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SYK model and the Schwarzian theory

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Contents

Introduction

The SYK (Sachdev-Ye-Kitaev) model was introduced by Kitaev,[5], in a talk at KITP in 2015 and was presented as a possible simple example of quantum holography, namely it was thought to be a quantum mechanical theory encoding, as a boundary theory, the properties of a two dimensional black hole.

The main reason for this conjecture lies in the emergence of a conformal symmetry in the strong coupling limit of the model, whose breaking pattern showed immediately an analogy with Jackiw-Teitelboim dilaton gravity because in both models it is realized by the arising of an action having the form of a Schwarzian derivative of a mode representing a reparametrization of the circle.

This fact relived the interest in the study of theories having the Schwarzian as action, which is an object deeply related to uniformization theory of Riemann surfaces. The aim of the thesis will be to review the role of the Schwarzian in the aforementioned topics, in the attempt of trying to understand if the arising of this peculiar action can be a hint to understand the AdS_2/CFT_1 duality through a geometric approach, something that still has to be done, but in our mind can surely shed some light on the nature of this holographic correspondence.

The topics will be presented as follows:

- In the first chapter we will discuss some basic concepts of conformal field theory, especially the two dimensional one, stressing out its relations with the theory of holomorphic functions.

- In the second one Riemann surfaces will be introduced along with their topological classification and then we will show how the Schwarzian derivative arises in a geometrical way in the framework of uniformization theory.

- The third chapter will be a review of the SYK model focused on the role of the Schwarzian action in its low energy dynamics. During the review we will stress out the properties of the model which led people to think that SYK, or some similar model having the same emergent conformal symmetry, might be the holographic dual of a two dimensional black hole.

- In the fourth and last chapter the relations between the Schwarzian dynamics and different other theories. The relation with Liouville gravity will be discussed in detail, but the last sections will be dedicated to the guessed holographically dual model: Jackiw-Teitelboim dilaton gravity. In particular we will see the Schwarzian action arising in its boundary dynamics and discuss its relation with SYK model.

Chapter 1

Conformal Field Theory and Riemann surfaces

The first feature of SYK model which hinted at the existence of an holographic dual, along with other facts that will be discussed later, was the emergence of a conformal symmetry.

In this chapter we will briefly review what conformal symmetry is and how to treat a theory having a conformal invariance. Conformal theories are of great importance in theoretical physics for several reasons. First of all the scale invariance is a property developed by systems at critical points where phase transitions happen and the system develops long range correlations: scale invariance often generalizes to a full conformal invariance. Moreover 2d conformal field theories have a lot of peculiar mathematical properties and naturally appear related to the reparametrization invariance of the Polyakov action defined on the worldsheet in string theory.

Furthermore a still undefined class of conformal field theories defined on the boundary of AdS spacetimes are believed to encode holographically, through AdS/CFT duality, all information about string theories defined on the curved space-time bulk. A simple realization of a slightly modified form of this duality is precisely what brought a lot of researchers to work on the SYK model.

We believe that uniformization theory of Riemann surfaces, for which we refer to [9], might be the right framework to understand this duality, so a short review of the main features of Riemann surfaces will be discussed in the second half of this chapter.

For the quoted basic CFT results [3], [4] and [15] have been used; while for Riemann surfaces the main source is [14].

1.1 The conformal group

Let's take a (pseudo)Riemannian differentiable manifold M with metric g. In d dimensions the conformal group is defined as the set of transformations of the coordinates, i.e. local diffeomorphisms, which leave the metric invariant up to a rescaling that depends on the points

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{1.1.1}$$

This condition, from a geometric point of view, just says that the angles formed by the integral curves of two vector fields intersecting at the point $P \in M$ is left invariant. This is interesting because one of the immediate consequences is that the causal structure of the manifold, used to model spacetime, remains the same after performing the transformation.

The Poincaré group is trivially a subgroup of the conformal group with conformal factor $\Lambda(x) = 1$.

The equation defining the conformal transformations can be linearized to obtain a local counterpart in term of a conformal generalization of Killing vectors by writing the transformation as $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ and $\Lambda(x) = 1 + f(x)$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{1.1.2}$$

Tracing this equation we obtain

$$\partial^\mu \epsilon_\mu = \frac{d}{2} f(x) \tag{1.1.3}$$

From now on we will assume to work in a d dimensional flat spacetime $\mathbb{R}^{1,d-1}$ with the standard Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. Taking another derivative of the conformal Killing equation we obtain

$$\partial_\alpha \partial_\mu \epsilon_\nu + \partial_\alpha \partial_\nu \epsilon_{mu} = \eta_{\mu\nu} \partial_\alpha f(x) \tag{1.1.4}$$

We now add to this equation the same one but with the indices cyclically permuted, and then subtract again the same equation after taking another cyclic permutation to get

$$2\partial_\mu \partial_\nu \epsilon_\alpha = \eta_{\nu\alpha} \partial_\mu f(x) + \eta_{\alpha\mu} \partial_\nu f(x) - \eta_{\mu\nu} \partial_\alpha f(x) \tag{1.1.5}$$

which can be traced in the μ and ν indices to obtain

$$2\partial^2 \epsilon_\mu = (2-d) \partial_\mu f(x) \tag{1.1.6}$$

Manipulating the obtained equations one gets the last important relation we are going to discuss

$$(d-1) \partial^2 f(x) = 0 \tag{1.1.7}$$

From this equation we immediately understand that in just 1 dimension the form of the infinitesimal scale factor is not constrained and, whatever we choose as f, we can fix the ϵ integrating the previous equation which becomes quite easy. This is related to the fact the obviously in one dimension there is no notion of angle and so every differentiable function is a local reparametrization of the real line.

So diffeomorphism invariance of $\mathbb{R}$, or of any other 1-manifold, is some kind of trivial conformal symmetry. It will be precisely the case of the SYK model.

1.2 Conformal symmetry in two dimensions

In two dimensions conformal transformations have a very special form, in fact if the pair $(x, y) \in \mathbb{R}^2$ represents the coordinates of a point in euclidean plane and $(\tilde{x}(x, y), \tilde{y}(x, y))$ is a generic local diffeomorphism, the conformality condition reduces to the equations

$$\Big\{ (\partial_x \tilde{x})^2 + (\partial_y \tilde{x})^2 = (\partial_x \tilde{y})^2 + (\partial_y \tilde{y})^2 + \quad \partial_x \tilde{x} \partial_x \tilde{y} + \partial_x \tilde{x} \partial_x \tilde{y} = 0 \tag{1.2.1}$$

whose solution is given by

$$\begin{cases} \partial_x \tilde{y} = -\partial_y \tilde{x} \\ \partial_y \tilde{y} = \partial_y \tilde{x} \end{cases} \tag{1.2.2}$$

or equivalently by

$$\begin{cases} \partial_x \tilde{y} = \partial_y \tilde{x} \\ \partial_y \tilde{y} = -\partial_y \tilde{x} \end{cases} \tag{1.2.3}$$

Now defining the complex variable $z = x + iy$ and writing the coordinate transformation as $\tilde{z} = \tilde{x} + i\tilde{y}$ we immediately recognize the Cauchy-Riemann equations for holomorphic or anti-holomorphic functions

$$\partial_{\bar{z}} \tilde{z} = 0 \qquad \partial_z \tilde{z} = 0 \tag{1.2.4}$$

This also suggests that a conformal field theory might be defined on any two dimensional manifold, by glueing different open patches by bi-holomorphic transition functions. It is indeed a very relevant case, from both mathematical and physical point of view, and is the basic idea of Riemann surfaces.

1.3 The Witt and Virasoro algebras

So holomorphic and antiholomorphic functions of the complex variables z and $\bar{z}$ define the local conformal maps of some open subset U of the euclidean plane modeled as the metric space $\mathbb{C}$, provided that the first derivative of the function is different from 0 to guarantee the local invertibility (univalence).

We can just focus on the holomorphic ones and all the results for the antiholomorphic ones will just follow by complex conjugation.

These functions can be represented locally by a Laurent series in the complex variable z. Let's focus on the algebraic structure of this set. The holomorphic vector fields basis generating the local action of one of these transformations on scalar functions is given by

$$L_n = -z^{n+1} \partial_z, \qquad n \in \mathbb{Z}, \qquad L_n \in T_{\mathbb{C}}(\Sigma) \tag{1.3.1}$$

From now on we will focus for simplicity on $\mathbb{C}$ or, if specified, its projective compactification as the Riemann sphere. On a generic surface Σ all these properties might be difficult to extend globally.

The form of the generators can be easily checked by directly computing the action of a conformal map on fields: we take for example a scalar one, i.e. a function, $\Phi(z,\bar{z}) \in \mathcal{C}^{\infty}(\mathbb{C})$ and perform an holomorphic transformation of the coordinates. The latter has a local representation on some open set U as a Laurent series with non null linear coefficient, so we have

$$z' = z + \epsilon(z) = z + \sum_{n=-\infty}^{+\infty} c_n z^{n+1} \tag{1.3.2}$$

and the field transforms according to the usual law:

$$\begin{aligned} \Phi'(z',\bar{z}') &= \Phi(z,\bar{z}) = \Phi(z' - \epsilon(z), \bar{z}' - \bar{\epsilon}(\bar{z})) \\ &= \Phi(z',\bar{z}') - \epsilon(z)\partial_z\Phi - \bar{\epsilon}(\bar{z})\partial_{\bar{z}}\Phi \end{aligned} \tag{1.3.3}$$

Defining the infinitesimal field variation

$$\delta\Phi = \Phi(z',\bar{z}') - \Phi(z,\bar{z}) = \Phi(z',\bar{z}') - \Phi'(z',\bar{z}') \tag{1.3.4}$$

we get:

$$\delta\Phi = -\epsilon(z)\partial\Phi - \bar{\epsilon}(\bar{z})\partial_{\bar{z}}\Phi \tag{1.3.5}$$

So we get two sets of vector fields: one made of the holomorphic vector fields given in (1.3.1) and another one which are the antiholomorphic vector fields and can be obtained by the previous ones just by complex conjugation.

Focusing on just the holomorohic action and treating the fields as derivations of the ring of holomorphic functions $O(\Sigma)$, they satisfy the commutation rules defining the so called Witt algebra $\mathcal{W}$:

$$[L_n, L_m] = (n-m)L_{n+m} \tag{1.3.6}$$

All fields defined in a 2d CFT will belong to different representations of the infinite dimensional symmetry algebra $\mathcal{W} \oplus \bar{\mathcal{W}}$, as any field has to belong to a representation of the symmetry algebra of the theory.

L_0 and $L1$ are globally defined on $\mathbb{C}$ and generate its complex automorphisms.

$$z \to az + b \tag{1.3.7}$$

with a,$b \in \mathbb{C}$. On the Riemann sphere $\mathbb{C} \cup \{\infty\}$ the generators L_{-1}, L_0, L_1 are defined and can be inverted globally and generate the group of fractional transformations $PSL2, \mathbb{C}$, which is precisely the set of all conformal automorphisms of the sphere

$$z \to \frac{az+b}{cz+d} \tag{1.3.8}$$

with a,b,c,$d \in \mathbb{C}$ and $ad - bc = 1$.

To close the section we mention that everything we've done up to know has been considered just at classical level: it can be useful to ask oneself what is the quantum analog.

The quantum conformal algebra is the unique central extension of the Witt algebra and is called Virasoro algebra, which differs from its classical counterpart by a central charge C, such that

$$\begin{aligned} [C, L_n] &= 0 \forall n \in \mathbb{Z} \\ [L_n, L_m] &= (n-m)L_{n+m} + \frac{C}{12}\delta_{n+m,0} \end{aligned} \tag{1.3.9}$$

1.4 Riemann surfaces

Up to now we stressed the importance of studying conformal field theories in two spacetime dimensions, where the natural identification of geometric conformal invariance and the Cauchy-Riemann conditions set the complex plane $\mathbb{C}$ as the natural representation of the euclidean spacetime.

However it was often stressed out that on a generic manifold which locally looked like the complex plane, all the discussed results would be true just locally on a patch: that's because a lot of global properties should depend on the topology and on the chosen atlas.

Riemann surfaces are precisely the manifolds which locally look like $\mathbb{C}$, so they are one complex dimension manifolds.
Especially we will focus on the compact ones, which are important in theoretical physics, especially in string theory, and will introduce uniformization theory

Compact Riemann surfaces are first of all compact topological spaces, so it is natural to ask oneself what classifies compact two dimensional topological manifolds.

So we start from a topological space Σ with topology $\mathcal{T}$ and we build an atlas choosing a family of open subsets $U_i \subseteq \Sigma$ such that $\bigcup_i U_i = \Sigma$ and to each one of them we associate a homeomorphism φ_i with an open set $V_i \subset \mathbb{R}^2$.
The existence of such an atlas allows us to associate a dimension to the topological space, in this case it is just 2.

For now we just ask that the transition functions are homeomorphisms, but of course we are going to build on it a conformal (or holomorphic) structure. This immediately constrains the topology to just orientable surfaces.

That's because if we start from the volume form, written in complex coordinates, in a patch related to the open set U_1 we have $dz \wedge d\bar{z}$, if we move to a different patch related to U_2, in the intersection $U_1 \cap U_2$ we have that the coordinates are related by $z = f(w)$ with f holomorphic.

$$dz \wedge d\bar{z} = f'(w)\bar{f}'(w) dw \wedge d\bar{w} = \left|f'(w)\right|^2 dw \wedge d\bar{w} \tag{1.4.1}$$

which means that moving along the surface there is no way to reverse the orientation of the volume form: namely the manifold must be orientable.

This immediately eliminates from our discussions the Moebius strip, the Klein bottle and the projective plane, which are compact non orientable topological spaces.

So we the topological classification is left to the Euler characteristic, which is defined as:

$$\chi(\Sigma) = \sum_{i=0}^{2} (-1)^i b_i \tag{1.4.2}$$

where b_i is the i-th Betti number of the manifold, namely the dimension as vector space of the i-th homology module $H^i(\Sigma, \mathbb{Z})$.

The coefficients could also be taken in $\mathbb{R}$, making the module a vector space: that's because we are not interested in torsion which does not influence the Betti numbers. However torsion will be trivial in our case because we got rid of the non-orientable surfaces for which the homology modules $H^i(\Sigma, \mathbb{Z})$ can contain discrete groups (i.e. torsion elements) as Z_2.

For compact orientable surfaces without boundary it is known, by Poincaré duality, that $b_0 = b_2$. b_0 is trivial to determine because it just measures the number of connected components of the manifold and restricting to just connected surfaces, from which any non connected one can be built by disjoint union, we have that $b_0 = 1$. This means that

$$\chi(\Sigma) = 2 - b_1 \tag{1.4.3}$$

Our last unknown variable is b_1.

$H^1(\Sigma, \mathbb{Z})$ is the abelianization of the fundamental group $\pi_1(\Sigma)$, so it is isomorphic to the direct sum of a number of $\mathbb{Z}$ which equals the number of generators of π_1. This leads to $b_1 = 2h$ where $h \in \mathbb{N}$ is called genus and represents the number of holes.

This result is easy to visualize because for every hole in the surface, we have two homotopically inequivalent ways of looping around it, this is trivial to understand in the case of the torus: $T = \mathcal{S}^1 \times \mathcal{S}^1$ and in this case the two independent homology cycles just represent the loops around the two circles.

For example:

- $\mathcal{S}^2$ is simply connected, so π_1 is trivial and the genus is zero;
- for a torus T^1 it is 1
- a generic compact orientable surface can be obtained as a connected sum of h tori: its genus is precisely h.

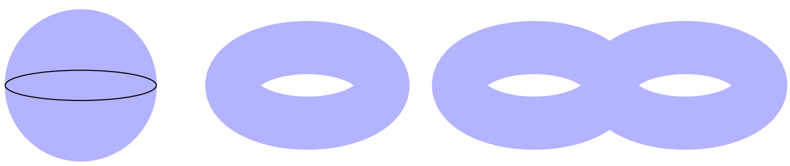


Figure 1.1: From left to right: examples of genus 0, 1 and 2 surfaces

So the Euler characteristic is just a function of the genus, which completely classifies this surfaces

$$\chi(\Sigma) = 2 - 2h \tag{1.4.4}$$

We are now ready to define the Riemann surface structure

Theorem 1. *A Riemann surface is a 2d topological space $(\Sigma, \mathcal{T})$ with manifold structure and charts taking values in the complex numbers $\varphi_i : \Sigma \supset U_i \to \mathbb{C}$, such that $\forall i \neq j$ the transition functions $\phi_i^{-1} \circ \phi_j$ are bi-holomorphic maps.*

Bi-holomorphic maps are used in general to identify the topology and conformal structures, that's because, from a geometric point of view, bi-holomorphic functions are just invertible and differentiable maps preserving the angle between two curves.

1.5 The uniformization theorem

One of the most important results classifying Riemann surfaces from the topological point of view is surely the uniformization theorem, but its usefulness goes beyond: it allows us to build complex Riemannian structures

Theorem 2 (Uniformization theorem 1)**.** *Every Riemann surface Σ admits a universal covering, called* ***uniformization map*** *and denoted by $J_\Sigma : S \to \Sigma$, such that S is of the three simply connected Riemmann surfaces: the complex plane $\mathbb{C}$, the Riemann sphere $\mathcal{S}^2$ or the upper-half plane $\mathbb{H}$.*

Every time we will consider the upper-half plane, it might be substituted by the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. In fact these two Riemann surfaces are conformally equivalent through the Cayley map $C : \mathbb{H} \to \Delta$, which will be useful when we will have to move from 0 temperature to the finite temperature description of the SYK model.

$$C(z) = \frac{i - z}{i + z}, \qquad z \in \mathbb{H} \tag{1.5.1}$$

Moreover, restricting to compact Riemann surfaces, classified by the number h (or equivalently $\chi(\Sigma)$), we can give a deeper statement

Theorem 3 (Uniformization theorem 2)**.** *A compact Riemann surface Σ admits as universal covering the sphere $\mathcal{S}^2$ if and only if its genus is $h = 0$, the complex plane $\mathbb{C}$ if and only if $h = 1$ ($\chi(\Sigma) = 0$), the upper-half plane $\mathbb{H}$ if and only if $h > 1$.*
Furthermore Σ is conformally equivalent (or bi-holomorphic) to the quotient $S \backslash \Gamma$, with Γ a discrete and freely acting subgroup of the conformal automorphisms of S such that $\Gamma \cong \pi_1(\Sigma)$.

The first fact is trivial because the sphere is simply connected, so of course admits itself as universal covering, the second one is easy because, as said before, the torus is just the product of two circles whose universal covering is precisely the plane. The third result is much more subtle and has an important geometric consequence because it allows us to transport the Poincaré hyperbolic model on $\mathbb{H}$.

Theorem 4. *Every compact Riemann surface with $h < 1$ admits a unique Riemannian metric g compatible with the complex structure such that the Ricci scalar curvature is constant $R_g = -1$*

This is quite impressive because from the uniformization theorem we have obtained that higher genus Riemann surfaces admit a natural hyperbolic geometry as a consequence of the fact that they can be represented inside the upper-half plane and so they locally look like an hyperbolic maximally symmetric space.

It is a well known fact that, through uniformization theory, $2d$ manifolds can be equipped with a geometric structure inherited from their universal covering. This geometry is highly constrained, more details about this topic will be given in the last chapter when we will discuss two dimensional gravity.

In the $h > 1$ case the group of the automorphisms of $\mathbb{H}$ is $PSL(2, \mathbb{R})$. This means that $\Gamma \subset PSL(2, \mathbb{R})$: these groups are called Fuchsian groups. They can be classified by studying the fixed point equation for real linear fractional transformations

$$w \in H, \gamma \in \Gamma \qquad \gamma \cdot w = \frac{aw + b}{cw + d} \in H \tag{1.5.2}$$

The fixed point equation reads:

$$w_{\pm} = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}, \tag{1.5.3}$$

so the elements of a Fuchsian group $\gamma \neq I$ can be classified according to the value of $|\operatorname{Tr} \gamma| = a + d$.

Furthermore this theorem tells us that the Liouville equation, which is equivalent to the constant negative curvature condition, has a unique solution.
In fact the Ricci scalar can be written as a function of the only relevant complex component of the metric written in conformal gauge $g^{z\bar{z}} = 2\mathrm{e}^{-\sigma(z,\bar{z})}$. The expression is

$$R_g = -g^{z\bar{z}} \partial_z \partial_{\bar{z}} \log g_{z\bar{z}} \tag{1.5.4}$$

and then the condition $R_g = -1$ is equivalent to the Liouville equation:

$$\partial_z \partial_{\bar{z}} \sigma(z, \bar{z}) = \frac{1}{2} \mathrm{e}^{\sigma(z,\bar{z})} \tag{1.5.5}$$

A metric of generic negative curvature $-\mu$ is then related to a field $\tilde{\sigma} = \sigma + \log \mu$.

The Poincaré metric on the upper-half plane is

$$ds^2 = \frac{|dw|^2}{(\operatorname{Im} w)^2} \tag{1.5.6}$$

The universal covering cannot be inverted because it's not one to one, but we can still define the polymorphic inverse of the uniformization map to write a general expression for the Poincaré metric on a generic Riemann surface as

$$ds_\sigma^2 = e^{\sigma(z,\bar{z})}|dz|^2 = \frac{|J_H^{-1}(z)'|^2|dz|^2}{(\mathrm{Im}\, J_H^{-1}(z))^2} \tag{1.5.7}$$

The function $J_H^{-1} : \Sigma \to \mathbb{H}$ is holomorphic on the whole compact surface, this means that it has a nontrivial monodromy, in particular a non abelian monodromy, if one loops around an homology cycle of the surface, otherwise by Liouville's theorem it would just be a constant.

After winding around a non trivial cycle the point $J_H^{-1}(z) \in \mathbb{H}$ moves from a representative of the point of the surface to another representative of the same point in $\mathbb{H}$.

$$J_H^{-1}(z) \to \gamma \cdot J_H^{-1}(z) = \frac{aJ_H^{-1}(z) + b}{cJ_H^{-1}(z) + d} \tag{1.5.8}$$

where $\gamma \in \Gamma$.
Actually the inverse of the uniformization map is a holomorphic covering, so it is locally univalent, which means that even the derivative $J_H^{-1}(z)$ has no zeroes and no poles.

1.6 The Schwarzian and the uniformization equation

The non abelian monodromy of the inverse uniformization functions can be eliminated through the Schwarzian uniformization equation, which relates $J_H^{-1}(z)$ to the σ function appearing in the Liouville equation through the Liouville stress-energy tensor$T^F(z)$:

$$\{J_H^{-1}(z), z\} = T^F(z) = \partial_z\partial_z\sigma - \frac{1}{2}\partial_z\sigma^2 \tag{1.6.1}$$

where the expression on the left hand side is called *Schwarzian derivative* and will be the main object on which we will focus during the thesis, because is precisely the form of the action which relates SYK model to two dimensional dilaton gravity.

This derivative is defined as

$$\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right) = -2(f')^{\frac{1}{2}}((f')^{-\frac{1}{2}})'' \tag{1.6.2}$$

Solving the Schwarzian equation allows one to find the uniformization map up to $PSL(2,\mathbb{C})$ transformations.

One of the important properties of the Schwarzian equation is that it can be linearized. In fact defining the covariant Schwarzian operator

$$\mathcal{S}_f^{(2)} = 2(f')^{\frac{1}{2}}\partial_z(f')^{-1}\partial_z(f')^{\frac{1}{2}} \tag{1.6.3}$$

which acts as

$$\mathcal{S}_f^{(2)} \cdot \psi = \left(2\partial_z^2 + \{f, z\}\right)\psi \tag{1.6.4}$$

one can consider the equation

$$\mathcal{S}_f^{(2)} \cdot \psi = 0 \tag{1.6.5}$$

which one can immediately check that is solved by two linearly independent objects

$$\psi_1 = k_1 (f')^{-\frac{1}{2}}, \qquad \psi_2 = k_2 f (f')^{-\frac{1}{2}} \tag{1.6.6}$$

where k_1 and k_2 admit a $\bar{z}$ dependence. Of course these two functions are the two linearly independent solutions of the left hand side of (1.6.4) and, if the Schwarzian of the function is known, $\frac{\psi_2}{\psi_1} \propto f$ solves the Schwarzian equation.

This means that we can find the inverse map by solving the *uniformization equation*:

$$\left(\partial_z^2 + \frac{1}{2} T^F(z)\right) \psi(z) = 0 \tag{1.6.7}$$

for a given stress-energy tensor. Then the inverse uniformization function, of course up to $PSL(2, \mathbb{C})$ transformations, is

$$J_H^{-1} = \frac{\psi_2}{\psi_1} \tag{1.6.8}$$

Chapter 2

The SYK model

In this chapter we will investigate the recently introduced Sachdev-Ye-Kitaev (briefly SYK) model, whose properties have captured the attention of theoretical physicists from different research areas, as said in the introduction.

In this chapter we will review the main properties of the model and will try to understand as the Schwarzian emerges in its IR dynamics and why this behavior hints at the possibility that SYK, or some similar model, might arise as the holographic dual of a 2d gravity theory.

To be more precise we observe that this atypical Schwarzian acrion slightly breaks an emergent conformal symmetry of the SYK model through a pattern also appearing in the dilaton gravity model, however only the effective field theory approach will be satisfying and will allow us to suppose that the Schwarzian theory dynamics is a universal feature of this breaking pattern.

MAin references will be: [7], [12], [13].

2.1 General Properties

The SYK model is a quantum mechanical (QFT in d=0+1) model for a many body theory involving N Majorana fermions interacting through an all to all quartic vertex. It can be described by means of the Lagrangian

$$L = -\frac{1}{2}\sum_{i=1}^{N} \chi_i \partial_\tau \chi_i - \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l \tag{2.1.1}$$

The $\{\chi^i\}$ fields have to be treated as real Grassmann functions of the euclidean (or imaginary) time if one wants to study the model in the path integral formulation, while in the canonical quantization formalism they become operators acting on a suitable Hilbert space satisfying the Clifford algebra

$$\{\chi_i, \chi_j\} = \delta_{ij} \tag{2.1.2}$$

The anticommutation rules just come from noticing that, by looking at L as a classical Grassmann Lagrangian, every field is proportional to its conjugate momentum: the Poisson brackets can be evaluated and fix the quantum relations (2.1.2).

SYK model has **quenched disorder**: the introduced coupling constants are time independent uncorrelated **Gaussian random variables** with zero mean ($\overline{J_{ijkl}} = 0$) and variance given by

$$\overline{J_{ijkl}J_{mnpq}} = 3!\frac{J^2}{N^3}\delta_{im}\delta_{jn}\delta_{kp}\delta_{lq} \tag{2.1.3}$$

therefore the larger is the number of fermion sites, the sharper the Gaussian functions describing the couplings will be.
Furthermore the anticommutation of the fermions belonging to different sites imposes that the J_{ijkl} must be totally antisymmetric.

The Majorana field is dimensionless, so the random coupling carries the dimension of a mass as any other possible coupling. This means that this theory is subject to a *relevant* interaction and its action will be important at low energies, namely the model has a sort of asymptotic freedom: it will be strongly coupled in the IR and free in the UV.
Therefore in the UV the theory will be the one given by the Lagrangian of a free real fermion

$$\begin{aligned} S_{UV} = \int d\tau L_{UV} = \int d\tau \frac{1}{2}\sum_{i=1}^{N} \chi_i \partial_\tau \chi_i = \\ = \int d\omega \sum_{k=1}^{N} \hat{\chi}_k\left(\omega\right) i\omega \hat{\chi}_k\left(-\omega\right) \end{aligned} \tag{2.1.4}$$

We can read off the propagator in the frequency domain from the Lagrangian by eye

$$G_{UV ij}(\omega) = -\frac{1}{i\omega}\delta_{ij} \tag{2.1.5}$$

The time domain two point function is given by its Fourier transform, just remembering to treat it as the principal value distribution

$$G_{UV ij}(\tau,\tau') = \frac{1}{2}\,\mathrm{sign}(\tau-\tau')\delta_{ij} = G_0(\tau,\tau')\delta_{ij} = \; i \underline{\qquad\qquad} j \tag{2.1.6}$$

Before moving forward in analyzing the model we stress out that this $1d$ fermionic propagator in frequency representation if convoluted with an arbitrary function $f(\omega)$ gives back its Hilbert transform $\mathcal{H}(f)$, namely its harmonic conjugate: this means that the function $f(\omega) + i\mathcal{H}(f)$ admits an holomorphic extension from the real line to the upper-half plane $\mathbb{H}$. This fact has not found any concrete application yet, but it may be related to the possibility of finding informations about the dual theory starting from the boundary.

If we want to obtain any physical information about the interacting system we have to keep in mind that the SYK model is a whole ensemble of quantum systems depending on the possible different realizations of the values of the Gaussian coupling: we have to take into account the randomness.

Of course, one could consider a specific representation of the Clifford algebra for fixed N which is made of square matrices acting on a finite dimensional Hilbert space isomorphic to $\mathbb{C}^{\frac{N}{2}}$. The fermion product in the interaction becomes in this way a product of four Clifford generating matrices.
When $N = 2K$, $K \in \mathbb{N}$, an explicit recursion relation to build an explicit hermitian representation using Pauli matrices as building blocks, unique up to unitary transformations, can be

$$\begin{aligned} \chi_i^{(K)} &= \chi_i^{K-1} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \chi_{N-1}^{(K)} &= \frac{1}{\sqrt{2}} \mathbb{1}_{2^{K-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \chi_N^{(K)} &= \frac{1}{\sqrt{2}} \mathbb{1}_{2^{K-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned} \tag{2.1.7}$$

for $i = 1, .., N-2$. In the odd N case one can just introduce the analogous of γ_5 through the product of all the Clifford generators.

Then the quantum dynamics is studied by diagonalizing the Hamiltonian operator $\hat{H}$ for a specific choice of the couplings and then by looking at the ensemble properties of the system by direct computation: a work much more suited for numerical simulations. The questioned Hamiltonian is easily obtained by Legendre transforming L

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl} \chi_i \chi_j \chi_k \chi_l \tag{2.1.8}$$

Here we will get rid of the randomness by averaging over the quenched disorder J_{ijkl}.

The reason why this model has inspired a lot of work in the high energy community is its strong coupling regime, which was presented as a simple candidate in which one should fully understand the Maldacena AdS/CFT duality due to the following main properties:

- **Solvability at strong coupling**, namely the possibility of easily sum all Feynman diagrams at large N;
- **Emergent conformal symmetry**, which suggests the existence of holographic dual;
- **Maximal chaos**, quantified by a Lyapunov exponent which can be extracted by the study of an out-of-time-order four point correlator.

These three properties, as will be largely explained in the following sections, place the model in a quite peculiar class. Firstly there are not many nontrivial models which can be solved at strong coupling, additionally we know that the classical solvability is understood as integrability, which rules out the possibility of finding chaotic behaviors: SYK model provides an example of how, at quantum level, there is no such a restriction. The quantum chaos is maximal in the sense described by Shenker and Stanford: the Lyapunov exponent of the model is precisely the one expected for a black hole: this fact, combined with the emergent conformal symmetry, has therefore been interpreted from the beginning as the possible boundary feature of a 2d bulk gravity theory.

All this and much more will be clear in the next sections: the first thing we will deal with, in analogy with the t'Hooft limit of matrix models, is the large N behavior of the theory.

Before moving on we just quote two interesting remarks:

- the Gaussian nature of the couplings is not important for the general properties of the model, because, as se will see, they are related to the dependence of the coupling probability distribution on the averaged coupling J and on N: the Gaussian is just the distribution which has the easiest momenta structure;

- as noticed by Witten, if we want to think about the AdS/CFT duality as it was formulated, the boundary theory we want to find should be a true quantum system, while the holographic behavior of SYK just arises once we average over all possible realizations of the system. However other tensor models, as for example the Gurau-Witten one, exhibit very similar properties avoiding the introduction of the disorder, right now the SYK model is still important because it was the first studied example and so is the most understood one;

- All the properties of the model can be generalized for an interaction with a more general q-fermions interactions: a lot of interesting physical properties of the model can be computed for $q = 2$ or setting up an expansion in $\frac{1}{q}$ as $q \to \infty$, but here we will focus on the first version of the model which was firstly proposed by Kitaev in which $q = 4$.

2.2 Large N expansion

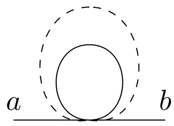
The most important feature of the disordered average is that the N-dependent variance allows us to evaluate a perturbative Feynman diagrams expansion in the small parameter $\frac{1}{N}$, and it is precisely in this thermodynamic limit that the theory exhibits the gravitational features.

A good starting point is the study of the full two point function of the model by formally expanding the exponential of the interaction Lagrangian

$$
\begin{aligned}
G_{a,b}(\tau_1,\tau_2) &= \langle \mathrm{T}(\chi_a(\tau_1)\chi_b(\tau_2))\rangle \\
&= \int \mathcal{D}\chi e^{-\int d\tau (L_{UV}+L_I)}\chi_a(\tau_1)\chi_b(\tau_2) \\
&= \int \mathcal{D}\chi e^{-\int d\tau L_{UV}}\chi_a(\tau_1)\chi_b(\tau_2)\left[1-\int d\tau L_I+\frac{1}{2!}\left(\int d\tau L_I\right)^2+...\right]
\end{aligned}
\tag{2.2.1}
$$

The first term in the sum is trivially the free propagator. The second one becomes zero after the disordered average because the Gaussian coupling has zero mean value. Explicitly, implying from now the summation over repeated indices, we have

$$
-\int \mathcal{D}\chi \mathrm{e}^{-\int L_{UV}d\tau}\chi_a(\tau_1)\chi_b(\tau_2)\frac{\overline{J_{ijkl}}}{4!}\int \chi_i\chi_j\chi_k\chi_l d\tau \tag{2.2.2}
$$

$=$  $= 0$

Where diagrammatically the disordered average is indicated by connecting the vertices through the dotted line

- - - - - - - - - - - - (2.2.3)

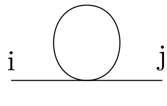
It's not necessary to specify the time indices on diagrams because they can just correspond to an integral over time domain for every vertex, otherwise they label the external legs, which can be identified just by the flavor index. We will omit them in the next graphics.

But the diagram was zero even before the disordered average. This can be seen noticing that, by Wick's theorem, the number of external legs always forces us to contract two Majorana fields inside the integral: $\underbrace{\chi_i\chi_j} \propto \delta_{ij}$. The Kronecker delta contracted with the fully antisymmetric coupling gives zero.

By (1.6) we've learned that:

- The euclidean time space vertex rule is $-\frac{J_{ijkl}}{4!}$;
- A disordered average must be performed contracting different vertices to give a non zero result, as a consequence a non zero disordered average must involve an even number of vertices;
- Diagrams containing fermion tadpoles are null even before the disordered average, i.e. fermions are always contracted from different vertices

$$\underbrace{\chi_k\chi_l} \propto \delta_{kl}$$

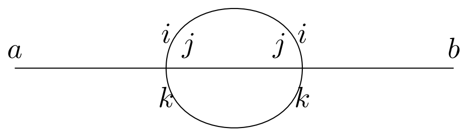
 $= 0$ (2.2.4)

By $\langle - \rangle_0$ we just indicate the free **quantum average** to simplify the notation and not write again the functional integral.
The first interesting contribution comes from the second order term of the expansion, namely

$$\begin{aligned}\frac{1}{2}\int d\tau d\tau' \Big\langle \chi_a(t_1)\chi_b(t_2)\frac{J_{jklm}}{4!}\chi_j(\tau)\chi_k(\tau)\chi_l(\tau)\chi_m(\tau)\times \\ \times\frac{J_{npqr}}{4!}\chi_n(\tau')\chi_p(\tau')\chi_q(\tau')\chi_r(\tau')\Big\rangle_0\end{aligned} \tag{2.2.5}$$

This term contains a sum over all possible contractions of the fermions, but we will not consider the ones containing vacuum diagrams (bubbles) because they are ruled out by the LSZ (Lehmann, Symanzik, Zimmermann) normalization and are not physically relevant in correlators. In other words their removal can be seen as an effect of dividing the second line of (2.2.1) by the partition function.

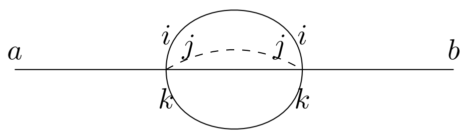
The only interesting diagram at second order in the coupling, according to the previous remarks, is



The permutation of the two vertices cancels the $\frac{1}{2}$ factor in the Taylor expansion, then we have a topological factor of $4^2 \cdot 3!$ coming from all possible contractions of the fermions giving rise to this diagram: 4 ways to attach each external leg to the two vertices and 3! ways to contract the remaining propagators to form the double loop.
The result is so

$$\frac{1}{3!}J_{aijk}J_{bijk}\int d\tau \int d\tau' G_{UV}(\tau_1,\tau)\left(G_{UV}(\tau,\tau')\right)^3 G_{UV}(\tau',\tau_2) \tag{2.2.6}$$

Repeated indices are again summed over. Now, the only non zero disordered average contraction links the two vertices



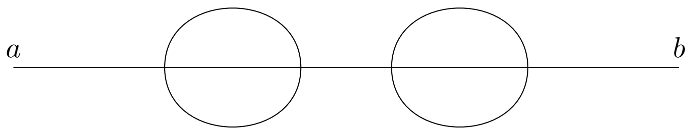
The average operation acts only on the coupling dependent part of the evaluated contribution giving

$$\overline{J_{aijk}J_{bijk}} = \delta_{ab}\delta_{ii}\delta_{jj}\delta_{kk}3!\frac{J^2}{N^3} = \delta_{ab}N^3 3!\frac{J^2}{N^3} = 3!J^2 \tag{2.2.7}$$

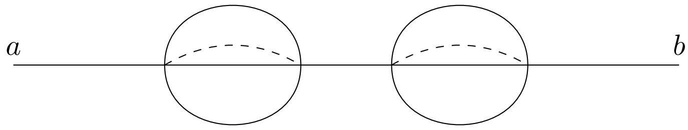
The important thing to notice is that the N dependence coming from the coupling variance has disappeared due to the delta traces coming from the three internal legs: at large N this contribution will dominate.
The whole averaged diagram results in

$$J^2 \int d\tau \int d\tau' G_{UV}(\tau_1, \tau) \left(G_{UV}(\tau, \tau')\right)^3 G_{UV}(\tau', \tau_2) \tag{2.2.8}$$

It's now natural to try to understand how the N dependence behaves for different disordered contractions.
At third order in J we expect a zero mean, but at fourth order we immediately find a 1PR graph which provides a good example

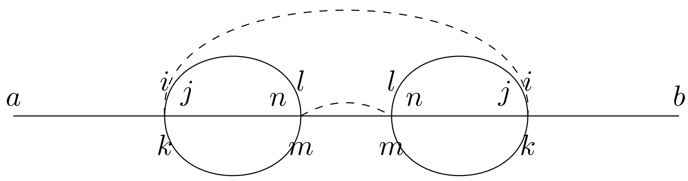


Here the disordered average can be performed in two topologically inequivalent ways. The first one is just the matrix product of two copies of the previous result connected through a free propagator, so it will be at the same order in $\frac{1}{N}$



The argument can be iterated to include the whole 1PR series generated by the diagram in the leading order large N expansion.

The second contraction pattern is more tricky and actually gives rise to a term which is subleading



To focus immediately on the N power present in the graph we can just apply a few rules, which can be deduced from the previous calculations and translated in a diagrammatic language as follows:

- Every fermion line appearing in the graph corresponds to a Kronecker delta of the indices labelling the points it connects;

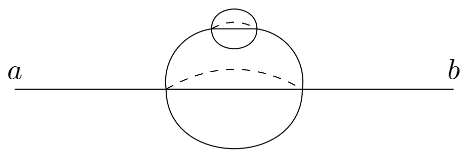
- Every vertex carries the coupling J_{ijkl} and four different indices which are all summed over, the external indices a and b are not summed;
- Every disordered average transforms products of couplings in products of Kronecker deltas, identifying the indices in the vertices it connects, as the previous diagrams show (of course it also brings a $\frac{J^2}{N^3}$ factor for every dotted line).

Applying these rules we immediately obtain

$$\left(\frac{J^2}{N^3}\right)^2 \delta_{il}\delta_{jm}\delta_{kn}\delta_{cc}\delta_{li}\delta_{mj}\delta_{nk} = \left(\frac{J^2}{N^3}\right)^2 N^4 \tag{2.2.9}$$

The different averaging pattern has brought to a contribution which is zero in the $N \to \infty$ limit and can be ignored at leading order. What happened?
Every time a dotted line connects two points which were already bound by a fermion propagator, the dotted line/undotted line loop brings a factor of N for every fermion propagator, which is just the trace of a delta. Otherwise in the second diagram we have just one N factor every two fermion lines, because two deltas are first multiplied and then traced, except for the propagator binding the two 1PI pieces which gives a delta trace as before.
This means that the power of N exactly equals the one coming from the variance if and only if we perform the disordered average between adjacent vertices: all other terms will be subleading. This property is also known as **melonic dominance** because of the shape of the 1PI diagram: the melons in the SYK model two point function play the same role of the genus zero (planar) term in the t'Hooft large N expansion of matrix models.

If we wanted to sum the whole series, there are other leading order contributions to take into account. For example, the one given by the following diagram and all other possible ways of nesting melons increasing the loop order: if we keep averaging just adjacent vertices the contribution will still be leading order



It's better to write the Schwinger-Dyson consistency equations for the exact leading order contribution to the two point function and analyze the result.
So we define $\Sigma(\tau, \tau')$ as the exact averaged self energy (i.e. the 1PI amputated two point Green function except for the inverse free propagator),

$$\Sigma(\tau, \tau') = J^2 G(\tau, \tau')^3 \tag{2.2.10}$$

Here $G(\tau, \tau')$ is the exact averaged interacting propagator containing the whole geometric series resummation of the 1PI terms, which can be represented in Fourier space by means of the second consistency equation

$$\frac{1}{-i\omega - \Sigma(\omega)} = G(\omega) \tag{2.2.11}$$

Unfortunately there is no analytic solution to those equations, but numerical calculations can be performed to understand the shape of the function, by knowing that as the energy grows they just have to reduce to the ones of the free theory. Is there any other limit which makes them solvable? The answer is yes and will be discussed in detail in the next section, after showing that the large N SYK can be mapped to a simpler theory involving just two fields.

2.3 The Bilocal Action and the IR limit

One of the most interesting things we can calculate is the free energy that is defined as the logarithm of the euclidean partition function, which contains the contributions of all the connected vacuum diagrams.

For now we consider the zero temperature many body model; switching to the finite temperature case we obtain the usual thermodynamic potential of the canonical ensemble at fixed bath temperature $-\beta F$: the Helmholtz free energy.

$$Z(J_{ijkl}) = \int \mathcal{D}\chi \exp\left[\int d\tau \left(-\frac{1}{2}\sum_{i=1}^{N}\chi_i\partial_\tau\chi_i + \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l\right)\right] \tag{2.3.1}$$

Even in this case we have to get rid of the random coupling by the J-disordered average, defined as

$$\overline{\log Z} = \int dJ_{ijkl}P(J_{ijkl})\log Z(J_{ijkl}) \tag{2.3.2}$$

where the integration is performed over all couplings and $P(J_{ijkl})$ is the Gaussian probability distribution defined above.

To avoid the complications in calculating the mean value of the logarithm, it's common for quenched disordered models to use the so called **replica method**, which consists in assuming the identity

$$\overline{\log Z} = \lim_{n\to 0}\frac{\overline{Z^n}-1}{n} = \lim_{n\to 0}\frac{\log\overline{Z^n}}{n} \tag{2.3.3}$$

The simplification lies in the possibility of reducing the calculation to the mean value of the product of n replicas of the original model, completing the square and then integrating out the J_{ijkl}. From a physical point of view the n-th power of Z is the partition function of a system made up of n identical and non-interacting copies of the same random realization of the system.

There is still the subtle problem of taking the limit sending n to zero as if it was a continuous variable, even though it actually isn't, but we will not focus on this point which would require an ad hoc discussion, because the results perfectly fit the ones found through simulations.

Therefore we evaluate the Gaussian integral in J_{ijkl} absorbing, as usual, all constants in the path integral measure step by step

$$\begin{aligned} \overline{Z^n} = \int \mathcal{D}\chi dJ_{ijkl} \exp\left(\sum_{a=1}^{n} \int d\tau \left(-\frac{1}{2}\sum_{i=1}^{N} \chi_i^a \partial_\tau \chi_i^a + \right.\right. \\ \left.\left. + \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i^a\chi_j^a\chi_k^a\chi_l^a\right) - \frac{N^3}{12J^2}J_{ijkl}^2\right) = \\ = \int \mathcal{D}\chi \exp\left(-\int d\tau \sum_{a=1}^{n} \frac{1}{2}\sum_{i=1}^{N} \chi_i^a \partial_\tau \chi_i^a + \frac{NJ^2}{8}\int d\tau d\tau' \sum_{a,b=1}^{n} \left(G^{ab}\left(\tau,\tau'\right)\right)^4\right) \end{aligned} \tag{2.3.4}$$

where a and b are the replica indices.

We observe that, removing the disordered coupling and the fermions, we have paid the price that now we have to deal with a **bilocal field**

$$G^{ab}\left(\tau,\tau'\right) = -\frac{1}{N}\sum_{i=1}^{N} \chi_i^a\left(\tau\right)\chi_i^b\left(\tau'\right) \tag{2.3.5}$$

Our task, as mentioned at the end of the previous chapter, in the integration of the N fermionic degrees of freedom: the memory of the fermionic nature of the initial fields will remain as an antisymmetry of the G field in the two times, which is manifest from the definition above. However, the new fields are commuting variables, so in someway we have mapped *couples of fermions* to *bilocal bosons*.

So let's move to rewriting the partition function as an integral over the above-mentioned bilocal field G and integrate out the fermions, but in order to be sure that the theory remains the same we will need to introduce another field Σ which will work as a Lagrange multiplier, enforcing G to contain the informations about its definition as function of the Majorana fermions.

To achieve this we insert in the path integral the identity

$$\begin{aligned} 1 = \int \mathcal{D}G\delta\left(G^{ab}\left(\tau,\tau'\right) + \frac{1}{N}\sum_{i=1}^{N} \chi_i^a\left(\tau\right)\chi_i^b\left(\tau'\right)\right) = \\ = \int \mathcal{D}G\mathcal{D}\Sigma \exp\left(\frac{1}{2}\sum_{a,b=1}^{n}\int d\tau d\tau' \Sigma^{ab}\left(\tau,\tau'\right)\left(NG^{ab}\left(\tau,\tau'\right) - \chi_i^a\left(\tau\right)\chi_i^b\left(\tau'\right)\right)\right) \end{aligned} \tag{2.3.6}$$

and finally integrate out the fermions using standard results on Gaussian integrals for fermionic variables to get

$$\overline{Z^n} = \int \mathcal{D}G\mathcal{D}\Sigma \mathrm{e}^{-S_n[G,\Sigma]} \tag{2.3.7}$$

where the action for n replicas of the system is now:

$$\begin{gathered} -S_n\left[G,\Sigma\right] = \frac{N}{2}\sum_{a,b=1}^{n}\log\det\left(\delta^{ab}\partial_\tau + \Sigma^{ab}\left(\tau,\tau'\right)\right) + \\ +\frac{N}{2}\sum_{a,b=1}^{n}\int d\tau d\tau'\left(\Sigma^{ab}\left(\tau,\tau'\right)G^{ab}\left(\tau,\tau'\right) - \frac{J^2}{4}\left(G^{ab}\left(\tau,\tau'\right)\right)^4\right) \end{gathered} \tag{2.3.8}$$

The determinant in the first line must obviously be performed separately in the replica indices and in the continuous variables.

It's manifest that the $\frac{1}{N}$ small parameter plays the same role as $\hbar$ in the path integral formulation of the loop expansion in field theories: we can expand in powers of $\frac{1}{N}$ and the leading contribution will come from the saddle point of the new bilocal action.

Assuming there's no replica symmetry breaking we can neglect the off-diagonal replica terms imposing $G^{ab}\left(\tau,\tau'\right) = G\left(\tau,\tau'\right)\delta^{ab}$ and $\Sigma^{ab}\left(\tau,\tau'\right) = \Sigma\left(\tau,\tau'\right)\delta^{ab}$ and the path integral factors as

$$\overline{Z^n} = \left(\int \mathcal{D}G\mathcal{D}\Sigma \mathrm{e}^{-S[G,\Sigma]}\right)^n = \overline{Z}^n \tag{2.3.9}$$

In this way the limit becomes trivial

$$\overline{\log Z} = \log\overline{Z} = \log\left(\int \mathcal{D}G\mathcal{D}\Sigma \mathrm{e}^{-S[G,\Sigma]}\right) \tag{2.3.10}$$

Where the action is identical to the previous one, except for the fact that it contains just one replica of the system

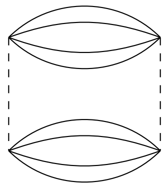
$$\begin{gathered} -S_{Bil}\left[G,\Sigma\right] = -S_1\left[G,\Sigma\right] = \frac{N}{2}\log\det\left(\partial_\tau + \Sigma\left(\tau,\tau'\right)\right) + \\ +\frac{N}{2}\int d\tau d\tau'\left(\Sigma\left(\tau,\tau'\right)G\left(\tau,\tau'\right) - \frac{J^2}{4}\left(G\left(\tau,\tau'\right)\right)^4\right) \end{gathered} \tag{2.3.11}$$

It can be useful to notice that in all these passages we have used divergent identities which are meaningful, at this level, just formally: the new bilocal SYK action will capture just the perturbative features of the averaged SYK model (of course under the assumption of unbroken replica symmetry), but it's not a big deal because it is precisely what we are interested in.

In practice we have justified the substitution $\overline{\log Z} \to \log\overline{Z}$. It's now natural to ask which contributions we are excluding by this approximation and at which order they appear in the expansion.

Actually what happens is that $\log Z$ contains, as said at the beginning, all connected vacuum graphs, but still no average operation has been performed and the J_{ijkl} coupling is just a set of random numbers, this means we are extracting diagrams which are connected through fermionic lines. Then we perform the disordered average.

By following this procedure the first diagram we could never have, for example, is one like this



where the solid lines are again exact (but still leading order in $\frac{1}{N}$) fermionic propagators.

But if we focus on Z, which contains the disconnected vacuum diagrams too, the different points become connected by a disorder line and the logarithm does not remove it, the approximated free energy we are calculating will contain this term, while it should not.

The $\frac{1}{N}$ order of the diagram will quantify the accuracy of the replica trick, of the symmetry ansatz and of the unbroken symmetry ansatz.

A direct application of the previously discussed rules allows us to easily calculate it and we obtain the same result as in (2.2.9): $\frac{1}{N^2}$. At leading order everything is safe, but at second order we immediately have to keep track of the terms we have to subtract if we are interested in the average free energy.

The saddle, in this theory of just two fields instead of a large N number, perfectly reproduces the *Schwinger-Dyson* equations for G, the leading order propagator, and Σ, the self energy, in the diagrammatic expansion for the Majorana fields action in the large N limit described in the previous section.

The two equations which annihilate the first variations of the action are therefore

$$\begin{gathered}\Sigma\left(\tau,\tau'\right)=J^2\left[G\left(\tau,\tau'\right)\right]^3\\ -\int d\tau'\left(\delta\left(\tau-\tau'\right)\partial_\tau+\Sigma\left(\tau,\tau'\right)\right)G\left(\tau'',\tau'\right)=\delta(\tau-\tau'')\end{gathered}\tag{2.3.12}$$

Exploiting the time translation invariance of the theory, the second equation can be seen as a convolution and has a far nicer shape written in Fourier transform, where the derivative term becomes a frequency and allows us to easily move to the *IR limit*

$$-i\omega-\Sigma\left(\omega\right)=\frac{1}{G\left(\omega\right)}\qquad\xrightarrow[\omega\to0]{}\qquad-\Sigma\left(\omega\right)=\frac{1}{G\left(\omega\right)}\tag{2.3.13}$$

The IR limit can also be interpreted as a *strong coupling* limit in which the self energy Σ overcomes the kinetic energy of the propagating Majorana field: this is desirable because makes the model one of the few know models which are exactly solvable in such regime.

In fact the two equations are now analytically *solvable*, returning an exact IR two point function

$$G\left(\tau,\tau'\right)_{IR} = b\frac{\text{sgn}\left(\tau-\tau'\right)}{\left|J\left(\tau-\tau'\right)\right|^{2\Delta}} \tag{2.3.14}$$

where $b^4 = \frac{1}{\pi}$, $\Delta = \frac{1}{4}$ is related to the $q = 4$ fields interaction by $\Delta = \frac{1}{q}$ and represents the IR *anomalous dimension* acquired by the fermions.

As it perfectly reproduces the previously discussed shape of a $PSL\left(2,\mathbb{R}\right)$ invariant two point function, this the first sign dropping the hint that we might be dealing with an emerging **conformal symmetry**.

2.4 Conformal symmetry and its breaking pattern

As discussed in the first chapter the conformal group in one dimension is made of all possible monotonic differentiable functions $Diff\left(\mathbb{R}\right)$. Specifically in our case they represent the euclidean time reparametrizations $f\left(\tau\right)$.

We see that this is precisely, as expected from the shape of the exact IR propagator, a symmetry of the saddle point equations, provided that the bilocal fields obey the following transformation rules under $\tau \to f(\tau)$

$$\begin{aligned} G\left(\tau,\tau'\right) &\to \left[f'\left(\tau\right)f'\left(\tau'\right)\right]^{\Delta} G\left(f\left(\tau\right),f\left(\tau'\right)\right) \\ \Sigma\left(\tau,\tau'\right) &\to \left[f'\left(\tau\right)f'\left(\tau'\right)\right]^{1-\Delta} \Sigma\left(f\left(\tau\right),f\left(\tau'\right)\right) \end{aligned} \tag{2.4.1}$$

In other words the propagator is a bilocal Δ-differential, while the self energy is a $1-\Delta$-differential on the real axis. Equivalently these transformation rules are precisely the ones defining the primary fields of a conformal field theory.

Assuming that $\Sigma(\tau,\tau')$ and $G(\tau,\tau')$ are solutions, for example the ones written down in the previous section, we see that performing a reparametrization the first equation gets sent to

$$\left(f'(\tau)f'(\tau')\right)^{\frac{3}{4}}\Sigma(f(\tau),f(\tau')) = J^2\left[\left(f'(\tau)f'(\tau')\right)^{\frac{1}{4}}G(\tau,\tau')\right]^3 \tag{2.4.2}$$

The differential transformation law of the bilinear field causes the cancellation of the f' factors, which are surely nonzero, and up to the change of variable $f(\tau) = \sigma$ the equation is precisely the same as before. So it's automatically solved

$$\Sigma(\sigma,\sigma') = J^2 G^3(\sigma,\sigma') \tag{2.4.3}$$

However for this result it was not necessary that the fields had the introduced differential transformation laws. The only thing that mattered was that, assuming the transformation law for the propagator field, the self energy field has an anomalous dimension which was three times the other one.

Now let's do the same thing for the second equation written in time space: this will dictate the anomalous dimensions of the fields to be precisely the used ones. Again we start by assuming

$$\int d\tau'' G(\tau,\tau'')\Sigma(\tau'',\tau') = -\delta(\tau-\tau') \tag{2.4.4}$$

Then we ask ourselves if the reparametrized fields satisfy the equation.

$$\begin{aligned}\int d\tau'' \left[f'(\tau)f'(\tau'')\right]^{\frac{1}{4}} G(f(\tau), f(\tau'')) \left[f'(\tau'')f'(\tau')\right]^{\frac{3}{4}} \Sigma(f(\tau''), f(\tau'))\\ = f'(\tau)^{\frac{1}{4}} f'(\tau')^{\frac{3}{4}} \int df(\tau'') G(f(\tau), f(\tau''))\Sigma(f(\tau''), f(\tau'))\end{aligned} \tag{2.4.5}$$

By exploiting the invertibility of the diffeomorphism we can change the variable, $f(\tau) = \sigma$, and use the fact that the fields satisfy (2.4.4) to get

$$\begin{aligned} f'(\tau)^{\frac{1}{4}} f'(\tau')^{\frac{3}{4}} \int d\sigma'' G(\sigma, \sigma'')\Sigma(\sigma'', \sigma') \\ = -f'(\tau)^{\frac{1}{4}} f'(\tau')^{\frac{3}{4}} \delta(\sigma - \sigma') \\ = -f'(\tau)^{\frac{1}{4}} f'(\tau')^{\frac{3}{4}} \delta(\sigma - \sigma') \\ = -f'(\tau)^{\frac{1}{4}} f'(\tau')^{\frac{3}{4}} \delta(f(\tau) - f(\tau') \\ = -f'(\tau)^{\frac{1}{4}} f'(\tau)^{\frac{3}{4}} \frac{\delta(\tau - \tau')}{f'(\tau')} = -\delta(\tau - \tau') \end{aligned} \tag{2.4.6}$$

where we have just used delta function properties and the fact that $f(\tau)$ is a monotonic function with positive derivative.
This definitely proves that the theory in the IR exhibits the emergent behavior of a one dimensional conformal field theory. By identical arguments we might show that this is indeed a symmetry of the bilocal action after getting rid of the time derivative in the determinant.

By defining $\sigma(\tau, \tau') = \delta(\tau - \tau\prime)\partial_\tau$ and changing the variable $\Sigma(\tau, \tau\prime) \to$ we can split the bilocal action in the conformal piece and a UV correction coming from the derivative.

$$S_{Bil}\left[G, \Sigma\right] = S_{CFT}\left[G, \Sigma\right] + S_{UV}\left[G, \Sigma\right] \tag{2.4.7}$$

where

$$\begin{aligned} S_{CFT} = -\frac{N}{2} \log\det\left(\Sigma\left(\tau, \tau'\right)\right) \\ +\frac{N}{2} \int d\tau d\tau' \left(\Sigma\left(\tau, \tau'\right) G\left(\tau, \tau'\right) - \frac{J^2}{4}\left(G\left(\tau, \tau'\right)\right)^4\right) \\ S_{UV} = \frac{1}{2} \int d\tau d\tau\prime \sigma(\tau, \tau') G(\tau, \tau\prime) \end{aligned} \tag{2.4.8}$$

It's now crucial to make a few comments about this symmetry. The equation of motion invariance allows us, given a specific leading order solution as the one found in the previous paragraph, to use the reparametrization to explore a whole infinite dimensional manifold of possible consistent choices of G and Σ whose contribution to the path integral is the same, that's why it is often called *soft mode*.
For example, if we choose the conformal propagator of (1.3.14), $\forall f(\tau) \in Diff^+(\mathbb{R})$ the

component of the aforementioned manifold connected to G_{IR}, which might be equivalently called G_{CFT}, can be explicitly parametrized as

$$G\left(\tau,\tau'\right) = b\,\mathrm{sgn}\left(f(\tau)-f(\tau\prime)\right)\frac{f'\left(\tau\right)f'\left(\tau'\right)}{\left|J\left(f(\tau)-f(\tau\prime)\right)\right|^{2\Delta}} \tag{2.4.9}$$

Furthermore, the process of choosing *one* of these solutions **spontaneously breaks** the reparametrization invariance: $f(\tau)$ plays the role of the Goldstone boson of the conformal symmetry breaking pattern.

Is there a subgroup which the conformal symmetry breaks to? The answer is subtle. What actually happens is that we are left with a symmetry in the soft modes which is isomorphic to the largely discussed real Möbius group $PSL\left(2,\mathbb{R}\right)$.
That's because after fixing a reparametrization of the conformal propagator through some $f(\tau)$, we can still change it through a $PSL\left(2,\mathbb{R}\right)$, namely $\frac{af(\tau)+b}{cf(\tau)+d}$, still leaving the reparametrized propagator invariant.
This means that the original physical $PSL\left(2,\mathbb{R}\right)$ symmetry in the time variable is actually broken, but it gets restored as an isomorphic **global symmetry** in the new fields: this observation suggests that the Goldstone manifold is the *right coset* $PSL(2,\mathbb{R})\backslash Diff^{+}(\mathbb{R})$.
So we are dealing with some kind of **infinite dimensional non linear sigma model**.

The path integral can be formally split in an integration over the $Diff^{+}(\mathbb{R})$-inequivalent G and an integration over the Goldstone manifold, unfortunately the infinite number of possible Goldstone directions will lead to diverging correlation functions performing the f path integral, but this is not compatible with the original quantum mechanical model we were treating

$$Z = \int \mathcal{D}f\mathcal{D}G'\mathcal{D}\Sigma' \mathrm{e}^{-S_{CFT}[G',\Sigma',f(\tau)]} \tag{2.4.10}$$

Two questions are mandatory now:

- Which right path integral measure one needs to introduce to be sure that this change of variables in the space of fields is actually consistent?
- As said before, it's clear that the naive flat f functional integration leads to a divergence, since $S_{CFT}\left[G',\Sigma',f(\tau)\right] = S_{CFT}\left[G',\Sigma'\right]$. Can the right integration measure cure automatically this infinite?

In the original SYK action the conformal symmetry was *explicitly broken* by the presence of the τ-derivative: the most natural thing to avoid this infinite now is finding the first UV correction to the action using $f(\tau)$ to perturb the chosen saddle solution. Of course this requires a *small* explicit breaking of the reparametrization symmetry with the soft modes becoming *pseudo Nambu-Goldstone* bosons with an action cost for variations of the fields along the Goldstone trajectories which will still be small, but non zero.
It will become clear soon that this effective field theory approach will return the same answer of the measure issues described previously.

2.5 The Schwarzian action from the SYK model

We will now try to understand which might be the aforementioned first small UV correction in the soft modes to the deep IR action.
In order to do this we first reprise the discussion about our pseudo Nambu-Goldstone bosons. The starting point was the conformal two point function G_c which exhibited the exact $PSL(2,\mathbb{R})$ symmetry in the euclidean time, while the whole $Diff(\mathbb{R})$ changed the solution, whilst leaving the saddle equations invariant.
The explicit choice of the solution breaks the whole $Diff(\mathbb{R})$, but we observe that the $PSL(2,\mathbb{R})$ symmetry appears again as a global symmetry in the soft modes, which will belong to the quotient manifold $\frac{Diff^+(\mathbb{R})}{PSL(2,\mathbb{R})}$.

That's because, after fixing $f(\tau)$ we can perform the transformation

$$f(\tau) \to \frac{af(\tau)+b}{cf(\tau)+d} \tag{2.5.1}$$

which still leaves the new G field invariant.

If one wants to write down an effective action including the first UV correction in the soft modes, the new piece must be the function of the reparametrization $f(\tau)$ of the lowest possible order in derivatives reflecting the properties of this symmetry breaking pattern. We propose a way of figuring this out inspired to [7].

In other words, we are searching for an operator $O(f)$ acting on the space of monotonically increasing functions and whose action is one-to-one after one quotients out the kernel $PSL(2,\mathbb{R})$, so such that

$$O\left[M \circ f\left(\tau\right)\right] = O[f\left(\tau\right)] \qquad \forall M \in PSL(2,\mathbb{R}) \tag{2.5.2}$$

The invariance under the addition of an arbitrary constant implies that the operator can just be a function of $f^{(n)}(\tau)$, the n-th derivatives of the function, while the scale invariance forces it to be just a function of ratios of equal numbers of arbitrary derivatives.

Moreover we are searching for the first correction in J^{-1}: the integral has to carry energy dimension 1 to assure the adimensionality of the action, which means that the searched operator must be of dimension 2. Therefore the allowed ratios will be such that the sum of the number of derivatives in the numerator will exceed the one in the denominator by 2.

Since we are searching for the expression with the lowest possible number of derivatives (and of the lowest possible order), we focus on linear combinations of these ratios

$$O[f\left(\tau\right)] = \sum_{i=1}^{k} a_i \frac{f^{(i+2)}}{f^{(i)}} + \sum_{i=1}^{k} b_i \left(\frac{f^{(i+1)}}{f^{(i)}}\right)^2 \tag{2.5.3}$$

where $k+2$ will be the highest desired derivation order we wish to stop at: up to second order derivative there is just $\left(\frac{f^{(2)}}{f^{(1)}}\right)^2$, up to third order we have $\frac{f^{(3)}}{f^{(1)}}$ and $\left(\frac{f^{(3)}}{f^{(2)}}\right)^2$.

Now we can write a generic linear combination of those three and try to fix the coefficients imposing the Möbius invariance. Explicit computation gives

$$\left(\frac{f^{(2)}}{f^{(1)}}\right)^2 \longrightarrow \left(\frac{f^{(2)}}{f^{(1)}}\right)^2 - \frac{4f^{(2)}(\tau)f^{(1)}(\tau)}{cf(\tau)+d} + \frac{4c^2\left(f^{(1)}(\tau)\right)^2}{(cf(\tau)+d)^2}$$
$$\frac{f^{(3)}}{f^{(1)}} \longrightarrow \frac{f^{(3)}}{f^{(1)}} - \frac{6cf^{(2}(\tau)}{cf(\tau)+d} + \frac{6c^2\left(f^{(1)}(\tau)\right)^2}{(cf(\tau)+d)^2} \tag{2.5.4}$$

and the questioned object is found out to be precisely the largely discussed Schwarzian derivative

$$\frac{f^{(3)}}{f^{(1)}} - \frac{3}{2}\left(\frac{f^{(2)}}{f^{(1)}}\right)^2 = \{f(\tau), \tau\} \tag{2.5.5}$$

The effective action of reparametrizations is so

$$S_{Sch} = \frac{N\alpha}{J}\int d\tau\,\{f(\tau), \tau\} \tag{2.5.6}$$

where the alpha constant can be computed numerically interpolating between the UV free two point function and the IR conformal and strongly coupled one. This is done imposing consistency with the general solution of the Schwinger-Dyson equations, which nobody was able to express analytically,.

This means that the partition function of the theory in (2.4.10) can be upgraded to

$$Z = \int \mathcal{D}f \mathrm{e}^{-S_{Sch}[f(\tau)]}\int \mathcal{D}G'\mathcal{D}\Sigma'\mathrm{e}^{-S[G',\Sigma']} \tag{2.5.7}$$

Higher order corrections might involve powers of this Schwarzian action. A theory like this one with a small conformal symmetry breaking is called a **nearly conformal field theory** or **nCFT**.

The Schwarzian action is also found in the boundary dynamics of the **Near** AdS_2, or **N**AdS_2, dilaton Jackiw-Teitelboim (or Almheiri-Polchinski) gravity theory, and appears through the same symmetry breaking pattern.

Actually there's more. The first UV corrections has been derived just by dimensional and symmetry arguments: this suggests that this Schwarzian behavior is a **universal property** of 0+1 dimensional theories exhibiting an emergent conformal symmetry by means of a full reparametrization invariance.

However the previous argument had an untold important assumption: we took for granted that the UV correction could be expressed through a **local** action, even though we were analyzing a theory which was **bilocal** in the two fields. This is not obvious and requires an explicit check, even though it is desirable due to the UV nature of the correction: if it had to depend on two times, they should be very close to each other.

In the SYK model different possibilities have been explored to do this in literature.

We could imagine to derive the Schwarzian as a feature of the explicit expansion of the determinant in the full effective action around the IR limit ($\partial_\tau \to 0$) and at first order in $\frac{1}{N}$ using the leading order result $\Sigma^{-1} = -G_{IR}$.

$$
\begin{aligned}
&\operatorname{Tr}\left[\log\left(\partial_\tau+\Sigma\right)-\log(\Sigma)\right]=\operatorname{Tr}\log\left(1-\partial_\tau G_{IR}\right)\approx\\
&\approx\operatorname{Tr}\left(-\partial_\tau G_{IR}-\frac{1}{2}\partial_\tau G_{IR}\partial_\tau G_{IR}\right)=-\frac{1}{2}\operatorname{Tr}\left(\partial_\tau G_{IR}\partial_\tau G_{IR}\right)=\\
&=-\frac{1}{2}\int d\tau_1 d\tau_2\left(\partial_{\tau_1}\left[f'(\tau_1)^{\frac{1}{4}}G_{IR}\left(f(\tau_1),f(\tau_2)\right)f'(\tau_2)^{\frac{1}{4}}\right]\times\right.\\
&\left.\times\partial_{\tau_2}\left(f'(\tau_2)^{\frac{1}{4}}G_{IR}\left(f(\tau_2),f(\tau_1)\right)f'(\tau_1)^{\frac{1}{4}}\right)\right)
\end{aligned}
\tag{2.5.8}
$$

Every term in this formal expansion is divergent and different regularization procedures have been used to try to get rid of the divergence and obtain the local Schwarzian, but for now none of them is fully satisfactory. We will follow one of these possibilities: [1].

In the last line we have inserted the generic expression of an element in the manifold of the saddle point solutions which is in the orbit of the reparametrization group containing the conformal propagator G_{IR}.
To simplify the notation we change the integration variable by $t_i = f(\tau_i)$ and define $a_t = f'(f^{-1}(t))^{\frac{1}{4}}$ to get

$$
\int dt_1 dt_2 \partial_{t_1}\left(a_{t_1}G_{IR}(t_1-t_2)a_{t_2}\right)\partial_{t_2}\left(a_{t_2}G_{IR}(t_2-t_1)a_{t_1}\right) \tag{2.5.9}
$$

The $PSL(2,\mathrm{R})$ invariance is manifest in the correction because it is a symmetry of $G_{IR}[f]$: to treat the UV-divergent integral given above one needs a suitable regularization procedure by introducing a UV cut-off Λ, or equivalently in time domain $\delta = \frac{1}{\Lambda}$, which in turn has to preserve the invariance.

We won't report the details of the calculation, however after regularization the action takes the form

$$
S[f] = \int d\omega b(-\omega)\Pi_\Lambda(\omega)b(\omega) \tag{2.5.10}
$$

where the regularized "polarization operator" evaluated to logarithmic accuracy reads

$$
\Pi_\Lambda(\omega) = \frac{b^2 N}{16J}\left(\omega^2\log\frac{\Lambda}{|\omega|}+4\Lambda^2\right) \tag{2.5.11}
$$

$$
S[f] = -b^2 N\int d\tau_1 d\tau_2 \frac{f'(\tau_1)f'(\tau_2)}{\left(\mid f(\tau_1)-f(\tau_2)\mid +\delta\right)^3} \tag{2.5.12}
$$

with $\delta = \frac{1}{J}f'\left(\frac{\tau_1+\tau_2}{2}\right)$ and where the UV divergence has been cured.

Going back to the Fourier space polarization operator and cutting the logarithm at $\omega = \Delta(f^{-1})'(s)$, the remaining powers of ω become derivatives and we get a local approximation for the action

$$
S^{loc}[f] \propto \int ds(\partial_s b_s)^2 \propto \int d\tau\{f,\tau\} \tag{2.5.13}
$$

2.6 Finite temperature

Now we will move to the finite temperature regime, wich is the interesting one having in mind a black hole dual: we will briefly review how to do this.
The thermodynamic partition function of a canonical ensemble of quantum field theories at fixed temperature can be written as

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_q \langle q| \, e^{-\beta \hat{H}} \, |q\rangle \tag{2.6.1}$$

where β is just the inverse temperature, having chosen natural units in which the Boltzmann constant is equal to 1; $\hat{H}$ is the Hamiltonian operator of the quantum theory and the summation is performed over a complete set of vectors.

The key observation comes now from the fundamental identity relating the operator and the path integral formulations of quantum mechanics giving the transition amplitude for two states of a system described by H

$$\langle q_f| \, e^{-i\hat{H}(t_f - t_i)} \, |q_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t) e^{i\int_{t_i}^{t_f} L[q(t),\dot{q}(t)]} \tag{2.6.2}$$

Performing the transformation $i(t_f - t_i) \to \beta$, i.e Wick rotating and changing the time variable through the time-dimensional parameter β, on the left hand side one gets an expression very similar to the one in (1.6.1). The difference is that the initial and the final state have to be identified introducing the periodic boundary conditions $q(t_i) = q(t_f) = q(t_i + \beta) = q$ and we have to sum the result over all possible states, which in the path integral perspective is equivalent to integrate over all periodic functions

$$\mathrm{Tr}\, e^{-\beta \hat{H}} = \int_{q(t)=q(t+\beta)} \mathcal{D}q(t) e^{-\int_0^\beta L_E[q(t),\dot{q}(t)]} \tag{2.6.3}$$

where L_E is the euclidean Lagrangian.

This means that the finite temperature QFT on the SYK lattice can be recovered studying the theory in euclidean time, by making the latter a coordinate on a circle whose length is linear in the inverse temperature. As $T \to 0$, and then $\beta \to +\infty$, the infinite circle recovers the whole partition function of the original field theory, which was called *zero temperature* because of this.

The circle, on which the euclidean periodic time variable is defined as a coordinate, is also called **thermal circle**: one can perform every calculation in the euclidean time and then switch to the circle by a map which topologically induces the Alexandroff one point compactification $g : \mathbb{R} \to \mathcal{S}^1$.
After changing the topology, all the SYK properties and calculations are the same discussed before, but with the important difference that the analytic reparametrization invariance is valid for functions which are globally defined on the circle, namely functions belonging to $Diff(\mathcal{S}^1)$.
The euclidean SYK has to be thought as the boundary dual of a bulk theory defined on the AdS_2 geometry, whose euclidean counterpart is the Poincaré disk model: so by means

of the Cayley transform we can build a globally biholomorphic map relating the open disk, whose removed boundary is the thermal circle, and the upper-half plane, which is a open set and has been met before as the universal covering of all compact orientable Riemann surfaces with $h > 1$

$$C(z) = \frac{i - z}{i + z}, \qquad z \in \mathbb{H} \tag{2.6.4}$$

So, having in mind the duality, we compactify the circle extrapolating the Cayley map on the real line, taking values on the unit circle with one point removed. The removed point gets recovered by adding ∞ as a single point at infinity, which is compatible with the map, since $\lim_{x\to+\infty} C(x) = \lim_{x\to-\infty} C(x) = -1, x \in \mathbb{R}$, this can be easily checked by writing the real Cayley transform in exponential form

$$C(x) = \frac{i - x}{i + x} = e^{-2i \arctan x} \in \mathcal{S}^1 \backslash \{-1\} \tag{2.6.5}$$

which winds precisely once along the unit circle without one point.

So we can treat $\phi = 2 \arctan x \in]-\pi, \pi[$ as a chart on the circle, after removing one point in the standard way used to build the usual atlas of $\mathcal{S}^1$. To rewrite on the circle everything we have calculated in the SYK model up to now, we need to invert $C(z)$, a procedure which yields

$$C^{-1}(z) = i\frac{1 - z}{1 + z} \tag{2.6.6}$$

Now, let z be a point on the unit circle, it can be represented as $z = e^{i\phi}$, $\phi \in]-\pi, \pi[$, therefore we rewrite the map as a function of ϕ

$$\begin{aligned} F(\phi) &:= C^{-1}(e^{i\phi}) = i\frac{1 - e^{i\phi}}{1 + e^{i\phi}} \\ &= i\frac{e^{-i\frac{\phi}{2}} - e^{i\frac{\phi}{2}}}{e^{-i\frac{\phi}{2}} + e^{i\frac{\phi}{2}}} = \tan\frac{\phi}{2} \end{aligned} \tag{2.6.7}$$

which is coherent with the previous result.

We remark that the inverse tangent function is a honest diffeomorphism sending $\mathbb{R}$ to a diffeomorphic open set: the SYK model emergent $Diff^+(\mathbb{R})$ symmetry guarantees that this transformation cannot modify its physical IR/strong coupling properties. The open set becomes the circle after taking its closure and gluing the endpoints (which is equivalent to the one-point compactification), which can be done applying the complex exponential function, as we have just proved.
The conformal symmetry on the finite temperature theory will now be $Diff^+(\mathcal{S}^1)$.

Let's see what happens to the Schwarzian action, which has just the unbroken symmetry $PSL(2, \mathbb{R})$, choosing such a reparametrization.

We will write the action of the Schwarzian theory, which was proved to represent the dynamics of the $h = 2$ enhanced pseudo Goldstone mode of SYK model, by renaming the coupling $\frac{1}{\lambda^2} = \frac{N\alpha}{J}$.

The τ variable is now a coordinate on the circle, $\phi(\tau) \in Diff^+(\mathcal{S}^1)$ is a further generic reparametrization of the circle, and the compactification is taken after reparametrizing through $\tan \frac{\pi\phi(\tau)}{\beta}$;

$$S_{Sch} = \frac{N\alpha}{J}\int_0^\beta d\tau\{\tan\frac{\pi\phi}{2\beta},\tau\} = \frac{1}{g^2}\left(\int_0^\beta d\tau\{\phi,\tau\} + \frac{1}{2}\phi'^2\right) \tag{2.6.8}$$

where the g coupling is defined as to $\frac{1}{g^2} = \frac{N\alpha}{J\beta}$ and we have just used the Schwarzian derivative chain rule. This means that in a finite temperature theory we can select a regime in which the Schwarzian is not just dependent on the coupling of the zero temperature theory: the expansion is now on the parameter βJ and, as one can imagine, a theory with high temperatures ($\beta \to 0$), at strong fixed J, might correspond to a small βJ coupling and move the theory to a regime in which the Schwarzian is not enough to describe the SYK model. So the temperature is now finite, but it is still such that $J >> \frac{1}{T}$, to remain in the Schwarzian (or *holographic*) limit.

The study of the Schwarzian action at finite temperature, which is a UV term, will be the main topic of the next sections, to close this one we report some finite temperature properties of the SYK model.

For example the two point function, taking $f(\tau) = \tan\frac{\pi\tau}{\beta}$, according to the differential transformation law (1.4.1) gets mapped to

$$\begin{aligned} G_\beta\left(\tau,\tau'\right) &= b\frac{\mathrm{sgn}\left(\tan\frac{\pi\tau}{\beta} - \tan\frac{\pi\tau'}{\beta}\right)}{\left|J\left(\tan\frac{\pi\tau}{\beta} - \tan\frac{\pi\tau'}{\beta}\right)\right|^{\frac{1}{2}}}\left(\frac{\pi^2}{\beta^2\cos^2\frac{\pi\tau}{\beta}\cos^2\frac{\pi\tau'}{\beta}}\right)^{\frac{1}{4}} \\ &= b\left(\frac{\pi}{\sin\pi\frac{\tau-\tau'}{\beta}}\right)^{\frac{1}{2}}\mathrm{sgn}(\tau-\tau') \end{aligned} \tag{2.6.9}$$

where just trivial goniometric identities have been used.

The periodic finite temperature two point function is obviously related to the fact that an immediate consequence of compactifying time is the discretization of the spectrum, so for example the Fourier representation Schwinger-Dyson equations won't be defined for all values of a continuous frequency, but just for a set of modes with fermionic Matsubara frequencies $\omega_n = \frac{2\pi}{\beta}(n+\frac{1}{2})$ because of the periodic boundary conditions.

2.7 The four point function

In this section, following [8], [11] and [13], we will briefly review why SYK is a quantum chaotic model by studying the four point function of the theory: we won't go into technical details, instead we will discuss the interesting physical informations about the model emerging in the process, especially in the nearly conformal limit.

2.7.1 Chaos

But what does the four point correlator have to do with probing chaos? And why is it interesting in the holographic picture?

In general for a dynamical system one talks about chaos whenever a small perturbation of the initial conditions can produce a big change in the phase space trajectory, which translates mathematically in an exponential divergence of two paths starting from very close initial conditions, namely

$$\frac{\partial q(t)}{\partial q(0)} = \frac{\partial q(t)}{\partial q(0)}\frac{\partial p(0)}{\partial p(0)} - \frac{\partial q(t)}{\partial p(0)}\frac{\partial p(0)}{\partial q(0)} = \{q(t), p(0)\}_P \sim \mathrm{e}^{\lambda_L t} \tag{2.7.1}$$

where $\{\cdot, \cdot\}_P$ is the Poisson bracket and q and p is the usual coordinate and momentum pair. The number λ_L is called Lyapunov exponent and quantifies how much two different trajectories actually diverge. The above relation might hold just for a finite time interval $[t_i, t_f]$, after which the system saturates.

In quantum mechanics the quantity which in the classical limit reduces to the Poisson bracket is the commutator of the operators associated by correspondence principle to the observables under analysis, with the difference that to extract a function from a time dependent operator we have to take a quantum average over a state: assuming that the system starts at a thermal equilibrium state and then thermalizes again, we can consider

$$C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta = -\frac{\mathrm{Tr}\,[W(t), V(0)]^2\, \mathrm{e}^{-\beta H}}{Z} \tag{2.7.2}$$

where $W(t)$ is the Heisenberg time evolution of some local perturbation, $V(0)$ is another local perturbation, H is the Hamiltonian and Z the partition function. The square avoids possible phase cancellations: in a chaotic system $C(t)$ is the function probing the chaotic behavior of the system. Explicitly it reads

$$\begin{aligned} C(t) = &\langle V(0)W(t)W(t)V(0)\rangle_\beta + \langle W(t)V(0)V(0)W(t)\rangle_\beta \\ &-\langle V(0)W(t)V(0)W(t)\rangle_\beta - \langle W(t)V(0)W(t)V(0)\rangle_\beta \end{aligned} \tag{2.7.3}$$

The first two terms are just norms: the exponential function, end therefore the Lyapunov exponent, depends on the other two out-of-time-order four point correlators.
So the fundamental object we need to analyze in the SYK to search for signs of chaos is, in euclidean time

$$\langle T\left(\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\right)\rangle. \tag{2.7.4}$$

In the holographic picture one knows that in a black hole the Lyapunov exponent saturates getting a maximal value $\frac{2\pi}{\beta}$, and this is precisely what one finds for the finite temperature out-of-time-order four point correlator of SYK: this, together with the conformal emerging symmetry, was one of the features that made the model interesting for the high energy community.

We won't do the explicit evaluation, which is quite involved, here: but, to follow out purposes, we will analyze some aspects of the four point correlator which give more information about the interpretation of the Schwarzian Goldstone modes.

2.7.2 One loop diagrammatics

If we average the four point function over N the Majorana contractions give rise to the exact two point propagators of the averaged theory at leading order in $\frac{1}{N}$. The indices have been taken equal in pairs because we have in mind to average over the random coupling which identifies the indices in the in-going line to the ones in the outgoing ones, as shown in the diagrammatic picture.
Before going on we want to remark two things:

- Even if we are calculating the standard quantum averages, we can always switch to the canonical finite temperature case, by the Cayley map which has been largely discussed;
- Up to now we have not yet made any expansion in the J (βJ at finite T) coupling, so everything discussed up to now is true at every energy scale.

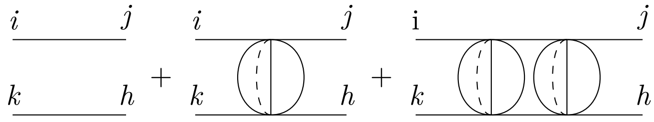
And still at any energy scale it is true that one of the main features of the large N limits is that every correlator disconnects in products of two point functions at leading order: it is indeed a classical limit

$$\begin{aligned}\Gamma(\tau_1,\tau_2,\tau_3,\tau_4) &= \frac{1}{N^2}\sum_{i,j=1}^{N}\langle T\left(\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\right)\rangle \\ &= \langle G(\tau_1,\tau_2)\rangle\langle G(\tau_1,\tau_2)\rangle + \frac{1}{N}\Gamma_c(\tau_1,\tau_2,\tau_3,\tau_4)\end{aligned} \tag{2.7.5}$$

This correlator can of course be interpreted as the quantum correction to the relevant fermion vertex of SYK model, but has another nice interpretation in term of the effective bilocal action describing the averaged SYK model where it is just the two field fluctuation around the saddle of the bilocal G fields as introduced in (3.3.5), without of course the replica indices which have been dropped, but, as argued, this is still a completely safe approximation at $\frac{1}{N}$

$$\begin{aligned}\Gamma(\tau_1,\tau_2,\tau_3,\tau_4) &= \frac{1}{N^2}\sum_{i,j=1}^{N}\langle T\left(\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\right)\rangle \\ &= \langle G(\tau_1,\tau_2)G(\tau_3,\tau_4)\rangle\end{aligned} \tag{2.7.6}$$

Thus to get some non-trivial informations we are moving at one loop level in the large N expansion, where the structure of the leading diagrams is given by a sum of iterated products of ladders

$\langle\Gamma(\tau_1,\tau_2,\tau_3,\tau_4)\rangle =$  (2.7.7) and that's because, as seen before, any disorder line connecting non adjacent vertices would give a subleading contribution in $\frac{1}{N}$: so the ladders are the quantum analogous of the iterated melons. The single rung ladder, i.e. the J^2 order term, is also known as fish diagram.

To each diagram one should also sum the one obtained by switching the two outgoing legs (u channel), with the relative minus sign due to the anticommutation rules of fermions.

We remark that this is the first quantum property of the model we study up to now, where by quantum we mean that the $\frac{1}{N}$ contributions are fluctuations around the classical saddle which has been studied in previous sections.

The first nontrivial term of the summation follows immediately from Feynman rules

$$\Gamma_1 = 3J^2 \int d\tau_a d\tau_b \left[G(\tau_{1a})G(\tau_{3b})G(\tau_{ab})^2 G(\tau_{a2})G(\tau_{b4}) - (2 \leftrightarrow 4)\right] \tag{2.7.8}$$

where a and b are the time indices of the two vertices and, to simplify formulas, the notation $G(\tau_{ij}) = G(\tau_i, \tau_j) = G(\tau_i - \tau_j)$ has been introduced.

The diagrammatic picture shows that we can write down a Schwinger-Dyson consistency equation: calling Γ_k the ladder with k rungs, namely the $(k+1)$-th iteration in the picture obey the recursive relation

$$\Gamma_{k+1}(\tau_1, \tau_2, \tau_3, \tau_4) = \int d\tau_a d\tau_b \hat{K}(\tau_1, tau_2; \tau_a, \tau_b)\Gamma_n(\tau_a, \tau_b, \tau_3, \tau_4) = \hat{K} \cdot \Gamma_k \tag{2.7.9}$$

where we have defined the kernel as

$$\hat{K}(\tau_1, \tau_2; \tau_a, \tau_b) = 3J^2 G(\tau_{1a})G(\tau_{2b})G(\tau_{ab})^2 \tag{2.7.10}$$

and introduced the shortcut $\cdot$ for the bilocal matrix product notation. Then the whole function can be written as a series

$$\Gamma = \sum_{k=0}^{\infty} \Gamma_k \tag{2.7.11}$$

where $\Gamma_0 = G(\tau_{12})G(\tau_{34}) - (2 \leftrightarrow 4)$ is just the disconnected piece.
So the connected correlator can be written as

$$\Gamma_c = \sum_{k=0}^{n} \Gamma_{k+1} = \sum_{k=0}^{n} \hat{K}^k \Gamma_0 = \frac{1}{1-\hat{K}}\Gamma_0 \tag{2.7.12}$$

The last expression can be evaluated by expanding Γ_0 in eigenstates of the kernel, to write the full correlator as a series with coefficients depending on the respective eigenvalues.

Now we move to the conformal limit and try to compute the questioned correlator: something interesting happens.
Substituting the conformal saddle, the kernel becomes

$$\hat{K}(\tau_1,\tau_2,\tau_3,\tau_4) = -\frac{1}{3J^2b^4}\frac{\text{sgn}(\tau_{13})\,\text{sgn}(\tau_{24})}{|\tau_{13}|^{\frac{1}{2}}|\tau_{24}|^{\frac{1}{2}}|\tau_{34}|} \tag{2.7.13}$$

The key step now is noticing that this kernel, as we can imagine in a conformal field theory, commutes with the $SL(2,\mathbb{R})$ representation given by

$$\hat{D}_i = -\tau_i\partial_{\tau_i} - \Delta, \qquad \hat{P} = \partial_{\tau_i}, \qquad \hat{R} = \tau_i^2\partial_{\tau_i} + 2\tau_i\Delta \tag{2.7.14}$$

with $\Delta = \frac{1}{4}$ the fermion anomalous dimension. The algebra structure constants can be immediately be checked by explicit calculation and are the desired ones.
This commutation has to be meant in the sense

$$\left(\hat{D}_1 + \hat{D}_2\right)\hat{K}(\tau_1,\tau_2,\tau_3,\tau_4) = (\tau_1,\tau_2,\tau_3,\tau_4)\left(\hat{D}_3 + \hat{D}_4\right) \tag{2.7.15}$$

up to total derivatives with respect to τ_3 and τ_4. So the ladders, because of conformal invariance, will just be functions of the invariant cross ratio $\theta = \frac{\tau_{12}\tau_{34}}{\tau_13\tau_{24}}$ In particular the kernel also commutes with the Casimir

$$\hat{C_{1+2}} = (\hat{D}_1 + \hat{D}_2)^2 - \frac{1}{2}(\hat{R}_1 + \hat{R}_2)(\hat{P}_1 + \hat{P}_2) - \frac{1}{2}(\hat{R}_1 + \hat{R}_2)(\hat{P}_1 + \hat{P}_2) \tag{2.7.16}$$

which means that we can write the correlator using eigenvectors and eigenvalues of this operator. In this way the series becomes

$$\Gamma(\theta) = \frac{1}{1-\hat{K}}\Gamma_0 = \sum_h \Psi_h(\theta)\frac{1}{1-k(h)}\frac{\langle\Psi_h,\Gamma_0\rangle}{\langle\Psi_h,\Psi_h\rangle} \tag{2.7.17}$$

where the sum runs over the spin related Casimir eigenvalues, labeled by h, and $k(h)$ is the corresponding conformal kernel eigenvalue.
Skipping all the tedious technical details the correlator results

$$\Gamma(\theta) = \frac{1}{3J^2b^4}\int_0^\infty \frac{ds}{2\pi}\frac{(2h-1)}{\pi\tan(\pi h)}\frac{k(h)}{1-k(h)}\Psi_h(\theta) + \frac{1}{3J^2b^4}\sum_{n=1}^{\infty}\left[\frac{(2h-1)}{\pi^2}\frac{k(h)}{1-k(h)}\Psi_h(\theta)\right]_{h=2n} \tag{2.7.18}$$

where in the integral $h = \frac{1}{2} + is$ and the eigenvalues of the kernel are

$$k(h) = -\frac{3}{2}\frac{\tan\frac{\pi(h-\frac{1}{2})}{2}}{h-\frac{1}{2}} \tag{2.7.19}$$

and is clear that $h = 2$ implies $k(2) = 1$ which makes the contribution of this mode divergent and so highly enhanced at quantum level, but out of control, if compared to all the others which are finite.

2.7.3 The gravitational mode

The above divergence is nothing else that another symptom suggesting us that the deep IR behavior leads to an approximation that is too strong to control the properties of the model, so we can think about lifting the diverging modes with a small UV correction: but this is not surprising if compared to the diverging f path integral we found if we didn't add the first UV corrections.

Despite all other modes however lead to finite contributions to the correlator, the $h = 2$ has to be studied aside from them: it precisely represents the bilocal field modes associated to the reparametrization, so we expect this infinite to be naturally cured by the Schwarzian action.

The small breaking of the conformal symmetry leads to the necessity of studying the correction starting from the finite temperature conformal two point function and study what happens slightly away from the conformal limit, where the Goldstone directions now have a finite action contribution.

If we move the conformal G along a small linearized reparametrization $\alpha \to \alpha + \epsilon(\alpha)$ the infinitesimal field variation

$$\delta_\epsilon G = \left(\frac{1}{4}\epsilon'(\alpha_1) + \frac{1}{4}\epsilon'(\alpha_2) + \epsilon(\alpha_1)\partial_{\alpha_1} + \epsilon(\alpha_2)\partial_{\alpha_2}\right) G \tag{2.7.20}$$

still has to satisfy the Schwinger-Dyson conformal saddle equation.

Denoting the integral matrix product (which actually is just a convolution in the second time entry of the bilocal field) by $F * H = \int dz F(x,z)H(z,y)$

$$\begin{gathered} 0 = \delta_\epsilon G * \Sigma + G * \delta_\epsilon \Sigma \\ 0 = \delta_\epsilon G + G * \left(3J^2 G^2 \delta_\epsilon G\right) * G = (1 - \hat{K}_c)\delta_\epsilon G \end{gathered} \tag{2.7.21}$$

where the leading order approximations have been used.

Now it's manifest that the variation of the field along the reparametrization is annihilated by the $1 - \hat{K}_c$ operator, so is an eigenvector of the conformal kernel with eigenvalue one and so $h = 2$.
We notice that this is true if and only if the variation is not trivial: this happens along the $PSL(2,\mathbb{R})$ directions, for which $\delta_{PSL} G = 0$.

So the linearized $f(\tau)$ mode is precisely associated to the enhanced contribution in the nearly conformal limit and is highly dominating, moreover it has the right Casimir eigenvalue to be thought as the boundary conformal block related to the bulk graviton: that's why it is also called gravitational mode.

Chapter 3

The Schwarzian theory

It was shown in previous chapter that the dynamics of SYK model is dominated in the IR by an action of a soft mode shaped as a Schwarzian theory. This one can be embedded in 2d Liouville gravity, which has been studied in detail and can emerge holographically as the boundary of a AdS_3 gravity, whose dimensional reduction leads to a dilaton gravity model in AdS_2, which has been proposed as a possible bulk dual of the IR SYK.

In this chapter we will discuss in detail some aspects of this picture of theories related one to each other, focusing mainly on the conjectured possible bulk dual of SYK or of some SYK-like model sharing the same symmetry breaking pattern: the nearly AdS_2 Jackiw-Teitelboim dilaton gravity model.

The section has been mainly based on [8] and [13].

3.1 The Schwarzian theory versus Liouville theory

So the Schwarzian theory describes the strong coupling (βJ) behavior of a SYK lattice, which is precisely the important one in holography. The aim of this section will be showing that the Schwarzian theory has a relation with another important model in CFT, string theory and holography: Liouville theory. The main references for this sections will be [10] and [1].

The first thing to notice is that the Schwarzian action can be rewritten through a new field variable $\sigma(t)$ such that

$$f'(\tau) = \exp\{\sigma(\tau)\} \tag{3.1.1}$$

which is legit due to the monotonicity of the reparametrization f.
It can be integrated to obtain

$$f(\tau) - f(\tau^p) = \int_{\tau^p}^{\tau} dt \exp\{\sigma(t)\} \tag{3.1.2}$$

After the transform the Schwarzian becomes

$$-\int d\tau\{f,\tau\} = -\int d\tau\left(\sigma'' - \frac{1}{2}\left(\sigma'\right)^2\right) = \int d\tau\left(\frac{1}{2}\left(\sigma'\right)^2\right) \tag{3.1.3}$$

where in the last step the total derivative has been omitted.

The reparametrized two point function G_σ of the conformal deep IR limit of SYK model in the new variable assumes the form

$$\begin{gathered} G_\sigma\left(\tau,\tau'\right) = \left[f'\left(\tau\right)f'\left(\tau'\right)\right]^{\frac{1}{4}} G\left(f\left(\tau\right),f\left(\tau'\right)\right) = \\ \frac{b}{\sqrt{J}}\left(f'(\tau)f'(\tau)\right)\frac{\operatorname{sign}\left(f(\tau)-f(\tau')\right)}{|f(\tau)-f(\tau')|} = \\ \frac{b}{\sqrt{J}}\frac{\mathrm{e}^{\frac{1}{4}\sigma(\tau)}\mathrm{e}^{\frac{1}{4}\sigma(\tau')}}{\left[\int_{\tau^p}^{\tau} dt\mathrm{e}^{\sigma(t)}\right]^{\frac{1}{2}}}\operatorname{sign}(\tau-\tau') \end{gathered} \tag{3.1.4}$$

Now we can perform the functional integration over the σ field weighted by the exponential of the Schwarzian action expressed in the new variable: this corrects the deep IR two point function including the contribution of the pseudo-Goldstones. Naming this corrected propagator $\tilde{G}$ this means:

$$\tilde{G}(\tau-\tau') = \int \mathcal{D}\sigma G_\sigma(\tau-\tau')\mathrm{e}^{-M\int d\tau[\sigma']^2} \tag{3.1.5}$$

where the Schwarzian coupling has been called M. Now we exploit the well known gamma function identity

$$\left[\int_{\tau^p}^{\tau} dt\exp\{\sigma(t)\}\right]^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty d\alpha\alpha^{\frac{1}{2}-1}\exp\left\{-\alpha\int_{\tau^p}^{\tau} dt\exp\{\sigma(t)\}\right\} \tag{3.1.6}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

So the two point SYK function, including correction slightly away from the conformal limit but still at leading order in the large N expansion, becomes

$$\tilde{G} = \frac{b}{\sqrt{\pi J}}\int_0^\infty \frac{d\alpha}{\sqrt{\alpha}}\int \mathcal{D}\sigma\mathrm{e}^{\frac{1}{4}}\sigma(\tau)\mathrm{e}^{\frac{1}{4}}\sigma(\tau')\mathrm{e}^{-M\int d\tau[\sigma']^2-\alpha\int_{\tau^p}^{\tau} dt\exp\{\sigma(t)\}} \tag{3.1.7}$$

and the effective action in the σ field becomes the euclidean quantum mechanical action of a theory with a Liouville potential containing the exponential of the field: our two point function can be sees as some sort of quantum Liouville expectation value of a symmetric bilocal operator $\mathrm{e}^{\frac{1}{4}\sigma(\tau)}\mathrm{e}^{\frac{1}{4}\sigma(\tau')}$.

However there are two important differences with a pure Liouville quantum mechanics one has to notice:

- after performing the Liouville path integral, here one has to integrate again over all possible positive values of the coupling α through the singular measure $\alpha^{-\frac{1}{2}}$: fortunately the integral factors out of matrix elements and can be ignored, even though it is formally divergent;

- the Liouville potential is time dependent, because it does not act always, but just in a time window limited between τ' and τ.

This fact is interesting because the euclidean Liouville quantum mechanics can be seen as the 1 dimensional analogue of Liouville $2d$ gravity, a model which can emerge holographically as the boundary of a 3 dimensional AdS gravity, but this aspect will be clarified later.

Conversely Liouville gravity can be manipulated to give back a $1d$ Schwarzian theory. We start from the Hamiltonian density of Liouville gravity on a surface with boundary

$$H_L = \frac{1}{8\pi b^2}\left(\frac{\pi_\sigma^2}{2} + \frac{\sigma_r^2}{2} + \mathrm{e}^\sigma - \sigma_{rr}\right) \tag{3.1.8}$$

where π_σ is the conjugate momentum to the Liouville field and σ_r is the derivative of the latter in the spatial variable.
The parameters of the theory are the central charge $c = 1 + 6Q^2$ and $Q = b + b^{-1}$.

To extract the Schwarzian we have to perform the non-canonical Gervais-Neveu field redefinition

$$\begin{aligned} \mathrm{e}^\sigma &= -8\frac{A_r B_r}{(A-B)^2} \\ \pi_\sigma &= \frac{A_{rr}}{A_r} - \frac{B_{rr}}{B_r} - 2\frac{A_r + B_r}{A - B} \end{aligned} \tag{3.1.9}$$

the first equation dictates that the two fields must be monotonic with opposite sign derivatives.
This transformation is invertible up to a double $PSL(2,\mathbb{R})$ simultaneous transformation in the new fields

$$A \to \frac{\alpha A + \beta}{\gamma A + \delta} \qquad B \to \frac{\alpha B + \beta}{\gamma B + \delta} \tag{3.1.10}$$

In the large central charge limit the Hamiltonian density in the new fields gets rewritten as the sum of the Schwarzian derivatives of the new fields

$$H_L = -\frac{c}{24\pi}\left\{(A(t,\tau), r\right\} + \left\{(B(t,\tau), r\right\}) \tag{3.1.11}$$

where the aforementioned symmetry is now manifest.

The transformation, being non canonical, modifies the symplectic form measure ω of the path integral, with the added prescription to quotient out the $PSL(2,\mathbb{R})$ redundancy.

$$\omega = \int_0^\pi dr \delta\pi_\sigma \wedge \delta\sigma = \int_0^\pi \left[\frac{\delta A_{rr} \wedge \delta A_r}{{A_r}^2} - \frac{\delta B_{rr} \wedge \delta B_r}{{B_r}^2}\right] \tag{3.1.12}$$

where the r coordinate represents the linear coordinate on a finite cylinder $[0,\pi] \times \mathcal{S}^1$ on which we define the theory, the circular coordinate being the periodic euclidean time.
Moreover we perform a thermal reparametrization, through the tangent map as discussed in the previous section, of the two fields

$$A(r,\tau) = \tan\frac{a(r,\tau)}{2}, \qquad B(r,\tau) = \tan\frac{b(r,\tau)}{2} \tag{3.1.13}$$

this is done to implement automatically the circular boundary conditions on the two fields defined on the cylinder, in fact the classical solution of the configuration is

$$\mathrm{e}^{\sigma} = -2\frac{f'(u)f'(v)}{\sin\left(\frac{f(u)-f(v)}{2}\right)^2} \tag{3.1.14}$$

where the function satisfies the $f(x+2\pi) = f(x) + 2\pi$ monodromy on the circle and so can be mapped to a circle reparametrization by conjugating f through the tangent function and, if we have to remove the $PSL(2,\mathbb{R})$ ambiguity, becomes nothing more that an element of the manifold where the IR SYK model lives.

So in terms of a, b the equation relating the Liouville potential to the Gervais Neveu fields becomes

$$e^{\sigma} = -2\frac{a_r b_r}{\sin\left(\frac{a-b}{2}\right)^2} \tag{3.1.15}$$

The boundary conditions of the theory are fixed in a way such that the Liouville fields is divergent ad $r = 0$ and $r = \pi$, so that $a(0,\tau) = b(0,\tau)$ and $a(\pi,\tau) = b(\pi,\tau) + 2\pi$.

We can introduce in this case a field defined on the circle, $[-\pi,\pi]$ with endpoints glued via the following *doubling trick*

$$f(r) = \begin{cases} a(r) & \text{0<r<}\pi \\ b(-r) & \text{-}\pi\text{<r<0} \end{cases} \tag{3.1.16}$$

so f is continuous, monotonic increasing and invertible on the circle, so we can ask $f \in Diff^{+}(\mathcal{S}^1)$.

We now write the expectation value of a generic correlator of n primaries $\mathrm{e}^{l_i\sigma(r_i,\tau_i)}$ which exhaust the nontrivial vertex operators of the theory: their transformation law as a $(1,1)$ differential is consistent with the chosen form of the Liouville Hamiltonian

$$\begin{aligned} \langle\prod_{i=1}^{n}\mathrm{e}^{l_i\sigma(r_i,\tau_i)}\rangle &= \int \mathcal{D}\sigma\mathcal{D}\pi_\sigma \prod_{i=1}^{n}\mathrm{e}^{l_i\sigma(r_i,\tau_i)}\mathrm{e}^{\frac{c}{28\pi}\int_0^\pi dr\int_0^T d\tau(i\pi_\sigma\dot{\sigma}-H)} \\ = \int \mathcal{D}A\mathcal{D}B\,\mathrm{Pf}(\omega)\prod_{i=1}^{n}&\left(\frac{A_{r_i}B_{r_i}}{(A-B)^2}\right)^{l_i}\mathrm{e}^{\frac{c}{28\pi}\int dr\int_0^T d\tau(i\pi_\sigma(A,B)\dot{\sigma}(A,B)+2\{A,r\}+2\{B,r\})} \end{aligned} \tag{3.1.17}$$

where the prescribed boundary conditions for the fields are assumed. $\mathrm{Pf}(\omega)$ is just the Pfaffian of the symplectic two-form and arises as Jacobian of the Gervais Neveu transformation.

The Schwarzian theory makes its appearance taking the limit in which the temperature goes to 0, but to keep the action finite the limit has to be taken such that $\frac{cT}{24\pi} = C$ remains fixed, which means that $c \to \infty$. The path integral gets modified to

$$\int \mathcal{D}A\mathcal{D}B\,\mathrm{Pf}(\omega)\prod_{i=1}^{n}\left(\frac{A_{r_i}B_{r_i}}{(A-B)^2}\right)^{l_i}\mathrm{e}^{\frac{c}{28\pi}\int_0^\pi dr\int_0^\tau(i\pi_\sigma(A,B)\dot{\sigma}(A,B)+2\{A,r\}+2\{B,r\})} \tag{3.1.18}$$

We remark that this limit has removed the τ dimension, leaving a theory defined just on the linear direction of the cylinder and, via the doubling trick, on a circle.

To write these correlators in terms of the doubled field f we first have to transform the symplectic two form applying the thermal reparametrization mapping $(A,B)\to(a,b)$

$$\begin{aligned}\omega &= \int_0^\pi\left[\frac{\delta a_{rr}\wedge\delta a_r}{{a_r}^2}-\left(\frac{2\pi}{\beta}\right)^2\delta a'(r)\wedge\delta a(r)\right]-(a\leftrightarrow b)\\ &= \int_{-\pi}^\pi\left[\frac{\delta f_{rr}\wedge\delta f_r}{{f_r}^2}-\left(\frac{2\pi}{\beta}\right)^2\delta f'(r)\wedge\delta f(r)\right]\end{aligned} \tag{3.1.19}$$

and the path integral finally is

$$\int_{Diff^+(\mathcal{S}^1)\backslash PSL(2,\mathbb{R})}\frac{\mathcal{D}f}{\dot{f}}\prod_{i=1}^{n}\left(\frac{\dot{f}(r_i)\dot{f}(-r_i)}{2\sin\left(\frac{1}{2}\left(f(r_i)-f(-r_i)\right)\right)^2}\right)^{l_i}\mathrm{e}^{C\int_{-\pi}^{\pi}dr\{F,r\}} \tag{3.1.20}$$

with $F=\tan\frac{1}{2}f$.
The doubling trick has transformed the local vertex operators of Liouville gravity to bilocal operators in the Schwarzian theory.

3.2 Jackiw-Teitelboim dilaton gravity

Now it's time to move to the conjectured bulk dual model, the aim of the next sections will be showing that a completely analogous of the SYK symmetry breaking pattern can be found in the, apparently unrelated, boundary dynamics of AdS_2 Jackiw-Teitelboim (or Almheiri-Polchinski) gravity, which is defined through the action

$$S_{JT}=\frac{1}{16\pi G_2}\left[\int_\Sigma d^2x\sqrt{g}\Phi^2(R_g+2)+2\int_{\partial\Sigma}du\sqrt{h}\Phi_b^2K\right] \tag{3.2.1}$$

The meaning and the importance of the scalar field Φ (with boundary value Φ_b), called dilaton, will be cleared soon, R_g is the Ricci scalar, G_2 is the two-dimensional gravitational constant, g and h are the determinants of the metric and of the induced metric on the boundary respectively and K is the trace of the second fundamental form and will be proved to be the most important connection between this theory and SYK.

In fact the last term in the action, except the ϕ_b, is precisely the Gibbons-Hawking term which is necessary to have a well defined variational problem for gravitational theories on manifolds with boundary returning the Einstein equations, provided that the metric has null variation on the boundary. In the $nAdS/nCFT$ picture this term describes the boundary dynamics from the bulk perspective, and it will have to agree with the one described by the candidate boundary nearly conformal theory.

Naturally a bulk theory for SYK model should be a gravitational theory in two dimensions, but why should we choose this action? The Riemann tensor symmetries dictate that it has a number of independent components given by $\frac{D^2(D^2-1)}{12}$.

The number of independent components in two dimensions just reduces to one: this means that the Ricci scalar encodes all information about the curvature of the manifold.

But if we try to write down the Hilbert-Einstein action in $D = 2$ we actually find the absence of dynamics, in fact by Gauss-Bonnet theorem that integral is topological and the action becomes a constant depending just on the topology of the two dimensional euclidean space-time

$$S_{H-E} = \frac{1}{16\pi G_2}\int_\Sigma d^2x\sqrt{-g}R + S_{G-H} = \frac{1}{8G_2}\chi(\Sigma) \tag{3.2.2}$$

Here $\chi(\Sigma)$ is just the Euler characteristic of the manifold we integrate over and so, being a topological invariant and moreover a *homotopical* invariant, it cannot tell us anything about the geometry: the variation of the action is null and gives rise to no Euler-Lagrange Einstein equations.

In this case the Gibbons-Hawking term just contributes to the Euler characteristic.

If we want a two dimensional theory which actually propagates gravitational degrees of freedom the action hat to be changed. One possibility is considering the Hilbert-Einstein action in 3 dimensions and integrate away one of the coordinates, using some suitable ansatz, to perform what is called a *dimensional reduction*.

In $3D$ of course the H-E action has precisely the same form, and by plugging in the spherical ansatz

$$ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \lambda^{-2}\Phi^4 d\phi^2 \tag{3.2.3}$$

where λ is a mass scale, Φ is related to the dilaton and everything is independent of the angular coordinate $\phi \in [0, 2\pi[$, so it does not contribute to the scalar curvature ($R^{(3)} = R^{(2)}$), which we are integrating away. The direct computation gives rise to a JT gravity theory

$$S_{HE} = \frac{1}{16\pi G_3}\int_\Sigma d^2x d\phi\sqrt{-g^{(3)}}\frac{\Phi^2}{\lambda}(R^{(3)} - \Lambda) = \frac{1}{8\lambda G_3}\int_\Sigma d^2x\sqrt{-g^{(2)}}\left(\Phi^2 R^{(2)} - \Lambda\right) \tag{3.2.4}$$

This spherical reduction sets an interesting relation between two different theories: AdS_3 gravity and JT-gravity. In fact when Λ, the usual cosmological constant, is negative we have an AdS_3 solution in the unreduced theory and a dilaton theory in the reduced one.

However this is not the only possibility we have to obtain dilaton gravity models, especially the JT model: actually it can emerge from a much more physical framework, i.e. the study of the near horizon geometry of a near extremal magnetically charged Reissner-Nordstrom black hole. This subject evades from our purposes, so we will just briefly sketch the whole picture without technical details and explicit calculation.

The action under exam just accounts gravity and an electromagnetic field and, of course, takes its name from Einstein and Maxwell

$$S_{EM} \approx \frac{1}{l_p^2} \int d^4x \sqrt{-g} \left(R_g - \frac{l_p^2}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{3.2.5}$$

Now l_p is the Planck length, f is the electromagnetic field and R_g is the Ricci scalar relative to the metric g.

If we look for static and spherically symmetric solutions, as above, the action can be dimensionally reduced integrating out the angular variables under the ansatz

$$\begin{gathered} ds^2 = g_{ij} dx^i dx^j + \Phi^2 d\Omega^2 \\ F = Q \sin\theta d\phi \wedge d\theta \end{gathered} \tag{3.2.6}$$

where the electromagnetic field F describes a magnetic charge Q and the Latin indices in g label the time and the radial coordinate.

The reduction, after some manipulations, gives back a dilaton gravity model described by the action

$$S_{EM} \propto \frac{4\pi}{l_p^2} \int dt dr \sqrt{-g} \left[\Phi \left(R_g + 2(\partial\Phi)^2 + 2 - \frac{1}{2} \Phi^{-2} Q^2 l_p^2 \right) \right] \tag{3.2.7}$$

If one starts from a theory like the Einstein-Maxwell one with more matter content and performs dimensional reductions through similar procedures, the general form of the dilaton gravity action one obtains is

$$S_dilaton = \frac{1}{16\pi G_N} \int d^2x \sqrt{-g} \left[\Phi^2 R_g + \lambda (\partial\Phi)^2 - U\left(\frac{\Phi^2}{d^2} \right) \right] \tag{3.2.8}$$

with U some scalar potential, λ a dimensionless coefficient and d a parameter of length dimension used to cancel the dimension of the dilaton in the argument of the potential.

Let's take a constant $\Phi^2 = \phi_0 > 0$ for the dilaton field. To study solutions which contain only fluctuations of this value, so $\Phi^2 = \phi_0 + \phi$. Under this assumption the action (3.2.1), up to a constant term, reduces to

$$S_{JT} = \frac{\phi_0}{16\pi G_2} \int_\Sigma d^2x \sqrt{-g} R_g + \frac{1}{16\pi G_2} \int_\Sigma d^2x \sqrt{-g} \phi (R_g + 2) \tag{3.2.9}$$

In the case of a manifold with boundary we need to restore the Gibbons-Hawking term to obtain

$$\begin{gathered} S_{JT} = \frac{1}{16\pi G_2} \left[\phi_0 \int_\Sigma d^2x \sqrt{-g} R_g \right. \\ \left. + 2\phi_0 b \int_{\partial\Sigma} du \sqrt{-h} K \right] + \frac{1}{16\pi G_2} \left[\int_\Sigma d^2x \sqrt{-g} \phi (R_g + 2) + \int_{\partial\Sigma} du \sqrt{-h} \phi_b K \right] \end{gathered} \tag{3.2.10}$$

This is the action we are going to analyze.

3.3 Nearly AdS_2 geometry

Moving forth to finally understand why SYK model, under the discussed limits, can be important from a geometric point of view, namely how the Schwarzian dynamics arises on the boundary of a two dimensional black hole, we will now give a dual geometric interpretation of the conformal limit of SYK and of the nearly conformal limit.

If we extremize the JT action with respect to the dilaton field with constant value on the boundary of Σ to get

$$0 = \delta_\phi S_{JT} = \int_\Sigma \sqrt{-g}\delta_\phi(R_g + 2) \tag{3.3.1}$$

where the topological piece has obviously null variation and the overall constant has been omitted from the action variation.

The solution is trivial: $R_g = -2$.
So the dilaton field just acts a Lagrange multiplier enforcing the local metric to be a pure AdS_2 if we want our coordinates to be global, otherwise it can be any negatively curved 2-manifold equipped with a constant negative curvature, moreover this means that the only allowed classical dynamics for the model, fixed the manifold, is given by the boundary.

In one patch the metric can be written in the conformal form

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} \tag{3.3.2}$$

The dual quantum theory is defined through a path integral, which converges just in euclidean signature, so the euclidean gravity theory is the one we have to study. Every computed correlator can be taken back to Minkowskian signature by analytic continuation.

By Wick rotating to euclidean time $\tau = it$ we get for the metric

$$ds_E^2 = \frac{d\tau^2 + dz^2}{z^2} \tag{3.3.3}$$

And this just becomes the Poincaré metric on the upper-half plane, where $\zeta \in \mathbb{H}$ can be written as $\zeta = \tau + iz$, with $\tau \in \mathbb{R}$ in a zero temperature case and $\tau \in [0, 2\pi]$ with the endpoints glued in the finite temperature one. The z coordinate instead has to be positive and non-null. Thus this metric can represent globally any subset of $\mathbb{H}$.

However the negative curvature constraint restricts our possibilities to any constant negative curvature 2-manifold, both in euclidean and Minkowskian signature: in the first case for example all compact Riemann surfaces with $g \geq 2$, with or without punctures, endowed with the Poincaré metric through the inverse uniformizing function.

The existence of such a metric on a generic compact Riemann surface is guaranteed by the discussed uniformization theorem which states that they all admit $\mathbb{H}$ as universal covering. The latter has been showed to be conformally equivalent to the Poincaré disk Δ through the Cayley map.

The action is also topologically dependent, so even if all the discussed possibilities are allowed extrema, if we searched for an absolute minimum of the action, in the no boundary case, it would mean searching for a surface with minimum Euler characteristic and so a surface with infinite genus, punctures or branch points.

We choose for now the pure euclidean AdS_2, i.e. the Poincaré disk, which being simply connected ($H_0(\Delta,\mathbb{R})=\mathbb{R}, H_n(\Delta,\mathbb{R})=0\forall n\geq 1$) has Euler characteristic $\chi(\Delta)=1$.

We immediatly notice that the Hilbert-Einstein metric becomes problematic in this limit: the Poincaré disk is intrinsically uncompact and has constant negative curvature, but infinite surface. This means that Gauss-Bonnet theorem does not hold and we cannot remove the constant topological piece from the action: it clearly diverges (taking the Hilbert-Einstein piece to its absolut and, being an invariant, there is no possible diffeomorphism to remove this singularity.

$$\int_\Sigma d^2x\sqrt{g}R_g = -4\pi\int_0^1 \frac{\rho d\rho}{\left(1-\rho^2\right)^2} = -\infty \tag{3.3.4}$$

where a -2 factor comes from the constant Ricci scalar, the 2π from the angular integration and the radial integral is infinite.

So this choice must be discarded: we cannot really take the near horizon limit because it gives rise to an infinite action, which might be avoided by integrating up to $1-\epsilon$, so cutting out a region from the interior of the disk. Otherwise one should define the theory on a compact negatively curved surface.

There is a physical way of visualizing this: anytime we try to couple the model to matter this would imply a boundary divergent dilaton field: this is the *back-reaction problem.*

In fact taking the generic dilaton gravity action and writing the metric in conformal gauge $ds^2=-\mathrm{e}^\sigma du^+du^-$, from the equation of motion for the h^{++} component

$$-\mathrm{e}^\sigma\partial_+\left(\mathrm{e}^{-\sigma}\partial_+\Phi^2\right) = T^{matter}_{++} \tag{3.3.5}$$

This equation can be integrated along a null curve $u^-=0$ from $u^+=0$ to $u^+=\pi$

$$\int_0^\pi du^+\mathrm{e}^{-\sigma}T^{matter}_{++} = \lim_{u^+\to 0}\left[\mathrm{e}^{-\sigma}\partial_+\Phi^2\right] - \lim_{u^+\to\pi}\left[\mathrm{e}^{-\sigma}\partial_+\Phi^2\right] \tag{3.3.6}$$

But we know that the conformal factor under those limits, which approach the boundary in these coordinates, diverges quadratically and so its inverse approaches zero quadratically. This means that the dilaton field has to diverge linearly if we want its derivative to compensate and give a non null (positive) integral: from a physical point of view the null stress-energy tensor would mean that we cannot couple this model to anything. But if the dilaton field blows up at the boundary, the boundary/dilaton term gives a divergent action contribution and the boundary is destroyed: this reminds of the diverging path integral of SYK model in the deep IR limit, which forced us to search for the first UV contribution!

From the holographic renormalization group point of view the radial dimension z on the euclidean AdS_2 should be interpreted as a energy scale on a desirable boundary theory, when $z\to 0$ gets arbitrarily closer to the boundary the dual theory is strongly coupled, which is precisely the limit in which SYK model develops the conformal symmetry.

The geometric framework of the dilaton theory cannot be the whole Poincaré disk, but we should move away from this limit. A way to do this is cutting out a closed simply connected subset $\Sigma_\epsilon \subset \Delta$ which will be seen as a open set $U_\epsilon(\Delta) \subset \Delta$ together with its boundary. The latter is just a curve $\gamma(u)$ in hyperbolic space and ϵ parametrizes its distance from the true boundary of the disk.

The hyperbolic geometry in this space, when going back to Lorentzian signature, will be called nearly AdS_2, or $nAdS_2$.

Now we take coordinates (t, z) and parametrize a curve which is a boundary of the AdS_2 region which cuts out the set U_ϵ: $\gamma(u) = (t(u), z(u))$. We normalize the proper length such that the induced metric on the boundary satisfies $g_{\partial\Sigma} = \frac{1}{\epsilon^2}$:

$$\epsilon^2 \frac{t'^2 + z'^2}{z^2} = 1 \tag{3.3.7}$$

Inverting this relation we can find $z(u)$, the *euclidean distance* of a point of the new boundary from the true boundary of the disk

$$z = \epsilon\sqrt{t'^2 + z'^2} = \epsilon t' + o(\epsilon^3) \tag{3.3.8}$$

As $\epsilon \to 0$ the cut out space tends to the whole disk and we get close to the actual boundary and the single function $t(u)$, representing the time on the boundary theory, determines the curve, its extrinsic curvature and consequentially the gravitational action, which will be explicitly evaluated in the next section.

Now we want to stress the analogy between the small ϵ limit and the strong coupling limit of SYK:

- This parameter works in this theory as an IR cut-off, when we take the whole disk, the action is independent from the $t(u)$ function: changing it just reparametrizes the boundary circle.
- Reparametrizations of the circle was precisely the symmetry group of the deep IR/strong coupling SYK, but now it arose through a geometric description.
- A parametrization choice for the boundary in Δ spontaneously breaks this symmetry, as a saddle choice in SYK.
- The different possible shapes of the boundary of Σ_ϵ explicitly breaks this symmetry because the $t(u)$ geometric modes get lifted, acquiring a non null contribute in the action.

The euclidean distance ϵ of our new boundary curve from the true boundary of the disk is the UV cutoff we put to our compact theory Our next and final aim is studying the functional form of the action in term of the function $t(u)$ to compare its residual symmetry to the $PSL(2, \mathbb{R})$ in SYK.

3.4 Geometric Schwarzian theory

Now everything is ready to study the only nontrivial gravitational dynamics of the model, which is given by the boundary action that remains after integrating out the bulk dilaton field, except for the constant topological term.

$$S_{boundary} = \frac{1}{8\pi G} \int_{\partial\Sigma} du\sqrt{-h}\phi_b K \tag{3.4.1}$$

This action just depends on the trace of the second fundamental form, coupled to the boundary dilaton field. First we regularize the field separating the u dependent finite piece from the diverging part defining

$$\phi_b = \frac{\phi_r(u)}{\epsilon} \tag{3.4.2}$$

The trace of the extrinsic curvature tensor for a one dimensional boundary can be evaluated as

$$K = -\frac{h\left(T, \nabla_T n\right)}{h(T, T} \tag{3.4.3}$$

where $T^a = (t', z')$ is the tangent vector to the γ boundary curve pushed forward to the tangent space of Δ through the embedding of the curve, $n^a = \frac{z}{\sqrt{t'^2+z'^2}}(-z', t')$ is a normalized vector orthogonal to the curve respect to Poincaré metric, so such that $h(T, n) = 0$, ∇_T is just the covariant derivative along the tangent vector field.

The resulting extrinsic curvature follows from direct computation

$$K = \frac{t'\left(t'^2 + z'^2 + z'z''\right) - zz't''}{\left(t'^2 + z'^2\right)^{\frac{3}{2}}} \tag{3.4.4}$$

Taking the small ϵ limit we finally find that the action reduces to the Schwarzian derivative of the function describing the reparametrization.

$$K = 1 + \{t(u), u\}\epsilon^2 + o(\epsilon^4) \tag{3.4.5}$$

And by plugging all these results inside the action we get the same Schwarzian action which governed the reparametrization pseudo Goldstone bosonic modes of SYK and breaks the full reparametrization of the boundary invariance of Δ to a $PSL(2, \mathbb{R})$ transformation in the field $t(u)$.

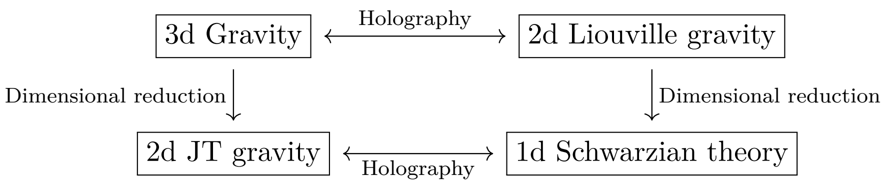
$$S_{Sch} = \frac{1}{8\pi G} \int du\phi_r(u)\{t(u), u\} \tag{3.4.6}$$

This fact puts the JT $nAdS_2$ gravity in an interesting picture which relates it to different theories:

- Previously we proved that the aforementioned action can be obtained after an angular dimensional reduction of AdS_3 Einstein gravity, on the euclidean boundary of a physically consistent version of the reduced theory we found that the only gravitational degrees of freedom obey the Schwarzian action dynamics;
- Moreover the Schwarzian dynamics has been argued to appear for all quantum mechanical models developing a conformal symmetry in the deep IR, all pathological signs of this limit, as we've seen in the study of the 4-point SYK correlator, ask by themselves to be cured moving to a small UV limit in which the symmetry breaking pattern gives back precisely a Schwarzian action.
- However even if there is no explicit procedure which gives rise to the Schwarzian in SYK model, the latter exhibits a deep connection to Liouville quantum mechanics and from Liouville field theory we've shown that one can reproduce a Schwarzian theory.

Now we make a last important statement, just quoting [6] and [10]: Liouville field theory can emerge holographically from AdS_3 gravity.

This creates a nice diagram relating different theories through holography and dimensional reductions.



Conclusions

In this thesis we have discussed in detail the properties of the SYK model, focusing mainly on the role of the Schwarzian action.

Then we have discussed how this unusual theory seems to be related to many interesting and studied models in theoretical physics, mainly the Jackiw-Teitelboim dilaton gravity, whose only dynamical gravitational (and so geometric) mode describes a fluctuating boundary of the spacetime of the theory. The slice has always been considered to be an arbitrary piece of the hyperbolic Poincaré disk cut out through a closed curve.

However the only prescription coming from the action is that the manifold has negative Ricci scalar curvature and this property is true, as was discussed in the first chapter, even for generic compact Riemann surfaces with $\chi(\Sigma) < 0$. The appearance of the Schwarzian, which is deeply related to uniformization theory, in the cut out boundary action enforces the suspect that there might be a relation between this gravity theory, its boundary dual and uniformization theory.

It's been noticed that the Hilbert-Einstein action on the disk is divergent and does not just equal a topological number: this leads us to understand immediatly that the pure

Poincaré disk cannot host a nontrivial gravity theory. Cutting out the space regularizes this singularity and gives rise to the peculiar Schwarzian action. But the cutting, instead of being done in arbitrary way, might be performed along the boundary of a fundamental domain for some Riemann surface uniformized by the disk and, as we've seen, the Schwarzian is deeply related to uniformization theory.

Moreover the Schwarzian can also be interpreted as an invariant curvature, [2], for a special class of curves in projective space $\mathbb{P}^2$. All these facts induce us to believe that the dilaton gravity and the related Schwarzian mode could be studied just by a purely algebraic geometric point of view.

Keeping in mind the duality perspective we've also suggested that the birth of a bulk theory starting from a boundary theory might be related to the functional interpretation of the massless Majorana propagator as a Hilbert transform.

All these aspects still need a deeper understanding, but, in our opinion, might shed light on the not clear relation between SYK and SYK-like models to the $2d$ nearly AdS_2 black hole.

To conclude we stress that the Schwarzian nature of the IR SYK pseudo Goldstone mode still lacks a satisfactory and regulator-independent explicit evaluation. We still think that the symmetry related effective field theory approach, discussed previously in a slightly modified way and originally proposed by Maldacena and Stanford, is the most interesting one at the moment because it makes the Schwarzian theory a universal feature of the class of all models exhibiting this symmetry breaking pattern.

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