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Tesi di Laurea

Invarianza per Inversione Temporale delle Interazioni Forti e Materia Oscura Assionica

Time Reversal Invariance of Strong Interactions and Axion Dark Matter

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Riassunto

L'evidenza di materia oscura non lascia spazio a dubbi riguardo la necessità di fisica oltre il Modello Standard. Un candidato fortemente motivato dalla fisica delle particelle è un nuovo ipotetico grado di libertà leggero denominato assione. Lo scopo di questa tesi è quello di analizzare come si comportano le interazioni forti, descritte dalla Cromodinamica Quantistica (QCD), una volta che viene invertita la freccia del tempo. L'invarianza sotto questa operazione è assicurata solo a patto di porre un parametro della QCD ad un valore innaturalmente piccolo. L'ipotesi dell'assione risolve questo problema. Nella seconda parte della tesi, si andrà a studiare come funziona questo meccanismo e come l'assione possa spiegare l'abbondanza osservata di materia oscura.

Introduction

The Standard Model (SM) of particle physics is a quantum field theory developed during the second half of the 20th century unifying our present understanding of 3 (strong, weak and electromagnetic interactions) out of the 4 known fundamental forces (the previous 3 interactions and gravity). In the proceeding we will discuss how adding a new global symmetry to the Standard Model could at once solve the strong CP problem, see later, and provide a viable dark matter candidate, the *axion.*
The SM strongly relies on the principle of gauge symmetry. Its Lagrangian is given writing the most general $SU(3) \times SU(2) \times U(1)$ gauge invariant (renormalizable) Lagrangian. Therefore we have a total amount of 12 gauge bosons: 8 generators of the $SU(3)$ group describing strong interactions, and $3+1$ gauge bosons relative to the electroweak sector. This gauge fields couple to different kinds of matter that can be divided into three main categories:

- **Quarks**: quarks are fermions charged under all factors of the SM gauge group (to be precise only left-handed quarks are charged under all interactions while right-handed ones do not interact through the weak charged currents). Quarks are the only particles to be charged under the $SU(3)$ strong interactions (they are said to carry color charge). There are 6 quark flavours (and 6 corresponding anti-quark flavours) divided into 3 generations of doublets of the $SU(2)$ weak factor.
- **Leptons**: leptons are also fermions, but they do not interact through strong interactions (so they do not present color charge). As for quarks, there are 6 different flavours of leptons again divided into 3 generations of doublets.
- **Higgs**: the Higgs field is a scalar which is not charged under the strong interactions. This boson has the role to give mass to both weak bosons (by means of spontaneous symmetry breaking) and fermions. Its goldstone bosons are "eaten" from the gauge fields acquiring mass and leaving just one real scalar field h known as the Higgs boson.

The Standard Model has an outstanding accord with experimental testing, with tensions only where really complex measurements or calculations have to be carried out. Between its major successes, we have:

- The prediction of the existence of the gluon and its properties in 1973 (by Fritzsch, Leutwyler and Gell-Mann), six years before its experimental detection at PETRA (1979).
- The prediction of the existence and properties of weak $W^{\pm}$ and Z bosons by the Glashow-Salam-Weinberg electroweak model (around 1968) and then discovered at CERN in 1983.
- The prediction of a third generation of quarks by Kobayashi and Maskawa in 1973, to explain CP violation in the weak interactions. The bottom quark was detected at Fermilab in 1977 for the first time but the top quark had to wait until 1995.
- The last piece of the Standard Model to be detected was the Higgs boson (incorporated in the standard model in 1967) which was found at CERN in 2012.

Besides all the questions the SM answers successfully, it also leaves other questions about its structure open, such as:

Can we unify the three interactions of the SM?
As we have summarized the SM contains three gauge symmetries and 15 different fermionic representations. In the final theory, we would like to have just one gauge group that descends to the three groups of the SM through spontaneous symmetry breaking and just one type of matter. This desire is further motivated by the fact that the electroweak sector of the gauge group has already been unified. This idea that there exists a fundamental gauge group goes under the name of *grand unification.* Although there is an attractive approach to pursue grand unification that is to regard $SU(3) \times SU(2) \times U(1)$ as a subgroup of $SO(10)$ or $SU(5)$, to this day great unification theories (GUT) still present problems like predicting the proton to decay, which has not yet been observed, and wrong couplings.

Why does the quark and lepton spectrum have this structure?
In the Standard Model, the masses of the quarks range from about a MeV for the quark up to about 100GeV for the quark top. The model explains how fermions gain mass through their interaction with the Higgs field but they accomplish this by means of a series of arbitrary couplings that have to be entered as parameters. Therefore we do not have any insight into why quark or lepton masses vary over such a large spectrum (5 orders of magnitude for the quarks and 3 for the leptons).

Why do some parameters of the SM assume extremely small values?
We would expect all the parameters of the standard model to assume values of order one (in natural units), but some of them assume some extremely small/big values, this is referred to as *naturalness problem*. The most important naturalness problems of the SM are the Strong CP problem (which we will cover later) and the SM hierarchy problem (regarding why the weak interactions are 10^{34} times stronger than gravitational ones).

There is then a series of questions left unanswered about how the knowledge of particle physics we have acquired relates to the picture of the universe emerging from cosmology:

Why is the universe full of structure and not homogeneous?
If the universe started out in a homogeneous thermal equilibrium state it would stay uniform during its evolution and we will not see all the structures we see today: stars, galaxies, galaxy clusters... In order for these inhomogeneities to be present we must provide a mechanism for them to form and grow. Clearly gravity could be a mean for structure to grow: slightly denser regions of the universe could attract other matter and become even denser.
An hypothesis proposed by Alan Guth in 1981 to solve the problem of inhomogeneities in the early universe and other related problems (such as why the Cosmic Microwave Background is so homogeneous despite coming from very far regions of the universe that could not have been able to reach thermal equilibrium) is given by *inflation* (basically an exponential expansion of the very early universe), but it would require at least to add new features to the Standard Model.

Why does the universe contain more matter than antimatter?
Why do we live in a world made mainly out of matter and anti-matter is so rare? It appears that when the temperature of the universe fell behind 1 GeV quark and anti-quark annihilated and it remained a small excess of quarks (it would be sufficient a leftover quark every 10^9 photons). We could assume that the universe started with this little asymmetry, but if we assume inflation to have happened it would have made the universe expand by roughly 10^{43} times diluting this asymmetry to become unnoticeable. The Standard Model provides a way to create a potential asymmetry in matter and anti-matter violating CP symmetry but it would produce an asymmetry 10^8 times smaller than observed.

What is Dark Matter?
There is a bunch of evidence for the so called *dark matter*: invisible weakly interacting matter that interacts mainly gravitationally with ordinary matter. This matter makes up 85% of the total mass of the universe and no particle in the SM constitutes a convincing dark matter candidate.

What is Dark Energy?
In 1988 it was discovered that the universe is not only expanding but that this expansion is accelerated. This has required adding to the Standard Model of cosmology a new term called the *cosmological constant*. This term behaves like a vacuum energy, however it is not known how to compute the SM contribution to the vacuum energy and rough estimates give a result 120 orders of magnitude too large.

How does gravity fit together with the Standard Model?
The SM does not contain the fourth fundamental interaction of nature: gravity. To this day it is not known any completely successful way to incorporate gravity to the other three interactions.

Chapter 1

Gauge theory

In this chapter, we will introduce the concept of gauge theory searching for a generalization of classical electrodynamics.

1.1 Gauge theory Lagrangian

Studying electrodynamics we immediately encounter the concept of gauge symmetry: a local continuous transformation of the field that leaves the classical action invariant.

Inspired by this concept we will choose a Lie group and write the most general Lagrangian that is left invariant under local transformations of the fields belonging to this group. The resulting theory is called a gauge theory and its Lie group is called gauge group.

We want to construct a theory of interacting fermions, exactly like electrodynamics. We start by taking the Dirac Lagrangian, which describes the dynamics of a spin-$\frac{1}{2}$ free field.

$$\mathcal{L} = \bar{\boldsymbol{\psi}}(x)(i\gamma^\mu\partial_\mu - m)\boldsymbol{\psi}(x) \, , \tag{1.1}$$

where $\bar{\boldsymbol{\psi}} = \boldsymbol{\psi}^\dagger\gamma^0$ and γ^μ are the Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.
This Lagrangian is clearly invariant under a gauge transformation that does not depend on x (a so called global gauge transformation) because for Wigner theorem every symmetry of a quantum theory has a unitary representation and ψ transforms under this representation of the gauge group as

$$\boldsymbol{\psi} \to U\boldsymbol{\psi} \, , \qquad \bar{\boldsymbol{\psi}} \to \bar{\boldsymbol{\psi}}U^\dagger \, . \tag{1.2}$$

Both the derivative and the γ matrices do not act on U and therefore the Lagrangian density does not change under these transformations. But if we allow the transformation U to depend on the 4-space coordinates $U(x) = e^{i\theta^a(x)R^a}$ (where R^a are the generators of the unitary representation of the Lie group) the derivative will now act on this transformation and the Lagrangian density will be no longer invariant.
We want to modify the derivative operator to obtain a sort of generalized derivative D_μ that is covariant also under local gauge transformations (we will call this operator the covariant derivative), so we want the gauge transformation to send

$$D_\mu\boldsymbol{\psi} \to UD_\mu\boldsymbol{\psi} \, . \tag{1.3}$$

We can modify the derivative in various ways introducing different connections. In the proceeding we will use a spin-1 connection (as we know classical electrodynamics must be a spin-1 theory). We introduce a gauge field potential $A_\mu(x) = A^a_\mu(x)R^a$ (with a potential for every generator of the Lie group) and we let:

$$D_\mu = \partial_\mu - igA_\mu \, . \tag{1.4}$$

Imposing equation (1.3) we get:

$$U(\partial_\mu - igA_\mu)\boldsymbol{\psi} = (\partial_\mu - igA'_\mu)U\boldsymbol{\psi}$$
$$U\partial_\mu\boldsymbol{\psi} - igUA_\mu\boldsymbol{\psi} = U\partial_\mu\boldsymbol{\psi} + (\partial_\mu U)\boldsymbol{\psi} - igA'_\mu U\boldsymbol{\psi}$$

$$A'_\mu = UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \, . \tag{1.5}$$

We have obtained how the potential A_μ transformations under gauge transformation to make the covariant derivative indeed covariant. We can now easily write a Lagrangian density that is invariant under gauge transformation:

$$\mathcal{L} = \bar{\boldsymbol{\psi}}(i\gamma^\mu D_\mu - m)\boldsymbol{\psi} \, .$$

This Lagrangian naturally presents a coupling between the gauge field A_μ and the fermionic field (which is no longer free). This is already an impressive result! If for example we take $U(1)$ as gauge group, to have a theory with just one potential, we can see (calling $j^\mu = g\boldsymbol{\psi}^\dagger\gamma^0\gamma^\mu\boldsymbol{\psi}$) that we have recovered exactly the radiation-matter interaction term of electrodynamics ($\mathcal{L}_{int} = A_\mu j^\mu$)!
We have basically obtained back a Lagrangian that looks just like a quantum-mechanical generalization of the classical electrodynamics Lagrangian and all we postulated was a local symmetry and the covariant derivative to have a relatively simple form.
The last thing to do (just like we did in classical electrodynamics) is to add to the Lagrangian a kinetic term to give each potential A^a_μ a dynamics of its own. This term has to be gauge invariant, a function of the potential A_μ only and obviously covariant under Lorentz transformations.
Recalling the transformation law of D_μ from equations (1.3) and (1.2), we can now construct a manifestly covariant object

$$[D_\mu, D_\nu] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]) = -igG_{\mu\nu}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \, . \tag{1.6}$$

The tensor $G_{\mu\nu}$ obviously transforms covariantly under gauge transformation and is called *field strength*. A standard way to obtain an invariant object from a covariant one is to take its trace (thanks to its cyclical properties). Therefore

$$\mathcal{L}_{field} = -\frac{1}{2}\operatorname{tr}(G_{\mu\nu}G^{\mu\nu})$$

is invariant under gauge transformation and covariant under Lorentz transformation. Note that if the gauge group is $U(1)$ we recover exactly the kinetic term of classical electrodynamics. Finally we have the complete gauge theory Lagrangian for a Dirac fermion field.

$$\mathcal{L} = \bar{\boldsymbol{\psi}}(i\gamma_\mu D_\mu - m)\boldsymbol{\psi} - \frac{1}{2}\operatorname{tr}(G_{\mu\nu}G^{\mu\nu}) \, . \tag{1.7}$$

Besides the Dirac fermion we can see in this Lagrangian some spin-1 bosonic fields corresponding to the different gauge bosons.

1.2 Discrete symmetries

The gauge theory Lagrangian we have just written has three important discrete symmetries: parity, charge conjugation and time reversal. To show these results we will need some preliminary facts.

The Dirac field can be written, choosing the chiral representation for the γ matrices (see [3] p.52-63), as a sum of plane waves

$$\boldsymbol{\psi} = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1,2} (a_{\vec{p},s}u^s(p)e^{-ip\cdot x} + b^\dagger_{\vec{p},s}v^s(p)e^{ip\cdot x}) \, , \tag{1.8}$$

where

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\zeta^s \\ \sqrt{p_\mu \tilde{\sigma}^\mu}\zeta^s \end{pmatrix} , \qquad v^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\eta^s \\ -\sqrt{p_\mu \tilde{\sigma}^\mu}\eta^s \end{pmatrix} , \qquad \zeta^1 = \eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \zeta^2 = \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.9}$$

and $a_{\vec{p},s}$ and $b^\dagger_{\vec{p},s}$ are respectively the annihilation operator for electrons and the creation operator for positrons. These operators (as electrons and positrons are fermions) must obey the anti-commutation relations $\{a_{\vec{p},s}, a^\dagger_{\vec{q},r}\} = \{b_{\vec{p},s}, b^\dagger_{\vec{q},r}\} = (2\pi)^3\delta^{(3)}(p-q)\delta_{rs}$. Furthermore:

$$\sqrt{2E_p} a^\dagger_{\vec{p},s} |0\rangle = |\vec{p}, s, e^-\rangle , \qquad \sqrt{2E_p} b^\dagger_{\vec{p},s} |0\rangle = |\vec{p}, s, e^+\rangle ,$$

$$\{\boldsymbol{\psi}_a(\vec{x}, t), \boldsymbol{\psi}^\dagger_b(\vec{y}, t)\} = \delta^{(3)}(\vec{x} - \vec{y})\delta_{a,b} , \tag{1.10}$$

where $|\vec{p}, s, n\rangle$ denotes the state of a free particle with momentum $\vec{p}$, spin s and eventually other quantum numbers n.

Parity

Parity is a transformation that acts on the space components of 4-vectors sending

$$x^\mu = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix} \xrightarrow{\pi} x'^\mu = \begin{pmatrix} x^0 \\ -\vec{x} \end{pmatrix} .$$

Clearly as $|\vec{p}, s\rangle \xrightarrow{\pi} P|\vec{p}, s\rangle = \eta |-\vec{p}, s\rangle$ with η arbitrary phase, we must have (assuming the vacuum state is parity invariant and non-degenerate)

$$P a_{\vec{p},s} P = \eta_a a_{-\vec{p},s} , \qquad P b_{\vec{p},s} P = \eta_b b_{-\vec{p},s} .$$

Noting that $u(p) = \gamma^0 u(p')$ and $v(p) = -\gamma^0 v(p')$ and changing the integration variable of equation (1.8) to $-\vec{p}$, we see that if $\eta_a = -\eta_b^*$ we get:

$$\boldsymbol{\psi}(x) \xrightarrow{\pi} \boldsymbol{\psi}'(x') = P\boldsymbol{\psi}(x)P = \eta_a \gamma_0 \boldsymbol{\psi}(x') .$$

Then we can compute how fermion bilinears transform under parity

$$P\bar{\boldsymbol{\psi}}\boldsymbol{\psi}P = |\eta_a|^2 \bar{\boldsymbol{\psi}}\gamma^0\gamma^0\boldsymbol{\psi} = \bar{\boldsymbol{\psi}}\boldsymbol{\psi} , \qquad P\bar{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi}P = |\eta_a|^2 \bar{\boldsymbol{\psi}}\gamma^0\gamma^\mu\gamma^0\boldsymbol{\psi} = \eta(\mu)\bar{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi} ,$$

$$\eta(\mu) = \begin{cases} 1 \text{ if } \mu = 0 \\ -1 \text{ if } \mu = 1, 2, 3 \end{cases} .$$

Where in the last line we used the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Using that A_μ and ∂_μ transform as vectors it is clear that all the fermionic part of (1.7) is invariant under parity. To extend parity invariance to the full (1.7) we must look to how $G_{\mu\nu}$ transforms under parity. Using again the transformation properties of A_μ and ∂_μ

$$G_{0i} \xrightarrow{\pi} -G_{0i} , \qquad G_{ij} \xrightarrow{\pi} G_{ij} ,$$

from which its invariance follows immediately.

Charge conjugation

Charge conjugation symmetry does not act on space-time, instead it maps a physical system into another physical system in which each particle is replaced by its antiparticle. This implies

$$C a_{\vec{p},s} C = \eta_C b_{\vec{p},s} , \qquad C b_{\vec{p},s} C = \eta_C a_{\vec{p},s} .$$

Now, using $u^s(p) = -i\gamma^2(v^s(p))^*$, it is easy to see that

$$C\boldsymbol{\psi}C = -i\eta_C\gamma^2\boldsymbol{\psi}^* = (-i\eta_C\boldsymbol{\psi}^\dagger\gamma^2)^{\mathrm{T}} .$$

Therefore charge conjugation acts on fermion bilinears in the following way:

$$C\bar{\boldsymbol{\psi}}\boldsymbol{\psi}C = (-i\eta_C^*\gamma^0\gamma^2\boldsymbol{\psi})^{\mathrm{T}}(-i\eta_C\bar{\boldsymbol{\psi}}\gamma^0\gamma^2)^{\mathrm{T}} = -\gamma^0_{ab}\gamma^2_{bc}\boldsymbol{\psi}_c\bar{\boldsymbol{\psi}}_d\gamma^0_{de}\gamma^2_{ea} = \bar{\boldsymbol{\psi}}_d\gamma^0_{de}\gamma^2_{da}\gamma^0_{ab}\gamma^2_{bc}\boldsymbol{\psi}_c = \bar{\boldsymbol{\psi}}\boldsymbol{\psi}\,.$$

Where we have used the fermion anti-commutation relation (1.10). In the same way, noting that in the chiral representation γ^0 and γ^2 are symmetric while γ^1 and γ^3 are antisymmetric

$$C\bar{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi}C = -\bar{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi}\,.$$

After an integration by parts (needed to have the derivative act on $\boldsymbol{\psi}$, as C exchanges $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^\dagger$) we can clearly see that the free Dirac Lagrangian is indeed invariant under charge conjugation.

We now want to examine how charge conjugation acts on the gauge fields. The interaction term of the Lagrangian can be shown, using the previous result, to transform as

$$C\bar{\boldsymbol{\psi}}\gamma^\mu A^a_\mu R^a\boldsymbol{\psi}C = (C\,A^a_\mu\,C)\,\cdot\bar{\boldsymbol{\psi}}\gamma^\mu(-R^a)^{\mathrm{T}}\boldsymbol{\psi}\,.$$

To preserve the invariance of the interaction term we have to impose:

$$C\,A^a_\mu\,C = \begin{cases} -A^a_\mu \text{ if } R^a \text{ is symmetric} \\ A^a_\mu \text{ if } R^a \text{ is antisymmetric} \end{cases}\,.$$

In the adjoint representation, where all the generators are antisymmetric, the gauge potential A^μ is left invariant under charge conjugation. This makes sense as the gauge bosons are their own antiparticle. It immediately follows that also the gauge kinetic term of the Lagrangian is left invariant by charge conjugation.

Time reversal

Time reversal symmetry is a discrete symmetry that reverses the orientation of the time axis. It is implemented both in QFT and in QM by an anti-unitary antilinear operator T.

$$x^\mu = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix} \xrightarrow{T} x'^\mu = Tx^\mu T = \begin{pmatrix} -x^0 \\ \vec{x} \end{pmatrix}\,.$$

By analogy with the classical case, we would like the time reversal operator to flip both momentum and spin of every particle. If we define flipped spinor creation and annihilation operators it can be shown (see [3] p.67-69) that

$$T\boldsymbol{\psi}(t,\vec{x})T = \gamma^1\gamma^3\boldsymbol{\psi}(-t,\vec{x})\,.$$

And we have the following actions on fermion bilinears:

$$T\bar{\boldsymbol{\psi}}(t,\vec{x})\boldsymbol{\psi}(t,\vec{x})T = -\bar{\boldsymbol{\psi}}(-t,\vec{x})\gamma^1\gamma^3\gamma^1\gamma^3\boldsymbol{\psi}(-t,\vec{x}) = \bar{\boldsymbol{\psi}}(-t,\vec{x})\boldsymbol{\psi}(-t,\vec{x})\,,$$

$$T\,i\bar{\boldsymbol{\psi}}(t,\vec{x})\gamma^\mu\partial_\mu\boldsymbol{\psi}(t,\vec{x})\,T = -i\boldsymbol{\psi}^\dagger(-t,\vec{x})\gamma^1\gamma^3(\gamma^\mu)^*\gamma^1\gamma^3\partial'_\mu\boldsymbol{\psi}(-t,\vec{x}) = i\bar{\psi}(-t,\vec{x})\gamma^\mu\partial_\mu\boldsymbol{\psi}(-t,\vec{x})\,.$$

As we did before for charge conjugation we examine how the interaction term transforms

$$T\bar{\boldsymbol{\psi}}\gamma^\mu A^a_\mu R^a\boldsymbol{\psi}T = (T\,A^a_\mu\,T)\,\cdot\bar{\boldsymbol{\psi}}\gamma^\mu(\eta(\mu)R^a)^*\boldsymbol{\psi}$$

which gives

$$T\,A^a_\mu\,T = \begin{cases} \eta(\mu)A^a_\mu \text{ if } R^a \text{ is real} \\ -\eta(\mu)A^a_\mu \text{ if } R^a \text{ is imaginary} \end{cases}\,.$$

In the adjoint representation all the generators are purely imaginary, you can see this by taking the adjoint of the commutation relation and using that the generators are hermitian for Wigner theorem. Then under time reversal A^μ transforms as a vector. Recalling that ∂_μ also transforms as a vector we immediately have

$$G_{\mu\nu} \xrightarrow{T} \eta(\mu)\eta(\nu)G_{\mu\nu}\,.$$

From this follows the invariance of gauge theory Lagrangian under time inversion.

Chapter 2

QCD and the strong CP problem

Quantum Chromodynamics (QCD) is a spin-1 gauge theory with gauge group $SU(3)$, the Lie group of unitary 3×3 matrices with determinant one. The $SU(3)$ Lie algebra has 8 generators, corresponding to 8 different gauge particles called gluons.

Chosen a representation of $SU(3)$ equation (1.7) gives immediately the Lagrangian of 1-quark QCD. To add new fermions coupled to the gauge field we just need to add to the Lagrangian another Dirac Lagrangian with the derivative substituted by the covariant derivative. The most general expression you can write taking into account weak interactions contains a complex mass matrix, but we will consider just low energy QCD in which we have only the up, down and strange quarks and a real mass matrix. The QCD Lagrangian takes the form

$$\mathcal{L} = \sum_j \bar{\boldsymbol{\psi}}_{\boldsymbol{j}}(i\gamma^\mu D_\mu - m_j)\boldsymbol{\psi}_{\boldsymbol{j}} - \frac{1}{2}tr(G_{\mu\nu}G^{\mu\nu}) \, ,$$

where $j \in \{u, d, s\}$ is the flavour of the quark.
The gluon kinetic part of the Lagrangian can be written in a more explicit way because

$$\mathrm{tr}[AB] = (A, B)$$

is a scalar product on the gauge Lie algebra. Therefore we can apply a Gram-Schmidt process and obtain a set of generators with the property

$$\mathrm{tr}\Big[R^a R^b\Big] = C(R)\delta^{ab} \, .$$

Where we put R to stress the dependence of this coefficient from the representation.
We will conventionally choose the generators of the fundamental representation in such a way that

$$\mathrm{tr}\Big[R^a R^b\Big] = \frac{1}{2}\delta^{ab} \, .$$

Then the Lagrangian is

$$\mathcal{L} = \sum_j \bar{\boldsymbol{\psi}}_{\boldsymbol{j}}(i\gamma^\mu D_\mu - m_j)\boldsymbol{\psi}_{\boldsymbol{j}} - \frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} \, .$$

2.1 Color from baryon spectrum

We have just stated that QCD is a $SU(3)$ gauge theory but we did not explain why should Strong interactions be invariant under local $SU(3)$ transformations and on what space these gauge transformations act.
In this section we will just sketch how the light baryon spectrum gives a hint of the presence of 3 colors and a color exchange symmetry (for a full account of the experimental evidence that led to QCD see

[10]). Baryons are bound states of 3 quarks, as we will look only at light baryons they can contain just quarks up, down or strange (the other three quarks have a mass greater than 1 GeV). Experimentally the light baryon spectrum is divided in 8 spin-$\frac{1}{2}$ and 10 spin-$\frac{3}{2}$ particles.
To build the total baryon state we must take the tensorial product of three of the following quark states:

$$|u\uparrow\rangle\ , \qquad |d\uparrow\rangle\ , \qquad |s\uparrow\rangle\ , \qquad |u\downarrow\rangle\ , \qquad |d\downarrow\rangle\ , \qquad |s\downarrow\rangle\ .$$

If we take the possible symmetric arrangements of 3 of the 6 states listed above we get by simple combinatorics 56 possible states. But this is just the total number of light baryons accounting for possible S^3 spin numbers: $4 \cdot 10 + 2 \cdot 8 = 56$. While taking an antisymmetric combination of these states gives a lower number of possible light baryons. Now we have a problem because quarks are fermions so the quantum baryon states are required to be totally antisymmetric, Han and Nambu were the first to suggest in 1965 that this problem could be solved by adding a new quantum number to quarks: *color.* We can see that in order for states like $|s\uparrow \quad s\uparrow \quad s\uparrow\rangle$ to be antisymmetric we need color quantum number to assume at least 3 different values. Furthermore if there were more than 3 possible colors or QCD is not invariant under color exchange we would have differently colored particles which correspond to the same observed particle. For these reasons quarks should possess an additional quantum number (which can assume 3 values: the three colors red, green and blue) and an $SU(3)$ symmetry on color space that swaps colors.

2.2 The θ term

There is an additional term we can add to the QCD Lagrangian (2) to get the most general $SU(3)$ and Lorentz invariant Lagrangian. This additional contribution is called θ *term*:

$$\mathcal{L}_\theta = \theta \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\alpha\beta} G^a_{\mu\nu} G^a_{\alpha\beta} = \theta \frac{g^2}{32\pi^2} \tilde{G}^{a\,\mu\nu} G^a_{\mu\nu}\ . \tag{2.1}$$

At the end of this chapter we will discuss why this term has to appear in the Lagrangian with this coupling constant.

As can be easily deduced by the discussion in the previous chapter, this θ term clearly violates parity and time reversal symmetry while it is still invariant under charge conjugation. It then violates also CP symmetry (this is analogous to violating T, as CPT is known to be a symmetry of every Lorentz covariant theory in QFT).

The θ term can be shown to be the total derivative of K^μ:

$$K^\mu = \epsilon^{\mu\alpha\beta\gamma} A^a_\alpha \left(G^a_{\beta\gamma} - \frac{g}{3} f^{abc} A^b_\beta A^c_\gamma \right)\ . \tag{2.2}$$

In fact

$$\begin{aligned}
\partial_\mu K^\mu &= \partial_\mu \epsilon^{\mu\alpha\beta\gamma} A^a_\alpha \left(G^a_{\beta\gamma} - \frac{g}{3} f^{abc} A^b_\beta A^c_\gamma \right) = \\
&= \epsilon^{\mu\alpha\beta\gamma} \left[(\partial_\mu A^a_\alpha)(\partial_\beta A^a_\gamma - \partial_\gamma A^a_\beta) + \frac{2g}{3} f^{abc} \left((\partial_\mu A^a_\alpha) A^b_\beta A^c_\gamma + A^a_\alpha (\partial_\mu A^b_\beta) A^c_\gamma + A^a_\alpha A^b_\beta (\partial_\mu A^c_\gamma) \right) \right] = \\
&= \frac{\epsilon^{\mu\alpha\beta\gamma}}{2} (\partial_\mu A^a_\alpha - \partial_\alpha A^a_\mu) \left[G^a_{\beta\gamma} + g f^{abc} A^b_\beta A^c_\gamma \right] = \frac{\epsilon^{\mu\alpha\beta\gamma}}{2} \left(G^a_{\mu\alpha} - g f^{abc} A^b_\mu A^c_\alpha \right) \left(G^a_{\beta\gamma} + g f^{ade} A^d_\beta A^e_\gamma \right) = \\
&= \tilde{G}^{a\,\mu\nu} G^a_{\mu\nu} - \frac{\epsilon^{\mu\alpha\beta\gamma}}{2} g^2 f^{abc} f^{ade} A^b_\mu A^c_\alpha A^d_\beta A^e_\gamma = \tilde{G}^{a\,\mu\nu} G^a_{\mu\nu}\ .
\end{aligned}$$

In the last line we have used that, thanks to the Jacobi identity,

$$\epsilon^{\mu\alpha\beta\gamma} f^{abc} f^{ade} A^b_\mu A^c_\alpha A^d_\beta A^e_\gamma = -\epsilon^{\mu\alpha\beta\gamma} (f^{adb} f^{ace} + f^{acd} f^{abe}) A^b_\mu A^c_\alpha A^d_\beta A^e_\gamma = -2\epsilon^{\mu\alpha\beta\gamma} f^{abc} f^{ade} A^b_\mu A^c_\alpha A^d_\beta A^e_\gamma = 0.$$

Therefore it can seem that the θ *term*, being a 4-divergence, can be integrated out and does not give any contribution to the action. This is not true due to the peculiar vacuum structure of QCD.

2.3 QCD vacuum structure

We want to focus on the vacuum states of QCD, i.e. the field configurations in which we have $G_{\mu\nu} = 0$. The requirement to impose to obtain such a condition is not $A_\mu = 0$ but rather A_μ to be a pure gauge field (i.e. to be obtained from $A_\mu = 0$ through a gauge transformation), from equation (1.5) we have

$$A_\mu = -\frac{i}{g}(\partial_\mu U(x))U^\dagger(x)\ .$$

Two vacuum states are equivalent if the gauge transformation going from one state to the other is connected to the identity, as in this case we can effectively go from one field configuration to the other without undergoing any change in the system observables. This equivalence relation partitions the gauge group into equivalence classes. We want to emphasize that all equivalence classes correspond indeed to the same physical state (as this is what we required constructing the theory from gauge symmetry) but a nontrivial process is needed to go from one vacuum class to the other. The problem is similar to a particle moving on a circle on which is defined a potential with only a minimum: we expect this kind of problem to have a tunnelling solution in which the particle returns to the same physical state after going around the circle.

Choosing the temporal gauge $A_0 = 0$ (as it can be shown that the transformation that fixes this condition is connected to the identity and hence does not change the vacuum equivalence classes) so that the residual gauge transformations are now time-independent and

$$A_j = -\frac{i}{g}(\partial_j U(\vec{x}))U^\dagger(\vec{x})\ .$$

We want to study the transition probability from two vacua belonging to different equivalence classes

$$\langle \overline{vac}|\, e^{-iHT}\, |vac\rangle\ .$$

The two vacuum states can be simultaneously gauge transformed without changing the transition amplitude, we may then choose without loss of generality $|\overline{vac}\rangle$ to be in the same equivalence class of the identity and then to be characterized by the following boundary conditions at spatial infinity

$$\vec{A}(\vec{x}) \to 0 \qquad U(\vec{x}) \to U_0\ .$$

Where U_0 is independent of the spatial direction.
From such vacuum, a transition is possible only to other states with the same type of boundary conditions in fact if

$$\vec{A}(\vec{x}) \not\to 0 \quad \text{for} \quad |\vec{x}| \to +\infty$$

we will have an infinite volume in which $F_{0i} = \dot{A}_i \neq 0$ entering the path integral and this transition must have zero amplitude.

Being pure gauge every vacuum is determined by the mapping $U(\vec{x})$ from $\mathbb{R}^3$ to the gauge group. But the 3-dimensional space has the point at infinity identified (as they correspond to the same gauge group element and can be thought of as the same point). With this identification it acquires the same topological structure of the 3-sphere $\mathcal{S}^3$ (this can be seen by means of a stereographic projection).

Choosing a gauge transformation at each point in space is then equivalent to choosing a mapping from $\mathcal{S}^3$ to the gauge group $SU(3)$. These mappings fall into different classes of equivalence (in which two mappings are identified if they can be obtained one from the other via a continuous transformation, that is precisely the same equivalence relation as before) that form the third homotopy group of $SU(3)$. Each class is identified by an integer number (the *winding number*) that expresses the number of times that the mapping *winds around* $SU(3)$. The winding number is given by:

$$n = \frac{g^2}{32\pi^2}\int d^3\vec{x}\, K^0 = \frac{ig^3}{24\pi^2}\int d^3\vec{x}\, \epsilon^{ijk}\, tr(A_iA_jA_k)\ .$$

In the last equality we exploited that A_μ is a pure gauge field ($G_{\mu\nu} = 0$) and the identity $f^{abc} = -2i\,tr([t^a, t^b]t^c)$ of the fundamental representation.
The number n can be shown to be integer and invariant under continuous deformations, but we will not cover this here (to have an idea of the proof see [2] that covers this topic for the instanton winding number, a slightly different winding number we will later mention, but the proof is exactly the same).
We have therefore infinite classes of inequivalent vacua labelled by an integer winding number n, we will call a vacuum field configuration characterised by w.n. n as $A_\mu^{(n)}$ and the corresponding vacuum state as $|n\rangle$.

A 2-dimensional analogue

The concepts introduced in the last section could be a little complicated to grasp at first, in particular the concept of a winding number of a mapping going from $\mathcal{S}^3$ to $SU(3)$. Let us concentrate on the simple example of a $U(1)$ gauge theory (QED) in 2 dimensions (a spatial and a temporal dimension). Repeating the above reasoning the 1-dimensional space is topologically equivalent to the circle $\mathcal{S}^1$, and also the gauge group is now equivalent to $\mathcal{S}^1$, therefore the previous mapping becomes a continuous mapping from a circle to a circle. It is now quite clear what the winding number represents: it is just the number of times that the mapping *goes around* the $U(1)$ circle! In such a theory the analogue of the parity breaking $\mathcal{L}_\theta$ term will be

$$\mathcal{L}_\theta^{(2)} = \frac{1}{2}\epsilon^{\mu\nu}G_{\mu\nu} = \partial_\mu(\epsilon^{\mu\nu}A_\nu) = \partial_\mu K^\mu \ ,$$

with the obvious definition of K^μ. Using again the $A_0 = 0$ gauge we have

$$n = \frac{ig}{2\pi}\int dx\, K^0 = \frac{ig}{2\pi}\int dx\, A_1 \ .$$

If now we take a representation of $U(x) = e^{i\alpha(x)}$ to exploit the equivalence of $U(1)$ to $\mathcal{S}^1$ we obtain

$$n = \frac{1}{2\pi}\int dx\, \alpha'(x) = \frac{\alpha_{+\infty} - \alpha_{-\infty}}{2\pi} \ ,$$

with the clear significance of a winding number ($\alpha_{+\infty}$ and $\alpha_{-\infty}$ must differ of a multiple of 2π for the boundary conditions on $U(x)$ to be respected).

2.4 Instanton contributions

Returning back to 4-dimensional QCD we want to compute the contribution to the action given by the *θ term* of the so called *instanton* solutions. Instanton solutions are minimum action solutions corresponding to non-perturbative tunnelling between two minima [1] such that

$$A_\mu(\vec{x}, t = +\infty) = A_\mu^{(n_+)}(\vec{x}) \ , \qquad A_\mu(\vec{x}, t = -\infty) = A_\mu^{(n_-)}(\vec{x}) \ .$$

Computing the action for an instanton configuration

$$\begin{aligned} S_\theta &= \int d^4x\, \mathcal{L}_\theta = \theta\int d^4x\, \frac{g^2}{32\pi^2}\partial_\mu K^\mu = \theta\frac{g^2}{32\pi^2}\left(\int d^3\vec{x}\int dt\, \partial_0 K^0 + \int dt\int d^3\vec{x}\,\partial_i K^i\right) = \\ &= \theta\frac{g^2}{32\pi^2}\int d^3x\, K^0(\vec{x}, t)|_{t\to-\infty}^{t\to+\infty} = (n_+ - n_-)\theta = \nu\theta \ . \end{aligned}$$

The spatial integral vanishes because $\mathbb{R}^3$ is equivalent to $\mathcal{S}^3$ and therefore has no border. This statement can be made precise using stereographic mapping and some differential geometry.
The integer number ν is the instanton winding number and can also be seen as the label of a homotopy group from the $\mathcal{S}^3_\infty$ sphere that is the border of $\mathbb{R}^4$ to the gauge group, for further discussion see [2].

[1] Instantons can be thought of as classical solutions of the Euclidean action i.e. the action in which we make the substitution $t \to t_E = it$. You can easily see that this change of variable makes the Lorentz metric Euclidean.

Previously we argued that the θ *term* should appear in the Lagrangian because it can be there as it respects all the symmetries we have imposed on the theory. Now we want to show that due to the vacuum structure of QCD, this term needs to be there. The vacuum states we have discussed so far are not physical in fact if we take a gauge transformation U_1 belonging to the equivalence class with winding number 1, this transformation acts on the vacuum states as

$$U_1 |n\rangle = |n+1\rangle \ .$$

We seek a physical vacuum $|\theta\rangle$ that is an eigenstate of U_1, being a unitary matrix we have

$$U_1 |\theta\rangle = e^{i\theta}\theta$$

$$\Rightarrow |\theta\rangle = \sum_{-\infty}^{+\infty} e^{in\theta} |n\rangle$$

up to an arbitrary phase. We must therefore add the theta term to the Lagrangian to account for the vacuum-vacuum $\langle\theta_+|\theta_-\rangle$ transitions amplitude to the computation of which almost every transition amplitude can be reduced. In the path integral formalism:

$$\langle\theta_+|\theta_-\rangle = \sum_{m,n} e^{i(n-m)\theta} \langle m|n\rangle = \sum_{\nu} e^{i\nu\theta} \sum_{m} \langle m|m+\nu\rangle =$$
$$= \sum_{\nu} \int \mathcal{D}A \ e^{i\int d^4x\,(\mathcal{L}+\mathcal{L}_\theta)} \delta\left(\nu - \frac{g^2}{32\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde{G}^{a\,\mu\nu}\right) \ .$$

2.5 Neutron EDM

We have seen that we must add to the QCD Lagrangian a new term $\mathcal{L}_\theta$ that gives a non-zero contribution to the action and can therefore violate badly CP symmetry if the coupling $\theta \sim 1$ [2].

An observable that can be used to measure CP violation is the electric dipole moment (EDM) in fact if a particle possesses an EDM we could watch its projection on spin and this projection changes sign under a CP transformation. Neutron EDM has been used to measure CP violation in QCD, in fact [4] theoretical calculations give

$$d_n \sim 2.4 \cdot 10^{-16} \theta \, e\, cm \ .$$

However the neutron EDM has been bounded experimentally [5] to be less than

$$|d_n| < 2.9 \cdot 10^{-26} \, e\, cm$$

implying that

$$|\theta| \lesssim 10^{-10} \ .$$

The Strong CP problem addresses exactly this issue: why is the θ parameter so small making CP an almost perfect symmetry of strong interactions?

[2] In general if, in QCD with N_f flavours, we have a complex mass matrix this gives, through a chiral (anomalous) transformation of quark fields, an additional θ_q term. We could group the CP violating terms defining: $\bar{\theta} = \theta + N_f\theta_q$. In the proceeding we will not make any difference between θ and $\bar{\theta}$

Chapter 3

The Peccei-Quinn solution

An elegant solution to the strong CP problem was proposed in 1977 by Peccei and Quinn. The fundamental element of this solution is the introduction of a new global $U_{PQ}(1)$ symmetry that has to satisfy two conditions:

- The symmetry must be spontaneously broken (in an unknown way) at energies of order f_a, a symmetry of the Lagrangian is said spontaneously broken if it is not a symmetry of the vacuum state of the theory.
- The symmetry must be anomalous under the strong interactions. I.e. U_{PQ} is a symmetry of the classical Lagrangian but fails to be a true symmetry of the theory because it is not a symmetry of the measure of the path integral.

Without further specifying any other feature of the theory we can discuss some low-energy behaviours of the PQ theory. At energies smaller than f_a the PQ symmetry is spontaneously broken and, being a global symmetry, the Goldstone theorem ensures that a massless boson will emerge, the axion ϕ. Now the second feature of PQ theory comes into play. The anomaly ensures the axion couples the gluon in the Lagrangian through a term of the form

$$\mathcal{L}_{\phi g} = \frac{\phi}{f_a} \frac{g}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\,\mu\nu} \,. \tag{3.1}$$

This coupling breaks explicitly the U_{PQ} symmetry down to a $Z(N)$ unbroken subgroup (where N depends on the number of quarks and the precise features of the model), making the axion gain a mass m_ϕ, we will soon see these concepts applied to a simple toy model to clarify them.

3.1 Toy model of spontaneous symmetry breaking

We want to stress that spontaneous symmetry breaking (SSB) is a feature only of theories with an infinite number of degrees of freedom, in fact in such a theory the tunnelling barriers go to infinity preventing tunnelling to restore the broken symmetry, in theories with a finite number of degrees of freedom such as non-relativistic quantum mechanics the vacuum state is never degenerate.

Let us consider a real parameter η and a complex scalar field ξ with Lagrangian

$$\mathcal{L} = \partial_\mu \xi \, \partial^\mu \xi^\dagger - \lambda^2 (|\xi|^2 - \eta^2)^2 \,. \tag{3.2}$$

This Lagrangian clearly enjoys a global $U(1)$ symmetry $\xi \to e^{i\alpha}\xi$, but this symmetry is not a symmetry of the vacuum solution, the vacuum in fact will be located in a random state between the ξ configurations with modulo η. Choosing a polar parametrization for ξ (we can do it until we stay away from the origin where this set of coordinates is degenerate) with the phase ϕ chosen in such a way that the vacuum state has phase $\phi = 0$ and expanding the radial coordinate around the radius of the vacuum the field can be written

$$\xi(x) = (\eta + \chi(x)) e^{i\phi(x)}$$

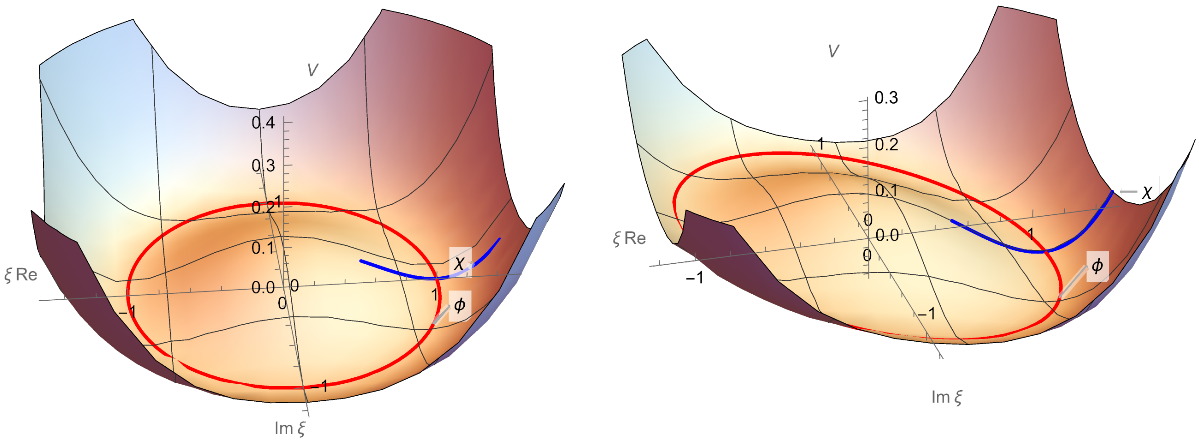

Figure 3.1: Plot of the spontaneously broken potential $V(\xi)$ before and after the explicit breaking of the $U(1)$ symmetry.

with χ and ϕ real fields [1]. The Lagrangian takes then the form

$$\mathcal{L} = \eta^2 \partial_\mu \phi\, \partial^\mu \phi + \partial_\mu \chi\, \partial^\mu \chi - \lambda^2 (2\eta\chi + \chi^2)^2 + (2\eta\chi + \chi^2)\partial_\mu \phi\, \partial^\mu \phi\,.$$

The Lagrangian has been divided into 3 parts: the free-ϕ Lagrangian, the free-χ and an interaction term. The field χ has a mass term of the form $m_\chi = 2\eta\lambda$, but the field ϕ completely lacks a mass term and corresponds to a massless degree of freedom, the Goldstone boson.

If we want the field ϕ to acquire a mass, we must break explicitly the $U(1)$ global symmetry. To accomplish this we add a new term with real coupling ω^2 (the simplest we can think of) to (3.2)

$$\mathcal{L} = \partial_\mu \xi\, \partial^\mu \xi^\dagger - \lambda^2 (|\xi|^2 - \eta^2)^2 + \omega^2\, \frac{\xi + \xi^\dagger}{2}\,.$$

This term gets rid of the degeneration of the vacuum state that will now be at a point on the real axis $\xi = \rho_0(\eta, \lambda, \omega)$. Writing as before $\xi(x) = (\rho_0 + \chi(x))e^{i\phi(x)}$, we get

$$\mathcal{L} = \rho_0^2\, \partial_\mu \phi\, \partial^\mu \phi + \omega^2 \rho_0\, cos(\phi) + \partial_\mu \chi\, \partial^\mu \chi - \lambda^2((\rho_0 + \chi)^2 - \eta^2)^2 + (2\rho_0\chi + \chi^2)\partial_\mu \phi\, \partial^\mu \phi + \chi\omega^2\, cos(\phi)$$

the field ϕ has now a mass term $m_\phi = \frac{\omega}{\sqrt{2\rho_0}}$ and as expected this mass term goes to 0 for $\omega \to 0$.

3.2 The axion potential

Analogously to what we have seen above, at low energy, where the U_{PQ} symmetry is broken both spontaneously and explicitly, we can write an effective Lagrangian that describes the behaviour of the theory

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \left(\theta + \frac{\phi}{f_a}\right) \frac{g}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\,\mu\nu} + \mathcal{L}_{int}\,.$$

The coupling of the axion to QCD generates a potential for the axion. This potential can be calculated in the zero energy approximation via chiral Lagrangian techniques (a low energy approximation of the QCD Lagrangian that exploits the high degree of symmetry, $SU(3)_L \times SU(3)_R \times U(1)$ of the Lagrangian in the limit where the masses of the three light quarks go to zero and the masses of the three heavy quarks go to infinity) obtaining:

$$V(\phi) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} sin^2\left(\frac{\theta}{2} + \frac{\phi}{2 f_a}\right)} \tag{3.3}$$

[1]Notice that the broken symmetry being global is crucial for the emergence of a field ϕ that corresponds to the Goldstone boson, if for example the symmetry were a local $U(1)$ gauge symmetry we could use the gauge freedom to eliminate the phase ϕ. After this redefinition, it can be seen that the gauge boson A_μ acquires a mass term and a degree of freedom (it is said that the gauge boson eats the Goldstone boson), this is the basis of the Higgs mechanism.

where $m_\pi \simeq 135\,\text{MeV}$ is the pion mass and $f_\pi \simeq 93\,\text{MeV}$ is the so called pion decay constant, we have also used that $m_s \gg m_u, m_d$.
It is now obvious that this potential has its minimum for $\theta + \frac{\phi}{f_a} = 0$ hence the QCD potential dynamically drives the axion towards the state in which the total parity violating term $G\tilde{G}$ in the Lagrangian is cancelled, solving the strong CP problem! This is the crucial point of the axion solution to the strong CP problem and is not true just at low energies. The Vafa-Witten theorem in fact assures that in a vector-like theory like QCD parity is not spontaneously broken, i.e. parity is a symmetry of the vacuum state [2].

Furthermore from this potential we can compute the zero-temperature mass of the axion

$$m_\phi(0) = \frac{\partial^2 V}{\partial \phi^2}\bigg|_{\phi=-\theta f_a} = \frac{\sqrt{m_u m_d}}{(m_u + m_d)} \frac{m_\pi f_\pi}{f_a} \simeq 6\,\mu\text{eV} \cdot \left(\frac{10^{12}\text{GeV}}{f_a}\right) ,$$

with $\frac{m_u}{m_d} \simeq 0.56$, $f_\pi \simeq 93$ MeV and $m_\pi \simeq 135$ MeV.
This result is qualitatively expected as the axion mass is due to equation (3.1) so it has to go to zero if either the QCD effects ($m_\pi f_\pi$) go to zero or f_a goes to infinity as in this limit the axion decouples. The axion mass can be shown through DIGA (dilute instanton gas approximation) to have a strong dependence on temperature that will later be of great importance

$$m_\phi(T) = \begin{cases} m_\phi(0) & \text{if } T < \Lambda_{QCD} \\ m_\phi(0)\left(\frac{\Lambda_{QCD}}{T}\right)^n & \text{if } T > \Lambda_{QCD} \end{cases} .$$

With n is in between 3.4 and 3.5. This behaviour of the axion mass is due to a phase transition at $T = \Lambda_{QCD} \simeq 200\,\text{MeV}$.

3.3 Mean life of the axion

For typical values of the PQ symmetry breaking scale $f_a \sim 10^{11}$ GeV, we will see later why this is a typical value, the axion is very light $m_\phi \sim 10^{-4}$eV. The axion is, therefore, lighter than all the particles of the Standard Model besides gluons and photons which are massless, therefore these are the only particles the axion can decay on.
We have previously described the coupling of the axion with gluons due to the anomaly of PQ symmetry, furthermore the decay of the axion to gluons is forbidden by confinement. We now want to describe the possible couplings that allow the axion to decay on photons. The process $\phi \to \gamma$ is prohibited by the conservation of 4-momentum, we will therefore look to couplings that allow $\phi \to \gamma\gamma$, considering rare the axion decay on a greater number of photons.
This term of the Lagrangian must be invariant under $U_{PQ}(1)$ symmetry, invariant under the $U(1)$ gauge group of QED, Lorentz covariant and quadratic in the fields. The $U_{PQ}(1)$ symmetry acts on the axion field like $\phi \to \phi + kf$, therefore the couplings must take the form

$$\mathcal{L}_{\phi\gamma\gamma} = \frac{G_{\phi\gamma\gamma}}{2} \phi\, \partial_\mu h(A)$$

where $h(A)$ is a function to be determined. Equivalently the partial derivative could be applied to ϕ after integration by parts (QED has a trivial vacuum structure in 4 dimensions so total divergences do not give contributions to the action). The function $h(A)$ can have only one form in order for the coupling to respect the previous conditions

$$\mathcal{L}_{\phi\gamma\gamma} = \frac{G_{\phi\gamma\gamma}}{4} \phi\, \tilde{F}^{\mu\nu} F_{\mu\nu} = \frac{G_{\phi\gamma\gamma}}{2} \phi\, \partial_\mu(\tilde{F}^{\mu\nu} A_\nu) .$$

We can now compute the rate of the process $\phi \to \gamma\gamma$, from Fermi golden rule:

$$\Gamma_{\phi\to\gamma\gamma} = \frac{|\mathcal{M}|^2}{32\pi m_\phi} = \frac{G^2_{\phi\gamma\gamma} m^3_\phi}{64\pi} .$$

[2]The proof of this theorem is based on the positive definiteness of the measure of the QCD euclidean path integral once we integrate out the fermion terms.

Experimentally we have $G_{\phi\gamma\gamma} \lesssim 6 \cdot 10^{-11}\,\mathrm{GeV}^{-1}$ from helioscopes searching for axions produced in the Sun. This bound gives:

$$\tau_{\phi\to\gamma\gamma} = \Gamma^{-1}_{\phi\to\gamma\gamma} \gtrsim 1.7 \cdot 10^{27} \cdot \left(\frac{f_a}{10^{10}\mathrm{GeV}}\right)^3 \mathrm{yrs}\,.$$

This number for typical values of f_a is orders of magnitude greater than the age of the Universe. The axion is therefore stable on cosmological time scales.

3.4 The axion in cosmology

In the last sections we showed how the PQ solution fixes the strong CP problem. This solution implies the presence of a new stable particle, the axion, that will be produced in the Universe.

In this section we will employ basic cosmology concepts and formulas that are introduced in the appendix to study the evolution of the axion field. If you have no confidence with this topic we suggest reading the appendix before going further.

Our main goal is to compute the axion field energy density, to do so we must study its evolution in a non-trivial space-time metric.

Coupling to gravity of the axion field

Using here and below the "$+\,-\,-\,-$" signature of the metric, the effective Lagrangian of the axion in a flat Minkowski space-time is written as

$$\mathcal{L}_\phi = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,.$$

The action takes then the form

$$S = \int d^4x\,\mathcal{L}_\phi = \int d^4x\,\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\,,$$

this form changes slightly if instead of flat Minkowski space-time we consider a less trivial metric. In this case (since the laws of physics must be the same in every system of coordinates) we must ensure that the action is covariant under arbitrary coordinate transformations, hence we must operate the substitutions:

- $\partial \to \nabla$ where ∇ is the so called covariant derivative. In our case, since the axion is a scalar field, there will be no difference between these two derivatives.
- $\eta_{\mu\nu} \to g_{\mu\nu}$ replacing the Minkowski metric with a more general metric.
- $d^4x \to \sqrt{|\det(g)|}d^4x$ replacing the measure of the integral with an invariant one.

$$S = \int d^4x\,\sqrt{|\det(g)|}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi - V(\phi)\partial_\nu\phi\right)\,. \tag{3.4}$$

We will now take the metric to be the Friedman-Lemaitre-Robertson-Walker (FLRW) metric from the cosmological principle and also approximate the metric to be flat (k=0) since $\Omega_k \simeq 7 \cdot 10^{-4}$.

$$S_\phi = \int d^4x\,a^3\left(\frac{1}{2}\partial_0\phi\,\partial_0\phi - \frac{1}{2a^2}\partial_i\phi\,\partial_i\phi - V(\phi)\right)\,. \tag{3.5}$$

This expression for the action can be simplified if we again recall that according to the cosmological principle we expect the field ϕ to be almost homogeneous

$$\phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x})\,.$$

We will neglect the fluctuations of the field (corresponding to the Fourier non-zero modes), these fluctuations will eventually help to create structures in the universe but, at least to a first approximation, will not contribute to the total axion abundance. This means that spatial derivative in equation (3.5) can be disregarded. The action is then

$$S_\phi = \int d^4x\, a^3 \left(\frac{1}{2}\partial_0\phi\,\partial_0\phi - V(\phi)\right) \tag{3.6}$$

and the E-L equations of motion

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0\,. \tag{3.7}$$

This equation represents a damped oscillation in the potential $V(\phi)$ with a dumping parameter described by three times the Hubble parameter!
In a radiation-dominated universe $H \simeq \frac{1}{2t}$, therefore during the expansion of the universe we have a transition between an extremely big and an extremely small Hubble parameter. This transition will identify three main regimes for the solutions to equation (3.7):

- **Field stuck by Hubble friction**: In this regime, the Hubble friction dominates on the potential force so the equations of motion become

  $$\ddot{\phi} = -3H\dot{\phi}\,.$$

  The field is said to be stuck by Hubble friction because if it starts its evolution at rest it remains at rest. Even if in the beginning $\dot{\phi} \neq 0$ solving the equations of motion we find that (taking $f_a \sim 10^{10}\, GeV$) the speed of the field at the end of this regime is approximately $\sim 10^{-32}$ times its initial value.

- **Transition period**: In this period the potential force and the dumping parameter are of the same order of magnitude

  $$H \simeq \frac{\partial V}{\partial \phi}\,.$$

  The field overcomes the Hubble friction and starts to run down towards the minimum of the potential.

- **Damped oscillations**: This is the regime of our solution when the Hubble friction becomes much smaller than the potential force. This is the most important regime to describe axion energy density since, as we have seen, in the last two regimes there is little or no change of the field value. Also in this regime we cannot completely disregard the Hubble friction as in the long run also small friction will change significantly the energy density of the field.

Energy Momentum tensor

To compute the axion energy density we must compute the energy-momentum tensor as we know that it corresponds to the time-time component of this tensor. The definition of the symmetric energy-momentum tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{|\det(g)|}}\frac{\delta(\sqrt{|\det(g)|}\,\mathcal{L}_\phi)}{\delta g^{\mu\nu}} \qquad \Rightarrow T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\,.$$

Disregarding the spatial variations of ϕ, the energy-momentum tensor takes the form

$$T_{\mu\nu} = \delta^0_\mu\,\delta^0_\nu\,(\partial_0\phi\,\partial_0\phi) - g_{\mu\nu}\mathcal{L}\,.$$

We will now work in a locally Lorentz frame in order to compare the properties of the axion with those of the cosmological perfect fluids discussed in the appendix. In this frame the metric $g_{\mu\nu} = \eta_{\mu\nu}$ and therefore

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \qquad T_{ij=} = \delta_{ij}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\,, \qquad T_{0i} = 0\,.$$

The derivation we have followed is valid for every scalar field ϕ and every arbitrary potential $V(\phi)$. Hence we have just seen that a scalar homogeneous field in a FLRW background naturally has an isotropic energy-momentum tensor in its locally Lorentz frame. Therefore it behaves like a perfect fluid with rest energy density ρ and pressure p.

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \qquad w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}\,. \tag{3.8}$$

We can identify 3 main behaviours of the field ϕ, according to the values of the parameter w_ϕ, calling K the kinetic energy:

- $K \simeq V \Rightarrow \boldsymbol{w_\phi \simeq 0}$ the field behaves like non-relativistic (cold) matter.
- $K \gg V \Rightarrow \boldsymbol{w_\phi \simeq \frac{1}{3}}$ the field behaves like relativistic (hot) matter.
- $V \gg K \Rightarrow \boldsymbol{w_\phi \simeq -1}$ the field is dominated by vacuum energy density and hence behaves like a cosmological constant.

Small damped oscillations as cold matter

From now on we will redefine the axion field in such a way that the minimum of its potential occurs at $\phi = 0$. For small oscillation of the field we Taylor-expand the expression of the potential (3.3) around its minimum. The potential can be written as:

$$V(\phi) \simeq \frac{1}{2}m_\phi^2\phi^2\,.$$

In this expression we disregarded a possible vacuum energy contribution that could be taken into account by including it in the cosmological constant. With this approximation equations (3.7), (3.14) take the simple form

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi = 0$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\phi^2\phi^2$$

We can now exploit the fact that $V(\phi) \propto \phi^2$ to use the virial theorem. Averaging between two successive passages through $\phi = 0$ we get

$$0 = \frac{1}{T}\int_{t^*}^{t^*+T} dt\frac{d}{dt}(\phi\,\dot{\phi}) = \langle\dot{\phi}^2\rangle + \langle\phi\ddot{\phi}\rangle = \langle\dot{\phi}^2\rangle - \langle m_\phi^2\phi^2\rangle - 3\langle H\phi\dot{\phi}\rangle\,.$$

In the damped oscillations regime $m_\phi \gg H$, therefore the period of one oscillation is $T \simeq \frac{2\pi}{m_\phi}$. At $T \gtrsim 1\text{eV}$ we have approximately $\dot{H} \simeq 2H^2$. We can take the Hubble parameter to be constant over a period to first order as its variation is $\Delta H \simeq \dot{H}T \simeq H\frac{4\pi H}{m_\phi} \ll H$. We thus have

$$\langle\dot{\phi}^2\rangle - \langle m_\phi^2\phi^2\rangle \simeq 0\,.$$

This equation tells us that *on average* the kinetic and potential energies are equal. This means that the axion field while oscillating from a vacuum energy behaviour (at the inversion point) to a hot matter behaviour (in the minimum of the potential) behaves on average as cold matter! Obviously, its evolution is dominated by its averaged behaviour.
As equation (3.9) shows $\dot{\rho} \propto H$, repeating the same reasoning as above we may regard ρ as constant over an oscillation period, obtaining:

$$\rho \simeq \langle\rho\rangle = \left\langle\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\phi^2\phi^2\right\rangle \simeq \langle\dot{\phi}^2\rangle\,,$$

where we used the condition obtained from the virial theorem. We are now ready to compute the energy density evolution in a small damped oscillations regime

$$\frac{d\rho}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\phi^2\phi^2\right) = \dot{\phi}(\ddot{\phi} + m_\phi^2\phi) = -3H\dot{\phi}^2\,. \tag{3.9}$$

In the second equality we used $\dot{m}_\phi = 0$, which means it holds for temperatures $T < \Lambda_{QCD}$. Averaging again we obtain

$$\frac{d\rho}{dt} = -3H\rho \,, \qquad \rho \propto a^{-3} \tag{3.10}$$

this means that the axion energy density redshifts as cold matter.

Exploiting an adiabatic invariant

Equation (3.10) can be derived in a more elegant and general way using adiabatic invariants from classical mechanics.

To get rid of the dependence from f_a inside $V(\phi)$, we scale equation (3.7) taking

$$\ddot{\theta} + 3H\dot{\theta} + \frac{1}{f_a^2}\frac{\partial V(\theta)}{\partial \theta} = 0 \quad \text{with} \quad \theta = \frac{\phi}{f_a} \,.$$

The potential $V(\theta)$ has now no dependence on f_a neither in its explicit expression (3.3) nor in its domain $-\pi \leq \theta \leq \pi$.
We can now consider this equation as a classical mechanics equation of motion given by the Lagrangian

$$L = a^3\left(\frac{1}{2}\dot{\theta}^2 - \frac{1}{f_a^2}V(\theta)\right) \quad \Rightarrow p_\theta = a^3\dot{\theta} \,.$$

Adiabatic invariance assures that if the Hamiltonian of the system is dependent on a slowly varying parameter we will have a conserved quantity in its evolution

$$I = \frac{1}{2\pi}\int p_\theta d\theta \,.$$

Where the integral is taken over an orbit during which the parameter is held fixed. This procedure can be done for every form of the potential $V(\theta)$ but for the majority of the potentials it implies the computation of non-elementary integrals. To do an analytical calculation we will approximate again the potential to a harmonic one

$$L = a^3\left(\frac{1}{2}\dot{\theta}^2 - \frac{1}{2}m_\phi^2\theta^2\right) \,.$$

This Lagrangian (and the corresponding Hamiltonian) depends on the scale factor a and on the mass of the axion m_ϕ, thus we must impose these parameters to be slowly varying:

$$H \ll m_\phi \qquad \text{and} \qquad \frac{\dot{m}_\phi}{m_\phi} \ll m_\phi \,. \tag{3.11}$$

When a and m_ϕ are fixed, the energy E is conserved (the Lagrangian has no explicit dependence on time), hence

$$p_\theta = \sqrt{a^3(2E - a^3 m_\phi^2\theta^2)} = a^3 m_\phi\sqrt{\theta_{max}^2 - \theta^2}$$

where we used $E = \frac{1}{2}a^3 m_\phi^2 \theta_{max}^2$. The adiabatic invariant is therefore

$$cost = I = \frac{1}{\pi}\, m_\phi a^3 \int_{-\theta_{max}}^{\theta_{max}} d\theta\sqrt{\theta_{max}^2 - \theta^2} = \frac{a^3 m_\phi \theta_{max}^2}{2} \,. \tag{3.12}$$

Using $\rho = \frac{1}{2}m_\phi^2\theta_{max}^2 f^2$ yields the same result as before when $T < \Lambda_{QCD}$ but is valid also at higher temperatures as long as conditions (3.11) hold.
Let us check when the adiabatic conditions are valid:

$$H = \frac{\pi}{3\sqrt{10}}\sqrt{g_*(T)}\frac{T^2}{M_P} \,,$$

$$\frac{\pi}{3\sqrt{10}}\sqrt{g_*(T)}\frac{T^2}{M_P} \ll m_\phi \qquad \text{and} \qquad n\frac{g_{*s}(T)}{g_{*c}(T)}\frac{\pi}{3\sqrt{10}}\sqrt{g_*(T)}\frac{T^2}{M_P} \ll m_\phi \,.$$

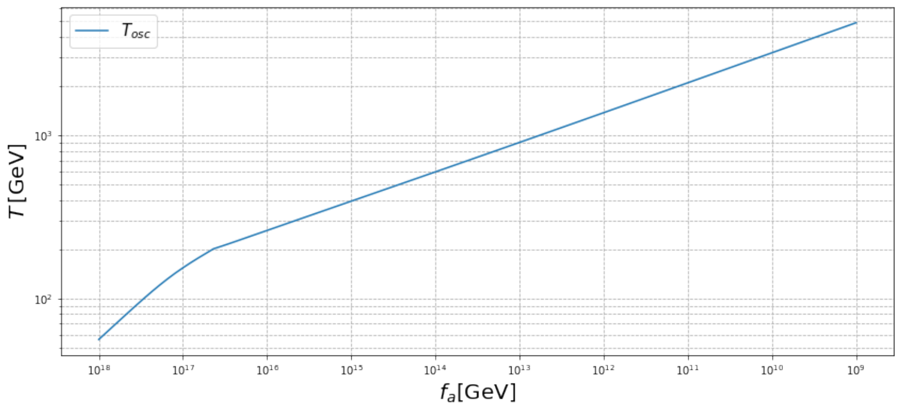

Figure 3.2: T_{osc} as a function of the PQ parameter f_a.

We easily note that the two conditions really are the same condition, calling T_{osc} the temperature for which $3H(T_{osc}) = m_\phi$ (that is the temperature for which the two coefficients in the equation of motion (3.7) are equal) we have a nice form for the adiabatic condition

$$T \ll T_{osc} \,,$$

$$T_{osc}{}^{n+2}\sqrt{g_*(T_{osc})} = \frac{\sqrt{10\, m_u m_d}}{\pi(m_u+m_d)}\frac{m_\pi f_\pi}{f_a}\,\Lambda_{QCD}{}^n\, M_P\,. \tag{3.13}$$

From the above equation we have that T_{osc} is an implicit function of f_a, this function has been reproduced in 3.2 taking n to be the best fit to lattice data $n = 3.42$. It can be seen the change in the behaviour of T_{osc} at the QCD phase transition as a consequence of the dependence of the axion mass from the temperature and the dependence of H from g_*.

We now have quite a precise picture of the evolution of the field: at temperature $T \gg T_{osc}$ the field is stuck to a constant value by Hubble friction, at temperature $T \ll T_{osc}$ the field undergoes damped oscillations conserving the adiabatic invariant computed above and for $T \sim T_{osc}$ we have a transition period in which the field begins to move.

Misalignment mechanism and axion relic density

Let us take θ_i as the initial misalignment angle of the rescaled axion field at $T \gg T_{osc}$. As a first approximation we can disregard the transition period and take the field to be stuck until the temperature T_{osc} and then let immediately start the damped oscillations. We will later address the transition period with a numerical simulation but for now this approximation allows us to keep our calculations simple while showing all the basic ideas.
In this approximation equation (3.12) immediately allows us to compute the maximum misalignment angle today

$$\theta_0 = \left(\frac{a_{osc}}{a_0}\right)^{\frac{3}{2}}\left(\frac{m_\phi(T_{osc})}{m_\phi(T_0)}\right)^{\frac{1}{2}}\theta_i = \left(\frac{g_{*s}(T_0)\,T_0^3}{g_{*s}(T_{osc})\;T_{osc}{}^3}\right)^{\frac{1}{2}}\left(\frac{\Lambda_{QCD}}{T_{osc}}\right)^{1.71}\theta_i\,.$$

For typical values of f_a the misalignment angle is $\theta_0 \ll 10^{-10}$ solving the strong CP problem. This is not indeed true because the harmonic potential does not take into account that in the real potential the field could in principle remain stuck for ever if $\theta_i = \pi$, but this seems a very unnatural value for the initial misalignment angle that would require further explanations.

Analogously we can compute the axion relic density

$$\begin{aligned}\rho_0 &= \frac{1}{2}\left(\frac{a_{osc}}{a_0}\right)^3 m_\phi(T_{osc}) m_\phi(T_0) f_a^2 \theta_i^2 = \\ &= \frac{m_\phi^2(0)}{2} \frac{g_{*s}(T_0)\, T_0^3}{g_{*s}(T_{osc})\ {T_{osc}}^3} \left(\frac{\Lambda_{QCD}}{T_{osc}}\right)^n f_a^2 \theta_i^2 = \\ &= \frac{m_u m_d}{(m_u+m_d)^2} \frac{(m_\pi f_\pi)^2}{2} \frac{g_{*s}(T_0)}{g_{*s}(T_{osc})} T_0^3 \Lambda_{QCD}^{3.42} T_{osc}^{-6.42} \cdot \theta_i^2 \,.\end{aligned}$$

Therefore, the relic axion density is determined by θ_i and T_{osc}, which is a function of the PQ symmetry breaking scale f_a.

The initial value of the field ϕ is selected randomly when the temperature of the universe drops under f_a at the PQ phase transition, we then expect it to be misaligned with the minimum of the axion potential. In the early universe the size of the casual horizon was much smaller than today because a short time had passed since the Big Bang, the casual horizon that we see today was therefore divided into patches that had not already had time to interact with each other. Therefore at the PQ phase transition we expect different initial conditions to be selected in all of these patches. We must now distinguish two different cases:

1. Inflation occurs with temperature $T_I = \frac{H_I}{2\pi} > f_a$, where H_I is the value of the Hubble parameter during inflation. In this case, after the PQ phase transition has taken place, inflation homogenizes the universe on a large scale thus selecting a unique misalignment angle θ_i. Therefore the calculation we made before is quite accurate as all the nonzero modes of the field are negligible and we have:
$$\Omega_\phi = \frac{m_u m_d}{(m_u+m_d)^2} \frac{(m_\pi f_\pi)^2}{6 H_0^2 M_P^2} \frac{g_{*s}(T_0)}{g_{*s}(T_{osc})} T_0^3 \Lambda_{QCD}^{3.42} T_{osc}^{-6.42} \cdot \theta_i^2 \,.$$

2. Inflation occurs with temperature $T_I < f_a$. In this case the universe starts out non-homogeneous, with a different value of the misalignment angle in each casual horizon, therefore non-zero modes of the field and topological defects [1] significantly contribute to the axion relic density.
We can compute the energy contribution of the field zero mode by averaging the previous expression over all possible misalignment angles $-\pi \leq \theta \leq \pi$.
$$\langle \theta^2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \theta^2 = \frac{\pi^2}{3} \,,$$
$$\Omega_\phi = \frac{\pi^2}{18} \frac{m_u m_d}{(m_u+m_d)^2} \frac{(m_\pi f_\pi)^2}{H_0^2 M_P^2} \frac{g_{*s}(T_0)}{g_{*s}(T_{osc})} T_0^3 \Lambda_{QCD}^{3.42} T_{osc}^{-6.42} \,. \tag{3.14}$$
It can be shown [6] that nonzero modes of the field give a contribution of the same order of magnitude to the energy density.

Axions as cold Dark Matter

In the previous sections we described the cosmological properties of axions:

- they are **stable on cosmological scales**.
- they **have tiny couplings to the SM** as we expect all its couplings to be suppressed as $\propto \frac{1}{f_a}$.
- they **redshift like cold matter**.
- their relic energy density is given by equation (3.14).

Any particle with the first three properties behaves like cold dark matter. Therefore if there is a global $U_{PQ}(1)$ spontaneously broken and anomalous under the strong interactions. It naturally produces a Goldstone boson that could constitute at least part of the dark matter observed in the universe.
Furthermore if $f_a \simeq 4.3 \cdot 10^{11}$GeV the post inflationary axion energy density (3.14) reproduces the

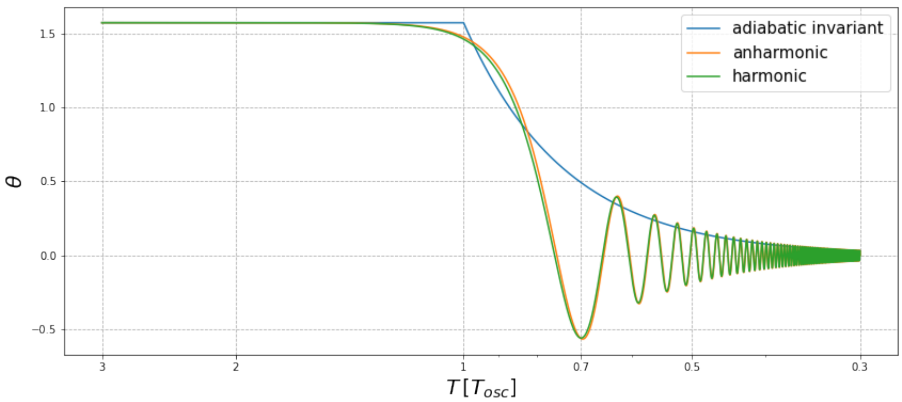

Figure 3.3: Numerical simulation of the transition period for $\theta_i = \frac{\pi}{2}$ and $f_a = 10^{11}\text{GeV}$.

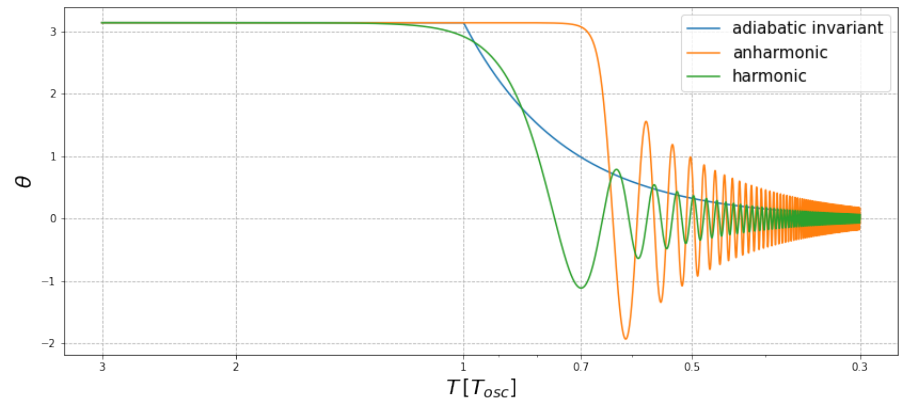

Figure 3.4: Numerical simulation of the transition period for $\theta_i = 0.9999\pi$ and $f_a = 10^{11}\text{GeV}$.

same order of magnitude of the dark matter energy density as measured by the cosmic microwave background

$$\Omega_{DM} h^2 \simeq 0.12 \,.$$

To this day the possibility that $f_a \sim 10^{11}$ GeV has not been ruled out by experiment therefore the possibility remains open for axions to constitute almost all the dark matter in the universe.

Numerical corrections to the axion relic density

In the previous computation of axion energy density we have made two main approximations:

- We neglected the transition period.
- We approximated the axion potential to be harmonic.

We expect in particular the second of these approximations to influence the estimate of axion relic density by defect. In fact approximating the potential with a parabola does not take into account that the real potential possesses an unstable equilibrium for $\theta = \pi$. When the field starts near π it will be stuck by Hubble friction much longer than it will for a harmonic potential resulting in less energy being dissipated for these starting values.

We simulate numerically equation (3.7) for $\theta_i = \frac{\pi}{2}$ and $\theta_i = 0.9999\pi$ and plot the result in 3.3 and 3.4. From these graphics it can be clearly seen that despite the harmonic approximation being excellent also for quite large angles (in figure 3.3 the harmonic and anharmonic evolution of the system are basically indistinguishable), it fails by defect (as expected) to reproduce the behaviour of the field

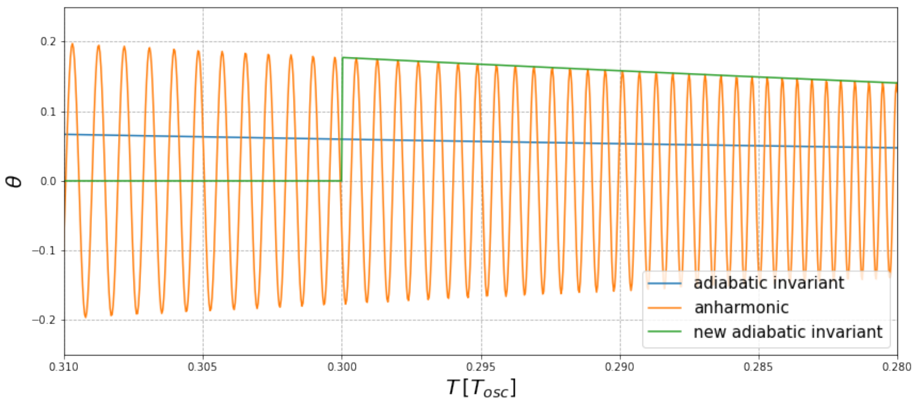

Figure 3.5: Comparison of the new adiabatic invariant starting from $T = 0.3 \cdot T_{osc}$ with the old one starting from $T = T_{osc}$, for $\theta_i = 0.9999\pi$ and $f_a = 10^{11}$GeV.

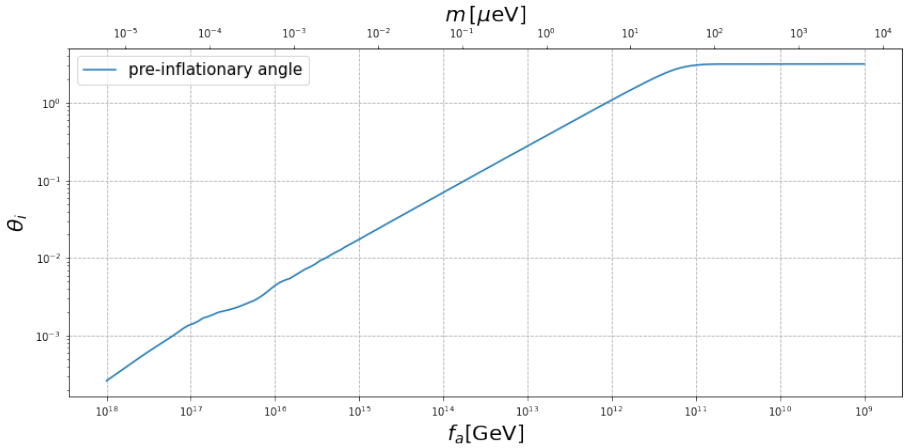

Figure 3.6: Graphic of the relation between f_a and the initial conditions θ_i in the pre-inflationary scenario.

when the starting values are near π.
Furthermore we notice that the correction due to the anharmonic potential seems much more important than disregarding the transition period, in fact also at a very large angle the adiabatic invariant seems to be in good agreement with the simulation of the harmonic potential.
In order to take into account both effects we numerically simulate the evolution of the field until $T = 0.3 \cdot T_{osc}$ and let the adiabatic invariant start from there. In figure 3.5 is shown the comparison between the old adiabatic invariant and the new one improved by the numerical simulation. With this numerical computation, the energy density for Case 2 is corrected by a factor $\simeq 1.8$ resulting in

$$f_a \simeq 3.3 \cdot 10^{11}\,\text{GeV}$$

to give the correct dark matter relic density. This factor is not so important if we take into account that we are neglecting other contributions of the same order such as the non-zero modes of the field.

We can also compute numerically the value of f_a as a function of the initial misalignment angle θ_i for Case 1, the result is reported in 3.6. In this graph we can quite clearly see that theoretically arbitrary small values of f_a are achievable if θ_i is really near to π, but as we said earlier this seems a very unnatural solution. The main part of the graphic shows that the dependence of f_a from θ_i is a power law: $\theta_i^2 \propto f_a^{-1.185}$ (this dependence could be shown analytically with the rough approximation that $g_*(T)$ and $g_{*s}(T)$ are constant functions). Another interesting feature of the graphic is the slight bend around at $f_a \sim 3 \cdot 10^{16}$ corresponding to the QCD phase transition.

Conclusions

In this work we have shown how the Peccei-Quinn solution to the Strong CP problem provides automatically a viable dark matter candidate.
The PQ solution is based on the introduction of a $U_{PQ}(1)$ global symmetry that is braked at scale f_a and is anomalous under the Strong Interactions. This anomaly automatically provides a coupling of the Goldstone boson of this symmetry, called the axion, to the CP-violating term in the effective Lagrangian. This coupling, as guaranteed by the Vafa-Witten theorem, dynamically drives to zero the coupling of this term.

We then focused on the cosmological consequences of the introduction of this symmetry and in particular of the axion. Axions are produced in the early universe when T drops under the PQ breaking scale f_a. The axion behaves like a cosmological dumped oscillator

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \,.$$

The axion field is stuck in the early universe as the Hubble parameter is big and dominates the other parameters in this equation. Oscillations start when the temperature of the universe drops below $T_{osc} \sim 1\,\text{GeV}$. After a short transition period, the Hubble parameter becomes small with respect to the other parameters and can be considered slowly varying. This enables us to employ an adiabatic invariant to study the axion energy density evolution

$$a^3 m_\phi \theta_{max}^2 = cost \,.$$

Furthermore, the axion behaves like cold dark matter as it is long-lived, with tiny couplings to the SM and redshifts as a^{-3} at $T < \Lambda_{QCD}$. We have therefore computed the value of f_a in order for axions to constitute all DM in the universe. To do this we must distinguish two cases:

- The pre-inflationary case in which a unique initial misalignment angle θ_0 is selected by inflation. In this case the value of f_a is given as a function of θ_0 by graphic 3.6.
- The post-inflationary case in which the PQ symmetry breaking occurs after the inflation and then the initial misalignment angle θ_i is randomly selected in every Hubble casual patch. We can therefore average the axion relic density on the initial values of θ_i and obtain

  $$f_a \simeq 3.3 \cdot 10^{11}\text{GeV} \,.$$

  In the post-inflationary scenario, the different values assumed by the axion field in every casual patch give rise to topological defects such as strings and domain walls which introduce some uncertainty in the determination of the energy density and, therefore, in the parameter f_a [1].

As can be seen in figure 3.7 are reported the constraints obtained by experiments on the axion mass (which is inversely proportional to f_a) and on the axion coupling to photons there is still room for axion mass $m_a \sim 60\mu\text{eV}$.

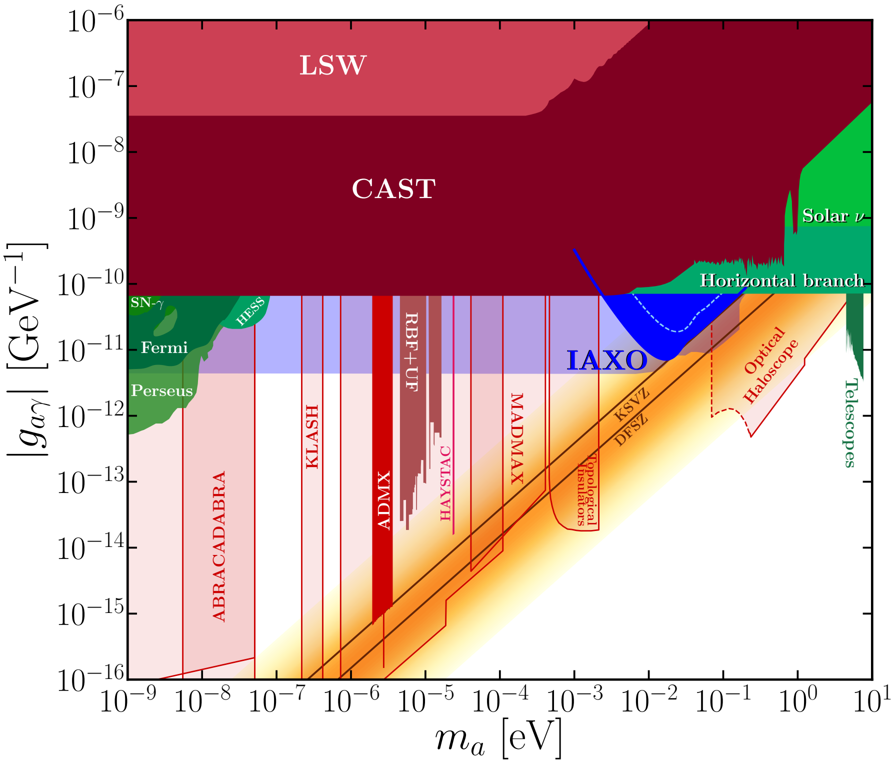

Figure 3.7: Axion parameter space within various hues of red (green) experimental (astrophysical) exclusion limits (opaque) and projections (transparent), taken from [8].

Appendix A

The cosmological SM

In this appendix, we will discuss some basic ingredients of the standard cosmological model called, for reasons that will be clear later, ΛCDM.

A.1 The cosmological principle

The milestone of modern cosmology is the assumption that there is no privileged place in the universe, that is the universe is *homogeneous* and *isotropic*.

This notion seems strange at first being used to all the anisotropic and inhomogeneous structures around us: clearly the solar system is not homogeneous, and neither is the Milky Way... The cosmological principle in fact regards the *large scale behaviour* of the universe at the typical scales of $\sim 100\,\text{Mpc} \simeq 3 \cdot 10^{24}\,\text{m}$, for comparison the visible part of the Milky Way has a radius of $\simeq 12.5\,\text{kpc}$. The cosmological principle then states that if we watch the universe at a large enough scale where the anisotropies that have formed through gravitational collapse become negligible the universe is isotropic and homogeneous.

A.2 Hubble's law and the scale factor

Crucial observational evidence in cosmology is that the universe is expanding. This was first recognised by the astronomer Edwin Hubble in 1929. Hubble measured the velocities of different galaxies (using the redshift of their light spectrum), their distance from us (using the so called standard candles) and realized that this two quantities were linearly proportional. Hubble's law is then

$$\vec{v} = H_0\,\vec{r}$$

where $H_0 = 100\,h\,\frac{\text{km}}{\text{Mpc}\,\text{s}}$ (with $h = 0.674 \pm 0.005$) is Hubble's constant. In hindsight, it is easy to show that this expansion law is the only one compatible with the cosmological principle and hence must have been valid not only today but at every age of the universe

$$\vec{v} = H(t)\,\vec{r}\,.$$

The Hubble's constant is no more a constant as it can now depend on time and takes the name of Hubble's parameter. Above (and throughout the second part of the thesis) we used the convention to indicate with the subscript "0" the parameters calculated at the present time.
We can now use a more convenient set of coordinates, the *comoving coordinates* these coordinates are carried with the expansion in the way that an object in the position $\vec{x}$ with respect to such coordinates at a given instant of time remains at $\vec{x}$ in the evolution of the universe. Because the expansion is uniform we can write the old coordinates as a function of the new ones

$$\vec{r}(t) = a(t)\vec{x}\,.$$

The parameter $a(t)$ is the so called scale factor and is usually chosen to be $a_0 = 1$.

A.3 Perfect fluids

When watching a bunch of interacting particles we can approximate them with a fluid provided that the typical temporal and spatial scales we are interested in are much bigger respectively than the mean time between two collisions and the mean free path of the particles, indeed this is the fundamental idea of thermodynamics. On cosmological scales these conditions are satisfied and we can approximate the matter content of the universe with *perfect fluids*. A perfect fluid is a fluid that is isotropic and homogeneous in its local rest frame.
We now want to study the structure of the energy-momentum tensor of these fluids as we know that this tensor determines how space-time is curved through the Einstein equation. The energy-momentum tensor is defined as

$$T^{\mu\nu} = \{\text{flux of } p^\mu \text{ through the surface } x^\nu = cost\} \,.$$

For a fluid composed of N particles we then have in total generality

$$T^{\mu\nu} = \sum_{i=1}^{N} \frac{p_i^\mu p_i^\nu}{p_i^0} \delta^4(x - x_i) \,. \tag{A.1}$$

Returning now to perfect fluids and exploiting again homogeneity and isotropy, in the local rest frame the energy-momentum tensor of the fluid does not have shear components and the pressure must be the same in all directions. Hence

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \,.$$

We can immediately put this tensor in a manifestly covariant form

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu - P\eta^{\mu\nu} \,.$$

We can now discuss two main types of cosmological fluids:

- **Cold matter**: we take $p_i^0 \simeq m_i$ in equation (A.1) . The equation then becomes

$$\rho \simeq \sum_{i=1}^{N} m \,, \qquad P \simeq 0 \,, \qquad w = \frac{P}{\rho} \simeq 0 \,.$$

- **Hot matter** or **radiation**: we take $p_i^0 \gg m_i$ in equation (A.1). The equation then becomes:

$$\rho = \sum_{i=1}^{N} p_i^0 \,, \qquad P = \sum_{i=1}^{N} \frac{p_i^2}{p_i^0} \simeq \frac{1}{3} \sum_{i=1}^{N} p_i^0 \,, \qquad w = \frac{P}{\rho} \simeq \frac{1}{3} \,.$$

A.4 The Friedman equation

The most general metric that respects the cosmological principle is [9]

$$g = dt \otimes dt - a^2(t) \left[\frac{dr \otimes dr}{1 + kr^2} + r^2 (d\theta \otimes d\theta + sin^2\theta \, d\phi \otimes d\phi) \right] \,.$$

If we now use this metric to compute the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}(R + \Lambda) g_{\mu\nu} = \frac{T_{\mu\nu}}{M_P^2}$$

where $M_P = \frac{1}{8\pi G}$ is the reduced Planck mass, $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature and Λ is the famous cosmological constant. We notice that, if we take the constituents of the universe to be perfect fluids, the Einstein equation has just two independent components due to isotropy and the energy-momentum tensor being diagonal. With some effort, these two components can be rewritten as

$$H^2 = \frac{\rho}{3M_P^2} + \frac{\Lambda}{3} - \frac{k}{a^2} \,, \tag{A.2}$$

$$\dot{\rho} = -3H(\rho + P) \,. \tag{A.3}$$

These two equations are respectively the *Friedman equation* and the *fluid equation* and they describe the expansion of the universe. These two equations are not sufficient on their own, but once we specify the relation between pressure and energy density of the constituents of the universe (as we did for the hot and cold matter), they uniquely determine the universe's dynamics.

A.5 Dark Energy

We want to discuss a radically different perfect cosmological fluid from the ones we described earlier: dark energy. If we look at equation (A.2) we can think of the parameter Λ to be given by an energy density

$$\rho_\Lambda = \Lambda M_P^2 \,.$$

This energy density is constant, for this reason dark energy is thought to be kind of a vacuum energy. Using equation (A.3) we get

$$0 = -3H(\rho + P) \quad \Rightarrow \quad \rho = -P \,.$$

We have therefore found that the cosmological constant can be substituted with a new constituent of the universe, dark energy, that behaves like a perfect fluid with negative pressure.

$$P_\Lambda = -\Lambda M_P^2 \,.$$

This is quite a big difference from the perfect matter fluids described above. With dark energy in mind we can include the cosmological constant in the universe's energy density. Friedman equation then takes the simpler form

$$H^2 = \frac{\rho_{tot}}{3M_P^2} - \frac{k}{a^2} \tag{A.4}$$

with $\rho_{tot} = \rho_{mat} + \rho_\Lambda + \rho_{rad}$.

A.6 Space-time geometry and density parameters

The parameter k in equations (A.2) and (A.4) expresses the curvature of space-time: $k > 0$ corresponds to a spherical universe, $k = 0$ to a flat universe and $k < 0$ to a hyperbolic universe.
These three geometries correspond to different fates of the universe. The Friedman equation (A.4) in fact tells that

- If $k > 0$ then the expansion of the universe will come to an end when $\rho_{tot} = 3M_P^2 \frac{k}{a^2}$ and then start shrinking back. This can be seen deriving the Friedman equation with respect to time and obtaining the so called acceleration equation.
- If $k = 0$ the universe will never stop expanding but asymptotically as the energy density becomes smaller $\dot{a}$ will go to zero.
- If $k < 0$ the universe will never stop expanding and asymptotically we have $\dot{a} = \sqrt{-k}$, therefore the universe will asymptotically approach a constant expansion speed.

Fixed H, there is a special value for the energy density for which the universe has a flat geometry. This value is called *critical density*

$$\rho_{crit} = 3H^2 M_P^2 \,.$$

With this definition, we can divide both sides the Friedman equations for H^2 and obtain a relation between dimensionless quantities

$$\Omega_{tot} + \Omega_k = 1 \,,$$

$$\Omega_{tot} = \frac{\rho_{tot}}{\rho_{crit}}\,, \qquad \Omega_k = -\frac{k}{a^2H^2}\,.$$

The parameters Ω are called *density parameters*. In 2018 the Planck collaboration obtained $\Omega_k = 0.0007 \pm 0.0019$. This measure is compatible with a flat universe and tells us that for most applications we can neglect the curvature of the universe. It is common in the literature to express the density parameters multiplied by h^2 (the factor we introduced early talking about the Hubble constant) to obtain quantities without errors due to the measurement of the Hubble constant.
From the cosmic microwave background (CMB) we can measure with great precision the density parameters of the different cosmological fluids [7]

$$\Omega_{mat,0} = 0.315 \pm 0.007\,, \qquad \Omega_{\Lambda,0} = 0.6847 \pm 0.0073\,, \qquad \Omega_{rad,0} \simeq 5.43 \cdot 10^{-5}\,.$$

A.7 Energy densities evolution

As we have seen above dark energy density is constant. Using equation (A.3) we can obtain also the energy density evolution for cold matter and for radiation.

$$\dot{\rho} = -3H\rho(1+w) \quad \Rightarrow \quad \rho \propto a^{-3(1+w)}\,.$$

Hence using w we previously computed

$$\rho_{cold} \propto a^{-3}\,, \qquad \rho_{rad} \propto a^{-4}\,.$$

We see that cold matter, as we would expect, dilutes in the universe as its number density. Radiation instead is redshifted by an additional factor $\frac{1}{a}$, this effect takes the name of gravitational redshift.

A.8 Dark Matter

An interesting thing we can measure is the amount of matter that was not interacting with radiation (called dark matter) when the CMB was formed. Dark matter (DM) is related to the anisotropies of the CMB: not interacting with light this matter undergoes gravitational collapse before the ordinary matter and starts structure formation creating brighter regions in the CMB. The current estimate for the DM density parameter according to the ΛCDM (CDM stands for "Cold Dark Matter") model is

$$\Omega_{DM,0}\, h^2 = 0.1200 \pm 0.0012\,.$$

This means that the amount of ordinary matter interacting with light is approximately 15.6% of the total matter in the universe.
It is worth noting that the presence of dark matter in the galaxies and galaxy clusters was discovered before the existence of the CMB was even proposed. The first evidence of DM was recorded by F. Zwicky in 1933. Studying the coma cluster he observed a significant discrepancy between the mass of the stars in the cluster and the mass he could infer using the virial theorem.

A.9 Statistical mechanics

Obviously, no real particle field, besides massless ones behaves like the above-mentioned fluids.
Particles in fact are divided between bosons and fermions that obey respectively the Bose-Einstein or Fermi-Dirac statistics, meaning that their density in phase space is (assuming the chemical potential $\mu = 0$) given by

$$f(E,T) = \frac{1}{e^{\frac{E}{T}} \mp 1}\,,$$

where the (-) sign refers to BE and the (+) to FD statistics.
From the phase space density, we can immediately compute the energy density

$$\rho_i = g_i \int \frac{d^3p}{(2\pi)^3}\, E\, f(E,T) = \frac{g_i}{2\pi^2} \int_{E=m_i}^{E=\infty} dE\, \frac{E^2\sqrt{E^2-m^2}}{e^{\frac{E}{T}} \mp 1}\,.$$

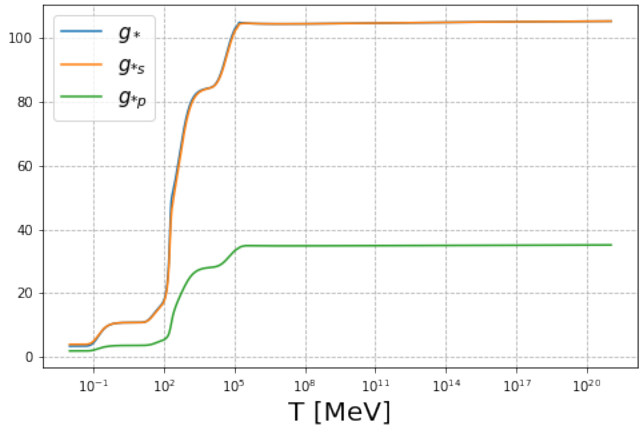

Figure A.1: Effective number of degrees of freedom of the Standard Model

This integral can be solved in the limit $T \gg m_i$, for which we obtain

$$\rho_i = \frac{\pi^2}{30} g_i T^4 \begin{cases} 1 & \text{for bosons} \\ \frac{7}{8} & \text{for fermions} \end{cases} .$$

We can then define

$$\delta g_*|_i(T) = \frac{30}{\pi^2 T^4} \frac{g_i}{2\pi^2} \int_{E=m_i}^{E=\infty} dE \, \frac{E^2\sqrt{E^2 - m^2}}{e^{\frac{E}{T}} \mp 1} .$$

This function counts the weighted number of degrees of freedom of the particle in the cosmic bath. To have a compact notation for the total energy density we define a function $g_*(T)$ that counts the effective number of degrees of freedom of the total bath

$$g_*(T) = \sum_{i=BE} \delta g_*|_i + \frac{7}{8} \sum_{i=FD} \delta g_*|_i \, , \qquad \rho = \sum_i \rho_i = \frac{\pi^2}{30} g_*(T) \, T^4 .$$

We can do the same thing for the pressure

$$P_i = g_i \int \frac{d^3p}{(2\pi)^3} f(E,T) \frac{|\vec{p}|^2}{3E} \, , \qquad \delta g_{*p}|_i = \frac{30}{\pi^2 T^4} g_i \int \frac{d^3p}{(2\pi)^3} f(E,T) \frac{|\vec{p}|^2}{3E} \, ,$$

$$g_{*p}(T) = \sum_{i=BE} \delta g_{*p}|_i + \frac{7}{8} \sum_{i=FD} \delta g_{*p}|_i \, , \qquad P = \frac{\pi^2}{30} g_{*p} \, T^4 .$$

Recalling now the Euler relation (when the chemical potential is $\mu = 0$) we have

$$s = \frac{\rho + P}{T} \, , \qquad g_{*s}(T) = \frac{3}{4}(g_*(T) + g_{*p}(T)) \, ,$$

$$s = \frac{\pi^2}{30}(g_*(T) + g_{*p}(T)) \, T^3 = \frac{2\pi^2}{45} g_{*s}(T) \, T^3 .$$

This relation is very important as (due to isotropy and homogeneity) the expansion of the universe is adiabatic, otherwise we will have the privileged direction of heat transfer or some privileged points that are sources of heat. The entropy in a comoving volume element needs then to be invariant

$$s \cdot a^3 = const \Rightarrow g_{*s}(T) \, T^3 a^3 = const .$$

We have a one-to-one relation between the temperature of the universe and the scale factor. The expansion of the universe can then be described as a function of its temperature.

Another useful quantity we used to compute $\dot{T}$ is

$$g_{*c}(T) = \frac{15}{2\pi^2} c(T)$$

where $c(T) = \frac{d\rho}{dT}$ is the heat capacity per unit volume.

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