$$\Lambda \left| \sum_{i=1}^{I} |F^i| - \omega_N \right| \leq \Lambda \left| |\tilde{E}| - \omega_N \right| + \delta. \tag{2.6}$$

Moreover, by [18, Lemma 15.12], for all $i \geq 1$ for almost every $R > 0$ we have:

$$P(E^i \cap B_R) = P(E^i, B_R) + \mathcal{H}^{N-1}(E^i \cap \partial B_R).$$

Also, Coarea Formula implies:

$$\int_0^{+\infty} \mathcal{H}^{N-1}(E^i \cap \partial B_R)\, dR = |E^i| \qquad \Longrightarrow \qquad \int_R^{2R} \mathcal{H}^{N-1}(E^i \cap \partial B_R)\, dR \leq |E^i|.$$

In particular, for all $i \geq 1$ there exists $R^i \in (R, 2R)$ such that $\mathcal{H}^{N-1}(E^i \cap \partial B_{R^i}) \leq \frac{|E^i|}{R}$. Combining the two properties, we get for all I:

$$\begin{aligned} P(\tilde{F}) &= \sum_{i=1}^{I} P(F^i) \leq \sum_{i=1}^{I} P(E^i \cap B_{R^i}) \leq \sum_{i=1}^{I} P(E^i, B_{R^i}) + \mathcal{H}^{N-1}(E^i \cap \partial B_{R^i}) \\ &\leq \sum_{i=1}^{I} P(E^i) + \frac{I \max_i |E^i|}{R} \leq P(\tilde{E}) + \delta \end{aligned} \tag{2.7}$$

again for R large enough. We only need to treat the energy term: let $\tilde{\mu} = \{\mu^i\}_{i\geq 1}$ be the optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$ given by Lemma 2.5. Then, we set $\nu^i = \frac{\mu^i|_{F^i}}{\sum_{i=1}^{I} \mu^i(F^i)}$ for $i \leq I$ and $\nu^i = 0$ otherwise, so that $\tilde{\nu} = \{\nu^i\}_{i\geq 1}$ is a competitor for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})$. By construction of $\tilde{F}$, we have that $\sum_{i=1}^{I} \mu(F^i)$ converges to 1 as both I and R goes to $+\infty$, so we can also assume as well that I and R are chosen large enough in order to have, apart from 2.6 and 2.7:

$$Q^2 I_{\alpha,\varepsilon}(\tilde{\nu}) = \frac{Q^2}{(\sum_{i=1}^{I} \mu(F^i))^2} \left(\int_{F^i \times F^i} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} + \varepsilon \int_{\mathbb{R}^N} (\mu^i)^2 \right) \leq Q^2 \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) + \delta. \tag{2.8}$$

Now We are finally ready to build the required set E by rearranging the sets F^i together with their associated measures ν^i for all $i \leq I$ in $\mathbb{R}^N$. First of all, we choose some points $\{x^i\}_{i=1}^{I} \subset \mathbb{R}^N$ such that $\min_{i\neq j} |x^i - x^j| \gg R$ and we define:

$$E = \bigcup_{i=1}^{I} (F^i + x^i) \qquad \text{and} \qquad \nu(x) = \sum_{i=1}^{I} \nu^i(x - x^i).$$

Since $F^i \subset B_R$ by construction for all $i \leq I$, the sets $F^i + x^i$ are pairwise disjoint, so the perimeter and the volume of their union decouple. In particular, from 2.6 and 2.7 we have:

$$P(E) + \Lambda \left||E| - \omega_N\right| = \sum_{i=1}^{I} P(F^i) + \Lambda \left| \sum_{i=1}^{I} |F^i| - \omega_N \right| \leq P(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right| + 2\delta.$$

Again, by construction, ν is admissible for $\mathcal{I}_{\alpha,\varepsilon}(E)$. Reasoning in the same exact way as we did in 1.14, we are able to estimate the remaining term:

$$\begin{aligned}
Q^2 I_{\alpha,\varepsilon}(\nu) &= Q^2 \int_{(\bigcup_{i=1}^I F^i)\times(\bigcup_{j=1}^I F^j)} \frac{d(\sum_{i=1}^K \nu^i)(x)\, d(\sum_{j=1}^K \nu^j)(y)}{|x-y|^{N-\alpha}} \\
&= Q^2 I_{\alpha,\varepsilon}(\tilde{\nu}) + Q^2 \sum_{i\neq j} \int_{F^i\times F^j} \frac{d\nu^i(x)\, d\nu^j(y)}{|x-y|^{N-\alpha}} \\
&\leq Q^2 I_{\alpha,\varepsilon}(\tilde{E}) + \delta + \frac{Q^2}{\min_{i\neq j}|x^i - x^j|} \\
&\leq Q^2 I_{\alpha,\varepsilon}(\tilde{E}) + \delta + \delta.
\end{aligned}$$

In the last inequality we used 2.8 and we assumed again R large enough. Eventually, we find as anticipated:

$$\begin{aligned}
\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) &\leq P(E) + Q^2 I_{\alpha,\varepsilon}(\nu) + \Lambda\, ||E| - \omega_N| \\
&\leq P(\tilde{E}) + Q^2 I_{\alpha,\varepsilon}(\tilde{E}) + \Lambda \left| |\tilde{E}| - \omega_N \right| + 4\delta \\
&= \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}) + 4\delta.
\end{aligned}$$

□

We are now ready to prove existence of generalized minimizers for the functional $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. The proof is one of the most significative of the dissertation and, as already anticipated, it relies on a concentration-compactness argument, aimed at preventing the loss of both mass and charge at $+\infty$. Starting from a minimizing sequence for $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$ made of classical sets, the idea is to split $\mathbb{R}^N$ into a partition of cubes, each one carrying its own mass and charge. After proving convergence of the values of such quantities, we build a generalized set which encompassing all the information obtained, together with its associated measure. Finally, we show that the set we constructed is actually a minimizer, thanks to the lower semicontinuity under L^1_{loc} convergence of all the terms composing the functional $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$.

Theorem 2.7. For every $\alpha \in (0,1]$, $Q > 0$, $\varepsilon > 0$ and $\Lambda \gg 1 + Q^2$, there exist generalized minimizers of the functional $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$.

Proof. Let $\{E_n\}_{n\in\mathbb{N}}$ be a classical minimizing sequence for $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, namely E_n is measurable for all $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E_n) = \inf_{0<|E|<+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E).$$

By Lemma 2.6, it is a minimizing sequence for generalized sets as well. Using the unit ball B_1 as a competitor we have:

$$\inf_{|E|<+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E) \leq \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(B_1) \lesssim 1 + Q^2 \qquad \Longrightarrow \qquad \sup_{n\in\mathbb{N}} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E_n) \lesssim 1 + Q^2. \tag{2.9}$$

In particular, we infer that $\Lambda\, ||E_n| - \omega_N| \lesssim 1 + Q^2$ for all $n \in \mathbb{N}$, so there exists a constant $C(N,\Lambda) > 0$ such that $m_n := |E_n| < C(N,\Lambda)$ for all $n \in \mathbb{N}$. Therefore, there exists $m \in]0,+\infty[$ such that, up to extraction of a subsequence, $m_n \longrightarrow m$ as $n \to +\infty$.

Now we proceed with the concentration-compactness argument. We fix a positive number $L \gg m^{\frac{1}{N}}$. For all $n \in \mathbb{N}$, we consider the lattice $(L\mathbb{Z})^N = \{z_{i,n}\}_{i\geq 1}$, thanks to which we construct a partition of $\mathbb{R}^N$ into cubes $\{Q_{i,n}\}_{i\geq 1}$, where $Q_{i,n} = [0,L]^N + z_{i,n}$. We let $m_{i,n} := |E_n \cap Q_{i,n}|$ (so that $\sum_i m_{i,n} = m_n$) and we assume without loss of generality that $\{m_{i,n}\}_{i\geq 1}$ is decreasing in i for all $n \in \mathbb{N}$, namely $m_{i,n} \geq m_{j,n}$ for all $i \leq j$. The procedure we are about to carry out is aimed at preventing the loss of mass when passing the limit as $n \to +\infty$, hence we can seamlessly consider from now on only the indexes i with measure $m_{i,n} > 0$. Finally, letting μ_n be the optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(E_n)$, we set $q_{i,n} := \mu_n(Q_{i,n})$, so that $\sum_i q_{i,n} = \mu_n(E_n) = 1$.

Now, we want to prove that there exist two sequences $\{m_i\}_{i\geq 1}$ and $\{q_i\}_{i\geq 1}$ such that, up to subsequences:

$$\{m_{i,n}\}_{i\geq 1} \rightharpoonup \{m_i\}_{i\geq 1} \quad \text{and} \qquad \{q_{i,n}\}_{i\geq 1} \rightharpoonup \{q_i\}_{i\geq 1} \quad \text{in } \ell^1 \text{ as } n \to +\infty,$$

which is:

$$\lim_{n\to+\infty} \sum_{i=1}^{+\infty} m_{i,n} f_i = \sum_{i=1}^{+\infty} m_i f_i \qquad \text{for all } f = \{f_i\}_{i\geq 1} \in \ell^\infty$$

and similarly for $\{q_i\}_{i\geq 1}$. In this way, applying the definition of weak convergence with the sequence $g = \{1\}_{i\geq 1}$, we get automatically:

$$m = \lim_{n\to+\infty} m_n = \lim_{n\to+\infty} \sum_{i=1}^{+\infty} m_{i,n} = \sum_{i=1}^{+\infty} m_i, \qquad 1 = \lim_{n\to+\infty} \mu_n(E_n) = \lim_{n\to+\infty} \sum_{i=1}^{+\infty} q_{i,n} = \sum_{i=1}^{+\infty} q_i.$$

In order to prove the desired weak convergence, we prove tightness of the two sequences of sequences $\{\{m_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$ and $\{\{q_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$. For the first one, we use the relative isoperimetric Inequality [18, Proposition 12.37] (more precisely its version employing cubes instead of balls), exploiting the fact that with our choice of $L \gg m^{\frac{1}{N}}$ we have $m_{i,n} = |Q_{i,n} \cap E_n| \leq |Q_{i,n}|/2$, so $\min\{|Q_{i,n} \cap E_n|, |Q_{i,n} \setminus E_n|\} = |Q_{i,n} \cap E_n|$. We compute:

$$\sum_i m_{i,n}^{\frac{N-1}{N}} \lesssim \sum_i P(E_n, Q_{i,n}) = P(E_n) \lesssim 1 + Q^2.$$

where the last inequality derive from 2.9. Using the assumptions that $\{m_{i,n}\}_{i\geq 1}$ is decreasing in i, for all $I \in \mathbb{N}$ we have $m_{i,n} \leq m_{I,n} \leq \frac{m_n}{I}$ for $i \geq I$, thus:

$$\sum_{i\geq I} m_{i,n} = \sum_{i\geq I} (m_{i,n})^{\frac{1}{N}} (m_{i,n})^{\frac{N-1}{N}} \leq \left(\frac{m_n}{I}\right)^{\frac{1}{N}} \sum_{i\geq I} (m_{i,n})^{\frac{N-1}{N}} \lesssim (1+Q^2) \left(\frac{m_n}{I}\right)^{\frac{1}{N}},$$

which tends to 0 as $I \to +\infty$. Concerning $\{\{q_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$, on the other hand, we apply twice Cauchy-Schwarz and we find:

$$\begin{aligned}\sum_{i\geq I} q_{i,n} &= \sum_{i\geq I} \int_{E_n\cap Q_{i,n}} \mu_n \leq \sum_{i\geq I} m_{i,n}^{\frac{1}{2}} \left(\int_{E_n\cap Q_{i,n}} \mu_n^2\right)^{\frac{1}{2}} \leq \left(\sum_{i\geq I} m_{i,n}\right)^{\frac{1}{2}} \left(\sum_{i\geq I} \int_{E_n\cap Q_{i,n}} \mu_n^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i\geq I} m_{i,n}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \mu_n^2\right)^{\frac{1}{2}} \lesssim \varepsilon^{-\frac{1}{2}} \mathcal{I}_{\alpha,\varepsilon}^{\frac{1}{2}}(E_n)(1+Q^2)^{\frac{1}{2}} \left(\frac{m_n}{I}\right)^{\frac{1}{2N}}.\end{aligned}$$

Being $\mathcal{I}_{\alpha,\varepsilon}^{\frac{1}{2}}(E_n)$ uniformly bounded in n by 2.9, the right-hand side of the last computation tends to 0 as $I \to +\infty$. Therefore, we proved tightness for both of the sequences $\{\{m_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$ and $\{\{q_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$: by applying Prohorov Theorem, we finally deduce the required weak convergence, up to extraction of subsequences.

The next step of the proof is to construct the generalized set $\tilde{E} = \{E^i\}_{i\geq 1}$, which will turn out to be our generalized minimizer. First of all, for all $i \geq 1$ we have, by 2.9:

$$P(E_n - z_{i,n}) = P(E_n) \lesssim 1 + Q^2 \qquad \Longrightarrow \qquad \sup_{n\in\mathbb{N}} P(E_n - z_{i,n}) < +\infty.$$

Therefore, by [18, Corollary 12.27], there exist a measurable set E^i of locally finite perimeter such that, up to extraction of a subsequence, $E_n - z_{i,n} \longrightarrow E^i$ in $L^1_{loc}(\mathbb{R}^N)$. Moreover, setting $\mu_n^i := \mu_n(\cdot + z_{i,n})$, by Banach-Alaoglu Theorem ($\|\mu_n^i\|_{L^2(\mathbb{R}^N)} = 1$ for all $n \in \mathbb{N}$) there exists $\mu^i \in L^2(\mathbb{R}^N)$ such that, up to another extraction, we have $\mu_n^i \rightharpoonup \mu^i$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$. Thanks to a diagonal argument, up to the extraction of an ulterior subsequence, this occurs simultaneously for all $i \geq 1$.

Now, noticing that for every i and j there holds $\limsup_{n\to+\infty} |z_{i,n} - z_{j,n}| = a_{ij} \in [0, +\infty]$, we define the equivalence relation $i \sim j$ if $a_{ij} < +\infty$ and we denote $[i]$ the equivalence class of i. In particular, if $i \sim j$ then E^i and E^j are translated of each other by construction. For each equivalence class we denote:

$$m_{[i]} = \sum_{j\sim i} m_j \quad \text{and} \quad q_{[i]} = \sum_{j\sim i} q_j \qquad \Longrightarrow \qquad \sum_{[i]\in\mathbb{N}/\sim} m_{[i]} = m \quad \text{and} \quad \sum_{[i]\in\mathbb{N}/\sim} q_{[i]} = 1.$$

Next, we show the key point of the proof, which consists in linking the L^1_{loc} convergence of sets we just established with the weak convergence of the sequences $\{\{m_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$ and $\{\{q_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$, namely:

$$|E^i| = m_{[i]} \qquad \text{and} \qquad \mu^i(E^i) = q_{[i]} \qquad\qquad \text{for all } i \geq 1.$$

In non-mathematical words, the measure of the limit sets E^i is given by the sum of the limits of the measures $m_{j,n} = |E_n \cap Q_{j,n}|$ for all $j \sim i$. In particular, each set E^i is defined by the union of the L^1_{loc} limits of the sets $Q_{j,n}$ for all $j \sim i$. We start with the first equality and we fix a class $[i]$ and $M \in \mathbb{N}$. Recalling that the cardinality of $[i]$ may be infinite (and thus the set E^i unbounded), we consider the finite family $\{i_1, \dots, i_M\}$. By construction of $[i]$, there exists a compact set K_M such that we have:

$$\bigcup_{k=1}^{M} Q_{i_k,n} \subset K_M + z_{i,n} \qquad \text{for all } n \in \mathbb{N}, \tag{2.10}$$

whence:

$$\sum_{k=1}^{M} m_{i_k,n} = \sum_{k=1}^{M} |E_n \cap Q_{i_k,n}| = |E_n \cap \bigcup_{k=1}^{M} Q_{i_k,n}| \leq |E_n \cap (K_M + z_{i,n})| = |(E_n - z_{i,n}) \cap K_M|.$$

We send $n \to +\infty$: using the L^1_{loc} convergence $E_n - z_{i,n} \longrightarrow E^i$ and the weak convergence of the sequence $\{\{m_{i,n}\}_{i\geq 1}\}_{n\in\mathbb{N}}$, we find

$$\sum_{k=1}^{M} m_{i_k} \leq |E^i \cap K_M| \leq |E^i| \qquad \text{for all } n \in \mathbb{N}.$$

Passing to the limit as $M \to +\infty$, we get $m_{[i]} \leq |E^i|$. Conversely, if we show that $\sum_{[i]\in\mathbb{N}/\sim} |E^i| \leq m$, we are able to conclude:

$$m = \sum_{[i]\in\mathbb{N}/\sim} m_{[i]} \leq \sum_{[i]\in\mathbb{N}/\sim} |E^i| \leq m \qquad \Longrightarrow \qquad |E^i| = m_{[i]} \text{ for all } i \geq 1.$$

Hence, we choose M integers $\{i_1, \dots, i_M\}$ belonging to different equivalence classes: given $R > 0$, again by construction of the equivalence relation, we have that $B_R(z_{i_k,n}) \cap B_R(z_{i_l,n}) = \emptyset$ for all $k \neq l$ for n large enough. Therefore:

$$m_n = |E_n| \geq \left| E_n \cap \bigcup_{k=1}^{M} B_R(z_{i_k,n}) \right| = \sum_{k=1}^{M} |E_n \cap B_R(z_{i_k,n})| \geq \sum_{k=1}^{M} |(E_n - z_{i_k,n}) \cap B_R|.$$

Passing to the limit as $n \to +\infty$, again L^1_{loc} convergence yields:

$$m \geq \sum_{k=1}^{M} |(E^{i_k}) \cap B_R|,$$

hence we conclude our thesis by letting first $R \to +\infty$ and finally $M \to +\infty$.

The proof of $\mu^i(E^i) = q_{[i]}$ exploits the exact same ideas but is a bit more convoluted. We begin by noticing that, for each $K \subset \mathbb{R}^N$ compact, obviously $\mu^i_n \rightharpoonup \mu^i$ in $L^2(\mathbb{R}^N)$ implies $\mu^i_n \rightharpoonup \mu^i$ in $L^2(K)$ as $n \to +\infty$ as well. Moreover, $E_n - z_{i,n} \longrightarrow E^i$ in $L^1_{loc}(\mathbb{R}^N)$ implies $E_n - z_{i,n} \longrightarrow E^i$ in $L^2_{loc}(\mathbb{R}^N)$, since $|\chi_{E_n - z_{i,n}} - \chi_{E^i}| \in \{0, 1\}$ so $|\chi_K(\chi_{E_n - z_{i,n}} - \chi_{E^i})|^2 = |\chi_K(\chi_{E_n - z_{i,n}} - \chi_{E^i})| \longrightarrow 0$ as $n \to +\infty$. Therefore, using the same notation as before and recalling 2.10 we deduce:

$$\begin{aligned}
\sum_{k=1}^{M} q_{i_k,n} &= \sum_{k=1}^{M} \mu_n(E_n \cap Q_{i_k,n}) = \mu_n(E_n \cap \bigcup_{k=1}^{M} Q_{i_k,n}) \leq \mu_n(E_n \cap (K_M + z_{i,n})) \\
&= \int_{E_n \cap K_M + z_{i,n}} \mu_n(x) = \int_{E_n - z_{i,n} \cap K_M} \mu_n(x + z_{i,n}) = \int_{K_M} \chi_{E_n - z_{i,n}}(x) \mu_n(x + z_{i,n}).
\end{aligned}$$

Notice that in the second variable we changed variable with the translation $x \longmapsto x + z_{i,n}$. Passing the right-hand side to the limit as $n \to +\infty$, we obtain by weak-strong convergence in duality in $L^2(K_M) \times L^2(K_M)$:

$$\sum_{k=1}^{M} q_{i_k} \leq \int_{K_M} \chi_{E^i} \mu^i = \int_{E^i \cap K_M} \mu^i = \mu^i(E^i \cap K_M) \leq \mu^i(E^i) \qquad \text{for all } n \in \mathbb{N}.$$

Passing to the limit as $M \to +\infty$, we get $q_{[i]} \leq \mu^i(E^i)$. Conversely, showing that $\sum_{[i] \in \mathbb{N}/\sim} \mu^i(E^i) \leq 1$, allows us to conclude:

$$1 = \sum_{[i] \in \mathbb{N}/\sim} q_{[i]} \leq \sum_{[i] \in \mathbb{N}/\sim} \mu^i(E^i) \leq 1 \qquad \Longrightarrow \qquad \mu^i(E^i) = q_{[i]} \text{ for all } i \geq 1.$$

Given the proof of previous inequality, the computations for the last one follow from a simple readaptation of what we did before.

Up to relabelling the indexes, we may now assume that each equivalence class $[i]$ is made of a single element. If we set $\tilde{E} = \{E^i\}_{i \geq 1}$ and $\tilde{\mu} = \{\mu^i\}_{i \geq 1}$, thanks to the previous section of the proof we have just shown that $\tilde{\mu}$ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$. Hence, it remains to prove:

$$P(\tilde{E}) + \mathcal{I}_{\alpha,\varepsilon}(\tilde{\mu}) + \Lambda \left| |\tilde{E}| - \omega_N \right| \leq \liminf_{n \to +\infty} P(E_n) + \mathcal{I}_{\alpha,\varepsilon}(\mu_n) + \Lambda \left| |E_n| - \omega_N \right|.$$

We consider separately each term of the energy. Since $|\tilde{E}| = m = \lim_{n \to +\infty} m_n = \lim_{n \to +\infty} |E_n|$, continuity of the function $\Lambda| \cdot - \omega_N|$ yields immediately:

$$\Lambda \left| |\tilde{E}| - \omega_N \right| \leq \liminf_{n \to +\infty} \Lambda \left| |E_n| - \omega_N \right|.$$

On the other hand, regarding the perimeter term, fix $I \in \mathbb{N}$ and $R > 0$. For n large enough, we can assume that $|z_{i,n} - z_{j,n}| \gg R$ for $i, j \leq I$ with $i \neq j$ (this is possible because we are dealing with a finite number number of equivalence classes and thus a finite number of "reference" points $z_{i,n}$). By the Coarea Formula we have, for every such n:

$$\int_{R}^{2R} \sum_{i \leq I} \mathcal{H}^{N-1}(E_n \cap \partial B_R(z_{i,n}))\, dR \leq |E_n|$$

Arguing in the same way as Lemma 2.6, we find that for every n large enough there exists a radius $R_n \in (R, 2R)$ such that:

$$\sum_{i\leq I} \mathcal{H}^{N-1}(E_n \cap \partial B_{R_n}(z_{i,n})) \lesssim \frac{1}{R}.$$

We set $E^{i,R_n} = (E_n - z_{i,n}) \cap B_{R_n}$ and, recalling again [18, Lemma 15.12], we have:

$$\begin{aligned}
\sum_{i\leq I} P(E^{i,R_n}) &= \sum_{i\leq I} P(E_n - z_{i,n}, B_{R_n}) + \sum_{i\leq I} \mathcal{H}^{N-1}((E_n - z_{i,n}) \cap \partial B_{R_n}) \\
&= \sum_{i\leq I} P(E_n, B_{R_n}(z_{i,n})) + \sum_{i\leq I} \mathcal{H}^{N-1}(E_n \cap \partial B_{R_n}(z_{i,n})) \\
&\leq P(E_n) + \frac{C}{R}.
\end{aligned}$$

The first inequality is a consequence of translation invariance of the perimeter and of $\mathcal{H}^{N-1}$ (indeed E^{i,R_n} is a translation of $E^n \cap B_{R_n}(z_{i,n})$), whereas the last one is given by the assumption $|z_{i,n} - z_{j,n}| \gg R$ and the fact that $B_{R_n}(z_{i,n}) \subset B_{2R}(z_{i,n})$. In this way, since $B_{2R}(z_{i,n}) \cap B_{2R}(z_{j,n}) = \emptyset$ for $i \neq j$ and $i, j \leq I$, the sum of the perimeters decouples and it can be majorized with $P(E_n)$. Now, considering the sequence $\{E^{i,R_n}\}_{n\in\mathbb{N}}$, we have that $E^{i,R_n} \subset B_{2R}$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} P(E^{i,R_n}) < +\infty$, by the last computation and exploiting again 2.9. Therefore, by [18, Theorem 12.26], there exists a set of finite perimeter $E^{i,R} \subset B_{2R}$ such that, up to extraction, $E^{i,R_n} \longrightarrow E^{i,R}$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$. Anyway, as we have $E_n - z_{i,n} \longrightarrow E^i$ in $L^1_{loc}(\mathbb{R}^N)$ too, there holds $(E_n - z_{i,n}) \cap B_{2R} \longrightarrow E^i \cap B_{2R}$ in $L^1(\mathbb{R}^N)$ as well. Being the first a subsequence of the latter, we must necessarily have $E^{i,R} \subset E^i \cap B_{2R}$. Moreover, we had $R_n \in (R, 2R)$ so in particular $(E_n - z_{i,n}) \cap B_R \subset E^{i,R_n}$ for all $n \in \mathbb{N}$: hence, by L^1 convergence, we deduce $E^i \cap B_R \subset E^{i,R}$. Putting everything together, we get:

$$E^i \cap B_R \subset E^{i,R} \subset E^i \cap B_{2R},$$

letting $R \to +\infty$ we infer $E^{i,R} \longrightarrow E^i$ in $L^1_{loc}(\mathbb{R}^N)$. Thus, by lower semicontinuity of the perimeter under L^1_{loc} convergence and by superadditivity of the inferior limit, we can conclude:

$$\begin{aligned}
\sum_{i\leq I} P(E^i) &\leq \sum_{i\leq I} \liminf_{R\to+\infty} P(E^{i,R}) \leq \sum_{i\leq I} \liminf_{R\to+\infty} \liminf_{n\to+\infty} P(E^{i,R_n}) \\
&\leq \liminf_{R\to+\infty} \liminf_{n\to+\infty} \sum_{i\leq I} P(E^{i,R_n}) \leq \liminf_{R\to+\infty} \liminf_{n\to+\infty} P(E_n) + \frac{C}{R} \\
&= \liminf_{n\to+\infty} P(E_n)
\end{aligned}$$

This is true for all $I \in \mathbb{N}$, so passing to the limit as $I \to +\infty$ we get:

$$P(\tilde{E}) = \sum_{i=1}^{+\infty} P(E^i) = \lim_{I\to+\infty} \sum_{i\leq I} \leq \liminf_{n\to+\infty} P(E_n).$$

It remains to estimate the regularized Riesz energy term. Similarly, we fix $I \in \mathbb{N}$ and $R > 0$: for n large enough, we can assume that $|z_{i,n} - z_{j,n}| \gg R$ for $i, j \leq I$ with $i \neq j$. Exploiting the weak convergence $\mu^i_n \rightharpoonup \mu^i$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$ and the lower semicontinuity of $I_{\alpha,\varepsilon}$ under weak convergence we get:

$$\begin{aligned}
\sum_{i\leq I} I_{\alpha,\varepsilon}(\mu^i|_{B_R}) &\leq \sum_{i\leq I} \liminf_{n\to+\infty} I_{\alpha,\varepsilon}(\mu^i_n|_{B_R}) \leq \liminf_{n\to+\infty} \sum_{i\leq I} I_{\alpha,\varepsilon}(\mu^i_n|_{B_R}) \\
&\leq \liminf_{n\to+\infty} I_{\alpha,\varepsilon}\left(\sum_{i\leq I} \mu^i_n|_{B_R(z_{i,n})}\right) \leq \liminf_{n\to+\infty} I_{\alpha,\varepsilon}(\mu_n).
\end{aligned}$$

Since $\mu^i|_{B_R} \rightharpoonup \mu^i$ in $L^2(\mathbb{R}^N)$ as $R \to +\infty$, we have:

$$\sum_{i\leq I} I_{\alpha,\varepsilon}(\mu^i) \leq \sum_{i\leq I} \liminf_{R\to+\infty} I_{\alpha,\varepsilon}(\mu^i|_{B_R}) \leq \liminf_{R\to+\infty} \sum_{i\leq I} I_{\alpha,\varepsilon}(\mu^i|_{B_R}) \leq \liminf_{n\to+\infty} I_{\alpha,\varepsilon}(\mu_n)$$

and we can find the estimate we need by letting $I \to +\infty$ like before. Finally, putting everything together we have:

$$\begin{aligned} P(\tilde{E}) + \mathcal{I}_{\alpha,\varepsilon}(\tilde{\mu}) + \Lambda \left||\tilde{E}| - \omega_N\right| &\leq \liminf_{n\to+\infty} P(E_n) + \liminf_{n\to+\infty} \mathcal{I}_{\alpha,\varepsilon}(\mu_n) + \lim_{n\to+\infty} \Lambda \left||E_n| - \omega_N\right| \\ &\leq \liminf_{n\to+\infty} P(E_n) + \mathcal{I}_{\alpha,\varepsilon}(\mu_n) + \Lambda \left||E_n| - \omega_N\right| \\ &= \inf_{|E|<+\infty} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(E), \end{aligned}$$

concluding the proof. □

2.4 First almost minimality property and density estimates

After proving existence of minimizers for the problem 2.5, our ultimate goal is to show that they actually enjoy many regularity properties (which will finally lead us to solve problem 1.16) and the current section is devoted to doing it. However, the path is quite long and it requires some preparatory arguments: we begin by defining (Λ, r_0)-perimeter minimality and by proving density estimates for finite perimeter sets enjoying such property in Lemma 2.9. Then, after another preliminary result explained in Lemma 2.10, we present the first almost minimality property characterizing minimizers of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, which is nothing but a way to say that they are (Λ, r_0)-minimizer for some parameter Λ. At that point, proving density estimates for our minimizers is just a formality and we will finally infer their regularity properties in Proposition 2.13.

First thing first, the very general notion of (Λ, r_0)-perimeter minimality. Be careful not to confuse the parameter Λ employed here (notation we decided to adopt in adherence with the reference) with its use when relaxing the volume constraint in Lemma 2.1.

Definition 2.8 ((Λ, r_0)-minimality)**.** Given two parameters $\Lambda, r_0 > 0$, a set of locally finite perimeter E is a (Λ, r_0)-*minimizer* of the perimeter if, for every $x \in \mathbb{R}^N$ and for all $r \leq r_0$, we have:

$$P(E) \leq P(F) + \Lambda r^{N-1} \qquad \text{for all } E\Delta F \subset B_r(x).$$

As highlighted in [14], such sets enjoy many interesting properties, but we are especially interested in their density estimates, which uniformly compare at small scales the size of the perimeter and of the volume of a (Λ, r_0)-minimizer with those of a ball.

Lemma 2.9 (Density estimates)**.** There exists a universal and small enough constant $\overline{\Lambda} > 0$ such that, if $\Lambda \leq \overline{\Lambda}$ and E is a (Λ, r_0)-minimizer, then for every $x \in \partial E$ and every $0 < r \leq r_0$ we have:

$$\min\{|E \cap B_r(x)|, |B_r(x) \setminus E|\} \gtrsim r^N \tag{2.11}$$

and

$$r^{N-1} \lesssim P(E, B_r(x)) \lesssim r^{N-1}. \tag{2.12}$$

Proof. By translation and density, we may assume without loss of generality that $x = 0$ and $0 \in \partial^* E$. We begin by proving the upper bound in 2.12. Noticing that $P(E \setminus B_r) = P(E, B_r^c) + \mathcal{H}^{N-1}(\partial B_r \cap E)$, we use the (Λ, r_0)-minimality property with the set $E \setminus B_r$:

$$P(E, B_r) + P(E, B_r^c) = P(E) \leq P(E, B_r^c) + \mathcal{H}^{N-1}(\partial B_r \cap E) + \Lambda r^{N-1}.$$

Therefore:

$$P(E,B_r) \le \mathcal{H}^{N-1}(\partial B_r \cap E) + \Lambda r^{N-1} \lesssim r^{N-1},$$

and the upper bound is proved.

Concerning the lower bound, we assume by contradiction that there exists some $r \le r_0$ such that, for all $\theta \in \left(0, \frac{1}{2}\right)$ there exists $\eta = \eta(\theta)$ small to be chosen precisely below). We claim that if there holds:

$$\frac{1}{r^{N-1}} P(E,B_r) \le \eta, \tag{2.13}$$

then there exists $C > 0$ such that:

$$\frac{1}{(\theta r)^{N-1}} P(E,B_{\theta r}) \le \theta \frac{1}{r^{N-1}} P(E,B_r) + C\Lambda. \tag{2.14}$$

Indeed, if 2.13 holds, then by relative Isoperimetric Inequality we find:

$$\min\left\{\frac{|E\cap B_r|}{r^N}, \frac{|B_r \setminus E|}{r^N}\right\} \lesssim \left(\frac{1}{r^{N-1}}P(E,B_r)\right)^{\frac{N}{N-1}} \lesssim \eta^{\frac{1}{N-1}} \frac{1}{r^{N-1}} P(E,B_r).$$

Hence, we assume first that $|E \cap B_r| \le |B_r \setminus E|$. Now, we can choose $t \in (\theta r, 2\theta r)$ such that, applying Coarea Formula we find:

$$\begin{aligned}\mathcal{H}^{N-1}(\partial B_t \cap E) &\le \frac{1}{\theta r}\int_{\theta r}^{2\theta r} \mathcal{H}^{N-1}(\partial B_s \cap E)\, ds \lesssim \frac{|E\cap B_{2\theta r}|}{\theta r} - \frac{|E\cap B_{\theta r}|}{\theta r}\\ &\le \frac{|E\cap B_{2\theta r}|}{\theta r} \le \frac{|E\cap B_r|}{\theta r} \lesssim \theta^{-1}\eta^{\frac{1}{N-1}} P(E,B_r).\end{aligned}$$

We test the (Λ, r_0)-minimality property with $E \setminus B_t$ to deduce (recalling $t \lesssim \theta r$):

$$P(E,B_t) \le \mathcal{H}^{N-1}(\partial B_t \cap E) + \Lambda t^{N-1} \lesssim \theta^{-1}\eta^{\frac{1}{N-1}} P(E,B_r) + \Lambda(\theta r)^{N-1}.$$

Now, we have that $P(E,B_{\theta r}) \le P(E,B_t)$ and dividing the last expression by $(\theta r)^{N-1}$ we have:

$$\frac{1}{(\theta r)^{N-1}} P(E,B_{\theta r}) \le C\left(\theta^{-N}\eta^{\frac{1}{N-1}}\frac{1}{r^{N-1}}P(E,B_r) + \Lambda\right).$$

On the other hand, if $|E\cap B_r| \ge |B_r \setminus E|$, we argue exactly in the same way noticing that, by the equality between the measures $P(E,\cdot)$ and $P(E^c,\cdot)$, the (Λ, r_0)-minimality property holds true for E^c as well. Then, we choose:

$$\eta = \min\left\{\frac{\omega_{N-1}}{2}, \left(\frac{1}{C}\theta^{N+1}\right)^{N-1}\right\} \qquad \text{so} \qquad C\theta^{-N}\eta^{\frac{1}{N-1}} \le \theta$$

and 2.14 follows. In particular, notice that $\eta \to 0$ as $\theta \to 0$. Now, we assume that:

$$\Lambda \le \frac{\eta(1-\theta)}{C}$$

(namely we are sufficiently decreasing Λ as reported in the statement), where C is the constant appearing in 2.14. Thus, if 2.13 holds, by 2.14 and our assumption on Λ we find:

$$\frac{1}{(\theta r)^{N-1}} P(E,B_{\theta r}) \le \theta\eta + C\Lambda \le \eta.$$

Iterating the previous argument we obtain:

$$\limsup_{k\to+\infty} \frac{1}{(\theta^k r)^{N-1}} P(E, B_{\theta^k r}) \leq \eta.$$

Since $\eta \leq \omega_{N-1}$ by our previous choice, this contradicts:

$$\lim_{r\to 0} \frac{1}{r^{N-1}} P(E, B_r) = \omega_{N-1},$$

which follows from the hypothesis $0 \in \partial^* E$ from [18, Corollary 15.8]. Hence, for all $r \leq r_0$, there exists some $\overline{\theta} \in \left(0, \frac{1}{2}\right)$ such that:

$$\limsup_{r\to 0} \frac{1}{r^{N-1}} P(E, B_r) \geq \eta(\overline{\theta}),$$

which yields the lower bound in 2.12. In other words, there exists $C_1 > 0$ such that for all $r \leq r_0$ we have $P(E, B_r) \geq C_1 r^{N-1}$.

Regarding 2.11, we reason in a similar way, namely we assume by contradiction that there exists some $r \leq r_0$ such that, for all $\theta \in \left(0, \frac{1}{2}\right)$ there exists $\eta' = \eta'(\theta)$ small (to be chosen precisely below) such that:

$$\min\{|E \cap B_r|, |B_r \setminus E|\} \leq \eta' r^N, \tag{2.15}$$

If 2.15 holds, then there exists $t \in (\theta r, 2\theta r)$ such that:

$$\mathcal{H}^{N-1}(\partial B_t \cap E) \leq \frac{|E \cap B_r|}{\theta r} \leq \frac{\eta'}{\theta} r^{N-1},$$

so, testing again the (Λ, r_0)-minimality property with $E \setminus B_t$ we infer

$$P(E, B_t) \leq \frac{\eta'}{\theta} r^{N-1} + \Lambda(2\theta)^{N-1} r^{N-1}.$$

We call C_1 the constant appearing in the upper bound of 2.12 and we make the choice:

$$\eta' < C_1 \frac{\theta^N}{2} \qquad \text{and} \qquad \Lambda \leq \frac{\eta'}{\theta(2\theta)^{N-1}}.$$

Again, we are sufficiently decreasing Λ as reported in the statement. In his way, we get:

$$P(E, B_t) \leq \mathcal{H}^{N-1}(\partial B_t \cap E) + \Lambda t^{N-1} \leq \frac{\eta'}{\theta} r^{N-1} + \Lambda(2\theta)^{N-1} r^{N-1} \leq \frac{2\eta'}{\theta} r^{N-1}.$$

In particular:

$$P(E, B_t) \leq \frac{2\eta'}{\theta^N} (\theta r)^{N-1} \leq C_1 t^{N-1},$$

which is a contradiction with 2.12, thus, in conclusion, 2.11 follows. □

As we will see later on, finite perimeter sets with density estimates enjoy some good regularity properties. Therefore, it is rather natural to ask ourselves whether minimizers of problem 2.5 enjoy such kind of estimates or not. Of course, we need to prove that they are (Λ, r_0)-minimizer for some couple (Λ, r_0) and the tool to make such conclusion, the first almost minimality property for minimizers of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, is presented in the next proposition. Anyway, we must prove first another preliminary lemma.

Lemma 2.10. For every generalized set $\tilde{E} = (E \cup F) \times \{E^i\}_{i\geq 2}$ with E and F sets with positive measure such that $|E \cap F| = 0$, if we define $\tilde{F} = F \times \{E^i\}_{i\geq 2}$ we have:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \geq \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})^2}{\mathcal{I}_{\alpha,\varepsilon}(E)}.$$

Proof. Let $\tilde{\mu} = \{\mu^i\}_{i\geq 1}$ be optimal for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$. We may assume without loss of generality that both $\mu^1(E) \neq 0$ and $\mu^1(F) + \sum_{i\geq 2} \mu^i(E^i) \neq 0$. Indeed, in the first case we would have $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \mathcal{I}_{\alpha,\varepsilon}(\tilde{F})$ so trivially:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) \geq \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})^2}{\mathcal{I}_{\alpha,\varepsilon}(E)}.$$

Whereas, in the second one we would have $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) = \mathcal{I}_{\alpha,\varepsilon}(E)$, from which we find the implication:

$$\mathcal{I}_{\alpha,\varepsilon}(E)^2 + \mathcal{I}_{\alpha,\varepsilon}(\tilde{F})^2 \geq \mathcal{I}_{\alpha,\varepsilon}(E)\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) \qquad \Longrightarrow \qquad \mathcal{I}_{\alpha,\varepsilon}(E) \geq \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})^2}{\mathcal{I}_{\alpha,\varepsilon}(E)}.$$

Hence, we define:

$$\mu = \frac{\mu^1|_E}{\mu^1(E)}, \qquad \nu^1 = \frac{\mu^1|_F}{1-\mu^1(E)} \qquad \text{and} \qquad \nu^i = \frac{\mu^i}{1-\mu^1(E)} \quad \text{for all } i \geq 2.$$

In this way, μ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(E)$ and $\tilde{\nu} = \{\nu^i\}_{i\geq 1}$ is admissible for $\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})$ and we have:

$$\begin{aligned} I_{\alpha,\varepsilon}(\mu^1) &= I_{\alpha,\varepsilon}(\mu^1|_E + \mu^1|_F) = I_{\alpha,\varepsilon}(\mu^1|_E) + I_{\alpha,\varepsilon}(\mu^1|_F) + I_{\alpha,\varepsilon}(\mu^1|_E, \mu^1|_F) \\ &\geq I_{\alpha,\varepsilon}(\mu^1|_E) + I_{\alpha,\varepsilon}(\mu^1|_F) = (\mu^1(E))^2 I_{\alpha,\varepsilon}(\mu) + (1-\mu^1(E))^2 I_{\alpha,\varepsilon}(\nu^1), \end{aligned}$$

by 2-homogeneity of the regularized interaction energy $I_{\alpha,\varepsilon}$. Therefore, by definition of $\mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$:

$$\begin{aligned} \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) &= I_{\alpha,\varepsilon}(\mu^1) + \sum_{i=2}^{+\infty} I_{\alpha,\varepsilon}(\mu^i) \\ &\geq (\mu^1(E))^2 I_{\alpha,\varepsilon}(\mu) + (1-\mu^1(E))^2 I_{\alpha,\varepsilon}(\nu^1) + (1-\mu^1(E))^2 \sum_{i=2}^{+\infty} I_{\alpha,\varepsilon}(\nu^i) \\ &= (\mu^1(E))^2 I_{\alpha,\varepsilon}(\mu) + (1-\mu^1(E))^2 I_{\alpha,\varepsilon}(\tilde{\nu}) \\ &\geq (\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(E) + (1-\mu^1(E))^2 \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}). \end{aligned}$$

Since $\theta := \mu^i(E) \in [0,1]$ (here we include again the trivial cases), we have as well:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \geq \min_{\theta\in[0,1]} \theta^2 \mathcal{I}_{\alpha,\varepsilon}(E) + (1-\theta)^2 \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}).$$

Optimizing in θ, we find $\overline{\theta} = \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})}{\mathcal{I}_{\alpha,\varepsilon}(E)+\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})}$, which in turn yields:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \geq \frac{\mathcal{I}_{\alpha,\varepsilon}(E)\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})}{\mathcal{I}_{\alpha,\varepsilon}(E) + \mathcal{I}_{\alpha,\varepsilon}(\tilde{F})} = \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) \left(1 + \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})}{\mathcal{I}_{\alpha,\varepsilon}(E)}\right)^{-1}.$$

We conclude thanks to the inequality $(1+x)^{-1} \geq 1-x$ for $x \geq 0$. □

We are now ready to prove the first almost minimality property for generalized minimizers of the functional $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$.

Proposition 2.11. For all $\alpha \in (0,1]$, $Q > 0$ and $\Lambda \gg 1 + Q^2$ such that Lemma 2.1 applies, there exist $C = C(N,\alpha,\Lambda) > 0$ and $0 < r_0 \ll 1$ such that for all $\varepsilon > 0$ every generalized minimizer $\tilde{E} = \{E^i\}_{i\geq 1}$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$ is an almost minimizer of the perimeter, in the sense that for every $i \geq 1$, $x \in \mathbb{R}^N$ and $r \leq r_0$:

$$P(E^i) \leq P(F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha} \qquad \text{for all } E^i \Delta F \subset B_r(x). \tag{2.16}$$

Proof. Without loss of generality, we assume $i = 1$ and $x = 0$, denoting $E = E^1$. Using $\tilde{F} = F \times \{E^i\}_{i\geq 2}$ as a competitor, $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}) \leq \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{F})$ yields:

$$P(E) \leq P(F) + Q^2\left(\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\tilde{E})\right) + \Lambda\left||\tilde{F}| - \omega_N\right| - \Lambda\left||\tilde{E}| - \omega_N\right|.$$

The function $x \longmapsto \Lambda|x - \omega_N|$ is Λ-Lipschitz, therefore:

$$\Lambda\left||\tilde{F}| - \omega_N\right| - \Lambda\left||\tilde{E}| - \omega_N\right| \leq \Lambda\left||\tilde{E}| - |\tilde{F}|\right| = \Lambda\left||F| - |E|\right|,$$

moreover, for all $E, F \subset \mathbb{R}^N$ we have:

$$\begin{aligned} |E| - |F| &= |E \setminus F| + |E \cap F| - |F \setminus E| - |F \cap E| \\ &\leq |E \setminus F| + |F \setminus E| = |E \Delta F|, \end{aligned}$$

so we deduce:

$$P(E) \leq P(F) + Q^2\left(\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\tilde{E})\right) + \Lambda|E\Delta F|.$$

By the property $P(E \cap F) + P(E \cup F) \leq P(E) + P(F)$ ([18, Lemma 12.22]), it is enough to prove the thesis under the additional condition $E \subset F$ or $F \subset E$. Indeed, if it is not the case, then we still have $E \cap F \subset E$ and $E \subset E \cup F$, so:

$$\begin{aligned} P(E) &\leq P(E \cap F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha} \\ P(E) &\leq P(E \cup F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha}. \end{aligned}$$

Summing up:

$$\begin{aligned} 2P(E) &\leq P(E \cap F) + P(E \cup F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha} \\ &\leq P(E) + P(F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha}, \end{aligned}$$

So the thesis follows by subtracting $P(E)$ from both sides.

The case $E \subset F$ is easy: if μ is a probability measure supported in E, then $\mu(E) = 1$ implies $\mu(F) = 1$, so by definition $\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) \leq \mathcal{I}_{\alpha,\varepsilon}(\tilde{E})$. As we have $E\Delta F \subset B_r$, then $|E\Delta F| \lesssim r^N$, hence:

$$\begin{aligned} P(E) &\leq P(F) + Q^2\left(\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\tilde{E})\right) + \Lambda|E\Delta F| \\ &\leq P(F) + Cr^N \leq P(F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha}. \end{aligned}$$

We are left with the case $F \subset E$: writing $E = F \cup (E \setminus F)$ and applying 2.10 with $E \setminus F$ instead of F, we get:

$$\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \geq \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})^2}{\mathcal{I}_{\alpha,\varepsilon}(E \setminus F)} \qquad \Longrightarrow \qquad \mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \leq \frac{\mathcal{I}_{\alpha,\varepsilon}(\tilde{F})^2}{\mathcal{I}_{\alpha,\varepsilon}(E \setminus F)}.$$

By Lemma 2.1 (precisely its readaptation with the regularized Riesz energy $\mathcal{I}_{\alpha,\varepsilon}$), since $\tilde{E}$ is a minimizer then $|\tilde{E}| = |E| + \sum_{i\geq 2} |E^i| = \omega_N$. The right choice of $r_0 \ll 1$ implies that for all $r \leq r_0$:

$$|\tilde{F}| = |F| + \sum_{i\geq 2} |E^i| = \omega_N - |E \setminus F| \gtrsim 1,$$

so we infer $\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) \lesssim 1$, by the upper bound of Lemma 2.4. On the other hand, since $E \setminus F \subset B_r$ we have:

$$\mathcal{I}_{\alpha,\varepsilon}(E \setminus F) \geq \mathcal{I}_{\alpha,\varepsilon}(B_r) \geq \mathcal{I}_\alpha(B_r) + \varepsilon \inf_{\mu(B_r)=1} \int_{B_r} \mu^2.$$

Now, by Cauchy-Schwarz we get:

$$1 = \left(\int_{B_r} \mu\right)^2 \leq |B_r| \int_{B_r} \mu^2 \lesssim r^N \int_{B_r} \mu^2 \qquad \Longrightarrow \qquad \varepsilon \inf_{\mu(B_r)=1} \int_{B_r} \mu^2 \gtrsim \varepsilon r^{-n}.$$

In addition, $\mathcal{I}_\alpha(B_r) = c(N,\alpha) r^{-(N-\alpha)}$, thus we can say:

$$\mathcal{I}_{\alpha,\varepsilon}(E \setminus F) \gtrsim r^{-(N-\alpha)} + \varepsilon r^{-n} \geq r^{-(N-\alpha)}.$$

Hence, we get $\mathcal{I}_{\alpha,\varepsilon}(\tilde{F}) - \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}) \lesssim r^{N-\alpha}$, which allows us to conclude (recalling $|E\Delta F| \lesssim r^N = r^\alpha r^{N-\alpha}$):

$$P(E) \leq P(F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha}.$$

□

Apart from proving regularity of minimizers of 2.5, we must not forget that our first intent was to show existence of minimizers for the original problem with Riesz energy $\mathcal{I}_\alpha$. A first step towards this direction is the fact that the first minimality property is uniform in ε. Subsequently, every conclusion we will infer from it holds true regardless of the parameter ε chosen, so we restrict ourselves to $\varepsilon \in (0,1]$ for convenience. We will eventually be able to pass to the limit as $\varepsilon \to 0$ in order to recover solutions of 1.16. However, before doing that, the first direct consequence of estimate 2.16 is the following corollary, where we also start noticing the difference between the cases $\alpha > 1$ and $\alpha \leq 1$.

Corollary 2.12. For every $\alpha \in (0,1]$ and $Q > 0$ let $\Lambda \gg 1 + Q^2$ be such that Proposition 2.11 applies. Then, for every $\varepsilon \in (0,1]$ and every generalized minimizer $\tilde{E} = \{E^i\}_{i\geq 1}$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$ there exists $r_0 \ll 1$ such that, if $\max\{Q^2 r^{1-\alpha}, r\} \leq r_0$ and $x \in \partial E^i$, then:

$$\min\{|E^i \cap B_r(x)|, |B_r(x) \setminus E^i|\} \gtrsim r^N \tag{2.17}$$

and

$$P(E^i, B_r(x)) \sim r^{N-1}. \tag{2.18}$$

Proof. Rearranging 2.16, we get that for every $r \leq r_0$, for some $r_0 \ll 1$:

$$P(E^i) \leq P(F) + C\left(Q^2 r^{1-\alpha} + r\right) r^{N-1} \qquad \text{for all } E^i \Delta F \subset B_r(x).$$

In particular, we see that E^i is a (Λ, r_0)-minimizer for all $i \geq 1$, with $\Lambda = \Lambda(r) = C\left(Q^2 r^{1-\alpha} + r\right)$. Applying Lemma 2.9, we reach the conclusion by imposing $\max\{Q^2 r^{1-\alpha}, r\}$ to be small enough. Hence, changing in turn the value of r_0 in case it is necessary, for all $r \leq r_0$ estimates 2.17 and 2.18 hold. □

As anticipated, thanks to density estimates we are able to infer many good properties of our generalized minimizers, furthermore distinguishing between the cases $\alpha \in (0,1)$ and $\alpha = 1$. From now on, we denote $\partial^M E$ the measure theoretical boundary (otherwise called essential boundary) of a set of locally finite perimeter, according to the definition given in [18, Chapter 16].

Proposition 2.13. Consider $\alpha \in (0,1]$, $Q > 0$, $\Lambda \gg 1 + Q^2$ such that Proposition 2.11 applies, $\varepsilon \in (0,1]$ and a generalized minimizer $\tilde{E} = \{E^i\}_{i\geq 1}$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. Then:

- if $\alpha \in (0,1)$, for every $Q \leq \overline{Q}$ for any $\overline{Q} > 0$;
- if $\alpha = 1$ and Q is not too large, there exists $Q^* \geq Q$ such that for all $Q \leq \overline{Q} \leq Q^*$;

up to the choice of a representative, $\tilde{E}$ is made of finitely many E^i, each of which is connected with $E^i \in \mathcal{S}$ and for which $\partial E^i = \partial^M E^i$. Moreover, the number of such components as well as their diameter depends only on $\overline{Q}$.

Proof. The difference between the cases $\alpha \in (0,1)$ and $\alpha = 1$ is given by condition $\max\{Q^2 r^{1-\alpha}, r\} \leq r_0$ for some $r_0 \ll 1$ from the last Corollary. Indeed, when $\alpha \in (0,1)$ then $r = o(Q^2 r^{1-\alpha})$ as $r \to 0$. Thus, we need not impose any condition of Q, as it suffices to choose only some r small enough to make $\max\{Q^2 r^{1-\alpha}, r\} \leq r_0$ true, regardless of the value of Q. On the other hand, if $\alpha = 1$, we must impose the condition $\max\{Q^2, r\} \leq r_0$. Hence, there exists $Q^* > 0$ small enough not to be exceeded in order to have proper density estimates. Therefore, we fix $\overline{Q}$ for the rest of the proof, according to the just motivated conditions from the statement of the Proposition.

Before starting the actual proof, we notice that there is an uniform bound in i and ε for the masses and the perimeters of the sets composing generalized minimizers. Indeed, using the unit ball B_1 as a competitor for $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$ as we already did before, we get $P(E^i_\varepsilon) \lesssim 1 + Q^2$ for all $i \geq 1$ and $\varepsilon \in (0,1]$. Moreover, as we highlighted in the proof of 2.16, $|\tilde{E}_\varepsilon| = \omega_N$ for all $\varepsilon \in (0,1]$. Therefore there exists some $C > 0$ such that $P(E^i_\varepsilon) < C$ and $|E^i_\varepsilon| < C$ for all $i \geq 1$ and $\varepsilon \in (0,1]$.

We fix $i \geq 1$, $\varepsilon > 0$ and we set $E^i_\varepsilon = E$. First of all, we show that E is open up to the choice of a representative. By Lebesgue points Theorem [18, Theorem 5.16] $E = E^{(1)}$ almost everywhere, where:

$$E^{(1)} = \left\{ x \in \mathbb{R}^N : \lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1 \right\}.$$

For all $x \in E^{(1)}$, we show that there exists $r_x > 0$ such that $B_{r_x}(x) \subset E^{(1)}$; up to translations it is enough to do it for $0 \in E^{(1)}$. By contradiction, we assume the converse, namely that there exists $\{r_n\}_{n\in\mathbb{N}}$ such that $r_n \to 0$ as $n \to +\infty$ and $B_{r_n} \setminus E^{(1)} \neq \emptyset$ for all $n \in \mathbb{N}$.

Thus, we claim $|B_{r_n} \setminus E^{(1)}| > 0$ for all $n \in \mathbb{N}$. We assume, again by contradiction that there exists $n \in \mathbb{N}$ such that $|B_{r_n} \setminus E^{(1)}| = 0$. Since $B_{r_n} \setminus E^{(1)} \neq \emptyset$, there exists $x \in B_{r_n} \setminus E^{(1)}$. As $x \in B_{r_n}$, there exists $\bar{r} > 0$ such that $B_{\bar{r}}(x) \subset B_{r_n}$. Hence:

$$B_{\bar{r}}(x) \setminus E^{(1)} \subset B_{r_n} \setminus E^{(1)} \qquad \Longrightarrow \qquad |B_{\bar{r}}(x) \setminus E^{(1)}| = 0.$$

Since $B_{\bar{r}}(x) = (B_{\bar{r}}(x) \cap E^{(1)}) \cup (B_{\bar{r}}(x) \setminus E^{(1)})$, we must have for all $r \leq \bar{r}$:

$$|B_r(x)| = |B_r(x) \cap E^{(1)}| \qquad \Longrightarrow \qquad \lim_{r\to 0} \frac{|B_r(x) \cap E^{(1)}|}{|B_r(x)|} = 1,$$

so $x \in E^{(1)}$ which is absurd.

Hence, we have $|B_{r_n} \cap E^{(1)}| > 0$ and $|B_{r_n} \setminus E^{(1)}| > 0$ for all $n \in \mathbb{N}$ and, by relative Isoperimetric Inequality, we get $P(E^{(1)}, B_{r_n}) = \mathcal{H}^{N-1}(\partial^* E^{(1)} \cap B_{r_n}) > 0$. Therefore, there exists $y_n \in \partial^* E^{(1)} \cap B_{r_n}$ for which we have density estimates, so, in particular, $|B_{r_n}(y_n) \setminus E^{(1)}| \gtrsim r_n^N$ for all $r_n \le r_0$. Namely, as $r_n \to 0$, there exists some $\bar{n} \in \mathbb{N}$ such that this is true for $n \ge \bar{n}$. Now, since $y_n \in B_{r_n}$, by triangle inequality $B_{r_n}(y_n) \subset B_{2r_n}(0)$, hence $n \ge \bar{n}$:

$$|B_{2r_n}(0) \setminus E^{(1)}| \ge |B_{r_n}(y_n) \setminus E^{(1)}| \gtrsim r_n^N \qquad \Longrightarrow \qquad \lim_{r\to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0$$

so $0 \notin E^{(1)}$, which is a contradiction. Thus, E is open up to a representative and we fix E to be equal to this representative from now on.

Next, we prove that E is bounded and connected by following [20]. If E is not bounded, then, fixing $r \le r_0$ (with r_0 appearing in the density estimates) there exists $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $x_n \to +\infty$ as $n \to +\infty$ and $|x_n - x_{n'}| > 2r$ for all n and n'. Clearly, as $B_r(x_n) \cap B_r(x'_n) = \emptyset$ for $n \ne n'$:

$$\omega_N = |\tilde{E}_\varepsilon| \ge |E| \ge \sum_{k\in\mathbb{N}} |E \cap B_r(x_n)| \gtrsim \sum_{k\in\mathbb{N}} r^N = +\infty,$$

which is impossible, so E is bounded. On the other hand, if E is not connected, then $E = E_1 \cup E_2$ with $E_1 \ne \emptyset$, $E_2 \ne \emptyset$ and $E_1 \cap E_2 = \emptyset$. We define $E_R = E_1 \cup (E_2 + e_1 R)$, noticing that $|E_R| = \omega_N$ and $P(E_R) = P(E) = P(E_1) + P(E_2)$ for some $R > 0$ sufficiently large. At the same time, the Riesz interaction energy decreases, since the interaction term between E_1 and $E_2 + e_1 R$ becomes negligible as R increases. Specifically:

$$\begin{aligned}\liminf_{R\to+\infty} \mathcal{F}_{\alpha,Q,\varepsilon}(E_R) &= P(E_R) + Q^2\mathcal{I}_{\alpha,\varepsilon}(E_1) + Q^2\mathcal{I}_{\alpha,\varepsilon}(E_2) \\ &< P(E) + Q^2\mathcal{I}_{\alpha,\varepsilon}(E_1) + Q^2\mathcal{I}_{\alpha,\varepsilon}(E_2) + 2Q^2 \int_{E_1\times E_2} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} = \mathcal{F}_{\alpha,Q,\varepsilon}(E)\end{aligned}$$

where μ denotes the optimal measure for $\mathcal{I}_{\alpha,\varepsilon}(E)$. Thus, choosing R sufficiently large, and calling $\tilde{F}_\varepsilon = \{E_\varepsilon^j\}_{j\ne i} \times E_R$, we obtain $\mathcal{F}_{\alpha,Q,\varepsilon}(\tilde{F}_\varepsilon) < \mathcal{F}_{\alpha,Q,\varepsilon}(\tilde{E}_\varepsilon)$, a contradiction with the minimality property of $\tilde{E}_\varepsilon$. Thus, E is connected as well.

It is time to show that $\partial E = \partial^M E$. Trivially $\partial^M E \subset \partial E$, so we choose $0 \in \partial E$ (without loss of generality by translation like before) and exploit the fact that E is open, so for all $r > 0$, there exist $y \in B_r \cap E$ and $z \in B_r \setminus E$. By an easy consequence of density estimates, we have:

- for all $y \in E$ and for all $r \le r_0$: $|E \cap B_r(y)| \gtrsim r^N$;
- for all $z \in E^c$ and for all $r \le r_0$: $|B_r(z) \setminus E| \gtrsim r^N$.

Indeed, we proved the previous properties for all $x \in \partial E$, but if we move the point x outside or inside E density estimates keep holding true, because the measure of the set $E \cap B_r(x)$ and $B_r(x) \setminus E$ can only increase. In particular, they hold for all $x \in \mathbb{R}^N$. Therefore, since $B_r(y) \subset B_{2r}$ and $B_r(z) \subset B_{2r}$, we have $|B_{2r} \cap E| \ge |E \cap B_r(y)| \gtrsim r^N$ and $|B_{2r} \setminus E| \ge |E \setminus B_r(z)| \gtrsim r^N$ for all $r \le r_0$, so:

$$\liminf_{r\to 0^+} \frac{|B_{2r} \cap E|}{|B_{2r}|} > 0 \qquad \text{and} \qquad \liminf_{r\to 0^+} \frac{|B_{2r} \setminus E|}{|B_{2r}|} > 0$$

hence $0 \in \partial^M E$ and $\partial E = \partial^M E$.

Now, we show that $P(E) = \mathcal{H}^{N-1}(\partial E) < +\infty$ and $|\partial E| = 0$. The proof of these two facts is easy: by De Giorgi's structure theorem for sets of finite perimeter [18, Theorem 15.9] we have $P(E,\cdot) = \mathcal{H}^{N-1}|_{\partial^* E}$, so, by minimality of $\tilde{E}_\varepsilon$, we have that $\mathcal{H}^{N-1}(\partial^* E) = P(E) < +\infty$. Whereas, by Federer's theorem

[18, Theorem 16.2] we have $\mathcal{H}^{N-1}(\partial^M E \setminus \partial^* E) = 0$ as well, so putting everything together we infer $P(E,\cdot) = \mathcal{H}^{N-1}|_{\partial^M E}$ and

$$\mathcal{H}^{N-1}(\partial E) = \mathcal{H}^{N-1}(\partial^M E) = P(E) < +\infty.$$

$\mathcal{H}^{N-1}(\partial E) < +\infty$ implies immediately $|\partial E| = \mathcal{H}^N(\partial E) = 0$, by properties of Hausdorff measure. In particular, being ∂E Lebesgue negligible, $\overline{E}$ is a closed and bounded representative of E, thus we are allowed to consider E to be compact and from now on we will do it. In this way, combining compactness with the fact that $P(E) = \mathcal{H}^{N-1}(\partial E) < +\infty$, we finally get $E \in \mathcal{S}$.

It remains to prove that $\tilde{E}_\varepsilon = \{E^i_\varepsilon\}$ is composed by finitely many components, each of which has bounded diameter uniformly in ε. We fix $\varepsilon \in (0,1]$ and write $\tilde{E} = \tilde{E}_\varepsilon$: first of all, we can trivially get rid of all the components E^i such that $|E^i| = 0$. Indeed, $E^i = \emptyset$ for all such indexes: if otherwise $|E^i| = 0$ and $E^i \neq \emptyset$ then there exists $x \in E^i$ for which we have density estimates, so $|E^i| \geq |E^i \cap B_r(x)| \gtrsim r^N > 0$, impossible. Since the empty set is not seen by $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$, we can avoid considering all indexes i such that $E^i = \emptyset$. We assume by contradiction there are infinitely many indexes with $|E^i| > 0$. If this is the case, then there exists $x_i \in E^i$ for which we have density estimates, so:

$$\omega_N = |\tilde{E}| = \sum_{i\geq 1} |E^i| \geq \sum_{i\geq 1} |E^i \cap B_r(x_i)| \gtrsim \sum_{i\geq 1} r^N = +\infty$$

which is impossible. Hence, there exists $I \in \mathbb{N}$ such that $\tilde{E} = \{E^i\}_{i=1}^I$. The uniform bound on diameters follows easily. We fix $r \leq r_0$: by compactness, there exist $M \in \mathbb{N}$ and $\{x_j\}_{j\leq M}$ such that there holds $E^i \subset \cup_{j\leq M} B_r(x_j)$. By density estimates, $|E^i \cap B_r(x_j)| \gtrsim r^N$, thus the number of balls M must be limited uniformly in I. Therefore, for all $i \leq I$

$$\operatorname{diam}(E^i) \leq \operatorname{diam}(\cup_{j\leq M} B_r(x_j)) \leq \sum_{j\leq M} \operatorname{diam}(B_r(x_j)) = C < +\infty.$$

We conclude the proof highlighting the fact that I (and so the bound C on the diameters) is uniform in ε and depends only on $\overline{Q}$. Indeed, I can be derived from the coefficient appearing in density estimate 2.17, which in turns comes from condition $\max\{Q^2 r^{1-\alpha}, r\} \leq r_0$. Hence, fixing $\overline{Q}$ as we did at the beginning of the proof makes everything uniform for each $Q \leq \overline{Q}$. □

2.5 Existence and regularity of minimizers in the case $\alpha < 1$

While the previous sections were devoted to accurately setting the stage, in this one, thanks to Theorem 2.14, we are finally ready to prove existence of solutions to the problem 1.16

$$\min_{|\tilde{E}|=\omega_N, \tilde{E}\in\mathcal{S}} \mathcal{F}_{\alpha,Q}(\tilde{E}).$$

Afterwards, we restrict ourselves to the case $\alpha < 1$ and we focus on regularity: in Theorem 2.15 we infer first the usual conclusions from classical regularity theory of perimeter almost minimizers and then we see that, for Q small enough, generalized minimizers are actually classical minimizers, whose boundary is $C^{1,\gamma}$ regular for some $\gamma \in \left(0, \frac{1}{2}\right)$.

Theorem 2.14. Depending on the value of α, let Q^* given by Proposition 2.13 and set $Q^* = +\infty$ if $\alpha \in (0,1)$. Then, for every $0 < Q \leq \overline{Q} \leq Q^*$ there exist generalized minimizers $\tilde{E} = \{E^i\}_{i=1}^I \in \mathcal{S}^{\mathbb{N}}$ of

$$\min_{\tilde{E}\in\mathcal{S}^{\mathbb{N}}} \left\{ \mathcal{F}_{\alpha,Q}(\tilde{E}) : |\tilde{E}| = \omega_N \right\}.$$

Moreover, for each $i \leq I$, E^i is a perimeter almost minimizer in the sense of 2.16 and both I and $\operatorname{diam}(E^i)$ are bounded by a constant depending only on $\overline{Q}$.

Proof. Let Λ such that both 2.1 and 2.7 apply. By the latter, for every $\varepsilon \in (0,1]$ and $Q \le \overline{Q}$, there exists a generalized minimizer $\tilde{E}_\varepsilon$ of $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}$. Moreover, by 2.13, $\tilde{E}_\varepsilon = \{E^i_\varepsilon\}_{i=1}^I$ for some connected sets $E^i_\varepsilon \in \mathcal{S}$, with both I and their diameters depending only on $\overline{Q}$. In particular, there exists $R > 0$ such that E^i is strictly contained in B_R for all $i \le I$. Moreover, from what we highlighted at the beginning of the proof of 2.13, there exists $C > 0$ such that $P(E^i_\varepsilon) < C$ for all $\varepsilon \in (0,1]$. Therefore, for all $i \le I$, by [18, Theorem 12.26], there exists a set of finite perimeter $E^i \subset B_R$ such that, along some sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to +\infty$, $E^i_{\varepsilon_n} \longrightarrow E^i$ in L^1 and almost everywhere as $n \to +\infty$. Without loss of generality, up to other $I-1$ extractions, we can assume that the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ does the job simultaneously for all $i \le I$. We call $\tilde{E} = \{E^i\}_{i=1}^I$.

First of all we show that also E^i enjoys the minimality property 2.16. To simplify notation, we set $E_n = E^i_{\varepsilon_n}$ and $E = E^i$. Then, we select a decreasing sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ with $\sigma_n \to 0$ and $\sigma_n < r$ for all $n \in \mathbb{N}$ such that, considering the quantities $\{|(E\Delta E_n) \cap (B_r \setminus B_{r-\sigma_n})|\}_{n\in\mathbb{N}}$, we impose the fact that $|(E\Delta E_n) \cap (B_r \setminus B_{r-\sigma_n})| = o_{\sigma=0}(\sigma_n)$. For all $n \in \mathbb{N}$ we have, by Coarea Formula:

$$\int_{r-\sigma_n}^{r} \mathcal{H}^{N-1}((E\Delta E_n) \cap \partial B_s)\, ds = |(E\Delta E_n) \cap (B_r \setminus B_{r-\sigma_n})|.$$

In particular, there exists some $s_n \in (r-\sigma_n, r)$ (so clearly $s_n \to r$) such that:

$$\mathcal{H}^{N-1}((E\Delta E_n) \cap \partial B_{s_n}) \le \frac{|(E\Delta E_n) \cap (B_r \setminus B_{r-\sigma_n})|}{\sigma_n} \longrightarrow 0 \qquad \text{as } n \to +\infty$$

Now, we fix some finite perimeter set F such that $E\Delta F \subset B_{r-\sigma_1}$. With our choice of s_n, we define:

$$F_n = (F \cap B_{s_n}) \cup (E_n \setminus B_{s_n}),$$

noticing that $P(F_n) = P(F, B_s) + P(E_n, B^c_{s_n}) + \mathcal{H}^{N-1}((E \cap \partial B_{s_n})\Delta(E_n \cap \partial B_{s_n}))$. Moreover, by construction $F_n \Delta E_n \subset B_r$ because $s_n < r$, so 2.16 yields:

$$P(E_n) \le P(F_n) + C(Q^2 + r^\alpha) r^{N-\alpha}$$

which in turn gives:

$$P(E_n, B_{s_n}) \le P(F, B_{s_n}) + \mathcal{H}^{N-1}((E\Delta E_n) \cap \partial B_{s_n}) + C(Q^2 + r^\alpha) r^{N-\alpha}.$$

Sending $n \to +\infty$, the second term in the right-hand side tends to 0 by our previous argument. Moreover, since $B_r = \cup_{n\in\mathbb{N}} B_{s_n}$:

$$P(F, B_{s_n}) = \mathcal{H}^{N-1}(\partial^* F \cap B_{s_n}) \xrightarrow{n\to+\infty} \mathcal{H}^{N-1}(\partial^* F \cap B_r) = P(F, B_r),$$

by continuity of the measure $\mathcal{H}^{N-1}$ over Borel sets. Regarding the left-hand side, for all $\delta > 0$ we have definitely $P(E_n, B_{r-\delta}) \le P(E_n, B_{s_n})$ and, by lower semicontinuity of the perimeter under L^1 convergence:

$$P(E, B_{r-\delta}) \le \liminf_{n\to+\infty} P(E_n, B_{r-\delta}) \le \liminf_{n\to+\infty} P(E_n, B_{s_n}) \le \lim_{n\to+\infty} P(E_n, B_{s_n}).$$

In conclusion, letting $\delta \to 0$ we obtain:

$$P(E, B_r) \le P(F, B_r) + C(Q^2 + r^\alpha) r^{N-\alpha}$$

and adding $P(E, B^c_r) = P(F, B^c_r)$ to both sides we find:

$$P(E) \le P(F) + C(Q^2 + r^\alpha) r^{N-\alpha} \qquad \text{for all } E\Delta F \subset B_{r-\sigma_1}$$

Sending $\sigma_1 \to 0$, we get the desired inequality for all $E\Delta F \subset B_r$ for all $r \leq r_0$.

Arguing like we did in Proposition 2.13, density estimates imply nice properties for $\tilde{E} = \{E^i\}_{i=1}^I$ as well. In particular, for all $i \leq I$, $E^i \in \mathcal{S}$ and $\partial E^i = \partial^M E^i$. Moreover, we underline that the convergence obviously maintains both the number I and the uniform bound R on the diameters $\text{diam}(E^i)$ and these constants keep depending only on $\overline{Q}$. Notice that we are not able to prove connectedness yet, but it will be clear at the end of the proof, by minimality of the set $\tilde{E}$ for the functional $\mathcal{F}_{\alpha,Q}$. Indeed, it suffices to argue in the same way as we did when proving minimality in Proposition 2.13.

Now, we prove that $E_n^i \xrightarrow{H} E^i$ as $n \to +\infty$ too, namely the convergence happens in the Hausdorff sense as well. We show first $\partial E_n \xrightarrow{H} \partial E$ and we start by noticing that, reasoning like we did in Corollary 2.12, E^i enjoys density estimates too, as a consequence of the previous argument. We fix $x \in E^i \cap \{y : d(y, \partial E) > r\}$ and we assume by contradiction $x \notin E_n$ for all $n \in \mathbb{N}$. By density estimates we have:

$$|E_n \Delta E| \geq |B_r(x) \setminus E_n| \geq Cr^N.$$

Indeed, since $x \in E^i \cap \{y : d(y, \partial E) > r\}$, then $B_r(x) \subset E$, so $B_r(x) \setminus E_n \subset E \setminus E_n \subset E\Delta E_n$. We reach a contradiction letting $n \to +\infty$, because $|E_n \Delta E| \longrightarrow 0$. Therefore, for n big enough, all the points of $E \cap \{y : d(y, \partial E) > r\}$ are inside E_n. Similarly, we can show that for n big enough all the points of $E^c \cap \{y : d(y, \partial E) > r\}$ are outside E_n. As a consequence, for all $r \leq r_0$, for n big enough we have that $\partial E_n \subset \{y : d(y, \partial E) \leq r\}$. Inverting the roles of E and E_n (here we need density estimates for E), the same argument shows that for all $r \leq r_0$, for n big enough we have that $\partial E \subset \{y : d(y, \partial E_n) \leq r\}$. Putting everything together, we conclude $\partial E_n \xrightarrow{H} \partial E$ as $n \to +\infty$. Therefore, it remains to show Hausdorff convergence for points in the interior of E. Since we are dealing with closed sets and the space we are working in can be chosen compact ($\overline{B_R}$ for example), it is enough to show Kuratowski convergence, as stated in [1, Proposition 4.4.14]. In particular, we have to check the following two conditions:

- for every sequence $x_n \to x$ such that $x_n \in E_n$ for all $n \in \mathbb{N}$, we have $x \in E$;
- if $x \in E$ then there exists $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in E_n$ for all $n \in \mathbb{N}$ such that $x_n \to x$.

The second one is an easy consequence of the L^1 convergence. Instead, to prove the first one, we appeal again to density estimates. In particular for all $r \leq r_0$, for each converging sequence $x_n \to x$ there holds the condition $|B(x_n, r) \cap E_n| \geq Cr^N$. It implies, together with the L^1 convergence, that the limit point x must be in $\overline{E}$. As we have already proven Hausdorff convergence for the boundaries, in this way we showed that $E_n^i \xrightarrow{H} E^i$ as $n \to +\infty$ for all $i \leq I$.

Now, for all $i \leq I$, we focus our attention on the convergence of the family of measures $\{\mu_n^i\}_{n\in\mathbb{N}}$, associated to the sets $\{E_n^i\}_{n\in\mathbb{N}}$. In particular, the aim of the theorem is to come back from the regularized Riesz energy $\mathcal{I}_{\alpha,\varepsilon}$ to classical Riesz energy $\mathcal{I}_\alpha$. Both of the variational problems have a measure as minimizer, nevertheless we are sure it is an L^2 function only in the first case. As a consequence, when passing to the limit as $\varepsilon \to 0$, we cannot apply Banach-Alaoglu theorem as we did before and we are obliged to treat the $L^2(B_R)$ functions $\{\mu_n^i\}_{n\in\mathbb{N}}$ as measures. Being minimizing measures for the functionals $\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon_n}$, we have for all $n \in \mathbb{N}$:

$$\sum_{i=1}^I \mu_n^i(E_n^i) = 1,$$

therefore $\mu_n^i(E_i^n) \leq 1$ for all $i \leq I$; moreover $\text{spt}(\mu_n^i) \subset E_n^i \subset B_R$. By compactness of the set $\overline{B_R}$, the set of measures supported in $\overline{B_R}$ with mass less or equal to 1 is compact with respect to the weak convergence of measures. Therefore, for all $i \leq I$ there exists a measure μ^i supported on $\overline{B_R}$ such that $\mu_n^i \rightharpoonup \mu^i$ in the space of finite measures up to extraction. Again, we can assume without loss of generality by performing other $N-1$ extractions that the sequence we chose does the job simultaneously for all $i \leq I$. At this point, we need to prove that $\tilde{\mu} := \{\mu^i\}_{i=1}^I$ is a competitor for $\mathcal{I}_\alpha(\tilde{E})$, namely that $\mathrm{spt}(\mu^i) \subset E^i$ for all $i \leq I$ and that:

$$\sum_{i=1}^{I} \mu^i(E^i) = 1. \tag{2.19}$$

Fixed $i \leq I$ and calling $\mu = \mu^i$, $\mu_n = \mu_n^i$, $E = E^i$ and $E_n = E_n^i$, the first condition is a consequence of the convergence $E_n \xrightarrow{H} E$ as $n \to +\infty$. Indeed, by [16, Lemma 0.1, Corollary 1] we have that $\mu(A) \leq \liminf_{n\to+\infty} \mu_n(A)$ for all open sets $A \subset \mathbb{R}^N$. Hence, being E closed and considering the measures μ, μ_n to be defined on the whole of $\mathbb{R}^N$, we infer by Fatou's Lemma:

$$\begin{aligned}\mu(E^c) = \int_{\mathbb{R}^N} \chi_{E^c}\, d\mu &\leq \liminf_{k\to+\infty} \int_{\mathbb{R}^N} \chi_{(\overline{E_{1/k}})^c}\, d\mu = \liminf_{k\to+\infty} \mu\left((\overline{E_{1/k}})^c\right) \\ &\leq \liminf_{k\to+\infty}\, \liminf_{n\to+\infty} \mu_n\left((\overline{E_{1/k}})^c\right) = 0.\end{aligned}$$

In the last computations, we denoted by $E_{1/k}$ the open $1/k$-neighbourhood of E. By Hausdorff convergence, for all $k \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$ we have $E_n \subset \overline{E_{1/k}}$. Since $\mathrm{spt}(\mu_n) \subset E_n$, for all $n \geq \bar{n}$ we have $\mu_n\left((\overline{E_{1/k}})^c\right) = 0$. Therefore, passing everything to the inferior limit, we obtain $\mu(E^c) = 0$, namely $\mathrm{spt}(\mu) \subset E$. To prove 2.19, this time we use [16, Lemma 0.1, Corollary 3]: $\mu(A) = \lim_{n\to+\infty} \mu_n(A)$ for all A Borel such that $\mu(\partial A) = 0$. We just showed that for all $i \leq I$ $\mathrm{spt}(\mu^i) \subset E^i$, so in particular $\mu^i(B_{2R}) = 0$, hence:

$$1 = \lim_{n\to+\infty} \sum_{i=1}^{I} \mu_n^i(B_{2R}) = \sum_{i=1}^{I} \lim_{n\to+\infty} \mu_n^i(B_{2R}) = \sum_{i=1}^{I} \mu^i(B_{2R}).$$

Using again the condition $\mathrm{spt}(\mu^i) \subset E^i$, we are able to infer 2.19.

It is time to show that:

$$\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}_\varepsilon). \tag{2.20}$$

To make the notation coherent with the final part of the proof, we restarted using $\varepsilon \to 0$ to index the convergence we established instead of $n \to +\infty$. For all $i \leq I$, by L^1 convergence of the components we have automatically :

$$|E^i| = \lim_{\varepsilon\to 0} |E_\varepsilon^i| \qquad \Longrightarrow \qquad \Lambda \left| |\tilde{E}| - \omega_N \right| = \lim_{\varepsilon\to 0} \Lambda \left| |\tilde{E}_\varepsilon| - \omega_N \right|,$$

by continuity of the function $\Lambda\, |\cdot - \omega_N|$. On the other hand, by lower semicontinuity of the perimeter it is easy to notice:

$$P(E^i) \leq \liminf_{\varepsilon\to 0} P(E_\varepsilon^i) \qquad \Longrightarrow \qquad P(\tilde{E}) \leq \liminf_{\varepsilon\to 0} P(\tilde{E}_\varepsilon).$$

Therefore, if we show:

$$Q^2\mathcal{I}_\alpha(\tilde{E}) \leq \liminf_{\varepsilon\to 0} Q^2\mathcal{I}_{\alpha,\varepsilon}(\tilde{E}_\varepsilon) \tag{2.21}$$

we can deduce 2.20 putting everything together and using the superlinearity of the inferior limit. To show 2.21, we begin by noticing that $I_\alpha(\mu_\varepsilon^i) \leq I_\alpha(\mu_\varepsilon^i) + \varepsilon \int_{\mathbb{R}^N} (\mu_\varepsilon^i)^2 = \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}_\varepsilon)$. For all $i \leq I$ we have $\mu_\varepsilon^i \rightharpoonup \mu^i$ as $\varepsilon \to 0$, by weak lower semicontinuity of the functional I_α we get:

$$\begin{aligned}I_\alpha(\tilde{\mu}) = \sum_{i=1}^{I} I_\alpha(\mu^i) &\leq \sum_{i=1}^{I} \liminf_{\varepsilon\to 0} I_\alpha(\mu_\varepsilon^i) \leq \liminf_{\varepsilon\to 0} \sum_{i=1}^{I} I_\alpha(\mu_\varepsilon^i) = \liminf_{\varepsilon\to 0} \sum_{i=1}^{I} I_{\alpha,\varepsilon}(\mu_\varepsilon^i) \\ &= \liminf_{\varepsilon\to 0} \mathcal{I}_{\alpha,\varepsilon}(\tilde{E}_\varepsilon).\end{aligned}$$

Now, by 2.19 the measure $\tilde{\mu}$ is a competitor for $\mathcal{I}_\alpha$. Passing the left-hand side to the minimum over the class measures of this kind, we finally infer 2.21.

Next, we prove that for every $F \in \mathcal{S}$, there exists a sequence $\{F_\varepsilon\}_{\varepsilon\in(0,1]}$ such that:

$$\mathcal{F}_{\alpha,Q,\Lambda}(F) \geq \limsup_{\varepsilon\to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(F_\varepsilon). \tag{2.22}$$

By [26, Theorem 1.1] applied to F^c, for all $\delta > 0$ we can find smooth compact sets F_δ such that $F \subset F_\delta$, $P(F_\delta) \leq P(F) + \delta$ and $||F| - |F_\delta|| \leq \delta$. The first condition implies $\mathcal{I}_\alpha(F) \geq \mathcal{I}_\alpha(F_\delta)$. Considering also the effects of the second and the third condition on the two remaining terms of $\mathcal{F}_{\alpha,Q,\Lambda}$, we find, passing to the superior limit as $\delta \to 0$:

$$\mathcal{F}_{\alpha,Q,\Lambda}(F) \geq \limsup_{\delta\to 0} \mathcal{F}_{\alpha,Q,\Lambda}(F_\delta).$$

Thus, we can further assume that F is smooth in the proof of 2.22. For smooth sets, by [12, Proposition 2.16], we can find for every $\delta > 0$ a function $f_\delta \in L^\infty(F)$ with $\int_F f_\delta = 1$ and such that:

$$I_\alpha(f_\delta) \leq \mathcal{I}_\alpha(F) + \delta.$$

Precisely, the statement holds only for connected sets, but it can be easily readapted for disconnected sets too. At this point, for every $\delta > 0$, we clearly have $\lim_{\varepsilon\to 0} I_{\alpha,\varepsilon}(f_\delta) = I_\alpha(f_\delta)$. Hence, a diagonal argument shows that $\mathcal{I}_\alpha(F) = \lim_{\varepsilon\to 0} \mathcal{I}_{\alpha,\varepsilon}(F)$. The proof of 2.22 is concluded once setting $F_\varepsilon = F$ for all $\varepsilon \in (0,1]$, considering the superior limit as $\varepsilon \to 0$ instead of the limit and finally adding the other two terms of the functional $\mathcal{F}_{\alpha,Q,\Lambda}$ (which are not perturbed by ε) to the inequality.

Now, by lemma 2.6, passing the right-hand side of 2.22 to the infimum over $F \in \mathcal{S}$ yields the same value as passing the same functional but defined on generalized sets to the infimum over $\tilde{F} \in \mathcal{S}^{\mathbb{N}}$. Therefore:

$$\inf_{\tilde{F}\in\mathcal{S}^{\mathbb{N}}} \left\{\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{F})\right\} \geq \limsup_{\varepsilon\to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(F_\varepsilon) \geq \limsup_{\varepsilon\to 0} \inf_{\tilde{F}\in\mathcal{S}^{\mathbb{N}}} \left\{\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{F})\right\}.$$

This last relation, combined with 2.20 allows us to finish: indeed, in this way we have

$$\begin{aligned}\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{E}) &\geq \inf_{\tilde{F}\in\mathcal{S}^{\mathbb{N}}} \left\{\mathcal{F}_{\alpha,Q,\Lambda}(\tilde{F})\right\} \geq \limsup_{\varepsilon\to 0} \inf_{\tilde{F}\in\mathcal{S}^{\mathbb{N}}} \left\{\mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{F})\right\} = \limsup_{\varepsilon\to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}_\varepsilon) \\ &\geq \liminf_{\varepsilon\to 0} \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}_\varepsilon) \geq \mathcal{F}_{\alpha,Q,\Lambda,\varepsilon}(\tilde{E}).\end{aligned}$$

Since $\tilde{E}$ is a generalized minimizer of $\mathcal{F}_{\alpha,Q,\Lambda}$, Lemma 2.1 implies that $|\tilde{E}| = \omega_N$ and thus $\tilde{E}$ is also a volume-constrained generalized minimizer of $\mathcal{F}_{\alpha,Q}$. □

After showing existence of minimizers, we are now ready to deal with their regularity. We begin by focusing on the case $\alpha \in (0,1)$ and we appeal to classical regularity theory for almost minimizers of the perimeter first pioneered by De Giorgi. The topic is too broad to be treated with the worthy precision in this dissertation, so we decided to outline just the basic ideas needed in the proof of the following theorem, avoiding some technicalities. Instead, the case $\alpha = 1$ will be analyzed in the next section.

Theorem 2.15. For $\alpha \in (0,1)$ and $Q > 0$, let $\tilde{E} = \{E^i\}_{i=1}^I$ be a volume-constrained generalized minimizer of $\mathcal{F}_{\alpha,Q}$. Then, for all $i \leq I$, $\partial^* E^i$ are $C^{1,\frac{(1-\alpha)}{2}}$ regular. Denoting by $\Sigma_i = \partial E^i \setminus \partial^* E^i$, we have that:

- $\Sigma_i = \emptyset$ if $N \leq 7$;
- Σ_i is at most finite if $N = 8$;
- Σ_i satisfies $\mathcal{H}^s(\Sigma_i) = 0$ if $s > N - 8$ and $N \geq 9$.

In addition, for $Q \ll 1$, $\tilde{E} = E_Q$ is a classical volume-constrained minimizer of $\mathcal{F}_{\alpha,Q}$ with uniformly bounded $C^{1,\frac{(1-\alpha)}{2}}$ norm, with no singularities (namely $\Sigma(E_Q) = \emptyset$) and, for every $\beta < \frac{(1-\alpha)}{2}$, E_Q converges to B_1 in $C^{1,\beta}$ as $Q \to 0$.

Idea of the proof. For all $i \leq I$, the set E^i satisfies 2.16. Therefore, the first part of the statement is a direct consequence of [27, Theorem 1], which itself given by a readaptation of the classical De Giorgi's regularity theory for almost minimizers of the perimeter.

The second part is more interesting. First of all we summon the Sharp Quantitative Isoperimetric Inequality from [10, Theorem 1.1]. However, we are not able to apply it straightaway, since it involves the perimeter and the volume of only classical sets. Thus, starting from the components E^i of our generalized set, recalling they are all uniformly bounded by some $R > 0$, we build a classical set E made up by such components positioned far enough from each other. In this way $P(E) = P(\tilde{E})$ and $|E| = |\tilde{E}|$. Hence, up to translation and relabelling, the Sharp Quantitative Isoperimetric Inequality yields:

$$\left(|E^1 \Delta B_1| + \sum_{i=2}^{I} |E^i|\right)^2 \lesssim P(\tilde{E}) - P(B_1) \leq Q^2 \mathcal{I}_\alpha(B_1) - Q^2 \mathcal{I}_\alpha(\tilde{E}) \lesssim Q^2,$$

where in the second inequality we used minimality of $\tilde{E}$ for $\mathcal{F}_{\alpha,Q}$, namely $\mathcal{F}_{\alpha,Q}(\tilde{E}) \leq \mathcal{F}_{\alpha,Q}(B_1)$. By translation we can assume without loss of generality $E^1 \cap B_1 \neq \emptyset$. Thus, density estimates from Corollary 2.12 imply that, for Q small enough, $E^i = \emptyset$ for $i \geq 2$, so that $\tilde{E} = E^1$ is a classical minimizer. Moreover, it is clear that $|E^1 \Delta B_1| \longrightarrow 0$ as $Q \to 0$, namely $E^1 \longrightarrow B_1$ in L^1.

Now, we exploit again regularity theory for almost minimizers of the perimeter (this is the portion of the proof where we give just the idea). First of all, given a set of finite perimeter E, we define its spherical excess at $x \in \partial E$ at scale r as:

$$\mathbf{e}(E, x, r) = \min_{\nu \in \mathbb{S}^{N-1}} \int_{\partial^* E \cap B(x,r)} \frac{|\nu_E(y) - \nu|}{2} d\mathcal{H}^{N-1}(y),$$

where $\mathbb{S}^{N-1}$ is the surface of $B_1 \subset \mathbb{R}^N$ and ν_E the measure theoretic normal of the set E appearing in its definition of reduced boundary. Now, we consider a set of finite perimeter E satisfying an estimate of the kind:

$$P(E) \leq P(F) + \Lambda r^{N-\alpha} \qquad \text{for all } E \Delta F \subset B_r(x)$$

for all $x \in \mathbb{R}^N$ for some $r_0 > 0$ for all $r \leq r_0$. From a readaptation of [18, Theorem 26.3], there exists $\varepsilon > 0$ such that, if for all $x \in \partial E$ we have

$$\mathbf{e}(E, x, r) + \Lambda r^{1-\alpha} \leq \varepsilon, \tag{2.23}$$

(for some r smaller that a certain constant), then $\partial E \cap B_{r/2}$ is the graph of a $C^{1,\frac{(1-\alpha)}{2}}$ function f, with:

$$[f]_{C^{1,\frac{(1-\alpha)}{2}}} \lesssim \mathbf{e}(E, x, r) + \Lambda r^{1-\alpha}.$$

Choosing the right coordinate system for f (for example a system such that $f(0) = 0$ and $\nabla f(0) = 0$), we have also $[f]_{C^{1,\frac{(1-\alpha)}{2}}} = \|f\|_{C^{1,\frac{(1-\alpha)}{2}}}$, the bound is uniform in $x \in \partial E$ and depends only on Λ.

We would like to apply this regularity result to our situation: we have a sequence $\{E_Q\}_{Q>0}$ of almost minimizers in the sense of 2.16, namely there exists $r_0 > 0$ such that for all $x \in \partial E$ and $r \leq r_0$ we have:

$$P(E_Q) \leq P(F) + C\left(Q^2 + r^\alpha\right) r^{N-\alpha} \qquad \text{for all } E_Q \Delta F \subset B_r(x).$$

In addition, $E_Q \longrightarrow B_1$ in L^1 as $Q \to 0$ and obviously B_1 has C^1 boundary. For every $\varepsilon > 0$, it can be shown that the estimate on the excess 2.23 holds for the right r (which we did not specify). Moreover, for almost minimizers of the perimeter in our sense 2.16, from the density estimates and the convergence $P(E_Q) \longrightarrow P(B_1)$, the excess is continuous with respect to L^1 convergence. In particular, choosing again the right scale, for Q small enough we have for all $x \in \partial E_Q$:

$$\mathbf{e}(E_Q, x, r) + \Lambda r^{1-\alpha} \leq \frac{\varepsilon}{2}.$$

Therefore, for Q small enough, ∂E_Q is of class $C^{1,\frac{(1-\alpha)}{2}}$ and in particular it has no singularities. Moreover, calling $\{f_x\}_{x \in \partial E}$ the corresponding functions defined on $\partial E \cap B_{r/2}(x)$ for $x \in \partial E$, their seminorms are uniformly bounded by:

$$[f_x]_{C^{1,\frac{(1-\alpha)}{2}}} \lesssim \mathbf{e}(E, x, r) + \Lambda r^{1-\alpha}$$

Hence, by compactness of the Hölder embedding over the right compact set K:

$$C^{1,\frac{(1-\alpha)}{2}}(K) \longrightarrow C^{1,\beta}(K) \qquad \text{for all } 0 < \beta < \frac{(1-\alpha)}{2},$$

we can conclude $E_Q \longrightarrow B_1$ in $C^{1,\beta}$ as $Q \to 0$ for all $0 < \beta < \frac{(1-\alpha)}{2}$. □

2.6 Second almost minimality property and regularity of minimizers in the case $\alpha = 1$

Proving a counterpart of Theorem 2.15 in the case $\alpha = 1$ is much more difficult: indeed, it is known that when $\alpha = 1$ the first minimality condition 2.16 does not even imply C^1 regularity. Therefore, we are not in the position to apply the same ideas employed before and the aim of this section is to develop a way to reach the same regularity conclusions of the case $\alpha < 1$. Clearly, from now on we assume that $\alpha = 1$.

The first step consists in introducing the notion of Reifenberg flat set.

Definition 2.16 (Reifenberg Flatness)**.** Let $E \subset \mathbb{R}^N$, $\delta, r_0 > 0$ and $x \in \mathbb{R}^N$. We say that E is (δ, r_0)-*Reifenberg flat* in $B_{r_0}(x)$ if for every $B_r(y) \subset B_{r_0}(x)$ there exists a hyperplane $H_{y,r}$ containing y and such that:

- we have (denoting d_H the Hausdorff distance):

$$d_H(\partial E \cap B_r(y), H_{y,r} \cap B_r(y)) \leq \delta r;$$

- one of the connected components of the set $\{d(\cdot, H_{y,r} \geq 2\delta r)\} \cap B_r(y)$ is included in E and the other in E^c.

We say that E is *uniformly* (δ, r_0)-*Reifenberg flat* if the above condition hold for every $x \in \partial E$.

It is immediate to see that C^1 regular sets are trivially Reifenberg flat at each point of their boundary, just by considering their tangent hyperplane as H_r. Using the same technique of Theorem2.15, in the next result we prove first L^1 convergence of a minimizer to the unit ball B_1 as $Q \to 0$ and then we exploit it to show that, for Q small enough, our minimizer is actually uniformly Reifenberg flat.

Theorem 2.17. Let $\alpha = 1$. There exists Q^* such that for every $Q \leq Q^*$, every volume-constrained generalized minimizer of $\mathcal{F}_{\alpha,Q}$ is a classical minimizer. Moreover, for every $\delta > 0$, there exist $Q_\delta > 0$ and $r_\delta > 0$ such that for every $Q \leq Q_\delta$, every volume-constrained minimizer E_Q of $\mathcal{F}_{\alpha,Q}$ is uniformly (δ, r_δ)-Reifenberg flat and, up to translation, $|E_Q \Delta B_1| \lesssim Q^2$.

Proof. The first part of the statement can be recovered as we did in Theorem 2.15 employing the Sharp Quantitative Isoperimetric Inequality. In particular, we highlight that $E_Q \longrightarrow B_1$ in L^1 as $Q \to 0$ and that E_Q is a (Λ, r_0)-minimizer in the sense that there exist r_0 such that for all $x \in \mathbb{R}^N$ and $r \leq r_0$, there holds:

$$P(E_Q) \leq P(F) + C(Q^2 + r)r^{N-1} \qquad \text{for all } E_Q \Delta F \subset B_r(x),$$

where $\Lambda_Q = C(Q^2 + r)$. Moreover, $\max\{Q^2, r\} \ll 1$, in accordance to Corollary 2.12.

Since B_1 is a C^1 regular and compact set, after fixing $\delta > 0$ we can apply [14, Corollary 1.5]: there exists $\bar{\Lambda} = \bar{\Lambda}(\delta)$ such that, if $\limsup_{Q\to 0} \Lambda_Q \leq \bar{\Lambda}$, then there exists $r_\delta > 0$ such that, if Q is small enough, E_Q is uniformly (δ, r_δ)-Reifenberg flat. In our case we have:

$$\limsup_{Q\to 0} \Lambda_Q = \limsup_{Q\to 0} C(Q^2 + r) = Cr.$$

So, choosing r small enough, the quantity $\limsup_{Q\to 0} \Lambda_Q$ can be made arbitrarily small and the thesis of [14, Corollary 1.5] follows. □

At this point, we would like to pass from Reifenberg flatness of volume-constrained minimizers of $\mathcal{F}_{1,Q}$ to their $C^{1,\gamma}$ regularity, for some γ to be determined. As one might expect from what we did before, we rely on a second almost minimality property for minimizers E of $\mathcal{F}_{1,Q}$ (depending on its optimal measure μ_E), thanks to which we will be able to draw the required regularity result. We highlight that in the next proposition Reifenberg flatness of E is not used.

Proposition 2.18. There exists a number $C = C(N) > 0$ such that if $Q \leq 1$ and E is a volume-constrained minimizer of $\mathcal{F}_{1,Q}$, whose corresponding 1/2-harmonic measure (namely the measure such that $\mathcal{I}_1(E) = I_1(\mu_E)$) is μ_E, then for every $x \in \mathbb{R}^N$ and $r \ll 1$ there holds:

$$P(E) \leq P(F) + C\left(Q^2\left(\int_{B_r(x)} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} + r^N\right) \qquad \text{for all } E\Delta F \subset B_r(x). \tag{2.24}$$

Proof. Without loss of generality we may assume that $x = 0$ and $\mu_E \in L^{\frac{2N}{N+1}}(B_r)$, since otherwise there is nothing to prove. In particular, we require this hypothesis for all $x \in \mathbb{R}^N$, according to the statement we want to prove, therefore we are asking $\mu_E \in L^{\frac{2N}{N+1}}_{loc}(\mathbb{R}^N)$. By Lemma 2.1, there exists an universal constant $\Lambda > 0$ (recall that we assumed $Q \leq 1$) such that E is a minimizer of

$$\mathcal{F}_{1,Q}(E) + \Lambda \left||E| - \omega_N\right|.$$

We choose F such that $E\Delta F \subset B_r(0)$ and we argue as in the proof of Proposition 2.16 to get:

$$P(E) \leq P(F) + Q^2\left(\mathcal{I}_1(F) - \mathcal{I}_1(E)\right) + \Lambda|E\Delta F|.$$

Again, using $P(E\cap F) + P(E\cup F) \leq P(E) + P(F)$, we can assume without loss of generality that either $E \subset F$ or $F \subset E$: in the first case $\mathcal{I}_1(F) - \mathcal{I}_1(E) \leq 0$, so we conclude immediately. Therefore, it is enough to prove that for every $F \subset E$ such that $E \setminus F \subset B_r$ we have:

$$\mathcal{I}_1(F) \leq \mathcal{I}_1(E) + C\left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}.$$

Noticing that $F \subset E$ implies $F \cap E = F$ and $F \cup E = E$, we use the measure:

$$\mu = \left(\mu_E + \frac{\mu_E(E\setminus F)}{|F|}\right)\chi_F$$

as a competitor for $\mathcal{I}_1(F)$, indeed:

$$\mu(F) = \int_{\mathbb{R}^N} \chi_F \left(\mu_E + \frac{\mu_E(E \setminus F)}{|F|} \right) = \int_F \mu_E + \int_F \frac{\mu_E(E \setminus F)}{|F|} = \mu_E(E \cap F) + \mu_E(E \setminus F) = 1.$$

Now, we consider the respective potentials:

$$u_E(x) = \int_E \frac{d\mu_E(y)}{|x-y|^{N-1}} \qquad \text{and} \qquad u(x) = \int_F \frac{d\mu(y)}{|x-y|^{N-1}},$$

which, by Proposition 1.12, solve respectively the non-local elliptic equations:

$$(-\Delta)^{\frac{1}{2}} u_E = C'(N,1)\mu_E \qquad \text{and} \qquad (-\Delta)^{\frac{1}{2}} u = C'(N,1)\mu.$$

Moreover, 1.10 yields:

$$\frac{1}{C'(N,1)}[u_E]^2_{H^{\frac{1}{2}}} = \int_E u_E \, d\mu_E = I_1(\mu_E) \qquad \text{and} \qquad \frac{1}{C'(N,1)}[u]^2_{H^{\frac{1}{2}}} = \int_E u \, d\mu = I_1(\mu)$$

Since $\mathcal{I}_1(F) \leq I_1(\mu)$ and $\mathcal{I}_1(E) = I_1(\mu_E)$, we have:

$$\begin{aligned} \mathcal{I}_1(F) - \mathcal{I}_1(E) \leq \int_E u \, d\mu - \int_E u_E \, d\mu_E &= \int_E (u - u_E) d(\mu - \mu_E) + \int_E u_E \, d\mu \\ &+ \int_E u \, d\mu_E - \int_E u_E \, d\mu_E - \int_E u_E \, d\mu_E. \end{aligned}$$

It is useful to notice that $\mu(E) = 1$ as well, since $\mu(F) = 1$ and $\operatorname{spt}(\mu) \subset F$. Recalling that $u_E(x) = \mathcal{I}_1(E)$ for every $x \in E$ by Proposition 1.12, this implies:

$$\int_E u_E \, d(\mu - \mu_E) = 0.$$

In addition, by Fubini:

$$\int_E u \, d\mu_E = \int_E u_E \, d\mu = \int_E u_E \, d\mu_E,$$

so the last four terms in the right-hand side of the previous computation erase and we get:

$$\mathcal{I}_1(F) - \mathcal{I}_1(E) \leq \int_E (u - u_E) d(\mu - \mu_E) \approx [u - u_E]^2_{H^{\frac{1}{2}}(E)}.$$

Being both u_E and u functions by assumptions and construction respectively, we estimate the $H^{\frac{1}{2}}$ seminorm using Hölder inequality and the classical Sobolev embedding $\|f\|_{L^q(\Omega)} \lesssim [f]_{H^s(\Omega)}$ with $\frac{N}{q} = \frac{N}{2} - s$ (highlighting that here $\Omega = B_r$ is a bounded C^1 domain). In particular, if $q = \frac{2N}{N-1}$ and $s = \frac{1}{2}$ we apply the inequality to a finite collection of balls covering E and we get:

$$\begin{aligned} [u - u_E]^2_{H^{\frac{1}{2}}(E)} \approx \int_E (u - u_E) d(\mu - \mu_E) &\leq \|u - u_E\|_{L^{\frac{2N}{N-1}}(E)} \|\mu - \mu_E\|_{L^{\frac{2N}{N+1}}(E)} \\ &\leq [u - u_E]_{H^{\frac{1}{2}}(E)} \|\mu - \mu_E\|_{L^{\frac{2N}{N+1}}(E)} \end{aligned}$$

Using Young's Inequality generalization $ab \leq \frac{a^2}{2\theta} + \theta \frac{b^2}{2}$ with the right value of θ, we find:

$$\mathcal{I}_1(F) - \mathcal{I}_1(E) \lesssim [u - u_E]^2_{H^{\frac{1}{2}}(E)} \lesssim \|\mu - \mu_E\|^2_{L^{\frac{2N}{N+1}}(E)},$$

so we are left with estimating $\|\mu - \mu_E\|_{L^{\frac{2N}{N+1}}(E)}$. By definition of μ, we have

$$\mu - \mu_E = \frac{\mu_E(E \setminus F)}{|F|}\chi_F - \mu_E \chi_{E\setminus F},$$

and thus, using that $F \cap E \setminus F = \emptyset$ and $(a+b)^{\frac{N+1}{N}} \lesssim a^{\frac{N+1}{N}} + b^{\frac{N+1}{N}}$ for $a, b \geq 0$ we find:

$$\begin{aligned}
\|\mu - \mu_E\|^2_{L^{\frac{2N}{N+1}}(E)} &= \left(\int_E \left|\frac{\mu_E(E \setminus F)}{|F|}\chi_F - \mu_E\chi_{E\setminus F}\right|^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \\
&= \left(\int_F \left(\frac{\mu_E(E \setminus F)}{|F|}\right)^{\frac{2N}{N+1}} + \int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \\
&= \left(|F|\frac{\mu_E(E \setminus F)^{\frac{2N}{N+1}}}{|F|^{\frac{2N}{N+1}}} + \int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} \\
&\lesssim \frac{\mu_E(E \setminus F)^2}{|F|^{\frac{N-1}{N}}} + \left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}.
\end{aligned}$$

Moreover, $|F| \gtrsim 1$ implies:

$$\|\mu - \mu_E\|^2_{L^{\frac{2N}{N+1}}(E)} \lesssim \mu_E(E \setminus F)^2 + \left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}}.$$

Finally, again by Hölder inequality:

$$\mu_E(E \setminus F)^2 \leq \left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}} |E \setminus F|^{\frac{N+1}{N}} \lesssim \left(\int_{E\setminus F} \mu_E^{\frac{2N}{N+1}}\right)^{\frac{N+1}{N}},$$

which concludes the proof. □

Notice that we do not know whether the measure μ_E actually belongs to the space $L^{\frac{2N}{N+1}}_{loc}(\mathbb{R}^N)$ or not. The remainder of the chapter is devoted to showing that not only this integrability property is true, but also that we can find a good decay estimate for $\|\mu_E\|_{L^{2N/(N+1)}(B_r(x))}$ as $r \to 0$ for all $x \in \partial E$, which will finally yields us an expression similar to the first minimality property 2.16.

Awfully, the procedure is rather involved and it takes quite a long time to be presented in detail. Essentially, in Lemma 2.21 we show a Hölder estimate on the potential u_E which will be crucial to prove the required decay bound for $\|\mu_E\|_{L^{2N/(N+1)}(B_r(x))}$. In particular, thanks to some tools from elliptic PDEs theory and to Alt-Caffarelli-Friedman monotonicity formula, the proof of the next lemma may be seen as an extension to Reifenberg flat domains of the boundary regularity theory for the half Laplacian.

Before diving into the lemma, let E be a minimizer of the functional $\mathcal{F}_{1,Q}$ for some Q small enough. Given its optimal measure μ_E and its associated potential u_E, we define the function $u : \mathbb{R}^N \longmapsto \mathbb{R}$ as

$$u(x) = 1 - \mathcal{I}_1^{-1}(E)u_E(x) = 1 - \mathcal{I}_1^{-1}(E)\int_E \frac{d\mu_E(y)}{|x-y|^{N-1}}$$

and we take some time studying its properties. First of all, by Proposition 1.12, we know that $u_E \geq 0$, $u_E = \mathcal{I}_1(E)$ Lebesgue almost everywhere on E and $u_E \leq \mathcal{I}_1(E)$ everywhere in $\mathbb{R}^N$. Hence, $u \in [0, 1]$

almost everywhere and $u(x) = 0$ for all $x \in E$. Now, we consider its harmonic extension v on $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\}$, namely the function $v : \mathbb{R}^N \longmapsto \overline{\mathbb{R}^{N+1}_+}$ solving the elliptic PDE with Dirichlet boundary condition:

$$\begin{cases} -\Delta v = 0 & \text{on } \mathbb{R}^{N+1}_+ \\ v = u & \text{on } \partial\,\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\}. \end{cases}$$

Normally, we would need to prove existence of a solution of such problem, however in this case the construction of v yields immediately the solution:

$$v(z) = 1 - \mathcal{I}_1^{-1}(E) \int_E \frac{d\mu_E(y)}{|z-y|^{N-1}} \qquad z \in \overline{\mathbb{R}^{N+1}_+}.$$

Indeed, clearly $v = u$ on $\mathbb{R}^N \times \{0\}$. To prove $-\Delta v = 0$ on $\mathbb{R}^{N+1}_+$, it is enough to show $-\Delta v = -\mu_E$ in $\mathcal{D}'(\mathbb{R}^{N+1})$ (thus considering v's natural extension to $\mathbb{R}^{N+1}$). Indeed, in this way, being μ_E supported in $\mathbb{R}^N \times \{0\}$, we have that $\Delta v = 0$ on the open set $\mathbb{R}^{N+1}_+$. This is easy once noticed that $\Gamma(x) = |x|^{1-N}$ is the Green function for the Laplacian on $\mathbb{R}^{N+1}$, so in particular $-\Delta\,\Gamma = \delta_0$, where δ_0 denotes the Dirac's delta measure centered in $x = 0$. Therefore:

$$-\Delta\,(\tilde{u_E}(x)) - \;\Delta\,(\Gamma(x) * \mu_E) = (-\Delta\,\Gamma(x)) * \mu_E = \delta_0(x) * \mu_E = \mu_E(x)$$

and so $-\Delta v = -\mu_E$ in $\mathcal{D}'(\mathbb{R}^{N+1})$. By classical properties of harmonic functions, we immediately infer $v \in C^\infty(\mathbb{R}^{N+1}_+)$ as well. In addition, we can easily prove some decay estimates on both $\tilde{u_E}$ and $\nabla\tilde{u_E}$ (calling $\tilde{u_E}$ the harmonic extension of u_E to $\mathbb{R}^{N+1}$):

$$\tilde{u_E}(z) \approx_{+\infty} \frac{1}{|z|^{N-1}} \qquad \text{and} \qquad |\nabla\tilde{u_E}(z)| \approx_{+\infty} \frac{1}{|z|^N}.$$

First of all, we find the right expression for $\nabla\tilde{u_E}$. We begin by noticing that $\tilde{u_E} \in C^\infty(\mathbb{R}^{N+1} \setminus (E \times \{0\}))$ for the same reasons as it is v: for our purposes we would like to compute its gradient where it is well defined. First of all $\tilde{u_E} \in L^1_{loc}(\mathbb{R}^{N+1})$. Indeed, for all $R > 0$ we have by Fubini:

$$\int_{|z|\le R} \int_{\mathbb{R}^{N+1}} \mu_E(y)|z-y|^{-(N-1)}\,dz = \int_{\mathbb{R}^{N+1}} \mu_E(y) \int_{|z|\le R} |z-y|^{-(N-1)}\,dz < +\infty.$$

In a similar way, since $\nabla_z|z-y|^{-(N-1)} \approx |z-y|^{-(N+1)}(z-y)$ belongs to $L^1_{loc}(\mathbb{R}^{N+1})$, $\nabla\tilde{u_E} \in L^1_{loc}(\mathbb{R}^{N+1})$ by the same proof as above, so we can derive under the integral sign and get:

$$\nabla\tilde{u_E}(z) \approx \int_E \frac{z-y}{|z-y|^{N+1}} d\mu_E(y) \qquad z \in \mathbb{R}^{N+1}.$$

Now, E is compact so for every $\varepsilon > 0$ there exists $|z| \gg \operatorname{diam}(E) = R$ big enough such that, for $y \in E$:

- $|z| \le |z-y| + |y|$ so $|z-y| \ge |z| - |y| \ge |z| - R \ge (1-\varepsilon)|z|$;
- $|z-y| \le |z| + |y| \le (1+\varepsilon)|z|$.

Hence, for all such $z \in \mathbb{R}^{N+1}$, $|z-y| \approx |z|$ and we get the desired estimates, among which:

$$|\nabla v(z)| \le \mathcal{I}_1^{-1}(E) \int_E \frac{d\mu_E(y)}{|z-y|^N} \approx_{+\infty} \frac{\mathcal{I}_1^{-1}(E)}{|z|^N}. \tag{2.25}$$

By the expression we derived for ∇v, we immediately deduce as well:

$$(\partial_{N+1} v)(z) = -\mathcal{I}_1^{-1}(E) \int_E \frac{(z_{N+1} - y_{N+1})}{|z-y|^{N+1}} d\mu_E(y) = -\mathcal{I}_1^{-1}(E) \int_E \frac{(z_{N+1})}{|z-y|^N} d\mu_E(y),$$

since $y \in E \subset \mathbb{R}^N \times \{0\}$. In particular, we have $\partial_{N+1}(z) = 0$ for all $z \in E^c \times \{0\}$. If instead $z \in E \times \{0\}$, we do not know whether v is differentiable in $\mathbb{R}^{N+1}$ or not, but, since $v = 0$ on $E \times \{0\}$, we can at least conclude:

$$v(z)(\partial_{N+1}v)(z) = 0 \qquad \text{for all } z \in \mathbb{R}^N \times \{0\}. \tag{2.26}$$

Moreover, as it happens for u, we have that $v \in [0,1]$. To understand it, it is enough to prove that $u_E : \mathbb{R}^{N+1} \longmapsto \mathbb{R}$ satisfies $\tilde{u_E} \in [0, \mathcal{I}_1(E)]$. By the variational formulation of the Dirichlet problem:

$$\begin{cases} -\Delta \tilde{u_E} = 0 & \text{on } \mathbb{R}^{N+1}_+ \\ \tilde{u_E} = u_E & \text{on } \partial\, \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\}, \end{cases}$$

$\tilde{u_E}$ satisfies it if and only if , for all $K \subset \mathbb{R}^{N+1}_+$ compact, it attains the minimum in the variational problem:

$$\min \left\{ \int_K |\nabla w|^2 : w \in H^1_{loc}(\mathbb{R}^{N+1}_+),\, w = u_E \text{ on } \mathbb{R}^N \times \{0\},\, w(z) \to 0 \text{ as } |z| \to +\infty \right\}.$$

Since $u_E \in [0, \mathcal{I}_1(E)]$ almost everywhere, if $\tilde{u_E} > \mathcal{I}_1(E)$ over some set A, we would be able to decrease the Dirichlet energy by just setting $\tilde{u_E} = \mathcal{I}_1(E)$ over such set. Therefore, since $\tilde{u_E}, v \in C^\infty(\mathbb{R}^{N+1}_+)$, then $\tilde{u_E} \in [0, \mathcal{I}_1(E)]$ everywhere on $\mathbb{R}^{N+1}_+$, so by construction $v \in [0,1]$ everywhere on $\mathbb{R}^{N+1}_+$.

Now, before introducing one last preliminary result concerning the function v, we set some useful notation. For every $x \in \mathbb{R}^{N+1}_+$ and every $r > 0$, we let $B^+_r(x) = B_r(x) \cap \mathbb{R}^{N+1}_+$ and $\partial^+ B_r(x) = \partial B_r(x) \cap \mathbb{R}^{N+1}_+$, so $\partial B_r(x)^+ = \partial^+ B_r(x) \cup (B_r(x) \cap (\mathbb{R}^N \times \{0\}))$. Finally, we will use also the set $\partial B_r(x) \cap (\mathbb{R}^N \times \{0\})$. If x is omitted, all the sets just defined are centered in $0 \in \mathbb{R}^{N+1}$. Finally, we define a regularization of the Green function Γ.

Definition 2.19. We define $\Gamma_1 \in C^1(\mathbb{R}^{N+1}_+, \mathbb{R}^+)$ by:

$$\Gamma_1 = \begin{cases} \frac{1}{|z|^{N-1}} & |z| \geq 1 \\ \frac{N+1}{2} - \frac{N-1}{2}|z|^2 & |z| < 1. \end{cases}$$

We also let $\Gamma_\varepsilon(z) = \Gamma_1(z/\varepsilon)\varepsilon^{1-N}$, so that $\Gamma_\varepsilon \nearrow \Gamma = |z|^{1-N}$ as $\varepsilon \to 0$.

We highlight that Γ_ε is radial, and, since for Γ we have:

$$\partial_{N+1}\, |z|^{-(N-1)} = (1-N) z_{N+1} |z|^{-(N+1)} \qquad \Longrightarrow \qquad \partial_{N+1}\Gamma(z) = 0 \text{ on } \mathbb{R}^N \times \{0\},$$

we get as well $\partial_{N+1}\Gamma_\varepsilon = 0$ on $\mathbb{R}^N \times \{0\}$. Moreover, another computation shows that Γ_ε is superharmonic in $\mathbb{R}^{N+1}$, namely $-\Delta\Gamma_\varepsilon \geq 0$. Now, we closely follow [28].

Lemma 2.20. Given the function v defined above, for all $\gamma \in]0,1[$ the function $\Phi : (0,1) \longmapsto (0,+\infty)$ given by

$$\Phi(r) = \int_{\partial^+ B_r} \frac{|\nabla v|^2}{|z|^{N-1}}$$

is well defined and bounded in $(0,1)$.

Proof. We denote $z = (x,y) \in \mathbb{R}^N \times \mathbb{R}$. Given any non negative $\varphi \in C^\infty_c(\mathbb{R}^{N+1})$ we have, integrating by parts on the domain $\mathbb{R}^{N+1}_+$:

$$\int_{\mathbb{R}^{N+1}_+} (-\Delta v)\, v\varphi \, dz + \int_{\mathbb{R}^N} (\partial_{N+1} v) v\varphi \, dx = \int_{\mathbb{R}^{N+1}_+} \nabla v \cdot \nabla(v\varphi)\, dz. \tag{2.27}$$

By harmonicity of v and 2.26, every term of the above expression is equal to 0. We notice that, developing the gradient, we have:

$$\int_{\mathbb{R}^{N+1}_+} \nabla v \cdot \nabla(v\varphi)\, dz = \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \varphi\, dz + \int_{\mathbb{R}^{N+1}_+} \frac{1}{2}\nabla(v)^2 \cdot \nabla\varphi\, dz = 0 \tag{2.28}$$

as well. Now, we let $\varepsilon, \delta > 0$ and $\eta_\delta \in C_c^\infty(B_{r+\delta})$ be a smooth, radial cut-off function such that $0 \le \eta_\delta \le 1$ and $\eta_\delta = 1$ on B_r. Choosing $\varphi = \eta_\delta \Gamma_\varepsilon$ in 2.28 we get:

$$\int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \eta_\delta \Gamma_\varepsilon\, dz + \int_{\mathbb{R}^{N+1}_+} \frac{1}{2}\nabla(v)^2 \cdot \nabla(\eta_\delta\Gamma_\varepsilon)\, dz = 0.$$

If we denote by ν the outward unit normal vector to the half sphere $\partial^+ B_\rho$, we find by developing, rearranging and passing to polar coordinates:

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} \left[|\nabla v|^2\Gamma_\varepsilon + \frac{1}{2}\nabla(v)^2\cdot\nabla\Gamma_\varepsilon\right]\eta_\delta\, dz &= -\int_{\mathbb{R}^{N+1}_+} \frac{1}{2}\Gamma_\varepsilon \nabla(v)^2 \cdot \nabla\eta_\delta\, dz \\ &= -\int_r^{r-\delta} \left[-\eta_\delta'(\rho)\int_{\partial^+ B_\rho} \Gamma_\varepsilon\, v\partial_\nu v\, d\sigma\right] d\rho.\end{aligned}$$

Sending the limit as $\delta \to 0$ we get, for almost every $r \in]0,1[$:

$$\int_{B_r^+} |\nabla v|^2\Gamma_\varepsilon + \frac{1}{2}\nabla(v)^2 \cdot \nabla\Gamma_\varepsilon\, dz = \int_{\partial^+ B_r} \Gamma_\varepsilon\, v\partial_\nu v\, d\sigma.$$

Integrating by parts over B_r^+ we get:

$$\int_{B_r^+} |\nabla v|^2\Gamma_\varepsilon + (-\Delta\Gamma_\varepsilon)\frac{v^2}{2}\, dz + \int_{\partial^+ B_r} \frac{v^2}{2}\partial_\nu\Gamma_\varepsilon\, d\sigma = \int_{\partial^+ B_r} \Gamma_\varepsilon\, v\partial_\nu v\, d\sigma,$$

recalling that $\partial_{N+1}\Gamma_\varepsilon(x,0) = 0$, so it is enough to integrate the third term of the left-hand side over $\partial^+ B_r$ instead of the whole ∂B_r^+. Now, we use the fact that $-\Delta\Gamma_\varepsilon \ge 0$ to get:

$$\int_{B_r^+} |\nabla v|^2\Gamma_\varepsilon\, dz \le \int_{\partial^+ B_r} \Gamma_\varepsilon\, v\partial_\nu v - \frac{v^2}{2}\partial_\nu\Gamma_\varepsilon\, d\sigma.$$

Noticing that $\Gamma(z) = r^{-(N-1)}$ over $\partial^+ B_r$, we let $\varepsilon \to 0$ and infer, by monotone convergence:

$$\Phi(r) = \int_{\partial^+ B_r} \frac{|\nabla v|^2}{|z|^{N-1}} \le \frac{1}{r^{N-1}}\int_{\partial^+ B_r} v\,\partial_\nu v\, d\sigma + \frac{N-1}{2r^N}\int_{\partial^+ B_r} v^2\, d\sigma. \tag{2.29}$$

Since $v^2 \le 1$ and 2.25 hold, the previous expression concludes the Lemma. □

We are finally ready to prove the Hölder estimate we are interested in. The motivation behind the statement is to find a growth estimate for the function u locally near the Reifenberg flat (so in particular not smooth domain) set E, which later on will allow us to majorize the integral term in 2.24. The idea of the proof is to add a dimension to the problem and, exploiting the fact that the function u is given by a convolution between the measure μ_E and the Green kernel Γ in $\mathbb{R}^{N+1}$, use some tools from elliptic PDE theory (culminating with an Alt-Caffarelli-Friedman type formula) in order to prove the desired estimate over $\mathbb{R}^{N+1}$. At that point, the conclusion follows straightforwardly by restricting ourselves back to $\mathbb{R}^N$.

Lemma 2.21. For every small enough $\delta > 0$ there exists $\gamma \in (0,1)$ with $\gamma \to \frac{1}{2}$ as $\delta \to 0$ such that, if E is a bounded (δ, r_0)-Reifenberg flat domain, then:

$$|1 - \mathcal{I}_1^{-1}(E)u_E(x)| \leq \frac{d^\gamma(x, \partial E)}{r_0^\gamma} \qquad \text{for all } x \in \mathbb{R}^N, \tag{2.30}$$

where $u_E(x) = \int_E \frac{d\mu_E(y)}{|x-y|^{N-1}}$ and μ_E is such that $\mathcal{I}_1(E) = I_1(\mu_E)$.

Proof. By scaling we may assume $r_0 = 1$. Let $u = 1 - \mathcal{I}_1^{-1}(E)u_E(x)$ and v its harmonic extension to $\mathbb{R}^{N+1}$ the functions we defined above: we keep in mind all of their property we listed. In particular, since $u \leq 1$, it is enough to prove 2.30 when $d(\cdot, \partial E) \ll 1$. The proof consists in three claims, from which we can draw the conclusion:

- (1) For every $z \in \overline{\mathbb{R}^{N+1}_+}$, have:

$$\frac{1}{r^{N-1}} \int_{B_r^+(z)} |\nabla v|^2 \lesssim r^{2\gamma} \sup_{z \in \overline{\mathbb{R}^{N+1}_+}} \int_{B_1^+(z)} \frac{|\nabla v|^2}{|z-y|^{N-1}} \qquad \text{for all } 0 < r \ll 1 \tag{2.31}$$

  for some exponent $0 \leq \gamma \leq 1$.

- (2) We have:

$$\sup_{z \in \overline{\mathbb{R}^{N+1}_+}} \int_{B_1^+(z)} \frac{|\nabla v|^2}{|z-y|^{N-1}} \lesssim 1. \tag{2.32}$$

- (3) $\gamma \in (0,1)$ and $\gamma \to \frac{1}{2}$ as $\delta \to 0$.

We start by the conclusion of the proof once assumed the three claims. First of all, by a step in the proof of Poincaré Inequality [6, Theorem 2, Section 4.5], for all $z \in \overline{\mathbb{R}^{N+1}_+}$ we have:

$$\int_{B_r^+(z)} |v - \fint_{B_r^+(z)} v|^2 \lesssim r^2 \int_{B_r^+(z)} |\nabla v|^2.$$

Thus, if 2.31 and 2.32 hold, we deduce for r small:

$$\frac{1}{r^{N+1}} \int_{B_r^+(z)} |v - \fint_{B_r^+(z)} v|^2 \lesssim \frac{1}{r^{N-1}} \int_{B_r^+(z)} |\nabla v|^2 \lesssim r^{2\gamma} \sup_{z \in \overline{\mathbb{R}^{N+1}_+}} \int_{B_1^+(z)} \frac{|\nabla v|^2}{|z-y|^{N-1}} \lesssim r^{2\gamma},$$

for a number γ close to 1/2. In particular:

$$\left(\frac{1}{r^{N+1}} \int_{B_r^+(z)} |v - \fint_{B_r^+(z)} v|^2 \right)^{\frac{1}{2}} \lesssim r^\gamma.$$

Hence, by Campanato's criterion [18, Theorem 6.1], $v \in C^{0,\gamma}(\overline{\mathbb{R}^{N+1}_+})$, namely there exists $C > 0$ such that for all $z, z' \in \overline{\mathbb{R}^{N+1}_+}$ there holds:

$$|v(z) - v(z')| \leq C|z - z'|^\gamma.$$

In particular, if $z' = (x', 0) \in E \times \{0\}$ and $z = (x, 0) \in E^c \times \{0\}$, then $v(z') = u(z') = 0$, so:

$$|u(x)| \lesssim |x - x'|^\gamma \quad \text{for all } x' \in E \quad \iff \quad |1 - \mathcal{I}_1^{-1}(E)u_E(x)| \leq d^\gamma(x, \partial E).$$

Since this is true for all $x \in E^c$, the thesis of the Lemma follows and it remains to prove the there claims.

(1) To show 2.31, we assume first $z \in \partial E \times \{0\}$. Without loss of generality we may assume that $z = 0$ and from now we denote by ∇_τ the tangential gradient on the sphere, by ∇_ν the normal derivative and by Δ_S the restriction of the Laplacian over the surface of the sphere. For every $0 < r \leq 1$, we consider the variational problem:

$$\lambda(r) = \min \left\{ \frac{\int_{\partial^+ B_r} |\nabla_\tau v|^2}{\int_{\partial^+ B_r} v^2} : v \in H^1(\partial^+ B_r),\ v = 0 \ \text{ on } (E \times \{0\}) \cap \partial B_r^+ \right\}. \tag{2.33}$$

First of all, we notice that the function v we are working with is a competitor for the problem. By direct method of Calculus of Variations, we can easily prove that there exist a minimizer v with $\|v\|_{L^2(\partial^+ B_r)} = 1$. Moreover, $\lambda(r) > 0$, because if $\lambda(r) = 0$ then the minimizer would be constant on $\partial^+ B_r$, hence equal to 0 by continuity of the trace operator, contradicting the condition $\|v\|_{L^2(\partial^+ B_r)} = 1$. In a similar way, choosing the function $w(x, y) = y$ with $(x, y) \in \mathbb{R}^N \times \mathbb{R}$, after a computation we can observe that $\lambda(r) \leq N$. In addition, thanks to a change of variable, by considering $r^2\lambda(r)$ instead of $\lambda(r)$, we can assume the problem to be set on $\partial^+ B_1$, with the condition $v = 0$ on $(E_{1/r} \times \{0\}) \cap \partial B_1^+$. Finally, from basic facts of Spectral Theory, we can equivalently characterize minimizers $v \in H^1(\partial^+ B_r)$ as weak solutions of the PDE problem with Robin boundary condition:

$$\begin{cases} -\Delta_S\, v = \lambda(r) v & \text{on } \partial^+ B_r \\ v = 0 & \text{on } E \times \{0\} \\ \partial_\nu v = 0 & \text{on } E^c \times \{0\}. \end{cases} \tag{2.34}$$

Now, we can define the function $\gamma : [0, +\infty) \longmapsto [0, +\infty)$ as

$$\gamma(\lambda) = \sqrt{\left(\frac{N-1}{2}\right)^2 + \lambda} - \frac{N-1}{2}$$

and then the number $\bar{\gamma} = \bar{\gamma}_E$ as:

$$\bar{\gamma} = \inf_{0<r\leq 1} \gamma(r^2\lambda(r)).$$

We will prove later this is the number γ which appears in the statement of the Lemma. We preliminary observe that obviously $\bar{\gamma} \geq 0$ and, by monotonicity of the function γ:

$$\bar{\gamma} \leq \gamma(r^2\lambda(r)) \leq \gamma(\lambda(1)) \leq \gamma(N) = 1.$$

Now, for $r \in (0, 1]$, using the function v we started the proof with, we define the function:

$$\Psi(r) = \frac{1}{r^{2\bar{\gamma}}} \int_{B_r^+} \frac{|\nabla v|^2}{|z|^{N-1}}.$$

By Lemma 2.20, Ψ is well defined: we want to prove that it is increasing. To do so, we compute it logarithmic derivative, with the help of Coarea Formula:

$$\frac{\Psi'}{\Psi} = -2\frac{\bar{\gamma}}{r} + \left(\int_{\partial^+ B_r} \frac{|\nabla v|^2}{|z|^{N-1}}\right)\left(\int_{B_r^+} \frac{|\nabla v|^2}{|z|^{N-1}}\right)^{-1}.$$

Since $\Psi \geq 0$, if we show that $\Psi'/\Psi \geq 0$ then $\Psi' \geq 0$, so Ψ is increasing and we are done; therefore it is enough to prove that:

$$\left(\int_{\partial^+ B_r} \frac{|\nabla v|^2}{|z|^{N-1}}\right)\left(\int_{B_r^+} \frac{|\nabla v|^2}{|z|^{N-1}}\right)^{-1} \geq 2\frac{\bar{\gamma}}{r}. \tag{2.35}$$

We have:

$$f(r) := \left(\int_{\partial^+ B_r} \frac{|\nabla v|^2}{|z|^{N-1}}\right)\left(\int_{B_r^+} \frac{|\nabla v|^2}{|z|^{N-1}}\right)^{-1} = \left(r^{1-N}\int_{\partial^+ B_r} |\nabla v|^2\right)\left(\int_{B_r^+} |\nabla v|^2\Gamma(z)\right)^{-1},$$

and, by formula 2.29:

$$\begin{aligned}
f(r) &\geq \left(r^{1-N}\int_{\partial^+ B_r} |\nabla v|^2\right)\left(r^{1-N}\left(\int_{\partial^+ B_r} v\partial_\nu v + \frac{N-1}{2r}\int_{\partial^+ B_r} v^2\right)\right)^{-1}\\
&\geq \left(\int_{\partial^+ B_r} |\nabla_\tau v|^2 + \int_{\partial^+ B_r} |\partial_\nu v|^2\right)\left(\left(\int_{\partial^+ B_r} v^2\right)^{\frac{1}{2}}\left(\int_{\partial^+ B_r} (\partial_\nu v)^2\right)^{\frac{1}{2}} + \frac{N-1}{2r}\int_{\partial^+ B_r} v^2\right)^{-1}\\
&= \left(\frac{\int_{\partial^+ B_r} |\nabla_\tau v|^2}{\int_{\partial^+ B_r} v^2} + \frac{\int_{\partial^+ B_r} |\partial_\nu v|^2}{\int_{\partial^+ B_r} v^2}\right)\left(\left(\frac{\int_{\partial^+ B_r} |\partial_\nu v|^2}{\int_{\partial^+ B_r} v^2}\right)^{\frac{1}{2}} + \frac{N-1}{2r}\right)^{-1}\\
&\geq \min_{t>0} \frac{\lambda(r)+t^2}{t+\frac{N-1}{2r}}.
\end{aligned}$$

Another direct computation shows that the above minimum is attained for $t_{\min} = \frac{1}{r}\gamma(r^2\lambda(r))$ and that $\min_{t>0} \frac{\lambda(r)+t^2}{t+\frac{N-1}{2r}} = 2t_{\min} = \frac{2}{r}\gamma(r^2\lambda(r))$, so that eventually:

$$\left(\int_{\partial^+ B_r} \frac{|\nabla v|^2}{|z|^{N-1}}\right)\left(\int_{B_r^+} \frac{|\nabla v|^2}{|z|^{N-1}}\right)^{-1} \geq \frac{2}{r}\gamma(r^2\lambda(r)) \geq 2\frac{\bar{\gamma}}{r},$$

concluding the proof of 2.35. By monotonicity of Ψ and recalling that $z \in B_r^+$ implies $|z| \leq r$, we deduce:

$$\frac{1}{r^{2\bar{\gamma}+N-1}}\int_{B_r^+} |\nabla v|^2 \leq \frac{1}{r^{2\bar{\gamma}}}\int_{B_r^+} \frac{|\nabla v|^2}{|z|^{N-1}} = \Psi(r) \leq \Psi(1) = \int_{B_1^+} \frac{|\nabla v|^2}{|z|^{N-1}} \tag{2.36}$$

and the proof of (1) with the exponent $\bar{\gamma}$ in the case $z \in \partial E \times \{0\}$ is concluded. From now on, we denote $\gamma = \bar{\gamma}$: we will able to distinguish the use of γ to denote either the parameter or the function defined before from the context.

On the other hand, if $z \notin \partial E \times \{0\}$, using either an odd or even reflection with respect to the hyperplane $z_{N+1} = 0$, we may assume that v is harmonic in $B_z(r)$ for every $r \leq \bar{r} = \min\{1, d(z, \partial E \times \{0\})\}$. Indeed, depending on the position of z with respect $E \times \{0\}$, it is possible to perform both: if $d(z, E \times \{0\}) < d(z, E^c \times \{0\})$ then we can use an odd reflection exploiting the fact that $v = 0$ on E, whereas, in the opposite case, we can use an odd one as $\partial_{N+1} v = 0$ on E^c. Moreover, being $|\nabla v|^2$ subharmonic, the function:

$$r \longmapsto \frac{1}{r^{N+1}}\int_{B_r(z)} |\nabla v|^2$$

is increasing by mean value formula. Therefore, since the parameter γ we selected before satisfies $\gamma \leq 1$, we have:

$$\frac{1}{r^{N+1}}\int_{B_r^+(z)} |\nabla v|^2 \leq \left(\frac{r}{\bar{r}}\right)\frac{1}{\bar{r}^{N-1}}\int_{B_{\bar{r}}^+(z)} |\nabla v|^2 \leq \left(\frac{r}{\bar{r}}\right)^{2\gamma}\frac{1}{\bar{r}^{N-1}}\int_{B_{\bar{r}}^+(z)} |\nabla v|^2. \tag{2.37}$$

If $\bar{r} \gtrsim 1$, then the conclusion is easy, since first of all $(1/\bar{r})^{2\gamma} \lesssim 1$ and then, for all $y \in B_{\bar{r}}^+(z)$ we have $|z-y| \leq \bar{r}$, so $1/\bar{r} \leq |x-y|^{-1}$. In this way:

$$\left(\frac{r}{\bar{r}}\right)^{2\gamma}\frac{1}{\bar{r}^{N-1}}\int_{B_{\bar{r}}^+(z)} |\nabla v|^2 \lesssim r^{2\gamma}\int_{B_{\bar{r}}^+(z)} \frac{|\nabla v|^2}{|z-y|^{N-1}} \leq r^{2\gamma}\int_{B_1^+(z)} \frac{|\nabla v|^2}{|z-y|^{N-1}},$$

where after the last inequality we used $B^+_{\bar r}(z) \subset B^+_1(z)$, as $\bar r \le 1$ by definition. Instead, if $\bar r \ll 1$ we argue as follows: we select $\bar z \in \partial E \times \{0\}$ of minimal distance, namely the point such that $|z - \bar z| = d(z, \partial E \times \{0\}) = \bar r \ll 1$. By 2.36, we have for $\bar z \in \partial E \times \{0\}$ and for all $r \in (0, 1/2)$:

$$\frac{1}{r^{2\gamma+N-1}} \int_{B^+_r(\bar z)} |\nabla v|^2 \le \Psi(r) \le \Psi(1/2) \lesssim \int_{B^+_{1/2}(\bar z)} \frac{|\nabla v|^2}{|\bar z - y|^{N-1}}. \tag{2.38}$$

In particular, choosing $r^* = 3\bar r \ll 1/2$, we have that $B^+_{\bar{(r)}}(z) \subset\subset B^+_{r^*}(\bar z)$, so coming back to our computations 2.37, we have, $\bar z \in \partial E \times \{0\}$:

$$\left(\frac{r}{\bar r}\right)^{2\gamma} \frac{1}{\bar r^{N-1}} \int_{B^+_{\bar r}(z)} |\nabla v|^2 \le \left(\frac{r}{\bar r}\right)^{2\gamma} \frac{1}{\bar r^{N-1}} \int_{B^+_{r^*}(\bar z)} |\nabla v|^2 \approx r^{2\gamma} \frac{1}{r^{*\,2\gamma+N-1}} \int_{B^+_{r^*}(\bar z)} |\nabla v|^2.$$

Applying 2.38 with $r^* \ll 1/2$ and majorizing the integration area with $B^+_{1/2}(\bar z) \subset B^+_1(z)$ (since we have that $|z - \bar z| \ll 1$), we finally get:

$$r^{2\gamma} \frac{1}{r^{*\,2\gamma+N-1}} \int_{B^+_{r^*}(\bar z)} |\nabla v|^2 \lesssim r^{2\gamma} \int_{B^+_{1/2}(\bar z)} \frac{|\nabla v|^2}{|\bar z - y|^{N-1}} \le r^{2\gamma} \int_{B^+_1(z)} \frac{|\nabla v|^2}{|\bar z - y|^{N-1}}.$$

So recalling where we started in 2.37 and passing the last result to the supremum over $z \in \overline{\mathbb{R}^{N+1}_+}$ we get:

$$\frac{1}{r^{N+1}} \int_{B^+_r(z)} |\nabla v|^2 \lesssim r^{2\gamma} \sup_{z \in \overline{\mathbb{R}^{N+1}_+}} \int_{B^+_1(z)} \frac{|\nabla v|^2}{|\bar z - y|^{N-1}}.$$

(2) We fix $z \in \overline{\mathbb{R}^{N+1}_+}$. By 2.29, for $R \gg 1$ we have:

$$\int_{B^+_1(z)} \frac{|\nabla v|^2}{|z - y|^{N-1}} \le \int_{B^+_R(z)} \frac{|\nabla v|^2}{|z - y|^{N-1}} \lesssim \frac{1}{R^{N-1}} \int_{\partial^+ B_R(z)} v\, \partial_\nu v + \frac{1}{R^N} \int_{\partial^+ B_R(z)} v^2.$$

Using Cauchy-Schwarz and that $|v| \le 1$ we find:

$$\int_{B^+_1(z)} \frac{|\nabla v|^2}{|z - y|^{N-1}} \lesssim \frac{1}{R^{N-1}} \int_{\partial^+ B_R(z)} |\partial_\nu v| + \frac{1}{R^N} \int_{\partial^+ B_R(z)} d\sigma \lesssim \frac{1}{R^{N-1}} \int_{\partial^+ B_R(z)} |\partial_\nu v| + 1.$$

For the second inequality, we used the fact that $\mathcal{H}^N(\partial^+ B_R(z)) \approx R^N$. We need to estimate the first term, if R is large enough we can apply 2.25 and get:

$$\frac{1}{R^{N-1}} \int_{\partial^+ B_R(z)} |\partial_\nu v| \lesssim \frac{1}{R^{N-1}} \int_{\partial^+ B_R(z)} \frac{\mathcal{I}_1^{-1}(E)}{|z|^N} \lesssim \frac{\mathcal{I}_1^{-1}(E)}{R^{N-1}},$$

again because $\mathcal{H}^N(\partial^+ B_R(z)) \approx R^N$. Sending $R \to +\infty$ the proof of (2) is concluded, once observing that the computations hold true for every $z \in \overline{\mathbb{R}^{N+1}_+}$.

(3) To prove this point, we finally use the fact that E is an uniformly (δ, r_0)-Reifenberg flat set, recalling that we assumed $\delta_0 = 1$ by scaling at the beginning of the proof. In particular, for all $x \in \partial E$ and for all $0 < r \le 1$ there exist two hyperplanes H^+_r and H^-_r such that $H^-_r \cap B_r(x) \subset E \cap B_r(x) \subset H^+_r \cap B_r(x)$. The idea to prove the claim is to exploit this property to infer the desired asymptotic estimate on the parameter γ, specifically working on the variational problem 2.33. Since the condition given by Reifenberg flatness is uniform in $x \in \partial E$, we can assume without loss of generality that $x = 0$.

Given a generic function $v \in H^1(\partial^+ B_r)$ we have that, if $F \subset E$, $v = 0$ on $(E \times \{0\}) \cap \partial B_r^+$ implies $v = 0$ on $(F \times \{0\}) \cap \partial B_r^+$, so in particular $\lambda_F(r) \leq \lambda_E(r)$ for every $r \leq 1$, namely $\lambda(r)$ is monotone under inclusion. Thus, we define $H^+_{\delta r} = \{x \in \mathbb{R}^N : x_1 \leq \delta r\}$ and $H^-_{\delta r} = \{x \in \mathbb{R}^N : x_1 \leq -\delta r\}$; combining monotonicity of λ with the hyperplane condition (in the right coordinate system according to the choice of $H^+_{\delta r}$ and $H^-_{\delta r}$) yielded by $(\delta, 1)$-Reifenberg flatness, we get $\lambda_{H^-_{\delta r}}(r) \leq \lambda_E(r) \leq \lambda_{H^+_{\delta r}}(r)$. Since the function $\gamma(\lambda)$ is increasing monotone:

$$\gamma(r^2 \lambda_{H^-_{\delta r}}(r)) \leq \gamma(r^2 \lambda_E(r)) \leq \gamma(r^2 \lambda_{H^+_{\delta r}}(r)),$$

so passing to the infimum over $0 < r \leq 1$ we infer $\gamma_{H^-_{\delta r}} \leq \gamma_E \leq \gamma_{H^+_{\delta r}}$. Therefore, if we show that $\gamma_{H^-_{\delta r}} \to \frac{1}{2}$ as $\delta \to 0$, we can immediately deduce that $\gamma_E \to \frac{1}{2}$ as $\delta \to 0$ too, $\gamma_E \in \left(\frac{1}{4}, \frac{3}{4}\right)$ for δ small enough. Before going on, we notice that both $\gamma_{H^-_{\delta r}}$ and $\gamma_{H^+_{\delta r}}$ actually do not depend on r: indeed, by changing variables in 2.33:

$$r^2 \lambda_{H^\pm_{\delta r}}(r) = \min \left\{ \frac{\int_{\partial^+ B_1} |\nabla_\tau v|^2}{\int_{\partial^+ B_1} v^2} : v \in H^1(\partial^+ B_1),\ v = 0 \text{ on } (H^\pm_{\delta r / r} \times \{0\}) \cap \partial B_r^+ \right\}.$$

Since $H^\pm_{\delta r/r} = H^\pm_\delta$, the quantities $r^2 \lambda_{H^\pm_{\delta r}}(r)$ do not depend on r, so from now on we just write $\gamma_{H^\pm_\delta}$ and $\gamma_{H^\pm_\delta}$ instead.

We begin by showing that, if $\delta = 0$, then $\gamma_{H_0} = \frac{1}{2}$. Since γ_{H_0} does not depend on r we can consider $\lambda = \lambda_{H_0}(1)$ and work on $\partial^+ B_1$. The trick is to use another characterization of the solution of 2.33 given by [28, Remark 2.3]: a function v competitor for 2.33 (namely such that $v \in H^1(\partial^+ B_1)$ and $v = 0$ on H_0) achieves the minimum λ if and only if it is of one sign and its $\gamma(\lambda)$-homogeneous extension to $\mathbb{R}^{N+1}_+$ is harmonic. We give an almost complete proof of this statement: the key is to consider the Laplacian in $\mathbb{R}^{N+1}$ in polar coordinates

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{N}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_S u.$$

If v is a solution of 2.33 then it solves the elliptic problem 2.34 as well. Its $\gamma(\lambda)$-homogeneous extension to $\mathbb{R}^{N+1}_+$ is given in polar coordinates by $w(r, \sigma) = r^{\gamma(\lambda)} v(\sigma)$ and computing its Laplacian yields

$$\Delta u = r^{\gamma(\lambda) - 2}\left[(\gamma(\lambda)^2 + (N-1)\gamma(\lambda))v + \Delta_S v\right].$$

Therefore, $\Delta u = 0$ if and only if:

$$-\Delta_S v = \left[\gamma(\lambda)^2 + (N-1)\gamma(\lambda)\right] v = \lambda\, v,$$

because $\lambda(\gamma) = \gamma^2 + (N-1)\gamma$ happens to be the inverse function of $\gamma(\lambda)$ defined above. Since we had $-\Delta_S v = \lambda\, v$ by hypothesis, we conclude the first implication. Conversely, if a competitor for 2.33 has its $\gamma(\lambda)$-homogeneous extension to $\mathbb{R}^{N+1}_+$ harmonic, then it is actually its minimizer if it is a solution of the elliptic problem 2.34. The computations we just did allow us to conclude almost everything: indeed it just remains to show that $\partial_\nu v = 0$ on $H_0^c \times \{0\}$, but we omit it.

Therefore, if we can find a positive function v with $v = 0$ on H_0 and such that its 1/2-homogeneous extension to $\mathbb{R}^{N+1}_+$ is harmonic, we immediately deduce $\gamma(\lambda) = \gamma_{H_0} = \frac{1}{2}$. We consider the function $v(x, 0) = \sqrt{x_1^+}$ defined on $\mathbb{R}^N$ (notice that $v = 0$ on H_0) and we take its 1/2-homogeneous harmonic extension to $\mathbb{R}^{N+1}_+$:

$$v(x, x_{N+1}) = \sqrt{\frac{\sqrt{x_1^2 + x_{N+1}^2} + x_1}{2}}.$$

Another computation allows us to check that $\partial_\nu v = 0$ on $H_0^c \times \{0\}$. Since $v > 0$ on $\mathbb{R}^{N+1}_+$, the previous argument implies that $\gamma(\lambda) = \frac{1}{2}$.

The proof of (3) is concluded once proved that the function $\delta \longrightarrow \lambda_{H_\delta^-}$ is continuous as $\delta \to 0$. Thus, by continuity of the function γ, we have that $\gamma_{H_\delta^-} = \gamma(\lambda_{H_\delta^-}) \longrightarrow \gamma(\lambda_{H_0}) = \gamma_{H_0} = \frac{1}{2}$. We consider $\{u_\delta\}_{\delta>0}$ a sequence such that u_δ is a minimizer for $\lambda_{H_\delta^-}$ with $\|u_\delta\|_{L^2(\partial^+ B_1)} = 1$. In (1) we highlighted the fact that $\lambda \leq N$: more in general this is true for all set E, since the function $w(x,y) = y$ is a valid competitor for λ_E for all E. In other words:

$$\int_{\partial^+ B_1} |\nabla_\tau u_\delta|^2 \leq \int_{\partial^+ B_1} |\nabla_\tau w|^2 \leq N \qquad \text{for all } \delta > 0.$$

In particular, we deduced that u_δ is bounded in $H^1(\partial^+ B_1)$ uniformly in δ. By compactness of the trace operator

$$\mathbf{tr} : H^1(\partial^+ B_1) \longrightarrow H^{\frac{1}{2}}(\partial B_1 \cap \{x_{N+1} = 0\})$$

and of the embedding

$$\mathbf{i} : H^{\frac{1}{2}}(\partial B_1 \cap \{x_{N+1} = 0\}) \longrightarrow L^2(\partial B_1 \cap \{x_{N+1} = 0\}),$$

we have that $\{\mathbf{tr} u_\delta\}_{\delta>0}$ is bounded in $L^2(\partial B_1 \cap \{x_{N+1} = 0\})$ uniformly in δ as well. Therefore, by Banach-Alaoglu, up to extraction, there exist a function $u_0 \in H^1(\partial^+ B_1)$ and a subsequence $\{u_{\delta_k}\}_{k\geq 1}$ with $\delta_k \to 0$ as $k \to +\infty$ such that:

- $u_{\delta_k} \rightharpoonup u_0$ in $H^1(\partial^+ B_1)$ as $k \to +\infty$;
- $\mathbf{tr} u_{\delta_k} \longrightarrow \mathbf{tr} u_0$ in $L^2(\partial B_1 \cap \{x_{N+1} = 0\})$ and $\mathcal{H}^{N-1}$-a.e. as $k \to +\infty$.

By convergence almost everywhere of the trace, we infer that $\mathbf{tr} u_0 = 0$ on $\partial B_1 \cap (H_0 \times \{0\})$ so u_0 is a competitor for λ_{H_0}. Therefore, by weak lower semicontinuity of the norm and since $\lambda_{H_0} \geq \lambda_{H_\delta^-}$, we have that:

$$\lambda_{H_0} \geq \liminf_{\delta_k \to 0} \lambda_{H_\delta^-} = \liminf_{\delta_k \to 0} \int_{\partial^+ B_1} |\nabla_\tau u_\delta|^2 \geq \int_{\partial^+ B_1} |\nabla_\tau u_0|^2 \geq \lambda_{H_0},$$

so u_0 is a minimizer for λ_{H_0}, and we have that $\lambda_{H_\delta^-} \longrightarrow \lambda_{H_0}$ as well. □

Now, we are ready to turn the Hölder property 2.30 on the potential into the desired decay estimate for the function μ_E. Notice that the assumption $|\gamma - \frac{1}{2}| \ll 1$ comes directly from the statement of 2.21.

Lemma 2.22. For every $|\gamma - \frac{1}{2}| \ll 1$, there exists $\delta_0 > 0$ such that for every $r_0 > 0$ and every (δ, r_0)-Reifenberg flat domain E with $\delta \leq \delta_0$, $\mu_E \in L^{\frac{2N}{N+1}}_{loc}(\mathbb{R}^N)$ and for every $x \in \mathbb{R}^N$ and $r < r_0/2$ there holds:

$$\left(\int_{B_r(x)} \mu_E^{\frac{2N}{N+1}} \right)^{\frac{N+1}{N}} \lesssim r^{N-1+2\gamma}, \tag{2.39}$$

where the implicit constant depends on N, γ, r_0 and $|E|$.

Proof. Let $\gamma = \gamma(\delta)$ be given by Lemma 2.21. First, we derive from 2.30 the following estimate on μ_E:

$$\mu_E \lesssim d^{-(1-\gamma)}(\cdot, \partial E). \tag{2.40}$$

Denoting u_E the associated potential, by Proposition 1.12 we get for $x \in E$:

$$C'(N,1)\mu_E(x) = (-\Delta)^{\frac{1}{2}} u_E(x) = C(N,1/2) \int_{\mathbb{R}^N} \frac{u_E(x) - u_E(y)}{|x-y|^{N+1}} dy$$

By Proposition 1.12, we have $u_E(x) = \mathcal{I}_1(E)$ and $\mathcal{I}_1(E) - u_E(y) = 0$ for $y \in E$, so the right-hand side reduces to:

$$C'(N,1)\mu_E(x) = C(N,1/2)\int_{E^c} \frac{\mathcal{I}_1(E) - u_E(y)}{|x-y|^{N+1}}dy \le \frac{\mathcal{I}_1(E)}{r_0^\gamma}\int_{E^c} \frac{d^\gamma(y,\partial E)}{|x-y|^{N+1}}dy,$$

where we used 2.30 in the last inequality. At this point, recalling $d(y,A) = d(Y,\partial A)$ for every $A \subset \mathbb{R}^N$ closed, since our minimizer E is closed we have $d(y,\partial E) = \min_{x^* \in E}|y - x^*| \le |x-y|$. Thus:

$$\int_{E^c} \frac{d^\gamma(y,\partial E)}{|x-y|^{N+1}}dy \le \int_{E^c} \frac{|x-y|^\gamma}{|x-y|^{N+1}}dy \le \int_{B^c_{d(x,\partial E)}(x)} \frac{1}{|x-y|^{N+1-\gamma}}dy = \int_{B^c_{d(x,\partial E)}(0)} \frac{1}{|z|^{N+1-\gamma}}dz$$

and solving the integral we get $C(\gamma)d(x,\partial E)^{-(1-\gamma)}$. Hence:

$$\mu_E(x) \lesssim \frac{\mathcal{I}_1(E)}{r_0^\gamma}d(x,\partial E)^{-(1-\gamma)} \lesssim d(x,\partial E)^{-(1-\gamma)},$$

where in the last line we used that if B is a ball of measure $|E|$ then $\mathcal{I}_1(E) \le \mathcal{I}_1(B)$, as stated in Proposition 1.9. Notice that the constant appearing in the estimate depends on N, γ, r_0 and $|E|$, as all of them appear in the computations now or later.

After the preliminary argument, we now prove 2.39. For $P > 0$, we set $\mu_P = \min\{\mu_E, P\}$. Clearly μ_P is an integrable function (being bounded and supported in E) and $\mu_P \to \mu_E$ a.e. in E. Moreover, since $0 \le \mu_P \le \mu_E$, it satisfies 2.40 as well. We claim that there exist $C_0, C_1 > 0$ such that for every $x \in \partial E$ and every $r \le r_0/2$, there exists a set $A(x) \subset \partial E$ such that, denoting by # the cardinality of a set:

$$\#A(x) \le C_1\delta^{1-N} \tag{2.41}$$

and

$$\int_{B_r(x)} \mu_P^{\frac{2N}{N+1}} \le C_0 r^{N - \frac{2N}{N+1}(1-\gamma)} + \sum_{y \in A(x)} \int_{B_{6\delta r}(y)} \mu_P^{\frac{2N}{N+1}}. \tag{2.42}$$

Again, up to translations, we may assume without loss of generality $x = 0$. By definition, since E is (δ, r_0)-Reifenberg flat, for every $r \le r_0/2$, there exists a hyperplane H_r such that $d_H(\partial E \cap B_r, H_r \cap B_r) \le \delta r$. We set $N_r = \{y \in B_r : d(y,H_r) > 2\delta r\}$: then, for $y \in N_r$, we have $d(y,\partial E) \sim d(y,H_r)$. Indeed, as $y \in N_r$, we have $\delta r \le \frac{1}{2}d(y,H_r)$, and moreover, by construction, we infer $d(y,\partial E) \in d(y,H_r) + (-\delta r, +\delta r)$. Thus:

- $d(y,\partial E) \le d(y,H_r) + \delta r \le \frac{3}{2}d(y,H_r)$;
- $d(y,\partial E) \ge d(y,H_r) - \delta r \ge \frac{1}{2}d(y,H_r)$.

Therefore, we can compute by 2.40:

$$\int_{N_r} \mu_P^{\frac{2N}{N+1}} \lesssim \int_{N_r} d(x,\partial E)^{-\frac{2N}{N+1}(1-\gamma)}dx \simeq \int_{N_r} d(x,H_r)^{-\frac{2N}{N+1}(1-\gamma)}dx.$$

Now, we majorize the last integral by integrating over the set $C(r) \setminus C(2\delta r)$, where both $C(r)$ and $C(2\delta r)$ are cylinders whose axes pass through 0 (center of the ball) and are orthogonal to H_r, both with base radius r and heights $2r$ and $2\delta r$ respectively. Notice that $C(r) \setminus C(2\delta r)$ clearly contains N_r. Highlighting that the quantity $d(x,H_r)$ is constant along hyperplanes parallel to H_r, we can compute the integral by performing the change of variable $d(x,H_r) = t$. Hence, recalling that $\omega_{N-1}r^{N-1}$ is the volume of the $N-1$ ball of radius r:

$$\begin{aligned}\int_{N_r} \mu_P^{\frac{2N}{N+1}} &\le C\int_{C(r)\setminus C(2\delta r)} d(x,H_r)^{-\frac{2N}{N+1}(1-\gamma)}dx = C\omega_{N-1}r^{N-1}\int_{2\delta r}^{r} \frac{dt}{t^{\frac{2N}{N+1}(1-\gamma)}} \\ &\le C_0 r^{N-1}\left(r^{-\frac{2N}{N+1}(1-\gamma)+1} - (2\delta r)^{-\frac{2N}{N+1}(1-\gamma)+1}\right) \\ &= C_0 r^{N-\frac{2N}{N+1}(1-\gamma)}.\end{aligned}$$

In particular, it is important to highlight that we need the quantity $-\frac{2N}{N+1}(1-\gamma)+1$ to be positive in order to make true the previous computations. This is possible by the assumption $|\gamma - \frac{1}{2}| \ll 1$, choosing γ such that $1-\gamma < \frac{N+1}{2N}$. Once established the constant C_0, we infer:

$$\int_{B_r} \mu_P^{\frac{2N}{N+1}} = \int_{N_r} \mu_P^{\frac{2N}{N+1}} + \int_{B_r \cap N_r} \mu_P^{\frac{2N}{N+1}} \leq C_0 r^{N-\frac{2N}{N+1}(1-\gamma)} + \int_{B_r \cap N_r} \mu_P^{\frac{2N}{N+1}}$$

and we need to estimate the second term. For all $x \in N_r^c \cap B_r$ we have $d(x, H_r) \leq 2\delta r$: therefore condition $d_H(\partial E \cap B_r, H_r \cap B_r) \leq \delta r$ implies $d(x, \partial E \cap B_r) \leq 3\delta r$. Now, clearly $\{B_{\delta r}(y)\}_{y \in \partial E \cap B_r}$ is a covering of $\partial E \cap B_r$. By the compact version of Vitali covering Lemma [6, Section 1.5.1], we can extract a finite subset of points $A \subset \partial E \cap B_r$ such that:

- $\{B_{\delta r}(y)\}_{y \in A}$ is made up of pairwise disjoint balls;
- $\{B_{3\delta r}(y)\}_{y \in A}$ is still a covering of $\partial E \cap B_r$.

Thanks to these information, we claim that $\{B_{6\delta r}(y)\}_{y \in A}$ is a covering of $N_r^c \cap B_r$. To prove it we fix $y \in N_r^c \cap B_r$: since $d(y, \partial E \cap B_r) \leq 3\delta r$ there exists $x^* \in \partial E \cap B_r$ such that $|y - x^*| \leq 3\delta r$. For such $x^* \in \partial E \cap B_r$, by the second property implied by Vitali Lemma, there exists $x_i \in A$ such that $|x^* - x_i| \leq 3\delta r$. Thus, for all $y \in N_r^c \cap B_r$ there exists $x_i \in A$ such that $|y - x_i| \leq 6\delta r$ by triangle inequality, so the claim follows.

Now, we estimate the cardinality of A in order to obtain 2.41. Fixing $y \in A$, by geometry of the problem we have:

$$\begin{aligned} 2\delta r = d(H_r \cap B_r, N_r) &\leq d(H_r \cap B_r, \partial E \cap B_r) + d(\partial E \cap B_r, N_r) \\ &\leq d(H_r \cap B_r, y) + d(y, N_r) < \delta r + d(y, N_r), \end{aligned}$$

so $d(y, N_r) > \delta r$ and in particular $B_{\delta r}(y) \subset N_r^c$. Moreover, as $y \in B_r$, it is clear that $B_{\delta r}(y) \subset B_{(1+\delta)r} \subset B_{2r}$ thus $\{B_{\delta r}(y)\}_{y \in A} \subset N_r^c \cap B_{2r}$. Majorizing again, our collection of disjoint balls is strictly contained in a closed cylinder $C(2\delta r, 2r)$, whose axes passes through 0 and is orthogonal to H_r, with base radius $2r$ and height $2\delta r$. Since the volume of one of the balls is $\omega_N(\delta r)^N$, we deduce:

$$\omega_N(\delta r)^N \# A \leq |C(2\delta r, 2r)| = \omega_{N-1}(2r)^{N-1}(2\delta r) \simeq \delta r^N.$$

Simplifying, we get $\# A \leq C_1 \delta^{1-N}$, which is 2.41. On the other hand, the fact that $\{B_{6\delta r}(y)\}_{y \in A}$ is a covering of $N_r^c \cap B_r$ implies:

$$\int_{B_r \cap N_r} \mu_P^{\frac{2N}{N+1}} = \sum_{y \in A} \int_{B_{6\delta r}(y))} \mu_P^{\frac{2N}{N+1}},$$

which in turn yields

$$\int_{B_r} \mu_P^{\frac{2N}{N+1}} =\leq C_0 r^{N-\frac{2N}{N+1}(1-\gamma)} \sum_{y \in A} \int_{B_{6\delta r}(y))} \mu_P^{\frac{2N}{N+1}},$$

concluding the proof of 2.42.

For $k \geq 0$, we set $r_k = (6\delta)^k r$ and define recursively $A_0 = \{0\}$ and $A_k = \cup_{x \in A_{k-1}} A(x)$. From 2.41, we have

$$\# A_k \leq (C_1 \delta^{1-N})^k \tag{2.43}$$

and thus applying recursively 2.42 we find for $K > 0$:

$$\int_{B_r} \mu_P^{\frac{2N}{N+1}} \leq C_0 \sum_{k=0}^{K} (\# A_k) r_k^{N-\frac{2N}{N+1}(1-\gamma)} \sum_{y \in A_{K+1}} \int_{B_{r_{K+1}}(y)} \mu_P^{\frac{2N}{N+1}}.$$

By definition of μ_P, we have:

$$\sum_{y\in A_{K+1}}\int_{B_{r_{K+1}}(y)}\mu_P^{\frac{2N}{N+1}} \le (\#A_{K+1})|B_{r_{K+1}}|P^{\frac{2N}{N+1}} \lesssim (C_1\delta^{1-N})^{K+1}(6\delta)^{N(K+1)}r^{N+1}P^{\frac{2N}{N+1}}$$
$$= (6^N C_1\delta)^{K+1}r^{N+1}P^{\frac{2N}{N+1}}.$$

Therefore, choosing δ small enough such that $6^N C_1\delta < 1$, we can send $K \to +\infty$ to make the second term in the right-hand side of 2.43 vanish and to obtain:

$$\begin{aligned}
\int_{B_r}\mu_P^{\frac{2N}{N+1}} &\le C_0\sum_{k=0}^{+\infty}(C_1\delta^{1-N})^k((6\delta)^k r)^{N-\frac{2N}{N+1}(1-\gamma)}\\
&= C_0\sum_{k=0}^{+\infty}(C_1 6^{N-\frac{2N}{N+1}(1-\gamma)})^k\delta^{k(1-N)}\delta^{kN}\delta^{-k\frac{2N}{N+1}(1-\gamma)}r^{N-\frac{2N}{N+1}(1-\gamma)}\\
&= C_0\sum_{k=0}^{+\infty}(C_1 6^{N-\frac{2N}{N+1}(1-\gamma)}\delta^{1-\frac{2N}{N+1}(1-\gamma)})^k r^{N-\frac{2N}{N+1}(1-\gamma)}\\
&= C_0\left(\sum_{k=0}^{+\infty}(C_2\delta^{1-\frac{2N}{N+1}(1-\gamma)})^k\right)r^{N-\frac{2N}{N+1}(1-\gamma)},
\end{aligned}$$

where we set $C_2 = C_1 6^{N-\frac{2N}{N+1}(1-\gamma)}$. By our previous choice of γ, we highlight again that $\frac{2N}{N+1}(1-\gamma) < 1$ so $1 - \frac{2N}{N+1}(1-\gamma) > 0$ and hence, provided δ is small enough , the sum converges and we have (noticing that all the constants are independent of P):

$$\int_{B_r}\mu_P^{\frac{2N}{N+1}} \lesssim r^{N-\frac{2N}{N+1}(1-\gamma)} \le r^{N-1+2\gamma}.$$

The second in quality is true if and only if $-\frac{2N}{N+1}(1-\gamma) \le -1+2\gamma$. After some rearrangements, we see that it is equivalent to the condition $1-2\gamma < N$, which clearly holds. Finally, sending $P \to +\infty$ concludes the proof of 2.39. □

Combining all the results we gathered in this section, in the last theorem of the chapter we deduce that, for small charges Q, every volume-constrained minimizer of $\mathcal{F}_{1,Q}$ is also a perimeter almost minimizer for which the classical theory applies, finally obtaining the desired counterpart of Theorem 2.15. Before going on, we can assume without loss of generality that $r_0/2 < 1$ in the statement of Lemma 2.22.

Theorem 2.23. Let $\alpha = 1$. For every $\gamma \in (0, \frac{1}{2})$ there exists $Q(\gamma, N) > 0$ such that for every $Q \le Q(\gamma, N)$, every volume-constrained minimizer E_Q of $\mathcal{F}_{1,Q}$ is $C^{1,\gamma}$ with uniformly bounded $C^{1,\gamma}$ norm. As a consequence, for every $\beta < \gamma$, up to translation, E_Q converges in $C^{1,\beta}$ to B_1 as $Q \to 0$.

Proof. For every $\delta > 0$, for Q small enough and some $r_0 = r_0(\delta)$, the set E is a (δ, r_0)-Reifenberg flat domain. Let $\gamma = \gamma(\delta)$ be given by 2.21: since $\gamma \to \frac{1}{2}$ as $\delta \to 0$ we can get to any γ close to $\frac{1}{2}$ just by diminishing enough the value of δ. Combining the second almost minimality property 2.24 with estimate 2.39 from Lemma 2.22, there exists $r_0 > 0$ such that for all $r \le r_0/2$ and for all $x \in \partial E$ we have:

$$P(E_Q) \le P(F) + C(Q^2 r^{N-1+2\gamma} + r^N) \qquad \text{for all } E_Q\Delta F \subset B_r(x). \tag{2.44}$$

Now, if $\gamma < \frac{1}{2}$, we have:

$$P(E_Q) \le P(F) + C(Q^2 + r^{1-2\gamma})r^{N-1+2\gamma} \qquad \text{for all } E_Q\Delta F \subset B_r(x).$$

Hence, we can reason exactly as we did in 2.15 to obtain both $C^{1,\gamma}$ regularity with uniformly bounded $C^{1,\gamma}$ norm and $C^{1,\beta}$ convergence of E_Q to B_1 as $Q \to 0$. Conversely. if $\gamma \geq \frac{1}{2}$ we exploit the fact that 2.44 holds for $r < 1$, so for any $\gamma^* \in \left(0, \frac{1}{2}\right)$ we have:

$$r^{N-1+2\gamma} < r^{N-1+2\gamma^*}.$$

In particular, we can reduce to the previous case choosing any $\gamma \in \left(0, \frac{1}{2}\right)$ by majorizing 2.44.

In conclusion, $C^{1,\gamma}$ regularity can be achieved by E_Q for all $\gamma \in \left(0, \frac{1}{2}\right)$, either by diminishing enough Q or by choosing directly the desired $0 < \gamma < \frac{1}{2}$. □

2.7 A non-existence result in dimension 2

We conclude the chapter by presenting a non-existence result for our problem 1.14, valid in dimension $N = 2$ when the charge Q is large enough.

Theorem 2.24. Let $N = 2$ and $\alpha \in (0, 1]$. Then, for $Q \gg 1$ the minimum problem:

$$\min \{\mathcal{F}_{\alpha,Q}(E) \,:\, |E| = \omega_N \,,\, E \in \mathcal{S}\} \tag{2.45}$$

admits no minimizers.

Proof. First of all, we set some notation. For $\nu \in \partial B_1$ and $t \in \mathbb{R}$, we let:

$$H^+_{\nu,t} = \{x \cdot \nu \geq t\}, \qquad H^-_{\nu,t} = \{x \cdot \nu < t\} \qquad \text{and} \qquad H_{\nu,t} = \{x \cdot \nu = t\}.$$

Then, we define for any measure μ and set E:

$$\mu^{\pm}_{\nu,t} = \mu|_{H^{\pm}_{\nu,t}} \qquad \text{and} \qquad E^{\pm}_{\nu,t} = E \cap H^{\pm}_{\nu,t}.$$

We assume by contradiction that E is a minimizer of 2.45 and we compare its energy with the one of a competitor made by two infinitely far apart copies of $E^+_{\nu,t}$ and $E^-_{\nu,t}$, with associated measure the suitable translations of $\mu^+_{\nu,t}$ and $\mu^-_{\nu,t}$ respectively. Notice that both the perimeter and the Riesz energy decouple by construction of our competitor:

$$P(E) + Q^2\mathcal{I}_{\alpha,Q}(E) \leq P(E^+_{\nu,t}) + P(E^-_{\nu,t}) + Q^2 I_\alpha(\mu^+_{\nu,t}) + Q^2 I_\alpha(\mu^-_{\nu,t}). \tag{2.46}$$

Now we need to estimate the left-hand side, taking into account the geometry of the construction. First, we claim that:

$$P(E^+_{\nu,t}) + P(E^-_{\nu,t}) = P(E) + 2\mathcal{H}^1(E \cap H_{\nu,t}). \tag{2.47}$$

To prove it, we follow the argument from [7, Lemma p. 1034]. For any set E of finite perimeter we denote by $\mu_E = -\nabla\chi_E$ its distributional outer unit normal, so that its associated perimeter measure is given by $|\mu_E| = P(E, \cdot)$. By [18, Ex. 15.13], we have that for almost every $t \in \mathbb{R}$:

$$\mu_{E^-_{\nu,t}} = \mu_E|_{H^-_{\nu,t}} + \nu\mathcal{H}^1|_{E \cap H_{\nu,t}}.$$

As it happens in the proof of [18, Lemma 15.12], the measures on the right-hand side are mutually singular, so $|\mu_{E^-_{\nu,t}}| = |\mu_E|\,|_{H^-_{\nu,t}} + \mathcal{H}^1|_{E \cap H_{\nu,t}}$ and in particular:

$$P(E^-_{\nu,t}) = P(E, H^-_{\nu,t}) + \mathcal{H}^1(E \cap H_{\nu,t})$$

Adding the corresponding equality for $-\nu$ and $-t$, since $P(E) = P(E, H^+_{\nu,t}) + P(E, H^-_{\nu,t})$, we obtain 2.47. In addition, for the Riesz energy term of $\mathcal{F}_{\alpha,Q}(E)$, we have:

$$\mathcal{I}_\alpha(E) = I_\alpha(\mu^+_{\nu,t}) + I_\alpha(\mu^-_{\nu,t}) + 2I_\alpha(\mu^+_{\nu,t}, \mu^-_{\nu,t}) \tag{2.48}$$

Therefore, after some rearrangements, we find $Q^2 I_\alpha(\mu^+_{\nu,t}, \mu^-_{\nu,t}) \le \mathcal{H}^1(E \cap H_{\nu,t})$ by replacing 2.47 and 2.48 in 2.46. Now, by Coarea Formula applied to the function $f(x) = x \cdot \nu - t$, we get:

$$|E| = \int_{\mathbb{R}} \mathcal{H}^1(E \cap H_{\nu,t})\, dt.$$

Therefore:

$$\begin{aligned}
|E| &\gtrsim \int_{\partial B_1} \int_{\mathbb{R}} \mathcal{H}^1(E \cap H_{\nu,t})\, dt\, d\nu \ge Q^2 \int_{\partial B_1} \int_{\mathbb{R}} \mathcal{I}_\alpha(\mu^+_{\nu,t}, \mu^-_{\nu,t})\, dt\, d\nu \\
&= Q^2 \int_{\partial B_1} \int_{\mathbb{R}} \int_{H^+_{\nu,t} \times H^-_{\nu,t}} \frac{d\mu(x) d\mu(y)}{|x-y|^{2-\alpha}}\, dt\, d\nu \\
&= Q^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\int_{\partial B_1} \int_{\mathbb{R}} \chi_{H^+_{\nu,t} \times H^-_{\nu,t}}(x,y)\, dt\, d\nu \right) \frac{d\mu(x) d\mu(y)}{|x-y|^{2-\alpha}},
\end{aligned}$$

by Fubini. Now, exploiting the definition of characteristic function and of $H^+_{\nu,t}$ and $H^-_{\nu,t}$, we have that:

$$\chi_{H^+_{\nu,t} \times H^-_{\nu,t}}(x,y) = \chi_{\nu \cdot x \ge t > \nu \cdot y}(t) \qquad \text{and} \qquad \int_{\mathbb{R}} \chi_{\nu \cdot x \ge t > \nu \cdot y}(t)\, dt = [\nu \cdot (x-y)]_+$$

Moreover, we can show that:

$$\int_{\partial B_1} [\nu \cdot (x-y)]_+\, d\nu \approx |x-y|. \tag{2.49}$$

The upper bound is easy, by Cauchy-Schwarz $\int_{\partial B_1} [\nu \cdot (x-y)]_+\, d\nu \le \int_{\partial B_1} |x-y|\, d\nu = 2\pi|x-y|$. On the other hand:

$$\begin{aligned}
\int_{\partial B_1} [\nu \cdot (x-y)]_+\, d\nu &\ge \int_{\partial B_1} \chi_{\nu \cdot (x-y) \ge \frac{|x-y|}{2}} [\nu \cdot (x-y)]_+\, d\nu \ge \int_{\nu \cdot (x-y) \ge \frac{|x-y|}{2}} \frac{|x-y|}{2}\, d\nu \\
&= \int_{\nu \in \left[\frac{x-y}{|x-y|} - \frac{\pi}{6}, \frac{x-y}{|x-y|} + \frac{\pi}{6}\right]} \frac{|x-y|}{2}\, d\nu = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{|x-y|}{2}\, d\nu = \frac{|x-y|}{6},
\end{aligned}$$

yields the lower bound. Now, for all $x, y \in E$ we have $|x-y| \le \operatorname{diam}(E) =: d$ so it holds the relation $|x-y|^{-(1-\alpha)} \ge d^{-(1-\alpha)}$. Therefore:

$$|E| \gtrsim Q^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d\mu(x) d\mu(y)}{|x-y|^{1-\alpha}} \ge \frac{Q^2}{d^{1-\alpha}} \qquad \Longrightarrow \qquad Q^2 \lesssim d^{1-\alpha}.$$

If $\alpha = 1$ we find immediately the contradiction $Q \lesssim 1$, so we are left with the case $\alpha < 1$.

Since $P(E) \gtrsim d$ for $N = 2$, we infer: $Q^{\frac{2}{1-\alpha}} \lesssim d \lesssim P(E) \le \mathcal{F}_{\alpha,Q}(E)$. Now, we build a generalized set $\tilde{E}_r = \{E^i_r\}_{i=1}^n$ made of n copies of the ball B_r with radius $r = n^{-1/2}$. In this way, we estimate its perimeter:

$$P(\tilde{E}_r) = \sum_{i=1}^n P(B_r) = 2\pi n r \lesssim nr = r^{-1}$$

Concerning the Riesz energy, we notice first that $\mathcal{I}_\alpha(B_r) = r^{-(2-\alpha)}\mathcal{I}_\alpha(B_1)$ by 1.3. In addition, by a readaptation of Lemma 2.3 for Riesz interaction energy $\mathcal{I}_\alpha$, we have:

$$\mathcal{I}_\alpha(\tilde{E}_r) = \inf\left\{\sum_i q_i^2\,\mathcal{I}_\alpha(B_r)\,:\,\sum_i q_i = 1\right\} \leq \sum_{i=1}^{n} \frac{1}{n^2}\,\mathcal{I}_\alpha(B_r) \lesssim \frac{r^{-(2-\alpha)}}{n} = r^\alpha.$$

Putting the two estimates together, we find $\mathcal{F}_{\alpha,Q}(\tilde{E}_r) \lesssim r^{-1} + Q^2 r^\alpha$. Choosing $r = Q^{-\frac{2}{1+\alpha}}$, we get by minimality of E:

$$Q^{\frac{2}{1-\alpha}} \lesssim \mathcal{F}_{\alpha,Q}(E) \leq \mathcal{F}_{\alpha,Q}(\tilde{E}_r) \lesssim Q^{\frac{2}{1+\alpha}} + Q^{(1-\alpha)\frac{2}{1+\alpha}} \lesssim Q^{\frac{2}{1+\alpha}}$$

which is a contradiction when $Q \gg 1$. Thus, we conclude that the variational problem 2.45 has no minimizers for very large charge Q.

□

Chapter 3

Minimality of the ball for small charges

The purpose of the third chapter is to prove that, for every $\alpha \in (0,2)$ and small enough charge Q, the ball B_1 is the unique minimizer of the functional $\mathcal{F}_{\alpha,Q}$ under volume constraints in the class of nearly spherical sets. In this way, after what we showed in the previous chapter, we are able to conclude that in the case $\alpha \in (0,1]$, when the charge Q is small enough, the ball B_1 is the unique minimizer for the variational problem 1.14:

$$\min_{|E|=\omega_N, E\in\mathcal{S}} \mathcal{F}_{\alpha,Q}(E).$$

The result we are interested in is stated and proved in Theorem 3.7, at the end of the chapter and the rest is organized as follows. We define first the notion of nearly spherical sets, highlighting its relation with Riesz interaction energy and drawing some preliminary conclusions useful for what follows. Then, as usual, we pave the way for Theorem 3.7 by stating and proving some technical lemmas. The first three of them are more general, whereas from the last two we begin to glimpse our final goal.

Hence, we start by fixing some notation about nearly spherical sets.

Definition 3.1 (Nearly spherical sets)**.** A set $E \subset \mathbb{R}^N$ is said to be *nearly spherical* if $|E| = |B_1|$, E has barycenter in $x = 0$ and there exists $\gamma \in (0,1)$ and $\varphi : \partial B_1 \longrightarrow \mathbb{R}$ with $\|\varphi\|_{C^{1,\gamma}(\partial B_1)} \leq 1$ such that

$$\partial E = \{(1+\varphi(x))x \,:\, x \in \partial B_1\}.$$

We fix a compact nearly spherical set E and we will work with it for the rest of the chapter. As usual, let $\mu = \mu_E$ be its optimal measure for $\mathcal{I}_\alpha(E)$ and $u_E = \int_E \frac{d\nu_E}{|x-y|^{N-\alpha}}$ its potential: we write just μ and u respectively when there is no risk of confusion. Let φ be its associated function: we assume $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ and we directly write $\varphi_x = \varphi(x)$ in order to simplify. With an abuse of notation, we keep denoting by φ its 0-homogeneous extension outside ∂B_1, namely the function

$$\begin{cases} \varphi(\frac{x}{|x|}) & \text{if } x \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

which is 0-homogeneous by definition. In this way, the parametrization of ∂E through φ naturally extends to the whole of $\mathbb{R}^N$ as the function $T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with $T(x) = (1+\varphi_x)x$. Since we have that $T(\partial B_1) = \partial E$, we immediately see that $T(B_1) = E$ from the construction of T. In addition, φ is a $C^{1,\gamma}$ function, $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ and $\frac{x}{|x|} = \frac{T(x)}{|T(x)|}$, so the function T is a C^1 diffeomorphism. Its inverse is $T^{-1} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ writes $T(y) = (1+\varphi_y)^{-1}y$. Clearly $T^{-1}(E) = B_1$, so we are able both to pushforward measures defined on E to measures defined on B_1 using the function T^{-1} and to do the converse with T. Moreover, being T a diffeomorphism, there holds $T^{-1}_{\#}\, T_{\#}\nu = \nu$ for every $\nu \in \mathcal{M}^+(B_1)$ and $T_{\#}\, T^{-1}_{\#}\lambda = \lambda$ for any $\lambda \in \mathcal{M}^+(E)$. Therefore, starting from the probability measure μ_E, we define the measure $g = T^{-1}_{\#}\mu_E$ on B_1. By basic properties of pushforward, g is still a probability measure and, since μ_E is absolutely continuous and T^{-1} is a C^1 diffeomorphism, g is a absolutely continuous too and, after some computations, writes as:

$$
\begin{aligned}
g(x) &= \mu((T^{-1})^{-1}(x))|\det(\mathrm{J}(T^{-1})^{-1}(x))| = \mu(T(x))|\det(\mathrm{J}T(x))| \\
&= (1+\varphi_x)^N \mu((1+\varphi_x)x).
\end{aligned}
$$

In addition, exploiting the fact that $\mu_E = T_{\#}\, T^{-1}_{\#}\mu_E = T_{\#}\, g$, we can find an alternative expression for $\mathcal{I}_\alpha(E)$:

$$
\begin{aligned}
\mathcal{I}_\alpha(E) &= \int_{E\times E} \frac{d\mu_E(x)d\mu_E(y)}{|x-y|^{N-\alpha}} = \int_{T(B_1)\times T(B_1)} \frac{d(T_{\#}\, g)(x)d(T_{\#}\, g)(y)}{|x-y|^{N-\alpha}} \\
&= \int_{B_1\times B_1} \frac{dg(x)dg(y)}{|T(x)-T(y)|^{N-\alpha}}.
\end{aligned} \tag{3.1}
$$

Finally, we recall the explicit expression of the optimal measure μ_{B_1} for $\mathcal{I}_\alpha(B_1)$, given by Proposition 1.13:

$$
\mu_{B_1}(x) = \frac{C_\alpha}{(1-|x|^2)^{\frac{\alpha}{2}}} \approx \frac{1}{d(x,\partial B_1)^{\frac{\alpha}{2}}}.
$$

Before going on, we need two more preliminary results regarding nearly spherical sets. First of all, we have that:

$$
\left|\int_{\partial B_1} \varphi\right| \lesssim \int_{\partial B_1} \varphi^2. \tag{3.2}
$$

Indeed, since $E = \{(1+\varphi_x)x : x \in B_1\}$, we pass to polar coordinate to find:

$$
\frac{1}{N}\int_{\partial B_1} d\sigma = \int_{\partial B_1}\int_0^1 r^{N-1}\,dr\,d\sigma = |B_1| = |E| = \int_{\partial B_1}\int_0^{1+\varphi_x} r^{N-1}\,dr\,d\sigma = \frac{1}{N}\int_{\partial B_1}(1+\varphi_x)\,d\sigma.
$$

Hence:

$$
\int_{\partial B_1}(1+\varphi_x)^N - 1 = 0 \qquad \Longrightarrow \qquad N\int_{\partial B_1}\varphi = \sum_{k=2}^N \binom{N}{k}\int_{\partial B_1}\varphi^k.
$$

In particular, passing to the modulus and recalling that $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$, we conclude:

$$
\left|\int_{\partial B_1}\varphi\right| \lesssim \sum_{k=2}^N \int_{\partial B_1}|\varphi|^k \lesssim \int_{\partial B_1}\varphi^2.
$$

From 3.2 we can infer another useful property for what will follow. If we set $\bar{\varphi} = \frac{1}{P(B_1)}\int_{\partial B_1}\varphi$, we have for $s \in (0,1)$:

$$
\int_{\partial B_1}\varphi^2 \lesssim \int_{\partial B_1}(\varphi - \bar{\varphi}^2) \lesssim [\varphi]^2_{H^s(\partial B_1)}. \tag{3.3}
$$

The second inequality is given by Proposition 1.6, so we need to show just the first one:

$$\int_{\partial B_1} (\varphi - \bar{\varphi}^2) = \int_{\partial B_1} \varphi^2 - 2\bar{\varphi} \int_{\partial B_1} \varphi + \int_{\partial B_1} \bar{\varphi}^2 = \int_{\partial B_1} \varphi^2 - 2P(B)\bar{\varphi}^2 + P(B)\bar{\varphi}^2$$
$$= \int_{\partial B_1} \varphi^2 - P(B)\bar{\varphi}^2.$$

Now, we appeal to 3.2 to get:

$$\bar{\varphi}^2 \approx \Big(\int_{\partial B_1} \varphi\Big)^2 \lesssim \Big(\int_{\partial B_1} \varphi^2\Big)^2 \ll \int_{\partial B_1} \varphi^2$$

where the last inequality is given by $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$. Therefore:

$$\frac{1}{2} \int_{\partial B_1} \varphi^2 \leq \int_{\partial B_1} \varphi^2 - P(B)\bar{\varphi}^2 = \int_{\partial B_1} (\varphi - \bar{\varphi}^2)$$

yields 3.3.

After setting the stage, we begin proving all the technical results we need to demonstrate what we want. From the previous discussion, we remark that for every nearly spherical set E with $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ there exist two probability measures naturally defined on B_1, that are μ_{B_1} and g. First of all, we prove that g has the same behaviour as μ_{B_1} close to ∂B_1.

Lemma 3.2. Let $\alpha \in (0, 2)$ and E be a nearly spherical set with $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$. Then, its associated probability measure g defined on B_1 satisfies, for $x \in B_1$:

$$g(x) \lesssim \frac{1}{d(x, \partial B)^{\frac{\alpha}{2}}} \sim \mu_{B_1}(x) \tag{3.4}$$

Proof. The first part of the proof is similar to Lemma 2.22, as we need to show an estimate for $\mu = \mu_E$:

$$\mu(x) \lesssim \frac{1}{d(x, \partial E)^{\frac{\alpha}{2}}} \qquad \text{for all } x \in E.$$

From the statement of Proposition 1.12, since μ is supported in E we have that:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = 0 & x \in E^c \\ u(x) - \mathcal{I}_\alpha(E) = 0 & x \in E. \end{cases}$$

Thus, we can appeal to the boundary regularity theory for the fractional Laplacian developed by Ros-Oton and Serra. In particular, we are in the position to apply [25, Theorem 1.2] in order to get:

$$u(x) - \mathcal{I}_\alpha(E) \lesssim d^{\frac{\alpha}{2}}(x, \partial E).$$

Now, we can repeat the steps of the proof of 2.40 from Lemma 2.22, applying the estimate we just got instead of 2.30. Given $x \in E$:

$$C'(N, \alpha)\mu(x) = (-\Delta)^{\frac{\alpha}{2}} u(x) = C(N, \frac{\alpha}{2}) \int_{E^c} \frac{\mathcal{I}_\alpha(E) - u(y)}{|x - y|^{N+\alpha}}\, dy \lesssim \int_{E^c} \frac{d^{\frac{\alpha}{2}}(y, \partial E)}{|x - y|^{N+\alpha}}\, dy$$
$$\lesssim \int_{B^c_{d(x,\partial E)}(x)} \frac{dz}{|z|^{N+\alpha-\frac{\alpha}{2}}} \lesssim d(x, \partial E)^{-\frac{\alpha}{2}}.$$

The function g writes as $g(x) = (1+\varphi_x)^N \mu((1+\varphi_x)x)$. Since we have that $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$, we can bound the term $(1+\varphi_x)^N$ and, for $x \in B_1$, we obtain the estimate:

$$g(x) \lesssim \mu((1+\varphi_x)x) \lesssim \frac{1}{d^{\frac{a}{2}}((1+\varphi_x)x, \partial E)}.$$

Therefore, we only need to show $d((1+\varphi_x)x, \partial E) \approx |1-|x||$ to conclude 3.4. The upper bound is easy, by definition we have that:

$$d((1+\varphi_x)x, \partial E) = \min_{y\in\partial B_1} |(1+\varphi_x)x - (1+\varphi_y)y| \,;$$

choosing $y = \frac{x}{|x|}$, we find (recall 0-homogeneity of φ):

$$d((1+\varphi_x)x, \partial E) \le \left|(1+\varphi_x)x - (1+\varphi_x)\frac{x}{|x|}\right| = (1+\varphi_x)\,|1-|x|| \lesssim |1-|x||\,.$$

Conversely, to get the lower bound we can assume that $|1-|x|| \ll 1$. Indeed, if $|\,|1-|x|| \gtrsim 1$ then both $d((1+\varphi_x)x, \partial E)$ and $|\,|1-|x|| \gtrsim 1$ are uniformly bounded from above and below, so the required estimate trivially follows. Hence, squaring we get

$$\begin{aligned} f(x) = d((1+\varphi_x)x, \partial E)^2 &= \min_{y\in\partial B_1} |(1+\varphi_x)x - (1+\varphi_y)y|^2 \\ &= \min_{y\in\partial B_1} \left\{(1+\varphi_x)^2|x|^2 - 2(1+\varphi_x)(1+\varphi_y)x\cdot y + |1+\varphi_y|^2\right\}. \end{aligned}$$

Writing $-x\cdot y = -|x| + x\cdot\left(\frac{x}{|x|} - y\right)$, we find:

$$\begin{aligned} f(x) &= \min_{y\in\partial B_1} \left\{(1+\varphi_x)^2|x|^2 - 2(1+\varphi_x)(1+\varphi_y)|x| + |1+\varphi_y|^2 + 2(1+\varphi_x)(1+\varphi_y)x\cdot\left(\frac{x}{|x|} - y\right)\right\} \\ &= \min_{y\in\partial B_1} \left\{(1+\varphi_x)^2|\left||x| - \frac{1+\varphi_y}{1+\varphi_x}\right|^2 + 2(1+\varphi_x)(1+\varphi_y)x\cdot\left(\frac{x}{|x|} - y\right)(|x| - y\cdot x)\right\}. \end{aligned}$$

Using again the fact that $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ we have $(1+\varphi_x)^2 \gtrsim 1$ and $(1+\varphi_x)(1+\varphi_y) \gtrsim 1$, so:

$$f(x) \gtrsim \min_{y\in\partial B_1} \left\{\left||x| - 1 + \frac{\varphi_x - \varphi_y}{1+\varphi_x}\right|^2 + (|x| - y\cdot x)\right\}.$$

For every $y \in \partial B_1$ either $(|x| - y\cdot x) \gtrsim ||x|-1|^2$ or $(|x| - y\cdot x) \ll ||x|-1|^2$. The first case directly leads to the conclusion $d((1+\varphi_x)x, \partial E)^2 \gtrsim ||x|-1|^2$. In the second case instead, we write $x = r\sigma$ with $\sigma \in \partial B_1$ and we compute:

$$|\sigma - y| = |\sigma|^2 + |y|^2 - 2\sigma\cdot y = 2 - 2\sigma\cdot y = \frac{2}{r}(r - r\sigma\cdot y) = \frac{2}{r}(|x| - x\cdot y) \ll ||x|-1|^2.$$

Therefore, since φ is Lipschitz over ∂B_1 we have:

$$|\varphi_x - \varphi_y| \lesssim |\sigma - y| \ll ||x|-1|^2$$

Therefore, exploiting the last two relations we get:

$$\left||x| - 1 + \frac{\varphi_x - \varphi_y}{1+\varphi_x}\right|^2 + (|x| - y\cdot x) \gtrsim ||x|-1|^2$$

which yields $d((1+\varphi_x)x, \partial E)^2 \gtrsim ||x|-1|^2$. □

Now, we prove other two preparatory lemmas which basically serve to carry out the Taylor expansion of $|T(x)-T(y)|^{-(N-\alpha)}$ appearing in 3.1.

Lemma 3.3. For $x, y \in B$, we have:

$$|T(x)-T(y)|^2 = |x-y|^2(1+\varphi_x+\varphi_y+\varphi_x\varphi_y+\psi(x,y)) \tag{3.5}$$

where

$$\psi(x,y) \leq \frac{1}{2}(|x|^2+|y|^2)\left(\frac{\varphi_x-\varphi_y}{|x-y|}\right)^2 + (|x|+|y|)\left(1+\frac{1}{2}(\varphi_x+\varphi_y)\right)\frac{\varphi_x-\varphi_y}{|x-y|}. \tag{3.6}$$

Proof. First we notice that:

$$\begin{aligned} |T(x)-T(y)|^2 &= |x+\varphi_x x - y - \varphi_y y|^2 = |(x-y)+(\varphi_x x-\varphi_y y)|^2 \\ &= \left|(x-y)+\frac{1}{2}((x+y)(\varphi_x-\varphi_y)+(x-y)(\varphi_x+\varphi_y))\right|^2. \end{aligned}$$

Expanding the last term we get:

$$\begin{aligned} &|x-y|^2 + (|x|^2-|y|^2)(\varphi_x-\varphi_y) + |x-y|^2(\varphi_x+\varphi_y) + \frac{1}{4}|x+y|^2\,|\varphi_x-\varphi_y|^2 + \\ &\frac{1}{4}|x-y|^2\,|\varphi_x+\varphi_y|^2 + \frac{1}{2}(|x|^2-|y|^2)(\varphi_x^2-\varphi_y^2) \end{aligned}$$

Factoring out $|x-y|^2$ and comparing with 3.5, we see that the first and the third addend matches the first three terms in 3.5. On the other hand, to handle the fourth and the fifth together we use $(\varphi_x+\varphi_y)^2 = (\varphi_x-\varphi_y)^2 + 4\varphi_x\varphi_y$ and we get:

$$\begin{aligned} \frac{1}{4}|x+y|^2\,|\varphi_x-\varphi_y|^2 + \frac{1}{4}|x-y|^2\,|\varphi_x+\varphi_y|^2 &= \frac{1}{4}\,|\varphi_x-\varphi_y|^2\left(|x+y|^2+|x-y|^2\right) + |x-y|^2\varphi_x\varphi_y \\ &= \frac{1}{2}\,(\varphi_x-\varphi_y)^2\left(|x|^2+|y|^2\right) + |x-y|^2\varphi_x\varphi_y. \end{aligned}$$

The last term matches the fourth addend in 3.5, so we are left to estimate:

$$\psi(x,y) := (|x|^2-|y|^2)(\varphi_x-\varphi_y) + \frac{1}{2}(|x|^2-|y|^2)(\varphi_x^2-\varphi_y^2) + \frac{1}{2}\,(\varphi_x-\varphi_y)^2\left(|x|^2+|y|^2\right).$$

The last term corresponds with 3.6, therefore we only have to deal with the first two. We factor out the common terms and divide by $|x-y|^2$:

$$(|x|+|y|)\frac{|x|-|y|}{|x-y|}\frac{\varphi_x-\varphi_y}{|x-y|}\left(1+\frac{1}{2}(\varphi_x+\varphi_y)\right) \leq (|x|+|y|)\frac{\varphi_x-\varphi_y}{|x-y|}\left(1+\frac{1}{2}(\varphi_x+\varphi_y)\right)$$

and we are done. □

As a consequence we get the following Taylor expansion for $|T(x)-T(y)|^{-(N-\alpha)}$. From now on, we set $\widehat{\alpha} = N-\alpha$ for briefness' sake.

Lemma 3.4. If $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ then for $x, y \in B_1$ we have:

$$|T(x)-T(y)|^{-(N-\alpha)} = |x-y|^{-(N-\alpha)}\left(\left(1-\frac{\widehat{\alpha}}{2}\varphi_x\right)\left(1-\frac{\widehat{\alpha}}{2}\varphi_y\right) - \frac{\widehat{\alpha}}{2}\psi(x,y) + \zeta(x,y)\right), \tag{3.7}$$

where

$$|\zeta(x,y)| \lesssim \varphi_x^2 + \varphi_y^2 + \psi^2(x,y) \tag{3.8}$$

and where ψ is the function defined above.

Proof. We begin by showing that $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ implies $\|\psi\|_{L^\infty(B_1)} \ll 1$. To do so, we need to estimate the two terms at the right-hand side of 3.6, using later that both $\|\varphi\|_{L^\infty(\partial B_1)} \ll 1$ and $\|\varphi'\|_{L^\infty(\partial B_1)} \ll 1$. One could be tempted to use straightaway the Lipschitz property of φ to estimate $\frac{\varphi_x - \varphi_y}{|x-y|}$, but we must remember that φ is Lipschitz only over ∂B_1 and not over the whole of B_1. Thus, we need to make preliminary computations, for which we write $x = r\sigma$ and $y = sv$, with $r, s \in [0,1]$ and $\sigma, v \in \partial B_1$. In particular:

$$\begin{aligned}|x-y|^2 = |r\sigma - sv|^2 = r^2 + s^2 - 2rs\sigma\cdot v - 2rs + 2rs = |r-s|^2 + rs(2 - 2\sigma\cdot v)\\ = |r-s|^2 + rs|\sigma - v|^2\end{aligned}$$

Hence, we start with the first term of 3.6:

$$\begin{aligned}\frac{|x|^2+|y|^2}{|x-y|^2} &= \frac{r^2+s^2}{|r-s|^2 + rs|\sigma-v|^2} = \frac{|r-s|^2}{|r-s|^2 + rs|\sigma-v|^2} + \frac{2rs}{|r-s|^2 + rs|\sigma-v|^2}\\ &\le 1 + \frac{2}{|\sigma - v|^2}.\end{aligned}$$

In this way, since $\sigma = \frac{x}{|x|}$ and $v = \frac{y}{|y|}$ we find:

$$\frac{1}{2}(|x|^2+|y|^2)\left(\frac{\varphi_x-\varphi_y}{|x-y|}\right)^2 \le \frac{1}{2}(\varphi_x - \varphi_y)^2 + \left(\frac{\varphi_x - \varphi_y}{|\frac{x}{|x|} - \frac{y}{|y|}|}\right)^2, \tag{3.9}$$

which we can easily estimate using the Lipschitz property of φ and $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$. The procedure to estimate the second term of 3.6 is similar:

$$\begin{aligned}\left(\frac{|x|+|y|}{|x-y|}\right)^2 &= \frac{r^2+s^2+2rs}{|r-s|^2 + rs|\sigma-v|^2} = \frac{|r-s|^2}{|r-s|^2 + rs|\sigma-v|^2} + \frac{4rs}{|r-s|^2 + rs|\sigma-v|^2}\\ &\le 1 + \frac{4}{|\sigma-v|^2},\end{aligned}$$

so

$$\frac{|x|+|y|}{|x-y|} \le \left(1 + \frac{4}{|\sigma - v|^2}\right)^{\frac{1}{2}} \le 1 + \frac{2}{|\sigma - v|}.$$

Again, we find:

$$\begin{aligned}(|x|+|y|)\left(1 + \frac{1}{2}(\varphi_x+\varphi_y)\right)\frac{\varphi_x-\varphi_y}{|x-y|} &\le \left(1 + \frac{1}{2}(\varphi_x+\varphi_y)\right)(\varphi_x - \varphi_y)\\ &\quad + 2\left(1 + \frac{1}{2}(\varphi_x+\varphi_y)\right)\frac{\varphi_x-\varphi_y}{|\frac{x}{|x|} - \frac{y}{|y|}|}\end{aligned}$$

and we conclude like before. Thus, we proved $\|\psi\|_{L^\infty(B_1)} \ll 1$.

Now, since $\|\varphi_x + \varphi_y + \varphi_x\varphi_y + \psi(x,y)\|_{L^\infty} \ll 1$, we can perform the Taylor expansion:

$$(1+t)^{-\frac{\widehat{\alpha}}{2}} = 1 - \frac{\widehat{\alpha}}{2}t + o(t) = 1 - \frac{\widehat{\alpha}}{2}t + O(t^2)$$

to get:

$$(1 + \varphi_x + \varphi_y + \varphi_x\varphi_y + \psi(x,y))^{-\frac{\widehat{\alpha}}{2}} = 1 - \frac{\widehat{\alpha}}{2}\varphi_x - \frac{\widehat{\alpha}}{2}\varphi_y - \frac{\widehat{\alpha}}{2}\varphi_x\varphi_y - \frac{\widehat{\alpha}}{2}\psi(x,y) + O(\varphi_x^2 + \varphi_y^2 + \psi(x,y)^2).$$

Adding and subtracting $\frac{\widehat{\alpha^2}}{4}\varphi_x\varphi_y$ first and estimating $\varphi_x\varphi_y \lesssim \varphi_x^2 + \varphi_y^2$, we finally get:

$$(1+\varphi_x+\varphi_y+\varphi_x\varphi_y+\psi(x,y))^{-\frac{\widehat{\alpha}}{2}} = \left(1-\frac{\widehat{\alpha}}{2}\varphi_x\right)\left(1-\frac{\widehat{\alpha}}{2}\varphi_y\right) - \frac{\widehat{\alpha}}{2}\psi(x,y) + \zeta(x,y)$$

with $|\zeta(x,y)| \lesssim \varphi_x^2 + \varphi_y^2 + \psi^2(x,y)$. □

After gathering some useful preliminaries in the previous lemmas, we can focus on the key linearization estimates which will lead us to prove the rigidity result we are interested in. There a lot of computations in the proofs of the following two lemmas and we make heavy use of majorization techniques to derive the estimates we need. When we write $\lesssim_\varepsilon$ we stress the fact that the implicit constant $C = C(\varepsilon)$.

Lemma 3.5. Let E be a nearly spherical set with $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$. Then, for every $\alpha \in (0,2)$ and $\varepsilon > 0$ there holds:

$$\left|\mathcal{I}_\alpha(E) - I_\alpha\left((1-\frac{\widehat{\alpha}}{2}\varphi)g\right)\right| \lesssim_\varepsilon [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)} \tag{3.10}$$

where $\left(1-\frac{\widehat{\alpha}}{2}\varphi\right) g$ is intended as a measure on B_1 and $\widehat{\alpha} = N - \alpha$.

Proof. We write the alternative expression for $\mathcal{I}_\alpha(E)$ we deduced in 3.1 and then we use Taylor expansion 3.7. The first of the three resulting terms at the numerator and $-I_\alpha\left((1-\frac{\widehat{\alpha}}{2}\varphi)g\right)$ erase each other, so to finish the proof it is enough showing that:

$$\left|\int_{B_1\times B_1}\frac{\psi(x,y)}{|x-y|^{N-\alpha}}dg_x dg_y\right| + \left|\int_{B_1\times B_1}\frac{\zeta(x,y)}{|x-y|^{N-\alpha}}dg_x dg_y\right| \lesssim_\varepsilon [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)} \tag{3.11}$$

The second term is easier to handle: applying 3.8 it is enough to provide an estimate for φ_x^2 and $\psi^2(x,y)$. By 3.4 we have $g(x) \le \mu_{B_1}(x)$, so we get:

$$\int_{B_1\times B_1}\frac{\varphi_x^2}{|x-y|^{N-\alpha}}dg_x dg_y \lesssim \int_{B_1\times B_1}\varphi_x^2\frac{d\mu_{B_1}(x)d\mu_{B_1}(y)}{|x-y|^{N-\alpha}}$$

To evaluate the last integral, we use Fubini Theorem, we recall that $u_{B_1}(x) = \mathcal{I}_\alpha(B_1)$ for all $x \in B_1$ and we pass to polar coordinates exploiting that μ_{B_1} is radial and φ is 0- homogeneous:

$$\begin{aligned}
\int_{B_1\times B_1}\varphi^2\left(\frac{x}{|x|}\right)\frac{d\mu_{B_1}(x)d\mu_{B_1}(y)}{|x-y|^{N-\alpha}} &= \int_{B_1}\varphi^2\left(\frac{x}{|x|}\right)\int_{B_1}\frac{d\mu_{B_1}(y)}{|x-y|^{N-\alpha}}\,d\mu_{B_1}(x)\\
&= \frac{\mathcal{I}_\alpha(B_1)}{\mathcal{H}^{N-1}(\partial B_1)}\int_{\partial B_1}d\tau\int_{\partial B_1}\varphi^2(\sigma)\int_0^1 r^{N-1}\mu_{B_1}(r)\,dr\,d\sigma\\
&= \frac{\mathcal{I}_\alpha(B_1)}{\mathcal{H}^{N-1}(\partial B_1)}\int_{\partial B_1}\varphi^2(\sigma)\,d\sigma\int_{\partial B_1}\int_0^1 r^{N-1}\mu_{B_1}(r)\,dr\,d\tau \qquad (3.12)\\
&= \frac{\mathcal{I}_\alpha(B_1)}{\mathcal{H}^{N-1}(\partial B_1)}\int_{B_1}d\mu(x)\int_{\partial B_1}\varphi^2\\
&= \frac{\mathcal{I}_\alpha(B_1)}{\mathcal{H}^{N-1}(\partial B_1)}\int_{\partial B_1}\varphi^2.
\end{aligned}$$

Thus, by 3.3 we deduce:

$$\int_{B_1\times B_1}\varphi_x^2\frac{d\mu_{B_1}(x)d\mu_{B_1}(y)}{|x-y|^{N-\alpha}} \le \frac{\mathcal{I}_\alpha(B_1)}{\mathcal{H}^{N-1}(\partial B_1)}\int_{\partial B_1}\varphi^2 \lesssim [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)}. \tag{3.13}$$

Now we need to estimate the third term from 3.8: by the proof of Lemma 3.4, $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$ implies $\|\psi\|_{L^\infty(B_1)} \ll 1$. Therefore, $\psi^2(x,y) \ll \psi(x,y)$ and we can majorize the corresponding term with the first member at the left-hand side of 3.11. In particular, as we have not estimated it yet, we are left with the proof of:

$$\left|\int_{B_1\times B_1} \frac{\psi(x,y)}{|x-y|^{N-\alpha}} dg_x dg_y\right| \lesssim_\varepsilon [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)}.$$

We appeal to 3.6 and we treat separately the two terms. We start by the second, which in turn counts two cases since there is a sum $1+\frac{1}{2}(\varphi_x+\varphi_y)$. The first one is harmless, as by symmetry in x and y we get:

$$\int_{B_1\times B_1} (\varphi_x-\varphi_y)\frac{|x|+|y|}{|x-y|^{N-\alpha+1}}\, dg_x dg_y = 0,$$

so it does not contribute. About the second case of the second term, Young inequality reiterated many times yields:

$$(|x|+|y|)|\varphi_x+\varphi_y|\frac{\varphi_x-\varphi_y}{|x-y|} \lesssim \left(|x|^2+|y|^2\right)\left(\frac{\varphi_x-\varphi_y}{|x-y|}\right)^2 + \varphi_x^2+\varphi_y^2.$$

We get rid of $\varphi_x^2+\varphi_y^2$ repeating the same computations of 3.13. Instead, what remains is incorporated in the estimate of the first term of 3.6 we are about to figure out, namely:

$$A := \int_{B_1\times B_1} \left(|x|^2+|y|^2\right)\left(\frac{\varphi_x-\varphi_y}{|x-y|}\right)^2 \frac{dg_x dg_y}{|x-y|^{N-\alpha}} \lesssim_\varepsilon [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)}.$$

By 3.9 and 3.4 we get:

$$A \lesssim \int_{B_1\times B_1} \left[(\varphi_x-\varphi_y)^2 + \left(\frac{\varphi_x-\varphi_y}{|\frac{x}{|x|}-\frac{y}{|y|}|}\right)^2\right] \frac{d\mu_{B_1}(x)d\mu_{B_1}(y)}{|x-y|^{N-\alpha}}.$$

The term $(\varphi_x-\varphi_y)^2$ is estimated by Young Inequality and then we argue again like 3.13.

Thus, we further reduced the proof of 3.10 to the trickiest part:

$$B := \int_{B_1\times B_1} \left(\frac{\varphi_x-\varphi_y}{|\frac{x}{|x|}-\frac{y}{|y|}|}\right)^2 \frac{d\mu_{B_1}(x)d\mu_{B_1}(y)}{|x-y|^{N-\alpha}} \lesssim_\varepsilon [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)} \tag{3.14}$$

To estimate it, we recall that $\mu_{B_1}(x) \lesssim (1-|x|)^{\frac{\alpha}{2}}$ and we switch to polar coordinates setting again $x = r\sigma$ and $y = sv$, with $r,s\in\mathbb{R}$ and $\sigma, v \in \partial B_1$. In this way we get:

$$\begin{aligned} B &\lesssim \int_{\partial B_1\times\partial B_1} \left(\frac{\phi(\sigma)-\phi(v)}{|\sigma-v|}\right)^2 \left[\int_0^1\int_0^1 \frac{r^{N-1}s^{N-1}}{|1-r|^{\frac{\alpha}{2}}|1-s|^{\frac{\alpha}{2}}} \frac{drds}{\left(|r-s|^2+rs|\sigma-v|^2\right)^{\frac{N-\alpha}{2}}}\right] d\sigma dv \\ &= \int_{\partial B_1\times\partial B_1} \left(\frac{\phi(\sigma)-\phi(v)}{|\sigma-v|}\right)^2 F(|\sigma-v|) d\sigma dv, \end{aligned}$$

where

$$F(\theta) = \int_0^1\int_0^1 \frac{r^{N-1}s^{N-1}}{|1-r|^{\frac{\alpha}{2}}|1-s|^{\frac{\alpha}{2}}} \frac{drds}{\left(|r-s|^2+rs\theta^2\right)^{\frac{N-\alpha}{2}}}.$$

Now, we fix $\varepsilon > 0$ and we claim that for $\theta\in(0,2)$:

$$F(\theta) \lesssim_\varepsilon \frac{1}{\theta^{N-\alpha-1}} + \frac{1}{\theta^{N-2+2\varepsilon}}. \tag{3.15}$$

As the function F is decreasing in θ, it is enough to prove the claim for $\theta \ll 1$. We split the integral in two and we separately estimate each term, starting with:

$$C := \int_0^{\frac{1}{2}} \int_0^1 \frac{r^{N-1} s^{N-1}}{|1-r|^{\frac{\alpha}{2}} |1-s|^{\frac{\alpha}{2}}} \frac{drds}{(|r-s|^2 + rs\theta^2)^{\frac{N-\alpha}{2}}}.$$

Since $r \in [0, 1/2]$, we have that $|1-r|^{-\frac{\alpha}{2}}$ is uniformly bounded. The, we use the facts that if $s \in [0, 3/4]$ then $|1-s|^{-\frac{\alpha}{2}}$ is uniformly bounded too and, if $r \in [0, 1/2]$ and $s \in [3/4, 1]$ then $|r-s| \in [1/4, 1]$, so $(|r-s|^2 + rs\theta^2)^{-\frac{N-\alpha}{2}}$ is uniformly bounded too. Putting everything together we find:

$$\begin{aligned} C &\lesssim \int_0^{\frac{1}{2}} r^{N-1} \left[\int_0^1 \frac{1}{|1-s|^{\frac{\alpha}{2}}} \frac{ds}{(|r-s|^2 + rs\theta^2)^{\frac{N-\alpha}{2}}} \right] dr \\ &\lesssim \int_0^{\frac{1}{2}} r^{N-1} \left[\int_0^{\frac{3}{4}} \frac{ds}{(|r-s|^2 + rs\theta^2)^{\frac{N-\alpha}{2}}} \right] dr + \int_0^{\frac{1}{2}} \left[\int_{\frac{3}{4}}^1 \frac{ds}{|1-s|^{\frac{\alpha}{2}}} \right] dr \\ &\lesssim \int_0^{\frac{1}{2}} r^{N-1} \left[\int_0^{\frac{3}{4}} \frac{ds}{(|r-s|^2 + rs\theta^2)^{\frac{N-\alpha}{2}}} \right] dr + 1, \end{aligned}$$

as the second term before the last inequality is integrable. Now, we first change variables $s = rt$ and the we split again the integral:

$$\begin{aligned} \int_0^{\frac{3}{4}} \frac{ds}{(|r-s|^2 + rs\theta^2)^{\frac{N-\alpha}{2}}} &= r^{1-N+\alpha} \int_0^{\frac{3}{4r}} \frac{dt}{(|1-t|^2 + t\theta^2)^{\frac{N-\alpha}{2}}} \\ &\lesssim r^{1-N+\alpha} \left[\int_0^{\frac{1}{2}} \frac{dt}{t^{\frac{N-\alpha}{2}}} + \int_{\frac{1}{2}}^2 \frac{dt}{(|1-t| + \theta)^{N-\alpha}} + \int_2^{\frac{3}{4r}} \frac{dt}{t^{N-\alpha}} \right] \\ &\lesssim r^{1-N+\alpha} \left[1 + \frac{1}{\theta^{N-\alpha-1}} + r^{N-\alpha-1} \right], \end{aligned}$$

where we majorized each of the three terms in different ways. Therefore, we conclude that:

$$\lesssim 1 + \int_0^{\frac{1}{2}} r^{\alpha} \left[1 + \frac{1}{\theta^{N-\alpha-1}} + r^{N-\alpha-1} \right] dr \lesssim 1 + \frac{1}{\theta^{N-\alpha-1}}. \tag{3.16}$$

It remains to majorize the integral between 1/2 and 1:

$$D := \int_{\frac{1}{2}}^1 \int_0^1 \frac{r^{N-1} s^{N-1}}{|1-r|^{\frac{\alpha}{2}} |1-s|^{\frac{\alpha}{2}}} \frac{drds}{(|r-s|^2 + rs\theta^2)^{\frac{N-\alpha}{2}}}.$$

Again, we split once more: if $r \in [1/2, 1]$ and $s \in [0, 1/4]$ the both $(|r-s|^2 + rs\theta)^{-\frac{N-\alpha}{2}}$ and $|1-s|^{-\frac{\alpha}{2}}$ are uniformly bounded. In addition, if $r \geq 1/2$ and $s \geq 1/4$ then $\theta^2 rs \gtrsim \theta^2$, so we get:

$$\begin{aligned} D &\lesssim \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{4}} \frac{1}{|1-r|^{\frac{\alpha}{2}}} drds + \int_{\frac{1}{2}}^1 \int_{\frac{1}{4}}^1 \frac{1}{|1-r|^{\frac{\alpha}{2}} |1-s|^{\frac{\alpha}{2}}} \frac{drds}{(|r-s|^2 + \theta^2)^{\frac{N-\alpha}{2}}} \\ &\lesssim 1 + \int_0^{\frac{1}{2}} \int_{-1}^1 \frac{1}{t^{\frac{\alpha}{2}} |t-w|^{\frac{\alpha}{2}}} \frac{dtdw}{(w^2 + \theta^2)^{\frac{N-\alpha}{2}}}. \end{aligned}$$

where in the last line we performed the change of variables $r = 1 - t$ and $s = 1 - t + w$. We now prove that for every $w \in (-1, 1)$,

$$\int_0^{\frac{1}{2}} \frac{dt}{t^{\frac{\alpha}{2}}|t-w|^{\frac{\alpha}{2}}} \lesssim |w|^{1-\alpha} + 1 + \chi_{\alpha=1}|\log|w||. \tag{3.17}$$

The left-hand side of 3.17 increases when we replace w by $|w|$, so it is enough to prove it for $w > 0$. We split once again and we find:

$$\int_0^{\frac{1}{2}} \frac{dt}{t^{\frac{\alpha}{2}}|t-w|^{\frac{\alpha}{2}}} \le \int_0^{\frac{w}{2}} \frac{dt}{t^{\frac{\alpha}{2}} w^{\frac{\alpha}{2}}} + \int_{\frac{w}{2}}^{2w} \frac{dt}{w^{\frac{\alpha}{2}}|t-w|^{\frac{\alpha}{2}}} + \int_{2w}^{2} \frac{dt}{t^{\alpha}} \lesssim w^{1-\alpha} + 1 + \chi_{\alpha=1}|\log w|,$$

where each time we majorized in the right way exploiting the choice of the integration intervals. Thus, plugging 3.17 into the last expression we obtained, we find:

$$\begin{aligned}\int_0^{\frac{1}{2}}\int_{-1}^{1} \frac{1}{t^{\frac{\alpha}{2}}|t-w|^{\frac{\alpha}{2}}} \frac{dtdw}{(w^2+\theta^2)^{\frac{N-\alpha}{2}}} &\lesssim \int_{-1}^{1} \frac{w^{1-\alpha}}{(w^2+\theta^2)^{\frac{N-\alpha}{2}}} + \frac{1}{(w^2+\theta^2)^{\frac{N-\alpha}{2}}} + \frac{|\log w|}{(w^2+\theta^2)^{\frac{N-\alpha}{2}}}\, dw \\ &\lesssim_\varepsilon \frac{1}{\theta^{N-\alpha-1}} + \frac{1}{\theta^{N-2+2\varepsilon}}.\end{aligned}$$

We used ε for the first time here to determine an uniform majorization for the integral with the logarithmic term. The choice of the exponent $N - 2 + 2\varepsilon$ will be clear in the conclusion of the proof. Hence, we finally showed:

$$D \lesssim_\varepsilon \frac{1}{\theta^{N-\alpha-1}} + \frac{1}{\theta^{N-2+2\varepsilon}},$$

which, together with 3.16, concludes the proof of 3.15. At this point, we are finally ready to conclude the estimate of B together with the proof of the Lemma. By applying 3.15 we find:

$$\begin{aligned}B &\lesssim_\varepsilon \int_{\partial B_1 \times \partial B_1} \frac{(\phi(\sigma)-\phi(v))^2}{|\sigma - v|^{N-1+(2-\alpha)}}\, d\sigma\, dv + \int_{\partial B_1 \times \partial B_1} \frac{(\phi(\sigma)-\phi(v))^2}{|\sigma - v|^{N-1+(1+2\varepsilon)}}\, d\sigma\, dv \\ &= [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)},\end{aligned}$$

which is 3.14. □

We may now conclude the proof of the stability inequality for nearly spherical set. In the next lemma, we exploit many times the fact that I_α is a positive bilinear operator over the space of measures.

Lemma 3.6. Let E be a nearly spherical set with $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$. Then, for every $\alpha \in (0, 2)$ and $\varepsilon > 0$ there holds:

$$\mathcal{I}_\alpha(B_1) - \mathcal{I}_\alpha(E) \lesssim_\varepsilon [\varphi]^2_{H^{\frac{\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)} + [\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)}. \tag{3.18}$$

As a consequence we have:

$$\mathcal{I}_\alpha(B_1) - \mathcal{I}_\alpha(E) \lesssim P(E) - P(B_1) \tag{3.19}$$

Proof. We denote u_{B_1} the potential associated to μ_{B_1}. Since I_α is a bilinear operator over the space of measures, we can write $I_\alpha(g) = I_\alpha(g - \mu_{B_1}) + 2I_\alpha(g - \mu_{B_1}, \mu_{B_1}) + I_\alpha(\mu_{B_1})$. Therefore:

$$\begin{aligned}\mathcal{I}_\alpha(B_1) - \mathcal{I}_\alpha(E) &= I_\alpha(\mu_{B_1}) - \mathcal{I}_\alpha(E) = I_\alpha(\mu_{B_1}) - I_\alpha(g) - I_\alpha(g) - \mathcal{I}_\alpha(E) \\ &= -I_\alpha(g - \mu_{B_1}) - 2I_\alpha(g - \mu_{B_1}, \mu_{B_1}) + I_\alpha(g) - \mathcal{I}_\alpha(E)\end{aligned}$$

Since μ_{B_1} is optimal for $\mathcal{I}_\alpha(B_1)$, by Proposition 1.12 its potential u_{B_1} is constant over B_1. Thus, using the fact that $\int_{B_1} \mu_{B_1} = \int_{B_1} g = 1$, we get:

$$I_\alpha(g-\mu_{B_1},\mu_{B_1}) = \int_{B_1} u_{B_1}(g-\mu_{B_1}) = u_{B_1}(0)\int_{B_1}(g-\mu_B) = 0,$$

hence:

$$\begin{aligned}\mathcal{I}_\alpha(B)-\mathcal{I}_\alpha(E)+I_\alpha(g-\mu_{B_1}) &= I_\alpha(g)-\mathcal{I}_\alpha(E)\\ &= I_\alpha(g)-I_\alpha\left((1-\frac{\widehat{\alpha}}{2}\varphi)g\right)+I_\alpha\left((1-\frac{\widehat{\alpha}}{2}\varphi)g\right)-\mathcal{I}_\alpha(E).\end{aligned}$$

By bilinearity, the first two terms write as $-\frac{\widehat{\alpha}^2}{2}I_\alpha(\varphi g)+\widehat{\alpha}I_\alpha(g,\varphi g)\leq \widehat{\alpha}I_\alpha(g,\varphi g)$, whereas the other two can be estimated by 3.10. Putting everything together we find:

$$\mathcal{I}_\alpha(B_1)-\mathcal{I}_\alpha(E)+I_\alpha(g-\mu_{B_1}) \lesssim_\varepsilon I_\alpha(g,\varphi g)+[\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)}+[\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)}.$$

We further decompose the term $I_\alpha(g,\varphi g)$ as follows:

$$\begin{aligned}I_\alpha(g,\varphi g) &= I_\alpha(\mu_{B_1},\varphi g)+I_\alpha(g-\mu_{B_1},\varphi g)\\ &= I_\alpha(\mu_{B_1},\varphi\mu_{B_1})+I_\alpha(\mu_{B_1},\varphi(g-\mu_{B_1}))+I_\alpha(g-\mu_{B_1},\varphi g)\end{aligned}.$$

Since μ_{B_1} and φ is 0-homogeneous, we can argue as in 3.12 and in 3.13 (with φ_x instead of φ_x^2) and we infer:

$$I_\alpha(\mu_{B_1},\varphi\mu_{B_1}) = C\int_{\partial B_1}\varphi \lesssim \int_{\partial B_1}\varphi^2 \lesssim [\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)}.$$

In the first inequality we used 3.2 and in the second one estimate 3.3 as usual. Therefore, we currently have:

$$\mathcal{I}_\alpha(B_1)-\mathcal{I}_\alpha(E)+I_\alpha(g-\mu_{B_1}) \lesssim I_\alpha(\mu_{B_1},\varphi(g-\mu_{B_1}))+I_\alpha(g-\mu_{B_1},\varphi g)+[\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)}+[\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)} \tag{3.20}$$

We start by estimating $I_\alpha\left(g-\mu_{B_1},\varphi g\right)$. By Cauchy-Schwarz Inequality in L^2 first and then again by 3.12 and 3.13, we get:

$$I_\alpha(\varphi g)\leq \left(\int_{B_1\times B_1}\frac{\varphi_x^2 g_x g_y}{|x-y|^{N-\alpha}}\right)^{\frac{1}{2}}\left(\int_{B_1\times B_1}\frac{\varphi_y^2 g_x g_y}{|x-y|^{N-\alpha}}\right)^{\frac{1}{2}} \lesssim \int_{\partial B_1}\varphi^2 \lesssim [\varphi]^2_{H^{\frac{\alpha}{2}}(\partial B_1)}.$$

Notice that this time we used $[\varphi]^2_{H^{\frac{\alpha}{2}}(\partial B_1)}$ instead of $[\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)}$. Thus, by Cauchy-Schwarz inequality for I_α we find:

$$I_\alpha\left(g-\mu_{B_1},\varphi g\right)\leq I_\alpha^{\frac{1}{2}}\left(g-\mu_{B_1}\right)I_\alpha^{\frac{1}{2}}(\varphi g)\lesssim I_\alpha^{\frac{1}{2}}\left(g-\mu_{B_1}\right)[\varphi]_{H^{\frac{\alpha}{2}}(\partial B_1)}. \tag{3.21}$$

We now turn to $I_\alpha\left(\mu_{B_1},\varphi\left(g-\mu_{B_1}\right)\right)$, using that u_{B_1} is constant on B_1 to write:

$$\begin{aligned}I_\alpha\left(\mu_B,\varphi\left(g-\mu_B\right)\right) &= \int_{B_1}\int_{B_1}\frac{d\mu_{B_1}(y)}{|x-y|^{N-\alpha}}\varphi\left(g-\mu_{B_1}\right)dx = \int_{B_1}u_{B_1}(y)\,\varphi\left(g-\mu_{B_1}\right)dx\\ &= u_{B_1}(0)\int_{B_1}\varphi\left(g-\mu_{B_1}\right).\end{aligned}$$

Let ρ be a smooth, positive cut-off function with $\rho = 1$ on B_1 and $\rho = 0$ on B_2^c. We set $\Phi = \varphi\rho$ so that, by Cauchy-Schwarz and relation 1.10 we infer:

$$\int_{B_1} \varphi\,(g-\mu_{B_1}) = \int_{\mathbb{R}^N} \Phi\,(g-\mu_{B_1}) \le [\Phi]_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)}\,[g-\mu_{B_1}]_{H^{-\frac{\alpha}{2}}(\mathbb{R}^N)}$$
$$\lesssim [\Phi]_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)} I_\alpha^{\frac{1}{2}}\,(g-\mu_{B_1})\,.$$

Thus, in order to recover an estimate similar to 3.21, we need to show that:

$$[\Phi]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)} \lesssim [\varphi]^2_{H^{\frac{\alpha}{2}}(\partial B_1)} + \int_{\partial B_1} \varphi^2 \lesssim [\varphi]^2_{H^{\frac{\alpha}{2}}(\partial B_1)} \tag{3.22}$$

The second inequality is trivial by now, so we just have to prove the first one. By Young Inequality and the facts that ρ is bounded and Lipschitz, for every $x, y \in \mathbb{R}^N$ we have:

$$(\Phi_x - \Phi_y)^2 = (\varphi_x\rho_x - \varphi_y\rho_y)^2 = ((\varphi_x-\varphi_y)\rho_x - \varphi_y(\rho_x-\rho_y))^2 \lesssim (\varphi_x-\varphi_y)^2\rho_x^2 + \varphi_y^2\,(\rho_x-\rho_y)^2$$
$$\lesssim (\varphi_x-\varphi_y)^2 + \varphi_y^2(x-y)^2.$$

In this way, we get from 1.5:

$$[\Phi]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)} = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(\Phi_x-\Phi_y)^2}{|x-y|^{N+\alpha}} = \int_{B_3\times B_3} \frac{(\Phi_x-\Phi_y)^2}{|x-y|^{N+\alpha}} + 2\int_{B_3\times B_3^c} \frac{\Phi_x^2}{|x-y|^{N+\alpha}}$$

As usual, we deal with the two terms at the right-hand side separately and we start with the second one. Since $\Phi_x = 0$ over $B_3 \setminus B_2$ and $\Phi^2(\sigma) = \varphi^2(\sigma)$ for all $\sigma \in \partial B_1$, we get by Fubini:

$$\int_{B_3\times B_3^c} \frac{\Phi_x^2}{|x-y|^{N+\alpha}} dx\,dy = \int_{B_2} \Phi_x^2 \int_{B_2^c} \frac{1}{|x-y|^{N+\alpha}} dy\,dx \le \int_{B_2} \Phi_x^2 \int_{B_1^c} \frac{1}{|z|^{N+\alpha}} dz\,dx \lesssim \int_{B_2} \Phi_x^2$$
$$= \int_{\partial B_1} \Phi^2(\sigma) \int_0^2 r^{N-1} dr\,d\sigma \lesssim \int_{\partial B_1} \varphi^2,$$

so we are done. Instead, for the first term we apply the estimate on $(\Phi_x - \Phi_y)^2$ and we find:

$$\int_{B_2\times B_2} \frac{(\Phi_x-\Phi_y)^2}{|x-y|^{N+\alpha}} \lesssim \int_{B_2\times B_2} \frac{(\varphi_x-\varphi_y)^2}{|x-y|^{N+\alpha}} + \int_{B_2\times B_2} \frac{\varphi_y^2}{|x-y|^{N+\alpha-2}}.$$

Once again, we consider separately the two terms. Concerning the second one:

$$\int_{B_2\times B_2} \varphi_y^2 \frac{dx\,dy}{|x-y|^{N+\alpha-2}} = \int_{B_2} \varphi_y^2 \int_{B_2} \frac{dx}{|x-y|^{N+\alpha-2}}\,dy \le \int_{B_2} \varphi_y^2 \int_{B_4} \frac{1}{|z|^{N+\alpha-2}} dz\,dy$$
$$\lesssim \int_{B_2} \varphi_y^2 \lesssim \int_{\partial B_1} \varphi^2$$

It remains to estimate the first term. First, we switch to polar coordinates:

$$\int_{B_2\times B_2} \frac{(\varphi_x-\varphi_y)^2}{|x-y|^{N+\alpha}} = \int_{\partial B_1\times\partial B_1} (\varphi(\sigma)-\varphi(v))^2 \left[\int_0^2\int_0^2 r^{N-1}s^{N-1} \frac{dr\,ds}{(|r-s|^2+rs|\sigma-v|^2)^{\frac{N+\alpha}{2}}}\right] d\sigma dv.$$

Arguing in a similar way as we did for 3.15 in Lemma 3.5, we get:

$$\int_0^2\int_0^2 r^{N-1}s^{N-1} \frac{dr\,ds}{(|r-s|^2+rs|\sigma-v|^2)^{\frac{N+\alpha}{2}}} \le \frac{1}{|\sigma-v|^{N-1+\alpha}}$$

so we can recover the $H^{\frac{\alpha}{2}}(\partial B_1)$ norm of φ concluding the proof of 3.22.

Therefore, coming back to what we were doing, we find:

$$I_\alpha\left(\mu_{B_1}, \varphi\left(g-\mu_{B_1}\right)\right) \lesssim I_\alpha^{\frac{1}{2}}\left(g-\mu_{B_1}\right)[\varphi]_{H^{\frac{\alpha}{2}}(\partial B_1)}. \tag{3.23}$$

Plugging 3.21 and 3.23 in 3.20, we deduce:

$$\mathcal{I}_\alpha(B_1)-\mathcal{I}_\alpha(E)+I_\alpha(g-\mu_{B_1}) \lesssim I_\alpha^{\frac{1}{2}}\left(g-\mu_{B_1}\right)[\varphi]_{H^{\frac{\alpha}{2}}(\partial B_1)}+[\varphi]^2_{H^{\frac{2-\alpha}{2}}(\partial B_1)}+[\varphi]^2_{H^{\frac{1}{2}+\varepsilon}(\partial B_1)}$$

Applying the general version of Young Inequality $ab \leq \frac{a^2}{2\delta}+\frac{\delta b^2}{2}$ with the correct δ, the third addend of the left-hand side is erased and we conclude the proof of formula 3.18.

It remains to show 3.19. Since $\alpha \in (0,2)$ and ε is small, we have that $\frac{\alpha}{2}$, $\frac{2-\alpha}{\alpha}$ and $\frac{1}{2}+\varepsilon$ are all strictly less than 1. Therefore, by Proposition 1.6 we can estimate:

$$\mathcal{I}_\alpha(B_1)-\mathcal{I}_\alpha(E) \lesssim \int_{\partial B_1}|\nabla\varphi|^2.$$

Hence, as $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$, we appeal to the theory developed by Fuglede in [9] and we get:

$$\int_{\partial B_1}|\nabla\varphi|^2 \lesssim P(E)-P(B_1),$$

finishing the proof of 3.19. Notice that this is the only place where we use that the barycenter of E is in $x=0$. □

Now, we have at our disposal all the necessary tools to state and prove minimality of the ball B_1 among the class of nearly spherical sets.

Theorem 3.7. Let $N \geq 2$ and $\alpha \in (0,2)$. There exists a charge $\overline{Q}=\overline{Q}(N,\alpha,\gamma)>0$ and a parameter $\varepsilon=\varepsilon(N,\alpha,\gamma)>0$ such that, for every nearly spherical set E with $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \leq \varepsilon$ and every $Q \leq \overline{Q}$, we have:

$$\mathcal{F}_{\alpha,Q}(B_1) \leq \mathcal{F}_{\alpha,Q}(E).$$

Moreover, equality is attained only if $E=B_1$.

Proof. Let E be a nearly spherical set with $\|\varphi\|_{W^{1,\infty}(\partial B_1)} \ll 1$. Assuming that $\mathcal{F}_{\alpha,Q}(E) \leq \mathcal{F}_{\alpha,Q}(B_1)$, rearranging the terms of the functional and applying 3.19 we find:

$$P(E)-P(B_1) \leq Q^2(\mathcal{I}_\alpha(B_1)-\mathcal{I}_\alpha(E)) \lesssim Q^2(P(E)-P(B_1))$$

This implies that either $Q^2 \gtrsim 1$ or $P(E)-P(B_1) \leq 0$. In the first case we reach a contradiction with the fact that $Q^2 \ll 1$. Thus, since $P(E)-P(B_1) \geq 0$ for every E nearly spherical, we must necessarily have $P(E)=P(B_1)$. Thanks to Isoperimetric Inequality, we finally conclude $E=B_1$, proving minimality of the ball for the functional $\mathcal{F}_{\alpha,Q}$ in the class of nearly spherical sets sufficiently close to the ball itself in the $C^{1,\gamma}$ topology. □

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