Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Corso di Laurea in Control Systems Engineering

Asymptotic statistics of repeated indirect quantum measurements

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MATRICOLA: 2020376

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18 Luglio 2022
Dipartimento di Ingegneria dell'Informazione
Università di Padova
Anno Accademico 2021/2022

Abstract

This research project analyzes the asymptotic behaviour of a quantum system subject to a sequence of indirect measurements. These quantum measurements give rise to a stochastic process, called *quantum trajectory*, which describes the state of the system after each measurement. Using martingale techniques we will prove that this quantum trajectory converges non-deterministically to one of the minimal invariant subspaces determined by the quantum channel, which is a linear map that describes the mean evolution of the state. The probability of convergence to each subspace depends on the initial state of the system. The convergence can be steered towards a chosen target subspace, modifying the dynamics with a feedback control scheme properly designed using Lyapunov techniques and graph-theoretic ideas, generalizing the control scheme preseneted in [2]. Preparation of quantum states in a target subspace finds one of its applications in cooling techniques and in state preparation in quantum information.

The other focus of this research project is on the derivation of some *statistical asymptotic laws* (Law of Large Numbers - Central Limit Theroem - Law of Iterated Logarithms) for the stochastic process describing the measurement outcomes, without requiring any ergodicity assumption on the quantum channel, and thus generalizing the results obtained in [3]. These statistical asymptotic laws can be used for solving estimation problems like process tomography.

This research project puts together probability theory and control theory in order to prove asymptotic results on quantum stochastic processes and in order to design a feedback control scheme that is able to prepare a quantum system in a precise target subspace. A rigorous mathematical treatment is employed in deriving results having important applications in quantum engineering problems, like information encoding or parameter estimation.

Sommario

In questa tesi viene analizzato il comportamento asintotico di un sistema quantistico soggetto ad una sequenza di misurazioni indirette. Tali misurazioni quantistiche danno vita ad un processo stocastico, chiamato traiettoria quantistica, il quale descrive lo stato del sistema dopo ogni misurazione. Tale progetto ricorre a tecniche che utilizzano martingale, per dimostrare che questa traiettoria quantistica converge in modo non deterministico in uno dei sottospazi invarianti minimali determinati dal canale quantistico, ossia una mappa lineare che descrive l'evoluzione media dello stato. La probabilità di convergenza a ogni sottospazio dipende dallo stato iniziale del sistema. Questa convergenza può essere orientata verso un preciso sottospazio target grazie ad uno schema di controllo con retroazione, il cui design sfrutta tecniche di Lyapunov e idee dalla teoria dei grafi. La preparazione di stati quantistici in un preciso sottospazio target trova applicazione in tecniche di cooling e preparazione degli stati in informazione quantistica.

L'altro focus di tale progetto di ricerca viene posto sulla derivazione di alcune leggi statistiche asintotiche (Legge dei Grandi Numeri - Teorema del Limite Centrale - Legge dei Logaritmi Iterati) per il processo stocastico che descrive i risultati delle misurazioni, senza richiedere alcuna particolare assunzione ergodica sul canale quantistico, e dunque generalizzando i risultati ottenuti in [3]. Tali leggi statistiche asintotiche possono essere utilizzate per risolvere problemi di stima come quello di tomografia del processo.

Questo progetto di ricerca mette insieme teoria della probabilità, teoria del controllo per derivare risultati asintotici per sistemi stocastici quantistici e per il design di uno schema di controllo con retroazione in grado di preparare un sistema quantistico in un preciso sottospazio target. Viene impiegato in formalismo matematico rigoroso per derivare risultati aventi importanti applicazioni in problemi di ingegneria quantistica, come codifica di informazione o stima di parametri.

Contents

Introduction

The potential of quantum technologies has been demonstrated in several applications requiring a high level of computing power. By exploiting the laws of quantum physics to process binary information, quantum computing circuits can already do calculations that can't easily be simulated on classical supercomputers: in 2019 Google claimed the famous "quantum supremacy", achieved by its 53 qubits quantum computer that executed, in about 200 seconds, a specific task that would have taken a classical computer 10 000 years [16]. At the end of 2021 IBM launched the 127 qubit Eagle quantum processor, which represents a step towards its goal of creating a 1121 qubit Condor quantum processor by 2023 [4]. On the other hand, solving real world problems, such as simulating drug molecules or materials using quantum chemistry, will require quantum computers to get drastically bigger (millions of physical qubits) and more powerful. That's why at the moment we can only talk about *potential* applications of quantum computers, as clearly presented in `https://www.youtube.com/watch?v=-UlxHPIEVqA`.

The main one regards *quantum simulations*, since obviously in this area quantum computers have an exponential speedup over classical ones. Indeed simulating quantum systems with as few as 30 particles is difficult even on the world's most powerful supercomputers. We also can't do this on quantum computers yet, but their promise is to be able to simulate larger and larger quantum systems. Quantum simulations concerns simulating chemical reactions or how electrons behave in different materials. This could permit a step towards understanding what makes some materials superconduct, or study important chemical reactions to improve their efficiency. One example aims to produce fertiliser in a way that emits way less carbon dioxide, as fertiliser production contributes to around 2% of global CO_2 emissions. In general quantum simulations would mean that we would be able to rapidly prototype many different materials inside a quantum computer and test all their physical parameters, instead of having to physically make them and test them in a laboratory, which is a much more laborious and expensive process. This could be a lot faster and save a huge amount of time and money.

Other applications of quantum computers can be found in optimization problems, machine learning, financial modelling and cybersecurity. This last one is due to Shor's algorithm, that can efficiently find the factors of large integers, breaking current cryptography techniques based on RSA algorithm. With the motivation provided by this algorithm, the topic of quantum computing has gathered momentum, supported also by many national government and military funding agencies, and researchers around the world are racing to be the first to create a working quantum computer.
What is keeping us away from the implementation of a working quantum computer?

Adding qubits to a quantum circuit represents a very challenging task, due to the fragility of quantum correlations, which represent the key advantage of Quantum Information (QI), and that are easily destroyed by the unavoidable interaction of the quantum system with its surrounding environment. The consequence of this interaction is that the information encoded in the qubits will start to leak away, leading to a phenomena called *decoherence*. For a deep understanding of decoherence we refer the reader to the chapter 8 of Nielsen and Chuang's book [22].,

To overcome this unwanted effect, many quantum control techniques have been developed, with the aim of cancelling the undesirable parts of the interaction Hamiltonian, which represents the energy exchanges between the quantum system and the environment and that is responsible of the leak of information. This type of control could be implemented through unitary pulses applied instantaneously and equidistantly separated in time to the quantum system, exploiting a technique called *bang-bang control* ([28], [9]).

These techniques are based on open-loop control, while in this project we will recur to feedback actions to fight decoherence. Decoherence control through feedback techniques can be implemented in two different ways: through *active manipulation* of the quantum state and through *passive stabilization* of it. In the first approach loss of information is corrected by monitoring the system and conditionally carrying on suitable feedback operations. On the other hand, the second approach relies on the existence of a subspace of states that, owing to special symmetry properties, are dynamically decoupled from the environment. This project deals more with the second approach, realizing a control feedback scheme that stabilizes the system in a target subspace.

The project is organized as follows: in chapter 3 we will show that a quantum system, subject to a sequence of indirect measurements, converges to one of the minimal invariant subspaces in a *non-deterministic* way. To demonstrate this and to find the probability of convergence to a specific subspace, we will exploit some tools of probability theory related to Markov Chains, martingales and ergodic theory, which are briefly presented in chapter 1. To this sequence of indirect measurements we can associate two stochastic processes: the first one describes the state of the system after each indirect measurement, while the second one counts how many times you get a certain outcome till a certain step n. In chapter 4 we will show that a *Law of Large Numbers* (LLN), a *Central Limit Theorem* (CLT) and a *Law of Iterated Logarithm* (LIL) result can be obtained on the second process, without requiring the quantum channel to be ergodic and thus generalizing [3]. Finally in chapter 5 we will present our feedback control scheme that stabilizes the quantum system in a target subspace, that we want to be a *Decoherence Free Subspace* (DFS) [20]. This could lead to a "passive" error-prevention scheme, in which logical qubits are encoded within subspaces which do not decohere thanks to symmetry reasons.

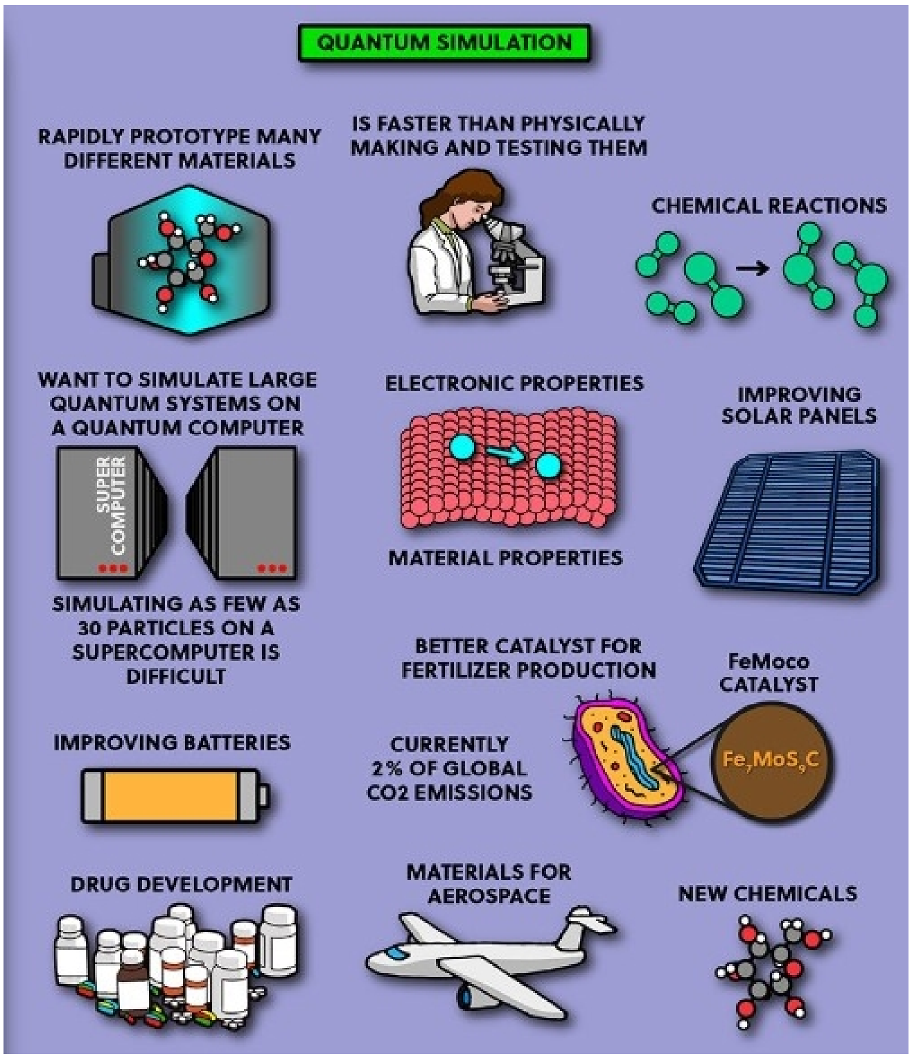

Figure 1: Overview of how quantum simulations can be exploited to solve real world problems: `https://www.youtube.com/watch?v=-UlxHPIEVqA`

Chapter 1

Probability Theory

There is a deep interconnection between quantum mechanics and probability, due to the intrinsic random nature of quantum measurements. The state of our quantum system subject to a sequence of indirect measurements will turn out to be a Markov Chain, and its stochastic evolution will be analyzed through a special martingale, that will permit us to prove the main convergence theorem. Therefore in this chapter we will review some concepts and tools of probability theory, that will lead us to our main results. For a deeper understanding of these probability tools we refer the reader to the book [18] and to the Van Handel lecture notes [27].

1.1 Probability space & types of convergence

In this section we will recap the basic concepts needed for building a probability space where the probabilistic evolution of an experiment can be modelled. Firstly we introduce the set of events Ω, where each element $\omega \in \Omega$ represents a possible fate of the experimental system. Once we have specified Ω, any yes-no question is represented by the subset of Ω consisting of those realizations $\omega \in \Omega$ for which the answer is yes. We will collect all sensible yes-no questions in a set $\mathcal{F} \subset \Omega$.

Definition 1. A $\sigma - algebra$ $\mathcal{F}$ is a collection of subsets of Ω such that

1. if $A_n \in \mathcal{F}$ for countable n, then $\bigcup_n A_n \in \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $A^c = \Omega \backslash A \in \mathcal{F}$,
3. $\Omega \in \mathcal{F}$.

An element $A \in \mathcal{F}$ is called an *$\mathcal{F}$-measurable set* or an *event*.

It remains to assign a probability to every event in $\mathcal{F}$. This has to be done in a consistent way: if A and B are two mutually exclusive events ($A \cap B = \varnothing$), then the probability of A *or* B has to be the sum of the individual probabilities. This leads to the following definition.

Definition 2. A *probability measure* is a map $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that:

1. for countable $\{A_n\}$ such that $A_n \cap A_m = \varnothing$ for $n \neq m$, $\mathbb{P}(\bigcup_n A_n) = \sum_n \mathbb{P}(A_n)$ (σ-additivity),
2. $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$.

We now have all the objects for building a probability space.

Definition 3. A *probability space* is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$.

The next most important ingredient in probability theory is the *random variable*. If $(\Omega, \mathcal{F}, \mathbb{P})$ describes all possible fates of the experimental system and their probabilities, then random variables describe concrete *observations* that we can make on the system. For example the outcome of a measurement of our experimental system is described by specifying what value it takes for every possible fate $\omega \in \Omega$ of the system.

Definition 4. A *random variable* is a measurable function $f : \Omega \to \mathbb{R}$, namely $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$ for every $A \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ represents the *Borel* σ-algebra.

For these functions defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there are different notions of convergence, since the usual notion of *pointwise convergence of functions* [1] is useless in probability. Indeed typically we do not have convergence for *all* ω, but we have convergence for *almost all* ω (i.e. the set of all ω where we do have convergence has probability one). Moreover, just as we introduced almost sure (a.s.) convergence because it naturally occurs when "pointwise convergence" (for all "points") fails, we need to introduce two more types of convergence, which arise naturally when a.s. convergence fails, but they are also useful as tools to help to show that a.s. convergence holds. These three different types of convergence of random variables can be thought as variants of standard pointwise convergence. There is yet another notion of convergence which is profoundly different from the previous three. This convergence, known as *weak convergence* (in distribution), is fundamental to the study of probability and statistics. For this new type of convergence the actual values of the random variables themselves are not important. It is simply the probabilities that they will assume those values that matter.

Definition 5 (Types of Convergence)**.** Let $\{f_n\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $f_n \xrightarrow{\mathbb{P}} f$: $\{f_n\}$ *converge in probability* to f if given $\varepsilon > 0$ there exists an index N such that
$$\mathbb{P}(\{x \in \Omega : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon, \ \forall n > N$$
2. $f_n \xrightarrow{a.s.} f$: $\{f_n\}$ *converge almost surely* to f if $f_n(x) \to f(x)$ except on a set A with $\mathbb{P}[A] = 0$
3. $f_n \xrightarrow{L^p} f$: $\{f_n\}$ *converge in L^p* (mean convergence) to $f > 0$ if
$$\lim_{n\to\infty} \mathbb{E}|f_n - f|^p = 0$$
4. $f_n \xrightarrow{\mathcal{D}} f$: $\{f_n\}$ *converge in distribution* to f if for every bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$ it holds
$$\lim_{n\to\infty} \mathbb{E}[g(f_n)] = \mathbb{E}[g(f)]$$
Convergence in distribution can be shown to be equivalent to convergence of the cumulative distribution function (CDF): $F_{f_n}(x) \to F_f(x)$ at each continuity point of F_f.

Example 1 (weak LLN)**.** *Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $m = \mathbb{E}X_i < \infty$. Then, the weak law of large numbers asserts that the empirical mean converges in probability to the the expectation m, namely*
$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathbb{P}} m$$

[1] pointwise convergence: let $f_n : \mathbb{R} \to \mathbb{R}$ then $\lim_{n\to\infty} f_n = f$ if $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in \mathbb{R}$

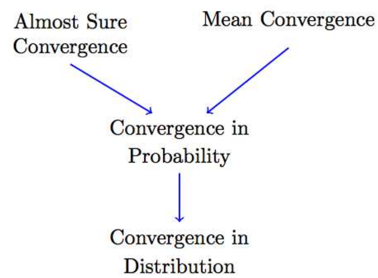

Figure 1.1: Implications between different types of convergence

Example 2 (strong LLN)**.** *Let* $\{X_n\}$ *be a sequence of i.i.d. random variables defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $m = \mathbb{E}X_i < \infty$*. Then, the strong law of large numbers asserts that the empirical mean converges in almost surely to the the expectation* m*, namely*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} m$$

Figure 1.1 illustrates the various implications between different types of convergence, from which follows that the strong law of large numbers does imply the weak law.

Actually the *strong Law of Large Numbers* (LNN) represents one of the fundamental results in probability, since it helps to justify our intuitive notions of what probability actually is, and it has many direct applications, such as Monte Carlo estimation theory. Another impressive achievements of probability theory is the *Central Limit Theorem* (CLT), which serves as the basis for much of statistical theory.

Example 3 (CLT)**.** *Let* $\{X_n\}$ *be a sequence of i.i.d. random variables defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with finite second moment* $\mathbb{E}X_n^2 < \infty$*. Let* $\mu = \mathbb{E}X_n,\ \sigma^2 = \mathbb{E}(X - \mathbb{E}X)^2$*. Let* $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$*. Then*

$$\frac{\bar{X}_n - n\mu}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

Thanks to this asymptotic law we can use data to do statistical tests to estimate μ and σ^2 which fully determine the asymptotic distribution of the statistical average of our random variables.

Another important asymptotic law is represented by the *Law of Iterated Logarithm* (LIL), that describes the magnitude of the fluctuations of a random walk.

Example 4 (LIL)**.** *Let* $\{X_n\}$ *be a sequence of i.i.d. random variables defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with finite second moment* $\mathbb{E}X_n^2 < \infty$*. Let* $\mu = \mathbb{E}X_n,\ \sigma^2 = \mathbb{E}(X - \mathbb{E}X)^2$*. Let* $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$*. Then*

$$\lim_{n\to\infty} \sup \frac{|\bar{X}_n - n\mu|}{\sqrt{2n\sigma^2 \log\log n\sigma^2}} = 1 \quad a.s.$$

On the other hand these three asymptotic laws require the random variables to be i.i.d., which is a strong assumption that doesn't hold for our process. Actually we will show that they all hold also for our process, and we will use martingale techniques to show that.

1.2 The Radon-Nikodym derivative

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. It is often interesting to try to find other measures on $\mathcal{F}$ with different properties, since calculations which are difficult under one measure can often become very simple if we change to a suitably modified measure. For example, if $\{X_n\}$ is a collection of random variables with some complicated dependencies under $\mathbb{P}$, it may be advantageous to pass to a modified measure $\mathbb{Q}$ under which the X_n are independent. In the following we present a technique that will permit us to generate a large family of measures related to the starting measure $\mathbb{P}$.

Let f be a nonnegative random variable with unit expectation $\mathbb{E}f = 1$. For any $A \in \mathcal{F}$, define

$$\mathbb{Q}(A) := \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A f) = \int_A f(\omega)\mathbb{P}(\omega)$$

which is a probability measure, and moreover

$$\mathbb{E}_{\mathbb{Q}}(g) = \int g(\omega)\mathbb{Q}(d\omega) = \int g(\omega)f(\omega)\mathbb{P}(d\omega) = \mathbb{E}_{\mathbb{P}}(gf)$$

for any random variable g for which either side is well defined.

Definition 6. (Density) A probability measure $\mathbb{Q}$ is said to have a *density* with respect to a probability measure $\mathbb{P}$ if there exists a nonnegative random variable f such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A f)$ for every measurable set A. The density f is denoted as $d\mathbb{Q}/d\mathbb{P}$.

Suppose that $\mathbb{Q}$ has a density f with respect to $\mathbb{P}$. Then these measures must satisfy an important *consistency* condition: if $\mathbb{P}(A) = 0$ for some event A, then $\mathbb{Q}(A)$ must also be zero. Evidently, the use of a density to transform a probability measure $\mathbb{P}$ into another probability measure $\mathbb{Q}$ "respects" those events that happen for sure or never happen at all.

Definition 7 (absolutely continuity)**.** A measure $\mathbb{Q}$ is said to be *absolutely continuous* with respect to a measure $\mathbb{P}$, denoted as $\mathbb{Q} \ll \mathbb{P}$, if $\mathbb{Q}(A) = 0$ for all events A such that $\mathbb{P}(A) = 0$.

We have seen that if $\mathbb{Q}$ has a density with respect to $\mathbb{P}$, then $\mathbb{Q} \ll \mathbb{P}$. It turns out that the converse is also true: if $\mathbb{Q} \ll \mathbb{P}$, then we can always find some density f such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A f)$. Hence the existence of a density is completely equivalent to absolute continuity of the measures. This is a deep result, known as the *Radon- Nikodym theorem.*

Theorem 1 (Radon-Nikodym)**.** *Suppose that $\mathbb{Q} \ll \mathbb{P}$ are two probability measures on the space $(\Omega, \mathcal{F})$. Then there exists a nonnegative $\mathcal{F}$-measurable function f with $\mathbb{E}_{\mathbb{P}}(f) = 1$, such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A f)$ for every $A \in \mathcal{F}$. Moreover, f is unique (i.e. if f' is another $\mathcal{F}$-measurable function with this property, then $f = f'$ $\mathbb{P} - a.s.$). Hence it makes sense to speak of 'the' density, or Radon-Nikodym derivative, of $\mathbb{Q}$ with respect to $\mathbb{P}$, and this density is denoted as $d\mathbb{Q}/d\mathbb{P}$.*

1.3 Martingale, Supermartingale & Submartingale

This section introduces the notion of a martingale, which will play a fundamental role in this project. To talk about martingales we need the notion of *stochastic process*, which is just a sequence of random variables $\{X_n\}$, indexed by time n. We will deal with discrete time stochastic processes (i.e. $n \in \mathbb{N}$). Stochastic processes start leading a life of their own once we build a notion of time into our probability space. To this aim we need to specify which sub-σ-algebra of questions in $\mathcal{F}$ can be answered by time n. If we label this σ-algebra by $\mathcal{F}_n$, we obtain the following notion.

Definition 8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A (discrete time) *filtration* is an increasing sequence $\{\mathcal{F}_n\}$ of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}$. The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is called a *filtered probability space.*

Note that the sequence $\mathcal{F}_n$ must be increasing, since a question that can be answered by time n can also be answered at any later time. We can now introduce a notion of *causality* for stochastic processes.

Definition 9. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ be a filtered probability space. A stochastic process $\{X_n\}$ is:

- *adapted* if $X_n \in \mathcal{F}_n$,
- *predictable* if $X_n \in \mathcal{F}_{n-1}$,

Hence if $\{X_n\}$ is adapted, then X_n represents a measurement of something in the past or present (up to and including time n), while in the predictable case X_n represents a measurement of something in the past (before time m). To introduce the definition of a martingale we need the notion of *conditional expectation*, which play a fundamental role in much of probability theory.

Definition 10. A σ-algebra $\mathcal{F}$ is said to be *finite* if it is generated by a finite number of sets $\mathcal{F} = \sigma\{A_1, ..., A_m\}$.

Definition 11. Let $\mathcal{F}$ be a separable σ-algebra (i.e. $\mathcal{F} = \sigma\{\mathcal{F}_n\}_{n>0}$) with $\mathcal{F}_n$ finite and generated by the partition $\{A_k\}_{k=1}^m$. Let $X \in L^1$. Then we define

$$\mathbb{E}(X|\mathcal{F}) = \lim_{n\to\infty} \mathbb{E}(X|\mathcal{F}_n)$$

with

$$\mathbb{E}(X|\mathcal{F}_n) = \sum_{k=1}^{m} \mathbb{E}(X|A_k) I_{A_k}$$

where $\mathbb{E}(X|A) = \mathbb{E}(XI_A)/\mathbb{P}(A)$ if $\mathbb{P}(A) > 0$ and $\mathbb{E}(X|A) = 0$ if $\mathbb{P}(A) = 0$.

We have now all the tools to introduce the notion of a martingale, supermartingale and submartingale.

Definition 12. Let us consider a stochastic process $\{X_n\}$ which is adapted and with $X_n \in L^1(\Omega)$ $\forall n \in \mathbb{N}$. Then

1. $\{X_n\}$ is a *martingale* if $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n$,
2. $\{X_n\}$ is a *supermartingale* if $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \leqslant X_n$,
3. $\{X_n\}$ is a *submartingale* if $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \geqslant X_n$,

We can think about a martingale as a constant process plus some stochastic fluctuations. On the other hand a supermartingale is a decreasing process plus some stochastic fluctuations, while a submartingale is an increasing process plus some stochastic fluctuations.

Given a martingale, we can easily obtain a supermartingale from it by simply using a concave function, as a consequence of the Jensen inequality [2] Same for a submartingale but with a convex function.

Proposition 1. *Let $\{X_n\}$ be a martingale. Let φ be a function such that $\varphi(X_n) \in L^1(\Omega)$ $\forall n \in \mathbb{N}$. Then*

1. *if φ is* concave *then $Y_n = \varphi(X_n)$ is a* supermartingale*,*
2. *if φ is* convex *then $Y_n = \varphi(X_n)$ is a* submartingale*.*

Moreover if a martingale X_n is *bounded*, then it cannot fluctuate forever; in other words, it must converge to some random variable X_∞.

[2]For a real *convex* function φ, numbers $x_1, x_2, ..., x_n$ in its domain, and positive weights a_i, Jensen inequality can be stated as $\varphi\left(\frac{\sum_i a_i x_i}{\sum a_i}\right) \leqslant \frac{\sum_i a_i \varphi(x_i)}{\sum_i a_i}$. The inequality is reversed if φ is *concave*.

Theorem 2 (Martingale convergence)**.** *Let $\{X_n\}$ be a martingale such that* $\sup_n \mathbb{E}[X_n^-] \leqslant \infty$*, where* X_n^- *is the negative part of* X_n*, i.e.* $X_n^- = -\min(X_n, 0)$*. Then there exists a random variable* $X_\infty \in L^1(\Omega)$ *such that*

$$X_n \xrightarrow{a.s.} X_\infty$$

Moreover we have convergence in L^1 *to* X_∞ *if* $\sup_n \mathbb{E}[|X_n|] \leqslant \infty$*.*

Martingales play an important role in this project since they will permit us to derive limit theorems (LLN-CLT-LIL) for a process that is neither i.i.d. nor Markovian. Moreover we will construct a special martingale to prove the convergence of our quantum trajectory to an invariant subspace.

1.4 Dynamical systems and ergodic theory

Dynamical systems are systems evolving in time, often governed by differential equations, but also perhaps by other continuous or discrete formulae. Ergodic theory is the study of statistical properties of that evolution. Originally created to connect thermodynamics to statistical mechanics, it has been extended to connect with many branches of mathematics, including Markov Chains and their stochastic evolution.

In our probabilistic setting a *dynamical system* is defined as a quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, T)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, while $T : \Omega \to \Omega$ is a transformation that preserves the measure $\mathbb{P}$, i.e. $\mathbb{P} \circ T^{-1} = \mathbb{P}$. Ergodic theory analyzes the long time behaviour and the average behaviour of $(T^n(\omega))$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

Definition 13. Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be a dynamical system. We say that $A \in \mathcal{F}$ is T-invariant if $T^{-1}(A) = A$. We say that $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is *ergodic* if all T-invariant sets have measure 0 or 1.

Let us recall the most important ergodic theorems which concern some convergence results for the averages $(\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k)_{n\in\mathbb{N}}$, for $f \in L^p(\Omega),\ p \geqslant 1$. The first one requires the notion of a *contraction mapping.*

Definition 14. A *contraction mapping* on a metric space (M, d) is a function $f : M \to M$, with the property that there is some real number $0 \leqslant k < 1$ such that

$$d(f(x), f(y)) \leqslant k d(x, y) \quad \forall x, y \in M$$

The smallest such value of k is called the *Lipschitz constant* of f.

Theorem 3 (Von Neumann's mean ergodic theorem, 1932)**.** *Let ϕ be a contraction map on a Hilbert space $\mathcal{H}$ and $\Pi_{\mathcal{F}_\phi}$ be the orthogonal projection on the set of fix points of ϕ, i.e.* $\mathcal{F}_\phi := \{x \in \mathcal{H} \mid \phi(x) = x\}$*. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi^k(x) \xrightarrow[n\to\infty]{} \Pi_{\mathcal{F}_\phi} x$$

for every $x \in \mathcal{H}$.

Theorem 4 (Birkhoff's ergodic theorem, 1931)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be an ergodic dynamical system. Then for every $f \in L^1(\Omega)$*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{} \mathbb{E}[f] \quad \mathbb{P} - a.s.$$

Now that we have recapped all the probability tools that we will need in this project, we pass to the presentation of the main character of this work: quantum mechanics.

Chapter 2

Quantum Mechanics

In this chapter we will recall the principal features and notions of quantum mechanics. We will provide the mathematical formalism needed to describe a quantum system and its time evolution, and we will link it to the notion of *attractive* and *transient* subspace. The main role in this connection is played by the set of invariant states of the quantum channel, namely the linear map that is dictating the time evolution of the system. Finally we will analyze the structure of this set and the spectral properties related to the quantum channel. For a detailed introduction on Quantum Mechanics and on quantum channel we refer the reader to the Nielsen and Chuang's book [22] and to the Wolf's lecture notes [29] respectively.

2.1 A look into the Quantum world

At the subatomic level nature makes strange jokes, and randomness comes into play. For example a photon could have horizontal and vertical polarization at the same time, in a perfect *quantum superposition*. Now the question follows naturally: what happens if we measure its polarization? With a certain probability, we would observe horizontal polarization, and with "the remaining probability" we would observe vertical polarization.

A new mathematical framework is needed to describe the quantum world. A *Hilbert space* sets up the arena in which quantum mechanics takes place. Hence given a k-dimensional quantum system, its state space is represented by the Hilbert space $\mathcal{H} \simeq \mathbb{C}^k$. In the Dirac notation $|\psi\rangle$ represents a state vector in $\mathcal{H}$, while $\langle\psi| = |\psi\rangle^\dagger$ represents its dual, and $\langle\psi|\phi\rangle$ is the inner product of $\mathcal{H}$. Turning back to the photon example, its quantum state could be described in the Horizontal-Vertical basis $\{|H\rangle, |V\rangle\}$ as

$$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To understand the meaning of the coefficients α and β, we have to introduce the notion of *observable*, which defines a *projective measurement*.

Definition 15 (Observable)**.** A linear operator represents an Observable $\hat{O}$ iff it is Hermitian, hence it

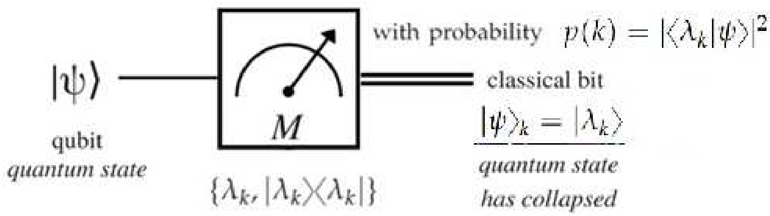

Figure 2.1: Projective measurement defined by the observable $\{\lambda_k, |\lambda_k\rangle\langle\lambda_k|\}$

can be associated to a physical observable. It can be represented through its spectral decomposition

$$\hat{O} = \sum_{k=1}^{d} \lambda_k |\lambda_k\rangle\langle\lambda_k|$$

hence it is completely characterised by its eigenvalues and eigenvectors $\{\lambda_k, |\lambda_k\rangle\langle\lambda_k|\}$.

Every measurement can be associated to an observable $\hat{O}$, and its possible outcomes correspond to the eigenvalues λ_k, while $|\lambda_k\rangle\langle\lambda_k|$ represent the rank-one projectors onto the eigenspaces of the operator $\hat{O}$.

Definition 16. A *projective measurement* with input $|\psi\rangle$ and defined by the observable $\{\lambda_k, |\lambda_k\rangle\langle\lambda_k|\}$, gets as output λ_k with probability $p(k) = |\langle\lambda_k|\psi\rangle|^2$, and projects the input into the k-th eigenspace of the observable, with a final collapse of the quantum state into $|\psi\rangle_k = |\lambda_k\rangle$.
The *measurement basis* is represented by the eigenvectors $\{|\lambda_k\rangle\}$ of the observable that we want to measure.

Turning back to our photon, if we want to measure its Horizontal-Vertical polarization, we have to measure the observable

$$\sigma_z = +1|H\rangle\langle H| - 1|V\rangle\langle V|$$

which gives as output $|\psi\rangle_{+1} = |H\rangle$ with probability $p(+1) = |\alpha|^2$, and $|\psi\rangle_{-1} = |V\rangle$ with probability $p(-1) = |\beta|^2$. Hence if we let N identical photons pass through a H-V *polarazing beam-splitter* (PBS), they will be detected $N|\alpha|^2$ times in the *H-path*, and $N|\beta|^2$ times in the *V-path*. What if they pass through a Diagonal-Antidiagonal PBS? Notice that

$$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle = (\alpha+\beta)|D\rangle + (\alpha-\beta)|A\rangle^1$$

hence they will be detected $N|\alpha+\beta|^2$ times in the *D-path*, and $N|\alpha-\beta|^2$ times in the *A-path*.
In this way we have measured the observable

$$\sigma_x = +1|D\rangle\langle D| - 1|A\rangle\langle A|$$

which gives as output $|\psi\rangle_{+1} = |D\rangle$ with probability $p(+1) = |\alpha+\beta|^2$, and $|\psi\rangle_{-1} = |A\rangle$ with probability $p(-1) = |\alpha-\beta|^2$.

This shows that before the measurement the photon is in a *quantum superposition* of possibilities, till its interaction with the measurement apparatus, which makes it collapse in one of the states of the measurement basis. Therefore the act of measurement of an observable of the system perturbs it, changing its state. In consequence of this the order of the measurements matters: changing the order of the measuring filters changes the way the system is perturbed. Therefore measuring filters do not commute! Hence we move to a *matrix notation*, to take into account this non commutative nature.

[1]Remember that $|D\rangle = \frac{|H\rangle+|V\rangle}{\sqrt{2}}$, while $|A\rangle = \frac{|H\rangle-|V\rangle}{\sqrt{2}}$

Notice that before the measurement we only know the different outputs probabilities, but we cannot predict in advance what will be the result of our measurement. This explains why there is an intrinsic random nature of the quantum realm.

The problem is that in practice the state of the quantum system is not known. What is known is that it can be with probability p_i in one of the states $|\psi_i\rangle$. Therefore what we have is an *ensemble* of *pure states*: $\{p_i, |\psi_i\rangle\}$. To this ensemble we can associate a quantum state, represented by a *density operator* ρ, namely

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

which has three properties: it is Hermitian, it is positive semidefinite and it has unitary trace. Notice that if the state is *pure* (known), then $\rho = |\psi\rangle\langle\psi|$, otherwise it is said to be *mixed*.

We introduce the set of states of $\mathcal{H}$

$$\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geqslant 0, tr(\rho) = 1\}$$

which is convex and has as extreme points the *pure states*, which are rank-one projectors. We will refer to $\mathcal{B}(\mathcal{H})$ as the set of linear and bounded operators on $\mathcal{H}$.

The last thing to formalize is the description of a system composed by n quantum subsystems. For simplicity we present the interaction of two subsystems, the case of $n > 2$ is easily obtained by iteration. The Hilbert space associated to the whole system is given by the *tensor product* between the Hilbert spaces associated to the two subsystems, namely

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \text{ with } \ d = d_1 \times d_2$$

In this composite setting $\mathcal{H}$ is endowed with the inner product $\langle u_1, u_2|v_1, v_2\rangle := \langle u_1|v_1\rangle\langle u_2|v_2\rangle$. We now introduce the concept of *partial trace* over one tensor factor of $\mathcal{H}$, that will permit us to analyze the behaviour of one subsystem, tracing out the other.

Definition 17. Let $\mathcal{L}(\mathcal{H})$ denote the set of linear operators on $\mathcal{H}$. Let $\{|e_i\rangle\}_{i=1}^{d_1}$ be an *orthonormal basis* of $\mathcal{H}_1$, with $E_{ij} := |e_i\rangle\langle e_j| \in \mathcal{L}(\mathcal{H}_1)$. We can rewrite any operator $A \in \mathcal{L}(\mathcal{H})$ in the following way

$$A = \sum_{i,j=1}^{d_2} A_{ij} \otimes E_{ij} = \begin{pmatrix} A_{11} & \dots & A_{1d_2} \\ \vdots & \ddots & \vdots \\ U_{d_2 1} & \dots & U_{d_2 d_2} \end{pmatrix}, \quad A_{ij} \in \mathcal{B}(\mathcal{H}_2)$$

Then the *partial trace* of A over $\mathcal{H}_1$ is an operator $\bar{A}_2 \in \mathcal{L}(\mathcal{H}_2)$ defined as

$$\bar{A}_2 = tr_{\mathcal{H}_1}(A) = \sum_{i=1}^{d_2} A_{ii}$$

Using the partial trace we obtain the statistical description of the two subsystems, namely their partial densities, just by "tracing-out" the other subsystem, namely

$$\bar{\rho}_1 = tr_{\mathcal{H}_2}[\rho], \quad \bar{\rho}_2 = tr_{\mathcal{H}_1}[\rho]$$

This formalism of composite systems will come out in chapter 3 to describe *indirect measurements*, where the system of interest $\mathcal{H}$ interacts with a probe $\mathcal{H}_p$, and the whole system $\mathcal{H} \otimes \mathcal{H}_p$ undergoes a unitary evolution followed by a direct projective measurement of $\mathcal{H}_p$. We are interested in the *back-action* of these measurements on our system of interest, that's why we will need the partial trace to trace out the probe and analyze the evolution of the state in the tensor factor $\mathcal{H}$.

2.2 Temporal Evolution & Quantum Channel

Quantum mechanics divides every physical process into preparation of a state and measurement of an observable, therefore there are different (but at the end equivalent) 'pictures' depicting time evolution: the *Schrodinger picture* describes the state evolution, while the *Heisenberg picture* describes the observable evolution.

For a given time the evolution is described by a *linear* transformation on observables $A \mapsto u(A)$ or on states $\rho \mapsto u^*(\rho)$, where consistency imposes the relation

$$tr[\rho u(A)] = tr[u^*(\rho)A] \tag{2.1}$$

namely u^* is the dual of the map u with respect to the usual Hilbert–Schmidt inner product.

Firstly we introduce *closed systems*, which are isolated systems whose evolution is physically *reversible*, hence it should be described by a mathematically reversible transformation. Moreover the concatenation of evolutions is naturally considered to be associative (i.e. $(uv)w = u(vw)$). Consequently, the set of reversible evolutions is described by a semigroup of *linear and unitary* transformations, and we have that

$$A \mapsto u(A) = U^\dagger A U \quad or \quad \rho \mapsto u^*(\rho) = U\rho U^\dagger \tag{2.2}$$

where $U \in \mathcal{U}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ is an element of the unitary operator set $\mathcal{U}(\mathcal{H})$. Notice that these types of maps being unitary are *spectrum preserving*, hence the spectrum of the state ρ will be preserved along the evolution.

What if our particle is interacting with the environment? It becomes an *open system* which undergoes a potentially *irreversible* dynamics. But if we consider it together with its interacting environment, the whole system becomes closed and so we turn back to the *reversible* dynamics. This is what stays back the Stinespring dilatation theorem 5, stated in the following.

Consider an evolution which, in the Schrodinger picture, is described by a map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. As highlighted in the first chapter of Wolf's lecture notes [29], when describing a physically meaningful evolution, ϕ should fulfil the following three conditions:

- *linearity*: it is a quantum mechanical requirement related to *locality*, namely the fact that a spatially localized action does not instantaneously influence distant regions. Any sort of non-linearity would imply a breakdown of locality. Notice that locality is a crucial ingredient when we want to talk about small systems without always having to take into account the entire rest of the universe.

- *Trace Preservation* (TP) to ensure that density operators are mapped into density operators, hence preserving its unitary trace.

- *Complete Positivity* (CP): another consequence of linearity together with asking ϕ to map density operators onto density operators is that it has to be a *positive map*, namely

  $$A \geqslant 0 \longrightarrow \phi(A) \geqslant 0 \quad \forall A \in \mathcal{B}(\mathcal{H})$$

  However positivity alone is not sufficient: consider $\mathcal{H}$ as part of a bipartite system so that the evolution of the larger system is described by $\phi \otimes I$. That is, the additional system merely plays the role of a spectator. Yet $\phi \otimes I$ should again be a positive map - a requirement which is stronger than positivity. So the relevant condition is complete positivity of ϕ which means positivity of the map $\phi \otimes I_n, \ \forall n \in \mathbb{N}$.

For the map $\phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ describing the same evolution in the Heisenberg picture via

$$tr[\phi(\rho)A] = tr[\rho\phi^*(A)] \tag{2.3}$$

the conditions linearity and complete positivity remain the same, only the trace preserving condition translates to *unitality*: $\phi^*(I) = I$, as showed in the following.

A mapping which fulfills the above three conditions (either in Heisenberg or Schrodinger picture) is called a *quantum channel*. Quantum channels are the most general framework in which general input-output relations (i.e. black-box devices) are described within quantum mechanics. It is crucial, however, that the mapping itself does neither depend on the input nor on its history. If such correlations appear, then the above black-box description becomes inappropriate and either a larger system (including 'the environments memory') has to be taken into account. When talking about quantum channels in the following we will always mean the *Markovian*, or synonymously memory-less case, in which such correlations do not occur. Let us present two important descriptions of a quantum channel: the Stinespring dilation and the Kraus decomposition.

Theorem 5. *(Stinespring dilation) Let ϕ be a CPTP map on $\mathcal{B}(\mathcal{H})$, there exists a Hilbert space $\mathcal{K}$, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ and a state $\beta \in \mathcal{D}(\mathcal{K})$ such that*

$$\phi(\rho) = tr_{\mathcal{K}}[U(\rho \otimes \beta)U^\dagger] \quad \forall \rho \in \mathcal{D}(\mathcal{H})$$

where $tr_{\mathcal{K}}$ denotes the partial trace over $\mathcal{K}$.

Theorem 6. *(Kraus representation) Let ϕ be a CPTP map on $\mathcal{B}(\mathcal{H})$. Then there exists a family of operators in $\mathcal{B}(\mathcal{H})$ denoted by $\{V_i\}_{i=1}^r$ satisfying*

$$\sum_{i=1}^{r} V_i^\dagger V_i = I_k \tag{2.4}$$

such that

$$\phi(\rho) = \sum_{i=1}^{r} V_i \rho V_i^\dagger, \quad \forall \rho \in \mathcal{D}(\mathcal{H}) \tag{2.5}$$

This decomposition is called a Kraus decomposition and the operators $\{V_i\}_{i=1}^r$ are called Kraus operators.

In general let us note that $r \leqslant k^2$, where r represents the *Kraus rank*, namely the minimal number of Kraus operators, while $k = \dim \mathcal{H}$. Moreover this decomposition is not unique. In particular we get the following characterization.

Proposition 2. *Let ϕ be a CPTP map on $\mathcal{B}(\mathcal{H})$. Let $\{V_i\}_{i=1}^r$ and $\{W_i\}_{i=1}^r$ be two family of operators such that*

$$\phi(\rho) = \sum_{i=1}^{r} V_i \rho V_i^\dagger = \sum_{i=1}^{r} W_i \rho W_i^\dagger \quad \forall \rho \in \mathcal{D}(\mathcal{H})$$

Then there exists a $r \times r$ unitary operator $U = (u_{ij})$ such that $W_i = \sum_{j=1}^r u_{ij} V_j$

Note that in this proposition some operators V_i or W_i can be equal to zero.
This project will consider discrete-time quantum dynamics described by sequences of trace-preserving quantum operations in Kraus representation. This assumption implies the Markovian character of the evolution, which, along with a forward composition law, ensures a semigroup structure.

Definition 18 (DQDS)**.** A discrete *Quantum Dynamical Semigroup* is a one-parameter family of CPTP maps $\{\phi_n(\cdot)\}_{n \in \mathbb{N}}$ that satisfies the following properties:

1. $\phi_0 = Id$,
2. $\phi_t \circ \phi_s = \phi_{t+s}, \ \forall t, s > 0$.

Therefore, ϕ can be seen as the generator of a DQDS in the Schrödinger picture by considering iterated applications of the map:

$$\rho_n = \phi(\rho_{n-1}) = \phi^n(\rho_0), \ \forall \rho_0 \in \mathcal{D}(\mathcal{H})$$

Due to the trace and positivity preserving assumptions, a QDS is a semigroup of contractions, meaning that ϕ is a *contraction map* in trace norm:

$$tr(|\phi(\rho_1) - \phi(\rho_2)|) \leqslant tr(|\rho_1 - \rho_2|), \ \forall \rho_1, \rho_2 \in \mathcal{D}(\mathcal{H}) \tag{2.6}$$

as showed in theorem 8.16 of Wolf's lecture notes [29]. This implies that

$$tr(|\phi^n(\rho) - \rho_\infty|) \leqslant tr(|\rho - \rho_\infty|)$$

for any invariant state ρ_∞ of ϕ. This means that in the state space the initial distance from an invariant state can never increase under the system evolution dictated by a quantum channel.

Each semigroup describing the time evolution of an open quantum system on a finite dimensional Hilbert space is related to a special structure of this space by the invariant states of the evolution, as deeply presented in [6], [15] and in the Wolf's lecture notes [29] (section 6.4). We will exploit this special structure of the state space to prove our main convergence result.

Notice that given a Kraus decomposition for ϕ, it follows from (2.3) that ϕ^* acts on an element $A \in \mathcal{B}(\mathcal{H})$ by

$$\phi^*(A) = \sum_{i=1}^{r} V_i^\dagger A V_i \tag{2.7}$$

Thus, the dual of a CP map is still CP. However, it is immediate to show that the dual of a CPTP map does not need to be TP, but it must be *unital*, as a consequence of condition (2.4).

2.3 Quantum Subspaces: invariance and stability

In this section, we recall some definitions related to quantum subspaces, which represent a key mathematical structure for this project.

Definition 19. A quantum subspace $\mathcal{R}$ of a system with associated Hilbert space $\mathcal{H}$, is a quantum system whose Hilbert space $\mathcal{H}_R$ is a subspace of $\mathcal{H}$, which can be written as the orthogonal direct sum of $\mathcal{H}_R$ and a remainder space $\mathcal{H}_T$:

$$\mathcal{H} = \mathcal{H}_R \oplus \mathcal{H}_T \tag{2.8}$$

To our aim, it is useful to introduce appropriate 'block' representations of maps and operators with respect to the decomposition (2.8) of the underlying Hilbert space. If we choose this orthonormal basis for $\mathcal{H}$:

$$\{|\psi_l\rangle\}_l = \{|\psi_j^R\rangle\}_j \cup \{|\psi_k^T\rangle\}_k$$

where $\{|\psi_j^R\rangle\}_j$ is an orthonormal basis for $\mathcal{H}_R$ and $\{|\psi_k^T\rangle\}_k$ for $\mathcal{H}_T$, the following block structure is induced on any matrix representing an element $X \in \mathcal{B}(\mathcal{H})$ in this orthonormal basis:

$$X = \begin{bmatrix} X_R & X_P \\ X_Q & X_T \end{bmatrix}$$

where X_R, X_T, X_P, X_Q are operators from $\mathcal{H}_R$ to $\mathcal{H}_R$, from $\mathcal{H}_T$ to $\mathcal{H}_T$, from $\mathcal{H}_T$ to $\mathcal{H}_R$, and from $\mathcal{H}_R$ to $\mathcal{H}_T$, respectively.

We shall also need the notion of support of a state. Let ρ be a state, the support of ρ is defined as

$$supp(\rho) = ker(\rho)^{\perp}$$

In a matrix notation: ρ is supported on the subspace $\mathcal{H}_R$ if it has the following block structure

$$\rho = \Pi_R \rho \Pi_R = \begin{bmatrix} \bar{\rho}_R & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{\rho}_R \in \mathcal{D}(\mathcal{H}_R)$$

where Π_R is an orthogonal projector onto $\mathcal{H}_R$. Studying how the support of the state evolves under iterations of the quantum channel ϕ, is instrumental for studying the properties of *invariant* subspaces. Firstly we will characterize invariance and stability for the mean evolution of our quantum system, dictated by the quantum channel ϕ. In section 3.1 we will clarify why the quantum channel dictates the mean evolution of our quantum system subject to indirect measurements.

Definition 20 (Invariance)**.** Let ϕ be a quantum channel. A subspace $\mathcal{H}_R$ of $\mathcal{H}$ is said to be *invariant* iff any trajectory starting from a state with support in it, has its support on $\mathcal{H}_R$ for all times, namely

$$supp(\rho) \subset \mathcal{H}_R \rightarrow supp(\phi^n(\rho)) \subset \mathcal{H}_R, \ \forall n \in \mathbb{N}$$

An invariant subspace is called *minimal* if it does not contain other non trivial invariant subspaces.

Definition 21 (Irreducibility)**.** A quantum channel is called *irreducible* if the only invariant subspaces are the trivial ones, i.e. $\{0\}$ and $\mathcal{H}$.

Let $\mathcal{V}$ be an invariant subspace of $\mathcal{H}$. If we consider the restriction of the quantum channel to $\Pi_{\mathcal{V}}\mathcal{D}(\mathcal{H})\Pi_{\mathcal{V}} \sim \mathcal{D}(\mathcal{V})$, we obtain again a quantum channel, that we shall denote $\phi_{|\mathcal{V}}$, having Kraus operators $V_{i|\mathcal{V}} := \Pi_{\mathcal{V}} V_i \Pi_{\mathcal{V}}$. If $\mathcal{V}$ is *minimal*, then the associated quantum channel $\phi_{|\mathcal{V}}$ is *irreducible* by construction.

A particular type of invariant subspace is the globally asymptotically stable (GAS) subspace, meaning that it is globally attractive for the dynamics.

Definition 22 (GAS)**.** Let ϕ be a quantum channel. A subspace $\mathcal{H}_R$ of $\mathcal{H}$ is *globally asymptotically stable* if it is *invariant* and

$$\lim_{n \to \infty} \| \phi^n(\rho) - \Pi_R \phi^n(\rho) \Pi_R \| = 0, \ \forall \rho \in \mathcal{D}(\mathcal{H})$$

where Π_R represents the orthogonal projection onto $\mathcal{H}_R$.

This represent the subspace into which the evolution converges.

2.4 Invariant states & state space decomposition

This section is devoted to the presentation of the structure of the invariant states of a quantum channel, that will play an fundamental role in this project. For a quantum channel ϕ, we denote by $\mathcal{F}_\phi$ the set of invariant states of ϕ, that is

$$\mathcal{F}_\phi = \{X \in \mathcal{B}(\mathcal{H}) \mid \ \phi(X) = X\}$$

In order to express the result let us introduce the set

$$\mathcal{T} := \{x \in \mathcal{H} \mid \langle x, \phi^n(\rho) x \rangle \xrightarrow[n \to \infty]{} 0, \ \forall \rho \in \mathcal{D}(\mathcal{H})\} \tag{2.9}$$

and its orthogonal $\mathcal{R} = \mathcal{T}^{\perp}$.

Theorem 7. *The set $\mathcal{T}$ defined in eq. (2.9) is a subspace of $\mathcal{H}$. It is the largest subspace such that for each $\rho \in \mathcal{D}(\mathcal{H})$ we have:*

$$\lim_{n\to\infty} tr(\Pi_{\mathcal{T}}\ \phi^n(\rho)) = 0, \quad \lim_{n\to\infty} tr(\Pi_{\mathcal{R}}\ \phi^n(\rho)) = 1$$

where $\Pi_{\mathcal{T}}$ and $\Pi_{\mathcal{R}}$ denote the orthogonal projectors onto $\mathcal{T}$ and $\mathcal{R}$ respectively. The decay in both limits is monotonus.

This theorem, proved in [6], represents the first step towards the investigation of the structure of $\mathcal{H}$ with respect to the time evolution. Loosely speaking, the subspace $\mathcal{T}$ contains the supports of all decaying states, and for this reason it is called *transient*. The subspace $\mathcal{R}$, containing the supports of all invariant states, it is called *recurrent* and can be further decomposed according to the structure of the set of invariant states, as will be shown in the following. Being $\mathcal{R}$ the subspace where the dynamics converges, it is GAS. The resultant orthogonal decomposition of $\mathcal{H}$ reads

$$\mathcal{H} = \mathcal{R} \oplus \mathcal{T} \tag{2.10}$$

The next property characterizes the invariance in terms of the Kraus Operators of ϕ, as presented and proved in [11] (proposition 1).

Proposition 3. *Let ϕ be a CPTP map on $\mathcal{B}(\mathcal{H})$ and $\mathcal{H} = \mathcal{R} \oplus \mathcal{T}$. Then $\mathcal{R}$ is* invariant $\iff$ *in any Kraus Decomposition of ϕ, the associated Kraus operators V_i's have the following block structure:*

$$V_i = \begin{bmatrix} V_{i,R} & V_{i,P} \\ 0 & V_{i,T} \end{bmatrix} \tag{2.11}$$

Theorem 6.14 of Wolf's lecture notes [29] proves that the set of invariant states of a quantum channel has the following structure

$$\mathcal{F}_\phi = U \left(\sum_{\alpha=1}^{d} \rho_\alpha \otimes \mathcal{B}(\mathbb{C}^{m_\alpha}) \oplus 0 \right) U^\dagger$$

with $U \in \mathcal{U}(\mathcal{H})$, for an appropriate decomposition of $\mathcal{H} = \mathcal{R} \oplus \mathcal{T}$ as $\mathbb{C}^k = \bigoplus_{\alpha=1}^{d} \mathbb{C}^{n_\alpha} \oplus \mathbb{C}^{n_T}$. This follows from the following facts:

1. being $\mathcal{R} \simeq \sum_{\alpha=1}^{d} \mathbb{C}^{n_\alpha}$ GAS, it supports all the invariant states;
2. each $\mathbb{C}^{n_\alpha}$ has a canonical tensor product structure, namely $\mathbb{C}^{n_\alpha} = \mathbb{C}^{k_\alpha} \otimes \mathbb{C}^{m_\alpha}$, $n_\alpha = k_\alpha m_\alpha$, with $\rho_\alpha > 0$ being a full rank density operator on $\mathbb{C}^{k_\alpha}$ such that

$$0 \oplus ... \oplus U_\alpha(\rho_\alpha \otimes I_{\mathbb{C}^{m_\alpha}})U_\alpha^\dagger \oplus ... \oplus 0 \in \mathcal{F}_\phi \tag{2.12}$$

Therefore this structure of the invariant states of ϕ induces a decomposition of the GAS subspace $\mathcal{R}$ into orthogonal *minimal invariant subspaces* $\mathcal{V}_\alpha$:

$$\begin{aligned} \mathcal{R} &= \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_d \\ &\simeq \mathbb{C}^{n_1} \oplus ... \oplus \mathbb{C}^{n_d} \end{aligned} \tag{2.13}$$

as deeply discussed also in [15]. In the general case the decomposition of $\mathcal{R}$ into minimal invariant subspaces is not unique, as reported in the following theorem, proved in [6].

Theorem 8 (Equivalence of splittings)**.** *If there are several possibilities to decompose $\mathcal{R}$ into orthogonal minimal invariant subspaces, then all these decompositions, together with the invariant states, are unitarily equivalent under unitary transformations commuting with the time evolution restricted to states having support in $\mathcal{R}$.*

The decomposition (2.13) induces the following properties on the Kraus operators characterizing ϕ:

1. the restriction of V_i to this subspace $\mathcal{R}$ is block diagonal with respect to the decomposition (2.13), which results in the following block structure for the Kraus operators:
$$V_i = \begin{bmatrix} V_{i,R}^{(1)} & 0 & 0 & * \\ 0 & \ddots & 0 & * \\ 0 & 0 & V_{i,R}^{(d)} & * \\ 0 & 0 & 0 & V_{i,T} \end{bmatrix} \tag{2.14}$$
This follows by the fact that the $\mathcal{V}_\alpha$'s are minimal invariant subspaces of $\mathcal{H}$, which implies no dynamical connections between them, while the last zero blocks derive from the invariance of $\mathcal{R}$, as stated in proposition 3;

2. each $\mathbb{C}^{n_\alpha}$ has a canonical tensor product structure, namely $\mathbb{C}^{n_\alpha} = \mathbb{C}^{k_\alpha} \otimes \mathbb{C}^{m_\alpha}$ with respect to which each $V_{i,R}^{(\alpha)}$ can be written as
$$V_{i,R}^{(\alpha)} = U_\alpha(\tilde{V}_i^{(\alpha)} \otimes I_{\mathbb{C}^{m_\alpha}})U_\alpha^\dagger \tag{2.15}$$
where U_α is a unitary operator on $\mathbb{C}^{n_\alpha}$, while $\tilde{V}_i^{(\alpha)}$ is an operator on $\mathbb{C}^{k_\alpha}$;

This shows the connection between the orthogonal decomposition of the recurrent subspace $\mathcal{R}$ into minimal invariant subspaces and the structure of the set of invariant states of our quantum channel ϕ.

Let us consider the following setting where $m_\alpha = 1$ for all $\alpha = 1, ..., d$, namely
$$\mathcal{R} \simeq \mathbb{C}^{k_1} \oplus ... \oplus \mathbb{C}^{k_d}$$
with minimal invariant subspaces $\mathcal{V}_\alpha \simeq \mathbb{C}^{k_\alpha}$. This is required for having *identifiability* of the invariant states from their relative probability measures, which will represent the main assumption of the convergence theorem for a quantum trajectory. Actually at the end of chapter 3 we will show that the convergence theorem holds even when $m_\alpha \geqslant 1$.

Let $\rho_{\infty,\alpha}$ be the unique invariant state with full support on $\mathcal{V}_\alpha$, namely $\bar{\rho}_{\infty,\alpha} = U_\alpha \rho_\alpha U_\alpha^\dagger \in \mathcal{D}(\mathcal{V}_\alpha)$ of eq. (2.12). Being the map ϕ linear, its invariant states are closed under convex combination, hence they form an operator subspace of $\mathcal{D}(\mathcal{H})$ defined as follows
$$\mathcal{F}_\phi = conv\{\rho_{\infty,\alpha} \in \mathcal{D}(\mathcal{H})\}$$
with $\rho_{\infty,\alpha}\rho_{\infty,\beta} = \rho_{\infty,\beta}\rho_{\infty,\alpha} = 0 \; \forall \alpha \neq \beta$ since they are supported in different orthogonal subspaces.

Remark (Unitary evolutions)**.** For unitary evolutions there is no decay, namely $\mathcal{T} = \{0\}$ and $\mathcal{H} = \mathcal{R}$. Each minimal invariant subspace $\mathcal{V}_\alpha$ is one-dimensional and contains an eigenvector of the relative Hamiltonian. Therefore if the spectrum of the Hamiltonian is *non-degenerate*, then the decomposition of $\mathcal{R}$ into minimal invariant subspaces is unique.

Unitary evolution may be part of a Markovian evolution in two ways:

1. $\mathcal{H}$ may present a subspace on which the time evolution is unitary. In such a subspace the minimal invariant subspaces are one-dimensional, supporting eigenstates of "energy". Decoherence does not take place on them, that's why their union is called *Decoherence Free Subspace* (DFS).

2. $\mathcal{H}$ may present a subspace which could be factorizable into two spaces, where the dynamics is factorized into a product of a unitary evolution on one factor-space with a non-invertible Markovian evolution on the other one. The subsystem that is evolving unitarily is called *Noiseless Subsystem* (NS). As an example one can think of a system containing two atoms, where one of the atoms is decaying, the other one not.

DFS and NS represents *Information Preserving Structures* (IPS) [10] that allow to store and preserve quantum information.

2.5 Spectral analysis

In the following section we will recall some properties of the spectrum of a CPTP map derived in the lecture notes of M. Wolf [29].

Being $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a linear map having equal input and output space, we can assign a *spectrum* to it, which represents the set of scalars $\lambda \in \mathbb{C}$ that are roots of the *characteristic polynomial* $\Delta_\phi(\lambda) = (\lambda I - \phi)$, namely

$$sp(\phi) := \{\lambda \in \mathbb{C} \mid \Delta_\phi(\lambda) = 0\}$$

Recall that the spectral radius of the linear map ϕ is defined as $\varrho(\phi) := sup\{|\lambda| \mid \lambda \in sp(\phi)\}$, namely as the radius of a disc that encapsulates the whole spectrum.

Proposition 4 (Spectral radius of positive maps)**.** *If ϕ is a positive map on $\mathcal{B}(\mathcal{H})$, then its spectral radius satisfies*

$$\varrho(\phi) \leqslant \|\phi(I)\|_\infty \tag{2.16}$$

Proof. By the theorem of Russo and Dye we have $\|\phi(X)\|_\infty \leqslant \|\phi(I)\|_\infty \|X\|_\infty$. Therefore if $\phi(X) = \lambda X$, then

$$|\lambda| \|X\|_\infty = \|\phi(X)\|_\infty \leqslant \|\phi(I)\|_\infty \|X\|_\infty$$

which implies (2.16). □

The spectrum of the quantum channel ϕ has the following properties:

1. being ϕ a CP map, it preserves hermicianity, which implies that eigenvalues are either real or they come in complex conjugate pairs, indeed

$$\phi(X) = \lambda X \to [\phi(X)]^\dagger = \phi(X^\dagger) = \lambda^* X^\dagger$$

2. from the previous property, from proposition 4 and from the unitality of ϕ^* follows that

$$\varrho(\phi) \leqslant 1$$

   indeed $\varrho(\phi) = \varrho(\phi^*) \leqslant \|\phi^*(I)\|_\infty = \|I\|_\infty = 1$. This means that the spectrum of a quantum channel lies in the unit disc;

3. from Brouwer's fixed point theorem follows that an invariant state of ϕ always exists;

4. Let the *peripheral spectrum* of ϕ be the set of its eigenvalues lying on the boundary of the unit disc, namely $|\lambda| = 1$, then all their Jordan blocks (defined in eq. (2.17)) are one-dimensional, i.e. $J(\lambda) = \lambda$.

Remark. Notice that being any CPTP map a map from the compact set of density operators $\mathcal{D}(\mathcal{H})$ to itself, any trajectory is bounded. This explains property 4.

Being a linear map, ϕ admits a *Jordan decomposition*:

$$\phi = V \left(\bigoplus_{k=1}^{K} J(\lambda_k) \right) V^{-1}, \quad J(\lambda_k) := \begin{pmatrix} \lambda_k & 1 & \\ & \ddots & 1 \\ & & \lambda_k \end{pmatrix} \tag{2.17}$$

where $J(\lambda_k)$ are the *Jordan blocks* of size d_k (with $\sum_k d_k = k$) and the number K of Jordan blocks equals the number of different eigenvectors. Subdividing each Jordan block into a projection and a nilpotent part we get

$$\phi = \sum_{k=1}^{K} \lambda_k \Pi_k + N_k, \quad N_k^{d_k} = 0,\ \Pi_k N_k = N_k \Pi_k = N_k, \tag{2.18}$$
$$\Pi_k \Pi_l = \delta_{k,l} \Pi_k,\ tr(\Pi_k) = d_k,\ \sum_k \Pi_k = I_d$$

where Π_k is the projection onto the generalized eigenspace in $\mathcal{B}(\mathcal{H})$ relative to the eigenvalue λ_k. The number of Jordan blocks with eigenvalue λ_k is the *geometric multiplicity* ν_k of λ_k, while their joint dimension $\sum_{k':\lambda_{k'}=\lambda_k} d_{k'}$ is its *algebraic multiplicity* n_k. If these two multiplicities are equal for every eigenvalue, then ϕ is called *non-defective*. In the *non-defective* case there exist a complete basis of $\mathbb{C}^k$ of eigenvectors, and our quantum channel has the following *spectral decomposition*:

$$\phi(.) = \sum_{\alpha=1}^{d} |\rho_{\infty,\alpha}\rangle\langle M_\alpha| + \sum_{\alpha=d+1}^{k} \lambda_\alpha |R_\alpha\rangle\langle L_\alpha|, \quad \lambda_\alpha \neq 1, |\lambda_\alpha| \leqslant 1 \tag{2.19}$$

where $\{\rho_{\infty,\alpha}, R_\alpha\}_{\alpha=1}^k$ are the eigenoperators of ϕ, while $\{M_\alpha, L_\alpha\}_{\alpha=1}^k$ are the eigenoperators of the dual ϕ^*. Therefore the quantum channel action on the state space can be rewritten in the following way

$$\phi(\rho) = \sum_{\alpha=1}^{d} tr(M_\alpha \rho)\rho_{\infty,\alpha} + \sum_{\alpha=d+1}^{k} \lambda_\alpha tr(L_\alpha^\dagger \rho) R_\alpha \tag{2.20}$$

This operator basis is *biorthogonal*, therefore

$$tr(M_\alpha \rho_{\infty,\beta}) = \delta_{\alpha,\beta}, \quad \forall \alpha, \beta \tag{2.21}$$
$$tr(M_\alpha R_{\alpha'}) = 0,\ \forall \alpha = 1, ..., d; \forall \alpha' = d+1, ..., k$$

which implies that

$$\sum_{\alpha=1}^{d} tr(M_\alpha R_{\alpha'}) = tr(\sum_{\alpha=1}^{d} M_\alpha R_{\alpha'}) = tr(R_{\alpha'}) = 0,\ \forall \alpha' = d+1, ..., k \to R_{\alpha'} \notin \mathcal{D}(\mathcal{H})$$

This means that the evolution of a quantum channel on the state space will never end up in an eigenoperator with associated eigenvalue $\lambda_\alpha \neq 1, |\lambda_\alpha| \leqslant 1$.

Moreover a precise form of the invariant operators M_α of the dual ϕ^* is derived in [14] (proposition 2.5), and reads

$$M_\alpha = \Pi_{\mathcal{V}_\alpha} + M_{\mathcal{T}_\alpha} \tag{2.22}$$

with $M_{\mathcal{T}_\alpha} = \Pi_{\mathcal{V}_\alpha^\perp} M_\alpha \Pi_{\mathcal{V}_\alpha^\perp}$ and with $0 \leqslant M_\alpha \leqslant 1$. From eq. (2.21) follows that:

1. M_α has no support on $\mathcal{V}_\beta$ for $\alpha \neq \beta$, therefore $M_{\mathcal{T}_\alpha} = \Pi_{\mathcal{T}} M_\alpha \Pi_{\mathcal{T}}$;
2. $\sum_\alpha tr(M_\alpha \rho_{\infty,\beta}) = \sum_\alpha \delta_{\alpha,\beta} \iff tr(\sum_\alpha M_\alpha \rho_{\infty,\beta}) = 1 \iff \sum_\alpha M_\alpha = Id_k$, since $tr(\rho_{\infty,\beta}) = 1$ and $0 \leqslant M_\alpha \leqslant 1$. Consequently $\sum_\alpha M_{\mathcal{T}_\alpha} = \Pi_{\mathcal{T}}$.

The resulting set of invariant operators for the dual map ϕ^* reads

$$\mathcal{F}_{\phi^*} = span\{M_\alpha \in \mathcal{B}(\mathcal{H}) \mid\ 0 \leqslant M_\alpha \leqslant 1, \sum_\alpha M_\alpha = Id_k\} \tag{2.23}$$

In the following proposition we introduce a map ϕ_∞ that projects onto the set of invariant states $\mathcal{F}_\phi$ (see proposition 6.3 of Wolf's notes [29]), that we will use in section 3.3.

Proposition 5 (Cesaro mean). *Let us introduce the following map:*

$$\phi_\infty := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi^n \tag{2.24}$$

Then it is a projector onto the set of invariant states $\mathcal{F}_\phi$, namely

$$\phi_\infty = \sum_{\alpha=1}^{d} \Pi_{\infty,\alpha} \tag{2.25}$$

having spectral decomposition

$$\phi_\infty(\rho) = \sum_{\alpha=1}^{d} tr(M_\alpha \rho)\rho_{\infty,\alpha}$$

Proof. The Jordan decomposition of ϕ can be subdivided into three parts: the first one relative to the invariant states, the second one associated with the eigenvalues of the peripheral spectrum different from one, and the third one associated with the remaining eigenvalues inside the unit circle:

$$\phi = \sum_{\alpha=1}^{d} \Pi_{\infty,\alpha} + \sum_{\substack{k:|\lambda_k|=1\\ \lambda_k\neq 1}} \lambda_k \Pi_k + \sum_{k:|\lambda_k|<1} \lambda_k \Pi_k + N_k$$

where in the first and in the second term the nilpotent part is not present due to property 4. Putting this decomposition in eq. (2.24) and exploiting the properties of the projectors and the nilpotent matrices in (2.19) we end up with

$$\phi_\infty = \sum_{\alpha=1}^{d} \Pi_{\infty,\alpha} + \lim_{N\to\infty} \left(\sum_{\substack{k:|\lambda_k|=1\\ \lambda_k\neq 1}} \frac{1}{N} \sum_{n=1}^{N} \lambda_k^n \Pi_k + \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{k:|\lambda_k|<1} \lambda_k \Pi_k + N_k \right)^n \right) \tag{2.26}$$

where

$$\left(\sum_{k:|\lambda_k|<1} \lambda_k \Pi_k + N_k \right)^n = \sum_{k:|\lambda_k|<1} \lambda_k^n \Pi_k + \sum_{l=1}^{n-1} \binom{n}{l} \lambda_k^{n-l} N_k^l + N_k^n$$

Notice that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_k^n = \lim_{N\to\infty} \frac{1}{N} \frac{1-\lambda^{N+1}}{1-\lambda} = 0, \quad \forall k \ s.t. \ \lambda_k \neq 1$$

where we have used the geometric sum. Therefore eq. (2.26) becomes

$$\begin{aligned} \phi_\infty &= \sum_{\alpha=1}^{d} \Pi_{\infty,\alpha} + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{k:|\lambda_k|<1} \sum_{l=1}^{n-1} \binom{n}{l} \lambda_k^{n-l} N_k^l + N_k^n \right) \\ &= \sum_{\alpha=1}^{d} \Pi_{\infty,\alpha} \end{aligned}$$

$\square$

With the spectral analysis of a quantum channel ends the part of the project devoted to the presentation of the preliminaries needed to develop the main subject of the elaborate: indirect measurements of a quantum system and the analysis of the asymptotic statistics of the stochastic process related to measurement outcomes. In the following chapter we will present the setting of indirect measurements, which give rise to a stochastic process called *quantum trajectory*. This process describes the state of the system subject to this sequence of indirect measurements. Afterwards we will analyze the asymptotic behaviour of the quantum trajectory, proving that it converges to one of the minimal invariant subspaces $\mathcal{V}_\alpha$, making use of martingale techniques.

Chapter 3

Indirect measurements & Quantum Trajectory

Due to the laws of quantum physics, a direct projective measurement of an observable brings to the state collapse of the system in one of the eigenstates of the measured observable. Moreover, measuring directly a small quantum sized physical system is done by letting it interact with a macroscopic instrument. This procedure can result in the destruction of the measured system. To avoid this destruction of the measured system, we recur to a sequence of indirect measurements, which aims at getting partial information on the quantum system with minimal impact on it. This setup of repeated quantum measurements, based on the repeated quantum interactions, corresponds to actual important physical experiments such as the ones performed by S. Haroche's team on the indirect observation of photons in a cavity ([17], [23]). In the following we will briefly recap the procedure of repeated indirect measurements.

We let the quantum system of interest $\mathcal{H}$ interact with another quantum system $\mathcal{H}_p \simeq \mathbb{C}^r$, called the *probe*, during a time interval $[0, t]$, following an Hamiltonian

$$H_{tot} = H_s \otimes \mathbb{I}_p + \mathbb{I}_s \otimes H_p + H_{sp}$$

where H_s and H_p represent the Hamiltonians dictating the evolution of the system and of the probe respectively, while H_{sp} models the energy exchanges between the two. This Hamiltonian H_{tot} give rise to a unitary evolution on $\mathcal{H} \otimes \mathcal{H}_p$

$$U = e^{-itH_{tot}}$$

Let $\{|e_i\rangle\}_{i=1}^r$ be an *orthonormal basis* of $\mathcal{H}_p$, with $E_{ij} := |e_i\rangle\langle e_j| \in \mathcal{B}(\mathcal{H}_p)$. We can rewrite the unitary evolution in the following way

$$U = \sum_{i,j=1}^{r} U_{ij} \otimes E_{ij} = \begin{pmatrix} U_{11} & \dots & U_{1r} \\ \vdots & \ddots & \vdots \\ U_{r1} & \dots & U_{rr} \end{pmatrix}, \quad U_{ij} \in \mathcal{B}(\mathcal{H}) \tag{3.1}$$

If we prepare the probe in the *pure* state $|e_1\rangle\langle e_1|$, while our system is in the state ρ_0, and we let them interact in $[0,t]$, then their joint state at time t will be described by the following density matrix:

$$\begin{aligned}\rho_{joint}(t) &= U(\rho_0 \otimes |e_1\rangle\langle e_1|)U^\dagger \\ &= \begin{pmatrix} U_{11}\rho_0 U_{11}^\dagger & \dots & U_{11}\rho_0 U_{r1}^\dagger \\ \vdots & \ddots & \vdots \\ U_{r1}\rho_0 U_{11}^\dagger & \dots & U_{r1}\rho_0 U_{r1}^\dagger \end{pmatrix}\end{aligned}$$

If we look at the evolution in $\mathcal{H}$, taking the partial trace over the probe system $\mathcal{H}_p$, we get

$$\rho_t = tr_{\mathcal{H}_p}[\rho_{joint}(t)] = \sum_{i=1}^{r} U_{i1}\rho_0 U_{i1}^\dagger =: \phi(\rho_0) \tag{3.2}$$

where ϕ is a quantum channel having Kraus operators $V_i = U_{i1}$, which corresponds to the blocks of the first "column" of U in 3.1. Indeed ϕ is CP and TP thanks to the unitarity if U: $tr(\phi(\rho)) = tr\left(\sum_{i=1}^{r} U_{i1}^\dagger U_{i1}\rho_0\right) = tr(\mathbb{I}_s\rho) = 1$. Eq. (3.2) represents the Stinespring dilation, recalled in theorem 5) of the quantum channel dictating the evolution of our system.

Then we perform a *direct projective measurement* on $\mathcal{H}_p$, with measurement basis $\{P_i := |e_i\rangle\langle e_i|\}_{i=1}^r$, which will produce the i^{th} outcome with probability

$$\pi_i := tr[(\mathbb{I}_s \otimes P_i)\rho_{joint}(t)] = tr[V_i\rho_0 V_i^\dagger]$$

and with consequent state collapse in

$$\rho_{joint|i}(t) = \frac{(\mathbb{I}_s \otimes P_i)\rho_{joint}(t)(\mathbb{I}_s \otimes P_i)}{\pi_i} = \frac{V_i\rho_0 V_i^\dagger}{\pi_i} \otimes |e_i\rangle\langle e_i|$$

where the collapse in a measurement eigenstate interests only the probe, while our system only undergoes a measurement back-action. The crucial point is that thanks to the correlation between the probe and the system (developed during their interaction), indirect measurements are able to gain information about the system avoiding its state collapse. Therefore if we look at what happens to our system after the probe's measurement, we get this random equation

$$\rho_1 = \frac{V_i\rho_0 V_i^\dagger}{tr(V_i\rho_0 V_i^\dagger)} \text{ with probability } tr(V_i\rho_0 V_i^\dagger)$$

where the randomness comes from the measurement process, that selects the index i of the Kraus operator. If we repeat this procedure on ρ_1 and with a new copy of the probe, we will get

$$\rho_2 = \frac{V_i\rho_1 V_i^\dagger}{tr(V_i\rho_1 V_i^\dagger)} \text{ with probability } tr(V_i\rho_1 V_i^\dagger)$$

and so on. This repeated sequence of indirect measurements of our system gives rise to a *quantum trajectory* $(\rho_n)_{n\in\mathbb{N}}$, which is characterized by the *random equation*

$$\rho_{n+1} = \frac{V_i\rho_n V_i^\dagger}{tr(V_i\rho_n V_i^\dagger)} \tag{3.3}$$

that holds with probability $tr(V_i\rho_n V_i^\dagger)$. Notice that being the Kraus operators V_i's not time dependent and being ρ_{n+1} only dependent on ρ_n, the quantum trajectory $(\rho_n)_{n\in\mathbb{N}}$ is a *Markov Chain.*

3.1 Equivalence between invariance and stability in mean and almost surely

The focus of this project is on the analysis of the asymptotic behaviour of the *stochastic* evolution of our quantum system subject to indirect measurements, which is governed by eq. (3.3). In the previous section we have characterized invariance (20) and stability (22) for the mean evolution of the quantum trajectory, namely

$$\hat{\rho}_{n+1} := \mathbb{E}[\rho_{n+1}] = \phi(\hat{\rho}_n)$$

which is *deterministic* and it is dictated by the quantum channel ϕ.

Now we have to characterize invariance and stability for the quantum trajectory, and thus in a stochastic setting.

Definition 23 (Invariance)**.** A subspace $\mathcal{H}_R$ of $\mathcal{H}$ is said to be *invariant almost surely* if

$$supp(\rho_0) \subset \mathcal{H}_R \to supp(\rho_n) \subset \mathcal{H}_R, \ \forall n \in \mathbb{N} \ a.s.$$

Definition 24 (GAS)**.** A subspace $\mathcal{H}_R$ of $\mathcal{H}$ is GAS *almost surely* if $\forall \rho_0 \in \mathcal{D}(\mathcal{H})$

$$\lim_{n \to \infty} \| \rho_n - \Pi_R \rho_n \Pi_R \| = 0 \quad a.s.$$

with Π_R being the usual orthogonal projector onto the subspace $\mathcal{R}$.

Taking inspiration from [8] we will show that also in the discrete setting there is an equivalence between invariance and stability in mean and almost surely. The proof is based on the following *linear Lyapunov function*:

$$V(\rho) := tr(\Pi_T \rho), \ 0 \leqslant V(\rho) \leqslant 1$$

with Π_T being the usual orthogonal projector onto the subspace $\mathcal{T}$. Notice that

$$V(\rho) = 0 \iff supp(\rho) \subset \mathcal{H}_R \tag{3.4}$$

Lemma 9. *If the subspace $\mathcal{H}_R$ is invariant then the process $(V(\rho_n))_{n \in \mathbb{N}}$ is a positive supermartingale, namely*

$$\mathbb{E}[V(\rho_{n+1}) \mid \rho_n = \rho] \leqslant V(\rho), \ \forall \rho \in \mathcal{D}(\mathcal{H})$$

Proof. The proof relies on the block structure of the Kraus operators (2.11), induced by the invariance of $\mathcal{H}_R$, which implies that

$$\begin{aligned} \Pi_T V_i &= \Pi_T V_i \Pi_T =: V_i^{(T)} \\ V_i \Pi_T &= \begin{bmatrix} 0 & V_{i,P} \\ 0 & V_{i,T} \end{bmatrix} \end{aligned} \tag{3.5}$$

Therefore

$$\begin{aligned} \mathbb{E}[V(\rho_{n+1}) \mid \rho_n = \rho] &= \sum_i tr\left(\Pi_T \frac{V_i \rho V_i^\dagger}{tr(V_i \rho V_i^\dagger)}\right) tr(V_i \rho V_i^\dagger) \\ &= \sum_i tr\left(V_i^{(T)} \rho V_i^{\dagger (T)}\right) \\ &= tr\left(\phi^{(T)}(\rho_{|_T})\right) \end{aligned}$$

where $\phi^{(T)}$ is the map defined by the Kraus operators $\{V_i^{(T)}\}$, while $\rho_{|_T} = \Pi_T \rho \Pi_T$. What we want to show is that the map $\phi^{(T)}$ is *trace non-increasing*, namely:

$$tr\left(\phi^{(T)}(\rho_{|_T}) - \rho_{|_T}\right) \leqslant 0$$

This can be seen in the following way, exploiting the block structure (3.5):

$$\begin{aligned} tr\left(\phi(\rho_{|_T})\right) &= tr\left(\sum_i V_i \Pi_T \rho_{|_T} \Pi_T V_i^\dagger\right) \\ &= \sum_i tr\left(V_{i,P}\rho_T V_{i,P}^\dagger\right) + tr\left(V_{i,T}\rho_T V_{i,T}^\dagger\right) \\ &= \sum_i tr\left(V_{i,P}\rho_T V_{i,P}^\dagger\right) + tr\left(\phi^{(T)}(\rho_{|_T})\right) \end{aligned}$$

with $\rho_T \in \mathcal{D}(\mathcal{H}_T)$ being the non-zero block of $\rho_{|_T}$. Being $\rho_T \geqslant 0$, we have that $\sum_i tr\left(V_{i,P}\rho_T V_{i,P}^\dagger\right) \geqslant 0$, which implies that

$$tr\left(\phi^{(T)}(\rho_{|_T})\right) \leqslant tr\left(\phi(\rho_{|_T})\right) = tr(\rho_{|_T})$$

that shows that

$$\mathbb{E}[V(\rho_{n+1}) \mid \rho_n = \rho] = tr\left(\phi^{(T)}(\rho_{|_T})\right) \leqslant tr(\Pi_T \rho) = V(\rho)$$

Finally the positivity comes by the fact that $\rho \geqslant 0, \forall \rho \in \mathcal{D}(\mathcal{H})$ and by the monotonicity of the trace map. ☐

Now we have all the tools to state and prove the equivalence theorem.

Theorem 10. *(Invariance and stability in mean iff almost surely) The subspace $\mathcal{H}_R$:*

- *is invariant* in mean $\iff$ *it is invariant* almost surely

- *is GAS* in mean $\iff$ *it is GAS* almost surely

Proof. We start by the invariance. Given condition (3.4) it is sufficient to prove that

$$V(\hat{\rho}_n) = 0 \ \forall n \in \mathbb{N} \iff V(\rho_n) = 0 \ \forall n \in \mathbb{N} \ a.s.$$

The implication $\Leftarrow$ derives by the linearity of V, which implies that

$$V(\hat{\rho}_n) = \mathbb{E}[V(\rho_n)]$$

For showing the other direction ($\Rightarrow$), let us recall that $V(\rho_n) \geqslant 0, \ \forall n \in \mathbb{N}$, from which follows that

$$if \ V(\hat{\rho}_n) = \mathbb{E}[V(\rho_n)] = 0 \ \forall n \in \mathbb{N} \ \ then \ \ V(\rho_n) = 0 \ \forall n \in \mathbb{N} \ a.s.$$

and the invariance equivalence holds.

We now move to the GAS property. Given condition (3.4) it is sufficient to prove that

$$\lim_{n\to\infty} V(\hat{\rho}_n) = 0 \iff \lim_{n\to\infty} V(\rho_n) = 0 \ a.s.$$

The implication $\Leftarrow$ follows by the dominated convergence theorem applied on V. Indeed if we have $\lim_{n\to\infty} V(\rho_n) = 0 \ a.s.$ and by the fact that $V(\rho_n) \leqslant 1, \ \forall n \in \mathbb{N}$ follows that

$$\lim_{n\to\infty} V(\hat{\rho}_n) = \lim_{n\to\infty} \mathbb{E}[V(\rho_n)] = \mathbb{E}\left[\lim_{n\to\infty} V(\rho_n)\right] = 0$$

The other direction ($\Rightarrow$) starts by assuming that $\lim_{n\to\infty} \mathbb{E}[V(\rho_n)] = 0$. Since $V(\rho_n) \geqslant 0$, this convergence corresponds to a L^1 convergence to 0. On the other hand, since $0 \leqslant V(\rho_n) \leqslant 1$ and by lemma 9, the process $(V(\rho_n))$ is a positive bounded supermartingale. It follows from bounded supermartingale convergence theorem, that this process converges almost surely and in L^1 to a random variable V_∞. The uniqueness of the L^1 limit implies $V_\infty = 0$ almost surely. ☐

3.2 Measurement outcomes

If we don't look at the measurement outcomes, a sequence of n indirect measurements gives rise to a random sequence $X_n := (i_1, i_2, ..., i_n) \in \{1, ..., r\}^{\otimes n}$, where i_1 is the outcome of the first measurement, i_2 of the second one and so on. A sequence of outcomes $(i_1, i_2, ..., i_n)$ appears with probability

$$\mathbb{P}_{\rho_0}[(i_1, i_2, ..., i_n)] := tr(V_{i_n}...V_{i_1}\rho_0 V_{i_1}^\dagger ...V_{i_n}^\dagger) \tag{3.6}$$

which depends on the initial state ρ_0 of the system. This can be easily shown for $n = 2$ in the following way

$$\mathbb{P}_{\rho_0}[(i_1, i_2)] = \mathbb{P}_{\rho_0}[i_2|i_1]\mathbb{P}_{\rho_0}[i_1] = tr(V_{i_2}\rho_{1|i_1}V_{i_2}^\dagger)tr(V_{i_1}\rho_0 V_{i_1}^\dagger) = tr(V_{i_2}V_{i_1}\rho_0 V_{i_1}^\dagger V_{i_2}^\dagger)$$

and the same procedure can be iterated to show that eq. (3.6) holds for any $n \in \mathbb{N}$. Let Ω be the space of events, namely of infinite sequences $(i_1, i_2, ...)$, where $i_l \in \{1, .., r\}$ for every measurement l. For a finite sequence $(i_1, ..., i_n)$ we can define $B_{i_1,...,i_n}$ as the subset of Ω made of all those realizations ω's whose first n components are $i_1, ..., i_n$. Let $\mathcal{F}_n$ be the σ-algebra generated by all the $B_{i_1,...,i_n}$. For convenience we define $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$. Then $\mathbb{F} := (\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, ...)$ is an increasing sequence of σ-algebras. We take $\mathcal{F}$ to be the smallest σ-algebra on Ω containing all the $\mathcal{F}_n$'s, making $(\Omega, \mathcal{F}, \mathbb{F})$ a filtered measurable space.

An initial state ρ_0 induces a probability measure $\mathbb{P}_{\rho_0}$ on Ω which is uniquely determined by the condition (3.6). Indeed, we see that the Kolmogorov consistency criterion is fulfilled:

$$\sum_{i_{n+1}=1}^{r} tr(V_{i_{n+1}}V_{i_n}...V_{i_1}\rho V_{i_1}^\dagger ...V_{i_n}^\dagger V_{i_{n+1}}^\dagger) = tr(V_{i_n}...V_{i_1}\rho V_{i_1}^\dagger ...V_{i_n}^\dagger)$$

The resultant probability space is $(\Omega, \mathcal{F}, \mathbb{P}_\rho)$, where $(\rho_n)_{n\in\mathbb{N}}$ is a unobserved Markov Chain defined by the random equation

$$\rho_n = \frac{V_{i_n}...V_{i_1}\rho V_{i_1}^\dagger ...V_{i_n}^\dagger}{tr(V_{i_n}...V_{i_1}\rho V_{i_1}^\dagger ...V_{i_n}^\dagger)} \tag{3.7}$$

which holds with probability $tr(V_{i_n}...V_{i_1}\rho V_{i_1}^\dagger ...V_{i_n}^\dagger)$. The only handy observation is the sequence of the measurement results (X_n). Bayes law maps the information of (X_n) in the evolution of the Markov chain. An important thing that we want to point out is that in general the sequence of random variables (X_n) is not i.i.d. (even not Markovian). Therefore the statistical analysis of the asymptotic behaviour of (X_n) cannot fully rely on standard results on i.i.d. models. That's why we will make use of martingale's asymptotic laws to develop our analysis.

The following proposition highlights some properties of the map $\rho \to \mathbb{P}_\rho$.

Proposition 6. *The map $\rho \to \mathbb{P}_\rho$ is:*

1. *affine:* $\lambda\mathbb{P}_\rho + (1-\lambda)\mathbb{P}_\sigma = \mathbb{P}_{\lambda\rho+(1-\lambda)\sigma} \quad \forall\lambda \in [0,1],\ \forall\rho,\sigma \in \mathcal{D}(\mathcal{H})$
2. *k-Lipschitz and consequently it is continuous in total variation with respect to the norm* 1*, namely*

$$\text{if } \rho_n \xrightarrow[n\to\infty]{} \rho \text{ then } \|\mathbb{P}_{\rho_n} - \mathbb{P}_\rho\|_{TV} \xrightarrow[n\to\infty]{} 0$$

with $\|\mathbb{P}_{\rho_n} - \mathbb{P}_\rho\|_{TV} := \sup_{A\in\mathcal{F}}|\mathbb{P}_{\rho_n}(A) - \mathbb{P}_\rho(A)|$.

Proof. 1 is trivially true by the linearity of the trace and by the fact that λ is a scalar. To prove 2, we define the quantity

$$M_n(i_1, ..., i_n) := \frac{V_{i_1}^\dagger ...V_{i_n}^\dagger V_{i_n}...V_{i_1}}{tr(V_{i_1}^\dagger ...V_{i_n}^\dagger V_{i_n}...V_{i_1})}$$

We want to prove that $\rho \to \mathbb{P}_\rho$ is *k-Lipschitz*, where k is the dimension of our quantum system, ie. $\mathcal{H} \sim \mathbb{C}^k$. We have that

$$\begin{aligned}
|\mathbb{P}_{\rho_n}(i_1,...,i_n) - \mathbb{P}_\rho(i_1,...,i_n)| &= |tr[V_{i_n}...V_{i_1}(\rho_n - \rho)V_{i_1}^\dagger...V_{i_n}^\dagger]| \\
&= |tr[V_{i_1}^\dagger...V_{i_n}^\dagger V_{i_n}...V_{i_1}(\rho_n - \rho)]| \\
&= tr(V_{i_1}^\dagger...V_{i_n}^\dagger V_{i_n}...V_{i_1})|tr[M_n(i_1,...,i_n)(\rho_n - \rho)]| \\
&\leqslant tr(V_{i_1}^\dagger...V_{i_n}^\dagger V_{i_n}...V_{i_1})\|M_n\|_\infty \|\rho - \rho_n\|_1 \qquad (3.8) \\
&\leqslant tr(V_{i_1}^\dagger...V_{i_n}^\dagger V_{i_n}...V_{i_1})\|\rho - \rho_n\|_1 = k\mathbb{P}_{I_k/k}(i_1,...,i_n)\|\rho - \rho_n\|_1 \qquad (3.9)
\end{aligned}$$

where (3.8) holds by inequality (17) proved in the appendix, while inequality (3.9) holds since $M_n \in \mathcal{D}(\mathcal{H})$ implies that $\|M_n\|_\infty \leqslant 1$. Now taking the sup over all possible infinite sequences in $\mathcal{F}$ we end up with

$$\begin{aligned}
\|\mathbb{P}_{\rho_n} - \mathbb{P}_\rho\|_{TV} = \sup_{A \in \mathcal{F}} |\mathbb{P}_{\rho_n}(A) - \mathbb{P}_\rho(A)| &\leqslant k \sup_{A \in \mathcal{F}} \mathbb{P}_{I_k/k}(A)\|\rho - \rho_n\|_1 \\
&\leqslant k\|\rho - \rho_n\|_1
\end{aligned}$$

which shows that $\rho \to \mathbb{P}_\rho$ is *k-Lipschitz* and consequently it is *continuous in total variation.* □

Let us focus on the probability measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ induced by the invariant states $\rho_{\infty,\alpha}$. The *identifiability* assumption on these measures represent the main assumption of this project, since if it is satisfied then the convergence theorem for the quantum trajectory holds and consequently also the asymptotic statistical analysis.

Assumption. *(ID) For any $\alpha \neq \beta$ there exists an $I := (i_1,...,i_n) \in \mathcal{F}_n$ such that*

$$\mathbb{P}_{\rho_{\infty,\alpha}}(I) \neq \mathbb{P}_{\rho_{\infty,\beta}}(I)$$

This assumption practically means that we are able to discriminate between the different invariant states from their relative probability measures. In the following we will show that when **ID** is satisfied then the probability measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ are *all mutually singular*, that will represent the key point to prove the main convergence theorem.

Definition 25 (Mutual singularity)**.** The measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ are *all mutually singular* iff there exists a collection of disjoint subsets (Ω_α) of Ω such that

$$\mathbb{P}_{\rho_{\infty,\alpha}}(\Omega_\beta) = \delta_{\alpha,\beta} \quad \forall \alpha, \beta \in \{1,...,d\}$$

Let us define the *process* $(\mathcal{J}, \rho)$ induced by the Kraus operators $\mathcal{J} = \{V_i\}_{i=1}^r$ and by a state ρ. This process is called the *unravelling* of our quantum channel ϕ. The *left shift* on the events space is defined as

$$\begin{aligned}
\varphi: \qquad & \Omega \to \Omega \\
(\omega_1, \omega_2, ...) &\mapsto (\omega_2, \omega_3, ...)
\end{aligned}$$

is a continuous surjection. The process $(\mathcal{J}, \rho_{\infty,\alpha})$, induced by the Kraus operators $\mathcal{J} = \{V_i^{(\alpha)} = \Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{V}_\alpha}\}_{i=1}^r$ defines a *dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}_{\rho_{\infty,\alpha}}, \varphi)$. Notice that the Kraus operators $\{V_i^{(\alpha)}\}_{i=1}^r$ defines the irreducible component $\phi^{(\alpha)}$ of our quantum channel, having a unique invariant state $\rho_{\infty,\alpha}$, which induces a probability measure $\mathbb{P}_{\rho_{\infty,\alpha}}$ that is $\varphi -$ *invariant*, namely

$$\mathbb{P}_{\rho_{\infty,\alpha}}(\varphi^{-1}(A)) = \mathbb{P}_{\rho_{\infty,\alpha}}(A), \ \forall A \in \mathcal{F}$$

We recall the following theorem which applies to our setting and shows that $(\Omega, \mathcal{F}, \mathbb{P}_{\rho_{\infty,\alpha}}, \varphi)$ is *ergodic.*

Theorem 11. *If $\rho_\infty > 0$ is the unique invariant state of ϕ, then $(\Omega, \mathcal{F}, \mathbb{P}_{\rho_\infty}, \varphi)$ is ergodic.*

Therefore we can apply the *Birkhoff Theorem* 4 for ergodic dynamical systems: for any function $f \in L^1(\Omega, \mathcal{F}, \mathbb{P}_{\rho_{\infty,\alpha}})$ we have that

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ \varphi^k \xrightarrow[n\to+\infty]{} \mathbb{E}_{\mathbb{P}_{\rho_{\infty,\alpha}}}(f), \quad \mathbb{P}_{\rho_{\infty,\alpha}} - a.s.$$

If we take $f = \mathbb{1}_I$, with $I = (i_1, ..., i_p) \in \mathcal{F}_p$, we can define the function $N_I(n) : \Omega \to \mathbb{N}$ as $N_I(n) := \sum_{k=0}^{n-1} \mathbb{1}_I \circ \varphi^k$, which counts the number of times the finite sequence I appears in a realization $\omega \in \Omega$ applying n shifts, with

$$\mathbb{1}_I(\omega) = \begin{cases} 1 \text{ if } \omega = (I, ...) \\ 0 \text{ otherwise} \end{cases}$$

Applying *Birkhoff* it turns out that

$$\frac{N_I(n)}{n} \xrightarrow[n\to+\infty]{} \mathbb{E}_{\mathbb{P}_{\rho_{\infty,\alpha}}}(\mathbb{1}_I) = \mathbb{P}_{\rho_{\infty,\alpha}}(I), \quad \mathbb{P}_{\rho_{\infty,\alpha}} - a.s. \tag{3.10}$$

and we will need this to prove the following lemma, which shows that when **ID** holds, then the measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ are *mutually singular*.

Lemma 12. *If **ID** holds then there exists a collection of disjoint subsets $(\Omega_\alpha) \subset \Omega$ such that*

$$\mathbb{P}_{\rho_{\infty,\alpha}}(\Omega_\beta) = \delta_{\alpha,\beta} \quad \forall \alpha, \beta = 1, ..., d$$

Proof. Exploiting eq. (3.10), we define the subsets Ω_α in the following way

$$\Omega_\alpha := \left\{ w \in \Omega \mid \frac{N_I(n)}{n} \xrightarrow[n\to+\infty]{} \mathbb{P}_{\rho_{\infty,\alpha}}(I) \ \forall I = (i_1, ..., i_{|I|}) \right\} \tag{3.11}$$

which are disjoint by ***ID***, and $\varphi -$ *invariant*, i.e. $\varphi^{-1}(\Omega_\alpha) = \Omega_\alpha$. By the definition of Ω_α follows that $\mathbb{P}_{\rho_{\infty,\alpha}}(\Omega_\beta) = 1$ if $\alpha = \beta$ and zero otherwise, which proves the lemma. □

In the following section we will construct a special martingale that will permit us to prove that, under the ID assumption, the support of the quantum trajectory converges *non-deterministically* to one of the minimal invariant subspaces $\mathcal{V}_\alpha$.

3.3 Convergence of the quantum trajectory

In the following section we will show that the quantum trajectory will asymptotically converge to one of the minimal invariant subspaces $\mathcal{V}_\alpha$. This convergence is *non-deterministic*, so we will derive the probability of convergence to a certain subspace, that will depend on the initial state of the quantum system and on the invariant states of the dual map ϕ^*.

The starting point of this analysis is the following random variable

$$\begin{aligned} Q_\alpha(n) :&= tr(M_\alpha \rho_n) \\ &= tr[\Pi_{\mathcal{V}_\alpha}\rho_n] + tr[M_{\mathcal{T}_\alpha}\rho_n] \end{aligned}$$

with $0 \leqslant Q_\alpha(n) \leqslant 1$ and $\sum_{\alpha=1}^d Q_\alpha(n) = 1$. This quantity represents the probability in ρ_n distributed in the $\mathcal{V}_\alpha$ subspace after the n-th measurement, plus a term $tr[M_{\mathcal{T}_\alpha}\rho_n]$ that is converging to zero by theorem 7. This random variable $Q_\alpha(n)$ gives rise to a sequence that turns out to be a *martingale*, and thanks to the convergence theorem for martingales, it will permit us to analyze the asymptotic behaviour of our quantum trajectory $(\rho_n)_{n\in\mathbb{N}}$.

Proposition 7. *The sequence of random variables $\{Q_\alpha(n)\}_{n\in\mathbb{N}}$ is a martingale which converges almost surely and in L^1 to a random variable $Q_\alpha(\infty)$.*

Proof. We have

$$\begin{aligned}
\mathbb{E}[Q_\alpha(n+1)|\mathcal{F}_n] &= \sum_{\alpha=1}^{d} \frac{tr(M_\alpha V_i \rho_n V_i^\dagger)}{tr(V_i \rho_n V_i^\dagger)} tr(V_i \rho_n V_i^\dagger) \\
&= \sum_{i=1}^{d} tr(V_i^\dagger M_\alpha V_i \rho_n) \\
&= tr(\phi^*(M_\alpha)\rho_n) = tr(M_\alpha \rho_n) = Q_\alpha(n)
\end{aligned}$$

where we have used the fact that M_α is an invariant operator of the dual ϕ^*. As $Q_\alpha(n)$ is also bounded, the martingale convergence theorem 2 ensures that $Q_\alpha(n) \to Q_\alpha(\infty)$ $\mathbb{P}_\rho$-almost surely and in $L^1(\Omega, \mathcal{F}, \mathbb{P}_\rho)$. □

Under the ID assumption we will show that the asymptotic random variable $Q_\alpha(\infty) = \lim_{n\to\infty} Q_\alpha(n)$ is either zero or one, namely $tr(\Pi_{\mathcal{V}_\alpha}\rho_n)$ is converging to either zero or one, while the other term $tr(M_{\mathcal{T}_\alpha}\rho_n)$ is converging to zero by theorem 7. Let $\{|\alpha\rangle\}_{\alpha=1}^{n_\alpha}$ be an orthonormal basis of $\mathcal{V}_\alpha$, and $\{|\beta\rangle\}_{\beta=1}^{n_\beta}$ of $\mathcal{V}_\beta$. By the *Cauchy-Schwarz* inequality

$$|\langle\alpha|\rho|\beta\rangle|^2 \leqslant \langle\alpha|\rho|\alpha\rangle\langle\beta|\rho|\beta\rangle$$

the off-diagonal blocks of the density matrix converges to zero, since either $\langle\alpha|\rho|\alpha\rangle$ or $\langle\beta|\rho|\beta\rangle$ is asymptotically equal to zero. Finally this will demonstrate that the support of our quantum trajectory either converges to the subspace $\mathcal{V}_\alpha$ or it converges to another minimal invariant subspace $\mathcal{V}_\beta$, with $\beta \neq \alpha$.

This asymptotic analysis will proceed as follows: we will start by the simpler case that do not consider the transient part $\mathcal{T}$ and we will firstly consider the subspaces $\mathcal{V}_\alpha$ to be *one-dimensional*, and secondly we will generalize the result to *multi-dimensional* subspaces $\mathcal{V}_\alpha$. Finally we will introduce the transient part, proving that everything holds also in this general case. The same asymptotic result was obtained in [5], which considers the transient part but one-dimensional subspaces $\mathcal{V}_\alpha$. In this case the indirect measurements are called *non-demolition*, and the convergence of the quantum trajectory is to one of the *pointer states* $\{\Pi_{\mathcal{V}_\alpha} = |\alpha\rangle\langle\alpha|\}_{\alpha=1}^{d}$.

3.3.1 No transient part: $\mathcal{T} = 0$

Our analysis starts from the case that do not consider the transient part:

$$\mathcal{H} = \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_d \tag{3.12}$$

which implies that for any $\alpha \in \{1, ..., d\}$:

1. $M_\alpha = \Pi_{\mathcal{V}_\alpha}$,
2. $\Pi_{\mathcal{V}_\alpha} V_i = V_i \Pi_{\mathcal{V}_\alpha}$ for all $i \in \{1, , ..., r\}$.
3. $V_i = \sum_{\alpha=1}^{d} \Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{V}_\alpha}$, since by condition 1 the Kraus operators are block diagonal w.r.t. the decomposition (3.12).

Given a sequence $I = (i_1, ..., i_n) \in \mathcal{F}_n$ and defining $V_I := V_{i_n}...V_{i_1}$, from these three properties follows that the probability measure induced by any state ρ reads as follows

$$\begin{aligned}\mathbb{P}_\rho(I) = tr(V_I \rho V_I^\dagger) &= \sum_{\alpha,\alpha'} tr(\Pi_{\mathcal{V}_\alpha} V_I \Pi_{\mathcal{V}_\alpha} \rho \Pi_{\mathcal{V}_{\alpha'}} V_I^\dagger \Pi_{\mathcal{V}_{\alpha'}}) \\ &= \sum_{\alpha,\alpha'} tr(\delta_{\alpha,\alpha'} \Pi_{\mathcal{V}_\alpha} V_I \Pi_{\mathcal{V}_\alpha} \rho \Pi_{\mathcal{V}_{\alpha'}} V_I^\dagger) \\ &= \sum_\alpha tr(V_I \Pi_{\mathcal{V}_\alpha} \rho \Pi_{\mathcal{V}_\alpha} V_I^\dagger) \\ &= \mathbb{P}_{\rho_0}(I)\end{aligned}$$

with

$$\rho_0 := \sum_{\alpha=1}^{d} \Pi_{\mathcal{V}_\alpha} \rho \Pi_{\mathcal{V}_\alpha} \tag{3.13}$$

namely the non-diagonal blocks of ρ w.r.t. the decomposition (3.12) do not influence the probability measure induced by it. This will permit us to rewrite the measure $\mathbb{P}_\rho$ as a convex combination of some measures, where each one is induced by a state supported only on one subspace $\mathcal{V}_\alpha$.

This section is divided in two parts: the first one considers one-dimensional subspaces $\mathcal{V}_\alpha$, while the second one generalizes to the case of multidimensional subspaces.

One-dimensional subspaces $\mathcal{V}_\alpha$

Let $\{|\alpha\rangle\}_{\alpha=1}^k$ be an orthonormal basis of $\mathcal{H}$. In this first case the invariant states are *pure* states, i.e. $\rho_{\infty,\alpha} = |\alpha\rangle\langle\alpha|$, namely rank one projectors onto the one-dimensional subspaces $\mathcal{V}_\alpha$, with $d = k$. In this setting the state ρ_0 of eq. (3.13) can be written as follows:

$$\rho_0 = \sum_{\alpha=1}^{k} Q_\alpha(0) \rho_{\infty,\alpha} \tag{3.14}$$

where $Q_\alpha(0) = tr(\Pi_{\mathcal{V}_\alpha} \rho_0)$ represents the probability that the initial state ρ_0 is in the subspace $\mathcal{V}_\alpha$. By the fact that the map $\rho \to \mathbb{P}_\rho$ is affine by 1, follows that the probability measure $\mathbb{P}_{\rho_0} = \mathbb{P}_\rho$ can be written as a convex combination of the probability measures induced by the invariant states $\rho_{\infty,\alpha}$'s:

$$\mathbb{P}_{\rho_0} := \sum_{\alpha=1}^{d} Q_\alpha(0) \mathbb{P}_{\rho_{\infty,\alpha}} \tag{3.15}$$

Observe that from eq. (3.14) and by the fact that $\Pi_{\mathcal{V}_\alpha}$ and V_I commutes follows that

$$Q_\alpha(n, I) = tr(\Pi_{\mathcal{V}_\alpha} \rho_n) = \frac{tr(\Pi_{\mathcal{V}_\alpha} V_I \rho_0 V_I^\dagger)}{tr(V_I \rho_0 V_I^\dagger)} = \frac{tr(V_I \Pi_{\mathcal{V}_\alpha} \rho_0 \Pi_{\mathcal{V}_\alpha} V_I^\dagger)}{tr(V_I \rho_0 V_I^\dagger)} = Q_\alpha(0) \frac{\mathbb{P}_{\rho_{\infty,\alpha}}(I)}{\mathbb{P}_{\rho_0}(I)} \tag{3.16}$$

for all $I = (i_1, ..., i_n) \in \mathcal{F}_n$.

We have now all the tools to prove the main convergence theorem for a quantum trajectory.

Theorem 13. *If **ID** holds then there exists a collection of disjoint subsets (Ω_α) of Ω such that $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}$ $\mathbb{P}_{\rho_0}$-a.s., namely*

$$Q_\alpha(\infty) = \lim_{n\to\infty} \frac{tr(M_\alpha V_{I_n} \rho_0 V_{I_n}^\dagger)}{tr(V_{I_n} \rho_0 V_{I_n}^\dagger)} = \begin{cases} 1 \text{ if } I_\infty \in \Omega_\alpha \\ 0 \text{ otherwise} \end{cases}$$

Proof. By lemma 12 **ID** implies that the measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ are all mutually singular. Therefore what we want to prove now is that this mutual singularity of the measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ is equivalent to the fact that there exists a collection of disjoint subsets (Ω_α) of Ω such that $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}$ $\mathbb{P}_{\rho_0}$-a.s.

We start by showing that $Q_\alpha(n)$ is the *Radon-Nikodym* derivative of $Q_\alpha(0)\mathbb{P}_{\rho_{\infty,\alpha}}$ with respect to $\mathbb{P}_{\rho_0}$. Indeed, under the assumption that $Q_\alpha(0) > 0$ $\forall \alpha \in \{1, ..., d\}$, any set of $\mathbb{P}_{\rho_0}$-measure 0 has also $\mathbb{P}_{\rho_{\infty,\alpha}}$-measure 0, namely $\mathbb{P}_{\rho_{\infty,\alpha}} \ll \mathbb{P}_{\rho_0}$. Therefore for all α there exists the *Radon-Nikodym* derivative of $Q_\alpha(0)\mathbb{P}_{\rho_{\infty,\alpha}}$ with respect to $\mathbb{P}_{\rho_0}$, which we will call $Q_n(\alpha)$. This $Q_n(\alpha)$ is a $\mathbb{P}_{\rho_0}$-integrable non-negative random variable on $(\Omega, \mathcal{F}_n)$, such that $\mathbb{P}_{\rho_0}$-a.s. $\sum_\alpha Q_n(\alpha) = 1$, so that each $Q_n(\alpha) \leqslant 1$. Moreover

$$Q_\alpha(0)\mathbb{E}_{\mathbb{P}_{\rho_{\infty,\alpha}}}[X] = \mathbb{E}_{\mathbb{P}_{\rho_0}}[Q_n(\alpha)X] \tag{3.17}$$

for every $\mathbb{P}_{\rho_{\infty,\alpha}}$-integrable random variable X on $(\Omega, \mathcal{F}_n)$. But this implies that

$$Q_n(\alpha, I) = Q_\alpha(0)\frac{\mathbb{P}_{\rho_{\infty,\alpha}}[I]}{\mathbb{P}_{\rho_0}[I]}$$

for every $I \in \mathcal{F}_n$, which is exactly the expression of our random variable $Q_\alpha(n)$ as showed in eq. (3.16). This argument holds also at the limit since $Q_\alpha(n)$ is bounded, ensuring that the random variable $Q_\alpha(\infty)$ on $(\Omega, \mathcal{F})$ is the *Radon-Nikodym* derivative of $Q_\alpha(0)\mathbb{P}_{\rho_{\infty,\alpha}}$ with respect to $\mathbb{P}_{\rho_0}$, which implies that

$$\mathbb{E}_{\mathbb{P}_{\rho_0}}[Q_\alpha(\infty)\mathbb{1}_{\Omega_\beta}] = Q_\alpha(0)\mathbb{E}_{\mathbb{P}_{\rho_{\infty,\alpha}}}[\mathbb{1}_{\Omega_\beta}] = Q_\alpha(0)\mathbb{P}_{\rho_{\infty,\alpha}}[\Omega_\beta] = Q_\alpha(0)\delta_{\alpha,\beta}$$

where the last equation holds by the mutual singularity of the measures $\mathbb{P}_{\rho_{\infty,\alpha}}$. Being $0 \leqslant Q_\alpha(\infty) \leqslant 1$ follows that if $\alpha \neq \beta$ then $Q_\alpha(\infty)\mathbb{1}_{\Omega_\beta} = 0$ $\mathbb{P}_{\rho_0}$-a.s.. Therefore

$$Q_\alpha(\infty)\mathbb{1}_{\Omega_\beta} = Q_\alpha(\infty)\delta_{\alpha,\beta}$$

$\mathbb{P}_{\rho_0}$-a.s.. If we take the sum over α from both sides of the previous equation, we end up with

$$\sum_\alpha Q_\alpha(\infty)\mathbb{1}_{\Omega_\beta} = \mathbb{1}_{\Omega_\beta}, \quad \sum_\alpha Q_\alpha(\infty)\delta_{\alpha,\beta} = Q_\beta(\infty)$$

from which follows that $Q_\beta(\infty) = \mathbb{1}_{\Omega_\beta}$. The converse is also true: if there exists a collection of disjoint subsets (Ω_α) of Ω such that $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}$ $\mathbb{P}_{\rho_0}$-a.s. then the measures $\mathbb{P}_{\rho_{\infty,\alpha}}$ are all mutually singular and concentrated on the Ω_α's. □

This results proves that the quantum trajectory asymptotically selects one of the subspaces $\mathcal{V}_\alpha$. Moreover it permits us to compute the probability of selection of a certain subspace:

$$\mathbb{P}_{\rho_0}[Q_\alpha(\infty) = 1] = \mathbb{P}_{\rho_0}[\Omega_\alpha] = \sum_{\beta=1}^{d} Q_\beta(0)\mathbb{P}_{\rho_{\infty,\beta}}[\Omega_\alpha] = Q_\alpha(0)$$

This shows that the probability of convergence into the subspace $\mathcal{V}_\alpha$ is $Q_\alpha(0) = tr(M_\alpha\rho_0)$, which depends on the initial state of our system.

Multi-dimensional subspaces $\mathcal{V}_\alpha$

In this more general setting, where the invariant states are no more rank one projectors onto the subspaces $\mathcal{V}_\alpha$, the state ρ_0 of eq. (3.13) can be written as follows

$$\rho_0 = \sum_{\alpha=1}^{d} Q_\alpha(0)\rho_\alpha$$

with states ρ_α defined as

$$\rho_\alpha := \frac{\Pi_{\mathcal{V}_\alpha}\rho_0\Pi_{\mathcal{V}_\alpha}}{tr(\Pi_{\mathcal{V}_\alpha}\rho_0\Pi_{\mathcal{V}_\alpha})} \tag{3.18}$$

and supported on $\mathcal{V}_\alpha$. As before by the fact that the map $\rho \to \mathbb{P}_\rho$ is affine by 1, follows that the probability measure $\mathbb{P}_{\rho_0} = \mathbb{P}_\rho$ can be written as a convex combination of the probability measures induced by states ρ_α's:

$$\mathbb{P}_{\rho_0} = \sum_{\alpha=1}^{d} Q_\alpha(0)\mathbb{P}_{\rho_\alpha} \tag{3.19}$$

Remark. Notice that ρ_α represents the state of our system, after its projection onto the subspace $\mathcal{V}_\alpha$. This happens when you perform on your system a projective measurement having measurement basis $\{\Pi_{\mathcal{V}_\alpha}\}$, and you get α as outcome, which happens with probability $tr(\Pi_{\mathcal{V}_\alpha}\rho) = Q_\alpha(0)$. Being $\mathcal{V}_\alpha$ a minimal invariant subspace of $\mathcal{H}$, projecting first on it is completely equivalent to conditioning on the limit being α, namely

$$\mathbb{P}_{\rho_\alpha} = \mathbb{P}_\rho[\cdot \mid Q_\alpha(\infty) = 1]$$

because if the initial state of our system is already supported in $\mathcal{V}_\alpha$, then it will remain inside it during the sequence of measurements by the invariance of $\mathcal{V}_\alpha$.

In the following lemma we will show that when **ID** holds then also the measures $\mathbb{P}_{\rho_\alpha}$ are all mutually singular.

Lemma 14. *If **ID** holds then there exists a collection of disjoint subsets $(\Omega_\alpha) \subset \Omega$ such that*

$$\mathbb{P}_{\rho_\alpha}(\Omega_\beta) = \delta_{\alpha,\beta} \quad \forall \alpha, \beta = 1, ..., d$$

with ρ_α defined in eq. (3.18).

Proof. We start by showing that $\mathbb{P}_\rho(\varphi^{-1}(A)) = \mathbb{P}_{\phi(\rho)}(A)$, $\forall A \in \mathcal{F}$ and $\forall \rho \in \mathcal{D}(\mathcal{H})$. Let $B_{i_1,...,i_n} = \{\omega \in \Omega \mid \omega_1 = i_1, ..., \omega_n = i_n\}$ be a cylinder subset of Ω, then

$$\begin{aligned}
\mathbb{P}_\rho(\varphi^{-1}(C_{i_1,...,i_n})) = \mathbb{P}_\rho(\omega_2 = i_1, ..., \omega_{n+1} = i_n) &= \sum_{\omega_1=1}^{r} \mathbb{P}_\rho(\omega_1, \omega_2 = i_1, ..., \omega_{n+1} = i_n) \\
&= \sum_{\omega_1=1}^{r} tr(V_{i_n}...V_{i_1}V_{\omega_1}\rho V_{\omega_1}^\dagger V_{i_1}^\dagger ... V_{i_n}^\dagger) \\
&= tr\left(V_{i_n}...V_{i_1}\sum_{\omega_1=1}^{r} V_{\omega_1}\rho V_{\omega_1}^\dagger V_{i_1}^\dagger ... V_{i_n}^\dagger\right) \\
&= \mathbb{P}_{\phi(\rho)}(C_{i_1,...,i_n})
\end{aligned}$$

and this equality holds also for infinite sequences in $\mathcal{F}$. Let us apply this results to the Ω_α's, together with the fact that they are $\varphi - invariat$:

$$\mathbb{P}_{\rho_\alpha}(\Omega_\beta) = \mathbb{P}_{\phi(\rho_\alpha)}(\varphi^{-1}(\Omega_\beta)) = \mathbb{P}_{\phi(\rho_\alpha)}(\Omega_\beta) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}_{\phi^k(\rho_\alpha)}(\Omega_\beta) = \mathbb{P}_{\frac{1}{n}\sum_{k=1}^{n}\phi^k(\rho_\alpha)}(\Omega_\beta) \tag{3.20}$$

where the last equality holds since the function $\rho \to \mathbb{P}_\rho$ is *affine* by proposition 6. Recall that $\phi_\infty = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \phi^k$ is a projector onto the set of invariant states $\mathcal{F}_\phi$, namely $\phi \circ \phi_\infty(\rho) = \phi_\infty(\rho)$. The function $\rho \to \mathbb{P}_\rho$ is *continuous in total variation* by proposition 6, so we can take the limit:

$$\lim_{n\to\infty} \mathbb{P}_{\frac{1}{n}\sum_{k=1}^{n}\phi^k(\rho_\alpha)}(\Omega_\beta) = \mathbb{P}_{\phi_\infty(\rho_\alpha)}(\Omega_\beta) = \mathbb{P}_{\rho_{\infty,\alpha}}(\Omega_\beta) = \delta_{\alpha,\beta}$$

where we have used the fact that by eq. (2.25) $\phi_\infty(\rho_\alpha) = \sum_{\alpha=1}^{d} \Pi_{\infty,\alpha}\rho_\alpha = \rho_{\infty,\alpha}$ since ρ_α is supported only on $\mathcal{V}_\alpha$, while the last equation holds by lemma 12. ☐

Now we have all the tools to prove that the theorem of the convergence of the quantum trajectory to an invariant subspace holds also in this setting.

Theorem 15. *If **ID** holds then $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}\ \mathbb{P}_{\rho_0}$-almost surely.*

Proof. We start by rewriting $Q_\alpha(n)$ as follows

$$Q_\alpha(n,I) = \frac{tr(\Pi_{\mathcal{V}_\alpha} V_I \rho_0 V_I^\dagger)}{tr(V_I\rho_0 V_I^\dagger)}\frac{Q_\alpha(0)}{Q_\alpha(0)} = Q_\alpha(0) tr\left(V_I \Pi_{\mathcal{V}_\alpha}\frac{\rho_0}{Q_\alpha(0)}\Pi_{\mathcal{V}_\alpha} V_I^\dagger\right)\frac{1}{tr(V_I\rho_0 V_I^\dagger)} = Q_\alpha(0)\frac{\mathbb{P}_{\rho_\alpha}(I)}{\mathbb{P}_{\rho_0}(I)} \tag{3.21}$$

for all $I = (i_1, ..., i_n) \in \mathcal{F}_n$. If **ID** holds then the measures $\mathbb{P}_{\rho_\alpha}$ have a disjoint support by the previous lemma. Let us focus on the current decomposition of the probability measure $\mathbb{P}_{\rho_0}$ of eq. (3.19). Assuming that $Q_\alpha(0) > 0$ for all $\alpha \in \{1, ..., d\}$, and then applying the same reasoning of Lemma 13 and by eq. (3.21), we get that the random variable $Q_\alpha(\infty)$ on $(\Omega, \mathcal{F})$ is the *Radon-Nikodym* derivative of $Q_\alpha(0)\mathbb{P}_{\rho_\alpha}$ with respect to $\mathbb{P}_{\rho_0}$, which implies that

$$\mathbb{E}_{\mathbb{P}_{\rho_0}}[Q_\alpha(\infty)\mathbb{1}_{\Omega_\beta}] = Q_\alpha(0)\mathbb{E}_{\mathbb{P}_{\rho_\alpha}}[\mathbb{1}_{\Omega_\beta}] = Q_\alpha(0)\mathbb{P}_{\rho_\alpha}[\Omega_\beta] = Q_\alpha(0)\delta_{\alpha,\beta}$$

and following the same procedure as in theorem 13 we get that $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}\ \mathbb{P}_{\rho_0} - a.s.$ ☐

The following step will be to generalize the convergence theorem to the case that considers also the transient part.

3.3.2 Transient part: $\mathcal{T} \neq 0$

Let us come back to the general case where also the transient part is present, therefore the state can have a part of its support on the transient subspace $\mathcal{T}$, that asymptotically decays. Recall that the resultant Hilbert space decomposition reads

$$\mathcal{H} = \bigoplus_{\alpha=1}^{d} \mathcal{V}_\alpha \oplus \mathcal{T}$$

where $\mathcal{V}_\alpha$ is the subspace where the invariant state $\rho_{\infty,\alpha}$ has its full support. When the transient part is present, the operators M_α take the form of eq. (2.22), hence they are no more orthogonal projectors, which brings to their possible non commutativity with the Kraus operators:

$$M_\alpha V_i \neq V_i M_\alpha$$

Moreover observe that in this case the measure induced by any initial state ρ is no more equal to the measure induced by ρ_0 (i.e. $\mathbb{P}_\rho \neq \mathbb{P}_{\rho_0}$), as it was in the previous subsection. Therefore we need to find a new decomposition of the probability measure $\mathbb{P}_\rho$. We start by defining a new probability measure $\mathbb{P}_{\alpha,\rho}$ on Ω, induced by an initial state ρ having $tr(M_\alpha\rho) \neq 0$:

$$\mathbb{P}_{\alpha,\rho}(I) := \frac{1}{tr(M_\alpha\rho)} tr(M_\alpha V_I \rho V_I^\dagger) \tag{3.22}$$

The normalization condition reads

$$\mathbb{P}_{\alpha,\rho}(\Omega) = \sum_{I\in\Omega}\mathbb{P}_{\alpha,\rho}(I) = \frac{1}{tr(M_\alpha\rho)}\sum_{I\in\Omega} tr(M_\alpha V_I\rho V_I^\dagger) = 1$$

since

$$\sum_{I\in\Omega} tr(M_\alpha V_I \rho V_I^\dagger) = tr\left(\sum_{I\in\Omega} V_I^\dagger M_\alpha V_I \rho\right) = tr(M_\alpha\rho)$$

by the fact that M_α is an invariant operator of ϕ^*. The positivity condition is also trivially satisfied. The map $\rho \to \mathbb{P}_{\alpha,\rho}$ is neither affine nor continuous in total variation. But we still have continuity in total variation for a particular sequence $\bar{\phi}_n(\rho)$, as the following proposition shows.

Proposition 8. *Let $\bar{\phi}_n(\rho) := \frac{1}{n}\sum_{k=1}^n \phi^k(\rho)$. If $\bar{\phi}_n(\rho) \xrightarrow[n\to\infty]{} \rho$ then the measure $\mathbb{P}_{\alpha,\bar{\phi}_n(\rho)}$ converges in total variation to the measure $\mathbb{P}_{\alpha,\rho}$, namely there exists a constant $c_\alpha > 0$ such that*

$$\|\mathbb{P}_{\alpha,\bar{\phi}_n(\rho)} - \mathbb{P}_{\alpha,\rho}\|_{TV} \leqslant c_\alpha k \|\bar{\phi}_n(\rho) - \rho\|_1$$

for every state $\rho \in \mathcal{D}(\mathcal{H})$ such that $tr(M_\alpha \rho) \neq 0$.

Proof. We start by defining a new quantity $M_{\alpha,n}(i_1, ..., i_n)$, which is equal to

$$M_{\alpha,n}(i_1,...,i_n) := \frac{V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}}{tr(V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1})}$$

when $tr(V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}) \neq 0$, while it is 0 when the denominator is equal to zero. We note that the two measures $\mathbb{P}_{\alpha,\bar{\phi}_n(\rho)}$ and $\mathbb{P}_{\alpha,\rho}$ have the same denominator:

$$tr(M_\alpha \bar{\phi}_n(\rho)) = \frac{1}{n}\sum_{k=1}^n tr(M_\alpha \phi^k(\rho)) = \frac{1}{n}\sum_{k=1}^n tr(\phi *^k (M_\alpha)\rho) = \frac{1}{n}\sum_{k=1}^n tr(M_\alpha \rho) = tr(M_\alpha \rho)$$

Therefore we have that

$$\begin{aligned}
|\mathbb{P}_{\alpha,\bar{\phi}_n(\rho)}(i_1,...,i_n) - \mathbb{P}_{\alpha,\rho}(i_1,...,i_n)| &= \frac{1}{tr(M_\alpha\rho)} |tr[M_\alpha V_{i_n} ... V_{i_1}(\bar{\phi}_n(\rho) - \rho) V_{i_1}^\dagger ... V_{i_n}^\dagger]| \\
&= \frac{1}{tr(M_\alpha\rho)} |tr[V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}(\bar{\phi}_n(\rho) - \rho)]| \\
&= \frac{1}{tr(M_\alpha\rho)} tr(V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}) |tr[M_{\alpha,n}(i_1,...,i_n)(\bar{\phi}_n(\rho) - \rho)]| \\
&\leqslant \frac{1}{tr(M_\alpha\rho)} tr(V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}) \|M_{\alpha,n}\|_\infty \|\bar{\phi}_n(\rho) - \rho\|_1 \\
&\leqslant \frac{1}{tr(M_\alpha\rho)} tr(V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}) \|\bar{\phi}_n(\rho) - \rho\|_1
\end{aligned}$$

with

$$\frac{1}{tr(M_\alpha\rho)} tr(V_{i_1}^\dagger ... V_{i_n}^\dagger M_\alpha V_{i_n} ... V_{i_1}) \|\bar{\phi}_n(\rho) - \rho\|_1 = c_\alpha k \; Q_{\alpha,I_k/k}(i_1,...,i_n) \|\bar{\phi}_n(\rho) - \rho\|_1$$

and where $c_\alpha := \frac{1}{tr(M_\alpha\rho)}$ while $Q_{\alpha,I_k/k}(i_1,...,i_n) = tr(M_\alpha V_{i_n} ... V_{i_1} \frac{I_k}{k} V_{i_1}^\dagger ... V_{i_n}^\dagger) \leqslant 1$. The first inequality holds by inequality (17) proved in the appendix, while the last one holds since $\|M_{\alpha,n}\|_\infty \leqslant 1$. Now taking the sup over all possible infinite sequences in $\mathcal{F}$ we end up with

$$\begin{aligned}
\sup_{A\in\mathcal{F}} |\mathbb{P}_{\alpha,\bar{\phi}_n(\rho)}(A) - \mathbb{P}_{\alpha,\rho}(A)| &\leqslant c_\alpha k \sup_{A\in\mathcal{F}} Q_{\alpha,I_k/k}(A) \|\bar{\phi}_n(\rho) - \rho\|_1 \\
&\leqslant c_\alpha k \|\bar{\phi}_n(\rho) - \rho\|_1
\end{aligned}$$

□

In line with what we have done in the previous section, we decompose the measure $\mathbb{P}_\rho$ as follows

$$\mathbb{P}_\rho := \sum_{\alpha=1}^d Q_\alpha(0) \mathbb{P}_{\alpha,\rho} \tag{3.23}$$

and we want to show that if **ID** holds, then also the measures $\mathbb{P}_{\alpha,\rho}$ are all mutually singular.

Lemma 16. *If **ID** holds then there exists a collection of disjoint subsets* $(\Omega_\alpha) \subset \Omega$ *such that*

$$\mathbb{P}_{\alpha,\rho}(\Omega_\beta) = \delta_{\alpha,\beta} \quad \forall \alpha, \beta = 1, ..., d \tag{3.24}$$

for all $\rho \in \mathcal{D}(\mathcal{H})$ *such that* $tr(M_\alpha \rho) \neq 0$.

Proof. We define the subsets Ω_α as in (3.11), which remain disjoint by ***ID***, and $\varphi - invariant$. We will show that also with the measures $\mathbb{P}_{\alpha,\rho}$ we have that $\mathbb{P}_{\alpha,\rho}(\varphi^{-1}(A)) = \mathbb{P}_{\alpha,\phi(\rho)}(A)$, $\forall A \in \mathcal{F}$ and $\forall \rho \in \mathcal{D}(\mathcal{H})$ such that $tr(M_\alpha \rho) \neq 0$. As before

$$\begin{aligned}
\mathbb{P}_{\alpha,\rho}(\varphi^{-1}(C_{i_1,...,i_n})) = \mathbb{P}_{\alpha,\rho}(\omega_2 = i_1, ..., \omega_{n+1} = i_n) &= \sum_{\omega_1=1}^{r} \mathbb{P}_{\alpha,\rho}(\omega_1, \omega_2 = i_1, ..., \omega_{n+1} = i_n) \\
&= \sum_{\omega_1=1}^{r} \frac{1}{tr(M_\alpha \rho)} tr(M_\alpha V_{i_n} ... V_{i_1} V_{\omega_1} \rho V_{\omega_1}^\dagger V_{i_1}^\dagger ... V_{i_n}^\dagger) \\
&= \frac{1}{tr(M_\alpha \rho)} tr\left(M_\alpha V_{i_n} ... V_{i_1} \sum_{\omega_1=1}^{r} V_{\omega_1} \rho V_{\omega_1}^\dagger V_{i_1}^\dagger ... V_{i_n}^\dagger\right) \\
&= \mathbb{P}_{\alpha,\phi(\rho)}(C_{i_1,...,i_n})
\end{aligned}$$

Let us apply this results to the Ω_α's, together with the fact that they are $\varphi - invariat$:

$$\mathbb{P}_{\alpha,\rho}(\Omega_\beta) = \mathbb{P}_{\alpha,\phi(\rho)}(\varphi^{-1}(\Omega_\beta)) = \mathbb{P}_{\alpha,\phi(\rho)}(\Omega_\beta) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}_{\alpha,\phi^k(\rho)}(\Omega_\beta) = \mathbb{P}_{\alpha,\frac{1}{n}\sum_{k=1}^{n}\phi^k(\rho)}(\Omega_\beta) \tag{3.25}$$

where the last equality holds because

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} \mathbb{P}_{\alpha,\phi^k(\rho)}(C_{i_1,...,i_n}) &= \frac{1}{n}\sum_{k=1}^{n} \frac{1}{tr(M_\alpha \phi^k(\rho))} tr(M_\alpha V_{i_n} ... V_{i_1} \phi^k(\rho) V_{i_1}^\dagger ... V_{i_n}^\dagger) \\
&= \frac{1}{n}\sum_{k=1}^{n} \frac{1}{tr(\phi^{*k}(M_\alpha)\rho)} tr(M_\alpha V_{i_n} ... V_{i_1} \phi^k(\rho) V_{i_1}^\dagger ... V_{i_n}^\dagger) \\
&= \frac{1}{tr(M_\alpha \rho)} tr\left(M_\alpha V_{i_n} ... V_{i_1} \frac{1}{n}\sum_{k=1}^{n} \phi^k(\rho) V_{i_1}^\dagger ... V_{i_n}^\dagger\right) \\
&= \frac{1}{tr(\frac{1}{n}\sum_{k=1}^{n}\phi^{*k}(M_\alpha)\rho)} tr\left(M_\alpha V_{i_n} ... V_{i_1} \frac{1}{n}\sum_{k=1}^{n} \phi^k(\rho) V_{i_1}^\dagger ... V_{i_n}^\dagger\right) \\
&= \frac{1}{tr(M_\alpha \frac{1}{n}\sum_{k=1}^{n}\phi^k(\rho))} tr\left(M_\alpha V_{i_n} ... V_{i_1} \frac{1}{n}\sum_{k=1}^{n} \phi^k(\rho) V_{i_1}^\dagger ... V_{i_n}^\dagger\right) \\
&= \mathbb{P}_{\alpha,\frac{1}{n}\sum_{k=1}^{n}\phi^k(\rho)}(C_{i_1,...,i_n})
\end{aligned}$$

Since the measure $\mathbb{P}_{\alpha,\bar{\phi}_n(\rho)}$ converges in total variation by proposition 8, we can take the limit:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}_{\alpha,\frac{1}{n}\sum_{k=1}^{n}\phi^k(\rho)}(I) &= \lim_{n\to\infty} \frac{1}{tr(M_\alpha \rho)} tr\left[M_\alpha V_I \frac{1}{n}\sum_{k=1}^{n} \phi^k(\rho) V_I^\dagger\right] \\
&= \frac{1}{tr(M_\alpha \rho)} tr\left[M_\alpha V_I \phi_\infty(\rho) V_I^\dagger\right]
\end{aligned} \tag{3.26}$$

where ϕ_∞ is the projector onto the set of invariant states, defined in eq. (2.24). Recall that the spectral decomposition of ϕ_∞ reads $\phi_\infty(\cdot) = \sum_{\alpha=1}^{d} tr(M_\alpha \cdot)\rho_{\infty,\alpha}$. Turning back to eq. (3.26) and using this last equality we end up with

$$
\begin{aligned}
tr\left[M_\alpha V_I \phi_\infty(\rho) V_I^\dagger\right] &= tr\left[(\Pi_{\mathcal{V}_\alpha} + M_{\mathcal{T}_\alpha}) V_I \left(\sum_{\alpha'=1}^{d} tr(M_{\alpha'}\rho)\rho_{\infty,\alpha'}\right) V_I^\dagger\right] \\
&= \sum_{\alpha'=1}^{d} tr(M_{\alpha'}\rho)\left[tr(\Pi_{\mathcal{V}_\alpha} V_I \rho_{\infty,\alpha'} V_I^\dagger) + tr(M_{\mathcal{T}_\alpha} V_I \rho_{\infty,\alpha'} V_I^\dagger)\right] \\
&= \sum_{\alpha'=1}^{d} tr(M_{\alpha'}\rho)\left[tr(\Pi_{\mathcal{V}_\alpha} V_I \Pi_{\mathcal{V}_{\alpha'}} \rho_{\infty,\alpha'} \Pi_{\mathcal{V}_{\alpha'}} V_I^\dagger \Pi_{\mathcal{V}_\alpha}) + tr(M_{\mathcal{T}_\alpha} V_I \rho_{\infty,\alpha'} V_I^\dagger)\right] \\
&= tr(M_\alpha \rho) tr(V_I \rho_{\infty,\alpha} V_I^\dagger)
\end{aligned}
$$

where the last equation holds since $\Pi_{\mathcal{V}_\alpha} V_I \Pi_{\mathcal{V}_{\alpha'}} = \delta_{\alpha,\alpha'} \Pi_{\mathcal{V}_\alpha} V_I \Pi_{\mathcal{V}_\alpha}$, and since $tr(M_{\mathcal{T}_\alpha} V_I \rho_{\infty,\alpha'} V_I^\dagger) = 0,\ \forall \alpha, \alpha'$. To understand why $tr(M_{\mathcal{T}_\alpha} V_I \rho_{\infty,\alpha'} V_I^\dagger) = 0,\ \forall \alpha, \alpha'$, we have to recall that $M_{\mathcal{T}_\alpha}$ has support only on the $\mathcal{T}$ subspace, while $V_I \rho_{\infty,\alpha'} V_I^\dagger$ has support only on the $\mathcal{V}_{\alpha'}$ subspace, due to the block structure of the Kraus operators and to the fact that $\rho_{\infty,\alpha'}$ is supported only on $\mathcal{V}_{\alpha'}$. Finally turning back to eq. (3.26) we have that:

$$
\begin{aligned}
\frac{1}{tr(M_\alpha \rho)} tr\left[M_\alpha V_I \phi_\infty(\rho) V_I^\dagger\right] &= \frac{1}{tr(M_\alpha \rho)} tr(M_\alpha \rho) tr(V_I \rho_{\infty,\alpha} V_I^\dagger) \\
&= \mathbb{P}_{\rho_{\infty,\alpha}}(I)
\end{aligned}
$$

Therefore we end up with

$$
\mathbb{P}_{\alpha,\rho}(\Omega_\beta) = \mathbb{P}_{\rho_{\infty,\alpha}}(\Omega_\beta) = \delta_{\alpha,\beta}
$$

where the last equation holds by lemma 12. ☐

We have now all the arguments to prove that the main convergence theorem for a quantum trajectory holds also in this more general setting that considers the transient part.

Theorem 17. *If **ID** holds then* $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}\ \mathbb{P}_\rho - a.s.$

Proof. We start by rewriting $Q_\alpha(n)$ as follows

$$
Q_\alpha(n, I) = \frac{tr(M_\alpha V_I \rho V_I^\dagger)}{tr(V_I \rho V_I^\dagger)} \frac{Q_\alpha(0)}{Q_\alpha(0)} = Q_\alpha(0) \frac{\mathbb{P}_{\alpha,\rho}(I)}{\mathbb{P}_\rho(I)} \tag{3.27}
$$

for all $I = (i_1, ..., i_n) \in \mathcal{F}_n$. If **ID** holds then the measures $\mathbb{P}_{\alpha,\rho}$ have a disjoint support by the previous lemma. Let us focus on the current decomposition of the probability measure $\mathbb{P}_\rho$ of eq. (3.23). Assuming that $Q_\alpha(0) > 0$ for all $\alpha \in \{1, ..., d\}$, and then applying the same reasoning of Lemma 13 and by eq. (3.27), we get that the random variable $Q_\alpha(\infty)$ on $(\Omega, \mathcal{F})$ is the *Radon-Nikodym* derivative of $Q_\alpha(0)\mathbb{P}_{\alpha,\rho}$ with respect to $\mathbb{P}_\rho$, which implies that

$$
\mathbb{E}_{\mathbb{P}_\rho}[Q_\alpha(\infty)\mathbb{1}_{\Omega_\beta}] = Q_\alpha(0)\mathbb{E}_{\mathbb{P}_{\alpha,\rho}}[\mathbb{1}_{\Omega_\beta}] = Q_\alpha(0)\mathbb{P}_{\alpha,\rho}[\Omega_\beta] = Q_\alpha(0)\delta_{\alpha,\beta}
$$

and following the same procedure of theorem 13 we get that $Q_\alpha(\infty) = \mathbb{1}_{\Omega_\alpha}\ \mathbb{P}_\rho - a.s.$. ☐

This finally proves that also when the transient part is present, the quantum trajectory asymptotically converges to one of the minimal invariant subspace $\mathcal{V}_\alpha$, with probability of converging to $\mathcal{V}_\alpha$ that can be derived as follows

$$
\mathbb{P}_\rho[Q_\alpha(\infty) = 1] = \mathbb{P}_\rho[\Omega_\alpha] = \sum_{\beta=1}^{d} Q_\beta(0)\mathbb{P}_{\beta,\rho}[\Omega_\alpha] = Q_\alpha(0)
$$

as in the case of no transient part. The analysis of the asymptotic behaviour of the quantum trajectory concludes ad follows: it will converge to the minimal invariant subspace $\mathcal{V}_\alpha$ with probability $Q_\alpha(0) = tr(M_\alpha \rho_0)$. But then what happens inside $\mathcal{V}_\alpha$? Maassen and Kümmerer explain in [21] that asymptotically the quantum trajectory performs a random walk between *dark subspaces* of the same dimension, i.e. spaces from which no information can leak out. In the trivial case that the dimension of the dark subspace is 1, *purification* has occurred. This means that either the quantum trajectory purifies or it continues to move about in a random fashion between the dark subspaces, thus continuing to produce 'quantum noise'.

Let $\Upsilon \in \{1, ..., d\}$ be the index of the subspace asymptotically selected by the quantum trajectory, namely

$$\Upsilon = \sum_{\alpha=1}^{d} \alpha Q_\alpha(\infty) \tag{3.28}$$

We recall a technical lemma proved in [7] and readapted to our decomposition (3.23) of the measure $\mathbb{P}_\rho$, that we will need for the formulation of the asymptotic laws of the following section.

Lemma 18. *Let (X_n^α) be a sequence of random variables depending on $\alpha \in \{1, ..., d\}$ and assume that **ID** holds. Then*

1. ***almost sure convergence**: if for any $\alpha \in \{1, ..., d\}$*

$$X_n^\alpha \xrightarrow{\mathbb{P}_{\alpha,\rho}-as} X^\alpha$$

then

$$X_n^\Upsilon \xrightarrow{\mathbb{P}_{\rho}-as} X^\Upsilon$$

2. ***convergence in distribution**: if for any $\alpha \in \{1, ..., d\}$*

$$X_n^\alpha \xrightarrow{\mathcal{D}-\mathbb{P}_{\alpha,\rho}-as} X^\alpha$$

then

$$X_n^\Upsilon \xrightarrow{\mathcal{D}-\mathbb{P}_{\rho}-as} X^\Upsilon$$

Finally we conjecture that for n large enough

$$Q_\alpha(n) \simeq e^{-nr(\alpha,\Upsilon)}, \ \alpha \neq \Upsilon \tag{3.29}$$

namely the random variable $Q_\alpha(n)$ with $\alpha \neq \Upsilon$ decays to zero exponentially fast, with a rate of convergence $r(\alpha, \Upsilon)$ that depends on α and on Υ. An extension of this project could look in this direction: prove this conjecture and find the expression of the rate of convergence.

A similar analysis on the rate of convergence of the quantum trajectory can be found in [5], which considers quantum non demolition measurements and proves that in this case the rate of convergence is $r(\alpha, \Upsilon) = \bar{S}(\Upsilon|\alpha)$, with $\bar{S}(\Upsilon|\alpha)$ being the *relative entropy* of the single measurement outcome distribution conditioned on the system being in the state $\rho_{\infty,\Upsilon} = |\Upsilon\rangle\langle\Upsilon|$ w.r.t. the one conditioned on the system being in the state $\rho_{\infty,\alpha} = |\alpha\rangle\langle\alpha|$, namely $\bar{S}(\Upsilon|\alpha) = \sum_{i=1}^{r} tr(V_i \rho_{\infty,\Upsilon} V_i^\dagger) \ln \frac{tr(V_i \rho_{\infty,\Upsilon} V_i^\dagger)}{tr(V_i \rho_{\infty,\alpha} V_i^\dagger)}$.

Another article that considers a similar problem but in continuous time is [8], where the authors provide sharp bounds on the rate of convergence of the quantum trajectory to the GAS subspace $\mathcal{R}$, making use of the Lyapunov exponents.

3.4 Case $m_\alpha \geqslant 1$

We recall that in the general case $\mathcal{V}_\alpha \simeq \mathbb{C}^{n_\alpha}$ has a canonical tensor product structure $\mathbb{C}^{n_\alpha} = \mathbb{C}^{k_\alpha} \otimes \mathbb{C}^{m_\alpha},\ n_\alpha = k_\alpha m_\alpha$, with respect to which each $V_{i,R}^{(\alpha)}$ can be written as

$$V_{i,R}^{(\alpha)} = \widetilde{V}_i^{(\alpha)} \otimes I_{\mathbb{C}^{m_\alpha}}$$

with $\widetilde{V}_i^{(\alpha)}$ being an operator on $\mathbb{C}^{k_\alpha}$. In this section we analyze the case that considers $m_\alpha \geqslant 1$. The resultant structure of the invariant states of ϕ reads:

$$\mathcal{F}_\phi = \bigoplus_{\alpha=1}^{d} \rho_\alpha \otimes \mathcal{B}(\mathbb{C}^{m_\alpha}) \oplus 0$$

where $\rho_\alpha > 0$ is a full rank positive operator in $\mathcal{D}(\mathbb{C}^{k_\alpha})$. It follows that in this setting the invariant states supported in $\mathcal{V}_\alpha$ have the following form

$$\bar{\rho}_{\infty,\alpha,i} = \rho_\alpha \otimes \rho_{\alpha,i}, \quad \rho_{\alpha,i} \in \mathcal{D}(\mathbb{C}^{m_\alpha})$$

from which follows that

$$\mathbb{P}_{\rho_{\infty,\alpha,i}}(I) = \mathbb{P}_{\rho_{\infty,\alpha,j}}(I), \quad \forall I \in \mathcal{F}_n, \forall \rho_{\alpha,i} \neq \rho_{\alpha,j} \in \mathcal{D}(\mathbb{C}^{m_\alpha})$$

which represents an identifiability problem. Indeed

$$\begin{aligned}
\mathbb{P}_{\rho_{\infty,\alpha,i}}(I) &= tr(V_{i,R}^{(\alpha)}(\rho_\alpha \otimes \rho_{\alpha,i})V_{i,R}^{(\alpha)\dagger}) \\
&= tr(\widetilde{V}_i^{(\alpha)}\rho_\alpha \widetilde{V}_i^{(\alpha)\dagger} \otimes \rho_{\alpha,i}) \\
&= tr(\widetilde{V}_i^{(\alpha)}\rho_\alpha \widetilde{V}_i^{(\alpha)\dagger}) = \mathbb{P}_{\rho_{\infty,\alpha,j}}(I)
\end{aligned}$$

This means that the state $\rho_{\alpha,i}$ of the subsystem of dimension m_α has no impact on the probability measure $\mathbb{P}_{\rho_{\infty,\alpha,i}}$ induced by the invariant state $\rho_\alpha \otimes \rho_{\alpha,i}$, and therefore from this measure is not possible to discriminate between the different states of the second subsystem. Therefore we need to extend the definition of identifiability to *sectors*, which are equivalence classes.

Let us define an equivalence relation among invariant states: two invariant states $\rho_{\infty,\alpha,i}$ and $\rho_{\infty,\alpha,j}$ are said to be equivalent (denoted $\rho_{\infty,\alpha,i} \sim \rho_{\infty,\alpha,j}$) if, for any $I \in \mathcal{F}_n$ and for any $n \in \mathbb{N}$

$$\mathbb{P}_{\rho_{\infty,\alpha,i}}(I) = \mathbb{P}_{\rho_{\infty,\alpha,j}}(I)$$

We define the sector $\underline{\alpha}$ as the equivalence class of the invariant states supported in $\mathcal{V}_\alpha$, with

$$\mathbb{P}_{\underline{\alpha}}(I) = tr(\widetilde{V}_i^{(\alpha)}\rho_\alpha \widetilde{V}_i^{(\alpha)\dagger})$$

The consequent identifiability assumption reads as follows.

Assumption (ID')**.** *For any $\underline{\alpha} \neq \underline{\beta}$ there exists an $I = (i_1, ..., i_n) \in \mathcal{F}_n$ such that*

$$\mathbb{P}_{\underline{\alpha}}(I) \neq \mathbb{P}_{\underline{\beta}}(I)$$

Notice that in this setting $\mathcal{V}_\alpha$ is no more the support of a unique invariant state, that was a key point for constructing the subsets Ω_α of lemma 12. Therefore we define a new space $\mathcal{H}' \simeq \sum_\alpha \mathbb{C}^{n_\alpha} \oplus \mathbb{C}^{n_t}$, having a lower dimension than $\mathcal{H}$. Here each subspace $\mathbb{C}^{n_\alpha}$ is the support of a unique invariant state ρ_α. On this new space the blocks $V_{i,R}^{(\alpha)}$ of the Kraus operators becomes $V_{i,R}^{(\alpha)} = \widetilde{V}_i^{(\alpha)}$, and their corresponding projection on $\mathcal{V}_\alpha$, i.e. $\mathcal{J} = \{V_i^{(\alpha)}\}$, induce a process $(\mathcal{J}, \rho_\alpha)$ that defines a dynamical system $(\Omega, \mathcal{F}, \mathbb{P}_{\rho_\alpha}, \varphi)$ which is ergodic by theorem 11. But notice that $\mathbb{P}_{\rho_\alpha} = \mathbb{P}_{\underline{\alpha}}$, therefore the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}_{\underline{\alpha}}, \varphi)$ is ergodic too. Therefore we can inherit our previous analysis and by the same reasonings of the case that considers $m_\alpha = 1$, we have that when **ID'** holds then our quantum trajectory converges to the minimal invariant subspace $\mathcal{V}_\alpha$ with probability $Q_\alpha(0)$.

Chapter 4

The Central Limit Theorem

In this chapter we will present the Central Limit Theorem (CLT) applied to the recording of successive measurements in quantum trajectories. Its first version appeared in [3], but it only tackles the case that considers a unique invariant state and the one that considers multiple invariant states but no transient part. Our aim is to establish a central limit type result without any restriction on the quantum channel ϕ. We will show that in this more general case the process associated to the recording of successive measurements asymptotically approaches a *mixture of Gaussians*, and we will give a complete description of their parameters.

In [3] was established a central limit theorem and a law of large numbers for Open Quantum Random Walks (OQRW) on lattices. Such processes are a possible noncommutative generalization of classical Markov Chains and have applications in quantum computing, as presented in [24]. A generalization of [3] is presented in [14], where a large deviations and a central limit theorem result were proved by making use of deformation techniques and spectral theory. We will apply these results to the recording of successive measurements in quantum trajectories, and we will exploit the Poisson equation and the CLT theorem for martingales to prove that also in the case that considers multiple invariant states and a transient part, the process asymptotically approaches a mixture of Gaussians. Everything is also showed with some simulations. Finally using the same techniques used to prove the CLT, we will derive a Law of Iterated Logarithm (LIL), which is then compared with the previously derived CLT.

This chapter is structured as follows: firstly we will treat the case that considers a quantum channel having a unique invariant state, showing that the process asymptotically approaches a single Gaussian. Secondly we will generalize this to the case of multiple invariant states but no transient part, showing that the process asymptotically approaches a mixture of Gaussians. Finally we will show that everything holds even if a transient part is present, ending the chapter with the derivation of a LIL result, without requiring any assumption on the quantum channel.

4.1 Single Gaussian

We start by considering a quantum channel ϕ that presents a unique invariant state. The following decomposition is induced in the state space:

$$\mathcal{H} = \mathcal{V}_1 \oplus \mathcal{T}$$

where $\mathcal{V}_1$ is the unique minimal invariant subspace supporting the unique invariant state ρ_∞. Therefore the first hypothesis reads:

(H1). ϕ admits a unique invariant state ρ_∞.

Let us introduce a vector $m \in \mathbb{R}^r$, where its r components are defined in the following way:

$$m_i = m \cdot e_i = tr(V_i \rho_\infty V_i^\dagger)$$

namely it contains the probabilities of the measurement outcomes, induced by the invariant state ρ_∞. The following lemma is proved in [3], and we will exploit it to demonstrate our main result.

Lemma 19. *For every $l \in \mathbb{R}^r$, the equation*

$$L - \phi^*(L) = \sum_{i=1}^r V_i^\dagger V_i (e_i \cdot l) - (m \cdot l) I \tag{4.1}$$

admits a solution L^*. The difference between any two solutions of (4.1) is a multiple of the identity.

In the following we will denote by L_l a solution of (4.1) associated to $l \in \mathbb{R}^r$, and with L_i a solution associated to $l = e_i$. Hence L_l can be rewritten in the following way

$$L_l = \sum_{i=1}^r l_i L_i$$

where l_i are the coordinates of the vector l. Let $N_i(n)$ be the random variable that counts how many times a sequence of n measurements produces the i-th outcome. Let $N(n) \in \mathbb{N}^r$ be the corresponding vector. The process $(\rho_n, N(n))_{n\in\mathbb{N}}$ is a Markov Chain (MC) that takes values in $\mathcal{D}(\mathcal{H}) \times \mathbb{N}^r$, and it is described as follows: from any position $(\rho_n, N(n))$, the MC can jump in one of the r different values

$$\left(\frac{V_i \rho_n V_i^\dagger}{tr(V_i \rho_n V_i^\dagger)}, N(n) + e_i \right)$$

with probability $tr(V_i \rho_n V_i^\dagger)$. Now we have all the tools to state and prove our main result: the Central Limit Theorem applied to measurement records.

Theorem 20. *Consider the quantum channel*

$$\phi(\rho) = \sum_{i=1}^r V_i \rho V_i^\dagger$$

on $\mathcal{H}$, for which H1 holds. Consider the random vector $N(n)$ associated to the successive measurements which give rise to the quantum trajectory of ϕ. Then

$$\text{LLN: } \frac{N(n)}{n} \xrightarrow{\mathbb{P}_\rho - as} m$$
$$\text{CLT: } \frac{N(n) - nm}{\sqrt{n}} \xrightarrow{\mathcal{D} - \mathbb{P}_\rho} \mathcal{N}(0, C)$$

with covariance matrix C having elements

$$C_{ij} = \delta_{ij} m_i - m_i m_j + \left(tr(L_j V_i \rho_\infty V_i^\dagger) + tr(L_i V_j \rho_\infty V_j^\dagger) \right) - (m_i tr(L_j \rho_\infty) + m_j tr(L_i \rho_\infty)) \tag{4.2}$$

Proof. Consider the MC $(\rho_n, N(n))_{n\in\mathbb{N}}$ and let $\Delta N(n) = N(n) - N(n-1),\ \forall n > 0$. The stochastic process $(\rho_n, \Delta N(n))_{n>0}$ is also a MC, but with values in $\mathcal{D}(\mathcal{H}) \times \{e_1, ..., e_r\}$ and with *Markov Kernel*

$$Pf(\rho, x) = \sum_{i=1}^{r} f\left(\frac{V_i\rho V_i^\dagger}{tr(V_i\rho V_i^\dagger)}, e_i\right) tr(V_i\rho V_i^\dagger)$$

Given a fixed $l \in \mathbb{R}^r$, we want to write a CLT for $(N(n)\cdot l)_{n\in\mathbb{N}}$. Our first step is to find a solution of the *Poisson equation*:

$$(I - P)f(\rho, x) = x\cdot l - m\cdot l \tag{4.3}$$

namely we wish to find a function $f : \mathcal{D}(\mathcal{H}) \times \{e_1, ..., e_r\} \to \mathbb{R}$.

Lemma 21. *A solution of the Poisson equation (4.3) is given by*

$$f(\rho, x) = tr(\rho L_l) + x\cdot l \tag{4.4}$$

Proof. To prove it we just need to put (4.4) in (4.3):

$$\begin{aligned}
(I-P)f(\rho,x) &= tr(\rho L_l) + x\cdot l - P(tr(\rho L_l) + x\cdot l)\\
&= tr(\rho L_l) + x\cdot l - \left[\sum_{i=1}^{r} tr\left(\frac{V_i\rho V_i^\dagger}{tr(V_i\rho V_i^\dagger)}L_l\right) + e_i\cdot l\right] tr(V_i\rho V_i^\dagger)\\
&= tr(\rho L_l) + x\cdot l - \left[\sum_{i=1}^{r} tr(\rho V_i^\dagger L_l V_i) + (e_i\cdot l) tr(\rho V_i^\dagger V_i)\right]\\
&= tr\left(\rho L_l - \sum_{i=1}^{r}(\rho V_i^\dagger L_l V_i + (e_i\cdot l)\rho V_i^\dagger V_i)\right) + x\cdot l\\
&= tr\left(\rho\left(L_l - \sum_{i=1}^{r}(V_i^\dagger L_l V_i + (e_i\cdot l)V_i^\dagger V_i)\right)\right) + x\cdot l\\
&= tr\left(\rho\left(L_l - \phi^*(L_l) - \sum_{i=1}^{r} V_i^\dagger V_i(e_i\cdot l)\right)\right) + x\cdot l\\
&= tr\left(\rho\left(-(m\cdot l)I\right)\right) + x\cdot l\\
&= -(m\cdot l) + x\cdot l
\end{aligned}$$

where the second last equation is true by eq. (4.1). ☐

The second step of the proof consists in translating the problem of our CLT to a CLT for a martingale. With the help of the Poisson equation, we have

$$\begin{aligned}
N(n)\cdot l - n(m\cdot l) &= N(1)\cdot l - N(1)\cdot l + ... + N(n-1)\cdot l - N(n-1)\cdot l + N(n)\cdot l - n(m\cdot l)\\
&= \sum_{k=2}^{n}((N(k) - N(k-1)) - m)\cdot l\\
&= \sum_{k=2}^{n}(I-P)f(\rho_k, \Delta N(k))\\
&= f(\rho_2, \Delta N(2)) + \sum_{k=3}^{n}[f(\rho_k, \Delta N(k)) - Pf(\rho_{k-1}, \Delta N(k-1))] - Pf(\rho_n, \Delta N(n))
\end{aligned}$$

Let

$$M_n := \sum_{k=3}^{n}[f(\rho_k, \Delta N(k)) - Pf(\rho_{k-1}, \Delta N(k-1))] \tag{4.5}$$

which defines a process $(M_n)_{n\geqslant 3}$ that is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 3}$, where $\mathcal{F}_n = \sigma\{(\rho_k, \Delta N(k)); k \leqslant n\}$ is the σ-algebra generated by all the realizations of $(\rho_k, \Delta N(k))_{k=3}^n$. Indeed

$$\mathbb{E}[\Delta M_n \mid \mathcal{F}_n] = \mathbb{E}[f(\rho_n, \Delta N(n)) \mid (\rho_{n-1}, \Delta N(n-1))] - Pf(\rho_{n-1}, \Delta N(n-1)) = 0$$

by the definition of the Markov Kernel P. Let

$$R_n := f(\rho_2, \Delta N(2)) - Pf(\rho_n, \Delta N(n))$$

hence

$$N(n) \cdot l - n(m \cdot l) = R_n + M_n$$

We claim that $(|R_n|)_{n>0}$ is bounded. Indeed by eq.s (4.3) and (4.4) we have

$$Pf(\rho_n, \Delta N(n)) = f(\rho_n, \Delta N(n)) - x \cdot l + m \cdot l = tr(\rho_n L_l) + m \cdot l$$

and $|tr(\rho_n L_l)|$ is bounded independently of n:

$$|tr(\rho_n L_l)| \leqslant \|\rho_n\|_1 \|L_l\|_\infty = \|L_l\|_\infty$$

where the first inequality holds by lemma 33 of the appendix. This means that the term R_n has no contribution to the LLN or to the CLT. It is thus sufficient to obtain a LLN and a CLT for the martingale $(M_n)_{n\geqslant 3}$. We recall the CLT for martingales, presented in [12], that we shall use here.

Theorem 22 (CLT for martingales)**.** *Let $(M_n)_{n\in\mathbb{N}}$ be a square integrable, real martingale for the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. If for all $\varepsilon > 0$ we have the following convergences in probability:*

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2 \mathbb{1}_{|\Delta M_k| \geqslant \varepsilon\sqrt{n}} \mid \mathcal{F}_{k-1}] \xrightarrow[n\to\infty]{} 0 \tag{4.6}$$

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}] \xrightarrow[n\to\infty]{} \sigma^2 \tag{4.7}$$

for some $\sigma \geqslant 0$, then

$$CLT: \ \frac{M_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

For the class of martingales for which (4.7) holds, the Lindeberg condition is defined by 4.6, where we recall that the classical Lindeberg condition is a sufficient condition for the CLT to hold for a sequence of independent random variables. Therefore we have to prove that our martingale $(M_n)_{n\geqslant 3}$ satisfies these two conditions. We have

$$\begin{aligned}\Delta M_k &= f(\rho_k, \Delta N(k)) - Pf(\rho_{k-1}, \Delta N(k-1)) \\ &= tr(\rho_k L_l) + \Delta N(k) \cdot l - m \cdot l - tr(\rho_{k-1} L_l)\end{aligned}$$

which permits us to show that ΔM_k is bounded independently of k:

$$\begin{aligned}|\Delta M_k| &\leqslant \|\rho_k\|_1 \|L_l\|_\infty + \|\Delta N(k)\| \|l\| - \|m\| \|l\| + \|\rho_{k-1}\|_1 \|L_l\|_\infty \\ &\leqslant 2\|L_l\|_\infty + \|l\| + \|m\| \|l\|\end{aligned}$$

Concerning the LLN, since M_n has bounded increments it implies that

$$LLN: \ \frac{M_n}{n} \xrightarrow{a.s.} 0$$

by Azuma's inequality and Borel Cantelli lemma, as showed in [3]. This implies the LLN for $(N(n))_{n\in\mathbb{N}}$ since $|R_n|$ is bounded and demonstrates that condition 4.6 is satisfied as $\mathbb{1}_{|\Delta M_k| \geqslant \varepsilon\sqrt{n}}$ vanishes for n

large enough. The last step of the proof consists in computing the quantity $\mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}]$ in order to verify that also condition (4.7) is satisfied.

We have

$$\Delta M_k = tr(\rho_k L_l) - tr(\rho_{k-1} L_l) + (\Delta N(k) - m) \cdot l$$

so that

$$\begin{aligned}(\Delta M_k)^2 &= tr(\rho_k L_l)^2 - tr(\rho_{k-1} L_l)^2 \\ &\quad - 2tr(\rho_{k-1} L_l)[tr(\rho_k L_l) - tr(\rho_{k-1} L_l) + (\Delta N(k) - m) \cdot l] \\ &\quad + (\Delta N(k) \cdot l - m \cdot l)^2 + 2tr(\rho_k L_l)(\Delta N(k) \cdot l - m \cdot l)\end{aligned}$$

We denote by T_1, T_2 and T_3, respectively, the three lines appearing in the right hand side above. We have

$$\mathbb{E}[T_1 \mid \mathcal{F}_{k-1}] = \mathbb{E}[tr(\rho_k L_l)^2 \mid \mathcal{F}_{k-1}] - tr(\rho_k L_l)^2 + tr(\rho_k L_l)^2 - tr(\rho_{k-1} L_l)^2$$

- the first term $\mathbb{E}[tr(\rho_k L_l)^2 \mid \mathcal{F}_{k-1}] - tr(\rho_k L_l)^2 =: \Delta Y_k$ represents the increment of a martingale (Y_n) and it is bounded independently of k (using the same kind of estimates as for $|R_n|$ above). Hence $\frac{Y_n}{n} \xrightarrow{a.s.} 0$
- the second term gives $\frac{1}{n} \sum_{k=2}^n tr(\rho_k L_l)^2 - tr(\rho_{k-1} L_l)^2 = \frac{tr(\rho_n L_l)^2 - tr(\rho_1 L_l)^2}{n} \xrightarrow[n \to \infty]{} 0$

Then we have

$$\mathbb{E}[T_2 \mid \mathcal{F}_{k-1}] = -2tr(\rho_{k-1} L_l)\mathbb{E}[\Delta M_k \mid \mathcal{F}_{k-1}] = 0$$

by the fact that M_n is a martingale. Finally we have

$$\begin{aligned}\mathbb{E}[T_3 \mid \mathcal{F}_{k-1}] &= \mathbb{E}[(\Delta N(k) \cdot l)^2 + (m \cdot l)^2 - 2(\Delta N(k) \cdot l)(m \cdot l) + 2tr(\rho_k L_l)(\Delta N(k) \cdot l - m \cdot l) \mid \mathcal{F}_{k-1}] \\ &= \sum_{i=1}^r \Big[tr(V_i \rho_{k-1} V_i^\dagger)[(e_i \cdot l)^2 + (m \cdot l)^2 - 2(e_i \cdot l)(m \cdot l)] + 2tr(V_i \rho_{k-1} V_i^\dagger L_l)(e_i \cdot l - m \cdot l)\Big] \\ &= tr\left[\rho_{k-1}\left(\sum_{i=1}^r V_i^\dagger V_i (e_i \cdot l - m \cdot l)^2 + 2V_i^\dagger L_l V_i (e_i \cdot l - m \cdot l)\right)\right] \\ &= tr[\rho_{k-1} \Gamma_l]\end{aligned}$$

where we have defined

$$\Gamma_l := \left(\sum_{i=1}^r V_i^\dagger V_i (e_i \cdot l - m \cdot l)^2 + 2V_i^\dagger L_l V_i (e_i \cdot l - m \cdot l)\right)$$

Finally by an ergodic theorem for quantum trajectories presented in [19], we have that

$$\frac{1}{n} \sum_{k=1}^n \rho_k \xrightarrow{a.s.} \rho_\infty$$

and putting everything together we have that condition (4.7) holds:

$$\frac{1}{n} \sum_{k=3}^n \mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}] \xrightarrow{a.s.} tr[\rho_\infty \Gamma_l] =: \sigma_l^2$$

and the CLT is proved. The explicit form of the covariance matrix C is derived in [3] by simply rewriting σ_l as

$$\sigma_l^2 = \sum_{i,j=1}^r l_i l_j C_{i,j}$$

□

4.2 Mixture of Gaussians with $\mathcal{T} = 0$

In the following we will generalize the previously presented CLT to the case where ϕ shows *several invariant states*, but no transient part is present, namely

$$\mathcal{H} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_d \tag{4.8}$$

where $\mathcal{V}_\alpha$ are the orthogonal minimal invariant subspaces of $\mathcal{H}$. Let us recall that in this scenario we have decomposed the probability measure $\mathbb{P}_\rho$ in the following way:

$$\mathbb{P}_\rho = \sum_{\alpha=1}^{d} Q_\alpha(0)\mathbb{P}_{\rho_\alpha}, \quad \rho_\alpha = \frac{\Pi_{\mathcal{V}_\alpha}\rho\Pi_{\mathcal{V}_\alpha}}{tr(\Pi_{\mathcal{V}_\alpha}\rho\Pi_{\mathcal{V}_\alpha})} = \frac{\Pi_{\mathcal{V}_\alpha}\rho\Pi_{\mathcal{V}_\alpha}}{Q_\alpha(0)}$$

When there is no transient part, $M_\alpha = \Pi_{\mathcal{V}_\alpha}$, with $\Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{V}_\alpha} = \Pi_{\mathcal{V}_\alpha} V_i = V_i \Pi_{\mathcal{V}_\alpha}$, namely the Kraus Operators V_i are block diagonal with respect to the decomposition (4.8). Let us define the Projection of the Kraus Operators V_i's onto the $\mathcal{V}_\alpha$ subspace:

$$V_i^{(\alpha)} := \Pi_{\mathcal{V}_\alpha} V_i \tag{4.9}$$

which are associated to the map $\phi^{(\alpha)}$, which is CP but not TP, indeed

$$\sum_{i=1}^{r} V_i^{(\alpha)\dagger} V_i^{(\alpha)} = \sum_{i=1}^{r} V_i^\dagger \Pi_{\mathcal{V}_\alpha} V_i = \Pi_{\mathcal{V}_\alpha} \sum_{i=1}^{r} V_i^\dagger V_i = \Pi_{\mathcal{V}_\alpha}$$

Notice that $\Pi_{\mathcal{V}_\alpha}$ still represents the identity on the $\mathcal{V}_\alpha$ subspace, meaning that the map $\phi^{(\alpha)}$ is a quantum channel only on $\mathcal{V}_\alpha$. What we have found is a decomposition of our quantum channel ϕ into its *irreducible components* $\phi^{(\alpha)}$'s, and each of them admits a unique invariant state $\rho_{\infty,\alpha}$, having full support on $\mathcal{V}_\alpha$. From these considerations the following proposition holds.

Proposition 9. *Let*

$$\rho_\alpha := \frac{\Pi_{\mathcal{V}_\alpha}\rho\Pi_{\mathcal{V}_\alpha}}{tr(\Pi_{\mathcal{V}_\alpha}\rho\Pi_{\mathcal{V}_\alpha})} \tag{4.10}$$

Under the law $\mathbb{P}_{\rho_\alpha}$, the Markov Chain $(\rho_n^{(\alpha)}, N(n))_{n\in\mathbb{N}}$ originated by the initial state ρ_α, has the law of the quantum trajectories associated to the family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^{r}$, namely

$$\rho_n = \frac{V_I^{(\alpha)}\rho_\alpha V_I^{(\alpha)\dagger}}{tr\left(V_I^{(\alpha)}\rho_\alpha V_I^{(\alpha)\dagger}\right)}$$

with probability $tr\left(V_I^{(\alpha)}\rho_\alpha V_I^{(\alpha)\dagger}\right)$, for every $I \in \mathcal{F}_n$.

We finally define the quantities

$$m_i(\alpha) = tr(V_i^{(\alpha)}\rho_{\infty,\alpha}V_i^{(\alpha)}) = tr(V_i\rho_{\infty,\alpha}V_i^\dagger)$$

which represents the r components of the vector $m(\alpha) \in \mathbb{R}^r$, and we present some hypothesis under which we can reconduce ourself to the previous scenario that considers a unique invariant state, and inherit the CLT:

(H1'). There exists a decomposition of $\mathcal{H}$ into orthogonal subspaces

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_d$$

such that all the V_i's are block-diagonal with respect to this decomposition.

In our scenario this hypothesis holds with $\mathcal{H}_\alpha = \mathcal{V}_\alpha$.

(H2). Each of the mappings $\phi^{(\alpha)}$ admits a unique invariant state $\rho_{\infty,\alpha}$.

This hypothesis also holds, being $\phi^{(\alpha)}$ a quantum channel on $\mathcal{V}_\alpha$, which admits a unique fixed point having full support on it.

(H3). ID

Under these hypothesis the CLT holds in each subspace $\mathcal{V}_\alpha$, as stated in the following theorem.

Theorem 23. *Under the hypotheses (H1'), (H2), (H3) we have these asymptotic laws*

$$\begin{aligned} &\textit{LLN: } \frac{N(n)}{n} \xrightarrow{\mathbb{P}_\rho - as} m(\Upsilon) \\ &\textit{CLT: } \frac{N(n) - nm(\Upsilon)}{\sqrt{n}} \xrightarrow{\mathcal{D}-\mathbb{P}_\rho} \mathcal{N}(0, C^{(\Upsilon)}) \end{aligned}$$

with Υ defined in eq. (3.28), and where the covariance matrix $C^{(\Upsilon)}$ is given by the same formula as in eq. (4.2), but with $V_i^{(\Upsilon)}$ instead of V_i.

Proof. Under the hypotheses (H1'), (H2), (H3) and conditionally to $Q_\alpha(\infty) = 1$ (i.e. under the measure $\mathbb{P}_\rho[\cdot \mid Q_\alpha(\infty) = 1] = \mathbb{P}_{\rho_\alpha}$), we have that $(\rho_n^{(\alpha)}, N(n))_{n\in\mathbb{N}}$ has the law of the quantum trajectories associated to the family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^{r}$ defined in eq. (4.9) by the previous proposition. In particular under this conditional law we have that

$$\begin{aligned} &\text{LLN: } \frac{N(n)}{n} \xrightarrow{\mathbb{P}_{\rho_\alpha} - as} m(\alpha) \\ &\text{CLT: } \frac{N(n) - nm(\alpha)}{\sqrt{n}} \xrightarrow{\mathcal{D}-\mathbb{P}_{\rho_\alpha}} \mathcal{N}(0, C^{(\alpha)}) \end{aligned}$$

and this by lemma 18 proves our theorem. □

What this theorem is telling us is that with probability $Q_\alpha(0)$ the process $(N(n))_{n\in\mathbb{N}}$ follows the law of quantum trajectories associated to the family $\left(V_i^{(\alpha)}\right)_{i=1}^{r}$, and then satisfies the corresponding CLT with mean $m(\alpha)$ and covariance matrix $C^{(\alpha)}$. This means that each entry $N_i(n)$ of the process $(N(n))_{n\in\mathbb{N}}$ will asymptotically distribute as a mixture of Gaussians, having means $m_i(\alpha)$ and variances $C_{i,i}^{(\alpha)}$, $\alpha = 1, ..., d$.

4.3 Mixture of Gaussians with $\mathcal{T} \neq 0$

Finally we present our main result, that shows that the CLT holds even when the quantum channel presents multiple invariant states and a transient part, which represents the most general case, namely

$$\mathcal{H} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_d \oplus \mathcal{T} \tag{4.11}$$

Recall that here $M_\alpha = \Pi_{\mathcal{V}_\alpha} + M_{\mathcal{T}_\alpha}$, with $M_\alpha V_i \neq V_i M_\alpha$.

In this scenario the probability measure $\mathbb{P}_\rho$ has been decomposed as follows:

$$\mathbb{P}_\rho = \sum_{\alpha=1}^{d} Q_\alpha(0)\mathbb{P}_{\alpha,\rho}, \quad \mathbb{P}_{\alpha,\rho}(I) = \frac{tr(M_\alpha V_I \rho V_I^\dagger)}{tr(M_\alpha \rho)}$$

We define $\Pi_\alpha = \Pi_{\mathcal{V}_\alpha} + \Pi_{\mathcal{T}_\alpha}$ to be the orthogonal projector onto the support of M_α, which brings to the following equality

$$\Pi_\alpha V_i = \Pi_\alpha V_i \Pi_\alpha \tag{4.12}$$

indeed from the block structure (2.14) of V_i we have that:

$$\begin{aligned} \Pi_\alpha V_i &= \Pi_{\mathcal{V}_\alpha} V_i + \Pi_{\mathcal{T}_\alpha} V_i = \Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{V}_\alpha} + \Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{T}_\alpha} + \Pi_{\mathcal{T}_\alpha} V_i \Pi_{\mathcal{T}_\alpha} \\ \Pi_\alpha V_i \Pi_\alpha &= \Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{V}_\alpha} + \Pi_{\mathcal{V}_\alpha} V_i \Pi_{\mathcal{T}_\alpha} + \Pi_{\mathcal{T}_\alpha} V_i \Pi_{\mathcal{T}_\alpha} \end{aligned}$$

Now we define a new family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^r$ as

$$V_i^{(\alpha)} := \sqrt{M_\alpha} V_i \left(\sqrt{M_\alpha}\right)^+ \tag{4.13}$$

where $\left(\sqrt{M_\alpha}\right)^+$ represents the *Moore–Penrose pseudo-inverse*, namely $\sqrt{M_\alpha}\left(\sqrt{M_\alpha}\right)^+ = \Pi_\alpha$. Notice that

$$\begin{aligned} \sum_{i=1}^r V_i^{(\alpha)\dagger} V_i^{(\alpha)} &= \sum_{i=1}^r \left(\sqrt{M_\alpha}\right)^+ V_i^\dagger \sqrt{M_\alpha}\sqrt{M_\alpha} V_i \left(\sqrt{M_\alpha}\right)^+ \\ &= \left(\sqrt{M_\alpha}\right)^+ \sum_{i=1}^r V_i^\dagger M_\alpha V_i \left(\sqrt{M_\alpha}\right)^+ \\ &= \left(\sqrt{M_\alpha}\right)^+ M_\alpha \left(\sqrt{M_\alpha}\right)^+ = \Pi_\alpha \Pi_\alpha = \Pi_\alpha \end{aligned}$$

Therefore the family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^r$ can be associated to a map $\phi^{(\alpha)}$, that is a quantum channel on $\mathcal{V}_\alpha \oplus \mathcal{T}_\alpha$, where $\mathcal{T}_\alpha$ represents the subspace of $\mathcal{T}$ on which M_α has part of its support. Indeed it is CP by construction and TP on $\mathcal{V}_\alpha \oplus \mathcal{T}_\alpha$. Moreover it admits a unique invariant state $\rho_{\infty,\alpha}$, having full support on $\mathcal{V}_\alpha$. The next step will be to show that the previously stated CLT holds on each subspace $\mathcal{V}_\alpha$, associated to the quantum channel $\phi^{(\alpha)}$.

Proposition 10. *Let*

$$\rho_\alpha := \frac{\sqrt{M_\alpha}\rho\sqrt{M_\alpha}}{tr(\sqrt{M_\alpha}\rho\sqrt{M_\alpha})} \tag{4.14}$$

Under the law $\mathbb{P}_{\alpha,\rho}$, the sequence $(\rho_n^{(\alpha)}, N(n))_{n\in\mathbb{N}}$ originated by the initial state ρ_α, has the law of the quantum trajectories associated to the family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^r$ defined in eq. (4.13), namely

$$\rho_n^{(\alpha)} = \frac{V_I^{(\alpha)} \rho_\alpha V_I^{(\alpha)\dagger}}{tr\left(V_I^{(\alpha)} \rho_\alpha V_I^{(\alpha)\dagger}\right)}$$

with probability $tr\left(V_I^{(\alpha)} \rho_\alpha V_I^{(\alpha)\dagger}\right)$, for every $I \in \mathcal{F}_n$.

Proof. What we need to prove is that $\mathbb{P}_{\alpha,\rho}(I) = tr\left(V_I^{(\alpha)}\rho_\alpha V_I^{(\alpha)\dagger}\right)$, indeed

$$
\begin{aligned}
tr\left(V_I^{(\alpha)}\rho_\alpha V_I^{(\alpha)\dagger}\right) &= tr\left(\sqrt{M_\alpha}V_I\left(\sqrt{M_\alpha}\right)^+ \frac{\sqrt{M_\alpha}\rho\sqrt{M_\alpha}}{tr(\sqrt{M_\alpha}\rho\sqrt{M_\alpha})}\left(\sqrt{M_\alpha}\right)^+ V_I^\dagger\sqrt{M_\alpha}\right) \\
&= \frac{tr\left(\sqrt{M_\alpha}V_I\Pi_\alpha\rho\Pi_\alpha V_I^\dagger\sqrt{M_\alpha}\right)}{tr(\sqrt{M_\alpha}\rho\sqrt{M_\alpha})} \\
&= \frac{tr\left(\sqrt{M_\alpha}\Pi_\alpha V_I\Pi_\alpha\rho\Pi_\alpha V_I^\dagger\Pi_\alpha\sqrt{M_\alpha}\right)}{tr(\sqrt{M_\alpha}\rho\sqrt{M_\alpha})} \qquad (4.15) \\
&= \frac{tr\left(\sqrt{M_\alpha}\Pi_\alpha V_I\rho V_I^\dagger\Pi_\alpha\sqrt{M_\alpha}\right)}{tr(\sqrt{M_\alpha}\rho\sqrt{M_\alpha})} \qquad (4.16) \\
&= \frac{tr(M_\alpha V_I\rho V_I^\dagger)}{tr(M_\alpha\rho)}
\end{aligned}
$$

where in (4.15) we have used the fact that $\sqrt{M_\alpha} = \sqrt{M_\alpha}\Pi_\alpha$, while in (4.16) we have used (4.12). $\square$

What we have proved is that $\mathbb{P}_{\alpha,\rho} = \mathbb{P}_{\rho_\alpha}$, which brings to the following consideration: if the initial state of our system is ρ_α (defined in eq. (4.14)), namely the state is initially supported in $\mathcal{V}_\alpha \oplus \mathcal{T}$, then the part supported in $\mathcal{T}$ will asymptotically decay and the quantum trajectory will converge to the subspace $\mathcal{V}_\alpha$, namely

$$\mathbb{P}_{\rho_\alpha} = \mathbb{P}_\rho[\cdot \mid Q_\alpha(\infty) = 1]$$

as before, but this time ρ_α is also supported in $\mathcal{T}$. We have showed that ,also when the transient part is present, we can inherit the CLT, with a little change of H1', that in this case holds on the recurrent subspace $\mathcal{R}$, namely

(H1"). There exists a decomposition of $\mathcal{R}$ into orthogonal subspaces

$$\mathcal{R} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \dots \oplus \mathcal{V}_d$$

such that all the V_i's are block-diagonal with respect to this decomposition.

Instead H2 and H3 hold also with the new family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^r$, defined in (4.13).

Theorem 24. *Under the hypotheses (H1"), (H2), (H3) we have these asymptotic laws*

$$
\begin{aligned}
&\textit{LLN:}\ \frac{N(n)}{n} \xrightarrow{\mathbb{P}_\rho - as} m(\Upsilon) \\
&\textit{CLT:}\ \frac{N(n) - nm(\Upsilon)}{\sqrt{n}} \xrightarrow{\mathcal{D}-\mathbb{P}_\rho} \mathcal{N}(0, C^{(\Upsilon)})
\end{aligned}
$$

with Υ defined in eq. (3.28), and where the covariance matrix $C^{(\Upsilon)}$ is given by the same formula as in eq. (4.2), but with $V_i^{(\Upsilon)}$ instead of V_i.

Proof. Under the hypotheses (H1"), (H2), (H3) and conditionally to $Q_\alpha(\infty) = 1$ (i.e. under the measure $\mathbb{P}_\rho[\cdot \mid Q_\alpha(\infty) = 1] = \mathbb{P}_{\rho_\alpha}$), we have that $(\rho_n^{(\alpha)}, N(n))_{n\in\mathbb{N}}$ has the law of the quantum trajectories associated to the family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^r$ defined in eq. (4.13) by the previous proposition. In particular under this conditional law we have that

$$
\begin{aligned}
&\text{LLN:}\ \frac{N(n)}{n} \xrightarrow{\mathbb{P}_{\rho_\alpha} - as} m(\alpha) \\
&\text{CLT:}\ \frac{N(n) - nm(\alpha)}{\sqrt{n}} \xrightarrow{\mathcal{D}-\mathbb{P}_{\rho_\alpha}} \mathcal{N}(0, C^{(\alpha)})
\end{aligned}
$$

and this by lemma 18 proves our theorem. $\square$

What we have shown is that, also when a transient part is present, the process $(N(n))_{n\in\mathbb{N}}$ converges to a limit distribution that is a mixture of Gaussian distributions.

This type of limit theorem could be useful for a *process tomography* purpose, namely when we want to estimate the quantum channel ϕ that is describing the mean evolution of our system. The goal would be to find an estimate of the parameter θ that parameterizes the Kraus operators $\{V_i(\theta)\}_i$ describing ϕ.

If we prepare the system in the state ρ_α (defined in (4.14)) and we perform a sequence of indirect measurements, by the CLT previously stated we have that for n large

$$\frac{N(n)}{n} = m_\theta(\alpha) + e(\alpha), \quad e(\alpha) \sim \mathcal{N}\left(0, \frac{C^{(\alpha)}}{n}\right) \tag{4.17}$$

where $y_i = \frac{N_i(n)}{n}$ represents the data that we can collect from the outcomes of the measurements, while $[m_\theta(\alpha)]_i = tr(V_i(\theta)\rho_{\infty,\alpha}V_i(\theta)^\dagger)$ depends on the parameter θ that we aim to estimate. The invariant states $\rho_{\infty,\alpha}$ of ϕ can be found by letting the quantum trajectory evolve till convergence. Same for the invariant states M_α of ϕ^* but in the dual picture. Let $f_\theta(y)$ be the probability density of y. From (4.17) we have that

$$f_\theta(y) \sim \mathcal{N}\left(m_\theta(\alpha), \frac{C^{(\alpha)}}{n}\right)$$

We define the *likelihood* of the data y as $L(y,\theta) := f_\theta(y)$, and the correspondent *negative log-likelihood* as $l(y,\theta) := -\log L(y,\theta)$. The *Maximum Likelihood* estimator of the parameter θ reads

$$\hat{\theta}_{ML}(y) := \operatorname*{argmin}_{\theta\in\Theta} l(y,\theta)$$

and it selects the parameters that give the observed data *a posteriori* more likely.

4.4 Law of Iterated Logarithm

We turn back to the same setting of section 4.1, with the aim of establishing a Law of Iterated Logarithm (LIL) result for the process $(N(n))_{n\in\mathbb{N}}$. More precisely, given a fixed $l \in \mathbb{R}^r$, we want to write a LIL for $(N(n)\cdot l)_{n\in\mathbb{N}}$, exploiting the LIL for martingales. We recall that

$$N(n)\cdot l - n(m\cdot l) = R_n + M_n$$

where $(|R_n|)_{n>0}$ is bounded, thus it gives no contribution to the LIL, while $(M_n)_{n\geqslant 3}$ (defined in eq. (4.5)) is a martingale. Thus it will be sufficient to obtain a LIL for the martingale M_n.

Let

$$s_n^2 := \sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}] \tag{4.18}$$

What we have proved in section 4.1 is that

$$\frac{s_n^2}{n} \xrightarrow{a.s.} \sigma^2, \quad \sigma^2 \geqslant 0 \tag{4.19}$$

We now define the random variable

$$u_n := \sqrt{2\log\log s_n^2} \tag{4.20}$$

We recall the LIL for martingales established by Stout in [25].

Theorem 25. *(LIL for martingales) Let $\{M_n\}_{n\in\mathbb{N}}$ be an $\mathbb{F}$-martingale defined on the filtered measurable space $(\Omega, \mathcal{F}, \mathbb{F})$, with $\mathbb{F} := (\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, \ldots)$ being an increasing sequence of σ-algebras. Let K_n be $\mathcal{F}_{n-1}$ measurable functions $\forall n \geqslant 1$ with $K_n \to 0$, a.s. . Let s_n^2 and u_n be defined as in equations (4.18) and (4.20) respectively. If $s_n^2 \to \infty$, a.s. and for $n \geqslant 1$*

$$|\Delta M_n| \leqslant K_n \frac{s_n}{u_n} \quad a.s. \tag{4.21}$$

then

$$\lim_{n\to\infty} \sup \frac{M_n}{s_n u_n} = 1 \ a.s.$$

What we need to prove is that these two hypothesis hold in our setting. By (4.19) we have that if $\sigma^2 > 0$ then $s_n^2 \to \infty$, which represents the first hypothesis of the theorem. We recall that in our setting

$$|\Delta M_k| \leqslant 2\|L_l\|_\infty + \|l\| + \|m\|\|l\| \tag{4.22}$$

Therefore to prove that also the inequality (4.21) holds, we choose

$$K_n = (2\|L_l\|_\infty + \|l\| + \|m\|\|l\|)\frac{u_n}{s_n}$$

which is an $\mathcal{F}_{n-1}$ measurable function and it converges to zero almost surely, since

$$\frac{u_n}{s_n} = \sqrt{\frac{2\log\log s_n^2}{s_n^2}} \xrightarrow{a.s.} 0$$

by the first hypothesis. Then by (4.22) we have that (4.21) holds, and the second hypothesis (inequality (4.21)) also holds. This proves the following theorem.

Theorem 26 (Unique invariant state)**.** *Consider the quantum channel*

$$\phi(\rho) = \sum_{i=1}^{r} V_i \rho V_i^\dagger$$

on $\mathcal{H}$, for which H1 holds. Consider the stochastic process $N(n)$ associated to the successive measurements which give rise to the quantum trajectory of ϕ. Then for every $i \in \{1, \ldots, r\}$

$$\textit{LIL: } \lim_{n\to\infty} \sup \frac{|N_i(n) - nm_i|}{\sqrt{2nC_{i,i} \log\log nC_{i,i}}} = 1 \quad \mathbb{P}_\rho - as$$

with $C_{i,i}$ defined in eq. (4.2) and $m_i = tr(V_i \rho_\infty V_i^\dagger)$.

To extend this results to the case that considers multiple invariant states, we will proceed as previously done for the CLT, and we obtain the following theorem.

Theorem 27 (Multiple invariant states)**.** *Under the hypotheses (H1"), (H2), (H3) we have that*

$$\textit{LIL: } \lim_{n\to\infty} \sup \frac{|N_i(n) - nm_i(\Upsilon)|}{\sqrt{2nC_{i,i}^{(\Upsilon)} \log\log nC_{i,i}^{(\Upsilon)}}} = 1 \quad \mathbb{P}_\rho - as$$

where $C_{i,i}^{(\Upsilon)}$ is given by the same formula as in eq. (4.2), but with $V_i^{(\Upsilon)}$ instead of V_i, while $m_i(\Upsilon) = tr(V_i \rho_{\infty,\Upsilon} V_i^\dagger)$.

Proof. Under the hypotheses (H1"), (H2), (H3) and conditionally to $Q_\alpha(\infty) = 1$ (i.e. under the measure $\mathbb{P}_\rho[\cdot \mid Q_\alpha(\infty) = 1] = \mathbb{P}_{\rho_\alpha}$), we have that $(\rho_n^{(\alpha)}, N(n))_{n\in\mathbb{N}}$ has the law of the quantum trajectories associated to the family of operators $\left(V_i^{(\alpha)}\right)_{i=1}^r$ defined in eq. (4.13) by proposition 10. In particular under this conditional law we have that

$$\text{LIL:} \ \lim_{n\to\infty} \sup \frac{|N_i(n) - nm_i(\alpha)|}{\sqrt{2\sigma_i^2(n,\alpha)\log\log\sigma_i^2(n,\alpha)}} = 1 \quad \mathbb{P}_{\rho_\alpha} - as$$

and this by lemma 18 proves our theorem. □

Comparing the CLT and the LIL results we have that for n large enough

$$\frac{N_i(n) - nm_i(\Upsilon)}{\sqrt{nC_{i,i}^{(\Upsilon)}}} \sim \mathcal{N}(0,1)$$
$$\frac{|N_i(n) - nm_i(\Upsilon)|}{\sqrt{nC_{i,i}^{(\Upsilon)}\log\log nC_{i,i}^{(\Upsilon)}}} \leqslant \sqrt{2} + o(1) \quad \mathbb{P}_\rho - as$$

where the first convergence (in distribution) is weaker than the second one (almost sure convergence). Moreover the first process is asymptotically distributed around 0, with a unitary variance as a Gaussian bell, but if we go a bit faster than $\sqrt{n}$ we obtain the second process which is asymptotically bounded between $\pm\sqrt{2}$.

4.5 Simulations

In this section we present the results obtained by some simulations of a quantum system having dimension $k = 8$, namely $\mathcal{H} \simeq \mathbb{C}^8$, subject to indirect measurements having $r = 4$ possible outcomes. We will consider two different state space decomposition:

1. $\mathcal{H} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{T}$, having dimensions $n_1 = 3$, $n_2 = 2$, $n_1 = 1$ and $n_T = 2$ respectively;
2. $\mathcal{H} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{T}$, having dimensions $n_1 = 3$, $n_2 = 3$ and $n_T = 2$ respectively.

4.5.1 Three subspaces $\mathcal{V}_\alpha$ with different dimensions

Firstly we consider the state space decomposition into orthogonal minimal invariant subspaces $\mathcal{V}_\alpha$ plus a transient part that reads

$$\begin{aligned}\mathcal{H} &= \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{T} \\ &\simeq \mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}^1 \oplus \mathbb{C}^2\end{aligned}$$

that induces the following block structure in the Kraus operators that describe the measurement

$$V_i = \begin{bmatrix} V_{i,R}^{(1)} & 0 & 0 & * \\ 0 & V_{i,R}^{(2)} & 0 & * \\ 0 & 0 & V_{i,R}^{(3)} & * \\ 0 & 0 & 0 & V_{i,T} \end{bmatrix}$$

The Kraus operators used for the simulations are:

$$V_1 = \begin{pmatrix} 0.216 & -0.104 & -0.086 & 0 & 0 & 0 & -0.025 & 0.117 \\ 0.012 & -0.149 & 0.105 & 0 & 0 & 0 & -0.027 & -0.034 \\ -0.209 & 0.329 & -0.027 & 0 & 0 & 0 & 0.199 & 0.030 \\ 0 & 0 & 0 & 0.124 & -0.196 & 0 & 0.041 & -0.163 \\ 0 & 0 & 0 & -0.107 & -0.199 & 0 & 0.028 & -0.161 \\ 0 & 0 & 0 & 0 & 0 & 0.068 & 0.222 & 0.015 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.113 & 0.101 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.098 & 0.362 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} -0.093 & -0.271 & 0.118 & 0 & 0 & 0 & 0.240 & 0.201 \\ 0.026 & -0.061 & -0.124 & 0 & 0 & 0 & -0.027 & -0.275 \\ -0.012 & -0.251 & 0.014 & 0 & 0 & 0 & -0.299 & -0.028 \\ 0 & 0 & 0 & -0.076 & -0.084 & 0 & -0.118 & -0.169 \\ 0 & 0 & 0 & 0.042 & 0.069 & 0 & 0.190 & 0.407 \\ 0 & 0 & 0 & 0 & 0 & 0.104 & -0.150 & 0.116 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.135 & 0.193 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.017 & -0.148 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} -0.066 & -0.039 & -0.050 & 0 & 0 & 0 & 0.049 & -0.162 \\ -0.038 & 0.098 & -0.115 & 0 & 0 & 0 & -0.042 & -0.075 \\ 0.154 & -0.287 & -0.221 & 0 & 0 & 0 & 0.003 & -0.280 \\ 0 & 0 & 0 & 0.071 & 0.005 & 0 & -0.037 & 0.135 \\ 0 & 0 & 0 & 0.039 & -0.187 & 0 & -0.245 & 0.073 \\ 0 & 0 & 0 & 0 & 0 & 0.158 & -0.040 & -0.003 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.116 & -0.005 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.137 & -0.112 \end{pmatrix}$$

$$V_4 = \begin{pmatrix} 0.933 & 0.120 & 0.056 & 0 & 0 & 0 & 0.073 & 0.039 \\ 0 & 0.781 & -0.022 & 0 & 0 & 0 & -0.110 & -0.072 \\ 0 & 0 & 0.940 & 0 & 0 & 0 & -0.031 & -0.133 \\ 0 & 0 & 0 & 0.979 & 0.001 & 0 & -0.007 & -0.040 \\ 0 & 0 & 0 & 0 & 0.935 & 0 & -0.059 & -0.100 \\ 0 & 0 & 0 & 0 & 0 & 0.980 & 0.007 & -0.013 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.725 & -0.270 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.389 \end{pmatrix}$$

which describes a quantum channel ϕ having the following fixed points:

$$\bar{\rho}_{\infty,1} = \begin{pmatrix} 0.403 & -0.004 & 0.107 \\ -0.004 & 0.077 & -0.064 \\ 0.107 & -0.064 & 0.521 \end{pmatrix}, \ \bar{\rho}_{\infty,2} = \begin{pmatrix} 0.770 & -0.027 \\ -0.027 & 0.230 \end{pmatrix}, \ \bar{\rho}_{\infty,1} = 1$$

where for simplicity we have reported only the non-zero blocks of the fixed points $\rho_{\infty,1}$, $\rho_{\infty,2}$, $\rho_{\infty,3}$ supported on $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ respectively.
The invariant states of the dual ϕ^* take the form $M_\alpha = \Pi_{\mathcal{V}_\alpha} \oplus M_{\mathcal{T}_\alpha}$ with

$$M_{\mathcal{T}_1} = \begin{pmatrix} 0.523 & 0.008 \\ 0.008 & 0.481 \end{pmatrix}, \ M_{\mathcal{T}_2} = \begin{pmatrix} 0.304 & 0.048 \\ 0.048 & 0.452 \end{pmatrix}, \ M_{\mathcal{T}_3} = \begin{pmatrix} 0.173 & -0.056 \\ -0.056 & 0.067 \end{pmatrix}$$

The initial state of the system reads

$$\rho_0 = \begin{pmatrix} 0.176 & 0.141 & -0.038 & 0.093 & 0.022 & -0.017 & 0.009 & 0.014 \\ 0.141 & 0.177 & -0.041 & 0.119 & 0.023 & -0.062 & 0.025 & 0.050 \\ -0.038 & -0.041 & 0.205 & -0.094 & -0.005 & 0.042 & -0.027 & 0.050 \\ 0.093 & 0.119 & -0.094 & 0.167 & 0.024 & -0.042 & 0.055 & 0.015 \\ 0.022 & 0.023 & -0.005 & 0.024 & 0.050 & -0.004 & 0.012 & -0.037 \\ -0.017 & -0.062 & 0.042 & -0.042 & -0.004 & 0.060 & -0.022 & -0.020 \\ 0.009 & 0.025 & -0.027 & 0.055 & 0.012 & -0.022 & 0.045 & -0.022 \\ 0.014 & 0.050 & 0.050 & 0.015 & -0.037 & -0.020 & -0.022 & 0.119 \end{pmatrix}$$

therefore the quantities $Q_\alpha(0) = tr(M_\alpha \rho_0)$, which represent the probability for the quantum trajectory to converge to the subspace $\mathcal{V}_\alpha$, take the values

$$Q_1(0) = 0.638, \; Q_2(0) = 0.283, \; Q_3(0) = 0.079$$

from which we can deduce that the evolving quantum trajectory has an high probability of converging into the $\mathcal{V}_1$ subspace, indeed it is the biggest minimal invariant subspace between the three. The upper plot of Figure 4.1 shows the evolution of a realization of the random variables $Q_\alpha(n)$. As expected from the theory, only one of them converges to one, while the others converge to zero. The lower plot of Figure 4.1 shows the exponential decay of a realization of the random variables $tr(M_{\mathcal{T}_\alpha}\rho_n)$, which proves the exponential *decay* of the quantum trajectory from the transient subspace $\mathcal{T}$ to the GAS subspace $\mathcal{R}$.

Finally we looked at the empirical distribution of the random variable $N_i(n)/n, \; i = 1, ..., 4$, repeating a sequence of $n = 500$ indirect measurements $F = 5 \times 10^4$ times, and getting the results depicted in Figure 4.2. The four histograms resembles a mixture of three Gaussians, having means $m_\alpha(i) = tr(V_i \rho_{\infty,\alpha} V_i^\dagger)$:

$$m_1 = \begin{pmatrix} 0.057 \\ 0.030 \\ 0.040 \\ 0.872 \end{pmatrix}, \; m_2 = \begin{pmatrix} 0.039 \\ 0.008 \\ 0.014 \\ 0.940 \end{pmatrix}, \; m_3 = \begin{pmatrix} 0.005 \\ 0.011 \\ 0.025 \\ 0.960 \end{pmatrix}$$

confirming the theory result.

4.5.2 Two subspaces $\mathcal{V}_\alpha$ with same dimension

Secondly we consider the state space decomposition into orthogonal minimal invariant subspaces $\mathcal{V}_\alpha$ plus a transient part that reads

$$\begin{aligned} \mathcal{H} &= \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{T} \\ &\simeq \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^2 \end{aligned}$$

that induces the following block structure in the Kraus operators that describe the measurement

$$V_i = \begin{bmatrix} V_{i,R}^{(1)} & 0 & * \\ 0 & V_{i,R}^{(2)} & * \\ 0 & 0 & V_{i,T} \end{bmatrix}$$

The Kraus operators used for the simulations are:

$$V_1 = \begin{pmatrix} -0.039 & 0.044 & 0.041 & 0 & 0 & 0 & 0.211 & -0.097 \\ 0.021 & 0.130 & -0.149 & 0 & 0 & 0 & 0.182 & -0.012 \\ -0.067 & 0.052 & 0.183 & 0 & 0 & 0 & -0.084 & 0.235 \\ 0 & 0 & 0 & -0.046 & -0.078 & -0.098 & -0.067 & 0.036 \\ 0 & 0 & 0 & -0.160 & -0.013 & 0.130 & -0.295 & -0.101 \\ 0 & 0 & 0 & 0.014 & 0.040 & -0.050 & -0.113 & 0.188 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.008 & 0.030 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.206 & 0.114 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0.009 & -0.093 & -0.101 & 0 & 0 & 0 & -0.015 & 0.324 \\ 0.030 & -0.038 & 0.046 & 0 & 0 & 0 & 0.247 & -0.076 \\ -0.192 & 0.073 & 0.057 & 0 & 0 & 0 & 0.189 & -0.188 \\ 0 & 0 & 0 & -0.112 & 0.196 & 0.021 & 0.207 & 0.009 \\ 0 & 0 & 0 & -0.214 & -0.058 & 0.108 & 0.047 & -0.214 \\ 0 & 0 & 0 & 0.221 & 0.122 & -0.174 & 0.185 & 0.100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.080 & -0.056 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.079 & -0.144 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0.062 & 0.224 & 0.085 & 0 & 0 & 0 & 0.263 & 0.051 \\ -0.026 & -0.073 & 0.008 & 0 & 0 & 0 & 0.201 & -0.118 \\ 0.192 & -0.171 & 0.180 & 0 & 0 & 0 & 0.076 & -0.021 \\ 0 & 0 & 0 & 0.166 & -0.366 & -0.120 & -0.112 & 0.064 \\ 0 & 0 & 0 & -0.026 & -0.066 & -0.207 & 0.229 & -0.099 \\ 0 & 0 & 0 & 0.042 & -0.061 & 0.178 & 0.120 & -0.129 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.087 & 0.073 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.117 & -0.065 \end{pmatrix}$$

$$V_4 = \begin{pmatrix} 0.956 & 0.037 & -0.013 & 0 & 0 & 0 & 0.002 & -0.028 \\ 0 & 0.936 & 0.010 & 0 & 0 & 0 & -0.070 & 0.013 \\ 0 & 0 & 0.942 & 0 & 0 & 0 & -0.029 & 0.007 \\ 0 & 0 & 0 & 0.914 & 0.042 & 0.096 & -0.041 & -0.100 \\ 0 & 0 & 0 & 0 & 0.890 & -0.035 & -0.092 & -0.022 \\ 0 & 0 & 0 & 0 & 0 & 0.911 & 0.068 & 0.094 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.586 & 0.228 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.720 \end{pmatrix}$$

which describes a quantum channel ϕ having the following fixed points:

$$\bar{\rho}_{\infty,1} = \begin{pmatrix} 0.231 & -0.013 & -0.102 \\ -0.013 & 0.174 & -0.074 \\ -0.102 & -0.074 & 0.596 \end{pmatrix}, \ \bar{\rho}_{\infty,2} = \begin{pmatrix} 0.532 & -0.111 & 0.224 \\ -0.111 & 0.226 & -0.203 \\ 0.224 & -0.203 & 0.242 \end{pmatrix}$$

where for simplicity we have reported only the non-zero blocks of the fixed points $\rho_{\infty,1}$, $\rho_{\infty,2}$ supported on $\mathcal{V}_1$, $\mathcal{V}_2$ respectively.
The invariant states of the dual ϕ^* take the form $M_\alpha = \Pi_{\mathcal{V}_\alpha} \oplus M_{\mathcal{T}_\alpha}$ with

$$M_{\mathcal{T}_1} = \begin{pmatrix} 0.527 & -0.076 \\ -0.076 & 0.531 \end{pmatrix}, \ M_{\mathcal{T}_2} = \begin{pmatrix} 0.473 & 0.076 \\ 0.076 & 0.469 \end{pmatrix}$$

The initial state of the system reads

$$\rho_0 = \begin{pmatrix} 0.114 & -0.018 & -0.034 & -0.013 & -0.029 & 0.036 & -0.010 & 0.018 \\ -0.018 & 0.083 & 0.008 & -0.015 & -0.027 & -0.052 & -0.035 & 0.023 \\ -0.034 & 0.008 & 0.071 & -0.001 & -0.036 & -0.005 & 0.002 & -0.016 \\ -0.013 & -0.015 & -0.001 & 0.116 & -0.037 & 0.010 & -0.023 & -0.083 \\ -0.029 & -0.027 & -0.036 & -0.037 & 0.260 & 0.009 & 0.128 & 0.106 \\ 0.036 & -0.052 & -0.005 & 0.010 & 0.009 & 0.097 & 0.001 & 0.004 \\ -0.010 & -0.035 & 0.002 & -0.023 & 0.128 & 0.001 & 0.131 & 0.051 \\ 0.018 & 0.023 & -0.016 & -0.083 & 0.106 & 0.004 & 0.051 & 0.129 \end{pmatrix}$$

therefore the quantities $Q_\alpha(0) = tr(M_\alpha \rho_0)$ take the values

$$Q_1(0) = 0.601, \; Q_2(0) = 0.399$$

from which we can deduce that the evolving quantum trajectory has an high probability of converging into the $\mathcal{V}_1$ subspace. As before we report in Figure 4.3 the plot of the evolution of a realization of the random variables $Q_\alpha(n)$ and of its transient part.

Finally we looked at the empirical distribution of the random variable $N_i(n)/n, \; i = 1, ..., 4$, which gave the four histograms depicted in Figure 4.4, which resembles a mixture of two Gaussians, having means $m_\alpha(i) = tr(V_i \rho_{\infty,\alpha} V_i^\dagger)$:

$$m_1 = \begin{pmatrix} 0.044 \\ 0.022 \\ 0.042 \\ 0.892 \end{pmatrix}, \; m_2 = \begin{pmatrix} 0.013 \\ 0.057 \\ 0.061 \\ 0.869 \end{pmatrix}$$

confirming another time the theory result.

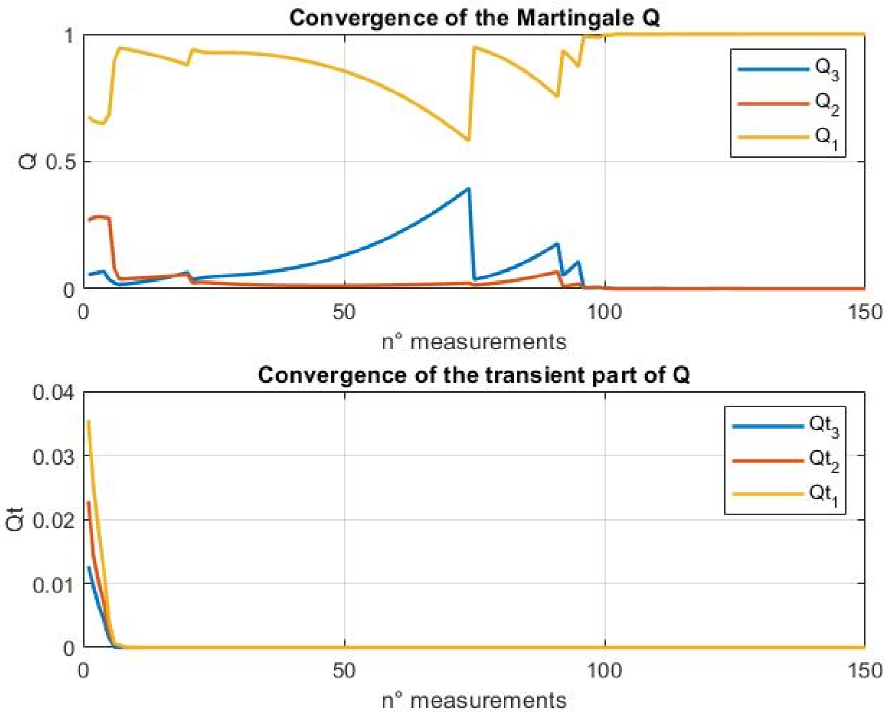

Figure 4.1: The upper plot shows the behaviour of the random variables $Q_\alpha(n)$. The lower plot shows the transient part of the $Q_\alpha(n)$'s, namely $Qt_\alpha(n) = tr(M_{\mathcal{T}_\alpha}\rho_n)$

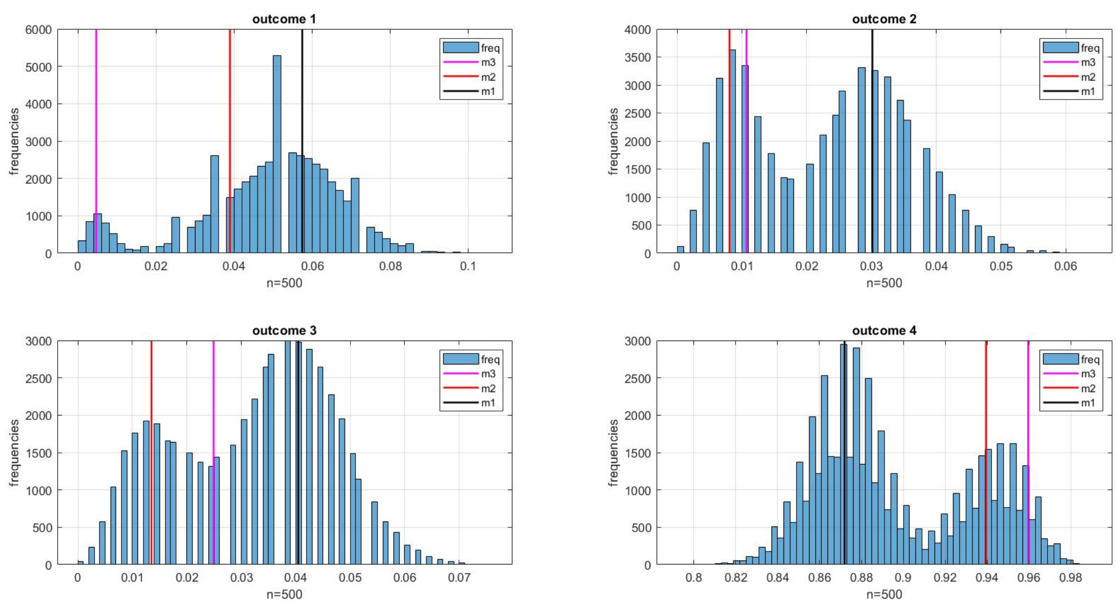

Figure 4.2: The upper plot shows the behaviour of the random variables $Q_\alpha(n)$. The lower plot shows the transient part of the $Q_\alpha(n)$'s, namely $Qt_\alpha(n) = tr(M_{\mathcal{T}_\alpha}\rho_n)$

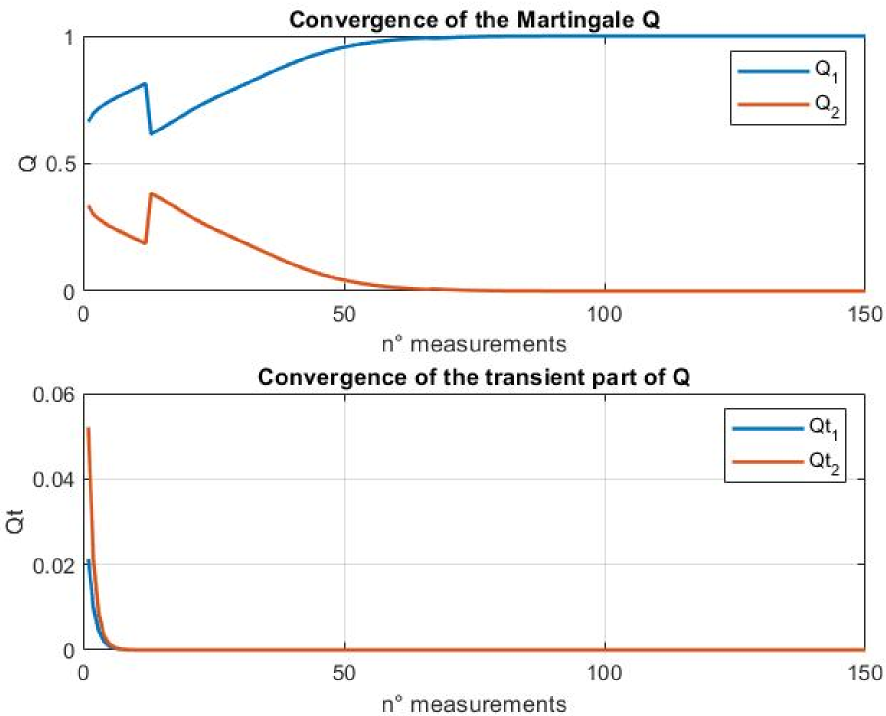

Figure 4.3: The upper plot shows the behaviour of the random variables $Q_\alpha(n)$. The lower plot shows the transient part of the $Q_\alpha(n)$'s, namely $Qt_\alpha(n) = tr(M_{\mathcal{T}_\alpha}\rho_n)$

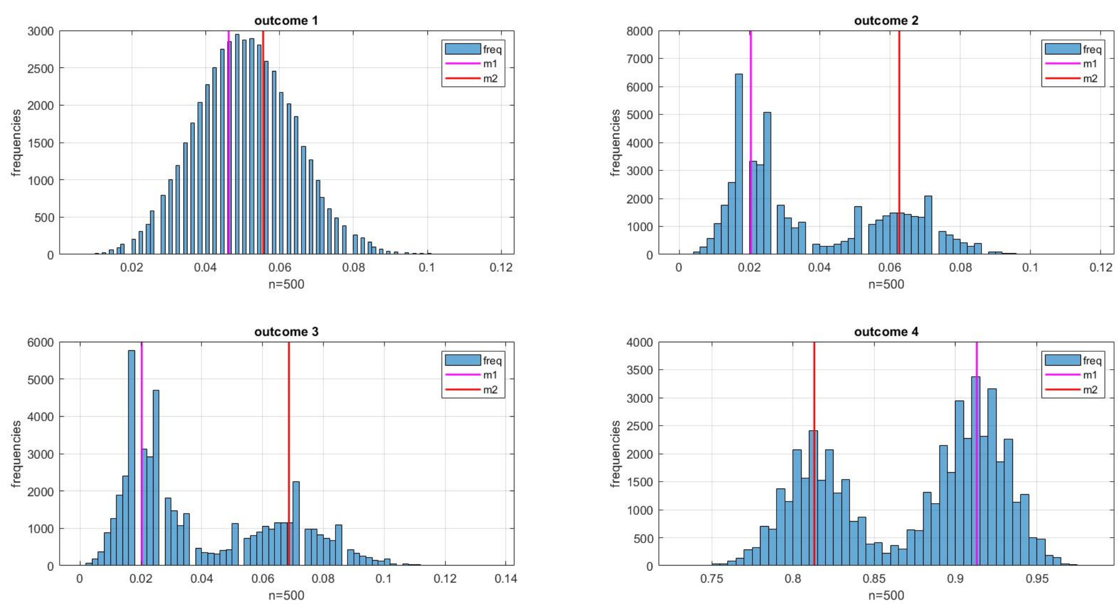

Figure 4.4: The upper plot shows the behaviour of the random variables $Q_\alpha(n)$. The lower plot shows the transient part of the $Q_\alpha(n)$'s, namely $Qt_\alpha(n) = tr(M_{\mathcal{T}_\alpha}\rho_n)$

Chapter 5

Feedback control scheme

What we have showed in chapter 3 is that the support of the state of a quantum system, subject to repeated indirect quantum measurements, converges to one of the minimal invariant subspaces $\mathcal{V}_\alpha$, selecting $\mathcal{V}_\alpha$ with probability $Q_\alpha(0) = tr(M_\alpha \rho_0)$. Therefore repeated indirect quantum measurements can be used to design a *non-deterministic* protocol for preparing the quantum system in one of the minimal invariant subspaces $\mathcal{V}_\alpha$. This could be of particular interest in quantum information applications, since to each $\mathcal{V}_\alpha$ remains associated an Information Preserving Structure (IPS) [10]: the subspace $\mathcal{V}_\alpha \simeq \mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{m_\alpha}$ contains a perfectly Noiseless Subsystem (NS) of dimension m_α (previously presented in point 2) where the dynamics is unitary by eq. (2.15), that specifies the form of the blocks $V_{i,R}^{(\alpha)}$ of the Kraus operators. Thanks to the unitary evolution, the subsystem $\mathbb{C}^{m_\alpha}$ is able to store, preserve and protect quantum information from the noise's action.

In this chapter we will design a feedback control scheme that ensures convergence towards a *chosen* target subspace $\mathcal{V}_{\alpha^*}$, $\alpha^* \in \{1, ..., d\}$, namely the task of this scheme is to make $\mathcal{V}_{\alpha^*}$ GAS. The control design is made using Lyapunov techniques and some graph theory tools. We refer the reader to Bullo's lecture notes [13] for a better overview on graph theory from the viewpoint of dynamical and control systems. This control technique can be used to stabilize the state of the system in a *Decoherence Free Subspace* [20] (previously presented in point 1), which is a subspace that is unitarily evolving. When $\mathcal{V}_{\alpha^*}$ supports the minimum energy eigenstate, this control scheme can be used for cooling problems, otherwise it founds its application in the realization of a "passive" error-prevention scheme, since quantum information is preserved inside a DFS.

This feedback control scheme that realizes a deterministic convergence towards a chosen subspace opens the door to a new problem: if the quantum trajectory converges to the subspace $\mathcal{V}_{\alpha^*}$, in which case does it converge to a specific state? Finding an answer to this question could lead to a new way of doing quantum information encoding exploiting such types of feedback schemes.

5.1 Feedback stabilization of a target subspace $\mathcal{V}_{\alpha^*}$

The starting point will be [2], where a Lyapunov technique is used to design a feedback scheme that stabilizes the quantum system in a target pure state $\rho_{\infty,\alpha} = |\alpha\rangle\langle\alpha|$. Therefore in that work the subspaces

$\mathcal{V}_\alpha$ are one-dimensional. Our goal is to generalize that feedback scheme in the case of multi-dimensional subspaces $\mathcal{V}_\alpha$ using *unitary control*: the system is controlled by an adjustable unitary evolution between two successive indirect measurements.

Let us start by defining the Kraus operators describing the closed-loop dynamics as the family of operators $(V_i^u)_{i=1}^r$, which depend on a scalar control input $u \in \mathbb{R}$ and satisfy the constraint $\sum_{i=1}^r V_i^{u\dagger} V_i^u = I_k$. We will consider

$$V_i^u = U_u V_i, \quad U_u = e^{-iuH}$$

which represents an instance of *Hamiltonian control*, where the amplitude of the Hamiltonian H is adjusted by the scalar control input u. The consequent dynamics is described by a *non-linear* controlled Markov chain $(\rho_n)_{n\in\mathbb{N}}$, modelled through the random equation:

$$\rho_{n+1} = \mathbb{V}_{i_n}^{u_n}(\rho_n) = \frac{V_{i_n}^{u_n} \rho_n V_{i_n}^{u_n\dagger}}{tr(V_{i_n}^{u_n} \rho_n V_{i_n}^{u_n\dagger})} \tag{5.1}$$

that holds with probability $tr(V_{i_n}^{u_n} \rho_n V_{i_n}^{u_n\dagger})$, and defined by the super-operator $\mathbb{V}_i^u : \rho \to \frac{V_i^u \rho V_i^{u\dagger}}{tr(V_i^u \rho V_i^{u\dagger})}$.

Let $\mathbb{K}^u$ be the Kraus map defined as

$$\mathbb{K}^u(\rho) := \sum_{i=1}^r V_i^u \rho V_i^{u\dagger} \in \mathcal{D}(\mathcal{H})$$

We suppose throughout this paper that the two following assumptions are verified by the system under consideration. Notice that when there is no control input ($u = 0$), we turn back to the original setting where generalized measurements are used, i.e. $V_i^0 = V_i$.

Assumption 1 (ID)**.** *For any $\alpha \neq \beta$ in $\{1, ..., d\}$, there exists an $I := (i_1, ..., i_n) \in \Omega$ such that $tr(V_I \rho_{\infty,\alpha} V_I^\dagger) \neq tr(V_I \rho_{\infty,\beta} V_I^\dagger)$*

We will resort to a technique that uses an *open-loop supermartingale* to design a Lyapunov function for the closed-loop system.

Definition 26. An *open-loop supermartingale* is a function $W : \mathcal{D}(\mathcal{H}) \to \mathbb{R}$ that satisfies

$$\mathbb{E}[W(\rho_{n+1}) \mid \rho_n = \rho, u_n = 0] \leqslant W(\rho), \ \forall \rho \in \mathcal{D}(\mathcal{H}) \tag{5.2}$$

namely is a function of the state that at each step decreases in expectation.

Secondly we define the feedback law: at each time-step n, the control input u_n is chosen by minimizing this supermartingale W knowing the state ρ_n:

$$u_n = \hat{u}(\rho) := \underset{u\in[-\bar{u},\bar{u}]}{\operatorname{argmin}} \{\mathbb{E}[W(\rho_{n+1}) \mid \rho_n = \rho, u_n = u]\} \tag{5.3}$$

where $\bar{u}$ is a small positive number that needs to be determined. The state ρ_n is estimated by a quantum filter from indirect measurements, namely if at step $n-1$ the outcome of the measurement is $i_{n-1} \in \{1, ..., r\}$ then

$$\rho_n = \frac{V_{i_{n-1}}^{u_{n-1}} \rho_{n-1} V_{i_{n-1}}^{u_{n-1}\dagger}}{tr(V_{i_{n-1}}^{u_{n-1}} \rho_{n-1} V_{i_{n-1}}^{u_{n-1}\dagger})}$$

where ρ_{n-1} represents the estimate of the state at the previous step. We want to point out that here the discrete-time behaviour is crucial for a possible real-time implementation of such controllers.

Being $0 \in [-\bar{u}, \bar{u}]$ and being the control input u_n chosen to minimize W at each step, we directly have that W is also a *closed-loop supermartingale*, namely

$$Q(\rho) := \mathbb{E}[W(\rho_{n+1}) \mid \rho_n = \rho, u_n = \hat{u}(\rho)] - W(\rho) \leqslant 0, \ \forall \rho \in \mathcal{D}(\mathcal{H}) \tag{5.4}$$

If this supermartingale W is bounded from below, then by the following convergence theorem for a Lyapunov function of a Markov Chain (proved in [1]), the state of the system ρ_n converges almost-surely to the set

$$I_\infty := \{\rho \in \mathcal{D}(\mathcal{H}) \mid Q(\rho) = 0\} \tag{5.5}$$

Theorem 28. *Let X_n be a Markov chain on the compact state space S. Suppose there exists a continuous function $W(X)$ satisfying*

$$\mathbb{E}[W(X_{n+1}) \mid X_n] - W(X_n) = -Q(X_n)$$

where $Q(X)$ is a non-negative continuous function of X, then the ω-limit set Ω (in the sense of almost sure convergence) of X_n is contained by the following set $I_\infty := \{X \mid Q(X) = 0\}$

Therefore we have to design the supermartingale W in such a way that the set I_∞ is restricted to the set of states having support in the target subspace $\mathcal{W}_t$, namely $I_\infty = \{\rho \in \mathcal{D}(\mathcal{H}) \mid supp(\rho) \subset \mathcal{V}_{\alpha^*}\}$. Thus we want W to be a *strict* supermartingale:

$$\begin{aligned} \mathbb{E}[W(\rho_{n+1}) \mid \rho_n = \rho, u_n = \hat{u}(\rho)] &\leqslant W(\rho) \\ &= W(\rho) \iff supp(\rho) \subset \mathcal{V}_{\alpha^*} \end{aligned}$$

We propose the following Lyapunov function:

$$\begin{aligned} W_\varepsilon(\rho_n) &= \sum_{\alpha=1}^{d} \sigma_\alpha Q_\alpha(n) - \varepsilon \frac{1}{2} \sum_{\alpha=1}^{d} Q_\alpha(n)^2 \\ &= W_0(\rho_n) - \varepsilon \Gamma(\rho_n) \end{aligned} \tag{5.6}$$

where ε and the weights σ_α are strictly positive numbers, except for $\sigma_t = 0$. This function W is a concave function of our original (open-loop) martingales $Q_\alpha(n) = tr(M_\alpha \rho_n)$, and therefore by proposition 1 is an open-loop supermartingale, namely

$$\mathbb{E}[W_\varepsilon(\rho_{n+1}) \mid \rho_n = \rho, u_n = 0] - W_\varepsilon(\rho) \leqslant 0, \ \forall \rho \in \mathcal{D}(\mathcal{H}) \tag{5.7}$$

and by the reasoning of before W_ε is also a *closed-loop supermartingale.*

The following proposition will show that the convergence in open-loop could be also proved through the Lyapunov function $\Gamma(\rho)$.

Proposition 11 (Open-loop convergence)**.** *In open-loop and when **ID** holds, the convergence of our quantum trajectory to one of the minimal invariant subspaces $\mathcal{V}_\alpha$ can also be proven through the Lyapunov function*

$$\Gamma(\rho) = \frac{1}{2} \sum_{\alpha=1}^{d} tr(M_\alpha \rho)^2$$

which is a submartingale, with

$$\begin{aligned} \bar{Q}(\rho) := \mathbb{E}[\Gamma(\rho_{n+1}) \mid \rho_n = \rho] - \Gamma(\rho) &\geqslant 0, \ \forall \rho \in \mathcal{D}(\mathcal{H}) \\ &= 0 \iff \rho = \rho^{(\alpha)} \ \forall \alpha \in \{1, ..., d\} \end{aligned}$$

where $\rho^{(\alpha)} = \Pi_{\mathcal{V}_\alpha} \rho^{(\alpha)} \Pi_{\mathcal{V}_\alpha}$ can be any state supported in $\mathcal{V}_\alpha$.

Proof. Being $\Gamma(\rho)$ a convex function of a martingale, it is a submartingale.

To prove that $\bar{Q}(\rho) = 0 \iff \rho = \rho^{(\alpha)}\ \forall \alpha \in \{1, ..., d\}$ we refer to a technique used in [1] (in the proof of theorem 1.3), which in our setting brings to the following equation

$$\bar{Q}(\rho) = \frac{1}{2}\sum_{\alpha}\sum_{i,j} tr(V_i\rho V_i^\dagger)tr(V_j\rho V_j^\dagger)\left(\frac{tr(M_\alpha V_i\rho V_i^\dagger)}{tr(V_i\rho V_i^\dagger)} - \frac{tr(M_\alpha V_j\rho V_j^\dagger)}{tr(V_j\rho V_j^\dagger)}\right)^2$$

implying that

$$\bar{Q}(\rho) = 0 \iff \frac{tr(M_\alpha V_i\rho V_i^\dagger)}{tr(V_i\rho V_i^\dagger)} = \frac{tr(M_\alpha V_j\rho V_j^\dagger)}{tr(V_j\rho V_j^\dagger)} \quad \forall \alpha, i, j$$

Rearranging the terms and taking the sum over i we get

$$\begin{aligned}\sum_i tr(V_j\rho V_j^\dagger)tr(M_\alpha V_i\rho V_i^\dagger) &= \sum_i tr(M_\alpha V_j\rho V_j^\dagger)tr(V_i\rho V_i^\dagger) \\ tr(V_j\rho V_j^\dagger)tr(M_\alpha\rho) &= tr(M_\alpha V_j\rho V_j^\dagger) \\ \mathbb{P}_\rho(j) &= \mathbb{P}_{\alpha,\rho}(j)\end{aligned}$$

where by eq.(3.23), the last equality holds iff $Q_\alpha(0) = 1$, and this is true by theorem 17 iff $\rho = \rho^{(\alpha)}$. ☐

Let us go back to the closed-loop scheme. A state ρ is in the set I_∞, defined in eq. (5.5) with our Lyapunov function W_ε, iff $\forall u \in [-\bar{u}, \bar{u}]$ we have

$$\mathbb{E}[W_\varepsilon(\rho_{n+1}) \mid \rho_n = \rho, u_n = u] - W_\varepsilon(\rho) \geqslant 0 \tag{5.8}$$

The design of the weights σ_α of our Lyapunov function is based on the Hamiltonian H underlying the controlled unitary evolution and relies on the connectivity of the graph attached to H. They are obtained by inverting a Metzler matrix derived from H and the quantum states that are supported in the subspaces $\mathcal{V}_\alpha$. In Lemma 30 we will prove that given any $\alpha^* \in \{1, ..., d\}$, we can always choose the weights $\sigma_1, ..., \sigma_d$ so that W determines a function $f(u, \rho^{(\alpha)})$ of the control input u and of the state $\rho^{(\alpha)}$ supported in $\mathcal{V}_\alpha$, defined as

$$f(u, \rho^{(\alpha)}) := \mathbb{E}[W_\varepsilon(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = u]$$

and having the following properties:

1. $f(u, \rho^{(\alpha)})$ has a strict local minimum at $u = 0$ for $\alpha = \alpha^*$;
2. $f(u, \rho^{(\alpha)})$ has a strict local maxima at $u = 0$ for $\alpha \neq \alpha^*$.

This ensures that the feedback law (5.3) sets $u = 0$ when the quantum trajectory finishes in the subspace $\mathcal{V}_{\alpha^*}$, while it sets $u \neq 0$ when the quantum trajectory finishes in a subspace $\mathcal{V}_\alpha,\ \alpha \neq \alpha^*$, preventing it to stabilize in the wrong subspace. Indeed for $\alpha \neq \alpha^*$, property 2 implies that

$$\mathbb{E}[W_\varepsilon(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = u] < \mathbb{E}[W_\varepsilon(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = 0] \leqslant W_\varepsilon(\rho^{(\alpha)})$$

which by eq. (5.8) demonstrates that

$$\rho^{(\alpha)} \in I_\infty \iff \alpha = \alpha^*$$

The next step will be to prove that the states $\rho^{(\alpha^*)}$, supported in the target subspace $\mathcal{V}_{\alpha^*}$, are the unique states contained in the set I_∞, showing that W_ε is a *strict* Lyapunov function.

Proposition 12. *Let*

$$Q(\rho) = \mathbb{E}[W_\varepsilon(\rho_{n+1}) \mid \rho_n = \rho, u_n = \hat{u}(\rho)] - W_\varepsilon(\rho) \leqslant 0, \ \forall \rho \in \mathcal{D}(\mathcal{H})$$

Then

$$Q(\rho) = 0 \iff \rho = \rho^{(\alpha*)}$$

for every state $\rho^{(\alpha*)}$, supported in the target subspace $\mathcal{V}_{\alpha*}$.

Proof. Firstly we decompose $Q(\rho)$ in the following way:

$$\begin{aligned}
Q(\rho) &= \sum_{i=1}^{r} \left(W_\varepsilon \left(\mathbb{V}_i^{\hat{u}(\rho)}(\rho) \right) - W_\varepsilon(\rho) \right) tr(V_i \rho V_i^\dagger) \\
&= \sum_{i=1}^{r} \left(W_\varepsilon \left(\mathbb{V}_i^0(\rho) \right) - W_\varepsilon(\rho) + \min_{u \in [-\bar{u}, \bar{u}]} W_\varepsilon \left(\mathbb{V}_i^u(\rho) \right) - W_\varepsilon \left(\mathbb{V}_i^0(\rho) \right) \right) tr(V_i \rho V_i^\dagger) \\
&= Q_1(\rho) + Q_2(\rho)
\end{aligned}$$

with

$$\begin{aligned}
Q_1(\rho) &:= \sum_{i=1}^{r} \left(W_\varepsilon \left(\mathbb{V}_i^0(\rho) \right) - W_\varepsilon(\rho) \right) tr(V_i \rho V_i^\dagger) \\
Q_2(\rho) &:= \sum_{i=1}^{r} \left(\min_{u \in [-\bar{u}, \bar{u}]} W_\varepsilon \left(\mathbb{V}_i^u(\rho) \right) - W_\varepsilon \left(\mathbb{V}_i^0(\rho) \right) \right) tr(V_i \rho V_i^\dagger)
\end{aligned}$$

To conclude the proof we will show that

$$Q_1(\rho) = 0 \iff \rho = \rho^{(\alpha)}, \quad \forall \alpha \in \{1, ..., d\} \tag{5.9}$$

$$Q_2(\rho) = 0 \iff \rho = \rho^{(\alpha*)} \tag{5.10}$$

Let us start with $Q_1(\rho)$:

$$\begin{aligned}
Q_1(\rho) &= \sum_{i=1}^{r} W_\varepsilon \left(\mathbb{V}_i^0(\rho) \right) tr(V_i \rho V_i^\dagger) - W_\varepsilon(\rho) \\
&= \sum_{\alpha=1}^{d} \sigma_\alpha \sum_{i=1}^{r} tr(M_\alpha V_i \rho V_i^\dagger) - \varepsilon \sum_{i=1}^{r} \Gamma(\mathbb{V}_i^0(\rho)) tr(V_i \rho V_i^\dagger) - \sum_{\alpha=1}^{d} \sigma_\alpha tr(M_\alpha \rho) + \varepsilon \Gamma(\rho) \\
&= -\varepsilon \sum_{i=1}^{r} \left(\Gamma \left(\mathbb{V}_i^0(\rho) \right) - \Gamma(\rho) \right) tr(V_i \rho V_i^\dagger) = -\varepsilon \bar{Q}(\rho)
\end{aligned}$$

which by proposition 11 shows (5.9).

Finally notice that

$$Q_2(\rho) = 0 \iff min_{u \in [-\bar{u}, \bar{u}]} W_\varepsilon \left(\mathbb{V}_i^u(\rho) \right) = W_\varepsilon \left(\mathbb{V}_i^0(\rho) \right) \quad \forall i \in \{1, ..., r\}$$

where the second equality hols iff $\rho = \rho^{(\alpha*)}$ by property 2. □

To show that properties 1 and 2 hold, we start by decomposing $f(u, \rho^{(\alpha)})$ into two different terms, to analyze separately the two derivatives:

$$\begin{aligned}
f(u, \rho^{(\alpha)}) &= \mathbb{E}[W_0(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = u] - \varepsilon \mathbb{E}[\Gamma(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = u] \\
&= f_0(u, \rho^{(\alpha)}) - \varepsilon f_\Gamma(u, \rho^{(\alpha)})
\end{aligned}$$

with

$$\begin{aligned}
f_0(u, \rho^{(\alpha)}) &:= \mathbb{E}[W_0(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = u] \\
&= \sum_\beta \sigma_\beta tr(M_\beta \sum_i V_i^u \rho^{(\alpha)} V_i^{u\dagger}) \\
&= W_0(\mathbb{K}^u(\rho^{(\alpha)})) \\
f_\Gamma(u, \rho^{(\alpha)}) &:= \mathbb{E}[\Gamma(\rho_{n+1}) \mid \rho_n = \rho^{(\alpha)}, u_n = u] \\
&= \frac{1}{2} \sum_i \sum_\beta \frac{tr(M_\beta V_i^u \rho^{(\alpha)} V_i^{u\dagger})^2}{tr(V_i \rho^{(\alpha)} V_i^\dagger)}
\end{aligned}$$

We begin with the analysis of the first term $f_0(u, \rho^{(\alpha)})$:

$$f_0(u, \rho^{(\alpha)}) = W_0(\mathbb{K}^u(\rho^{(\alpha)})) = \sum_{\beta=1}^d \sigma_\beta \sum_{i=1}^r tr(M_\beta V_i^u \rho^{(\alpha)} V_i^{u\dagger}) = \sum_{\beta=1}^d \sigma_\beta P_\beta^u$$

with $P_\beta^u := \sum_{i=1}^r tr(M_\beta V_i^u \rho^{(\alpha)} V_i^{u\dagger})$. Then taking the first and the second derivative with respect to u we obtain:

$$\begin{aligned}
\frac{dP_\beta^u}{du} &= \sum_{i=1}^r tr\left(M_\beta \frac{dV_i^u}{du} \rho^{(\alpha)} V_i^{u\dagger}\right) + tr\left(M_\beta V_i^u \rho^{(\alpha)} \frac{dV_i^{u\dagger}}{du}\right) \\
\frac{d^2 P_\beta^u}{du^2} &= \sum_{i=1}^r tr\left(M_\beta \frac{d^2 V_i^u}{du^2} \rho^{(\alpha)} V_i^{u\dagger}\right) + 2tr\left(M_\beta \frac{dV_i^u}{du} \rho^{(\alpha)} \frac{dV_i^{u\dagger}}{du}\right) + tr\left(M_\beta V_i^u \rho^{(\alpha)} \frac{d^2 V_i^{u\dagger}}{du^2}\right)
\end{aligned}$$

with

$$\frac{dV_i^u}{du} = -iHe^{-iuH} V_i, \quad \frac{d^2 V_i^u}{du^2} = -H^2 e^{-iuH} V_i$$

Evaluating the first and the second derivative in zero and using the following property

$$M_\beta V_i \rho^{(\alpha)} V_i^\dagger = \delta_{\alpha,\beta} V_i \rho^{(\alpha)} V_i^\dagger$$

that derives by the fact that $supp(V_i \rho^{(\alpha)} V_i^\dagger) \subset \mathcal{V}_\alpha$, we get:

$$\begin{aligned}
\left.\frac{dP_\beta^u}{du}\right|_{u=0} &= \sum_i -itr(M_\beta H V_i \rho^{(\alpha)} V_i^\dagger) + itr(M_\beta V_i \rho^{(\alpha)} V_i^\dagger H) \\
&= \sum_i -i\delta_{\alpha,\beta} tr(H V_i \rho^{(\alpha)} V_i^\dagger) + i\delta_{\alpha,\beta} tr(V_i \rho^{(\alpha)} V_i^\dagger H) = 0 \qquad (5.11) \\
\left.\frac{d^2 P_\beta^u}{du^2}\right|_{u=0} &= 2\sum_{i=1}^r -\delta_{\alpha,\beta} tr\left(H^2 V_i \rho^{(\alpha)} V_i^\dagger\right) + tr\left(M_\beta H V_i \rho^{(\alpha)} V_i^\dagger H\right) \\
&= -2\delta_{\alpha,\beta} tr\left(H^2 \phi(\rho^{(\alpha)})\right) + 2tr\left(M_\beta H \phi(\rho^{(\alpha)}) H\right) =: R_{\alpha,\beta} \qquad (5.12)
\end{aligned}$$

from which follows that

$$\begin{aligned}
\left.\frac{df_0(u, \rho^{(\alpha)})}{du}\right|_{u=0} &= 0 \quad \forall \alpha \\
\left.\frac{d^2 f_0(u, \rho^{(\alpha)})}{du^2}\right|_{u=0} &= \sum_{\beta=1}^d \sigma_\beta R_{\alpha,\beta}
\end{aligned}$$

Let us move to the second term:

$$f_\Gamma(u,\rho^{(\alpha)}) = \frac{1}{2}\sum_i\sum_\beta \frac{tr(M_\beta V_i^u \rho^{(\alpha)} V_i^{u\dagger})^2}{tr(V_i\rho^{(\alpha)}V_i^\dagger)} = \frac{1}{2}\sum_i\sum_\beta \frac{(P_\beta^u(i))^2}{tr(V_i\rho^{(\alpha)}V_i^\dagger)}$$

with $P_\beta^u(i) := tr(M_\beta V_i^u \rho^{(\alpha)} V_i^{u\dagger})$. Then from eq.s (5.11) and (5.12) follows that

$$\begin{aligned}
\frac{df_\Gamma(u,\rho^{(\alpha)})}{du}\bigg|_{u=0} &= \sum_i\sum_\beta \frac{P_\beta^u(i)}{tr(V_i\rho^{(\alpha)}V_i^\dagger)}\frac{dP_\beta^u(i)}{du}\bigg|_{u=0} \\
&= \sum_i\sum_\beta \frac{P_\beta^0(i)}{tr(V_i\rho^{(\alpha)}V_i^\dagger)}(-i\delta_{\alpha,\beta}tr(HV_i\rho^{(\alpha)}V_i^\dagger) + i\delta_{\alpha,\beta}tr(V_i\rho^{(\alpha)}V_i^\dagger H)) \\
&= 0 \quad \forall\alpha \\
\frac{d^2f_\Gamma(u,\rho^{(\alpha)})}{du^2}\bigg|_{u=0} &= \sum_i\sum_\beta \frac{1}{tr(V_i\rho^{(\alpha)}V_i^\dagger)}\left(\frac{dP_\beta^u(i)}{du}\frac{dP_\beta^u(i)}{du} + P_\beta^u(i)\frac{d^2P_\beta^u(i)}{du^2}\right)\bigg|_{u=0} \\
&= \sum_i\sum_\beta \frac{P_\beta^u(i)}{tr(V_i\rho^{(\alpha)}V_i^\dagger)}\frac{d^2P_\beta^u(i)}{du^2}\bigg|_{u=0} \\
&= \sum_i\sum_\beta \frac{\delta_{\alpha,\beta}tr(V_i\rho^{(\alpha)}V_i^\dagger)}{tr(V_i\rho^{(\alpha)}V_i^\dagger)}\frac{d^2P_\beta^u(i)}{du^2}\bigg|_{u=0} \\
&= 2\sum_i -tr\left(H^2V_i\rho^{(\alpha)}V_i^\dagger\right) + tr\left(M_\alpha HV_i\rho^{(\alpha)}V_i^\dagger H\right) = R_{\alpha,\alpha}
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\frac{df(u,\rho^{(\alpha)})}{du}\bigg|_{u=0} &= 0 \quad \forall\alpha \\
\frac{d^2f(u,\rho^{(\alpha)})}{du^2}\bigg|_{u=0} &= \sum_{\beta=1}^d \sigma_\beta R_{\alpha,\beta} - \varepsilon R_{\alpha,\alpha}
\end{aligned}$$

which proves that the first derivative vanishes in $u = 0$.

The following step will be to design the weights σ_β in such a way that the second derivative, evaluated in $u = 0$, is positive for $\alpha = \alpha^*$ and negative for $\alpha \neq \alpha^*$.

5.2 Graph theory for weights design

Let G^H be the directed graph associated to the Hamiltonian H defining the controlled unitary evolution $U_u = e^{-iuH}$. Its adjacency matrix $P = I - R/tr(R)$ is constructed through the Metzler matrix R (as shown in the following lemma), and having elements

$$P_{\alpha,\alpha} = 1 - \frac{R_{\alpha,\alpha}}{tr(R)}, \quad P_{\alpha,\beta} = -\frac{R_{\alpha,\beta}}{tr(R)} \quad \alpha \neq \beta \tag{5.13}$$

This graph is formed by d vertices labelled by $\alpha \in \{1, ..., d\}$. Given (5.12) two different vertices $\alpha \neq \beta$ are linked by an edge iff $tr\left(M_\beta H\phi(\rho^{(\alpha)})H\right) \neq 0$.

Lemma 29. *Consider the $d \times d$ matrix R with elements defined as*

$$R_{\alpha,\beta} = 2\left(tr\left(M_\beta H\phi(\rho^{(\alpha)})H\right) - \delta_{\alpha,\beta}tr\left(H^2\phi(\rho^{(\alpha)})\right)\right)$$

When $R \neq 0$, the matrix $P = I - R/tr(R)$ is a non-negative, row stochastic matrix, i.e. $P\mathbb{1} = \mathbb{1}$, $\mathbb{1} = (1, ..., 1)^T$.

Proof. For $\alpha \neq \beta$,

$$R_{\alpha,\beta} = 2tr\left(M_\beta H\phi(\rho^{(\alpha)})H\right) \geqslant 0$$

being $M_\beta \geqslant 0$ and $H\phi(\rho^{(\alpha)})H \geqslant 0$. Thus R is a Metzler matrix, namely a matrix having non-negative off-diagonal components. What we need to prove now is that $R\mathbb{1} = 0$, namely $\sum_\beta R_{\alpha,\beta} = 0$. This follows by the fact that

$$\sum_\beta P_\beta^u = \sum_\beta \sum_{i=1}^r tr(M_\beta V_i^u \rho^{(\alpha)} V_i^{u\dagger}) = \sum_{i=1}^r tr(V_i^u \rho^{(\alpha)} V_i^{u\dagger}) = 1$$

being $\sum_\beta M_\beta = I$. Therefore deriving two times the previous expression we get

$$\sum_\beta \frac{d^2 P_\beta^u}{du^2} = 0 \rightarrow \sum_\beta R_{\alpha,\beta} = 0$$

By the fact that the rows of R sums to zero, while its off-diagonal elements are non-negative follows that the diagonal elements of R are non positive. Therefore if $R \neq 0$, then $tr(R) < 0$ and the matrix $P = I - R/tr(R)$ is well defined with non-negative entries. Since the sum of each row of R vanishes, the sum of each row of P is equal to 1. Thus P is a row stochastic matrix. □

The following lemma represents the main result of this chapter, since it proposes a technique for the design of the weights σ_α of our Lyapunov function.

Lemma 30. *Assume that the directed graph G^H associated to the row stochastic matrix P defined in the previous lemma is strongly connected. Then, there exist $d-1$ strictly positive real numbers $e_\alpha > 0$, $\alpha \in \{1, ..., d\}\backslash\{t\}$, such that:*

- *$\forall \lambda_\alpha \in \mathbb{R}$, $\alpha \in \{1, ..., d\}\backslash\{t\}$, there exists a unique vector $\sigma \in \mathbb{R}^d$, with $\sigma_t = 0$ such that $R\sigma = \lambda$, where $\lambda \in \mathbb{R}^d$ with components λ_α and $\lambda_t = -\sum_{\alpha\neq\alpha^*} e_\alpha \lambda_\alpha$.*
  If additionally $\lambda_\alpha < 0$ for all $\alpha \in \{1, ..., d\}\backslash\{t\}$, then $\sigma_\alpha > 0$ for all $\alpha \in \{1, ..., d\}\backslash\{t\}$.
- *for any $\sigma \in \mathbb{R}^d$, solution of $R\sigma = \lambda \in \mathbb{R}^d$, the function $W_0(\rho_n)$ satisfies*

$$\left.\frac{d^2 W_0(\mathbb{K}^u(\rho^{(\alpha)}))}{du^2}\right|_{u=0} = \lambda_\alpha, \ \forall \alpha \in \{1, ..., d\}$$

Proof. Being G^H strongly connected, its associated matrix P is irreducible. Being a row stochastic matrix, its spectral radius is equal to 1. By Perron-Frobenius theorem for non-negative irreducible matrices, this spectral radius is also an eigenvalue of P and of P^T , with multiplicity one and associated to eigenvectors having strictly positive entries: the right eigenvector $\mathbb{1}$, being $P\mathbb{1} = \mathbb{1}$, and the left eigenvector e ($P^T e = e \rightarrow e^T R = 0$). Then

- let $\lambda \in Im(R) \rightarrow e^T\lambda = \sum_{\alpha\neq\alpha^*} e_\alpha\lambda_\alpha + e_t\lambda_t = 0 \rightarrow e$ can be chosen s.t. $e_t = 1 \rightarrow \sum_{\alpha\neq\alpha^*} e_\alpha\lambda_\alpha = -\lambda_t$.
  Therefore there exists σ solution of $R\sigma = \lambda$
- $\ker(R) = span(\mathbb{1})$ and $rank(R) = d-1 \rightarrow$ there exists a unique σ solution of $R\sigma = \lambda$ s.t. $\sigma_t = 0$

The fact that $\sigma_\alpha > \sigma_t = 0$ when $\lambda_\alpha < 0$ for $\alpha \neq \alpha^*$, comes from elementary manipulations of $P\sigma = \sigma - \lambda/tr(R)$ showing that $\min_{\alpha\neq\alpha^*} \sigma_\alpha > \sigma_t$. Finally from eq. 5.12 we have that

$$\left.\frac{d^2 W_0(\mathbb{K}^u(\rho^{(\alpha)}))}{du^2}\right|_{u=0} = \sum_\beta \sigma_\beta \left.\frac{d^2 P_\beta^u}{du^2}\right|_{u=0} = \sum_\beta \sigma_\beta R_{\alpha,\beta} = \lambda_\alpha$$

with $\lambda_\alpha < 0$ for $\alpha \neq \alpha^*$, while $\lambda_t = -\sum_{\alpha\neq\alpha^*} e_\alpha\lambda_\alpha > 0$. □

Therefore if the graph G^H associated with the Hamiltonian H is strongly connected, then we can define the weights σ_α is such a way that $\lambda_\alpha < 0$ for $\alpha \neq \alpha^*$, which brings to

$$\left.\frac{d^2 f(u,\rho^{(\alpha)})}{du^2}\right|_{u=0} = \lambda_\alpha - \varepsilon R_{\alpha,\alpha} \begin{cases} < 0 \text{ for } \alpha \neq \alpha^*, \varepsilon \in \left]0, \min_{\alpha\neq\alpha^*} \frac{\lambda_\alpha}{R_{\alpha,\alpha}}\right] \\ > 0 \text{ for } \alpha = t \end{cases}$$

and finally proves that

1. $f(u,\rho^{(\alpha)})$ has a strict local minimum at $u = 0$ for $\alpha = \alpha^*$;
2. $f(u,\rho^{(\alpha)})$ has a strict local maxima at $u = 0$ for $\alpha \neq \alpha^*$.

5.3 Connectivity of the graph G^H

Given a Hamiltonian H we can easily check if the associated graph G^H is strongly connected using *Tarjan's algorithm* [26], having running time that is linear in the number of nodes and edges in $G^H = (V, E)$, i.e. $O(|V| + |E|)$. On the other hand, another problem regards the necessary and sufficient conditions on the Hamiltonian H for having G^H strongly connected, and consequently for having a working control scheme. We recall that G^H is strongly connected iff its adjacency matrix P satisfies

$$\sum_{k=1}^{d-1} P^k > 0 \tag{5.14}$$

This condition requires every couple of nodes α and β to be linked at least by one path of length $k \in \{1, ..., d-1\}$, since $[P^k]_{\alpha,\beta} > 0$ iff there exists a path of length k that links the two nodes. Finding the necessary and sufficient conditions on H such that condition (5.14) holds is not easy in general, therefore we will start from the case that considers two nodes, i.e. $d = 2$. Afterwards we will slightly generalize that result to the case that considers $d > 2$ nodes, finding a sufficient condition.

Notice that by the definition of the adjacency matrix P (5.13), a node α is linked to another node β iff $R_{\alpha,\beta} \neq 0$. Thus firstly we derive the conditions on H for having $R_{\alpha,\beta} \neq 0$.

We recall that when $\alpha \neq \beta$ $R_{\alpha,\beta} = 2tr\left(M_\beta H \phi(\rho^{(\alpha)}) H\right)$, that is different from zero when

$$tr(\Pi_{\mathcal{V}_\beta} H \phi(\rho^{(\alpha)}) H) + tr(\Pi_{\mathcal{T}} H \phi(\rho^{(\alpha)}) H) \neq 0 \tag{5.15}$$

Moreover being $\mathcal{V}_\alpha$ an invariant subspace, we have that $\phi(\rho^{(\alpha)})$ remains supported only on $\mathcal{V}_\alpha$, and we define $\bar{\varphi}_\alpha$ to be its unique non-zero block (i.e. $\phi(\rho^{(\alpha)}) = \bar{\varphi}_\alpha \oplus 0$). Then we introduce another decomposition of the state space $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_R \oplus \mathcal{V}_\alpha \oplus \mathcal{V}_\beta \oplus \mathcal{T}$$

where $\mathcal{H}_R$ represents the reminder subspace, namely $\mathcal{H}_R = \bigoplus_{\gamma\neq\alpha,\beta} \mathcal{V}_\gamma$. The block structure of H with respect to this decomposition reads

$$H = \begin{bmatrix} H_R & h_{R,\alpha} & h_{R,\beta} & h_{R,T} \\ h_{\alpha,R} & H_\alpha & h_{\alpha,\beta} & h_{\alpha,T} \\ h_{\beta,R} & h_{\beta,\alpha} & H_\beta & h_{\beta,T} \\ h_{T,R} & h_{T,\alpha} & h_{T,\beta} & H_T \end{bmatrix}$$

From this block structure of H and from eq. (5.15) follows that

$$R_{\alpha,\beta} \neq 0 \iff tr(h_{\beta,\alpha} \bar{\varphi}_\alpha h_{\alpha,\beta}) + tr(h_{T,\alpha} \bar{\varphi}_\alpha h_{\alpha,T}) \neq 0 \tag{5.16}$$

where $h_{\alpha,\beta} = h^\dagger_{\beta,\alpha}$ and $h_{\alpha,T} = h^\dagger_{T,\alpha}$ by the hermitianity of H. Let $\{|x\rangle\}_{x=1}^{n_\beta}$ be an orthogonal basis for $\mathcal{V}_\beta$, and $\{|y\rangle\}_{y=1}^{n_t}$ for $\mathcal{T}$. We can rewrite the blocks $h_{\beta,\alpha}$ and $h_{T,\alpha}$ in the following way

$$h_{\beta,\alpha} = \sum_{x=1}^{n_\beta} |x\rangle\langle h_{\beta\alpha,x}|, \quad h_{T,\alpha} = \sum_{y=1}^{n_t} |y\rangle\langle h_{T\alpha,y}|$$

where $\langle h_{\beta\alpha,x}|$ and $\langle h_{T\alpha,y}|$ represent the rows of the blocks $h_{\beta,\alpha}$ and $h_{T,\alpha}$ respectively. We have now all the tools to state and prove the following lemma on the conditions for having $R_{\alpha,\beta} \neq 0$.

Lemma 31. *Let G^H be a graph having d nodes, with adjacency matrix P defined in eq. (5.13), associated to the Hamiltonian H. Then $R_{\alpha,\beta} \neq 0$ iff at least one of the two conditions holds:*

1. *there exists an index $x \in \{1, ..., n_\beta\}$ such that $\langle h_{\beta\alpha,x}|\bar{\varphi}_\alpha|h_{\beta\alpha,x}\rangle \neq 0$;*
2. *there exists an index $y \in \{1, ..., n_t\}$ such that $\langle h_{T\alpha,y}|\bar{\varphi}_\alpha|h_{T\alpha,y}\rangle \neq 0$.*

Proof. Using the fact that $tr(B) = \sum_{x=1}^{n_\beta}\langle x|B|x\rangle$ for any $n_\beta \times n_\beta$ matrix B, and $tr(T) = \sum_{y=1}^{n_t}\langle y|T|y\rangle$ for any $n_t \times n_t$ matrix T, condition (5.16) reads

$$R_{\alpha,\beta} \neq 0 \iff \sum_{x=1}^{n_\beta}\langle h_{\beta\alpha,x}|\bar{\varphi}_\alpha|h_{\beta\alpha,x}\rangle + \sum_{y=1}^{n_t}\langle h_{T\alpha,y}|\bar{\varphi}_\alpha|h_{T\alpha,y}\rangle \neq 0$$

namely $R_{\alpha,\beta} \neq 0$ if at least one term of the above two sums is different from zero, since $\bar{\varphi}_\alpha \geqslant 0$ and consequently the terms of the above two sums are non-negative. This proves the stated lemma. □

Our main result is stated in the following theorem, that highlights which edges $R_{\alpha,\beta}$ have to be different from zero for having the connectivity of the graph. Notice that this condition on the edges $R_{\alpha,\beta}$ is reflected into a condition on the Hamiltonian H by the previous lemma. Therefore this theorem, together with the previous lemma, gives a sufficient condition on the Hamiltonian H for having G^H strongly connected.

Theorem 32. *Let G^H be a graph having d nodes, with adjacency matrix P defined in eq. (5.13), associated to the Hamiltonian H. We have that*

- *when $d = 2$: $G^H = (V, E)$, $V = (\alpha, \beta)$ is strongly connected iff $R_{\alpha,\beta} \neq 0$ and $R_{\beta,\alpha} \neq 0$;*
- *when $d > 2$: if all the nodes of G^H form a cycle, namely there exists a path of length d from a node α' to itself that touches all the other nodes $\beta' \neq \alpha'$, then G^H is strongly connected. This path that forms the cycle is defined by the edges $R_{\alpha,\beta}$ that have to be different from zero.*

This result represents a first step towards the harder problem of finding a necessary and sufficient condition on the Hamiltonian H for having G^H strongly connected. Moreover, once found this mathematical condition on H, it would be interesting to translate it into a physical condition on the Hamiltonian.

5.4 Simulations

Let us turn back to the setting of the simulations of subsection 4.5.1, that considers three minimal invariant subspaces. In open-loop the quantum trajectory of that simulation converged to the subspace $\mathcal{V}_1$. What we will show in this subsection is that if we choose as target subspace $\mathcal{V}_2$, and we close the loop using the feedback scheme presented in this chapter, the state of our quantum system will actually converge to the target subspace $\mathcal{V}_2$.

The unitary control is realized through the fixed Hamiltonian

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

while the resulting Metzler matrix R reads

$$R = \begin{pmatrix} -0.600 & 0.600 & 0 \\ 0.974 & -2.000 & 1.026 \\ 1.046 & 2.607 & -3.653 \end{pmatrix}$$

from which we can observe that the associated graph G^H is strongly connected. Indeed every node is linked by an edge to all the other nodes, except for the link that goes from node 1 to node 3 (i.e. $R_{1,3} = 0$). Therefore by lemma 30, we can design the weights σ_α is such a way that conditions 1 and 2 holds. We have chosen the vector λ to be

$$\lambda = \begin{pmatrix} -0.642 & 1.474 & -0.425 \end{pmatrix}^T$$

and the corresponding solution of the equation $R\sigma = \lambda$ reads

$$\sigma = \begin{pmatrix} 1.069 & 0 & 0.422 \end{pmatrix}^T$$

At the $n = 376$ iteration the quantum trajectory has converged to the subspace $\mathcal{V}_2$. The comparison between the evolution of the martingale $Q_\alpha(n)$ in open-loop and in closed-loop is reported in Figure 5.1. From this plot we can deduce that the feedback scheme is kicking the quantum trajectory away from the "most attractive" subspace $\mathcal{V}_3$, leading the convergence to the target subspace $\mathcal{V}_2$.

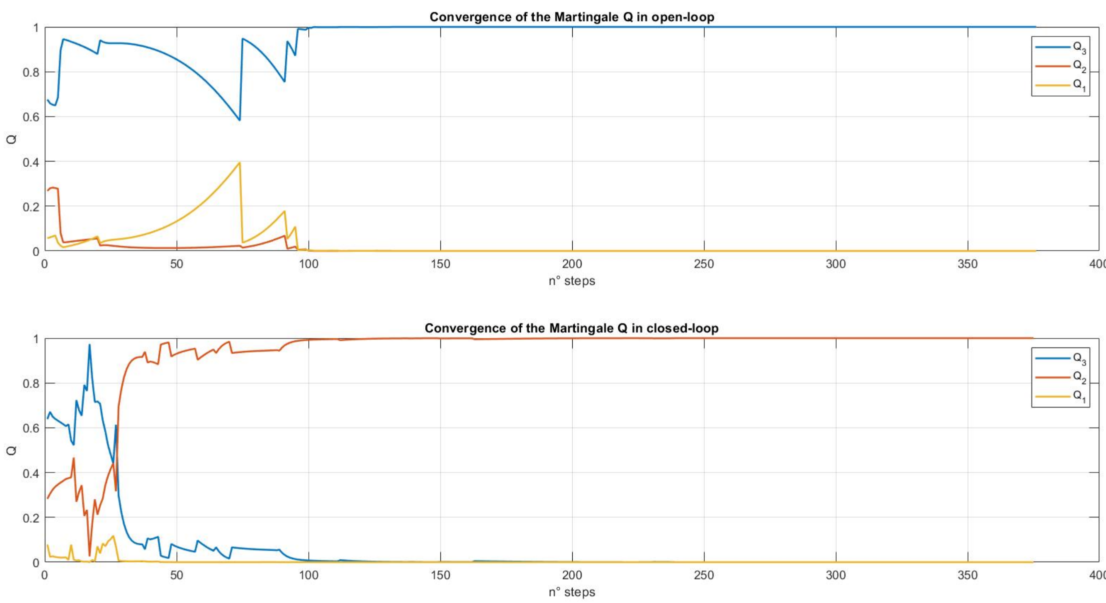

Figure 5.1: Comparison between the evolution of the martingale $Q_\alpha(n)$ in open-loop and in closed-loop

Conclusions

In this project we have analyzed the large time behaviour of a quantum system subject to a sequence of indirect measurements, constructing a special martingale and exploiting the related convergence theorem. The asymptotic value of this martingale tells us in which subspace the quantum trajectory is converged, while its initial value sets the different probabilities of convergence to the various subspaces. Thus this martingale contains all the information regarding the stochastic evolution of the quantum trajectory. The second main topic treated by this project regards the central limit theorem applied to the stochastic process related to the measurement outcomes. What we have showed is that the distribution of that process converges to a mixture of Gaussians, with parameters that depends on the invariant states of the quantum channel. Moreover we have presented how this CLT could be applied for solving a process tomography problem. Finally we have designed a state feedback scheme, which exploits a Lyapunov technique to realize a deterministic convergence of the quantum trajectory to a specific target subspace. Two different simulations demonstrate the applicability of our results.

Therefore our control scheme is able to prepare a quantum system in a precise subspace, but it does not take into account possible measurements imperfections or a non correct initialization of the quantum filter or a delay between the measurement process and the control process. Hence a possible extension of this project goes in the direction of the design of a *robust* control scheme.

For what concerns the rate of convergence to a subspace in open-loop and in closed-loop, it remains to be proved that the convergence is exponential, and to find the relative rate of convergence. This further analysis could lead to the design of a control scheme that maximizes the rate of convergence, which could be useful in devising cooling and state preparation strategies in feedback quantum control.

Appendix

Lemma 33. *Given two matrices $A, B \in \mathbb{C}^{k\times k}$ such that $A \geqslant 0$ and $B = B^\dagger$, having spectral decomposition $B = \sum_i \lambda_i(B)|\psi_i(B)\rangle\langle\psi_i(B)|$ we have that*

$$|tr(AB)| \leqslant \|A\|_\infty \|B\|_1 \tag{17}$$

where we recall that

$$\|A\|_1 := tr(|A|),\ \|A\|_\infty := \max_i \sum_j |a_{ij}| = \max_i \lambda_i(A)$$

Proof. We directly have that

$$\begin{aligned}
|tr(AB)| = |tr\left(\sum_i \lambda_i(B) A |\psi_i(B)\rangle\langle\psi_i(B)|\right)| &= |\sum_i \lambda_i(B)\langle\psi_i(B)|A\psi_i(B)\rangle| \\
&\leqslant \sum_i |\lambda_i(B)||\langle\psi_i(B)|A\psi_i(B)\rangle| \qquad (18)\\
&\leqslant \sum_i |\lambda_i(B)|\|A\|_\infty = tr(|B|)\|A\|_\infty = \|B\|_1\|A\|_\infty
\end{aligned}$$

where (18) holds by the *Cauchy–Schwarz* inequality. ☐

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