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On Finiteness Properties of Local Cohomology Modules

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Al Catria e al Cesano,
A coloro che vi hanno abitato.

To Catria and to Cesano,
To those who have lived there.

Contents

Introduction

Local cohomology was introduced by Grothendieck in the early 1960s, in part to answer a conjecture of Pierre Samuel about when certain types of commutative rings are unique factorization domains. In the area of algebraic geometry, local cohomology modules arise in a natural way. Indeed, given a function defined on an open subset of an algebraic variety, they measure the obstruction to extending that function to a larger domain. But in addition to this, local cohomology allows us to answer many seemingly difficult problems such as the minimum number of defining equations of algebraic sets or their connectedness properties.

There are many equivalent ways to define local cohomology. From the homological algebra perspective, one of the easiest definitions is the following: given an ideal $\mathfrak{a}$ in a commutative ring, for each module we consider the submodule of elements annihilated by some power of $\mathfrak{a}$. This operation is functorial but is not exact and so the theory of derived functors leads us to define the local cohomology functors $H^i_{\mathfrak{a}}(\_)$, which measure the failure of this exactness. Typically, local cohomology modules are not finitely generated, and in this sense they may seem "big" and difficult to work with. However, they frequently have properties that make them manageable objects of study. Our main interest is to study their finiteness properties.

In the first chapter we will study the local cohomology functors for modules over Noetherian local rings. In this context, it turns out that all the local cohomology modules with respect to the maximal ideal $H^i_{\mathfrak{m}}(M)$ are Artinian when the module M is finitely generated [Theorem 1.38].

The main problem we consider in the following is how to extend this result to the non-local case and to non-maximal ideals: local cohomology modules are torsion, and for $\mathfrak{m}$-torsion modules N over Noetherian local rings, one may show that being Artinian is equivalent to $\mathrm{Ext}^i_R(R/\mathfrak{m}, N)$ being finitely generated for all i [Theorem 1.39]. Modules which satisfies this latter property with respect to an ideal $\mathfrak{a}$ are called $\mathfrak{a}$-cofinite modules, and the above led Grothendieck to conjecture that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite whenever M is a finitely generated module over a Noetherian ring. Hartshorne showed in [Har70] that this conjecture is false, and was able to construct a counterexample. At the end of the chapter we will present this counterexample to Grothendieck's conjecture, and in the remaining of this thesis we will describe a way to correct it, using the language of derived categories, whose construction and basic properties will be recalled in the second chapter.

The idea is to move our set-up from the category of modules over a ring $\mathbf{Mod}(\mathrm{R})$ to its derived category $\mathbf{D}(\mathrm{R})$ and instead of considering the classical derived functors $H^i_{\mathfrak{a}}(M)$, we will consider the total right derived functor $\mathrm{R}\Gamma_{\mathfrak{a}}(M)$, which is a complex whose i^{th} cohomology is equal to $H^i_{\mathfrak{a}}(M)$. In this new setup we will define the triangulated subcategory of cohomologically $\mathfrak{a}$-adically complete complexes $\mathbf{D}(R)_{\mathfrak{a}-com}$ and the one of cohomologically $\mathfrak{a}$-torsion complexes $\mathbf{D}(R)_{\mathfrak{a}-tor}$. In the section dedicated to the MGM equivalence we will show that the total right derived functor $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ is actually an equivalence between these two triangulated categories. Then, we will define cohomologically $\mathfrak{a}$-adically cofinite complexes as bounded complexes which are images through the functor $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ of bounded complexes with finitely generated cohomologies. Since these latter complexes are cohomologically $\mathfrak{a}$-adically complete, it follows that the cohomologically $\mathfrak{a}$-adically cofinite complexes are cohomologically $\mathfrak{a}$-torsion.

Finally, in the end of the third chapter, we will prove that over a Noetherian and $\mathfrak{a}$-adically complete ring R, the bounded cohomologically $\mathfrak{a}$-adically cofinite complexes are exactly the cohomologically $\mathfrak{a}$-torsion complexes $N^\bullet$ such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, N^\bullet)$ is finitely generated for every i [Theorem 3.47]. So, we can conclude that given a finitely generated module M over a Noetherian and $\mathfrak{a}$-adically complete ring R the complex $\mathrm{R}\Gamma_{\mathfrak{a}}(M)$ is cohomologically $\mathfrak{a}$-adically cofinite. Indeed, we will end the chapter understanding this result in the context of the Hartshorne's counterexample.

Chapter 1

The Local Case

All the rings R are supposed to be commutative.

1.1 Local Cohomology Functor

Definition 1.1 (**$\mathfrak{a}$-torsion functor**).
Let R be a commutative ring, $\mathfrak{a} \subseteq R$ an ideal. Denote by $\mathfrak{a}^n = \{a_1 \cdot \ldots \cdot a_n \mid a_i \in \mathfrak{a}\ \forall i = 1, \ldots, n\}$.
Given an R-module M define its $\mathfrak{a}$-torsion as

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \mid \mathfrak{a}^n \cdot m = 0 \ \textit{for some}\ n \in \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n)$$

where $(0 :_M \mathfrak{a}^n)$ is the set of elements m in M such that $\mathfrak{a}^n \cdot m = 0$.

Remark 1.2.
Clearly, $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. When $\Gamma_{\mathfrak{a}}(M) = M$, M is said to be an $\mathfrak{a}$-*torsion module.*
Moreover, given a homomorphism of R-modules $f : M \to M'$ we have that $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(M')$, and so we can define the homomorphism

$$\Gamma_{\mathfrak{a}}(f) := f|_{\Gamma_{\mathfrak{a}}(M)} : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M')$$

Proposition 1.3. $\Gamma_{\mathfrak{a}}(\_)$ *is a covariant left-exact functor.*

Proof. Obviously, $\Gamma_{\mathfrak{a}}(\mathbf{id}_M) = \mathbf{id}_{\Gamma_{\mathfrak{a}}(M)}$ and $\Gamma_{\mathfrak{a}}(g \circ f) = \Gamma_{\mathfrak{a}}(g) \circ \Gamma_{\mathfrak{a}}(f)$ for every R-module M and pair of morphism $M \xrightarrow{f} M' \xrightarrow{g} M''$. So, $\Gamma_{\mathfrak{a}}(\_)$ is a covariant functor.
Given a short exact sequence

$$0 \to M \xrightarrow{f} N \xrightarrow{g} K \to 0$$

notice that $\Gamma_{\mathfrak{a}}(f)$ is still injective and $\mathrm{Im}(\Gamma_{\mathfrak{a}}(f)) = \mathrm{Im}(f) \cap \Gamma_{\mathfrak{a}}(N) = \mathrm{Ker}(g) \cap \Gamma_{\mathfrak{a}}(N) = \mathrm{Ker}(\Gamma_{\mathfrak{a}}(g))$. □

Remark 1.4. *In general* $\Gamma_{\mathfrak{a}}(\_)$ *is not exact.*
Take $R = \mathbb{Z}$ and $\mathfrak{a} = (2)$. Notice that $\Gamma_{(2)}(\mathbb{Z}) = 0$ and $\Gamma_{(2)}(\mathbb{Z}/(2)) = \mathbb{Z}/(2)$. Applying $\Gamma_{(2)}(\_)$ to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$$

gives the sequence

$$0 \to 0 \to 0 \to \mathbb{Z}/(2) \to 0$$

which is not exact.

Definition 1.5 (**Local Cohomology Functor**).
The j^{th} right derived functor of $\Gamma_{\mathfrak{a}}(\_)$ is denoted by $H^j_{\mathfrak{a}}(\_)$ for every $j \geq 0$.
Given an R-module M, we call j^{th} *local cohomology module of* M *with respect to* $\mathfrak{a}$

$$H^j_{\mathfrak{a}}(M) = H^j(\Gamma_{\mathfrak{a}}(E^{\bullet}))$$

where $E^\bullet$ is an injective resolution of M and $\Gamma_\mathfrak{a}(E^\bullet)$ is the complex with components $\Gamma_\mathfrak{a}(E^n)$ and differentials $\Gamma_\mathfrak{a}(d^n)$.

Example 1.6.
Take $R = \mathbb{Z}$ an $\mathfrak{p} = (p)$, with $p \in \mathbb{Z}$ prime. We want to compute $H^j_\mathfrak{p}(\mathbb{Z})$ for $j \geq 0$.
An injective resolution of $\mathbb{Z}$ is

$$E^\bullet = (0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0)$$

By a computation we can show that

$$\Gamma_\mathfrak{p}(\mathbb{Q}) = 0 \textit{ and } \Gamma_\mathfrak{p}(\mathbb{Q}/\mathbb{Z}) = \left\{ \overline{q} \in \mathbb{Q}/\mathbb{Z} \mid q = \tfrac{r}{p^k} \textit{ for some } r \in \mathbb{Z},\ k > 0 \right\} \cong \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$$

and so we have that the sequence $\Gamma_\mathfrak{p}(E^\bullet) = (0 \to 0 \to \Gamma_\mathfrak{p}(\mathbb{Q}/\mathbb{Z}) \to 0)$ gives

$$H^j_\mathfrak{p}(\mathbb{Z}) = 0 \textit{ for } j \neq 1 \textit{ and } H^1_\mathfrak{p}(\mathbb{Z}) \cong \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$$

Now we want to summarize the main properties of these functors. The following corresponds to Proposition 7.3 and Theorem 7.8 of [ILL+07]:

Proposition 1.7. *Let R be a ring, $\mathfrak{a} \subseteq R$ an ideal and $M \in \mathbf{Mod}(R)$. Then:*

1. $H^0_\mathfrak{a}(M) \cong \Gamma_\mathfrak{a}(M)$.
2. $H^j_\mathfrak{a}(M)$ *is* $\mathfrak{a}$*-torsion* $\forall j \geq 0$.
3. $H^j_\mathfrak{a}(M) \cong H^j_{\sqrt{\mathfrak{a}}}(M)$ $\forall j \geq 0$.
4. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ a family of R-modules:* $H^j_\mathfrak{a}(\bigoplus_{\lambda \in \Lambda} M_\lambda) \cong \bigoplus_{\lambda \in \Lambda} H^j_\mathfrak{a}(M_\lambda)$ $\forall j \geq 0$.
5. *There is a natural isomorphism of functors* $H^j_\mathfrak{a}(\_) \cong \varinjlim_n \operatorname{Ext}^j_R(R/\mathfrak{a}^n, \_)$.
6. *Every short exact sequence $0 \to M \to M' \to M'' \to 0$ induces a long exact sequence*

$$\ldots \to H^{j-1}_\mathfrak{a}(M'') \to H^j_\mathfrak{a}(M) \to H^j_\mathfrak{a}(M') \to H^j_\mathfrak{a}(M'') \to H^{j+1}_\mathfrak{a}(M) \to \ldots$$

1.2 Injective Modules over Noetherian Rings

Since the local cohomology is computed using injective resolutions, it is fair to ask if (or under which assumptions) there is a classification of injective modules and, in general, of injective resolutions.

Theorem 1.8 (**Baer's Criterion** [ILL+07, Theorem A.2])**.**
Let R be a ring and E an R-module. Then E is injective if and only if every R-linear map $f : \mathfrak{a} \to E$ can be extended to $\tilde{f} : R \to E$ for every ideal $\mathfrak{a} \subseteq R$.

Proposition 1.9. *$E_1 \oplus E_2$ is an injective R-module if and only if E_1, E_2 are injective R-modules.*

Example 1.10 (**Divisible Modules**)**.**
Let R be an integral domain. An R-module M is called *divisible* if for every non-zero element $r \in R$ and $m \in M$ there's an element $n \in M$ such that $r \cdot n = m$.

1) If M is injective then M is divisible.
Let $r \in R$ and $m \in M$ be two non-zero elements. Consider the ideal $\mathfrak{a} = r \cdot R$ and the R-linear map $f : \mathfrak{a} \to M$ such that $f(r \cdot s) = s \cdot m$. By injectivity of M exists $\tilde{f} : R \to M$ extending f and $m = f(r) = \tilde{f}(r) = r \cdot \tilde{f}(1)$. Taking $n := \tilde{f}(1)$ we have the thesis.

2) If R is a PID: M injective if and only if M is divisible.
Let $\mathfrak{a} \subseteq R$ be a non-zero ideal and $f : \mathfrak{a} \to M$ an R-linear map. By hypothesis there is $r \in R$ generating $\mathfrak{a} = (r)$ and by divisibility there is $n \in M$ such that $r \cdot n = f(r)$. Define $\tilde{f}(s) := s \cdot n$, this extend f and so M is injective.

Definition 1.11 (**Essential Extension**).
Given two R-modules E and M. E is an *essential extesion of* M, denoted by $M \leq_e E$, if :

(i) $M \subseteq E$ is a submodule,

(ii) For every non-zero submodule $N \subseteq E$: $M \cap N \neq 0$.

An essential extension E of M is *maximal* when it is not contained in another essential extension E' of M.

Example 1.12.
Let $\mathfrak{p} \subseteq R$ be a prime ideal, the set $R \setminus \mathfrak{p}$ is a multiplicatively closed set of nonzerodivisors in R. We denote by $R_{\mathfrak{p}} = \left\{ \frac{r}{s} \mid r \in R,\ s \in R \setminus \mathfrak{p} \right\}$ the localization of R at $\mathfrak{p}$.
It turns out that $R \leq_e R_{\mathfrak{p}}$ is an essential extension. Indeed, the localization map $\pi : R \to R_{\mathfrak{p}}$ is injective and any ideal $\mathfrak{b} \subseteq R_{\mathfrak{p}}$ is of the form $\mathfrak{b} = \pi(\mathfrak{a})R_{\mathfrak{p}}$ for some ideal $\mathfrak{a} \subseteq R$ and since $\pi(\mathfrak{a})$ contains a copy of $\mathfrak{a}$ by injectivity, $R \cap \mathfrak{b} \neq 0$.

Remark 1.13.

1) *Let $M \subseteq L_1 \subseteq L_2$ be non-zero R-modules. Then $M \leq_e L_2$ if and only if $M \leq_e L_1$ and $L_1 \leq_e L_2$.*

2) *M is an injective R-module if and only if M has no proper essential extensions.*

Proof of 2. If M is injective, we can prove that for every module M' which strictly contains M there is a non-zero submodule of M' disjoint from M. In fact, in such a situation, the short exact sequence

$$0 \to M \hookrightarrow M' \twoheadrightarrow M'/M \to 0$$

splits by injectivity of M. So, $M' \cong M \oplus S$ with $S \cong M'/M$ and $M \cap S = 0$.

On the other hand, by the fact that every module M embeds in an injective module I, let $\varphi : M \to I$ be the embedding and S be the maximal submodule of I disjoint from $\varphi(M)$, which exists by Zorn's lemma. Notice that $M \cong \varphi(M) \cong \varphi(M)/S$ and for every non-zero submodule $N \subseteq I/S$, since $N \cong N'/S$ for some $S \subseteq N' \subseteq I$ and $\varphi(M) \cap N' \neq 0$ by maximality of S:

$$(\varphi(M)/S) \cap N \neq 0$$

So, $M \cong \varphi(M)/S \leq_e I/S$ and by hypothesis $I/S \cong M$. It follows that the short exact sequence

$$0 \to S \hookrightarrow I \xrightarrow{\pi} M \to 0$$

is right split trough φ, indeed $\pi\varphi(M) = \varphi(M)/S \cong M$. So, $I \cong S \oplus M$ and M is injective. $\square$

From these last remarks we can guess that, when a maximal essential extesion of a module exists it is injective. We will show a deeper relation between injectives an essential extensions with the following theorem.

Theorem 1.14. *Let R be a ring and M an R-module. The following are equivalent:*

1. *I is a maximal essential extension of M.*

2. *I is injective and $M \leq_e I$.*

3. *I is minimal among the injective modules that contain M.*

Moreover, such a module always exists and it is unique up to isomorphism.

Proof. The equivalence of the statements is proved in [Lam99, Theorem 3.30]. For the existence and the uniqueness see [Lam99, Lemma 3.29, Corollary 3.32]. $\square$

Definition 1.15 (**Injective Hull**).
A module satisfying the above conditions is called *injective hull of* M and it is denoted by $E_R(M)$.

Remark 1.16. An R-module M is called *indecomposable* if it cannot be written as a direct sum of two proper non-zero submodules. For injective hulls holds the following property:

M is indecomposable if and only if $E_R(M)$ is indecomposable

Proof. Let M be indecomposable. Suppose by contradiction we have the decomposition $E_R(M) = J \oplus J'$, then by definition of injective hull $M \cap J$ and $M \cap J'$ are both non-zero submodules of M and $M = (M \cap J) \oplus (M \cap J')$.

On the other hand, let $E_R(M)$ be indecomposable. If $M = N \oplus N'$, then $E_R(N)$ and $E_R(N')$ are injective submodules of $E_R(M)$, in particular they are direct summands and so they are both equal to $E_R(M)$. It follows that $E_R(N) = E_R(N')$ and so $N \subseteq E_R(N')$, that means $N \cap N' \neq 0$. □

These last objects are very useful to construct a standard injective resolution.

Definition 1.17 (Minimal Injective Resolution).
Let M be an R-module. We call *minimal injective resolution of M* the complex

$$0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \dots$$

where $E^0 = E_R(M)$, $E^1 = E_R(E^0/M)$ and $E^{i+1} = E_R(E^i/E^{i-1})$ for $i \geq 1$ and the differentials are the composition of the natural projection onto the quotient and the injection into the injective hull. Indeed, $E^{i+1} \cong E_R(\mathrm{Im}(d^i))$ for $i \geq 0$.

In general, injective modules are not finitely generated neither have a standard direct sum decomposition. So, without further assumptions on the ring there is no hope to get some nice results. But we will discover that, in Noetherian rings, injective modules, and hence minimal injective reslutions, have a complete characterization. The following results are proved in [Lam99, Theorem 3.48].

Theorem 1.18 (Bass - Papp Characterization).
A ring R is Noetherian if and only if every direct sum of injective R-modules is injective over R.

Corollary 1.19.
Every injective module over a Noetherian ring is a direct sum of indecomposable injective modules.

It is immediate, from this corollary, that the most important injective modules over a Noetherian ring are the indecomposable ones. We can say much more about them using *associated primes* of a module.

Definition 1.20.
Given an R-module M, a prime ideal $\mathfrak{p}$ is said to be *associated to M* if $\mathfrak{p} = (0 :_R m)$ for some $m \in M$. We will denote the set of the prime ideals associated to M with $Ass(M)$. While, the set of all prime ideals will be denoted with $Spec(R)$.

Lemma 1.21. *Let R be a Noetherian ring and M a non-zero R-module. Then:*

1. $Ass(M) \neq \emptyset$.

2. $\bigcup_{\mathfrak{p} \in Ass(M)} \mathfrak{p} = \bigcup_{m \in M} (0 :_R m)$.

Proof.
1. Let $\mathcal{S}$ be the set of ideals of the form $(0 :_R m)$ for some non-zero element $m \in M$. Let $\mathfrak{p} = (0 :_R x)$ be a maximal element in $\mathcal{S}$, which exists since R is Noetherian. We claim that $\mathfrak{p}$ is a prime ideal of R.
Let $rs \in \mathfrak{p}$ and suppose that s is not in $\mathfrak{p}$. Then $r \in (0 :_R sx)$ and $\mathfrak{p} \subseteq (0 :_R sx)$. By the maximality of $\mathfrak{p}$, it holds the equality and therefore r is in $\mathfrak{p}$.
2. Clearly, any element of an associated prime is a zero-divisor for M. On the other hand, if $rm = 0$, then (r) is contained in $(0 :_R m)$ which is contained in a maximal element of $\mathcal{S}$, i.e. in a prime $\mathfrak{p} \in Ass(M)$. □

We have to mention now a few basic results about how injective modules behave with localizations, and, surprisingly, everything turns out to work in a very easy way. The following lemma is about the localization at a prime ideal, but it can be generalized to every multiplicatively closed subset. The general version can be found in [ILL+07, Proposition A.22].

Lemma 1.22. *Let R be a Noetherian ring and $\mathfrak{p} \in Spec(R)$:*

1. *If E is an injective R-module, then $E_{\mathfrak{p}}$ is an injective $R_{\mathfrak{p}}$-module.*
2. *If $M \leq_e E$ is an essential extension of R-modules (resp. $E = E_R(M)$), then $M_{\mathfrak{p}} \leq_e E_{\mathfrak{p}}$ is an essential extension of $R_{\mathfrak{p}}$-modules (resp. $E_{\mathfrak{p}} = E_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$).*
3. *If $E^\bullet$ is a minimal injective resolution of M over R, then $E^\bullet_{\mathfrak{p}}$ is a minimal injective resolution of $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$.*

Theorem 1.23.
Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$ and set $\mathbb{K}(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $E = E_R(R/\mathfrak{p})$. Then:

1. *For every $x \in R \setminus \mathfrak{p}$ the map $E \xrightarrow{x\cdot} E$ is an isomorphism. Moreover, $E_{\mathfrak{p}} \cong E$.*
2. *$Ass(E) = \{\mathfrak{p}\}$ and E is $\mathfrak{p}$-torsion. In particular, if $\mathfrak{q} \nsubseteq \mathfrak{p}$ then $E_R(R/\mathfrak{q})_{\mathfrak{p}} = 0$.*
3. *$\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E) \cong \mathbb{K}(\mathfrak{p})$ and $\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E_R(R/\mathfrak{q})_{\mathfrak{p}}) = 0$ for every prime ideal $\mathfrak{q} \neq \mathfrak{p}$.*

Proof.
1. Let $x \in R \setminus \mathfrak{p}$ and consider the map $E \xrightarrow{x\cdot} E$. For every $r \neq 0 \in R/\mathfrak{p}$ we have that $x \cdot r \neq 0$, otherwise $x \cdot r \in \mathfrak{p}$ which implies that $r \in \mathfrak{p}$. So $R/\mathfrak{p} \cap \operatorname{Ker}(x\cdot) = 0$. Since E is an essential extension of $R/\mathfrak{p}$, it follows that $\operatorname{Ker}(x\cdot) = 0$ and so $\operatorname{Im}(x\cdot) \cong E$. Thus $\operatorname{Im}(x\cdot)$ is an injective module containing a copy of $R/\mathfrak{p}$ as a submodule. We can conclude that $\operatorname{Im}(x\cdot) = E$ and the map is surjective.
Moreover, the localization map $E \to E_{\mathfrak{p}}$ is an isomorphism. Indeed, $\frac{e}{1} = \frac{0}{1}$ if and only if there is $x \in R \setminus \mathfrak{p}$ such that $x \cdot (e \cdot 1 - 0 \cdot 1) = x \cdot e = 0$, by injectivity of $x\cdot$ it happens if and only if $e = 0$. To prove the surjectivity, let $\frac{e}{x} \in E_{\mathfrak{p}}$, by surjectivity of multiplication by x, e can be write as $e = x \cdot e'$ and so $\frac{e}{x} = \frac{x \cdot e'}{x} = \frac{e'}{1}$.

2. Let $\mathfrak{q} = (0 :_R e) \in Ass(E)$ be a prime ideal associated to E. By definition of E: $R/\mathfrak{p} \cap R \cdot e \neq 0$ and so there is $r \in R \setminus \mathfrak{q}$ such that $r \cdot e \in R/\mathfrak{p}$, which implies that $\mathfrak{p} = (0 :_R r \cdot e)$. Since R is commutative, it holds that $(0 :_R r \cdot e) = ((0 :_R e) :_R r)$. We can conclude that $\mathfrak{p} = (\mathfrak{q} :_R r) = \mathfrak{q}$ and so $Ass(E) = \{\mathfrak{p}\}$.
Pick an element $e \in E$ and let $\mathfrak{a} = (0 :_R e)$, one can notice that $R/\mathfrak{a} \cong R \cdot e \subseteq E$ and so $Ass(R/\mathfrak{a}) \subseteq Ass(E) = \{\mathfrak{p}\}$, which is equivalent to say $\sqrt{\mathfrak{a}} = \mathfrak{p}$ [Mus, Proposition 2.3]. It follows that $e \in \Gamma_{\mathfrak{a}}(E) = \Gamma_{\mathfrak{p}}(E)$ by Proposition 1.7 (3), so $\Gamma_{\mathfrak{p}}(E) = E$.
In particular, given a prime ideal $\mathfrak{q}$ such that $\mathfrak{q} \nsubseteq \mathfrak{p}$ then exists $x \in \mathfrak{q} \setminus \mathfrak{p}$ and for every $e \in E_R(R/\mathfrak{q})$ $\mathfrak{q}^t \cdot e = 0$ for some $t \in \mathbb{N}$. It follows that for every $e \in E_R(R/\mathfrak{q})_{\mathfrak{p}}$: $e = \frac{x^t \cdot e}{x^t} = 0$ and so $E_R(R/\mathfrak{q})_{\mathfrak{p}} = 0$.

3. Let $\mathfrak{q} \in Spec(R)$. If $\mathfrak{q} \nsubseteq \mathfrak{p}$ then $E_R(R/\mathfrak{q})_{\mathfrak{p}} = 0$ and so $\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E_R(R/\mathfrak{q})_{\mathfrak{p}}) = 0$.
If $\mathfrak{q} \subseteq \mathfrak{p}$ then every homomorphism $\varphi : \mathbb{K}(\mathfrak{p}) \to E_R(R/\mathfrak{q})_{\mathfrak{p}}$ is uniquely determinated by $\varphi(1)$ and it has to be such that $\varphi(\mathfrak{p}R_{\mathfrak{p}}) = \mathfrak{p}R_{\mathfrak{p}} \cdot \varphi(1) = 0$. It follows that

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E_R(R/\mathfrak{q})_{\mathfrak{p}}) \cong (0 :_{E_R(R/\mathfrak{q})_{\mathfrak{p}}} \mathfrak{p}R_{\mathfrak{p}})$$

If this is not 0, then $\exists\, e \in E_R(R/\mathfrak{q})$ such that $\mathfrak{p} \cdot e = 0$, i.e. $\mathfrak{p} \subseteq (0 :_R e) \subseteq \mathfrak{q}$, and so $\mathfrak{p} = \mathfrak{q}$.
If $\mathfrak{p} = \mathfrak{q}$: $\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E_{\mathfrak{p}}) \cong (0 :_{E_{\mathfrak{p}}} \mathfrak{p}R_{\mathfrak{p}})$ and this is a $\mathbb{K}(\mathfrak{p})$-vector space, since every module over a field is both injective and projective, the inclusion $\mathbb{K}(\mathfrak{p}) \hookrightarrow (0 :_{E_{\mathfrak{p}}} \mathfrak{p}R_{\mathfrak{p}})$ splits and since $\mathbb{K}(\mathfrak{p}) \cong (R/\mathfrak{p})_{\mathfrak{p}} \subseteq E_{\mathfrak{p}}$ is essential, it follows that $(0 :_{E_{\mathfrak{p}}} \mathfrak{p}R_{\mathfrak{p}}) \cong \mathbb{K}(\mathfrak{p})$. □

Proposition 1.24.
Let R be a Noetherian ring. The assignment which sends a prime ideal $\mathfrak{p}$ to $E_R(R/\mathfrak{p})$ is a bijection between $Spec(R)$ and the set of isoclasses of indecomposable injective R-modules.

Proof. We first show that the map is well-defined, i.e. $E_R(R/\mathfrak{p})$ is indecomposable for every $\mathfrak{p} \in Spec(R)$. By contradiction let $E_R(R/\mathfrak{p}) = J_1 \oplus J_2$, it follows that $\exists x_i \neq 0 \in R/\mathfrak{p} \cap J_i$ for i=1,2. Since $R/\mathfrak{p}$ is a domain $x_1 \cdot x_2 \neq 0$ but lies in $J_1 \cap J_2 = 0$.
To prove the surjectivity, let M be an indecomposable injective R-module. By Lemma 1.21 (1), we can find $\mathfrak{p} \in Ass(M)$ and so an injection $i : R/\mathfrak{p} \hookrightarrow M$. We can suppose $R/\mathfrak{p} \subseteq M$ and so $E_R(R/\mathfrak{p}) \subseteq E_R(M)$, since both M and $E_R(R/\mathfrak{p})$ are injective submodules, they are direct summand of $E_R(M)$, which is indecomposable. It follows that $M \cong E_R(R/\mathfrak{p})$.
For the injectivity, let $\mathfrak{p}, \mathfrak{q} \in Spec(R)$ such that there is an isomorphism $E_R(R/\mathfrak{p}) \cong E_R(R/\mathfrak{q})$, it follows that $\{\mathfrak{p}\} = Ass(E_R(R/\mathfrak{p})) = Ass(E_R(R/\mathfrak{q})) = \{\mathfrak{q}\}$. □

Corollary 1.25 ([ILL+07, Theorem A.21])**.**
Every injective module E over a Noetherian ring R has a direct sum decomposition of the form

$$E = \bigoplus_{\mathfrak{p} \in Spec(R)} E_R(R/\mathfrak{p})^{\oplus \mu_{\mathfrak{p}}}$$

and the numbers $\mu_{\mathfrak{p}}$ are independent of the decomposition.

Definition 1.26 (**Bass Numbers**)**.**
Let R be a Noetherian ring, M an R-module and $E^\bullet$ a minimal injective resolution of M over R. According to Corollary 1.25, for each component E^i we have the following decomposition

$$E^i = \bigoplus_{\mathfrak{p} \in Spec(R)} E_R(R/\mathfrak{p})^{\oplus \mu_i(\mathfrak{p}, M)}$$

The number $\mu_i(\mathfrak{p}, M)$ is called i^{th} *Bass number of M with respect to* $\mathfrak{p}$.

Theorem 1.27.
Let R be a Noetherian ring, M an R-module and $\mathfrak{p} \in Spec(R)$, set $\mathbb{K}(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

$$\mu_i(\mathfrak{p}, M) = \operatorname{rank}_{\mathbb{K}(\mathfrak{p})}(\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), M_{\mathfrak{p}})) \; for\ every\ i \geq 0$$

where the action of an element $k \in \mathbb{K}(\mathfrak{p})$ is induced by the map $\mathbb{K}(\mathfrak{p}) \xrightarrow{k\cdot} \mathbb{K}(\mathfrak{p})$.

Proof. Let $E^\bullet$ be a minimal injective resolution of M over R. From Lemma 1.22 (3), $E^\bullet_{\mathfrak{p}}$ is a minimal injective resolution of $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ and by Theorem 1.23 (2) and the fact that localization commutes with direct sums

$$E^i_{\mathfrak{p}} = \bigoplus_{\substack{\mathfrak{q} \in Spec(R) \\ \mathfrak{q} \subseteq \mathfrak{p}}} E_R(R/\mathfrak{q})_{\mathfrak{p}}^{\oplus \mu_i(\mathfrak{q}, M)}$$

i.e. the Bass numbers of M and $M_{\mathfrak{p}}$ are the same for the ideals $\mathfrak{q} \subseteq \mathfrak{p}$.
Now consider the complex $\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E^\bullet_{\mathfrak{p}})$:

$$0 \to \operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E^0_{\mathfrak{p}}) \xrightarrow{d^0 \circ} \operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E^1_{\mathfrak{p}}) \xrightarrow{d^1 \circ} \operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E^2_{\mathfrak{p}}) \xrightarrow{d^2 \circ} \dots$$

We claim that all the maps in this complex are the zero maps, i.e. for every homomorphism $\varphi : \mathbb{K}(\mathfrak{p}) \to E^i_{\mathfrak{p}}$ the composition $d^i \circ \varphi = 0$. Let $\varphi(x) \neq 0 \in E^i_{\mathfrak{p}}$, by the minimality of $E^\bullet_{\mathfrak{p}}$ we have that $E^i_{\mathfrak{p}} \cong E_{R_{\mathfrak{p}}}(\operatorname{Im}(d^{i-1}))$, so $r \cdot \varphi(x) \neq 0 \in \operatorname{Im}(d^{i-1})$ for some $r \in R_{\mathfrak{p}} \setminus \mathfrak{p}R_{\mathfrak{p}}$. Since r has an inverse in $R_{\mathfrak{p}}$, it follows that $\varphi(x) \in \operatorname{Im}(d^{i-1})$ and so $d^i(\varphi(x)) = 0$.
By this last claim $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), M_{\mathfrak{p}}) = \operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E^i_{\mathfrak{p}})$ and since $\mathbb{K}(\mathfrak{p})$ is finitely generated over $R_{\mathfrak{p}}$ we have that $\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), \_)$ commutes with direct sums, and so

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E^i_{\mathfrak{p}}) = \bigoplus_{\substack{\mathfrak{q} \in Spec(R) \\ \mathfrak{q} \subseteq \mathfrak{p}}} \operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E_R(R/\mathfrak{q})_{\mathfrak{p}})^{\oplus \mu_i(\mathfrak{q}, M)}$$

The thesis follows from Theorem 1.23 (3):

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), M_{\mathfrak{p}}) = \operatorname{Hom}_{R_{\mathfrak{p}}}(\mathbb{K}(\mathfrak{p}), E_R(R/\mathfrak{p}))^{\mu_i(\mathfrak{p}, M)} \cong \mathbb{K}(\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}.$$

□

Lemma 1.28. *Let R be a Noetherian ring, $\mathfrak{a} \subseteq R$ an ideal and M an R-module such that $\mathfrak{a}M = 0$. Then M is also an $R/\mathfrak{a}$-module and M is finitely generated over R if and only if M is finitely generated over $R/\mathfrak{a}$.*

Proof. For every $m \in M$ and $r \in R$, since $\mathfrak{a} \cdot m = 0$, it follows that $rm = \overline{r}m$ in M. So, for every $m_1, \dots, m_k \in M$ we have that $Rm_1 + \dots + Rm_k = (R/\mathfrak{a})m_1 + \dots + (R/\mathfrak{a})m_k$. □

Corollary 1.29. *Let M a finitely generated module over a Noetherian ring, then for every prime ideal $\mathfrak{p} \in Spec(R)$ all the Bass numbers $\mu_i(\mathfrak{p}, M)$ are finite.*

Proof. Let R be a Noetherian ring and $\mathfrak{p} \in Spec(R)$. The ring $R_\mathfrak{p}$ is a Noetherian local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$ and $M_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module. Denote as usual the residue field as $\mathbb{K}(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Since $\mathbb{K}(\mathfrak{p})$ and $M_\mathfrak{p}$ are both finitely generated over $R_\mathfrak{p}$, it turns out that $\text{Ext}^i_{R_\mathfrak{p}}(\mathbb{K}(\mathfrak{p}), M_\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module for every $i \geq 0$ [Rot09, Theorem 7.36]. Since every element of $\mathbb{K}(\mathfrak{p})$ is annihilated by the ideal $\mathfrak{p}R_\mathfrak{p}$, the same holds for the modules $\text{Ext}^i_{R_\mathfrak{p}}(\mathbb{K}(\mathfrak{p}), M_\mathfrak{p})$. So, by Lemma 1.28, they are finitely generated also as $\mathbb{K}(\mathfrak{p})$-modules. Hence, the Bass numbers $\mu_i(\mathfrak{p}, M)$ are finite for every $i \geq 0$. □

1.3 Back to Local Cohomology

Now that we have a deep characterization of the injective modules and of the minimal injective resolutions, we can obtain nice results also in the context of local cohomology. Indeed, as we mentioned in the first section, the torsion functor commutes with direct sums and so, for example, to compute this functor on an injective module over a Noetherian ring, it is sufficent to compute it on the indecomposable injectives.

Theorem 1.30. *Let R be a Noetherian ring, $\mathfrak{a} \subseteq R$ an ideal and $\mathfrak{p} \in Spec(R)$.*

$$\Gamma_\mathfrak{a}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & if\ \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & otherwise \end{cases}$$

Proof. By theorem 1.3, $E = E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion and $\Gamma_\mathfrak{p}(E) \subseteq \Gamma_\mathfrak{a}(E)$ for every $\mathfrak{a} \subseteq \mathfrak{p}$.
If $\mathfrak{a} \nsubseteq \mathfrak{p}$, the intersection $\mathfrak{a} \cap R \setminus \mathfrak{p}$ is non-empty and multiplicatively closed, and so $\exists x \in \mathfrak{a}^t \cap R \setminus \mathfrak{p}$ for every $t \in \mathbb{N}$ and multiplication by x in $E_R(R/\mathfrak{p})$ is injective. It follows that $\mathfrak{a}^t \cdot e \neq 0$ for every $e \in E$, i.e. $\Gamma_\mathfrak{a}(E) = 0$. □

Corollary 1.31. *Let R be a Noetherian ring, $\mathfrak{a} \subseteq R$ an ideal. If E is an injective R-module, then the module $\Gamma_\mathfrak{a}(E)$ is still injective.*

Proof. By Corollary 1.25, E is an injective R-module if and only if it can be written as

$$E = \bigoplus_{\mathfrak{p} \in Spec(R)} E_R(R/\mathfrak{p})^{\oplus \mu_\mathfrak{p}}$$

From Proposition 1.7 (4) and the previous theorem, it follows that

$$\Gamma_\mathfrak{a}(E) = \bigoplus_{\substack{\mathfrak{p} \in Spec(R) \\ \mathfrak{a} \subseteq \mathfrak{p}}} E_R(R/\mathfrak{p})^{\oplus \mu_\mathfrak{p}}$$

□

Remark 1.32. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M an R-module.*
Some of the proprierties of the local cohomology modules $H^j_\mathfrak{m}(M)$ depend on the properties of the injective hull of the residue field $E_R(\mathbb{K})$, where $\mathbb{K} = R/\mathfrak{m}$, in particular on the properties which are inherited by direct sums, quotients and submodules.
Indeed, let $E^\bullet$ be a minimal injective resolution of M, by the previous theorem follows that

$$\Gamma_\mathfrak{m}(E^\bullet) = 0 \to E_R(\mathbb{K})^{\mu_0(\mathfrak{m},M)} \to E_R(\mathbb{K})^{\mu_1(\mathfrak{m},M)} \to E_R(\mathbb{K})^{\mu_2(\mathfrak{m},M)} \to \dots$$

and so all the modules $H^j_\mathfrak{m}(M)$ are quotients of submodules of $E_R(\mathbb{K})^{\mu_j(\mathfrak{m},M)}$.
If the module M has finite Bass numbers $\mu_j(\mathfrak{m}, M)$, we can look at the properties inherited by finite direct sums, like Artinianity, Noetherianity, having finite length, etc.

1.4 Matlis Duality

By the previous remark, we know that of particular importance is the injective hull of the residue field of a Noetherian local ring.
We will write $(R, \mathfrak{m})$ to mean a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and we will denote with $\mathbb{K} = R/\mathfrak{m}$ the residue field.
The following results about completions can be found in [AM69].

Definition 1.33. Let $(R, \mathfrak{m})$ be a Noetherian local ring, the *Matlis dual* of an R-module M is the module $M^\vee = \mathrm{Hom}_R(M, E_R(\mathbb{K}))$.

Lemma 1.34. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M an R-module. Then, the map*

$$\nu : M \longrightarrow M^{\vee\vee}, \text{ where } x \mapsto \nu_x \text{ and } \nu_x(f) = f(x)$$

is injective. In particular, $(\_)^\vee$ is faithful, i.e. $M^\vee = 0$ if and only if $M = 0$.

Proof. Let $x \in M$ be a non-zero element and set $L = R \cdot x \subseteq M$. Notice that $L/\mathfrak{m}L$ is an $R/\mathfrak{m}$-vector space generated by one element and so $L/\mathfrak{m}L \cong \mathbb{K}$, in particular $\overline{x} \mapsto 1$. Consider the composed map

$$f = L \twoheadrightarrow \mathbb{K} \overset{i}{\hookrightarrow} E$$

Since E is an injective module, f can be extended to a map $\tilde{f} : M \to E$. It follows that

$$\nu_x(\tilde{f}) = \tilde{f}(x) = f(x) = i(1) \neq 0.$$

□

Definition 1.35 ($\mathfrak{a}$-adic Completion).
Let R be a commutative ring, $\mathfrak{a}$ an ideal of R and M an R-module.
The natural surjections $\ldots \twoheadrightarrow M/\mathfrak{a}^3M \twoheadrightarrow M/\mathfrak{a}^2M \twoheadrightarrow M/\mathfrak{a}M \twoheadrightarrow 0$ give an inverse system, whose inverse limit is called $\mathfrak{a}$-*adic completion of* M, denoted by $\widehat{M}$ or $\Lambda_\mathfrak{a}(M)$ when we want to underlying the functorial property of this operation. More formally:

$$\widehat{M} = \varprojlim_i (M/\mathfrak{a}^iM) = \{(\ldots, \overline{m}_3, \overline{m}_2, \overline{m}_1) \in \prod_i M/\mathfrak{a}^iM \mid m_{i+1} - m_i \in \mathfrak{a}^iM\}$$

There is a natural homomorphism $\tau_M : M \longrightarrow \widehat{M}$ where $\tau_M(m) = (\overline{m})_{i\in\mathbb{N}}$.

Now we will mention some basic results about completions, which can be useful for the understanding of what follows:

(1) $\mathrm{Ker}(\tau_M) = \bigcap_i \mathfrak{a}^iM$.

(2) $\widehat{R}$ is a ring and $\tau_R : R \longrightarrow \widehat{R}$ is a flat ring homomorphism, i.e. it makes $\widehat{R}$ a flat R-module.

(3) $\widehat{M}$ is an $\widehat{R}$-module and τ_R is compatible with the structures, i.e. $\tau_R(r) \cdot (\overline{m_i})_{i\in\mathbb{N}} = r \cdot (\overline{m_i})_{i\in\mathbb{N}}$ for every $r \in R$ and $(\overline{m_i})_{i\in\mathbb{N}} \in \hat{M}$.

(4) If R is a Noetherian ring, then also $\widehat{R}$ is Noetherian as a ring.

(5) If $(R, \mathfrak{m})$ is a local ring, then $(\widehat{R}, \mathfrak{m}\widehat{R})$ is local.

(6) If M is a finitely generated R-module, then $\widehat{M} \cong \widehat{R} \otimes_R M$.

In particular, $\widehat{M}$ is a finitely generated $\widehat{R}$-module and if R is $\mathfrak{a}$-adically complete, then $\widehat{M} \cong M$.

(7) $\widehat{R}/\mathfrak{a}^i\widehat{R} \cong R/\mathfrak{a}^i$ for every $i \geq 0$, and so:

If M is $\mathfrak{a}$-torsion, then $M \cong \widehat{R} \otimes_R M$. In particular, it has an $\widehat{R}$-module structure.

If M, N are both $\mathfrak{a}$-torsion, then $\mathrm{Hom}_{\widehat{R}}(M, N) = \mathrm{Hom}_R(M, N)$.

Theorem 1.36 ([ILL+07, Theorem A.31])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\widehat{R}$ its $\mathfrak{m}$-adic completion, $\mathbb{K} = R/\mathfrak{m}$ and $E = E_R(\mathbb{K})$.*
The map $\widehat{R} \longrightarrow \mathrm{Hom}_R(E, E)$, where $r \longmapsto (e \mapsto r \cdot e)$, is an isomorphism.

Corollary 1.37. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathbb{K} = R/\mathfrak{m}$, then $E_R(\mathbb{K})$ is an Artinian module.*

Proof. Consider a descending chain of submodules $\ldots \subseteq E_3 \subseteq E_2 \subseteq E_1 \subseteq E_R(\mathbb{K})$.
Applying the Matlis functor, it gives a sequence of surjections

$$\widehat{R} \twoheadrightarrow E_1^\vee \twoheadrightarrow E_2^\vee \twoheadrightarrow E_3^\vee \twoheadrightarrow \ldots$$

and, since $\widehat{R}$ is a Noetherian ring, the ascending chain

$$\mathrm{Ker}(\widehat{R} \twoheadrightarrow E_1^\vee) \subseteq \mathrm{Ker}(\widehat{R} \twoheadrightarrow E_2^\vee) \subseteq \mathrm{Ker}(\widehat{R} \twoheadrightarrow E_3^\vee) \subseteq \ldots$$

stabilizes. It follows that $E_i^\vee \cong E_{i+1}^\vee$ for every $i \geq n$ for some $n \in \mathbb{N}$.
By exactness of $(\_)^\vee$, we have the two short exact sequences

$$0 \to E_{i+1} \to E_i \to E_i/E_{i+1} \to 0$$

$$0 \to (E_i/E_{i+1})^\vee \to E_i^\vee \xrightarrow{\cong} E_{i+1}^\vee \to 0$$

from the last one, we can conclude that $(E_i/E_{i+1})^\vee = 0$ and so by Lemma 1.34 $E_i/E_{i+1} = 0$.
It follows that $E_{i+1} \cong E_i$ and also the descending sequence stabilizes. □

Theorem 1.38. *For every finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ all the cohomology modules $H^j_\mathfrak{m}(M)$ are Artinian.*

Proof. As it is proved in Corollary 1.29 all the Bass numbers $\mu_i(\mathfrak{m}, M)$ are finite. By Remark 1.32 all the cohomology modules $H^j_\mathfrak{m}(M)$ are quotients of submodules of finite direct sums of $E_R(\mathbb{K})$, so the Artinianity of the latter is inherited by all the cohomologies. □

Theorem 1.39. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathbb{K} = R/\mathfrak{m}$ and M an $\mathfrak{m}$-torsion R-module. Then M is Artinian if and only if all the Bass numbers $\mu_i(\mathfrak{m}, M)$ are finite.*

Proof. In the case in which all the Bass numbers with respect to $\mathfrak{m}$ are finite, we have the same situation as for a finitely generated module, i.e. all the local cohomology modules $H^j_\mathfrak{m}(M)$ are Artinian. Moreover, since M is $\mathfrak{m}$-torsion, we have that $M = \Gamma_\mathfrak{m}(M) = H^0_\mathfrak{m}(M)$.
On the other hand, let M be Artinian. Since M is $\mathfrak{m}$-torsion, we have that $(0 :_M \mathfrak{m}) \leq_e M$ is an essential extension, and so $E_R((0 :_M \mathfrak{m})) = E_R(M)$. Indeed, for any non-zero element $x \in M$, let t be the smallest integer such that $\mathfrak{m}^t x = 0$, then $\mathfrak{m}^{t-1}x \subseteq (0 :_M \mathfrak{m}) \cap Rx$. Moreover, $(0 :_M \mathfrak{m})$ is Artinian and it has a $\mathbb{K}$-vector space structure (it is isomorphic to $\mathrm{Hom}_R(\mathbb{K}, M)$), so it is a finite dimensional $\mathbb{K}$-vector space, i.e. $(0 :_M \mathfrak{m}) \cong \mathbb{K}^n$ for some $n \in \mathbb{N}$. In particular, $E_R(M) = E_R(\mathbb{K})^n$ and the Bass number $\mu_0(\mathfrak{m}, M) = n$ is finite.
Notice that $E_R(M)$ is $\mathfrak{m}$-torsion by Theorem 1.23 (2) and Artinian by Corollary 1.37, so it is also $E_R(M)/M$. Since, in the minimal injective resolution $E^\bullet$, $E^1 = E_R(E_R(M)/M)$, it follows that E^1 is again a finite direct sum of $E_R(\mathbb{K})$, hence also the Bass number $\mu_1(\mathfrak{m}, M)$ is finite. By induction on i we have the finiteness for all the Bass numbers. □

Corollary 1.40. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. Then $\mathrm{Ext}^i_R(\mathbb{K}, H^j_\mathfrak{m}(M))$ is finitely generated over R for every $i, j \geq 0$.*

Proof. Since the localization by $\mathfrak{m}$ is trivial, in the sense that $M_\mathfrak{m} \cong M$ for every module M, in general the Bass numbers are $\mu_i(\mathfrak{m}, M) = \mathrm{rank}_\mathbb{K}(Ext^i_R(\mathbb{K}, M))$. Combining Theorem 1.38 and 1.39, we have that $Ext^i_R(\mathbb{K}, M)$ is finitely generated over $\mathbb{K}$ for every i. So, by Lemma 1.28, they are finitely generated R-modules. □

Since we mentioned the Matlis duality, we have to state also the following important theorem, which will explain what the word "duality" stands for: the Matlis functor gives an anti-equivalence between the categories of Artinan and Noetherian R-modules.

Theorem 1.41 (**Matlis Duality** [ILL+07, Theorem A.35])**.**
Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and M an R-module:

1. *If M is Noetherian (resp. Artinian), then $M^\vee$ is Artinian (resp. Noetherian).*

2. *If M is Artinian or Noetherian, then the injective map $M \hookrightarrow M^{\vee\vee}$ is an isomorphism.*

1.5 The Non-Local Case: A Counterexample

The first attempt to generalize the result in Theorem 1.38 to a non-local ring R and a non-maximal ideal $\mathfrak{a}$ was by replacing the Artinianity with the weaker condition to have finitely generated $\mathrm{Ext}^i_R(R/\mathfrak{a}, \_)$-modules [Corollary 1.40]. The new aim was to find when for a module M the local cohomology modules $H^j_{\mathfrak{a}}(M)$ have this finiteness property.
In particular, Grothendieck conjectured that for a finitely generated module M over a Noetherian ring R the modules $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ are finitely generated for every $i, j \geq 0$. But Hartshorne showed with the folllowing counterexample that this was false.

Example 1.42 (**Hartshrone [Har70]**)**.**
Let $R = \mathbb{K}[x, y][[u, v]]$ be the ring of formal power series in the variables u, v with coefficents in the ring of polynomials over a field $\mathbb{K}$ in the variables x, y. Choose $\mathfrak{a} = (u, v)$ be the ideal generated by the elements u, v and $M = R/(xu + yu)$. Notice that R is Noetherian and M is a finitely generated R-module. We want to show that $\mathrm{Hom}_R(R/\mathfrak{a}, H^2_{\mathfrak{a}}(M))$ is not finitely generated, i.e. the Grothendieck's conjecture fails for $i = 0$ and $j = 2$.
It is well known that there is a natural isomorphism of functors [Proposition 1.7 (5)]

$$H^i_{\mathfrak{a}}(\_) \cong \varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, \_)$$

but we can find a further characterization in our example which will be useful for the following computations.

Proposition 1.43. *Denote by $\mathfrak{a}^{(n)} = (u^n, v^n)$. Then*

$$\varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^{(n)}, \_) \cong \varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, \_)$$

Proof. We first show that for any $n \in \mathbb{N}$, it holds that

$$\mathfrak{a}^{2n-1} \subseteq \mathfrak{a}^{(n)} \subseteq \mathfrak{a}^n$$

Indeed, an element of $\mathfrak{a}^{2n-1}$ is of the form $a = \sum_{i=0}^{2n-1} r_i u^i v^{(2n-1)-i}$ with $r_i \in R$. By splitting the sum in $k \leq n-1$ and $i \geq n$ we find $s, t \in R$ such that $a = su^n + tv^n$, so $a \in \mathfrak{a}^{(n)}$. The containment $\mathfrak{a}^{(n)} \subseteq \mathfrak{a}^n$ is strightforward.
From the identification $\mathrm{Hom}_R(R/\mathfrak{a}, M) \cong (0 :_M \mathfrak{a})$, it follows that for every R-module M there is a containment of Hom-modules

$$\mathrm{Hom}_R(R/\mathfrak{a}^n, M) \subseteq \mathrm{Hom}_R(R/\mathfrak{a}^{(n)}, M) \subseteq \mathrm{Hom}_R(R/\mathfrak{a}^{2n-1}, M)$$

Since the index set $\{2n-1\}_{n \in \mathbb{N}}$ is a cofinal subset of $\mathbb{N}$ (i.e. for any $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $2m - 1 \geq n$), from the theory of direct limits we have that

$$\varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^{2n-1}, \_) \cong \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n, \_) =: L$$

Since the direct limit is an exact functor: it respects the above inclusions

$$L = \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n, M) \subseteq \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^{(n)}, M) \subseteq \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^{2n-1}, M) = L$$

And it commutes with the cohomology functors H^i. So, given an injective resolution $M \to I^\bullet$, for every $i \geq 0$ we have that

$$\varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^{(n)}, M) \cong \varinjlim_n H^i(\operatorname{Hom}_R(R/\mathfrak{a}^{(n)}, I^\bullet)) \cong H^i(\varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^{(n)}, I^\bullet)) \cong$$
$$H^i(\varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n, I^\bullet)) \cong \varinjlim_n H^i(\operatorname{Hom}_R(R/\mathfrak{a}^n, I^\bullet)) \cong \varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$$

□

We start computing $H^i_\mathfrak{a}(R)$ and to do this we want to use the latter proposition. Let us take

$$P^\bullet_n = (0 \to R \xrightarrow{\left[\begin{smallmatrix}-v^n\\ u^n\end{smallmatrix}\right]} R \oplus R \xrightarrow{[u^n\ v^n]} R \to 0)$$

as a projective resolution of $R/\mathfrak{a}^{(n)}$. Applying the functor $\operatorname{Hom}_R(\_, R)$ we get the complex concetrated in degrees $[0, 2]$

$$\operatorname{Hom}_R(P^\bullet_n, R) = (0 \to R \xrightarrow{\left[\begin{smallmatrix}u^n\\ v^n\end{smallmatrix}\right]} R \oplus R \xrightarrow{[-v^n\ u^n]} R \to 0)$$

It follows that

$$\operatorname{Ext}^i_R(R/\mathfrak{a}^{(n)}, R) = \begin{cases} R/\mathfrak{a}^{(n)} & for\ i = 2 \\ 0 & for\ i \neq 2 \end{cases} \quad and\ so \quad H^i_\mathfrak{a}(R) = \begin{cases} \varinjlim_n R/\mathfrak{a}^{(n)} & for\ i = 2 \\ 0 & for\ i \neq 2 \end{cases}$$

Remark 1.44. Recall that the direct limit is defined as follows

$$\varinjlim_n R/\mathfrak{a}^{(n)} = \left(\oplus R/\mathfrak{a}^{(n)}\right) \Big/ S$$

where $S = \left\langle \left\{ (\dots, 0, r_i, \dots, -u^{j-i}v^{j-i} \cdot r_i, 0, \dots) \mid i \in \mathbb{N}, r_i \in R/\mathfrak{a}^{(i)}, j \geq i \right\} \right\rangle_R$.
So, every element of the direct limit can be represented by just one element of $R/\mathfrak{a}^{(n)}$ for n sufficiently large. Indeed, let n be the last non-zero entry of the sequence, we have that

$$[(r_1, \dots, r_n, 0, \dots)] = \left[(0, \dots, \sum_{i=1}^n u^{n-i}v^{n-i}r_i, 0, \dots)\right].$$

Moreover, notice that the element $\sum_{i=1}^n u^{n-i}v^{n-i}r_i$ can be written uniquely as $\sum_{s,t=0}^{n-1} u^s v^t k_{s,t}$ with $k_{s,t} \in \mathbb{K}[x, y]$.

We can find a more menageable identification for $H^2_\mathfrak{a}(R)$. Let N be the free $\mathbb{K}[x, y]$-module generated by the set $\{u^i v^j \mid i, j < 0\}$ where the R-module structure is given by the followng rules:

$$u \cdot u^i v^j = \begin{cases} u^{i+1}v^j & if\ i + 1 < 0 \\ 0 & if\ i + 1 = 0 \end{cases}$$

$$v \cdot u^i v^j = \begin{cases} u^i v^{j+1} & if\ j + 1 < 0 \\ 0 & if\ j + 1 = 0 \end{cases}$$

We achieve the identification $N \cong \varinjlim_n R/\mathfrak{a}^{(n)}$ identifying each element $u^i v^j$ with the class of the element $(\dots, 0, u^{i+n}v^{j+n}, 0, \dots)$ for n sufficiently large. Indeed, since for every $n \geq \max\{|i|, |j|\}$ all the elements $(\dots, 0, u^{i+n}v^{j+n}, 0, \dots)$, with the only non-zero entry at positon n, belong to the same class, the assignment is well defined and injective. By the previous remark every element of the direct limit can be written as $\left[(0, \dots, \sum_{s,t=0}^{n-1} u^s v^t k_{s,t}, 0, \dots)\right]$ and it is the image of $\sum_{s,t=0}^{n-1} u^{s-n}v^{t-n}k_{s,t}$, so the assignment is also surjective.

Now consider the short exact sequence

$$0 \to R \xrightarrow{(xu+yv)\cdot} R \to M \to 0$$

Applying the functor $\Gamma_{\mathfrak{a}}(\_)$ we obtain the long exact sequence in cohomology

$$0 \to H^1_{\mathfrak{a}}(M) \to N \xrightarrow{\varphi} N \to H^2_{\mathfrak{a}}(M) \to 0$$

where φ is again the multiplication by $xu + yv$ and so $H^2_{\mathfrak{a}}(M) = N/\operatorname{Im}(\varphi)$.

Taking the elements $a_n = y^{n-1}u^{-n}v^{-1}$ for any $n \in \mathbb{N}$, notice that $a_n \notin \operatorname{Im}(\varphi)$, indeed the only element $x \in N$ which satisfies $a_n = \varphi(x)$ would be the element $\sum_{s=0}^{n-1}(-1)^s x^s y^{n-2-s} u^{-n+s} v^{-2-s}$, but obviously it does not exist, otherwise for $s = n-1$ we would have $x^{n-1}y^{-1}u^{-1}v^{-n-1} \in R$. However, we have $(x, y, u, v) \cdot a_n \subset \operatorname{Im}(\varphi)$:

$$x \cdot a_n = \varphi(y^{n-1}u^{-n-1}v^{-1}),$$

$$y \cdot a_n = \varphi\left(\sum_{s=0}^{n-1}(-1)^s x^s y^{n-1-s} u^{-n+s} v^{-2-s}\right),$$

$$u \cdot a_n = \varphi\left(\sum_{s=0}^{n-2}(-1)^s x^s y^{n-2-s} u^{-n+1+s} v^{-2-s}\right) \text{ for } n \geq 2,$$

$$v \cdot a_n = u \cdot a_1 = 0 = \varphi(0).$$

In other words $\overline{a_n} \neq 0$ in $H^2_{\mathfrak{a}}(M)$ and for a non-zero element $r \in R$ we have that

$$r \cdot \overline{a_n} \neq 0 \textit{ if and only if } r \in \mathbb{K}$$

It follows that the submodule of $H^2_{\mathfrak{a}}(M)$ generated by the set $\{a_n \mid n \in \mathbb{N}\}$ is an infinite dimensional $\mathbb{K}$-vector space and so $\bigoplus_{i=1}^{\infty} \operatorname{Hom}_R(R/\mathfrak{a}, \mathbb{K}) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}, H^2_{\mathfrak{a}}(M))$ as a submodule. We can conclude that the latter is not finitely generated since is not Noetherian.

Chapter 2

Preliminaries on Derived Categories

To obtain results analogue to the ones in the first chapter in the setting of non-local rings, we have to move our set-up from the category of modules over the ring R to the derived category of the category of modules over R. Derived categories are the natural framework where to use derived functors. Until now, we computed derived functors on a module in three steps: taking a resolution of the module, applying the functor to the resolution and taking the cohomologies of the resulting complex. This last step is very important because it solves the problem of non-uniqueness of the resolutions. Indeed, all the resolutions of a module are homotopy equivalent but in general they are not isomorphic to each other. And so, if we take different resolutions and we apply a functor to them we obtain different complexes but still homotopy equivalent, hence they have isomorphic cohomology modules.
Clearly, there is a loss of information in focusing only on the cohomologies instead on the whole complex and to avoid this problem we need an environment where things work better. In particular, we want to construct a category where two homotopy equivalent complexes are isomorphic and each module is isomorphic to its resolutions. In this way applying a derived functor to a module, or more in general to a complex of modules, is the same as applying it to one of its resolutions and this operation will be independent on the choice of the latter. Moreover, the resulting complex carries all the information with it, indeed taking its cohomologies will give us the same modules resulting from the "previous" derived functor. For this reason we will call this the total derived functor.
Let us show briefly how derived categories are constructed.

2.1 Derived Categories

Let $\mathbf{M}$ be an abelian category (for example $\mathbf{M} = \mathbf{Mod}(R)$ the category of modules over a commutative ring R). We will denote by $\mathbf{C}(\mathbf{M})$ the category of complexes over $\mathbf{M}$, where the objects are the complexes

$$M^\bullet = (\dots \xrightarrow{d_M^{i-1}} M^i \xrightarrow{d_M^i} M^{i+1} \xrightarrow{d_M^{i+1}} \dots)$$

and the morphisms $\varphi^\bullet : M^\bullet \to N^\bullet$ are sequences of morphism in $\mathbf{M}$ such that $\varphi^i : M^i \to N^i$ and $\varphi^{i+1} \circ d_M^i = d_N^i \circ \varphi^i$ for every i.
We will omit the symbol $^\bullet$ when it is unnecessary.

Proposition 2.1 ([Yek12, Definition 4.1.1])**.** *If* $\mathbf{M}$ *is abelian then* $\mathbf{C}(\mathbf{M})$ *is still an abelian category.*

Recall that two morphisms of complexes $\varphi^\bullet, \psi^\bullet : M \to N$ are *homotopic* if there is a sequence of morphism $h^i : M^i \to N^{i-1}$ in $\mathbf{M}$ such that $\varphi^i - \psi^i = h^{i+1} \circ d_M^i + d_N^{i-1} \circ h^i$. Being homotopic is an equivalence relation in every Hom-set $\mathrm{Hom}_{\mathbf{C}(\mathbf{M})}(M, N)$. Two complexes M,N are *homotopy equivalent* if there are two morphisms $\varphi : M \to N$ and $\psi : N \to M$ such that their compositions are homotopic to the identities, in this case the morphisms are called *homotopy equivalences*. We will denote with $\mathbf{K}(\mathbf{M})$ the homotopy category of complexes over $\mathbf{M}$. In particular, it has the same object of $\mathbf{C}(\mathbf{M})$ and the morphisms in $\mathbf{K}(\mathbf{M})$ are homotopy classes of morphisms in $\mathbf{C}(\mathbf{M})$. Notice that with this construction two complexes which are homotopic equivalent in $\mathbf{C}(\mathbf{M})$ are isomorphic in the category $\mathbf{K}(\mathbf{M})$.

Example 2.2. If we consider $\mathbf{M} = \mathbf{Mod}(\mathbb{Z})$ and the module $M = \mathbb{Z}/2\mathbb{Z}$, we can take two different projective resolutions of M: $P^\bullet = (0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to 0)$ and $Q^\bullet = (0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]} \mathbb{Z} \oplus \mathbb{Z} \to 0)$. Clearly this two complexes are not isomorphic, but there are two morphism $\varphi : P^\bullet \to Q^\bullet$ and $\psi : Q^\bullet \to P^\bullet$, which are homotopic equivalences inverse to each other:

$$\varphi = \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 \end{array} \quad \textit{and} \quad \psi = \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle [1\ 0]} & & \downarrow{\scriptstyle [1\ 0]} & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Indeed one can check that $\psi \circ \varphi = \mathbf{id}_{P^\bullet}$ and $\varphi \circ \psi$ is homotopy equivalent to $\mathbf{id}_{Q^\bullet}$ through the homotopy $h^\bullet$ such that $h^0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $h^i = 0$ for every $i \neq 0$.

So we have that $P^\bullet \cong Q^\bullet$ in $\mathbf{K}(\mathbf{Mod}(\mathbb{Z}))$, while the same is not true in $\mathbf{C}(\mathbf{Mod}(\mathbb{Z}))$.
More in general this result holds for every pair of projective (or injective) resolutions of a module in the homotopy category.

The homotopy category $\mathbf{K}(\mathbf{M})$ is no longer an abelian category but it has a new important structure called triangulated category.

Definition 2.3. Let $\mathbf{T}$ be an additive category. A *shift functor* on $\mathbf{T}$ is an additive functor $\Sigma : \mathbf{T} \to \mathbf{T}$ which is an auto-equivalence of categories. A *triangle* in $\mathbf{T}$ is a diagram of the form $X \to Y \to Z \to \Sigma(X)$ and a *morphism of triangles* is a triple of maps (u, v, w) such that the following diagram commutes

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

If u, v, w are isomorphism, it is an isomorphism of triangles.

A *triagulated category* structure on $\mathbf{T}$ consists of a shift functor Σ and a set of triangles called *distinguished triangles* which satisfies the following properties:

(TR1) Any triangle isomorphic to a distinguished triangle is a distinguished triangle. For any object $X \in \mathbf{T}$,

$$X \xrightarrow{\mathbf{id}_X} X \to 0 \to \Sigma(X)$$

is a distinguished triangle. For any morphism $f : X \to Y$ in $\mathbf{T}$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma(X)$$

(TR2) The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is distinguished if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y)$$

is distinguished.

(TR3) Let

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle \Sigma(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

be a diagram where the rows are distinguished triangles and the first square is commutative. Then there exists a morphism $w : Z \to Z'$ such that the triple (u, v, w) is a morphism of distinguished triangles.

(TR4) Let f, g and $h = g \circ f$ be morphisms in $\mathbf{T}$. Then the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{a} & A & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle \mathbf{id}_X} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma(\mathbf{id}_X)} \\ X & \xrightarrow{h} & Z & \xrightarrow{b} & B & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbf{id}_Z} & & & & \downarrow{\scriptstyle \Sigma(f)} \\ Y & \xrightarrow{g} & Z & \xrightarrow{c} & C & \longrightarrow & \Sigma(Y) \end{array}$$

where the rows are distinguished triangles can be completed to the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{a} & A & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle \mathbf{id}_X} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \Sigma(\mathbf{id}_X)} \\ X & \xrightarrow{h} & Z & \xrightarrow{b} & B & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbf{id}_Z} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \Sigma(f)} \\ Y & \xrightarrow{g} & Z & \xrightarrow{c} & C & \longrightarrow & \Sigma(Y) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \mathbf{id}_C} & & \downarrow{\scriptstyle \Sigma(a)} \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma(A) \end{array}$$

where all four rows are distinguished triangles and the vertical arrows are morphisms of triangles.

One of the main results regarding distinguished triangles is the following lemma.

Lemma 2.4 ([Mil, Lemma 1.4.2])**.** *Let*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

be a morphism of distinguished triangles. If two morphisms among u, v, w are isomorphisms, then so is the third.

The triangulated structure of $\mathbf{K}(\mathbf{M})$ is given by the usual shifting $\Sigma(\_) = (\_)[1]$, where on objects $M[1]^i = M^{i+1}$ and $d^i_{M[1]} = -d^{i+1}_M$, while on morphisms $\varphi[1]^i = \varphi^{i+1}$. We will use the notation $\Sigma^n(M) = M[n]$ for any $n \in \mathbb{Z}$. The set of distinguished triangles is the set of triangles isomorphic in $\mathbf{K}(\mathbf{M})$ to

$$M \xrightarrow{\alpha} N \to Cone(\alpha) \to M[1]$$

where $\alpha \in \mathrm{Hom}_{\mathbf{C}(\mathbf{M})}(M, N)$ and $Cone(\alpha) = M[1] \oplus N$ with differential $d_{Cone(\alpha)} = \begin{bmatrix} d_{M[1]} & 0 \\ \alpha[1] & d_N \end{bmatrix}$.

Definition 2.5. Given two triangulated categories $(\mathbf{T}, \Sigma)$, $(\mathbf{T}', \Sigma')$. A *triangulated functor* is an additive functor $F : \mathbf{T} \to \mathbf{T}'$ which commutes with the shifts, i.e. $\Sigma' \circ F = F \circ \Sigma$, and sends distinguished triangles to distinguished triangles.

Remark 2.6. Let $F : \mathbf{M} \to \mathbf{M}'$ an additive functor between abelian categories. This can be extended to an additive functor $\mathbf{C}F : \mathbf{C}(\mathbf{M}) \to \mathbf{C}(\mathbf{M}')$ where $(\mathbf{C}F(M^\bullet))^i = F(M^i)$. Moreover, since the functor $\mathbf{C}F$ respect homotopy equivalences, we can get another additive functor $\mathbf{K}F : \mathbf{K}(\mathbf{M}) \to \mathbf{K}(\mathbf{M}')$. This last functor is a triangulated functor, indeed it commutes with the shift $(\_)[1]$ and it preserves cones, i.e. $\mathbf{K}F(Cone(\alpha)) = Cone(\mathbf{K}F(\alpha))$.

We said in the introduction that our aim was to construct a category in which all the resolutions of the same module are isomorphic to each other and to that module. Is the homotopic category sufficent enough ?

Example 2.7. Go back to the Example 2.2. We can think at the module $M = \mathbb{Z}/2\mathbb{Z}$ as a complex $M^\bullet = (0 \to \mathbb{Z}/2\mathbb{Z} \to 0)$ concentrated in degree 0. Clearly $M^\bullet$ is not isomorphic to $P^\bullet = (0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to 0)$ in $\mathbf{C}(\mathbf{Mod}(\mathbb{Z}))$. It turns out that they cannot be isomoprphic either in $\mathbf{K}(\mathbf{Mod}(\mathbb{Z}))$. Indeed all the morphisms between them have to be 0 at degree -1 since $M^{-1} = 0$, so to proof the homotopy equivalence we need a $\mathbb{Z}$-linear map $h^0 : \mathbb{Z} \to \mathbb{Z}$ such that $\mathbf{id}_{\mathbb{Z}} = h^0 \circ 2\cdot$. In particular, if we set $h^0(1) = n$, it must satisfy $1 = h^0(2 \cdot 1) = 2 \cdot n$ which has no solution in $\mathbb{Z}$.

In general the morphisms between modules and their resolutions are almost never homotopic equivalences, but they are always quasi-isomorphisms. Now, what we need to do is to construct another category in which these quasi-isomorphisms are isomorphisms. The general idea is to formally invert them using localization.

Recall that the cohomology functors $H^i : \mathbf{C}(\mathbf{M}) \to \mathbf{M}$ send homotopy equivalences to isomorphisms, so they are well-defined functors also in the homotopy category as $H^i : \mathbf{K}(\mathbf{M}) \to \mathbf{M}$.

Definition 2.8. A morphism $\varphi : M \to N$ in $\mathbf{K}(\mathbf{M})$ (or in $\mathbf{C}(\mathbf{M})$) is called *quasi-isomorphism* if it induces isomorphisms it the cohomologies, i.e. $H^i(\varphi) : H^i(M) \to H^i(N)$ is an isomorphism for every $i \in \mathbb{Z}$.

Let S be the set of all quasi-isomorphisms of $\mathbf{K}(\mathbf{M})$, clearly S is a multiplicatively closed subset, i.e. it contains all the identities and it is closed under composition. Moreover S satisfies other axioms, which make it a *localizing class* (see [Mil, Section 1.6 and Proposition 3.1.2]).

We define the localization of $\mathbf{K}(\mathbf{M})$ at S being the category $S^{-1}\mathbf{K}(\mathbf{M})$. It has the same objects of $\mathbf{K}(\mathbf{M})$ and it is naturally endowed with a *localization functor* $Q : \mathbf{K}(\mathbf{M}) \to S^{-1}\mathbf{K}(\mathbf{M})$, which is the identity on objects and sends quasi-isomorphisms to isomorphisms. A morphism of $S^{-1}\mathbf{K}(\mathbf{M})$ is the composition of a morphism in $\mathbf{K}(\mathbf{M})$ and the formal inverse of a morphism in S. More formally, $\varphi \in \mathrm{Hom}_{S^{-1}\mathbf{K}(\mathbf{M})}(M, N)$ can be written as

$$\varphi = Q(s_1)^{-1} \circ Q(\phi_1) = Q(\phi_2) \circ Q(s_2)^{-1}$$

for some $\phi_i \in \mathbf{K}(\mathbf{M})$ and $s_i \in S$, where the latter equality is guaranteed by the conditions on S. Morphisms can be also represented by diagrams called *right* and *left roofs*:

$$\varphi = \begin{array}{ccc} & L & \\ {}^{\phi_1}\nearrow & & \nwarrow^{s_1} \\ M & & N \end{array} = \begin{array}{ccc} & K & \\ {}^{s_2}\swarrow & & \searrow^{\phi_2} \\ M & & N \end{array}$$

Remark 2.9 ([Mil, Theorem 3.2.1])**.** Since all the homotopy equivalences in $\mathbf{C}(\mathbf{M})$ are quasi-isomorphism, if we denote with $S_{\mathbf{C}}$ the set of all quasi isomorphism of $\mathbf{C}(\mathbf{M})$, it turns out that the two localizations $S_{\mathbf{C}}^{-1}\mathbf{C}(\mathbf{M})$ and $S^{-1}\mathbf{K}(\mathbf{M})$ are equivalent categories. With our construction we wanted to emphasize the two-step process, which reflects the fact that resolutions can be in some sense unique, still remaining different from the original module.

The *derived category* of $\mathbf{M}$ is the category $\mathbf{D}(\mathbf{M}) = S^{-1}\mathbf{K}(\mathbf{M})$. This category inherits a triangulated structure from $\mathbf{K}(\mathbf{M})$ and the localization functor $Q : \mathbf{K}(\mathbf{M}) \to \mathbf{D}(\mathbf{M})$ is a triangulated functor. Moreover, it satisfies the following universal property.

Proposition 2.10. *Given a triangulated category* $\mathbf{T}$ *and a triangulated functor* $F : \mathbf{K}(\mathbf{M}) \to \mathbf{T}$ *such that* $F(s)$ *is an isomorphism for every* $s \in S$*, then there exists a unique triangulated functor* $S^{-1}F : \mathbf{D}(\mathbf{M}) \to \mathbf{T}$ *such that* $S^{-1}F \circ Q = F$.

Example 2.11. Go back to the Example 2.2. The morphism of complexes $\varepsilon : P^\bullet \to M^\bullet$ which is the natural projection $\varepsilon^0 : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ at degree $i = 0$ and $\varepsilon^i = 0$ for every $i \neq 0$ is a quasi-isomorphism. Indeed, $H^0(P^\bullet) = H^0(M^\bullet) = \mathbb{Z}/2\mathbb{Z}$ and $H^0(\varepsilon) = \mathbf{id}_{\mathbb{Z}/2\mathbb{Z}}$ and $H^i(P^\bullet) = H^i(M^\bullet) = 0$ and $H^i(\varepsilon) = 0$ for $i \neq 0$, so the cohomology functors induce isomorphisms in every degree. If we consider the complexes in the derived category, the quasi-isomorphism ε is an isomorphism with inverse $Q(\varepsilon)^{-1}$.
We have that $P^\bullet \cong M^\bullet$ in $\mathbf{D}(\mathbf{Mod}(\mathbb{Z}))$, while the same is not true in the other categories.
More in general this result holds for every projective (or injective) resolution of a module in the derived category.

2.2 Total Derived Functors

In the following we will restrict our work to the category of modules over a ring R. We will denote the derived category of $\mathbf{Mod}(R)$ with $\mathbf{D}(R)$, similarly for $\mathbf{C}(R)$ and $\mathbf{K}(R)$.
By Remark 2.6, given an additive functor $F : \mathbf{Mod}(R) \to \mathbf{Mod}(S)$ we constructed a triangulated functor $\mathbf{K}F : \mathbf{K}(R) \to \mathbf{K}(S)$. When the functor $\mathbf{K}F$ preserves quasi-isomorphism we can consider the functor $Q \circ \mathbf{K}F : \mathbf{K}(R) \to \mathbf{D}(S)$ and apply the Proposition 2.10 which allow us to define the lifted functor to derived category $\mathbf{D}F : \mathbf{D}(R) \to \mathbf{D}(S)$. Sadly, this does not happen very often. There are some relevant cases in which $\mathbf{K}F$ preserves quasi-isomorphisms $s : M^\bullet \to N^\bullet$, like when F is exact, or $M^\bullet$ and $N^\bullet$ are "special" complexes, for example they are bounded above complexes of projective modules (more generally K-projective complexes) or bounded below complexes of injective modules (more generally K-injective complexes). In this section we will show briefly how to construct these lifted functors. As we restricted our setting to the category of modules, it may be necessary to introduce some basic definitions and notation that could be used in the following.

Definition 2.12. Given a non zero complex $M^\bullet \in \mathbf{C}(R)$, define:

$$\sup(M^\bullet) = \sup\{i \in \mathbb{Z} \mid M^i \neq 0\} \in \mathbb{Z} \cup \{+\infty\}$$

$$\inf(M^\bullet) = \inf\{i \in \mathbb{Z} \mid M^i \neq 0\} \in \mathbb{Z} \cup \{-\infty\}$$

$$\operatorname{amp}(M^\bullet) = \sup(M^\bullet) - \inf(M^\bullet) \in \mathbb{N} \cup \{+\infty\}$$

The complex $M^\bullet$ is called *bounded above* if $\sup(M^\bullet) < +\infty$, *bounded below* if $\inf(M^\bullet) > -\infty$ and *bounded* if it is both bounded above and bounded below (if and only if $\operatorname{amp}(M^\bullet) < +\infty$). We will denote with $\mathbf{C}^{-,+,b}(R)$ the full subcategories of $\mathbf{C}(R)$ consisting of bounded above, bounded below and bounded complexes, respectively. Similarly, we write $\mathbf{K}^{-,+,b}(R)$ and $\mathbf{D}^{-,+,b}(R)$.
Analogous notation is used for the boundedness type of the cohomologies: the definitions of *cohomologically bounded* can be obtained from the above substituting $M^\bullet$ with $H(M^\bullet)$ and M^i with $H^i(M^\bullet)$. In this case, we will denote the full subcategories with $\mathbf{C}(R)^{-,+,b}$ (or $\mathbf{K}(R)^{-,+,b}$, $\mathbf{D}(R)^{-,+,b}$).
It can be checked that the inclusions $\mathbf{D}^{-,+,b}(R) \subseteq \mathbf{D}(R)^{-,+,b}$ are equivalences.

Before starting with the construction, let us see an example about why if the functor $\mathbf{K}F$ does not preserves quasi-isomorphism a lifted functor $\mathbf{D}F$ would have no meaning.

Example 2.13. Take $R = S = \mathbb{Z}$, $F = \Gamma_{(p)} : \mathbf{Mod}(\mathbb{Z}) \to \mathbf{Mod}(\mathbb{Z})$ the (p)-torsion functor, $M^\bullet = (0 \to \mathbb{Z} \to 0)$ and an injective resolution $I^\bullet = (0 \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0)$ with the quasi-isomorphism of complexes $\mu : M^\bullet \to I^\bullet$, which is the natural injection $\mu^0 : \mathbb{Z} \to \mathbb{Q}$ at degree $i = 0$ and $\mu^i = 0$ for every $i \neq 0$.
By the computation on Example 1.6 we have that $\mathbf{K}\Gamma_{(p)}(M^\bullet) = (0 \to 0 \to 0)$ while $\mathbf{K}\Gamma_{(p)}(I^\bullet) = (0 \to 0 \to \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to 0)$. Notice that in $\mathbf{K}(\mathbb{Z})$ the complexes $M^\bullet$ and $I^\bullet$ are quasi-isomophic but not isomorphic, while the complexes $\mathbf{K}\Gamma_{(p)}(M^\bullet)$ and $\mathbf{K}\Gamma_{(p)}(I^\bullet)$ are not even quasi-isomorphic.
If it would exist the lifted functor $\mathbf{D}\Gamma_{(p)} : \mathbf{D}(\mathbb{Z}) \to \mathbf{D}(\mathbb{Z})$ such that $\mathbf{D}\Gamma_{(p)} \circ Q = Q \circ \mathbf{K}\Gamma_{(p)}$ we would have a contradiction. Indeed on one hand $Q(M^\bullet)$ would be isomorphic to $Q(I^\bullet)$ through $Q(\mu)$, and so $\mathbf{D}\Gamma_{(p)}(Q(M^\bullet)) \cong \mathbf{D}\Gamma_{(p)}(Q(I^\bullet))$ but on the other hand we do not have any isomorphism between $Q(\mathbf{K}\Gamma_{(p)}(M^\bullet))$ and $Q(\mathbf{K}\Gamma_{(p)}(I^\bullet))$.

To construct another kind of lifted functors we need the notions of K-projective and K-injective resolutions of a complex of modules.

Definition 2.14.
Given two complexes M and N in $\mathbf{C}(R)$, we can define a new complex $\mathrm{Hom}^{\bullet}_R(M,N)$ where

$$\mathrm{Hom}^i_R(M,N) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(M^n, N^{n+i})$$

Given an element $\varphi = (\varphi_n)_{n\in\mathbb{Z}} \in \mathrm{Hom}^i_R(M,N)$ the differential is

$$d^i_{\mathrm{Hom}^{\bullet}_R(M,N)}(\varphi) = (d^{n+i}_N \circ \varphi_n - (-1)^i \varphi_{n+1} \circ d^n_M)_{n\in\mathbb{Z}}$$

Definition 2.15.
A complex $P \in \mathbf{K}(R)$ is called *K-projective* if for any acyclic (i.e. exact) complex $M \in \mathbf{K}(R)$ the complex $\mathrm{Hom}^{\bullet}_R(P,M)$ is acyclic. A *K-projective resolution* of a complex $M \in \mathbf{K}(R)$ is a quasi-isomorphism $\varepsilon_M : P_M \to M$ where $P_M \in \mathbf{K}(R)$ is K-projective.

The following results about K-projective resolutions can be found in [Spa88].

Proposition 2.16.

1. *Bounded above complexes of projective modules are K-projective.*
2. $\mathbf{K}(R)$ *has enough K-projectives, i.e. every complex M has a K-projective resolution. Moreover, if the complex is bounded above a K-projective resolution can be chosen of the type at point 1.*
3. *Given two complexes M, N with K-projective resolutions $\varepsilon_M : P_M \to M$ and $\varepsilon_N : P_N \to N$, for any morphism $\varphi : M \to N$ there exist a unique morphism $P_\varphi : P_M \to P_N$ such that there is a commutative diagram*

$$\begin{array}{ccc} P_M & \xrightarrow{\varepsilon_M} & M \\ {\scriptstyle P_\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ P_N & \xrightarrow{\varepsilon_N} & N \end{array}$$

   Moreover, if φ is a quasi-isomorphism then P_φ is an homotopy equivalence, i.e. an isomorphism.
4. *Let P be a K-projective complex. Then for every complex M the canonical map $\mathrm{Hom}_{\mathbf{K}(R)}(P,M) \to \mathrm{Hom}_{\mathbf{D}(R)}(P,M)$ is an isomorphism.*

Definition 2.17.
A complex $I \in \mathbf{K}(R)$ is called *K-injective* if for any acyclic complex $M \in \mathbf{K}(R)$ the complex $\mathrm{Hom}^{\bullet}_R(M,I)$ is acyclic. A *K-injective resolution* of a complex $M \in \mathbf{K}(R)$ is a quasi-isomorphism $\mu_M : M \to I_M$ where $I_M \in \mathbf{K}(R)$ is K-injective.

Proposition 2.18.

1. *Bounded below complexes of injective modules are K-injective.*
2. $\mathbf{K}(R)$ *has enough K-injectives, i.e. every complex M has a K-injective resolution. Moreover, if the complex is bounded below a K-injective resolution can be chosen of the type at point 1.*
3. *Given two complexes M, N with K-injective resolutions $\mu_M : M \to I_M$ and $\mu_N : N \to I_N$, for any morphism $\varphi : M \to N$ there exist a unique morphism $I_\varphi : I_M \to I_N$ such that there is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\mu_M} & I_M \\ \big\downarrow{\scriptstyle \varphi} & & \vdots{\scriptstyle I_\varphi} \\ N & \xrightarrow{\mu_N} & I_N \end{array}$$

Moreover, if φ is a quasi-isomorphism then I_φ is an homotopy equivalence, i.e. an isomorphism.

4. *Let I be a K-injective complex. Then for every complex M the canonical map $\mathrm{Hom}_{\mathbf{K}(R)}(M, I) \to \mathrm{Hom}_{\mathbf{D}(R)}(M, I)$ is an isomorphism.*

Remark 2.19. Similar results can be obtained for free and K-flat resolutions.
A *free resolution* of a complex $M \in \mathbf{K}(R)$ is a quasi-isomorphism $\varepsilon_M : F_M \to M$ where F_M is a bounded above complex of free modules. Every bounded above complex has a free resolution such that $\sup(F_M) = \sup(H(M))$.

The definition of K-flat resolution needs the definition of the tensor product of complexes.
Given two complexes M and N in $\mathbf{C}(R)$, we define the complex $(M \otimes_R N)^\bullet$, where

$$(M \otimes_R N)^i = \bigoplus_{p+q=i} M^p \otimes_R N^q$$

Given an elementary tensor $m \otimes n \in M^p \otimes_R N^q$ the differential is

$$d^{p+q}_{(M \otimes_R N)^\bullet}(m \otimes n) = d^p_M(m) \otimes n + (-1)^p m \otimes d^q_N(n)$$

A complex $P \in \mathbf{K}(R)$ is called *K-flat* if for any acyclic complex $M \in \mathbf{K}(R)$ the complex $(M \otimes_R P)^\bullet$ is acyclic. A *K-flat resolution* of a complex $M \in \mathbf{K}(R)$ is a quasi-isomorphism $\varepsilon_M : P_M \to M$ where $P_M \in \mathbf{K}(R)$ is K-flat. For this resolutions it holds a proposition analogue to the Proposition 2.16, in particular, bounded above complexes of flat modules are K-flat.

Definition 2.20. Given a functor $F : \mathbf{Mod}(R) \to \mathbf{Mod}(S)$, we have a well defined triangulated functor $F : \mathbf{K}(R) \to \mathbf{D}(S)$ (with some abuse of notation we call it F instead of $Q \circ \mathbf{K}F$).
For any complex M take a K-projective resolution $\varepsilon_M : P_M \to M$. The *total left derived functor of F* is the triangulated functor

$$\mathrm{L}F : \mathbf{D}(R) \to \mathbf{D}(S)$$

where $\mathrm{L}F(M) = F(P_M)$ and $\mathrm{L}F(\varphi) = \mathrm{L}F(\phi \circ s^{-1}) = F(P_\phi) \circ F(P_s)^{-1}$.
For any complex M take a K-injective resolution $\mu_M : M \to I_M$. The *total right derived functor of F* is the triangulated functor

$$\mathrm{R}F : \mathbf{D}(R) \to \mathbf{D}(S)$$

where $\mathrm{R}F(M) = F(I_M)$ and $\mathrm{R}F(\varphi) = \mathrm{R}F(\phi \circ s^{-1}) = F(I_\phi) \circ F(I_s)^{-1}$.

Moreover, this two total derived functors comes together with two natural transformation, which makes them unique in some sense. They are:

$$\eta : \mathrm{L}F \circ Q \to F \textit{ where } \eta_M = F(\varepsilon_M) : F(P_M) \to F(M)$$

$$\rho : F \to \mathrm{R}F \circ Q \textit{ where } \rho_M = F(\mu_M) : F(M) \to F(I_M)$$

It follows that the pair $(\mathrm{L}F, \eta)$ is terminal among all such pairs [Yek12, Definition 5.2.3], i.e. for every other pair (G, η') exist a unique natural transormation $\theta : G \to \mathrm{L}F$ such that $\eta' = \eta \circ \theta$. While the pair $(\mathrm{R}F, \rho)$ is initial among all such pairs [Yek12, Definition 5.2.1], i.e. for every other pair (H, ρ') exist a unique natural transormation $\theta : \mathrm{R}F \to H$ such that $\rho' = \theta \circ \rho$.

Remark 2.21. In general, total derived functors do not exist for every triangulated functor $F : \mathbf{K}(\mathbf{M}) \to \mathbf{T}$, where $\mathbf{M}$ is an abelian category and $\mathbf{T}$ is a triangulated category. In our setting the existence of the total derived functors $\mathrm{L}F$ and $\mathrm{R}F$ is guaranteed [Yek12, Corollary 6.3.6] by the fact that the category $\mathbf{K}(R)$ has enough K-projectives and enough K-injectives, respectively.

Example 2.22.
The complexes defined in Definition 2.14 and in Remark 2.19 are actually additive functors in $\mathbf{C}(R)$. So they can be extended to

$$\mathrm{Hom}^\bullet_R(\_,\_),\ \_\otimes_R\_^\bullet : \mathbf{K}(R)\times\mathbf{K}(R)\to\mathbf{D}(R)$$

Their total derived functors are

$$\mathrm{R}\,\mathrm{Hom}_R(\_,\_),\ \_\otimes^{\mathrm{L}}_R\_ : \mathbf{D}(R)\times\mathbf{D}(R)\to\mathbf{D}(R)$$

The first one can be computed using either K-projective or K-injective resolutions [Yek12, Lemma 14.5.1]:

$$\mathrm{R}\,\mathrm{Hom}_R(M,N)=\mathrm{Hom}^\bullet_R(P_M,N)\cong\mathrm{Hom}^\bullet_R(M,I_N)\cong\mathrm{Hom}^\bullet_R(P_M,I_N)$$

and the "usual" Ext-modules can be calculated also on complexes defining them as

$$\mathrm{Ext}^i_R(M,N)=H^i(\mathrm{R}\,\mathrm{Hom}_R(M,N))\ \textit{for every } i$$

The second one can be computed using K-projective (or K-flat) resolutions [Yek12, Lemma 14.4.4]:

$$M\otimes^{\mathrm{L}}_R N=(P_M\otimes_R N)^\bullet\cong(M\otimes_R P_N)^\bullet\cong(P_M\otimes_R P_N)^\bullet$$

We can not only extend the calculation of Ext-modules to complexes, but we can also obtain long exact sequences of Ext-modules from distinguished triangles, as was the case with short exact sequences. Indeed:

Proposition 2.23. *Given a distinguished triangle in* $\mathbf{D}(R)$

$$X\to Y\to Z\to X[1]$$

We get the long exact sequence in cohomology in $\mathbf{Mod}(R)$

$$\ldots\to H^{-1}(Z)\to H^0(X)\to H^0(Y)\to H^0(Z)\to H^1(X)\to\ldots$$

In particular, for any complex $M\in\mathbf{D}(R)$*, since* $\mathrm{R}\,\mathrm{Hom}_R(M,\_)$ *and* $\mathrm{R}\,\mathrm{Hom}_R(\_,M)$ *are triangulated functors, we get the long exact sequences*

$$\ldots\to\mathrm{Ext}^{-1}_R(M,Z)\to\mathrm{Ext}^0_R(M,X)\to\mathrm{Ext}^0_R(M,Y)\to\mathrm{Ext}^0_R(M,Z)\to\mathrm{Ext}^1_R(M,X)\to\ldots$$

$$\ldots\to\mathrm{Ext}^{-1}_R(X,M)\to\mathrm{Ext}^0_R(Z,M)\to\mathrm{Ext}^0_R(Y,M)\to\mathrm{Ext}^0_R(X,M)\to\mathrm{Ext}^1_R(Z,M)\to\ldots$$

Ext-modules of complexes over R are an important object of study, since they are able to detect morphisms in the derived category $\mathbf{D}(R)$.

Proposition 2.24. *Given two complexes* $M,N\in\mathbf{D}(R)$*. Then for every* $i\in\mathbb{Z}$

$$\mathrm{Ext}^i_R(M,N)\cong\mathrm{Hom}_{\mathbf{D}(R)}(M,N[i])$$

Proof. Given a K-projective resolution of $P\to M$ and a K-injective resolution $N\to I$, we know that $\mathrm{Ext}^i_R(M,N)\cong H^i(\mathrm{Hom}^\bullet_R(P,I))$.
Consider the differential $d^i=d^i_{\mathrm{Hom}^\bullet_R(P,I)}:\prod_{n\in\mathbb{Z}}\mathrm{Hom}_R(P^n,I^{n+i})\to\prod_{n\in\mathbb{Z}}\mathrm{Hom}_R(P^n,I^{n+i+1})$.

$$\varphi\in\mathrm{Ker}(d^i)\iff d^{n+i}_I\circ\varphi_n=(-1)^i\varphi_{n+1}\circ d^n_P\ \textit{for every } n\iff$$

$$\begin{array}{ccc} P^n & \xrightarrow{d^n_P} & P^{n+1}\\ \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n+1}}\\ I[i]^n & \xrightarrow{d^n_{I[i]}} & I[i]^{n+1}\end{array}\ \textit{commutes for every } n\iff\varphi\in\mathrm{Hom}_{\mathbf{C}(R)}(P,I[i])$$

Moreover, consider the differential $d^{i-1}:\prod_{n\in\mathbb{Z}}\mathrm{Hom}_R(P^n,I^{n+i-1})\to\prod_{n\in\mathbb{Z}}\mathrm{Hom}_R(P^n,I^{n+i})$.

$$\varphi\in\mathrm{Im}(d^{i-1})\iff\varphi_n=d^{n+i-1}_I\circ\psi_n-(-1)^{i-1}\psi_{n+1}\circ d^n_P\ \textit{for every } n\iff\text{Ignoring the sign}$$
$$\varphi_n=d^{n-1}_{I[i]}\circ\psi_n+\psi_{n+1}\circ d^n_P\ \textit{where } \psi_n:P^n\to I[i]^{n-1}\iff\varphi\text{ is homotopic to }0$$

It follows that

$$H^i(\mathrm{Hom}^\bullet_R(P,I))\cong\mathrm{Hom}_{\mathbf{K}(R)}(P,I[i])$$

And so by Proposition 2.16 (4), $\mathrm{Ext}^i_R(M,N)\cong\mathrm{Hom}_{\mathbf{D}(R)}(P,I[i])\cong\mathrm{Hom}_{\mathbf{D}(R)}(M,N[i])$ ☐

Chapter 3

The Global Case

The aim in this chapter is to study the finiteness properties of the complexes $\mathrm{R}\Gamma_{\mathfrak{a}}(M^{\bullet})$ in the category $\mathbf{D}(R)$, trying to generalize the results of the first chapter to any Noetherian ring R and any ideal $\mathfrak{a}$. In the end of the chapter we will focus on the failure of the Grothendieck's conjecture, explaining why given a finitely generated module M, the results which are not true for local cohomology modules $H^i_{\mathfrak{a}}(M)$ are instead true for the complex $\mathrm{R}\Gamma_{\mathfrak{a}}(M)$. Moreover, in this chapter we will study properties and relations between some triangulated subcategories of $\mathbf{D}(R)$, here is a glossary:

- $\mathbf{D}(R)_{\mathfrak{a}-com}$ is the full triangulated subcategory of *cohomologically* $\mathfrak{a}$*-adically complete* complexes.
- $\mathbf{D}(R)_{\mathfrak{a}-tor}$ is the full triangulated subcategory of *cohomologically* $\mathfrak{a}$*-torsion* complexes.
- $\mathbf{D}(R)_{\mathfrak{a}-cof}$ is the full triangulated subcategory of *cohomologically* $\mathfrak{a}$*-adically cofinite* complexes.
- $\mathbf{D}_{\mathfrak{a}-com}(R)$ is the full triangulated subcategory of complexes with $\mathfrak{a}$*-adically complete* cohomologies.
- $\mathbf{D}_{f}(R)$ is the full triangulated subcategory of complexes with *finitely generated* cohomologies.
- $\mathbf{D}_{\mathfrak{a}-cof}(R)$ is the full triangulated subcategory of complexes with $\mathfrak{a}$*-adically cofinite* cohomologies.

In the following we will suppose R to be a Noetherian ring and $\mathfrak{a}$ an ideal in it.

3.1 Completion and Torsion

In this section we will introduce two important total derived functors, constructed starting from two foundamental operations associated to an ideal $\mathfrak{a}$: the $\mathfrak{a}$-adic completion and the $\mathfrak{a}$-torsion. Recall that the $\mathfrak{a}$-adic completion of an R-module M, as defined in Definition 1.35, is

$$\Lambda_{\mathfrak{a}}(M) = \varprojlim_{i}(M/\mathfrak{a}^i M)$$

This operation defines an additive functor $\Lambda_{\mathfrak{a}} : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ and a functorial homomorphism $\tau_M : M \to \Lambda_{\mathfrak{a}}(M)$ for any $M \in \mathbf{Mod}(R)$, i.e. a natural transformation between the identity functor $\mathbf{id}_{\mathbf{Mod}(R)}$ and $\Lambda_{\mathfrak{a}}(\_)$. Given a complex $M^{\bullet}$ we will call $\tau_{M^{\bullet}}$ the morphism of complexes such that $\tau^i_{M^{\bullet}} = \tau_{M^i}$ for every i, which is again a natural transformation between the identity functor $\mathbf{id}_{\mathbf{C}(R)}$ and $\mathbf{C}\Lambda_{\mathfrak{a}}(\_)$.
A module M is called $\mathfrak{a}$*-adically complete* if τ_M is an isomorphism. We denote by $\mathbf{Mod}(R)_{\mathfrak{a}-com}$ the full subcategory of $\mathbf{Mod}(R)$ consisting of a-adically complete modules.
The functor $\Lambda_{\mathfrak{a}}$ is idempotent, in the sense that $\tau_{\Lambda_{\mathfrak{a}}(M)} : \Lambda_{\mathfrak{a}}(M) \to \Lambda_{\mathfrak{a}}(\Lambda_{\mathfrak{a}}(M))$ is an isomorphism for every module [Yek11, Corollary 3.5].

Remark 3.1. Idempotence of $\Lambda_{\mathfrak{a}}$ follows from Noetherianity of R. Indeed, it is always the case when the ideal $\mathfrak{a}$ is finitely generated. Moreover, if the ring R is both Noetherian and $\mathfrak{a}$-adically complete, then all finitely generated modules are $\mathfrak{a}$-adically complete [Definition 1.35 (6)].

As for any additive functor, the functor $\Lambda_{\mathfrak{a}}$ has a left derived functor

$$\mathrm{L}\Lambda_{\mathfrak{a}} : \mathbf{D}(R) \to \mathbf{D}(R)$$

constructed using K-projective resolutions, together with a homomorphism [Definition 2.20]

$$\eta_M : \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \to \Lambda_{\mathfrak{a}}(M^\bullet)$$

which is an isomorphism when $M^\bullet$ is a K-projective complex.

Theorem 3.2. *Let $M^\bullet \in \mathbf{D}(R)$. There is a functorial morphism*

$$\tau_M^{\mathrm{L}} : M^\bullet \to \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \text{ such that } \eta_M \circ \tau_M^{\mathrm{L}} = \tau_M$$

Proof. Choose a K-projective resolution $\varepsilon : P^\bullet \to M^\bullet$. Since both η_P and ε are isomorphisms in $\mathbf{D}(R)$, we can define

$$\tau_M^{\mathrm{L}} = \mathrm{L}\Lambda_{\mathfrak{a}}(\varepsilon) \circ \eta_P^{-1} \circ \tau_P \circ \varepsilon^{-1} : M^\bullet \to \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet)$$

which is the composition of two left roofs:

$$M \xleftarrow{\ \varepsilon\ } P \xrightarrow{\ \tau_P\ } \Lambda_{\mathfrak{a}}(P) \xleftarrow{\ \eta_P\ } \mathrm{L}\Lambda_{\mathfrak{a}}(P) \xrightarrow{\ \mathrm{L}\Lambda_{\mathfrak{a}}(\varepsilon)\ } \mathrm{L}\Lambda_{\mathfrak{a}}(M) .$$

□

Definition 3.3.
A complex $M^\bullet \in \mathbf{D}(R)$ is called *cohomologically $\mathfrak{a}$-adically complete* if the morphism τ_M^{L} is an isomorphism.
The full subcategory of $\mathbf{D}(R)$ consisting of cohomologically $\mathfrak{a}$-adically complete complexes is denoted by $\mathbf{D}(R)_{\mathfrak{a}-com}$.

A full subcategory $\mathbf{C}$ of a triangulated category $(\mathbf{T}, \Sigma)$ is called *triangulated subcategory* if it is invariant under the shift functor, i.e. $\Sigma(\mathbf{C}), \Sigma^{-1}(\mathbf{C}) \subseteq \mathbf{C}$, and for every distinguished triangle $X \to Y \to Z \to \Sigma(X)$ in $\mathbf{T}$ such that X and Y are in $\mathbf{C}$, then so is Z. This last condition is equivalent to the 2-out-of-3 version, which says that if two among X, Y, Z are in $\mathbf{C}$, then so is the third.

Theorem 3.4. *The subcategory $\mathbf{D}(R)_{\mathfrak{a}-com}$ is triangulated.*

Proof. Since the functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ commutes with the shifting, if $M \cong \mathrm{L}\Lambda_{\mathfrak{a}}(M)$ then for any $n \in \mathbb{Z}$ $M[n] \cong \mathrm{L}\Lambda_{\mathfrak{a}}(M)[n] \cong \mathrm{L}\Lambda_{\mathfrak{a}}(M[n])$ and so $\mathbf{D}(R)_{\mathfrak{a}-com}$ is closed under shifting.
Now suppose that $L \to M \to N \xrightarrow{[1]} L[1]$ is a distinguished triangle in $\mathbf{D}(R)$ such that L and M are cohomologically $\mathfrak{a}$-adically complete. The morphism τ^{L} induces a morphism of triangles

$$\begin{array}{ccccccc}
L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L[1] \\
\big\downarrow \tau_L^{\mathrm{L}} & & \big\downarrow \tau_M^{\mathrm{L}} & & \big\downarrow \tau_N^{\mathrm{L}} & & \big\downarrow \tau_L^{\mathrm{L}}[1] \\
\mathrm{L}\Lambda_{\mathfrak{a}}(L) & \longrightarrow & \mathrm{L}\Lambda_{\mathfrak{a}}(M) & \longrightarrow & \mathrm{L}\Lambda_{\mathfrak{a}}(N) & \longrightarrow & \mathrm{L}\Lambda_{\mathfrak{a}}(L)[1]
\end{array}$$

Since both τ_L^{L} and τ_M^{L} are isomorphisms, then by Lemma 2.4 so is τ_N^{L}. □

Proposition 3.5 ([PSY14, Proposition 3.6])**.** *If $P^\bullet$ is a K-flat complex then the morphism*

$$\eta_P : \mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet) \to \Lambda_{\mathfrak{a}}(P^\bullet)$$

is an isomorphism in $\mathbf{D}(R)$. Thus we can calculate $\mathrm{L}\Lambda_{\mathfrak{a}}$ using K-flat resolutions.

In what concerns the $\mathfrak{a}$-torsion operation, we have analogous results. Recall that the definition of the $\mathfrak{a}$-torsion of an R-module M is

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n)$$

This leads to an additive functor $\Gamma_{\mathfrak{a}} : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ and a functorial homomorphism $\sigma_M : \Gamma_{\mathfrak{a}}(M) \to M$, which is just the inclusion, for any $M \in \mathbf{Mod}(R)$, this gives a natural transformation between $\Gamma_{\mathfrak{a}}(\_)$ and the identity functor $\mathbf{id}_{\mathbf{Mod}(R)}$. Given a complex $M^\bullet$ we will call $\sigma_{M^\bullet}$ the morphism of complexes such that $\sigma^i_{M^\bullet} = \sigma_{M^i}$ for every i, which is again a natural transformation between $\mathbf{C}\Lambda_{\mathfrak{a}}(\_)$ and the identity functor $\mathbf{id}_{\mathbf{C}(R)}$.
Also in this case we denote with $\mathbf{Mod}(R)_{\mathfrak{a}-tors}$ the full subcategory of all $\mathfrak{a}$-*torsion* modules, i.e. the modules for which σ_M is an isomorphism.
The functor $\Gamma_{\mathfrak{a}}$ is always left-exact and idempotent, without any assumption on R neither on $\mathfrak{a}$.

Like every additive functor, the functor $\Gamma_{\mathfrak{a}}$ has a right derived functor

$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathbf{D}(R) \to \mathbf{D}(R)$$

constructed using K-injective resolutions, together with a homomorphism [Definition 2.20]

$$\rho_M : \Gamma_{\mathfrak{a}}(M^\bullet) \to \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)$$

which is an isomorphism when $M^\bullet$ is a K-injective complex.

Theorem 3.6. *Let $M^\bullet \in \mathbf{D}(R)$. There is a functorial morphism*

$$\sigma^{\mathrm{R}}_M : \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \to M^\bullet \text{ such that } \sigma^{\mathrm{R}}_M \circ \rho_M = \sigma_M$$

Proof. Choose a K-injective resolution $\mu : M^\bullet \to I^\bullet$. Since both ρ_I and μ_M are isomorphisms in $\mathbf{D}(R)$, we can define

$$\sigma^{\mathrm{R}}_M = \mu_M^{-1} \circ \sigma_I \circ \rho_I^{-1} \circ \mathrm{R}\Gamma_{\mathfrak{a}}(\mu_M) : \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \to M^\bullet$$

which is the composition of two right roofs:

$$\mathrm{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\mathrm{R}\Gamma_{\mathfrak{a}}(\mu_M)} \mathrm{R}\Gamma_{\mathfrak{a}}(I) \xleftarrow{\rho_I} \Gamma_{\mathfrak{a}}(I) \xrightarrow{\sigma_I} I \xleftarrow{\mu_M} M .$$ ☐

Definition 3.7.
A complex $M^\bullet \in \mathbf{D}(R)$ is called *cohomologically* $\mathfrak{a}$-*torsion* if the morphism σ^{R}_M is an isomorphism. The full subcategory of $\mathbf{D}(R)$ consisting of cohomologically $\mathfrak{a}$-torsion complexes is denoted by $\mathbf{D}(R)_{\mathfrak{a}-tor}$.

Theorem 3.8. *The subcategory $\mathbf{D}(R)_{\mathfrak{a}-tor}$ is triangulated.*

Proof. Same as proof of Theorem 3.4 (with some modifications). ☐

3.2 Cohomologically Complete Nakayama

Now we want to get a better understanding of cohomologically $\mathfrak{a}$-adically complete complexes and of the objects in the categories $\mathbf{Mod}(R)_{\mathfrak{a}-com}$ and $\mathbf{D}(R)_{\mathfrak{a}-com}$. The aim of this section is to prove a version of the Nakayama's Lemma in the latter category. To do this we will introduce some new objects.

Definition 3.9. Let Z be a set.
We denote by $F(Z, R)$ the set of all functions from $f : Z \to R$. This is an R-module, precisely

$$F(Z, R) \cong \prod_{z \in Z} R$$

Given such a function, the *support of* f is the subset of Z consisting of all the $z \in Z$ such that $f(z) \neq 0$. The subset of finite support functions is denoted by $F_{fin}(Z, R)$, this is a free R-module with basis the set $\{\delta_z\}_{z \in Z}$ of delta functions, namely $\delta_z(z) = 1$ and $\delta_z(z') = 0$ for any $z' \neq z$. In particular

$$F_{fin}(Z, R) \cong \bigoplus_{z \in Z} R$$

Let $\widehat{R} = \Lambda_{\mathfrak{a}}(R)$ and $\widehat{\mathfrak{a}} = \mathfrak{a} \cdot \widehat{R}$. In general, $\widehat{\mathfrak{a}}$ is an ideal of $\widehat{R}$ and if the ring R is Noetherian then $\widehat{R}$ is Noetherian and $\widehat{\mathfrak{a}}$-adically complete (see Definition 1.35 for other properties). Given an element $r \in \widehat{R}$, its $\mathfrak{a}$-*adic order* is

$$\mathrm{ord}_{\mathfrak{a}}(r) = \sup\{i \in \mathbb{N} \mid r \in \widehat{\mathfrak{a}}^i\} \in \mathbb{N} \cup \{\infty\}$$

A function $f : Z \to \widehat{R}$ is called $\mathfrak{a}$-*adically decaying* if for every $i \in \mathbb{N}$ the set

$$\{z \in Z \mid \mathrm{ord}_{\mathfrak{a}}(f(z)) \leq i\}$$

is finite. The subset of $\mathfrak{a}$-adically decaying functions is denoted by $F_{dec}(Z, \widehat{R})$, it is called the *module of decaying functions* and it is an $\widehat{R}$-submodule of $F(Z, \widehat{R})$. An R-module M is called $\mathfrak{a}$-*adically free* if it is isomorphic to $F_{dec}(Z, \widehat{R})$ for some set Z.

The following results show that the $\mathfrak{a}$-adically free module $F_{dec}(Z, \widehat{R})$ is actually the free object on Z in the category $\mathbf{Mod}(R)_{\mathfrak{a}-com}$. The proofs are presented in [Yek11, Section 3].

Theorem 3.10 ([PSY15, Theorem 1.5, Corollary 1.6]). *Let Z be a set:*

1. *$F_{dec}(Z, \widehat{R})$ is the $\mathfrak{a}$-adic completion of $F_{fin}(Z, R)$, i.e. $F_{dec}(Z, \widehat{R}) \cong \Lambda_{\mathfrak{a}}(F_{fin}(Z, R))$.*

2. *$F_{dec}(Z, \widehat{R})$ is a flat and $\mathfrak{a}$-adically complete R-module.*

3. *For any $\mathfrak{a}$-adically complete module M and any function $f : Z \to M$, there is a unique homomorphism $\phi : F_{dec}(Z, \widehat{R}) \to M$ such that $\phi(\delta_z) = f(z)$.*

In particular, any $\mathfrak{a}$-adically complete R-module is a quotient of an $\mathfrak{a}$-adically free R-module.

Definition 3.11.
An R-module P is called $\mathfrak{a}$-*adically projective* if it is a projective object in $\mathbf{Mod}(R)_{\mathfrak{a}-com}$.
In particular:

(i) P is an $\mathfrak{a}$-adically complete module,

(ii) Given two $\mathfrak{a}$-adically complete modules M and N, for any surjection $q : M \to N$ and any homomorphism $\varphi : P \to N$ there is a homomorphism $\tilde{\varphi} : P \to M$ such that $q \circ \tilde{\varphi} = \varphi$.

Corollary 3.12 ([PSY15, Corollary 1.7, Corollary 1.8]).

1. *An R-module is $\mathfrak{a}$-adically projective if and only if it is a direct summand of an $\mathfrak{a}$-adically free module.*

2. *Any $\mathfrak{a}$-adically projective R-module is flat over R.*

3. *Any $\mathfrak{a}$-adically complete R-module is a quotient of an $\mathfrak{a}$-adically projective R-module.*

4. *If P is a projective R-module then its $\mathfrak{a}$-adic completion $\Lambda_{\mathfrak{a}}(P)$ is $\mathfrak{a}$-adically projective.*

Here it is a characterization of bounded above cohomologically $\mathfrak{a}$-adically complete complexes.

Theorem 3.13. *The following conditions are equivalent for $M^\bullet \in \mathbf{D}(R)^-$:*

1. *$M^\bullet$ is cohomologically $\mathfrak{a}$-adically complete.*

2. *There is an isomorphism $M^\bullet \cong P^\bullet$ in $\mathbf{D}(R)$, where $P^\bullet$ is a bounded above complex of $\mathfrak{a}$-adically free modules and $\sup(P^\bullet) = \sup(H(M^\bullet))$.*

3. *There is an isomorphism $M^\bullet \cong P^\bullet$ in $\mathbf{D}(R)$, where $P^\bullet$ is a bounded above complex of $\mathfrak{a}$-adically projective modules.*

Proof. [1.⇒2.] Assume $M^\bullet \in \mathbf{D}(R)_{\mathfrak{a}-com}$, choose a free resolution $\varepsilon_M : F^\bullet \to M^\bullet$ and let $P^\bullet = \Lambda_\mathfrak{a}(F^\bullet)$. Notice that, for a suitable choice of $F^\bullet$, we have $\sup(P^\bullet) = \sup(F^\bullet) = \sup(H(M^\bullet))$. Since each component F^i is a free R-module it can be viewed as an $F_{fin}(Z_i, R)$ where Z_i is a basis of F^i, it follows that each P^i is an $\mathfrak{a}$-adically free module by Theorem 3.10 (1). Because $F^\bullet \cong M^\bullet$ in $\mathbf{D}(R)$, $F^\bullet$ is also cohomologically $\mathfrak{a}$-adically complete and so $\tau^{\mathrm{L}}_F : F^\bullet \to \mathrm{L}\Lambda_\mathfrak{a}(F^\bullet)$ is an isomorphism. Because $F^\bullet$ is also K-projective $\eta_F : \mathrm{L}\Lambda_\mathfrak{a}(F^\bullet) \to \Lambda_\mathfrak{a}(F^\bullet)$ is an isomorphism, and so

$$M^\bullet \cong F^\bullet \cong \mathrm{L}\Lambda_\mathfrak{a}(F^\bullet) \cong \Lambda_\mathfrak{a}(F^\bullet) = P^\bullet$$

[2.⇒3.] Any $\mathfrak{a}$-adically free module is also $\mathfrak{a}$-adically projective by Corollary 3.12 (1).

[3.⇒1.] Let $P^\bullet$ be a bounded above complex of $\mathfrak{a}$-adically projective modules. By definition each P^i is also $\mathfrak{a}$-adically complete and so $P^\bullet \cong \Lambda_\mathfrak{a}(P^\bullet)$ via τ_P. By Corollary 3.12 (2) each P^i is flat over R and so, by Remark 2.19, the complex $P^\bullet$ is K-flat. From the Proposition 3.5 follows that $\mathrm{L}\Lambda_\mathfrak{a}(P^\bullet) \cong \Lambda_\mathfrak{a}(P^\bullet)$ via η_P. So $\tau^{\mathrm{L}}_P = \eta_P^{-1} \circ \tau_P$ is an isomorphism in $\mathbf{D}(R)$, i.e. $P^\bullet$ is cohomologically $\mathfrak{a}$-adically complete, hence so is $M^\bullet$. □

Lemma 3.14.
Let R be a Noetherian ring, $\mathfrak{a}$-adically complete with respect to some ideal $\mathfrak{a}$. Let $R_0 = R/\mathfrak{a}$ and $\phi : M \to N$ be a homomorphism between $\mathfrak{a}$-adically complete R-modules. Then ϕ is surjective if and only if the induced homomorphism

$$\mathbf{id}_{R_0} \otimes \phi : R_0 \otimes_R M \to R_0 \otimes_R N$$

is surjective.

Proof. The first implication follows from the fact that the tensor product is a right exact functor. For the converse, assume that $\phi_0 = \mathbf{id}_{R_0} \otimes \phi$ is surjective. Choose a surjection $\psi : F_{dec}(Z, R) \to M$ for some set Z [Theorem 3.10]. We get a commutative diagram

$$\begin{array}{ccccc} F_{dec}(Z,R) & \xrightarrow{\psi} & M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle \pi_N} \\ F_{fin}(Z,R_0) & \xrightarrow{\psi_0} & R_0 \otimes_R M & \xrightarrow{\phi_0} & R_0 \otimes_R N \end{array}$$

Since ϕ_0 is surjective, then so is $\phi_0 \circ \psi_0 \circ \pi$. So, let $(\phi \circ \psi)(\delta_z) = n_z$, it follows that the set $\{\pi_N(n_z)\}_{z\in Z}$ generetes the R_0-module $R_0 \otimes_R N$.
According to [Yek11, Definition 2.10, Theorem 2.11], for every $n \in N$ exists a decaying function $g \in F_{dec}(Z, R)$ such that

$$n = \sum_{z\in Z} g(z) n_z = \sum_{z\in Z} g(z)(\phi\circ\psi)(\delta_z) = (\phi\circ\psi)\left(\sum_{z\in Z} g(z)\delta_z\right) = (\phi\circ\psi)(g)$$

i.e. $\phi \circ \psi$ is surjective. Hence ϕ is surjective. □

In the following there are the main results of this section. We will write $R_0 = R/\mathfrak{a}$.

Theorem 3.15 (**Cohomologically Complete Nakayama** [PSY15, Theorem 2.2])**.**
Let R be a Noetherian ring, $\mathfrak{a}$-adically complete with respect to some ideal $\mathfrak{a}$.
Let $M^\bullet \in \mathbf{D}(R)^-_{\mathfrak{a}-com}$ and $i_0 = \sup(H(M^\bullet))$. Assume that $H^{i_0}(R_0 \otimes^{\mathrm{L}}_R M^\bullet)$ is a finitely generated R_0-module. Then $H^{i_0}(M^\bullet)$ is a finitely generated R-module.

Proof. By shifting if needed, we may assume without loss of generality that $i_0 = 0$. By Theorem 3.13, we can replace $M^\bullet$ with a complex $P^\bullet$ of $\mathfrak{a}$-adically free modules such that $\sup(P^\bullet) = 0$. There is an exact sequence of R-modules

$$P^{-1} \xrightarrow{d} P^0 \xrightarrow{\xi} H^0(P^\bullet) \to 0$$

Since each P^i is also $\mathfrak{a}$-adically projective [Corollary 3.12 (1)] and $P^\bullet$ is bounded above, by Theorem 3.13, $P^\bullet$ is K-flat. Consider the functor

$$R_0 \otimes^{\mathrm{L}}_R \_ : \mathbf{D}(R) \to \mathbf{D}(R_0)$$

and notice that $R_0 \otimes^{\mathrm{L}}_R M^\bullet \cong R_0 \otimes_R P^\bullet$ in $\mathbf{D}(R_0)$. Setting $L_0 = H^0(R_0 \otimes_R P^\bullet)$ we have an exact sequence of R_0-modules

$$R_0 \otimes_R P^{-1} \xrightarrow{\mathbf{id}_{R_0} \otimes d} R_0 \otimes_R P^0 \xrightarrow{\nu} L_0 \to 0$$

Let $\{\overline{p_z}\}_{z \in Z}$ be a finite collection of elements of $R_0 \otimes_R P^0 \cong P^0/\mathfrak{a}P^0$ such that $\{\nu(\overline{p_z})\}_{z\in Z}$ generates L_0. Let

$$\theta_0 : F_{fin}(Z, R_0) \to R_0 \otimes_R P^0 \textit{ such that } \theta_0(\delta_z) = \overline{p_z}$$

Then the homomorphism

$$\psi_0 = (\mathbf{id}_{R_0} \otimes d, \theta_0) : (R_0 \otimes_R P^{-1}) \oplus F_{fin}(Z, R_0) \to R_0 \otimes_R P^0$$

is surjective.
For any $z \in Z$ choose a representative $p_z \in P^0$ of the element $\overline{p_z}$. We get the corresponding homomorphisms of R-modules $\theta : F_{fin}(Z, R) \to P^0$, such that $\theta(\delta_z) = p_z$, and $\psi = (d, \theta) : P^{-1} \oplus F_{fin}(Z, R) \to P^0$. Which fit into a commutative diagram

$$\begin{array}{ccc} P^{-1} \oplus F_{fin}(Z,R) & \xrightarrow{\quad\psi\quad} & P^0 \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ (R_0 \otimes_R P^{-1}) \oplus F_{fin}(Z, R_0) & \xrightarrow{\psi_0} & R_0 \otimes_R P^0 \end{array}$$

Where $(R_0 \otimes_R P^{-1}) \oplus F_{fin}(Z, R_0) \cong R_0 \otimes_R (P^{-1} \oplus F_{fin}(Z, R))$ and π, π' are the canonical surjections induced by $R \to R_0$. Since both $P^{-1} \oplus F_{fin}(Z, R)$ and P^0 are $\mathfrak{a}$-adically complete, by previous lemma the homomorphism ψ is surjective. So, we can conclude that $H^0(P^\bullet)$ is finitely generated by the collection $\{\xi(p_z)\}_{z\in Z}$. □

Lemma 3.16 (**Künneth Trick** [CFH22, Proposition 7.6.8 (b)]).
Let $M^\bullet$, $N^\bullet \in \mathbf{D}(R)^-$. If $i \geq \sup(H(M^\bullet))$ and $j \geq \sup(H(N^\bullet))$, then there is a canonical isomorphism of R-modules

$$H^{i+j}(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet) \cong H^i(M^\bullet) \otimes_R H^j(N^\bullet)$$

Proof. Take two K-flat resolutions $P \to M$ and $Q \to N$ such that $\sup(P^\bullet) = \sup(H(M^\bullet))$ and $\sup(Q^\bullet) = \sup(H(N^\bullet))$. Recall that $H^{i+j}(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet) \cong H^{i+j}(P^\bullet \otimes_R Q^\bullet)$. We have to compute this last cohomology on the sequence

$$(P^\bullet \otimes_R Q^\bullet)^{i+j-1} \xrightarrow{d^{i+j-1}} (P^\bullet \otimes_R Q^\bullet)^{i+j} \xrightarrow{d^{i+j}} 0$$

Which is

$$(P^{i-1} \otimes_R Q^j) \oplus (P^i \otimes_R Q^{j-1}) \xrightarrow{(d_P^{i-1} \otimes \mathbf{id}_{Q^j}, (-1)^i \mathbf{id}_{P^i} \otimes d_Q^{j-1})} P^i \otimes Q^j \to 0$$

It follows that

$$H^{i+j}(P^\bullet \otimes_R Q^\bullet) = (P^i \otimes Q^j)\big/S$$

where $S = \left\langle \{d_P^{i-1}(p') \otimes q'' + (-1)^i p'' \otimes d_Q^{j-1}(q') \mid p' \in P^{i-1},\ q' \in Q^{j-1},\ p'' \in P^i,\ q'' \in Q^j\} \right\rangle_R$.
With this identification the map $\varphi : H^i(P^\bullet) \otimes_R H^j(Q^\bullet) \to H^{i+j}(P^\bullet \otimes_R Q^\bullet)$ such that $\varphi(\overline{p} \otimes \overline{q}) = \overline{p \otimes q}$ is an homomorphism with inverse $\varphi^{-1}(\overline{p \otimes q}) = \overline{p} \otimes \overline{q}$. So,

$$H^{i+j}(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet) \cong H^{i+j}(P^\bullet \otimes_R Q^\bullet) \cong H^i(P^\bullet) \otimes_R H^j(Q^\bullet) \cong H^i(M^\bullet) \otimes_R H^j(N^\bullet).$$

□

Remark 3.17 ([Aut, Lemma 00DV]). *Classical version of Nakayama's lemma.* (We refer to Remark 3.33 for the defintion of Jacobson radical).
Let R be a commutative ring, $\mathfrak{a} \subseteq R$ an ideal, $J(R)$ its Jacobson radical and M an R-module. Then, if $M/\mathfrak{a}M = 0$, M is finitely generated and $I \subseteq J(R)$ then $M = 0$.

Corollary 3.18. *Let R be a Noetherian ring, $\mathfrak{a}$-adically complete with respect to some ideal $\mathfrak{a}$. Let $M^\bullet \in \mathbf{D}(R)^-_{\mathfrak{a}-com}$. If $R/\mathfrak{a} \otimes^{\mathrm{L}}_R M^\bullet = 0$ then $M^\bullet = 0$.*

Proof. Assume by contradiction that $R_0 \otimes^{\mathrm{L}}_R M^\bullet = 0$ and $M^\bullet \neq 0$. Let $i_0 = \sup(H(M^\bullet))$.
By Lemma 3.16 $H^{i_0}(R/\mathfrak{a} \otimes^{\mathrm{L}}_R M^\bullet) = R/\mathfrak{a} \otimes_R H^{i_0}(M^\bullet)$. Hence $R/\mathfrak{a} \otimes_R H^{i_0}(M^\bullet) = 0$ and by the cohomologically complete Nakayama $H^{i_0}(M^\bullet)$ is a finitely generated R-module. So, by the Classical version of Nakayama's lemma and by Proposition 3.34, we can conclude that $H^{i_0}(M^\bullet) = 0$. This is a contradiction. □

3.3 MGM Equivalence

In this section we will show an important result concerning the categories of bounded cohomologically $\mathfrak{a}$-adically complete complexes $\mathbf{D}(R)^b_{\mathfrak{a}-com}$ and the category of bounded cohomologically $\mathfrak{a}$-torsion complexes $\mathbf{D}(R)^b_{\mathfrak{a}-tor}$. In particular, it turns out that the two categories are equivalent [Theorem 3.29].
We start the section showing that the two derived functors $\mathrm{L}\Lambda_{\mathfrak{a}}(\_)$ and $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ are idempotent.

Theorem 3.19. *1. For any $M^\bullet \in \mathbf{D}(R)^-$ the morphism*

$$\tau^{\mathrm{L}}_{\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet)} : \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \to \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet))$$

is an isomorphism. So the functor

$$\mathrm{L}\Lambda_{\mathfrak{a}}(\_) : \mathbf{D}(R)^- \to \mathbf{D}(R)^-_{\mathfrak{a}-com}$$

is idempotent.

2. For any $M^\bullet \in \mathbf{D}(R)^+$ the morphism

$$\sigma^{\mathrm{R}}_{\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)} : \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \to \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet))$$

is an isomorphism. So the functor

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\_) : \mathbf{D}(R)^+ \to \mathbf{D}(R)^+_{\mathfrak{a}-tor}$$

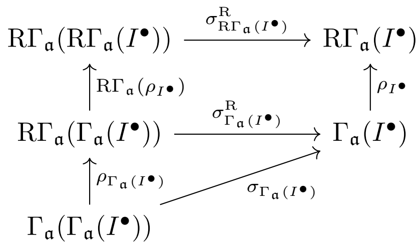
is idempotent.

Proof of 1. By Proposition 2.16, we can replace $M^\bullet$ with a bounded above complex of projectives $P^\bullet$. Consider the commutative diagram in $\mathbf{D}(R)$

$$\begin{array}{ccc}
\mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet) & \xrightarrow{\tau^{\mathrm{L}}_{\mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet)}} & \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet)) \\
\downarrow \eta_{P^\bullet} & & \downarrow \mathrm{L}\Lambda_{\mathfrak{a}}(\eta_{P^\bullet}) \\
\Lambda_{\mathfrak{a}}(P^\bullet) & \xrightarrow{\tau^{\mathrm{L}}_{\Lambda_{\mathfrak{a}}(P^\bullet)}} & \mathrm{L}\Lambda_{\mathfrak{a}}(\Lambda_{\mathfrak{a}}(P^\bullet)) \\
& \searrow^{\tau_{\Lambda_{\mathfrak{a}}(P^\bullet)}} & \downarrow \eta_{\Lambda_{\mathfrak{a}}(P^\bullet)} \\
& & \Lambda_{\mathfrak{a}}(\Lambda_{\mathfrak{a}}(P^\bullet))
\end{array}$$

Since $P^\bullet$ is a K-projective complex, the morphism $\eta_{P^\bullet}$ is an isomorphism and so is $\mathrm{L}\Lambda_{\mathfrak{a}}(\eta_{P^\bullet})$. By Corollary 3.12 (4) $\Lambda_{\mathfrak{a}}(P^\bullet)$ is a bounded above complex of $\mathfrak{a}$-adically projective modules and so by (2) it is a K-flat complex. From Proposition 3.5 we can conclude that also $\eta_{\Lambda_{\mathfrak{a}}(P^\bullet)}$ is an isomorphism. Since the completion functor is idempotent $\tau_{\Lambda_{\mathfrak{a}}(P^\bullet)}$ is an isomorphism too. It follows that $\tau^{\mathrm{L}}_{\Lambda_{\mathfrak{a}}(P^\bullet)}$ and $\tau^{\mathrm{L}}_{\mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet)}$ are both isomorphism.

Proof of 2. The logic of the proof is the same, just consider a bounded below complex of injectives $I^\bullet$, consider the following diagram and use the fact that over a Noetherian ring $\Gamma_{\mathfrak{a}}(\_)$ of an injective module is again injective [Corollary 1.31].



□

Recall that a triangulated functor $F : \mathbf{D}(R) \to \mathbf{D}(R)$ is said to have finite *cohomological dimension* if $\operatorname{amp}(H(F(M^\bullet))) \leq \operatorname{amp}(H(M^\bullet)) + d$ for some $d \in \mathbb{N}$, for all $M^\bullet \in \mathbf{D}(R)$.

Corollary 3.20. *For any bounded complex $M^\bullet \in \mathbf{D}(R)^b$ one has*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \in \mathbf{D}(R)^b_{\mathfrak{a}-tor} \text{ and } \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \in \mathbf{D}(R)^b_{\mathfrak{a}-com}$$

Proof. This is a consequence of the fact that $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ and $\mathrm{L}\Lambda_{\mathfrak{a}}(\_)$ have finite cohomological dimenson [PSY14, Corollary 4.28, Corollary 5.27]. □

Before continuing with further results we have to talk about *Koszul complexes.*

Definition 3.21. Given an element $x \in R$ the *Koszul complex of R on x* is

$$K(R;x) = (0 \to R \xrightarrow{x\cdot} R \to 0)$$

concentrated in degrees $[-1,0]$. Given a finite sequence $\mathbf{x} = (x_1,\ldots,x_n)$ of elements of R, the Koszul complex associated to $\mathbf{x}$ is

$$K(R;\mathbf{x}) = K(R;x_1) \otimes_R \cdots \otimes_R K(R;x_n)$$

This is a complex of finitely generated free R-modules, concentrated in degrees $[-n,0]$.
In particular, $K^{-i} = R^{\binom{n}{i}}$ for $0 \leq i \leq n$.

Example 3.22.
The complexes $P_n^\bullet$ in Example 1.42 are Koszul complexes, precisely $P_n^\bullet = K(R;u^n,v^n)$.

Two main properties of Koszul complexes are presented in the following, see [ILL+07, Chapter 6, Sections 1-3] for more details.

Proposition 3.23.
Let $\mathfrak{a} = (a_1,\ldots,a_n)$ be an ideal in R and $K^\bullet = K(R;a_1,\ldots,a_n)$ the associated Koszul complex. Then:

1. *$H^0(K^\bullet) \cong R/\mathfrak{a}$ and $\mathfrak{a} \subseteq (0:_R H^i(K^\bullet))$ for any $i \in \mathbb{Z}$.*

   In particular, $H^i(K^\bullet)$ is a finitely generated $R/\mathfrak{a}$-module for every i.

2. *Let $K^{\bullet\vee} := \operatorname{Hom}^\bullet_R(K^\bullet,R)$. The Koszul complex has the following self-duality property*

$$K^\bullet \cong K^{\bullet\vee}[n]$$

Example 3.24.
Consider the Koszul complex, concetrated in degrees $[-2,0]$, associated to two elements x,y

$$K(R;x,y) = (0 \to R \xrightarrow{\begin{bmatrix}-y\\ x\end{bmatrix}} R \oplus R \xrightarrow{[x\ y]} R \to 0)$$

The dual complex $\mathrm{Hom}_R(K(R;x,y),R)$, concetrated in degrees $[0,2]$, is

$$K^\vee(R;x,y) = (0 \to R \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} R \oplus R \xrightarrow{[-y\ x]} R \to 0)$$

The homomorphism of complexes $\varphi : K(R;x,y) \to K^\vee(R;x,y)[2]$ such that

$$\varphi^{-2} = id_R,\ \varphi^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \varphi^0 = -id_R$$

is an isomorphism.

For any $i \geq 1$, if $\mathbf{x} = (x_1,\ldots,x_n)$, we define $\mathbf{x}^{(i)} = (x_1^i,\ldots,x_n^i)$. If $j \geq i$ then there is a homomorphism of complexes

$$p_{i,j} : K(R;\mathbf{x}^{(j)}) \to K(R;\mathbf{x}^{(i)})$$

which, in degree 0, corresponds to the canonical surjection $R/\mathbf{x}^{(j)} \twoheadrightarrow R/\mathbf{x}^{(i)}$.
Thus the set $\{K(R;\mathbf{x}^{(i)})\}_{i\in\mathbb{N}}$ forms an inverse system of complexes, while $\{K^\vee(R;\mathbf{x}^{(i)})\}_{i\in\mathbb{N}}$ forms a direct system of complexes. We call *infinite dual Koszul complex* associated to $\mathbf{x}$, the complex

$$K_\infty^\vee(R;\mathbf{x}) = \varinjlim_i K^\vee(R;\mathbf{x}^{(i)})$$

Here is a characterization of infinite dual Koszul complexes, see [PSY14, Section 4] for more details.

Proposition 3.25.
Given an element $x \in R$ the infinite dual Koszul complex on x is

$$K_\infty^\vee(R;x) = (0 \to R \xrightarrow{d} R[x^{-1}] \to 0)$$

where R is in degree 0, $R[x^{-1}]$ is the localization of R by the multiplicatively closed subset $S = \{1,x,x^2,\ldots\}$ and the differential $d : R \to R[x^{-1}]$ is the localization map.

Moreover, given a finite sequence $\mathbf{x} = (x_1,\ldots,x_n)$ of elements of R, the infinite dual Koszul complex associated to this sequence is

$$K_\infty^\vee(R;\mathbf{x}) = K_\infty^\vee(R;x_1) \otimes_R \cdots \otimes_R K_\infty^\vee(R;x_n)$$

In particular, infinite dual Koszul complexes are bounded complexes of flat R-modules, thus they are K-flat.

The infinite dual Koszul complexes turn out to be very useful for the computation of derived torsion and, moreover, they have nice properties in the context of completions. Indeed, there are some isomorphisms involving them which are worth to be mentioned. We want to summarize them in the following.

Proposition 3.26. *Given an ideal $\mathfrak{a} = (a_1,\ldots,a_n)$ of R and a complex $M^\bullet$, there is a canonical isomorphisms in $\mathbf{D}(R)$:*

$$\text{(1)}\ \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \cong K_\infty^\vee(R;a_1,\ldots,a_n) \otimes_R M^\bullet$$

Moreover, if $P^\bullet$ is a K-flat complex, then:

$$\text{(2)}\ \Lambda_{\mathfrak{a}}(K_\infty^\vee(R;a_1,\ldots,a_n) \otimes_R P^\bullet) \cong \Lambda_{\mathfrak{a}}(P^\bullet)$$

$$\text{(3)}\ K_\infty^\vee(R;a_1,\ldots,a_n) \otimes_R \Lambda_{\mathfrak{a}}(P^\bullet) \cong K_\infty^\vee(R;a_1,\ldots,a_n) \otimes_R P^\bullet$$

Proof. Taking into account the result in [PSY14, Theorem 4.34]:

(1) follows form [PSY14, Corollary 4.26].

(2) follows from [PSY14, Equation 7.3 in Lemma 7.2].

(3) follows from [PSY14, Equation 7.7 in Lemma 7.6] combined with [PSY14, Lemma 5.7, Corollary 5.21].

□

There are isomorphisms involving the two derived functors $\mathrm{L}\Lambda_{\mathfrak{a}}(\_)$ and $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ which will allow us to show that they are actually inverse to each other. We will give a sketch of the proofs, showing that the isomorphisms hold without writing them explicitly. The complete results are proved in [PSY14, Lemma 7.2, Lemma 7.6].

Proposition 3.27. *For any $M \in \mathbf{D}(R)^b$ the morphism*

$$L\Lambda_{\mathfrak{a}}(\sigma^{\mathrm{R}}_{M^\bullet}) : \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) \to \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet)$$

is an isomorphism.

Proof. Let $\mathfrak{a} = (a_1, \dots, a_n)$ and $\varepsilon : P^\bullet \to M^\bullet$ a K-flat resolution, with $P^\bullet$ a bounded above complex of flat modules. Since direct sums and tensor products of flat modules are flat, $K^\vee_\infty(R; a_1, \dots, a_n) \otimes_R P^\bullet$ is still K-flat. By Proposition 3.26 (1), (2)

$$\mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(P^\bullet)) \cong \mathrm{L}\Lambda_{\mathfrak{a}}(K^\vee_\infty(R; a_1, \dots, a_n) \otimes_R P^\bullet) = \Lambda_{\mathfrak{a}}(K^\vee_\infty(R; a_1, \dots, a_n) \otimes_R P^\bullet) \cong \Lambda_{\mathfrak{a}}(P^\bullet)$$

and so we can conclude that

$$\mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) \cong \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(P^\bullet)) \cong \Lambda_{\mathfrak{a}}(P^\bullet) = \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet)$$

□

Proposition 3.28. *For any $M \in \mathbf{D}(R)^b$ the morphism*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\tau^{\mathrm{L}}_{M^\bullet}) : \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \to \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet))$$

is an isomorphism.

Proof. Let $\mathfrak{a} = (a_1, \dots, a_n)$ and $\varepsilon : P^\bullet \to M^\bullet$ a K-flat resolution, with $P^\bullet$ a bounded above complex of flat modules. By Proposition 3.26 (1), (3)

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet)) \cong K^\vee_\infty(R; a_1, \dots, a_n) \otimes_R \Lambda_{\mathfrak{a}}(P^\bullet) \cong K^\vee_\infty(R; a_1, \dots, a_n) \otimes_R P^\bullet \cong \mathrm{R}\Gamma_{\mathfrak{a}}(P^\bullet)$$

and so we can conclude that

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet)) \cong \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(P^\bullet)) \cong \mathrm{R}\Gamma_{\mathfrak{a}}(P^\bullet) \cong \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)$$

□

Now we can prove the main result of this section.

Theorem 3.29 (MGM equivalence).
The functor $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ induces an equivalence of triangulated categories

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\_) : \mathbf{D}(R)^b_{\mathfrak{a}-com} \to \mathbf{D}(R)^b_{\mathfrak{a}-tor}$$

with quasi-inverse $\mathrm{L}\Lambda_{\mathfrak{a}}(\_)$.

Proof. The two maps are well-defined by Corollary 3.20.
Moreover, let $N^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-com}$, i.e. $N^\bullet \cong \mathrm{L}\Lambda_{\mathfrak{a}}(N^\bullet)$. Applying Proposition 3.27

$$\mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(N^\bullet)) \cong \mathrm{L}\Lambda_{\mathfrak{a}}(N^\bullet) \cong N^\bullet.$$

On the other hand, let $M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-tor}$, i.e. $M^\bullet \cong \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)$. Applying Proposition 3.28

$$\mathrm{R}\Gamma_{\mathfrak{a}}(L\Lambda_{\mathfrak{a}}(M^\bullet)) \cong \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \cong M^\bullet.$$

□

Remark 3.30.
With some extra work, using the way-out argument [PSY14, Proposition 2.9], one can prove that this result is true also without the boundedness assumptions. A stronger version of the MGM equivalence can be found in [PSY14, Theorem 7.11 (2)].

3.4 Cohomologically Cofinite Complexes

In this section we want to go deeper in the understanding of the MGM equivalence and show how this can be restricted to two subcategories of complexes which have important finiteness properties. We will end the chapter with two results which are a generalization of Theorem 1.39 and Corollary 1.40 in the category $\mathbf{D}(R)$. In the next, all the rings R are supposed to be Noetherian and $\mathfrak{a}$-adically complete.

Remark 3.31 ([Yek11, Section 1 and 2])**.** Before starting the section, we want to give a topological interpretation of $\mathfrak{a}$-adic completion. Given an ideal $\mathfrak{a} \subseteq R$, any R-module M can be naturally endowed with a structure of a topological module, called $\mathfrak{a}$-*adic topology*, in which the collection of submodules $\{\mathfrak{a}^i M\}_{i\geq 0}$ is a basis of open neighborhoods of the element 0. It turns out that any homomorphism of R-modules $f : M \to N$ is continuous with respect to the $\mathfrak{a}$-adic topology.

Now consider an R-module M such that $\bigcap_{i\geq 0} \mathfrak{a}^i M = 0$, i.e. $\tau_M : M \to \widehat{M}$ is injective.

Given an element $m \in M$, we can define its $\mathfrak{a}$-adic order as

$$\mathrm{ord}_{\mathfrak{a}}(m) = \sup\{i \in \mathbb{N} \mid m \in \mathfrak{a}^i M\}$$

For any two elements $m, n \in M$, define

$$\mathrm{dist}_{\mathfrak{a}}(m, n) = \left(\frac{1}{2}\right)^{\mathrm{ord}_{\mathfrak{a}}(m-n)}$$

The function $\mathrm{dist}_{\mathfrak{a}}$ is a metric on M, which we call the $\mathfrak{a}$-*adic metric*.
The module M is $\mathfrak{a}$-adically complete if and only if it is a complete metric space with respect to the $\mathfrak{a}$-adic metric.
Given a set Z and a function $f : Z \to M$, one says that the series $\sum_{z\in Z} f(z)$ *converge* to some element $m \in M$ for the $\mathfrak{a}$-adic topology, if for any natural number $i \geq 0$ there is a finite subset $Z_i \subseteq Z$, such that

$$f(z) \in \mathfrak{a}^{i+1}M \textit{ for all } z \notin Z_i \textit{ and } m - \sum_{z\in Z_i} f(z) \in \mathfrak{a}^{i+1}M$$

In this case, one can write $m = \sum_{z\in Z} f(z)$.

Proposition 3.32 ([Yek11, Proposition 2.5])**.**
Let M be an $\mathfrak{a}$-adically complete R-module and let $f : Z \to M$ be a function. Then the series $\sum_{z\in Z} f(z)$ converges in M for the $\mathfrak{a}$-adic topology if and only if f is decaying.

Remark 3.33.
Recall that the *Jacobson radical* $J(R)$ of a ring R is the intersection of all maximal ideals of R. Given an ideal $\mathfrak{a} \subseteq R$, it turns out that $\mathfrak{a} \subseteq J(R)$ if and only if for every $x \in \mathfrak{a}$ the element $1 - x$ is invertible in R [Aut, Lemma 0AME].

Proposition 3.34.
Let R be an $\mathfrak{a}$-adically complete ring, then the ideal $\mathfrak{a}$ is contained in the Jacobson radical $J(R)$.

Proof. Given an element $x \in \mathfrak{a}$, let $Z = \{1, x, x^2, \dots\}$ and $f : Z \to R$ the inclusion. Notice that the order of each element is $\mathrm{ord}_{\mathfrak{a}}(x^i) \geq i$ and so, for every $i \geq 0$ the set $\{z \in Z \mid \mathrm{ord}_{\mathfrak{a}}(f(z)) \leq i\}$ is finite, i.e. f is decaying. We can conclude that, by the previous proposition, the series $\sum_{z\in Z} f(z)$ converges, and so there is a well defined element $r = 1 + x + x^2 + \dots$ in R, which is the inverse of $1 - x$. □

Remark 3.35 ([Yek11, Corollary 2.6])**.** Recall that from Theorem 3.10 (3), given an $\mathfrak{a}$-adically complete module M and a function $f : Z \to M$, there is a unique homomorphism $\phi : F_{dec}(Z, \widehat{R}) \to$

M such that $\phi(\delta_z) = f(z)$. It can be proved that for any decaying function $g : Z \to \widehat{R}$ the image of g is the series $\phi(g) = \sum_{z \in Z} g(z)f(z)$ and this converges in M for the $\mathfrak{a}$-adic topology, i.e. there is a well defined element $m = \sum_{z \in Z} g(z)f(z) \in M$.

The next result is a particular case of Theorem 3.13 (3) stated for $\mathfrak{a}$-adically complete modules, but using the topological interpretation of $\mathfrak{a}$-adic completion and the notion of convergence in Remark 3.31, it can be proved without referring to it.

Proposition 3.36. *Let M be an $\mathfrak{a}$-adically complete R-module. Then there is a quasi-isomorphism $P^\bullet \to M$, where $P^\bullet$ is a bounded above complex of $\mathfrak{a}$-adically projective R-modules.*

Proof. By Corollary 3.12 (3) there is an $\mathfrak{a}$-adically projective module P^0 and a surjection $\pi : P^0 \to M$. The module $K^0 = \mathrm{Ker}(\pi)$ is a closed submodule of P^0 with respect to the $\mathfrak{a}$-adic topology. Choose a collection of elements $\{n_z\}_{z \in Z}$ in K^0 that generates K^0 as an R-module. Consider the function $f : Z \to K^0$ such that $f(z) = n_z$ and the homomorphism $\phi : F_{dec}(Z, \widehat{R}) \to M$. Because K^0 is closed, it follows that the elements $\phi(g) = \sum_{z \in Z} g(z)n_z$ are in K^0. Writing $P^{-1} = F_{dec}(Z, \widehat{R})$, we have constructed a surjection $\phi : P^{-1} \to K^0$. Iterating this last step, we obtain a bounded above complex $P^\bullet$ of $\mathfrak{a}$-adically projective modules such that $H^i(P^\bullet) = 0$ for all $i \neq 0$ and $H^0(P^\bullet) \cong M$. So, the natural projection π induce a quasi-isomorphism. □

In the following we will prove some results using a very common tecnique in homological algebra which allow us to apply induction on complexes. In particular, it is the induction on $\mathrm{amp}(H(M^\bullet))$. To do this we need to know *smart truncations*. Given a complex $M \in \mathbf{D}(R)$ and $n \in \mathbb{Z}$, we define the complexes $\chi_{\leq n}(M)^\bullet$ and $\chi_{>n}(M)^\bullet$, where:

$$\chi_{\leq n}(M)^i = \begin{cases} M^i & if\ i < n \\ \mathrm{Ker}(d^n) & if\ i = n \\ 0 & if\ i > n \end{cases} \quad and \quad \chi_{>n}(M)^i = \begin{cases} 0 & if\ i < n \\ \mathrm{Im}(d^n) & if\ i = n \\ M^i & if\ i > n \end{cases}$$

The morphisms $\iota : \mathrm{Ker}(d^n) \hookrightarrow M^n$ and $d^n : M^n \to \mathrm{Im}(d^n)$ induce the distinguished triangle

$$\chi_{\leq n}(M)^\bullet \to M^\bullet \to \chi_{>n}(M)^\bullet \xrightarrow{[1]}$$

Moreover,

$$H^i(\chi_{\leq n}(M)^\bullet) = \begin{cases} H^i(M^\bullet) & if\ i \leq n \\ 0 & if\ i > n \end{cases} \quad and \quad H^i(\chi_{>n}(M)^\bullet) = \begin{cases} 0 & if\ i \leq n \\ H^i(M^\bullet) & if\ i > n \end{cases}$$

and so,

$$H^i(M^\bullet) = H^i(\chi_{\leq n}(M)^\bullet) \oplus H^i(\chi_{>n}(M)^\bullet)$$

Recall that we denote by $\mathbf{D}_{\mathfrak{a}-com}(R)^b$ the subcategory of bounded complexes $M^\bullet$ whose cohomologies $H^i(M^\bullet)$ are $\mathfrak{a}$-adically complete R-modules.

Corollary 3.37. *Let $M^\bullet \in \mathbf{D}_{\mathfrak{a}-com}(R)^b$. Then $M^\bullet$ is a cohomologically $\mathfrak{a}$-adically complete complex, i.e. $\mathbf{D}_{\mathfrak{a}-com}(R)^b \subseteq \mathbf{D}(R)^b_{\mathfrak{a}-com}$.*

Proof. By induction on $\mathrm{amp}(H(M^\bullet))$. If the amplitude of $H(M^\bullet)$ is 0, then we can assume $M^\bullet = (0 \to M \to 0)$ is a single $\mathfrak{a}$-adically complete module (sitting in degree 0). Applying Proposition 3.36 and Theorem 3.13 it follows that $M^\bullet$ is cohomologically $\mathfrak{a}$-adically complete.
If $\mathrm{amp}(H(M^\bullet)) > 0$, let $n = \sup(H(M^\bullet)) - 1$ and using smart truncations we get a distinguished triangle

$$M' \to M^\bullet \to M'' \xrightarrow{[1]}$$

where $M' = \chi_{\leq n}(M^\bullet)$ and $M'' = \chi_{>n}(M^\bullet)$ have smaller amplitudes and their cohomologies are still $\mathfrak{a}$-adically complete modules, i.e. M' and M'' are in $\mathbf{D}(R)_{\mathfrak{a}-com}$. By Theorem 3.4, $\mathbf{D}(R)_{\mathfrak{a}-com}$ is a triangulated subcategory of $\mathbf{D}(R)$ and so it contains M too. □

Recall that we denote by $\mathbf{D}_f(R)^b$ the subcategory of bounded complexes $M^\bullet$ whose cohomologies $H^i(M^\bullet)$ are finitely generated R-modules.

Corollary 3.38. *Let $M^\bullet \in \mathbf{D}_f(R)^b$. Then $M^\bullet$ is a cohomologically $\mathfrak{a}$-adically complete complex, i.e. $\mathbf{D}_f(R)^b \subseteq \mathbf{D}(R)^b_{\mathfrak{a}-com}$.*

Proof. Since we assume R be Noetherian and $\mathfrak{a}$-adically complete, any finitely generated R-module is $\mathfrak{a}$-adically complete [Definition 1.35 (6)]. So $\mathbf{D}_f(R)^b \subseteq \mathbf{D}_{\mathfrak{a}-com}(R)^b$ and by the last corollary we get the thesis. □

Definition 3.39.
A complex $M^\bullet \in \mathbf{D}(R)^b$ is called *cohomologically $\mathfrak{a}$-adically cofinite* if $M^\bullet \cong \mathrm{R}\Gamma_{\mathfrak{a}}(N^\bullet)$ for some $N^\bullet \in \mathbf{D}_f(R)^b$. We will denote the full subcategory of cohomologically $\mathfrak{a}$-adically cofinite complexes with $\mathbf{D}(R)^b_{\mathfrak{a}-cof}$.
By idempotence of the functor $\mathrm{R}\Gamma_{\mathfrak{a}}$, it follows that $\mathbf{D}(R)^b_{\mathfrak{a}-cof} \subseteq \mathbf{D}(R)^b_{\mathfrak{a}-tor}$

Proposition 3.40. *The subcategory $\mathbf{D}_f(R)^b$ is triangulated.*

Proof. Obviously the subcategory $\mathbf{D}_f(R)^b$ is closed under shifting.
Now suppose that $L \to M \to N \xrightarrow{[1]} L[1]$ is a distinguished triangle in $\mathbf{D}(R)$ such that L and M have finitely generated cohomology in every degree. By Propostion 2.23, for every $i \in \mathbb{Z}$ we obtain the exact sequences

$$H^i(M) \to H^i(N) \to H^{i+1}(L)$$

Since R is a Noetherian ring, we can conclude that $H^i(N)$ is finitely generated. So, $N \in \mathbf{D}_f(R)$. □

Corollary 3.41. *The subcategory $\mathbf{D}(R)^b_{\mathfrak{a}-cof}$ is triangulated.*

Proof. This follows from the fact that $\mathbf{D}(R)^b_{\mathfrak{a}-cof} \cong \mathrm{R}\Gamma_{\mathfrak{a}}(\mathbf{D}_f(R)^b)$ and that $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ is a triangulated fully faithful functor. □

We will see in Remark 3.50 that the word *cofinite* is used also for modules and that the two notions are essentially the same. Here is a characterization of cohomologically $\mathfrak{a}$-adically cofinite complexes.

Proposition 3.42. *Let $M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-tor}$, then:*

$$M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-cof} \text{ if and only if } \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \in \mathbf{D}_f(R)^b$$

Proof. By MGM equivalence $\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \in \mathbf{D}(R)^b_{\mathfrak{a}-com}$ and $M^\bullet \cong \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet))$. Moreover, for any $N^\bullet$ such that $M^\bullet \cong \mathrm{R}\Gamma_{\mathfrak{a}}(N^\bullet)$, we have that $\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \cong N^\bullet$. Thus $M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-cof}$ if and only if $\mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \in \mathbf{D}_f(R)^b$. □

Corollary 3.43 (Restricted MGM equivalence).
The functor $\mathrm{R}\Gamma_{\mathfrak{a}}(\_)$ induces an equivalence of triangulated categories

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\_) : \mathbf{D}_f(R)^b \longrightarrow \mathbf{D}(R)^b_{\mathfrak{a}-cof}$$

with quasi-inverse $\mathrm{L}\Lambda_{\mathfrak{a}}(\_)$.

Proof. Immediate from the MGM equivalence and the previous proposition. □

The characterization of cohomologically $\mathfrak{a}$-adically cofinite complexes in Proposition 3.42 is theoretically powerful but it is not really close to our purpose. In the following we will try to get another characterization of the category $\mathbf{D}(R)^b_{\mathfrak{a}-cof}$. We will write $R_0 = R/\mathfrak{a}$.

Lemma 3.44. *Let $L^\bullet,\ K^\bullet \in \mathbf{D}(R)^b$. Assume that $\mathrm{Ext}^i_R(R_0, L^\bullet)$ and $H^i(K^\bullet)$ are finitely generated R_0-modules for every $i \in \mathbb{Z}$. Then $\mathrm{Ext}^i_R(K^\bullet, L^\bullet)$ are finitely generated R_0-modules for all i.*

Proof. By induction on $\operatorname{amp}(H(K^\bullet))$. If the amplitude of $H(K^\bullet)$ is 0, then we can assume $K^\bullet = (0 \to K \to 0)$ is a single R-module (sitting in degree 0). Then, by hypothesis K is finitely generated over R_0. Let

$$M^\bullet = \mathrm{R}\operatorname{Hom}_R(R_0, L^\bullet) \in \mathbf{D}_f(R_0)^+$$

By the derived Hom-Tensor adjunction [Yek12, Proposition 14.5.8]

$$\mathrm{R}\operatorname{Hom}_R(A^\bullet \otimes^{\mathrm{L}}_S B^\bullet, C^\bullet) \cong \mathrm{R}\operatorname{Hom}_S(A^\bullet, \mathrm{R}\operatorname{Hom}_R(B^\bullet, C^\bullet))$$

for any two rings R, S such that there is a ring homomorphism $R \to S$ and complexes $A^\bullet,\ B^\bullet \in \mathbf{D}(S)$ and $C^\bullet \in \mathbf{D}(R)$. So we get

$$\mathrm{R}\operatorname{Hom}_R(K, L^\bullet) = \mathrm{R}\operatorname{Hom}_R(K \otimes^{\mathrm{L}}_{R_0} R_0, L^\bullet) \cong \mathrm{R}\operatorname{Hom}_{R_0}(K, \mathrm{R}\operatorname{Hom}_R(R_0, L^\bullet)) = \mathrm{R}\operatorname{Hom}_{R_0}(K, M^\bullet)$$

Now, we want to show that $\mathrm{R}\operatorname{Hom}_{R_0}(K, M^\bullet)$ has finitely generated cohomologies. To compute it, we need a projective resolution of K. Since it is a finitely generated R_0-module, by [Yek12, Theorem 13.3.9] we can replace it with a bounded above complex $P^\bullet$ of finitely generated projective R_0-modules. This is a projective resolution of K and we have that $P^i = 0$ for all $i > 0$. Since M is bounded below, we can assume that $M^i = 0$ for all $i < m$, for some integer m. The complex

$$X^\bullet = \mathrm{R}\operatorname{Hom}_{R_0}(K, M^\bullet) \cong \operatorname{Hom}^\bullet_{R_0}(P^\bullet, M^\bullet)$$

in degree i, is given by $X^i = \prod_{n\in\mathbb{Z}} \operatorname{Hom}_{R_0}(P^n, M^{n+i})$ and, by our boundedness assumptions, we have that $P^n = 0$ for $n > 0$ and $M^{n+i} = 0$ for $n + i < m$. So, the components of X^i which are non-zero are precisely those indexed by $k \in Z$ such that $m \leq k + i \leq i$. In other words, in each degree i, only finitely many components of $M^\bullet$ play a role in the computation of X^i. What this implies is that when we want to compute the i^{th} cohomology of the complex $X^\bullet$, we do not need to consider the whole complex $M^\bullet$, but we can consider the smart truncation $\chi_{\leq i+1}(M^\bullet) \in \mathbf{D}(R_0)^b$ and the result will not change.
So, we can assume that $M^\bullet \in \mathbf{D}_f(R_0)^b$ is a bounded complex with finitely generated cohomologies and, by [Yek12, Theorem 13.3.9], it is isomorphic to a complex $Q^\bullet$ consisting of finitely generated R_0-modules. It follows that

$$\mathrm{R}\operatorname{Hom}_{R_0}(K, M^\bullet) \cong \operatorname{Hom}^\bullet_{R_0}(P^\bullet, Q^\bullet) \in \mathbf{D}_f(R_0)^+$$

This shows that the modules $\operatorname{Ext}^i_R(K, L^\bullet)$ are finitely generated over R_0 [Rot09, Theorem 7.36].

If $\operatorname{amp}(H(K^\bullet)) > 0$, let $n = \sup(H(K^\bullet)) - 1$ and using smart truncations we get a distinguished triangle

$$K' \to K^\bullet \to K'' \xrightarrow{[1]}$$

where $K' = \chi_{\leq n}(K^\bullet)$ and $K'' = \chi_{>n}(K^\bullet)$ have smaller amplitudes and their cohomologies are still finitely generated. By applying the triangulated functor $\mathrm{R}\operatorname{Hom}_R(\_, L^\bullet)$ we obtain the distinguished triangle

$$\mathrm{R}\operatorname{Hom}_R(K'', L^\bullet) \to \mathrm{R}\operatorname{Hom}_R(K^\bullet, L^\bullet) \to \mathrm{R}\operatorname{Hom}_R(K', L^\bullet) \xrightarrow{[1]}$$

By Proposition 2.23, for every i we get the exact sequence

$$\operatorname{Ext}^i_R(K'', L^\bullet) \xrightarrow{\varphi} \operatorname{Ext}^i_R(K^\bullet, L^\bullet) \xrightarrow{\psi} \operatorname{Ext}^i_R(K', L^\bullet)$$

and so, the short exact sequences

$$0 \to \operatorname{Ext}^i_R(K'', L^\bullet)/\operatorname{Ker}(\varphi) \hookrightarrow \operatorname{Ext}^i_R(K^\bullet, L^\bullet) \twoheadrightarrow \operatorname{Im}(\psi) \to 0$$

where $\operatorname{Ext}^i_R(K'', L^\bullet)/\operatorname{Ker}(\varphi)$ and $\operatorname{Im}(\psi)$ are finitely generated R_0-modules, hence so it is $\operatorname{Ext}^i_R(K^\bullet, L^\bullet)$. □

Lemma 3.45. *Let $L^\bullet \in \mathbf{D}(R)^b$ and $i_0 = \sup(H(L^\bullet))$. Assume that $\operatorname{Ext}^i_R(R_0, L^\bullet)$ is a finitely generated R_0-module for every $i \in \mathbb{Z}$. Then $H^{i_0}(R_0 \otimes^{\mathrm{L}}_R L^\bullet)$ is a finitely generated R_0-module.*

Proof. Let $a_1, \dots, a_n$ be a generating sequence for the ideal $\mathfrak{a}$ and $K^\bullet = K(R; a_1, \dots, a_n)$ the associated Koszul complex [Definition 3.21]. Consider the complex

$$M^\bullet = \mathrm{R}\operatorname{Hom}_R(K^\bullet, L^\bullet)$$

by Lemma 3.44 and Proposition 3.23 (1), $H^i(M^\bullet)$ is a finitely generated R_0-module for every $i \in \mathbb{Z}$. Since $K^\bullet$ is a bounded complex of finitely generated free R-modules by [CFH22, Corollary 12.3.23]

$$M^\bullet \cong \mathrm{R}\operatorname{Hom}_R(K^\bullet, R \otimes^{\mathrm{L}}_R L^\bullet) \cong \mathrm{R}\operatorname{Hom}_R(K^\bullet, R) \otimes^{\mathrm{L}}_R L^\bullet \cong K^{\bullet\vee} \otimes^{\mathrm{L}}_R L^\bullet$$

So, by the Künneth trick [Lemma 3.16] (applied twice)

$$H^{n+i_0}(M^\bullet) = H^{n+i_0}(K^{\bullet\vee} \otimes^{\mathrm{L}}_R L^\bullet) \cong H^n(K^{\bullet\vee}) \otimes_R H^{i_0}(L^\bullet) \cong R_0 \otimes_R H^{i_0}(L^\bullet) \cong H^{i_0}(R_0 \otimes^{\mathrm{L}}_R L^\bullet)$$

Hence $H^{i_0}(R_0 \otimes^{\mathrm{L}}_R L^\bullet)$ is a finitely generated R_0-module. □

Proposition 3.46. *Let* $N^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-com}$*, then:*

$$N^\bullet \in \mathbf{D}_f(R)^b \textit{ if and only if } \operatorname{Ext}^i_R(R_0, N^\bullet) \textit{ is a finitely genenerated } R_0\textit{-module for every } i \in \mathbb{Z}$$

Proof. Let $N^\bullet \in \mathbf{D}_f(R)^b$. Since the modules $\operatorname{Ext}^i_R(R_0, N^\bullet)$ are $\mathfrak{a}$-torsion, by Lemma 1.28 it is sufficent to prove that they are finitely generated R-modules.
We can proceed by induction on $\operatorname{amp}(H(N^\bullet))$. If the amplitude is 0, we can assume $N^\bullet = (0 \to N \to 0)$ is a single R-module, finitely generated by hypothesis. Since R_0 is also a finitely generated R-module, it turns out that $\operatorname{Ext}^i_R(R_0, N)$ is a finitely generated R-module for every i [Rot09, Theorem 7.36].
If $\operatorname{amp}(H(N^\bullet)) > 0$, the proof is similar to the one of Lemma 3.44. Let $n = \sup(H(N^\bullet)) - 1$ and using smart truncations we get the distinguished triangle

$$N' \to N^\bullet \to N'' \xrightarrow{[1]}$$

where $N' = \chi_{\le n}(N^\bullet)$ and $N'' = \chi_{>n}(N^\bullet)$ have smaller amplitudes and their cohomologies are still finitely generated. By applying the triangulated functor $\mathrm{R}\operatorname{Hom}_R(R_0, \_)$ we obtain the exact sequences

$$\operatorname{Ext}^i_R(R_0, N') \to \operatorname{Ext}^i_R(R_0, N^\bullet) \to \operatorname{Ext}^i_R(R_0, N'')$$

where $\operatorname{Ext}^i_R(R_0, N')$ and $\operatorname{Ext}^i_R(R_0, N'')$ are finitely generated R-modules, hence so it is $\operatorname{Ext}^i_R(R_0, N^\bullet)$.

Conversely, suppose that $\operatorname{Ext}^i_R(R_0, N^\bullet)$ is a finitely genenerated R_0-module for every i and let $i_0 = \sup(H(N^\bullet))$. We are going to prove that $H^i(N^\bullet)$ is finitely generated over R by descending induction on i, starting from $i = i_0 + 1$ (which is trivial). Take $i \le i_0$ and suppose that $H^j(N^\bullet)$ is finitely generated for all $j > i$.
Using smart truncations at i we get the distinguished triangle $N' \to N^\bullet \to N'' \xrightarrow{[1]}$, where

$$H^j(N') = 0 \textit{ and } H^j(N'') \cong H^j(N^\bullet) \textit{ for all } j > i$$

$$H^j(N'') = 0 \textit{ and } H^j(N') \cong H^j(N^\bullet) \textit{ for all } j \le i$$

So $N'' \in \mathbf{D}_f(R)^b$ has finitely generated cohomologies by induction hypothesis. By Corollary 3.38 $N'' \in \mathbf{D}(R)^b_{\mathfrak{a}-com}$.
Since $N^\bullet$ is cohomologically $\mathfrak{a}$-adically complete too and $\mathbf{D}(R)^b_{\mathfrak{a}-com}$ is a triangulated category, it follows that also $N' \in \mathbf{D}(R)^b_{\mathfrak{a}-com}$. Moreover, by the first part of the proof we know that $\operatorname{Ext}^j_R(R_0, N'')$ is a finitely genenerated R_0-module for every $j \in \mathbb{Z}$.
By applying the triangulated functor $\mathrm{R}\operatorname{Hom}_R(R_0, \_)$ and by Proposition 2.23, for every j we obtain the exact sequence

$$\operatorname{Ext}^{j-1}_R(R_0, N'') \to \operatorname{Ext}^j_R(R_0, N') \to \operatorname{Ext}^j_R(R_0, N^\bullet)$$

So, we have that $\operatorname{Ext}^j_R(R_0, N')$ is a finitely genenerated R_0-module for every $j \in \mathbb{Z}$. From the previous lemma it follows that $H^i(R_0 \otimes^{\mathrm{L}}_R N')$ is a finitely generated R_0-module and applying the cohomologically complete Nakayama $H^i(N^\bullet) \cong H^i(N')$ is finitely generated over R. □

The main result of this section is the following characterization of cohomologically $\mathfrak{a}$-adically cofinite complexes, this is a generalization of Theorem 1.39 in the category $\mathbf{D}(R)^b$.

Theorem 3.47. *Let $M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-tor}$, then:*

$M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-cof}$ *if and only if* $\mathrm{Ext}^i_R(R_0, M^\bullet)$ *is a finitely genenerated R_0-module for every $i \in \mathbb{Z}$*

Proof. By Propositions 3.42 and 3.46, we know that $M^\bullet \in \mathbf{D}(R)^b_{\mathfrak{a}-cof}$ if and only if $N^\bullet = \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet) \in \mathbf{D}_f(R)^b$ if and only if $\mathrm{Ext}^i_R(R_0, N^\bullet)$ is a finitely genenerated R_0-module for every $i \in \mathbb{Z}$.
By MGM equivalence [Theorem 3.29] $\mathrm{L}\Lambda_{\mathfrak{a}}$ is a fully faithfull functor and since R_0 is $\mathfrak{a}$-adically complete

$$\mathrm{Hom}_{\mathbf{D}(R)}(R_0, M^\bullet) \cong \mathrm{Hom}_{\mathbf{D}(R)}(\mathrm{L}\Lambda_{\mathfrak{a}}(R_0), \mathrm{L}\Lambda_{\mathfrak{a}}(M^\bullet)) \cong \mathrm{Hom}_{\mathbf{D}(R)}(R_0, N^\bullet)$$

Finally applying Propositon 2.24

$$\mathrm{Ext}^i_R(R_0, N^\bullet) \cong \mathrm{Hom}_{\mathbf{D}(R)}(R_0, N^\bullet[i]) \cong \mathrm{Hom}_{\mathbf{D}(R)}(R_0, M^\bullet[i]) \cong \mathrm{Ext}^i_R(R_0, M^\bullet)$$

So we get the result. □

There is also a very important generalization of Corollary 1.40.

Corollary 3.48. *Let $M^\bullet \in \mathbf{D}_f(R)^b$. Then $\mathrm{Ext}^i_R(R_0, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet))$ is a finitely genenerated R-module for every $i \in \mathbb{Z}$.*

Proof. From the restricted MGM equivalence we have that $\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)$ is a bounded complex cohomologically $\mathfrak{a}$-adically cofinite, and so, by the last theorem, the modules $\mathrm{Ext}^i_R(R_0, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet))$ are all finitely genenerated over R_0. Morever, since they are annihilated by $\mathfrak{a}$, from Lemma 1.28 they are finitely generated also over R. □

Corollary 3.49. *Let $M \in \mathbf{Mod}(R)$ be a finitely generated module. Then $\mathrm{Ext}^i_R(R_0, \mathrm{R}\Gamma_{\mathfrak{a}}(M))$ is a finitely genenerated R-module for every $i \in \mathbb{Z}$.*

Remark 3.50 (Back to the Local Case).
Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and take $\mathfrak{a} = \mathfrak{m}$, its maximal ideal. An R-module is called *cofinite* if it is Artinian. We proved Theorem 1.39 that an $\mathfrak{m}$-torsion module M is cofinite if and only if all the Bass numbers $\mu_i(\mathfrak{m}, M)$ are finite, i.e. if and only if $\mathrm{Ext}^i_R(R/\mathfrak{m}, M)$ is a finitely genenerated $R/\mathfrak{m}$-module for every $i \in \mathbb{Z}$. Looking at a module M as a complex $M^\bullet = (0 \to M \to 0) \in \mathbf{D}(R)^b$, we conclude that if M is a cofinite module, then $M^\bullet$ is cohomologically $\mathfrak{m}$-adically cofinite.
Furthermore, given a finitely generated module M we proved in Theorem 1.38 that all the local cohomology modules $H^i_{\mathfrak{m}}(M) = H^i(\mathrm{R}\Gamma_{\mathfrak{m}}(M^\bullet))$ are cofinite and by restricted MGM equivalence we know that the whole complex $\mathrm{R}\Gamma_{\mathfrak{m}}(M^\bullet)$ is cohomologically $\mathfrak{m}$-adically cofinite. Indeed, this is a particular case of the Theorem 3.52, which is a new result.

Definition 3.51. Let $\mathfrak{a} \subseteq R$ be an ideal. An R-module M is called $\mathfrak{a}$-*adically cofinite* if $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a finitely genenerated $R/\mathfrak{a}$-module for every $i \in \mathbb{Z}$.
Notice that Artinian modules over a Noetherian local ring are $\mathfrak{m}$-adically cofinite.

Recall that we denote by $\mathbf{D}_{\mathfrak{a}-cof}(R)^b$ the subcategory of bounded complexes $M^\bullet$ whose cohomologies $H^i(M^\bullet)$ are $\mathfrak{a}$-adically cofinite R-modules.

Theorem 3.52. *Let $M^\bullet \in \mathbf{D}_{\mathfrak{a}-cof}(R)^b$. Then $M^\bullet$ is a cohomologically $\mathfrak{a}$-adically cofinite complex, i.e. $\mathbf{D}_{\mathfrak{a}-cof}(R)^b \subseteq \mathbf{D}(R)^b_{\mathfrak{a}-cof}$.*

Proof. We will prove the result by induction on $\mathrm{amp}(H(M^\bullet))$ using the characterization of Theorem 3.47. Let us write $R_0 = R/\mathfrak{a}$.
If the amplitude of $H(M^\bullet)$ is 0, we can assume $M^\bullet = (0 \to M \to 0)$ is a single R-module, $\mathfrak{a}$-adically cofinite by hypothesis. We have that $\mathrm{Ext}^i_R(R_0, M^\bullet) = \mathrm{Ext}^i_R(R_0, M)$ is finitely generated over R_0 for every $i \in \mathbb{Z}$.

If $\mathrm{amp}(H(N^\bullet)) > 0$, the proof is similar to the one of Lemma 3.44. Let $n = \sup(H(M^\bullet)) - 1$ and using smart truncations we get a distinguished triangle

$$M' \to M^\bullet \to M'' \xrightarrow{[1]}$$

where $M' = \chi_{\leq n}(M^\bullet)$ and $M'' = \chi_{>n}(M^\bullet)$ have smaller amplitudes and $\mathfrak{a}$-adically cofinite cohomologies, so $\mathrm{Ext}^i_R(R_0, M')$ and $\mathrm{Ext}^i_R(R_0, M'')$ are finitely generated R_0-modules for every $i \in \mathbb{Z}$. By Proposition 2.23, applying the triangulated functor $\mathrm{R}\,\mathrm{Hom}_R(R_0, \_)$, we obtain the exact sequences

$$\mathrm{Ext}^i_R(R_0, M') \to \mathrm{Ext}^i_R(R_0, M^\bullet) \to \mathrm{Ext}^i_R(R_0, M'')$$

Hence the R_0-module $\mathrm{Ext}^i_R(R_0, N^\bullet)$ is finitely generated for every $i \in \mathbb{Z}$. □

Remark 3.53. The key point to understand what is the difference between what happens in $\mathbf{Mod}(R)$ and in $\mathbf{D}(R)$, in particular why the Grothendieck conjecture fails, is that the containment of Theorem 3.52 is strict. There are cohomologically $\mathfrak{a}$-adically cofinite complexes whose cohomologies are not $\mathfrak{a}$-adically cofinite modules, that is the case of the Hartshrone's counterexample [Example 1.42].

Example 3.54. We continue with the set-up of Example 1.42.
Consider the complex $M^\bullet = (0 \to M \to 0)$. Recall that

$$H^i_{\mathfrak{a}}(M) = H^i(\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet))$$

we already showed that the complex $\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)$ does not lie in $\mathbf{D}_{\mathfrak{a}-cof}(R)^b$. Now, we will show that it belongs to $\mathbf{D}(R)^b_{\mathfrak{a}-cof}$. By Theorem 3.47, it is sufficent to prove that $\mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet))$ is a finitely genenerated $R/\mathfrak{a}$-module for every $i \in \mathbb{Z}$.
Let $M^\bullet \to I^\bullet$ be an injective resolution, since $M^\bullet$ is a bounded complex then $I^\bullet$ is a bounded complex of injective modules. Let $X^\bullet := \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) = \Gamma_{\mathfrak{a}}(I^\bullet)$. From Corollary 1.31, we know that $X^\bullet$ is still a bounded complex of injectives. Moreover, from computations in Example 1.42, we have that

$$H^i(X^\bullet) = H^i_{\mathfrak{a}}(M) \cong \begin{cases} \mathrm{Ker}(\varphi) & for\ i = 1 \\ N/\mathrm{Im}(\varphi) & for\ i = 2 \\ 0 & otherwise \end{cases}$$

where N is the R-module generated by the set $\{u^i v^j \mid i, j < 0\}$ and $\varphi : N \to N$ is the multiplication by $xu + yv$.
We can suppose that

$$X^\bullet = (0 \to X^1 \xrightarrow{d^1_X} X^2 \to 0)$$

Moreover, there is a solid diagram with exact rows, which can be completed to a commutative diagram

$$\begin{array}{ccccccc} H^1_{\mathfrak{a}}(M) & \xhookrightarrow{i_N} & N & \xrightarrow{\varphi} & N & \xrightarrow{p_N} & H^2_{\mathfrak{a}}(M) \\ \downarrow{\cong} & & \vdots\, s^1 & & \vdots\, s^2 & & \downarrow{\cong} \\ H^1_{\mathfrak{a}}(M) & \xhookrightarrow{i_X} & X^1 & \xrightarrow{d^1_X} & X^2 & \xrightarrow{p_X} & H^2_{\mathfrak{a}}(M) \end{array}$$

where the existence of s^1 follows from the injectivity of X^1, since it is the lifting of i_X; while the existence of s^2 follows from the injectivity of X^2, indeed there is a short exact sequence

$$\mathrm{Hom}_R(N, X^2) \xrightarrow{\_\circ\varphi} \mathrm{Hom}_R(N, X^2) \xrightarrow{\_\circ i_N} \mathrm{Hom}_R(H^1_{\mathfrak{a}}(M), X^2)$$

and since the composition $(d^1_X \circ s^1) \circ i_N = 0$, it follows that $d^1_X \circ s^1 \in \mathrm{Ker}(\_ \circ i_N) = \mathrm{Im}(\_ \circ \varphi)$.
Let us define the complex $N^\bullet = (0 \to N \xrightarrow{\varphi} N \to 0)$ and the morphism of complexes $s : N^\bullet \to X^\bullet$ which is $s^i = 0$ in degrees $i \neq 1, 2$. One can check that both $H^1_{\mathfrak{a}}(s)$ and $H^2_{\mathfrak{a}}(s)$ are isomorphisms (respectively the left one and the right one in the above diagram), it follows that s is a quasi-isomorphism and so $\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet) \cong N^\bullet$ in $\mathbf{D}(R)$.
Taking as a projective resolution of $R/\mathfrak{a}$ the complex

$$P^\bullet_1 = (0 \to R \xrightarrow{\left[\begin{smallmatrix} -v \\ u \end{smallmatrix}\right]} R \oplus R \xrightarrow{[u\ v]} R \to 0)$$

It can be shown by a calculation that the complex $\mathrm{R}\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) \cong \operatorname{Hom}_R(P_1^\bullet, N^\bullet)$, concetrated in degrees $[1,4]$, is

$$\operatorname{Hom}_R(P_1^\bullet, N^\bullet) = (0 \to N \xrightarrow{d^1 = \begin{bmatrix} u \\ v \\ \varphi \end{bmatrix}} N \oplus N \oplus N \xrightarrow{d^2 = \begin{bmatrix} v & -u & 0 \\ \varphi & 0 & -u \\ 0 & \varphi & -v \end{bmatrix}} N \oplus N \oplus N \xrightarrow{d^3 = [\varphi \ -v \ u]} N \to 0)$$

Let us compute the Ext-modules $\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet))$ for $i = 1, 2, 3, 4$.

(1) $\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) = \operatorname{Ker}(d^1) = \{n \in N \mid (un, vn, \varphi(n)) = 0\} = \operatorname{Ker}(u\cdot) \cap \operatorname{Ker}(v\cdot) \cap \operatorname{Ker}(\varphi)$
Since $\operatorname{Ker}(u\cdot) \cap \operatorname{Ker}(v\cdot) = \langle u^{-1}v^{-1} \rangle \subseteq \operatorname{Ker}(\varphi)$ is the submodule of N generated by the element $u^{-1}v^{-1}$, it follows that $\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) = \langle u^{-1}v^{-1} \rangle$ is finitely generated.

(2) $\operatorname{Ker}(d^2) = \left\{ (n_1, n_2, n_3) \in N \oplus N \oplus N \mid \begin{pmatrix} vn_1 - un_2 \\ \varphi(n_1) - un_3 \\ \varphi(n_2) - vn_3 \end{pmatrix} = 0 \right\}$
For an element $n = \sum_{i,j<0} r_{ij} u^i v^j$, denote with $u^{-1}n = \sum_{i,j<0} r_{ij} u^{i-1} v^j$ and notice that $(n_1, n_2, n_3) \in \operatorname{Ker}(d^2)$ if and only if $(n_1, n_2, n_3) = d^1(u^{-1}n_1)$. So, $\operatorname{Ext}^2_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) = 0$.

(3) $\operatorname{Ker}(d^3) = \{(n_1, n_2, n_3) \in N \oplus N \oplus N \mid \varphi(n_1) - vn_2 + un_3 = 0\}$
For an element $n = \sum_{i,j<0} r_{ij} u^i v^j$, denote with $v^{-1}n = \sum_{i,j<0} r_{ij} u^i v^{j-1}$ and notice that $(n_1, n_2, n_3) \in \operatorname{Ker}(d^3)$ if and only if $(n_1, n_2, n_3) = d^2\big((v^{-1}n_1, 0, -v^{-1}n_3)\big)$. So, $\operatorname{Ext}^3_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) = 0$.

(4) Notice that d^3 is surjective, indeed for every element $n \in N$ it holds $n = d^3\big((0, 0, u^{-1}n)\big)$. So, $\operatorname{Ext}^4_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)) = 0$.

We can conclude that $\mathrm{R}\Gamma_{\mathfrak{a}}(M^\bullet)$ is a cohomologically $\mathfrak{a}$-adically cofinite complex.

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