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Quantum Cohomology of $\mathbf{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and Enumerative Applications

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Introduzione

Lo schema di Hilbert $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ parametrizza i sottoschemi chiusi zero dimensionali di lunghezza due di $\mathbb{P}^1 \times \mathbb{P}^1$ e risulta essere liscio, irriducibile e 4-dimensionale. In questa tesi diamo una presentazione esplicita della sua Coomologia Quantum Piccola. Inoltre elaboriamo un algoritmo (parziale) che ci permetta di calcolarne anche la Coomologia Quantum Grande, pur non essendo in grado di darne una presentazione esplicita.
Entrambe le coomologie quantum sono una deformazione dell'usuale anello di coomologia $H^*(\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1), \mathbb{Q})$. Si ottengono aggiungendo opportune variabili formali e definendo un prodotto $*$ che estende il prodotto $\cup$ dell'anello di coomologia stesso.
Per ottenere i suddetti risultati utilizziamo la teoria degli spazi di moduli di mappe stabili, che sono degli stack nel senso di Deligne-Mumford. In particolare usiamo tecniche tipiche della teoria delle deformazioni oltre che calcoli di classi fondamentali virtuali per stack di Deligne-Mumford. Tutto ciò è giustificato dal fatto che i coefficienti del prodotto $*$ sono gli invarianti di Gromov-Witten dello schema di Hilbert in esame. In questo caso, essi hanno un significato enumerativo, i.e. contano il numero di curve razionali che soddisfano certe proprietà di intersezione, come ad esempio passare per un fissato numero di punti. In particolare mentre la Coomologia Quantum Grande coinvolge gli invarianti corrispondenti ad un numero $n \geq 3$ di condizioni di incidenza, per quella Piccola $n = 3$.
Infine abbiamo dimostrato come si possano contare le curve iperellittiche su $\mathbb{P}^1 \times \mathbb{P}^1$, di genere $g \geq 2$ e bi-grado (d_1, d_2) fissati, che passano per un certo numero di punti per mezzo degli invarianti di Gromov-Witten di $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$. Quest'ultimo risultato è un'applicazione dei calcoli di coomologia quantum ed estende l'analogo risultato ottenuto da Tom Graber per le curve iperellittiche piane in [Gr].
Riteniamo che il metodo usato per trovare questi risultati abbia raggiunto il suo limite naturale con lo studio di $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$. Il tentativo di estenderlo allo schema di Hilbert di due punti sul blowup di $\mathbb{P}^2$ in un punto o su $\mathbb{P}^n$ si è rivelato inefficace a causa della più complicata struttura degli spazi di moduli da prendere in considerazione, per i quali non disponiamo di una buona descrizione geometrica.

Introduction

Over the last decades a great interest in the Quantum Cohomology of a manifold has grown out of the work of physicists (see [W1], [W2]), providing a rich field of investigation for mathematicians. In particular given a smooth complex projective variety X (or a symplectic manifold), there are two different objects which can be called Quantum Cohomology of X; these are the *Big Quantum Cohomology ring* and the *Small Quantum Cohomology ring*.
The Big Quantum Cohomology ring is a $*$-product structure on $V \otimes R$, where $V = H^*(X, \mathbb{Q})$ and R is a power series ring, which makes $V \otimes R$ into a R-algebra and reduces to the cup product when putting all the variables to zero. The Small Quantum Cohomology ring is defined by setting equal to zero some of the formal variables, for more details see [F-P], [G-P].
The $*$-product is defined in terms of the (genus zero) *Gromov-Witten invariants* of X, i.e. the virtual number of genus zero m-pointed stable maps $\mu : C \to X$ with prescribed $\mu_*[C]$ that meet m general cycles on X. We use the word "virtual" because the Gromov-Witten invariants need not have enumerative significance in general. In the Small Quantum Cohomology ring only the 3-point Gromov-Witten invariants appear. The quantum product can be shown to be commutative, associative, with unit. From the associativity relations one gets a system of quadratic equations known as the WDVV-equations (so named after E. Witten, R. Dijkgraaf, H. Verlinde, E. Verlinde by B. Dubrovin). Kontsevich and Manin in [K-M] remark that, under good hypotheses on X, the WDVV-system admits a unique solution once a few starting data are known, and it is in fact very overdetermined.
Quantum Cohomology can be explicitly computed using various tools. When $H^*(X, \mathbb{Q})$ is generated by $H^2(X, \mathbb{Q})$ the same authors prove the *First Reconstruction Theorem*: it gives an algorithm to find recursively all the genus zero Gromov-Witten invariants from the 2-point invariants by means of the WDVV-equations. The most famous application is due to Kontsevich [Kon]. He calculates the number of rational curves of degree d in $\mathbb{P}^2$ going through $3d - 1$ points. He only needs as starting datum the number of lines through two points. Other examples of computations exploiting the WDVV-equations can also be found in [DF-I].
There are some examples of varieties for which the Big and/or the Small

Quantum Cohomology rings have been computed, such as $\mathbb{P}^n$, $\mathbb{P}^1 \times \mathbb{P}^1$ [F-P], the blowup of $\mathbb{P}^2$ in r points [G-P], Grassmannians [Ber], flag varieties [CF], rational surfaces [C-M], some complete intersections [B], the moduli space of stable bundles over Riemann surfaces [Mu], some projective bundles [Q-R] and some blowups of projective bundles [Ma].
A smooth variety X is called *convex* if $H^1(\mathbb{P}^1, f^*T_X) = 0$ for all genus zero stable maps $f : \mathbb{P}^1 \to X$. Convexity ensures that the Gromov-Witten invariants are enumerative. Only few of the varieties mentioned above are non-convex.
A significant example of a non-convex variety is represented by the *Hilbert scheme* $\mathrm{Hilb}^n(X)$ of n points on a smooth complex projective surface X. It parametrizes the closed 0-dimensional subschemes of X of length n; it is smooth, projective, $2n$-dimensional. For $n = 1, 2$ it is easy to describe; $\mathrm{Hilb}^1(X)$ is X itself and $\mathrm{Hilb}^2(X)$ is obtained by blowing up $X \times X$ along the diagonal and then taking the quotient by the obvious lifted action of the involution. The case where $n = 2$, $X = \mathbb{P}^2$ has been studied by Graber in [Gr]. The author gives a presentation of the Small Quantum Cohomology ring of the Hilbert scheme by means of quantum deformations of the relations defining the Chow ring $A^*(\mathrm{Hilb}^2(\mathbb{P}^2), \mathbb{Q})$. Moreover he gets enumerative results on the hyperelliptic plane curves passing through an opportune number of points by studying the moduli space of genus zero stable maps into $\mathrm{Hilb}^2(\mathbb{P}^2)$.
The aim of this thesis is to study the Quantum Cohomology of the Hilbert scheme $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and to give some enumerative applications extending Graber's results to the case of hyperelliptic curves on $\mathbb{P}^1 \times \mathbb{P}^1$. The structure of the work is the following.
Chapter 1 is devoted to describing the Hilbert scheme we are working on. In §1.1 we follow the above mentioned construction of the Hilbert scheme as a quotient by the action of an involution and we give the corresponding presentation of its Chow ring which is isomorphic to the cohomology ring. In §1.2 we prove that $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ can be seen as a blowup of the Grassmannian of lines in $\mathbb{P}^3$ along two lines and also in this case we give the corresponding presentation of its Chow ring. In particular it turns out that the Chow ring is not generated by the divisor classes, but we need to add a cycle class in codimension two to get a complete set of generators. Then we study the induced action of the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$. The Hilbert scheme $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ is not homogeneous but only *almost-homogeneous*, i.e. it has a finite number of orbits forming a stratification. This property is good enough to make enumerative geometry on it, as shown in §1.3. In paragraph 1.4 we analyse the homogeneous part of degree 1 of the Chow ring of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$. In the following §1.5 we give the generators of the effective cone, postponing a detailed description of some connected effective curves to §1.8. Paragraphs 1.6 and 1.7 are dedicated to the description of two special divisors on the Hilbert scheme which are related to the orbit stratification.

Finally in §1.9 we study the cycle whose points are closed subschemes of dimension zero and length 2 incident to a given point of $\mathbb{P}^1 \times \mathbb{P}^1$. It represents a cycle class in $A^2(\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1))$ which will be of crucial importance for applications in Chapter 4.
In **Chapter 2** we recall the notion of moduli space of stable maps (§2.1) with a brief review of deformation theory in §2.2. Paragraph 2.3 collects some results about the virtual fundamental class of a moduli space and in §2.4, §2.5, §2.6 we apply the general theory to some moduli spaces of genus zero stable maps into $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$. The chapter finishes with the general definition of the Gromov-Witten invariants (§2.7) and the calculation (§2.8) of some invariants on the Hilbert scheme we are interested in. In particular, we carry out some excess calculations on the moduli spaces mentioned above involving their *obstruction bundles*.
Chapter 3 collects some of the main results. We recall the definition of the Big Quantum Cohomology ring of a non-convex variety and the techniques we want to use in order to obtain a presentation of it for $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ (§3.1). Then after fixing the notations in §3.2, we construct the Small Quantum Cohomology ring and give a presentation of it in §3.3 and §3.4. This is possible only after making some explicit computation of Gromov-Witten invariants using both techniques from classical enumerative geometry and the WDVV-equations. We conclude the chapter restricting our attention to the subalgebra $\mathbf{S}$ of the Chow ring generated by the divisors classes. This allows us to write a (partial) algorithm calculating recursively all the genus zero Gromov-Witten invariants of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ starting from few initial data. The idea is to divide the problem into two parts. The invariants with all the arguments in the subring $\mathbf{S}$ are known by the First Reconstruction theorem, only those involving the generating cycle class in codimension two are left and for them we use the WDVV-equations.
Chapter 4 presents our main result (theorem 4.3.1) which solves the problem of counting the hyperelliptic curves of given genus and bi-degree on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through a certain number of points which may also be hyperelliptical conjugated (theorems 4.3.5, 4.3.10). In particular in §4.1 we construct a space parametrizing maps from a hyperelliptic curve to $\mathbb{P}^1 \times \mathbb{P}^1$ with good properties. In §4.2 we prove it is canonically isomorphic to the space of stable maps from irreducible rational curves into $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ with good intersection properties with the stratification. This means that we can reduce an enumerative problem in higher genus to a question about rational curves. Finally our main theorem is stated and proved in the last paragraph 4.3. It extends the result obtained by Graber in [Gr], Theorem 2.7, as well as its applications to the enumerative problem.
The main technical differences between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are related to the problem of finding a presentation of the Quantum Cohomology rings, since the Chow ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is not generated by the divisor classes. As said above, we (partially) succeeded in solving the problem dividing it in two

parts and using two powerful tools as the First Reconstruction Theorem and the WDVV-equations. Moreover the description of the effective cone of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ is more complicated, and requires to give two geometrical descriptions of the Hilbert scheme. We also need to consider more effective curves for the calculation of the initial data of the algorithm computing (almost) all the Gromov-Witten invariants. This is because the group of automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ has more orbits. In particular we have to be careful about intersection properties of curves with the induced stratification (theorem 2.4.5).
We think that the techniques we used in this thesis have reached their natural limits and they can not be successfully applied to find any enumerative result for example in the case where $X = \mathbb{P}^n, n \geq 3$, or $Bl_p\mathbb{P}^2$. In fact we considered $\mathbb{P}^n$, and found out that problems arise from studying the components of excess dimension of the moduli space of genus zero stable maps into $\mathrm{Hilb}^2(\mathbb{P}^n)$. Instead for the blowup of $\mathbb{P}^2$ in a point we were not able to find a simple geometrical description of the effective cone of the corresponding Hilbert scheme. Moreover also in this case the Chow ring of $\mathrm{Hilb}^2(Bl_p\mathbb{P}^2)$ is not generated by the divisor classes. Finally the orbits of the induced action of the automorphisms group of $Bl_p\mathbb{P}^2$ give a stratification with no good intersection properties.

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Chapter 1

Some properties of $\mathbf{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

In this chapter we will fix notations and present some results on the Hilbert scheme $\mathbf{H} :=\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ whose points are represented by 0-dimensional length-2 closed subschemes Z of $\mathbb{P}^1 \times \mathbb{P}^1$. There are two possible geometric descriptions of $\mathbf{H}$, as a desingularization of the second symmetric product $\mathrm{Sym}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ (see [Fo]) and as a blow up of the Grassmannian $\mathrm{Grass}(2,4)$ of lines in $\mathbb{P}^3$ (see §1.2). We will give a description of both constructions with the corresponding Chow rings. Then we will study how some particular divisors and effective curves on $\mathbf{H}$ look like, so that we will have a detailed picture of the ambient space we are going to work on.

Notations and conventions: we work over $\mathbb{C}$ and we identify the variety $\mathbb{P}^1 \times \mathbb{P}^1$ with its image under the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, i.e. the smooth quadric Q in $\mathbb{P}^3$. We have two rulings on Q, if q_1, q_2 are the two projections on $\mathbb{P}^1$, then $q_1^{-1}(p)$ represents the first ruling and $q_2^{-1}(p)$ the second one.
We consider Chow rings with $\mathbb{Q}$-coefficients. All the varieties under consideration in this chapter have a cellular decomposition, hence their Chow rings are isomorphic to their even-codimension cohomology rings, [Ful] Example 19.1.11. In particular we can identify them.
Given a vector bundle E we denote by $\mathbb{P}(E)$ the projective bundle $\mathrm{Proj}(\mathrm{Sym}\mathcal{E})$ where $\mathcal{E}$ is the sheaf of sections of E. Geometrically, points of $\mathbb{P}(E)$ correspond to hyperplanes in the fibers of E.
We indicate a non-reduced 0-dimensional subscheme Z of length 2 of Q as a pair (p, v) where $p \in Q$ is the support of Z and $v \in \mathbb{P}(T_{Q,p})$ is a direction. We call it a non-reduced point of $\mathbf{H}$.

1.1 The Hilbert scheme as a quotient

The following description of the Hilbert scheme $\mathbf{H}$ is valid for all Hilbert schemes of 2 points on a smooth variety [F-G].
Let U be the product $Q \times Q$, pr_1, pr_2 the two projections, $\tilde{U}$ the blowup of U along the diagonal $\delta \subseteq U$. The group $\mathbb{Z}_2$ acts on U fixing δ, so there is an induced action on the blowup $\tilde{U}$. The Hilbert scheme $\mathbf{H}$ is the quotient scheme $\tilde{U}/\mathbb{Z}_2$, hence it is smooth, projective, irreducible and 4-dimensional. We have the following diagram:

$$\begin{tikzcd}
\tilde{\delta} \arrow[r, "j"] \arrow[d, "bl|_\delta"'] & \tilde{U} \arrow[r, "\theta"] \arrow[d, "bl"] & \mathbf{H} \\
\delta \arrow[r, "i"'] & U \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & Q
\end{tikzcd}$$

with i, j the natural inclusions, bl the blowup map, θ the quotient map and $\tilde{\delta}$ the exceptional divisor.

Remark 1.1.1. Given the quotient map $\theta : \tilde{U} \to \mathbf{H} = \tilde{U}/\mathbb{Z}_2$, we have two induced homomorphisms:

$$\begin{aligned}
\theta^* &: A^*(\mathbf{H}) \longrightarrow (A^*(\tilde{U}))^{\mathbb{Z}_2} \subseteq A^*(\tilde{U}) \\
\theta_* &: A^*(\tilde{U}) \longrightarrow A^*(\mathbf{H})
\end{aligned}$$

They are such that $\theta_*\theta^* = 2\ id = \theta^*\theta_*$. More precisely:

$$\begin{array}{rlcl}
\theta_*\theta^* & : A^*(\mathbf{H}) & \longrightarrow & A^*(\mathbf{H}) \\
 & \gamma & \longrightarrow & 2\gamma \\
 & & & \\
\theta^*\theta_* & : A^*(\tilde{U}) & \longrightarrow & A^*(\tilde{U})^{\mathbb{Z}_2} \\
 & \alpha & \longrightarrow & \alpha + \sigma^*(\alpha)
\end{array}$$

where $\sigma : \tilde{U} \to \tilde{U}$ is the natural involution defined by $\sigma(\alpha \otimes \beta) = \beta \otimes \alpha$. It follows that the map $\theta^*\theta_*|_{A^*(\tilde{U})^{\mathbb{Z}_2}}$ is the multiplication by 2 homomorphism. Note that θ^* is an isomorphism of $\mathbb{Q}$-algebras which does not respect the degree:

$$\begin{tikzcd}
A^4(\mathbf{H}) \arrow[r, "deg_{\mathbf{H}}"] \arrow[d, "\theta^*"'] & \mathbb{Q} \arrow[d, "\cdot 2"] \\
A^4(\tilde{U})^{\mathbb{Z}_2} \arrow[r, "deg_{\tilde{U}}"] & \mathbb{Q}
\end{tikzcd}$$

Moreover by projection formula θ_* is $A^4(\mathbf{H})$-linear, where $A^4(\tilde{U})$ is made into an $A^4(\mathbf{H})$-algebra via θ^*.

First description of $A^*(\mathbf{H})$

As pointed out in (1.1.1), θ^* induces an isomorphism of $A^*(\mathbf{H})$ with $A^*(\tilde{U})^{\mathbb{Z}_2}$. Then to write down explicitly the Chow ring of $\mathbf{H}$ we need to know $A^*(\tilde{U})$.

Lemma 1.1.2. *Let ζ be the class $c_1(\mathcal{N}_{\tilde{\delta}|\tilde{U}}) = [\tilde{\delta}]|_{\tilde{\delta}}$ which has degree -1 on a fiber of the blowup map over δ. Then:*

$$A^*(\tilde{\delta}) = \frac{A^*(\delta)[\zeta]}{\zeta^2 + \sum_{i=1}^{2}(-1)^i c_i(T_Q)\zeta^{2-i} = 0}$$

Proof. As $\tilde{\delta}$ is the projectivization of the rank-2 vector bundle $\mathcal{N}_{\delta/U}$ we can use [G-H] p.606. Moreover by the isomorphism $\delta \cong Q$ and the exact sequence:

$$0 \to T_\delta \to i^* T_U \to \mathcal{N}_{\delta|U} \to 0$$

we have $c_i(\mathcal{N}_{\delta|U}) = c_i(T_Q)$. □

Lemma 1.1.3. *Let ξ be the class of the exceptional divisor $\tilde{\delta}$ in $\tilde{U}$. Set γ_s to be such that $j^*(\gamma_s) = (-1)^s c_s(T_Q)$, for $s = 1, 2$. Then:*

$$A^*(\tilde{U}) = \frac{A^*(Q)^{\otimes 2}[\xi]}{\begin{array}{c}(\alpha \otimes \beta - \beta \otimes \alpha)\xi = 0 \;\; \forall \alpha, \beta \in A^*(Q) \\ \xi^2 + \sum_{s=1}^{2} \gamma_s \xi^{2-s} = 0\end{array}} \tag{1.1}$$

In particular, as a vector space $A^*(\tilde{U})$ is simply $A^*(Q) \otimes A^*(Q) \oplus A^*(Q)\xi$.

Proof. The exact sequence (see [Ful] p.114-115):

$$0 \to A^*(\delta) \to A^*(U) \oplus A^*(\tilde{\delta}) \to A^*(\tilde{U}) \to 0$$

gives the equality:

$$A^*(\tilde{U}) = \frac{A^*(U) \oplus A^*(\tilde{\delta})}{A^*(\delta)} \tag{1.2}$$

By the Künneth formula, the Chow rings of δ and U are isomorphic to $A^*(\mathbb{P}^1) \otimes A^*(\mathbb{P}^1)$ and $A^*(Q) \otimes A^*(Q)$, respectively. Let h_1, h_2 be the cycle classes of the two rulings on Q. We write $h_0 = [Q]$, h_1, h_2, $h_3 := h_1 h_2$ for the basis of $A^*(Q)$ and $h_r \otimes h_s$, with $0 \le r, s \le 3$, for the basis of $A^*(U)$. Then:

$$A^*(\delta) = A^*(Q) = \frac{\mathbb{Z}[h_1]}{h_1^2} \otimes \frac{\mathbb{Z}[h_2]}{h_2^2}$$

By Lemma 1.1.2:

$$A^*(\tilde{\delta}) = \frac{A^*(Q)[\zeta]}{\zeta^2 - (2h_1 + 2h_2)\zeta + 4h_1 h_2 = 0} \tag{1.3}$$

The pullback of the divisor class ξ via the natural embedding j is exactly the class ζ in $A^1(\tilde{\delta})$. Moreover the quotient (1.2) means that for each α, β

in $A^*(Q)$ we have to identify the element $\alpha \otimes \beta \in A^*(U)$ pulled back to δ with the product class $\alpha\beta \in A^*(Q)$, i.e. $(\alpha \otimes \beta - \beta \otimes \alpha)\xi = 0$. Then we get the formula (1.1). $\square$

Remark 1.1.4. Writing explicitly the second relation at denominator in (1.1) we get:

$$\xi^2 = (2h_1 \otimes 1 + 2h_2 \otimes 1)\xi - (h_3 \otimes 1 + 1 \otimes h_3 + h_1 \otimes h_2 + h_2 \otimes h_1)$$

Let $[\delta] \in A^*(U)$ be the class of the diagonal and $\check{h}_i$ be the dual basis with respect to the intersection pairing on $A^*(Q)$. Then:

$$[\delta] = \sum_{i=0}^{3} \check{h}_i \otimes h_i$$

Moreover $i^*[\delta] = 4h_3 = c_2(T_Q)$, so that we can write $\xi^2 + \gamma_1\xi + [\delta] = 0$.

Remark 1.1.5. We can identify $\theta^*\theta_*(h_i \otimes 1)$ with the element $h_i \otimes 1 + 1 \otimes h_i$ in $A^*(\tilde{U})$, obviously $\theta^*\theta_*(\xi) = 2\xi$ since it is invariant under involution.

Proposition 1.1.6. *A basis for $A^*(\mathbf{H})$ is given by the elements:*

$$h_i \otimes h_j + h_j \otimes h_i, \ (h_i \otimes 1)\xi$$

with $0 \leq i, j, \leq 3$.

Proof. A basis is given by all the elements in $A^*(\tilde{U})$ which are invariant for the $\mathbb{Z}_2$-action. $\square$

We can write the following table:

$A^0(\mathbf{H})$	$A^1(\mathbf{H})$	$A^2(\mathbf{H})$	$A^3(\mathbf{H})$	$A^4(\mathbf{H})$
$S_0 = 1$	S_1	S_3	S_7	S_9
	S_2	S_4	S_8	
	S_{10}	S_5	S_{13}	
		S_6		
		S_{11}		
		S_{12}		

The cycle classes are defined to be:

$$
\begin{aligned}
&S_0 = [\mathbf{H}]\\
&S_1 = h_1 \otimes 1 + 1 \otimes h_1\\
&S_2 = h_2 \otimes 1 + 1 \otimes h_2\\
&S_3 = h_3 \otimes 1 + 1 \otimes h_3\\
&S_4 = h_1 \otimes h_2 + h_2 \otimes h_1\\
&S_5 = h_1 \otimes h_1\\
&S_6 = h_2 \otimes h_2\\
&S_7 = h_1 \otimes h_3 + h_3 \otimes h_1\\
&S_8 = h_2 \otimes h_3 + h_3 \otimes h_2\\
&S_9 = h_3 \otimes h_3\\
&S_{10} = \xi\\
&S_{11} = (h_1 \otimes 1)\xi\\
&S_{12} = (h_2 \otimes 1)\xi\\
&S_{13} = (h_3 \otimes 1)\xi
\end{aligned}
$$

Remark 1.1.7. We work with coefficients in $\mathbb{Q}$ so $\theta^* : A^*(\mathbf{H}) \to A^*(\tilde{U})^{\mathbb{Z}_2}$ is an isomorphism and we can identify the class $S_j \in A^*(\mathbf{H})$ with the corresponding element $\theta^* S_j \in A^*(\tilde{U})^{\mathbb{Z}_2}$, being careful about the degrees.

Poincaré Duality on $\tilde{U}$ gives:

	S_1	S_2	S_{10}
S_8	2	0	0
S_7	0	2	0
S_{13}	0	0	-1

	S_3	S_4	S_5	S_6	S_{11}	S_{12}
S_3	2	0	0	0	0	0
S_4	0	2	0	0	0	0
S_5	0	0	0	1	0	0
S_6	0	0	1	0	0	0
S_{11}	0	0	0	0	0	1
S_{12}	0	0	0	0	1	0

Dividing by 2 the above values we obtain the coefficients for the intersection pairing on $\mathbf{H}$.

Proposition 1.1.8. *As a $\mathbb{Q}$-algebra $A^*(\mathbf{H})$ is generated by S_1, S_2, S_3, S_{10} and it is defined by the relations:*

$$
\begin{array}{ll}
S_1^3 = S_2^3 = S_3^3 = 0 & S_1S_3S_{10} = S_2S_3S_{10} = 0\\
S_1^2S_3 = S_2^2S_3 = 0 & S_1S_2S_3 = S_3^2\\
S_1^2S_{10} = S_2^2S_{10} = 0 & S_1S_2S_{10} = 2S_3S_{10}\\
S_3^2S_1 = S_3^2S_2 = S_3^2S_{10} = 0 & S_1^2S_2 = 2S_1S_3\\
S_{10}^2 = (S_1 + S_2)S_{10} - S_1S_2 & S_2^2S_1 = 2S_2S_3
\end{array}
$$

Proof. The following equalities hold:

$$\begin{array}{l} S_4 = S_1S_2 - S_3 \\ S_5 = \frac{1}{2}S_1^2 \\ S_6 = \frac{1}{2}S_2^2 \\ S_7 = S_1S_3 \\ S_8 = S_2S_3 \\ S_9 = \frac{1}{2}S_3^2 \\ S_{11} = \frac{1}{2}S_1S_{10} \\ S_{12} = \frac{1}{2}S_2S_{10} \\ S_{13} = \frac{1}{2}S_3S_{10} \end{array}$$

It goes straightforward that the relations in the statement define $A^*(\mathbf{H})$ has a $\mathbb{Q}$-algebra. $\square$

1.2 The Hilbert scheme as a blow up

In the following we will prove that the Hilbert scheme $\mathbf{H}$ can be obtained as a blow up of the smooth projective 4-dimensional Grassmannian Grass$(2,4)$ of lines in $\mathbb{P}^3$. We use the symbol $\mathbf{G}$ to denote such a Grassmannian.

Lemma 1.2.1. *There exists a surjective morphism $\varphi : \mathbf{H} \to \mathbf{G}$ defined by mapping a point $Z \in \mathbf{H}$ to its associated line l_Z.*

Proof. Let $Grass(2,4)$ be the functor represented by $\mathbf{G}$ and $\mathcal{Z}_\mathbf{H} \subseteq \mathbf{H} \times \mathbb{P}^3$ be the universal family with projections p_1, p_2 to $\mathbf{H}$ and $\mathbb{P}^3$ respectively. Denote by $\mathcal{L}$ the sheaf $p_2^*\mathcal{O}_{\mathbb{P}^3}(1)$. The natural morphism $\psi : (p_1)_*\mathcal{L} \to (p_1)_*\mathcal{L}|_{\mathcal{Z}_\mathbf{H}}$ is surjective. Moreover $(p_1)_*\mathcal{L}$ is a trivial bundle since it is flasque ([G-D] 3.2.1), with fiber over $Z \in \mathbf{H}$ canonically isomorphic to $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Also $(p_1)_*\mathcal{L}|_{\mathcal{Z}_\mathbf{H}}$ is a vector bundle on $\mathbf{H}$, it has rank 2. Then ψ is an element of $Grass(2,4)(\mathbf{H})$. On the fiber over $Z \in \mathbf{H}$, ψ is the surjection:

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(Z, \mathcal{O}_Z(1))$$

with kernel $H^0(\mathbb{P}^3, \mathcal{I}_Z(1))$ the space of homogeneous linear forms which vanish on Z. It corresponds to the surjective morphism $\varphi : \mathbf{H} \to \mathbf{G}$ which maps each point Z to its associated line l_Z. $\square$

There are two special lines $W_1, W_2 \subseteq \mathbf{G}$ which are disjoint. A point $l_i \in W_i$ represents a line on the i-th ruling of Q, $i = 1, 2$. Denote by W the disjoint union of these special lines, i.e. $W = \{l \in \mathbf{G} : l \subseteq Q\}$. Let V be the open subset $\mathbf{G} - W$.

Lemma 1.2.2. *The morphism φ is birational.*

Proof. The inverse map φ^{-1} is well defined on the open subset V. It is given by $\varphi^{-1}(r) = r \cap Q$, for all $r \in V$. Since $\mathbf{G}$ is the Hilbert scheme of lines in $\mathbb{P}^3$, there is a universal family $\mathcal{Z}_{\mathbf{G}} \subseteq \mathbf{G} \times \mathbb{P}^3$. The morphism $\pi : \mathcal{Z}_{\mathbf{G}} \cap (\mathbf{G} \times Q) \to V$ is a flat family of 0-dimensional length-2 subschemes of Q, then by the universal property of $\mathbf{H}$, there exists a unique morphism $V \to \mathbf{H}$ which has to be φ^{-1}. This shows that φ is birational. $\square$

In particular 1.2.2 says that there is an isomorphism between $\mathbf{H} - \varphi^{-1}(W)$ and $\mathbf{G} - W$. If $r \in W$ then the inverse image $\varphi^{-1}(r)$ is $\mathrm{Sym}^2(r) \cong \mathbb{P}^2$, so that $\varphi^{-1}(W)$ is a Cartier divisor in $\mathbf{H}$. Hence we have a commutative diagram:

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\exists!\ \alpha"] \arrow[dr, "\varphi"'] & Bl_W\mathbf{G} \arrow[d, "\rho"] \\ & \mathbf{G} \end{tikzcd}$$

where ρ is the blowup morphism.

Lemma 1.2.3. *The morphism α is an isomorphism.*

Proof. Since both $\mathbf{H}$ and $Bl_W\mathbf{G}$ are smooth, α is an isomorphism if and only if it is bijective. It is obviously bijective on V. To verify bijectivity on the exceptional locus it is enough to look at the restriction

$$\alpha_1 : \varphi^{-1}(W_1) \to \rho^{-1}(W_1) = \mathbb{P}(\mathcal{N}_{W_1|\mathbf{G}})$$

If $r \in W_1$, then $\alpha_1 : \mathrm{Sym}^2(r) \to \mathbb{P}(\mathcal{N}_{W_1|\mathbf{G}})_r$ is a morphism from $\mathbb{P}^2$ into itself. Then it is defined by a triple of homogeneous polynomials of some degree n without common zeros. By explicit calculations it can be verified that $n = 1$. This implies that the generic fiber of α_1 is a point, i.e. α_1 is a bijection. $\square$

Theorem 1.2.4. *The Hilbert scheme* $\mathbf{H}$ *is isomorphic to the blow up of the Grassmannian* $\mathbf{G}$ *along* W.

Proof. It follows from Lemmas 1.2.1, 1.2.2, 1.2.3. $\square$

This result permits us to write the Chow ring of $\mathbf{H}$ by means of $A^*(\mathbf{G})$.

The Chow ring of the Grassmannian

We recall the description of $A^*(\mathbf{G})$ by Schubert cycles [Ful] §14.7.
Fix a flag in $\mathbb{P}^3$:

$$p \in r \subseteq \pi \subseteq \mathbb{P}^3$$

where p is a point, r a line and π a plane. Then:

. $A^0(\mathbf{G})$ has basis: $\sigma_{0,0} = \{l \in \mathbf{G} : l \cap \pi \neq \emptyset\} = [\mathbf{G}]$

. $A^1(\mathbf{G})$ has basis: $\sigma_{1,0} = \{l \in \mathbf{G} : l \cap r \neq \emptyset\}$

. $A^2(\mathbf{G})$ has basis: $\sigma_{1,1} = \{l \in \mathbf{G} : l \subseteq \pi\}$, $\sigma_{2,0} = \{l \in \mathbf{G} : p \in l\}$

. $A^3(\mathbf{G})$ has basis: $\sigma_{2,1} = \{l \in \mathbf{G} : p \in l \subseteq \pi\}$

. $A^4(\mathbf{G})$ has basis: $\sigma_{2,2} = \{l \in \mathbf{G} : p \in l = r\} = [pt]$

Proposition 1.2.5. *$A^*(\mathbf{G})$ is generated by $\sigma_{1,0}, \sigma_{2,0}$, as a $\mathbb{Q}$-algebra. The ring structure is defined by the relations:*

$$\begin{array}{ll} \sigma_{1,0}\sigma_{2,0} = \sigma_{1,0}\sigma_{1,1} = \sigma_{2,1} & \sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0} \\ \sigma_{2,0}^2 = \sigma_{1,1}^2 = \sigma_{1,0}\sigma_{2,1} = 1 & \sigma_{2,0}\sigma_{1,1} = 0 \end{array}$$

Proof. See [G-H] Chap.1 §5. □

Second description of $A^*(\mathbf{H})$

In the following we will refer to the diagram:

$$\begin{tikzcd} \tilde{W}_k \arrow[r, "j_k"] \arrow[d, "\varphi_k = \varphi_{|W_k}"'] & \mathbf{H} \arrow[d, "\varphi"] \\ W_k \arrow[r, "i_k"'] & \mathbf{G} \end{tikzcd}$$

where φ is our blowup map, i_k is the inclusion and $\tilde{W}_k$ is $\mathbb{P}(\mathcal{N}_{W_k|\mathbf{G}})$, for $k = 1, 2$. We denote by $\tilde{W}$ the exceptional divisor $\tilde{W}_1 \sqcup \tilde{W}_2$. As in the previous section, to write down $A^*(\mathbf{H})$ we use the short exact sequence:

$$0 \to A^*(W) \to A^*(\mathbf{G}) \oplus A^*(\tilde{W}) \to A^*(\mathbf{H}) \to 0$$

So we have to calculate the quotient:

$$\begin{aligned} A^*(\mathbf{H}) &= A^*(\mathbf{G}) \oplus A^*(\tilde{W})/A^*(W) \\ &= A^*(\mathbf{G}) \oplus A^*(\tilde{W}_1) \oplus A^*(\tilde{W}_2)/A^*(W_1) \oplus A^*(W_2) \end{aligned} \quad (1.4)$$

We need to know the rings $A^*(W_k), A^*(\tilde{W}_k)$. W_k is a projective line then $A^*(W_k) = A^*(\mathbb{P}^1)$. Denote by l_k the pullback via φ_k of the generator of this Chow ring.

Lemma 1.2.6. *Let $\xi_k = c_1(\mathcal{N}_{\tilde{W}_k|\mathbf{H}})$ be such that its pullback to a fiber $\varphi_k^{-1}(r)$ is represented by a line of degree -1. Let $\mathcal{N}_{W_k|\mathbf{G}}$ be the normal bundle of W_k in $\mathbf{G}$. Then:*

$$\begin{aligned} A^*(\tilde{W}_k) &= \frac{A^*(W_k)[\xi_k]}{\xi_k^3 + \sum_{i=1}^3 (-1)^i \; c_i(\mathcal{N}_{W_k|\mathbf{G}}) \; \xi_k^{3-i} = 0} \\ &= \frac{\mathbb{Z}[l_k, \xi_k]}{(l_k^2, \xi_k^3 - 6 l_k \xi_k^2)} \end{aligned} \quad (1.5)$$

Proof. As $\tilde{W}_k$ is a projectivization of the vector bundle $\mathcal{N}_{W_k|\mathbf{G}}$ the lemma follows by applying [G-H] p.606. □

By formula (1.4), we have the following description of $A^*(\mathbf{H})$:

$A^0(\mathbf{H})$	$A^1(\mathbf{H})$	$A^2(\mathbf{H})$	$A^3(\mathbf{H})$	$A^4(\mathbf{H})$
$[\mathbf{H}] = 1$	$\sigma_{1,0}$	$\sigma_{1,1}$ $\sigma_{2,0}$	$\sigma_{2,1}$	$\sigma_{2,2}$
	η_1	η_3	η_7	
	η_2	η_4	η_8	
		η_5		
		η_6		

The cycle classes are defined as:

$$
\begin{aligned}
&\sigma_{r,s} = \varphi^*(\sigma_{r,s}) \\
&\eta_1 = j_{1*}(1_{\tilde{W}_1}) \quad \text{and } \ j_1^*\eta_1 = \xi_1 \\
&\eta_2 = j_{2*}(1_{\tilde{W}_2}) \quad \text{and } \ j_2^*\eta_2 = \xi_2 \\
&\eta_3 = j_{1*}(l_1) \\
&\eta_4 = j_{2*}(l_2) \\
&\eta_5 = j_{1*}\xi_1 \\
&\eta_6 = j_{2*}\xi_2 \\
&\eta_7 = j_{1*}(l_1\xi_1) \\
&\eta_8 = j_{2*}(l_2\xi_2)
\end{aligned}
$$

There are two more cycle classes in codimension 3 which we are interested in: $\eta_9 = j_{1*}\xi_1^2$ and $\eta_{10} = j_{2*}\xi_2^2$.

Remark 1.2.7. Note that the class $-l_i\xi_i$ is represented by a line in the fiber of φ_i over a point $r \in W_i$ such that $\varphi_i^*[r] = l_i$, for $i = 1, 2$.

Theorem 1.2.8. *1) As a $\mathbb{Q}$-algebra $A^*(\mathbf{H})$ is generated by $\sigma_{1,0}, \sigma_{2,0}, \eta_1, \eta_2$.*
2) As a $A^*(\mathbf{G})$-algebra $A^*(\mathbf{H})$ is equal to the quotient:

$$A^*(\mathbf{H}) = \frac{A^*(\mathbf{G})[\eta_1, \eta_2]}{R}$$

where R is the set of relations:

$$
\begin{aligned}
&\eta_1 \cdot \eta_2 = 0 \\
&\eta_1 \cdot \sigma_{2,0} = \eta_2 \cdot \sigma_{2,0} = 0 \\
&\eta_1 \cdot (\sigma_{1,0}^2 - \sigma_{2,0}) = \eta_2 \cdot (\sigma_{1,0}^2 - \sigma_{2,0}) = 0 \\
&\eta_1^3 = 2\sigma_{1,0}\sigma_{2,0} + 3\eta_1^2\sigma_{1,0} \\
&\eta_2^3 = 2\sigma_{1,0}\sigma_{2,0} + 3\eta_2^2\sigma_{1,0}
\end{aligned}
$$

Proof. The relations among the elements of the basis of $A^*(\mathbf{G})$ hold also for the pulled back elements in $A^*(\mathbf{H})$. Moreover:

$$\begin{aligned} \eta_3 &= \tfrac{1}{2}\eta_1 \cdot \sigma_{1,0} \\ \eta_4 &= \tfrac{1}{2}\eta_2 \cdot \sigma_{1,0} \\ \eta_5 &= \eta_1^2 \\ \eta_6 &= \eta_2^2 \\ \eta_7 &= \tfrac{1}{2}\eta_1^2 \cdot \sigma_{1,0} \\ \eta_8 &= \tfrac{1}{2}\eta_2^2 \cdot \sigma_{1,0} \end{aligned}$$

Then the statement 1) is true.
Statement 2) follows from previous calculations. □

Finally the coefficients for the intersection pairing on $\mathbf{H}$ are:

	$\sigma_{1,2}$	η_7	η_8	η_9	η_{10}
$\sigma_{1,0}$	1	0	0	2	2
η_1	0	1	0	6	0
η_2	0	0	1	0	6

	$\sigma_{1,1}$	$\sigma_{2,0}$	η_3	η_4	η_5	η_6
$\sigma_{1,1}$	1	0	0	0	0	0
$\sigma_{2,0}$	0	1	0	0	0	0
η_3	0	0	0	0	1	0
η_4	0	0	0	0	0	1
η_5	0	0	1	0	6	0
η_6	0	0	0	1	0	6

Remark 1.2.9. In 1.4.1 and 1.8.1 we will make explicit the relationship between the two different sets of generators of $A^*(\mathbf{H})$ we found.
In §3.1 we will choose the more convenient basis of the Chow ring of $\mathbf{H}$ in order to make easier calculations in the (Small) Quantum Cohomolgy ring. The basis will consist of elements taken from both the presentations of $A^*(\mathbf{H})$ we gave.

1.3 The action of $Aut(Q)$ on H

Let $\mathcal{A}$ be the group of automorphisms of Q and $\mathcal{M}$ be the group of automorphisms of $\mathbb{P}^1$, i.e. $PGL(2)$. We denote by $\phi = (\phi_1, \phi_2)$ an element in $\mathcal{M} \times \mathcal{M}$ and by $\iota : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ the involution defined by $\iota(p, q) = (q, p)$. The group $\mathcal{A}$ acts on Q and then on U. Blowing up the diagonal, the action lifts to $\tilde{U}$. Since $\mathbf{H}$ is the quotient of $\tilde{U}$ by the involution, we have an induced action of $\mathcal{A}$ on it. In this section we study this induced action and exploit it to get a transversality result.

Proposition 1.3.1. *The connected component $\mathcal{A}_0 \subseteq \mathcal{A}$ containing the identity is exactly $\mathcal{M} \times \mathcal{M}$.*

Proof. There exists an embedding $Q \to \mathbb{P}^3$ given by a multiple of the anti-canonical divisor, i.e. such that $\mathbb{P}^3 = \mathbb{P}(H^0(Q, -\frac{1}{2}K_Q))$. Any element of $\mathcal{A}$ acts on $\mathbb{P}^3$ too (since K_Q is canonical). This implies that:

$$\mathcal{A} = \{\phi \in PGL(4) : \phi(Q) = Q\}$$

$\mathcal{A}$ is smooth because we work in characteristic zero, and $T_{\mathcal{A}}$ is a trivial bundle so dim $\mathcal{A}_0 =$ dim $T_{\mathcal{A},Id}$. To prove the statement it is enough to show that the tangent space $T_{\mathcal{A},Id}$ is 6-dimensional as well as $\mathcal{M} \times \mathcal{M}$.
If we think of Q as the set $\{v \in \mathbb{P}^3 : {}^t v \cdot v = 0\}$, an element of $PGL(4)$ in $\mathcal{A}$ has to satisfy $F(A) = {}^tA \cdot A - \lambda Id = 0$. Hence an element of the tangent space $T_{\mathcal{A},Id}$ is of type $Id + sB \pmod{s^2}$ with $B \in M(4 \times 4)$ and it has to fulfil:

$$F(Id + sB) \equiv 0 \pmod{s^2} \tag{1.6}$$

Put $\tilde{F}(A) = {}^tA \cdot A - Id = 0$. We can consider the following equation which is equivalent to (1.6), up to scalars:

$$\tilde{F}(Id + sB) = s({}^tB + B) = \lambda Id \quad \text{for } B \in M(4 \times 4)$$

In order to have a solution, λ must be divisible by s, that is to say there exists $a \in \mathbb{C}$ such that ${}^tB + B = a \cdot Id$. Consider the map $\varphi : M(4\times4) \to M(4\times4)$, defined by mapping a matrix B into the sum ${}^tB+B$. It is easy to see that the inverse image under φ of the subgroup generated by the identity matrix is 7-dimensional. Hence the tangent space $T_{\mathcal{A},Id}$ is 6-dimensional. We conclude that $\mathcal{A}_0$ has the same dimension as $\mathcal{M} \times \mathcal{M}$, then they coincide. □

Note that $\mathcal{A}_0 \neq \mathcal{A}$ since $\iota \in \mathcal{A}$ is not an element of $\mathcal{A}_0$. In fact it is easy to see that the group $\mathcal{A}$ has exactly two connected components: $\mathcal{A}_0$ and $\iota\mathcal{A}_0$.

Description of the orbits for the $\mathcal{A}$-action

We give a description of the orbits with respect to the $\mathcal{A}$-action on the varieties under consideration. We have three orbits on U:

$$\begin{aligned}
&\sigma_1 = \{(p,q) \times (a,b) : (p,q) = (a,b)\} = \delta \\
&\sigma_2 = \{(p,q) \times (a,b) : p = a, q \neq b\} \cup \{(p,q) \times (a,b) : p \neq a, q = b\} \\
&\sigma_3 = \{(p,q) \times (a,b) : p \neq a, q \neq b\}
\end{aligned}$$

where p, q, a, b are points on $\mathbb{P}^1$.
The lifted action on $\tilde{U}$ has the exceptional divisor $\tilde{\delta}$ (corresponding to the orbit σ_1) as invariant locus. Moreover $\tilde{\delta}$ is the disjoint union of two orbits:

$$\begin{aligned}
\tilde{\delta}_1 &= \{Z : \text{Supp } Z = p,\ l_Z \in Q\} \\
\tilde{\delta}_2 &= \{Z : \text{Supp } Z = p,\ l_Z \notin W_i,\ i = 1, 2\}
\end{aligned}$$

Note that in turn $\tilde{\delta}_1$ is the disjoint union of two closed subsets:

$$\tilde{\delta}_1^i = \{Z : \text{Supp } Z = p,\ l_Z \in W_i\},\ i = 1, 2$$

$\mathbf{H}$ is the quotient of $\tilde{U}$ by the involution so we have four orbits on it:

$$\Delta_2 \ (\rightsquigarrow \tilde{\delta}_1) \quad \Delta_3 \ (\rightsquigarrow \tilde{\delta}_2) \quad \Sigma_3 \ (\rightsquigarrow \sigma_2) \quad \Sigma_4 \ (\rightsquigarrow \sigma_3)$$

Here indexes are choosen equal to the dimensions of the orbits. We can give a description of them:

$$\begin{aligned}
\Sigma_4 &= \{Z \in \mathbf{H} : \text{Supp } Z = \{p, q\}, p \neq q, l_Z \nsubseteq Q\} \\
\Sigma_3 &= \{Z \in \mathbf{H} : \text{Supp } Z = \{p, q\}, p \neq q, l_Z \subseteq Q\} \\
\Delta_3 &= \{Z \in \mathbf{H} : \text{Supp } Z = p, l_Z \nsubseteq Q\} \\
\Delta_2 &= \{Z \in \mathbf{H} : \text{Supp } Z = p, l_Z \subseteq Q\}
\end{aligned}$$

The closed orbit Δ_2 is the disjoint union of two closed subvarieties Δ_2^i, $i = 1, 2$, corresponding to $\tilde{\delta}_1^i$ in $\tilde{U}$.
The closure $\overline{\Delta}_3 = \Delta_2 \sqcup \Delta_3$ is the subvariety of $\mathbf{H}$ of non-reduced points, i.e. Z such that Supp Z is only one point.
The orbit Σ_3 is the disjoint union $\Sigma_3^1 \sqcup \Sigma_3^2$ where

$$\Sigma_3^i = \{Z \in \mathbf{H} : \text{Supp } Z = \{p, q\}, p \neq q, \ l_Z \in W_i\}$$

In particular the closures $\overline{\Sigma}_3^1$, $\overline{\Sigma}_3^2$ are the two exceptional divisors $\tilde{W}_1$, $\tilde{W}_2$ respectively, of the blowup map $\varphi : \mathbf{H} \to \mathbf{G}$. Hence the closure $\overline{\Sigma}_3$ is equal to the disjoint union $\tilde{W}_1 \sqcup \tilde{W}_2$.
Finally the orbit Σ_4 is open and dense in $\mathbf{H}$.
These orbits form a stratification of $\mathbf{H}$:

$$\begin{tikzcd}
& \Delta_2 \arrow[dl] \arrow[dr] & \\
\Delta_3 \arrow[dr] & & \Sigma_3 \arrow[dl] \\
& \Sigma_4 &
\end{tikzcd}$$

where an arrow $A \to B$ means $\overline{A} \subseteq B$.

A transversality result

The action of $\mathcal{A}$ is obviously transitive on each orbit, but not on $\mathbf{H}$. We say that $\mathbf{H}$ is an *almost-homogeneuos space* since it has a finite number of orbits for the $\mathcal{A}$-action and they form a stratification. Note that the action is transitive on $\mathbf{H} - (\overline{\Delta}_3 \cup \Sigma_3)$.
A slightly modified version of the Kleiman-Bertini theorem holds for almost homogeneous spaces and gives us a transversality result.

Lemma 1.3.2. (Position Lemma) *Let A be a smooth, almost-homogeneous space under the action of an integral group G, $f : B \to A$ a morphism with B smooth. Let Γ be a smooth cycle on A which intersects the stratification properly, and Γ_{reg} be the locus in Γ where the intersection with the stratification is transversal. Then:*

1. *for a generic $g \in G$, $f^{-1}(g\Gamma)$ is of pure dimension equal to the expected one;*

2. *the open set (possibly empty) $f^{-1}(g\Gamma_{reg})$ is smooth.*

For a proof of this lemma see [Gr] Lemma 2.5.

Remark 1.3.3. If in the hypotheses of 1.3.2 we do not ask B smooth but only pure dimensional we can consider its desingularization $\nu : \tilde{B} \to B$. Then by applying the Position Lemma to the composition map $\tilde{f} : \tilde{B} \to A$ we get that cod $(\tilde{f}^{-1}(g\Gamma) \subseteq \tilde{B})$ is the expected one, i.e. equal to cod $(g\Gamma \subseteq A)$. Since:

$$\text{cod } (\tilde{f}^{-1}(g\Gamma) \subseteq \tilde{B}) \leq \text{cod } (f^{-1}(g\Gamma) \subseteq B)$$

we have that 1.3.2-1) holds with the inequality

$$\text{cod } (f^{-1}(g\Gamma) \subseteq B) \geq \text{cod } (g\Gamma \subseteq A)$$

Remark 1.3.4. The group $\mathcal{A}$ is not integral, so we can not apply the Position Lemma for any $\mathcal{A}$-action. But we can consider the connected component $\mathcal{A}_0 \subseteq \mathcal{A}$ containing the identity. Note that it defines a stratification of **H** with six orbits:

$$\begin{tikzcd}
\Delta_2^1 \arrow[dd] \arrow[dr] & & \Delta_2^2 \arrow[dd] \arrow[dl] \\
& \Delta_3 \arrow[dd] & \\
\Sigma_3^1 \arrow[dr] & & \Sigma_3^2 \arrow[dl] \\
& \Sigma_4 &
\end{tikzcd}$$

1.4 Divisor classes of H

We want to describe the Picard group Pic (**H**). We need to choose between the two possible sets of generators of $A^1(\mathbf{H})$ we presented in the previous sections. To do this we introduce some 3-codimensional cycle classes with "good" intersection properties.

Geometrical description of the divisor classes

In §1.1 we showed that S_1, S_2, S_{10} generate $A^1(\mathbf{H})$. We know a geometric description for each of these divisor classes:

- $S_1 = [\{Z : \text{Supp } Z \cap l_1 \neq \emptyset,\ l_1 \in W_1 \text{ fixed line}\}]$
- $S_2 = [\{Z : \text{Supp } Z \cap l_2 \neq \emptyset,\ l_2 \in W_2 \text{ fixed line}\}]$
- $2S_{10} = [\{Z : \text{Supp } Z = pt\}]$

In §1.2 the generating divisor classes are $\sigma_{1,0}, \eta_1, \eta_2$. The first class can be represented by the irreducible subvariety $\{Z : l_Z \cap r \neq \emptyset, r \subseteq \mathbb{P}^3 \text{ given line}\}$. The classes η_i, $i = 1, 2$ are such that their restriction to a fiber of the blowup maps φ_i, $i = 1, 2$, is represented by a line in the plane $\text{Sym}^2(l_i)$ for some $l_i \in W_i$, with structure sheaf $\mathcal{O}_{\text{Sym}^2(l_i)}(-1)$ (see §1.2 for notations).
Note that given a projective line l, $\text{Sym}^2(l)$ is the Hilbert scheme $\text{Hilb}^2(l)$.

The choice of a basis for $A^1(\mathbf{H})$ and $A^3(\mathbf{H})$

We define three 3-codimensional cycle classes and calculate their intersection product with all of the divisors.
Fix a point $l_1 \in W_1$ and let $C(l_1)$ be a line in the plane $\text{Sym}^2(l_1)$. We want to stress that all the points Z of $\mathbf{H}$ contained in $C(l_1)$ are such that $\text{Supp } Z \subseteq l_1$. We denote by C_1 the corresponding cycle class in $A^3(\mathbf{H})$. We do the same for the class C_2. Finally fix a point $p_0 \in Q$ and consider the line $C(p_0) = \mathbb{P}(T_{Q,p_0}) = \{Z : \text{Supp } Z = p_0\}$. Let F be the corresponding cycle class in $A^3(\mathbf{H})$. The curves $C(l_1)$, $C(l_2)$, $C(p_0)$ are effective in $\mathbf{H}$. We have the following intersection products:

	S_{10}	S_2	S_1	$\sigma_{1,0}$	η_1	η_2
C_1	1	1	0	0	-1	0
C_2	1	0	1	0	0	-1
F	-1	0	0	1	1	1

From now on we will use the following notations:

- generators of $A^1(\mathbf{H})$: $T_1 := S_1$, $T_2 := S_2$, $T_3 := \sigma_{1,0}$
- generators of $A^3(\mathbf{H})$: C_1, C_2, F

	T_1	T_2	T_3
C_1	0	1	0
C_2	1	0	0
F	0	0	1

Throughout this work the symbol (a, b, c) will be intended as a curve in $\mathbf{H}$ of class $aC_1 + bC_2 + cF$.

Remark 1.4.1. From the first table we obtain:

$$\eta_1 = T_3 - T_2, \quad \eta_2 = T_3 - T_1, \quad S_{10} = T_1 + T_2 - T_3.$$

Remark 1.4.2. We described $\mathbf{H}$ as the blowup of $\mathbf{G}$ along W, so we can use the formula given in [G-H] p.608 to calculate the first Chern class $c_1(T_{\mathbf{H}})$:

$$c_1(T_{\mathbf{H}}) = \varphi^* c_1(T_{\mathbf{G}}) - (n-k-1)\tilde{W}$$

with $n =$dim $\mathbf{G} = 4$, $k =$dim $W = 1$. By 1.4.1 we obtain:

$$c_1(T_{\mathbf{H}}) = 2(T_1 + T_2)$$

The following proposition gives a complete description of the cone of effective curves in $\mathbf{H}$.

Proposition 1.4.3. *An effective curve in* $\mathbf{H}$ *is of class* $aC_1 + bC_2 + cF$ *with* $a, b, c \geq 0$.

Proof. The proof consists of two steps. First we show that the linear systems associated to T_1, T_2, T_3 are base-points-free and then we look at their intersection product with the effective classes C_1, C_2, F.
The linear system associated to T_3 is obviously base-points-free, because T_3 is the pullback of an ample divisor class of $\mathbf{G}$. Let $D_{l_i} \in |T_i|$, $i = 1, 2$, be the divisor represented by the set

$$\{Z \in \mathbf{H} : \text{Supp } Z \cap l_i \neq \emptyset,\ l_i \in W_i \text{ fixed line}\}$$

Given a point $Z \in \mathbf{H}$, we have two possibilities for its support either it consists of a single point p or of two distinct points p, q. In both cases there exists a divisor D_{l_i} with $Z \notin D_{l_i}$. In fact it is enough to choose $l_i \in W_i$ such that either $p \notin l_i$ or $l_i \in W_i - \{l_i(p), l_i(q)\}$, with $l_i(p), l_i(q)$ the only lines in W_i through p and q respectively. This shows that also T_1, T_2 are base-points-free divisor classes.
The intersection product between an arbitrary effective curve and a base-points-free divisor is always non-negative. Since:

$$C_1 \cdot T_2 = 1,\ C_2 \cdot T_1 = 1,\ F \cdot T_3 = 1$$

and all other possible intersections give zero, an effective curve in $\mathbf{H}$ is of class $aC_1 + bC_2 + cF$ with $a, b, c \geq 0$. □

1.5 The locus Δ of non-reduced points of H

In §1.3 we described the closure $\overline{\Delta}_3$ as the set of non-reduced points of $\mathbf{H}$. Since it is closed and 3-dimensional, it is a divisor. Let us denote it by Δ. We will use the same notation also for the associated cycle class in $A^1(\mathbf{H})$ and we will refer to it as to the diagonal of $\mathbf{H}$. In this section we will give a complete description of such a divisor and of its Chow ring.

Geometrical description

As a divisor class in $\mathbf{H}$, the diagonal Δ is $\theta_*\xi$, hence by 1.1.1 and 1.4.1:

$$\Delta = 2S_{10} = 2(T_1 + T_2 - T_3) \tag{1.7}$$

This means that $\Delta = \tilde{\delta} = \mathbb{P}(T_Q)$. In particular there exists a map $s : \Delta \to Q$ which is a $\mathbb{P}^1$-bundle. It maps a non reduced point to its support so we will call it the *support map*. We deduce that Δ is irreducible. Obviously it is invariant for the $\mathcal{A}$-action.

It will be useful for our subject to know how the intersection of Δ with a fiber of the blowup morphism φ looks like. Then call D the image of Δ in $\mathbf{G}$ via φ. It is a union $D_0 \sqcup D_1$, where $D_0 = \{l \in \mathbf{G} : l \cap Q = pt\}$ and $D_1 = W_1 \sqcup W_2$ are disjoint. The inverse image $\varphi^{-1}(D_0)$ is the orbit Δ_3 isomorphic to D_0 while for each $l \in D_1$, $\varphi^{-1}(l)$ is the intersection $\mathrm{Sym}^2(l) \cap \Delta$. We want to describe the inclusion map:

$$\mathbb{P}^1 \cong \mathrm{Sym}^2(l) \cap \Delta \hookrightarrow \mathrm{Sym}^2(l) \cong \mathbb{P}^2$$

Proposition 1.5.1. *The diagonal Δ defines a conic in a fiber over the blown up locus.*

Proof. We can look at $\mathrm{Sym}^2(l)$ as at the space of all the quadratic forms on the 2-dimensional vector space V defining l up to scalars. Then $\mathrm{Sym}^2(l) \cap \Delta$ is the space of linear forms on the same vector space V, up to scalars. The inclusion map has to be defined by mapping a form f to the square power f^2, i.e. into the sublocus of quadratic forms with a unique root. This implies that the inclusion maps a point $(x : y) \in \mathbb{P}^1$ to the point $(x^2 : xy : y^2) \in \mathbb{P}^2$. This defines a conic in the plane $\mathrm{Sym}^2(l)$. $\square$

Chow ring and effective curves in Δ

We will refer to the following commutative diagram:

$$\begin{tikzcd}
\Delta \arrow[r, "j"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{U} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] \arrow[d, "\theta"] & Q \\
\Delta \arrow[r, "i"'] & \mathbf{H} &
\end{tikzcd}$$

Note that $p_1 \circ j = p_2 \circ j$ is the support map.

In 1.1 we have already calculated the Chow ring $A^*(\mathbb{P}(T_Q)) = A^*(\Delta)$, we refer to that section for notations. A basis is given by:

$A^0(\Delta)$	$A^1(\Delta)$	$A^2(\Delta)$	$A^3(\Delta)$
$[\Delta]$	h_1	$h_1\zeta$	$h_1h_2\zeta$
	h_2	$h_2\zeta$	
	ζ	h_1h_2	

The classes $h_1\zeta$, $h_2\zeta$ are the liftings of the two rulings on Q and h_1h_2 is the class of a fiber of s, (here we identify $h_i = s^*(h_i)$). The sets:

$$\begin{aligned} L_1 &= \{Z = (p,v) : p \in l_Z = l_1,\ l_1 \in W_1 \text{ given line}\} \cong l_1 \\ L_2 &= \{Z = (p,v) : p \in l_Z = l_2,\ l_2 \in W_2 \text{ given line}\} \cong l_2 \\ L_3 &= \{Z = (p_0,v) : p_0 \text{ fixed point}\} = \mathbb{P}(T_{Q,p_0}) \end{aligned}$$

are effective curves in Δ. Moreover, they are such that the corresponding classes in $A_1(\mathbf{H})$ are the Poincaré dual classes of $2C_1, 2C_2, F$, since Δ defines a conic on a fiber $\mathrm{Sym}^2(l_i)$, $i = 1, 2$, and F is the class $[\mathbb{P}(T_{Q,p})]$ for some $p \in Q$. In particular $A_1(\Delta)$ is generated by these effective classes.
Finally we want to describe the effective curves in $\mathbf{H}$ which are actually completely contained into Δ. We need to find a relationship between the divisor classes in Δ and the pullbacks i^*T_1, i^*T_2, i^*T_3.

Proposition 1.5.2. *We can write:*

$$Pic(\Delta) = \langle \frac{1}{2}T_1, \frac{1}{2}T_2, T_3 \rangle$$

where $T_j = i^*T_j$, by abuse of notation.

Proof. Note that by definition:

$$h_1 = j^*p_1^*(h_1) = j^*(h_1 \otimes 1) = \frac{1}{2}j^*\theta^*\theta_*(h_1 \otimes 1) = \frac{1}{2}j^*(h_1 \otimes 1 + 1 \otimes h_1)$$

According to our notation $h_1 \otimes 1 + 1 \otimes h_1 = \theta^*T_1$ in $\tilde{U}$, so we conclude $h_1 = \frac{1}{2}i^*T_1$ and by symmetry $h_2 = \frac{1}{2}i^*T_2$. Then it is easy to verify that the intersection product gives:

	i^*T_1	i^*T_2	i^*T_3
L_1	0	2	0
L_2	2	0	0
L_3	0	0	1

The thesis follows. □

Remark 1.5.3. By the adjunction formula and 1.4.2 we get:

$$c_1(T_\Delta) = c_1(\mathbf{H})|_\Delta - \Delta|_\Delta = 2T_3$$

Proposition 1.5.4. *The effective curves in* $\mathbf{H}$ *which are contained into* Δ *are of type (a,b,c) with $a,b,c \geq 0$ and a,b even.*

Proof. Let $C \subseteq \Delta$ be an effective curve of class (α,β,γ), then i_*C is an effective curve in $\mathbf{H}$ of class (a,b,c) for some non negative integers a,b,c. By the projection formula, $\deg_\Delta \frac{1}{2}T_1 \cdot C = \frac{a}{2}$ is an integer number equal to α, hence a is even. The same is true for b, by symmetry. □

1.6 The divisor Σ

In this section we study the closure of the 3-dimensional orbit Σ_3. We will denote it by Σ throughout this work. It is the divisor given by the disjoint union $\tilde{W}_1 \sqcup \tilde{W}_2$ of the two exceptional divisors of the blowup map $\varphi : \mathbf{H} \to \mathbf{G}$, (see §1.3). Each $\tilde{W}_i$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ because it is the relative Hilbert scheme $Hilb^2(Q/\mathbb{P}^1)$.

Chow ring of $\tilde{W}_i$ and effective curves in Σ

Note that $\tilde{W}_1$ and $\tilde{W}_2$ are completely symmetric, so let us consider only $\tilde{W}_1$. We have a good description of $\tilde{W}_1$ as a divisor in $\mathbf{H}$ (see 1.4.1) and of its Chow ring by (1.5). In particular:

$$\begin{aligned} &\tilde{W}_1 = \eta_1 = T_3 - T_2 \\ &\mathrm{Pic}(\tilde{W}_1) = \langle l_1, \xi_1 \rangle \\ &A_1(\tilde{W}_1) = \langle -l_1\xi_1, \xi_1^2 \rangle \end{aligned}$$

Proposition 1.6.1. *We can write:*

$$Pic(\tilde{W}_1) = \langle T_3 - T_2, \frac{1}{2}T_3 \rangle$$

where $T_i = j_1^* T_i$, by abuse of notation.

Proof. By definition $j_1^*\eta_1 = \xi_1$ and as one can easily check $j_1^*T_3 \cdot (-l_1\xi_1) = 0$, $j_1^*T_3 \cdot \xi_1^2 = 2$. □

Remark 1.6.2. By the adjunction formula and 1.4.2, the first Chern classes $c_1(T_{\tilde{W}_1})$, $c_1(T_\Sigma)$ are:

$$c_1(T_{\tilde{W}_1}) = c_1(T_{\mathbf{H}})|_{\tilde{W}_1} - \tilde{W}_1|_{\tilde{W}_1} = 3T_2 + T_3$$

$$c_1(T_\Sigma) = c_1(T_{\mathbf{H}})|_\Sigma - (\tilde{W}_1 + \tilde{W}_2)|_\Sigma = 3(T_1 + T_2) - T_3$$

As in the previous section we are interested in the effective curves in $\mathbf{H}$ which are the pushforward of effective curves in Σ. So we need to know the effective cone of Σ, i.e. of $\tilde{W}_1$ and $\tilde{W}_2$.

Proposition 1.6.3. *The effective cone in $\tilde{W}_1$ is generated by $-l_1\xi_1 = (1,0)$ and $4(-l_1\xi_1) + \xi_1^2 = (4,1)$.*

Proof. The inclusion map j_1 induces a ring homomorphism $A_1(\tilde{W}_1) \xrightarrow{(j_1)_*} A_1(\mathbf{H})$. The classes $-(j_1)_* l_1\xi_1$, $(j_1)_*\xi_1^2$ are such that:

	T_1	T_2	T_3
$-(j_1)_* l_1\xi_1$	0	1	0
$(j_1)_*\xi_1^2$	2	4	2

It follows that $-(j_1)_* l_1\xi_1 = C_1$ and $(j_1)_*\xi_1^2 = -4C_1 + 2C_2 + 2F$. So the pushforward map is defined by:

$$(j_1)_*(-ah_1\xi_1 + b\xi_1^2) = (a-4b)C_1 + 2bC_2 + 2bC_3$$

Suppose that $-ah_1\xi_1 + b\xi_1^2$ is an effective curve, since $(j_1)_*$ maps effective curves to effective curves, it holds:

$$a \geq 4b, \quad b \geq 0$$

We already know that a curve of class $-h_1\xi_1 = (1,0)$ is effective, since it is a line in a fiber of φ_1 (see 1.2.7). We have to look for a second generator of the effective cone in $\tilde{W}_1$. Let us fix two lines $l_2^{'}, l_2^{''} \in W_2$ and let C be the following curve:

$$\{Z : \text{Supp } Z = \{p,q\},\ \exists\ l_1 \in W_1 \text{ with } p = l_1 \cap l_2^{'},\ q = l_1 \cap l_2^{''}\}$$

It is isomorphic to W_1. Write $[C] = -al_1\xi_1 + b\xi_1^2$, then:

$$(j_1)_*[C]\cdot T_3 = 2b = 2 \Rightarrow b = 1$$

$$(j_1)_*[C]\cdot T_2 = a - 4b = 0 \Rightarrow a = 4$$

$$(j_1)_*[C]\cdot T_1 = 2b = 2$$

The last equality tells us that C intersects a divisor in $|T_1|$ in a point with multiplicity. Since C is effective, we find that the effective cone in $\tilde{W}_1$ is the set $\{a(1,0) + b(4,1) :\ a,b \geq 0\}$. □

Proposition 1.6.4. *The effective curves in* $\mathbf{H}$ *which are actually effective curves in* $\tilde{W}_1$ *are of class* (a,b,c) *with* $a,b,c \geq 0,\ b = c$ *even.*
Symmetrically, the effective curves in $\mathbf{H}$ *contained into* $\tilde{W}_2$ *are of class* (a,b,c) *with* $a,b,c \geq 0,\ a = c$ *even.*
These conditions describe all effective curves in $\mathbf{H}$ *contained into* Σ.

Proof. It is enough to write the homomorphism $A_1^{eff}(\tilde{W}_1) \to A_1^{eff}(\mathbf{H})$:

$$\begin{array}{ccc} \mathbb{N}(1,0) \oplus \mathbb{N}(4,1) & \xrightarrow{(j_1)_*} & \mathbb{N}C_1 \oplus \mathbb{N}C_2 \oplus \mathbb{N}F \\ (1,0) & \mapsto & C_1 \\ (4,1) & \mapsto & 2C_2 + 2F \\ a(1,0) + b(4,1) & \mapsto & aC_1 + 2bC_2 + 2bF \end{array}$$

□

1.7 Description of some effective curves

We describe all the effective connected curves in some cycle classes in $A_1(\mathbf{H})$. In the following chapters we will make explicit calculations on the moduli spaces of stable maps involving such curves.

Notations : if $A_1, A_2, \ldots, A_n$ are classes in $A_1(\mathbf{H})$ we will say that an effective curve has class $A_1 \cup A_2 \cup \ldots \cup A_n$ to mean that these are the classes of its irreducible components (eventually counted with multiplicity). A linear combination $A_1 + A_2 + \ldots + A_n$ will denote the class of an irreducible curve. All the coefficients are understood to be non-negative.
If $p \in Q$ is a point we will denote by $l_i(p)$ the unique line of the i-ruling on Q going through p.
We will use Propositions 1.5.4 and 1.6.4 without explicit reference throughout.

Curves of class $(0, 0, c)$

i) Given a point $p_0 \in Q$, the curve $C(p_0) = \mathbb{P}(T_{Q,p_0})$ is of class $(0, 0, 1)$, (see §1.4). In particular it is entirely contained into Δ, because it is a fiber of the support map $s : \Delta \to Q$. Note that each point $Z \in C(p_0)$ has support Supp $Z = p_0$. Conversely a curve of class $(0, 0, 1)$ is irreducible and contained into Δ. Its pushforward to Q is zero. Hence it is completely contained in a fiber of the support map s, since it is connected. This shows that all the curves of such a class look like $C(p_0)$ for some p_0 in Q. Moreover also a curve of class $(0, 0, c)$ is contained into Δ. It is a c-cover of a $C(p_0)$ curve. The intersection product gives:

$$(0, 0, c) \cdot \Delta = -2c$$
$$(0, 0, c) \cdot \Sigma = 2c$$

Curves of class $(1, 0, c)$, $(0, 1, c)$

Since the classes $(1, 0, c), (0, 1, c)$ are symmetric under the involution we can analyse only one of them. We choose $(1, 0, c)$.

ii) A curve of class $\beta = (1, 0, 0)$ is necessarily irreducible.
Let $\tilde{\varphi} : \mathbf{H} \to Bl_{W_1}\mathbf{G}$ be the natural map. Then $\tilde{\varphi}_*(1, 0, 0) = 0$. Hence if C has class $(1, 0, 0)$ it must be contained in a positive dimensional fiber of $\tilde{\varphi}$. Such a fiber is Hilb$^2(l_1)$ for some $l_1 \subseteq Q$, so C is a line in it. We denote it by $C(l_1)$, (see §1.4). All $Z \in C(l_1)$ are such that Supp $Z \subseteq l_1$. The intersection product gives:

$$(1, 0, 0) \cdot \tilde{W}_1 = -1 \quad (1, 0, 0) \cdot \tilde{W}_2 = 0$$

It is contained into $\tilde{W}_1$. It intersects the diagonal Δ in at most two points, since Δ restricted to the fiber of the blowup map $\varphi : \mathbf{H} \to \mathbf{G}$ over l_1 is a conic (see §1.5).

iii) Fix a point $p_1 \in Q$ and a line $l_1 \in W_1$ such that $p_1 \notin l_1$. The curve $C(p_1, l_1) = \{Z \in \mathbf{H} : \text{Supp } Z = (p_1, q), q \in l_1\}$ is irreducible of class $\beta = (1,0,1) = C_1 + F$. It is disjoint from Δ. Since:

$$(1,0,1) \cdot \tilde{W}_1 = 0 \quad (1,0,1) \cdot \tilde{W}_2 = 1$$

we conclude that it is disjoint from $\tilde{W}_1$ and not contained into $\tilde{W}_2$ (by §1.6). It intersects Σ in a unique point $Z = (p_1, l_1 \cap l_2(p_1)) \in \tilde{W}_2$. It is easy to see that these are all the possible irreducible curves of this class. In fact let C be an irreducible curve of class $(1,0,1)$. Since it is disjoint from Δ we can consider the curve $\tilde{C}$ defined by the cartesian diagram:

$$\begin{tikzcd} \tilde{C} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\Box"] & \tilde{U} \arrow[r, "bl"] \arrow[d, "\theta"] & U = Q \times Q \\ C \arrow[r, hook] & \mathbf{H} & \end{tikzcd}$$

Also $\tilde{C}$ is disjoint from Δ, then it is isomorphic to the image curve $bl(\tilde{C})$. We can identify them and work on U.
$\tilde{C}$ has class $h_1 \otimes h_3 + h_3 \otimes h_1$ on U (see Lemma 1.1.3 for notations). It is symmetric under the natural involution. If it has two components then these are of class $h_1 \otimes h_3$, $h_3 \otimes h_1$ respectively. This implies that C is a curve $C(p_1, l_1)$ for some $p_1 \in Q$, $l_1 \in W_1$, $p_1 \notin l_1$.
If $\tilde{C}$ has only one component then there is a morphism $\tilde{C} \to l_1$, generically of degree 1, for some $l_1 \in W_1$. By symmetry, we conlcude that it is an irreducible curve contained in $l_1 \times l_1$ such that it does not intersects the diagonal. This is impossible.
Let $(1,0,1)$ be the class of a reducible curve C. Then C is the union $C(l_1) \cup C(p)$ for some $l_1 \in W_1$ and $p \in l_1$ with $p \in C(l_1) \cap \Delta|_{\varphi^{-1}(l_1)}$. It is contained into $\Delta \cup \Sigma$.

iv) A curve of class $\beta = (1,0,c)$, $c \geq 2$, can be written as a union of irreducible effective components. A priori we have three possibilities to do that:
- $C_1 + cF$ is the class of an irreducible curve;
- $C_1 \cup c_1 F \cup c_2 F$ with $c_1 + c_2 = c$
- $(C_1 + c_1 F) \cup c_2 F$ with $c_1 + c_2 = c$
The first case implies that a $(C_1 + cF)$-curve is contained into Δ and this is impossible because 1 is odd (see §1.5). So the third one is also impossible because we know that if $c_1 = 1$, a $(C_1 + F)$-curve does not intersect Δ. The second decomposition represents curves with support

$C(l_1) \cup C(p) \cup C(q)$ or $C(l_1) \cup C(p)$ for some $l_1 \in W_1$ and p, q points in $C(l_1) \cap \Delta|_{\varphi^{-1}(l_1)}$. We conclude that for $c \geq 2$ there are only reducible curves of class $(1, 0, c)$ entirely contained into $\Delta \cup \Sigma$.

Remark 1.7.1. To have a description of curves of type $(0, 1, c)$ it is enough to interchange each l_1 appearing in the above discussion with a line l_2. In particular, fixing $p_2 \in Q$ and $l_2 \in W_2$ such that $p_2 \notin l_2$:

$$C(p_2, l_2) = \{Z \in \mathbf{H} : \text{Supp } Z = (p_2, q), q \in l_2\}$$

is an irreducible curve of class $(0, 1, 1) = C_2 + F$ and all such curves are of this kind.

Curves of class $(1, 1, c)$

v) Connected curves of class $(1, 1, 0)$ do not exist. In fact if the curve is reducible then it is a union $C(l_1) \cup C(l_2)$, but such curves cannot intersect. If it is irreducible, then it is contained into Σ, since

$$(1, 1, 0) \cdot \Sigma = -2$$

but this is impossible by what we showed in §1.6.

vi) Let C be a reducible curve of type $(1, 1, 1)$. We have three possible decompositions:
- $C_1 \cup C_2 \cup F$ is represented by a curve contained into $\Delta \cup \Sigma$ of the form $C(l_1(p)) \cup C(l_2(p)) \cup C(p)$ with p a point of Q;
- $(C_1 + F) \cup C_2$ is the class of a curve $C(p, l_1) \cup C(l_2(p))$ for a given line l_1 and a given point $p \in Q$, with $C(l_2(p))$ a line in $\text{Hilb}^2(l_2(p))$ passing through (p, q), $q = l_1 \cap l_2(p)$;
- $(C_2 + F) \cup C_1$ similarly.

If C is irreducible, then it intersects Δ in two points with multiplicity and it is disjoint from Σ, in fact:

$$(1, 1, 1) \cdot \Delta = 2 \quad (1, 1, 1) \cdot \Sigma = 0$$

There are two possible families of irreducible curves of class $(1, 1, 1)$. In fact let C be irreducible of such a class and consider the blowup morphism $\varphi : \mathbf{H} \to \mathbf{G}$. The image curve $\varphi(C)$ is isomorphic to C because C is disjoint from Σ, and we can identify them. It is of class $\sigma_{2,1}$, then we know a geometrical description of it (see §1.2). Fix a plane $\Lambda \subseteq \mathbb{P}^3$ and a generic point $q \in \Lambda$, $q \notin Q$. Then the intersection $\Lambda \cap Q$ is a conic. There are two possibilities: the plane is generic and the conic is irreducible or Λ is tangent to Q at a point p and the conic is the union $l_1(p) \cup l_2(p)$. In the first case C corresponds to a line in $\mathrm{Hilb}^2(\Lambda \cap Q)$ whose points are the closed subschemes Z such that $\mathrm{Supp}\ Z \subseteq (\Lambda \cap Q)$, $q \in l_Z$. We will denote it as follows:

$$\Lambda(l) = \{Z \in \mathbf{H} : Z \in l,\ l \subseteq \mathrm{Hilb}^2(\Lambda \cap Q) \text{ a line}\}$$

In the second case we get an irreducible curve determined by choosing a plane tangent to Q and a point $q \in \Lambda$ such that $q \notin \Lambda \cap Q$. Its points are the closed subschemes Z such that $\mathrm{Supp}\ Z \cap l_1(p) \neq \emptyset$, $\mathrm{Supp}\ Z \cap l_2(p) \neq \emptyset$ and $q \in l_Z$. Such a curve has only 4 moduli, while the expected dimension is 5.
The class $C_1 + C_2 + F$ can be represented by a curve $\Lambda(l)$.

vii) If a curve of class $(1,1,2)$ is reducible, we have:
- $C_1 \cup C_2 \cup c_1 F \cup c_2 F$ with $c_1 + c_2 = 2$ is the class of some curve $C(l_1(p)) \cup C(l_2(p)) \cup c_1 C(p) \cup c_2 C(q)$ with q the support of a point in $C(l_i) \cap \Delta|_{\varphi^{-1}(l_i)}$ for $i = 1$ or 2. Such a curve is completely contained into $\Delta \cup \Sigma$;
- $(C_1 + F) \cup (C_2 + F)$ is either the class of $C(p_1, l_1(p_2)) \cup C(p_2, l_2(p_1))$, with $p_1 \neq p_2$, or $C(p, l_1) \cup C(p, l_2)$ with l_1, l_2 not passing through p. Note that each component is not contained into $\Delta \cup \Sigma$;
- $(C_1 + F) \cup C_2 \cup F$ with $C_1 + F = [C(p_1, l_1)]$, C_2 the class of a line in $\mathrm{Hilb}^2(l_2(p_1))$ through $(p_1, l_2(p_1) \cap l_1)$ and $F = [C(q)]$ where q is the support of $Z \in C(l_2) \cap \Delta|_{\varphi^{-1}(l_2)}$;
- $(C_2 + F) \cup C_1 \cup F$ similarly;
- $(C_1 + C_2 + F) \cup F$ is the class of a curve $\Lambda(l) \cup C(p)$ with $p \in l \cap \Delta$.
If a curve of type $(1,1,2)$ is irreducible, then it is disjoint from Δ, and not contained into Σ:

$$(1,1,2) \cdot \Delta = 0 \quad (1,1,2) \cdot \Sigma = 2$$

By considering the pushforward to the product $Q \times Q$, it can be shown that there are only two possible families of irreducible curves of class $(1,1,2)$. The first one is:

$$\Lambda(p) = \{Z \in \mathbf{H} : p \in \mathrm{Supp}\ Z, \Lambda \cap Q \cap \mathrm{Supp}\ Z \neq \emptyset\}$$

where p is a fixed point of Q and $\Lambda \subseteq \mathbb{P}^3$ a given generic plane, $p \notin \Lambda$. The second one is determined by the following data: one fixes two lines $l_1 \in W_1$ and $l_2 \in W_2$ on Q with $p = l_1 \cap l_2$ and an isomorphism $f : l_1 \to l_2$ such that $f(p) \neq p$. The curve is:

$$C(l_1, l_2, f) = \{(q, f(q)) : q \in l_1\}$$

It is isomorphic to $\mathbb{P}^1$ and intersects Σ in $(p, f(p))$ and $(f^{-1}(p), p)$.

viii) For $c > 2$, irreducible curves of type $(1,1,c)$ do not exist, because they should be contained into Δ being:

$$(1,1,c) \cdot \Delta = 2 - 2c < 0 \quad (1,1,c) \cdot \Sigma = 2c - 2$$

So they are reducible and because of connectedness they can only decompose as:
- $C_1 \cup C_2 \cup c_1F \cup c_2F \cup c_3F$ with $c_1 + c_2 + c_3 = c$;
- $(C_1 + F) \cup C_2 \cup c_1F \cup c_2F$ with $c_1 + c_2 = c - 1$;
- $(C_2 + F) \cup C_1 \cup c_1F \cup c_2F$ as above;
- $(C_1 + C_2 + F) \cup c_1F \cup c_2F$ with $c_1 + c_2 = c - 1$.
All the curves representing these classes are not completely contained into $\Delta \cup \Sigma$ but the first one.

Remark 1.7.2. Note that irreducible curves of class $C_1 + F$, $C_2 + F$, $C_1 + C_2 + F$ intersect the stratification properly.

Curves of class $(2, 0, c)$, $(0, 2, c)$

We conclude with the description of the connected curves of class $(2, 0, c)$, $(0, 2, c)$. As before these classes are symmetric under the natural involution, so we study only the class $(2, 0, c)$.

ix) A curve of class $(2, 0, 0)$ is always contained into Σ. It can be represented by a conic in the plane $\mathrm{Hilb}^2(l_1)$ for some $l_1 \in W_1$. So it can be irreducible or not. The intersection product gives:

$$(2, 0, 0) \cdot \Delta = 4 \quad (2, 0, 0) \cdot \Sigma = -2$$

x) There are no irreducible curves of class $(2, 0, 1)$, because the intersection product with $\tilde{W}_1$ gives -1, but such a curve can not be contained into $\tilde{W}_1$ because $1 \neq 0$ is odd. We have only one possibility for a reducible curve, it is a union of two components of class $2C_1$ and F respectively, hence it is completely contained into $\Delta \cup \Sigma$.

xi) If a $(2, 0, 2)$-curve is reducible we have two possibilities:
- $2C_1 \cup c_1F \cup c_2F$ with $2C_1$ the class of a conic in some $\mathrm{Hilb}^2(l_1)$ and $c_1 + c_2 = 2$; the corresponding curve is completely contained into $\Delta \cup \Sigma$;
- $(C_1 + F) \cup (C_1 + F)$ is the class of a curve $C(p_1, l_1(p_2)) \cup C(p_2, l_1(p_1))$, $p_1 \neq p_2$, by connectedness. It is disjoint from Δ and not in Σ.
Since we have:

$$(2, 0, 2) \cdot \Delta = 0 \quad (2, 0, 2) \cdot \Sigma = 2 \quad (2, 0, 2) \cdot \tilde{W}_1 = 0$$

an irreducible and reduced curve of class $2C_1 + 2F$ is disjoint from $\tilde{W}_1$ and it can be contained into Δ. In fact, denote by $\mathbb{F}_2$ the inverse image of a fixed $l_1 \subseteq Q$ via the support map $s : \Delta \to Q$. Then $\mathbb{F}_2$ is the rational ruled surface defined by the sheaf $\mathcal{O} \oplus \mathcal{O}(-2)$ on $\mathbb{P}^1$. There exist irreducible curves $C \subseteq \mathbb{F}_2$ of type $D_1 + 2D_3$, with $D_3 = F$ the class of a fiber of s and $\mathcal{L}(D_1) \cong \mathcal{O}_{\mathbb{F}_2}(1)$, [Har] Chap.V Cor. 2.18. Moreover we know that $2C_1 + 2F = D_1 + 2D_3$ in Δ.

We do not know a geometric description of such a curve. We can only say that if it is not contained into Δ then it is disjoint from it.
Irreducible non reduced curves of class $2(C_1 + F)$ are disjoint from Δ and not contained into Σ, their image in $\mathbf{H}$ is the same image of the corresponding $(C_1 + F)$-curves.

xii) Irreducible curves of class $(2, 0, c), c \geq 3$ are all reduced and contained into Δ, in fact:

$$(2, 0, c) \cdot \Delta = 4 - 2c < 0 \quad (2, 0, c) \cdot \Sigma = 2c - 2$$

The reducible ones can be decomposed as:
- $2C_1 \cup c_1F \cup c_2F \cup c_3F \cup c_4F$ with $\sum c_i = c$, this is the class of a curve completely contained into $\Delta \cup \Sigma$;
- $(2C_1 + c_1F) \cup C$, where C is the union of an opportune number of classes c_iF, $i \geq 2$, with $\sum_{i \geq 1} c_i = c$, $c_1 \geq 2$.

1.8 Subschemes incident to a given point

We conclude the chapter with the description of a 2-codimensional cycle on $\mathbf{H}$ which is of great interest for our work.
Let $\Gamma(p)$ be the set $\{[Z] \in \mathbf{H} : p \in \text{Supp } Z,\ p \in Q \text{ given point}\}$. It is the blowup of Q in p, so it is smooth and 2-dimensional. It represents the class $S_3 \in A^2(\mathbf{H})$. We fix once for all the following notation $T_4 := S_3 = [\Gamma(p)]$.

Lemma 1.8.1. *Then:*

$$T_4 = \sigma_{2,0} - \eta_3 - \eta_4$$

Proof. Consider the restriction $\overline{\varphi}$ of the blowup map $\varphi : \mathbf{H} \to \mathbf{G}$ to $\Gamma(p)$:

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{\varphi} & \mathbf{G} \\ \cup & & \cup \\ \Gamma(p) & \xrightarrow{\overline{\varphi}} & \sigma_{2,0}(p) \cong \mathbb{P}^2 \end{array}$$

Let $V = \sigma_{2,0}(p) \backslash \{l_1(p), l_2(p)\}$, where $l_1(p), l_2(p)$ are the two lines in Q through p. We have:

- $\overline{\varphi} : \Gamma(p) \to \sigma_{2,0}(p)$ is surjective
- $\overline{\varphi} : \varphi^{-1}(V) \to V$ is an isomorphism

$\Gamma(p)$ contains an open dense subset isomorphic to V via φ. For $k = 1, 2$ the intersection $\Gamma(p) \cap \varphi^{-1}(l_k(p))$ is the line $\{Z \in \text{Sym}^2(l_k(p)) : p \in \text{Supp } Z\}$. So $\Gamma(p)$ is isomorphic to the blowup of the projective plane in two points. Note that we have the following scheme-theoretical decomposition in irreducible

components $\varphi^{-1}(\sigma_{2,0}(p)) = \varphi^{-1}(V) \cup \varphi^{-1}(l_1) \cup \varphi^{-1}(l_2)$.
Since $[\varphi^{-1}(l_k(p))] = \eta_{k+2}$, with $k = 1, 2$, we have:

$$\sigma_{2,0} = [\Gamma(p)] + \eta_3 + \eta_4$$

Hence the lemma is proved. □

Remark 1.8.2. Since the degree of T_4^2 in $\mathbf{H}$ is half the degree that it has in $\tilde{U}$ (see 1.1.1), we deduce that $T_4^2 \in A^4(\mathbf{H})$ is the class of a point.

Remark 1.8.3. For any $p \in Q$ the cycle $\Gamma(p)$ intersects the stratification properly. In fact $\Gamma(p) \cap \Sigma_4 \cong Q - (l_1(p) \cup l_2(p))$ is obviously a proper intersection and $\Gamma(p) \cap \Delta_2 = \{(p, T_{l_1(p),p}),\ (p, T_{l_2(p),p})\}$ is 0-dimensional. Since these intersections are non-empty, it is also satisfied $\Gamma(p) \not\subseteq \Sigma_3 \sqcup \Delta_3$.

We set Γ_{reg} to be the locus of $\Gamma(p)$ where the intersection with the stratification is transversal.

Lemma 1.8.4. *Given a point $p \in Q$, the locus Γ_{reg} is the open subset of $\Gamma(p)$ of points with reduced support.*

Proof. We first prove that $\Delta_2^k \cap \Gamma(p)$, $k = 1, 2$, is not transversal. Δ_2^k is a surface in $\tilde{W}_k$, it is the pullback of the diagonal Δ via the inclusion map $j_k : \tilde{W}_k \hookrightarrow \mathbf{H}$. Then it is a divisor in $\tilde{W}_k$. By the projection formula we obtain $\Delta_2^k = 4l_k - 2\xi_k$. It is easy to verify that $\Gamma(p)$ intersects Δ_2^k only in one point, but the degree of the intersection product $T_4 \cdot (j_k)_*\Delta_2^k$ in $\mathbf{H}$ is 2, this means the intersection is not transversal.
We now consider the intersection $\Gamma(p) \cap \Sigma_3$. Since Σ_3 is open in Σ and $\Sigma = \tilde{W}_1 \sqcup \tilde{W}_2$, we can work with a divisor $\tilde{W}_k$. The quotient map $\theta : \tilde{U} \to \mathbf{H}$ is an isomorphism between $Q \times Q - \delta$ and $\mathbf{H} - \Delta$. The inverse image $\theta^{-1}(\Gamma(p) - (\Gamma(p) \cap \Delta))$ is isomorphic to the disjoint union of two copies of $Q - p$. So in order to study the differential of the map $\Gamma(p) \to \mathbf{H}$ away from Δ it is enough to study the differential of $Q - p \to Q \times Q - \delta$ defined by $q \mapsto (p, q)$. As $Q = \mathbb{P}^1 \times \mathbb{P}^1$ we can choose coordinates on both $\mathbb{P}^1$'s so that the above map becomes:

$$\begin{array}{ccc} \mathbb{A}^2 - \{(p_1, p_2)\} & \longrightarrow & \mathbb{A}^4 \\ (q_1, q_2) & \mapsto & (p_1, p_2, q_1, q_2) \end{array}$$

where $p = (p_1, p_2)$. We denote by x_1, x_2, y_1, y_2 the coordinates on $\mathbb{A}^4$. Then $\theta^{-1}(\tilde{W}_k)$ is the set $\{(x_1, x_2, y_1, y_2) : x_k = y_k\}$ in $\mathbb{A}^4$ so that the tangent space $T_{(p,q)}\tilde{W}_k$ is the 3-dimensional affine space defined by the equation $x_k - y_k = 0$. Besides $\Gamma(p) - \Delta$ is the set $\{(x_1, x_2, y_1, y_2) : x_1 = p_1, x_2 = p_2\}$, so it is isomorphic to $\mathbb{A}^2$ and the tangent space $T_{(p,q)}(\Gamma(p) - \Delta)$ is the 2-dimensional affine space defined by the equations $x_1 = 0, x_2 = 0$. Then for each $(p, q) \in \Gamma(p) \cap \tilde{W}_k - \Delta$, the space $T_{(p,q)}(\Gamma(p) - \Delta)$ is not contained in $T_{(p,q)}\tilde{W}_k$, that is to say $\Gamma(p)$ intersects Σ_3 transversally.

Finally consider the closed immersion $f : Q \to Q \times Q$, $f(q) = (p,q)$. By [Har], Chap.II Cor. 7.15, there is a unique closed immersion $\tilde{f}$ such that the following diagram is commutative:

$$\begin{tikzcd} Bl_pQ \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{U} \arrow[r, "\theta"] \arrow[d] & \mathbf{H} \\ Q \arrow[r, "f"] & Q \times Q & \end{tikzcd}$$

where $Bl_pQ = \theta^{-1}(\Gamma(p))$. Let $y_1, \dots, y_4$ be local coordinates in $\mathbf{H}$ and $x_1, \dots, x_4$ local coordinates in $\tilde{U}$ such that the diagonal Δ is the zero locus $\{y_4 = 0\}$ and $\tilde{\delta} = \{x_4 = 0\}$. Then the quotient map $\theta : \tilde{U} \to \mathbf{H}$ is given by $\theta(x_1, \dots, x_4) = (x_1, x_2, x_3, x_4^2)$ and the differential $d\theta$ has matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2x_4 \end{pmatrix}$$

For each $(p,v) \in \theta^{-1}(\Delta)$, the image $d\theta_p(T_{(p,v)}\tilde{U})$ is contained into $T_{\theta(p,v)}\Delta$. As $\tilde{f}$ is a closed immersion and Δ_3 is open dense in Δ, it follows that $\Gamma(p)$ does not intersect Δ_3 transversally. $\square$

Chapter 2

Gromov-Witten Invariants

This chapter is devoted to describing the more general set-up in which one can define the Gromov-Witten Invariants. Moreover we present some results about the way of computing some particular invariants we will need in the following.

2.1 Moduli space of stable maps

Fix a smooth projective variety X and a class $\beta \in A_1(X)$.

Definition 2.1.1. *A* n-pointed stable map to X of type β *consists of the following data:*

- *a connected projective reduced curve C with at most ordinary double singular points, arithmetic genus g and $n \geq 0$ pairwise distinct non-singular marked points $x_1, \ldots, x_n$;*
- *a map $\mu : C \to X$ such that $\mu_*[C] = \beta$,*

such that the tuple $(C, x_1, l \ldots, x_n; \mu)$ has only finitely many automorphisms.

There is a Deligne-Mumford stack $\overline{M}_{g,n}(X, \beta)$, called the *moduli space of stable maps*, which is a fine moduli space parametrizing these maps (see [B-M]). We will denote its points by $[C, x_1, \ldots, x_n, \mu]$.
Let $M_{g,n}(X, \beta)$ be the open substack parametrizing the stable maps from smooth irreducible curves, we can think of $\overline{M}_{g,n}(X, \beta)$ as a compactification of this subspace even if $M_{g,n}(X, \beta)$ does not need to be dense in $\overline{M}_{g,n}(X, \beta)$.
The moduli space of stable maps comes equipped with some natural morphism. For each marked point one can define:

$$ev_i : \overline{M}_{g,n}(X, \beta) \to X$$

by $ev_i[C, x_1, \ldots, x_n, \mu] = \mu(x_i)$. We will denote by:

$$ev = (ev_1, ev_2, \ldots, ev_n)$$

the morphism mapping to X^n and call it the *evaluation map*.
The flat morphism:

$$\pi : \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$$

which forgets the last marked point and eventually stabilizes the curve, realizes $\overline{M}_{g,n+1}(X,\beta)$ as the universal curve over $\overline{M}_{g,n}(X,\beta)$, with ev_{n+1} the universal map to X:

$$\begin{tikzcd} \overline{M}_{g,n+1}(X,\beta) \arrow[r, "ev_{n+1}"] \arrow[d, "\pi"] & X \\ \overline{M}_{g,n}(X,\beta) & \end{tikzcd}$$

In particular, for each subset I of the index set $\{1,\ldots,n\}$ one can define an analogous map forgetting only the points labelled by I. It is a composition of universal families, then it is flat of relative dimension equal to the cardinality of I. We are principally interested in the case $I = \{1,\ldots,n\}$ and we will denote such a map again by π.
Finally for $n + 2g - 3 \geq 0$ there is a morphism:

$$\overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$$

which simply forgets the map and stabilizes the curve if necessary. Let $\mathfrak{M}_{g,n}$ be the smooth Artin stack parametrizing quasi-stable curves of genus g with n markings. It has dimension equal to $3g-3+n$ and $\overline{M}_{g,n}$ is an open dense subset of $\mathfrak{M}_{g,n}$. One defines a natural morphism:

$$\eta : \overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$$

by forgetting the map to X (without stabilizing).

2.2 Deformation theory on $\overline{M}_{0,n}(\mathbf{H},\beta)$

The local structure of the moduli space $\overline{M}_{0,n}(\mathbf{H},\beta)$ can be studied by deformation theory. In [L-T] Li and Tian proved that to every point m in the moduli space one can associate two finite dimensional spaces: a tangent space T and an obstruction space E. In particular at a fixed point $[C,\mu] \in \overline{M}_{0,0}(\mathbf{H},\beta)$:

$$T = Ext^1(\mu^*\Omega_{\mathbf{H}} \to \Omega_C, \mathcal{O}_C)$$
$$E = Ext^2(\mu^*\Omega_{\mathbf{H}} \to \Omega_C, \mathcal{O}_C)$$

(see Propositions 1.4-1.5 in [L-T]). Moreover T and E fit into the exact sequence:

$$0 \to \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(C, \mu^* T_{\mathbf{H}}) \to T \xrightarrow{\phi} \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to \\ \to H^1(C, \mu^* T_{\mathbf{H}}) \to E \to 0 \qquad (2.1)$$

which we call the *tangent-obstruction sequence.*
The space $\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) = H^0(C, T_C)$ is the space of automorphisms of the nodal curve C while $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ is the space of first order deformations of C [H-M], Ch.2 §B.3. $H^0(C, \mu^* T_{\mathbf{H}})$ is the relative tangent space and $H^1(C, \mu^* T_{\mathbf{H}})$ the relative obstruction space of $\overline{M}_{0,0}(\mathbf{H}, \beta)$ over $[C] \in \mathfrak{M}_{0,0}$, by [K] Thm.II.1.7. They encode the possibility of deforming a map from a fixed nodal curve to $\mathbf{H}$. Note that if $H^1(C, \mu^* T_{\mathbf{H}}) = 0$ then the deformation is unobstructed, i.e. we can always lift a n-order deformation of the stable map to a $(n+1)$-order deformation. This vanishing condition implies that $\overline{M}_{0,0}(\mathbf{H}, \beta)$ is smooth at that point, because $E = 0$. In a more general setting the following result can be proved (see [K] Thm.II.1.7):

Theorem 2.2.1. *If $\mu : C \to \mathbf{H}$ is an n-pointed stable map such that $H^1(C, \mu^* T_{\mathbf{H}}) = 0$, then the forgetful morphism $\eta : \overline{M}_{g,n}(\mathbf{H}, \beta) \to \mathfrak{M}_{g,n}$ is smooth at $[C, x_1, \dots, x_n, \mu]$.*

Remark 2.2.2. A smooth variety X is called *convex* if $H^1(\mathbb{P}^1, f^* T_X) = 0$ for all genus zero stable maps $f : \mathbb{P}^1 \to X$. If X is convex then $H^1(C, f^* T_X) = 0$ for all maps $f : C \to X$, C a genus zero rational curve. Hence the moduli space $\overline{M}_{0,0}(X, \beta)$ is smooth of dimension equal to the expected one, [Al] I.3.

Remark 2.2.3. Consider $\overline{M}_{0,0}(\mathbf{H}, (a,b,c))$, with $(a,b,c) \neq 0$, and fix a point $[\mathbb{P}^1, \mu]$ in it, then:

$$T = H^0(\mathbb{P}^1, N_\mu)$$
$$E = H^1(\mathbb{P}^1, N_\mu)$$

where N_μ is defined as the cokernel of the differential map $d\mu : T_{\mathbb{P}^1} \to \mu^* T_{\mathbf{H}}$, [H-M] Chap.3 §B p.96. This follows by comparing (2.1) with the long exact sequence in cohomology associated to the exact sequence:

$$0 \to T_{\mathbb{P}^1} \to \mu^* T_{\mathbf{H}} \to N_\mu \to 0$$

Smoothening the nodes

Let C be a prestable curve. The space $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ of first order deformations of the nodal curve C fits into the exact sequence:

$$0 \to H^1(C, T_C) \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_{\mathcal{C}})) \to 0 \qquad (2.2)$$

where $H^1(C, T_C)$ is the space of first order deformations of C which do not smoothen the nodes. $H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_{\mathcal{C}}))$ parametrizes the first order

deformations of the nodes. It is isomorphic to $\bigoplus_{p \text{ node}} \mathcal{O}_{C,p}$.
Let $\mathcal{C}$ be the universal curve over the unpointed moduli space $\overline{M}_{0,0}(\mathbf{H},\beta)$. Let $\mathcal{M}_0$ be a smooth open subset of $\overline{M}_{0,0}(\mathbf{H},\beta)$ such that there exist a section $\nu : \mathcal{M}_0 \to \mathcal{C}$ such that $\nu(m)$ is a node on the corresponding curve $\mathcal{C}_m$ and the induced map $T_{\mathcal{M}_0,m} \to \mathrm{Ext}^1(\Omega_{\mathcal{C}_m}, \mathcal{O}_{\mathcal{C}_m})$ has as image the kernel K of the natural projection $h : \mathrm{Ext}^1(\Omega_{\mathcal{C}_m}, \mathcal{O}_{\mathcal{C}_m}) \to \mathcal{E}xt^1(\Omega_{\mathcal{C}_m}, \mathcal{O}_{\mathcal{C}_m})_{\nu(m)}$.
It is easy to see that the following proposition holds:

Proposition 2.2.4. *Fix a point $[C,\mu] \in \mathcal{M}_0$ and let $p = \nu([C,\mu])$ be the node which can not be smoothened. Consider the tangent-obstruction sequence (2.1). Then:*

1. *coker $\phi = \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_p$;*
2. *there exists an injective map $\varphi : \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_p \to H^1(C, \mu^* T_{\mathbf{H}})$ such that coker $\varphi = E$.*

Proof. By definition we get an exact sequence:

$$0 \to K \xrightarrow{g} \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \xrightarrow{h} \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_p \to 0 \tag{2.3}$$

The natural map $K \to H^1(C, \mu^* T_{\mathbf{H}})$ is identically zero, because the deformations of C in K do not smoothen the node p. Then there is no obstruction to extend μ to a given deformation of C locally trivial near p. There exists an injective map $\varphi : \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_p \to H^1(C, \mu^* T_{\mathbf{H}})$, because an element of the domain space corresponds in a unique way to an obstruction to extend the map μ, as it is a smoothing of the node. Finally there is a surjective map $f : T \to H^1(C, T_C)$, since $[C,\mu] \in \mathcal{M}_0$ is such that C can not be smoothened at p.

$$\begin{tikzcd}
\cdots \arrow[r] & T \arrow[r, "\phi"] \arrow[d, "f"'] & \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \arrow[r, "\psi"] \arrow[dr, "h"'] & H^1(C, \mu^* T_{\mathbf{H}}) \arrow[r] & E \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[ur, "g"'] \arrow[d] & & \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_p \arrow[r] \arrow[u, "\varphi"'] & 0 & \\
& 0 & & 0 \arrow[u] & &
\end{tikzcd}$$

Since (2.3) is exact, the proposition follows. $\square$

Remark 2.2.5. If we assume that we have n no-smoothenable nodes on C, we get dim coker $\phi = n$ and E fits into the exact sequence:

$$0 \to \bigoplus_{p \text{ node}} \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_p \xrightarrow{\varphi} H^1(C, \mu^* T_{\mathbf{H}}) \to E \to 0$$

We will only use the case $n = 2$ in 2.6.10.

2.3 The virtual fundamental class

The expected dimension of a moduli space may not coincide with the actual one. We may think of this as if the moduli space is a subspace of some ambient space and it is cut out by a set of equations whose vanishing loci do not meet properly. To define the Gromov-Witten invariants on $\mathbf{H}$ we need to work in the right dimension, so that we need the existence of a Chow homology class in $\overline{M}_{0,n}(\mathbf{H},\beta)$ of the expected dimension. It is called the *virtual fundamental class* and denoted by $[\overline{M}_{0,n}(\mathbf{H},\beta)]^{vir}$.
An algebraic approach to the problem of constructing such a class was developed by Behrend and Fantechi [B-F] as well as by Li and Tian [L-T]. The basic idea comes from the excess intersection theory [Ful].

Excess intersection theory

We recall some results from [Ful] Chap.6.
Given an algebraic variety X, a closed regular imbedding $i : Z \to X$ of codimension e and a morphism $f : V \to X$ from a purely k-dimensional scheme V, it happens that in general the scheme W defined by the cartesian diagram:

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

has not the expected dimension $\dim V - \operatorname{cod}(Z \subseteq X)$. Anyway the pullback bundle $N = g^*N_{Z/X}$ has rank e and it comes with a natural projection $p : N \to W$ inducing an isomorphism $p^* : A_d(W) \to A_{d-e}(N)$ for all d. In particular p^* as an inverse s^*, the Gysin map induced by the zero section of N, [Ful] Thm.3.3.a)-Def.3.3. There exists a closed imbedding of $C = C_W N$, the normal cone to W in N, as a subcone of N. Since C is purely k-dimensional, the class $[C]$ is a k-cycle on N. The intersection product $[Z]\cdot[V] \in A_*(W)$ is defined to be the class obtained by "intersecting $[C]$ with the zero section of N":

$$[Z]\cdot[V] = s^*[C]$$

In particular $s^*[C] \in A_{k-e}(W)$ has the expected dimension. One defines $s^*[C]$ to be the virtual fundamental class of W.

The intrinsic normal cone

In [B-F], [L-T] the authors define for Deligne-Mumford stacks an analogue of the normal cone. So on $\mathcal{M}_n := \overline{M}_{0,n}(\mathbf{H},\beta)$ one can use techniques similar to those seen above in order to construct the class $[\mathcal{M}_n]^{vir}$ of the right dimension.

Denote by $L^\bullet$ the cotangent complex of $\mathcal{M}_n$ (see [Il] for its definition on schemes and [L-MB] for its generalization to algebraic stacks). Recall that given a homomorphism $d : S^0 \to S^1$ of abelian sheaves on a Deligne-Mumford stack, one may consider it has a complex on the stack. One can define the quotient stack:

$$h^1/h^0(S^\bullet) = [S^1/S^0]$$

because S^0 acts on S^1 via d. If $S^\bullet$ is a complex of abelian sheaves of arbitrary length, one consider the two-term cut-off:

$$\tau_{[0,1]}S^\bullet = [\text{coker } (S^{-1} \to S^0) \to \ker\ (S^1 \to S^2)]$$

and defines $h^1/h^0(S^\bullet) := h^1/h^0(\tau_{[0,1]}S^\bullet)$.

Definition 2.3.1. *The stack* $\mathfrak{N} := h^1/h^0((L^\bullet)^\vee)$ *is the* intrinsic normal sheaf *of* $\mathcal{M}_n$.

To construct the analogue of the normal cone one needs to consider *local embeddings* of $\mathcal{M}_n$.

Definition 2.3.2. *A* local embedding *of* $\mathcal{M}_n$ *is a diagram:*

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "f"'] & X \\ \mathcal{M}_n & \end{tikzcd}$$

where:
- U is an affine scheme of finite type,
- f is an étale morphism,
- X is a smooth affine scheme of finite type,
- i is a local immersion.

There is a well defined normal cone $C_U X$ of U in X. The tangent space i^*T_X acts naturally on it by translation. There exists a unique closed subcone stack $\mathfrak{C} \subseteq \mathfrak{N}$ that locally is given by the stack quotients $\mathfrak{C}|_U = [C_U X/i^*T_X]$, [B-F] Cor.3.9. Moreover this construction is independent of the local embeddings. $\mathfrak{C}$ has pure dimension zero.

Definition 2.3.3. $\mathfrak{C}$ *is the* intrinsic normal cone *of* $\mathcal{M}_n$.

Let $F^\bullet$ be a complex of $\mathcal{O}_{\mathcal{M}_n}$-modules concentrated in degrees -1 and 0 such that $h^i(F^\bullet)$ is coherent for $i = -1, 0$.

Definition 2.3.4. *If there is a morphism* $\Phi : F^\bullet \to L^\bullet$ *in* $D^b_{Coh}(Qcoh_{\mathcal{M}_n})$ *such that* $h^0(\Phi)$ *is an isomorphism and* $h^{-1}(\Phi)$ *is surjective, the map* Φ *(or* $F^\bullet$*) is called a* perfect obstruction theory *for* $\mathcal{M}_n$.

Fix a perfect obstruction theory $\Phi : F^\bullet \to L^\bullet$. Let $F^{\bullet\vee} = [F_0 \xrightarrow{\varphi} F_1]$ with $F_0 = F^{0^\vee}$ and $F_1 = F^{-1^\vee}$. Then Φ induces a closed immersion [B-F] Prop.2.6:

$$\Phi^\vee : \mathfrak{N} \to h^1/h^0((F^\bullet)^\vee)$$

Hence the intrinsic normal cone is a closed subcone stack of $[F_1/F_0]$. This is a vector bundle stack on $\mathcal{M}_n$. Moreover F_1 is a presentation of it and contains a closed subcone $C(F^\bullet)$ with a map over $\mathfrak{C}$ smooth of relative dimension rkF_0.

Definition 2.3.5. *The* virtual fundamental class $[\mathcal{M}_n]^{vir}$ *is the intersection of* $C(F^\bullet)$ *with the zero section of* F_1.

Definition 2.3.6. *Let* $F^\bullet$ *be a perfect obstruction theory for* $\mathcal{M}_n$ *as above. For each closed point* $m \in \mathcal{M}_n$ *one defines* $Ker\ \varphi_m = T$ *and* $Coker\ \varphi_m = E$ *to be the* tangent space *and the* obstruction space *of* $\mathcal{M}_n$ *at* m*, respectively. The difference* $rkF_0 - rkF_1$ =dim T-dim E *is called the* expected dimension *of* $\mathcal{M}_n$.

The virtual class $[\mathcal{M}_n]^{vir}$ has the expected dimension $rkF_0 - rkF_1$, [B-F] p.76. Proposition 5.10 in [B-F] ensures that it behaves well under pullback:

Proposition 2.3.7. *If* π *is the flat morphism forgetting the marked points, then:*

$$[\overline{M}_{0,n}(\mathbf{H}, \beta)]^{vir} = \pi^*[\overline{M}_{0,0}(\mathbf{H}, \beta)]^{vir}$$

If the moduli space is smooth, there is an easier description of the virtual fundamental class.

Theorem 2.3.8. *If the moduli space* $\mathcal{M}_n$ *is smooth, given a perfect obstruction theory* $F^\bullet$ *with* $(F^\bullet)^\vee = [F_0 \xrightarrow{\varphi} F_1]$*, then the sheaf cohomology* $h^1(F^{\bullet\vee})$ =*Coker*(φ) *is locally free and:*

$$[\mathcal{M}_n]^{vir} = c_{top}(h^1(F^{\bullet\vee})) \cdot [\mathcal{M}_n] \tag{2.4}$$

This is Proposition 5.6 in [B-F]. Note that $h^1(F^{\bullet\vee})_m$ is the obstruction space at $m \in \mathcal{M}_n$.

Definition 2.3.9. *We will denote* $h^1(F^{\bullet\vee}) = \mathcal{E}$ *and we will call it the* obstruction bundle *of* $\mathcal{M}_n$.

Remark 2.3.10. Throughout the paper we will often refer to the obstruction bundle by simply naming its fibers. So we will possibly write the tangent-ostruction sequence (2.1) with $\mathcal{E}$ instead of E as last term.

As proved in [Beh], since there is a canonical map $\eta : \mathcal{M}_n \to \mathfrak{M}_{0,n}$ which is an open substack of a relative space of morphisms, there exists a *relative obstruction theory*. This consists of a map $\phi : F^\bullet \to L^\bullet_{\mathcal{M}_n/\mathfrak{M}_{0,n}}$ such that $L^\bullet_{\mathcal{M}_n/\mathfrak{M}_{0,n}}$ is the relative cotangent complex, $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

Proposition 2.3.11. *With respect to the diagram:*

$$\begin{tikzcd} \overline{M}_{0,1}(\mathbf{H},\beta) \arrow[r, "ev"] \arrow[d, "\pi"'] & \mathbf{H} \\ \overline{M}_{0,0}(\mathbf{H},\beta) & \end{tikzcd}$$

$F^\bullet = (R\pi_*(ev^* T_{\mathbf{H}}))^\vee$ *is a perfect relative obstruction theory for* $\overline{M}_{0,0}(\mathbf{H},\beta)$.

Proof. This is Proposition 5 in [Beh]. □

The intrinsic normal cone construction can be extended to the relative case, so that one can define the virtual fundamental class $[\overline{M}_{0,0}(\mathbf{H},\beta)]^{vir}$ as the intersection of the relative intrinsic normal cone with the zero section of $h^1/h^0(\mathrm{R}\pi_*(ev^*T_{\mathbf{H}}))$, [Beh] p.606. In particular it has the expected dimension [Beh] p.605. Note that the (relative) obstruction bundle is $\mathcal{E} = \mathrm{R}^1\pi_*(ev^*T_{\mathbf{H}})$.

Remark 2.3.12. The results stated in 2.3.7, 2.3.8 and 2.3.11 imply that in order to calculate the virtual fundamental class of $\overline{M}_{0,n}(\mathbf{H},\beta)$ it is enough to study the perfect obstruction theory $(\mathrm{R}\pi_*(ev^*T_{\mathbf{H}}))^\vee$ on the unpointed moduli space. Moreover, on the smooth locus of $\overline{M}_{0,0}(\mathbf{H},\beta)$ it is enough to calculate the top Chern class of the obstruction bundle $\mathcal{E} = \mathrm{R}^1\pi_*(ev^*T_{\mathbf{H}})$.

A formula for the expected dimension

We denote by $\mathrm{ed}_{\mathbf{H}}$ the expected dimension of $\overline{M}_{0,0}(\mathbf{H},(a,b,c))$. Choose a point $[C,\mu]$ of the moduli space. Then by the tangent-obstruction sequence (2.1) we know that $\mathrm{ed}_{\mathbf{H}}$ is given by:

$$\mathrm{ed}_{\mathbf{H}} = \chi(\mu^*T_{\mathbf{H}}) - (\dim \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) - \dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C))$$

We apply Riemann-Roch to calculate the first term of the algebraic sum, while the second one is known to be equal to $3 - 3g_C$. Hence we get:

$$\mathrm{ed}_{\mathbf{H}} = \dim \mathbf{H} + \int_{(a,b,c)} c_1(T_{\mathbf{H}}) - 3 = 2a + 2b + 1 \tag{2.5}$$

The actual dimension of the unpointed moduli space will always be denoted by $\mathrm{d}_{\mathbf{H}}$.

Remark 2.3.13. The map $\overline{M}_{0,n}(\mathbf{H},\beta) \to \overline{M}_{0,0}(\mathbf{H},\beta)$ has relative dimension equal to n because it is the composition of n universal families. Then the expected dimension of $\overline{M}_{0,n}(\mathbf{H},\beta)$, with $\beta = (a,b,c)$, is:

$$\dim \mathbf{H} + \int_\beta c_1(T_\mathbf{H}) - 3 + n = 2a + 2b + 1 + n$$

We will denote it again by $\mathrm{ed}_\mathbf{H}$, if no confusion arises.

2.4 A smoothness result

$\mathbf{H}$ is an almost-homogeneous space under the action of $\mathcal{A}$. We can exploit this action to get transversality results (as the Position Lemma) and to control the smoothness of the moduli space $\overline{M}_{0,n}(\mathbf{H},\beta)$.
Recall that as $\mathbf{H}$ is not a convex space, in general the moduli space does not have the expected dimension:

$$\mathrm{exp.dim}\ \overline{M}_{0,n}(\mathbf{H},(a,b,c)) = \mathrm{ed}_\mathbf{H} = 2a + 2b + 1 + n$$

Lemma 2.4.1. *Σ is convex.*

Proof. Recall that $\Sigma = \tilde{W}_1 \sqcup \tilde{W}_2$. Since $\tilde{W}_k$ is the relative Hilbert scheme $\mathrm{Hilb}^2(Q/\mathbb{P}^1)$, it is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ and it comes with two natural projections p_1, p_2. If $\mu : \mathbb{P}^1 \to \tilde{W}_k$ is a point in $\overline{M}_{0,n}(\tilde{W}_k,\beta)$, for some β, then $f^*(T_{\tilde{W}_k}) = f^*p_1^*T_{\mathbb{P}^2} \oplus f^*p_2^*T_{\mathbb{P}^1}$ is a fiber bundle of positive degree on $\mathbb{P}^1$ and the higher cohomology vanishes. This implies the thesis. □

Corollary 2.4.2. *The moduli space of n-pointed genus zero stable maps to Σ is smooth of the expected dimension which is equal to:*

$$d_\Sigma = \dim \Sigma + \int_\beta c_1(T_\Sigma) - 3 + n = 3(a+b) - 2c + n$$

for $\beta = (a,b,c)$.

Proof. It follows from 2.2.2 and 1.6.2. □

Lemma 2.4.3. *Δ is convex.*

Proof. The support map $s : \Delta \to Q$ gives the exact sequence:

$$0 \to T_{\Delta/Q} \to T_\Delta \to s^*T_Q \to 0$$

Let $\mu : \mathbb{P}^1 \to \Delta$ be a stable map, then to show that $H^1(\mathbb{P}^1, \mu^*T_\Delta) = 0$ it suffices to prove that $H^1(\mathbb{P}^1, \mu^*T_{\Delta/Q}) = 0$, since Q is homogeneous. We can think of Δ as the exceptional divisor $\tilde{\delta}$ in $\tilde{U}$. With notations as in §1.1:

$$\tilde{\delta} \hookrightarrow \tilde{U} \xrightarrow{bl} U \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} Q$$

Denote by p the restriction of pr_1, pr_2 to the diagonal $\delta \subseteq U$. By the adjunction formula and the exact sequence written above we get:

$$T_{\Delta/Q} = p^* K_Q^* \otimes (-2\tilde{\delta})|_{\tilde{\delta}} = 2h_1 + 2h_2 - 2\zeta$$

The generators of the cone of effective curves in Δ are such that the degree of $T_{\Delta/Q}$ restricted to each of them is non-negative, so deg $\mu^* T_{\Delta/Q} \geq 0$ and $H^1(\mathbb{P}^1, \mu^* T_{\Delta/Q}) = 0$. We conclude that Δ is convex. □

Corollary 2.4.4. *The moduli space of n-pointed genus zero stable maps to Δ is smooth of the expected dimension which is equal to:*

$$d_\Delta = \dim \Delta + \int_\beta c_1(T_\Delta) - 3 + n = 2c + n$$

for $\beta = (a, b, c)$.

Proof. As before it follows from 2.2.2 and 1.5.3. □

Since the expected dimension is the lowest possible dimension for a moduli space, whenever $\mathrm{d}_\Delta > \mathrm{ed}_\mathbf{H}$ or $\mathrm{d}_\Sigma > \mathrm{ed}_\mathbf{H}$ we should have components of $\overline{M}_{0,n}(\mathbf{H}, \beta)$ of excess dimension. Those inequalities are equivalent to the conditions $\beta \cdot \Delta < 0$ or $\beta \cdot \Sigma < 0$. Geometrically this means that the excess dimension is due to components entirely mapped into Δ or into Σ. The following theorem formalizes such a statement.

Theorem 2.4.5. *If $\mu : C \to \mathbf{H}$ is a stable map from a genus 0 curve such that no component of C is mapped entirely into $\Delta \cup \Sigma$, then the moduli space $\overline{M}_{0,0}(\mathbf{H}, \beta)$ is smooth at $[C, \mu]$ of the expected dimension.*

Proof. $\mathbf{H} - (\Delta \cup \Sigma)$ is Σ_4, the open dense orbit for the action on $\mathbf{H}$ induced by $\mathcal{A}$. The action on Σ_4 is transitive, so we can say that $T_\mathbf{H}$ is generically generated by global sections on $\mathbf{H}$. Let $\mu : C \to \mathbf{H}$ be as in the hypotesis, then $\mu^* T_\mathbf{H}$ is generically generated by global sections on C. This means that $H^1(C, \mu^* T_\mathbf{H}) = 0$ and the moduli space $\overline{M}_{0,0}(\mathbf{H}, \beta)$ is smooth at $[C, \mu]$ of the expected dimension by 2.2.1. □

2.5 The moduli space $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$

Here and in the following section we prove some results on the obstruction bundles of two moduli spaces which we will use later on to make explicit calculations.

For $c \geq 1$, a curve of class $(0, 0, c)$ in $\mathbf{H}$ is represented by a c-sheeted cover of $\mathbb{P}^1$ and it is contained into Δ which is convex. Then the moduli space $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$ is smooth of dimension $2c$ bigger than the expected one,

$\mathrm{ed}_{\mathbf{H}} = 1$. The obstruction bundle $\mathcal{E} = \mathrm{R}^1\pi_*(ev^*T_{\mathbf{H}})$ has rank $2c-1$ and by 2.3.8 the virtual fundamental class is:

$$[\overline{M}_{0,0}(\mathbf{H},(0,0,c))]^{vir} = [\overline{M}_{0,0}(\mathbf{H},(0,0,c))] \cdot c_{2c-1}(\mathcal{E})$$

Proposition 2.5.1. *If $c=1$ then $\mathcal{E} = -K_Q$ and we get:*

$$[\overline{M}_{0,0}(\mathbf{H},(0,0,1))]^{vir} = [Q] \cap c_1(-K_Q)$$

Proof. For $c=1$, $(0,0,1)$ is the class of a fiber of the support map $s:\Delta \to Q$. So we can work on Δ because of the diagram:

$$\begin{tikzcd} \Delta \cong \overline{M}_{0,1}(\mathbf{H},(0,0,1)) \arrow[r, "ev"] \arrow[d, "s"'] \arrow[d, shift left=8ex, "\pi"] & \mathbf{H} \\ Q \cong \overline{M}_{0,0}(\mathbf{H},(0,0,1)) & \end{tikzcd}$$

By the convexity of Δ and the exact sequences:

$$0 \to T_\Delta \to T_{\mathbf{H}}|_\Delta \to \mathcal{N}_{\Delta/\mathbf{H}} \to 0$$
$$0 \to T_{\Delta/Q} \to T_\Delta \to s^*T_Q \to 0$$

we obtain $\mathrm{R}^1s_*(ev^*T_{\mathbf{H}}) = \mathrm{R}^1s_*(\mathcal{N}_{\Delta/\mathbf{H}}) = \mathrm{R}^1s_*(\mathcal{O}_\Delta(-2))$. Finally [Har] Chap.III ex.8.4-c) gives:

$$\mathcal{E} = \mathrm{R}^1s_*(\mathcal{O}_\Delta(-2)) = \Lambda^2T_Q = -K_Q$$

□

Proposition 2.5.2. *Let $g:\overline{M}_{0,0}(\mathbf{H},(0,0,c)) \to Q$ be the map defined by $g([C,\mu]) = Supp\ \mu(C)$. In general it holds:*

$$c_{2c-1}(\mathcal{E}) = -g^*K_Q \cdot c_{2c-2}(\tilde{\mathcal{E}}) \tag{2.6}$$

where $\tilde{\mathcal{E}}$ is such that:

$$c_{2c-2}(\tilde{\mathcal{E}}|_{g^{-1}(p)}) = \frac{1}{c^3} \tag{2.7}$$

for any point $p \in Q$.

Proof. In the general case, $\mathcal{E} = \mathrm{R}^1\pi_*(ev^*T_{\mathbf{H}})$ has stalk $H^1(C,\mu^*T_{\mathbf{H}})$ at the point $[C,\mu] \in \overline{M}_{0,0}(\mathbf{H},(0,0,c))$. We get $H^1(C,\mu^*T_{\mathbf{H}}) = H^1(C,\mu^*\mathcal{O}_{\mathbb{P}^1}(-2))$. Let $\tilde{ev}:\overline{M}_{0,1}(\mathbf{H},(0,0,c)) \to \Delta$ be the evaluation map into Δ such that the composition with the inclusion $\Delta \hookrightarrow \mathbf{H}$ is $ev:\overline{M}_{0,1}(\mathbf{H},(0,0,c)) \to \mathbf{H}$. By [L-Q] Lemma 3.2, $\mathcal{E}$ sits in the exact sequence:

$$0 \to g^*\mathcal{O}_Q(-K_Q) \to \mathcal{E} \to \mathrm{R}^1\pi_*\tilde{ev}^*(s^*T_Q \otimes \mathcal{O}_\Delta(-1)) = \tilde{\mathcal{E}} \to 0$$

Hence we get:

$$c_{2c-1}(\mathcal{E}) = -g^*K_Q \cdot c_{2c-2}(\tilde{\mathcal{E}})$$

Note that the inverse image $g^{-1}(p)$, $p \in Q$, is isomorphic to $\overline{M}_{0,0}(\mathbb{P}^1, c)$, with $\mathbb{P}^1 \cong M_2(p)$ the punctual Hilbert scheme of points on Q at p.
With respect to the diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(\mathbb{P}^1, c) & \xrightarrow{ev_1} & \mathbb{P}^1 \\ {\scriptstyle f}\downarrow & & \\ \overline{M}_{0,0}(\mathbb{P}^1, c) & & \end{array}$$

the restriction $\tilde{\mathcal{E}}|_{g^{-1}(p)}$ is isomorphic to $\mathrm{R}^1 f_* ev_1^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ [L-Q] Rmk.3.1. By Theorem 3.2 in [Man]:

$$c_{2c-2}(\mathrm{R}^1 f_* ev_1^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) = \frac{1}{c^3}$$

This concludes the proof. □

2.6 The moduli space $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$

The moduli space $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$ has expected dimension $\mathrm{ed}_{\mathbf{H}} = 3$. In particular if $c = 0$, then $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 0))$ is smooth of the expected dimension, because $(1, 0, 0)$ is the class of a curve contained into Σ which is convex and $\overline{M}_{0,0}(\Sigma, (1, 0, 0))$ has the same expected dimension (see 2.4.1 and 2.4.2).

Remark 2.6.1. In §1.5 we showed that Δ defines a conic on a fiber over the blown up locus of the map $\varphi : \mathbf{H} \to \mathbf{G}$. A curve of class $(1, 0, 0)$ can be represented by a line in the projective plane $\mathrm{Hilb}^2(l_1)$ for a fixed line $l_1 \in W_1$ (see §1.4), so that it intersects Δ in at most two points.

If $c \geq 1$, the excess dimension comes from the components of the moduli space which parametrize stable maps with reducible domain. We know that the only irreducible curves are of type $(1, 0, 1)$ and disjoint from Δ (see §1.7 iii)).

Lemma 2.6.2. *The moduli space $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 1))$ is the disjoint union of two components both of the expected dimension.*

Proof. We have two possibilities for the source curves of a stable map in $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 1))$. The curve can be irreducible or not. Hence we have two components of the moduli space. One parametrizes stable maps from the irreducible curves and it is smooth of the expected dimension. The second one parametrizes stable maps with reducible domain and has the expected dimension. These two components are obviously disjoint. □

We need to study only the case $c \geq 2$. A curve of class $(1,0,c)$ with $c \geq 2$ is always reducible. We consider the morphism:

$$\tau : \overline{M}_{0,0}(\mathbf{H},(1,0,c)) \longrightarrow \overline{M}_{0,0}(\mathbf{H},(1,0,0))$$

It is defined by forgetting the components mapping to Δ.
The action induced by $\mathcal{A}_0$ (see §1.3) on $\overline{M}_{0,0}(\mathbf{H},(1,0,0))$ has two orbits corresponding to the geometry of the source curve for a stable map in that moduli space. By 2.6.1 the open dense orbit parametrizes the stable maps $[C,f]$ such that $f(C)$ is a line intersecting in two distinct points the conic defined by Δ on the corresponding fiber of the blowup map. The closed orbit parametrizes the stable maps from curves representing a line tangent to the conic defined by Δ since τ is $\mathcal{A}_0$-equivariant. The morphism τ is flat over the open orbit.

Definition 2.6.3. *Let $[C,f]$ be a point in the open orbit of $\overline{M}_{0,0}(\mathbf{H},(1,0,0))$. We denote by $\mathcal{M}_c$ the fiber $\tau^{-1}([C,f])$.*

Definition 2.6.4. *We denote by $M(c)$, $c \geq 1$, the space parametrizing the data of a degree c stable map to $\mathbb{P}^1$ with a marked point mapping to the origin. It is the fiber of the evaluation map $\overline{M}_{0,1}(\mathbb{P}^1,c) \to \mathbb{P}^1$ over the origin and it is smooth of dimension $2c-2$. Let $M(0)$ be a point.*

Lemma 2.6.5. *There is an isomorphism:*

$$\mathcal{M}_c \cong \coprod_{\substack{c_1+c_2=c \\ c_i \geq 0}} M(c_1) \times M(c_2)$$

In particular τ is smooth over the open dense orbit.

Proof. Let $\mathcal{M}_c$ be the fiber over $[C,f] \in \overline{M}_{0,0}(\mathbf{H},(1,0,0))$ as in 2.6.3. With notations as in §1.4, let $C(l_1)$ be the curve $f(C)$ of class $(1,0,0)$ such that it intersects $\Delta \cap \mathrm{Hilb}^2(l_1)$ in two points $p_1 \neq p_2$. Let $C(p_i) = \mathbb{P}(T_{Q,p_i})$, for $i = 1,2$, be contained into Δ. It is of class $(0,0,1)$. A point $[D,\mu] \in \mathcal{M}_c$ is a stable map from a nodal curve $D = D_0 \cup D_1 \cup D_2$ with $D \cap D_i = q_i$ such that:

$$\begin{aligned} &\mu : D_0 \xrightarrow{\cong} C(l_1) \\ &\mu : D_i \xrightarrow{c_i:1} C(p_i) \\ &\mu(q_i) = p_i \end{aligned}$$

Since $C(l_1)$ is fixed as well as its intersection points with the diagonal, the only moduli comes from the choice of the sheeted covers of the $(0,0,1)$-curves. In particular for $i = 1,2$, the curve $[D_i, \mu|_{D_i}]$ is a point of $M(c_i)$ with q_i mapping to the origin p_i of $C(p_i)$. $\square$

Remark 2.6.6. The composition of the inclusion:

$$M(c_1) \times M(c_2) \to \mathcal{M}_c \to \overline{M}_{0,0}(\mathbf{H}, (1,0,c))$$

with the forgetful map $\overline{M}_{0,0}(\mathbf{H}, (1,0,c)) \to \mathfrak{M}_{0,0}$ is smooth on its image which consists of:

- the divisor parametrizing curves with a node in p_1 if $c_2 = 0$ or in p_2 if $c_1 = 0$;
- the (smooth) locus of codimension 2 parametrizing curves with two nodes p_1, p_2 if $c_1, c_2 > 0$.

Remark 2.6.7. The space $M(c)$ is smooth of dimension $2c - 2$.
Then $M(0) \times M(c)$ has dimension $2c-2$ and $M(c_1) \times M(c_2)$ has dimension $2c - 4$.

Remark 2.6.8. Since $\overline{M}_{0,0}(\mathbf{H}, (1,0,c))$ has expected dimension $\mathrm{ed}_{\mathbf{H}} = 3$ and τ is smooth on the open dense orbit, a fiber $\mathcal{M}_c$ has expected dimension equal to zero. Since $\mathcal{M}_c$ is the disjoint union of the components $M(c_1) \times M(c_2)$, its virtual fundamental class $[\mathcal{M}_c]^{vir} := [\overline{M}_{0,0}(\mathbf{H}, (1,0,c))]^{vir} \tau^*[C, f]$ is equal to the sum of the virtual fundamental classes of all components. Moreover each of them must have expected dimension equal to zero.

To calculate the virtual fundamental class of $\mathcal{M}_c$ we can use the formula (2.4). Then we need to know the obstruction bundle $\mathcal{E}$ at a point $[D, \mu]$ in $\mathcal{M}_c$. The following lemma gives a description of the space $H^1(D, \mu^* T_{\mathbf{H}})$ which will permit us to express $\mathcal{E}$ as the cokernel of an injection (see 2.6.10). Let $[D, \mu] \in \mathcal{M}_c$ be as in the proof of 2.6.5:

$$\begin{aligned} &\mu : D_0 \xrightarrow{\cong} C(l_1) \\ &\mu : D_i \xrightarrow{c_i:1} C(p_i) \\ &\mu(q_i) = p_i \in C(l_1) \cap C(p_i) \end{aligned}$$

Lemma 2.6.9. *Let $\mathcal{L}_i$ be the invertible sheaf $\mu^* \mathcal{O}_{C(p_i)}(-2)$ of degree $-2c_i$, $i = 1, 2$. Then:*

$$H^1(D, \mu^* T_{\mathbf{H}}) \cong H^1(D_1, \mathcal{L}_1) \oplus H^1(D_2, \mathcal{L}_2)$$

Proof. We tensor by $- \otimes_{\mathcal{O}_D} \mu^* T_{\mathbf{H}}$ the following exact sequence:

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D_0} \oplus \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \to \mathcal{O}_{q_1} \oplus \mathcal{O}_{q_2} \to 0$$

We consider the long exact sequence in cohomology:

$$\ldots \to H^0(T_{q_1} \oplus T_{q_2}) \to H^1(D, \mu^* T_{\mathbf{H}}) \to \bigoplus_{i=1,2} H^1(D_i, \mu^* T_{\mathbf{H}}|_{D_i}) \to 0$$

where we use the convexity of Σ to deduce $H^1(D_0, \mu^* T_{\mathbf{H}}|_{D_0}) = 0$. Analogously, Δ is convex so $H^1(D_i, \mu^* T_\Delta|_{D_i}) = 0$, $i = 1, 2$. The support map $s : \Delta \to Q$ is a $\mathbb{P}^1$-bundle, so the tangent sheaf T_Δ restricted to a fiber l of s is $T_\Delta|_l = \mathcal{O}_l(2) \oplus \mathcal{O}_l^{\oplus 2}$. The usual exact sequence:

$$0 \to T_\Delta \to T_{\mathbf{H}}|_\Delta \to \mathcal{N}_{\Delta/\mathbf{H}} \to 0$$

restrected to the fiber l gives $\mathcal{N}_{\Delta/\mathbf{H}}|_l = \mathcal{O}_l(-2)$. Hence we get:

$$H^1(D_i, \mu^* T_{\mathbf{H}}|_{D_i}) = H^1(D_i, \mu^* \mathcal{O}_{C(p_i)}(-2))$$

Let $\mathcal{L}_i$, $i = 1, 2$ be as in the hypothesis, then $H^1(D_i, \mathcal{L}_i)$ has dimension $2c_i - 1$ and the above sequence becomes:

$$0 \to T_{q_1} \oplus T_{q_2} \to H^1(D, \mu^* T_{\mathbf{H}}) \xrightarrow{\vartheta} \bigoplus_{i=1,2} H^1(D_i, \mathcal{L}_i) \to 0$$

Then ϑ is a surjective morphism between two vector spaces of the same dimension, i.e. it is an isomorphism. $\square$

Proposition 2.6.10. *Let L_i, $i = 1, 2$, be the line bundle corresponding to the deformation which resolves the i-th node. Then the obstruction bundle $\mathcal{E}$ fits in the exact sequence:*

$$0 \to \bigoplus_{i=1,2} L_i \to \bigoplus_{i=1,2} H^1(D_i, \mathcal{L}_i) \to \mathcal{E} \to 0 \tag{2.8}$$

In particular it has rank $\sum_{i=1}^2 (2c_i - 2)$.

Proof. We can not smoothen all the nodes of D (see §2.2), then we know that the map $\phi : T \to \mathrm{Ext}^1(\Omega_D, \mathcal{O}_D)$ in the tangent-obstruction sequence (2.1) has a 2-dimensional cokernel and it factors through a surjective morphism f as in the diagram:

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[d, "f"'] & \mathrm{Ext}^1(\Omega_D, \mathcal{O}_D) \\ H^1(D, T_D) \arrow[ur, "g"'] & \end{tikzcd}$$

The map g sits in the exact sequence defining the space of first order deformations of the nodal curve D (2.2):

$$0 \to H^1(D, T_D) \xrightarrow{g} \mathrm{Ext}^1(\Omega_D, \mathcal{O}_D) \xrightarrow{h} H^0(\mathcal{E}xt^1(\Omega_D, \mathcal{O}_D)) \to 0$$

where $H^0(\mathcal{E}xt^1(\Omega_D, \mathcal{O}_D)) = \bigoplus_{i=1,2} L_i$, $L_i \cong \mathcal{O}_{D,q_i}$. By remark 2.2.5, coker $\phi = L_1 \oplus L_2$ and by 2.6.9 the sequence (2.8) is exact. $\square$

For each i, let $\mathcal{E}_{c_i}$ be the cokernel of the injection $L_i \to H^1(D_i, \mathcal{L}_i)$. It is a vector bundle of rank $2c_i - 2$ on $M(c_i)$. It is the one we find when we have only one node on D. Since $\mathcal{E}_{c_1} \oplus \mathcal{E}_{c_2}$ and $\mathcal{E}$ fit into the same exact sequence, it holds $c_{\mathrm{top}}(\mathcal{E}) = c_{\mathrm{top}}(\oplus \mathcal{E}_{c_i})$.
In [Gr], Graber constructs a variety X by blowing up $\mathbb{P}^2$ in a point and then blowing up a point on the exceptional divisor. He gets two exceptional divisors meeting in a node. Let A be the (-1)-curve, B the (-2)-curve and $\beta_c = A + cB$. Then he shows that the moduli space $\overline{M}_{0,0}(X, \beta_c)$ is smooth of expected dimension zero and isomorphic to $M(c)$. Besides its virtual fundamental class can be realized as the top Chern class of a vector bundle $\tilde{\mathcal{E}}_c$ which sits in the same exact sequence defining the bundle $\mathcal{E}_c$. Then $c_{\mathrm{top}}(\mathcal{E}_c) = c_{\mathrm{top}}(\tilde{\mathcal{E}}_c)$.

Proposition 2.6.11. (Graber) *For all $c \geq 2$, $c_{top}(\tilde{\mathcal{E}}_c) = 0$.*

Proof. This is Proposition 3.5 in [Gr]. □

Remark 2.6.12. Let $M^*(c)$ be the fiber over $(0, \infty)$ of the evaluation map $ev = (ev_1, ev_2) : \overline{M}_{0,2}(\mathbb{P}^1, c) \to \mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\mathcal{E}_c^*$ the obstruction bundle of $M^*(c)$. The following diagram is commutative:

$$\begin{array}{ccc} M^*(c) & \xrightarrow{f} & M(c) \\ \downarrow & & \downarrow \\ \overline{M}_{0,2}(\mathbb{P}^1, c) & \xrightarrow{g} & \overline{M}_{0,1}(\mathbb{P}^1, c) \end{array}$$

where g and f forget the point mapping to ∞.

Lemma 2.6.13. *With notations as in 2.6.12, $\mathcal{E}_c^*$ is the pullback bundle $f^*\mathcal{E}_c$ of the obstruction bundle of $M(c)$. In particular $c_{top}(\mathcal{E}_c^*) = 0$ for $c \geq 2$.*

Proof. We will prove that $H^1(C, \mu^* T_{\mathbf{H}})$ and $H^1(C', (\mu')^* T_{\mathbf{H}})$ are canonically isomorphic. It is enough to study what happens for a stable map $[C, \mu]$ in $\overline{M}_{0,2}(\mathbb{P}^1, c)$ such that C has a component contracted by f. We can write $C = D \cup D_1 \cup D_2$ with D the contracted component, $\mu(D) = x$, $p = D_1 \cap D$ and $q = D_2 \cap D$ two nodes. Let $[C' = D_1 \cup D_2, \mu']$ be the image under f, i.e. $\mu' \circ f = \mu$. There exists a morphism $\alpha : (\mu')^* T_{\mathbf{H}} \to f_* f^* (\mu')^* T_{\mathbf{H}} = f_* \mu^* T_{\mathbf{H}}$, [Har] Chap.II ex.1.18.

a) If α is an isomorphism then $H^1(C', (\mu')^* T_{\mathbf{H}}) \cong H^1(C', f_* \mu^* T_{\mathbf{H}})$.

b) If $\mathrm{R}^i f_*(\mu^* T_{\mathbf{H}}) = 0$ for all $i > 0$, $H^1(C', f_* \mu^* T_{\mathbf{H}}) \cong H^1(C, \mu^* T_{\mathbf{H}})$ by [Har] Chap.III ex.8.1

To prove our claim it is enough to verify the hypothesis of a), b). We can work in a small neighbourhood and assume D_1, D_2 affine, i.e. C' affine.
The exact sequence:

$$0 \to \mathcal{O}_{D_1}(-p) \to \mathcal{O}_C \to \mathcal{O}_{D \cup D_2} \to 0$$

gives $H^1(C,\mu^*T_{\mathbf{H}}) = H^1(D\cup D_2, \mu^*T_{\mathbf{H}}|_{D\cup D_2})$, because D_1 is affine. Since $\mu(D)$ is a point, the sheaf $\mu^*T_{\mathbf{H}}|_D$ is trivial: $H^0(D,\mu^*T_{\mathbf{H}}|_D) = H^0(q,(\mu^*T_{\mathbf{H}})_q)$. The exact sequence:

$$0 \to \mathcal{O}_{D\cup D_2} \to \mathcal{O}_{D_2} \oplus \mathcal{O}_D \to \mathcal{O}_q \to 0$$

induces an isomorphism $H^0(D\cup D_2, \mu^*T_{\mathbf{H}}|_{D\cup D_2}) \cong H^0(D_2,\mu^*T_{\mathbf{H}}|_{D_2})$. Then:

$$H^1(D\cup D_2, \mu^*T_{\mathbf{H}}|_{D\cup D_2}) \cong H^1(D_2,\mu^*T_{\mathbf{H}}|_{D_2}) \oplus H^1(D,\mu^*T_{\mathbf{H}}|_D)$$

We conclude that $H^1(C,\mu^*T_{\mathbf{H}}) = 0$, because D_2 is affine and $\mu^*T_{\mathbf{H}}|_D$ is trivial. Then $\mathrm{R}^1 f_*\mu^*T_{\mathbf{H}} = 0$ by [Har] Chap.III Prop. 8.5.
To verify a), we tensor by $-\otimes(\mu')^*T_{\mathbf{H}}$ the following exact sequence:

$$0 \to \mathcal{O}_{D_1}(-x) \to \mathcal{O}_{C'} \to \mathcal{O}_{D_2} \to 0$$

and we get:

$$0 \to H^0(D_1,(\mu')^*T_{\mathbf{H}}|_{D_1}\otimes\mathcal{O}_{D_1}(-x)) \to H^0(C',(\mu')^*T_{\mathbf{H}}) \to$$
$$\to H^0(D_2,(\mu')^*T_{\mathbf{H}}|_{D_2}) \to 0$$

Since the first vector space is isomorphic to $H^0(D_1,\mu^*T_{\mathbf{H}}|_{D_1}\otimes\mathcal{O}_{D_1}(-p))$ and the third one to $H^0(D_2,\mu^*T_{\mathbf{H}}|_{D_2})$, we get $H^0(C,\mu^*T_{\mathbf{H}}) \cong H^0(C',(\mu')^*T_{\mathbf{H}})$. Then $\mu^*T_{\mathbf{H}}$ and $f_*(\mu')^*T_{\mathbf{H}}$ have the same sections, i.e. α is an isomorphism. The claim is proved. □

Theorem 2.6.14. *The virtual fundamental class of a component of a fiber $\mathcal{M}_c$ of τ is given by:*

$$[M(c_1)\times M(c_2)]^{vir} = [M(c_1)\times M(c_2)] \quad \textit{if}\ \ 0\le c_1,c_2\le 1$$

$$[M(c_1)\times M(c_2)]^{vir} = 0 \qquad \textit{otherwise}$$

Proof. If c_1, c_2 are 0 or 1 then $M(c_1)\times M(c_2)$ is smooth of the expected dimension equal to zero and the virtual fundamental class coincide with the usual fundamental class. If c_1 or c_2 is bigger than or equal to 2 then by 2.6.11 the top Chern class of the obstruction bundle vanishes. □

Corollary 2.6.15. *If $c\ge 2$, then the virtual fundamental class of $\mathcal{M}_c$ is given by:*

$$[\mathcal{M}_2]^{vir} = [M(1)\times M(1)]$$

$$[\mathcal{M}_c]^{vir} = 0 \qquad \textit{if}\ \ c\ge 3$$

Proof. It follows from 2.6.8 and 2.6.14. □

2.7 Gromov-Witten Invariants

We recall the definition and some properties of the Gromov-Witten (GW) invariants on a d-dimensional smooth complex projective variety X.
Let $\beta \in A_1(X)$ be the class of an effective curve and consider the moduli space $\overline{M}_{g,n}(X,\beta)$ of genus g, n-pointed stable maps into X, $g, n \geq 0$ and $n + 2g - 3 \geq 0$. It has expected dimension equal to:

$$\mathrm{ed}_X = (\dim X - 3)(1-g) + \int_\beta c_1(T_X) + n$$

Now we fix n cycle classes $\gamma_1, \dots, \gamma_n \in A^*(X)$ and consider the cohomology class $ev^*(\gamma_1 \times \ldots \times \gamma_n)$, where $ev : \overline{M}_{g,n}(X,\beta) \to X^n$ is the usual evaluation map. We call *GW invariant* the number:

$$\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta := \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} ev^*(\gamma_1 \times \ldots \times \gamma_n)$$

This is the "virtual" number of curves of genus g and class β in X intersecting the homology cycles Γ_i, where the Poincaré dual of Γ_i is γ_i, for all $i = 1, \dots, n$. If $g = 0, n \geq 3$ we speak about *genus zero GW invariants*.
If no confusion arises, we will omit the symbol " $\cdot$ " among the arguments γ_i of the invariant.
The GW invariants have some nice properties such as to be invariant under deformation. Moreover they are zero if the following equality is not satisfied:

$$\sum_i \mathrm{cod}\ \gamma_i = \mathrm{ed}_X$$

Let $\gamma_n = 1_X \in A^0(X)$ be the fundamental class, then:

$$\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta = \begin{cases} \int_X \gamma_1 \cup \gamma_2 & \text{if } \beta = 0, n = 3 \\ 0 & \text{otherwise} \end{cases}$$

Finally we will often use the so called divisor axiom. Let $\gamma_1 \in A^1(X)$ and $\beta \neq 0$, then:

$$\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta = (\int_\beta \gamma_1) \cdot \langle \gamma_2 \cdot \ldots \cdot \gamma_n \rangle_\beta$$

For an exhaustive treatment of the invariants and their properties see [K-M].
In this general setting Proposition 5.6 in [B-F] (see 2.3.8) implies:

Theorem 2.7.1. *Let $\pi : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,0}(X,\beta)$ be the usual map forgetting the markings and $ev = (ev_1, \dots, ev_n)$ be the evaluation map. Let $\mathcal{E}$ be the obstruction sheaf on $\overline{M}_{g,0}(X,\beta)$. Choose cycles $\Gamma_1, \dots, \Gamma_n$ in X representing the cohomology classes $\gamma_1, \dots, \gamma_n$ such that $ev_i^{-1}(\Gamma_i)$ intersect* *generically transversally. Then if* $A = \pi_*(\cap_i ev_i^{-1}(\Gamma_i))$ *is a cycle in the smooth locus of* $\overline{M}_{g,0}(X,\beta)$*:*

$$\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta = \int_{[\overline{M}_{g,n}(X,\beta)]} ev^*(\gamma_1 \times \ldots \times \gamma_n) \cdot \pi^* c_{top}(\mathcal{E}) \tag{2.9}$$

Remark 2.7.2. The above integral is equal to the degree of the top Chern class of the obstruction sheaf restricted to the cycle A:

$$\langle \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta = \int_A c_{top}(\mathcal{E})$$

2.8 Some invariants

We prove some vanishing results for the GW invariants which are related to the particular geometry of the effective curves involved.

Let $S_3 = [\Gamma(p)], S_4, S_5, S_6, S_{11}, S_{12}$ be the cycle classes forming a basis for $A^2(\mathbf{H})$ which we found in §1.1.

Proposition 2.8.1. *For* $k = 3,4,5,6$, $\langle S_k \rangle_{(0,0,c)} = 0$.

Proof. Suppose $c = 1$. A curve $(0,0,1)$ is incident to the cycle $\Gamma(p)$ if it is the curve of non-reduced subschemes supported on p, i.e. if it is the fiber over p of the support map s:

$$\Delta \cong \overline{M}_{0,1}(\mathbf{H},(0,0,1)) \xrightarrow{s} \overline{M}_{0,0}(\mathbf{H},(0,0,1)) \cong Q$$

Let ev be the evaluation map $\overline{M}_{0,1}(\mathbf{H},(0,0,1)) \to \mathbf{H}$.
Since s is flat, $s^*(p) = s^{-1}(p)$ and it is of codimension 2 in $\overline{M}_{0,1}(\mathbf{H},(0,0,1))$.
As a set $ev^{-1}(\Gamma(p)) = s^{-1}(p)$, so $ev^*(S_3) = \lambda s^*(p)$ has codimension 2.

$$\langle S_3 \rangle_{(0,0,1)} = \int_{[\overline{M}_{0,1}(\mathbf{H},(0,0,1))]^{vir}} ev^*(S_3) = \lambda \int_{[\overline{M}_{0,1}(\mathbf{H},(0,0,1))]} s^*[p \cdot c_{top}(\mathcal{E})] = 0$$

where $\mathcal{E}$ is the obstruction bundle on $\overline{M}_{0,0}(\mathbf{H},(0,0,1))$ and it has rank $\dim \overline{M}_{0,0}(\mathbf{H},(0,0,1)) - 1 = 1$, so that $p \cdot c_{top}(\mathcal{E}) = 0$ on $\overline{M}_{0,0}(\mathbf{H},(0,0,1))$. Curves of type $(0,0,c)$ intersecting $\Gamma(p)$ are multiple covers of $(0,0,1)$, so $\langle S_3 \rangle_{(0,0,c)} = 0$.
The cycle class S_4 can be represented by the set of subschemes whose support is incident to two lines l_1, l_2 with $l_k \in W_k$, $k = 1,2$. A curve $(0,0,1)$ can meet such a cycle only if it is the curve supported on the incident point $l_1 \cap l_2$. The previous argument works and $\langle S_4 \rangle_{(0,0,c)} = 0$.
The cycle classes S_5, S_6 are represented by the sets of subschemes with support incident to two lines in the same ruling, so a curve $(0,0,1)$ can never meet these cycles. This concludes the proof. □

Let T_8 be the cycle class $S_1^2 + S_1S_2 - 2S_{11}$ and $T_9 = S_2^2 + S_1S_2 - 2S_{12}$. They are symmetric. We consider only T_8.

Lemma 2.8.2. *For each $c \geq 1$, $\langle T_8 \rangle_{(0,0,c)} = \frac{4}{c^2}$*

Proof. Consider the diagram:

$$\begin{tikzcd} \overline{M}_{0,1}(\mathbf{H},(0,0,c)) \arrow[r, "\tilde{ev}"] \arrow[d, "\pi"'] & \Delta \arrow[r, "i"] \arrow[d, "s"] & \mathbf{H} \\ \overline{M}_{0,0}(\mathbf{H},(0,0,c)) \arrow[r, "g"'] & Q & \end{tikzcd}$$

where $g([C,\mu]) = \text{Supp } \mu(C)$. Set ev to be the composition map $i \circ \tilde{ev}$.
We know that $c_{2c-1}(\mathcal{E}) = -g^*K_Q \cdot c_{2c-2}(\tilde{\mathcal{E}})$, where $\mathcal{E}$ is the obstruction sheaf on $\overline{M}_{0,0}(\mathbf{H},(0,0,c))$ and $\tilde{\mathcal{E}}$ is the sheaf defined in 2.5.2. So we have to calculate:

$$\langle T_8 \rangle_{(0,0,c)} = \int_{[\overline{M}_{0,1}(\mathbf{H},(0,0,c))]} ev^*T_8 \cdot \pi^*(g^*(-K_Q) \cdot c_{2c-2}(\tilde{\mathcal{E}}))$$

Note that a point $[C,x,\mu] \in \overline{M}_{0,1}(\mathbf{H},(0,0,c))$ is such that the support of $\mu(C) = \mu(x) = Z$ is a point $p \in Q$, because a curve of class $(0,0,c)$ is a multiple cover of a fiber of s.
The above diagram is commutative, let f be the composition $g \circ \pi = s \circ \tilde{ev}$.
It is easy to verify that $i^*T_8 = 2 \cdot s^*h_1 \cdot \zeta$ (with notations as in (1.3)). Then we have to calculate the degree:

$$\int_{[\overline{M}_{0,1}(\mathbf{H},(0,0,c))]} 2f^*(-K_Q \cdot h_1) \cdot \tilde{ev}^*\zeta \cdot \pi^*(c_{2c-2}(\tilde{\mathcal{E}}))$$

Since $-K_Q \cdot h_1 = 2h_3$ where h_3 is the point-class in $A^2(Q)$, we get:

$$f^*(-K_Q \cdot h_1) \cdot \tilde{ev}^*\zeta = 2\tilde{ev}^*(\zeta \cdot s^*h_3)$$

Let $x \in \Delta$ be a point, we denote by M_1 the inverse image $\tilde{ev}^{-1}(x)$ and by M_0 its image $\pi(M_1) = g^{-1}(s(x))$ in $\overline{M}_{0,0}(\mathbf{H},(0,0,c))$. The restricted morphism $\tilde{\pi} : M_1 \to M_0$ has degree c. In particular:

$$\tilde{\pi}_*[M_1] = c[M_0] = c \cdot g^*[s(x)]$$

Since $\zeta \cdot s^*h_3 = [x]$ is the point-class in Δ, by the projection formula and what we said in §2.5, our invariant is:

$$\begin{aligned} \langle T_8 \rangle_{(0,0,c)} &= \int_{[\overline{M}_{0,1}(\mathbf{H},(0,0,c))]} 4\tilde{ev}^*(\zeta \cdot s^*h_3) \cdot \pi^*c_{2c-2}(\tilde{\mathcal{E}}) \\ &= \int_{[M_1]} 4\tilde{\pi}^*c_{2c-2}(\tilde{\mathcal{E}}) = \int_{[M_0]} 4c \cdot c_{2c-2}(\tilde{\mathcal{E}}) \\ &= 4c \cdot c_{2c-2}(\tilde{\mathcal{E}}|_{g^{-1}(s(x))}) = \frac{4}{c^2} \end{aligned}$$

□

The study of the obstruction bundle in section 2.6 gives us the tools for proving another vanishing result.

Remark 2.8.3. With notations as in §2.6, let M^* be the closed subset of $\overline{M}_{0,0}(\mathbf{H},(1,0,c))$, $c \geq 2$, of stable maps $\mu : D \to \mathbf{H}$ where the domain curve is reducible and $\mu(D_0) = C(l_1)$ is tangent to the conic defined by Δ in $\mathrm{Hilb}^2(l_1)$, (see 2.6.1). Consider the following maps:

$$\overline{M}_{0,3}(\mathbf{H},(1,0,c)) \xrightarrow{\pi} \overline{M}_{0,0}(\mathbf{H},(1,0,c)) \xrightarrow{\tau} \overline{M}_{0,0}(\mathbf{H},(1,0,0))$$

The map π forgets the marked points and (eventually) stabilizes the curve. The map τ is defined by restricting the stable map to the $(1,0,0)$ component. In particular it is surjective. If we denote by U_{c_1,c_2} the open subset of $\overline{M}_{0,0}(\mathbf{H},(1,0,c))$ of points $[D,\mu]$ such that:

$$\begin{array}{l} \mu(D_0) \ \text{ is not tangent to } \ \Delta \\ [\mu(D_1)] = c_1 \cdot (0,0,1) \\ [\mu(D_2)] = c_2 \cdot (0,0,1) \end{array}$$

then $\overline{M}_{0,0}(\mathbf{H},(1,0,c))$ is the union over all $c_1 \geq c_2$, $c_1+c_2 = c$ of the closures $\overline{U}_{c_1,c_2}$. Fix $\overline{U}_{c_1,c_2}$ then the restricted map $\tau : \overline{U}_{c_1,c_2} \to \overline{M}_{0,0}(\mathbf{H},(1,0,0))$ is surjective with fibers:

$$\text{if } \ c_1, c_2 > 0 \quad \begin{cases} M(c_1) \times M(c_2) \amalg M(c_2) \times M(c_1) & \text{generic fiber} \\ M(c_1) \times M(c_2) \ \text{ fiber over } \ M^* \end{cases}$$

$$\text{if } \ c_2 = 0 \quad \begin{cases} M(c) \amalg M(c) & \text{generic fiber} \\ M(c) \ \text{ fiber over } \ M^* \end{cases}$$

Σ is convex and a curve of class $(1,0,0)$ is contained into it, then the moduli space $\overline{M}_{0,0}(\mathbf{H},(1,0,0))$ is smooth of the expected dimension $\mathrm{d}_{\mathbf{H}} = 3$. Moreover it is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1 \cong W_1$, with fibers the Hilbert scheme $\mathrm{Hilb}^2(l_1)$ over a point $l_1 \in W_1$. Then it is irreducible. Also $\overline{U}_{c_1,c_2}$ is irreducible, then all the fibers of τ are 3-codimensional.

Proposition 2.8.4. *If $c > 2$ then all GW invariants $\langle \gamma_1\gamma_2\gamma_3 \rangle_\beta$ for curves of type $(1,0,c),(0,1,c)$ vanish.*

Proof. The two cases are symmetric. We consider only $(1,0,c)$.
We have seen that such a curve is reducible. It has a component of class $(1,0,0)$ not contained into Δ and it decomposes as:

$$(1,0,c) = (0,0,c_1) + (0,0,c_2) + (1,0,0)$$

with $c_1, c_2 \geq 0,\ c_1 + c_2 = c$.
We are free to choose a basis of $A^*(\mathbf{H})$ such that every cycle class can be

represented by cycles intersecting the stratification properly. It is enough to prove that GW invariants involving such classes vanish. Choose three of them $\gamma_1, \gamma_2, \gamma_3$ satisfying $\sum \mathrm{cod}\ \gamma_i = 6$. This condition means we are looking at three possible 3-uples of elements whose codimensions, up to a permutation of indexes, are:

$$(1,1,4),\ (1,2,3),\ (2,2,2)$$

Consider the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \gamma_1 \times \gamma_2 \times \gamma_3 \arrow[d] \\ \overline{M}_{0,3}(\mathbf{H},(1,0,c)) \arrow[r, "ev"] & \mathbf{H}^3 \end{tikzcd}$$

where $A = ev^*(\gamma_1 \times \gamma_2 \times \gamma_3)$. By the Position Lemma cod $A \geq 6$ (see 1.3.3). Let π be the flat map forgetting points $\overline{M}_{0,3}(\mathbf{H},(1,0,c)) \to \overline{M}_{0,0}(\mathbf{H},(1,0,c))$ and $B = \pi(A)$. Then cod $B \geq 3$. If cod $B > 3$, the GW invariants vanishes for dimensional reasons, so we can assume cod $B = 3$.
Let $\tau : \overline{M}_{0,0}(\mathbf{H},(1,0,c)) \to \overline{M}_{0,0}(\mathbf{H},(1,0,0))$ be the map defined in 2.8.3. If the class of a map $[f]$ is in B, then all the points in $\tau^{-1}(\tau([f]))$ are in B, because they differ only by the choice of a multiple cover of $(0,0,1)$ and this does not affect incidence conditions. The codimension of a fiber of τ is already equal to 3, so B is a union of finitely many components of fibers of τ. With notations as in §2.6 the set B is:

$$B = \coprod_{\substack{c_1+c_2=c \\ c_i \geq 0}} M(c_1) \times M(c_2)$$

where $M(0)$ is a point. If $c > 2$ then there exists i such that $c_i > 1$. By 2.6.14:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{(1,0,c)} = 0$$

□

Chapter 3

Quantum Cohomology

Quantum Cohomology is a deformation of the cup product of $A^*(\mathbf{H})$ involving the genus zero GW invariants. Moreover from the associativity law we can get some formulas for computing these invariants recursively. In this chapter we recall how to define the new product (see e.g. [G-P]) and we give a description of the ring we obtain.

Notations: the cup product in $A^*(\mathbf{H})$ will be denoted by $\alpha \cup \beta$. We will use the symbol $\langle T^n \rangle_\beta$ to denote the GW invariant $\langle \underbrace{T \cdot \ldots \cdot T}_{n} \rangle_\beta$.

3.1 The Big Quantum Cohomology Ring

Let $T_0 = 1, T_1, \ldots, T_{13}$ be a homogeneous $\mathbb{Q}$-basis for $A^*(\mathbf{H})$ such that T_1, T_2, T_3 generate $A^1(\mathbf{H})$. We denote by (g_{ij}) the matrix $(\int_{\mathbf{H}} T_i \cup T_j)$ and by (g^{ij}) its inverse. We introduce formal variables $\{y_0, q_1, q_2, q_3, y_4, \ldots, y_{13}\}$ which we will abbreviate as q, y. For β an effective class in $A_1(\mathbf{H})$, the following expression defines a power series in the ring $\mathbb{Q}[[q,y]]$:

$$\Gamma(q,y) := \sum_{n_4+\cdots+n_{13}\geq 0} \sum_{\beta \neq 0} \langle T_4^{n_4} \cdot \ldots \cdot T_m^{n_{13}} \rangle_\beta \cdot q_1^{\int_\beta T_1} q_2^{\int_\beta T_2} q_3^{\int_\beta T_3} \prod_{i=4}^{m} \frac{y_i^{n_i}}{n_i!}$$

In the case of a homogeneous space, substituting $q_i = e^{y_i}$ we get the quantum part of the potential function of [K-M] modulo some relation in the y_i. The symbol ∂_i will denote $q_i \frac{\partial}{\partial q_i}$ if $i = 1, 2, 3$, the partial derivative $\frac{\partial}{\partial y_i}$ otherwise. If $f \in \mathbb{Q}[[q,y]]$ then we set $f_{ijk} = \partial_i \partial_j \partial_k f$.
Consider the free $\mathbb{Q}[[q,y]]$-module $A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q,y]]$ generated by $T_0, \ldots, T_{13}$. We define a $\mathbb{Q}[[q,y]]$-linear product on it, called the $*$-*product*:

$$T_i * T_j = T_i \cup T_j + \sum_{e,f=0}^{13} \Gamma_{ije} g^{ef} T_f$$

It yields a $\mathbb{Q}[[q,y]]$-algebra structure.

Definition 3.1.1. *The Big Quantum Cohomology ring of* $\mathbf{H}$ *is the ring:*

$$QH^*(\mathbf{H}) = (A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q,y]], *)$$

Remark 3.1.2. By formal calculation, using the divisor axiom and the linearity of the GW invariants, we obtain:

$$\Gamma_{ijk} = \sum_{n\geq 0}\sum_{\beta\neq 0} \frac{1}{n!}\langle \gamma^n T_i T_j T_k\rangle_\beta \cdot q_1^{\int_\beta T_1} q_2^{\int_\beta T_2} q_3^{\int_\beta T_3}$$

where $\gamma = y_4T_4 + \cdots + y_{13}T_{13}$. Note that if one of the indexes i, j, k is zero, then the expression vanishes, because of the condition $\beta \neq 0$.

Definition 3.1.3. *The symbol* Φ_{ijk} *is defined as the sum* $\langle T_iT_jT_k\rangle_0 + \Gamma_{ijk}$*. In the homogeneous case, it corresponds to the 3-partial derivative of the potential function of* [K-M]*.*

We can write the $*$-product in a more compact way:

$$T_i * T_j = \sum_{e,f=0}^{13} \Phi_{ije} g^{ef} T_f$$

Since the partial derivatives are symmetric in the subscripts and GW invariants are invariant under a permutation of the arguments, it is evident that the $*$-product is commutative. Moreover it has $T_0 = 1$ as unit element:

$$T_0 * T_j = \sum_{e,f=0}^{13} \Phi_{0je} g^{ef} T_f = \sum_{e,f=0}^{13} g_{je} g^{ef} T_f = T_j$$

The quantum product is also associative. A proof can be found in [K-M] or, in the homogeneous case, in [F-P]. Associativity is equivalent to the following equality:

$$\sum_{e,f=0}^{13} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f=0}^{13} \Phi_{ike} g^{ef} \Phi_{fjl}$$

Writing down explicitly what it means in terms of GW invariants and using the splitting principle (see [K-M]), it turns out that this equality holds since it translates the condition of linear equivalence between pairs of points on $\mathbb{P}^1$. For further porposes, it seems useful to write explicitly the general associativity equation in terms of the GW invariants.
Let $\gamma_1, \ldots, \gamma_n$ be cohomology classes on $\mathbf{H}$, $\beta \in A_1(\mathbf{H})$ the class of an effective curve and A, B sets of indexes. Then the associativity reads:

$$\begin{aligned}&\sum \langle T_i \cdot T_j \cdot T_e \cdot \prod_{a\in A} \gamma_a\rangle_{\beta_1}\, g^{ef} \langle T_k \cdot T_l \cdot T_f \cdot \prod_{b\in B} \gamma_b\rangle_{\beta_2} = \\ &= \sum \langle T_i \cdot T_k \cdot T_e \cdot \prod_{a\in A} \gamma_a\rangle_{\beta_1}\, g^{ef} \langle T_j \cdot T_l \cdot T_f \cdot \prod_{b\in B} \gamma_b\rangle_{\beta_2}\end{aligned} \tag{3.1}$$

where the sum is over all the possible partitions $A \cup B = [n]$ of n indexes, all possible sums $\beta_1 + \beta_2 = \beta$ with β_i effective and over $e, f = 0, \ldots, 13$.
On the left hand side, the terms corresponding to β_1 or β_2 equal to zero sum up to:

$$\langle T_i \cdot T_j \cdot T_k \cup T_l \cdot \prod_1^n \gamma_s \rangle_\beta + \langle T_k \cdot T_l \cdot T_i \cup T_j \cdot \prod_1^n \gamma_s \rangle_\beta \tag{3.2}$$

An analogous expression gives the $\beta_i = 0$ terms for the right hand side of the equality. By means of these equations and of the divisor axiom, in [K-M] Kontsevich and Manin proved the *First Reconstruction Theorem*: on a variety whose cohomology is generated by the divisor classes all the genus zero GW invariants can be uniquely reconstructed starting from the 2-point invariants $\langle \gamma_1 \cdot \gamma_2 \rangle_\beta$.
If the cohomology ring is not generated by divisors, we can restrict our attention to the subring $\mathbf{S}$ these classes generate. Then the cited theorem holds anyway and we can reconstruct all the GW invariants involving classes in $\mathbf{S}$ from the 2-point invariants. To know the complete tree level GW-system we have to calculate the invariants including the classes we disregarded. This is the technique we are going to explain in §3.5.

3.2 A good $\mathbb{Q}$-basis for $A^*(\mathbf{H})$

We choose once for all the following $\mathbb{Q}$-basis for the Chow ring $A^*(\mathbf{H})$:

$A^0(\mathbf{H})$	$A^1(\mathbf{H})$	$A^2(\mathbf{H})$	$A^3(\mathbf{H})$	$A^4(\mathbf{H})$
T_0	T_1	T_4	T_{10}	T_{13}
	T_2	T_5	T_{11}	
	T_3	T_6	T_{12}	
		T_7		
		T_8		
		T_9		

The cycles classes are defined by:

$$\begin{array}{ll}
T_0 = 1 & T_7 = S_2^2 \\
T_1 = S_1 & T_8 = S_1\sigma_{1,0} \\
T_2 = S_2 & T_9 = S_2\sigma_{1,0} \\
T_3 = \sigma_{1,0} & T_{10} = S_2S_3 \\
T_4 = S_3 & T_{11} = S_1S_3 \\
T_5 = S_1S_2 & T_{12} = S_3\sigma_{1,0} \\
T_6 = S_1^2 & T_{13} = S_3^2
\end{array}$$

By 1.1.8 and 1.4.1, we know that $A^*(\mathbf{H})$ is the $\mathbb{Q}$-algebra:

$$\frac{\mathbb{Q}[T_1, T_2, T_3, T_4]}{(f_i)_{i=1,\ldots,17}}$$

where the relations are:

$$
\begin{array}{ll}
1) & T_3^2 - (T_1 + T_2)T_3 - T_1T_2 = 0 \\
2) & T_1^3 = 0 \\
3) & T_2^3 = 0 \\
4) & T_1^2T_2 - 2T_1T_4 = 0 \\
5) & T_1^2T_3 - 2T_1T_4 = 0 \\
6) & T_2^2T_1 - 2T_2T_4 = 0 \\
7) & T_2^2T_3 - 2T_2T_4 = 0 \\
8) & T_1T_2T_3 - 2T_3T_4 = 0 \\
9) & T_4^3 = 0 \\
10) & T_4^2T_1 = 0 \\
11) & T_4^2T_2 = 0 \\
12) & T_4^2T_3 = 0 \\
13) & T_1^2T_4 = 0 \\
14) & T_2^2T_4 = 0 \\
15) & T_1T_2T_4 - T_4^2 = 0 \\
16) & T_1T_3T_4 - T_4^2 = 0 \\
17) & T_2T_3T_4 - T_4^2 = 0
\end{array}
$$

In particular $A^*(\mathbf{H})$ is not generated by the divisor classes.
The matrix (g^{ij}) is the inverse of $(g_{ij}) = (\int_{\mathbf{H}} T_i \cup T_j)$:

$$
(g^{ij}) = \begin{pmatrix} & & & & 1 \\ & & & A & \\ & & B & & \\ & A & & & \\ 1 & & & & \end{pmatrix}
$$

where $1 = \int_{\mathbf{H}} T_0 \cup T_{13}$, A and B are the matrices induced by the Poincaré duality $A^1(\mathbf{H}) - A^3(\mathbf{H})$ and $A^2(\mathbf{H}) - A^2(\mathbf{H})$ respectively.

$$
A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}
$$

$$
B = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix}
$$

Throughout the paper, we will understand that the sum in $T_i * T_j$ is over $e, f = 0, \ldots, 13$ and we will use the convention:

$$
q^\beta = q_1^{\int_\beta T_1} q_2^{\int_\beta T_2} q_3^{\int_\beta T_3}
$$

Remark 3.2.1. We note that all the classes T_i can be generated by cycles intersecting the stratification properly.

Remark 3.2.2. There are some symmetric cycle classes: T_1 and T_2, T_6 and T_7, T_8 and T_9, T_{10} and T_{11}. This is because they depend on the choice of one of the two rulings on Q.

Remark 3.2.3. The classes T_{10}, T_{11}, T_{12} are the classes $C_2 + F$, $C_1 + F$, $C_1 + C_2 + F$ respectively. They can be represented by the irreducible curves $C(p_2, l_2)$, $C(p_1, l_1)$, $\Lambda(l)$ respectively (see §1.7)

3.3 The Small Quantum Cohomology Ring

The Small Quantum Cohomology ring $QH^*_s(\mathbf{H})$ of $\mathbf{H}$ incorporates only the genus zero 3-point GW invariants in its product and it is defined by setting to zero all the formal variables except those corresponding to the divisor classes. This means that we consider $(A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, q_2, q_3]], *)$ with the product given by:

$$T_i * T_j = \sum_{e,f=0}^{13} \overline{\Phi}_{ije} g^{ef} T_f$$

where:

$$\overline{\Phi}_{ijk} = \int_{\mathbf{H}} T_i \cup T_j \cup T_k + \overline{\Gamma}_{ijk}$$

$$\overline{\Gamma}_{ijk} = \sum_{\beta \neq 0} \langle T_i T_j T_k \rangle_\beta \cdot q_1^{\int_\beta T_1} q_2^{\int_\beta T_2} q_3^{\int_\beta T_3}$$

The last equality is a consequence of putting $y_i = 0$ in Γ_{ijk} (see 3.1.2).
The product yields a commutative, associative graded $\mathbb{Q}[[q_1, q_2, q_3]]$-algebra structure with T_0 as unit. The variables q_i, T_j are graded by the following degrees:

$$\deg q_i = \int_{\beta_i} c_1(T_{\mathbf{H}})$$
$$\deg T_j = \operatorname{cod} T_j$$

where β_i is the dual class to T_i for $i = 1, 2, 3$, i.e. C_2, C_1, F, respectively. In particular q_1, q_2 have degree 2, while $\deg q_3 = 0$.

Lemma 3.3.1. *Since q_1, q_2 have positive degree, we have:*

$$T_i * T_j \in A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[q_1, q_2][[q_3]] \subseteq A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, q_2, q_3]]$$

Proof. In the product $T_i * T_j$, for a fixed e the invariant $\langle T_i T_j T_e \rangle_\beta$ is zero unless the sum of the codimensions is equal to $-\beta \cdot K_{\mathbf{H}} + 4$. For $\beta = (a, b, c)$ effective, the condition implies that $-\beta \cdot K_{\mathbf{H}} = 2a + 2b$ is a fixed number. Then there are only finitely many possible values for a, b which are the exponents of the variables q_2, q_1, respectively. The only exponent having no bound is that of q_3. □

Definition 3.3.2. *We define the Small Quantum Cohomology ring of* $\mathbf{H}$ *to be:*

$$QH_s^*(\mathbf{H}) = (A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[q_1, q_2][[q_3]], *)$$

It is a deformation of $A^*(\mathbf{H})$ in the usual sense, in fact we can recover the Chow ring of $\mathbf{H}$ by setting all the q_i equal to zero.
Let $\mathbb{Q}[Z] = \mathbb{Q}[Z_1, \dots, Z_4]$ and let

$$A^*(\mathbf{H}) = \frac{\mathbb{Q}[Z]}{(f_1, \dots, f_s)}$$

be a presentation with arbitrary homogeneous generators $f_1, \dots, f_s$ for the ideal of relations. Finally let $\mathbb{Q}(q, Z) = \mathbb{Q}[q_1, q_2, Z_1, \dots, Z_4][[q_3]]$. The following proposition is a slightly modified version of [F-P] §10 Prop.11.

Proposition 3.3.3. *Let* $f_1', \dots, f_s'$ *be any homogeneous elements in* $\mathbb{Q}(q, Z)$ *such that:*

(i) $f_i'(0, 0, 0, Z_1, \dots, Z_4) = f_i(Z_1, \dots, Z_4)$ *in* $\mathbb{Q}(q, Z)$,

(ii) $f_i'(q_1, q_2, q_3, Z_1, \dots, Z_4) = 0$ *in* $QH_s^*(\mathbf{H})$.

Then the canonical map

$$\frac{\mathbb{Q}(q, Z)}{(f_1', \dots, f_s')} \xrightarrow{\varphi} QH_s^*(\mathbf{H})$$

is an isomorphism.

Proof. As in [F-P] we can use a Nakayama-type induction. First we observe that given a homogeneous map $\psi : M \to N$ between two finitely generated $\mathbb{Q}(q, Z)$-modules such that the induced map:

$$\frac{M/(q_3)}{(q_1, q_2)} \xrightarrow{\psi_{1,2}} \frac{N/(q_3)}{(q_1, q_2)}$$

is surjective, then $\psi_3 : M/(q_3) \to N/(q_3)$ is surjective, because q_1, q_2 have positive degree. Since the ideal (q_3) is contained into the radical of Jacobson of $\mathbb{Q}(q, Z)$ and $N = \psi(M) + (q_3)N$, by surjectivity of ψ_3, it follows that ψ is surjective ([A-M] Cor. 2.7). Hence by hypothesis (i) our map φ is surjective. If $\tilde{T}_i$, $i = 0, \dots, 13$ are homogeneous lifts to $\mathbb{Q}[q_1, q_2][[q_3]]$ of a basis of $A^*(\mathbf{H})$, exactly the same argument of passing to the quotients shows that their images in $\mathbb{Q}(q, Z)/(f_1', \dots, f_s')$ generates this $\mathbb{Q}[q_1, q_2][[q_3]]$-module. But $QH_s^*(\mathbf{H})$ is free over $\mathbb{Q}$ of rank 14, so φ is an isomorphism. ∎

3.4 A presentation of $QH^*_s(\mathbf{H})$

According to Proposition 3.3.3, to have a presentation of the Small Quantum Cohomology ring of $\mathbf{H}$ we need to find some equations lifting the relations defining $A^*(\mathbf{H})$ and vanishing in $QH^*_s(\mathbf{H})$. Let $\{f_i\}_{i=1,\dots,17}$ be the relations listed at the end of §3.1 and denote by f_i^* the ith-relation calculated using the $*$-product. We will show that the f_i^*'s are the equations we are looking for.

We calculate all the monomials arising from the $*$-product of two generators of $A^*(\mathbf{H})$, disregarding $T_4 * T_4$ for the moment.
We distinguish different cases.

$$\begin{aligned} T_3 * T_3 \;&= T_3 \cup T_3 + \sum_{c\geq 1} 2\left\{2T_5 + T_6 + T_7 - T_8 - T_9\right\} q_3^c + \\ &+ \sum_{c\geq 0} c^2 \left\{\langle T_{13}\rangle_{(1,0,c)} q_2 q_3^c + \langle T_{13}\rangle_{(0,1,c)} q_1 q_3^c\right\} T_0 \end{aligned}$$

where we use 2.8.1, 2.8.2 and 3.2.2.
If T_i is a divisor class with $i \neq 3$:

$$T_i * T_3 = T_i \cup T_3 + \sum_{c\geq 0} c\langle T_{13}\rangle_\beta \cdot T_0 \cdot q^\beta \quad \text{with } \beta = (1,0,c) \text{ or } (0,1,c)$$

If T_i, T_j are divisor classes with $i, j \neq 3$:

$$T_i * T_j = T_i \cup T_j + \sum_{c\geq 0} \langle T_i T_j T_{13}\rangle_\beta \cdot T_0 \cdot q^\beta \quad \text{with } \beta = (1,0,c) \text{ or } (0,1,c)$$

If T_i is a divisor class with $i \neq 3$:

$$T_i * T_4 = T_i \cup T_4 + \sum_{\substack{\text{cod } T_e = 3 \\ c\geq 0}} \langle T_i T_4 T_e\rangle_\beta g^{ef} T_f \cdot q^\beta \quad \text{with } \beta = (1,0,c) \text{ or } (0,1,c)$$

Finally:

$$T_3 * T_4 = T_3 \cup T_4 + \sum_{\substack{\text{cod } T_e = 3 \\ c\geq 0}} c\langle T_4 T_e\rangle_\beta g^{ef} T_f \cdot q^\beta \quad \text{with } \beta = (1,0,c) \text{ and } (0,1,c)$$

where we use 2.8.1 again.
This list points out that we need to know the value of some GW invariants involving T_4, T_{13} in order to write down the f_i^*'s. By the vanishing result 2.8.4 it is enought to calculate $\langle T_{13}\rangle_\beta$ and $\langle T_4, \text{cod } 3\rangle_\beta$ with $\beta = (1,0,c), (0,1,c),\ 0 \leq c \leq 2$.
We will use the same notation fixed in §1.7.

The invariant $\langle T_{13}\rangle_{(1,0,c)}$

If $\mathcal{E}$ is the rank $\mathrm{d}_{\mathbf{H}}-3$ obstruction bundle on $\overline{M}_{0,0}(\mathbf{H},(1,0,c))$, we have to compute:

$$\begin{aligned}\langle T_{13}\rangle_{(1,0,c)} &= \int_{[\overline{M}_{0,1}(\mathbf{H},(1,0,c))]} ev^*T_{13}\cdot \pi^* c_{\mathrm{d}_{\mathbf{H}}-3}(\mathcal{E}) \\ &= \int_{ev^{-1}(Z)} \pi^* c_{\mathrm{d}_{\mathbf{H}}-3}(\mathcal{E})\end{aligned}$$

where Z is a generic point of $\mathbf{H}$ representing the class T_{13} and π is the map forgetting a point and stabilizing.

If $c=0$, we know that $\overline{M}_{0,0}(\mathbf{H},(1,0,0))$ is smooth of the expected dimension $\mathrm{d}_{\mathbf{H}}=3$, since each curve of class $(1,0,0)$ is contained into Σ. In particular the top Chern class of $\mathcal{E}$ gives 1. We can choose a representative Z of the class T_{13} such that $l_Z\notin W_1$, then the fiber $ev^{-1}(Z)$ is empty and the GW invariant vanishes.

If $c=1$, by 2.6.2 we have to analyse separately what happens on the two components of the moduli space. We can choose $Z\notin \Delta\cup\Sigma$, with Supp $Z=\{p_0,q_0\}$, so that reducible curves of type $C_1\cup F$ give no contribution to the invariant. Let us consider a stable map with image an irreducible curve of class C_1+F. It is a smooth point for the moduli space $\overline{M}_{0,1}(\mathbf{H},(1,0,1))$ which is 4-dimensional in it. Denote by M^{irr} the irreducible component parametrizing such maps, then $ev(\overline{M^{irr}})=\mathbf{H}$. The restricted map $ev:\overline{M^{irr}}\to\mathbf{H}$ has degree two, because an irreducible curve C of class $(1,0,1)$ is completely determined by choosing a line $l_1\in W_1$ and a point $p_1\notin l_1$ and all its points are reduced. The fiber over Z contains two points: the isomorphism classes $[\mathbb{P}^1,x,\mu]$ where $\mathbb{P}^1\cong C(p_0,l_1(q_0))$ or $\mathbb{P}^1\cong C(q_0,l_1(p_0))$. In the first case, $\mu(t)=(p_0,f(t))$, with $f:\mathbb{P}^1\to Q$ a parametrization of $l_1(q_0)$ such that $f(x)=q_0$. Similarly for the other map. Then we have a contribution equal to 2 to the GW invariant.

If $c=2$, all the curves of class $(1,0,2)$ are reducible contained into $\Delta\cup\Sigma$, choosing $Z\notin\Delta\cup\Sigma$ the fiber $ev^{-1}(Z)$ is empty and the GW invariant vanishes.

By symmetry the same results hold for $\langle T_{13}\rangle_{(0,1,c)}$.

The invariant $\langle T_4T_i\rangle_{(1,0,c)}$

We want to calculate:

$$\langle T_4T_i\rangle_{(1,0,c)} = \int_{[\overline{M}_{0,2}(\mathbf{H},(1,0,c))]} ev^*(T_4\times T_i)\cdot \pi^* c_{\mathrm{d}_{\mathbf{H}}-3}(\mathcal{E})$$

where T_i lives in codimension 3, π has relative dimension equal to 2 and $ev = (ev_1, ev_2)$ is the evaluation map with image in $\mathbf{H} \times \mathbf{H}$. By linearity of the GW invariants we can consider only the generators T_{10}, T_{11}, T_{12} of $A^3(\mathbf{H})$. Choose once for all a representative $\Gamma(p)$ for the class T_4, with $p \in Q$ a generic point.

If $c = 0$, as before $c_{top}(\mathcal{E}) = 1$. Let $C(p_1, l_1)$ represent T_{11} such that $p \neq p_1$ and $p \notin l_1$ (see 3.2.3). Since all the points of a curve of class $(1,0,0)$ have the same associated line, it never intersects a curve of class T_{11}. The GW invariant for maps $\mu : C \to \mathbf{H}$ with $\mu_*[C] = (1,0,0)$ vanishes. The same holds for the invariant involving T_{10}, because for a generic representative $C(p_2, l_2)$ of that class $p \neq p_2$ and $p \notin l_2$. Finally, if $\Lambda(l)$ represents T_{12}, then it is disjoint from Σ and also this invariant is zero.

Let $c = 1$ and consider the component of the moduli space parametrizing maps from irreducible curves. An irreducible curve of class $(1,0,1)$ never intersects both a $\Gamma(p)$ and a $C(p_1, l_1)$ cycle generically chosen. The contribution to the invariant $\langle T_4 T_{11} \rangle_{(1,0,1)}$ is zero.
If $C(p_2, l_2)$ represents T_{10}, then $ev^*(T_4 \times T_{10})$ is a unique reduced point, the class of the stable map $[\mathbb{P}^1, x_1, x_2, \mu]$ determined by $\mathbb{P}^1 \cong C(p_2, l_1(p))$, $\mu(t) = (p_2, f(t))$ with $f : \mathbb{P}^1 \to Q$ a parametrization of $l_1(p)$ such that $f(x_1) = p, f(x_2) = l_1(p) \cap l_2$.
Let $\Lambda(l)$ represent T_{12}, then for a general plane $p \notin \Lambda$. Moreover the line $l_1(p)$ intersects the hyperplane section $\Lambda \cap Q$ exactly in a point q_1. The set $\{Z \in \mathrm{Hilb}^2(\Lambda \cap Q) : \mathrm{Supp}\ Z \ni q_1\}$ is a line in $\mathrm{Hilb}^2(\Lambda \cap Q)$ tangent to the conic defined by Δ on that Hilbert scheme. It intersects l in a point (q_1, q_2) with $q_1 \neq q_2$, because we chose l generic. Hence the curve $\mathbb{P}^1 \cong C(q_2, l_1(p))$ intersects $\Gamma(p)$ in $Z = (q_2, p)$ and $\Lambda(l)$ in $Z = (q_2, q_1)$. We conclude that there is exactly one class $[\mathbb{P}^1, x_1, x_2, \mu]$ satisfying the incidence conditions. The stable map μ is defined by $\mu(t) = (q_2, f(t))$ where $f : \mathbb{P}^1 \to Q$ is a parametrization of $l_1(p)$ such that $f(x_1) = p$, $f(x_2) = q_1$.
Now we have to count the contribution coming from the reducible curves of class $(1,0,1)$. All these are contained into $\Delta \cup \Sigma$. Let $D = C(l) \cup C(q)$ be one of them. We want to intersect it with a cycle $\Gamma(p)$ and a cycle $C(p_1, l_1)$. Intersection points can not lie all on one component, because everything is generic. The only possibility is that the intersection with $\Gamma(p)$ is on the F-component and the other one on the C_1-component. Then $q = p$ and $l = l_1(p)$. The second equality prevents any other point of $C(l)$ from intersecting the cycle $C(p_1, l_1)$, since in general $p_1 \notin l_1(p)$. Then there are no reducible curves satisfying these incidence conditions, i.e. the contribution to the GW invariant is zero.
The same holds for the invariant involving T_{10}.
Finally, the cycle $\Lambda(l)$ representing T_{12} is such that the generic plane Λ does not contain any line of Q. Then there does not exist any point Z on the C_1-

component of D lying in $\mathrm{Hilb}^2(\Lambda \cap Q)$. The interseciton with $\Lambda(l)$ must be a point on the F-component. $\Lambda(l)$ intersects Δ in at most 2 points Z_i with support q_i. Suppose $q = q_1$, then $C(l)$ is a line in $\mathrm{Hilb}^2(l_1(q_1))$. Since $p \notin l_1(q_1)$, for general points, we conclude that there are no curves D satisfying both the incident conditions. The contribution to the GW invariant is again zero.

If $c = 2$, all the curves are of type $C_1 \cup c_1 F \cup c_2 F$, with $c_1 + c_2 = 2$. As before the intersection points do not lie on the same component. In particular the intersection with $\Gamma(p)$ is a point on a F-component. Besides the intersection with $C(p_1, l_1)$, $C(p_2, l_2)$ or $\Lambda(l)$ has to be on the C_1-component, because Λ does not contain any line in Q and all $Z \in C(p_i, l_i)$, $i = 1, 2$, are reduced. Then as for the reducible curve in the case $c = 1$, all the GW invariants vanish.

We can summarize our results in a table:

	$c=0$	$c=1$	$c=2$	$c>2$
$\langle T_{13}\rangle_{(1,0,c)}$	0	2	0	0
$\langle T_4 T_{10}\rangle_{(1,0,c)}$	0	1	0	0
$\langle T_4 T_{11}\rangle_{(1,0,c)}$	0	0	0	0
$\langle T_4 T_{12}\rangle_{(1,0,c)}$	0	1	0	0

Remark 3.4.1. For $\beta = (0, 1, c)$ we have the same table, interchanging the values obtained for the invariants involving T_{10}, T_{11}.

We find the following expressions:

$$\begin{aligned}
T_3 * T_3 &= T_3^2 + 2(q_1 + q_2)q_3 T_0 + \sum_{c \geq 1} 2\{2T_5 + T_6 + T_7 - T_8 - T_9\} q_3^c \\
T_1 * T_1 &= T_1^2 + 2q_1 q_3 T_0 \\
T_1 * T_2 &= T_1 T_2 \\
T_1 * T_3 &= T_1 T_3 + 2q_1 q_3 T_0 \\
T_2 * T_2 &= T_2^2 + 2q_2 q_3 T_0 \\
T_2 * T_3 &= T_2 T_3 + 2q_2 q_3 T_0 \\
T_1 * T_4 &= T_1 T_4 + q_1 q_3 T_2 \\
T_2 * T_4 &= T_2 T_4 + q_2 q_3 T_1 \\
T_3 * T_4 &= T_3 T_4 + q_2 q_3 T_1 + q_1 q_3 T_2
\end{aligned}$$

Applying associativity to the f_i^* equations of §3.2 will permit us to calculate almost all the GW invariants we need to write them explicitly. For example the identity $(T_1 * T_1) * T_2 = T_1 * (T_1 * T_2)$ gives:

$$2q_1q_3T_2 + \sum_{\substack{\text{cod } T_e=3\\ c\geq 0}} \langle T_6T_e\rangle_{(1,0,c)} g^{ef} T_f \cdot q_2 q_3^c = \sum_{\substack{\text{cod } T_e=3\\ c\geq 0}} \langle T_5T_e\rangle_{(0,1,c)} g^{ef} T_f \cdot q_1 q_3^c$$

By comparing the coefficients of the variables and by 2.8.4 we find:

$c=0$	$\langle T_5T_e\rangle_{(0,1,0)} = 0$	$\langle T_6T_e\rangle_{(1,0,0)} = 0$ for all $T_e \in A^3(\mathbf{H})$
$c=1$	$\langle T_5T_{10}\rangle_{(0,1,1)} = 0$ $\langle T_5T_{11}\rangle_{(0,1,1)} = 2$ $\langle T_5T_{12}\rangle_{(0,1,1)} = 2$	$\langle T_6T_e\rangle_{(1,0,1)} = 0$ for all $T_e \in A^3(\mathbf{H})$
$c\geq 2$	$\langle T_5T_e\rangle_{(0,1,c)} = 0$	$\langle T_6T_e\rangle_{(1,0,c)} = 0$ for all $T_e \in A^3(\mathbf{H})$

By symmetry we have:

$c=0$	$\langle T_5T_e\rangle_{(1,0,0)} = 0$	$\langle T_7T_e\rangle_{(0,1,0)} = 0$ for all $T_e \in A^3(\mathbf{H})$
$c=1$	$\langle T_5T_{10}\rangle_{(1,0,1)} = 2$ $\langle T_5T_{11}\rangle_{(1,0,1)} = 0$ $\langle T_5T_{12}\rangle_{(1,0,1)} = 2$	$\langle T_7T_e\rangle_{(0,1,1)} = 0$ for all $T_e \in A^3(\mathbf{H})$
$c\geq 2$	$\langle T_5T_e\rangle_{(1,0,c)} = 0$	$\langle T_7T_e\rangle_{(0,1,c)} = 0$ for all $T_e \in A^3(\mathbf{H})$

The values in the second table arise also from the associativity applied to $T_2 * T_2 * T_1$.

The only necessary invariants we can not compute with this technique are:

1)	$\langle T_8\rangle_{(0,0,c)}$		$c \geq 1$
2)	$\langle T_{11}T_6\rangle_{(0,1,c)}$	$\langle T_{10}T_7\rangle_{(1,0,c)}$	$2 \geq c \geq 0$
3)	$\langle T_4 \text{ cod } 3\rangle_{(0,1,c)}$	$\langle T_4 \text{ cod } 3\rangle_{(1,0,c)}$	$2 \geq c \geq 0$
4)	$\langle T_{13}\rangle_{(1,0,c)}$	$\langle T_{13}\rangle_{(0,1,c)}$	$2 \geq c \geq 0$
5)	$\langle T_{13}T_{10}\rangle_{(2,0,c)}$	$\langle T_{13}T_{11}\rangle_{(0,2,c)}$	$c \geq 0$
6)	$\langle T_{13}T_{10}\rangle_{(0,2,c)}$	$\langle T_{13}T_{11}\rangle_{(2,0,c)}$	$c \geq 0$
7)	$\langle T_{13} \text{ cod } 3\rangle_{(1,1,1)}$		
8)	$\langle T_{13}T_{10}\rangle_{(1,1,c)}$	$\langle T_{13}T_{11}\rangle_{(1,1,c)}$	$c \geq 2$
9)	$\langle T_4T_4T_4\rangle_{(1,0,c)}$	$\langle T_4T_4T_4\rangle_{(0,1,c)}$	$2 \geq c \geq 0$
10)	$\langle T_{13}T_4T_4\rangle_{(1,1,c)}$		$c \geq 1$
11)	$\langle T_{13}T_4T_4\rangle_{(2,0,c)}$	$\langle T_{13}T_4T_4\rangle_{(0,2,c)}$	$c \geq 1$
12)	$\langle T_{13}T_{13}T_4\rangle_{\beta}$	$\beta = (3,0,c),(0,3,c),(2,1,c),(1,2,c)$	$c \geq 0$

By symmetry, to calculate all the above invariants it is enough to consider only those on the left side of the list, in particular for the last one we have to study only the cases with $\beta = (3,0,c),(2,1,c)$. Note that we already know the values of the GW invariants 1), 3), 4) by previous calculations.
The last four values can be calculated using the associativity equation (3.1), (see §3.5). Since they are not so difficult, here we work them out by hand.

Lemma 3.4.2. *All the invariants number 12) are zero.*

Proof. We can choose generic representatives Z_0, $Z_0^{'}$ for the two point-classes. None of the $(3,0,0)$-curve can intersect both of them. Moreover there are only reducible curves of class $(3,0,c)$ for $c = 1,2$ and they are of type $3C_1 \cup F$, $3C_1 \cup c_1F \cup c_2F$, $c_1 + c_2 = 2$, respectively. Then for the same choice of generic $Z_0, Z_0^{'}$ none of them intersect such cycles. If $c = 3$ we choose Z_0 generic and $Z_0^{'} \in \Delta$. Reducible curves of class $(3,0,3)$ are disjoint from Δ or they live in the wrong dimension. Finally irreducible curves are disjoint from Δ. Then also in this case the invariant is zero. For $c \geq 4$ all the curves are reducible of type $3C_1 \cup C$, where C is a union of an appropriate number of c_iF-curves. Then for a generic representative we have no contribution to the GW invariant. An analogous argument shows that also for stable maps of class $(2,1,c)$, $c \geq 0$, everything vanishes. $\square$

Remark 3.4.3. Since in the proof we do not make use of the cycles representing T_4, we proved something more, precisely:

$$\langle T_{13}T_{13} \text{ cod } 2\rangle_\beta = 0$$

for $\beta = (3,0,c),(2,1,c)$.

A similar argument yields:

$$\langle T_{13}T_4 \text{ cod } 2\rangle_{(2,0,c)} = 0$$

for all 2-codimensional classes and $c \geq 0$. In particular $\langle T_{13}T_4T_4\rangle_{(2,0,c)} = 0$, for all $c \geq 1$.
By means of 2.8.4 and genericity assumptions it is very easy to see that for all $c \geq 0$:

$$\langle T_4T_4T_4\rangle_{(1,0,c)} = 0$$

Finally the invariant $\langle T_{13}T_4T_4\rangle_{(1,1,c)}$ gives 2 for $c = 2$ and zero otherwise [P].

The invariant $\langle T_{11}T_6\rangle_{(0,1,c)}$

We want to calculate:

$$\langle T_{11}T_6\rangle_{(0,1,c)} = \int_{[\overline{M}_{0,2}(\mathbf{H},(0,1,c))]} ev_1^*(C(p_1,l_1)) \cdot ev_2^*(\gamma) \cdot \pi^*(c_{\mathrm{d}_{\mathbf{H}}-3}(\mathcal{E}))$$

where $\gamma = \{Z \in \mathbf{H} : \text{Supp } Z \cap l_1^{'} \neq \emptyset, \text{Supp } Z \cap l_1^{''} \neq \emptyset\}$ is a cycle representing T_6, for fixed lines $l_1^{'}, l_1^{''} \in W_1$, and $\mathcal{E}$ is the obstruction bundle on $\overline{M}_{0,0}(\mathbf{H},(0,1,c))$. Both representatives of T_6 and T_{11} can be choosen generic.

It is very easy to see that for $c = 0$ the invariant gives 1, because of the geometry of a curve of class $(0,1,0)$.

If $c = 1$ we do not have any contribution from the irreducible curves by the genericity assumptions. Let C be a reducible curve of class $(0,1,1)$. It has to be a union $C(l_2) \cup C(p)$ for some $p \in Q$ and $l_2 \in W_2$. Since all the points on $C(p_1,l_1)$ and γ are reduced, C can intersects them only along $C(l_2)$. The line $l_2 = l_2(p_1)$ is then determined. The curve $C(l_2)$ is the line in $\mathrm{Hilb}^2(l_2(p_1))$ through $(p_1, l_2 \cap l_1)$ and $(l_2 \cap l_1^{'}, l_2 \cap l_1^{''})$. Moreover there are two possible points for attaching $C(p)$. This gives a contribution 2 to the invariant.

If $c = 2$ we know that each stable map μ has a reducible domain curve D, in particular $\mu(D)$ is a curve of class $C_2 \cup c_1F \cup c_2F$ with $c_1 + c_2 = 2$. The F-components are points of $M(c_1)$ and $M(c_2)$ respectively. As before the intersection points with $C(p_1, l_1)$ and γ lie on the C_2-component which is completely determined. It intersects Δ in at most two points Z_i, with Supp $Z_i = q_i$. Then by proposition 2.8.4 there is only a point satisfying all the incident conditions $[D, x_1, x_2, \mu] \in \overline{M}_{0,2}(\mathbf{H}, (0,1,2))$:

$$\begin{array}{ll} D = D_0 \cup D_1 \cup D_2 & y_i = D_0 \cap D_i,\ i = 1,2 \\ \mu_*[D_0] = C_2 & \mu_*[D_i] = [C(q_i)],\ i = 1,2 \\ \mu(x_1) = (p_1, l_2(p_1) \cap l_1) & \mu(x_2) = (l_2(p_1) \cap l_1^{'}, l_2(p_1) \cap l_1^{''}) \\ \mu(y_i) = q_i,\ i = 1,2 & \end{array}$$

It is a reduced point so it counts with multiplicity one.

We summarize our results in the following table:

	$c=0$	$c=1$	$c=2$
$\langle T_{11}T_6\rangle_{(0,1,c)}$	1	2	1

The invariants $\langle T_{13}T_{10}\rangle_{(2,0,c)}$ and $\langle T_{13}T_{11}\rangle_{(2,0,c)}$

To calculate $\langle T_{13}T_{10}\rangle_{(2,0,c)}$ and $\langle T_{13}T_{11}\rangle_{(2,0,c)}$ for $c \geq 0$ is equivalent to solve problems 5) and 6).

If $c = 0$, fixing a generic Z_0 representing T_{13}, there are no stable maps of type $(2,0,0)$ satisfying all the incident conditions in both cases. Then the GW invariants vanish.

If $c = 1$, curves of class $(2,0,1)$ are all reducible. Fix Z_0 as above, then no curves of type $2C_1 \cup F$ can intersect it. Also in this case our invariants are zero.

Let $c = 2$ and Z_0 be a non-reduced point with support p_0. Note that the points on the curve $C(p_1, l_1)$ are all reduced as well as those of $C(p_2, l_2)$. In order to calculate the GW invariants we have to add the contributions given by maps of type $2C_1 \cup c_1F \cup c_2F$, $(C_1+F) \cup (C_1+F)$, $2(C_1+F)$ and $2C_1 + 2F$. If a reducible curve $2C_1 \cup a_1F \cup a_2F$ intersects Z_0, then its support is completely determined by $l_1(p_0)$ and it does not intersect $C(p_1, l_1)$ or $C(p_2, l_2)$.
Curves of type $(C_1+F) \cup (C_1+F)$ are disjoint from Δ as well as $2(C_1+F)$, so they do not give any contribution.
Finally, if a curve $2C_1 + 2F$ is in Δ it does not intersects $C(l_i, p_i)$, $i = 1,2$,

otherwise it does not pass through Z_0.

If $c > 2$, we can choose Z_0 in Δ. Then we have to analyze only the contributions given by maps from irreducible curves of type $2C_1 + cF$ contained into Δ and from reducible ones of type $2C_1 \cup a_1F \cup a_2F \cup a_3F \cup a_4F$, $\sum a_i = c$. By the same argument used above, the GW invariants vanish.

$$\langle T_{13}T_{10}\rangle_{(2,0,c)} = 0 \quad \forall\, c \geq 0$$
$$\langle T_{13}T_{11}\rangle_{(2,0,c)} = 0 \quad \forall\, c \geq 0$$

The invariant $\langle T_{13}, \textit{cod } 3\rangle_{(1,1,1)}$

Choosing generic representatives for the classes T_{10}, T_{11}, T_{12}, T_{13}, stable maps from reducible curves of class $(1,1,1)$ give no contribution because the expected dimension of $\overline{M}_{0,2}(\mathbf{H},(1,1,1))$ is 7 while reducible curves have less moduli. Then we restrict to study what happens on the component $\overline{M}_{0,2}(\mathbf{H},(1,1,1))^{irr}$ parametrizing maps from irreducible curves of class $(1,1,1)$, which is smooth of the expected dimension. Fix a generic point Z_0 of $\mathbf{H}$ representing T_{13} with Supp $Z_0 = \{p_0, q_0\}$.

Lemma 3.4.4. *If* $(ev_1, ev_2) : \overline{M}_{0,2}(\mathbf{H},(1,1,1))^{irr} \to \mathbf{H}\times\mathbf{H}$ *is the evaluation map and* $A = \{Z \in \mathbf{H} : l_Z \cap l_{Z_0} \neq \emptyset\}$. *Then* $[A] = (ev_2)_* ev_1^*[Z_0]$.

Proof. Let $(C, x_1, x_2, \mu) \in ev_1^{-1}(Z_0)$ with $Z_0 = \mu(x_1)$ and $Z_1 = \mu(x_2)$. The map μ is an isomorphism with the image curve $\Lambda(l)$, which is a line l in $\mathrm{Hilb}^2(\Lambda\cap Q)$ for Λ generic plane in $\mathbb{P}^3$. Since both Z_0 and Z_1 are in $\Lambda \cap Q$, $l_{Z_1} \cap l_{Z_0} \neq \emptyset$, because they lie on the same plane, then $ev_2(ev_1^{-1}Z_0) \subseteq A$. The set $ev_1^{-1}(Z_0)$ is 3-dimensional as well as A, in particular $[A] = T_3$. The map ev_2 has degree 1 over A, in fact given a generic point $Z \in A$, the lines l_{Z_0}, l_Z generate a unique plane Λ. It cuts a section $\Lambda\cap Q$ on Q and there is a unique line $l \subseteq \mathrm{Hilb}^2(\Lambda \cap Q)$ through Z_0, Z. A curve $\Lambda(l)$ with two markings is uniquely determined. Hence the fiber over Z consists of a unique point $[\mathbb{P}^1, x_1, x_2, \mu]$ where $\mu : \mathbb{P}^1 \to \mathbf{H}$ is an isomorphism with $\Lambda(l)$ such that $\mu^{-1}(Z_0) = x_1$, $\mu^{-1}(Z) = x_2$. Then $ev_2(ev_1^{-1}(Z_0))$ is 3-dimensional. This proves the lemma. □

Corollary 3.4.5. *For all* $T_e \in A^3(\mathbf{H})$*:*

$$\langle T_{13}T_e\rangle_{(1,1,1)} = \int_{\mathbf{H}} T_3 \cdot T_e$$

Proof. It follows from 3.4.4 and 2.7.1. □

The invariant $\langle T_{13}T_{10}\rangle_{(1,1,c)}$ for $c \geq 2$

We can fix generic representatives for the classes T_{13}, T_{10}, in particular $Z_0 = \{p_0, q_0\}$, $C(p_2, l_2)$ with $p_0 \notin l_2$ and $p_0, q_0 \neq p_2$.

If $c = 2$, all reducible curves of class

$$\begin{array}{ll} C_1 \cup C_2 \cup c_1 F \cup c_2 F & \text{with } c_1 + c_2 = 2 \\ (C_1 + F) \cup (C_2 + F) & \\ (C_1 + F) \cup C_2 \cup F & \\ (C_2 + F) \cup C_1 \cup F & \end{array}$$

can not intersect Z_0, $C(p_2, l_2)$ since they have less moduli then the expected dimension. Instead reducible curves of type $(C_1 + C_2 + F) \cup F$ give a contribution equal to 2. In fact both the markings have to lie on the $C_1 + C_2 + F$-component and the previous calculation showed there is exactly one stable map of such a class fulfilling the incident conditions. Moreover we have two possible choices for adding the F-component. Instead irreducible curves $\Lambda(p)$ of type $C_1 + C_2 + 2F$ give contribution zero because the intersection with Z_0 fixes $p = p_0$ and $q_0 \in \Lambda \cap Q$ while the second evaluation map imposes $p_0 \in l_2$. This is impossible because of the hypothesis of genericity. A similar argument shows that also irreducible curves $C(l_1, l_2, f)$ do not give any contribution to the invariant.

A curve of type $(1, 1, c)$ with $c \geq 3$ is necessarily reducible. There can be a contribution only from those not completely contained into $\Delta \cup \Sigma$, for dimensional reasons. By means of the vanishing result 2.8.4, it is easy to see that the GW invariant is equal to 1 for $c = 3$ and vanishes otherwise.

We have the following table:

	$c = 2$	$c = 3$	$c \geq 4$
$\langle T_{13} T_{10} \rangle_{(1,1,c)}$	2	1	0

Relations defining $QH_s^*(\mathbf{H})$

The relations f_i^* defining $QH_s^*(\mathbf{H})$ are:

$$T_3 * T_3 - (T_1 + T_2) * T_3 + T_1 * T_2 - \sum_{c \geq 1} 2 q_3^c (2T_1 T_2 + T_1^2 + T_2^2 - T_1 T_3 - T_2 T_3) = 0$$

$$T_1 * T_1 * T_1 - q_1 (T_3 - T_1) + 2 q_1 q_3 (2T_1 + T_2) + q_1 q_3^2 (T_1 + 2T_2 - T_3) = 0$$

$$T_2 * T_2 * T_2 - q_2 (T_3 - T_2) + 2 q_2 q_3 (T_1 + 2T_2) + q_2 q_3^2 (2T_1 + T_1 - T_3) = 0$$

$$T_1 * T_1 * T_2 - 2T_1 * T_4 = 0$$

$$T_1 * T_1 * T_3 - 2T_1 * T_4 - 2 q_1 q_3 (T_1 + T_3) - 2 q_1 q_3^2 (T_1 + 2T_2 - T_3) = 0$$

$$T_2 * T_2 * T_1 - 2T_2 * T_4 = 0$$

$$T_2 * T_2 * T_3 - 2T_2 * T_4 - 2q_2q_3(T_2 + T_3) - 2q_2q_3^2(2T_1 + T_2 - T_3) = 0$$

$$T_1 * T_2 * T_3 - 2T_3 * T_4 = 0$$

$$T_4 * T_4 * T_4 - 2q_1q_2q_3^2T_4 = 0$$

$$T_4 * T_4 * T_1 - 2q_1q_3T_2T_4 - q_1q_2q_3T_3 + 2q_1q_2q_3^2(2T_1 + T_2) + \\ -q_1q_2q_3^3(2T_1 + 2T_2 - T_3) = 0$$

$$T_4 * T_4 * T_2 - 2q_2q_3T_1T_4 - q_1q_2q_3T_3 + 2q_1q_2q_3^2(T_1 + 2T_2) + \\ -q_1q_2q_3^3(2T_1 + 2T_2 - T_3) = 0$$

$$T_4 * T_4 * T_3 - 2(q_1q_3T_2T_4 + q_2q_3T_1T_4) - q_1q_2q_3T_3 - 2q_1q_2q_3^2(2T_1 + 2T_2 - T_3) + \\ -3q_1q_2q_3^3(2T_1 + 2T_2 - T_3) = 0$$

$$T_1 * T_1 * T_4 - \tfrac{1}{2}q_1(T_2T_3 - T_1T_2) - q_1q_3(2T_1T_2 + T_2^2) - \tfrac{1}{2}q_1q_3^2(T_1T_2 + 2T_2^2 - T_2T_3) + \\ -q_1q_2q_3(1 + 2q_3)T_0 = 0$$

$$T_2 * T_2 * T_4 - \tfrac{1}{2}q_2(T_1T_3 - T_1T_2) - q_2q_3(2T_1T_2 + T_1^2) - \tfrac{1}{2}q_2q_3^2(T_1T_2 + 2T_1^2 - T_1T_3) + \\ -q_1q_2q_3(1 + 2q_3)T_0 = 0$$

$$T_1 * T_2 * T_4 - T_4 * T_4 - q_1q_3T_2^2 - q_2q_3T_1^2 - q_1q_2q_3(1 + 2q_3)T_0 = 0$$

$$T_1 * T_3 * T_4 - T_4 * T_4 - q_1q_3(T_1T_2 + T_2^2 + T_2T_3) - q_2q_3T_1^2 - q_1q_3^2(T_1T_2 + 2T_2^2 - T_2T_3) + \\ -q_1q_2q_3(1 + 4q_3 + 3q_3^2)T_0 = 0$$

$$T_2 * T_3 * T_4 - T_4 * T_4 - q_2q_3(T_1T_2 + T_1^2 + T_1T_3) - q_1q_3T_2^2 - q_2q_3^2(T_1T_2 + 2T_2^2 - T_1T_3) + \\ -q_1q_2q_3(1 + 4q_3 + 3q_3^2)T_0 = 0$$

In fact they satisfy the hypothesis of 3.3.3. In particular we can write:

$$QH_s^*(\mathbf{H}) = \frac{\mathbb{Q}[q_1, q_2, T_1, T_2, T_3, T_4][[q_3]]}{(f_i^*)_{i=1,\cdots,17}}$$

Remark 3.4.6. In the ring $QH_s^*(\mathbf{H})$ the identity $T_4^2 = T_{13}$ corresponds to:

$$T_4 * T_4 = T_{13} + 2q_1q_2q_3^2T_0$$

3.5 The subring generated by the divisor classes

We apply the First Reconstruction Theorem (FRT) to the subalgebra of the Chow ring of $\mathbf{H}$ generated by the divisor classes. So we can calculate all the tree level GW invariants which do not have T_4 among the arguments. Then we present a (partial) algorithm which permit us to compute (almost) all the genus zero GW invariants for $\mathbf{H}$.

Let $\mathbf{S}$ denote the subalgebra of $A^*(\mathbf{H})$ generated by T_1, T_2, T_3. All the classes in the fixed basis of $A^*(\mathbf{H})$ can be written as some product of the divisor classes except T_4. Then T_4 is not in $\mathbf{S}$.
We can consider the associated subring $Q\mathbf{S}$ in $QH^*(\mathbf{H})$ and apply FRT to it. It says we can compute all the genus zero GW invariants with arguments in $\mathbf{S}$ by knowing few initial values. These are determined as follows. Let $ev : \overline{M}_{0,2}(\mathbf{H}, \beta) \to \mathbf{H}^2$ be the usual evaluation map For $\gamma_1, \gamma_2 \in \mathbf{S}$ we have to calculate:

$$\int_{[\overline{M}_{0,2}(\mathbf{H},\beta)]^{vir}} ev^*(\gamma_1 \times \gamma_2)$$

Since cod $ev^*(\gamma_1 \times \gamma_2)$ has to be equal to $2a + 2b + 3$ and cod $\gamma_i \leq 4$ for $i = 1, 2$, we find the upper-bound $a + b \leq 2$. So we have to consider only the following cases:

β	$(0,0,c)$	$(1,0,c)$	$(0,1,c)$	$(1,1,c)$	$(2,0,c)$	$(0,2,c)$
$(\text{cod } \gamma_1, \text{cod } \gamma_2)$	$(1,2)$	$(1,4)$	$(1,4)$	$(3,4)$	$(3,4)$	$(3,4)$
		$(2,3)$	$(2,3)$			

In §2.7 and §3.3 we calculated some of these invariants. The left ones are obtained by means of the associativity applied to the equations f_i^*. Then we know all of them. This implies that we can calculate all the GW invariants on $\mathbf{H}$ without T_4 among the arguments.

An algorithm for the tree level GW invariants

To have a complete knowledge of the genus zero GW-system on $\mathbf{H}$ we need an algorithm computing invariants of type $\langle T_4^m \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta$, with $\gamma_i \in \mathbf{S}$ such that $4 \geq \deg \gamma_1 \geq \ldots \geq \deg \gamma_n \geq 2$.
We use equation (3.1) and by induction we suppose to know all the invariants:

$$\langle T_4^r \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta \quad \text{with } r < m$$

$$\langle T_4^r \gamma_1 \cdot \ldots \cdot \gamma_s \rangle_\beta \quad \text{with } r + s < m + n$$

$$\langle T_4^m \tilde{\gamma}_1 \cdot \ldots \cdot \tilde{\gamma}_n \rangle_\beta \quad \text{with } \deg \tilde{\gamma}_n < \deg \gamma_n$$

$$\langle T_4^m \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_{\beta'} \quad \text{with } \beta - \beta' > 0 \text{ effective}$$

If $m = 0$, there is no problem because each γ_i is in $\mathbf{S}$.
If $m = 1$ and $n = 0$, then we know $\langle T_4 \rangle_{(0,0,c)} = 0$, for all $c \geq 1$.

If $m = 1$ and $n = 1$, then γ_1 lives necessarily in codimension 3 and we have already calculated all the invariants in §3.3.
If $m = 1$ and $n \geq 2$, we use (3.1):

$$\sum \langle T_i \cdot T_j \cdot T_e \cdot \prod_{a \in A} \gamma_a \rangle_{\beta_1} \; g^{ef} \langle T_k \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b \rangle_{\beta_2} =$$
$$= \sum \langle T_i \cdot T_k \cdot T_e \cdot \prod_{a \in A} \gamma_a \rangle_{\beta_1} \; g^{ef} \langle T_j \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b \rangle_{\beta_2}$$

By induction, we know all the invariants with $\beta_i \neq 0$, $i = 1, 2$. We look only to the terms with either β_1 or β_2 equal to zero, i.e. on the left-hand side:

$$\underbrace{\langle T_i \cdot T_j \cdot T_k \cup T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_1} + \underbrace{\langle T_i \cup T_j \cdot T_k \cdot T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_2}$$

on the right-hand side:

$$\underbrace{\langle T_i \cdot T_k \cdot T_j \cup T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_3} + \underbrace{\langle T_i \cup T_k \cdot T_j \cdot T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_4}$$

Since $\gamma_i \in \mathbf{S}$, there exists a decomposition $\gamma_n = \alpha \cup \alpha_1$ with $\alpha_1 \in A^1(\mathbf{H})$ and deg $\alpha =$ deg $\gamma_n - 1$. We choose:

$$T_i = T_4, \quad T_j = \gamma_1, \quad T_k = \alpha, \quad T_l = \alpha_1, \quad R = \gamma_2 \cdot \ldots \cdot \gamma_{n-1}$$

Then I_1 is the value $\langle T_4 \gamma_1 \cdot \ldots \cdot \gamma_n \rangle_\beta$ we want to know (this will always be the case). Up to a scalar (possibly zero) I_2 is $\langle T_4 \cup \gamma_1 \cdot \alpha \cdot R \rangle_\beta$, all its arguments are in $\mathbf{S}$. Analogously I_4 is proportional to the known invariant $\langle T_4 \cup \alpha \cdot \gamma_1 \cdot R \rangle_\beta$. Finally in $I_3 = \langle T_4 \cdot \alpha \cdot \gamma_1 \cup \alpha_1 \cdot R \rangle_\beta$ the minimal degree decreased by one. Then we can write I_1 as a combination of lower degree terms. After a finite number of steps we can reduce our problem to the previous case with $n = 1$.
If $m \geq 2$ and $n = 1$, then we have three possibilities for cod γ_1. If cod $\gamma_1 = 4$, we can suppose $\gamma_1 = T_{13}$. We choose:

$$\begin{aligned} &T_k, T_l \in A^2(\mathbf{H}) \cap \mathbf{S} \text{ with } T_k \cup T_l = T_{13} \\ &T_i = T_j = T_4 \\ &R = T_4^{m-2} \end{aligned}$$

We obtain that in $I_2 = \langle T_4^{m-2} T_{13} T_k T_l \rangle_\beta$ we have a lower number of T_4's as well as in I_3 and I_4, since $T_4 \cup T_k$, $T_4 \cup T_l$ are in $\mathbf{S}$. We can reduce the problem to find $\langle T_4 T_{13} \gamma \rangle_\beta$, with $\gamma \in A^2(\mathbf{H}) \cap \mathbf{S}$, i.e. $m = 1$.

If cod $\gamma_1 = 3$, then we can decompose it as $\gamma_1 = \alpha \cup \alpha_1$, with $\alpha_1 \in A^1(\mathbf{H})$ as above. Fixing:

$$T_i = T_4, \quad T_j = T_4, \quad T_k = \alpha, \quad T_l = \alpha_1, \quad R = T_4^{m-2}$$

we get I_2 proportional to $\langle T_4^{m-2} T_{13} \alpha \rangle_\beta$, and we know it by induction. The invariant $I_3 = \langle T_4^{m-1} \alpha \cdot T_4 \cup \alpha_1 \rangle_\beta$ has less T_4-classes and the minimal degree is lower. Finally I_4 is proportional to $\langle T_4^{m-1} T_{13} \rangle_\beta$, then it is known.
If cod $\gamma_1 = 2$, we use the same trick with:

$$T_i = T_4, \quad T_j = T_4, \quad T_k = \alpha_1, \quad T_l = \alpha_2, \quad R = T_4^{m-2}$$

where α_1, α_2 are two divisors such that $\alpha_1 \cup \alpha_2 = \gamma_1$. Also in this case we can reduce our problem to the case $m = 1$.
If $m \geq 2$ and $n \geq 2$, then we write $\gamma_n = \alpha \cup \alpha_1$, $\alpha_1 \in A^1(\mathbf{H})$ and we choose:

$$T_i = T_4, \quad T_j = \gamma_1, \quad T_k = \alpha, \quad T_l = \alpha_1, \quad R = T_4^{m-1} \gamma_2 \cdot \ldots \cdot \gamma_{n-1}$$

Then I_2, I_4 are invariants with less T_4's and in I_3 the minimal degree is deg $\alpha =$ deg $\gamma_n - 1$. By induction we reduce to the case $n = 1$ or $m = 1$.

To complete the algorithm we need to find the invariants with $m \geq 2$ and $n = 0$. For dimensional reasons m has to be odd. At the moment we are not able to give a recursion formula to evaluate these invariants.

Chapter 4

Enumerative applications

We use the results on the Small Quantum Cohomology obtained in the previous chapter to count how many hyperelliptic curves on Q of given genus and bi-degree pass through a fixed number of generic points. Basically we reduce a question in higher genus to a question about rational curves on the Hilbert scheme $\mathbf{H}$, as in [Gr]. To do this we need to find a relationship between our hyperelliptic curves and some rational curves on $\mathbf{H}$.
With the word hyperelliptic we will mean an irreducible curve with a choice of hyperelliptic involution.

4.1 The moduli space of hyperelliptic curves mapping to Q

We start with two lemmas, a proof of the first one can be found in [Gr].

Lemma 4.1.1. *If $f : C \to \mathbb{P}^r$ is a morphism from a hyperelliptic curve such that it does not factor through the hyperelliptic map $\pi : C \to \mathbb{P}^1$ then $H^i(C, f^*\mathcal{O}(1)) = 0$ for all $i > 0$.*

A similar result holds for maps to Q.

Lemma 4.1.2. *Let $p_i : Q \to \mathbb{P}^1$ be the two projections and $\mu : C \to Q$ be a morphism from a hyperelliptic curve such that $\mu_i := p_i \circ \mu : C \to \mathbb{P}^1$, $i = 1, 2$, does not factor through the hyperelliptic map.*
Then $H^i(C, \mu^* T_Q) = 0$ for all $i > 0$.

Proof. Consider the Euler sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 2} \to T_{\mathbb{P}^1} \to 0$$

Since $T_Q = p_1^*(T_{\mathbb{P}^1}) \oplus p_2^*(T_{\mathbb{P}^1})$, a surjection is defined:

$$H^1(C, \mu^* p_1^* \mathcal{O}^{\oplus 2}(1) \oplus \mu^* p_2^* \mathcal{O}^{\oplus 2}(1)) \to H^1(C, \mu^* T_Q) \to 0$$

By hypothesis $H^j(C, \mu^* p_i^* \mathcal{O}^{\oplus 2}(1)) = 0$ for $j > 0$, so $H^j(C, \mu^* T_Q) = 0$. $\square$

Let $M_{g,0}(Q,(d_1,d_2))$ be the moduli space of maps $\mu : C \to Q$ from a smooth irreducible projective curve C of genus g such that $\mu_*[C] = (d_1, d_2)$. Let M_g be the moduli space of semistable projective curves of genus g. We denote by H_g the sub-locus parametrizing hyperelliptic curves. If C is hyperelliptic then the cyclic group of order 2 acts on the space of universal deformations $\mathcal{U}$ of C. It can be proved that the fixed locus $V \subseteq \mathcal{U}$ is the universal deformation space of C as a hyperelliptic curve and it is obviously smooth. It follows that $H_g \subseteq M_g$ is a smooth substack. The cartesian diagram:

$$\begin{array}{ccc} \tilde{H}_g(Q,(d_1,d_2)) & \longrightarrow & H_g \\ \downarrow & & \downarrow \\ M_g(Q,(d_1,d_2)) & \longrightarrow & M_g \end{array}$$

defines the space $\tilde{H}_g(Q,(d_1,d_2))$ parametrizing maps $\mu : C \to Q$ from a hyperelliptic curve C of genus g with $\mu_*[C] = (d_1,d_2)$. We are interested in the open subset $H_g(Q,(d_1,d_2))$ of maps μ such that the composition maps $\mu_i = p_i \circ \mu : C \to \mathbb{P}^1$, $i = 1,2$ do not factor through the hyperelliptic map.

Theorem 4.1.3. *The natural morphism* $\nu : H_g(Q,(d_1,d_2)) \to H_g$ *is smooth.*

Proof. It follows from the vanishing result 4.1.2; for each $\mu : C \to Q$ in $H_g(Q,(d_1,d_2))$, we have $H^1(C,\mu^*T_Q) = 0$. Then by theorem 2.2.1, the forgetful morphism $\overline{M}_{g,0}(Q,(d_1,d_2)) \to \mathfrak{M}_g$ is smooth in $[\mu]$. Since smoothness is a local property, the theorem follows. □

Corollary 4.1.4. $H_g(Q,d)$ *is smooth and irreducible.*

Proof. Smoothness is a direct consequence of the theorem, since both H_g and ν are smooth.
Since H_g is irreducible, it is enough to prove the fibers of ν are irreducible of constant dimension. A fiber $\nu^{-1}(C)$ is the set of all $\mu : C \to Q$ of bidegree (d_1,d_2) such that both μ_1, μ_2 do not factor through the hyperelliptic map. They are two morphisms to the projective line, so they correspond to two line bundles on C of degree d_1, d_2 respectively. We get a morphism $f = (f_1, f_2) : \nu^{-1}(C) \to \mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C)$. By Lemma 4.1.1 $Im(f_i)$ is a subset of $\{\mathcal{L}_i : \mathcal{L}_i \text{ is spanned}, h^1(\mathcal{L}_i) = 0\}$. Conversely, for $i = 1,2$, let W_i be the subset of $\mathrm{Pic}^{d_i}(C)$ of sheaves $\mathcal{L}_i$ such that $\mathcal{L}_i$ is spanned, $h^1(\mathcal{L}_i) = 0$ and $\mathcal{L}_i$ is not a multiple of g^1_2. Then each $\mathcal{L}_i \in W_i$ is in the image $Im(f_i)$. W_i is open and dense (if not empty), because $\mathrm{Pic}^{d_i}(C)$ is irreducible. Hence $Im(f_i)$ contains the open subset W_i and therefore it is irreducible (because W_i is). It follows that $Im(f)$ is irreducible of dimension $2g$. Each fiber $f^{-1}(\mathcal{L}_1, \mathcal{L}_2)$, $\mathcal{L}_i \in W_i$, is a product $V_1 \times V_2$, where V_i is the open set of pairs of global sections (s_i^1, s_i^2) of $\mathcal{L}_i$ without common zeros, modulo scalars.

Hence these fibers are irreducible and they have the same dimension equal to $2(d_1+d_2)-2g$, because the first cohomology of $\mathcal{L}_i$ vanishes. Therefore $\nu^{-1}(C)$ is irreducible of dimension $2(d_1+d_2)$. □

4.2 The basic correspondence

An element in $H_g(Q,(d_1,d_2))$ is a diagram:

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\pi"', "2:1"] & Q \\ \mathbb{P}^1 & \end{tikzcd}$$

where π is the hyperelliptic map and $\mu_*[C]$ has bi-degree (d_1,d_2) on Q. Define $\alpha : C \to \mathbb{P}^1 \times Q$ by $\alpha(p) = (\pi(p), \mu(p))$. Then $\mathcal{Z} = Im(\alpha)$ is closed because α is proper and it is irreducible because α is regular. It comes with a natural map $pr_1 : \mathcal{Z} \to \mathbb{P}^1$ which is a flat morphism, by [Har] Chap. III Prop. 9.7. The generical fiber of pr_1 is a set with two distinct points, that is to say pr_1 is a flat family over $\mathbb{P}^1$ with fibers subschemes of Q of dimension zero and length two. By the universal property of $\mathbf{H}$ there exists a unique morphism g making the following diagram cartesian:

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, "pr_1"'] & \mathcal{U} \arrow[r] \arrow[d, "u"] & Q \\ \mathbb{P}^1 \arrow[r, "g"'] & \mathbf{H} & \end{tikzcd}$$

where $\mathcal{U}$ is the universal family over $\mathbf{H}$.
So we associate to μ a morphism $g : \mathbb{P}^1 \to \mathbf{H}$ canonically. This is well defined for each point $[C,\mu] \in H_g(Q,(d_1,d_2))$.
Conversely, given a map $g : \mathbb{P}^1 \to \mathbf{H}$ we can pull it back via u:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\pi"'] \arrow[rr, bend left, "\mu"] & \mathcal{U} \arrow[r] \arrow[d, "2:1"', "u"] & Q \\ \mathbb{P}^1 \arrow[r, "g"'] & \mathbf{H} & \end{tikzcd}$$

where $\mathcal{C} = \mathcal{U} \times_{\mathbf{H}} \mathbb{P}^1$ and π is a $2:1$ flat morphism. Then we get a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mu"] \arrow[d, "\pi"', "2:1"] & Q \\ \mathbb{P}^1 & \end{tikzcd}$$

If $g(\mathbb{P}^1) \subseteq \Delta$, then $\mathcal{C}$ is not a hyperelliptic curve because it would not be reduced. If $g(\mathbb{P}^1)$ meets Δ transversally, then $\mathcal{C}$ is a smooth hyperelliptic curve. Intersection points $g(\mathbb{P}^1) \cap \Delta$ correspond to branch points for the hyperelliptic map $\pi : \mathcal{C} \to \mathbb{P}^1$, because Δ is the branch locus of u.
The genus of $\mathcal{C}$ is given by the Hurwitz formula:

$$2g_{\mathcal{C}} - 2 = \deg\, \pi \cdot (2g_{\mathbb{P}^1} - 2) + [g(\mathbb{P}^1)] \cdot \Delta$$

Let $g(\mathbb{P}^1)$ be a curve of class (a, b, c), then $g_{\mathcal{C}} = a + b - c - 1$.

Remark 4.2.1. Since the genus is non-negative, $a + b > c$.

Finally we calculate the bi-degree of $\mu(\mathcal{C})$. It is given by the intersection of $\mu(\mathcal{C})$ with the generic conic $(1, 0) + (0, 1)$ on Q. It corresponds to the intersection product of the cycle class (a, b, c) with the two divisors T_1, T_2 of $\mathbf{H}$:

$$d_1 = (a, b, c) \cdot T_1 = b$$
$$d_2 = (a, b, c) \cdot T_2 = a$$

Remark 4.2.2. Since $[g(\mathbb{P}^1)] \cdot \Delta = 2(g_{\mathcal{C}} + 1) > 0$ there is ramification for the map π and $\mathcal{C}$ is connected.

Then a hyperelliptic curve C on Q of genus g and bi-degree (d_1, d_2) is represented by a rational curve in $\mathbf{H}$ of class $(d_2, d_1, d_1 + d_2 - g - 1)$. We will make this sentence more rigorous after fixing some more notation.
Let us consider particular rational curves, those parametrized by the open subset $M^{\mathrm{tr}}_{0,0}(\mathbf{H}, \beta) \subseteq M_{0,0}(\mathbf{H}, \beta)$ of maps from irreducible rational curves fulfilling:

i. they intersect Δ transversally

ii. they are not contained in Σ

iii. they are disjoint from Δ_2

Let $H^{\mathrm{tr}}_g(Q, d) \subseteq H_g(Q, d)$ be the open subset parametrizing maps μ such that:

a) C is a smooth hyperelliptic curve ($\leftrightsquigarrow$ i.)

b) both μ_i do not factor through π ($\leftrightsquigarrow$ ii.)

c) both differentials $d\mu_i$ are injective on ramification points of π ($\leftrightsquigarrow$ iii.).

Remark 4.2.3. Conditions defining $H^{\mathrm{tr}}_g(Q, d)$ are equivalent to say that $\alpha : C \to \mathbb{P}^1 \times Q$ is an embedding, i.e. $\mathcal{Z}$ is closed, reduced and irreducible.

Theorem 4.2.4. *There is a canonical isomorphism:*

$$H^{tr}_g(Q, (d_1, d_2)) \cong M^{tr}_{0,0}(\mathbf{H}, (d_2, d_1, d_1 + d_2 - g - 1))$$

Proof. The proof of theorem 2.4 in [Gr] never makes use of the fact that the curves are in $\mathbb{P}^2$, then it works also for hyperrelliptic curves on Q. $\square$

4.3 The main theorem

By what we have showed so far, to count hyperelliptic curves on Q of bidegree (d_1, d_2) and genus g passing through $r = 2d_1 + 2d_2 + 1$ general points is equivalent to count irreducible rational curves of type $(d_2, d_1, d_1 + d_2 - g - 1)$ in $\mathbf{H}$ which are transversal to Δ and meet r general translates of $\Gamma(p)$, $p \in Q$. So we might expect a relationship between the number we want to count and the Gromov-Witten invariants $\langle T_4^r \rangle_\beta$. In general, the moduli space $\overline{M}_{0,r}(\mathbf{H}, (d_2, d_1, d_1 + d_2 - g - 1))$ can have some components whose general element corresponds to a reducible curve, moreover these components can have dimension as large or larger than the expected one. Then there can be undesired contributions to the number we want to find. The following theorem gives us a picture of the curves in $\mathbf{H}$ we are counting.

Theorem 4.3.1. *Fix an effective class $\beta = (a, b, c) \in A_1(\mathbf{H})$, $a + b \geq 1$, and r general points $p_1, \ldots, p_r$ on Q with:*

$$r = 2a + 2b + 1$$

Then:

1. *there exists at most a finite number of irreducible rational curves of class β incident to all the cycles $\Gamma(p_i)$;*

2. *all such curves intersect $\Delta \cup \Sigma$ in points disjoint from the $\Gamma(p_i)$;*

3. *given any arbitrary stable map $\mu : C \to \mathbf{H}$ of class β incident to all the cycles $\Gamma(p_i)$, then C has a unique irreducible component which is not entirely mapped into $\Delta \cup \Sigma$, such a component is of class (a, b, c_0), where $c_0 \leq c$.*

Consequences : the theorem tells us that given a stable map $\mu : C \to \mathbf{H}$ satisfying all incident conditions, aside from the distinguished component of C of class (a, b, c_0), all other components are of type $(0, 0, c')$ and they are entirely mapped into Δ. So they are multiple covers of $\mathbb{P}^1$. Moreover, adding a component of type $(0, 0, c')$ to a stable map can never cause it to be incident to any extra $\Gamma(q)$, since it would force another component of the curve to meet the corresponding cycle. Finally, different $(0, 0, c')$ components are disjoint, since they are different fibers of the support map s, then they must be incident to the distinguished component, C been connected.
We conclude that the source curve looks like a comb, with the component of class (a, b, c_0) as the handle and the components of class $(0, 0, c')$ as the teeth. We get exactly the same picture obtained in [Gr].
There is a finite number of such curves. Infact, if C is irreducible, then the theorem confirms our assertion. If C is reducible, we have only a finite number of possibilities for the multiple covers of a $(0, 0, 1)$-curve and only a

finite number of points of intersection of the distinguished component with Δ. So there are only finitely many potential image curves for stable maps incident to all of the cycles. In particular, if we denote by A the locus in $\overline{M}_{0,0}(\mathbf{H},(a,b,c))$ defined by $\pi(ev_1^{-1}\Gamma(p_1)\cap\ldots\cap ev_r^{-1}\Gamma(p_r))$, where π is the usual map forgetting the markings (and stabilizing) and $ev=(ev_1,\ldots,ev_r)$ is the evaluation map on $\mathbf{H}^r$, then A is a union of finitely many components. In fact the theorem says that the only moduli in the choice of a stable map meeting all the $\Gamma(p_i)$ comes from the choice of multiple covers of the $(0,0,1)$ curve. Then as a set, each component of A decomposes as a product:

$$M(c_1)\times M(c_2)\times\ldots\times M(c_m)$$

with $c_1+\ldots+c_m=c$ and $M(c_i)$ as in §2.6. In paricular A is contained in the smooth locus of $\overline{M}_{0,0}(\mathbf{H},(a,b,c))$.

Before going on with the proof of the theorem we need a lemma.

Lemma 4.3.2. *With notations as in the theorem, let C be an irreducible rational curve meeting all the cycles $\Gamma(p_i)$ and the orbit Σ_4. Then it intersects $\Delta\cup\Sigma$ in points disjoint from all the $\Gamma(p_i)$.*

Proof. Let $r=2a+2b+1$ and $M\subseteq\overline{M}_{0,r}(\mathbf{H},(a,b,c))$ be the open subset of points $[C,\mu,x_j]_{j=1,\ldots,r}$ such that $C\cong\mathbb{P}^1$, $\mu(C)\cap\Sigma_4\neq\emptyset$. It is smooth of dimension $2r$. The map $M\to\overline{M}_{0,0}(\mathbf{H},(a,b,c))$ which forgets the markings and stabilizes factors through:

$$M\xrightarrow{\pi_i}\overline{M}_{0,1}(\mathbf{H},(a,b,c))\longrightarrow\overline{M}_{0,0}(\mathbf{H},(a,b,c))$$

where π_i is the map forgetting all the markings but x_i and stabilizing. It is surjective onto its image $\mathrm{Im}(\pi_i)=\mathcal{U}_1$ which is the universal curve over the smooth locus $\mathcal{U}_0$ of $\overline{M}_{0,0}(\mathbf{H},(a,b,c))$. Then $\pi_i:M\to\mathcal{U}_1$ is flat of relative dimension $r-1$. The set $N=\{[C,\mu,x]:\mu(x)\in\Delta\cup\Sigma\}$ is a closed subset of $\mathcal{U}_1$, as it is the inverse image $ev^{-1}(\Delta\cup\Sigma)$. Its complementary $\mathcal{U}_1\backslash N$ is open and intersects all the 1-dimensional fibers of $\mathcal{U}_1\to\mathcal{U}_0$, then it is dense. This implies that N is a proper closed subset, equivalently it has dimension lower than $r+1$. Moreover the inverse image $M_i=\pi_i^{-1}(N)$ has dimension dim $M_i<$ dim $M=2r$ because also the restricted map $\pi_i:M_i\to N$ is flat of relative dimension $r-1$. Let $\tilde{M}_i$ be the resolution of singularities of M_i. It has the same dimension as M_i. Set $\Gamma=\prod_{i=1}^r\Gamma(p_i)$, for generic fixed points $p_1,\ldots,p_r\in Q$ and consider the inverse image of Γ in $\tilde{M}_i$ via the evaluation map, i.e. the composition:

$$ev_i:\tilde{M}_i\twoheadrightarrow M_i\xrightarrow{ev}\mathbf{H}^r$$

We apply the Position Lemma to ev_i with the group $\mathcal{A}_0$, the connected component of $\mathcal{A}$ containing the identity, acting on $\mathbf{H}$. Then $ev_i^{-1}(\Gamma)$ has pure dimension equal to dim M_i $-$ cod$(\Gamma\subseteq\mathbf{H}^r)<0$, that is to say it is empty. In particular $ev^{-1}(\Gamma)\cap M_i=\emptyset$. □

Remark 4.3.3. Note that $\mathbf{H}^r$ is almost homogeneous because there is a group acting on each factor and globally we have a finite number of orbits.

Proof of the Main Theorem and applications

We are ready to give a proof of the theorem. We will use induction on the number of components of the source curve C and we will apply the Position Lemma with respect to the action of $\mathcal{A}_0$ on $\mathbf{H}$.

Proof. **STEP 1.** Let C be a rational, irreducible curve with r markings and let $\mu : C \to \mathbf{H}$ be a stable map of class $\beta = (a, b, c)$ such that $\mu(C) \nsubseteq \Delta \cup \Sigma$. Set $\Gamma = \prod_{i=1}^{r} \Gamma(p_i)$. Note that the moduli space $\overline{M}_{0,r}(\mathbf{H}, \beta)$ is smooth in $[C, \mu]$ of the expected dimension. Then we can consider the restriction of the evaluation map to the smooth open subset:

$$M = \{[C, \mu, x_j] : C \cong \mathbb{P}^1,\ \mu(C) \nsubseteq \Delta \cup \Sigma\} \subseteq \overline{M}_{0,r}(\mathbf{H}, \beta)$$

We can apply the Position Lemma:

$$\begin{tikzcd} ev^{-1}(\Gamma) \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ M \arrow[r, "ev"'] & \mathbf{H}^r \end{tikzcd}$$

It follows that dim $ev^{-1}(\Gamma) = 0$ since M is of the expected dimension $2r$.
STEP 2. Suppose that C is irreducible and $\mu(C) \subseteq \Delta$, then in Δ we have $\mu_*[C] = \tilde{\beta} = (\frac{a}{2}, \frac{b}{2}, c)$ by what we showed in §1.5. Let $a' = \frac{a}{2}, b' = \frac{b}{2}$. We have a curve D of class $a'[L_1] + b'[L_2] + c[L_3]$ in Δ and we can consider its projection to Q. The image B is a curve of genus zero and of bi-degree (a', b') on Q. In fact L_i maps to l_i for $i = 1, 2$ and L_3 maps to a point. The cycles $\Gamma(p_i)$ restricted to Δ have codimension 2 and the curve D is incident to all of them if and only if the image curve B goes through all the points p_i. A rational curve on Q of bi-degree (a', b') passes through at most s generical points of Q, where:

$$2s = \dim Q + \int_{(a',b')} c_1(T_Q) - 3 + s \Rightarrow s = 2a' + 2b' - 1 = a + b - 1$$

We have $s < r = 2a + 2b + 1$, so the irreducible curves $\mu(C) \subseteq \Delta$ give no contribution to our calculations.
If C is reducible and $\mu(C) \subseteq \Delta$ then we can write $C = C_1 \cup \ldots \cup C_k$. Every irreducible component C_j is such that $\mu(C_j)$ meets at most $s_j = a_j + b_j - 1$ cycles $\Gamma(p_i)$, where $\sum a_i = a, \sum b_i = b$. This means that $\mu(C)$ intersects at most $\sum s_i = a + b - k < r$ points. Also these curves give no contributions to our calculations.

STEP 3. Now we analyse the contribution from irreducible rational curves C such that $\mu(C) \subseteq \Sigma$. Since Σ is the disjoint union $\tilde{W}_1 \sqcup \tilde{W}_2$ and $\mu(C)$ is irreducible, it is enough to consider the case $\mu(C) \subseteq \tilde{W}_1$. The pushforward class $\mu_*[C]$ in $\tilde{W}_1$ is $(a, \frac{b}{2})$ with b even. Denote by φ_1^r the map $\tilde{W}_1^r \xrightarrow{\varphi_1 \times \ldots \times \varphi_1} W_1^r$ induced by the blowup $\mathbf{H} \to \mathbf{G}$. Then we have a composition map:

$$\overline{M}_{0,r}(\tilde{W}_1, (a, b/2)) \xrightarrow{ev} \tilde{W}_1^r \xrightarrow{\varphi_1^r} W_1^r \subseteq \mathbf{G}^r$$

If a curve of class $(a, b/2)$ intersects all the cycles $\Gamma(p_i)$ then its image via φ_1 is of class $(\varphi_1)_*(a, b/2) = \frac{b}{2}[W_1] = b[\sigma_{2,1}]$ because W_1 is a quadric in $\mathbf{G}$, and it goes through all the points $l_1(p_i) \in \mathbf{G}$. Such a curve passes through at most s fixed points in $\mathbf{G}$, with s given by the formula:

$$4s = \dim \mathbf{G} + \int_{b[\sigma_{2,1}]} c_1(T_{\mathbf{G}}) - 3 + s \Rightarrow s = \frac{1 + 4b}{3}$$

Since $s < r$ we verify that irreducible curves mapped into Σ give no contribution to our computation.
Suppose that C is the union of k irreducible components and $\mu(C) \subseteq \Sigma$. Since $\tilde{W}_1, \tilde{W}_2$ are disjoint, if an irreducible component is mapped into $\tilde{W}_i$ then all the components are actually mapped into the same divisor $\tilde{W}_i$, by connectedness. We can assume $\mu(C) \subseteq \tilde{W}_1$. The number k of components is bounded. In fact $\mu_*[C] = (a, b, b)$ in $\mathbf{H}$, with b even, then k is at most equal to $a + \frac{b}{2}$. This implies that $\mu(C)$ goes through at most $s = \frac{k+4b}{3} \leq \frac{2a+9b}{6}$ cycles. We get $s < r$ also in this case.
Lemma 4.3.2 conclude the proof of 1.-2.
STEP 4. Suppose C is reducible and $\mu(C) \subseteq \Delta \cup \Sigma$. In particular assume that C has k irreducible components C_i such that:

$$\begin{array}{ll} C_i \subseteq \Delta_3 & \text{for } 1 \leq i \leq k_1 \\ C_i \subseteq \tilde{W}_1 & \text{for } k_1 + 1 \leq i \leq k_2 \\ C_i \subseteq \tilde{W}_2 & \text{for } k_2 + 1 \leq i \leq k \end{array}$$

We fix the notations:

$$\begin{array}{ll} D_1 = \bigcup_{i=1}^{k_1} C_i & \text{is of class } (a_1, b_1, c_1) \\ \\ D_2 = \bigcup_{i=k_1+1}^{k_2} C_i & \text{is of class } (a_2, b_2, c_2) \\ \\ D_3 = \bigcup_{i=k_2+1}^{k} C_i & \text{is of class } (a_3, b_3, c_3) \end{array}$$

The conditions $\sum a_j = a$, $\sum b_j = b$, $\sum c_j = c$ hold. The image curve $\mu(D_j)$ intersects r_j cycles. By the previous results we know that $r_j \leq 2a_j + 2b_j$ for all j then $r_1 + r_2 + r_3 \leq 2a + 2b < r$. The curve C does not intersects all the cycles $\Gamma(p_i)$.

STEP 5. Note that a point $[C, \mu, x_j]$ such that $C = \bigcup C_i$ and each C_i intersects the dense orbit Σ_4 lives in the smooth locus of $\overline{M}_{0,r}(\mathbf{H}, (a, b, c))$. The subset R parametrizing all such stable maps is a proper closed subset of the smooth locus, then it has dimension lower than $2r$. Then for generic points p_i the intersection $R \cap ev^{-1}(\Gamma)$ is empty.
We have to analyse only the contribution from stable maps $\mu : C \to \mathbf{H}$ with rational reducible domain and $\mu(C) \nsubseteq \Delta \cup \Sigma$ but such that there is excess at the point $[C, \mu, x_i]$, i.e. there exists at least one component of C mapped into $\Delta \cup \Sigma$. We can write $C = C_0 \cup C_1$ with $C_0 \cap C_1 = \{p\}$ a point mapped in $\Delta \cup \Sigma$. Set $[\mu(C_i)] = (a_i, b_i, c_i)$ in $\mathbf{H}$, with $\sum a_i = a$, $\sum b_i = b$, $\sum c_i = c$. Suppose $(a_i, b_i) \neq (0, 0)$ for $i = 0, 1$ and let $\mu(C)$ be incident to all the cycles $\Gamma(p_i)$. We know that $\mu(C_i)$ intersects $r_i = 2a_i + 2b_i + 1 - k_i$ cycles, with $k_0 + k_1 \leq 1$. We can assume $r_0 = 2a_0 + 2b_0 + 1$ and $r_1 = 2a_1 + 2b_1$. By induction, C_0 has a unique irreducible component of class $(a_0, b_0, \bar{c}_0)$, with $\bar{c}_0 \leq c_0$, not entirely mapped into $\Delta \cup \Sigma$ and intersecting all the cycles. All the other components are of class $(0, 0, c_j)$. There is only a finite number of possible values for $\bar{c}_0$ then there are finitely many possible images of C_0. In particular $\mu(C_0) \cap (\Delta \cup \Sigma)$ can be contained in a finite number of cycles of the form $\Gamma(q_j)$ with $q_j \notin \{p_1, \dots, p_r\}$, by statement 2. The curve $\mu(C_1)$ meets at least one of the $\Gamma(q_j)$, because $\mu(p) \in \Delta \cup \Sigma$. So it intersects $r_1 + 1$ cycles and a point of intersection is in $\Delta \cup \Sigma$. This is impossible. Then $a_1 = b_1 = 0$. □

We can write explicitly the relationship between Gromov-Witten invariants on $\mathbf{H}$ and enumerative geometry of hyperelliptic curves on Q.

Definition 4.3.4. *Let $E((d_1, d_2), g)$ be the number of hyperelliptic curves of genus g and bi-degree (d_1, d_2) on Q passing through $2d_1 + 2d_2 + 1 = r$ general points, counted with multiplicity.*

Theorem 4.3.5. *With $\beta = (d_2, d_1, d_1 + d_2 - g - 1)$ and r as above, the enumerative numbers $E((d_1, d_2), g)$ satisfy the equation:*

$$\langle T_4^r \rangle_\beta = \sum_{h \geq g} \binom{2h+2}{h-g} E((d_1, d_2), h) \tag{4.1}$$

Proof. We write $\beta = (a, b, c)$ where $a = d_2, b = d_1, c = d_1 + d_2 - g - 1$.
Fixed r general points $p_1, \dots, p_r$ the invariant $\langle T_4^r \rangle_\beta$ is given by the degree $\deg(ev^{-1}(\prod \Gamma(p_i))$ where $ev^{-1}(\prod \Gamma(p_i))$ is a finite set of points by 4.3.1. Each of them corresponds to a hyperelliptic curve which then comes with a multiplicity. By the results of section 2.6, the only contribution to the invariant comes from the zero dimensional component of the moduli space $\overline{M}_{0,r}(\mathbf{H}, \beta)$ corresponding to stable maps from curves which look like combs. These are the union of an irreducible (a, b, c_0)-curve, $c_0 \leq c$, incident to all the cycles $\Gamma(p_i)$ with $c - c_0$ rational curves mapping isomorphically onto a

$(0,0,1)$-curve. Hence the number of stable maps is equal to the number of possible irreducible curves of class (a,b,c_0) times the number of choices for the attachment points of the $(0,0,1)$-curves. We have to choose $c-c_0$ points among the $2(a+b-c_0)$ ones in the intersection $(a,b,c_0)\cdot\Delta$. The formula then follows from the relationship between (a,b,c) and (d_1,d_2,g). □

Remark 4.3.6. We expect that for generic data the hyperelliptic curves of given genus and bi-degree passing through r points have always multiplicity equal to one. This is equivalent to the condition that a general irreducible rational curve in $\mathbf{H}$ intersects the stratification trasversally. At the moment we can only partially prove such a statement. Roughly speaking the idea is that given a stable map $\mu : C \to \mathbf{H}$ such that $\mu(C)$ is as in the hypothesis, the sheaf $\mu^*(T_{\mathbf{H}})$ is generated by global sections. We can move the curve away from any 2-dimensional closed orbit Δ_2^i and make it intersect transversally any 3-dimensional orbit. So let M be the smooth open subset of $\overline{M}_{0,0}(\mathbf{H},(a,b,c))$ of the expected dimension $r=2a+2b+1$, parametrizing the stable maps $\mu : C \to \mathbf{H}$ such that $C \cong \mathbb{P}^1$ and $\mu(C)\cap\Sigma_4 \neq \emptyset$. We denote by N_i the closed subsets of M defined by the condition $\mu(C)\cap\Delta_2^i \neq \emptyset$, $i=1,2$. We need to prove that each N_i has dimension lower than r, i.e. given a point in N_i there exists a deformation of it which is not in N_i. Each map $\mu \in M$ is a free morphism in the sense of [K], (Chap. II, Def. 3.1) and each orbit Δ_2^i is of codimension 2 in $\mathbf{H}$, then we can apply [K] Proposition II.3.7 and conclude that N_i is a closed proper subset. Transversality for the intersection with the orbits Δ_3 and Σ_3 is more subtle (see [P]).

Remark 4.3.7. We note that the sum in 4.3.5 is finite, in fact the values of h are equal to $d_1+d_2-c_0-1$ with $c_0 \le c$. We can recover the values $E((d_1,d_2),g)$ by knowledge of all the GW invariants $\langle T_4^r\rangle_{(d_2,d_1,d_1+d_2-g-1)}$.

Remark 4.3.8. At the moment we do not know how to compute all the invariants $\langle T_4^r\rangle_{(d_2,d_1,d_1+d_2-g-1)}$, (see §3.5).

Remark 4.3.9. The numbers $E((d_1,d_2),g)$ are zero for small values of d_1,d_2,g. In fact $E((d_1,d_2),g)$ is less then or equal to $S((d_1,d_1),g)$, the number of smooth curves of bi-degree (d_1,d_2) of genus g passing through r points. $S((d_1,d_1),g)$ is zero if $d_1d_2-d_1-d_2-1<0$. Then the first possibly nonzero GW invariants are $\langle T_4^{11}\rangle_{(3,2,2)}$ and $\langle T_4^{11}\rangle_{(2,3,2)}$.

We can also extend [Gr] Theorem 3.7 to our case, as follows. Fix k general points p_i on Q and l general pairs of points q_j, q_j' with $k+3l=r$, r as in the previous theorem. We want to count how many hyperelliptic curves on Q of genus g and bi-degree (d_1,d_2) pass through all the points and satisfy also the condition that for some choice of the hyperelliptic involution q_i is hyperelliptically conjugate to q_i' for all i. Let $E^l((d_1,d_2),g)$ be the solution of this problem. A hyperelliptic curve on Q will meet hyperelliptically conjugated points q,q' if and only if the corresponding rational curve on $\mathbf{H}$ will meet the point Z with support $\{q, q'\}$. Choosing a representative of the point class T_{13} outside $\Delta \cup \Sigma$, curves moving in excess dimension cannot satisfy this condition, so theorem 4.3.1 is true also for cycles representing T_{13}. The same argument used in the proof of [Gr] Thm.3.7. will hold for the Gromov-Witten invariants involving the point cycle class. In particular:

Theorem 4.3.10. *With notations as in theorem 4.3.5:*

$$\langle T_{13}^{l} \cdot T_{4}^{r-3l} \rangle_{\beta} = \sum_{h \geq g} \binom{2h+2}{h-g} E^{l}((d_1, d_2), h) \tag{4.2}$$

Remark 4.3.11. Also in this case the sum over h is finite. By what we showed in §3.5 if $l \geq 1$ then we can compute all the invariants $\langle T_{13}^{l} \cdot T_{4}^{r-3l} \rangle_{\beta}$. Therefore we can invert the formula (4.2) to get the numbers $E^{l}((d_1, d_2), h)$.

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