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Limit theorems for nearly unstable Hawkes processes and their applications

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Introduction

Hawkes processes were introduced by Alan Hawkes in 1971 (see **[13]** **[14]** and **[15]**) in order to model earthquakes and their aftershoks. Nowadays they are frequently used in finance. This is due to the observation of financial time series, where a long memory behaviour is registered; Hawkes processes naturally play the role of an autoregressive process. Moreover the branching structure of a Hawkes process allows to easily encode some well reported properties of high frequency trading markets, such as the high endogeneity of the buy and sell orders.

We can consider a one dimensional Hawkes process as a self exciting process $(N_t)_{t\geq 0}$, whose intensity at time t is given by

$$\lambda_t = \mu + \int_0^t \varphi(t-s)\, dN_s\,,$$

where μ is a positive real number and φ is a regression kernel. This simple process is used in **[4]** to build a model for a single asset price. In view of the interpretation of N_t as a branching process given in **[15]**, the norm $\|\varphi\|_1$ represent the degree of endogeneity of the market. As a matter of fact one may see μ as the number of orders due to a real economic reason and interpret $\|\varphi\|_1$ as the average number of orders triggered by each order. Under the assumption $\|\varphi\|_1 < 1$, the average number of orders triggered by a single order is $\sum_{k\geq 1}\|\varphi\|_1^k = \|\varphi\|_1/(1-\|\varphi\|_1)$, hence the proportion of triggered orders in the hole market is $\|\varphi\|_1/(1-\|\varphi\|_1)$ divided by $1+\|\varphi\|_1/(1-\|\varphi\|_1)$, that is equal to $\|\varphi\|_1$.

The condition $\|\varphi\|_1 < 1$ is crucial, not only to give this branching interpretation of the process, but also to provide other important features. In **[3]**, for example, Bacry, Delattre and Hoffmann show that this is necessary to obtain some ergodic results on a large scale observation.

Unfortunately, the condition $\|\varphi\|_1 < 1$ seems to be too restrictive. As a matter of fact empirical measurings (see for example **[2]**) shows that, trying to calibrate this kind of models on the financial time series, one usually gets values for $\|\varphi\|_1$ close to unity.

This is the starting point of our study: we are interested in the behaviour of Hawkes processes when the parameter $\|\varphi\|$ is close to one. In order to do that we will introduce an asymptotic framework and we will study a sequence of Hawkes processes N_t^T, indexed by T, each observed on the interval $[0,T]$ and such that the norm $\|\varphi^T\|_1$ of the autoregressive kernel of the intensity tends to 1 as T goes to infinity. We call these processes a sequence of *nearly unstable Hawkes processes.*

We will tackle both the one dimensional and the multidimensional problem and we will make different assumptions that will lead us to different scaling results. Finally we will come back to our motivation showing a financial application. Our conclusion finds a strong validation in the latest empirical observations of the rough nature of volatility in high frequency trading markets.

We briefly present the material contained in this thesis. In Chapter 1 the reader can find the basic notions about point processes and the results concerning Hawkes processes satisfying the "stability condition" $\|\varphi\|_1 < 1$. These results come from **[3]** and show that, in the case of a fixed kernel (not depending on T) with norm strictly smaller than one, it is possible to obtain a deterministic limit for the properly normalized sequence of Hawkes processes.

In Chapter 2 we start to study a sequence of nearly unstable Hawkes processes. It will be assumed that the kernel function φ exibits a light tails behaviour through the condition $\int_0^\infty s\varphi(s)\,ds < \infty$. First of all it will be shown that there is only one temporal scaling that allows to find a nondegenerate limit. It will be shown that, choosing a proper rescaling, it is possible to obtain in the limit a Cox-Ingersoll-Ross dynamics. In this chapter we will work on the sequence of intensity functions to obtain a limit intensity and we will use some theorems for stochastic differential equation to get the behaviour of the limit process.

In Chapter 3 we will drop the light tails assumption. A different rescaling in time will be necessary in order to obtain a nondegenerate limit. The most interesting fact will be that the heavy tails condition, that can be interpreted as a persistence of the memory of the process, is the basis to obtain a rough fractional diffusion in the limit behaviour. The approach to study this case will be different from the previous chapter, since we will work directly on the process itself rather than on its intensity.

In Chapter 4 we generalize the preceding results to the multidimensional case, where we have a sequence of multivariate nearly unstable Hawkes processes. The proofs are quite similar to those contained in Chapters 2 and 3. As a matter of fact, using some hypothesis on the matrix kernel of the intensity, it will be possible to project the processes along proper directions in a way that the nondegenerate part concentrate along one direction and thus we come back to tackle a known problem.

In Chapter 5 we use the preceding result in order to investigate a financial model for a single asset price. This model, based on a two dimensional Hawkes process, was introduced in [**4**] and leads to a stochastic dynamics for the volatility of the price. All the hypothesis made during the preceding chapters will get a financial meaning and the light or heavy tails assumption will get the biggest relevance. Under the light tails assumption a Heston model with leverage effect (negative correlation between the Brownian motion driving the asset and the one driving the volatility) will arise, while the heavy tails condition will lead to a rough Heston dynamics for the asset price.

Ringrazio il Professor Paolo Dai Pra per la guida attenta e disponibile. Ho imparato molto dalla semplicità e chiarezza con cui affronta ogni problema. Ringrazio anche il Professor David Barbato, il quale per primo ha risvegliato in me l'interesse per la probabilità.

CHAPTER 1

Hawkes processes and stationary limits

We start giving some notions about point processes and Hawkes processes in particular. It will be a brief presentation based on the book by Pierre Brémaud **[7]** and we redirect to that source for more details. We assume that the reader is already acquainted with basic theory of stochastic processes.

1. Point processes

A point process over the half line $[0,\infty)$ can be viewed in different ways. Here we look at it through its associated counting process.

1.1. Simple univariate point process. A realization of a point porcess over $[0,\infty)$ can be described by a sequence T_n in $[0,\infty]$ such that

$$T_0 = 0,$$
$$T_n < \infty \Rightarrow T_n < T_{n+1} .$$

This realization is, by definition *nonexplosive* if and only if

$$T_\infty = \lim_{n\to\infty} T_n = \infty .$$

To each realization corresponds a counting function N_t defined by

$$N_t = \begin{cases} n & \text{if } t \in [T_n, T_{n+1}),\ n \geq 0 , \\ +\infty & \text{if } t \geq T_\infty . \end{cases}$$

N_t is therefore a right-continuous step function such that $N_0 = 0$ and its jumps are upward jumps of magnitde 1.

If the above T_n are random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, one then calls the sequence T_n a *point process*. Since the random variables $(T_n)_n$ and the counting process $(N_t)_t$ carry the same information, with abuse of notation, we call also N_t a point process. Notice that, using the second point of view, nonexplosivity corresponds to have $N_t < \infty$ for any $t \geq 0$.

1.2. Multivariate point processes. Let T_n be a point process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(Z_n, n \geq 1)$ be a sequence of $\{1, \dots, k\}$-valued random variables, also defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Define for all i, $1 \leq i \leq k$ and all $t \geq 0$:

$$N_t(i) = \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}} \mathbb{1}_{\{Z_n = i\}} .$$

Both the k-vector process $N_t = (N_t(1), \dots, N_t(k))$ and the double sequence $(T_n, Z_n, n \geq 1)$ are called *k-variate point processes*. The limit $T_\infty = \lim_{n\to\infty} T_n$ is the explosion point of N_t. Note that the $N_t(i)$ have no common jumps.

1.3. Stochastic intensity and integration theorem.

DEFINITION 1.1. Let N_t be a point process adapted to some history $\mathcal{F}_t$ and let λ_t be a nonnegative $\mathcal{F}_t$-progressive process such that for all $t \geq 0$

$$\int_0^t \lambda_s \, ds < \infty \quad \mathbb{P}\text{-a.s.}$$

If for al nonnegative $\mathcal{F}_t$-predictable process C_t, the equality

$$\mathbb{E}\Big[\int_0^\infty C_s \, dN_s\Big] = \mathbb{E}\Big[\int_0^\infty C_s \lambda_s \, ds\Big]$$

is verified, then we say that N_t admits the $(\mathbb{P}, \mathcal{F}_t)$-intensity λ_t.

REMARK 1.2. Integrating with respect to dN_s has the following meaning:

$$\int_0^t C_s \, dN_s = \sum_{n \geq 1} C_{T_n} \mathbb{1}_{\{T_n \leq t\}}$$

and

$$\int_0^\infty C_s \, dN_s = \sum_{n \geq 1} C_{T_n} \mathbb{1}_{\{T_n < \infty\}} \,.$$

REMARK 1.3. Note that we didn't speak about uniqueness of the intensity process. As a matter of fact one may find different intensity processes, but if "the" intensity is constrained to be predictable, it is essentially unique. Moreover one can always find such a predictable version of the intensity. This fact is well detailed in [**7**, II-4]. For our purpose it will be enough to know that speaking about "the intensity process" is harmless.

Moreover this fact allows us to characterize a point process through its intensity function.

We now state, without proof, an integration theorem that is widely used in the next chapters.

THEOREM 1.4 (Integration Theorem). *If a point process N_t admits the $\mathcal{F}_t$-intensity λ_t (where $\int_0^t \lambda_s \, ds < \infty$ $\mathbb{P}$-a.s., $t \geq 0$), then N_t is $\mathbb{P}$-nonexplosive and*

(i) *$M_t = N_t - \int_0^t \lambda_s \, ds$ is an $\mathcal{F}_t$-local martingale;*

(ii) *if X_t is an $\mathcal{F}_t$-predictable process such that $\mathbb{E}[\int_0^t |X_s| \lambda_s \, ds] < \infty$, $t \geq 0$, then $\int_0^t X_s \, dM_s$ is an $\mathcal{F}_t$-martingale;*

(iii) *if X_t is an $\mathcal{F}_t$-predictable process such that $\int_0^t |X_s| \lambda_s \, ds < \infty$ $\mathbb{P}$-a.s., $t \geq 0$, then $\int_0^t X_s \, dM_s$ is an $\mathcal{F}_t$-local martingale.*

1.4. A useful isometry. We report here a very important result that will be frequently used in next chapters. It can be found in [**7**, III-4].

THEOREM 1.5 (Fundamental isometry for square-integrable point process martingales). *Let $(N_t(1), \dots, N_t(m))$ be an m-variate point process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with the $(\mathbb{P}, \mathcal{G}_t)$-intensity $(\lambda_t(1), \dots, \lambda_t(m))$, where $\mathcal{G}_t$ is the internal history of the point process. Let $\mathcal{M}_0^2$ be the Hilbert space of zero-mean square-integrable $(\mathbb{P}, \mathcal{G}_t)$-martingales over $[0, T]$ with the scalar product*

$$\langle M, M' \rangle_{\mathcal{M}_0^2} = \mathbb{E}[M_T M'_T] \,.$$

Let $\mathcal{H}$ be the Hilbert space of $\mathcal{G}_t$-predictable processes $C_t = (C_t(1), \dots, C_t(m))$ such that

$$\sum_{i=1}^m \mathbb{E}\Big[\int_0^T |C_s(i)|^2 \lambda_s(i) \, ds\Big] < \infty$$

with the scalar product

$$\langle C, C' \rangle_{\mathcal{H}} = \sum_{i=1}^{m} \mathbb{E}\left[\int_0^T C_s(i) C'_s(i) \lambda_s(i)\, ds \right].$$

Then $\mathcal{M}_0^2$ and $\mathcal{H}$ are isometric with respect to the mapping $\mathcal{H} \xrightarrow{\varphi} \mathcal{M}_0^2$ defined by

$$\varphi(C)_t = \sum_{i=1}^{m} \int_0^t C_s(i)\big(dN_s(i) - \lambda_s(i)\, ds\big).$$

2. Hawkes processes and stable limits

2.1. Definition of Hawkes process. As we already saw, the intensity process characterizes a point process. We use this fact to give the following definition.

Definition 1.6 (Hawkes process). A linear Hawkes process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a multivariate point process $N = (N(1), \dots, N(d))$ with intensity process

$$\lambda_t = \mu_t + \int_0^t \mathbf{\Phi}(t-s) \cdot dN_s\,,$$

where $\mu\colon \mathbb{R}_+ \to \mathbb{R}_+^d$ and $\mathbf{\Phi} = (\varphi_{i,j})_{i,j=1,\dots,d}$ with $\varphi_{i,j}$ positive locally integrable functions on $\mathbb{R}_+$.

Such a construction can be done according to Jacod (see[**17**]). Note that, if we use $J_n(i)$ to indicate the n-th jump time of the process $N(i)$, we can rewrite the intensity process as

$$\lambda_{t,i} = \mu_{t,i} + \sum_{j=1}^{d} \Bigg(\sum_{0<J_n(j)<t} \varphi_{i,j}(t - J_n(j)) \Bigg).$$

We have a non-esxplosion criterion which is proved in [**3**].

Lemma 1.7 (Non-explosion criterion). *Let $(J_n)_n$ be the sequence of jump times for the Hawkes process $(N_t)_{t\geq 0}$. Set $J_\infty = \lim_{n\to\infty} J_n$. Assume that the following holds:*

$$\int_0^t \varphi_{ij}(s)\, ds < +\infty \quad \forall i,j \text{ and } \forall t \geq 0\,.$$

Then $J_\infty = \infty$ almost surely.

2.2. Stability condition and scaling limits. We now report some results borrowed from [**3**]. We are not going to prove them, we just want to use them as a motivation for our study.

Consider a multivariate Hawkes process as in Definition 1.6 specified by the constant vector

$$\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}_+$$

and the $d \times d$-matrix valued function

$$\mathbf{\Phi} = (\varphi_{i,j})_{1\leq i,j\leq d}\,.$$

Furthermore, consider the following assumption.

Assumption 1.1. For all i,j we have $\int_0^\infty \varphi_{i,j}(t)\, dt < \infty$ and the spectral radius $\mathcal{S}(\mathbf{K})$ of the matrix $\mathbf{K} = \int_0^\infty \mathbf{\Phi}(t)\, dt$ satisfies $\mathcal{S}(\mathbf{K}) < 1$.

First we have a law of large numbers in the following sense:

THEOREM 1.8 (A law of large numbers). *Under Assumption 1.1, $N_t \in L^2$ for all $t \geq 0$ and*

$$\sup_{v\in[0,1]} \left\| \frac{N_{Tv}}{T} - v(\mathbf{Id} - \mathbf{K})^{-1}\mu \right\| \to 0 \quad \textit{as} \quad T \to +\infty$$

almost surely and in L^2.

Next we have a functional central-limit theorem.

Introduce the functions $\mathbf{\Phi}^{*k}$ defined on $\mathbb{R}_+$ and with values in the set of $d \times d$-matrices with entries in $[0, \infty]$ by

$$\mathbf{\Phi}^{*1} = \mathbf{\Phi}\,, \quad \mathbf{\Phi}^{*(k+1)}(t) = \int_0^t \mathbf{\Phi}(t-s)\mathbf{\Phi}^{*k}(s)\,ds\,, \quad n \geq 1\,.$$

Under Assumption 1.1 we have $\int_0^\infty \mathbf{\Phi}^{*n}(t)\,dt = \mathbf{K}^n$, hence the series $\sum_{n\geq 1} \mathbf{\Phi}^{*n}$ converges in $L^1(\mathbb{R})$. We set

$$\mathbf{\Psi} = \sum_{k\geq 1} \mathbf{\Phi}^{*k}\,.$$

THEOREM 1.9 (A central limit theorem). *Under Assumption 1.1,*

$$\mathbb{E}[N_t] = t\mu + \left(\int_0^t \mathbf{\Psi}(t-s)s\,ds\right)\mu.$$

Moreover, the process

$$\frac{1}{\sqrt{T}}\big(N_{Tv} - \mathbb{E}[N_{Tv}]\big), \quad v \in [0,1]$$

converges in law for the Skorokhod topology to

$$(\mathbf{Id} - \mathbf{K})^{-1}\mathbf{\Sigma}^{-\frac{1}{2}}W_v, \quad \textit{as} \quad T \to \infty,$$

where

$$\mathbf{\Sigma}_{ii} = \big((\mathbf{Id} - \mathbf{K})^{-1}\mu\big)_i\,, \quad \mathbf{\Sigma}_{ij} = 0 \quad \forall i \neq j$$

and W is a d-dimensional Brownian motion on $[0,1]$.

Consider now the following restriction on $\mathbf{\Phi}$:

ASSUMPTION 1.2.

$$\int_0^\infty \varphi(t)t^{\frac{1}{2}}\,dt < \infty \quad \text{componentwise.}$$

Using Theorem 1.8 and Assumption 1.2, we can replace $T^{-1}\mathbb{E}[N_{Tv}]$ by its limit in theorem 1.9 and obtain the following corollary.

COROLLARY 1.10. *Under Assumptions 1.1 and 1.2, the process*

$$\sqrt{T}\left(\frac{1}{T}N_{Tv} - v(\mathbf{Id} - \mathbf{K})^{-1}\mu\right), \quad v \in [0,1]$$

converges in law, for the Skorokhod topology, toward

$$(\mathbf{Id} - \mathbf{K})^{-1}\mathbf{\Sigma}^{\frac{1}{2}}W_v$$

as $T \to \infty$, where W is a d-dimensional Brownian motion on $[0,1]$.

Assumption 1.1 is very often called *stability condition* because it allows to obtain these nice properties for a sequence of properly rescaled Hawkes processes. Note that, if we focus on Theorem 1.8, we are observing a Hawkes process defined on the interval $[0, T]$ and rescaled by the length of the interval; the result tells us that, when the length of the interval approaches intinity, the rescaled process has a deterministic behaviour. At the same way, in Theorem 1.9 we read that, if we observe this Hawkes process on a large time scale, after a proper rescaling, it looks like a Brownian diffusion.

In next chapters we will try to get similar results for a Hawkes process that tends to violate the stability condition, i.e. with spectral radius of the kernel matrix that is *almost* one. We will do it introducing an asymptotic framework in which we study a sequence of Hawkes processes with spectral radius of the kernel matrix tending to one.

CHAPTER 2

Light tailed nearly unstable Hakwes processes

Here we start to study the limit behaviour of a sequence of nearly unstable Hawkes processes. We will do it in a constructive way, most of our assumptions will be given when they will be needed.

First of all we need to specify the meaning of nearly unstable Hawkes process and to define the asymptotic setting.

1. Asymptotic framework

We consider a sequence of Hawkes processes $(N_t^T)_{t\geq 0}$ indexed by T, where T goes to infinity. When it will be needed by the context, for example using convergence theorems, we will actually consider an increasing sequence of times T^n, with $T \to \infty$ for $n \to \infty$; we will not need to make this procedure explicit, it will be clear what $T \to \infty$ means. For a given T, (N_t^T) satisfies $N_0^T = 0$ and the process is observed on the time interval $[0, T]$. We give a process $(\lambda_t^T)_{t\geq 0}$ defined as follows:

$$\lambda_t^T = \mu + \int_0^t \varphi^T(t-s)\, dN_s^T,$$

where $\mu \in \mathbb{R}$, $\mu > 0$ and φ^T is a nonegative measurable function on $\mathbb{R}^+$ which satisfies $\|\varphi^T\|_1 < \infty$. For a given T the process (N_t^T) is defined on a probabilty space $(\Omega^T, \mathcal{F}^T, \mathbb{P}^T)$ equipped with the filtration $(\mathcal{F}_t^T)_{t\in[0,T]}$, where $\mathcal{F}_t^T$ is the σ-algebra generated by $(N_s^T)_{s\leq t}$. Moreover we assume that for any $0 \leq a < b \leq T$ and $A \in \mathcal{F}_a^T$

$$\mathbb{E}[(N_b^T - N_a^T)\mathbb{1}_A] = \mathbb{E}\left[\int_a^b \lambda_s^T \mathbb{1}_A\, ds\right],$$

which sets λ^T as the intensity of N^T. This construction can be done as it is explained in chapter 1 and we also know that, if we denote by $(J_n^T)_{n\geq 1}$ the jump times of (N_t^T), the process

$$N_{t\wedge J_n^T}^T - \int_0^{t\wedge J_n^T} \lambda_s^T\, ds$$

is a martingale and the law of N^T is characterized by λ^T.

Since it will be widely used in the following, we define the process

$$M_t^T := N_t^T - \int_0^t \lambda_s^T\, ds.$$

We now give more specific assumptions on the function φ^T. We denote by $\|\cdot\|_\infty$ the L^∞ norm on $\mathbb{R}^+$.

Assumption 2.1. For $t \in \mathbb{R}^+$,

$$\varphi^T(t) = a_T\varphi(t),$$

where $(a_T)_{T\leq 0}$ is a sequence of positive numbers converging to one, such that for all T, $a_T < 1$ and φ is a nonnegative measurable function such that

$$\int_0^{+\infty} \varphi(s)\,ds = 1 \quad \text{and} \quad \int_0^{+\infty} s\varphi(s)\,ds = m < \infty.$$

Moreover, φ is differentiable with derivative φ' such that $\|\varphi'\|_\infty < \infty$ and $\|\varphi'\|_1 < \infty$.

REMARK 2.1. Note that, under Assumption 2.1, $\|\varphi\|_\infty$ is finite, since $\forall t \geq 0$

$$\varphi(t) \leq \varphi(0) + \|\varphi'\|_1.$$

In chapter 1 we used to have $\|\varphi\|_1 < 1$ and this allowed some results on the limit behaviour of the process N. Here for a given T, $\|\varphi^T\|_1 = a_T < 1$, therefore the stability condition of chapter 1 is in force. Moreover, as remarked in that case, we have almost surely no explosions (see 1.7).

Since $\|\varphi^T\|_1 = a_T$ tends to one, this framework is a way to get close to instability. Hence we call our sequence of processes *nearly unstable Hawkes processes*. Note that the form of the function φ^T depends on T so that its shape is fixed, but its L^1 norm increases to 1 with $T \to \infty$. There are of course other ways to make the L^1 norm of φ converge to one that the multiplicative manner used here. Finally note that we call our processes *light tailed* because of the condition $\int_0^{+\infty} s\varphi(s)\,ds < \infty$.

REMARK 2.2. Under Assumption 2.1 we have the integrability of N^T thanks to theorem 1.9. This ensures that the process M^T is a martingale, since we can take the limit for $n \to \infty$ in the equality

$$\mathbb{E}\Big[N^T_{t\wedge J^T_n} - \int_0^{t\wedge J^T_n} \lambda^T_u\,du \Big| \mathcal{F}_s\Big] = N^T_{s\wedge J^T_n} - \int_0^{s\wedge J^T_n} \lambda^T_u\,du$$

Moreover we have that M^T is a square integrable martingale with quadratic variation process the process N^T. As a matter of fact, using the isometry given by theorem 1.5 we have

$$\mathbb{E}\Big[\big(M^T_t - M^T_s\big)^2 \Big| \mathcal{F}_s\Big] = \mathbb{E}\Big[\Big(\int_s^t dM^T_u\Big)^2 \Big| \mathcal{F}_s\Big] = \mathbb{E}\Big[\int_s^t \lambda^T_u\,du \Big| \mathcal{F}_s\Big] = \mathbb{E}\big[N^T_t - N^T_s \big| \mathcal{F}_s\big].$$

2. Observation scales

In our framework, two parameters degenerate to infinity, they are T and $(1-a_T)^{-1}$. The relationship between these two sequences will determine the scaling behaviour of the sequence of Hawkes processes. Keeping in mind the results of chapter 1, we would expect that, if $1-a_T$ tends slowly to zero, the process reaches the asymptotic regime for a T such that a_T is sufficiently far from unity and hence we get the same limit behaviour. This is exactly what happens, as it is stated in the next theorem.

THEOREM 2.3. *Assume $T(1-a_T) \to +\infty$. Then, under Assumption 2.1, the sequence of Hawkes processes defined in section 1 is asymptotically deterministic, in the sense that the following convergence holds:*

$$\sup_{v\in[0,1]} \frac{1-a_T}{T}\big|N^T_{Tv} - \mathbb{E}[N^T_{Tv}]\big| \to 0 \quad \text{in} \quad L^2.$$

Before giving the proof, we recall a result that we borrow from **[3]**. In order to understand this result we introduce a quantity that will play a fundamental role in the convergence of our processes.

Let define inductively, for $n \in \mathbb{N}$, the functions $(\varphi^T)^{*n}$ as follows:

$$(\varphi^T)^{*1} = \varphi^T, \quad (\varphi^T)^{*(n+1)}(t) = \int_0^t \varphi^T(t-s)(\varphi^T)^{*n}(s)\,ds.$$

We then set ψ^T to be the function defined on $\mathbb{R}^+$ by

$$\psi^T(t) = \sum_{n=1}^{\infty} (\varphi^T)^{*n}(t).$$

It will be easily shown that

$$\|\psi^T\|_1 = \frac{\|\varphi^T\|_1}{1-\|\varphi^T\|_1}. \tag{2.1}$$

Now we can state the lemma that we borrow from [**3**]:

LEMMA 2.4. *For all $v \in [0,1]$ and all $T \geq 0$ we have*

$$\mathbb{E}[N^T_{Tv}] = \mu T v + \mu \int_0^{Tv} \psi^T(Tv-s)s\,ds$$

and

$$N_{Tv} - \mathbb{E}[N^T_{Tv}] = M^T_{Tv} + \int_0^{Tv} \psi^T(Tv-s)M^T_s\,ds.$$

PROOF OF THEOREM 2.3. Using equation (2.1) and the second equation in lemma 2.4 we deduce

$$\frac{1-\|\varphi^T\|_1}{T}\left(N^T_{Tv} - \mathbb{E}[N^T_{Tv}]\right) \leq \frac{1-\|\varphi^T\|_1}{T}(1+\|\psi^T\|_1)\sup_{t\in[0,T]}\left|M^T_t\right| \leq \frac{1}{T}\sup_{t\in[0,T]}\left|M^T_t\right|.$$

Now recall that M^T is a square integrable martingale with quadratic variation process N^T. Thus we can apply Doob's L^p-inequality to get

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} M^T_t\Big)^2\Big] \leq 4\sup_{t\in[0,T]}\mathbb{E}\left[\left(M^T_t\right)^2\right] \leq 4\mathbb{E}\left[N^T_T\right] \leq 4\mu\frac{T}{1-\|\varphi^T\|_1}.$$

Therefore we finally obtain

$$\mathbb{E}\left[\sup_{v\in[0,1]}\left(\frac{1-\|\varphi^T\|_1}{T}\left(N^T_{Tv} - \mathbb{E}\big[N^T_{Tv}\big]\right)\right)^2\right] \leq \frac{4\mu}{T(1-\|\varphi^T\|_1)},$$

which gives the result since $T(1-\|\varphi^T\|_1) = T(1-a_T)$ tends to infinity. □

We have an opposite situation, when $1-a_T$ tends too rapidly to zero. In this case we expect that, for a given T, the Hawkes process N^T may already be very close to instability whereas T is not large enough to reach the asymptotic regime. We will see that it will be quite natural to require that $T(1-a_T)$ tends to a finite real number in order to get a nondegenerate scaling limit.

3. Result

In this section we introduce the last quantity that we need in order to state the main theorem of this chapter and we finally state this theorem. We also start the proof and give some heuristics about the derivation of this result before we tackle the complete and formal proof in the next section.

DEFINITION 2.5. Let ϱ^T be the function defined for $x \geq 0$ by

$$\varrho^T(x) = T\frac{1-a_T}{a_T}\psi^T(Tx) \tag{2.2}$$

We need to set a uniform bound on ϱ^T, this is our second assumption:

ASSUMPTION 2.2. There exists $K_\varrho > 0$ such that for all $x \geq 0$

$$\left|\varrho^T(x)\right| \leq K_\varrho.$$

We should think if this assumption is too restrictive. Note that, if the function φ is decreasing, then any ϱ^T is decreasing and since $|\varrho^T(0)|$ is bounded we get a uniform bound.

We now have all the ingredients to state the main theorem of this chapter:

THEOREM 2.6 (Convergence of light tailed Hawkes processes). *Assume there exists $\lambda > 0$ such that $T(1-a_T) \to \lambda$ for $T \to \infty$. Under Assumptions 2.1 and 2.2, the sequence of renormalized Hawkes intensities (C_t^T), defined as*

$$C_t^T := \lambda_{tT}^T(1-a_T), \tag{2.3}$$

converges in law, for the Skorohod topology, toward the law of the unique strong solution of the following Cox-Ingersoll-Ross stochastic differential equation on $[0,1]$*:*

$$X_t = \int_0^t (\mu - X_s)\frac{\lambda}{m}\,ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{X_s}\,dB_s.$$

Furthermore, the sequence of renormalized Hawkes processes

$$V_t^T := \frac{1-a_T}{T}N_{tT}^T$$

converges in law, for the Skorohod topology, toward the process

$$\int_0^t X_s\,ds, \quad t \in [0,1].$$

REMARK 2.7. We see in the formulation of the theorem that we actually study the convergence of the process $\lambda_{tT}^T(1-a_T)$. The temporal scaling is quite natural: we want all the intensities to be defined on the same interval and we renormalize it to $[0,1]$. For the scaling in space note that in the stationary case, the expectation of λ_t^T is $\mu/(1-a_T)$, thus the order of magnitude of intensity is $(1-a_T)^{-1}$. Thus a multiplcative factor $(1-a_T)$ is natural and we end up studying

$$C_t^T = \lambda_{tT}^T(1-a_T).$$

We will see that the asymptotic behaviour of C_t^T is closely connected to that of the function ϱ^T. About ϱ^T, one can remark that this function is the density of the random variable

$$H^T = \frac{1}{T}\sum_{i=1}^{I^T} H_i \tag{2.4}$$

where the $(H_i)_i$ are i.i.d random variable with density φ and I^T is a geometric random variable with parameter $1-a_T$, indipendent of H_i for any $i \in \mathbb{N}$. As a matter of fact recall that if H has density f then H/T has density $Tf(Tx)$. Now

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{I^T} H_i \in dx\Big) &= \mathbb{P}\Big(\bigcup_{k=1}^{\infty}\Big\{I^T = k, \sum_{i=1}^{k} H_i \in dx\Big\}\Big)\\ &= \sum_{k=1}^{\infty}(1-a_T)a_T^{k-1}\varphi^{*k}(x) = \frac{1-a_T}{a_T}\sum_{k=1}^{\infty}(\varphi^T)^{*k}(x)\\ &= \frac{1-a_T}{a_T}\psi^T(x).\end{aligned}$$

Hence H^T has density

$$T\frac{1-a_T}{a_T}\psi^T(Tx) = \varrho^T(x).$$

As a corollary from this computation we also get that, since $\int \varrho^T(x)\,dx = 1$ (ϱ^T is a density),

$$\|\psi^T\|_1 = \frac{a_T}{1-a_T}.$$

We now state a proposition that gives the asymptotic behaviour of the sequence of random variables $(H^T)_{T\geq 0}$.

Proposition 2.8. *Assume there exists $\lambda > 0$ such that $T(1-a_T) \to \lambda$ for $T \to \infty$. Under Assumption 2.1, the sequence of random variables H^T, defined in* (2.4), *converges in law toward an exponential random variable with parameter λ/m.*

Proof. Let $z \in \mathbb{R}$. The characteristic function of the random variable H^T, denoted by $\hat{\varrho}^T$, satisfies

$$\begin{aligned}\hat{\varrho}^T(z) = \mathbb{E}\big[e^{izH^T}\big] &= \mathbb{E}\Big[\sum_{k=1}^{\infty} e^{izH^T}\mathbb{1}_{\{I^T=k\}}\Big] = \sum_{k=1}^{\infty}\mathbb{P}\big(I^T = k\big)\mathbb{E}\Big[e^{i\frac{z}{T}\sum_{i=1}^{k} H_i}\Big]\\ &= \sum_{k=1}^{\infty}(1-a_T)a_T^{k-1}\Big(\mathbb{E}\Big[e^{i\frac{z}{T}H_1}\Big]\Big)^k = \sum_{k=1}^{\infty}(1-a_T)(a_T)^{k-1}\left(\hat{\varphi}\left(\frac{z}{T}\right)\right)^k\\ &= (1-a_T)\hat{\varphi}\left(\frac{z}{T}\right)\sum_{k=1}^{\infty}\left(a_T\hat{\varphi}\left(\frac{z}{T}\right)\right)^{k-1} = (1-a_T)\hat{\varphi}\left(\frac{z}{T}\right)\frac{1}{1-a_T\hat{\varphi}\left(\frac{z}{T}\right)}\\ &= \frac{\hat{\varphi}\left(\frac{z}{T}\right)}{\frac{1-a_T+a_T\left(1-\hat{\varphi}\left(\frac{z}{T}\right)\right)}{1-a_T}} = \frac{\hat{\varphi}\left(\frac{z}{T}\right)}{1-\frac{a_T}{1-a_T}\left(\hat{\varphi}\left(\frac{z}{T}\right)-1\right)},\end{aligned}$$

where $\hat{\varphi}$ denotes the characteristic function of H_1. Since

$$\mathbb{E}[H_1] = \int_0^{+\infty} s\varphi(s)\,ds = m > \infty,$$

the function $\hat{\varphi}$ is continuously differentiable with $\hat{\varphi}'(0) = im$. Therefore using taylor expansions we have

$$\hat{\varphi}\left(\frac{z}{T}\right) = \hat{\varphi}(0) + \frac{zim}{T} + o\left(\frac{z}{T}\right) = 1 + \frac{izm}{T} + o\left(\frac{z}{T}\right)$$

and hence

$$\lim_{T\to+\infty}\hat{\varrho}^T(z) = \frac{1}{1-\frac{izm}{\lambda}} = \frac{\lambda/m}{(\lambda/m)-iz},$$

that is the characteristic function of an exponential with parameter λ/m. □

REMARK 2.9. Notice that the light tails property was crucial in order to prove the previous proposition, together with the observation scale

$$T(1-a_T) \to \lambda \text{ for } T \to +\infty. \tag{2.5}$$

This gives a more rigorous motivation to our assumptions.

Assume from now on that condition (2.5) holds. Set

$$u_t := \frac{T(1-a_T)}{\lambda},$$

so that $u_T \to 1$ as $T \to \infty$. Proposition 2.8 gives us the asymptotic behaviour of ψ^T in this setting. Indeed we have

$$\psi^T(Tx) = \varrho^T(x)\frac{a_T}{\lambda u_T} \to \frac{\lambda}{m}e^{-x\frac{\lambda}{m}}\frac{1}{\lambda} = \frac{1}{m}e^{-x\frac{\lambda}{m}}. \tag{2.6}$$

In the sequel it will be convenient to work with another form of the intensity process $(\lambda_t^T)_t$. The following result holds:

PROPOSITION 2.10. *For all $t \geq 0$, we have*

$$\lambda_t^T = \mu + \int_0^t \psi^T(t-s)\mu\, ds + \int_0^t \psi^T(t-s)\, dM_s^T\,.$$

PROOF. From the definition of λ^T, using the fact that φ is bounded on $[0,t]$, we can write

$$\lambda_t^T = \mu + \int_0^t \varphi^T(t-s)\, dM_s^T + \int_0^t \varphi^T(t-s)\lambda_s^T\, ds \tag{2.7}$$

We now recall a classical lemma, that is a version of the renewal equation.

LEMMA 2.11. *Let h be a Borel and locally bounded function from $\mathbb{R}^+$ to $\mathbb{R}$. Then there exists a unique locally bounded function $f\colon \mathbb{R}^+ \to \mathbb{R}$, solution to*

$$f(t) = h(t) + \int_0^t \varphi^T(t-s)f(s)\, ds \quad \forall t \geq 0, \tag{2.8}$$

given by

$$f(t) = h(t) + \int_0^t \psi^T(t-s)h(s)\, ds.$$

PROOF. Since $\psi^T \in L^1(\mathbb{R}^+)$ and h is locally bounded, the function f is locally bounded. Let's show that f satisfies the equation (2.8):

$$\begin{aligned}
\int_0^t \varphi^T(t-s)f(s)\, ds &= \int_0^t \varphi^T(t-s)h(s) + \int_0^t \varphi^T(t-s)\left(\int_0^s \psi^T(s-r)h(r)\, dr\right) ds \\
&= \int_0^t \varphi^T(t-s)h(s)\, ds + \int_0^t h(r)\left(\int_r^t \varphi^T(t-s)\psi^T(s-r)\, ds\right) dr \\
&= \int_0^t \varphi^T(t-s)h(s)\, ds + \int_0^t h(r)\left(\int_0^{t-r} \varphi^T(t-r-u)\psi^T(u)\, du\right) dr \\
&= \int_0^t \varphi^T(t-s)h(s)\, ds + \int_0^t h(r)\psi^T(t-r)\, dr - \int_0^t h(r)\varphi^T(t-r)\, dr \\
&= \int_0^t \psi^T(t-r)h(r)\, dr,
\end{aligned}$$

where we used the following equation

$$\int_0^t \varphi^T(t-s)\psi^t(s)\,ds = \sum_{k=1}^{\infty}\int_0^t \varphi^T(t-s)(\varphi^T)^{*k}(s)\,ds = \sum_{k=1}^{\infty}(\varphi^T)^{*(k+1)}$$
$$= \psi^T(t) - \varphi^T(t).$$

Now let f_1 and f_2 be both solutions of our equation. We have

$$f_1(t) - f_2(t) = \int_0^t \varphi^T(t-s)(f_1(s) - f_2(s))\,ds.$$

Thus, if $g(t) = |f_1(t) - f_2(t)|$, one has

$$g(t) \le \int_0^t \varphi^T(t-s)g(s)\,ds,$$

which yields

$$\int_0^{\infty} g(t)\,dt \le \|\varphi^T\|_1 \int_0^t g(t)\,dt.$$

Since $\|\varphi^T\|_1 < 1$ for all $T > 0$, it follows that $f_1 = f_2$ almost everywhere. Therefore

$$\int_0^t \varphi^T(t-s)f_1(s)\,ds = \int_0^t \varphi^T(t-s)f_2(s)\,ds \text{ for all } t$$

and thus $f_1 = f_2$, since both the functions satisfies the equation (2.8). □

We apply this lemma to the equation (2.7) taking as function h the function

$$h(t) = \mu + \int_0^t \varphi^T(t-s)\,dM_s^T.$$

We thus obtain

$$\lambda_t^T = \mu + \int_0^t \varphi^T(t-s)\,dM_s^T + \int_0^t \psi^T(t-s)\left(\mu + \int_0^s \varphi^T(s-r)\,dM_r^T\right)ds. \tag{2.9}$$

Now, using Fubini and the fact that

$$\psi^T * \varphi^T = \psi^T - \varphi^T,$$

we get

$$\begin{aligned}\int_0^t \psi^T(t-s)\int_0^s \varphi^T(s-r)\,dM_r^T\,ds &= \int_0^t\int_r^t \psi^T(t-s)\varphi^T(s-r)\,ds\,dM_r^T\\ &= \int_0^t\int_0^{t-r} \psi^T(t-r-s)\varphi^T(s)\,ds\,dM_r^T\\ &= \int_0^t \psi^T * \varphi^T(t-r)\,dM_r^T\\ &= \int_0^t \psi^T(t-r)\,dM_r^T - \int_0^t \varphi^T(t-r)\,dM_r^T.\end{aligned}$$

Now we just need to rewrite (2.9) using the last equality. □

We now propose a heuristic proof of the theorem, working on the process C^T. Note that, using proposition 2.10, we can rewrite C_t^T in the following way

$$\begin{aligned} C_t^T &= \lambda_{tT}^T(1-a_T) \\ &= (1-a_T)\mu + \frac{u_T\lambda}{T}\int_0^t T\psi^T(Tt-s)\mu\, ds + \frac{u_T\lambda}{T}\int_0^{tT}\psi^T(Tt-s)\, dM_s^T. \end{aligned}$$

The first integral becomes

$$\frac{u_T\lambda}{T}\mu\int_0^t T\psi^T(y)\, dy = u_T\lambda\mu\int_0^t \psi^T(Ts)$$

and, after a change of variable, we rewrite the second integral as follows

$$\int_0^t T\frac{u_T\lambda}{T}\psi^T(Tt-s)\, dM_s^T = \int_0^t \sqrt{\lambda}\psi^T(T(t-s))\sqrt{C_s^T}\, dB_s^T,$$

where we set

$$B_t^T = \frac{1}{\sqrt{T}}\sqrt{u_T}\int_0^{tT}\frac{dM_s^T}{\sqrt{\lambda_s^T}}. \tag{2.10}$$

Finally we can write

$$C_t^T = (1-a_T)\mu + \mu\int_0^t u_T\lambda\psi^T(Ts)\, ds + \int_0^t \sqrt{\lambda}\psi^T(T(t-s))\sqrt{C_s^T}\, dB_s^T. \tag{2.11}$$

In the next paragraph we will show that B^T is a sequence of martingales converging to a Brownian motion for $T\to\infty$. Hence, heuristically replacing B^T by a brownian motion B and $\psi^T(Tx)$ by $(1/m)e^{-x(\lambda/m)}$ in (2.11), we get for $T\to\infty$

$$C_t^\infty = \mu\int_0^t \frac{\lambda}{m}e^{-(t-s)\frac{\lambda}{m}}\, ds + \frac{\sqrt{\lambda}}{m}\int_0^t e^{-(t-s)\frac{\lambda}{m}}\sqrt{C_s^\infty}\, dB_s.$$

Now note that $C_t^\infty = e^{-t(\lambda/m)}X_t$, where X_t is obviously defined. Applying Ito's formula we get:

$$\begin{aligned} dC_t^\infty &= -\frac{\lambda}{m}e^{-t\frac{\lambda}{m}}X_t\, dt + e^{-t\frac{\lambda}{m}}\, dX_t \\ &= -\frac{\lambda}{m}e^{-t\frac{\lambda}{m}}X_t\, dt + \mu\frac{\lambda}{m}\, dt + \frac{\sqrt{\lambda}}{m}\sqrt{C_t^\infty}\, dB_t \\ &= \left(-\frac{\lambda}{m}C_t^\infty + \mu\frac{\lambda}{m}\right)\, dt + \frac{\sqrt{\lambda}}{m}\sqrt{C_t^\infty}\, dB_t. \end{aligned}$$

Hence

$$C_t^\infty = \int_0^t (\mu - C_s^\infty)\frac{\lambda}{m}\, ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{C_s^\infty}\, dB_s,$$

which is precisely the stochastic differential equation satisfied by a CIR process.

4. Proof of the main result

In this section we make precise the heuristic statements given at the end of the last section. We start with some preliminary results about the function φ and the function ϱ^T defined in 2.5.

4.1. Preliminary lemmas.

LEMMA 2.12. *Let $\delta > 0$. There exists $\varepsilon > 0$ such that*

$$|1 - \hat{\varphi}(z)| \geq \varepsilon \quad \text{for any } z \text{ with } |z| \geq \delta.$$

Moreover there exists a constant $c > 0$ such that

$$|\hat{\varphi}(z)| \leq \frac{c}{|z|}.$$

PROOF. Since φ is bounded, $\hat{\varphi}$ tends to zero as z tends to infinity, by Riemann-Lebesgue lemma. Therefore there exists $b \geq \delta$ such that

$$|\hat{\varphi}(z)| \leq 1/2 \text{ for } |z| \geq b.$$

If $b = \delta$, we take $\varepsilon = 1/2$ and the proof is concluded. If $b > \delta$, let M denote the supremum of the real part of $\hat{\varphi}$ on $I = [-b, -\delta] \cup [\delta, b]$; since $\hat{\varphi}$ is continuous, this supremum is achieved at some point $z_0 \in I$. We have $M = \text{Re}(\hat{\varphi}(z)) = \mathbb{E}[\cos(z_0 X)]$, with X a random variable with density φ. Since φ is continuous, X does not belong to $2\pi/z_0\mathbb{Z}$ almost surely. Thus $M = \mathbb{E}[\cos(z_0 X)] < \|\varphi\|_1 = 1$. Therefore we have

$$|1 - \hat{\varphi}(z)| \geq |1 - \text{Re}(\hat{\varphi}(z))| \geq 1 - M \quad \text{for } z \in I.$$

We just need to take $\varepsilon = \min\{1/2; 1 - M\}$ to conclude.

In order to prove the second bound, we integrate by parts and use the Assumption 2.1, that is $\|\varphi'\|_1 < \infty$:

$$\hat{\varphi}(z) = \int_0^{+\infty} e^{isz} \varphi s \, ds = \frac{1}{iz} \varphi(s) \Big|_{s=0}^{s=+\infty} - \int_0^{+\infty} \frac{e^{isz}}{iz} \varphi'(s) \, ds$$
$$- \frac{1}{iz} \left(\varphi(0) + \int_0^{+\infty} e^{isz} \varphi'(s) \, ds \right),$$

which gives

$$|\hat{\varphi}(z)| \leq \frac{1}{|z|} \left(\varphi(0) + \int_0^{+\infty} |\varphi'(s)| \, ds \right).$$

□

We now want to show the L^2 convergence of the function ϱ^T toward the density of an exponential random variable

LEMMA 2.13. *Let $\varrho(x) = \frac{\lambda}{m} e^{-x\lambda/m}$ be the density of the exponential random variable with parameter λ/m. We have the following convergence:*

$$\varrho^T - \varrho \to 0 \quad \text{in } L^2.$$

PROOF. We first prove that there exists c>0 such that $\forall z \in \mathbb{R}$ and $\forall T > 1$,

$$|\hat{\varrho}^T(z)| \leq c \left(1 \wedge \frac{1}{|z|} \right). \tag{2.12}$$

Note that $|\hat{\varrho}^T| \leq 1$ because it is the characteristic function of a random variable. Using the fact that

$$\int_0^{+\infty} x\varphi(x) \, dx = m < +\infty,$$

given in Assumption 2.1, we get, for $x \to 0$,

$$\frac{\text{Im}(\hat{\varphi})(x)}{x} = \int_0^{+\infty} \frac{\sin(sx)}{x} \varphi(s) \, ds \to \int_0^{+\infty} s\varphi(s) \, ds = m,$$

where we used dominated convergence. This implies that there exists a $\delta > 0$ such that

$$\frac{|\mathrm{Im}(\hat{\varphi})(x)|}{|x|} \geq \frac{m}{2} \quad \forall x \text{ with } |x| \leq \delta.$$

Remember that from lemma 2.12 we also have that there exists $\varepsilon > 0$ such that

$$|1 - \hat{\varphi}(x)| \geq \varepsilon \quad \forall x \text{ with } |x| \geq \delta.$$

Therefore, using the explicit computation that we gave during the proof of proposition 2.8, we deduce that, if $|z/T| \leq \delta$, then

$$|\hat{\varrho}^T(z)| = \left| \frac{(1-a_T)\hat{\varphi}(z/T)}{1 - a_T \hat{\varphi}^T(z/T)} \right| \leq \frac{1-a_T}{a_T |\mathrm{Im}(\hat{\varphi})(z/T)|} \leq \frac{2(1-a_T)T}{a_T m |z|}.$$

The last quantity tends to $2\lambda/(a_T m|z|)$ for $T \to \infty$, hence there exists a positive constant c, such that

$$|\hat{\varrho}^T(z)| \leq c/|z|.$$

Moreover, thanks to the second assertion of lemma 2.12, if $|z/T| \geq \delta$,

$$|\hat{\varrho}^T(z)| \leq \frac{(1-a_T)|\hat{\varphi}(z/T)|}{|1 - \hat{\varphi}(z/T)|} \leq \frac{c(1-a_T)T}{|z|\varepsilon};$$

With the same argment used before we get, also for $|z/T| \geq \delta$, that $|\hat{\varrho}^T(z)| \leq c/|z|$. Thus the inequality (2.12) is proved.

Now we can focus on the L^2-convergence claimed in the statement. Using Fourier isometry we get

$$\|\varrho^T - \varrho\|_2 = \frac{1}{2\pi} \|\hat{\varrho}^T - \hat{\varrho}\|_2.$$

Thanks to proposition 2.8, we have that, for z fixed, $(\hat{\varrho}^T(z) - \hat{\varrho}(z)) \to 0$. Now, using inequality (2.12), we can apply dominated convergence theorem, which gives the convergence in L^2. □

We now show that, for all $T \geq 0$, ϱ^T is a lipschitz function, with lipschitz constant dependending proportionally on T.

Lemma 2.14. *There exists $c > 0$ such that for all $x \geq 0$, $y \geq 0$ and $T \geq 1$,*

$$|\varrho^T(z) - \varrho^T(y)| \leq cT|x - y|.$$

Proof. We simply compute the derivative of ϱ^T on $\mathbb{R}^+$. We have

$$\begin{aligned} (\psi^T)'(x) &= (\varphi^T)'(x) + \sum_{k \geq 2} \big((\varphi^T)^{*k}\big)'(x) \\ &= (\varphi^T)'(x) + (\varphi^T)' * \left(\sum_{k\geq 2} (\varphi^T)^{*(k-1)}(x) \right) \\ &= (\varphi^T)'(x) + (\varphi^T)' * \psi^T(x). \end{aligned}$$

Hence

$$\begin{aligned} (\varrho^T)'(x) &= T \frac{1-a_T}{a_T} T\big((\varphi^T)'(x) + (\varphi^T)' * \psi^T(Tx)\big) \\ &= T\big(T(1-a_T)\varphi'(x) + \varphi' * \varrho^T(x)\big). \end{aligned}$$

Since $T(1-a_T) \to \lambda$, there exists $c > 0$ such that

$$|(\varrho^T)'(x)| \leq T(c\|\varphi'\|_\infty + \|\varphi'\|_1 \|\varrho^T\|_\infty)$$

and we can conclude because our assumptions make all the quantities in the last inequality bounded. □

Let's give a definition that will make simplier the future computations.

DEFINITION 2.15. For every $T > 0$, we define the function f^T as

$$f^T(x) = \left(\frac{m}{\lambda} \frac{a_T}{u_T} \varrho^T(x) - e^{-x\frac{\lambda}{m}} \right) \mathbb{1}_{\{x \geq 0\}} \,.$$

The next corollary follows directly from the preceding lemmas.

COROLLARY 2.16. (i) *For any* $T \geq 1$

$$\int |f^T(x)|^2 \, dx \to 0 \text{ for } T \to \infty \,;$$

(ii) *There exists* $c > 0$ *such that, for any* $T \geq 1$ *and any* $x \geq 0$,

$$|f^T(x)| \leq c \,;$$

(iii) *There exists* $c > 0$ *such that, for any* $T \geq 1$ *and any* $w \geq 0$,

$$\widehat{f^T}(w) \leq c \left(\left| \frac{1}{w} \right| \wedge 1 \right) ;$$

(iv) *There exists* $c > 0$ *such that, for any* $T \geq 1$ *and all* $x, y \geq 0$,

$$|f^T(x) - f^T(y)| \leq cT|x - y| \,.$$

We can also give a lemma on the integrated difference associated to the function f^T.

LEMMA 2.17. *For any* $0 < \varepsilon < 1$, *there exists* c_ε *such that, for all* $t, s \geq 0$,

$$\int_{\mathbb{R}} (f^T(t-u) - f^T(s-u))^2 \, du \leq c_\varepsilon |t-s|^{1-\varepsilon} \,.$$

PROOF. Let's define $g^T_{t,s}(u) := f^T(t-u) - f^T(s-u)$. We can write the Fourier transform of $g^T_{t,s}$ as

$$|\widehat{g^T_{t,s}}(w)| = |e^{iwt} - e^{iws}||\widehat{f^T}(-w)| \,.$$

Using Fourier isometry and corollary 2.16(iii) we get

$$\begin{aligned}
\int_{\mathbb{R}} (f^T(t-u) - f^T(s-u))^2 \, du &= \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{g^T_{t,s}}(w)|^2 \, dw \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |e^{iwt} - e^{iws}|^2 |\widehat{f^T}(-w)|^2 \, dw \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}} |e^{iwt} - e^{iws}|^{1+\varepsilon} |e^{iwt} - e^{iws}|^{1-\varepsilon} c \left(\left| \frac{1}{w} \right|^2 \wedge 1 \right) dw \\
&\leq \frac{c}{2\pi} \int_{\mathbb{R}} 2^{1+\varepsilon} \left| \frac{e^{iwt} - e^{iws}}{w(t-s)} \right|^{1-\varepsilon} |w(t-s)|^{1-\varepsilon} \left(\left| \frac{1}{w} \right|^2 \wedge 1 \right) dw \\
&\leq c \frac{2^{1+\varepsilon}}{2\pi} |t-s|^{1-\varepsilon} \int_{\mathbb{R}} \underbrace{\left(\left| \frac{1}{w} \right|^{1+\varepsilon} \wedge |w|^{1-\varepsilon} \right)}_{\in L^1} dw \\
&= c_\varepsilon |t-s|^{1-\varepsilon} \,.
\end{aligned}$$

□

4.2. Core of the proof. We now begin with the proof of the first assertion in Theorem 2.6. We split this proof into several step.

4.2.1. *Convenient rewriting of the process C^T.* Our first goal is to obtain a suitable expression for the process C^T. From now on, set

$$d_0 = \frac{m}{\lambda}.$$

Recall equation (2.11):

$$C_t^T = (1-a_T)\mu + \mu \int_0^t u_T \lambda \psi^T(Ts)\, ds + \int_0^t \sqrt{\lambda}\psi^T(T(t-s))\sqrt{C_s^T}\, dB_s^T.$$

Keeping in mind the limiting behaviour of the function ψ^T, given by Proposition 2.8, we rewrite the last equation as

$$C_t^T = R_t^T + \mu(1-e^{-t/d_0}) + \frac{\sqrt{\lambda}}{m}\int_0^t e^{-(t-s)/d_0}\sqrt{C_s^T}\, dB_s^T\,, \tag{2.13}$$

where the process R_t^T is simply defined as

$$R_t^T = C_t^T - \mu(1-e^{-t/d_0}) - \frac{\sqrt{\lambda}}{m}\int_0^t e^{-(t-s)/d_0}\sqrt{C_s^T}\, dB_s^T\,.$$

Now consider the process X_t^T defined as

$$X_t^T = \frac{\sqrt{\lambda}}{m}\int_0^t e^{s/d_0}\sqrt{C_s^T}\, dB_s^T\,,$$

so that the third summand in equation (2.13) can be written as $e^{-t/d_0}X_t^T$. Applying Ito's formula to this quantity we get

$$\begin{aligned} d(e^{-t/d_0}X_t^T) &= -\frac{1}{d_0}e^{-t/d_0}X_t^T\, dt + e^{-t/d_0}\, dX_t^T \\ &= -\frac{\sqrt{\lambda}}{md_0}\left(\int_0^t e^{-(t-s)/d_0}\sqrt{C_s^T}\, dB_s^T\right) dt + \frac{\lambda}{m}\sqrt{C_t^T}\, dB_t^T\,. \end{aligned}$$

Thus we can write

$$\begin{aligned} C_t^T =& R_t^T + \frac{\mu}{d_0}\int_0^t e^{-u/d_0}\, du + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{C_u^T}\, dB_u^T \\ &- \frac{\sqrt{\lambda}}{md_0}\int_0^t\left(\int_0^u e^{-(u-s)/d_0}\sqrt{C_s^T}\, dB_s^T\right) du\,. \end{aligned}$$

Note that from equation (2.13) we have

$$\frac{\sqrt{\lambda}}{md_0}\int_0^u e^{-(u-s)/d_0}\sqrt{C_s^T}\, dB_s^T = \frac{1}{d_0}\big(C_u^T - R_u^T - \mu(1-e^{-u/d_0})\big)\,.$$

Hence, if we set

$$U_t^T := R_t^T + \frac{1}{d_0}\int_0^t R_s^T\, ds,$$

we finally derive

$$C_t^T = R_t^T + \frac{\mu}{d_0}\int_0^t e^{-u/d_0}\, du + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{C_u^T}\, dB_u^T - \frac{1}{d_0}\int_0^t \big(C_u^T - R_u^T - \mu + \mu e^{-u/d_0}\big)\, du\,,$$

that is

$$C_t^T = U_t^T + \frac{1}{d_0}\int_0^t (\mu - C_s^T)\, ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{C_u^T}\, dB_u^T\,. \tag{2.14}$$

We will therefore aim to show that the process U_t^T converges to zero so that expression (2.14) at the limit almost represents a stochastic differential equation.

4.2.2. *Convergence of the process U^T.* We want to show that the process $(U_t^T)_{t\in[0,1]}$ converges to zero in law, for the Skorohod topology. It is clear that showing the convergence of $(R_t^T)_{t\in[0,1]}$ gives also the convergence of U^T. Let us tackle R^T:

$$\begin{aligned} R_t^T &= \mu(1-a_T) + \mu\int_0^t T \underbrace{(1-a_T)}_{=a_T/\|\psi^T\|_1} \psi^T(Ts)\,ds + \sqrt{\lambda}\int_0^t \psi^T(T(t-s))\sqrt{C_s^T}\,dB_s^T \\ &\quad - \mu(1-e^{-t/d_0}) - \frac{\sqrt{\lambda}}{m}\int_0^t e^{-(t-s)/d_0}\sqrt{C_s^T}\,dB_s^T \\ &= \mu(1-a_T) - \mu\left(1-e^{-t/d_0} - a_T\int_0^t T\frac{\psi^T(Ts)}{\|\psi^T\|_1}\,ds\right) \\ &\quad + \sqrt{\lambda}\int_0^t \left(\psi^T(T(t-s)) - \frac{e^{-(t-s)/d_0}}{m}\right)\sqrt{C_s^T}\,dB_s^T\,. \end{aligned}$$

Now recalling the definition of B^T given by (2.10), we get

$$\begin{aligned} R_t^T &= \mu(1-a_T) - \mu\Big(1-e^{-t/d_0} - \int_0^t a_T T\frac{\psi^T(Ts)}{\|\psi^T\|_1}\,ds\Big) \\ &\qquad + \frac{1}{m}T(1-a_T)\int_0^t \Big(m\psi^T(T(t-s)) - e^{-(t-s)/d_0}\Big)\,\frac{dM_{sT}^T}{T}\,. \end{aligned}$$

Note that, since a_T tends to one, the first term tends to zero. Moreover, for $t\in[0,1]$, Proposition 2.8 gives the following convergence:

$$a_T\int_0^t T\frac{\psi^T}{\|\psi^T\|_1}(Ts)\,ds \to e^{-t/d_0}\,.$$

Since this convergence is monotone on a compact set and the limit is a continuous function, Dini's lemma gives that this convergence is in fact uniform on the interval $[0,1]$.

Thus, in order to prove the convergence of the process U^T to zero, it is enough to show the convergence of the process

$$Y_t^T := \int_0^t \Big(m\psi^T(T(t-s)) - e^{-(t-s)/d_0}\Big)\,d\overline{M}_s^T,$$

where we set $\overline{M}_s^T = M_{sT}^T/T$. Note that, using definition 2.15, we can rewrite the process Y^T as

$$Y_t^T = \int_0^t f^T(t-s)\,d\overline{M}_s^T\,.$$

As usual, we will first look for the convergence of the finite dimensional laws for the sequence of processes $(Y^T)_{T\geq 1}$ and then for its tightness.

PROPOSITION 2.18. *For any $(t_1,\ldots,t_n)\in[0,1]^n$, the following convergence in law holds:*

$$(Y_{t_1}^T,\ldots,Y_{t_n}^T)\to 0 \quad \textit{for} \quad T\to\infty\,.$$

PROOF. First note that the quadratic variation of $\overline{M}^T$ at time t is given by N_{tT}^T/T^2, whose predicable compensator process at time t is equal to

$$\frac{1}{T^2}\int_0^{tT}\lambda_s^T\,ds = \frac{1}{T}\int_0^t \lambda_{sT}^T\,ds\,.$$

Hence, using the isometry given by Theorem 1.5, we get

$$\mathbb{E}[(Y_t^T)^2] = \mathbb{E}\left[\int_0^t \left(f^T(t-s)\right)^2 \frac{\lambda_{sT}^T}{T}\, ds\right] = \int_0^t \left(f^T(t-s)\right)^2 \frac{\mathbb{E}[\lambda_{sT}^T]}{T}\, ds\,.$$

Now note that we can uniformly bound $\mathbb{E}[\lambda_t^T]$ by cT, with $c > 0$, since

$$\mathbb{E}[\lambda_t^T] = \mu + \mu \int_0^t \psi^T(t-s)\, ds \leq \mu + \mu \frac{a_T}{1-a_T} \leq cT\,.$$

Therefore we get

$$\mathbb{E}[(Y_t^T)^2] \leq c \int_0^t \left(f^T(t-s)\right)^2\, ds\,,$$

that, thanks to corollary 2.16 (i), leads to

$$\mathbb{E}[(Y_t^T)^2] \to 0\,,$$

which gives the result. □

4.2.3. *Tightness of the sequence* $(Y^T)_{T\geq 1}$. We now need to prove the tightness of the sequence $(Y^T)_{T\geq 1}$ We have the following inequality on the moments of the increments of Y^T, which is a first step in order to prove the tightness of the sequence.

Lemma 2.19. *For any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that for all $T \geq 1$, $0 \leq t, s \leq 1$,*

$$\mathbb{E}[(Y_t^T - Y_s^T)^4] \leq c_\varepsilon \left(|t-s|^{\frac{3}{2}-\varepsilon} + \frac{1}{T^2}|t-s|^{1-\varepsilon}\right)\,. \tag{2.15}$$

Proof. Using Lemma A.3 and A.17 in **[19]** we have

$$\begin{aligned}
\mathbb{E}[(Y_t^T - Y_s^T)^4] \leq & \frac{c}{T^3}\int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right|^4 du \\
& + \frac{c}{T^3}\int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right|^3 du \\
& \times \int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right| du \\
& + \frac{c}{T^2}\int_0^T \left|f^T(\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right|^2 du \\
& \times \int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right|^2 du \\
& + \frac{c}{T^3}\left(\int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right| du\right)^2 \\
& \times \int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right|^2 du\,.
\end{aligned}$$

Now using Cauchy-Schwarz inequality and Lemma 2.17, we get

$$\int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T(s-\frac{u}{T}\Big)\right| du \leq c_\varepsilon T\sqrt{|t-s|^{1-\varepsilon}}$$

and for $p = 2, 3, 4$, using also Corollary 2.16(ii),

$$\int_0^T \left|f^T\Big(t-\frac{u}{T}\Big) - f^T\Big(s-\frac{u}{T}\Big)\right|^p du \leq c_\varepsilon T|t-s|^{1-\varepsilon}\,.$$

Applying these inequalities allows us to conclude the proof. □

We will now use a linear interpolation of Y^T, namely $\tilde{Y}^T$, such that the difference between Y^T and $\tilde{Y}^T$ converges uniformly to zero in probability (u.c.p.). After that, we will just need to prove the tightness of $\tilde{Y}^T$ to conclude.

DEFINITION 2.20. Let $\tilde{Y}^T$ the linear interpolation of Y^T with mesh $1/T^4$, defined by

$$\tilde{Y}^T_t = Y^T_{\lfloor tT^4 \rfloor/T^4} + (tT^4 - \lfloor tT^4 \rfloor)(Y^T_{(\lfloor tT^4 \rfloor+1)/T^4} - Y^T_{\lfloor tT^4 \rfloor/T^4}) .$$

We immediately show the convergence of the difference between the process Y^T and its linear interpolation.

LEMMA 2.21. *We have the following convergence in probability:*

$$\sup_{|t-s|\leq 1/T^4} |Y^T_t - Y^T_s| \to 0 .$$

PROOF. *Ho modificato questa dimostrazione, la trova corretta?*
Recall that for $0 \leq s \leq t \leq 1$, since $Y^T_t = \int_0^t f^T(t-u)\, d\overline{M}^T_u$,

$$|Y^T_t - Y^T_s| = \left| \int_0^s f^T(t-u) - f^T(s-u)\, d\overline{M}^T_u + \int_s^t f^T(t-u)\, d\overline{M}^T_u \right| .$$

Thus we have

$$|Y^T_t - Y^T_s| \leq \int_0^{sT} |f^T(t-u/T) - f^T(s-u/T)|(dN^T_u + \lambda_u\, du)\frac{1}{T} + \int_{sT}^{tT} |f^T(t-u/T)|(dN^T_u + \lambda_u\, du)\frac{1}{T} .$$

Using Corollary 2.16 (ii) and (iv) we get

$$|Y^T_t - Y^T_s| \leq c|t-s| \left(N^T_T + \int_0^T \lambda^T_u\, du \right) + c \left(N^T_{tT} - N^T_{sT} + \int_{sT}^{tT} \lambda^T_u\, du \right) \frac{1}{T} .$$

Taking now $t > s$ such that $|t-s| \leq 1/T^4$, this gives

$$|Y^T_t - Y^T_s| \leq c\frac{1}{T^4} \left(N^T_T + \int_0^T \lambda^T_u\, du \right) + c \max_{|t-s|\leq 1/T^4} \left(N^T_{tT} - N^T_{sT} + \int_{sT}^{tT} \lambda^T_u\, du \right) \frac{1}{T} \tag{2.16}$$

From lemma 2.4 we easily deduce that there exists a constant $k_1 > 0$ such that

$$\mathbb{E}\left[N^T_T + \int_0^T \lambda^T_u\, du \right] \leq k_1 T^2 ,$$

in fact we have

$$\begin{aligned}
\mathbb{E}\Big[N^T_T + \int_0^T \lambda^T_u\, du\Big] &= 2\mathbb{E}[N^T_T] = 2\Big(\mu T + \mu \int_0^T \psi^T(T-s)s\, ds\Big) \\
&\leq 2\mu T + 2\mu T\Big(\int_0^T \psi^T(T-s)\, ds\Big) \\
&\leq 2\mu T\Big(1 + \frac{a_T}{1-a_T}\Big) = 2\mu T\Big(\frac{1}{1-a_T}\Big) \\
&= 2\mu T \frac{T}{T(1-a_T)} \leq k_1 T^2 ,
\end{aligned}$$

where, for the last inequality, we used that $T(1-a_T)$ is convergent, hence bounded.

Thus, using Markov inequality, for every $\varepsilon > 0$ we get

$$\mathbb{P}\left(c\frac{1}{T^4} \left(N^T_T + \int_0^T \lambda^T_u\, du \right) \geq \varepsilon \right) \leq \frac{ck_1}{\varepsilon T^2},$$

which gives the convergence in probability of the first term in the right-hand side of equation (2.16). In the same way, note that there exists $k_2 > 0$, such that

$$\mathbb{E}\left[N^T_{tT} - N^T_{sT} + \int_{sT}^{tT} \lambda^T_u \, du\right] \leq 2k_2 T^2 (t-s) \leq \frac{2k_2}{T^2}, .$$

Again using Markov inequality, for every $\varepsilon > 0$ we get

$$\mathbb{P}\left(c\frac{1}{T}\max_{|t-s|\leq 1/T^4}\left(N^T_{tT} - N^T_{sT} + \int_{sT}^{tT} \lambda^T_u \, du\right) > \varepsilon\right) \leq \frac{2ck_2}{\varepsilon T^3},$$

which gives the convergence in probability of the second term in (2.16) and completes the proof. □

Applying this lemma to definition 2.20, we immediately get the following corollary:

COROLLARY 2.22. *The following convergence in probability holds:*

$$\sup_{t\in[0,1]} |\tilde{Y}^T_t - Y^T_t| \to 0 \quad \textit{for} \quad T \to \infty .$$

The last step in order to prove the convergence u.c.p. of Y^T to zero is the tightness of the sequence $\tilde{Y}^T$.

LEMMA 2.23. *The sequence $(\tilde{Y}^T)$ is tight.*

PROOF. Our idea is to use the Kolmogorov criterion for tightness, that is to show that there exists $\gamma > 1$ and $c > 0$ such that for any $0 \leq s \leq t \leq 1$,

$$\mathbb{E}[|\tilde{Y}^T_t - \tilde{Y}^T_s|^4] \leq c|t-s|^\gamma .$$

Let $n^T_t = \lfloor tT^4 \rfloor$ and $n^T_s = \lfloor sT^4 \rfloor$. Let ε and ε' two constant such that $0 < \varepsilon, \varepsilon' < 1/4$ and let $T \geq 1$. There are three cases:

(1) $n^T_t = n^T_s$. Using Lemma 2.19 we get

$$\begin{aligned}\mathbb{E}[(\tilde{Y}^T_t - \tilde{Y}^T_s)^4] &= |t-s|^4 T^{16}\mathbb{E}[(Y^T_{(n^T_t+1)/T^4} - Y^T_{n^T_t/T^4})^4] \\ &\leq c_\varepsilon \frac{2}{T^{4(3/2-\varepsilon)}} T^{16}|t-s|^4 .\end{aligned}$$

Now, since

$$|t-s|^4 = |t-s|^{1+\varepsilon'}|t-s|^{3-\varepsilon'} \leq |t-s|^{1+\varepsilon'}\frac{1}{T^{4(3-\varepsilon')}},$$

we have

$$\mathbb{E}[(\tilde{Y}^T_t - \tilde{Y}^T_s)^4] \leq c_\varepsilon \frac{2}{T^{4(3/2-\varepsilon)}} T^{16}|t-s|^{1+\varepsilon'}\frac{1}{T^{4(3-\varepsilon')}} \leq c_\varepsilon |t-s|^{1+\varepsilon'} .$$

(2) $n^T_t = n^T_s + 1$. There exists a constant $c > 0$ such that

$$\mathbb{E}[(\tilde{Y}^T_t - \tilde{Y}^T_s)^4] \leq c\mathbb{E}[(\tilde{Y}^T_t - \tilde{Y}^T_{n^T_t/T^4})^4] + c\mathbb{E}[(\tilde{Y}^T_{n^T_t/T^4} - \tilde{Y}^T_s)^4] \leq c_\varepsilon |t-s|^{1+\varepsilon'} .$$

(3) $n^T_t \geq n^T_s + 2$. We use Lemma 2.19 again to get

$$\begin{aligned}\mathbb{E}[(\tilde{Y}^T_t - \tilde{Y}^T_s)^4] &\leq c\mathbb{E}[(\tilde{Y}^T_t - \tilde{Y}^T_{n^T_t/T^4})^4] + c\mathbb{E}[(\tilde{Y}^T_{(n^T_s+1)/T^4} - \tilde{Y}^T_s)^4] \\ &\quad + c\mathbb{E}[(\tilde{Y}^T_{n^T_t/T^4} - \tilde{Y}^T_{(n^T_s+1)T^4})^4] \\ &\leq c_\varepsilon\Big(\frac{1}{T^4}\Big)^{1+\varepsilon'} + c_\varepsilon\Big|\frac{n^T_t}{T^4} - \frac{n^T_s+1}{T^4}\Big|^{3/2-\varepsilon} \\ &\leq c_\varepsilon |t-s|^{\min(3/2-\varepsilon, 1+\varepsilon')} .\end{aligned}$$

Therefore the Kolmogorov criterion holds. □

Finally we can state the convergence of the sequence of processes (Y^T).

PROPOSITION 2.24. *The process* Y^T *converges u.c.p. to* 0 *on* $[0,1]$.

PROOF. We have

$$\sup_{t\in[0,1]} |Y^T_t| \le \sup_{t\in[0,1]} |\tilde{Y}^T_t| + \sup_{t\in[0,1]} |\tilde{Y}^T_t - Y^T_t| \,.$$

From Proposition 2.18 and Lemma 2.23 we have that $\tilde{Y}^T$ tends to zero in law for the Skorohod topology. This implies that the random variable $\sup_{t\in[0,1]} |\tilde{Y}^T_t|$ tends to zero in law and hence in probability. Applying Corrollary 2.22 to the second summand in the right-hand side of the equation we get the result. □

4.2.4. *Convergence of the intensity process* C^T. Here we complete the proof using the results illustrated in Appendix A. Recall that, thanks to equation (2.14), we can write the process C^T almost in the form of a SDE:

$$C^T_t = U^T_t + \frac{1}{d_0}\int_0^t (\mu - C^T_s)\, ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{C^T_u}\, dB^T_u \,,$$

where

$$B^T_t = \frac{1}{\sqrt{T}}\sqrt{u_T}\int_0^{tT} \frac{dM^T_s}{\sqrt{\lambda^T_s}} \,.$$

We want to apply Theorem A.1. First of all we study the convergence of B^T. Note that it is a sequence of martingales with jumps uniformly bounded by $c/\sqrt{\mu}$, since $1/\lambda^T_s$ is trivially bounded by $1/\mu$ for any $s \in [0,1]$. We now study its quadratic variation. Thanks to the isometry 1.5, for $t \in [0,1]$,

$$\langle B^T\rangle_t = \frac{u_T}{T}\int_0^{tT} \frac{dN^T_s}{\lambda^T_s} = \frac{u_T}{T}\left(\int_0^{tT}\left(ds + \frac{dM^T_s}{\lambda^T_s}\right)\right) = u_T\left(t + \int_0^{tT}\frac{dM^T_s}{T\lambda^T_s}\right) \,.$$

Now note that

$$\mathbb{E}\left[\left(\int_0^{tT} \frac{dM^T_s}{T\lambda^T_s}\right)^2\right] = \mathbb{E}\left[\int_0^{tT} \frac{1}{T^2\lambda^T_s}\, ds\right] \le \mathbb{E}\left[\int_0^{T} \frac{1}{T^2\lambda^T_s}\, ds\right] \le c/(T\mu) \,.$$

Hence, thanks to Markov inequality, for any $\varepsilon > 0$, we get for $T \to +\infty$

$$\mathbb{P}\left(\int_0^{tT} \frac{dM^T_s}{T\lambda^T_s} \ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\mathbb{E}\left[\left(\int_0^{tT} \frac{dM^T_s}{T\lambda^T_s}\right)^2\right] \le \frac{c}{T\varepsilon^2\mu} \to 0 \,.$$

We then have that the quadratic variation of the process B^T at time t converges to t in probability for any $t \ge 0$. Therefore, applying Theorem A.3 in Appendix A we get that B^T converges in law for the Skrohod topology toward a Brownian motion.

Since U^T converges to a deterministic limit, using Theorem 4.4 in Billingsley [**6**, pag. 27], we get the convergence in law, for the product topology, of the couple (U^T_t, B^T_t) to the couple $(0, B_t)$ for any $t \in [0,1]$, where B is a Brownian motion. Now, since the components of $(0, B_t)_t$ are continuous, by Proposition VI.2.2 in [**18**], the convergence also takes place for the Skorohod topology on the product space. It is easy to see that condition C1 of Theorem A.1 is satisfied by the sequence of processes B^T, as well as conditions C2 (i) and (ii) are satisfied by the square root function. Finally, according to [**8**], the CIR equation

$$X_t = \int_0 t(\mu - X_s)\frac{1}{d_0}\, ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{X_s}\, dB_s \tag{2.17}$$

admits a unique strong solution on $[0,1]$. Therefore all the hypotesis of theorem A.1 are satisfied and we get the convergence in law for the Skorohod topology of the process C^T toward the unique solution of the preceding CIR stochastic differential equation.

4.3. Convergence of the process $(V_t^T)_t$. We need to prove the second part of Theorem 2.6, that is the convergence of the renormalized Hawkes processes

$$V_t^T = \frac{1-a_T}{T} N_{tT}^T .$$

We can rewrite V_t^T in the following way:

$$V_t^T = \frac{1-a_T}{T} N_{tT}^T + \int_0^T (1-a_T)\lambda_{sT}^T \, ds - \int_0^{tT} \frac{1-a_T}{T} \lambda_s^T \, ds .$$

Hence, defining the martingale

$$\hat{M}_t^T = \frac{1-a_T}{T} \left(N_{tT}^T - \int_0^{tT} \lambda_s^T \, ds \right) ,$$

we have

$$V_t^T = \int_0^t C_s^T \, ds + \hat{M}_t^T$$

Using Doob's inequality, we obtain

$$\begin{aligned} \mathbb{E}\left[\left(\sup_{t \in [0,1]} \hat{M}_t^T \right)^2 \right] &\le 4\mathbb{E}[(\hat{M}_1^T)^2] = 4 \left(\frac{1-a_T}{T} \right)^2 \mathbb{E}[(M_T^T)^2] \\ &= 4 \left(\frac{1-a_T}{T} \right)^2 \mathbb{E}[N_T^T] \le \frac{4c(1-a_T)^2}{T} . \end{aligned}$$

Last inequality shows the convergence of $\hat{M}^T$ to zero u.c.p. Now note that (C^T, t) converges in law for the Skorohod topology to (C, t), where C is the unique strong solution of the CIR equation (2.17). We see that the hypotesis of theorem A.5 are trivially satisfied taking $X_t^n \equiv t$. Hence the sequence of stochastic integrals converges in law and Lemma VI.3.31 in [**18**] allows to conclude that

$$V_t^T \xrightarrow{\mathcal{D}} \int_0^t C_s \, ds .$$

4.4. Focus on the assumptions. Let's meditate a little bit about the importance of our assumptions.

First of all, Assumption 2.1 gives us a very treatable way to define the nearly unstability condition for the sequence of Hawkes processes. As we already remarked, that is not the only possible choice, but it seems reasonable to determine a dependance on a scalar constant rather than to change the shape of the intensity kernels φ^T at each T.

The very crucial part is the light tails condition $\int_0^\infty s\varphi(s)\, ds < \infty$. As a matter of fact, this was necessary to prove the covergence of the sequence ψ^T to an exponential in Proposition 2.8 (see equation (2.6)) and also to prove the boundedness of the Fourier transform of ϱ^T in Lemma 2.13. Hence, assuming that $\varphi(s) \sim s^{\alpha-1}$ with $\alpha \in (0,1)$, is the basis of the main resut. In the next chapter we will see that, dropping this hypotesis and assuming that $\varphi(s) \sim s^{-(1+\alpha)}$, we will be forced to change approach and we will get a very different result.

Differentiability of the function φ allowed us some integrations by parts and to use Taylor expansion for φ in Proposition 2.8. The same Proposition also shows us the importance of the limit of the real sequence $T(1-a_T)$, beyond the result of Theorem 2.3. As already remarked there, Theorem 2.3 gives a motivation to the choice of a proper observation scale.

Finally, Assumption 2.2 appears to be a rather technical assumption and it was used in proof of Lemma 2.14 to bound the derivative of ϱ. Nevertheless, that result is crucial to prove the convergence to zero of the process Y^T, which gives the possibility to write the rescaled intensity in the form of SDE.

CHAPTER 3

Heavy tailed nearly unstable Hawkes processes

As we remarked at the end of Chapter 2, the condition

$$\int_0^{+\infty} s\varphi(s)\,ds < \infty$$

was crucial in order to obtain the limiting behaviour of a CIR dynamics for the intensity process. We now want to inspect what happens if this condition is violated. We will then suppose that

$$\varphi(x) \underset{x\to+\infty}{\sim} \frac{K}{x^{1+\alpha}}\,,$$

where $\alpha \in (0,1)$ and K is a positive constant. We will address to this condition as the heavy tailed condition.

The main result of this chapter will be that, for $\alpha \in (1/2, 1)$, the law of a nearly unstable heavy tailed Hawkes process, properly rescaled, converges to that of a process which can be interpreted as an integrated fractional diffusion. We will closely follow the approach of Jaisson and Rosenbaum [**20**].

1. Setting and assumptions

The setting for this chapter is very similar to what was introduced in Section 1 of Chapter 2.

We still have a sequence of Hawkes processes $(N_t^T)_{t\geq 0}$, indexed by T and we study its behaviour for T that goes to infinity. When it will be required by the context (e.g. using convergence theorems) we will consider an increasing sequence of indeces $(T_n)_{n\in\mathbb{N}}$ such that $T_n \to \infty$ for $n \to \infty$.

For a given T, (N_t^T) is defined on the interval $[0,T]$, $N_0^T = 0$ and the associated intensity process is given by

$$\lambda_t^T = \mu_T + \int_0^t \varphi^T(t-s)\,dN_s^T\,, \quad \text{for } t \geq 0\,,$$

where μ_T is a sequence of positive real numbers and φ^T is a non-negative measurable function on $\mathbb{R}^+$ which satisfies $\|\varphi^T\|_1 < +\infty$. The process (N_t^T) again is defined on a probability space $(\Omega^T, \mathcal{F}^T, \mathbb{P}^T)$, equipped by the filtration $(\mathcal{F}_t^T)_{t\in[0,T]}$, where $\mathcal{F}_t^T$ is the σ-algebra generated by $(N_s^T)_{s\leq t}$. We will usually omit the index T in $\mathbb{P}^T$, writing $\mathbb{P}$, unless it is specifically required by the context.

We continue to call M_t^T the compensated process, that is

$$M_t^T = N_t^T - \int_0^t \lambda_s^T\,ds\,.$$

Recall that M_t^T is a local martingale since, if the process $(J_n^T)_{n\geq 1}$ represents the jump times of (N_T^T), then

$$N_{t\wedge J_n^T}^T - \int_0^{t\wedge J_n^T} \lambda_s^T\,ds$$

is a martingale for every n.

This was the general setting, we now start giving some more specific assumptions.

ASSUMPTION 3.1. For $t \in \mathbb{R}^+$,

$$\varphi^T(t) = a_T \varphi(t)\,,$$

where $(a_T)_{T\geq 0}$ is a sequence of positive real numbers converging to 1 such that for all T, $a_T < 1$ and φ is a non-negative measurable function such that $\|\varphi\|_1 = 1$. Moreover we assume there exist $\alpha \in (0,1)$ and $K > 0$ such that

$$\lim_{x\to+\infty} \alpha x^\alpha (1 - G(x)) = K\,,$$

where $G(x) = \int_0^x \varphi(s)\,ds$.

REMARK 3.1. Note that Assumption 3.1 is a way to violate the condition

$$\int_0^{+\infty} s\varphi(s)\,ds < +\infty$$

given in Assumption 2.1. As a matter of fact, dividing by x we get

$$\alpha x^{\alpha-1}(1 - G(x)) \underset{x\to\infty}{\sim} K/x\,.$$

Hence the integral of the preceding function above over $[0, +\infty)$ diverges. Integrating by parts we get

$$\begin{aligned}\int_0^{+\infty} \alpha x^{\alpha-1} \int_x^{+\infty} \varphi(s)\,ds\,dx &= \left[x^\alpha \int_x^{+\infty} \varphi(s)\,ds\right]_{x=0}^{x=+\infty} + \int_0^{+\infty} x^\alpha \varphi(x)\,dx \\ &= \frac{K}{\alpha} + \int_0^{+\infty} x^\alpha \varphi(x)\,dx = \infty\end{aligned}$$

and this implies $\int_0^\infty s\varphi(s)\,ds = \infty$.

Note that we continue to have almost surely no explosion, thanks to Lemma 1.7 in Chapter 1. Moreover Remark 2.2 holds also under Assumption 3.1, therefore the process (M_t^T) is a square integrable martingale with quadratic variation process the process (N_t^T).

The second assumption is related to the observation scale:

ASSUMPTION 3.2. There are two positive constant λ and μ^* such that

$$\lim_{T\to+\infty} T^\alpha(1 - a_T) = \lambda\delta$$

and

$$\lim_{T\to+\infty} T^{1-\alpha}\mu_T = \mu^*\delta^{-1}\,,$$

where we set

$$\delta = K\frac{\Gamma(1-\alpha)}{\alpha}\,,$$

with Γ the usual *gamma function*, $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$.

Finally, keeping the same notation of Chapter 2, we define $\psi^T : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\psi^T(t) = \sum_{k=1}^{\infty} (\varphi^T)^{*k}(t)\,.$$

2. Heuristics

In this section we prepare the main results giving some heuristic computation. We follow the ideas given in Section 3 of Chapter 2.

As it was proven in Proposition 2.10, we can write the intensity process in the following alternative way:

$$\lambda_t^T = \mu_T + \int_0^t \psi^T(t-s)\mu_T\, ds + \int_0^t \psi^T(t-s)\, dM_s^T \,.$$

Hence we get the same rescaled intensity on the interval $[0,1]$, that is

$$\lambda_{tT}^T = \mu_T + \int_0^{tT} \psi^T(Tt-s)\mu_T\, ds + \int_0^{tT} \psi^T(Tt-s)\, dM_s^T \,.$$

For the rescaling in space, recall from Theorem 1.9 that in the stationary case the expectation of λ_t^T is $\mu_T/(1-\|\varphi^T\|_1)$, hence a natural rescaling factor is $(1-a_T)/\mu_T$. Therefore we define the sequence of rescaled intensity

$$C_t^T = \frac{1-a_T}{\mu_T}\lambda_{tT}^T \,.$$

From the preceding alternative form of the intensity, after a change of variable, we get

$$C_t^T = (1-a_T) + \int_0^t T(1-a_T)\psi^T(Ts)\, ds + \sqrt{\frac{T(1-a_T)}{\mu_T}} \int_0^t \psi^T(T(t-s))\sqrt{C_s^T}\, dB_s^T \,, \tag{3.1}$$

with

$$B_t^T = \frac{1}{\sqrt{T}} \int_0^{tT} \frac{dM_s^T}{\sqrt{\lambda_s^T}} \,.$$

The process (B_t^T) is chosen in such a way that its quadratic variation tends to t as T tends to infinity; we will see this in detail in the following.

As we already remarked in Chapter 2, the asymptotic behaviour of (C_t^T) is closely related to that of function $\psi^T(T\cdot)$. We then study the rescaling

$$\varrho^T(x) = T\frac{\psi^T(Tx)}{\|\psi^T\|_1} \,.$$

Borrowing the computations from Chapter 2 we note that ϱ^T is the density of the random variable

$$H^T = \frac{1}{T}\sum_{i=1}^{I^T} H_i \,,$$

where the $(H_i)_i$ are i.i.d. random variable with density φ and I^T is a geometric random variable with success probability $1-a_T$, independent from H_i for every i. The Laplace transform of the random variable H^T is then the Laplace transform of function ϱ^T, that is

$$\begin{aligned}
\hat{\varrho}^T(z) = \mathbb{E}\Big[e^{-zH^T}\Big] &= \sum_{k=1}^{\infty}(1-a_T)(a_T)^{k-1}\mathbb{E}\Big[e^{-\frac{z}{T}\sum_{i=1}^k H_i}\Big] \\
&= \sum_{k=1}^{\infty}(1-a_T)(a_T)^{k-1}(\hat{\varphi}(z/T))^k = (1-a_T)\hat{\varphi}(z/T)\sum_{k=1}^{\infty}(a_T\hat{\varphi}(z/T))^{k-1} \\
&= (1-a_T)\hat{\varphi}(z/T)\frac{1}{1-a_T\hat{\varphi}(z/T)} = \frac{\hat{\varphi}(z/T)}{1-\frac{a_T}{1-a_T}(\hat{\varphi}(z/T)-1)} \,,
\end{aligned}$$

where $\hat{\varphi}$ is the Laplace transform of function φ. In Chapter 2 we used a Taylor expansion for φ thanks to the differentiability condition given by Assumption 2.1. Here we cannot use the same argument. However, integrating by parts, we can rewrite $\hat{\varphi}$.

$$\begin{aligned}\hat{\varphi}(z) &= \int_0^{+\infty} e^{-zx}\varphi(x)\,dx = \Big[e^{-zx}\int_0^x \varphi(s)\,ds\Big]_{x=0}^{x=+\infty} + z\int_0^{+\infty} e^{-zx}\int_0^x \varphi(s)\,ds\,dx \\ &= z\int_0^{+\infty} e^{-zx}G(x)\,dx\,.\end{aligned}$$

Now, note that $z\int_0^{+\infty} e^{-zx}\,dx = 1$. Hence we get

$$\begin{aligned}\hat{\varphi}(z) &= 1 - z\int_0^{+\infty} e^{-zx}(1-G(x))\,dx = 1 - z\int_0^{+\infty} e^{-zx}\int_x^{+\infty}\varphi(s)\,ds\,dx \\ &= 1 - \int_0^{+\infty} e^{-u}\int_{u/z}^{+\infty}\varphi(s)\,ds\,du = 1 - z^\alpha\int_0^{+\infty}\Big(\frac{u}{z}\Big)^\alpha\int_{u/z}^{+\infty}\varphi(s)\,ds\,u^{-\alpha}e^{-u}\,du\,.\end{aligned}$$

Using Assumption 3.1, taking the limit for $z \to 0$, we get

$$\hat{\varphi}(z) \underset{z\to 0}{\sim} 1 - z^\alpha\frac{K}{\alpha}\int_0^{+\infty} u^{-\alpha}e^{-u}\,du = 1 - z^\alpha K\frac{\Gamma(1-\alpha)}{\alpha}\,,$$

where the passage to the limit under the integral was justified by dominated convergence theorem and the fact that $\|\varphi\|_1 = 1$. Finally, using the notation in Assumption 3.1, we get for $|z|$ small enough

$$\hat{\varphi}(z) = 1 - z^\alpha\delta + o(z)\,.$$

Setting $v_T = \delta^{-1}T^\alpha(1-a_T)$, we get for T that goes to infinity

$$\begin{aligned}\hat{\varrho}^T(z) &= \frac{1-\delta\left(\frac{z}{T}\right)^\alpha + o\left(\frac{z}{T}\right)}{1-\frac{a_T}{1-a_T}\left(-\delta\left(\frac{z}{T}\right)^\alpha + o\left(\frac{z}{T}\right)\right)} = \frac{1-v_T^{-1}(1-a_T)z^\alpha + o\left(\frac{z}{T}\right)}{1+a_Tv_T^{-1}z^\alpha - \frac{a_T}{1-a_T}o\left(\frac{z}{T}\right)} \\ &\sim \frac{1}{1+v_T^{-1}\alpha} = \frac{v_T}{v_T+z^\alpha}\,.\end{aligned}$$

From [**12**, equation (19.4) page 32] we see that the function whose Laplace transform is $v_T/(v_T+z^\alpha)$ is

$$v_Tx^{\alpha-1}E_{\alpha,\alpha}(-v_Tx^\alpha)\,,$$

with $E_{\alpha,\beta}$ the Mittlag-Leffler function, with parameter α and β, given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\,.$$

We then know that, taking the limit for T that goes to infinity, we get

$$\varrho^T(x) \to vTx^{\alpha-1}E_{\alpha,\alpha}(-v_Tx^\alpha)\,,$$

hence

$$\psi^T(Ts) \to \frac{\|\psi^T\|_1}{T}v_Ts^{\alpha-1}E_{\alpha,\alpha}(-v_Ts^\alpha)\,.$$

Substituting heuristically this limit in equation (3.1), recalling that $\|\psi^T\|_1 = \frac{a_T}{1-a_T}$, we get

$$C_t^T \sim v_T\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-v_Ts^\alpha)\,ds + \gamma_Tv_T\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-v_T(t-s)^\alpha)\sqrt{C_s^T}\,dB_s^T\,,$$

where we put

$$\gamma_T = \frac{1}{\sqrt{\mu_TT(1-a_T)}}\,.$$

Using the fact that the maximum jump of the process B^T tends to zero and its quadratic variation tends to t, it will be shown that the process B^T converges in law toward a brownian motion B. Now note that, thanks to Assumption 3.2, we have for $T \to \infty$

$$v_T \to \lambda \quad \text{and} \quad \gamma_T v_T \to \sqrt{\frac{\lambda}{\mu^*}}\,.$$

Hence, substituting in a non rigorous way we get

$$C_t^\infty \sim \lambda \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda s^\alpha)\, ds + \sqrt{\frac{\lambda}{\mu^*}} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \sqrt{C_s^\infty}\, dB_s\,.$$

We got a non deterministic behaviour for the limiting law of the rescaled intensity. This form of the intensity reminds us of the fractional Brownian motion (B_t^H), with Hurst parameter H, that is the process

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \left(\int_0^t (t-s)^{H-1/2}\, dW_s + \int_{-\infty}^0 (t-s)^{H-1/2} - (-s)^{H-1/2}\, dW_s \right),$$

where $(W_t)_t$ is a standard Brownian motion. This is why we say that the limiting intensity process has the form of an integrated fractional diffusion.

Remark 3.2. We could have derived the preceding heuristic computation without stating Assumption 3.2. In that case we would have obtained the following limit for C_t^∞:

$$C_t^\infty \sim v_\infty \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-v_\infty s^\alpha)\, ds + \gamma_\infty v_\infty \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-v_\infty (t-s)^\alpha) \sqrt{C_s^\infty}\, dB_s\,.$$

Then it is clear that we want both v_∞ and γ_∞ to be strictly positive constant in order to get a non deterministic limiting behaviour. This is the reason for choosing the observation scale given by Assumption 3.2.

3. Convergence

We now tackle the problem of convergence for our sequence of heavy tailed nearly unstable Hawkes process. Let us start giving some notations. We will study the renormalized Hawkes process

$$X_t^T = \frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} N_{tT}^T\,, \quad t \in [0,1]$$

and its integrated intensity

$$\Lambda_t^T = \frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} \int_0^{tT} \lambda_s^T\, ds\,, \quad t \in [0,1]\,.$$

We already remarked that this normalization is chosen in a way that the process has expectation of order one. We define also the following martingale:

$$Z_t^T = \sqrt{\frac{T^\alpha \mu^* \delta^{-1}}{1-a_T}} (X_t^T - \Lambda_t^T)\,, \quad t \in [0,1]\,.$$

Next result is essential in order to obtain a representation of the limiting Hawkes process and it concernes the convergence in distribution of the sequence of processes (Z^T, X^T).

Proposition 3.3. *Under Assumptions 3.1 and 3.2, the sequence (Z^T, X^T) is tight. Furthermore, if (Z, X) is a limit point of (Z^T, X^T), then Z is a continous martingale with quadratic variation process the process X.*

PROOF. We will use many convergence results borrowed from [**18**] to prove the proposition. As firts step in this proof, we prove that the two sequences X^T and Λ^T are C-tight, that means that they are tight and every limit point is a continuous process. Thanks to Lemma 2.4 we have

$$\mathbb{E}[N_t^T] = \mu_T t + \mu_T \int_0^t \psi^T(t-s)s\,ds \le t\mu_T(1 + \|\psi^T\|_1)\,.$$

Therefore, since $\|\psi^T\|_1 = a_T/(1-a_T)$, we get that there exists a constant $c > 0$, such that uniformly in T we have

$$\mathbb{E}[\Lambda_1^T] = \mathbb{E}[X_1^T] = \mathbb{E}\left[\frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} N_T^T\right] \le \frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} \frac{T\mu_T}{1-a_T} = \frac{T^{1-\alpha}\mu_T}{\mu^*\delta^{-1}} \le c\,.$$

Since both X^T and Λ^T are increasing processes, their total variation on $[0,1]$ is simply given by X_1^T and Λ_1^T respectively, whose expected value is uniformly bounded. Hence, applying Corollary 9 in [**23**], both the processes are tight.

Moreover, since $(1-a_T)/T^\alpha$ tends to zero, the maximum jump size of X^T tends to zero as T goes to infinity, while the process Λ^T is already continuous. Thanks to Proposition VI.3.26 in [**18**], this implies the C-tightness of the sequences X^T and Λ^T.

Let us now study the sequence (Z^T, X^T). Notice that

$$\mathbb{E}[(Z_t^T)^2] = \frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} \mathbb{E}[(M_{tT}^T)^2] = \frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} \mathbb{E}[N_{tT}^T]\,.$$

Hence $\langle Z^T \rangle = X^T$. We can then apply Theorem VI.4.13 in [**18**] and use that the sequence of quadratic variation of the process Z^T is C-tight to deduce that Z^T is tight. Finally, using corollary VI.3.33 in [**18**], we get the joint tightness of the sequence (Z^T, X^T).

We now focus on the second part of the proposition. Consider a convergent subsequence (Z^{T_n}, X^{T_n}) with limit the process (Z, X). Proposition VI.6.26 in [**18**], together with the fact that the quadratic variation of the process Z^{T_n} is X^{T_n}, gives us that $\langle Z \rangle = X$. Now we just need to know that Z is a continuous martingale. It is continuous because $\sqrt{(1-a_T)/T^\alpha}$ tends to zero as T tends to infinity and this implies that the maximum jump size of Z^{T_n} tends to zero; therefore the sequence Z^{T_n} is C-tight and the limit Z is continuous. Now, since the maximum jump size of Z^{T_n} is bounded uniformly in n, because it converges to zero, Corollary IX.1.19 in [**18**] gives us that Z is a local martingale. Moreover its quadratic variation, that is X, has finite expected value, hence Z is a martingale. □

Having the tightness of the sequence of processes (Z^T, X^T), we can now take a pair of processes (Z, X), defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with law being one of the possible limit points of the sequence of distributions associated to the sequence (Z^T, X^T). We have the following results, giving an explicit expression for the pair (Z, X):

THEOREM 3.4. *There exists a Brownian motion B on $(\Omega, \mathcal{A}, \mathbb{P})$, such that for $t \in [0,1]$, $Z_t = B_{X_t}$ and for any $\varepsilon > 0$, X is Hölder continuous with Hölder exponent $(1 \wedge 2\alpha) - \varepsilon$ on $[0,1]$ and it can be written as*

$$X_t = \int_0^t s f^{\alpha,\lambda}(t-s)\,ds + \frac{1}{\sqrt{\mu^*\lambda}} \int_0^t f^{\alpha,\lambda}(t-s) B_{X_s}\,ds\,, \tag{3.2}$$

where we define

$$f^{\alpha,\lambda}(x) = \lambda x^{\alpha-1} E_{\alpha,\alpha}(-\lambda x^\alpha)\,.$$

3.1. Proof of theorem 3.4. We devote this section to prove the preceding theorem. We will need many step to reach the result.

We start with some preliminaries about the function $f^{\alpha,\lambda}$ and its fractional integrals and derivatives (see Appendix B for the basic definitions). We do not provide any proof of the next Proposition, we just recall that all the results are easily obtained applying the results of [**12**, Section 11] to the function $f^{\alpha,\lambda}$.

PROPOSITION 3.5. *The function* $f^{\alpha,\lambda}$ *is* C^∞ *on* $(0,1]$ *and*

$$f^{\alpha,\lambda}(x) \underset{x\to 0^+}{\sim} \frac{\lambda}{\Gamma(\alpha)} x^{\alpha-1}\,,$$

$$(f^{\alpha,\lambda})'(x) \underset{x\to 0^+}{\sim} \frac{\lambda(\alpha-1)}{\Gamma(\alpha)} x^{\alpha-2}\,.$$

Furthermore, $f^{\alpha,\lambda}(x)x^{1-\alpha}$ *has Hölder regularity* α *on* $(0,1]$*.*

For $\nu<\alpha$*,* $f^{\alpha,\lambda}$ *is* ν *fractionally differentiable and*

$$D^\nu f^{\alpha,\lambda}(x) = \lambda x^{\alpha-1-\nu} E_{\alpha,\alpha-\nu}(-\lambda x^\alpha)\,.$$

Therefore we have

$$D^\nu f^{\alpha,\lambda}(x) \underset{x\to 0^+}{\sim} \frac{\lambda}{\Gamma(\alpha-\nu)} \frac{1}{x^{1-\alpha+\nu}}\,,$$

$$(D^\nu f^{\alpha,\lambda})'(x) \underset{x\to 0^+}{\sim} \frac{\lambda(\alpha-1-\nu)}{\Gamma(\alpha-\nu)} \frac{1}{x^{2-\alpha+\nu}}\,.$$

For $\nu'>0$*,* $f^{\alpha,\lambda}$ *is* ν' *fractionally integrable and*

$$I^{\nu'} f^{\alpha,\lambda}(x) = \lambda \frac{1}{x^{1-\alpha-\nu'}} E_{\alpha,\alpha+\nu'}(-\lambda x^\alpha)\,.$$

Therefore we have

$$I^{\nu'} f^{\alpha,\lambda}(x) \underset{x\to 0^+}{\sim} \frac{\lambda}{\Gamma(\alpha+\nu')} \frac{1}{x^{1-\alpha-\nu'}}$$

and, for $\alpha+\nu'\neq 1$*,*

$$(I^{\nu'} f^{\alpha,\lambda})'(x) \underset{x\to 0^+}{\sim} \frac{\lambda(\alpha-1+\nu')}{\Gamma(\alpha+\nu')} \frac{1}{x^{2-\alpha-\nu'}}\,.$$

Finally, the following relation holds:

$$I^{1-\alpha} f^{\alpha,\lambda}(x) = \lambda\big(1-F^{\alpha,\lambda}(x)\big)\,.$$

3.1.1. *Proof of equation* (3.2). We now start to investigate the limiting expression of the process X^T. Next Lemma tells us that we can actually study the behaviour of the process Λ^T rather than the behaviour of the process X^T.

LEMMA 3.6. *The sequence of martingales* $X^T-\Lambda^T$ *tends to zero in probability, uniformly on* $[0,1]$*.*

PROOF. We have

$$X_t^T - \Lambda_t^T = \frac{1-a_T}{T^\alpha \mu^* \delta^{-1}} M_{tT}^T\,.$$

Applying Doob's inequality to the martingale M^T we get

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \{(X_t^T-\Lambda_t^T)^2\}\Big] \le c\Big(\frac{1-a_T}{T^\alpha}\Big)^2 \mathbb{E}\big[(M_T^T)^2\big]\,,$$

where c is a positive constant. Now, since the quadratic variation of the martingale M^T is given by the process N^T and

$$\mathbb{E}\big[N_T^T\big] \leq T\mu_T(1 + \|\psi^T\|_1) = T\mu_T\left(\frac{1}{1-a_T}\right),$$

we deduce

$$\mathbb{E}\bigg[\sup_{t\in[0,1]}\{(X_t^T - \Lambda_t^T)^2\}\bigg] \leq c\left(\frac{1-a_T}{T^\alpha}\right)^2 \mathbb{E}\big[N_T^T\big] \leq c\left(\frac{1-a_T}{T^\alpha}\right)^2 T\frac{\mu_T}{1-a_T}$$
$$= c\underbrace{\mu_T T^{1-\alpha}}_{\to\mu^*\delta^{-1}}\frac{1-a_T}{T^\alpha}.$$

Hence, using Markov inequality, for any $\varepsilon > 0$,

$$\mathbb{P}\bigg(\sup_{t\in[0,1]}\{(X_t^T - \Lambda_t^T)^2\} > \varepsilon\bigg) \leq \frac{1}{\varepsilon}\mathbb{E}\bigg[\sup_{t\in[0,1]}\{(X_t^T - \Lambda_t^T)^2\}\bigg] \leq \frac{c'}{\varepsilon}\frac{1-a_T}{T^\alpha}.$$

The last quantity goes to zero for T that goes to infinity, giving the convergence u.c.p of the sequence of martingales $X^T - \Lambda^T$. □

We now state a Lemma whose proof is already contained in Section 2. In particular it was showed there that the Laplace transform of the measure with density $\varrho^T(x)$ converges towards the Laplace transform of the measure with density $\lambda x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^\alpha)$. As a consequence we get the following:

LEMMA 3.7. *Let ϱ^T defined as in Section 2, that is*

$$\varrho^T(x) = T\frac{\psi^T(Tx)}{\|\psi^T\|_1}.$$

Then the sequence of measures with density ϱ^T converges weakly towards the measure with density $\lambda x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^\alpha)$. In particular, for $t \in [0,1]$, the function

$$F^T(t) = \int_0^t \varrho^T(x)\,dx$$

converges uniformly towards

$$F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(x)\,dx.$$

We focus now on equation (3.2). Let us consider a converging subsequence (Z^{T_n}, X^{T_n}) and write (Z, X) for its limit. By abuse of notation we continue to write (Z^T, X^T) instead of (Z^{T_n}, X^{T_n}). Thanks to Skorohod's representation theorem, there exists a probability space on which one can define copies in law of the couple (Z^T, X^T) converging almost surely for the Skorohod topology to a random variable with the same law as (Z, X). We now work with this sequence of variables converging almost surely and their limit. Recall that the process Z and X are continuous and that the Skorohod topology, relativized to C (the space of continuous functions on $[0,1]$), coincides with the uniform topology. Therefore we have

$$\sup_{t\in[0,1]}|X_t^T - X_t| \to 0, \quad \sup_{t\in[0,1]}|Z_t^T - Z_t| \to 0. \tag{3.3}$$

We work on a more suitable expression for the cumulated intensity.

$$\int_0^t \lambda_s^T\,ds = \int_0^t \Big\{\mu_T + \int_0^s \varphi^T(s-u)\big(dM_u^T + \lambda_u^T\,du\big)\Big\}\,ds$$
$$= t\mu_T + \int_0^t \int_0^s \varphi^T(s-u)\,dM_u^T\,ds + \int_0^t \int_0^s \varphi^T(s-u)\lambda_u^T\,du\,ds$$

For the first integral, applying Fubini theorem, we get

$$\int_0^t \int_0^s \varphi^T(s-u)\,dM_u^T\,ds = \int_0^t \int_u^t \varphi^T(s-u)\,ds\,dM_u$$
$$= \int_0^t \int_u^t \varphi^T(t-z)\,dz\,dM_u^T = \int_0^t \left(\int_0^z dM_u^T\right)\varphi^T(t-z)\,dz$$
$$= \int_0^t M_z^T \varphi^T(t-z)\,dz\,.$$

For the second one we get

$$\int_0^t \int_0^s \varphi^T(s-u)\lambda_u^T\,du\,ds = \int_0^t \left(\int_u^t \varphi^T(s-u)\,ds\right)\lambda_u^T\,du$$
$$= \int_0^t \left(\int_u^t \varphi^T(t-z)\,dz\right)\lambda_u^T\,du$$
$$= \int_0^t \left(\int_0^z \lambda_u^T\,du\right)\varphi^T(t-z)\,dz\,.$$

Hence we have the following expression:

$$\int_0^t \lambda_s^T\,ds = t\mu_T + \int_0^t \varphi^T(t-s)M_s^T\,ds + \int_0^t \varphi^T(t-s)\left(\int_0^s \lambda_u^T\,du\right)\,ds\,.$$

Now using Lemma 2.11 we get

$$\int_0^t \lambda_s^T\,ds = t\mu_T + \int_0^t \varphi^T(t-s)M_s^T\,ds + \int_0^t \psi^T(t-s)\left(s\mu_T + \int_0^s \varphi^T(s-u)M_u^T\,du\right)\,ds$$
$$= t\mu_T + \int_0^t \varphi^T(t-s)M_s\,ds + \int_0^t s\mu_T\psi^T(t-s)\,ds$$
$$+ \int_0^t \psi^T(t-s)\int_0^s \varphi^T(s-u)M_u^T\,du\,ds$$

We can rewrite the last summand using the fact that $\psi^T * \varphi^T = \psi^T - \varphi^T$:

$$\int_0^t \psi^T(t-s)\int_0^s \varphi^T(s-u)M_u^T\,du\,ds = \int_0^t \int_u^t \psi^T(t-s)\varphi^T(s-u)\,dsM_u^T\,du$$
$$= \int_0^t \int_0^{t-u} \psi^T(t-u-s)\varphi^T(s)\,dsM_u^T\,du$$
$$= \int_0^t \psi^T * \varphi^T(t-u)M_u^T\,du$$
$$= \int_0^t \psi^T(t-u)M_u^T\,du - \int_0^t \varphi^T(t-u)M_u^T\,du$$

Hence, using this in the previous equation for the cumulated intensity, we get

$$\int_0^t \lambda_s^T\,ds = t\mu_T + \int_0^t \psi^T(t-s)s\mu_T\,ds + \int_0^t \psi^T(t-s)M_s^T\,ds\,. \tag{3.4}$$

In order to get Λ^T we just need to replace t by tT and multiply by $(1-a_T)/(T^\alpha\mu^*\delta^{-1})$. Moreover we define the quantity

$$u_T = \frac{\mu_T}{\mu^*\delta^{-1}T^{\alpha-1}}\,,$$

that goes to 1 as T goes to infinity and will make our expression simplier. We then have

$$\begin{aligned}\Lambda_t^T &= (1-a_T)tu_T + T(1-a_T)u_T\int_0^t \psi^T(T(t-s))s\,ds \\ &\quad + T^{1-\alpha/2}\sqrt{\frac{1-a_T}{\mu^*\delta^{-1}}}\int_0^t \psi^T(T(t-s))Z_s^T\,ds\,.\end{aligned} \tag{3.5}$$

Since u_T goes to one, the first summand converges to zero as T goes to infinity. For the second summand we can rewrite and integrate by parts, getting

$$\begin{aligned}T(1-a_T)u_T\int_0^t \psi^T(T(t-s))s\,ds &= a_Tu_T\int_0^t \varrho^T(t-s)s\,ds = a_Tu_T\int_0^t \varrho^T(u)(t-u)\,du \\ &= a_Tu_T\Big[tF^T(t) - \int_0^t \varrho^T(u)u\,du\Big] \\ &= a_Tu_T\Big[tF^T(t) - tF^T(t) + \int_0^t F^T(u)\,du\Big] \\ &= a_tu_t\int_0^t F^T(t-s)\,ds\end{aligned}$$

Now, thanks Lemma 3.7, the last quantity tends uniformly to

$$\begin{aligned}a_Tu_T\int_0^t F^{\alpha,\lambda}(t-s)\,ds &= a_Tu_T\Big(\Big[F^{\alpha,\lambda}(t-s)s\Big]_{s=0}^{s=t} + \int_0^t f^{\alpha,\lambda}(t-s)s\,ds\Big) \\ &= a_Tu_T\int_0^t f^{\alpha,\lambda}(t-s)s\,ds\,.\end{aligned}$$

We now turn to the last summand in (3.5), that can be rewritten as

$$\frac{a_T}{\sqrt{T^\alpha(1-a_T)\mu^*\delta^{-1}}}\int_0^t \varrho^T(t-s)Z_s^T\,ds\,.$$

We would like to show that for any $t\in[0,1]$

$$\int_0^t \varrho^T(t-s)Z_s^T\,ds \to \int_0^t f^{\alpha,\lambda}(t-s)Z_s\,ds\,.$$

Using integration by parts for Lebesgue-Stiltjes integrals, we have

$$\begin{aligned}\int_0^t \varrho^T(t-s)Z_s^T\,ds &= \Big[\Big(\int_0^s \varrho^T(t-u)\,du\Big)Z_s^T\Big]_{s=0}^{s=t} - \int_0^t\int_0^s \varrho^T(t-u)\,du\,dZ_s^T \\ &= \int_0^t \varrho^t(t-u)\,du\,Z_t^T - \int_0^t\int_{t-s}^t \varrho^T(z)\,dz\,dZ_s^T \\ &= F^T(t)Z_t^T - \int_0^t \big(F^T(t)-F^T(t-s)\big)\,dZ_s^T \\ &= \int_0^t F^T(t-s)\,dZ_s^T\,.\end{aligned}$$

With the same computations we get also

$$\int_0^t f^{\alpha,\lambda}(t-s)Z_s^T\,ds = \int_0^t F^{\alpha,\lambda}(t-s)\,dZ_s^T\,.$$

Then we have

$$\begin{aligned}\left|\int_0^t \varrho^T(t-s)Z_s^T\,ds - \int_0^t f^{\alpha,\lambda}(t-s)Z_s\,ds\right| &= \left|\int_0^t F^T(t-s)\,dZ_s^T - \int_0^t f^{\alpha,\lambda}(t-s)Z_s^T\,ds\right.\\ &\quad\left.+\int_0^t f^{\alpha,\lambda}(t-s)Z_s^T\,ds - \int_0^t f^{\alpha,\lambda}(t-s)Z_s\,ds\right|\\ &\leq \left|\int_0^t \big(F^T(t-s)-F^{\alpha,\lambda}(t-s)\big)\,dZ_s^T\right| + \int_0^t f^{\alpha,\lambda}(t-s)\big|Z_s^T - Z_s\big|\,ds\,.\end{aligned}$$

The first integral goes to zero since

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \big(F^{\alpha,\lambda}(t-s)-F^T(t-s)\big)\,dZ_s^T\right)^2\right] &= \int_0^t \big(F^{\alpha,\lambda}(t-s)-F^T(t-s)\big)^2\mathbb{E}[X_s^T]\,ds\\ &\leq c\int_0^t \big(F^{\alpha,\lambda}(t-s)-F^T(t-s)\big)^2\,ds\end{aligned}$$

and the last quantity tends to zero thanks to Lemma 3.7. For the second integral, use equation (3.3) and again we have convergence to zero. Hence we finally obtain that the third summand in equation (3.5) converges, for any $t \in [0,1]$, to

$$\frac{1}{\sqrt{\mu^*\lambda}}\int_0^t f^{\alpha,\lambda}(t-s)Z_s\,ds\,.$$

In order to conclude the proof of equation (3.2) in Theorem 3.4, we need to show that $Z_t = B_{X_t}$ with B a Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$. This is given by Dambis-Dubin-Schwarz theorem (Theorem A.6), noticing that Z is a continuous martingale with quadratic variation the process X. See [**25**, V.1.6] for details on this theorem.

Putting all together the previous computations, we got the pointwise limit for the process Λ^T, that is exactly what we were looking for:

$$\Lambda_t^T \underset{T\to\infty}{\to} \int_0^t s f^{\alpha,\lambda}(t-s)\,ds + \frac{1}{\sqrt{\mu^*\lambda}}\int_0^t f^{\alpha,\lambda}(t-s)B_{X_s}\,ds\,.$$

Now we conclude using Lemma 3.6 and writing the process X^T as

$$X_t^T = \Lambda_t^T + X_t^T - \Lambda_t^T\,.$$

3.1.2. *Proof of the Hölder property for X.* After proving equation (3.2), we now devote our attention to prove the Hölder property for X in Theorem 3.4. We will use a contradiction argument based on the following technical Lemma.

LEMMA 3.8. *Let B be a Brownian motion and X a solution of* (3.2) *associated to B. Let $H \in [0,1]$. If X has Hölder regularity H on* $[0,1]$*, then for any $\varepsilon > 0$, X has also Hölder regularity* $((\alpha + H/2) \wedge 1) - \varepsilon$ *on* $[0,1]$.

Before giving the proof of this Lemma, we need a technical result about the smoothness of the convolution of a power type function with a continuous function. This result will be widely used in the following.

PROPOSITION 3.9. *Let f be a differentiable function on $(0,1]$ such that, for some $K>0$, $0<\beta<1$ and any x in $(0,1]$,*

$$|f(x)| \leq \frac{K}{x^\beta} \quad \text{and} \quad |f'(x)| \leq \frac{K}{x^{\beta+1}}\,,$$

and g a continuous function $[0,1]$. Then the convolution

$$f*g(t) = \int_0^t f(t-s)g(s)\,ds$$

is Hölder regular with exponent $1-\beta$.

PROOF. Let $G=\sup_{x\in[0,1]}|g(x)|$. We can split $f*g(t+h)-f*g(t)$ as follows:

$$\begin{aligned} f*g(t+h)-f*g(t) &= \int_0^{t+h} f(t+h-s)g(s)\,ds - \int_0^t f(t-s)g(s)\,ds \\ &= \int_0^{t-h} \big(f(t+h-s)-f(t-s)\big)g(s)\,ds \\ &\quad + \int_{t-h}^{t} \big(f(t+h-s)-f(t-s)\big)g(s)\,ds \\ &\quad + \int_t^{t+h} f(t+h-s)g(s)\,ds\,. \end{aligned}$$

We bound the three terms separately. For the first one we have

$$\begin{aligned} \int_0^{t-h} \big(f(t+h-s)-f(t-s)\big)g(s)\,ds &\leq \int_0^{t-h} \big|f(t+h-s)-f(t-s)\big| G\,ds \\ &\leq G\int_0^{t-h} \frac{K}{(t-s)^{\beta+1}}h\,ds \leq GK\frac{h}{\beta}\Big[\frac{1}{(t-s)^{\beta+1}}\Big]_{s=0}^{s=t-h} \\ &= \frac{GK}{\beta}h\Big(\frac{1}{h^\beta}-\frac{1}{t^\beta}\Big) \leq \frac{GK}{\beta}h^{1-\beta}\,. \end{aligned}$$

For the second one we use the bound on the modulus of f:

$$\begin{aligned} \int_{t-h}^{t} \big(f(t+h-s)-f(t-s)\big)g(s)\,ds &\leq G\int_{t-h}^{t} |f(t+h-s)-f(t-s)|\,ds \\ &\leq G\Big(\int_{t-h}^{t}\frac{K}{(t+h-s)^\beta}\,ds + \int_{t-h}^{t}\frac{K}{(t-s)^\beta}\,ds\Big) \\ & GK\frac{1}{1-\beta}\Big[\big((2h)^{1-\beta}-h^{1-\beta}+h^{1-beta}\big) = \frac{GK}{1-\beta}2^{1-\beta}h^{1-\beta}\,. \end{aligned}$$

Finally we easily bound the third term:

$$\begin{aligned} \int_t^{t+h} f(t+h-s)g(s)\,ds &\leq \int_t^{t+h}|f(t+h-s)|G\,ds \\ &\leq GK\int_t^{t+h}\frac{1}{(t+h-s)^\beta}\,ds = -GK\frac{1}{1-\beta}\Big[(t+h-s)^{1-\beta}\Big]_{s=t}^{s=t+h} \\ &= \frac{GK}{1-\beta}h^{1-\beta}\,. \end{aligned}$$

Therefore we have

$$f*g(t+h)-f*g(t) \leq GK\max\Big\{\frac{2^{1-\beta}}{1-\beta};\frac{1}{\beta}\Big\}h^{1-\beta}\,,$$

that gives the Hölder exponent for $f * g$.

□

PROOF OF LEMMA 3.8. Let $\varepsilon > 0$ and $Z_t = B_{X_t}$. Since the function

$$t \mapsto \int_0^t s f^{\alpha,\lambda}(t-s)\, ds$$

is C^1, it is enough to show that the function

$$t \mapsto \int_0^t f^{\alpha,\lambda}(t-s) Z_s\, ds$$

has the required Hölder regularity. Recalling that the Brownian motion has Hölder regularity $1/2 - \varepsilon'$ for every $\varepsilon' > 0$, we easily get that, for any $\varepsilon' > 0$, Z has Hölder regularity $(H/2 - \varepsilon')$. Therefore, by Proposition B.7, it is fractionally differentiable and $D^{H/2-\varepsilon} Z$ is continuous. Since $f^{\alpha,\lambda}$ is in $L^1([0,1])$, it is fractionally integrable and, by Corollary B.9, we get

$$\int_0^t f^{\alpha,\lambda}(t-s) Z_s\, ds = \int_0^t I^{H/2-\varepsilon} f^{\alpha,\lambda}(t-s) D^{H/2-\varepsilon} Z_s\, ds\,.$$

Finally, since Proposition 3.5 gives us that

$$I^{H/2-\varepsilon} f^{\alpha,\lambda}(x) \sim K/x^{1-\alpha-H/2+\varepsilon}$$

and

$$(I^{H/2-\varepsilon} f^{\alpha,\lambda})'(x) \sim K/x^{2-\alpha-H/2+\varepsilon}\,,$$

we have that, if $\alpha + H/2 - \varepsilon < 1$, we can apply Proposition 3.9 and get the result. □

Now let B be a Brownian motion and X a solution of (3.2) associated to B. We show that for any $\varepsilon > 0$, almost surely, the process X has Hölder regularity $(1 \wedge 2\alpha) - \varepsilon$ on $[0,1]$.

Let M be the supremum of Hölder exponents of X. The estimates on $f^{\alpha,\lambda}$ given by Proposition 3.5, together with the continuity of the Brownian motion, allow us to apply Proposition 3.9 to the second integral in the definition of X and conclude that $M \geq \alpha$.

Let us now assume that $M < (1 \wedge 2\alpha)$. Then we can find some $H < M$ and some $\varepsilon > 0$ such that

$$M < \big((\alpha + H/2) \wedge 1\big) - \varepsilon\,.$$

But, since $H < M$, X has Hölder regularity with exponent H and hence, by Lemma 3.8, has also Hölder regularity with exponent $\big((\alpha + H/2) \wedge 1\big) - \varepsilon$, which is a contradiction. Therefore we must have $M \geq (1 \wedge 2\alpha)$.

4. Derivative of process X

In this section we will find, under suitable conditions, an explicit expression for the derivative of the process X defined by (3.2). This expression, already obtained with the heuristic computations of section 2, gives us an interpretation of the process X as an integrated fractional diffusion.

The key point is that if the tail of the function φ is not too heavy, X is differentiable. The following resut indeed holds. We keep all the notations introduced during this chapter.

THEOREM 3.10. *Let $(X_t)_{t\in[0,1]}$ a process satisfying equation (3.2) and assume that $\alpha \in (1/2, 1)$. Then the process X is differentiable on $[0,1]$ and its derivative Y satisfies*

$$Y_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \int_0^t f^{\alpha,\lambda}(t-s) \sqrt{Y_s}\, dB^1_s\,, \tag{3.6}$$

where B^1 is a Brownian motion. Furthermore, for any $\varepsilon > 0$, Y has Hölder regularity $\alpha - 1/2 - \varepsilon$.

Proof. We start with a simple Lemma, giving us the Hölder regularity of process Z.

Lemma 3.11. *Let B be a Brownian motion, X a solution of (3.2) associated to B and $Z_t = B_{X_t}$. Then, for any $\varepsilon > 0$, almost surely, the process Z has Hölder regularity $(1/2 \wedge \alpha) - \varepsilon$ on $[0, 1]$.*

Proof. Using the Hölder regularity of the Brownian motion and that of the process X we get almost surely, for any $\varepsilon, \varepsilon' > 0$,

$$|Z_t - Z_s| = |B_{X_t} - B_{X_s}| \le k_1 |X_t - X_s|^{1/2-\varepsilon} \le k_2 \big(|t-s|^{(1\wedge 2\alpha)-\varepsilon'}\big)^{1/2-\varepsilon},$$

where k_1, k_2 are positive constant. □

Now let $\alpha > 1/2$, therefore let Z be Hölder continuous with exponent $1/2 - \varepsilon$, for any $\varepsilon > 0$. From Proposition 3.5 we have that the functon $x \mapsto f^{\alpha,\lambda}(x) x^{1-\alpha}$ is Hölder continuous with Hölder exponent α. Then Corollary B.10 implies that $D^\nu f^{\alpha,\lambda}$ exists for any $\nu \in (0, \alpha)$ and we can rewrite the second integral in equation (3.2) to get

$$X_t = \int_0^t s f^{\alpha,\lambda}(t-s)\, ds + \frac{1}{\sqrt{\mu^* \lambda}} \int_0^t D^\nu f^{\alpha,\lambda}(t-s) I^\nu Z_s \, ds\,.$$

Note that, integrating by parts in the first integral, we get

$$X_t = \int_0^t F^{\alpha,\lambda}(s)\, ds + \frac{1}{\sqrt{\mu^* \lambda}} \int_0^t D^\nu f^{\alpha,\lambda}(t-s) I^\nu Z_s \, ds\,.$$

Now, taking $\nu > 1/2$, since Z is $1-\nu$ Hölder continuous, Z is also $1-\nu$ fractionally differentiable and we have

$$Z_s = I^{1-\nu} D^{1-\nu} Z_s\,,$$

that gives, using the semigroup property of the fractional integral operator,

$$I^\nu Z_s = I D^{1-\nu} Z_s = \int_0^s D^{1-\nu} Z_u \, du\,.$$

We use this expression in the formula for X and apply Fubini's theorem:

$$\begin{aligned}
\int_0^t D^\nu f^{\alpha,\lambda}(t-s) I^\nu Z_s \, ds &= \int_0^t \int_0^s D^\nu f^{\alpha,\lambda}(t-s) D^{1-\nu} Z_u \, du\, ds \\
&= \int_0^t \int_u^t D^\nu f^{\alpha,\lambda}(t-s) D^{1-\nu} Z_u \, ds\, du \\
&= \int_0^t \int_u^t D^\nu f^{\alpha,\lambda}(s-u) D^{1-\nu} Z_u \, ds\, du \\
&= \int_0^t \int_0^s D^\nu f^{\alpha,\lambda}(s-u) D^{1-\nu} Z_u \, du\, ds\,.
\end{aligned}$$

Hence we got

$$X_t = \int_0^t Y_s \, ds\,,$$

where

$$Y_s = F^{\alpha,\lambda}(s) + \frac{1}{\sqrt{\mu^* \lambda}} \int_0^s D^\nu f^{\alpha,\lambda}(s-u) D^{1-\nu} Z_u \, du\,.$$

Applying again Proposition 3.9 together with the properties of $D^\nu f^{\alpha,\lambda}$ given by Proposition 3.5, we get that Y has Hölder regularity with exponent $(\alpha - \nu)$. Thus, taking ν close enough to $1/2$ (we can do that), we get that for any $\varepsilon > 0$, Y has Hölder regularity $(\alpha - 1/2 - \varepsilon)$.

Since $\alpha > 1/2$, this implies that Y is continuous and hence the derivative of X is defined for any $t \in [0,1]$.

We now need to find the right expression for Y. Since Z is a continuous martingale with bracket X, the condition to apply stochastic Fubini theorem are easily verified (see **[27]**). Hence we can obtain

$$\begin{aligned} D^{1-\nu} Z_s &= \frac{1}{\Gamma(\nu)} \frac{d}{ds} \int_0^s \frac{Z_s}{(s-v)^{1-\nu}}\, dv = \frac{1}{\Gamma(\nu)} \frac{d}{ds} \int_0^s \int_0^v \frac{1}{(s-v)^{1-\nu}}\, dZ_u\, dv \\ &= \frac{1}{\Gamma(\nu)} \frac{d}{ds} \int_0^s \int_u^s \frac{1}{(s-v)^{1-\nu}}\, dv\, dZ_u = \frac{1}{\Gamma(\nu+1)} \frac{d}{ds} \int_0^s (s-u)^{\nu}\, dZ_u \end{aligned}$$

Therefore we can write

$$Y_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^* \lambda}} \int_0^t D^{\nu} f^{\alpha,\lambda}(t-s) \frac{1}{\Gamma(\nu+1)} \frac{d}{ds} \int_0^s (s-u)^{\nu}\, dZ_u\, ds\,.$$

Now note that, for any two differentiable fuctions f and g on $\mathbb{R}^+$,

$$(f * g)'(t) = \frac{d}{dt} \int_0^{+\infty} g(t-s) f(s)\, ds = \int_0^{+\infty} g'(t-s) f(s)\, ds$$

so that $f * (g') = (f * g)'$. Using this fact together with Fubini's theorem we get

$$\begin{aligned} Y_t &= F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \frac{d}{dt} \int_0^t \frac{1}{\Gamma(\nu+1)} \int_u^t D^{\nu} f^{\alpha,\lambda}(t-s)(s-u)^{\nu}\, ds\, dZ_u \\ &= F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \frac{d}{dt} \int_0^t I^{\nu+1} D^{\nu} f^{\alpha,\lambda}(t-u)\, dZ_u \\ &= F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \frac{d}{dt} \int_0^t \int_0^{t-u} I^{\nu} D^{\nu} f^{\alpha,\lambda}(s)\, ds\, dZ_u \\ &= F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \frac{d}{dt} \int_0^t \int_u^t I^{\nu} D^{\nu} f^{\alpha,\lambda}(v-u)\, dv\, dZ_u \\ &= F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \frac{d}{dt} \int_0^t \int_0^v I^{\nu} D^{\nu} f^{\alpha,\lambda}(v-u)\, dZ_u\, dv \\ &= F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \int_0^t f^{\alpha,\lambda}(t-u)\, dZ_u\,. \end{aligned}$$

Using the representation result given by Theorem A.7, since $d\langle Z, Z\rangle_t = Y_t\, dt$, we have that there exists a Brownian motion B^1 such that

$$Z_t = \int_0^t \sqrt{Y_s}\, dB_s^1\,.$$

Let's consider the process $(\tilde{Y}_t)_t$ defined by

$$\tilde{Y}_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \int_0^t f^{\alpha,\lambda}(t-u) \sqrt{Y_s}\, dB_s^1\,.$$

going backward in the previous computations, substituting $\sqrt{Y_u}\, dB_u^1$ to dZ_u, we get that

$$\tilde{Y}_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \int_0^t D^{\nu} f^{\alpha,\lambda}(t-s) \frac{1}{\Gamma(\nu+1)} \frac{d}{ds} \int_0^s (s-u)^{\nu} \sqrt{Y_u}\, dB_u^1\, ds$$

and

$$
\begin{aligned}
\frac{1}{\Gamma(\nu+1)}\frac{d}{ds}\int_0^s (s-u)^\nu \sqrt{Y_u}\,dB_u^1 &= \frac{1}{\Gamma(\nu)}\frac{d}{ds}\int_0^s \frac{(s-u)^\nu}{\nu}\sqrt{Y_u}\,dB_u^1 \\
&= \frac{1}{\Gamma(\nu)}\frac{d}{ds}\int_0^s \Big(\int_0^{s-u} x^{\nu-1}\,dx\Big)\sqrt{Y_u}\,dB_u^1 \\
&= \frac{1}{\Gamma(\nu)}\frac{d}{ds}\int_0^s \Big(\int_u^{s} (s-v)^{\nu-1}\,dv\Big)\sqrt{Y_u}\,dB_u^1 \\
&= \frac{1}{\Gamma(\nu)}\frac{d}{ds}\int_0^s (s-v)^{\nu-1}\Big(\int_0^{v} \sqrt{Y_u}\,dB_u^1\Big)\,dv \\
&= \frac{1}{\Gamma(\nu)}\frac{d}{ds}\int_0^s \frac{Z_v}{(s-v)^{1-\nu}}\,dv \\
&= D^{1-\nu} Z_s\,.
\end{aligned}
$$

This implies

$$\tilde{Y}_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}}\int_0^t D^\nu f^{\alpha,\lambda}(t-s) D^{1-\nu} Z_s\,ds = Y_t$$

and we can conclude that

$$Y_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}}\int_0^t f^{\alpha,\lambda}(t-u)\sqrt{Y_s}\,dB_s^1\,,$$

which is the desired result. □

We eventually provide here an alternative form for equation (3.6), that shows that the process Y solves a Volterra-type stochastic differential equation, a sort of rough CIR equation. This result connects with the light tails case, where the limiting intensity process was solution of a CIR stochastic differential equation, showing that the heavy tails condition is the basis of the rough behaviour in our limiting process.

We state a general proposition, that will be useful in the following too.

PROPOSITION 3.12. *Let λ, ν, ϑ be positive constants, $\alpha \in (1/2, 1)$ and B a Brownian motion. The process V is solution of the rough stochastic differential equation*

$$V_t = \vartheta F^{\alpha,\lambda}(t) + \nu \int_0^t f^{\alpha,\lambda}(t-s)\sqrt{V_s}\,dB_s \tag{3.7}$$

if and only if it is solution of

$$V_t = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\lambda(\vartheta - V_s)\,ds + \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\sqrt{V_s}\,dB_s\,. \tag{3.8}$$

Furthermore, both equation admit a unique strong solution.

PROOF. We already proved the existence of a solution for (3.7) deriving equation (3.6). Let V be a solution to (3.7) and write

$$K = I^{1-\alpha} V\,.$$

Using stochastic Fubini theorem we get

$$K_t = \vartheta \int_0^t I^{1-\alpha} f^{\alpha,\lambda}(u)\,du + \nu \int_0^t I^{1-\alpha} f^{\alpha,\lambda}(t-u)\sqrt{V_u}\,dB_u\,.$$

Moreover, we know from Proposition 3.5 that $I^{1-\alpha}f^{\alpha,\lambda} = \lambda\big(1 - F^{\alpha,\lambda}(t)\big)$. Hence, using stochastic Fubini theorem again, we obtain

$$\begin{aligned}
K_t &= \vartheta \int_0^t \lambda\big(1 - F^{\alpha,\lambda}(u)\big)\,du + \nu\lambda \int_0^t \sqrt{V_u}\,dB_u - \lambda\nu \int_0^t F^{\alpha,\lambda}(t-u)\sqrt{V_u}\,dB_u \\
&= \vartheta \int_0^t \lambda\big(1 - F^{\alpha,\lambda}(u)\big)\,du + \nu\lambda \int_0^t \sqrt{V_u}\,dB_u - \lambda\nu \int_0^t \int_0^{t-u} f^{\alpha,\lambda}(s)\,ds\sqrt{V_u}\,dB_u \\
&= \vartheta \int_0^t \lambda\big(1 - F^{\alpha,\lambda}(u)\big)\,du + \nu\lambda \int_0^t \sqrt{V_u}\,dB_u - \lambda\nu \int_0^t \int_u^{t} f^{\alpha,\lambda}(s-u)\,ds\sqrt{V_u}\,dB_u \\
&= \vartheta \int_0^t \lambda\big(1 - F^{\alpha,\lambda}(u)\big)\,du + \nu\lambda \int_0^t \sqrt{V_u}\,dB_u - \lambda \int_0^t \nu \int_0^{s} f^{\alpha,\lambda}(s-u)\sqrt{V_u}\,dB_u\,ds \\
&= \vartheta \int_0^t \lambda\big(1 - F^{\alpha,\lambda}(u)\big)\,du + \nu\lambda \int_0^t \sqrt{V_u}\,dB_u - \lambda \int_0^t \big(V_s - \vartheta F^{\alpha,\lambda}(s)\big)\,ds \\
&= \lambda \int_0^t (\vartheta - V_u)\,du + \lambda\nu \int_0^t \sqrt{V_u}\,dB_u\,.
\end{aligned}$$

Recalling that we have $V_t = D^{1-\alpha}K_t$, we get

$$\begin{aligned}
V_t = \frac{1}{\Gamma(\alpha)}\lambda\frac{d}{dt}\int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_0^s (\vartheta - V_u)\,du\,ds \\
+ \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \lambda\nu\frac{1}{(t-s)^{1-\alpha}} \int_0^s \sqrt{V_u}\,dB_u\,ds\,.
\end{aligned}$$

We study each summand separately. For the first one we get

$$\begin{aligned}
\int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_0^s (\vartheta - V_u)\,du\,ds &= \int_0^t \int_u^t \frac{1}{(t-s)^{1-\alpha}}(\vartheta - V_u)\,ds\,du \\
&= \int_0^t \int_u^t \frac{1}{(s-u)^{1-\alpha}}(\vartheta - V_u)\,ds\,du \\
&= \int_0^t \int_0^s \frac{1}{(s-u)^{1-\alpha}}(\vartheta - V_u)\,du\,ds\,.
\end{aligned}$$

With the same method we get for the second one

$$\begin{aligned}
\int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_0^s \sqrt{V_u}\,dB_u\,ds &= \int_0^t \int_u^t \frac{1}{(t-s)^{1-\alpha}}\sqrt{V_u}\,ds\,dB_u \\
&= \int_0^t \int_u^t \frac{1}{(s-u)^{1-\alpha}}\sqrt{V_u}\,ds\,dB_u \\
&= \int_0^t \int_0^s \frac{1}{(s-u)^{1-\alpha}}\sqrt{V_u}\,dB_u\,ds\,.
\end{aligned}$$

Therefore we got

$$\begin{aligned}
V_t = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \lambda \int_0^s \frac{1}{(s-u)^{1-\alpha}}(\vartheta - V_u)\,du\,ds \\
+ \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \lambda\nu \int_0^s \frac{1}{(s-u)^{1-\alpha}}\sqrt{V_u}\,dBu\,ds\,,
\end{aligned}$$

which gives

$$V_t = \frac{1}{\Gamma(\alpha)}\lambda\int_0^t \frac{1}{(t-u)^{1-\alpha}}(\vartheta - V_u)\,du + \frac{1}{\Gamma(\alpha)}\lambda\nu\int_0^t \frac{1}{(t-u)^{1-\alpha}}\sqrt{V_u}\,dB_u\,.$$

Hence V is solution of (3.8). The uniqueness of such a solution is given by Theorem 2.5 in [**24**]. □

Now, a straightforward application of the last proposition gives us the following corollary:

COROLLARY 3.13. *The derivative Y of a process $(X_t)_{t\in[0,1]}$ satisfying equation* (3.2), *when it exists, satisfies the following rough CIR stochastic differential equation:*

$$Y_t = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\lambda(1-Y_s)\,ds + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{\lambda}{\mu^*}}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\sqrt{Y_s}\,dW_s\,,$$

where W is a Brownian motion on $[0,1]$.

CHAPTER 4

Multidimensional limit theorems

In this chapter we aim to extend the limit theorems obtained in the previous chapters to the multidimensional case. We will still have a separate approach for the light tails case and the heavy tails case.

1. General setting

We study the convergence of a sequence of nearly unstable d-dimensional Hawkes processes defined on $[0, T]$, with T tending to infinity. We need to define what *nearly unstable* means in a multidimensional context.

We keep the notation N^T for our d-dimensional Hawkes process whose intensity process λ^T is defined by

$$\lambda_t^T = \mu_T \mathbf{1} + \int_0^t \mathbf{\Phi}^T(t-s) \cdot dN_s^T ,$$

where $\mu_T > 0$ and $\mathbf{\Phi}^T = a_T \mathbf{\Phi}$, with a_T an increasing sequence of positive numbers converging to 1, and the matrix $\mathbf{\Phi} \colon \mathbb{R}_+ \to \mathcal{M}^d(\mathbb{R}_+)$ has positive and integrable components. We use $\varphi_{i,j}$ for the components of the matrix $\mathbf{\Phi}$. If $\mathcal{S}$ is the spectral radius operator, we impose

$$\mathcal{S}\left(\int_0^\infty \mathbf{\Phi}(s)\, ds\right) = 1 .$$

Hence the nearly unstability condition is given by the limit $\mathcal{S}\left(\int_0^\infty \mathbf{\Phi}^T(s)\, ds\right) \to 1$ as $T \to \infty$. We assume that for any $t \geq 0$, $\mathbf{\Phi}(t)$ is diagonalizable on $\mathbb{R}$. We write $\lambda_1(t) \geq \cdots \geq \lambda_d(t)$ for the eigenvalues of $\mathbf{\Phi}^*$ and $v_1, \ldots, v_d$ for the corresponding eigenvectors.[1] We assume that these eigenvectors do not depend on t. Note that, from Perron-Frobenius theorem, for $i \geq 2$, $|\lambda_i(t)| < \lambda_1(t) = \mathcal{S}(\mathbf{\Phi}(t))$ and v_1 can be taken in $\mathbb{R}_+^d$ (that is with positive entries). Moreover we choose an orthonormal basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$ such that $e_1^* \cdot v_1 > 0$ and

$$\operatorname{span}(e_2, \ldots, e_d) = \operatorname{span}(v_2, \ldots, v_d)$$

and set

$$v' = e_1 - \frac{1}{e_1^* \cdot v_1} v_1 .$$

Note that $v' \in \operatorname{span}(v_2, \ldots, v_d)$ since $e_1^* \cdot v' = 0$.

As usual we write M^T for the martingale process associated to N^T:

$$M_t^T = N_t^T - \int_0^t \lambda_s^T\, ds .$$

[1]Here the symbol * represents the transpose operator. Moreover the dot $\cdot$ represents the usual matrix product.

2. Light tails case

We now make more specific assumptions, focusing on the light tails case. It is clear the connection with the assumptions of Chapter 2.

ASSUMPTION 4.1. There exist positive parameters λ and μ such that

$$T(1-a_T) \underset{T\to\infty}{\to} \lambda, \qquad \mu_T = \mu$$

and

$$\int_0^\infty x\,\varphi_{i,j}(x)\,dx < \infty \quad \text{for any } i,j \in \{1,\dots,d\}\,.$$

We set $m := \int_0^\infty x\lambda_1(x)\,dx$.

Now, as in the previous chapter, but with an obvious multidimensional meaning, let

$$\mathbf{\Psi}^T = \sum_{k\geq 1} (\mathbf{\Phi}^T)^{*k}\,,$$

where $(\mathbf{\Phi}^T)^{*1} = \mathbf{\Phi}^T$ and for $k > 1$,

$$(\mathbf{\Phi}^T)^{*k}(t) = \int_0^t \mathbf{\Phi}^T(s)\cdot(\mathbf{\Phi}^T)^{*(k-1)}(t-s)\,ds\,.$$

We then need the following technical assumption:

ASSUMPTION 4.2. The function $\mathbf{\Psi}^T$ is uniformly bounded and $\mathbf{\Phi}$ is differentiable such that each component φ_{ij} satisfies $\|\varphi'_{ij}\|_\infty < \infty$ and $\|\varphi'_{ij}\|_1 < \infty$.

REMARK 4.1. Assumption 4.2 in particular implies that, for any $i \in \{1,\dots,d\}$,

$$\lim_{x\to\infty} \lambda_i(x) = 0\,.$$

2.1. Heuristics and theorem. We can provide some intuitions and heuristic computations to understand the general behaviour of a sequence of nearly unstable light tailed Hawkes processes. The ideas are very similar to the heuristics of chapter 2, but it is interesting to observe how the multidimensional case reduces to the one dimensional approach.

First of all, note that as we did in Chapter 2 we can rewrite the intensity process as

$$\lambda_t^T = \mu_T\mathbf{1} + \int_0^t \mathbf{\Phi}^T(t-s)\cdot dM_t^T + \int_0^t \mathbf{\Phi}^T(t-s)\cdot\lambda_s^T\,ds\,.$$

Now, using a straightforward multidimensional generalization of Lemma 2.11 and reproducing the proof of Proposition 2.10, we get

$$\lambda_t^T = \mu_T\mathbf{1} + \mu_T\int_0^t \mathbf{\Psi}^T(t-s)\,ds\cdot\mathbf{1} + \int_0^t \mathbf{\Psi}^T(t-s)\cdot dM_s^T\,. \tag{4.1}$$

Therefore

$$\mathbb{E}[\lambda_t^T] = \mu_T\mathbf{1} + \mu_T\int_0^t \mathbf{\Psi}^T(t-s)\,ds\cdot\mathbf{1}\,,$$

which gives

$$\mathbb{E}[\lambda_{tT}^T] = \mu_T\mathbf{1} + \mu_T\int_0^t \mathbf{\Psi}^T(T(t-s))\,T\,ds\cdot\mathbf{1}\,.$$

Since the function $\mathbf{\Psi}^T$ is uniformly bounded and $\mu_T \equiv \mu$, we get that λ_{tT}^T is of order T. Thus, it is natural to consider the following rescaling for $t \in [0,1]$:

$$C_t^T = \frac{1}{T}\lambda_{tT}^T\,.$$

From equation (4.1) we get for C_t^T

$$C_t^T = \frac{\mu}{T}\mathbf{1} + \mu \int_0^t \mathbf{\Psi}^T(T(t-s))\,ds \cdot \mathbf{1} + \int_0^t \mathbf{\Psi}^T(T(t-s)) \cdot d\overline{M}_s^T\,,$$

where $\overline{M}_t^T = M_{tT}^T/T$. Note that

$$\langle M^T, M^T\rangle_t = \operatorname{diag}\left(\int_0^t \lambda_s^T\,ds\right).$$

Hence we get that

$$\mathbb{E}[\langle \overline{M}^T, \overline{M}^T\rangle_t] = \frac{1}{T^2}\mathbb{E}\left[\operatorname{diag}\left(\int_0^{tT} \lambda_s^T\,ds\right)\right] = \operatorname{diag}\left(\int_0^t \mathbb{E}[C_s^T]\,ds\right)$$

is bounded. By an easy induction we can show that, for any $k \geq 1$, $v_i^* \cdot \mathbf{\Phi}^{*k} = \lambda_i^{*k}(t) v_i^*$. Consequently, defining

$$\psi_i^T = \sum_{k\geq 1} a_T^k \lambda_i^{*k}\,, \quad \text{for } i \in \{1,\dots,d\}\,,$$

we have

$$v_i^* \cdot \mathbf{\Psi}^T = \psi_i^T v_i^*\,.$$

This allows us to rewrite the dynamics of $v_i^* \cdot C_t^T$ in the following way

$$v_i^* \cdot C_t^T = \frac{\mu}{T}(v_i^* \cdot \mathbf{1}) + \mu(v_i^* \cdot \mathbf{1}) \int_0^t \psi_i^T(T(t-s))\,ds + \int_0^t \psi_i^T(T(t-s))(v_i^* \cdot d\overline{M}_s^T)\,. \tag{4.2}$$

We can observe that the functions ψ_i play here the role that the function ψ played in chapter 2. We need to study these functions in order to understand the asymptotic behaviour of $v_i^* \cdot C^T$ as T goes to infinity. As we already did before, we compute the Fourier transform of $\psi_j^T(T\cdot)$ for each $j \in \{1,\dots,d\}$. Since $\hat{\lambda}_i^{*k} = (\hat{\lambda}_i)^k$, it is

$$\hat{\psi}_j^T(T\cdot)(z) = \int_{x\in\mathbb{R}_+} \psi_j^T(Tx) e^{ixz}\,dx = \frac{1}{T}\sum_{k\geq 1} a_T^k (\hat{\lambda}_j(z/T))^k = \frac{a_T \hat{\lambda}_j(z/T)}{T\big(1 - a_T \hat{\lambda}_j(z/T)\big)}\,.$$

Thanks to dominated convergence theorem we have that

$$\lim_{T\to\infty} \hat{\lambda}_j(z/T) = \|\lambda_j\|_1$$

and, since $\|\lambda_j\|_1 < 1$ for $j \geq 2$, this tells us that $\psi_j^T(T\cdot)$ vanishes asymptotically. This implies that also $v_j^* \cdot C^T$ vanishes asymptoticaly for $j \geq 2$.

For $j = 1$ we get, using Taylor expansion for $\hat{\lambda}_1(z/T)$ and Assumption 4.1,

$$\lim_{T\to\infty} T(\hat{\lambda}_1(z/T) - 1) = iz \int_0^\infty x\lambda_1(x)\,dx = izm\,.$$

Therefore we have

$$\hat{\psi}_1^T(T\cdot)(z) = \frac{a_T \hat{\lambda}_1(z/T)}{T(1-a_T) - a_T T(\hat{\lambda}_1(z/T) - 1)} \underset{T\to\infty}{\to} \frac{1}{\lambda - izm}\,,$$

which is (in analogy to Chapter 2) the Fourier transform of

$$x \mapsto \frac{1}{m} e^{-\frac{\lambda}{m}x}\,, x \in \mathbb{R}_+\,.$$

This gives us the covergence of $\psi_1(Tx)$ to $\frac{1}{m}e^{-\frac{\lambda}{m}x}$.

We use this information to deduce the dynamics of $v_1^* \cdot C^T$. From (4.2) we get that

$$v_1^* \cdot C_t^T = \frac{\mu}{T}(v_1^* \cdot \mathbf{1}) + \mu(v_1^* \cdot \mathbf{1}) \int_0^t \psi_1^T\big(T(t-s)\big)\, ds + \int_0^t \psi_1^T\big(T(t-s)\big) \sqrt{(v_1^2)^* \cdot C_s^T}\, dB_s^T \,, \tag{4.3}$$

where $v_1^2 = (v_{1,i}^2)_{1 \le i \le d}$ and

$$B_t^T = \int_0^{tT} \frac{v_1^* \cdot dM_s^T}{\sqrt{T(v_1^2)^* \cdot \lambda_s^T}} \,.$$

It is clear at this stage that the sequence of processes B^T has been chosen in a way that the associated sequence of quadratic variations converges to identity. Thus the limit of B^T is a Brownian motion. We will make this statement precise in the following.

Now, decomposing v_1^2 in the basis $(e_1, \dots, e_d)$, we get

$$(v_1^2)^* \cdot C_t^T = \frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}(v_1^* \cdot C_t^T) + (e_1^* \cdot v_1^2)\big((v')^* \cdot C_t^T\big) + \sum_{2 \le i \le d} (e_i^* \cdot v_1^2)(e_i^* \cdot C_t^T) \,.$$

Thus, using that $v^* \cdot C_t^T$ converges to zero for any $v \in \mathrm{span}(v_2, \dots, v_d)$, we deduce that $(v_1^2)^* \cdot C_t^T$ behaves asymptotically as

$$(v_1^2)^* \cdot C_t^T = \frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}(v_1^* \cdot C_t^T) \,.$$

Therefore, letting T go to infinity in (4.3), we heuristically deduce that $v_1^* \cdot C_t^T$ is solution of the following stochastic differential equation:

$$X_t = \frac{\mu}{m}(v_1^* \cdot \mathbf{1}) \int_0^t e^{-\frac{\lambda}{m}(t-s)}\, ds + \frac{1}{m}\sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \int_0^t e^{-\frac{\lambda}{m}(t-s)} \sqrt{X_s}\, dB_s \,.$$

Now, note that we can write

$$X_t = W_t\, e^{-\frac{\lambda}{m}t} \,,$$

where

$$W_t = \frac{\mu}{m}(v_1^* \cdot \mathbf{1}) \int_0^t e^{\frac{\lambda}{m}s}\, ds + \frac{1}{m}\sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \int_0^t e^{\frac{\lambda}{m}s} \sqrt{X_s}\, dB_s \,.$$

Hence, applying Ito's formula, we get

$$\begin{aligned} dX_t &= dW_t e^{-\frac{\lambda}{m}t} - \frac{\lambda}{m} W_t e^{-\frac{\lambda}{m}t}\, dt \\ &= \frac{\lambda}{m}(v_1^* \cdot \mathbf{1})\, dt + \frac{1}{m}\sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \sqrt{X_t}\, dB_t - \frac{\lambda}{m} X_t\, dt \\ &= \frac{\lambda}{m}\left(\frac{\mu}{\lambda}(v_1^* \cdot \mathbf{1}) - X_t\right) dt + \frac{1}{m}\sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \sqrt{X_t}\, dB_t \,. \end{aligned}$$

Thus the process X_t satisfies a Cox-Ingersoll-Ross dynamics.

We now decompone C_t^T in the basis $(e_1, \dots, e_d)$

$$C_t^T = \frac{1}{e_1^* \cdot v_1}(v_1^* \cdot C_t^T) e_1 + \big((v')^* \cdot c_t^T\big) e_1 + \sum_{2 \le i \le d} (e_i^* \cdot C_t^T) e_i$$

and recall that $v_i^* \cdot C_t^T$ asymptoticaly vanishes for $i \ge 2$, to get the asymptotical behaviour of C^T.

Next theorem is the result of the heuristical computations provided in this section:

THEOREM 4.2. *In the setting described by Section 1, under the Assumptions 4.1 and 4.2, the multidimensional process* $(C_t^T, B_t^T)_{t\in[0,1]}$ *converges in law for the Skorohod topology to* $\left(\frac{1}{e_1^*\cdot v_1} X e_1, B\right)$, *where* B *is a Brownian motion and* X *satisfies the following Cox-Ingersoll-Ross dynamics:*

$$dX_t = \frac{\lambda}{m}\left(\frac{\mu}{\lambda}(v_1^*\cdot \mathbf{1}) - X_t\right) dt + \frac{1}{m}\sqrt{\frac{e_1^*\cdot v_1^2}{e_1^*\cdot v_1}}\sqrt{X_t}\, dB_t\,, \quad X_0 = 0\,.$$

2.2. Proof of Theorem 4.2. As it is suggested by the heuristic computations in last section, the proof of the theorem will be quite similar to the one dimensional case proof. The key point will be showing that $v_i^*\cdot C_t^T$ actually converges to zero for any $i \geq 2$. Then we will reproduce the arguments used in the one dimensional setting.

2.2.1. *A basic Lemma.* We start with a basic Lemma, that was crucial in Chapter 2 and we extend it to the multidimensional case.

LEMMA 4.3. *Let* $f^T : \mathbb{R}_+ \to \mathbb{R}$ *be a sequence of measurable functions such that for some* $c > 0$

(a) $f^T \in L^1(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$ *and* $\int_{\mathbb{R}_+} |f^T(x)|^2\, dx \underset{T\to\infty}{\to} 0$;
(b) $|f^T(x)| \leq c$ *for any* $x \geq 0$;
(c) $|\hat{f}^T(x)| \leq c\left(1\wedge \frac{1}{|x|}\right)$ *for any* $x\in\mathbb{R}$;
(d) $|f^T(x) - f^T(y)| \leq cT|x-y|$ *for any* $x, y \geq 0$.

Then, under Assumptions 4.1 and 4.2, the process

$$\left(\int_0^t f^T(t-s)\, d\overline{M}_s^T\right)_{t\in[0,1]}$$

converges to zero in probability as T *goes to infinity, uniformly over compact sets.*

PROOF. This result has already been proved in dimension one in Chapter 2. The tightness of the process

$$Y_t^T = \int_0^t f^T(t-s)\, d\overline{M}_s^T$$

holds in the same way that in chapter 2, working componentwise. Hence we just need to show the finite dimensional convergence of Y^T to zero.

Using that

$$\langle M^T, M^T\rangle_t = \mathrm{diag}\left(\int_0^t \lambda_s^T\, ds\right)$$

together with isometry 1.5, we get

$$\mathbb{E}\left[\|Y_t^T\|_2\right] = \mathbb{E}\left[\sum_{i=1}^d (Y_t^T)_i^2\right] = \frac{1}{T^2}\mathbb{E}\left[\int_0^{tT} \left(f^T(t-s/T)\right)^2 \sum_{i=1}^d \lambda_{s,i}^T\, ds\right]$$
$$= \frac{1}{T^2}\int_0^{tT} \left(f^T(t-s/T)\right)^2 \sum_{i=1}^d \mathbb{E}[\lambda_{s,i}^T]\, ds\,.$$

We look for an upper bound for $\mathbb{E}[\lambda_{s,i}^T]$. Using equation (4.1) and the fact that $v_i^*\cdot \mathbf{\Psi}^T = \psi_i^T v_i^*$, we obtain for any $i\in\{1,\dots,d\}$

$$\mathbb{E}[v_i^*\cdot\lambda_s^T] = \mu(v_i^*\cdot \mathbf{1})\left(1+\int_0^s \psi_i^T(u)\, du\right)\,.$$

Thus

$$
\begin{aligned}
|\mathbb{E}[v_i^* \cdot \lambda_s^T]| &\leq \mu |v_i^* \cdot \mathbf{1}| \Big(1 + \sum_{k \geq 1} \int_0^\infty a_T^k |\lambda_i|^{*k}(u)\, du \Big) \\
&\leq \mu |v_i^* \cdot \mathbf{1}| \frac{1}{1 - a_T \|\lambda_i\|_1} \leq \mu |v_i^* \cdot \mathbf{1}| \frac{T}{T(1-a_T)} \\
&\leq cT
\end{aligned}
$$

for some $c > 0$. Hence, decomposing the i-th vector of the canonical basis in the basis $(v_1, \dots, v_d)$, we get $\mathbb{E}[\lambda_{s,i}^T] \leq c' T$, for any $i \in \{1, \dots, d\}$ and a constant $c' > 0$. Therefore

$$
\mathbb{E}\big[\|Y_t^T\|_2^2\big] \leq \frac{1}{T^2} \int_0^{tT} \big(f^T(t - s/T)\big)^2 c' T\, ds = c \int_0^t \big(f^T(t-s)\big)^2\, ds \leq c' \int_0^\infty \big(f^T(s)\big)^2
$$

and last quantity tends to zero as T goes to infinity thanks to property (a). Now, applying Markov inequality, we get that Y_t^T tends to zero in probability, giving the finite dimensional convergence of the process. □

2.2.2. *Convergence of $v_i^* \cdot C^T$ for $i \in \{2, \dots, d\}$.* We now come to the convergence of the process C^T on the vector space $\operatorname{span}(v_2, \dots, v_d)$.

Proposition 4.4. *Let $2 \leq i \leq d$. Under Assumptions 4.1 and 4.2, the process $v_i^* \cdot C^T$ converges u.c.p. to zero as T goes to infinity.*

Proof. Recall equation (4.2):

$$
v_i^* \cdot C_t^T = \frac{\mu}{T}(v_i^* \cdot \mathbf{1}) + \mu (v_i^* \cdot \mathbf{1}) \int_0^t \psi_i^T(T(t-s))\, ds + \int_0^t \psi_i^T(T(t-s))(v_i^* \cdot d\overline{M}_s^T)
$$

and note that it can be rewritten as

$$
v_i^* \cdot C_t^T = \frac{\mu}{T}(v_i^* \cdot \mathbf{1}) + \mu (v_i^* \cdot \mathbf{1}) \frac{1}{T} \int_0^{tT} \psi_i^T(Tt - s))\, ds + \int_0^t \psi_i^T(T(t-s))(v_i^* \cdot d\overline{M}_s^T)\, .
$$

Since $\|\psi_i^T\|_1 < \infty$, we have that the first and the second summand goes to zero as T goes to infinity. For the third one it is enough to show that the family of functons $\big(\psi_i^T(T\cdot)\big)_{T>0}$ satisfies the hypotesis of Lemma 4.3. Point (b) easily comes from the fact that $v_i^* \cdot \mathbf{\Psi}^T = \psi_i^T \cdot v_i^*$ and the uniform boundedness of $\mathbf{\Psi}^T$, in fact

$$
|\psi_i^T(Tt)| = |\psi_i^T(Tt)(v_i^* \cdot v_i)| = |v_i^* \cdot \mathbf{\Psi}^T(Tt) \cdot v_i| \leq \|\mathbf{\Psi}^T(Tt)\|_{\mathcal{M}^d(\mathbb{R}_+)} \leq c\, .
$$

Now, as we saw in Remark 4.1, $\lambda_i(x)$ tends to zero as x goes to infinity. Then using integration by parts on the Fourier transform of λ_i together with Assumption 4.2, we get

$$
\begin{aligned}
\hat{\lambda}_i(w) &= \int_0^\infty \lambda_i(x) e^{iwx}\, dx \\
&= \frac{1}{iw}\Big[e^{iwx}\lambda_i(x)\Big]_{x=0}^{x=\infty} - \frac{1}{w}\int_0^\infty \lambda_i'(x) e^{iwx}\, dx \\
&- \frac{1}{iw}\Big(\lambda_i(0) + \int_0^\infty \lambda_i'(x) e^{iwx}\, dx\Big)
\end{aligned}
$$

and hence

$$
|\hat{\lambda}_i(w)| \leq \Big(\frac{1}{w}\Big(|\lambda_i(0)| + \int_0^\infty |\lambda_i'(x)|\, dx\Big)\Big) \wedge \|\lambda_i\|_1\, .
$$

We have $|\widehat{\lambda_i^{*k}}(w)| = |\hat\lambda_i(w)|^k \le |\hat\lambda_i(w)|$. Point (c) then follows since

$$\begin{aligned}|\hat\psi_i^T(T\cdot)(w)| &= \left|\int_0^\infty \psi_i^T(Tx)e^{iwx}\,dx\right| = \frac{1}{T}\left|\int_0^\infty \sum_{k\ge1} a_T\widehat{\lambda_i^{*k}}(x)e^{i\frac{w}{T}x}\,dx\right| \\ &\le \frac{1}{T}\sum_{k\ge1} a_T^k|\widehat{\lambda_i^{*k}}(w/T)| \le \frac{1}{T}\sum_{k\ge1} a_T^k|\hat\lambda_i(w/T)| \\ &= \frac{|a_T\hat\lambda_i(w/T)|}{|T\big(1-a_T\hat\lambda_i(w/T)\big)|} \le \frac{|a_T\hat\lambda_i(w/T)|}{T(1-\|\lambda_i\|_1)} \le c\Big(1\wedge\frac{1}{|w|}\Big)\,.\end{aligned}$$

This inequality also gives us that $\hat\psi_i^T(T\cdot)$ is square integrable and thus $\psi_i^T(T\cdot)$ is square integrable too. Moreover by Fourier isometry we have

$$\begin{aligned}\int_{\mathbb{R}_+}|\psi_i^T(Tx)|^2\,dx &= \frac{1}{2\pi}\int_{w\in\mathbb{R}}|\hat\psi_i^T(T\cdot)(w)|^2\,dw \le c\int_{w\in\mathbb{R}}\frac{|\hat\lambda_i(w/T)|^2}{T^2(1-\|\lambda_i\|_1)^2}\,dw \\ &\le \frac{c}{T}\int_{z\in\mathbb{R}}|\hat\lambda_i(z)|^2\,dz \underset{T\to\infty}{\to} 0\,.\end{aligned}$$

Thus (a) is obtained. Finally, in order to show (d), we use the relation

$$\begin{aligned}\psi_i^T &= \sum_{k\ge1} a_T^k\lambda_i^{*k} = a_T\lambda_i + \sum_{k\ge2} a_T^k\lambda_i^{*k} \\ &= a_T\lambda_i + \sum_{k\ge1} a_T^{k+1}\lambda_i^{*(k+1)} = a_T\lambda_i + a_T\lambda_i * \Big(\sum_{k\ge1} a_T^k\lambda_i^{*k}\Big) \\ &= a_T\lambda_i + a_T\lambda_i*\psi_i^T\,.\end{aligned}$$

In fact, noticing that

$$\begin{aligned}\frac{d}{dx}(\lambda_i*\psi_i^T)(x) &= \frac{d}{dx}\Big(\int_0^\infty \lambda_i(z)\psi_i^T(x-z)\,dz\Big) = \int_0^x \lambda_i(z)\big(\psi_i^T\big)'(x-z)\,dz \\ &= -\big[\lambda_i(z)\psi_i^T(x-z)\big]_{z=0}^{z=x} + \int_0^x \lambda_i'(z)\psi_i^T(x-z)\,dz \\ &= -\lambda_i(0)\psi_i^T(x) + \lambda_i' * \psi_i^T(x)\,,\end{aligned}$$

we can write

$$\begin{aligned}|(\psi_i^T)'(Tx)| &= T|a_T\lambda_i'(Tx) - a_T\lambda_i(0)\psi_i^T(Tx) + \lambda_i'*\psi_i^T(Tx)| \\ &\le T(\|\lambda_i'\|_\infty + |\lambda_i(0)|\|\psi_i^T\|_\infty + \|\lambda_i'\|_1\|\psi_i^T\|_\infty)\,.\end{aligned}$$

Thanks to Assumption 4.2 all the quantities in the last inequality are bounded and we can conclude, getting (d). ☐

2.2.3. *Convergence of the process $v_1^*\cdot C_T$.* We now treat the term $v_1^*\cdot C_T$. We observe that a different behaviour will arise because of the fact that $\|\lambda_1\|_1 = 1$, while the condition $\|\lambda_i\|_i < 1$ for $i\in\{2,\dots,d\}$ was essential to get convergence in the last Proposition, because it allowed the convergence of power series.

PROPOSITION 4.5. *Under Assumptions 4.1 and 4.2, the process $(v_1^*\cdot C_t^T, B_t^T)_{t\in[0,1]}$ converges in law for the Skorohod topology to (X,B), where B is a Brownian motion and X satisfies the following Cox-Ingersoll-Ross dynamic:*

$$dX_t = \frac{\lambda}{m}\Big(\frac{\mu}{\lambda}(v_1^*\cdot\mathbf{1}) - X_t\Big)\,dt + \frac{1}{m}\sqrt{\frac{e_1^*\cdot v_1^2}{e_1^*\cdot v_1}}\sqrt{X_t}\,dB_t\,,\quad X_0 = 0\,.$$

PROOF. We start rewriting the process $v_1^* \cdot C^T$ in a more suitable way. Recalling that $v' = e_1 - \frac{1}{e_1^* \cdot v_1} v_1$, let S^T the process defined by

$$S_t^T = \sum_{i=2}^{d} \big(e_i^* \cdot C_t^T\big)(e_i^* \cdot v_1^2) + \big((v')^* \cdot C_t^T\big)(e_1^* \cdot v_1^2)\,.$$

From Proposition 4.4 we get that S_t^T tends u.c.p. to zero. Decomposing v_1^2 in the basis $(e_i)_i$ we get

$$\begin{aligned}(v_1^2)^* \cdot C_t^T &= \big((v')^* \cdot C_t^T\big)\big(e_1^* \cdot v_1^2\big) + \sum_{i=2}^{d} \big(e_i^* \cdot C_t^T\big)\big(e_i^* \cdot v_1^2\big) + \frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}\big(v_1^* \cdot C_t^T\big) \\ &= S_t^T + \frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}\big(v_1^* \cdot C_t^T\big)\,,\end{aligned}$$

which, together with (4.3) gives

$$\begin{aligned}v_1^* \cdot C_t^T = \frac{\mu}{T}(v_1^* \cdot \mathbf{1}) &+ \mu(v_1^* \cdot \mathbf{1}) \int_0^t \psi_1^T\big(Ts\big)\, ds \\ &+ \int_0^t \psi_1^T\big(T(t-s)\big)\sqrt{S_s^T + \frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}\big(v_1^* \cdot C_s^T\big)}\, dB_s^T\,.\end{aligned}$$

We then focus on the convergence of the function $\psi_1^T(T\cdot)$. Let's define, for $x \geq 0$, the function

$$f^T(x) = \psi_1^T(Tx) - \frac{1}{m} e^{-\frac{\lambda}{m}x}\,.$$

In Section 2.1 we have studied the Fourier transform of $\psi_1^T(T\cdot)$ and as a result we get that f^T converges to zero as T goes to infinity. Moreover, with the same arguments used in Chapter 2, Section 4, we can prove the following proposition:

PROPOSITION 4.6. *Under Assumptions 4.1 and 4.2, the function f^T satisfies all the hypotesis of Lemma 4.3.*

We can hence rewrite the dynamics of $v_1^* \cdot C^T$ as

$$v_1^* \cdot C_t^T = R_t^T + \frac{\mu}{m}(v_1^* \cdot \mathbf{1}) \int_0^t e^{-\frac{\lambda}{m}s}\, ds + \frac{1}{m}\sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \int_0^t e^{-\frac{\lambda}{m}(t-s)} \sqrt{v_1^* \cdot C_s^T}\, dB_s^T\,, \tag{4.4}$$

where the process R^T is defined by

$$\begin{aligned}R_t^T &= \frac{\mu}{T}(v_1^* \cdot \mathbf{1}) + \mu(v_1^* \cdot \mathbf{1}) \int_0^t f^T(s)\, ds + \int_0^t f^T(t-s)(v_1^* \cdot d\overline{M}_s^T) \\ &\quad + \frac{1}{m}\int_0^t e^{-\frac{\lambda}{m}(t-s)}\left(\sqrt{S_s^T + \frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}(v_1^* \cdot C_s^T)} - \sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}(v_1^* \cdot C_s^T)}\right) dB_s^T\,.\end{aligned} \tag{4.5}$$

Now, using integration by parts for Lebesgue-Stieltjes integrals, we get

$$\begin{aligned}
\int_0^t e^{-\frac{\lambda}{m}(t-s)} \sqrt{v_1^* \cdot C_s^T}\, dB_s^T = \\
&= \int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T - \int_0^t \left(\int_0^s \sqrt{v_1^* \cdot C_u^T}\, dB_u^T \right) \frac{\lambda}{m} e^{-\frac{\lambda}{m}(t-s)}\, ds \\
&= \int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T - \frac{\lambda}{m} \int_0^t \int_u^t e^{-\frac{\lambda}{m}(t-s)} \sqrt{v_1^* \cdot C_u^T}\, ds\, dB_u^T \\
&= \int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T - \frac{\lambda}{m} \int_0^t \int_u^t e^{-\frac{\lambda}{m}(s-u)} \sqrt{v_1^* \cdot C_u^T}\, ds\, dB_u^T \\
&= \int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T - \frac{\lambda}{m} \int_0^t \int_0^s e^{-\frac{\lambda}{m}(s-u)} \sqrt{v_1^* \cdot C_u^T}\, dB_u^T\, ds\,.
\end{aligned}$$

From (4.4), we can rewrite last quantity as

$$\int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T - \lambda \sqrt{\frac{e_1^* \cdot v_1}{e_1^* \cdot v_1^2}} \int_0^t \left\{ v_1^* \cdot C_s^T - R_s^T - \frac{\mu}{\lambda}(v_1^* \cdot \mathbf{1})\big(1 - e^{-\frac{\lambda}{m}s}\big) \right\} ds\,.$$

Therefore we get

$$\begin{aligned}
v_1^* \cdot C_t^T &= R_t^T + \frac{\mu}{m}(v_1^* \cdot \mathbf{1}) \int_0^t e^{-\frac{\lambda}{m}s}\, ds + \frac{1}{m} \sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T \\
&\quad - \frac{\lambda}{m} \int_0^t \left\{ v_1^* \cdot C_s^T - R_s^T - \frac{\mu}{\lambda}(v_1^* \cdot \mathbf{1}) + \frac{\mu}{\lambda}(v_1^* \cdot \mathbf{1}) e^{-\frac{\lambda}{m}s} \right\} ds \qquad (4.6) \\
&= U_t^T + \int_0^t \frac{\lambda}{m} \Big(\frac{\mu}{\lambda}(v_1^* \cdot \mathbf{1}) - v_1^* \cdot C_s^T \Big)\, ds + \frac{1}{m} \sqrt{\frac{e_1^* \cdot v_1^2}{e_1^* \cdot v_1}} \int_0^t \sqrt{v_1^* \cdot C_s^T}\, dB_s^T\,,
\end{aligned}$$

where we set

$$U_t^T = R_t^T + \frac{\lambda}{m} \int_0^t R_s^T\, ds\,.$$

We now see that, in order to conclude, it is sufficient to prove the convergence to zero of the process U^T. We will show that indeed this process converges u.c.p. to zero. This vanishing behaviour is given by that of the funtion f^T and the process S^T. We actually just need to show the convergence to zero of the process R^T.

From Proposition 4.6 and Lemma 4.3 we get the convergence to zero for the first three summand in equation (4.5). We then tackle the last summand.

First, notice that, for any $\beta \in \mathbb{R}_+^d$,

$$\left| \sqrt{S_s^T + \beta v_1^* \cdot C_s^T} - \sqrt{\beta v_1^* \cdot C_s^T} \right| \leq \sqrt{|S_s^T|}\,,$$

which tends to zero as T goes to infinity, as remarked before.

Now, B^T is a sequence of martingales with uniformly bounded jumps, whose quadratic variation is given by

$$\begin{aligned}
[B^T, B^T]_t &= \int_0^{tT} \frac{(v1^2)^* \cdot dN_s^T}{T\big((v_1^2)^* \cdot \lambda_s^T\big)} = \int_0^{tT} \frac{(v_1^2)^* \cdot (\lambda_s^T\, ds + dM_s^T)}{T\big((v_1^2)^* \cdot \lambda_s^T\big)} \\
&= \int_0^{tT} \left\{ \frac{ds}{T} + \frac{(v_1^2)^* \cdot dM_s^T}{T\big((v_1^2)^* \cdot \lambda_s^T\big)} \right\} = t + \int_0^{tT} \frac{(v_1^2)^* \cdot dM_s^T}{T\big((v_1^2)^* \cdot \lambda_s^T\big)}\,.
\end{aligned}$$

From the trivial inequality $\lambda^T \geq \mu\mathbf{1}$ and recalling that $\langle M^T, M^T\rangle_t = \mathrm{diag}\big(\int_0^t \lambda_s^T\,ds\big)$, we have

$$\mathbb{E}\Bigg[\bigg(\int_0^{tT} \frac{(v_1^2)^*\cdot dM_s^T}{T(v_1^2)^*\cdot\lambda_s^T}\bigg)^2\Bigg] = \mathbb{E}\Bigg[\int_0^{tT} \frac{(v_1^4)^*\cdot\lambda_s^T\,ds}{T^2\big((v_1^2)^*\cdot\lambda_s^T\big)^2}\Bigg] \leq \mathbb{E}\Bigg[\int_0^{tT} \frac{(v_1^4)^*\cdot\lambda_s^T\,ds}{T^2\mu\big((v_1^4)^*\cdot\lambda_s^T\big)}\Bigg].$$

Last quantity is bounded by c/T for some $c > 0$ and hence tends to zero as T goes to infinity. Therefore, applying Markov inequality, we have that the quadratic variation of B^T converges in probability to the identity. Thus, using Theorem A.3, we get that B^T converges in law towards a Brownian motion B for the Skorohod topology. Finally from Theorem A.5 we get the convergence to zero in law, for the Skorohod topology, of the third summand in equation (4.5).

Now, looking at equation (4.6) and using that (B^T, U^T) converges in law for the Skorohod topology to $(B, 0)$, we can apply theorem A.1 to get Proposition 4.5. □

This allows us to conclude the proof of the theorem just decomposing C^T in the basis $(e_1, \ldots, e_d)$. As a matter of fact the following holds

$$C_t^T = \sum_{i=2}^d \big(e_i^*\cdot C_t^T\big)e_i + \big((v')^*\cdot C_t^T\big)e_1 + \frac{1}{e_1^*\cdot v_1}\big(v_1^*\cdot C_t^T\big)e_1$$

and we just need to apply Propositions 4.4 and 4.5.

3. Heavy tails case

We now turn to the heavy tails case. The argument will be similar to that used in the light tails case, we will find that the non-degenerating behaviour in the scale limit concetrates on the direction of v_1.

3.1. Assumption and theorem. Keeping the same setting introduced in Section 1, we want to replace Assumptions 4.1 and 4.2 in order to get a sowly decreasing behaviour for the kernel matrix $\mathbf{\Phi}^T$. This will imply a modification of the asymptotic setting in order to get a non-degenerate scaling limit.

We then introduce a new assumption.

Assumption 4.3. There exist $\alpha \in (1/2, 1)$ and $C > 0$ such that

$$\alpha x^\alpha \int_x^\infty \lambda_1(s)\,ds \underset{x\to\infty}{\to} C\,.$$

Moreover, for some $\lambda > 0$ and $\mu > 0$,

$$T^\alpha(1 - a_T) \underset{T\to\infty}{\to} \lambda > 0\,, \quad T^{1-\alpha}\mu_T \underset{T\to\infty}{\to} \mu\,.$$ [2]

Assumption 4.3 introduces a different temporal scaling and, as a consequence, we need to adjust the scaling factors for our nearly unstable Hawkes processes. We will hence consider the following renormalizations for $t \in [0, 1]$:

$$X_t^T = \frac{1-a_T}{T^\alpha\mu}N_{tT}^T\,, \quad \Lambda_t^T = \frac{1-a_t}{T^\alpha\mu}\int_0^{tT}\lambda_s^T\,ds\,, \quad Z_t^T = \sqrt{\frac{T^\alpha\mu}{1-a_T}}\big(X_t^T - \Lambda_t^T\big)\,.$$

The following theorem holds.

[2] A comparison with Assumption 3.2 shows that λ replaced $\lambda\delta$. This will not affect the results, but it will require small adjustments in the asymptotic setting.

THEOREM 4.7. *Under Assumption 4.3, the process* $\big(\Lambda_t^T, X_t^T, Z_t^T\big)_{t\in[0,1]}$ *converges in law for the Skorohod topology to* $\big(\Lambda, X, Z\big)$, *where*

$$\Lambda_t^T = X_t^T = \frac{1}{e_1^* \cdot v_1}\Big(\int_0^t Y_s\, ds\Big) e_1$$

and for $1 \le i \le d$,

$$Z_t^i = \int_0^t \sqrt{\frac{e_{1,i}}{e_1^* \cdot v_1}} \sqrt{Y_s}\, dB_s^i\,,$$

where $(B^1, \dots, B^d)$ *is a d-dimensional Brownian motion and* Y *is the unique solution of the following rough stochastic differential equation:*

$$Y_t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(v_1^*\cdot \mathbf{1} - Y_s)\, ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\sqrt{\mu\lambda}}\sqrt{\frac{e_1^*\cdot v_1^2}{e_1^*\cdot v_1}}\sqrt{Y_s}\, dB_s\,,$$

with

$$B = \frac{1}{\sqrt{e_1^*\cdot v_1^2}}\sum_{i=1}^d \sqrt{e_{1,i} v_{1,i}^2} B^i\,.$$

Furthermore, Y *has Hölder regularity* $\alpha - \frac{1}{2} - \varepsilon$ *for any* $\varepsilon > 0$.

3.2. Proof of Theorem 4.7. This proof is quite similar to proof of Theorems 3.4 and 3.10.

3.2.1. *Tightness.* We begin showing the tightness of the sequence of renormalized nearly unstable Hawkes processes.

PROPOSITION 4.8. *Under Assumption 4.3, the sequence* (Λ^T, X^T, Z^T) *is C-tight. Furthermore, if* (X, Z) *is a limit point of* (X^T, Z^T), *then* Z *is a continous martingale with quadratic variation process the process* X.

PROOF. Recall equation (4.1):

$$\lambda_t^T = \mu_T \mathbf{1} + \mu_T \int_0^t \mathbf{\Psi}^T(t-s)\, ds \cdot \mathbf{1} + \int_0^t \mathbf{\Psi}^T(t-s)\cdot dM_s^T\,.$$

This allows us to write

$$\begin{aligned}
\mathbb{E}\big[N_T^T\big] &= \mathbb{E}\Big[\int_0^T \lambda_s^T\, ds\Big] \\
&= \mathbb{E}\Big[T\mu_T\mathbf{1} + \mu_T\int_0^T\int_0^s \mathbf{\Psi}^T(s-u)\, du\cdot\mathbf{1}\, ds + \int_0^T\int_0^s \mathbf{\Psi}^T(s-u)\, dM_u^T\, ds\Big] \\
&= T\mu_T\mathbf{1} + \mu_T\int_0^T\int_0^s \mathbf{\Psi}^T(s-u)\, du\cdot\mathbf{1}\, ds \\
&= T\mu_T\mathbf{1} + \mu_T\int_0^T\int_0^s \mathbf{\Psi}^T(u)\, du\cdot\mathbf{1}\, ds \\
&= T\mu_T\mathbf{1} + \Big[\mu_T\int_0^s \mathbf{\Psi}^T(u) s\Big]_{s=0}^{s=T}\cdot\mathbf{1} - \mu_T\int_0^T \mathbf{\Psi}^T(s)\, ds\cdot\mathbf{1} \\
&= T\mu_T\cdot\mathbf{1} + \mu_T\Big(\int_0^T (T-s)\mathbf{\Psi}^T(s)\, ds\Big)\cdot\mathbf{1} \\
&= T\mu_T\mathbf{1} + \mu_T\Big(\int_0^T \mathbf{\Psi}^T(T-s)s\, ds\Big)\cdot\mathbf{1}\,.
\end{aligned}$$

Consequently,

$$\mathbf{1}^* \cdot \mathbb{E}\big[N_T^T\big] = T\mu_T d + \mu_T \mathbf{1}^* \cdot \left(\int_0^T s\mathbf{\Psi}^T(T-s)\,ds\right)\cdot \mathbf{1}$$

and therefore there exists a constant $c>0$ such that

$$\begin{aligned}\mathbf{1}^* \cdot \mathbb{E}\big[N_T^T\big] &\le cT\mu_T\left(1+\mathcal{S}\Big(\int_0^\infty \mathbf{\Psi}^T(s)\,ds\Big)\right) \le cT\mu_T\left(1+\frac{a_T}{1-a_T}\right) = c\frac{T\mu_T}{1-a_T}\\ &= c\frac{T^{1-\alpha}\mu_T}{T^\alpha(1-a_T)}T^{2\alpha} \le c' T^{2\alpha}\,.\end{aligned}$$

Thus we obtain

$$\mathbb{E}\big[X_1^T\big] = \frac{1-a_T}{T^\alpha \mu}\mathbb{E}\big[N_T^T\big] \le c'\frac{(1-a_T)T^\alpha}{\mu}$$

Hence there exists $c>0$ such that

$$\mathbb{E}\big[\Lambda_1^T\big] = \mathbb{E}\big[X_1^T\big] \le c\,.$$

Since each component of X^T and Λ^T is increasing, we deduce the tightness of each component of X^T, Λ^T using Corollary 9 in [**23**]. Moreover, the maximum jump size of X^T and Λ^T is $(1-a_T)/T^\alpha\mu$, which goes to zero as T goes to infinity. Hence, applying Proposition VI.3.26 in [**18**], we get the C-tightness of (X^T,Λ^T).

Now, for Z^T we use that

$$\mathbb{E}\big[(Z_{t,i}^T)^2\big] = \frac{1-a_T}{T^\alpha\mu}\mathbb{E}\big[(M_{tT,i}^T)^2\big] = \frac{1-a_T}{T^\alpha\mu}\mathbb{E}\Big[\int_0^{tT}\lambda_s^T\,ds\Big] = \Lambda_{t,i}^T$$

to get that

$$\langle Z^T, Z^T\rangle_t = \mathrm{diag}(\Lambda_t^T)\,,$$

which is C-tight. Thus, from theorem VI.4.13 in [**18**], we obtain the tightness of Z^T. The maximum jump size of Z^T vanishes as T goes to infinity, hence, as before, we conclude that Z^T is C-tight.

Now, let (X,Z) be a possible limit point of (X^T,Z^T). We know (X,Z) is continuous, therefore Corollary IX.1.19 of [**18**] gives us that Z is a local martingale. Moreover, using Theorem VI.6.26 in [**18**], together with the fact that

$$[Z^T,Z^T] = \mathrm{diag}(X^T)\,,$$

we get that $[Z,Z]$ is the limit of $[Z^T,Z^T]$ and $[Z,Z]=\mathrm{diag}(X)$. Finally, by Fatou's lemma, the expectation of $[Z,Z]$ is finite and hence Z is a martingale. □

We now provide a lemma that shows that we can actually work on the sequence Λ^T rather than on the sequence X^T.

LEMMA 4.9. *Under Assumption 4.3, the following convergence holds:*

$$\sup_{t\in[0,1]}\|\Lambda_t^T - X_t^T\| \underset{T\to\infty}{\to} 0 \quad \textit{in probability,}$$

where $\|\cdot\|$ *is the standard euclidean norm in* $\mathbb{R}^d$.

PROOF. We have

$$X_t^T - \Lambda_t^T = \frac{1-a_T}{T^\alpha\mu}M_{tT}^T\,.$$

Applying Doob's L^p-inequality, we get for each component

$$\mathbb{E}\Big[\sup_{t\in[0,1]}|\Lambda_{t,i}^T - X_{t,i}^T|^2\Big] \le cT^{-4\alpha}\mathbb{E}\big[(M_{T,i}^T)^2\big]\,.$$

Since $[M^T, M^T] = \text{diag}(N^T)$, recalling from the proof of the preceding Proposition that $\mathbb{E}\big[N^T_{T,i}\big] \le cT^{2\alpha}$, we deduce

$$\mathbb{E}\Big[\sup_{t\in[0,1]} |\Lambda^T_{t,i} - X^T_{t,i}|^2\Big] \le cT^{-4\alpha}\mathbb{E}\big[(M^T_{T,i})^2\big] = cT^{-4\alpha}\mathbb{E}\big[N^T_{T,i}\big] \le c'T^{-2\alpha}\,.$$

Hence, applying Markov inequality, we get the uniform convergence in probability:

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |\Lambda^T_{t,i} - X^T_{t,i}|^2 > \varepsilon\Big) \le \frac{1}{\varepsilon}\mathbb{E}\Big[\sup_{t\in[0,1]} |\Lambda^T_{t,i} - X^T_{t,i}|^2\Big] \le c'T^{-2\alpha} \underset{T\to\infty}{\to} 0\,,$$

for any $\varepsilon > 0$. □

3.2.2. *Convergence of the process $v_i^* \cdot X^T$ for $i \ge 2$.* Also in the heavy tails context we observe a vanishing behaviour in the direction of the eigenvectors v_i for $i \ge 2$.

PROPOSITION 4.10. *Under Assumption 4.3, if X is a possible limit point of X^T, then for $i \ge 2$ we have $v_i^* \cdot X = 0$.*

PROOF. With the same method used in Chapter 3 to get equation (3.4), we can rewrite the cumulated intensity as

$$\int_0^t \lambda^T_s\, ds = t\mu_T\mathbf{1} + \mu_T\int_0^t s\mathbf{\Psi}^T(t-s)\, ds\cdot\mathbf{1} + \int_0^t \mathbf{\Psi}^T(t-s)\cdot M^T_s\, ds\,.$$

Therefore, for $t \in [0,1]$, we have the following decomposition:

$$\Lambda^T_t = A_1(t) + A_2(t) + A_3(t)\,, \tag{4.7}$$

with

$$A_1(t) = (1-a_T)tu_T\mathbf{1}\,,$$
$$A_2(t) = T(1-a_T)u_T\int_0^t s\mathbf{\Psi}^T\big(T(t-s)\big)\, ds\cdot\mathbf{1}\,,$$
$$A_3(t) = T^{1-\alpha/2}\sqrt{\frac{1-a_T}{\mu}}\int_0^t \mathbf{\Psi}^T\big(T(t-s)\big)\cdot Z^T_s\, ds\,,$$

where we set $u_T = \mu_T/(\mu T^{\alpha-1})$. Note that u_T tends to one as T goes to infinity.

Now recall that for $1 \le i \le d$,

$$\psi^T_i = \sum_{k\ge1} a^k_T \lambda^{*k}_i$$

and let $\varrho^T_i : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\varrho^T_i = T(1-a_T)\psi^T_i(T\cdot)\,.$$

Moreover define

$$F^T_i(t) = \int_0^t \varrho^T_i(s)\, ds\,.$$

We want to bound $|F_i^T(t)|$. Note that it is easy to prove by induction that $\|\lambda_i^{*k}\|_1 \le \|\lambda_i\|_1^k$. Hence we can write, for $i \ge 2$

$$\begin{aligned}|F_i^T(t)| &\le \int_0^t T(1-a_T)|\psi_i^T(Ts)|\,ds \le (1-a_T)\int_0^\infty |\psi_i^T(s)|,ds\\ &\le (1-a_T)\sum_{k\ge1} a_T^k\|\lambda_i^{*k}\|_1 \le (1-a_T)\sum_{k\ge1}\|\lambda_i\|_1^k = (1-a_T)\frac{a_T\|\lambda_i\|_1}{1-a_T\|\lambda_i\|_1}\\ &\le (1-a_T)\frac{\|\lambda_i\|_1}{1-\|\lambda_i\|_1}\,.\end{aligned}$$

This gives us the uniform convergence to zero of F_i^T. Thanks to this we deduce the convergence to zero of $v_i^* \cdot A_2$, since

$$\begin{aligned}v_i^*\cdot A_2(t) &= T(1-a_T)u_T\int_0^t s v_i^*\cdot \mathbf{\Psi}^T\big(T(t-s)\big)\,ds\cdot \mathbf{1}\\ &= T(1-a_T)u_T\int_0^t s\psi_i^T\big(T(t-s)\big)\,ds(v_i^*\cdot\mathbf{1})\\ &= u_T(v_i^*\cdot\mathbf{1})\int_0^t s\varrho_i^T(t-s)\,ds\\ &= u_T(v_i^*\cdot\mathbf{1})\Big[sF_i^T(t-s)\Big]_{s=t}^{s=0} + u_T(v_i^*\cdot\mathbf{1})\int_0^t F_i^T(t-s)\,ds\\ &= u_T(v_i^*\cdot\mathbf{1})\int_0^t F_i^T(s)\,ds\,.\end{aligned}$$

For $v_i^* \cdot A_3$ we apply the same integration bt parts, this time for Lebesgue-Stieltjes integrals:

$$\begin{aligned}v_i^*\cdot A_3(t) &= T^{1-\alpha/2}\sqrt{\frac{1-a_T}{\mu}}\int_0^t v_i^*\cdot\mathbf{\Psi}^T\big(T(t-s)\big)\cdot Z_s^T\,ds\\ &= T^{1-\alpha/2}\sqrt{\frac{1-a_T}{\mu}}\int_0^t \psi_i^T\big(T(t-s)\big)(v_i^*\cdot Z_s^T)\,ds\\ &= \frac{1}{\sqrt{\mu(1-a_T)T^\alpha}}\int_0^t \varrho_i^T(t-s)(v_i^*\cdot Z_s^T)\,ds\\ &= \frac{1}{\sqrt{\mu(1-a_T)T^\alpha}}\bigg(\Big[F_i^T(t-s)(v_i^*\cdot Z_s^T)\Big]_{s=t}^{s=0} + \int_0^t F_i^T(t-s)(v_i^*\cdot dZ_s^T)\bigg)\\ &= \frac{1}{\sqrt{\mu(1-a_T)T^\alpha}}\int_0^t F_i^T(t-s)(v_i^*\cdot dZ_s^T)\,.\end{aligned}$$

Since the quadratic variation of Z^T is Λ^T, whose expectation is uniformly bounded, we have

$$\mathbb{E}\big[(v_i^*\cdot A_3(t))^2\big] \le \frac{c}{\mu(1-a_T)T^\alpha}\int_0^t \big(F_i^T(s)\big)^2\,ds\,.$$

Then the uniform convergence to zero of F_i^T gives the convergence to zero of $v_i^* \cdot A_3$. Finally, Lemma 4.9 tells us that if X is a limit point of X^T, then X is also a limit point of Λ^T. Therefore, we obtain $v_i^* \cdot X = 0$. □

3.2.3. *Convergence of the process $v_1^* \cdot X^T$.* We use the decomposition (4.7) and the same ideas used in Chapter 3. We start with a preliminary lemma about the convergence of function ϱ_1^T.

LEMMA 4.11. *Consider the function ϱ_1^T, defined by*

$$\varrho_1^T(x) = T(1-a_T)\psi_1^T(Tx)\,.$$

Then, under Assumption 4.3, the sequence of measures with density ϱ_1^T converges weakly towards the measure with density $f^{\alpha,\lambda} = \lambda x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^\alpha)$. In particular, for $t \in [0,1]$, the function

$$F_1^T(t) = \int_0^t \varrho_1^T(x)\,dx$$

converges uniformly towards

$$F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(x)\,dx\,.$$

PROOF. Following the approach of Chapter 3, we show that the Laplace transform of ϱ_1^T converges pointwise to the Laplace transform of $f^{\alpha,\lambda}$.

$$\hat{\varrho}_1^T(z) = \int_0^\infty \varrho_1^T(x)e^{-zx}\,dx = (1-a_T)\hat{\psi}_1^T(z/T) = (1-a_T)\frac{a_T\hat{\lambda}_1(z/T)}{1-a_T\hat{\lambda}_1(z/T)}\,.$$

Last equality is due to the fact that $\widehat{\lambda_1^{*k}} = (\hat{\lambda}_1)^k$. Now integrating by parts and using that $\|\lambda_1\|_1 = 1$ we get

$$\begin{aligned}\hat{\lambda}_1(z) &= \int_0^\infty \lambda_1(x)e^{-zx}\,dx = 1 - z\int_0^\infty \Big(\int_x^\infty \lambda_1(u)\,du\Big)e^{-zx}\,dx\,.\\ &= 1 - z^\alpha\int_0^\infty \Big(\frac{x}{z}\Big)^\alpha\Big(\int_{x/z}^\infty \lambda_1(u)\,du\Big)x^{-\alpha}e^{-x}\,dx\,.\end{aligned}$$

Using Assumption 4.3, together with dominated convergence theorem, we get

$$\hat{\lambda}_1(z) = 1 - \frac{C}{\alpha}\Gamma(1-\alpha)z^\alpha + \underset{z\to 0}{o}(z)$$

From this, we deduce that for $z > 0$,

$$\hat{\varrho}_1^T(z) \underset{T\to\infty}{\to} \frac{\lambda}{\lambda + z^\alpha}\,,$$

which is the Laplace transform of the function $f^{\alpha,\lambda}$, as it is shown in **[12]**. □

We now have all the ingredients to imitate the computations performed in Section 3.1.1 of Chapter 3. We then obtain for the terms in decomposition (4.7)

$$v_1^* \cdot A_2(t) \underset{T\to\infty}{\to} (v_1^*\cdot\mathbf{1})\int_0^t s f^{\alpha,\lambda}(t-s)\,ds$$

and

$$v_1^* \cdot A_3(t) \underset{T\to\infty}{\to} \frac{1}{\sqrt{\lambda\mu}}\int_0^t f^{\alpha,\lambda}(t-s)(v_1^+\cdot Z_s)\,ds\,.$$

Taking the limit for $T\to\infty$ in decomposition (4.7) we get the following proposition.

PROPOSITION 4.12. *Under Assumption 4.3, if (X,Z) is a possible limit point of (X^T, Z^T), then the process $v_1^*\cdot X$ satisfies the following equation:*

$$v_1^*\cdot X_t = (v_1^*\cdot\mathbf{1})\int_0^t s f^{\alpha,\lambda}(t-s)\,ds + \frac{1}{\sqrt{\lambda\mu}}\int_0^t f^{\alpha,\lambda}(t-s)(v_1^*\cdot Z_s)\,ds\,.$$

3.2.4. *End of the proof.* Let (X, Z) be a possible limit point of (X^T, Z^T). We can apply the same method used in Theorem 3.10 to show that

$$v_1^* \cdot X_t = \int_0^t Y_s \, ds$$

where Y satisfies

$$Y_t = (v_1^* \cdot \mathbf{1}) F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\lambda\mu}} \int_0^t f^{\alpha,\lambda}(t-s)(v_1^* \cdot dZ_s)\,.$$

We decompone X_t in the orthonormal basis $(e_i)_i$:

$$X_t = \sum_{i=2}^d (e_i^* \cdot X_t) e_i + \big((v')^* \cdot X_t\big) e_1 + \frac{1}{e_1^* \cdot v_1} (v_1^* \cdot X_t) e_1$$

and using Proposition 4.10 we get

$$X_t = \frac{1}{e_1^* \cdot v_1} (v_1^* \cdot X_t) e_1 = \frac{1}{e_1^* \cdot v_1} \bigg(\int_0^t Y_s \, ds \bigg) e_1\,.$$

From Proposition 4.8 we know that

$$[Z, Z] = \operatorname{diag}(X) = \frac{1}{e_1^* \cdot v_1} \bigg(\int_0^t Y_s \, ds \bigg) \operatorname{diag}(e_1)\,.$$

Hence we can use Theorems A.7 and A.8 to get the existence of a d-dimensional Brownian motion $(B^1, \ldots, B^d)$ such that, for $i \in \{1, \ldots, d\}$,

$$Z_t^i = \frac{1}{\sqrt{e_1^* \cdot v_1}} \sqrt{e_{1,i}} \int_0^t \sqrt{Y_s} \, dB_s^i\,.$$

Finally, in the same way as in the proof of Theorem 3.10, we get that Y satisfies the stochastic differential equation

$$Y_t = (v_1^* \cdot \mathbf{1}) F^{\alpha,\lambda}(t) + \sqrt{\frac{e_1^* \cdot v_1^2}{\lambda\mu(e_1^* \cdot v_1)}} \int_0^t f^{\alpha,\lambda}(t-s) \sqrt{Y_s} \, dB_s\,, \tag{4.8}$$

where B is a Brownian motion defined by

$$B = \frac{1}{\sqrt{e_1^* \cdot v_1^2}} \sum_{i=1}^d \sqrt{e_{1,i}} v_{1,i} B^i$$

and that Y has Hölder regularity $\alpha - 1/2 - \varepsilon$, for any $\varepsilon > 0$.

Now, equation (4.8) can be transformed into the rough stochastic differential equation written in Theorem 4.7 thanks to Proposition 3.12. This Proposition also gives the strong uniqueness of the solution.

CHAPTER 5

A microscopic model for single asset price

In this chapter we discuss a financial model for single asset price based on the preceding results about nearly unstable Hawkes processes. This kind of model was introduced in [**4**] as a partecipants based model for high frequency trading markets. We will see that this model, even if it is not built *ad hoc*, produces in the long run a leverage effect for the asset price dynamics. Moreover, under certain assumptions, we will get a rough dynamics for the volatility of the asset price and this fact agrees with the recent empirical results about high frequency trading markets (see [**11**]).

1. The basic model

We consider a tick-by-tick price model based on a bidimensional Hawkes process $N_t = (N_t^+, N_t^-)$, with intensity $\lambda_t = (\lambda_t^+, \lambda_t^-)$ defined by

$$\begin{pmatrix} \lambda_t^+ \\ \lambda_t^- \end{pmatrix} = \begin{pmatrix} \mu^+ \\ \mu^- \end{pmatrix} + \int_0^t \begin{pmatrix} \varphi_1(t-s) & \varphi_3(t-s) \\ \varphi_2(t-s) & \varphi_4(t-s) \end{pmatrix} \cdot \begin{pmatrix} dN_s^+ \\ dN_s^- \end{pmatrix} ,$$

where μ^+ and μ^- are positive constant and

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_1 & \varphi_3 \\ \varphi_2 & \varphi_4 \end{pmatrix} : \mathbb{R}_+ \to \mathcal{M}^2(\mathbb{R}_+)$$

is a kernel matrix whose components φ_i are positive and locally integrable.

We then consider the price process P_t that is simply given by

$$P_t = N_t^+ - N_t^- .$$

We then interpret N_t^+ as the number of upward jumps of one tick of the asset price in the time interval $[0, t]$ and, at the same way, N_t^- as the number of downward jumps.

The use of a self exciting point process helps modelling the fact that in modern high frequency trading markets the number of endogenous orders is much larger than the number of exogenous orders. We can see this interpreting the intensity process λ_t^+ (analogous considerations hold for λ_t^-). We can look at $\lambda_t^+ \, dt$ as the probability at time t to get a new one-tick upward jump between times t and $t + dt$. This quantity can be split into three terms:

- $\mu^+ \, dt$, which corresponds to the probability that the price goes up because of some exogenous reason;
- $\big(\int_0^t \varphi_1(t-s) \, dN_s^+ \big) \, dt$, which is the probability of upward jump induced by past upward jumps;
- $\big(\int_0^t \varphi_3(t-s) \, dN_s^- \big) \, dt$, which is the probability of upward jump induced by past downward jumps.

This shows that, working on the shape of the functions φ_i, we can reproduce many effects. For example the bid-ask bounce effect is given imposing a high probability of upward (resp. downward) jump right after a downward (resp. upward) jump.

We are actually interested in encoding in the model three features, we will then add a fourth feature in Section 3. Here are the three features we focus on at this stage:

(i) Markets are highly endogenous, meaning that most of the orders are sent in reaction to other orders, rather than being motivated by economical reason;
(ii) There's a kind of absence of arbitrage opportunity at high frequency scale, that is building strategies which are on average profitable is almost impossible.
(iii) Buying and selling are not symmetric actions: the ask side is more liquid than the bid side.

We start providing a specific structure on the intensity process so that properties (ii) and (iii) are satisfied. In a high frequency setting, the no arbitrage condition amounts to say that on average there should be as many upward as downward jumps on any given time-period. Noting that

$$\mathbb{E}[N_t^+] = \int_0^t \mathbb{E}[\lambda_s^+]\, ds\,, \quad \mathbb{E}[N_t^-] = \int_0^t \mathbb{E}[\lambda_s^-]\, ds$$

and

$$\mathbb{E}[\lambda_t^+] = \mu^+ + \int_0^t \varphi_1(t-s)\mathbb{E}[\lambda_s^+]\, ds + \int_0^t \varphi_3(t-s)\mathbb{E}[\lambda_s^-]\, ds$$
$$\mathbb{E}[\lambda_t^-] = \mu^- + \int_0^t \varphi_2(t-s)\mathbb{E}[\lambda_s^+]\, ds + \int_0^t \varphi_4(t-s)\mathbb{E}[\lambda_s^-]\, ds\,,$$

we choose to impose $\mathbb{E}[\lambda_t^+] = \mathbb{E}[\lambda_t^-]$ by setting

$$\mu^+ = \mu^- \quad \text{and} \quad \varphi_1 + \varphi_3 = \varphi_2 + \varphi_4\,.$$

Property (iii) can be restated as follows: the conditional probability to observe an upward jump right after an upward jump is smaller than the conditional probability to observe a downward jump right after a downward jump. In our framework this corresponds to have $\varphi_1(x) < \varphi_4(x)$, that is the same of $\varphi_3(x) > \varphi_2(x)$, when x is close to zero. We actually make a more restrictive assumption, setting, for some $\beta > 1$,

$$\varphi_3 = \beta\varphi_2\,.$$

Therefore we assume the following structure for the intensity process:

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \mu\cdot\begin{pmatrix}1\\1\end{pmatrix} + \int_0^t \mathbf{\Phi}(t-s)\cdot\begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix}\,, \tag{5.1}$$

where

$$\mathbf{\Phi} = \begin{pmatrix}\varphi_1 & \beta\varphi_2\\ \varphi_2 & \varphi_1 + (\beta-1)\varphi_2\end{pmatrix}\,,$$

and $\mu > 0$, $\beta \geq 1$.

We now come to property (i). In **[15]** a one dimensional Hawkes process $\tilde{N}_t$, with intensity $\tilde{\lambda}_t = \tilde{\mu} + \int_0^t \tilde{\varphi}(t-s)\, d\tilde{N}_s$, is seen as a population process where we have a number of migrants and each migrant gives birth to descendants according to an inhomogeneous Poisson process with intensity the kernel function $\tilde{\varphi}$. It is shown there that the L^1 norm $\|\tilde{\varphi}\|_1$ represents the average number of children of each migrant. Hence a migrant has on average

$$\sum_{k\geq 1} \|\tilde{\varphi}\|_1^k = \frac{\|\tilde{\varphi}\|_1}{1-\|\tilde{\varphi}\|_1}$$

descendants and the proportion of descendant in the whole population is simply given by $\|\tilde{\varphi}\|_1$. Coming back to our financial model, migrants corresponds to exogenous orders, given by real economic reasons, while descendants can be seen as endogenous orders, given by algorithmical reactions. Thus, in order to represent the fact that modern markets are strongly endogenous, we want $\|\tilde{\varphi}\|_1$ to be strictly smaller than but close to unity.

In our two dimensional process, with intensity given by (5.2), at the same way as in the one dimensional case, we can define the degree of endogeneity as the spectral radius of the kernel matrix integral:

$$\mathcal{S}\Big(\int_0^\infty \mathbf{\Phi}(s)\,ds\Big) = \|\varphi_1\|_1 + \beta\|\varphi_2\|_1\,.$$

In order to assume that this spectral radius is smaller but close to unity we introduce an asymptotic framework and, in the spirit of the preceding chapters, we work with a sequence of nearly unstable two dimensional Hawkes processes. Therefore we work on a sequence of probability spaces $(\Omega^T, \mathcal{F}^T, \mathbb{P}^T)$, indexed by $T > 0$, where we have a Hawkes process $N^T = (N^{T,+}, N^{T,-})$ defined on $[0, T]$, with intensity

$$\lambda_t^T = \begin{pmatrix}\lambda_t^{T,+}\\ \lambda_t^{T,-}\end{pmatrix} = \mu_T \cdot \begin{pmatrix}1\\1\end{pmatrix} + \int_0^t \mathbf{\Phi}^T(t-s)\cdot dN_s^T\,. \tag{5.2}$$

We then study this sequence of processes, properly rescaled, for T that goes to infinity.

We state an assumption that translate the discussion about properies (ii) and (iii) in the nearly unstable framework:

Assumption 5.1. We have $\mu_T > 0$ and $\mathbf{\Phi}^T = a_T\mathbf{\Phi}$, with

$$\mathbf{\Phi} = \begin{pmatrix}\varphi_1 & \beta\varphi_2\\ \varphi_2 & \varphi_1 + (\beta-1)\varphi_2\end{pmatrix}\,,$$

where $\beta \geq 1$, φ_1 and φ_2 are two positive measurable functions such that

$$\mathcal{S}\Big(\int_0^\infty \mathbf{\Phi}(s)\,ds\Big) = \|\varphi_1\|_1 + \beta\|\varphi_2\|_1 = 1$$

and a_T is an increasing sequence of positive numbers converging to one.

From now on, we consider the microscopic price

$$P_t^T = N_t^{T,+} - N_t^{T,-}\,.$$

Remark that, under Assumption 5.1, we are working in the nearly unstable case since

$$\mathcal{S}\Big(\int_0^\infty \mathbf{\Phi}^T(s)\,ds\Big) = a_T\,.$$

We now focus on the asymptotic behaviour of P^T.

2. Framework leading to leverage effect

In this section we will see the convergence of the rescaled microscopic price towards a Heston model with leverage effect. We need to recall Assumptions 4.1 and 4.2, that will allow us to use the convergence results we got for light tailed nearly unstable Hawkes processes.

Assumption 5.2. There exist positive constants λ, μ and m such that

$$T(1-a_T) \underset{T\to\infty}{\to} \lambda\,, \quad \mu_T = \mu$$

and

$$\mathcal{S}\Big(\int_0^\infty x\mathbf{\Phi}(x)\,dx\Big) = m < \infty\,.$$

Now, defining as usual

$$\mathbf{\Psi}^T = \sum_{k\geq 1} (\mathbf{\Phi}^T)^{*k},$$

where $(\mathbf{\Phi}^T)^{*k} = \mathbf{\Phi}^T$ and, for $k \geq 1$, $(\mathbf{\Phi}^T)^{*k}(t) = \int_0^t \mathbf{\Phi}^T(s)(\mathbf{\Phi}^T)^{*(k-1)}(t-s)\,ds$, we have

Assumption 5.3. The function $\mathbf{\Psi}^T$ is uniformly bounded and $\mathbf{\Phi}$ is differentiable such that each component φ_{ij} satisfies $\|\varphi'_{ij}\|_\infty < \infty$ and $\|\varphi'_{ij}\|_1 < \infty$.

In agreement with the renormalization of Chapter 4, we will study the following rescaling:

$$\frac{1}{T} P^T_{tT} = \frac{N^{T,+}_{tT} - N^{T,-}_{tT}}{T}.$$

Moreover we keep the same notations of Chapter 4 and we have

$$C^T_t = \frac{1}{T}\lambda^T_t$$

and

$$B^T_t = \int_0^{tT} \frac{v_1^* \cdot dM^T_s}{\sqrt{T(v_1^2)^* \cdot \lambda^T_s}}.$$

Finally, we state an immediate corollary of Theorem 4.2 that will lead us to the long term limit of our microscopic price model.

Corollary 5.1. *Under Assumption 5.1, 5.2 and 5.3, the process* $(C^{T,+}_t, C^{T,-}_t, B^T_t)_{t\in[0,1]}$ *converges in law for the Skorohod topology to* $(\frac{1}{\beta+1}X, \frac{1}{\beta+1}X, B)$*, where* B *is a Brownian motion and* X *satisfies the following Cox-Ingersoll-Ross dynamics:*

$$dX_t = \frac{\lambda}{m}\Big(\frac{\mu}{\lambda}(\beta+1) - X_t\Big)\,dt + \frac{1}{m}\sqrt{\frac{1+\beta^2}{1+\beta}}\sqrt{X_t}\,dB_t \quad X_0 = 0. \tag{5.3}$$

Proof. Keeping the notations from last chapter, we have that the two eigenvalues of $\mathbf{\Phi}^*$ are

$$\lambda_1 = \varphi_1 + \beta\varphi_2, \quad \lambda_2 = \varphi_1 - \varphi_2$$

and the associated eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ \beta \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We then apply these data to Theorem 4.2. □

We now state the main theorem of this section, concerning the limiting law of the rescaled microscopic price.

Theorem 5.2. *Under Assumptions 5.1, 5.2 and 5.3, as* T *tends to infinity, the rescaled microscopic price*

$$\frac{1}{T} P^T_{tT} = \frac{N^{T,+}_{tT} - N^{T,-}_{tT}}{T}, \quad t \in [0,1],$$

converges in law for the Skorohod topology to the following Heston model:

$$P_t = \frac{1}{1 - (\|\varphi_1\|_1 - \|\varphi_2\|_1)}\sqrt{\frac{2}{1+\beta}}\int_0^t \sqrt{X_s}\,dW_s$$

where

$$dX_t = \frac{\lambda}{m}\Big(\frac{\mu}{\lambda}(\beta+1) - X_t\Big)dt + \frac{1}{m}\sqrt{\frac{1+\beta^2}{1+\beta}}\sqrt{X_t}\,dB_t \quad X_0 = 0.$$

and (W, B) *is a correlated bidimensional Brownian motion with*

$$d\langle W, B\rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}}\, dt\,.$$

REMARK 5.3. Notice that, putting together properties (i),(ii) and (iii) we obtain in a natural way a stochastic volatility with leverage effect. Indeed the assumption $\beta > 1$, coming from the asymmetry in the liquidity of the bid and ask side, generates a negative correlation between the Brownian motion driving the asset price and the one driving the volatility. Property (i), that in our model is given by the nearly unstable framework, is essential to get a stochastic volatility and the failure of property (ii) would intuitively lead to a drift process.

PROOF. We now state the proof of the theorem, splitting it into several step.

Convenient rewriting of the process P^T. We want write in a more suitable way the rescaled price P^T_{tT}/T.

$$\frac{1}{T}P^T_{tT} = \frac{N^{T,+}_{tT} - N^{T,-}_{tT}}{T} = \int_0^{tT} \frac{dM^{T,+}_s - dM^{T,-}_s}{\sqrt{T(\lambda^{T,+}_s + \lambda^{T,-}_s)}}\sqrt{\frac{\lambda^{T,+}_s + \lambda^{T,-}_s}{T}} + \int_0^{tT} \frac{\lambda^{T,+}_s - \lambda^{T,-}_s}{T}\, ds\,.$$

Furthermore

$$\begin{aligned}\lambda^{T,+}_t - \lambda^{T,-}_t &= \int_0^t a_T\big(\varphi_1(t-s) - \varphi_2(t-s)\big)\big(dN^{T,+}_s - dN^{T,-}_s\big)\\ &= \int_0^t a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big) + \int_0^t \lambda_2(t-s)\big(\lambda^{T,+}_s - \lambda^{T,-}_s\big)\, ds\,.\end{aligned}$$

Thus, applying Lemma 2.11 and repeating the computations we are now used to, we get

$$\begin{aligned}\lambda^{T,+}_t - \lambda^{T,-}_t &= \int_0^t a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big)\\ &\quad + \int_0^t \psi^T_2(t-s)\Big\{\int_0^s a_T\lambda_2(s-u)\big(dM^{T,+}_u - dM^{T,-}_u\big)\Big\}\, ds\\ &= \int_0^t a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big)\\ &\quad + \int_0^t \Big(\int_u^t \psi^T_2(t-s)a_T\lambda_2(s-u)\, ds\Big)\big(dM^{T,+}_u - dM^{T,-}_u\big)\\ &= \int_0^t a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big) + \int_0^t \psi^T_2 * a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big)\\ &= \int_0^t a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big) + \int_0^t \psi^T_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big)\\ &\quad - \int_0^t a_T\lambda_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big)\\ &= \int_0^t \psi^T_2(t-s)\big(dM^{T,+}_s - dM^{T,-}_s\big)\,.\end{aligned}$$

Then, using Fubini theorem, we get

$$\int_0^t \big(\lambda^{T,+}_s - \lambda^{T,-}_s\big)\, ds = \int_0^t \Big(\int_0^{t-s} \psi^T_2(u)\, du\Big)\big(dM^{T,+}_s - dM^{T,-}_s\big)\,.$$

Hence, the rescaled price process P^T_{tT}/T can be finally written as

$$\int_0^t \sqrt{C_s^{T,+}+C_s^{T,-}}\,dW_s^T - \int_0^t \int_{T(t-s)}^\infty \psi_2^T(u)\,du\big(d\overline{M}_s^{T,+}-d\overline{M}_s^{T,-}\big) + \int_0^\infty \psi_2^T(u)\,du\big(\overline{M}_t^{T,+}-\overline{M}_t^{T,-}\big)\,,$$

where

$$W_t^T = \int_0^{tT} \frac{dM_s^{T,+}-dM_s^{T,-}}{\sqrt{T(\lambda_s^{T,+}+\lambda_s^{T,-})}}$$

and recall that $\overline{M}_s^T = M_{sT}^T/T$.

Now, note that by induction it is easy to prove that

$$\int_0^\infty \psi_2^T(u)\,du = \frac{a_T(\|\varphi_1\|_1-\|\varphi_2\|_1)}{1-a_T(\|\varphi_1\|_1-\|\varphi_2\|_1)}\,.$$

Moreover

$$\begin{aligned}\overline{M}_t^{T,+}-\overline{M}_t^{T,-} &= \frac{M_{tT}^{T,+}-M_{tT}^{T,-}}{T} = \int_0^{tT} \frac{dM_s^{T,+}-dM_s^{T,-}}{T} \\ &= \int_0^{tT} \frac{dM_s^{T,+}-dM_s^{T,-}}{\sqrt{T(\lambda_s^{T,+}+\lambda_s^{T,-})}}\sqrt{\frac{\lambda_s^{T,+}+\lambda_s^{T,-}}{T}} = \int_0^t \sqrt{C_s^{T,+}+C_s^{T,-}}\,dW_s^T\,.\end{aligned}$$

Therefore

$$\frac{1}{T}P_{tT}^T = \frac{1}{1-a_T(\|\varphi_1\|_1-\|\varphi_2\|_1)}\int_0^t \sqrt{C_s^{T,+}+C_s^{T,-}}\,dW_s^T - R_t^T\,, \tag{5.4}$$

with

$$R_t^T = \int_0^t \int_{T(t-s)}^\infty \psi_2^T(u)\,du\big(d\overline{M}_s^{T,+}-d\overline{M}_s^{T,-}\big)\,.$$

Convergence of the process R^T. We show here that , under Assumptions 5.1, 5.2 and 5.3, the process R^T tends u.c.p. to zero.

It is enough to show that the sequence of functions g^T, defined by

$$g^T(x) = \int_{Tx}^\infty \psi_2^T(u)\,du\,,$$

satisfies the hypotesis of Lemma 4.3. Point (b) is easy to be proven, since, recalling that $\|\lambda_2^{*k}\|_1 \le \|\lambda_2\|_1^k$,

$$\begin{aligned}|g^T(z)| &\le \int_0^\infty |\psi_2^T(x)|\,dx = \int_0^\infty \big|\sum_{k\ge1} a_T^k \lambda_2^{*k}(x)\big|\,dx \\ &\le \sum_{k\ge1} a_T^k \|\lambda_2^{*k}\|_1 \le \frac{\|\lambda_2\|_1}{1-\|\lambda_2\|_1}\,.\end{aligned}$$

Then we compute the Fourier transform of g^T:

$$\begin{aligned}\hat{g}^T(z) &= \int_0^\infty \Big(\int_{Tw}^\infty \psi_2^T(u)\,du\Big)e^{izw}\,dw = \int_0^\infty \Big(\int_0^{u/T} \psi_2^T(u)e^{izw}\,dw\Big)\,du \\ &= \int_0^\infty \psi_2^T(u)\frac{e^{izu/T}-1}{iz}\,du\,.\end{aligned}$$

Hence point (c) is satisfied since

$$|\hat{g}^T(z)| \leq \int_0^\infty |\psi_2^T(u)| \frac{2}{|z|} \, du \leq \frac{c}{|z|} \, .$$

Property (d) comes from the differentiability of g and the fact that

$$|(g^T)'(x)| = T |\psi_2^T(Tx)| \leq cT \, .$$

Finally, in order to show that property (b) holds, we use Fubin theorem to write

$$\int_0^\infty |g^T(x)|^2 \, dx = \int_{x \geq 0; y,z \geq Tx} \psi_2^T(y) \psi_2^T(z) \, dy \, dz \, dx = \int_{y,z \geq 0} \int_{[0, \frac{y \wedge z}{T})} \psi_2^T(y) \psi_2^T(z) \, dx \, dy \, dz$$
$$= \frac{1}{T} \int_{y,z \geq 0} (y \wedge z) \psi_2^T(y) \psi_2^T(z) \, .$$

Hence

$$\int_{x \geq 0} |g^T(x)|^2 \, dx \leq \frac{1}{T} \int_{y \geq 0} y |\psi_2^T(y)| \, dy \int_{z \geq 0} |\psi_2^T(z)| \leq \frac{c}{T} \sum_{k \geq 1} \int_{y \geq 0} y |\lambda_2^{*k}(y)| \, dy \, .$$

We now use a recursion to compute, for $k \geq 1$,

$$\begin{aligned}
\int_0^\infty y |\lambda_2^{*k}(y)| \, dy &= \int_0^\infty y \Big| \int_0^y \lambda_2(s) \lambda_2^{*(k-1)}(y-s) \, ds \Big| \, dy \\
&\leq \int_0^\infty \int_0^y y |\lambda_2(s) \lambda_2^{*(k-1)}(y-s)| \, ds \, dy \\
&= \int_0^\infty |\lambda_2(s)| \int_s^\infty y |\lambda_2^{*(k-1)}(x-s)| \, dy \, ds \\
&= \int_0^\infty |\lambda_2(s)| \, ds \int_0^\infty t |\lambda_2^{*(k-1)}(t)| \, dt + \int_0^\infty s |\lambda_2(s)| \, ds \int_0^\infty |\lambda_2^{*(k-1)}(t)| \, dt \\
&\leq \|\lambda_2\|_1 \int_0^\infty t |\lambda_2^{*(k-1)}(t)| \, dt + \|\lambda_2\|^{k-1} \int_0^\infty s |\lambda_2(s)| \, ds \\
&\leq \|\lambda_2\|_1 \Big(\|\lambda_2\|_1 \int_0^\infty s |\lambda_2^{*(k-2)}(s)| \, ds + \|\lambda_2\|^{k-2} \int_0^\infty s |\lambda_2(s)| \, ds \Big) \\
&\quad + \|\lambda_2\|^{k-1} \int_0^\infty s |\lambda_2(s)| \, ds \\
&= \|\lambda_2\|_1^2 \int_0^\infty s |\lambda_2^{*(k-2)}(s)| \, ds + 2 \|\lambda_2\|_1^{k-1} \int_0^\infty s |\lambda_2(s)| \, ds \\
&\leq \cdots \leq k \|\lambda_2\|_1^{k-1} \int_0^\infty s |\lambda_2(s)| \, ds \, .
\end{aligned}$$

Since, thanks to Assumption 5.2, the integral $\int_0^\infty s |\lambda_2(s)| \, ds$ is a finite quantity, we can bound the sum with

$$\sum_{k \geq 1} \int_{y \geq 0} y |\lambda_2^{*k}(y)| \, dy \leq \Big(\int_0^\infty s |\lambda_2(s)| \, ds \Big) \sum_{k \geq 1} k \|\lambda_2\|_1^{k-1}$$

and conclude that it is finite using the root test. Hence

$$\int_0^\infty |g^T(x)|^2 \, dx \leq \frac{c}{T}$$

and property (a) easily follows.

Then, we can apply Lemma 4.3 to state that the process R^T tends u.c.p. to zero.

Convergence of the process (W^T, B^T). As we did for the quadratic variation of the process B^T in the proof of Theorem 4.2, we get the following covergences in probability:

$$[W^T, W^T]_t \underset{T\to\infty}{\to} t\,, \qquad [B^T, B^T]_t \underset{T\to\infty}{\to} t\,.$$

We now show that, under Assumptions 5.1, 5.2 and 5.3,

$$[W^T, B^T]_t \underset{T\to\infty}{\to} \frac{1-\beta}{\sqrt{2(1+\beta^2)}}\, t \quad \text{in probability.}$$

Using the fact that $[M^T, M^T] = \mathrm{diag}(N^T)$, we can write

$$\begin{aligned}[W^T, B^T]_t &= \int_0^{tT} \frac{dN_s^{T,+} - \beta dN_s^{T,-}}{T\sqrt{\lambda_S^{t,+} + \lambda_s^{T,-}}\sqrt{\lambda_s^{T,+} + \beta^2\lambda_s^{T,-}}} \\ &= \int_0^t \frac{C_s^{T,+} - \beta C_s^{T,-}}{\sqrt{C_s^{T,+} + C_s^{T,-}}\sqrt{C_s^{T,+} + \beta^2 C_s^{T,-}}} + \varepsilon_t^T\,,\end{aligned}$$

where

$$\varepsilon_t^T = \int_0^{tT} \frac{dM_s^{T,+} - \beta dM_s^{T,-}}{T\sqrt{\lambda_s^{t,+} + \lambda_s^{T,-}}\sqrt{\lambda_s^{T,+} + \beta^2\lambda_s^{T,-}}}\,.$$

Now, using that $\langle M^T, M^T\rangle_t = \mathrm{diag}(\int_0^t \lambda^T)$ and that $\lambda^T \geq \mu\mathbf{1}$, we get

$$\mathbb{E}[(\varepsilon_t^T)^2] = \mathbb{E}\left[\int_0^{tT} \frac{1}{T^2(\lambda_s^{T,+} + \lambda_s^{T,-})}\right] \leq \frac{1}{2\mu T} \underset{T\to\infty}{\to} 0\,.$$

This implies convergence u.c.p. of ε^T to zero. Moreover, we know from Corollary 5.1 that $(C^{T,+}, C^{T,-})$ converges in law for the Skorohod topology to $(\frac{1}{1+\beta}X, \frac{1}{1+\beta}X)$, where X satisfies stochastic differential equation (5.3). Since the set of zeros of a Cox-Ingersoll-Ross process on a finite interval has Lebesgue measure zero, we deduce that

$$\frac{C_s^{T,+} - \beta C_s^{T,-}}{\sqrt{C_s^{T,+} + C_s^{T,-}}\sqrt{C_s^{T,+} + \beta^2 C_s^{T,-}}}$$

tends u.c.p. to

$$\frac{1-\beta}{\sqrt{2(1+\beta^2)}}\,.$$

Thus we got the following convergence in probability

$$[W^T, B^T]_t \underset{T\to\infty}{\to} \frac{1-\beta}{\sqrt{2(1+\beta^2)}}\, t\,.$$

Last step. We now consider equation (5.4). We already know that R^T tends u.c.p. to zero. Furthermore, applying Theorem VIII.3.11 in **[18]** together with the results of the preceding step, we get that the process (W^T, B^T) converges in law for the Skorohod topology to a correlated bi-dimensional Brownian motion (W, B) such that

$$\langle W, B\rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}}\, t\,.$$

From Corollary 5.1 we get that $(\sqrt{C^{T,+} + C^{T,-}}, B^T)$ converges in law for the Skorohod topology to $(\sqrt{\frac{2}{\beta+1}X}, B)$, where X satisfies equation (5.3). Then Theorem A.5 gives us that

$$\int_0^t \sqrt{C_s^{T,+} + C_s^{T,-}}\, dW_s^T$$

converges in law for the Skorohod topology to

$$\int_0^t \sqrt{\frac{2X_s}{1+\beta}}\, dW_s$$

and the proof is concluded. □

3. Framework leading to rough volatility

As we did in our study of nearly unstable Hawkes processes, we now want to drop the light tails assumption to tackle a more general situation. This actually encodes an economical aspect that we want to include in our model. This is the fourth property we spoke about in last section and can be described as follows:

(iv) Many transactions are due to large orders, called metaorders, which are not executed at once but split in time by trading algorithms.

This fact is well explained in [**1**] and [**22**]. We translate this property in our Hawkes framework by considering the model defined by Assumption 5.1, but under the condition that the kernel matrix exhibits a heavy tail. This is also observed in practise with empirical estimations, see for example [**5**].

Therefore, instead of Assumptions 5.2 and 5.3, we consider the next one:

Assumption 5.4. There exist $\alpha \in (1/2, 1)$ and $C > 0$ such that

$$\alpha x^\alpha \int_x^\infty \lambda_1(s)\, ds \underset{x\to\infty}{\to} C\,.$$

Moreover, for some $\lambda^* > 0$ and $\mu > 0$,

$$T^\alpha(1-a_T) \underset{T\to\infty}{\to} \lambda^*\,, \quad T^{1-\alpha}\mu_T \underset{T\to\infty}{\to} \mu\,.$$

As it happened in chapter 4, we need to change the scale in the asymptotic setting in order to get a nondegenerate scaling limit. We keep the notiations from last chapter and recall that we studied the following processes:

$$X_t^T = \frac{1-a_T}{T^\alpha \mu} N_{tT}^T\,, \quad \Lambda_t^T = \frac{1-a_t}{T^\alpha \mu}\int_0^{tT} \lambda_s^T\, ds\,, \quad Z_t^T = \sqrt{\frac{T^\alpha \mu}{1-a_T}}\big(X_t^T - \Lambda_t^T\big)\,.$$

From now on let $\lambda = \alpha\lambda^*/\big(C\Gamma(1-\alpha)\big)$. Reducing Theorem 4.7 to the two dimensional case and noticing that we have the eigenvalues and eigenvector of $\mathbf{\Phi}$ as in the proof of Corollary 5.1, we state the following result.

Corollary 5.4. *Under Assumptions 5.1 and 5.4, the process* $(\Lambda_t^T, X_t^T, Z_t^T)_{t\in[0,1]}$ *converges in law for the Skorohod topology to* (X, X, Z)*, with*

$$X_t = \frac{1}{\beta+1}\int_0^t Y_s\, ds\mathbf{1}\,, \quad Z_t = \int_0^t \sqrt{\frac{1}{\beta+1}Y_s}\begin{pmatrix} dB_s^1 \\ dB_s^2 \end{pmatrix}\,,$$

where (B^1, B^2) *is a bidimensional Brownian motion and* Y *is the unique solution of the following rough stochastic differential equation:*

$$Y_t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\big((1+\beta) - Y_s\big)\, ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\sqrt{\frac{1+\beta^2}{\lambda^*\mu(1+\beta)}}\sqrt{Y_s}\, dB_s\,,$$

with

$$B=\frac{B^1+\beta B^2}{\sqrt{1+\beta^2}}\,.$$

We can now state the main result about the convergence of the rescaled microscopic price in the heavy tails setting.

THEOREM 5.5. *Under Assumptions 5.1 and 5.4, as T tends to infinity, the rescaled microscopic price*

$$\sqrt{\frac{1-a_T}{\mu T^\alpha}}P^T_{tT},\quad t\in[0,1]$$

converges in the sense of finite dimensional laws to the following rough Heston model:

$$P_t=\frac{1}{1-(\|\varphi_1\|_1-\|\varphi_2\|_1)}\sqrt{\frac{2}{\beta+1}}\int_0^t\sqrt{Y_s}\,dW_s\,,$$

with Y the unique solution of

$$Y_t=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\lambda\big((1+\beta)-Y_s\big)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\lambda\sqrt{\frac{1+\beta^2}{\lambda^*\mu(1+\beta)}}\sqrt{Y_s}\,dB_s\,,$$

where (W,B) is a correlated bidimensional Brownian motion with

$$d\langle W,B\rangle_t=\frac{1-\beta}{\sqrt{2(1+\beta^2)}}\,dt\,.$$

Furthermore, the process Y_t has Hölder regularity $\alpha-1/2-\varepsilon$ for any $\varepsilon>0$.

PROOF. As we did in the proof of Theorem 5.2 to get equation (5.4) we can get a more suitabe expression for the rescaled price. As a matter of fact, recalling that

$$\int_0^t\big(\lambda^{T,+}_s-\lambda^{T,-}_s\big)\,ds=\int_0^t\Big(\int_0^{t-s}\psi_2^T(u)\,du\Big)\big(dM^{T,+}_s-dM^{T,-}_s\big)$$

and

$$\int_0^\infty\psi_2^T(u)\,du=\frac{a_T(\|\varphi_1\|_1-\|\varphi_2\|_1)}{1-a_T(\|\varphi_1\|_1-\|\varphi_2\|_1)}\,,$$

we can compute

$$\begin{aligned}\sqrt{\frac{1-a_T}{T^\alpha\mu}}P^T_{tT}&=\sqrt{\frac{1-a_T}{T^\alpha\mu}}\big(N^{T,+}_{tT}-N^{T,-}_{tT}\big)\\&=\sqrt{\frac{1-a_T}{T^\alpha\mu}}\Big(M^{T,+}_{tT}-M^{T,-}_{tT}+\int_0^{tT}\big(\lambda^{T,+}_s-\lambda^{T,-}_s\big)\,ds\Big)\\&=\sqrt{\frac{1-a_T}{T^\alpha\mu}}\Big(M^{T,+}_{tT}-M^{T,-}_{tT}+\int_0^{tT}\int_0^{Tt-s}\psi_2^T(u)\,du(dM^{T,+}_s-dM^{T,-}_s)\Big)\\&=Z^{T,+}_t-Z^{T,-}_t+\int_0^t\int_0^{T(t-s)}\psi_2^T(u)\,du(dZ^{T,+}_s-dZ^{T,-}_s)\\&=Z^{T,+}_t-Z^{T,-}_t+\int_0^t\Big(\int_0^\infty\psi_2^T(u)\,du\Big)(dZ^{T,+}_s-dZ^{T,-}_s)\\&\quad-\int_0^t\int_{T(t-s)}^\infty\psi_2^T(u)\,du(dZ^{T,+}_s-dZ^{T,-}_s)\\&=\frac{1}{1-a_T(\|\psi_1\|_1-\|\psi_2\|_1)}(Z^{T,+}_t-Z^{T,-}_t)-R^T_t\,,\end{aligned}$$

where

$$R_t^T = \int_0^t \int_{T(t-s)}^{\infty} \psi_2^T(u)\,du(dZ_s^{T,+} - dZ_s^{T,-})\,.$$

Using Corollary 5.4 we have that

$$\frac{1}{1 - a_T(\|\psi_1\|_1 - \|\psi_2\|_1)}(Z_t^{T,+} - Z_t^{T,-})$$

converges in law for the Skorohod topology to P, where P is defined by the rough Heston dynamics stated in the theorem.

We now prove the convergence of R^T to zero in the sense of finite dimensional laws. Recalling from the proof of Proposition 4.8 that $\langle Z^T, Z^T\rangle_t = \mathrm{diag}(\Lambda_t^T)$ and that there exists a $c > 0$ such that $\mathbb{E}[\Lambda_t^T] \le c$ for any $t \in [0,1]$, we have

$$\mathbb{E}[(R_t^T)^2] \le c \int_0^t \Big(\int_{Ts}^{\infty} \psi_2^T(u)\,du\Big)^2\,ds\,.$$

Let G be the function defined by

$$G(t) = \sum_{k\ge 1} |\varphi_1(t) - \varphi_2(t)|^{*k}\,.$$

Note that G is integrable since $\int_0^\infty |\varphi_1 - \varphi_2| < 1$. Hence

$$\begin{aligned}
\mathbb{E}[(R_t^T)^2] &\le c\int_0^t \Big(\int_{Ts}^{\infty} G(u)\,du\Big)^2\,ds \le \frac{c}{T}\int_0^T \Big(\int_s^\infty G(u)\,du\Big)^2\,ds\\
&= \frac{c}{T}\Big[T^{\frac12}\Big(\int_0^\infty G(u)\,du\Big)^2 + \big(T - T^{\frac12}\big)\Big(\int_{T^{\frac12}}^\infty G(u)\,du\Big)^2\Big]\\
&\le c\Big(T^{-\frac12}\Big(\int_0^\infty G\Big)^2 + \Big(\int_{T^{\frac12}}^\infty G\Big)^2\Big)\,.
\end{aligned}$$

which vanishes as T tends to infinity. This gives us the result. □

Remark 5.6. (a) Last theorem only proved the convergence of the microscopic price in the sense of finite dimensional laws. We notice that if we had $\varphi_1 = \varphi_2$, that is if we give up to property (iii) of our model, we would have $R_t^T \equiv 0$ and hence we would have the convergence in law for the Skorohod topology of the rescaled price to the process P.

(b) Note that, as we did in the end of the proof of Theorem 5.5, we get

$$\sup_{t\in[0,1]}\Big|\int_0^t\int_{Ts}^\infty \psi_2^T(u)\,du\,ds\Big| \le c\Big(T^{-\frac12}\int_0^\infty G(u)\,du + \int_{T^{\frac12}}^\infty G(u)\,du\Big)\,,$$

which vanishes as T goes to infinity. Then, using Fubini theorem, we get that

$$\begin{aligned}
\int_0^t R_s^T\,ds &= \int_0^t\int_0^s \Big(\int_{T(s-u)}^\infty \psi_2^T\Big)(dZ_u^{T,+} - dZ_u^{T,-})\,ds\\
&= \int_0^t\int_u^t \Big(\int_{T(s-u)}^\infty \psi_2^T\Big)\,ds(dZ_u^{T,+} - dZ_u^{T,-})\\
&= \int_0^t\int_0^{t-u} \Big(\int_{Ts}^\infty \psi_2^T\Big)\,ds(dZ_u^{T,+} - dZ_u^{T,-})
\end{aligned}$$

converges to zero u.c.p. Therefore the integrated rescaled price converges in law for the Skorohod topology to

$$\int_0^t P_s \, ds \, .$$

APPENDIX A

Technical results

1. Some useful limit theorems

In this appendix we report some limit theorems that are used in the proofs contained in Chapters 2 and 4.

1.1. Limit of a sequence of SDEs. We start with a result by Kurtz and Protter (see [**21**]) about the convergence of a sequence of SDEs. The meaning of this theorem is that if the functions and the processes defining the SDEs satisfy some convergence properties, then the laws of the solutions of the SDEs converge to the law of the solution of the limiting SDE. Let's state the setting for this theorem.

Let's $(X^n_t)_{t\geq 0}$ be adapted to the filtration $(\mathcal{F}^n_t)_{t\geq 0}$ and suppose it satisfies

$$X^n_t = U^n_t + \int_0^t F^n_{s-}(X^n)\, dY^n_s \tag{A.1}$$

where $F^n\colon D_{\mathbb{R}^k}[0,\infty) \to D_{M^{km}}[0,\infty)$ and

$$U^n \in D_{\mathbb{R}^k}[0,\infty)\,, \quad Y^n \in D_{\mathbb{R}^m}[0,\infty)\,,$$

both adapted to $(\mathcal{F}^n_t)$. Suppose Y^n is a semimartingale and F^n is non anticipating, that is $F^n_t(x) = F^n_t(x^t)$ for all $t \geq 0$ and $x \in D_{\mathbb{R}^k}[0,\infty)$, where we set $x^t(\cdot) := x(\cdot \wedge t)$.

We now give a condition on Y^n. Define, for $\delta > 0$, the function $h_\delta\colon [0,\infty) =\!\!\rightarrow [0,\infty)$, such that $h_\delta(r) = (1-\delta/r)^+$. Then we can define the functional J_δ in the following way:

$$J_\delta(x)(t) = \sum_{s\leq t} h_\delta(|x(s)-x(s-)|)(x(s)-x(s-))\,.$$

The functional J_δ measures the sum jumps of the function x up to time t giving a weight depending on the amplitude of each jump. Since Y^n is a semimartingale, we know that it can be decomposed as $Y^n = M^n + A^n$, where M^n is a local martingale and A^n is a process with bounded variation on bounded intervals. It can be proved that also $(Y^n)^\delta := Y^n - J_\delta(Y^n)$ is a semimartingale, with decomposition $(Y^n)^\delta = (M^n)^\delta + (A^n)^\delta$. The next condition on Y^n is expressed in term of the decomposition of $(Y^n)^\delta)$.

C1 For each $\alpha > 0$, there exist a sequence of stopping times (τ^α_n) such that $\mathbb{P}(\tau^\alpha_n \leq \alpha) \leq 1/\alpha$ and $\sup_n \mathbb{E}\left[(M^n)^\delta_{t\wedge\tau^\alpha_n} + V_{0,t\wedge\tau^\alpha_n}((A^n)^\delta)\right] < \infty$.

Recall that $V_{a,b}(f)$ is the total variation of the function f in the interval $[a,b]$, that is $V_{a,b} = \sup \sum |f(t_{i+1}) - f(t_i)|$, where the supremum is taken over all the subdivision $a = t_1, t_2, \ldots, t_k = b$ of the interval $[a,b]$.

Call $F\colon D_{\mathbb{R}_k}[0,\infty) \to D_{M^{km}}[0,\infty)$ the function that will play the role of F^n in the limiting SDE and that is non anticipating itself. We then need assumptions on the properties of the functions F^n and F under transformations of the time scale. Let $T_1[0,\infty)$ denote the collection of nondecreasing mappings $\lambda\colon [0,\infty) \to [0,\infty)$ (in particular, $\lambda(0) = 0$) such that $\lambda(t+h) - \lambda(h) \leq h$ for all $t, h \geq 0$. Let id denote the identity map $\mathrm{id}(s) = s$. We will assume that there exist mappings $G^n, G\colon D_{\mathbb{R}^k}[0,\infty) \times T_1[0,\infty) \to D_{\mathbb{M}^{km}}[0,\infty)$ such that

$F^n(x)\circ\lambda = G^n(x\circ\lambda,\lambda)$ and $F(x)\circ\lambda = G(x\circ\lambda,\lambda)$ for $(x,\lambda)\in D_{\mathbb{R}^k}[0,\infty)\times T_1[0,\infty)$. The second condition is then

- C2(i) For each compact subset $\mathcal{H}\subset D_{\mathbb{R}^k}[0,\infty)\times T_1[0,\infty)$ and $t>0$, $\sup_{(x,\lambda)\in\mathcal{H}}\sup_{s\le t}|G^n_s(x,\lambda)-G_s(x,\lambda)|\to 0$.
- C2(ii) For $\{(x_n,\lambda_n)\}\in D_{\mathbb{R}^k}[0,\infty)\times T_1[0,\infty)$, $\sup_{s\le t}|x_n(s)-x(s)|\to 0$ and $\sup_{s\le t}|\lambda_n(s)-\lambda(s)|\to 0$ for each $t>0$ implies $\sup_{s\le t}|G_s(x_n,\lambda_n)-G_s(x,\lambda)|\to 0$.

For our purpose we can simply note that any $F_t(x)=g(x(t),t)$ with $g:\mathbb{R}^k\times[0,\infty)\to\mathbb{M}^{km}$ continuous, has a representation in therm of a G satisfying C2(ii).

We now state a simplified version of Theorem 5.4 in [**21**], suitable for our purposes.

THEOREM A.1. *Suppose that* (U^n,X^n,Y^n) *satisfies* (A.1), $(U^n,Y^n)\Rightarrow(U,Y)$ *in the Skorohod topology and that* Y^n *satisfies* C1 *for some* $0<\delta\le\infty$. *Assume that* (F^n) *and* F *have representations in term of* (G^n) *and* G *satisfying* C2. *If there exists a global solution* X *of*

$$X_t = U_t + \int_0^t F_{s-}(X)\,dY_s \tag{A.2}$$

and weak local uniqueness holds, then $(U^n,X^n,Y^n)\Rightarrow(U,X,Y)$.

REMARK A.2. What does *weak local uniqueness* mean? We say that (X,τ) is a local solution of SDE (A.2) if there exists a filtration $(\mathcal{F}_t)_t$ to which X,U and Y are adapted, Y is an $(\mathcal{F}_t)$-semimartingale, τ is an $(\mathcal{F}_t)$-stopping time and

$$X_{t\wedge\tau} = U_{t\wedge\tau} + \int_0^{t\wedge\tau} F_{s-}(X)\,dY_s. \tag{A.3}$$

We say that *strong local uniqueness* holds for (A.2) if any two local solutions $(X^{(1)},\tau^{(1)})$, $(X^{(2)},\tau^{(2)})$ satisfy $X^{(1)}_t = X^{(2)}_t$, $t\le\tau^{(1)}\wedge\tau^{(2)}$, a.s. To define a notion of weak local uniqueness, that is uniqueness in distributions, we need to require the stopping time associated with the solution to be a meaurable function of the solution. We say that $(\hat U,\hat Y,\hat X,\hat\tau)$ is a weak local solution of (A.2) if $(\hat U,\hat Y)$ is a version of (U,Y) and (A.3) holds with (U,Y,X,τ) replaced by $(\hat U,\hat Y,\hat X,\hat\tau)$. We say that weak local uniqueness holds for (A.2) if for any two weak local solutions $(U^{(1)},Y^{(1)},X^{(1)},\tau^{(1)})$ and $(U^{(2)},Y^{(2)},X^{(2)},\tau^{(2)})$ with $\tau^{(1)}=h^{(1)}(X^{(1)})$ and $\tau^{(2)}=h^{(2)}(X^{(2)})$ for measurable functions $h^{(1)},h^{(2)}$ on $D_{\mathbb{R}^k}[0,\infty)$, $(X^{(1)},h^{(1)}\wedge h^{(2)}(X^{(1)}))$ and $(X^{(2)},h^{(1)}\wedge h^{(2)}(X^{(2)}))$ have the same distribution.

1.2. A condition for the convergence to a Brownian motion. We state a result that is an immediate corollary of Theorem VIII.3.11 in [**18**].

THEOREM A.3. *Assume* B^n *is a sequence of local martingales with* $|\Delta B^n|\le K$ *identically (i.e. with jumps uniformly bounded) and* B *is a one dimensional Brownian motion. If* D *is a dense subset of* $\mathbb{R}_+$ *the following are equivalent:*

(i) $B^n \xrightarrow{\mathcal{D}} B$;
(ii) *The quadratic variation of* B^n *at point* t *converges in probability to* t *for each* $t\in D$: $\langle B^n\rangle_t \xrightarrow{P} t$.

1.3. Convergence of a sequence of stochastic integrals. We state a result borrowed from [**16**] that gives sufficient condition for the convergence in law, for the Skorohod topology, of a sequence of stochastic integrals.

Recall that, given a filtered probabilty space $\mathcal{B}^n = (\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_t, \mathbb{P}^n)$, we denote by $\mathcal{H}^n$ the set of all predictable processes H^n on $\mathcal{B}^n$ having the form

$$H^n = Y^n_0 \mathbb{1}_0 + \sum_{i=1}^k Y^n_i \mathbb{1}_{(s_i, s_{i+1}]}(t)\,,$$

where $k \in \mathbb{N}$, $0 = s_0 < \ldots, < s_{k+1}$, and Y^n_i is $\mathcal{F}^n_{s_i}$-measurable with $|Y^n_i| \leq 1$.

If X^n is any 1-dimensional process on $\mathcal{B}^n$ and if $H^n \in \mathcal{H}^n$ is as above, we define the elementary stochastic integral process $H^n \bullet X^n$ by

$$H^n \bullet X^n_t = \sum_{i=1}^k Y^n_i (X^n_{t\wedge s_{i+1}} - X^n_{t \wedge s_i})\,.$$

We need the following definition.

Definition A.4 (UT processes). A sequence (X^n) of adapted cádlág processes, each X^n on $\mathcal{B}^n$, is said *UT* (for Uniformly Tight) if for every $t > 0$ the family of random variables $\{H^n \bullet X^n_t \colon n \in \mathbb{N}, H^n \in \mathcal{H}^n\}$ is tight in $\mathbb{R}$, that is

$$\lim_{a\to\infty} \sup_{H^n \in \mathcal{H}^n, n \in \mathbb{N}} \mathbb{P}^n(|H^n \bullet X^n_t| > a) = 0\,.$$

Finally we have the theorem by Jakubowski, Mémin, Pages:

Theorem A.5. *For any n, let X^n be a semimartingale on the space $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_t, \mathbb{P}^n)$ and K^n be a cadlag process on the same filtered space, both real valued, with (X^n) satisfying the UT condition.*

Then the convergence $(K^n, X^n) \xrightarrow{\mathcal{D}} (K, X)$ implies the following convergences in law, for the Skorohod topology:

$$K^n \bullet X^n \to K \bullet X^n \to K \bullet X \ \text{and}$$
$$(K^n \bullet X^n, K^n, X^n) \to (K \bullet X, K, X)$$

2. Brownian motion and continuous martingales

In this section we gather some results about Brownian motion and continuous martingales. They represent a very important amount of background material that is hidden in the proofs of chapter 3, but that is essential in order to work on nearly unstable Hawkes processes in the heavy tails case. All these results come from **[25]**, where they are proven in detail.

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$.

Theorem A.6 (Dambis, Dubin-Schwarz). *Let M be a $(\mathcal{F}_t)$-continuous local martingale vanishing at 0 and such that $\langle M, M\rangle_\infty = \infty$. If we set*

$$T_t = \inf\{s \colon \langle M, M\rangle_s > t\}\,,$$

then $B_t = M_{T_t}$ is a $(\mathcal{F})_t$-Brownian motion and $M_t = B_{\langle M,M\rangle_t}$.

Next theorem partially answer the question "which martingales can be written as a stochastic integral with respect to a Brownian motion?".

Theorem A.7. *If M is a continuous local martingale such that the measure $d\langle M, M\rangle_t$ is a.s. equivalent to the Lebesgue measure, there exists an $(\mathcal{F}_t)^M$-predictable process f_t which is strictly positive $dt \otimes d\mathbb{P}$-a.s. and an $(\mathcal{F}^M_t)$-Brownian motion B such that*

$$d\langle M, M\rangle_t = f_t\, dt \quad \text{and} \quad M_t = M_0 + \int_0^t f_s^{1/2}\, dB_s\,.$$

Proof. By Lebesgue's derivation theorem, the process

$$f_t = \lim_{n\to\infty} n\big(\langle M, M\rangle_t - \langle M, M\rangle_{t-1/n}\big)$$

satisfies the requirements in the statement. Moreover, $(f_t)^{-1/2}$ is clearly in $L^2_{\rm loc}(M)$ and the process

$$B_t = \int_0^t f_t^{-1/2}\, dM_s$$

is a continuous local martingale with increasing process t, hence a Brownian motion. □

We finally state a multidimensional version of the preceding result.

Theorem A.8. *Let $M = (M^1, \dots, M^d)$ be a continuous vector local martingale such that $d\langle M^i, M^i\rangle_t \ll dt$ for every i. Then there exist, possibly on an enlargement of the probability space, a d-dimensional Brownian motion B and a $d\times d$ matrix-valued predictable process α in $L^2_{loc}(B)$ such that*

$$M_t = M_0 + \int_0^t \alpha_s\, dB_s\,.$$

APPENDIX B

Fractional integrals and derivatives

1. Fractional integral and differential operators

For an n-fold integral next formula, proved by induction, holds

$$\int_0^x dt \int_0^x dt \cdots \int_0^x \varphi(t)\, dt = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \varphi(t)\, dt\,.$$

Writing $(n-1)! = \Gamma(n)$ we observe that the right-hand side of the preceding equation may have a meaning for non-integer values of n. So it is natural to define the integration of a non-integer order as follows.

Definition B.1 (Fractional integral). Let $\varphi \in L^1(0,\infty)$. The left-sided fractional integral of φ of order $\alpha > 0$ is

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt\,, \quad x > 0\,.$$

Proposition B.2 (Semigroup property). *The fractional integration has the semigroup property*

$$I^\alpha I^\beta \varphi = I^{\alpha+\beta} \varphi\,, \quad \alpha > 0,\, \beta > 0\,. \tag{B.1}$$

Proof. This property can be proved directly:

$$I^\alpha I^\beta \varphi = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \frac{dt}{(x-t)^{1-\alpha}} \int_0^t \frac{\varphi(\tau)\, d\tau}{(t-\tau)^{1-\beta}}$$

and, using Fubini theorem and setting $t = \tau + s(x-\tau)$, we have

$$I^\alpha I^\beta \varphi = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \frac{\varphi(\tau)\, d\tau}{(x-\tau)^{1-\alpha-\beta}}\,,$$

where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$ is the usual *beta function*. The proof is concluded recalling the relation

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

□

Now it comes natural to introduce an inverse operation to fractional integration. This is given by fractional differentiation.

Definition B.3 (Fractional derivative). For a function f given in the interval $[0,\infty)$,the left-sided fractional derivative of f of order α, for $0 < \alpha < 1$, is

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^\alpha}\, dt\,, \quad x > 0\,.$$

In **[26]** it is shown that a simple and sufficient condition for the existence a.e. of the fractional derivative of f is that $f \in AC([0,\infty))$.

We now clarify how the fractional integration and differentiation ar inverse operations. We need a definition.

DEFINITION B.4. Let $0 < \alpha < 1$. We denote by $I^\alpha(L^p)$ the space of functions f represented by the left-sided fractional integral of order α of a summable function: $f = I^\alpha \varphi$, $\varphi \in L^p([0,\infty))$, $1 \le p < \infty$.

We then have

THEOREM B.5. *Let* $0 < \alpha < 1$. *Then the equality*

$$D^\alpha I^\alpha \varphi(x) = \varphi(x)$$

is valid for any summable function φ, *while*

$$I^\alpha D^\alpha f(x) = f(x)$$

is satisfied for $f \in I^\alpha(L^1)$.

REMARK B.6. (i) We defined fractional integrals and derivatives in a simplified way, suitable to our applications. We could define them for functions f defined on an interval $[a,b]$. Moreover the term *left-sided* appears in contrast with the right-sided fractional integral and derivatives. As an example we report the definition of right-sided fractional integral of a function f defined on the interval $[a,b]$:

$$I^\alpha_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt\,, \quad x < b\,.$$

All the properties we stated about left-sided fractional operators are valid for right-sided fractional operators too.

(ii) We defined the fractional derivative of order α for $0 < \alpha < 1$. It is actually easy to extend this definition to $\alpha > 0$. We simply consider $\alpha = \lfloor \alpha \rfloor + \{\alpha\}$, then we define

$$D^\alpha f = \left(\frac{d}{dx}\right)^{\lfloor \alpha \rfloor} D^{\{\alpha\}} f = \left(\frac{d}{dx}\right)^{\lfloor \alpha \rfloor + 1} I^{1-\{\alpha\}} f\,.$$

(iii) Finally, it is possible to extend the former definitions to any complex order $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, but this is beyond our aim.

2. Some properties for fractional operators

We now report some results borrowed from **[26]**. The first one gives us a relation between the Hölder exponent of a function and the Hölder exponent of its fractional derivatives.

PROPOSITION B.7. *If* $f \in H^\lambda$ *and* $f(0) = 0$, *then for any* $\alpha < \lambda$, f *admits a fractional derivative of order* α *and* $D^\alpha f \in H^{\lambda-\alpha}$.

A fractional integration by parts formula follows:

PROPOSITION B.8. *If* $\varphi \in L^p$ *and* $\psi \in L^q$ *with* $1/p + 1/q \le 1 + \alpha$, *then* φ *and* ψ *have a fractional integral of order* α *and*

$$\int_0^t \varphi(t-s) I^\alpha \psi(s)\, ds = \int_0^t I^\alpha \varphi(t-s) \psi(s)\, ds\,.$$

As corollaries one gets the followings.

COROLLARY B.9. *Let $\varphi \in L^r$, with $r > 1$ and $\psi \in H^\beta$. Then, for any $\alpha < \beta$, $D^\alpha\psi$ exists, belongs to $H^{\beta-\alpha}$ and*

$$\int_0^t \varphi(t-s)\psi(s)\,ds = \int_0^t I^\alpha\varphi(t-s)D^\alpha\psi(s)\,ds\,.$$

COROLLARY B.10. *Let φ be continuous and ψ such that $x^\mu\psi(x) \in H^\lambda$ for some $\mu > 0$. Then, for any $\alpha < \min\{1-\mu;\lambda\}$, $D^\alpha\psi$ exists, belongs to L^r for some $r > 1$ and*

$$\int_0^t \varphi(t-s)\psi(s)\,ds = \int_0^t I^\alpha\varphi(t-s)D^\alpha\psi(s)\,ds\,.$$

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