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The concentration-compactness method in L^2 and its application to the Hartree equation with $L^{3/2,\infty}$ convolutional potential

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To my family and my dearest loved ones

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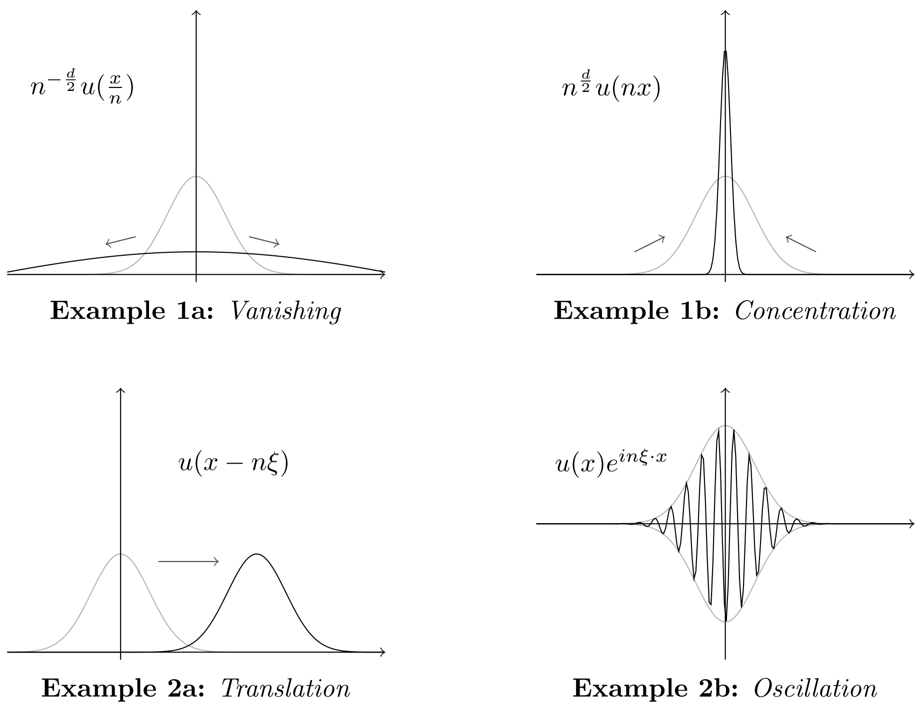
Introduction

When we consider a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ we usually think of a sequence which converges to u weakly in L^2 but does not converge in the strong topology. For istance, we can think of a sequence $u_n \rightharpoonup 0$ weakly in L^2 but such that $\|u_n\|_{L^2}^2 = \lambda > 0$ for every n. The four "non compact" typical behaviours of such a sequence can be summarized in the following:



Example 1a: *Vanishing* **Example 1b:** *Concentration*

Example 2a: *Translation* **Example 2b:** *Oscillation*

Figure 1: In all of these examples $u_n \rightharpoonup 0$ while $\|u_n\|_{L^2}^2 = \lambda$ for all n. Here u is a fixed smooth function such that $\|u\|_{L^2}^2 = \lambda$.

In Example 1a the sequence u_n spreads out but since the total L^2 mass is conserved the local mass goes to 0. Conversely, in Example 1b the sequence concentrates at one point, i.e. $|u_n|^2$ converges to a delta measure in 0. In Example 2a the sequence keeps the same shape for all n, but it runs off away from the origin. Finally in Example 2b we see that u_n oscillates so fast that it converges to 0 weakly in L^2 by the Riemann-Lebesgue Lemma.
The disposition of these examples in Figure 1 is not random. First of all, the behaviours on the same row are dual to each other via the Fourier transform: a sequence which oscillates very fast has a lack of compactness due to translations in the Fourier space, and vice versa. Similarly, a sequence that concentrates has a lack of compactness because its Fourier transform vanishes and conversely.
One difference between the two lines is the behaviour with respect to the L^p norms with $p \neq 2$: if u_n vanishes in the sense of Example 1a we have that $\|u_n\|_{L^p} \to 0$ for $2 < p \leq \infty$ and $\|u_n\|_{L^p} \to \infty$ for $1 \leq p < 2$, and the reverse happens in the case of a concentrating sequence. On the contrary, in both Examples 2a and 2b we have that all L^p norms are conserved. This tells us that using information on the behaviour of L^p norms we can detect vanishing and concentration, but not translation and oscillation. we will give a more precise statement to this idea in Section 1.2.
There are also crucial differences between examples of the first and second column: Examples 1a and 2a cannot happen in a bounded domain, hence the non compactness of these examples arises from the non

compactness of $\mathbb{R}^d$. However, these behaviours are *locally compact*, which will be one of the key factors that contributes to the concentration-compactness method functioning.

Another difference between the two columns is the behaviour of the derivatives: in both Examples 1b and 2b we have that $\|\nabla u_n\|_{L^2} \to \infty$ even in the case of a bounded domain, while in Examples 1a and 2a as $\|\nabla u_n\|_{L^2}$ is uniformly bounded. In particular, if we already know that the sequence u_n is bounded in $H^1(\mathbb{R}^d)$ we can rule out concentration and oscillation *a priori*.

It is clear that we can combine these examples however we want: we could, for instance, add two sequences which present different behaviours or compose them. One such example could be a sequence u_n which concentrates to a point x_n that is escaping to infinity. It is not obvious that the list of behaviours we discussed is, in some sense, universal; what we mean by universal is that a non-compact sequence should, up to a subsequence, be a (possibly infinite) sum of sequences having one or more of the above behaviours. Several works during the 1980s were presented tackling this problem; we have chosen to focus on the *concentration-compactness principle* developed by Lions [8, 9] in 1984. Lions' main result was Lemma 1.8, which we chose to state as in the original paper to underline its generality. This Lemma is very general and admits variants based on the setting; however, if one wants to apply it the underlying space must be locally compact, as in order for the concentration-compactness method to work we need the Rellich-Kondrachov Theorem (or something equivalent to it for non-Sobolev spaces). Indeed, one could say that the concentration-compactness method is an extension to functions defined on the whole $\mathbb{R}^d$ of the usual methods that only work on bounded domains: since we know that our problem can be solved in bounded domains, we prove that most of the mass does not escape to infinity and conclude. An example of this reasoning can be found in Appendix B of [1].

In this thesis we review the concentration-compactness method and the profile decomposition in Sobolev spaces, which allow to study and analyse lack of compactness of (minimizing) sequences in these spaces. We then apply these tools to prove that the *Hartree energy functional* without external potential and $L^{3/2,\infty}$ convolutional potential admits a minimizer with large L^2 mass, which was not previously known in this case. Finally, we apply some classical spectral analysis tools to prove basic properties of the minimizer.

In Chapter 1 we first present the *bubble decomposition* of a bounded sequence as proposed in [6]; then, we study the case where there are no bubbles, and the sequence *vanishes* in some sense. Finally, we state the *concentration-compactness principle* and how to apply it to minimizing sequences in Sobolev spaces. Next, in Chapter 2, we apply the method developed in the first chapter to the minimization of the Hartree energy functional without external potential 2.1; after a brief presentation of Lorentz spaces and some useful inequalities, we prove that the Hartree functional admits a minimizer with large L^2 mass, in the general case, and with positive mass for some specific convolutional potentials. This is done by tweaking the method presented in the first chapter in a new way to adapt it to a $L^{3/2,\infty}$ potential. Finally, using elliptic bootstrapping and a theorem for *positivity improvement* we prove that the minimizer is positive and smooth. We do not discuss uniqueness, as the concentration-compactness method is a tool fit only for existence.

In Appendix A we prove two technical statements from Chapter 1; to do so, we use standard calculus tools and we introduce the *Levy concentration functions* [5]. In Appendix B we give an heuristic explanation of the mathematical and physical importance of the Hartree functional.

1 Motivations and formulations of the concentration-compactness method

In this chapter we will briefly analyse how a sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ bounded in $L^2(\mathbb{R}^d)$ can behave up to subsequences; in particular, we will be interested in when $\mathbf{u}$ exhibits a lack of compactness. Then, we will use these results to state the *concentration-compactness principle* in the $H^1(\mathbb{R}^d)$ framework and see how it can be used to prove the relative compactness of minimizing sequences for energy problems. Our main references for this chapter will be [6] and [8].
Throughout the following two sections, we will always consider the case $d \geq 3$ during the proofs, since all computations are easier to follow in this case but the geist of the proofs stays the same.

1.1 Defining and extracting bubbles

The first problem we are interested in studying is the detection of pieces of mass which retain their shape and possibly escape to infinity, like in Example 2a. To this aim, we consider all possible limits (up to translations) of subsequences of $\mathbf{u}$ and define the maximum L^2 mass that these limits can have.

Definition 1.1. *Let $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$). We define*

$$\mathbf{m}(\mathbf{u}) = \sup\left\{\int_{\mathbb{R}^d} |u|^2 \ : \ \exists (x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d,\ u_{n_k}(\cdot + x_k) \rightharpoonup u \text{ weakly in } L^2(\mathbb{R}^d) \text{ (resp. } H^1(\mathbb{R}^d))\right\}. \tag{1.1}$$

First of all, notice that for every sequence of translation $\mathbf{x} \subset \mathbb{R}^d$ we have $\mathbf{m}(\mathbf{u}) = \mathbf{m}(\mathbf{u}(\cdot + \mathbf{x}))$. Moreover, for every subsequence $\mathbf{u}'$ of $\mathbf{u}$ we have $\mathbf{m}(\mathbf{u}') \leq \mathbf{m}(\mathbf{u})$. However, there does not necessarily exist a $u \in L^2$ realizing the supremum in (1.1), that is u such that $u_{n_k}(\cdot + x_k) \rightharpoonup u$ with $\|u\|_{L^2}^2 = \mathbf{m}(\mathbf{u})$. Indeed, letting $\psi_n(x) = n^{-d/2}\sqrt{1 - \frac{1}{n}}u(x/n)$ for some fixed $u \in L^2(\mathbb{R}^d)$ and defining u_n as follows

$$\begin{aligned} &u_1 = \psi_1,\\ &u_2 = \psi_1,\ u_3 = \psi_2\\ &u_4 = \psi_1,\ u_5 = \psi_2,\ u_6 = \psi_3\\ &u_7 = \psi_1,\ \dots \end{aligned}$$

one can easily verify that $\mathbf{m}(\mathbf{u}) = \|u\|_{L^2}^2$ but there is no subsequence which weakly converges (up to translations) to u.

Example 1.2. *We compute $\mathbf{m}(\mathbf{u})$ for the examples in Figure 1. We start from Example 1a: fix $u \in L^2(\mathbb{R}^d)$, and let $u_n(x) = n^{-d/2}u(\frac{x}{n})$; for every sequence of translations $(x_n)_{n\in\mathbb{N}}$ such that $u_n(\cdot + x_n) \rightharpoonup v$ weakly in L^2 (up to subsequences) we have that for every $\varphi \in L^2$*

$$\begin{aligned}\int_{\mathbb{R}^d} v\varphi &= \lim_{n\to\infty}\int_{\mathbb{R}^d} u_n(x + x_n)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_n(x)\varphi(x - x_n)\,dx = \lim_{n\to\infty} n^{-d/2}\int_{\mathbb{R}^d} u(\frac{x}{n})\varphi(x - x_n)\,dx\\ &= \lim_{n\to\infty} n^{d/2}\int_{\mathbb{R}^d} u(x)\varphi(nx - x_n)\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^d} u(x)[n^{d/2}\varphi(nx - x_n)]\,dx = 0\end{aligned}$$

since $n^{d/2}\varphi(nx - x_n) \rightharpoonup 0$ (it is a combination of Examples 1b and 2a). We've proved that

$$\int_{\mathbb{R}^d} v\varphi = 0 \text{ for every } \varphi \in L^2(\mathbb{R}^d),$$

so we have $v = 0$. This proves that for Example 1a $\mathbf{m}(\mathbf{u}) = 0$.
Following this line of reasoning one can tackle Examples 1b and 2b.

To tackle Example 2a we have to use a different strategy: using the sequence of translations $x_n = n\xi$ we have that $u_n(x + x_n) = u(x - n\xi + n\xi) = u(x)$, for every $x \in \mathbb{R}^d$, so $u_n(\cdot + x_n) \to u$ strongly (and in particular weakly) in L^2, hence $\mathbf{m}(\mathbf{u}) \geq \|u\|^2_{L^2}$. Next, since $\|\cdot\|_{L^2}$ is weakly lowersemicontinous we also have that for every sequence of translations $(x_n)_{n\in\mathbb{N}}$ such that $u_n(\cdot + x_n) \rightharpoonup v$ weakly in L^2 $\|v\|_{L^2} \leq \liminf_{n\to\infty} \|u_n(\cdot + x_n)\|_{L^2} = \liminf_{n\to\infty} \|u_n\|_{L^2} = \|u\|_{L^2}$, hence $\mathbf{m}(\mathbf{u}) \leq \|u\|^2_{L^2}$. In conclusion, we have that for Example 2a $\mathbf{m}(\mathbf{u}) = \|u\|^2_{L^2}$.

If our sequence $\mathbf{u}$ is bounded in $H^1(\mathbb{R}^d)$ we can give the following equivalent definition of $\mathbf{m}(\mathbf{u})$:

Lemma 1.3. *For every bounded sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^d)$ we have*

$$\mathbf{m}(\mathbf{u}) = \lim_{R\to\infty} \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2, \tag{1.2}$$

where $B_R(x)$ denotes the ball of radius $R > 0$ centered in $x \in \mathbb{R}^d$.

We will provide a proof of this statement in the Appendix.

The purpose of $\mathbf{m}(\mathbf{u})$ is to detect the largest piece of mass in the sequence $\mathbf{u}$; this piece of mass can escape to infinity if $|x_k| \to \infty$. Indeed, if $\mathbf{m}(\mathbf{u}) > 0$ one can find a subsequence u_{n_k}, translations $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ and $0 \neq u \in L^2(\mathbb{R}^d)$ such that $u_{n_k}(\cdot + x_k) \rightharpoonup u$ weakly in L^2 and $0 < \mathbf{m}(\mathbf{u}) - \epsilon \leq \|u\|^2_{L^2} \leq \mathbf{m}(\mathbf{u})$.
We have found the first "bubble" u; we could go on and try to find the next one by considering $r_k = u_{n_k} - u(\cdot - x_k)$ and the corresponding $\mathbf{m}(\mathbf{r})$. Proceeding by induction, we could find all the bubbles contained in the original sequence $\mathbf{u}$, as shown in the following

Lemma 1.4 (Extracting Bubbles)**.** *Let $\mathbf{u} = \{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$). Then there exists a sequence of functions $\{u^{(1)}, u^{(2)}, ...\}$ in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R})^d$) such that the following holds:*
For any $\epsilon > 0$ fixed, there exists

- $J \in \mathbb{N}$,
- *A subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $\mathbf{u}$,*
- *space translations $(x^{(j)}_k)_{k\geq 1} \subset \mathbb{R}^d$, $j = 1, ..., J$ such that $|x^{(j)}_k - x^{(j')}_k| \xrightarrow{k\to\infty} \infty$ for $j \neq j'$,*

such that

$$u_{n_k} = \sum_{j=1}^{J} u^{(j)}(\cdot - x^{(j)}_k) + r^{(J+1)}_k \tag{1.3}$$

where $r^{(J+1)}_k(\cdot + x^{(j)}_k) \xrightarrow{k\to\infty} 0$ weakly in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R})^d$) for all $j = 1, ..., J$ and $\mathbf{m}(\mathbf{r}^{(J+1)}) \leq \epsilon$. In particular, we have

$$u_{n_k}(\cdot + x^{(j)}_k) \rightharpoonup u^{(j)} \text{ weakly in } L^2(\mathbb{R}^d)(\text{resp. } H^1(\mathbb{R}^d))$$

and

$$\lim_{k\to\infty} \left(\|u_{n_k}\|^2_{L^2} - \|r^{J+1}_k\|^2_{L^2}\right) = \sum_{j=1}^{J} \|u^{(j)}\|^2_{L^2}. \tag{1.4}$$

Moreover, if the sequence $\mathbf{u}$ is also bounded in $H^1(\mathbb{R})^d$ we have

$$\lim_{k\to\infty} \left(\|\nabla u_{n_k}\|^2_{L^2} - \|\nabla r^{J+1}_k\|^2_{L^2}\right) = \sum_{j=1}^{J} \|\nabla u^{(j)}\|^2_{L^2} \tag{1.5}$$

and

$$\lim_{k\to\infty}\left(\|u_{n_k}\|^p_{L^p}-\|r_k^{J+1}\|^p_{L^p}\right)=\sum_{j=1}^J\|u^{(j)}\|^p_{L^p} \tag{1.6}$$

for every subcritical p, *i.e.* $2\le p\le 2^*$, $2^*=\infty$ *for* $d=1,2$ *and* $2^*=\frac{2d}{d-2}$ *for* $d\ge 3$.

The bubbles $u^{(j)}$ are the possible weak limits of subsequences of $\mathbf{u}$ up to the translations $x_k^{(j)}$. What (1.3) is telling us is that we can decompose $\mathbf{u}$ as a linear combination of these limits (translated in space) up to the reminder $r_k^{(J+1)}$. This reminder is not necessarily small in L^2-norm, because it can still have other compactness issues; however, we know that its maximal local mass is small. Let us also remark that the bubbles $u^{(j)}$ do not depend on the choice of ϵ: they can be constructed a priori for any sequence $\mathbf{u}$. However, the choice of ϵ influences the number of bubbles J, the translations $x_k^{(j)}$ and the choice of subsequence u_{n_k}.

Proof of Lemma 1.3. Let Let $\mathbf{u}=\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$). We can assume that $\mathbf{m}(\mathbf{u})>0$, otherwise there would be nothing to prove. Then, there exist a subsequence u_{n_k}, translations $(x_k)_{k\in\mathbb{N}}\subset\mathbb{R}^d$ and $u^{(1)}\in L^2(\mathbb{R}^d)$, $u^{(1)}\neq 0$ such that $\frac{\mathbf{m}(\mathbf{u})}{2}\le\int_{\mathbb{R}^d}|u^{(1)}|^2\le\mathbf{m}(\mathbf{u})$ and $u_{n_k}(\cdot+x_k^{(1)})\xrightarrow{k\to\infty}u^{(1)}$ weakly in L^2 (resp. H^1). We define $r_k^{(2)}:=u_{n_k}-u^{(1)}(\cdot-x_k^{(1)})$, so that $r_k^{(2)}(\cdot+x_k^{(1)})\rightharpoonup 0$ weakly in L^2 (resp. H^1). Moreover,

$$\|u_{n_k}\|^2_{L^2}=\|u^{(1)}(\cdot-x_k^{(1)})\|^2_{L^2}+\|r_k^{(2)}\|^2_{L^2}+2\Re\left\langle r_k^{(2)},u^{(1)}(\cdot-x_k^{(1)})\right\rangle_{L^2}.$$

Next, since $\|u^{(1)}(\cdot-x_k^{(1)})\|^2_{L^2}=\|u^{(1)}\|^2_{L^2}$ and $\left\langle r_k^{(2)},u^{(1)}(\cdot-x_k^{(1)})\right\rangle_{L^2}=\left\langle r_k^{(2)}(\cdot+x_k^{(1)}),u^{(1)}\right\rangle_{L^2}\to 0$ by the weak convergence of $r_k^{(2)}$, we conclude that

$$\lim_{k\to\infty}\left(\|u_{n_k}\|^2_{L^2}-\|r_k^{(2)}\|^2_{L^2}\right)=\|u^{(1)}\|^2_{L^2}.$$

If $\mathbf{u}$ is bounded also in H^1, reasoning in the same way we get in addition

$$\|\nabla u_{n_k}\|^2_{L^2}=\|\nabla u^{(1)}(\cdot-x_k^{(1)})\|^2_{L^2}+\|\nabla r_k^{(2)}\|^2_{L^2}+2\Re\left\langle\nabla r_k^{(2)},\nabla u^{(1)}(\cdot-x_k^{(1)})\right\rangle_{L^2}$$

that we use to conclude

$$\lim_{k\to\infty}\left(\|\nabla u_{n_k}\|^2_{L^2}-\|\nabla r_k^{(2)}\|^2_{L^2}\right)=\|\nabla u^{(1)}\|^2_{L^2}.$$

Now, if $\mathbf{m}(\mathbf{r}^{(2)})=0$ we are done. If not, we can find a further subsequence n_{k_l}, other space translations and $u^{(2)}\in L^2$, $u_2\neq 0$ such that $r_{k_l}^{(2)}(\cdot+x_l^{(2)})\rightharpoonup u^{(2)}$. We also extract a further subsequence $u_{n_{k_l}}$ and $x_{k_l}^{(1)}$. For ease of notation, we will only index further subsequences with the index k. We can can then write $u_{n_k}=u^{(1)}(\cdot-x_k^{(1)})+u^{(2)}(\cdot-x_k^{(2)})+r_k^{(3)}$ with $r_k^{(3)}(\cdot+x_k^{(2)})\rightharpoonup 0$.

Now, we prove that $|x_k^{(1)}-x_k^{(2)}|\to\infty$: by contradiction, if $x_k^{(1)}-x_k^{(2)}$ is bounded in $\mathbb{R}^d$, then there exists $v\in\mathbb{R}^d$ such that, up to a subsequence, $x_k^{(1)}-x_k^{(2)}\to v$. Next, $u_{n_k}(\cdot+x_k^{(1)})\rightharpoonup u^{(1)}+u^{(2)}(\cdot+v)$ because $r_k^{(3)}(\cdot+x_k^{(1)})=r_k^{(3)}(\cdot+v+x_k^{(2)})\rightharpoonup 0$ by the weak convergence of $r_k^{(3)}$. Since $u_{n_k}(\cdot+x_k^{(1)})\rightharpoonup u^{(1)}$ by construction, this would imply that $u^{(2)}=0$, which is not the case.

Now, $r_k^{(3)}(\cdot+x_k^{(1)})=u_{n_k}(\cdot+x_k^{(1)})-u^{(1)}-u^{(2)}(\cdot+x_k^{(1)}-x_k^{(2)})\rightharpoonup 0$ since $u_{n_k}(\cdot+x_k^{(1)})\rightharpoonup u^{(1)}$ and $u^{(2)}(\cdot-(x_k^{(2)}-x_k^{(1)}))\rightharpoonup 0$ because $|x_k^{(1)}-x_k^{(2)}|\to\infty$.

Finally, we could apply the same reasoning as before to get

$$\lim_{k\to\infty}\left(\|\nabla u_{n_k}\|^2_{L^2}-\|\nabla r_k^{(2)}\|^2_{L^2}\right)=\|\nabla u^{(1)}\|^2_{L^2}+\|\nabla u^{(2)}\|^2_{L^2}.$$

We could repeat this process until we reach a remainder r_k^{J+1} such that $\mathbf{m}(\mathbf{r}^{(J+1)}) = 0$; if that is not reached, we continue and construct $u^{(j)}$ and r^j for every $j \in \mathbb{N}$. Moreover, this construction satisfies

$$\lim_{k\to\infty} \left(\|u_{n_k}\|_{L^2}^2 - \|r_k^{J+1}\|_{L^2}^2\right) = \sum_{j=1}^{J} \|u^{(j)}\|_{L^2}^2$$

for every $J \in \mathbb{N}$. In particular, $\sum_{j\in\mathbb{N}} \|u^{(j)}\|_{L^2}^2 \leq \limsup_{n\to\infty} \|u_n\|_{L^2}^2 < \infty$, therefore $\|u^{(j)}\|_{L^2}^2 \to 0$ as $j \to \infty$. In addition, if the sequence is bounded in H^1, the construction also satisfies

$$\lim_{k\to\infty} \left(\|\nabla u_{n_k}\|_{L^2}^2 - \|\nabla r_k^{J+1}\|_{L^2}^2\right) = \sum_{j=1}^{J} \|\nabla u^{(j)}\|_{L^2}^2.$$

Finally, since by construcion $\mathbf{m}(\mathbf{r}^{J+1}) \leq 2\int_{\mathbb{R}^d} |u^{(j)}|^2$, $\mathbf{m}(\mathbf{r}^{(J)}) \to 0$ as $J \to \infty$.
To prove (1.6) in the H^1 framework, we use two functional analysis tools: the *Rellich-Kondrachov Theorem*, which states that if $u_n \rightharpoonup u$ weakly in H^1 then up to a subsequence $u_n \to u$ strongly in L^p_{loc} for every $2 \leq p < 2^*$ and almost everywhere, and the *Brezis-Lieb Lemma*, which states that if $u_n \to u$ almost everywhere, then $\lim_{n\to\infty} \left(\int_{\mathbb{R}^d} |u_n|^p - \int_{\mathbb{R}^d} |u - u_n|^p\right) = \int_{\mathbb{R}^d} |u|^p$ for every $p \geq 1$.
First, $u_{n_k}(\cdot + x_k^{(1)}) \rightharpoonup u^{(1)}$ weakly in H^1, so up to a subsequence $u_{n_k}(\cdot + x_k^{(1)}) \to u^{(1)}$ in L^2 and almost everywhere. In turn, this implies that

$$\begin{aligned}
\int_{\mathbb{R}^d} |u^{(1)}|^p &= \lim_{k\to\infty} \left(\int_{\mathbb{R}^d} |u_{n_k}(x + x_k^{(1)})|^p\, dx - \int_{\mathbb{R}^d} |u^{(1)}(x) - u_{n_k}(x + x_k^{(1)})|^p\, dx\right) \\
&= \lim_{k\to\infty} \left(\int_{\mathbb{R}^d} |u_{n_k}|^p - \int_{\mathbb{R}^d} |u_{n_k}(x) - u^{(1)}(x - x_k^{(1)})|^p\, dx\right) \\
&= \lim_{k\to\infty} \left(\int_{\mathbb{R}^d} |u_{n_k}|^p - \int_{\mathbb{R}^d} |u_{n_k}(x + x_k^{(2)}) - u^{(1)}(x - (x_k^{(1)} - x_k^{(2)}))|^p\, dx\right).
\end{aligned}$$

Since $u_{n_k}(\cdot + x_k^{(2)}) \rightharpoonup u^{(2)}$ and $u^{(1)}(\cdot - (x_k^{(1)} - x_k^{(2)})) \rightharpoonup 0$ weakly in H^1, we can repeat this argument to get $\lim_{k\to\infty} \left(\int_{\mathbb{R}^d} |u_{n_k}|^p - \int_{\mathbb{R}^d} \left|u_{n_k}(x + x_k^{(2)}) - u^{(1)}(x - (x_k^{(1)} - x_k^{(2)}) - u^{(2)}(x)\right|^p\, dx\right) = \int_{\mathbb{R}^d} \left|u^{(1)}\right|^p + \int_{\mathbb{R}^d} \left|u^{(2)}\right|^p$.
The same reasoning can be applied J times to finally get

$$\lim_{k\to\infty} \left(\|u_{n_k}\|_{L^p}^p - \|r_k^{J+1}\|_{L^p}^p\right) = \sum_{j=1}^{J} \|u^{(j)}\|_{L^p}^p.$$

Notice that all the terms are well defined because $H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for every $2 \leq p \leq 2^*$. □

As we will see in Section 2.2, sometimes it is useful to "isolate" the bubbles; this means writing u_{n_k} as a sum of "localized bubbles" $u_k^{(j)}$ with compact support such that $u_k^{(j)} \to u^{(j)}$ strongly and the distance between the supports diverges. This is more in spirit of Lions' original concentration-compactness principle seen in [8]; it is not always useful in the L^2 or H^1 framework, but it is the preferred technique in other spaces, in particular non-Hilbert ones. In our setting we provide the following theorem, which we will prove in the Appendix.

Theorem 1.5 (Extracting localized bubbles)**.** *Let* $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^1(\mathbb{R}^d)$ *and* $(u^{(j)})_{j\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$ *be the sequence given by Lemma 1.4. For any* $\epsilon > 0$ *and any fixed sequence* $0 \leq R_k \xrightarrow{k\to\infty} \infty$*, there exist*

- $0 \leq J \in \mathbb{N}$,
- *a subsequence* $(u_{n_k})_{k\in\mathbb{N}}$,
- *sequences of functions* $\mathbf{u}^{(1)} = (u_k^{(1)})_{k\in\mathbb{N}}, \ldots, \mathbf{u}^{(J)} = (u_k^{(J)})_{k\in\mathbb{N}}$, $\boldsymbol{\psi}^{(J+1)} = (\psi_k^{(J+1)})_{k\in\mathbb{N}}$ *in* $H^1(\mathbb{R}^d)$,
- *space translations* $\mathbf{x}^{(1)} = (x_k^{(1)})_{k\in\mathbb{N}}, \ldots, \mathbf{x}^{(J)} = (x_k^{(J)})_{k\in\mathbb{N}}$ *in* $\mathbb{R}^d$,

such that

$$\lim_{k\to\infty} \left\| u_{n_k} - \sum_{j=1}^{J} u_k^{(j)}(\cdot - x_k^{(j)}) - \psi_k^{(J+1)} \right\|_{H^1(\mathbb{R}^d)} = 0 \tag{1.7}$$

where

- $u_k^{(j)}$ *converges to* $u^{(j)}$ *weakly in* $H^1(\mathbb{R}^d)$ *and strongly in* $L^p(\mathbb{R}^d)$ *for all* $2 \leq p < 2^*$;
- *supp*$(u_k^{(j)}) \subset B_{R_k}(0)$ *for all* $j = 1, \ldots, J$ *and all* k;
- *supp*$(\psi_k^{(J+1)}) \subset \mathbb{R}^d \setminus \bigcup_{j=1}^{J} B_{2R_k}(x_k^{(j)})$ *for all* k;
- $|x_k^{(i)} - x_k^{(j)}| \geq 5R_k$ *for all* $i \neq j$ *and all* k;
- $\mathbf{m}(\boldsymbol{\psi}^{(J+1)}) \leq \epsilon$.

We remark that $\boldsymbol{\psi}^{(J+1)}$ is different from $\mathbf{r}^{(J+1)}$ defined in Lemma 1.4, even though they behave essentially the same in the limit $k \to \infty$. Once again, $\boldsymbol{\psi}^{(J+1)}$ is not necessarily small in L^2 norm, since it can still undergo vanishing, but we know that it does not contain local mass larger than ϵ.
Even though we cannot say anything of the L^2 masses of both $\mathbf{r}^{(J+1)}$ and $\boldsymbol{\psi}^{(J+1)}$, we can say something about their subcritical norms (provided $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1): indeed, at the end of section 1.2 we will prove the following

Lemma 1.6. *Let* $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^1(\mathbb{R}^d)$. *There exists a constant* C *depending only on* d *such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \leq C\mathbf{m}(\mathbf{u})^{\frac{2}{d}} \limsup_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^d)}^2. \tag{1.8}$$

Using Lemma 1.6 and Hölder's inequality we have that for every $2 < p < 2^*$

$$\|r_n^{(J+1)}\|_{L^p} \leq \|r_n^{(J+1)}\|_{L^2}^{\theta} \|r_n^{(J+1)}\|_{L^{2+4/d}}^{1-\theta} \lesssim \mathbf{m}(\mathbf{r}^{(J+1)})^{\frac{2-2\theta}{d}} \text{ if } 2 < p < 2 + \frac{4}{d},\ \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2+4/d},$$

$$\|r_n^{(J+1)}\|_{L^p} \leq \|r_n^{(J+1)}\|_{L^{2+4/d}}^{\theta} \|r_n^{(J+1)}\|_{2^*}^{1-\theta} \lesssim \mathbf{m}(\mathbf{r}^{(J+1)})^{\frac{2\theta}{d}} \text{ if } 2 + \frac{4}{d} < p < 2^*,\ \frac{1}{p} = \frac{\theta}{2+4/d} + \frac{1-\theta}{2^*},$$

and similarly for $\boldsymbol{\psi}^{(J+1)}$. This means that we can make the subcritical norms of the remainders $\mathbf{r}^{(J+1)}$ and $\boldsymbol{\psi}^{(J+1)}$ as small as we want, provided we take J large enough.

1.2 Vanishing sequences and subcritical L^p norms

In the previous section we have defined the highest mass that the weak limits can have up to translations; then, by Lemma 1.4 we proved that any bounded sequence in $L^2(\mathbb{R}^d)$ can be written as a linear combination of "bubbles" plus a remainder $\mathbf{r}^{(J+1)}$ such that $\mathbf{m}(\mathbf{r}^{(J+1)})$ is small. If we continue this process indefinetly, we are essentially left with a remainder $\mathbf{r}^{(\infty)}$ such that $\mathbf{m}(\mathbf{r}^{(\infty)}) = 0$. Motivated by our earlier remarks on L^p norms in the examples of Figure 1, we proceed in studying what happens when $\mathbf{m}(\mathbf{u}) = 0$ under the additional hypothesis of boundedness in $H^1(\mathbb{R}^d)$,

Lemma 1.7 (Vanishing)**.** *Let* $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^1(\mathbb{R}^d)$. *Then the following are equivalent:*

1. $\mathbf{m}(\mathbf{u}) = 0$;
2. *For all* $R > 0$, *we have* $\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2 = 0$;
3. $u_n \to 0$ *strongly in* L^p *for all* $2 < p < 2^*$.

Proof of Lemma 1.7.
(1)⇒(2): We start by proving that

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 \le \mathbf{m}(\mathbf{u}) \tag{1.9}$$

with $C_z = \prod_{j=1}^{d} [z^{(j)}, z^{(j+1)})$; here we used the notation $\mathbb{R}^d \ni z = (z^{(1)}, ..., z^{(d)})$. Consider a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\lim_{n\to\infty} \int_{C_{z_n}} |u_n|^2 = \limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2$. Then, the sequence $u_n(\cdot + z_n)$ is bounded in H^1, so up to a subsequence it converges to u weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^2(C_0)$. Next, since u is the weak limit of $(u_n)_{n\in\mathbb{N}}$ up to the translations $(z_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \int_{C_{z_n}} |u_n|^2 = \lim_{k\to\infty} \int_{C_0} |u_{n_k}(x + z_{n_k})|^2\, dx = \int_{C_0} |u|^2 \le \int_{\mathbb{R}^d} |u|^2 \le \mathbf{m}(\mathbf{u})$$

which proves our claim.
Next, from (1.9) and the hypothesis *(1)* we get that $\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 = 0$. Finally, since every ball $B_R(x) \subset \mathbb{R}^d$ can be covered by finitely many unitary cubes $C_{z_1}^{(x,R)}, ..., C_{z_k}^{(x,R)}$, we get

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2 \le \sum_{j=1}^{k} \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{C_{z_j}^{(x,R)}} |u_n|^2 \le k \limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 = 0.$$

(2)⇒(3): First, we prove that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \le C \left(\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 \right)^{\frac{2}{d}} \limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}. \tag{1.10}$$

By Hölder's inequality

$$\int_{\mathbb{R}^d} |u_n|^q = \sum_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^q \le \sum_{z\in\mathbb{Z}^d} \|u_n\|^{q\theta}_{L^2_{C_z}} \|u_n\|^{q(1-\theta)}_{L^{2^*}(C_z)} \text{ with } \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{2^*}.$$

Choosing $q = 2 + \frac{4}{d}$, we have $q(1-\theta) = 2$ and $q\theta = \frac{4}{d}$. Notice that with this choice we have $2 < q < 2^*$. Combining this with the Sobolev embedding inequality $\|u_n\|^2_{L^{2^*}(C_z)} \le C\|u_n\|^2_{H^1(C_z)}$ we get

$$\int_{\mathbb{R}^d} |u_n|^{2+4/d} \le C(\sup_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)})^{\frac{4}{d}} \sum_{z\in\mathbb{Z}^d} \|u_n\|_{H^1(C_z)} = C(\sup_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)})^{\frac{4}{d}} \|u_n\|_{H^1(\mathbb{R}^d)}$$

because the constant C depends only on the volume of the d-dimensional cube. Passing to the limsup $n \to \infty$ and recalling that the limsup of a product of non negative sequences is not greater than the

product of the limsups of the sequences, we deduce our claim.
Then, by (1.10) and the hypothesis *(2)*, we have that $\|u_n\|_{L^{2+4/d}} \to 0$. Recalling that by Sobolev embedding $(u_n)_{n\in\mathbb{N}}$ is bounded in both L^2 and L^{2^*}, we end using Hölder inequality in two different ways:

$$\|u_n\|_{L^p} \le \|u_n\|_{L^2}^{\theta}\|u_n\|_{L^{2+4/d}}^{1-\theta} \text{ if } 2 < p < 2+\frac{4}{d} \text{ with } \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2+4/d};$$
$$\|u_n\|_{L^p} \le \|u_n\|_{L^{2+4/d}}^{\theta}\|u_n\|_{L^{2^*}}^{1-\theta} \text{ if } 2+\frac{4}{d} < p < 2^* \text{ with } \frac{1}{p} = \frac{\theta}{2+4/d} + \frac{1-\theta}{2^*}.$$

(3)⇒(1): Let $(x_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k}(\cdot + x_{n_k}) \rightharpoonup u$ weakly in H^1 and strongly in L^p, $2 < p < 2^*$. Since $\|u_{n_k}(\cdot + x_{n_k})\|_{L^p} = \|u_{n_k}\|_{L^p} \to 0$ we have that $u_{n_k} \to 0$ strongly in L^p so by uniqueness of the weak limit $u = 0$. Since every weak limit of subsequences of $\mathbf{u}$ up to translations is 0, then $\mathbf{m}(\mathbf{u}) = 0$. □

Notice that the hypothesis $\mathbf{u}$ bounded in H^1 and not only in L^2 is really necessary: in Examples 1b and 2b we have $\mathbf{m}(\mathbf{u}) = 0$ but both *(2)* and *(3)* fail.

Finally, notice that the proof of Lemma 1.6 is a direct consequence of our proof of Lemma 1.7: indeed combining (1.9) and (1.10) we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \le C\mathbf{m}(\mathbf{u})^{\frac{2}{d}} \limsup_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^d)}^2.$$

1.3 The concentration compactness method in Hilbert spaces

The *concentration-compactness principle* was first stated by Lions in [8, 9]; we will briefly discuss his results and then focus on [6], which provides a more structured approach to the matter.
The concentration-compactness principle is a method which can be used to prove the compactness of sequences $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ (possibly up to translations); in particular, we are interested in applying it to minimizing sequences for variational problems. It is important to note that applying this method does not consist in merely checking the hypothesis of some abstract theorem; one has to adapt a general strategy to each practical case.
The main idea is to prove the compactness of $\mathbf{u}$ by showing that it must stay "concentrated", meaning that it does not split in two or more bubbles and it does not vanish. In practice, this is done proving that if the sequence is not compact then the energy is too high.

1.3.1 Setting

Let $\mathcal{E}$ be an energy functional on $H^1(\mathbb{R}^d)$, which is bounded from below, continous and coercive on

$$\mathcal{S}_{\le\lambda} = \left\{ u \in H^1(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |u|^2 \le \lambda \right\}.$$

Letting

$$\mathcal{S}_{\lambda} = \left\{ u \in H^1(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |u|^2 = \lambda \right\},$$

we are interested in the minimization problem

$$I(\lambda) = \inf_{u\in\mathcal{S}_\lambda} \mathcal{E}(u). \tag{M$_\lambda$}$$

We always assume that $I(0) = 0$.
To deal with vanishing and splitting (or, as Lions calls it, *dichotomy*) we introduce two auxiliary funcionals, together with corresponding minimization problems:

- To deal with vanishing, we define $\mathcal{E}^{\text{van}}$ which is the original energy $\mathcal{E}$ to which we remove all the "subcritical terms", i.e. all terms that go to 0 when $u_n \to 0$ in $L^p(\mathbb{R}^d)$ for $2 < p < 2^*$. Then, we define
$$I^{\text{van}}(\lambda) = \inf_{u \in \mathcal{S}_\lambda} \mathcal{E}^{\text{van}}(u) = \inf_{\substack{\mathbf{u}=(u_n)\subset \mathcal{S}_\lambda \\ \mathbf{m}(\mathbf{u})=0}} \liminf_{n\to\infty} \mathcal{E}(u_n) \tag{$\text{M}_\lambda^{\text{van}}$}$$
In most applications, $\mathcal{E}^{\text{van}}$ only contains the gradient terms of $\mathcal{E}$.
- To deal with dichotomy, we define a "problem at infinity": we consider $\mathcal{E}^\infty$, which is the original energy $\mathcal{E}$ to which we remove all the compact terms that converge to 0 when $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^d)$ without assuming a priori that $u_n \to 0$ in $L^p(\mathbb{R}^d)$. Then, we define
$$I^\infty(\lambda) = \inf_{u \in \mathcal{S}_\lambda} \mathcal{E}^\infty(u) = \inf_{\substack{(u_n)\subset \mathcal{S}_\lambda \\ u_n \rightharpoonup 0}} \liminf_{n\to\infty} \mathcal{E}(u_n) \tag{M_λ^∞}$$
In most examples, $\mathcal{E}^\infty$ consists in the translation-invariant terms of $\mathcal{E}$ and the ones which admit a "limit at infinity".

Notice that $I^\infty(\lambda) \le I^{\text{van}}(\lambda)$. Moreover, if the initial problem (M_λ) is translation invariant we have $I^\infty(\lambda) = I(\lambda)$.

1.3.2 A general view: the original concentration-compactness principle

In this section we adapt to our setting what Lions discusses in [8]; there one can find a general method for functionals defined on Hilbert and Banach spaces.
First of all, one can always prove that
$$I(\lambda) \le I(\alpha) + I^\infty(\lambda - \alpha) \text{ for every } 0 \le \alpha < \lambda.$$
We give an heuristic proof of why this inequality holds: let $\epsilon > 0$ and u_ϵ, v_ϵ such that
$$\begin{cases} I(\alpha) \le \mathcal{E}(u_\epsilon) \le I(\alpha) + \epsilon, & \|u_\epsilon\|_{L^2}^2 = \alpha \\ I^\infty(\lambda-\alpha) \le \mathcal{E}^\infty(v_\epsilon) \le I^\infty(\lambda - \alpha) + \epsilon, & \|v_\epsilon\|_{L^2}^2 = \lambda - \alpha \end{cases}$$
Without loss of generality u_ϵ and v_ϵ have compact support. For a fixed unit vector $\xi \in \mathbb{R}^d$ we define $v_\epsilon^{(n)} = v_\epsilon(\cdot + n\xi)$, so that $d_n := \text{dist}(\text{supp}(u_\epsilon), \text{supp}(v_\epsilon^{(n)})) \to \infty$. Hence we deduce
$$\begin{cases} \|u_\epsilon + v_\epsilon^{(n)}\|_{L^2}^2 - (\|u_\epsilon\|_{L^2}^2 + \|v_\epsilon^{(n)}\|_{L^2}^2) \to 0 \\ \mathcal{E}(u_\epsilon + v_\epsilon^{(n)}) - (\mathcal{E}(u_\epsilon) + \mathcal{E}^\infty(v_\epsilon^{(n)})) \to 0 \end{cases}$$
and since $\mathcal{E}^\infty$ is translation invariant
$$\begin{cases} I(\alpha) + I^\infty(\lambda - \alpha) \le \lim_n \mathcal{E}(u_\epsilon + v_\epsilon^{(n)}) = \lim_n (\mathcal{E}(u_\epsilon) + \mathcal{E}(v_\epsilon^{(n)})) \le I(\alpha) + I^\infty(\lambda-\alpha) + 2\epsilon \\ \|u_\epsilon\|_{L^2}^2 + \|v_\epsilon^{(n)}\|_{L^2}^2 \to \lambda \end{cases}$$
so we conclude
$$I(\lambda) \le I(\alpha) + I^\infty(\lambda - \alpha) + 2\epsilon.$$
We now describe the typical results that we can obtain using the concentration-compactness principle: in the case where $\mathcal{E}$ is not translation-invariant we have that for every fixed $\lambda > 0$ all minimizing sequences of (M_λ) are relatively compact if and only if
$$I(\lambda) < I(\alpha) + I^\infty(\lambda - \alpha) \text{ for every } 0 \le \alpha < \lambda. \tag{C1}$$

In the translation-invariant case, where (M_λ) and ($\mathrm{M}_\lambda^\infty$) are equivalent, we have that for every fixed $\lambda > 0$ all minimizing sequences of (M_λ) are relatively compact (up to translations) if and only if

$$I(\lambda) < I(\alpha) + I(\lambda - \alpha) \text{ for every } 0 \leq \alpha < \lambda \tag{C2}$$

since in this case $I(\alpha) = I^\infty(\alpha)$ for every $0 < \alpha \leq \lambda$. The fact that the condition (C1) (resp. (C2)) is necessary is a consequence of the argument we just used: for instance, if for some $0 < \alpha < \lambda$

$$I(\lambda) = I(\alpha) + I(\lambda - \alpha)$$

and we denote with u_n, v_n minimizing sequences with compact supports of (M_α) and ($\mathrm{M}_{\lambda-\alpha}$) respectivly, letting $\tilde{v}_n = v_n(\cdot + n\xi)$ we have that $w_n = u_n + \tilde{v}_n$ cannot be relatively compact. Indeed, since $\mathrm{dist}(\mathrm{supp}(u_n), \mathrm{supp}(v_n)) \to \infty$ we can find $\chi_n \in C_C^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} w_n \chi_n = 0$. However, we also find that

$$\begin{cases} \|w_n\|_{L^2}^2 \to \lambda \\ \mathcal{E}(w_n) = \lim_n (\mathcal{E}(u_n) + \mathcal{E}(v_n)) \to I(\lambda) \end{cases}$$

hence w_n is a non relatively compact minimizing sequence.
We now give an heuristic argument for proving that (C1) (resp. (C2)) is sufficient to ensure the relative compactness of the minimizing sequences. The argument will be based on the following Lemma, which admits variants based on the setting:

Lemma 1.8 (Concentration-compactness Lemma). *Let $(\rho_n)_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^d)$ such that $\|u_n\|_{L^1} = \lambda$ where $\lambda > 0$ is fixed. Then there exists a subsequence $(\rho_{n_k})_{k\in\mathbb{N}}$ such that one of the following three possibilities occurs:*

1. *(Compactness) There exists $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that for every $\epsilon > 0$ exists $0 < R < \infty$ such that*

$$\int_{B_R(y_k)} \rho_{n_k} \geq \lambda - \epsilon; \tag{1.11}$$

2. *(Vanishing) For every $0 < R < \infty$*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B_R(y)} \rho_{n_k} = 0; \tag{1.12}$$

3. *(Dichotomy) There exists $0 < \alpha < \lambda$ such that for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ and $0 \leq \rho_k^{(1)}, \rho_k^{(2)} \in L^1(\mathbb{R}^d)$ such that for every $k \geq k_0$*

$$\begin{cases} \|\rho_{n_k} - (\rho_k^{(1)} + \rho_k^{(2)})\|_{L^1}) \leq \epsilon \\ |\|\rho_k^{(1)}\|_{L^1} - \alpha| < \epsilon, \ |\|\rho_k^{(2)}\|_{L^1} - (\lambda - \alpha)| < \epsilon \\ \mathrm{dist}(\mathrm{supp}(\rho_k^{(1)}), \mathrm{supp}(\rho_k^{(2)})) \to \infty \end{cases} \tag{1.13}$$

The proof of this Lemma uses the *Levy concentration functions* we will introduce in the proof of Theorem 1.5; one can find it in [8].

We apply Lemma 1.8 to a minimizing sequence for (M_λ) with $\rho_n = |u_n|^2$ and find a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that *(1)*, *(2)* or *(3)* occurs. We then have to rule out possibilities *(2)* and *(3)*:
First of all, *(2)* cannot occur since by (C1) $I(\lambda) < I^\infty(\lambda)$ and $\|u_n\|_{L^2}^2 = \lambda$. Then, if *(3)* occurs we can split u_n as we split ρ_{n_k} and find, for all $\epsilon > 0$, $u_k^{(1)}, u_k^{(2)} \in H^1(\mathbb{R}^d)$ such that

$$\begin{cases} |\|u_k^{(1)}\|_{L^2}^2 - \alpha| < \epsilon, \ |\|u_k^{(2)}\|_{L^2}^2 - (\lambda - \alpha)| < \epsilon, \ \|v_k\|_{L^2}^2 < \epsilon \\ \mathrm{dist}(\mathrm{supp}(u_k^{(1)}), \mathrm{supp}(u_k^{(2)})) \to \infty \end{cases}$$

Since we can exchange α and $\lambda - \alpha$, we can assume that $\lim_k (\mathcal{E}(u_k^{(2)}) - \mathcal{E}^\infty(u_k^{(2)})) \geq 0$, and we obtain

$$I(\lambda) = \lim_k \mathcal{E}(u_{n_k}) \geq \lim_k (\mathcal{E}(u_k^{(1)}) + \mathcal{E}^\infty(u_k^{(2)})) - \delta_\epsilon \geq I(\alpha - \epsilon) + I^\infty(\lambda - \alpha - \epsilon) - \delta_\epsilon$$

and sending ϵ to 0 we obtain

$$I(\lambda) \geq I(\alpha) + I^\infty(\lambda - \alpha)$$

which contradicts (C1). To conclude in the case where $\mathcal{E}$ is not translation invariant, one should also check that once we are reduced to *(1)*, we do not have $|y_k| \to \infty$.
We once again remark that the argument we've presented is not at all rigorous, it will have to be adapted and perfected in each practical case.
It is also important to notice that in order to be able to apply this method a lot of assumptions on the energy $\mathcal{E}$ are required: this is because we need our problem to be solvable by "usual" arguments (like convexity-compactness) if it was posed in a bounded domain instead of the whole space. Then, the role of (C1) (resp. (C2)) is to ensure that when we utilize Lemma 1.8 cases *(2)* and *(3)* cannot occur, so that we can reduce our problem to the case of a bounded domain.

1.3.3 A more specific approach for the H^1 framework

In this section we utilize the tools of Sections 1.1 and 1.2 to get a more precise procedure to follow when we want to use the concentration-compactness method for bounded sequences in $H^1(\mathbb{R}^d)$. Our main reference for this part is [6].
The main idea of this method is to study what happens to the minimizing sequences when they undergo vanishing or splitting and find out how the total energy behaves in these cases.
We start by proving the energetic inequalities

$$I(\lambda) \leq I(\lambda - \alpha) + I^\infty(\alpha) \text{ for all } 0 \leq \alpha \leq \lambda \tag{1.14}$$

and

$$I(\lambda) \leq I(\lambda - \alpha) + I^{\mathrm{van}}(\alpha) \text{ for all } 0 \leq \alpha \leq \lambda \tag{1.15}$$

The proof of these inequalities usually follows a similar blueprint to what we've discussed in the previous section. Then, since we always assume $I(0) = I^{\mathrm{van}}(0) = I^\infty(0)$ we get that

$$I(\lambda) \leq I^{\infty/\mathrm{van}}(\lambda) \text{ for every } \lambda$$

For translation-invariant functionals we always have $\mathcal{E}^\infty = \mathcal{E}$; in these cases the best we can do is to prove relative compactness *up to translations*. We start by explaining what to do in this case:
The first thing to do is to rule out vanishing, usually by proving that

$$I(\lambda) = I^\infty(\lambda) < I^{\mathrm{van}}(\lambda) \text{ for all } \lambda > 0 \tag{1.16}$$

Unlike our first energetic inequalities, there is no general blueprint to prove this strict one; in the example we give below, the argument is based on how the different terms of $\mathcal{E}(u)$ scale when we scale u. It is worth mentioning that (1.16) is not the only way to rule out vanishing: by Lemma 1.7, if we already know that the subcritical norms of u_n do not go to 0 we know that $\mathbf{m}(\mathbf{u}) > 0$, so vanishing does not occur.
Now, by definition of $\mathbf{m}(\mathbf{u})$ we know that there exist a subsequence (still denoted with u_n for ease of notation), $0 \neq u^{(1)} \in H^1(\mathbb{R}^d)$ and $(x_n^{(1)})_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $u_n(\cdot - x_n^{(1)}) \rightharpoonup u^{(1)}$ in $H^1(\mathbb{R}^d)$. Using Lemma 1.4 we write $u_n(\cdot - x_n^{(1)}) = u^{(1)} + r_n^{(2)}$ and prove that

$$\mathcal{E}(u_n) = \mathcal{E}(u_n(\cdot - x_n^{(1)})) = \mathcal{E}(u^{(1)}) + \mathcal{E}(r_n^{(2)}) + o_n(1) \tag{1.17}$$

using properties (1.4), (1.5), (1.6) or other similar results, for other kinds of terms in the energy $\mathcal{E}$. Defining

$$\lambda^{(1)} := \int_{\mathbb{R}^d} |u^{(1)}|^2 > 0$$

we remark that by (1.4)

$$\int_{\mathbb{R}^d} |r_n^{(2)}|^2 \to \lambda - \lambda^{(1)}.$$

Using that

$$\mathcal{E}(r_n^{(2)}) \geq I\left(\int_{\mathbb{R}^d} |r_n^{(2)}|^2\right)$$

and passing to the limit $n \to \infty$ (here we use the continuity of $\lambda \mapsto I(\lambda)$) we get

$$I(\lambda) = \mathcal{E}(u^{(1)}) + I(\lambda - \lambda^{(1)}) \geq I(\lambda^{(1)}) + I(\lambda - \lambda^{(1)})$$

which together with (1.14) gives

$$I(\lambda) = \mathcal{E}(u^{(1)}) + I(\lambda - \lambda^{(1)}) = I(\lambda^{(1)}) + I(\lambda - \lambda^{(1)}).$$

This in particular implies that $u^{(1)}$ is a minimizer for $(\mathrm{M}_{\lambda^{(1)}})$ and that $\mathcal{E}(r_n^{(2)}) \to I(\lambda - \lambda^{(1)})$, i.e. $(r_n^{(2)})_{n\in\mathbb{N}}$ is a minimizing sequence for $(\mathrm{M}_{\lambda-\lambda^{(1)}})$.
If we already know that

$$I(\lambda) < I(\lambda - \alpha) + I(\alpha) \text{ for all } 0 < \alpha < \lambda \tag{1.18}$$

we conclude (since $\lambda^{(1)} > 0$) that $\lambda = \lambda^{(1)}$ and so $u^{(1)}$ is the sought minimizer of (M_λ).
In practice, proving (1.18) may be difficult without knowing more information on $I(\alpha)$ and $I(\lambda - \alpha)$. In the example provided in Section 2.2, to get this strict inequality we once again take advantage of the scaling properties of the Hartree energy functional through the application of the following

Lemma 1.9. *Let* $h\,;\,[0,\lambda] \to \mathbb{R}$ *such that for every* $x \in (0,\lambda)$ *and every* $\theta \in (1, \frac{\lambda}{x})$ $h(\theta x) < \theta h(x)$*. Then* $h(\lambda) < h(x) + h(\lambda - x)$ *for every* $0 < x < \lambda$*.*

Proof. If, $x \geq \lambda - x$, we have

$$h(\lambda) = h(\frac{\lambda}{x}x) < \frac{\lambda}{x}h(x) = h(x) + \frac{\lambda - x}{x}h(x) = h(x) + \frac{\lambda - x}{x}h(\frac{x}{\lambda - x}(\lambda - x)) < h(x) + h(\lambda - x).$$

Conversely, if $\lambda - x \geq x$ just exhange x and $\lambda - x$ in the previous computation. □

More generally, one possible strategy consists in extracting more bubbles and apply the previous argument to $\mathbf{r}^{(2)} = (r_n^{(2)})_{n\in\mathbb{N}}$, which we know to be a minimizing sequence for $I(\lambda - \lambda^{(1)})$ (assuming that $\lambda^{(1)} < \lambda$): by (1.16) $\mathbf{r}^{(2)}$ cannot vanish, so there exists $0 \neq u^{(2)} \in H^1(\mathbb{R}^d)$ and $x_n^{(2)}$ such that $u_n(\cdot + x_n^{(2)}) \rightharpoonup u^{(2)}$ (up to a subsequence). As before, we can prove that $u^{(2)}$ is a minimizer for $\mathrm{M}_{\lambda^{(2)}}$, where $\lambda^{(2)} := \int_{\mathbb{R}^d} |u^{(2)}|^2 > 0$. Moreover, we can write $r_n^{(2)}(\cdot - x_n^{(2)}) = u^{(2)} + r_n^{(3)}$ and $\mathbf{r}^{(3)}$ is a minimizing sequence for $I(\lambda - \lambda^{(1)} - \lambda^{(2)})$. As a consequence, we get

$$I(\lambda) = I(\lambda^{(1)}) + I(\lambda^{(2)}) + I(\lambda - \lambda^{(1)} - \lambda^{(2)})$$

and that $I(\lambda^{(1)})$ and $I(\lambda^{(2)})$ have $u^{(1)}$ and $u^{(2)}$ as minimizers. Next, by applying (1.14) twice

$$I(\lambda) \leq I(\lambda^{(1)} + \lambda^{(2)}) + I(\lambda - \lambda^{(1)} - \lambda^{(2)}) \leq I(\lambda^{(1)}) + I(\lambda^{(2)}) + I(\lambda - \lambda^{(1)} - \lambda^{(2)})$$

and since the first and last term are equal we have

$$I(\lambda^{(1)} + \lambda^{(2)}) = I(\lambda^{(1)}) + I(\lambda^{(2)}).$$

To get a contradiction, we have to prove the strict inequality (1.18); we've gained that without loss of generality we can assume that both $I(\alpha)$ and $I(\lambda - \alpha)$ have minimizers. In practice, one proceeds to put the minimizers "far away" at a distance R and then study the energy expansion as $R \to \infty$ to show that the bubbles must in fact attract each other.

If our functional is not translation-invariant, i.e. $\mathcal{E} \neq \mathcal{E}^\infty$, we need one more step in the beginning. We first need to show that

$$I(\lambda) < I^\infty(\lambda) \text{ for all } \lambda > 0,$$

which implies that a minimizing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ cannot have a vanishing limit up to subsequences, i.e. $u_n \rightharpoonup u^{(1)} \neq 0$. Notice that in this first step there is no translation. Then, we write $u_n = u^{(1)} + r_n^{(2)}$ and as before we need to show that the energy splits as

$$\mathcal{E}(u_n) = \mathcal{E}(u^{(1)}) + \mathcal{E}^\infty(r_n^{(2)}) + o_n(1).$$

Notice that the second term is $\mathcal{E}^\infty$ because $r_n^{(2)} \rightharpoonup 0$, so the local terms disappear.
Arguing as before we find

$$I(\lambda) = I(\lambda^{(1)}) + I^\infty(\lambda - \lambda^{(1)}).$$

The rest of the precedure is similar to the translation-invariant case, and the strict inequality one needs to prove is

$$I(\lambda^{(1)} + \lambda^{(2)}) < I(\lambda^{(1)}) + I^\infty(\lambda^{(2)}),$$

where both $I(\lambda^{(1)})$ and $I^\infty(\lambda^{(2)})$ can be assumed to have minimizers.

2 Existence of a minimizer for the Hartree equation without external potential

In this chapter, we will discuss the existence of a minimizer for the Hartree energy functional without external potential. In particular, we will study the functional

$$J(u) = \|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^3} (w * |u|^2)(x)|u(x)|^2 \, dx, \; u \in \mathcal{S}_\lambda \tag{2.1}$$

where $0 \leq w = w_1 + w_2 \in L^\infty(\mathbb{R}^3) + L^{3/2,\infty}(\mathbb{R}^3)$, $w \not\equiv 0$ and $\mathcal{S}_\lambda = \{u \in H^1(\mathbb{R}^3) \, : \, \|u\|_{L^2}^2 = \lambda\}$.
This energy functional arises from the study of the stationary states of the Hartree equation

$$i\partial_t u = -\Delta u + (w * |u|^2)u, \tag{HE}$$

which has many applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules. Indeed, HE appears in the study of the mean-field limit of such systems, i. e., of a regime where the number of bosons is very large, but the interactions between them are weak. We will give an heuristic explanation of this in Appendix B.

We are interested in finding a minimizer for

$$I(\lambda) = \inf_{u \in \mathcal{S}_\lambda} J(u). \tag{2.2}$$

In order to do so, we will need the following assumption on w:

Hypothesis 1. *The decomposition* $w = w_1 + w_2 \in L^\infty(\mathbb{R}^3) + L_\epsilon^{3/2,\infty}(\mathbb{R}^3)$ *is such that* $w_1(x) \to 0$ *as* $|x| \to \infty$. *In more precise terms, for every* $\epsilon > 0$ *there exist* $w_{1,\epsilon} \in L^\infty$ *and* $w_{2,\epsilon} \in L^{3/2,\infty}$ *such that*

$$\begin{cases} w = w_{1,\epsilon} + w_{2,\epsilon}, \\ w_{1,\epsilon}(x) \to 0 \text{ as } |x| \to \infty, \\ \|w_{2,\epsilon}\|_{L^{3/2,\infty}} \leq \epsilon \end{cases}$$

For example, it is not difficult to verify that for all $0 < \alpha < 2$, the potential $w(x) = \frac{1}{|x|^\alpha}$ belongs to $L^\infty + L_\epsilon^{3/2,\infty}$. Indeed, setting $w_{2,\epsilon} = \frac{1_{|x| \leq \delta_\epsilon}}{|x|^\alpha}$ and $w_{1,\epsilon} = \frac{1_{|x| \geq \delta_\epsilon}}{|x|^\alpha}$ with $\delta_\epsilon \leq \left((\frac{3}{4\pi})^{\frac{2}{3}} \epsilon \right)^{\frac{1}{2-\alpha}}$, we have that $w_{1,\epsilon}$ is bounded and vanishing at infinity, while a direct computation shows that $\|w_{2,\epsilon}\|_{L^{3/2,\infty}} \leq \epsilon$.

Proving the following theorem will be the main goal of this section; its proof relies mainly on the concentration-compactness principle stated in Chapter 1, along with some ideas from [8] and [1].

Theorem 2.1. *Let* $0 \leq w = w_1 + w_2 \in L^\infty(\mathbb{R}^3) + L_\epsilon^{3/2,\infty}(\mathbb{R}^3)$, $w \not\equiv 0$ *satisfy Hypothesis 1. Then there exists* $\lambda^* \geq 0$ *such that for every* $\lambda > \lambda^*$ *problem* (2.2) *has a minimizer* $u \in \mathcal{S}_\lambda$, *which is smooth and strictly positive.*

We will show that for the specific convolutional potential $w = \frac{1}{|x|^\alpha}$, $0 < \alpha < 2$ we have $\lambda^* = 0$. However, it is known that for some specific potentials that $\lambda^* > 0$.

We remark that Theorem 2.1 does not answer the question of *uniqueness* (up to translations) of the minimizer. There are, however, some results in this direction: E. Lieb [7] has proven uniqueness of the minimizer up to phases and translations in the case $w = \frac{1}{|x|}$.

2.1 Notation and Preliminaries

In the proofs contained in Section 2.2, we will use some functional inequalities in Lorentz spaces that we present in this section.

Definition 2.2. *For $1 \le p < \infty$, $1 \le q \le \infty$, we define the Lorentz space $L^{p,q}(\mathbb{R}^d)$ as the set of (equivalence classes of) measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that*

$$\|f\|_{L^{p,q}} = p^{1/q} \|\lambda(\{|f| > t\})^{1/p} t\|_{L^q((0,\infty),dt/t)}$$

is finite. Here λ is the Lebesgue measure on $\mathbb{R}^d$.
In particular, for $1 \le p < \infty$

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} \Big(t\lambda(\{|f| > t\})^{1/p} \Big).$$

Remark 2.3. *For $1 \le p < \infty$, $1 \le q_1 \le q_2 \le \infty$, we have $L^{p,q_1} \subset L^{p,q_2}$ and such embedding is continous. Moreover, we can identify $L^{p,p}$ with L^p.*

Remark 2.4. *For $1 < p < \infty$, if $f \in L^{p,\infty}$ then for every $\delta > 0$ $f \mathbb{1}_{|f| \ge \delta} \in L^q$ $\forall\, 1 \le q < p$.*

Lemma 2.5 (Hölder Inequality in Lorentz spaces)**.** *For $1 \le p, p_1, p_2 < \infty$, $1 \le q, q_1, q_2 \le \infty$, there exists a constant $C > 0$ such that*

$$\|f_1 f_2\|_{L^{p,q}} \le C \|f_1\|_{L^{p_1,q_1}} \|f_2\|_{L^{p_2,q_2}}, \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \tag{2.3}$$

whenever the right hand side is finite.

Lemma 2.6 (Young Inequality in Lorentz spaces)**.** *For $1 < p, p_1, p_2 < \infty$, $1 \le q, q_1, q_2 \le \infty$, there exists a constant $C > 0$ such that*

$$\|f_1 * f_2\|_{L^{p,q}} \le C \|f_1\|_{L^{p_1,q_1}} \|f_2\|_{L^{p_2,q_2}}, \qquad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \tag{2.4}$$

whenever the right hand site is finite. Moreover, for $1 < p < \infty$, $1 \le q \le \infty$ there exists $C > 0$ such that

$$\|f_1 * f_2\|_{L^\infty} \le C \|f_1\|_{L^{p,q}} \|f_2\|_{L^{p',q'}}, \qquad \frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'} \tag{2.5}$$

For a proof of these inequalities, see [4].

We also have the following estimates that will be used several times in Section 2.2. The first one is an obvious application of the usual Hölder and Young inequalities, while the second and third ones are close to the Hardy-Littlewood-Sobolev inequality but cannot be directly deduced from it. To be more concise, we introduce the following notation: $a \lesssim b$ if and only if there exists $C > 0$ such that $a \le Cb$. We also recall the definition of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d) \ : \ \nabla f \in L^2(\mathbb{R}^d)\}$.

Lemma 2.7.

1. *Let u_1, $u_2 \in L^2$ and $w \in L^\infty$. Then*

$$\|w * (u_1 u_2)\|_{L^\infty} \lesssim \|w\|_{L^\infty} \|u_1\|_{L^2} \|u_2\|_{L^2}. \tag{2.6}$$

2. *Let u_1, $u_2 \in \dot{H}^1$ and $w \in L^{3/2,\infty}$. Then*

$$\|w * (u_1 u_2)\|_{L^\infty} \lesssim \|w\|_{L^{3/2,\infty}} \|u_1\|_{\dot{H}^1} \|u_2\|_{\dot{H}^1}. \tag{2.7}$$

Proof.
(2.6) follows directly from the classical Young and Hölder inequalities:

$$\|w * (u_1 u_2)\|_{L^\infty} \leq \|w\|_{L^\infty} \|u_1 u_2\|_{L^1} \lesssim \|w\|_{L^\infty} \|u_1\|_{L^2} \|u_2\|_{L^2}.$$

To prove (2.7), we start applying Young and Hölder inequalities (2.4) and (2.3)

$$\|w * (u_1 u_2)\|_{L^\infty} \lesssim \|w\|_{L^{3/2,\infty}} \|u_1 u_2\|_{L^{3,1}} \lesssim \|w\|_{L^{3/2,\infty}} \|u_1\|_{L^{6,2}} \|u_2\|_{L^{6,2}}.$$

Moreover, $\dot{H}^1(\mathbb{R}^3) \subset L^{6,2}(\mathbb{R}^3)$: indeed,

$$\begin{aligned}\|u\|_{L^{6,2}} &\lesssim \|\bar{\mathcal{F}}(\mathcal{F}(u))\|_{L^{6,2}} \lesssim \|\mathcal{F}(u)\|_{L^{6/5,2}} = \|\frac{1}{|k|}|k|\mathcal{F}(u)\|_{L^{6/5,2}} \lesssim \|\frac{1}{|k|}\|_{L^{3,\infty}} \||k|\mathcal{F}(u)\|_{L^{2,2}} \\ &\lesssim \|\nabla u\|_{L^2} = \|u\|_{\dot{H}^1},\end{aligned}$$

where we've used (2.3) once more and that

$$\|\frac{1}{|k|}\|_{L^{3,\infty}} = \sup_{t>0} [t\lambda(\frac{1}{|k|} > t)^{\frac{1}{3}}] = \sup_{t>0} [t(\frac{4}{3}\pi\frac{1}{t^3})^{\frac{1}{3}}] < \infty.$$

Putting all of this together, we get (2.7). □

2.2 Proof of Theorem 2.1

2.2.1 Fundamental properties of the energy functional

In this section we will check the fundamental properties of the functional J required for the application of the concentration-compactness method as stated in Section 1.3.3, which can be sum up in the following.

Lemma 2.8. *Assume that w satisfies Hypothesis 1. Then J is well defined, translation invariant, continous, bounded from below on $H^1(\mathbb{R}^3)$ and coercive on $\mathcal{S}_{\leq\lambda}$ for all $\lambda \geq 0$, where $\mathcal{S}_{\leq\lambda} = \{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2}^2 \leq \lambda\}$.*

Proof. First of all, $\|\nabla u\|_{L^2}^2$ is finite for every $u \in H^1$; then, $|\int_{\mathbb{R}^3}(w * |u|^2)|u|^2| \leq \|w * |u|^2\|_{L^\infty} \|u\|_{L^2}^2$ and

$$\|w * |u|^2\|_{L^\infty} \leq \|w_1 * |u|^2\|_{L^\infty} + \|w_2 * |u|^2\|_{L^\infty} \lesssim \|w_1\|_{L^\infty} \| \|u\|_{L^2}^2 + \|w_2\|_{L^{3/2,\infty}} \|u\|_{\dot{H}^1}^2$$

by (2.6) and (2.7) In conclusion, we have that

$$\left| \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 \right| \lesssim (\|w_1\|_{L^\infty} \|u\|_{L^2}^2 + \|w_2\|_{L^{3/2,\infty}} \|u\|_{\dot{H}^1}^2) \|u\|_{L^2}^2, \tag{2.8}$$

which allows us to conclude that J is well defined on $H^1(\mathbb{R}^3)$.
Now, we prove that $J(u) = J(u(\cdot + z))$ for every $z \in \mathbb{R}^3$:
Clearly $\|\nabla u\|_{L^2}^2 = \|\nabla(u(\cdot + z))\|_{L^2}^2$; then,

$$\begin{aligned}\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |u(y+z)|^2 w(x-y)\, dy \right) |u(x+z)|^2\, dx &= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |u(y)|^2 w(x+z-y)\, dy \right) |u(x+z)|^2\, dx \\ &= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |u(y)|^2 w(t-y)\, dy \right) |u(t)|^2\, dt,\end{aligned}$$

so $J(u) = J(u(\cdot + z))$.

Using (2.8), we have $J(u) = \|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^3}(w * |u|^2)(x)|u(x)|^2\,dx \gtrsim -\|u\|_{H^1}^4 > -\infty$, so J is bounded from below.
To prove that J is continous from H^1 to $\mathbb{R}$, we just need to show that $u \mapsto \int_{\mathbb{R}^3}(w * |u|^2)(x)|u(x)|^2\,dx$ is: Let $(u_n)_{n\in\mathbb{N}}$ and u such that $u_n \to u$ in H^1; then,

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(w * |u_n|^2)|u_n|^2 - \int_{\mathbb{R}^3}(w * |u|^2)|u|^2\right| &= \left|\int_{\mathbb{R}^3}(w * (|u_n|^2 - |u|^2))|u_n|^2 + \int_{\mathbb{R}^3}(w * |u|^2)(|u_n|^2 - |u|^2)\right| \\ &\lesssim \|(w_1 * (|u_n|^2 - |u|^2))|u_n|^2\|_{L^1} + \|(w_2 * (|u_n|^2 - |u|^2))|u_n|^2\|_{L^1} \\ &+ \|(w_1 * |u|^2)(|u_n|^2 - |u|^2)\|_{L^1} + \|(w_2 * |u|^2)(|u_n|^2 - |u|^2)\|_{L^1}\end{aligned}$$

We handle the third and forth term by applying respectively (2.6) and (2.7).
For the first term, using the classical Young inequality, together with $\|u_n\|_{L^2} = 1$ for every n we get

$$\|(w_1 * (|u_n|^2 - |u|^2))|u_n|^2\|_{L^1} \lesssim \|w_1 * (|u_n|^2 - |u|^2)\|_{L^\infty}\|u_n\|_{L^2} \lesssim \|w_1\|_{L^\infty}\||u_n|^2 - |u|^2\|_{L^1}$$

Finally, for the second term we use (2.3), (2.4), the inclusion $\dot{H}^1 \subset L^{6,2}$ and that $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1:

$$\begin{aligned}\|(w_2 * (|u_n|^2 - |u|^2))|u_n|^2\|_{L^1} &\lesssim \|w_2 * (|u_n|^2 - |u|^2)\|_{L^{3/2,\infty}}\||u_n|^2\|_{L^{3,1}} \\ &\lesssim \|w_2\|_{L^{3/2,\infty}}\||u_n|^2 - |u|^2\|_{L^1}\|u_n\|_{L^{6,2}}^2 \lesssim \|w_2\|_{L^{3/2,\infty}}\||u_n|^2 - |u|^2\|_{L^1}.\end{aligned}$$

Putting all four terms together, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(w * |u_n|^2)|u_n|^2 - \int_{\mathbb{R}^3}(w * |u|^2)|u|^2\right| &\lesssim (\|w_1\|_{L^\infty} + \|w_2\|_{L^{3/2,\infty}})\||u_n|^2 - |u|^2\|_{L^1} \\ &\lesssim (\|w_1\|_{L^\infty} + \|w_2\|_{L^{3/2,\infty}})\|u_n + u\|_{L^2}\|u_n - u\|_{L^2} \\ &\lesssim (\|w_1\|_{L^\infty} + \|w_2\|_{L^{3/2,\infty}})\|u_n - u\|_{L^2},\end{aligned}$$

which proves continuity.
Lastly, to prove the coercivity on $\mathcal{S}_{\leq\lambda}$, by (2.8), we have that

$$J(u) \gtrsim \|\nabla u\|_{L^2}^2 - (\|w_1\|_{L^\infty}\|u\|_{L^2}^2 + \|w_2\|_{L^{3/2,\infty}}\|\nabla u\|_{L^2}^2)\|u\|_{L^2}^2 \geq -\|w_1\|_{L^\infty}\lambda^2 + (1 - \lambda\|w_2\|_{L^{3/2,\infty}})\|\nabla u\|_{L^2}^2$$

if $u \in \mathcal{S}_{\leq\lambda}$, so J is coercive on $\mathcal{S}_{\leq\lambda}$ under the assumption $1 - \lambda\|w_2\|_{L^{3/2,\infty}} > 0$, which we have thanks to Hypothesis 1. □

Remark 2.9. *The computations we did for the continuity of $u \mapsto \int_{\mathbb{R}^3}(w * |u|^2)(x)|u(x)|^2\,dx$ can be done with fewer assumptions: indeed, one can prove that if $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1 and $u_n \to u$ in $L^2(\mathbb{R}^3)$ then $\int_{\mathbb{R}^3}(w * |u_n|^2)(x)|u_n(x)|^2\,dx \to \int_{\mathbb{R}^3}(w * |u|^2)(x)|u(x)|^2\,dx$.*

2.2.2 The concentration-compactness method in action

Now that we've proved that J is coercive and continous on $\mathcal{S}_{\leq\lambda}$, we know that every minimizing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, so up to a subsequence there exists $u_\infty \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u_\infty$ weakly in H^1. We prove that the convegence is also strong in $L^2(\mathbb{R}^3)$ using the concentration-compactness method explained in section 1.3.
Our problem is translation invariant, we do not need to define a problem "at infinity". In order to rule out vanishing, we define

$$J^{\text{van}}(u) = \|\nabla u\|^2, \tag{2.9}$$

so that $J^{\mathrm{van}}(u_n) \to 0$ if $u_n \to 0$ in subcritical L^p. Next, we define the respective minimal energy

$$I^{\mathrm{van}}(\lambda) = \inf_{u \in \mathcal{S}_\lambda} J^{\mathrm{van}}(u). \tag{2.10}$$

It can be easily proved that $I^{\mathrm{van}}(\lambda) = 0$ for all $\lambda \geq 0$.
We start by proving inequality (1.14), i.e.

$$I(\lambda) \leq I(\lambda - \alpha) + I^\infty(\alpha) \text{ for all } 0 \leq \alpha \leq \lambda,$$

following the general reasoning we discussed in Section 1.3.2: for every $\epsilon > 0$ there exist $u_\epsilon \in \mathcal{S}_\alpha$ and $v_\epsilon \in \mathcal{S}_{\lambda-\alpha}$ such that

$$\begin{cases} I(\alpha) \leq J(u_\epsilon) \leq I(\alpha) + \epsilon \\ I(\lambda - \alpha) \leq J(v_\epsilon) \leq I(\lambda - \alpha) + \epsilon \end{cases}$$

By continuity of J and density of $C_C^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$, we can take u_ϵ and v_ϵ with compact support without loss of generality. For a fixed $\xi \in \mathbb{R}^3$ and $n \in \mathbb{N}$, define $v_\epsilon^{(n)} = v_\epsilon(\cdot - n\xi)$, so that for n large $\mathrm{supp}(v_\epsilon^{(n)}) \cap \mathrm{supp}(u_\epsilon) = \emptyset$. Then, for such n we have $u_\epsilon + v_\epsilon^{(n)} \in \mathcal{S}_\lambda$ and

$$\begin{aligned} J(u_\epsilon + v_\epsilon^{(n)}) &= \|\nabla(u_\epsilon + v_\epsilon^{(n)})\|_{L^2}^2 - \int_{\mathbb{R}^3} (w * (|u_\epsilon|^2 + |v_\epsilon^{(n)}))|u_\epsilon|^2 - \int_{\mathbb{R}^3} (w * (|u_\epsilon|^2 + |v_\epsilon^{(n)}|^2))|v_\epsilon^{(n)}|^2 \\ &= J(u_\epsilon) + J(v_\epsilon^{(n)}) - \int_{\mathbb{R}^3} (w * |u_\epsilon|^2)|v_\epsilon^{(n)}|^2 - \int_{\mathbb{R}^3} (w * |v_\epsilon^{(n)}|^2)|u_\epsilon|^2 \\ &\leq J(u_\epsilon) + J(v_\epsilon^{(n)}) \end{aligned}$$

because $w \geq 0$. As a consequence, we have

$$I(\lambda) \leq J(u_\epsilon + v_\epsilon^{(n)}) \leq J(u_\epsilon) + J(v_\epsilon^{(n)}) \leq I(\alpha) + I(\lambda - \alpha) + 2\epsilon$$

that allows us to conclude the proof of (1.14) by arbitrarity of ϵ. In exactly the same way, we can prove inequality (1.15).
As a consequence of (1.15), we also have that

$$I(\lambda) \leq I^{\mathrm{van}}(\lambda) \text{ for every } \lambda \geq 0 \tag{2.11}$$

Our next step is to prove that vanishing does not occur: to do so, we need to prove inequality (1.16), i.e.

$$I(\lambda) < 0 \text{ for every } \lambda > 0.$$

Since by (2.11) we already know that $I(\lambda) \leq 0$, we just need to prove that $I(\lambda) \neq 0$.
Assuming that for some $\lambda > 0$ we have $I(\lambda) = 0$: if this is the case, for every $\epsilon > 0$ there exists $u \in \mathcal{S}_\lambda$ such that $\|\nabla u\|_{L^2}^2 = \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 + \epsilon$. Then, for $\theta > 0$, we have that $\theta u \in \mathcal{S}_{\theta^2\lambda}$ and

$$J(\theta u) = \theta^2 \|\nabla u\|_{L^2}^2 - \theta^4 \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 < 0 \text{ for } \theta \gg 1,$$

which shows that there exists λ^* such that for every $\lambda \geq \lambda^*$ (1.16) holds.
It is important to remark that while in general we cannot do better than this, as there are convolution potentials for which $\lambda^* > 0$, for specific potentials we can do better: for instance, for the potential $w(x) = \frac{1}{|x|^\alpha}$, $0 < \alpha < 2$, which is in $L^\infty(\mathbb{R}^3) + L^{3/2,\infty}(\mathbb{R}^3)$, we can prove that $I(\lambda) < 0$ for every $\lambda > 0$ following a similar scaling argument as before:
for $\sigma > 0$ letting $u_\sigma(x) = \sigma^{-3/2} u(\frac{x}{\sigma})$, we have

$$J(u_\sigma) = \int_{\mathbb{R}^3} |\nabla u_\sigma|^2 - \int_{\mathbb{R}^3} (w * |u_\sigma|^2)|u_\sigma|^2 = \frac{1}{\sigma^2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{\sigma^\alpha} \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 < 0$$

for $\sigma \gg 1$, which proves that vanishing does not occur for every $\lambda > 0$ in this case.
We proceed in proving (1.18), i.e. the strict inequality

$$I(\lambda) < I(\lambda - \alpha) + I(\alpha) \text{ for all } 0 < \alpha < \lambda.$$

In order to establish this inequality, we will apply Lemma 1.9 to $h(\lambda) = I(\lambda)$; to do so, we need to prove

Lemma 2.10. *For every $\lambda > 0$ such that $I(\lambda) < I^{van}(\lambda) = 0$, $I(\theta\lambda) < \theta I(\lambda)$ for every $\theta > 1$.*

Proof. First, notice that

$$I(\theta\lambda) = \inf_{u \in \mathcal{S}_\lambda} \left\{ \theta \|\nabla u\|_{L^2}^2 - \theta^2 \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 \right\} = \theta \inf_{u \in \mathcal{S}_\lambda} \left\{ \|\nabla u\|_{L^2}^2 - \theta \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 \right\}$$

Then, notice that when defining problem (2.2) we can restrict ourselves to taking the inf over the set $\left\{u \in \mathcal{S}_\lambda \text{ such that } \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 \geq \alpha\right\}$ for a fixed $\alpha > 0$: indeed, if that wasn't the case, there would exist a minimizing sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{S}_\lambda$ such that $\int_{\mathbb{R}^3} (w * |v_n|^2)|v_n|^2 \to 0$. In turn, this would imply that $I(\lambda) = I^{\text{van}}(\lambda) = 0$, which contradicts our hypothesis.
To conclude, observe that since $\theta > 1$

$$I(\theta\lambda) = \theta \inf_{u \in \mathcal{S}_\lambda} \left\{ \|\nabla u\|_{L^2}^2 - \theta \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 \right\} < \theta \inf_{u \in \mathcal{S}_\lambda} \left\{ \|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 \right\} = \theta I(\lambda).$$

□

Finally, we need to prove that minimizing sequences do not split into two or more bubbles. Unlike what we've discussed in Section 1.3.3, we choose to rely on Theorem 1.5 instead of Lemma 1.4 and prove a similar result to (1.17).
Since we've already ruled out vanishing, we know that every minimizing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ is such that $\mathbf{m}(\mathbf{u}) > 0$. Then, by Theorem 1.5, we know that if we fix a sequence $0 \leq R_k \xrightarrow{k\to\infty} \infty$ there exist

- A subsequence $(u_{n_k})_{k\in\mathbb{N}}$,
- Sequences of functions $(u_k^{(1)})_{k\in\mathbb{N}}$, $(\psi_k^{(2)})_{k\in\mathbb{N}}$ in $H^1(\mathbb{R}^3)$
- A sequence of translations $(x_k^{(1)})_{k\in\mathbb{N}} \subset \mathbb{R}^3$

such that

$$u_{n_k} - u_k^{(1)}(\cdot - x_k^{(1)}) - \psi_k^{(J+1)} \to 0 \text{ in } H^1(\mathbb{R}^3), \tag{2.12}$$

and

- $u_k^{(1)}$ converges to $u^{(1)}$ weakly in $H^1(\mathbb{R}^3)$ and strongly in $L^2(\mathbb{R}^3)$,
- $\text{supp}(u_k^{(1)}) \subset B_{R_k}(0)$ and $\text{supp}(\psi_k^{(2)}) \subset \mathbb{R}^3 \backslash B_{2R_k}(x_k^{(1)})$.

Since our problem is translation invariant and we will only deal with a single bubble, without loss of generality we can choose the translations $x_k^{(1)} \equiv 0$.
As $u_k^{(1)} \to u^{(1)}$ strongly in L^2, we have that $\|u_k^{(1)}\|_{L^2}^2 \to \|u^{(1)}\|_{L^2}^2 =: \alpha > 0$. This, combined with (2.12), proves that $\|\psi_k^{(2)}\|_{L^2}^2 \to \lambda - \alpha$. We also have

$$\liminf_{k\to\infty} J(u_k^{(1)}) \leq J(u^{(1)}). \tag{2.13}$$

Indeed, since $u_k^{(1)}$ is bounded in H^1 and $u_k^{(1)} \to u^{(1)}$ strongly in L^2 we can apply Remark (2.9) to get $\int_{\mathbb{R}^3} (w * |u_k^{(1)}|^2)(x)|u_k^{(1)}(x)|^2\, dx \to \int_{\mathbb{R}^3} (w * |u^{(1)}|^2)(x)|u^{(1)}(x)|^2\, dx$. The gradient part comes from the weak

lower semicontinuity of the L^2 norm and the fact that $\nabla u_k^{(1)} \rightharpoonup \nabla u^{(1)}$.
Then, by the continuity of $\lambda \mapsto I(\lambda)$ we have

$$J(\psi_k^{(2)}) \geq I(\|\psi_k\|_{L^2}^2) \to I(\lambda - \alpha) \text{ as } k \to \infty. \tag{2.14}$$

Now, if we are able to prove that

$$J(u_{n_k}) = J(u_k^{(1)}) + J(\psi_k^{(2)}) + o_k(1), \tag{2.15}$$

combining (2.13), (2.14) and (2.15) and passing to the liminf we obtain

$$I(\lambda) \geq J(u^{(1)}) + I(\lambda - \alpha) \geq I(\alpha) + I(\lambda - \alpha).$$

Since the converse inequalty (1.14) holds, we have that

$$I(\lambda) = I(\alpha) + I(\lambda - \alpha),$$

which contradicts the strict energetic inequality (1.18) unless $\alpha = \lambda$.
Finally, since $u_{n_k} \rightharpoonup u^{(1)}$ weakly in L^2 by Lemma 1.4 and $\|u_{n_k}\|_{L^2}^2 = \lambda = \|u^{(1)}\|_{L^2}^2$ we have that the convergence is also strong in $L^2(\mathbb{R}^3)$, and by uniqueness of the limit $u_\infty = u^{(1)}$.
To prove (2.15), we start by noticing that as a consequence of (2.12) and the continuity of J we have

$$J(u_{n_k}) = J(u_k^{(1)} + \psi_k^{(2)}) + o_k(1).$$

Moreover, since the supports of $u_k^{(1)}$ and $\psi_k^{(2)}$ are disjoint, we have

$$\begin{aligned} J(u_k^{(1)} + \psi_k^{(2)}) &= \|\nabla u_k^{(1)} + \nabla \psi_k^{(2)}\|_{L^2}^2 - \int_{\mathbb{R}^3} (w * (|u_k^{(1)}|^2 + |\psi_k^{(2)}|^2))(|u_k^{(1)}|^2 + |\psi_k^{(2)}|^2) \\ &= J(u_k^{(1)}) + J(\psi_k^{(2)}) - \int_{\mathbb{R}^3} (w * |u_k^{(1)}|^2)|\psi_k^{(2)}|^2 - \int_{\mathbb{R}^3} (w * |\psi_k^{(2)}|^2)|u_k^{(1)}|^2. \end{aligned}$$

To tackle the integral terms, we define $w_\delta = w \mathbb{1}_{|w| \geq \delta}$ for a fixed $\delta > 0$, so that

$$\begin{aligned} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w(x-y)|u_k^{(1)}(x)|^2 |\psi_k^{(2)}(y)|^2 \, dxdy &\leq \delta\lambda^2 + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_\delta(x-y)|u_k^{(1)}(x)|^2 |\psi_k^{(2)}(y)|^2 \, dxdy \\ &\leq \delta\lambda^2 + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_\delta(x-y) \mathbb{1}_{|x-y| \geq R_k} |u_k^{(1)}(x)|^2 |\psi_k^{(2)}(y)|^2 \, dxdy \end{aligned}$$

since $\|u_k^{(1)}\|_{L^2}^2, \|\psi_k^{(2)}\|_{L^2}^2 \leq \|u_{n_k}\|_{L^2}^2 = \lambda$ and $\mathrm{dist}(\mathrm{supp}(u_k^{(1)}), \mathrm{supp}(\psi_k^{(2)})) \geq R_k$.
Finally, letting $w_{1,\delta} = w_1 \mathbb{1}_{|w_1| \geq \delta}$ and $w_{2,\delta} = w_2 \mathbb{1}_{|w_2| \geq \delta}$,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_{1,\delta}(x-y) \mathbb{1}_{|x-y| \geq R_k} |u_k^{(1)}(x)|^2 |\psi_k^{(2)}(y)|^2 \, dxdy \leq \|w_{1,\delta} \mathbb{1}_{|\cdot| \geq R_k}\|_{L^\infty} \lambda^2 \to 0$$

as $k \to \infty$ since by Hypothesis 1 $w_1 \to 0$ at infinity and

$$\begin{aligned} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_{2,\delta}(x-y) \mathbb{1}_{|x-y| \geq R_k} |u_k^{(1)}(x)|^2 |\psi_k^{(2)}(y)|^2 \, dxdy &\leq \||u_k^{(1)}|^2\|_{L^{\frac{2q}{2q-1}}} \||\psi_k^{(2)}|^2\|_{L^{\frac{2q}{2q-1}}} \|w_{2,\delta} \mathbb{1}_{|\cdot| \geq R_k}\|_{L^q} \\ &= \|u_k^{(1)}\|_{L^{\frac{4q}{2q-1}}}^2 \|\psi_k^{(2)}\|_{L^{\frac{4q}{2q-1}}}^2 \|w_{2,\delta} \mathbb{1}_{|\cdot| \geq R_k}\|_{L^q} \to 0 \end{aligned}$$

as $k \to \infty$ for every $1 \leq q < \frac{3}{2}$. Indeed, by Remark 2.4 $w_{2,\delta} \in L^q$ for $1 \leq q < \frac{3}{2}$, so $\|w_{2,\delta} \mathbb{1}_{|\cdot| \geq R_k}\|_{L^q} \to 0$ as $k \to \infty$. Moreover, for such q we have $3 < \frac{4q}{2q-1} \leq 4$, so $H^1(\mathbb{R}^3) \subset L^{\frac{4q}{2q-1}}(\mathbb{R}^3)$ and since in the proof of

Theorem 1.5 we built both $u_k^{(1)}$ and $\psi_k^{(2)}$ are proportional to u_{n_k}, which is bounded in H^1, we get that $\|u_k^{(1)}\|^2_{L^{\frac{4q}{2q-1}}}\|\psi_k^{(2)}\|^2_{L^{\frac{4q}{2q-1}}}$ is bounded.
By arbitrarity of $\delta > 0$ we can conclude the proof of (2.15), which, in turn, gives us that $u_{n_k} \to u_\infty$ strongly in L^2.
Lastly, we prove that u_∞ is in fact a minimizer for (2.2): since $u_j \to u_\infty$ in $L^2(\mathbb{R}^3)$, $\|u_\infty\|^2_{L^2} = \lambda$, so $u_\infty \in \mathcal{S}_\lambda$. Then, by weak lower semicontinuity of the L^2 norm we have,

$$\|\nabla u_\infty\|^2_{L^2} \leq \liminf_{j\to\infty} \|\nabla u_j\|^2_{L^2}.$$

Moreover, by Remark 2.9, we have that

$$\int_{\mathbb{R}^3} (w * |u_\infty|^2)|u_\infty|^2 = \lim_{j\to\infty} \int_{\mathbb{R}^3} (w * |u_j|^2)|u_j|^2.$$

Combining these, we obtain

$$I(\lambda) \leq J(u_\infty) \leq \liminf_{j\to\infty} J(u_j) = I(\lambda),$$

so u_∞ is the sought-after minimizer.

2.2.3 Properties of the minimizer

We start by proving that we can take a strictly positive minimizer. First of all, since $\|\nabla|u|\|_{L^2} \leq \|\nabla u\|_{L^2}$ we have that $J(|u|) \leq J(u)$ fore every $u \in H^1$. Thus, we can choose our minimizer u_∞ to be real valued and non negative. Introducing the Schrödinger operator $H = -\Delta + V$ with $V = 2w * u_\infty^2$, we know that u_∞ is an eigenfunction, i.e.

$$Hu_\infty = -\omega u_\infty,$$

which can be written as

$$e^{-H}u_\infty = e^{\omega}u_\infty.$$

We will prove that e^{-H} is *positivity improving*, i.e. that if $f \geq 0$ with $f \not\equiv 0$ then $e^{-H}f > 0$, which in turn will give us the positivity of the minimizer.
We just check the hypothesis of the following Theorem, whose proof can be found in the proof of Theorem 2 in [2].

Theorem 2.11. *Let $H = -\Delta + V$ be a Schrödinger operator defined on $\mathbb{R}^d$, with $0 \leq V \in L^1_{loc}(\mathbb{R}^d)$. Then the lowest eigenvalue of H is simple and e^{-tH} has strictly positive kernel for every $t > 0$.*

The fact that e^{-tH} has strictly positive kernel means that

$$e^{-tH}f(x) = \int_{\mathbb{R}^d} k(x,y)f(y)\,dy \text{ with } k > 0.$$

which in turn would imply that e^{-tH} is positivity improving.
The hypothesis of Theorem 2.11 are satisfied because H is self adjoint on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$ and by (2.6), (2.7) $V \in L^\infty(\mathbb{R}^3) \subset L^1_{loc}(\mathbb{R}^3)$.
To get the smoothness of the minimizer, we can use elliptic bootstrapping; here, we only choose to show the first step for the sake of brevity. We know that u_∞ solves the eigenproblem

$$Hu = -\omega u,\ u \in H^1(\mathbb{R}^3)$$

where $\omega > 0$ is as above. We write this as

$$-\Delta u = f(\cdot, u(\cdot)),\ u \in H^1(\mathbb{R}^3) \tag{2.16}$$

where $f(x,t) = -Vt - \omega t$. Now, since $-\Delta$ is an isometry between H^1 and H^{-1}, it is also an isometry between $W^{2,p} \cap H^1$ and $L^p \cap H^{-1}$; to get the integrability of $f(\cdot, u)$ we need in order to conclude, we use the following Lemma.

Lemma 2.12 (Continuity of the superposition operator)**.** *Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that for some $\theta \geq 1$ $|f(x,t)| \lesssim |t|^\theta$ for every x, t. Then, for every $\theta \leq p < \infty$ the superposition operator*

$$\begin{aligned} \Phi_f \,:\, L^p(\mathbb{R}^d) &\to L^{p/\theta}(\mathbb{R}^d) \\ u &\mapsto f(\cdot, u) \end{aligned}$$

is continous. In particular, for $\theta \leq p \leq 2^*$ $\Phi_f \,:\, H^1(\mathbb{R}^d) \to L^{p/\theta}(\mathbb{R}^d)$ is continous.

Proof. We compute

$$\int_{\mathbb{R}^d} |f(x, u(x))|^{p/\theta}\, dx \lesssim \int_{\mathbb{R}^d} |u|^p = \|u\|_{L^p}^p.$$

This is sufficient to prove our claim. To prove the continuity of $\Phi_f \,: H^1 \to L^{p/\theta}$ we just combine what we just proved with the usual Sobolev injections $H^1 \hookrightarrow L^p$, $2 \leq p \leq 2^*$:

$$H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow L^{p/\theta}(\mathbb{R}^d)$$

□

Since our f defined in (2.16) is clearly Caratheodory and $|f(x,t)| \leq (\|V\|_{L^\infty} + \omega)|t|$ we can apply Lemma 2.12 to get that $f(\cdot, u_\infty(\cdot)) \in L^{2^*} = L^6$, as $u_\infty \in H^1 \subset L^6$. As a consequence, we get that $u_\infty \in W^{2,6} \subset C^{1,1/2}$. One can proceed in this fashion to get that $u_\infty \in C^\infty(\mathbb{R}^3)$. □

Appendix A. Proofs from Chapter 1

In this section, we prove Lemma 1.3 and Theorem 1.5, whose proof we postponed during Chapter 1 for the sake of exposition. We start by proving the equivalent definition of $\mathbf{m}(\mathbf{u})$ for a sequence $\mathbf{u}$ bounded in $H^1(\mathbb{R}^d)$.

Proof of Lemma 1.3. To prove the equality

$$\mathbf{m}(\mathbf{u}) = \lim_{R\to\infty} \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2.$$

we prove that both inequalities hold.
$\geq$): We proceed as we did in Lemma 1.7. Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \int_{B_R(x_n)} |u_n|^2 = \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2.$$

Since u_n is bounded in H^1, up to a subsequence $u_n(\cdot + x_n)$ converges to u weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^2(B_R(0))$. Then,

$$\lim_{n\to\infty} \int_{B_R(x_n)} |u_n|^2 = \lim_{n\to\infty} \int_{B_R(0)} |u_n(x+x_n)|^2\,dx = \int_{B_R(0)} |u|^2 \leq \int_{\mathbb{R}^d} |u|^2 \leq \mathbf{m}(\mathbf{u}).$$

Since this estimate holds for every $R > 0$, it passes to the limit $R \to \infty$ to get us our desired inequality. Notice that the limit exists because $R \mapsto \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2$ is a non decreasing function.
$\leq$): We start from the definition of $\mathbf{m}(\mathbf{u})$: for every $\epsilon > 0$, there exist a subsequence u_{n_k} and translations $(x_k)_{k\in\mathbb{N}}$ such that $u_{n_k}(\cdot + x_k) \rightharpoonup u$ weakly in H^1 with $\mathbf{m}(\mathbf{u}) - \epsilon \leq \int_{\mathbb{R}^d} |u|^2 \leq \mathbf{m}(\mathbf{u})$. By the Rellich-Kondrachov Theorem, up to a further subsequence $u_{n_k}(\cdot + x_k) \to u$ strongly in L^2_{loc}. Then, for every $x \in \mathbb{R}^d$

$$\mathbf{m}(\mathbf{u}) - \epsilon \leq \int_{\mathbb{R}^d} |u|^2 = \lim_{R\to\infty} \int_{B_R(x)} |u|^2 = \lim_{R\to\infty} \lim_{n_k\to\infty} \int_{B_R(x)} |u_{n_k}|^2.$$

Then, since the limit along a subsequence is always not greater than the limsup and the inequality holds for every x in $\mathbb{R}^d$, we get that

$$\mathbf{m}(\mathbf{u}) - \epsilon \leq \lim_{R\to\infty} \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_R(x)} |u_n|^2$$

for every $\epsilon > 0$. Since the choice of ϵ is arbitrary, we get the desired inequality. □

Before tacklig the proof of Theorem 1.5, we state and prove a technical Lemma we will need:

Lemma A.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Consider two sequences of real numbers $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ such that $0 \leq a_k \leq b_k$ and $a_k \to \infty$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that*

$$\int_{|x-x_k^{(j)}|\leq a_k} |u_{n_k}(x)|^2\,dx \xrightarrow{k\to\infty} \int_{\mathbb{R}^d} |u^{(j)}(x)|^2\,dx \ \text{ and } \int_{a_k\leq|x-x_k^{(j)}|\leq b_k} (|u_{n_k}(x)|^2 + |\nabla u_{n_k}(x)|^2)\,dx \xrightarrow{k\to\infty} 0$$

for every $j = 1, ..., J$, where $x_k^{(j)}$ and $u^{(j)}$ are as in Lemma 1.4.

Proof. We define the *Levy concentration functions*

$$Q_k^{(j)}(R) = \int_{B_R(x_k^{(j)})} |u_{n_k}|^2 \text{ and } K_k^{(j)}(R) = \int_{B_R(x_k^{(j)})} |\nabla u_{n_k}|^2;$$

Clearly $Q_k^{(j)}(R) + K_k^{(j)}(R) \leq \|u_{n_k}\|^2_{H^1(\mathbb{R}^d)} < \infty$, so by the Rellich-Kondrachov Theorem for every $R > 0$ and every $j = 1, ...J$

$$Q_k^{(j)}(R) = \int_{B_R(0)} |u_{n_k}(x + x_k^{(j)})|^2 \, dx \xrightarrow{k\to\infty} \int_{B_R(0)} |u^{(j)}|^2 =: Q(R).$$

Then, since $K_k^{(j)}$ is a bounded sequence of non-decreasing non-negative functions, there exists a non-decreasing function $K^{(j)}(R)$ such that $K_k^{(j)}(R) \xrightarrow{k\to\infty} K^{(j)}(R)$ for every R. Moreover, for every fixed j $K^{(j)}(R)$ is bounded for every R so it has finite limit as $R \to \infty$. Next, since $Q_k^{(j)}(a_k) \to Q^{(j)}(a_k)$, $Q_k^{(j)}(b_k) \to Q^{(j)}(b_k)$, and the same for the $K_k^{(j)}$, for every j we have that up to a further subsequence

$$|Q_k^{(j)}(a_k) - Q^{(j)}(a_k)| + |Q_k^{(j)}(b_k) - Q^{(j)}(b_k)| + |K_k^{(j)}(a_k) - K^{(j)}(a_k)| + |K_k^{(j)}(b_k) - K^{(j)}(b_k)| \leq \frac{1}{k},$$

which is what we need to prove our claims:

$$\begin{aligned} |\int_{B_{a_k}(x_k^{(j)})} |u_{n_k}|^2 - \int_{\mathbb{R}^d} |u^{(j)}|^2| = |Q_k^{(j)}(a_k) - Q^{(j)}(\infty)| &\leq |Q_k^{(j)}(a_k) - Q^{(j)}(a_k)| + |Q^{(j)}(a_k) - Q^{(j)}(\infty)| \\ &\leq \frac{1}{k} + \int_{\mathbb{R}^d \setminus B_{a_k}(x_k^{(j)})} |u^{(j)}|^2 \xrightarrow{k\to\infty} 0 \end{aligned}$$

proves the first one. Similarly,

$$\int_{a_k \leq |x - x_k^{(j)}| \leq b_k} |u_{n_k}(x)|^2 = Q_k^{(j)}(b_k) - Q_k^{(j)}(a_k) \leq \frac{1}{k} + Q^{(j)}(b_k) - Q^{(j)}(a_k),$$

$$\int_{a_k \leq |x - x_k^{(j)}| \leq b_k} |\nabla u_{n_k}(x)|^2 = K_k^{(j)}(b_k) - K_k^{(j)}(a_k) \leq \frac{1}{k} + K^{(j)}(b_k) - K^{(j)}(a_k),$$

which converge both to 0 since both $Q^{(j)}$ and $K^{(j)}$ have finite limits at infinity. ☐

Remark A.2. *Under the hypothesis of Lemma A.1 we can also prove that* $u_{n_k} \mathbb{1}_{B_{a_k}(x_k^{(j)})} \to u^{(j)}$ *strongly in* $L^p(\mathbb{R}^d)$ *for* $2 \leq p < 2^*$*: first,* $u_{n_k} \mathbb{1}_{B_{a_k}(x_k^{(j)})} = u_{n_k}(\cdot + x_{n_k}^{(j)}) \mathbb{1}_{B_{a_k}(0)} \rightharpoonup u^{(j)}$ *weakly in* L^2 *and* $\|u_{n_k} \mathbb{1}_{B_{a_k}(x_k^{(j)})}\|^2_{L^2} \to \|u\|^2_{L^2}$*, so the convergence is also strong in* L^2*. Moreover, by Sobolev embeddings* $(u_n)_{n\in\mathbb{N}}$ *is also bounded in* $L^p(\mathbb{R}^d)$*, for every* $2 \leq p \leq 2^*$*, and as a consequence so is* $u_{n_k}(\cdot + x_k^{(j)}) \mathbb{1}_{B_{a_k}(0)}$*. Then, by Hölder inequality*

$$\|\mathbb{1}_{B_{a_k}(0)} u_{n_k}(\cdot + x_k^{(j)}) - u^{(j)}\|_{L^p} \leq \|\mathbb{1}_{B_{a_k}(0)} u_{n_k}(\cdot + x_k^{(j)}) - u^{(j)}\|^{\theta}_{L^2} \|\mathbb{1}_{B_{a_k}(0)} u_{n_k}(\cdot + x_k^{(j)}) - u^{(j)}\|^{1-\theta}_{L^{2^*}}$$

with $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2^*}$.

We now proceed with the proof of Theorem 1.5, which will mainly rely on the ideas used to prove Lemma 1.4 and on this last technical Lemma.

Proof of Theorem 1.5. First, fix $J \in \mathbb{N}$. By Lemma 1.4, there exist space translations $x_k^{(j)}$, $j = 1, ..., J$, and a subsequence u_{n_k} such that $u_{n_k}(\cdot + x_k^{(j)}) \rightharpoonup u^{(j)}$ weakly in $H^1(\mathbb{R}^d)$. Looking at the proof of Lemma 1.4, clearly we can choose $x_k^{(j)}$ such that $|x_k^{(i)} - x_k^{(j)}| \geq 5R_k$ for every $i \neq j$ and every k.
Now, we apply Lemma A.1 with $a_k = R_k/2$ and $b_k = 8R_k$: there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$\int_{|x - x_k^{(j)}| \leq R_k/2} |u_{n_k}(x)|^2 \, dx \to \int_{\mathbb{R}^d} |u^{(j)}(x)|^2 \, dx \tag{A.1}$$

and

$$\int_{R_k/2 \leq |x - x_k^{(j)}| \leq 8R_k} (|u_{n_k}(x)|^2 + |\nabla u_{n_k}(x)|^2) \, dx \to 0. \tag{A.2}$$

Next, let $\chi \in C^\infty(\mathbb{R}^+; [0,1])$ such that $0 \leq \chi' \leq 2$, $\chi_{|_{[0,1]}} \equiv 1$ and $\chi_{|_{[2,\infty)}} \equiv 0$ and define

$$\chi_k(x) = \chi\left(\frac{2|x|}{R_k}\right) \text{ and } \zeta_k(x) = 1 - \chi\left(\frac{|x|}{2R_k}\right)$$

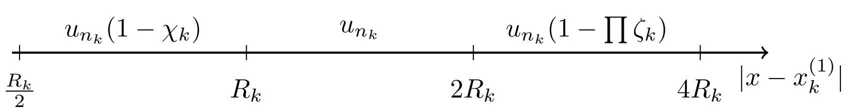
Finally, define

$$u_k^{(j)} = \chi_k u_{n_k}(\cdot + x_k^{(j)})$$

for $j = 1, ..., J$ and

$$\psi_k^{(J+1)} = \left(\prod_{j=1}^J \zeta_k(\cdot - x_k^{(j)})\right) u_{n_k}.$$

Notice that $\operatorname{supp}(u_k^{(j)}) \subset B_{R_k}(0)$ for every $j = 1, ..., J$ and $\operatorname{supp}(\psi_k^{(J+1)}) \subset \mathbb{R}^d \setminus \bigcup_{j=1}^j B_{2R_k}(x_k^{(j)})$ by construction, as the balls $B_{2R_k}(x_k^{(j)})$, $j = 1, ..., J$ are pairwise disjoint for every k.
To prove that $u_{n_k} - \displaystyle\sum_{j=1}^J u_k^{(j)}(\cdot - x_k^{(j)}) - \psi_k^{(J+1)} \xrightarrow{k\to\infty} 0$ in $H^1(\mathbb{R}^d)$, first notice that the support of this function is contained in the union of annuli $\{R_k/2 \leq |x - x_k^{(j)}| \leq 4R_k\}$. Next, let's analyse the behaviour of the function on, for instance, the annulus $\{R_k/2 \leq |x - x_k^{(1)}| \leq 4R_k\}$: here, our function is reduced to $u_{n_k}(1 - \chi_k - \prod_{j=1}^J \zeta_k(\cdot - x_k^{(j)}))$ which behaves like



In the region $S_1 = \{R_k/2 \leq |x - x_k^{(1)}| \leq R_k\}$ we have

$$|u_{n_k}(1 - \chi_k)|^2 \leq |u_{n_k}|^2 \tag{A.3}$$

and

$$|\nabla(u_{n_k}(1-\chi_k))|^2 = |-\nabla\chi_k u_{n_k} + \chi_k \nabla u_{n_k}|^2 \leq (|u_{n_k}||\nabla_{\chi_k}| + |\chi_k||\nabla_{u_{n_k}}|)^2 \leq \frac{32}{R_k^2}|u_{n_k}|^2 + 2|\nabla u_{n_k}|^2; \tag{A.4}$$

we've used that $|\chi_k| \leq 1$ and that $|\nabla\chi_k(x)| = |\chi'(\frac{2|x|}{R_k})\frac{2}{R_k}\frac{x}{|x|}| \leq \frac{4}{R_k}$ as $|\chi'| \leq 2$. Using A.3 and A.4, we get

$$\|u_{n_k}(1 - \chi_k)\|_{H^1(S_1)} \lesssim \|u_{n_k}\|_{H^1(S_1)} \to 0$$

by (A.2).
In the region $S_2 = \{R_k \le |x - x_k^{(1)}| \le 2R_k\}$ we directly have that $\|u_{n_k}\|_{H^1(S_2)} \to 0$ by (A.2).
In the region $S_3 = \{2R_k \le |x - x_k^{(1)}| \le 4R_k\}$ we can proceed as in region S_1.
Moreover, since by (A.1) $\lim_{k\to\infty} \int_{\mathbb{R}^d} |u_k^{(j)}|^2 = \lim_{k\to\infty} \int_{B_{R_k/2}(0)} |u_k^{(j)}|^2 = \int_{\mathbb{R}^d} |u^{(j)}|^2$ and $u_k^{(j)} \rightharpoonup u^{(j)}$ weakly in $H^1(\mathbb{R}^d)$ the convergence is also strong in L^p, $2 \le p < 2^*$. To prove this, we just use Hölder inequality and Sobolev embeddings as we've done in Remark A.2.
Notice that by construction $\mathbb{1}_{B_{8R_k}(x_k^{(j)})} \psi_k^{(J+1)} \to 0$ strongly in $L^2(\mathbb{R}^d)$ for every $j = 1, ...J$.
To prove that $\mathbf{m}(\boldsymbol{\psi}^{(J+1)}) \to 0$ as $J \to \infty$, we proceed in three steps:

1. We prove that if $\psi_k^{(J+1)}(\cdot - y_k) \rightharpoonup \psi \ne 0$ weakly in H^1 for $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$, then $u_{n_k}(\cdot - y_k) \rightharpoonup \psi$ weakly in H^1;
2. We prove that $\psi^{(J+1)}(\cdot + x_k^{(j)}) \rightharpoonup 0$ weakly in H^1 for every $1 \le j \le J$;
3. We conclude.

Step 1: Assume that $\psi_k^{(J+1)}(\cdot - y_k) \rightharpoonup \psi \ne 0$ weakly in H^1 for $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$: if, up to a subsequence, $|y_k| \le 6R_k$, then $B_{2R_k}(y_k) \subset B_{8R_k}(0)$ and $\psi_k^{(J+1)}(\cdot - y_k)\mathbb{1}_{B_{2R_k}(0)} \rightharpoonup 0 = \psi$ which contradicts the fact that $\psi \ne 0$. We can therefore assume that $|y_k| \ge 6R_k$ for k large; this in turn implies that $\zeta_K \equiv 1$ on $B_{2R_k}(y_k)$, thus $u_{n_k}(\cdot - y_k)(1)_{B_{2R_k}} = \psi_k^{(J+1)}(\cdot - y_k)(1)_{B_{2R_k}} \rightharpoonup \psi$, which concludes the proof of our claim.
Step 2: We have

$$\psi_k^{(J+1)}(\cdot + x_k^{(j)}) = \left(\prod_{l=1}^{J} \zeta_k(\cdot + x_k^{(j)} - x_j^{(l)})\right) u_{n_k} = \left(\zeta_k \prod_{l\ne j} \zeta_k(\cdot + x_k^{(j)} - x_j^{(l)})\right) u_{n_k} m,$$

hence $\psi_k^{(J+1)}(\cdot + x_k^{(j)})\mathbb{1}_{B_{2R_k}(0)} \equiv 0$ as $\mathrm{supp}(\zeta_k) \subset \mathbb{R}^d \backslash B_{2R_k}(0)$. This proves our claim.
Step 3: Since all the possible weak H^1 limits of $\psi_k^{(J+1)}(\cdot - y_k)$ are also the limits of $u_{n_k}(\cdot - y_k)$, they are the $u^{(j)}$, $j \in \mathbb{N}$ given by Lemma 1.4. Moreover, Step 2 implies that all the possible limits remaining are the $u^{(j)}$ with $j > J$, thus impliying that

$$\mathbf{m}(\boldsymbol{\psi}^{J+1}) = \sup_{j>J} \|u^{(j)}\|_{L^2}^2.$$

Since in Lemma 1.4 we proved that $\|u^{(j)}\|_{L^2}^2 \to 0$ for $j \to \infty$, we have $\mathbf{m}(\boldsymbol{\psi}^{(J+1)}) \to 0$ as $J \to \infty$.
This concludes the proof of the Theorem. $\square$

Appendix B. Heuristically deriving the Hartree equation as mean-field limit of large systems

In this section we demonstrate *heuristically* the emergence of the Hartree equation in the description of a bosonic system in its mean-field limit; our main reference for this is [3].
Consider a system of N identical, non-relativistic bosons with two-body interactions given by the potential κw, where $\kappa \geq 0$ is a coupling constant and w is a real valued function. To be coherent with the energy functional and equation given in Chapter 2, we work with no external potential. The dynamical evolution of the state is governed by the linear Schrödinger equation

$$i\partial_t \Psi_N = H_N \Psi_N, \tag{B.1}$$

where $\Psi_N = \Psi_N(t, x_1, ..., x_N)$, $x_j \in \mathbb{R}^3$, $1 \leq j \leq N$ and the N-particle Hamiltonian H_N is given by

$$H_N = \sum_{j=1}^{N} -\frac{1}{2m}\Delta_{x_j} + \kappa \sum_{i<j}^{N} w(x_i - x_j), \text{ on } L^2(\mathbb{R}^3)^{\otimes_s N}. \tag{B.2}$$

We have denoted with Δ_{x_j} the Laplacian associated with the jth copy of $\mathbb{R}^3$, with $\otimes_s$ the symmetric tensor product and with m the mass of the boson.
Since the energy of the system scales like $\mathcal{O}(N) + \kappa\mathcal{O}(N^2)$, the energy per particle is $\mathcal{O}(1)$ if the coupling obeys $\kappa = \mathcal{O}(N^{-1})$. Thus, we define the mean field limit as

$$N \to \infty, \text{ and } \kappa \to 0, \text{ such that } \nu = \kappa N \text{ is constant.}$$

We will operate under the assumption that all (except $o(N)$) bosons are in the same one-particle state described by the wave function $\psi \in L^2(\mathbb{R}^3)$. We pick an initial datum $\psi_0 \in L^2(\mathbb{R}^3)$ with $\|\psi_0\|_{L^2}^2 = 1$ and introduce the $N-$particle state

$$\Psi_N(t = 0, x_1, ..., x_N) = \prod_{j=1}^{N} \psi_0(x_j) \in L^2(\mathbb{R}^3)^{\otimes_s N}$$

We also assume that B.1 approximately preserves the norm of $\Psi_N(t)$ when N becomes large, so we can write

$$\Psi_N(t, x_1, ..., x_N) \approx \prod_{j=1}^{N} \psi(t, x_j) \in L^2(\mathbb{R}^3)^{\otimes_s N}.$$

Physically speaking, this means that correlation effects remain small.
When approaching the mean-field limit, we expect that the potential per particle V_{eff} given by

$$V_{eff}(t, x) = \frac{\nu}{N} \sum_{j=1}^{N} \int_{\mathbb{R}^3} w(x - x_j)|\psi(t, x_j)|^2 \, dx_j = \nu(w * |\psi(t)|^2)(x). \tag{B.3}$$

From this heuristic discussion we conclude that the dynamical evolution of the bosonic system in its mean-field regime is described by the Schrödinger equation for the one-particle wave function, $\psi(t, x)$ with a potential term given by $V_{eff}(t, x)$. Thus, we are led to the (nonlinear) Hartree equation as an effective description of the limiting dynamics:

$$\begin{cases} i\partial_t \psi = -\Delta\psi + \nu(w * |\psi|^2)\psi, \\ \psi(0, x) = \psi_0(x) \end{cases} \tag{HE}$$

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