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APERIODIC SET OF TILES AND CUT AND PROJECT METHOD

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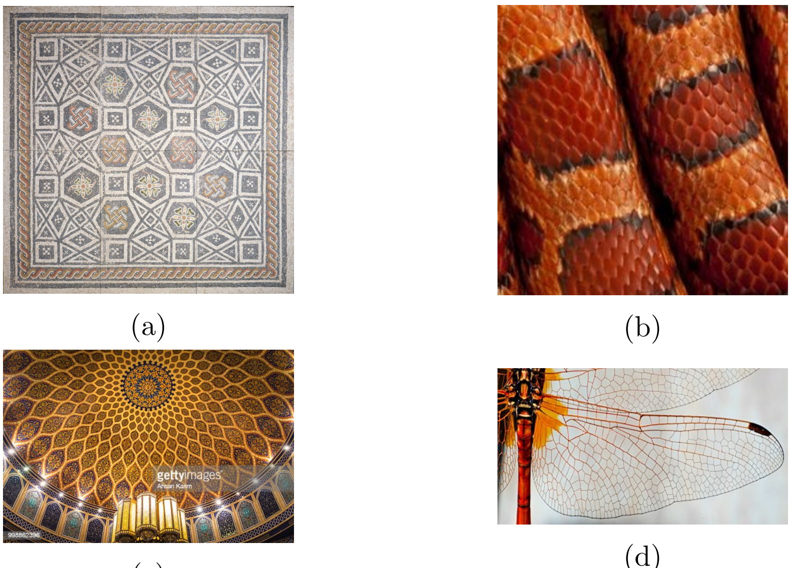
Introduction

The art of tilings and patterns has long been present in the history of civilizations. Used in almost any situation that requires some type of decoration, famous examples can be found in Roman temples, Persian, Celtic and Arabic constructions. From geometric shapes to animals and human figures are present in tilings. However, their use is not limited to aesthetics, they can also be found in nature such as in the honeycomb of a bee, the design of a spider's web, the wings of dragonflies and the skin of lizards. They have also been used in engineering for the production of components where it is more economical to put shapes together without any space between them.



(a) (b) (c) (d)

Figure 2: Some examples of tilings and patterns. (a) Roman mosaic, (b) snake skin tiling, (c) dragonflies wings tiling, (d) Persian tiling in a temple. Image credit: Image credit: [Bee].

Because we can find them all around us in different situations, mathematicians, physicists, engineers, artists, among others, have made significant contributions to the study of tilings and patterns.

In this work we will focus in the study of aperiodic set of tiles, and the project method. The last one it was first described by Bruijin [De 81] in 1981, proving that Penrose tilings can be seen as the projection of a 2-dimensional plane in a 5-dimensional space. Hence, the aim of this thesis is to try to answer the following question: Can we use the IFS (iterated function system) approach to give a description of Ammann tilings?

In order to accomplish this, we must first study the projection method, which allows us to describe certain tilings as model sets. With that goal in mind, Chapter 1 presents the basic tools we need to introduce ourselves into the study of tilings, such as number fields, Perron Frobenius Theory, lattices and theorems on the existence of fixed points in metric spaces. Chapter 2 defines the main objects of study in this work which are the tilings. We describe several classes of tilings and different examples. Further on, in Chapter 3, we study what a substitution is, and its relation with tilings. In Chapter 4 we explain the projection method, the model sets and we illustrate them with the silver mean substitution. Finally, in Chapter 5 we discuss the examples we have worked on during the study time dedicated to this thesis. Firstly, we provide the cut and project scheme of the 1-dimensional Fibonacci substitution, next we study the 19 self-similar Wang shift. Lastly we analyze if the tilings admitted by the Ammann aperiodic set A2 can be described as model sets.

Chapter 1

Preliminaries

In this introductory chapter, we recall some fundamental facts about fields and linear algebra to stablish a firm basis for the rest of the document.

1.1 Number-theoretic tools

In this section we set notation and recall briefly some important definitions and theorems about number fields which will be mainly used in the last two chapters of this document. For general reference on this topic and proofs of the theorems mentioned here refer to [BG13].

A field L is called a *field extension* of K if L is a larger field containing K and this will be denoted by $L : K$. If $L : K$ is a field extension, then L has a natural structure as a vector space over K with its dimension $[L : K]$ being the degree of the extension.

If $[L : K]$ is finite we say that L is a *finite extension* of K. Given a field extension $L : K$ and an element $\alpha \in L$, there may or may not exist a polynomial $P \in K[x]$ such that $P(\alpha) = 0$, $P \neq 0$. If not, we say that α is *transcendental* over K. If such a P exists, we say that α is *algebraic* over K. If α is algebraic over K, then there exists a unique monic polynomial Q of minimal degree such that $Q(\alpha) = 0$, and Q is called the *minimum polynomial* of α over K.

Let $\alpha_1, \dots, \alpha_k \in L$. We will denote $K(\alpha_1, \dots, \alpha_k) \subseteq L$ the smallest subfield of L containing K and $\alpha_1, \dots, \alpha_k$.

Theorem 1. *If $L : K$ is a field extension and $\alpha \in L$, then α is algebraic over K if and only if $K(\alpha)$ is a finite extension of K. In this case, $[K(\alpha) : K] = deg(P)$ where P is the minimum polynomial of α over K, and $K(\alpha) = K[\alpha]$.*

Now, we say that a complex number α is *algebraic* if it is algebraic over $\mathbb{Q}$. Let $\mathcal{A}$ denote the set of algebraic numbers. This set is a subfield of $\mathbb{C}$. If K is a subfield of $\mathbb{C}$ such that $[K : \mathbb{Q}]$ is finite then K is a *number field*, and by previous theorem this means that every element of K is algebraic, hence $K \subseteq \mathcal{A}$ and $K = \mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_n)$, for some $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{A}$

Theorem 2. *If K is a number field then $K = \mathbb{Q}(\theta)$ for some algebraic number θ.*

If $K = \mathbb{Q}(\theta)$ is a number field of degree n over $\mathbb{Q}$ and $Emb_K = \{\sigma : K \to \mathbb{C} : \sigma \text{ a monomorphism}\}$, then $card(Emb_K) = n$, where the elements $\sigma_i(\theta) = \theta_i$ are the distinct zeros in $\mathbb{C}$ of the minimum polynomial of θ over $\mathbb{Q}$. We call the θ_i's the *conjugates* of θ.

A complex number α is an *algebraic integer* if there is a monic polynomial P with integer coefficients such that $P(\alpha) = 0$. We will denote by $\mathcal{B}$ the set of algebraic integers. This set is a subring of $\mathcal{A}$ and we will write $\mathfrak{O} = K \cap \mathcal{B}$ for the ring of integers of a number field K. The following criterion allows us to determine when an algebraic number α is an algebric integer.

Proposition 1. *An algebraic number α is an algebraic integer if and only if its minimum polynomial over $\mathbb{Q}$ has coefficients in $\mathbb{Z}$.*

A *quadratic field* is a number field $K = \mathbb{Q}(\theta)$ of degree 2 over $\mathbb{Q}$. This means that θ is a zero of the polynomial $x^2 - ax + b$ for $a, b \in \mathbb{Z}$.

Theorem 3. *Let d be a square-free rational integer. Then the integers of $\mathbb{Q}(\sqrt{d})$ are:*

1. $\mathbb{Z}[\sqrt{d}]$ *if* $d \not\equiv 1$ *mod 4,*
2. $\mathbb{Z}[\frac{1}{2}(1+\sqrt{d})]$ *if* $d \equiv 1$ *mod 4.*

Example 1. The examples presented here are used in later chapters.

1. Since $\alpha = \sqrt{2}$ satisfies $P(x) = x^2 - 2$, then it is an algebraic integer with minimal polynomial P and correponding quadratic number field $K = \mathbb{Q}(\alpha)$. By Theorem 3, the ring of rational integers of K is $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.
2. $K = \mathbb{Q}(\sqrt{5})$ is a quadratic number field since $\alpha = \frac{1}{2}(1+\sqrt{5})$ is an algebraic integer with minimal polynimial $P(x) = x^2 - x - 1$. Again, using Theorem 3, the ring of rational integers of K is $\mathbb{Z}[\alpha]$.

1.2 Lattices

Lattices are used in the last two chapters to define a certain set of points that characterise the set of tiles we are working on. We recall definitions and important theorems from the book [BG13].

A set consisting of one point is called a *singleton set*, and countable unions of singleton sets are called *point sets*. We say that a point set $\Lambda \in \mathbb{R}^d$ is *discrete* if each element in Λ has an open neighbourhood in $\mathbb{R}^d$ that does not contain any other point of Λ. And Λ is *uniformly discrete* if there is an open neighbourhood U of $0 \in \mathbb{R}^d$ such that $(x+U) \cap (y+U) = \emptyset$ holds for every pair of distinct elements of Λ.

We define the *Minkowsky sum* of $U, V \subseteq \mathbb{R}^d$ as

$$U + V := \{u + v \mid u \in U, v \in V\}.$$

A point set Λ is *relatively compact* if its closure is compact, and it is *relatively dense* if exists a compact set $K \subseteq \mathbb{R}^d$ such that $\Lambda + K = \mathbb{R}^d$.

Definition 1. A lattice is a point set $\Gamma \subseteq \mathbb{R}^d$ such that

$$\Gamma = \mathbb{Z}b_1 \oplus \ldots \oplus \mathbb{Z}b_d$$

for some vectors $b_1, \ldots, b_d$, with the requirement that its $\mathbb{R}$-span is $\mathbb{R}^d$.

Example 2. The easiest lattice we can find is $\mathbb{Z}^d$ for $d \in \mathbb{N}$.

Let $K = \mathbb{Q}(\theta)$ be a quadratic number field with $\theta \in \mathbb{R}$ and consider the non trivial algebraic conjugation defined by $\theta \to -\theta$ together with its unique extension to a field automorphism. We define the *Minkowski embedding* of the ring of algebraic integers $\mathbb{Z}[\theta]$ of K as

$$\mathcal{L} = \{(x, x') \mid x \in \mathbb{Z}[\alpha]\} \subseteq \mathbb{R} \times \mathbb{R},$$

where x' is the image of x under the algebraic conjugation, with basis vectors $(1, 1)$ and (θ, θ').

Example 3.

1. The non trivial algebraic conjugation in the field $K = \mathbb{Q}(\sqrt{2})$ is $\star : \sqrt{2} \to -\sqrt{2}$. Then the Minkowsky embedding of $\mathbb{Z}[\sqrt{2}]$ is $\mathcal{L} = \{(x, x') \mid x \in \mathbb{Z}[\sqrt{2}]\}$.

2. The non trivial algebraic conjugation in the field $K = \mathbb{Q}(\sqrt{5})$ is $\star : \sqrt{5} \to -\sqrt{5}$. The Minkowsky embedding of $\mathbb{Z}[\tau]$ with $\tau = \frac{1}{2}(1 + \sqrt{5})$ being the golden ratio is $\mathcal{L} = \{(x, x') \mid x \in \mathbb{Z}[\tau]\}$.

1.3 Perron-Frobenius Theory

The Perron–Frobenius Theorem is a fundamental result for non-negative matrices that we will use later in chapter 2 where Perron Frobenius left and right eigenvectors play an important role in substitutions.

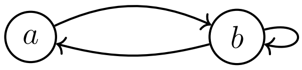
We start saying that a matrix $M \in M_d(\mathbb{R})$ is *non-negative* if all its entries are non-negative numbers. A non-negative matrix is *positive* when at least one entry is strictly greater than zero, and it is called *strictly positive* when all its entries are strictly greater than zero. We write, respectively, $M \geqslant 0$, $M > 0$ and $M \gg 0$.

Definition 2. A non-negative matrix $M = (M_{ij})_{1 \leq i,j \leq d}$ is called *irreducible* if, for each index pair (i, j), there is an integer $k \in \mathbb{N}$ with $(M^k)_{ij} > 0$.

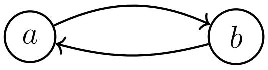
Definition 3. A non-negative matrix $M \in M_d(\mathbb{R})$ is called *primitive* if there exists an integer $k \in \mathbb{N}$ such that $M^k \gg 0$.

The directed graph $\mathcal{G}_M$ associated to a non-negative matrix $M \in M_d(\mathbb{R})$ is defined as the graph with d vertices labelled from 1 to d and a directed edge from vertex j to vertex i, for all index pairs (i, j) with $M_{ij} > 0$.

Example 4. It is clear from the definition that a primitive matrix is irreducible. For example, $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is primitive with $k = 2$, and therefore it is irreducible with associated graph



However, the converse is not true. For instance, the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is merely irreducible and it has associated graph



If $M \in M_d(F)$ is a matrix over a field F, its *adjoint matrix* $\mathrm{adj}(M)$ is the transpose of the $d \times d$ matrix whose (i, j)-th term is equal to $(-1)^{i+j}$ times the determinant of the matrix obtained from M by erasing the ith row and jth column. Let λ be a eigenvalue of M with associate eigenvector v. The *minimal polynomial* of M is the monic polynomial $q_\lambda \in F[\lambda]$ of smallest degree such that $q_\lambda(M) = 0$. Let g_λ be the monic greatest common divisor of all the coefficients of the matrix $\mathrm{adj}(\lambda I - M)$. The *reduced adjoint matrix* of $\lambda I - M$ is defined by $C_\lambda := adj(\lambda I - M)/g_\lambda$.

Theorem 4 (Perron-Frobenius Theorem)**.** *Let $M \in M_d(\mathbb{R})$ be an irreducible non-negative matrix. Its spectral radius is then strictly positive and a simple eigenvalue of M, called the Perron–Frobenius eigenvalue λ_{PF} of M. Moreover, the corresponding eigenvector v can be chosen to have all entries strictly positive, written as $v > 0$.*

To see the geometric intuition under Perron's theorem, take a d-dimensional closed positive cone C. Since the entries of the matrix are non-negative, the image of C under the transformation defined by M is itself. But the matrix is additionally primitive, then there exists $k \in \mathbb{N}$ such that after k iterations the image is contained in the interior of the cone because all the entries are positive, therefore the transformation is a contraction. And if we iterate the matrix we obtain a sequence of cones that approximate to a unique invariant direction that is PF eigenvector, as we can see in Figure 1.1 with the matrix of Example 4.

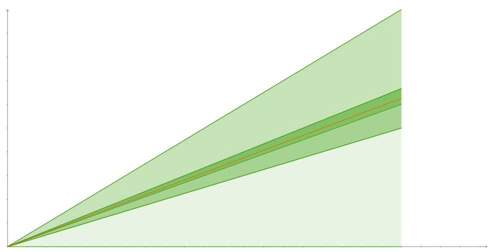


Figure 1.1: Sequence of cones obtained after the iteration of a primitive matrix and converging to the PF left eigenvector indicated in orange.

Theorem 5. *Let $M \in M_d(\mathbb{R})$ be primitive, and λ_{PF} be its Perron–Frobenius eigenvalue. Let C_λ be the reduced adjoint matrix of $\lambda I - M$, and q_λ be the minimal polynomial of M. Then:*

1. *Let u and v be a right and left eigenvector of M for the eigenvalue λ_{PF}. Suppose that u and v are normalized such that $vu = 1$. Then we have*

$$C_{\lambda_{PF}} = q'_{\lambda_{PF}} uv.$$

In particular, all the entries of the matrix $C_{\lambda_{PF}}/q'_{\lambda_{PF}}$ are positive.

2. The relation

$$\lim_{n\to\infty} \frac{1}{\lambda_{PF}^n} M^n = \frac{1}{q'_{\lambda_{PF}}} C_{\lambda_{PF}}$$

holds.

Proof. See [AS+03]. □

1.4 Groups

The content of this section is a short remainder of some definitions of topological abelian groups taken from [BG13].

Let G be a group, and assume that it is equipped with a topology. G is *locally compact* if each point of G possesses a compact neighbourhood. If G is also Abelian, it is called a *locally compact Abelian group*, or LCAG for short. For example, $\mathbb{R}^d$ with its usual topology is an LCAG, as is any finite Abelian group (equipped with the discrete topology) and any compact Abelian group, such as the unit circle $\mathbb{S}^1$. Further examples are discrete groups such as $\mathbb{Z}^n$ with the discrete topology.

An LCAG G is called *compactly generated* when there is a compact neighbourhood of $0 \in G$ that generates the entire group. For instance, $\mathbb{R}^d$, $\mathbb{Z}^n$ and compact Abelian groups K are examples.

1.5 Fixed point theorems

The existence of a unique fixed point of an iterated function system plays an important role in then last chapter of this document. Therefore this section intends to only present a selection of basic notions and main results. For further material refer to [Con14].

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *contraction* on X if there exists a positive constant $K < 1$ such that $d(T(x), T(y)) \leq Kd(x, y)$, for all $x, y \in X$. A *fixed point* of a mapping $T : X \to X$ of a set into itself is an $x \in X$ that satisfies $T(x) = x$. For instance, a rotation of the plane has as a unique fixed point the center of the rotation. A finite collection of contraction mappings on a complete metric space is called an *iterated function system (IFS).*

Theorem 6 (Banach's fixed point theorem)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$.*

Let (X, d) be a metric space. Let $H(X)$ be the set of non-empty closed and bounded subsets of X. To turn $H(X)$ into a metric space, a metric is defined by using the notion of expansion. Given a non-empty subset $A \subset X$ and $r \leq 0$, an *r-expansion* of A is defined to be the points in X within distance at most r of some point of A:

$$E_r(A) = \{x \in X \mid d(x, a) \leq r \text{ for some } a \in A\}.$$

The *Hausdorff distance* between $A, B \in H(X)$ defined by $d_H(A, B) = inf\{R \geq 0 : A \subset E_r(B),\ B \subset E_r(A)\}$, is a metric on $H(X)$ called the *Hausdorff metric* on $H(X)$.

Theorem 7. *Let $f : X \to X$ be a contraction with constant K. Then the induced map on $H(X)$ given by $A \to \overline{f(A)}$ is a contraction with the same constant: $d_H(\overline{f(A)}, \overline{f(B)}) \leq K \cdot d_H(A, B)$ for all $A, B \in H(X)$.*

Theorem 8 (Hutchinson)**.** *Let $f_1, \ldots, f_m : X \to X$ be contractions. Set $F : H(X) \to H(X)$ by $F(A) = \overline{f_1(A)} \cup \ldots \cup \overline{f_m(A)}$. If f_i has contraction constant K_i, F is a contraction with constant $max(K_1, \ldots, K_m)$.*

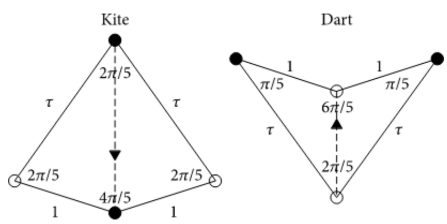
Corollary 1. *If X is a complete metric space and $f_1, \ldots, f_m$ are contractions on X, then there is a unique non-empty closed and bounded subset $A \subset X$ such that $\overline{f_1(A)} \cup \ldots \cup \overline{f_m(A)} = A$.*

Chapter 2

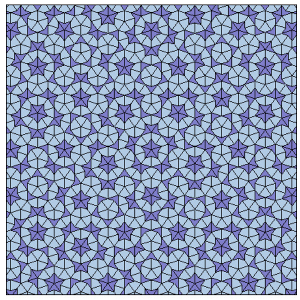
Tilings and Aperiodicity

In this chapter we define the principal notions of tilings and some of its properties. The main reference of this section is the book [GS87].

A *tile* in $\mathbb{R}^d$ is defined as a non-empty compact subset of $\mathbb{R}^d$ which is the closure of its interior. A *tiling* $\mathcal{T}$ in $\mathbb{R}^d$ is a countable set of tiles, which is covering as well as a packing of $\mathbb{R}^d$, i.e., the union of all tiles in $\mathcal{T}$ is $\mathbb{R}^d$, and the intersection of the interior of two different tiles in $\mathcal{T}$ is empty.



(a)



(b)

Figure 2.1: (a) Kite and dart Penrose tiles. (b) A tiling by Penrose kites and darts. Image credit: [Ouy+18].

We will say that two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathbb{R}^d$ are *congruent* if one of them may be made coincide with the other by a rigid motion. And we will say that these two tilings are the same if there is a similarity transformation in $\mathbb{R}^d$ that maps $\mathcal{T}_1$ onto $\mathcal{T}_2$.

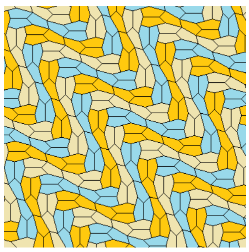


Figure 2.2: Monohedral convex pentagonal tiling. Image credit: [Wik21a]

If all the tiles of a tiling $\mathcal{T}$ are congruent to one of a minimal set $P = \{P_1, P_2, \dots, P_n\}$ of tiles, then the P_i's are called the *prototiles* of $\mathcal{T}$, and we say that they *admit* the tiling

$\mathcal{T}$. We say that a tiling $\mathcal{T}$ is *n-hedral* if there are n prototiles such that every tile of $\mathcal{T}$ is congruent directly or reflectively to one of the n prototiles. For instance, Figure 2.2 is a monohedral tiling discovered in 2015 by Casy Mann, Jennifer McLoud-Mann,and David Von Derau, mathematicians of the University of Washington Bothell, with unique prototile the convex pentagon of angles 60°, 150°, 90°, 105° and 135°.

2.1 Classes of tilings

When we talk about tilings we can refer to periodic or non-periodic tilings, aperiodic set of tiles, substitution tilings, cut and project tilings, and among others. Since the formal definition of these tilings is not uniform in the different bibliographies of the topic, we take as reference the notation used in [GS87].

An *isometry* or *congruence transformation* is a bijective distance preserving transformation between metric spaces. For example, in the Euclidean plane E^2 there are four types of isometries: rotation, translation, reflection and glide-reflection. We say that an isometry σ of a set S is a *symmetry* of S if $\sigma(S) = S$. The set of isometries of a metric space forms a group with respect to the composition called the *isometry group*.

The definition of isometry can be extended to tilings in the natural way. If $\mathcal{T}$ is a tiling, an isometry σ is a *symmetry of* $\mathcal{T}$ if it maps every tile of $\mathcal{T}$ onto a tile of $\mathcal{T}$. If $\mathcal{T}$ is a marking tiling, i.e., a tiling in which there is a marking or motif on each tile, a symmetry of the marked tiling is an isometry which maps the tiles and the markings of $\mathcal{T}$ onto tiles and markings of $\mathcal{T}$.

Definition 4. A tiling $\mathcal{T}$ is *periodic* if its symmetry group contains a non-trivial translation. And $\mathcal{T}$ is *non-periodic* if the only translation in its symmetry group is the trivial translation, i.e., the identity.

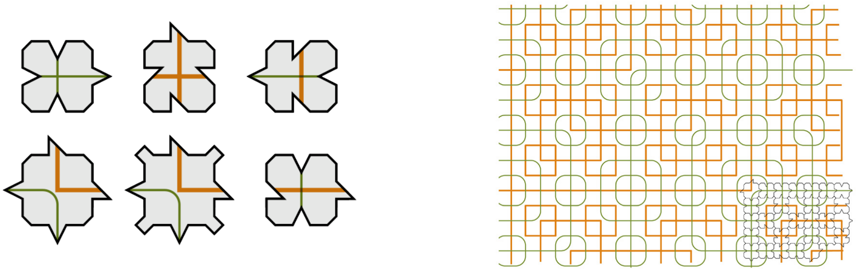
The monohedral tiling shown in Figure 2.2 is a periodic tiling as the tiling in (a) of Figure 2.3, and in (b) we have a non-periodic tiling. Notice that the group of symmetries of (b) contains rotations, however the unique translation that contains is the identity.

(a) (b)

Figure 2.3: (a) Periodic floret pentagonal tiling. Image credit [Wik21b]. (b) Non-periodic Gailiunas's spiral tiling. Image credit: [Ste17].

Definition 5. A set of prototiles in $\mathbb{R}^d$ is called *aperiodic* if it tiles $\mathbb{R}^d$, and every tiling admit by the prototiles is non-periodic.

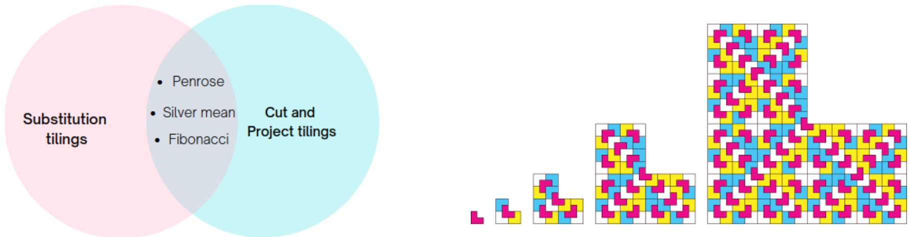
In the previous definition it is really important to make emphasis in the word *every*, since a set of prototiles can admitted non-periodic tilings and not necessarily be aperiodic. The set R1 of Robinson's tiles shown in Figure 2.4 is an example of an aperiodic set of tiles dicovered by Raphael Robinson in 1971.



(a) (b)

Figure 2.4: Robinson's aperiodic set R1 of six tiles and an admitted tiling by R1. Image credit: [Wik22].

Other classes of tilings different from periodic and non-periodic tilings are substitution and cut and project tilings. There are two classes of tiling substitution rules [Fra08]: geometric and combinatorial. The first one is based on a linear expansion map and the last one rely on a sort of concatenation of tiles. A tiling obtained from a geometric subsitution can be defined as a tiling whose tiles can be composed into larger tiles, called level-one tiles, whose level-one tiles can be composed into level-two tiles, and so on ad infinitum. In some cases it is necessary to partition the original tiles before composition. Figure 2.5 is an example of this class of tilings. Cut and project tilings are tilings that can be expressed as model sets of certains cut and project schemes. Later on, we define these classes of tilings in detail and we provide several examples. Although it seems that from every substitution tiling it is possible to obtain a CPS and viceversa, this is not true in general. There exist cut and project tilings that can not be obtained as substitution tilings and viceversa. In this document most of the examples provided in the last chapter will remain in the intersection of these two classes.



(a) (b)

Figure 2.5: (a) Relation between some classes of tilings. (b) Chair substitution tiling. Image credit: [Bla17].

Chapter 3

Substitutions

Some of the most simple examples of tilings are given using the theory of substitutions which we introduce here. The main reference of this section for the first section is [AS+03] and for the second section it will be the [BG13].

3.1 One dimensional substitutions

An *alphabet* $\mathcal{A}$ is a non empty set of symbols that can be finite or infinite. In this text we will focus on finite alphabets denoted by $\mathcal{A}_n$ if it contains n symbols. A *word* is a finite or infinite chain of symbols from $\mathcal{A}$, written by juxtaposing their symbols. Let $[m \dots n]$ denote the set of integers from m to n. A *finite word* is a map from $[1 \dots n]$ to $\mathcal{A}$. If $n = 0$, we get the *empty word*, which we denote by e, and we denote by $\mathcal{A}^*$ the set of all finite words made up of letters chosen from $\mathcal{A}$. If w is a finite word, then its length is the number of symbols it contains, which is denoted by $card(w)$. For example, if $w = 010101$, then $card(w) = 6$. The number of occurrences of a symbol a in w is denoted by $card_a(w)$. Thus, $card_0(w) = 3 = card_1(w)$.

The *composition* of two finite words u and v is defined as the concatenation of their symbols and denoted by uv. Composition of words is not commutative, but it is associative, so $\mathcal{A}^*$ together with the concatenation is a monoid. A word y is a *subword* of a word w if there exist words x, z such that $w = xyz$. We denote by $w_{[k,l]}$ for $k,\ l \in \mathbb{Z}$ and $k \geq l$ the finite subword of w from position k to l.

Definition 6. A *general substitution rule* on a finite alphabet $\mathcal{A}_n$ with n letters is a map ϱ from $\mathcal{A}_n^*$ to itself such that $\varrho(uv) = \varrho(u)\varrho(v)$ for $u,\ v \in \mathcal{A}_n^*$.

If there is a constant k such that $card(\varrho(a)) = k$ for all $a \in \mathcal{A}_n$, then we say that ϱ is *k-uniform.* A substitution is said to be *expanding* if $card(\varrho(a)) \geq 2$ for all $a \in \mathcal{A}$.

Definition 7. For a given general substitution rule ϱ, the matrix $M_\varrho \in M_n(\mathbb{Z})$ defined by $(M_\varrho)_{i,j} = card_{a_i}(\varrho a_j)$ is called the *substitution matrix* of ϱ

Example 5. (Fibonacci substitution). Consider the rule

$$\varrho : \begin{array}{l} a \to ab \\ b \to a. \end{array}$$

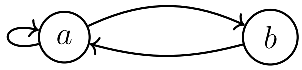
This substitution is called the Fibonacci substitution because it can be also defined by $f_0 = a$, $f_1 = ab$, $f_{n+2} = f_{n+1}f_n$, and the sequence of lengths of the words f_i is the Fibonacci sequence.

Definition 8. A substitution rule on a finite alphabet $\mathcal{A}_n$ is called *irreducible* when, for each index pair (i, j), there exists some $k \in \mathbb{N}$ such that a_j is a subword of $\varrho^k(a_i)$. And ϱ is *primitive* when some $k \in \mathbb{N}$ exists such that, for all $a_i, a_j \in \mathcal{A}_n$, a_i occurss in $\varrho^k(a_j)$.

From Section 1.3 we can deduce that a substitution rule ϱ is irreducible or primitive if and only if its substitution matrix M_ϱ is an irreducible or a primitive non-negative matrix, respectively.

Example 6. The substitution matrix and associated directed graph of the Fibonacci substitution are

$$M_\varrho = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$



Since M_ϱ is primitive then the Fibonacci substitution is also primitive.

Proposition 2. *Let ϱ be a substitution on a finite alphabet $\mathcal{A}_n$ and w be a finite word. Then, for any $l \in \mathbb{N}$,*

$$\begin{pmatrix} card_{a_1}\varrho^l(w) \\ \vdots \\ card_{a_n}\varrho^l(w) \end{pmatrix} = M_\varrho^l \begin{pmatrix} card_{a_1}(w) \\ \vdots \\ card_{a_n}(w) \end{pmatrix}.$$

Proof. By induction over l. For $l = 1$, we have that $card_{a_i}\varrho(w) = \sum_{1 \le a_i \le n} card_{a_i}\varrho(a_j)card_{a_j}(w)$ which is equal to

$$\begin{pmatrix} card_{a_i}\varrho(a_0) & \cdots & card_{a_i}\varrho(a_n) \end{pmatrix} \cdot \begin{pmatrix} card_{a_i}(w) \\ \vdots \\ card_{a_i}(w) \end{pmatrix} = {}^iM_\varrho \cdot \begin{pmatrix} card_{a_i}(w) \\ \vdots \\ card_{a_i}(w) \end{pmatrix},$$

where ${}^iM_\varrho$ is the i-th row of M_ϱ.

Suppose this is true for l and let's show it for $l + 1$.

$$\begin{pmatrix} card_{a_1}\varrho^{l+1}(w) \\ \vdots \\ card_{a_n}\varrho^{l+1}(w) \end{pmatrix} = \begin{pmatrix} card_{a_1}\varrho^l(\varrho(w)) \\ \vdots \\ card_{a_n}\varrho^l(\varrho(w)) \end{pmatrix} = M_\varrho^l \begin{pmatrix} card_{a_1}\varrho(w) \\ \vdots \\ card_{a_n}\varrho(w) \end{pmatrix} = M_\varrho^l M_\varrho \begin{pmatrix} card_{a_1}(w) \\ \vdots \\ card_{a_n}(w) \end{pmatrix}$$

□

Since we can associated a matrix to a substitution, we can use PF theory to describe some properties of non-negative substitutions. But before this we will introduce some important definitions and propositions.

Proposition 3. *Let ϱ be a primitive substitution on the finite alphabet $\mathcal{A}_n = \{a_1, \ldots, a_n\}$. Let M_ϱ be the substitution matrix of ϱ, and let λ_{PF} be the PF eigenvalue of M_ϱ. Then for each non-empty word w on $\mathcal{A}_n^*$, there exists a positive constant c_w such that*

$$\lim_{l \to \infty} \frac{card(\varrho^l(w))}{\lambda_{PF}^l} = c_w$$

Proof. If $w \in \mathcal{A}_n^*$ is a non empty word then by the previous proposition,

$$\lim_{l\to\infty} \frac{card(\varrho^l(w))}{\lambda_{PF}^l} = \lim_{l\to\infty} \frac{1}{\lambda_{PF}^l} \begin{pmatrix}1 & \cdots & 1\end{pmatrix} M_\varrho^l \begin{pmatrix} card_{a_1}(w) \\ \vdots \\ card_{a_n}(w) \end{pmatrix}.$$

Then, by Theorem 5, this limit exists and

$$\lim_{l\to\infty} \frac{card(\varrho^l(w))}{\lambda_{PF}^l} = \begin{pmatrix}1 & \cdots & 1\end{pmatrix} \frac{M_\varrho^l}{\lambda_{PF}^l} \begin{pmatrix} card_{a_1}(w) \\ \vdots \\ card_{a_n}(w) \end{pmatrix} = c_w.$$

Since the substitution ϱ is primitive, there exists an integer $k > 0$ such that, for each $i = 1, 2, \ldots, n$, we have $card_{a_i}\varrho^k(w) > 0$. Hence

$$c_w = \lim_{l\to\infty} \frac{card(\varrho^{l+k}(w))}{\lambda_{PF}^{l+k}} = \lim_{l\to\infty} \frac{card(\varrho^l(\varrho^k(w))}{\lambda_{PF}^{l+k}} = \begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix} \frac{C_r}{\lambda_{PF}^k q_r'} \begin{pmatrix} card_{a_1}\varrho^k(w) \\ \vdots \\ card_{a_n}\varrho^k(w) \end{pmatrix}.$$

But by Theorem 5, C_r/q_r' is positive, so $c_w > 0$. □

Infinite and *bi-infinite* words are sequences of the form $w = w_0w_1w_2w_3\ldots$ or $w = \ldots w_{-2}w_{-1}w_0w_1w_2$, with respective sets of symbols denoted by $\mathcal{A}_n^{\mathbb{N}}$ and $\mathcal{A}_n^{\mathbb{Z}}$. The image of a substitution ϱ in an infinite word w is defined by $\varrho(w) = \ldots\varrho(c_{-2})\varrho(c_{-1})\varrho(c_0)\varrho(c_1)\ldots$. Let $b \in \mathcal{A}_n$ and w be a word. If

$$lim_{n\to\infty} \frac{card_b(w_{[0,n-1]})}{n}$$

exists and equals to r, then the *frequency* of b in w is defined to be r and it is denoted by $Freq_b(w)$.

Definition 9. Let ϱ be a substitution rule on a finite alphabet $\mathcal{A}_n$. A finite word is called *legal* for ϱ, if it occurs as a subword of $\varrho^k(a_i)$ for some $1 \le i \le n$ and some $k \in \mathbb{N}$. A finite or infinite word w such that $\varrho(w) = w$ is called a fixed point of ϱ. A bi-infinite word ω is called a *fixed point* of a primitive substitution if $\varrho(\omega) = \omega$ and $\omega_{-1}|\omega_0$ is a legal two-letter word of ϱ.

A fixed point ω_a with legal seed $a \in \mathcal{A}_n^*$ is called a *morphic fixed point* if there exists a 1-uniform substitution $\tau : \mathcal{A}_n^* \to \mathcal{A}_m^*$ such that $\omega_a = \tau(w_a)$.

Example 7. Let $t = t_0t_1t_2\ldots$ be the fixed point starting with 0, of the morphism ϱ which maps $0 \to 01$, $1 \to 10$, called the Thue–Morse sequence. Then we can see that, for all $k \ge 0$, the subword $t_{[2k,2k+11]}$ has one 0 and one 1, and $Freq_0(t) = Freq_1(t) = 1/2$.

Theorem 9. *Let ω be a fixed point of a primitive substitution ϱ. If the frequency of all letters exists, then the vector of the frequencies is a non-negative normalized right eigenvector of the substitution matrix of ϱ, associated with the PF eigenvalue of this matrix.*

Proof. Let $\varrho : \mathcal{A}_n^* \to \mathcal{A}_n^*$ be a substitution with fixed point $w = w_{a_1}$ for $a_1 \in \mathcal{A}_n$, and M be the substitution matrix of ϱ. We have that

$$M \begin{pmatrix} freq_{a_1}(w) \\ \vdots \\ freq_{a_n}(w) \end{pmatrix} = M \lim_{l\to\infty} \frac{1}{card(\varrho^l(a_1))} M^l \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \lim_{l\to\infty} \frac{card(\varrho^{l+1}(a_1))}{card(\varrho^l(a_1))} \frac{1}{card(\varrho^{l+1}(a_1))} M^{l+1} \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} freq_{a_1}(w) \\ \vdots \\ freq_{a_n}(w) \end{pmatrix}.$$

The third equality holds because at least one of the rows of the vector of frequencies is not zero, then the limit to the infinity of $card(\varrho^{l+1}(a_1))/card(\varrho^l(a_1))$ exists and it is denoted by λ. The first equality holds because the frequency of all the letters exists, in particular

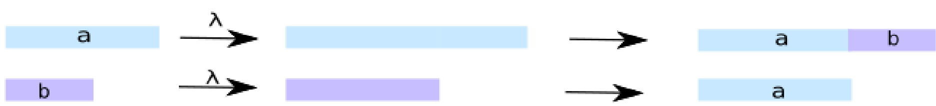
$$\begin{pmatrix} freq_{a_1}(w) \\ \vdots \\ freq_{a_n}(w) \end{pmatrix} = \lim_{l\to\infty} \frac{1}{card(\varrho^l(a_1))} M^l \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Additionally, there exists an integer $h \geq 1$ such that M^h has no eigenvalue diferent of λ_{PF}^h with modulus equal to λ_{PF}^h, and there exists an integer $p \geq 1$ and a positive number e such that $card(\varrho^{hl}(a_1)) = el^{p-1}\lambda_{PF}^{hn} + O(n^{p-1}r^{hn})$. From this we can deduce that $\lambda = \lambda_{PF}$. □

We have just seen the meaning of the right PF eigenvector for substitutions. Likewise, through some geometric realization using the left PF eigenvector of the substitution matrix, we will be able to tile the real line.

Definition 10. Consider a primitive substitution rule on a finite alphabet with substitution matrix M_ϱ and PF eigenvalue λ. The *associated geometric inflation rule* with inflation multiplier λ is obtained by turning the letters a_i into closed intervals (the prototiles) with lengths proportional to the entries of the left PF eigenvector of M_ϱ, and by dissecting the λ-inflated prototiles into copies of the original ones, respecting the order specified by ϱ.

Example 8. Consider the Fibonacci substitution defined in Example 5. Its substitution matrix M_ϱ has PF left eigenvector $\lambda = \frac{1+\sqrt{5}}{2}$. Then the substitution rule can be interpreted as an inflation rule for two prototiles as



3.2 Block substitutions

In the previous sections we were able to give a geometrical interpretation in one dimension of the substitutios. However, substitution rules can also be used to define objects in higher dimensions. For this, let's consider block substitutions.

Intuitively, we can think of a block substitution as a map that sends every letter of a finite alphabet to a d-dimensional rectangular array of letters.

Definition 11. Fix a dimension d and lengths $l_1, l_2, \ldots, l_d$,which are positive integers with each $l_i > 1$. Let I^d the set defined by $I^d = \{j = (j_1, \ldots, j_d) \mid j_i \in 0, 1, \ldots, l_i - 1 \text{ for all } i = 1, \ldots, d\}$. A *substitution* S is a map from $A \times I^d$ into A such that for each element $a \in A$, it assigns a map $S_a : I^d \to A$ and for $k \in I^d$, the mapping S restricted to the element k is a mapping from A to A.

As one dimensional substitutions, block substitutions also have an associated matrix defined in the same way than before.

Example 9. This example is taken from [Fra05].

1. The chair tiling in Figure 2.5 with set of tiles L-triominoes; three squares attached in a L shape, it can be obtained from the substitution

$$0 \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 \to \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad 2 \to \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad 3 \to \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}.$$

   Its substitution matrix is

$$M = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

2. The table tiling can be obtained from the substitution

$$0 \to \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}, \quad 1 \to \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad 2 \to \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad 3 \to \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix},$$

   and it has substitution matrix

$$M = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

3.3 Inflation rule of tilings

Definition 12. Consider a finite set $\{T_1, T_2, \ldots, T_n\}$ of tiles, where each $T_i \subset \mathbb{R}^d$ is a compact set with non-empty interior and $\overline{T_i^\circ} = T_i$. An *inflation rule* with inflation multiplier $\lambda > 1$ consists of the mappings

$$\lambda T_i \to \cup_{j=1}^n T_j + A_{ji} \tag{3.1}$$

with finite sets $A_{ji} \subset \mathbb{R}^d$, subject to the mutual disjointness of the interiors of the sets on the right hand side and to the individual volume consistency conditions $vol(\lambda T_i) = \sum_{j=1}^n vol(T_j) card(A_{ji})$, both for each $1 \leq i \leq n$.

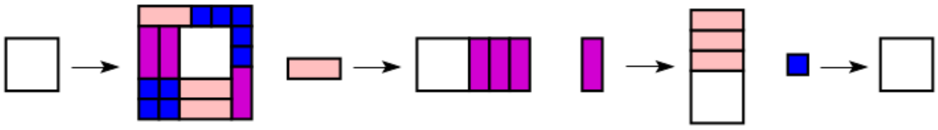


Figure 3.1: Example of a stone inflation. Image credit: [Fra08].

Given an inflation rule we can associate to it a matrix M defined by $(M_{ij})_{ij} = (card(A_{ij}))_{ij}$. This is a non-negative matrix, so the terms irreducible and primitive can be defined as in Section 3.1.

As we did with the prototiles obtained from the geometric rule of substitutions, we can choose a specified control point. In this text we work with *consistent* inflation rules, i.e., inflations tha map tiles with disjoint interiors in tiles with the same property. A tiling that is constructed by such an inflation rule will be called a *self-similar tiling*. When an inflation rule satisfies equality in Equation (3.1) between the left and the right hand sides, we call it *stone inflation*.

Example 10. Consider the set of tiles $T = \{T_1, T_2, T_3, T_4\}$ consisting of a square T_1 with edges of length $\alpha = (1+\sqrt{13})/2$, of two rectangles T_2 and T_3 of edges of size α and 1, and a square of edges of size 1. Consider the inflation rule in Figure 3.1 with inflation multiplier α.

Let's see explicitely which are the maps of the inflation rule and the corresponding sets A_{ij}. For this, let's take as control point the left bottom corner and identify it with $(0,0)$.

$$\lambda T_1 : \begin{array}{l} T_1 + \{(2,2)\} \\ T_2 + \{(2,0),(2,1),(0,\alpha+2)\} \\ T_3 + \{(\alpha+2,0),(0,2),(1,2)\} \\ T_4 + \{(0,0),(1,0),(0,1), \\ (1,1),(\alpha+2,\alpha),(\alpha+2,\alpha+1), \\ (\alpha+2,\alpha+2),(\alpha,\alpha+2), \\ (\alpha+1,\alpha+2)\} \end{array} \qquad \lambda T_2 : \begin{array}{l} T_1 + \{(0,0)\} \\ T_2 + \emptyset \\ T_3 + \{(\alpha,0),(\alpha+1,0),(\alpha+2,0)\} \\ T_4 + \emptyset \end{array}$$

$$\lambda T_3 : \begin{array}{l} T_1 + \{(0,0)\} \\ T_2 + \{(0,0),(0,\alpha),(0,\alpha+1),(0,\alpha+2)\} \\ T_3 + \emptyset \\ T_4 + \emptyset \end{array} \qquad \lambda T_4 : \begin{array}{l} T_1 + \{(0,0)\} \\ T_2 + \emptyset \\ T_3 + \emptyset \\ T_4 + \emptyset \end{array}$$

So, this rule has the inflation matrix M, and Figure 3.2 is a square-shaped patch of this stone inflation.

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 3 & 0 \\ 3 & 3 & 0 & 0 \\ 9 & 0 & 0 & 0 \end{pmatrix}$$

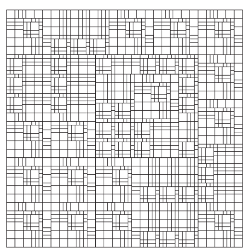


Figure 3.2: Iteration of the inflation rule. Image credit: [BG13].

Chapter 4

Cut and Project Schemes

In this chapter we will explain the projection method. This chapter is mainly based in chapter 7 of [BG13].

Definition 13. A cut and project scheme (CPS) is a triple $(\mathbb{R}^d, \mathcal{H}, \mathcal{L})$ with a (compactly generated) LCAG $\mathcal{H}$, a lattice $\mathcal{L}$ in $\mathbb{R}^d \times \mathcal{H}$ and the two natural projections $\pi : \mathbb{R}^d \times \mathcal{H} \to \mathbb{R}^d$ and $\pi_{int} : \mathbb{R}^d \times \mathcal{H} \to \mathcal{H}$, subject to the conditions that $\pi \mid_{\mathcal{L}}$ is injective and that $\pi_{int}(\mathcal{L})$ is dense in $\mathcal{H}$.

Let's denote $L = \pi(\mathcal{L})$. We define the *star map* of the CPS as the map $\star : L \to \mathcal{H}$ given by

$$x \mapsto x^\star := \pi_{int}((\pi \mid_{\mathcal{L}})^{-1}(x)),$$

with the image of L under the star map denoted by $L^\star$. The star map is well-defined since π is a bijection between $\mathcal{L}$ and L with $((\pi \mid_{\mathcal{L}})^{-1}(x))$ the unique point in the set $\mathcal{L} \cap \pi^{-1}(x)$. Additionally, we can see $\mathcal{L}$ as the Minkowski embedding of L,

$$\mathcal{L} = \{(x, x^\star) \mid x \in L\}.$$

We can sumarise the definition of CPS in the following diagram.

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathcal{H} & \xrightarrow{\;\pi_{\text{int}}\;} & \mathcal{H} \\
\cup & & \cup & & \cup \text{ dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\star\qquad} & & L^\star
\end{array}$$

For a given CPS $(\mathbb{R}^d, \mathcal{H}, \mathcal{L})$ and a set $A \subset \mathcal{H}$,

$$\curlywedge(A) = \{x \in \pi(\mathcal{L}) \mid x^\star \in A\}$$

denotes a projection set within the CPS. The set A is called a *window*.

Definition 14. Let $(\mathbb{R}^d, \mathcal{H}, \mathcal{L})$ be a CPS according to Definition 13. If $W \subset \mathcal{H}$ is a relatively compact set with non-empty interior, the projection set $\curlywedge(W)$, or any translate $t + \curlywedge(W)$ with $t \in \mathbb{R}^d$, is called a *model set*. A model set is regular when $\mu_{\mathcal{H}} = 0$ where $\mu_{\mathcal{H}}$ is the Haar measure of $\mathcal{H}$. If $L^\star \cup \partial W = \emptyset$, the model set is called generic.

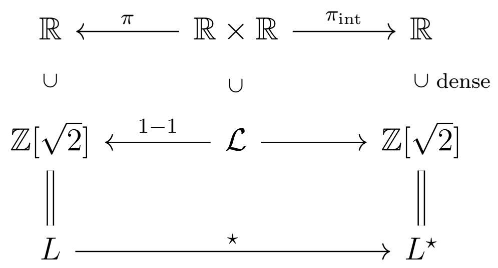


Figure 4.1: Cut and project scheme obtained from silver mean substitution.

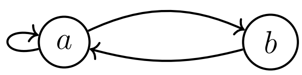
Example 11. Let $d = 1$, $\mathcal{H} = \mathbb{R}$, $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the projection into the first coordinate and $\pi_{int} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the projection in the second coordinate. Now, let's define the star map as the algebraic conjugation in the field $K = \mathbb{Q}(\sqrt{2})$. In Chapter 1 we saw that, with the algebraic conjugation, we can define a lattice generated by the vectors $(1, 1)$ and $(\sqrt{2}, -\sqrt{2})$. Let $\mathcal{L}$ be the Minkowski embedding. $\pi|_{\mathcal{L}}$ is injective, and $\mathbb{Z}[\sqrt{2}] \subset \mathbb{R}$ is dense in $\mathbb{R}$ as represented in Figure 4.1.

Now we will see how this CPS is related to substitutions and tilings. Consider the silver mean substitution given by

$$\varrho : \begin{array}{l} a \to aba \\ b \to a. \end{array}$$

Its substitution matrix and associated graph are

$$M_\varrho = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$



with left eigenvalue $\lambda_{PF} = 1 + \sqrt{2}$.

According to the theory of Section 3.1, we can construct a bi-infinite fixed point as follows. Consider the legal seed $\omega^{(1)} = a\,|\,a$ and define $\omega^{(i+1)} = \varrho(\omega^{(i)})$ for $i \geq 1$. Iterating we obtain

$$a\,|\,a \xrightarrow{\varrho} aba\,|\,aba \xrightarrow{\varrho} abaaaba\,|\,abaaaba \xrightarrow{\varrho} \ldots \xrightarrow{\varrho} \omega^{(i)} \xrightarrow{i\to\infty} \omega = \varrho(\omega).$$

Now, if we set $\lambda = \lambda_{PF}$ as the inflation multiplier of the geometric inflation rule according to Definition 10, we obtain an interval a of length λ_{PF} and an interval b of length 1 as in Figure 4.2.

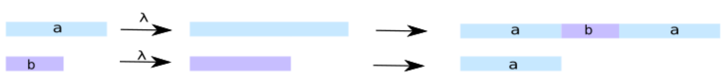


Figure 4.2: Geometric interpretation of the silver mean substitution.

Take the left endpoints of the intervals of length λ_{PF} and length 1 obtained in the image of the bi-infinite fixed point ω under the inflation rule, and denote them by Λ_a and Λ_b. By the property of the fixed point, we obtain the following equation system.

$$\begin{aligned} \Lambda_a &= \lambda\Lambda_a \,\dot{\cup}\, (\lambda\Lambda_a + (\lambda + 1)) \,\dot{\cup}\, \lambda\Lambda_b \\ \Lambda_b &= \lambda\Lambda_b + \lambda. \end{aligned} \tag{4.1}$$

Notice that Λ_a and Λ_b are subsets of $\mathbb{Z}[\sqrt{2}]$. By applying the star map to the previous equations, and taking closures we obtain the following equation

$$\begin{aligned} W_a &= \lambda^\star W_a \,\cup\, (\lambda^\star W_a + (\lambda^\star + 1)) \,\cup\, \lambda^\star W_b \\ W_b &= \lambda^\star W_b + \lambda^\star \end{aligned} \tag{4.2}$$

with $\overline{\Lambda_a^\star} = W_a$ and $\overline{\Lambda_b^\star} = W_b$

Since $|\lambda^\star| < 1$, Equation (4.2) is an iterated funtion system (IFS). By Corollary 1 of Hutchinson Theorem, this IFS has a unique solution. Let's try to compute the solution using SageMath. First we create a function describing the equation system given by the λ-inflation.

```
R.<sqrt2> = ZZ[sqrt(2)]
def stone_inflation(U,V):
    s=1+sqrt2
    Us=[s*i for i in U]
    U1_s=[(1+s)+i for i in Us]
    Vs=[s*i for i in V]
    EndPoints_a=Us+U1_s+Vs
    EndPoints_b=[s+i for i in Us]
    return(EndPoints_a+EndPoints_b)
```

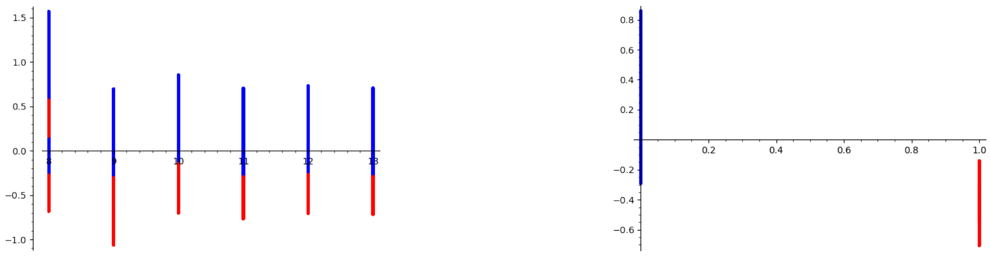
Afterwards, we apply the star map as in Equation (4.2).

```
R.<sqrt2> = ZZ[sqrt(2)]
def F(U,V):
    s=1-sqrt2
    Us=[s*i for i in U]
    U1_s=[(1+s)+i for i in Us]
    Vs=[s*i for i in V]
    EndPoints_a=Us+U1_s+Vs
    EndPoints_b=[s+i for i in Us]
    return([EndPoints_a,EndPoints_b])
```

Finally, we define the composition of the inflation map that allows us to take the limit to infinity. In Figure 4.3(a) we can appreciate the results after fourteen iterations, where each vertical line corresponds to an iteration. The figure only shows the last seven iterations. And in 4.3(b) we see the solution of the system which is $W_a = [\frac{\sqrt{2}-2}{2}, \frac{\sqrt{2}}{2}]$ and $W_b = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}-2}{2}]$.

```
g=Graphics()
U=[25]
V=[1000]
for i in range(1,14):
    U,V = F(U,V)
    #print("U=",[x.n() for x in U])
    #print("V=",[x.n() for x in V])
    if i > 7:
        g+=points([(i,x) for x in U], color='blue')
        g+=points([(i,x) for x in V], color='red')
g.show()
```

Let's denote $\Lambda_a \cup \Lambda_b = \Lambda$ and $W_a \cup W_b = W$. By definition we have that $\Lambda_a \subset \curlywedge(W_a)$ and $\Lambda_b \subset \curlywedge(W_b)$. Also, since the the boundary of W is not contained in L then $\curlywedge(W) =$



(a) (b)

Figure 4.3: Silver mean atractor.

$⅄(W^\circ)$. By the substitution we can deduce that the distance between neighbouring points in Λ is 1 or $1+\sqrt{2}$. But the same is true for $⅄(W^\circ)$.

Proposition 4. *If $x \in ⅄(W^\circ)$ is an arbitrary element of the silver mean point set, its successor to the right is either $x+1$ or $x+1+\sqrt{2}$.*

Proof. See [BG13]. □

We now can prove one of the principal results of this chapter.

Theorem 10. *The silver mean point set Λ satisfies $\Lambda = ⅄(W^\circ) = ⅄(W)$ within the CPS in Figure 4, with $W = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$. The corresponding identities hold for Λ_a and Λ_b as well, with windows W_a and W_b.*

Proof. The second equality is already proven. It remains to prove that $\Lambda = ⅄(W^\circ)$. With the previous results we showed that these two sets share the property that the distance between neighbouring points is either 1 or $1+\sqrt{2}$. If there exists an element $x \in ⅄(W^\circ)\backslash\Lambda$, this element will create new distances, which is a contradiction. So the sets have to be the same. □

Chapter 5

More examples

The aim of this chapter is to show different examples of cut and project schemes computed during the study of the theory of the previous chapter. We will start with the well known Fibonacci substitution, we will continue with the 19 self similar Wang shift and finish with the Ammann L-shape tile.

5.1 Fibonacci Cut and Project scheme

As we saw in Example 5, the Fibonacci substitution is a primitive substitution defined by $a \to ab$, $b \to a$, with substitution matrix $M_\varrho = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and PF left eigenvector $\lambda = \frac{1+\sqrt{5}}{2}$.

We want to construct a fixed point of the substitution to apply the inflation rule defined in Chapter 3, and in this way to obtain a tiling of the real line. Starting from the legal seed $a|a$, an iteration of ϱ gives

$$\begin{aligned} \underline{a}|a &\xrightarrow{\varrho} \underline{\underline{ab}}|ab \xrightarrow{\varrho} a\underline{ba}|aba \xrightarrow{\varrho} aba\underline{\underline{ab}}|abaab \xrightarrow{\varrho} abaaba\underline{ba}|abaababa \\ &\xrightarrow{\varrho} abaababaaba\underline{\underline{ab}}|abaababaabaab \\ &\xrightarrow{\varrho} abaababaabaababaaba\underline{ba}|abaababaabaababaababa \xrightarrow{\varrho} \cdots \end{aligned}$$

which converges on all positions except the underline one. The latter positions alternates between ab and ba, so that we have created a 2-cycle under ϱ, and hence two bi-infinite fixed points of ϱ^2. Let w be the fixed point of ϱ^2 obtained from the seed $a|a$. Choose intervals of length λ and 1 for a and b respectively, apply the inflation rule with inflation factor λ and take the left endpoints as their characteristic points denoted by Λ_a and Λ_b. The geometric fixed point equation for ϱ^2 implies the identities

$$\begin{aligned} \Lambda_a &= \lambda\Lambda_a \,\dot{\cup}\, \lambda\Lambda_b \\ \Lambda_b &= \lambda\Lambda_a + \lambda. \end{aligned}$$

Notice that $\Lambda_a,\ \Lambda_b \subset \mathbb{Z}\left[\frac{1+\sqrt{2}}{2}\right] \subset K = \mathbb{Q}(\sqrt{5})$, and $\mathbb{Z}\left[\frac{1+\sqrt{2}}{2}\right] \subset \mathbb{R}$ is a dense subset. Then considering the algebraic conjugation in K, $\star : K \to K$, by taking closures, we can transform the previous equations system in

$$W_a = \lambda^\star W_a \cup \lambda^\star W_b$$
$$W_b = \lambda^\star W_a + \lambda^\star.$$

with $W_a = \overline{\Lambda_a^\star}$ and $W_b = \overline{\Lambda_b^\star}$.

Since $|\lambda^\star| < 1$, each equation of the previous system is a contraction, therefore it is an IFS, and by Hutchinson theory it has a unique solution given by $W_a = [\lambda - 2, \lambda - 1]$ and $W_b = [-1, \lambda - 2]$. We can check this solution using SageMath in a similar way than in Example 11.

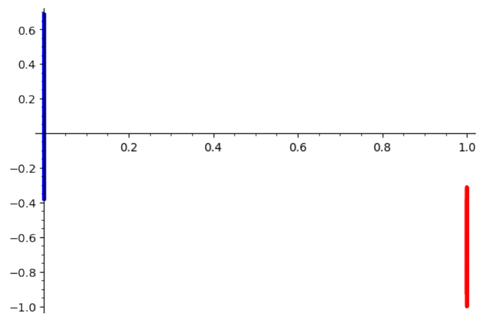


Figure 5.1: Fibonacci windows with W_a in blue and W_b in red.

As we did in Chapter 4, from the geometric inflation rule associated to the Fibonacci substitution we can define a CPS. For this let $d = 1$, $\mathcal{H} = \mathbb{R}$, and $\mathcal{L} = \{(x, x^\star) \mid x \in \mathbb{Z}[\lambda]\}$ be the lattice obtained defining the star map as the algebraic conjugation in the field $K = \mathbb{Q}(\sqrt{5})$. Let π and π_{int} be the natural projections in the first and second coordinates, respectively. Then it is clear that $\pi \mid_{\mathcal{L}}$ is injective and $\pi_{int}(\mathcal{L}) = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ is dense in $\mathbb{R}$. Then the CPS corresponding to the Fibonacci substitution is as in Figure 5.2.

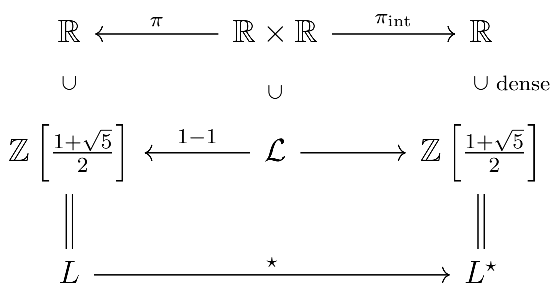


Figure 5.2: CPS of the Fibonacci substitution

By definition, $\Lambda_a \subset \curlywedge(W_a)$ and $\Lambda_b \subset \curlywedge(W_b)$. But in the same way as in the example of silver mean substitution, $\Lambda_a = \curlywedge(W_a)$ and $\Lambda_b = \curlywedge(W_b)$. This means that the tiling of the real line obtained from the geometric inflation rule of the Fibonacci substitution is a cut and project tiling.

Although this CPS describes the points of the Fibonacci geometric inflation, it is not unique. If we consider $d = 1$, $\mathcal{L} = \mathbb{Z}^2$, $\mathcal{H} = \mathbb{R}$, and the projections $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$\pi(x, y) = \lambda x + y$ and $\pi_{int}(x, y) = -\frac{x}{\lambda} + y$, then this also defines a CPS. In this case the star map $\star : \pi(\mathcal{L}) \to \mathcal{H}$ is defined by $\star : x \to x^\star := \pi_{int}((\pi|_{\mathcal{L}})^{-1}(x))$. Moreover,

$$\Lambda_a = \pi(\mathbb{Z}^2 \cap \pi_{int}^{-1}(W_a)) = \{\pi(x) \mid x \in \mathbb{Z}^2, \pi_{int}(x) \in W_a\},$$
$$\Lambda_b = \pi(\mathbb{Z}^2 \cap \pi_{int}^{-1}(W_b)) = \{\pi(x) \mid x \in \mathbb{Z}^2, \pi_{int}(x) \in W_b\}.$$

So, for this case the CPS is summarised in the following figure

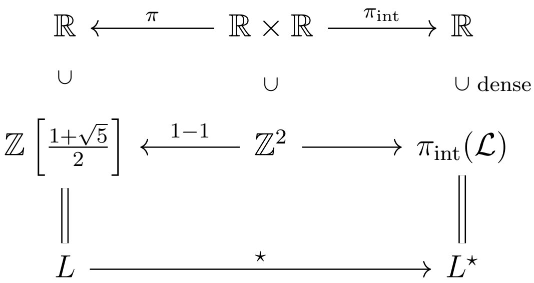


Figure 5.3: Another CPS associated to the Fibonacci substitution.

5.2 19 self-similar Wang shift

Wang tiles are square tiles with colored edges with matching condition being edge-to-edge, and colors on contiguous edges must match and only translations of the tiles are allowed. Wang tiles were created in 1960 by Wang. The first aperiodic set of Wang tiles was found by Wang's student, R. Berger, in 1966. This aperiodic set consisted of 20426 tiles. In the same year Berger was able to give a smaller aperiodic set of 104 Wang tiles. After that, several mathematicians have made efforts to find smaller aperiodic sets of Wang tiles. Some of them are mentioned in the following table.

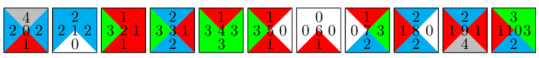


Figure 5.4: set of Wang tiles.

Authors	Size	Year
Berger	104	1966
Knuth	92	1968
Robinson	56	1971
Grunbaum et al.	24	1987
Ammann	16	1971
Kari	14	1996
Culik	13	1996
Jeandel and Rao	11	2015

In this section we study the stone inflation of 19 self-similar Wang shift shown in Figure 5.5, which was provided by Labbé in [Lab19].

Take as characteristic point the bottom left corner of each tile and denotes by Λ_i the set of characteristic points of the tile u_i for $i \in \{0, \dots, 18\}$. We can construct a fixed point of ϱ^2 starting from one of the last seven tiles. Applying the stone inflation to the fixed point we obtain the equation system

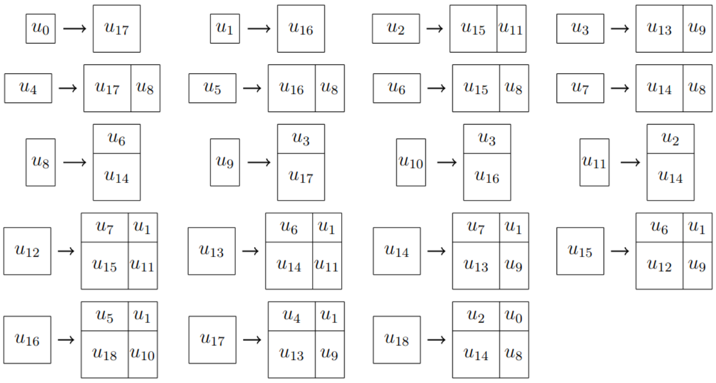


Figure 5.5: Stone inflation of the 19 self-similar Wang shift. Image credit: [Lab19].

$$
\begin{aligned}
\Lambda_0 &= \lambda\Lambda_{18} + v_1,\\
\Lambda_1 &= \lambda(\Lambda_{12} + v_1) \dot\cup \lambda(\Lambda_{13} + v_1) \dot\cup \lambda(\Lambda_{14} + v_1) \dot\cup \lambda(\Lambda_{15} + v_1),\\
&\quad \dot\cup \lambda(\Lambda_{16} + v_1) \dot\cup \lambda(\Lambda_{17} + v_1),\\
\Lambda_2 &= \lambda(\Lambda_{11} + v_2) \dot\cup \lambda(\Lambda_{18} + v_1),\\
\Lambda_3 &= \lambda(\Lambda_9 + v_2) \dot\cup \lambda(\Lambda_{10} + v_2),\\
\Lambda_4 &= \lambda\Lambda_{17} + v_2,\\
\Lambda_5 &= \lambda\Lambda_{16} + v_2,\\
\Lambda_6 &= \lambda(\Lambda_8 + v_2) \dot\cup \lambda(\Lambda_{13} + v_2) \dot\cup \lambda(\Lambda_{15} + v_2),\\
\Lambda_7 &= \lambda(\Lambda_{12} + v_1) \dot\cup \lambda(\Lambda_{14} + v_2),\\
\Lambda_8 &= \lambda(\Lambda_4 + v_3) \dot\cup \lambda(\Lambda_5 + v_3) \dot\cup \lambda(\Lambda_6 + v_3),\\
&\quad \dot\cup \lambda(\Lambda_7 + v_3) \dot\cup \lambda(\Lambda_{18} + v_3),\\
\Lambda_9 &= \lambda(\Lambda_3 + v_3) \dot\cup \lambda(\Lambda_{14} + v_3) \dot\cup \lambda(\Lambda_{15} + v_3) \dot\cup \lambda(\Lambda_{17} + v_3),\\
\Lambda_{10} &= \lambda(\Lambda_{16} + v_3),\\
\Lambda_{11} &= \lambda(\Lambda_2 + v_3) \dot\cup \lambda(\Lambda_{12} + v_3) \dot\cup \lambda(\Lambda_{13} + v_3),\\
\Lambda_{12} &= \lambda\Lambda_{15},\\
\Lambda_{13} &= \lambda\Lambda_3 \dot\cup \lambda\Lambda_{14} \dot\cup \lambda\Lambda_{17},\\
\Lambda_{14} &= \lambda\Lambda_7 \dot\cup \lambda\Lambda_{11} \dot\cup \lambda\Lambda_{13} \dot\cup \lambda\Lambda_{18} \dot\cup \lambda\Lambda_8,\\
\Lambda_{15} &= \lambda\Lambda_2 \dot\cup \lambda\Lambda_6 \dot\cup \lambda\Lambda_{12},\\
\Lambda_{16} &= \lambda\Lambda_1 \dot\cup \lambda\Lambda_5 \dot\cup \lambda\Lambda_{10},\\
\Lambda_{17} &= \lambda\Lambda_0 \dot\cup \lambda\Lambda_4 \dot\cup \lambda\Lambda_9,\\
\Lambda_{18} &= \lambda\Lambda_{16},
\end{aligned}
\tag{5.1}
$$

where $v_1 = (1,1)^T$, $v_2 = (0,1)^T$, $v_3 = (1,0)^T$.

The point sets Λ_i, $i \in \{0, \dots, 18\}$ are subsets of $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \times \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. If we consider the algebraic conjugation automorphism $i : \mathbb{Q}(\sqrt{5}) \to \mathbb{Q}(\sqrt{5})$ that sends $\sqrt{5}$ to $-\sqrt{5}$, we can define the star map $\star : \mathbb{Q}(\sqrt{5}) \times \mathbb{Q}(\sqrt{5}) \to \mathbb{Q}(\sqrt{5}) \times \mathbb{Q}(\sqrt{5})$ by $\star : (x, y) \to (i(x), i(y))$. So, the corresponding diagonal embedding is $\mathcal{L} = \left\{(x, y, x^\star, y^\star) \mid x, y \in \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]\right\}$. $\mathcal{L}$ is a discrete and co-compact set in $\mathbb{R}^4$, therefore it is a lattice and it is generated by the vectors $(1,0,1,0)$, $(0,1,0,1)$, $(\lambda, 0, \lambda^\star, 0)$, $(0, \lambda, 0, \lambda^\star)$. Let's consider the natural projection $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ that send a point into the first two coordinates, and $\pi_{int} : \mathbb{R}^4 \to \mathbb{R}^2$ that send a point into the last two coordinates. Clearly $\pi|_{\mathcal{L}}$ is injective, and since $\mathbb{Z}[\lambda]$ is dense in $\mathbb{R}$, the product space $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \times \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ is dense in $\mathbb{R}^2$. Then the CPS corresponding to the 19 self-similar Wang shift is $(\mathbb{R}^4, \mathbb{R}^2, \mathcal{L})$.

Using SageMath we can see where point sets Λ_i are locate. First we define the stone inflation and we find the IFS relative to the fixed point. Figure 5.6 shows the points in the Λ_i's sets.

```
from slabbe import Substitution2d
from slabbe import GraphDirectedIteratedFunctionSystem as GIFS
da = {0: [[17]], 1: [[16]], 2: [[15],[11]], 3: [[13],[9]],
       4: [[17],[8]], 5: [[16],[8]], 6: [[15],[8]],
       7: [[14],[8]], 8: [[14,6]], 9: [[17, 3]], 10: [[16, 3]],
      11: [[14, 2]], 12: [[15, 7],[11,1]], 13: [[14, 6],[11,1]],
      14: [[13, 7],[9,1]], 15: [[12, 6],[9,1]], 16: [[18, 5],[10,1]],
      17: [[13, 4],[9,1]], 18: [[14, 2],[8,0]]}

wangshift_stoneinflation = Substitution2d(da)

wang_selfsimilar = GIFS.from_two_dimensional_substitution(
    wangshift_stoneinflation)
```

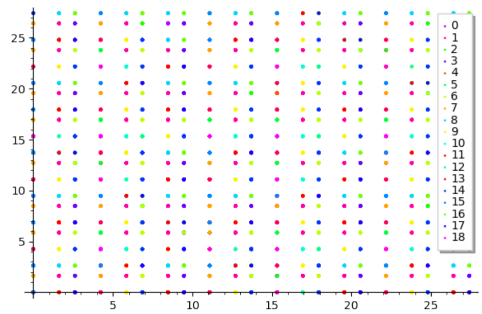


Figure 5.6: Location of the point sets Λ_i's.

Now we apply the Galois conjugation to the Equation (5.1) to obtain the unique windows W_i that satisfy the IFS.

```
wang_selfsimilar_conjugate =  wang_selfsimilar.galois_conjugate()
wang_selfsimilar_conjugate.plot(S=S, n_iterations=10)
```

The solution of the IFS, Figure 5.7, defines a partition of the plane, which it is really interesting because each region indicates us where to put the corresponding Wang tile.

Although it seems that the previous stone inflation is not related to the examples given so far, we will see that it is related with the Fibonacci substitution. Notice that the tiles obtained under the inflation can be gruped in four subsets of tiles with the same shape as in Figure 5.8. If we consider equivalence classes as follows "two letters are equivalent if they appear in the same column", then $a = [A]_{col} = [C]_{col} = \{A, C\}$ and $b = [B]_{col} = [D]_{col} = \{B, D\}$. So, the map $\{A, B, C, D\} \to \{a, b\}$ that sends each letter to the corresponding class is well defined, and in the same way the map that maps into the equivalence classes with respect to the rows.

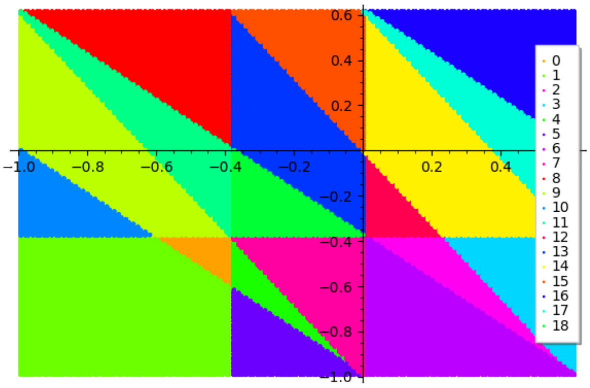


Figure 5.7: Solution sets of the IFS determined by (5.1)

With the previous setting a new CPS is defined as follows. Let $d = 4$, $\mathcal{H} = R^2$, $\mathcal{L} = \mathbb{Z}^4$ and the projections $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ and $\pi_{int} : \mathbb{R}^4 \to \mathbb{R}^2$ given by $\pi : (x_1, x_2, x_3, x_4) \mapsto (\lambda x_1 + x_2, \lambda x_3 + x_4)$, and $\pi_{int} : (x_1, x_2, x_3, x_4) \mapsto (-x_1/\lambda + x_2, -x_3/\lambda + x_4)$.

Figure 5.9 shows that we cannot obtain a fixed point of the stone inflation starting with the seed of four A squares. Instead, we are able to obtain a fixed point of the square of the inflation with such seed.

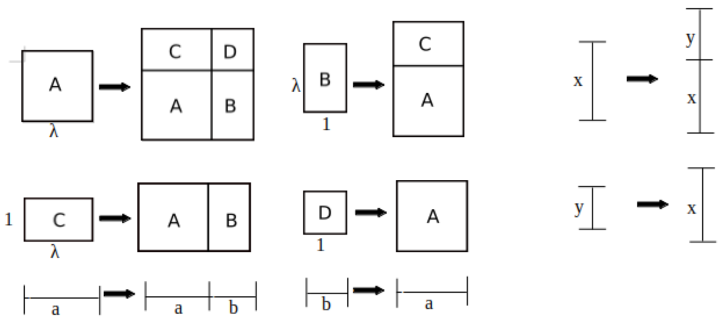


Figure 5.8: Equivalence classes obtained from 19 self-similar Wang shift stone inflation.

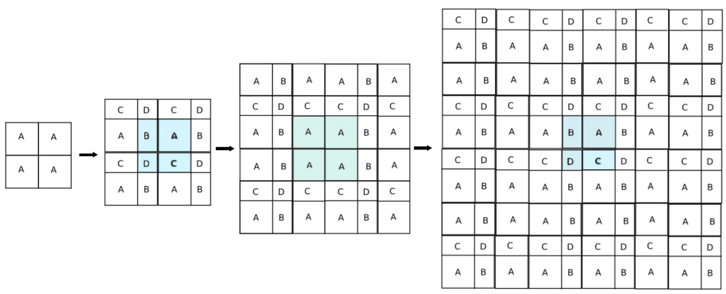


Figure 5.9: Fixed point of the square of the inflation of the 19 self-similar Wang shift.

Applying the stone inflation to the fixed point described above, we obtain the equation system

$$\begin{aligned}\Lambda_A &= \lambda\Lambda_A \,\dot\cup\, \lambda\Lambda_B \,\dot\cup\, \lambda\Lambda_C \,\dot\cup\, \lambda\Lambda_D\\ \Lambda_B &= \lambda(\Lambda_A + v_1) \,\dot\cup\, \lambda(\Lambda_C + v_1)\\ \Lambda_C &= \lambda(\Lambda_A + v_2) \,\dot\cup\, \lambda(\Lambda_B + v_2)\\ \Lambda_D &= \lambda(\Lambda_A + v_1 + v_2)\end{aligned} \tag{5.2}$$

with $v_1 = (1,0)^T$ and $v_2 = (0,1)^T$.

Given the projections, the star map is defined, so we can apply the star map and take closures in Equations (5.2) to obtain as unique solution the set $W_a \cup W_b = W = [-1, \lambda - 1] \times [-1, \lambda - 1]$.

5.3 Ammann L-Shape tile

Robert Ammann discovered four new sets of aperiodic tiles commonly denoted by A2, A3, A4 and A5 [GS87]. The first two sets of Ammann tiles have relation with the golden ratio $\lambda = \frac{1+\sqrt{5}}{2}$ and we will focus in the first set. The set A2 of Ammann tiles contains two tiles known as L-shape tiles denoted by A- and B-tile.

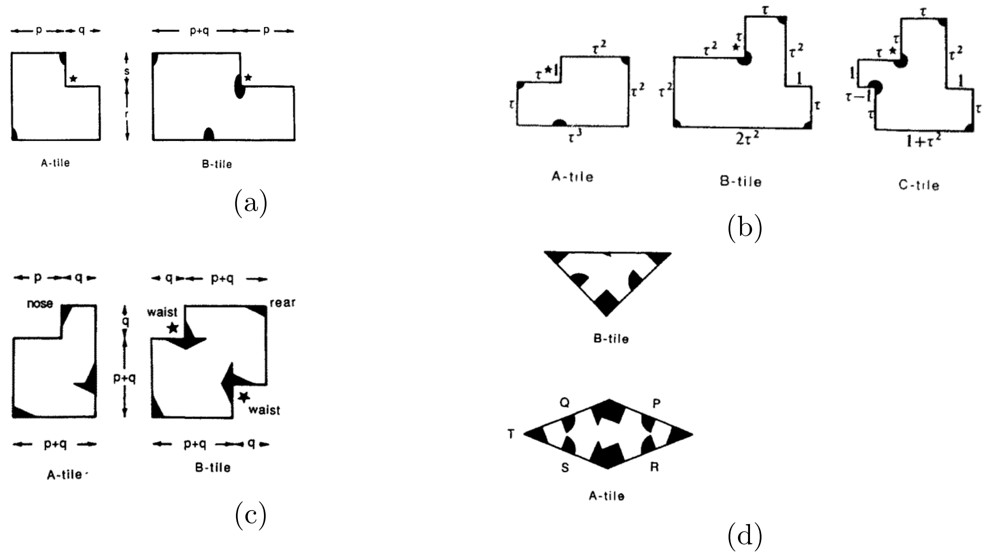


Figure 5.10: Ammann aperiodic sets of tiles: (a) A2, (b) A3, (c) A4, (d) A5. Image credit: [AGS92]

In 1992 Ammann, Grübaum, and Shepard showed in [AGS92] using the recognizability property that the set of supertiles is aperiodic. The *recognizability property* says that if a set **P** of prototiles satisfies:

1. in evering tiling admitted by **P** there is a *unique way* in which the tiles can be grouped into patches which lead to a tiling by supertiles; and
2. the markings on the supertiles, inherited from the original prototiles, imply a matching condition for the supertiles which is exactly equivalent to that originally specified for the prototiles,

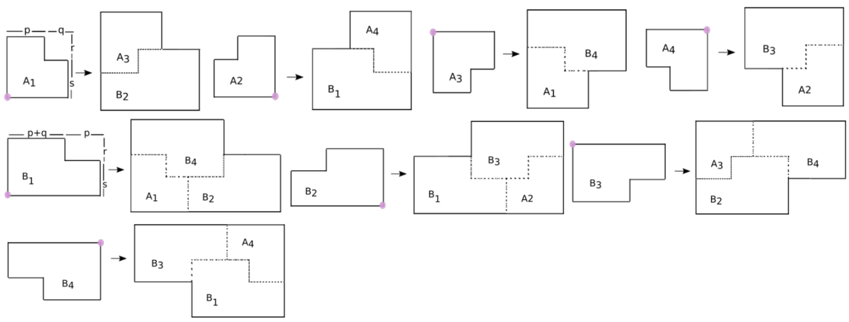


Figure 5.11: Inflation rule of the A-, B-tiles.

then **P** is aperiodic.

The goal of the master thesis is to know if we can use the IFS approach to give a CPS description of Ammann tilings admitted by the set A2. In this section it is presented what was achieve.

The set of supertiles or composed tiles has the property that are similar in shape to the original ones if and only if $p/q = (p+q)/p$ and $r/s = (r+s)/r$, that is, if $p/q = r/s = \lambda$. From [AGS92] we recover the substitution given by the inflation multiplier $\lambda = \frac{1+\sqrt{5}}{2}$. Since we consider only translations, the set of tiles that it will be consider has eight tiles which includes the vertical, horizontal and a double reflection of the two original A-, B-tiles.

From each tile take as characteristic point the point marked in pink in Figure 5.11. Using the fixed point of the substitution that starts with A_1 (see the code at the end to see the construction of the fixed point using SageMath) we obtain the following equation system.

$$
\begin{aligned}
\Lambda_{A1} &= (\lambda\Lambda_{A3} - v_1)\ \dot\cup\ \lambda\Lambda_{B1},\\
\Lambda_{A2} &= (\lambda\Lambda_{A4} - v_1)\ \dot\cup\ \lambda\Lambda_{B2},\\
\Lambda_{A3} &= (\lambda\Lambda_{A1} + v_1)\ \dot\cup\ \lambda\Lambda_{B3},\\
\Lambda_{A4} &= (\lambda\Lambda_{A2} + v_1)\ \dot\cup\ \lambda\Lambda_{B4},\\
\Lambda_{B1} &= (\lambda\Lambda_{A2} - v_2)\ \dot\cup\ (\lambda\Lambda_{B2} - v_3)\ \dot\cup\ (\lambda\Lambda_{B4} - v_1 - v_2),\\
\Lambda_{B2} &= (\lambda\Lambda_{A1} + v_2)\ \dot\cup\ (\lambda\Lambda_{B1} + v_3)\ \dot\cup\ (\lambda\Lambda_{B3} + v_2 - v_1),\\
\Lambda_{B3} &= (\lambda\Lambda_{A4} - v_2)\ \dot\cup\ (\lambda\Lambda_{B2} + v_1 - v_2)\ \dot\cup\ (\lambda\Lambda_{B4} - v_3),\\
\Lambda_{B4} &= (\lambda\Lambda_{A3} + v_2)\ \dot\cup\ (\lambda\Lambda_{B1} + v_1 + v_2)\ \dot\cup\ (\lambda\Lambda_{B3} + v_3),
\end{aligned}
\tag{5.3}
$$

where $v_1 = (0, 2r+s)^T$, $v_2 = (2p+q, 0)^T$, $v_3 = (3p+2q, 0)^T$.

To see where are located the points Λ_{A_i} and Λ_{B_i}, $i \in \{1,2,3,4\}$, we use SageMath. In the first lines of code we define some linear and affine maps and then we define the substitution of Figure 5.11 to values $p = r = \lambda$, and $q = s = 1$.

```
from slabbe import GraphDirectedIteratedFunctionSystem as GIFS
z   = polygen(QQ, 'z')
K = NumberField(z**2-z-1, 'phi', embedding=RR(1.6))
phi = K.gen()
```

```
F = AffineGroup(2,K)
p = phi; q = 1;s = 1;r = phi
phi_matrix = matrix([[phi, 0],[0, phi]])
f1 = F.linear(phi)
f2 = F(phi_matrix, vector([0, -2*r-s]))
f3 = F(phi_matrix, vector([0, 2*r+s]))
f4 = F(phi_matrix, vector([-2*p-q, -2*r-s]))
f5 = F(phi_matrix, vector([-2*p-q, 2*r+s]))
f6 = F(phi_matrix, vector([2*p+q, 2*r+s]))
f7 = F(phi_matrix, vector([2*p+q, 0]))
f8 = F(phi_matrix, vector([-2*p-q, 0]))
f9 = F(phi_matrix, vector([-3*p-2*q, 0]))
f10 = F(phi_matrix, vector([3*p+2*q, 0]))
f11 = F(phi_matrix, vector([2*p+q, -2*r-s]))

edges = [('A1','A3',f3),('A1','B2',f7),('A2','A4',f3),('A2','B1',f8),
         ('A3','A1',f2),('A3','B4',f7), ('A4','A2',f2),('A4','B3',f8),
         ('B1','B4',f6),('B1','B2',f10),('B1','A1',f1),('B2','B1',f9),
         ('B2','B3',f5),('B2','A2',f1),('B3','B2',f11),('B3','B4',f10),
         ('B3','A3',f1),('B4','B1',f4),('B4','B3',f9),('B4','A4',f1)]
ammann_IFS = GIFS(K^2,edges)
```

Once our equation system is defined as an IFS, we set our set of tiles as a the set of vertices with initial position in the origin. Then Figure 5.12 shows us the location of all the vertices satisfying the equations.

```
vertices = ammann_IFS.vertices()
S = {v:[vector((0,0))] for v in vertices}
ammann_IFS.plot(S=S, n_iterations=4)
```

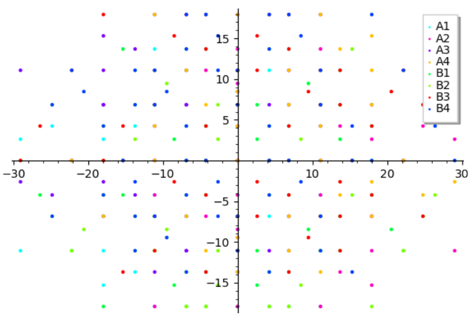


Figure 5.12

Since the sets of endpoints are contained in $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \times \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, then we can make the same construction that we did with 19 self-similar Wang shift. Applying the star map to the equation system (5.3) and taking closure as expressed in the following code, we are able to obtain the unique solution of the system.

```
ammann_IFS_conjugate = ammann_IFS.galois_conjugate()
ammann_IFS_conjugate.plot(S=S, n_iterations=6)
```

In Figure 5.13 we can see that the windows that we obtained from the IFS are not a partition of the plane as it is in Figure 5.7 for the 19 self-similar Wang shift case. So, from this IFS we can not deduce anything about the point sets Λ_{A_i}, Λ_{B_i}. If we want to

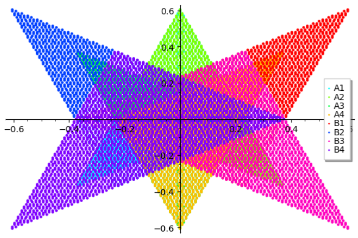


Figure 5.13

recover the n-th iteration of the inflation it is possible to do it using the definition of the inflation. Figure 5.14 is the example of the 5-th iteration.

```
from sage.misc.latex_standalone import TikzPicture

vertices = ammann_IFS.vertices()
S = {v:[] for v in vertices}
S['A1'] = [vector((0,0))]
d = ammann_IFS(S=S, n_iterations=5)

def tikz_from_dictionary(d):
    phi=1.6180
    p = phi; q = 1;s = 1;r = phi
    lines = []
    lines.append(r'\begin{tikzpicture}')
    for (x,y) in d['A1']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,{r+s}) -- ++ ({p},0) -- ++ (0,{-s}) -- ++ ({q},0) -- ++(0,{-r}) -- cycle;')
    for (x,y) in d['A2']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,{r+s}) -- ++ (-{p},0) -- ++ (0,{-s}) -- ++ (-{q},0) -- ++(0,{-r}) -- cycle;')
    for (x,y) in d['A3']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,-{r+s}) -- ++ ({p},0) -- ++ (0,{s}) -- ++ ({q},0) -- ++(0,{r}) -- cycle;')
    for (x,y) in d['A4']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,-{r+s}) -- ++ (-{p},0) -- ++ (0,{s}) -- ++ (-{q},0) -- ++(0,{r}) -- cycle;')
    for (x,y) in d['B1']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,{r+s}) -- ++ ({p+q},0) -- ++ (0,{-s}) -- ++ ({p},0) -- ++(0,{-r}) -- cycle;')
    for (x,y) in d['B2']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
```

```
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,{r+s}) -- ++ (-{p+q},0) -- ++ (0,{-s}) -- ++ (-{p},0) -- ++(0,{-r}) -- cycle;')
    for (x,y) in d['B3']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,-{r+s}) -- ++ ({p+q},0) -- ++ (0,{s}) -- ++ ({p},0) -- ++(0,{r}) -- cycle;')
    for (x,y) in d['B4']:
        x = numerical_approx(x, digits=5)
        y = numerical_approx(y, digits=5)
        lines.append(r'\draw[very thick]' + f' ({x},{y}) -- ++ (0,-{r+s}) -- ++ (-{p+q},0) -- ++ (0,{s}) -- ++ (-{p},0) -- ++(0,{r}) -- cycle;')
    lines.append(r'\end{tikzpicture}')
    return TikzPicture('\n'.join(lines))
```

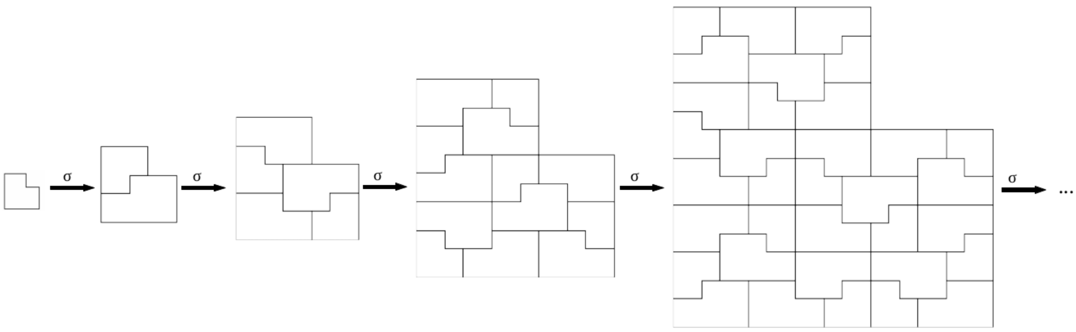


Figure 5.14: Iteration of Ammann substitution.

Conclusion

In this work we provided in Chapter 2 the basic notions about tilings mentioning the different classes that there exists and some well-known examples. In Chapter 3 we introduced symbolic and geometric substitutions in different dimensions. Moreover, we gave meaning to the left and right eigenvalue and eigenvector of a substitution matrix. In Chapter 4 was presented the project method and the cut and project tilings, illustrating the corresponding definitions with the silver mean substitution. Finally, in Chapter 5 the goal was to construct different CPS's using the theory of the previous chapter.

Basically we presented three examples of well-known substitution and set of tiles. In the last example, the aperiodic Ammann set of tiles A2 we take is the aperiodic set of supertiles constructed in [AGS92], and the respective inflation rule. From this we obtain an iteration function system that allowed us to obtained eight compact sets of $\mathbb{R}^2$ that contain all the characteristic points. However, we were not able to describe the sets of characteristic points of the supertiles as project sets in such a way that the corresponding sets obtained from the iterated function system (IFS) was a partition. Nevertheless, we notice that if we make a different choice of characteristic points, the IFS change and therefore different solutions of the system are obtained. For instance, if we only change the control point of A_1 to choose the point in the upper right corner of the tile, then the solution of the corresponding IFS is a fractal as shown in Figure 5.15.

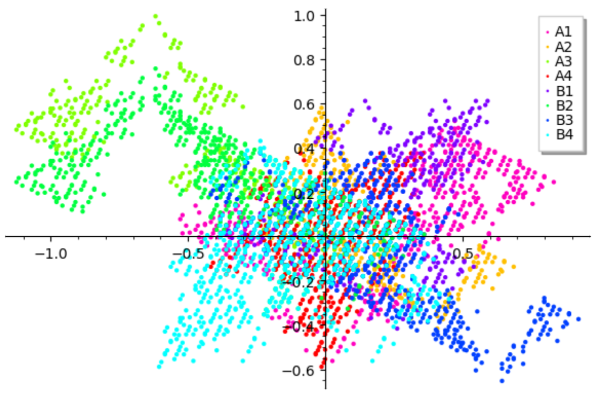


Figure 5.15: Solution of the iterated function system obtained from Ammann stone inflation.

So a question remains open. Does there exist a good selection of characteristic points such that the solution of the associated IFS is a partition?

Bibliography

[De 81] Nicolaas Govert De Bruijn. "Algebraic theory of Penrose's non-periodic tilings of the plane". In: *Kon. Nederl. Akad. Wetensch. Proc. Ser. A* 43.84 (1981), pp. 1–7.

[GS87] Branko Grünbaum and Geoffrey Colin Shephard. *Tilings and patterns*. Courier Dover Publications, 1987.

[AGS92] Robert Ammann, Branko Grünbaum, and Geoffrey C Shephard. "Aperiodic tiles". In: *Discrete & Computational Geometry* 8.1 (1992), pp. 1–25.

[AS+03] Jean-Paul Allouche, Jeffrey Shallit, et al. *Automatic sequences: theory, applications, generalizations*. Cambridge University Press, 2003.

[Fra05] Natalie Priebe Frank. "Multidimensional constant-length substitution sequences". In: *Topology and its Applications* 152.1-2 (2005), pp. 44–69.

[Fra08] Natalie Priebe Frank. "A primer of substitution tilings of the Euclidean plane". In: *Expositiones Mathematicae* 26.4 (2008), pp. 295–326.

[BG13] Michael Baake and Uwe Grimm. *Aperiodic order*. Vol. 1. Cambridge University Press, 2013.

[Con14] KEITH Conrad. "The Contraction Mapping Theorem, II". In: (2014).

[Bla17] Blackaller, Luis, Buza Kyle, Mazzola Paluska, Justin. *Piles of Tiles*. [Online; accessed 7-June-2022]. 2017. URL: `http://black.mitplw.com/tiles/aperiodic.html`.

[Ste17] Dieter Steemann. *Gailiunas's Spiral Tilings*. `http://demonstrations.wolfram.com/GailiunassSpiralTilings`. [Online; accessed 1-June-2022]. 2017.

[Ouy+18] Peichang Ouyang et al. "Fractal tilings based on successive adjacent substitution rule". In: *Complexity* 2018 (2018).

[Lab19] Sébastien Labbé. "A self-similar aperiodic set of 19 Wang tiles". In: *Geometriae Dedicata* 201.1 (2019), pp. 81–109.

[Wik21a] Wikipedia contributors. *Pentagonal tiling — Wikipedia, The Free Encyclopedia*. [Online; accessed 1-June-2022]. 2021. URL: `https://en.wikipedia.org/w/index.php?title=Pentagonal_tiling&oldid=1046436949`.

[Wik21b] Wikipedia contributors. *Snub trihexagonal tiling — Wikipedia, The Free Encyclopedia*. `https://en.wikipedia.org/w/index.php?title=Snub_trihexagonal_tiling&oldid=1041405448`. [Online; accessed 1-June-2022]. 2021.

[Wik22] Wikipedia contributors. *Aperiodic tiling — Wikipedia, The Free Encyclopedia*. `https://en.wikipedia.org/w/index.php?title=Aperiodic_tiling&oldid=1087844157`. [Online; accessed 1-June-2022]. 2022.

[Bee] Bee Clock. [Online; accessed 7-June-2022]. URL: https://ar.pinterest.com/pin/822469950676194981/.