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EXTINCTION TIME OF THE CONTACT PROCESS ON RANDOM GRAPHS

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*A Giacomo,
con cui avrei voluto
condividere questo
traguardo*

Extinction time of the contact process on random graphs

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1 Introduction

This thesis oversees the limiting behaviour of the extinction time of the contact process on random graphs.

The contact process was introduced by Harrys in the late '70s, and was then extensively studied by Liggett about twenty years later. In particular, in [4] Liggett deepens the description of the contact process on deterministic structures, such as the d -dimensional integer lattice \mathbb{Z}^d and the homogeneous tree \mathbb{T}^d .

The contact process is characterized by a Markovian dynamics and is an example of interacting particles system. Indeed, given a graph G fixed, and starting from a configuration in which to each vertex of G is assigned the value 0 or 1, corresponding to a healthy or an infectious site respectively, the contact process describes the evolution of the infection across the sites of G , that is determined by local rules.

The only absorbing state of the contact process is the configuration δ_0 that assigns 0 to every site, since it corresponds to the extinction of the infection (roughly speaking, if all the individuals of a population are healthy, no one can infect the others, so they will remain healthy from that moment on).

In order to understand how long it takes to reach such a configuration, it is necessary to introduce the concept of extinction time. The extinction time is a random variable that corresponds to the first time in which our system reaches the configuration δ_0 . It is also possible that the infection disappears locally but not globally, so it is necessary to distinguish between the extinction time and the first time in which just a certain subset of the vertices of G is healthy, which will be called local extinction time.

Therefore, the contact process is said to die out if the extinction time is finite almost surely, to die out locally but not globally if the local extinction time is finite almost surely, but the extinction time is not finite almost surely, and to survive if the local extinction time is not finite almost surely.

The first results obtained for the contact process on \mathbb{Z}^d and \mathbb{T}^d underline a phase transition depending on the rate λ at which a contagious site can infect the others. In particular, in the case of \mathbb{Z}^d there exists a critical value λ_c of λ such that for $\lambda \leq \lambda_c$ the contact process on \mathbb{Z}^d dies out, while for $\lambda > \lambda_c$, it survives. On the contrary, the process on \mathbb{T}^d exhibits an intermediate phase, in which the process survives locally but not globally.

In this thesis we will focus on the contact process on a specific class of random graphs.

Random graphs are random variables that take values in the set of all the possible graphs that can be drawn starting from a fixed number of vertices n . One of the simplest class of random graphs is the Erdős Renyi graph, in which each couple of vertices can connect with probability p independently of the others.

Another example of random graph is the random regular graph, which is a graph chosen uniformly at random among all graphs on n vertices with constant degree. It is not easy to compute the cardinality of this ensemble, and thus the corresponding uniform distribution, so in order to analyze the properties of random regular graphs, we introduce the configuration model, that gives a procedure to construct multigraphs at random with a given degree sequence.

Random regular graphs will serve as the structure of our contact process. This means that we will have to deal with a problem that is anything but trivial, since it will consist in the study of a double randomness, due to both the contact process and the random regular graph.

In our analysis we will follow the paper [5] by Mourrat and Valesin, whose main result is to prove that the limiting behaviour of the extinction time exhibits a phase transition on the infection rate λ . In particular, we will show that there exists a critical λ_g such that asymptotically almost surely for $\lambda < \lambda_g$ the extinction time will be less than $C \log n$, while for $\lambda > \lambda_g$ it will be greater than e^{Kn} . The critical value corresponds to the critical λ_c of global extinction on \mathbb{T}^d .

In the proof we will use probabilistic techniques, due to the fact that the problem has a Markovian structure. The local characterisation of the graph as cycle-free and its connection to branching processes here will play a crucial role.

The thesis is organized as follow:

- in Chapter 2 we provide the definition of continuous time Markov chain, of the corresponding semigroup and generator, and recall some basic facts about invariant measures and asymptotic convergence. We end the chapter with an example of a continuous time Markov chain, the Poisson process, since it will be used in the description of the contact process.
- in Chapter 3 we define the contact process on graph, introducing invariant measures, critical values and the concept of local and global extinction. We continue giving some basic results on \mathbb{Z}^d and \mathbb{T}^d and mentioning a result about the extinction time of the contact process on \mathbb{T}_l^d .
- in Chapter 4 we define and describe random graphs, starting from the Erdős Renyi random graph, continuing with the configuration model and ending with random regular graphs.
- in Chapter 5 we state our main result, we give a proof and mention some possible generalisations.

2 Continuous time Markov chains

[1][2]

Our aim here is to give a formal definition and a description of a *continuous time Markov chain*.

2.1 Definition, semigroups, and generators

We start by defining a discrete time Markov chain in the following way:

Definition 2.1. A sequence $(X_n)_{n \in \mathbb{N}}$ of random variables which take values in a finite or countable set I is called a *Markov chain* if, for each $n \geq 1$ and $i_1, \dots, i_{n+1} \in I$, it holds that

$$\begin{aligned} \mathbb{P}(X_{n+1} = i_{n+1} | X_n = i_n, \dots, X_1 = i_1, X_0 = i_0) &= \\ &= \mathbb{P}(X_{n+1} = i_{n+1} | X_n = i_n) \end{aligned}$$

In other words a Markov chain is a stochastic process that retains no memory of where it has been in the past, so that only the current state will influence where it goes next.

We are interested in the case in which $\mathbb{P}(X_{n+1} = i | X_n = j)$ does not depend on n : when this holds we say that the chain is *time-homogeneous* and the matrix P defined by

$$P := (p_{ij})_{i,j \in I}, p_{ij} := \mathbb{P}(X_{n+1} = i | X_n = j)$$

is called the *transition matrix*.

By time-homogeneity we have that

$$\begin{aligned} \mathbb{P}(X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) &= \\ &= p_{i_{n-1}i_n} p_{i_{n-2}i_{n-1}} \dots p_{i_0i_1} \mathbb{P}(X_0 = i_0) \end{aligned}$$

and this tells us that the law of the process is completely determined by its initial distribution and its transition matrix.

Notation 2.1. A discrete time Markov chain $(X_n)_{n \in \mathbb{N}}$ with initial distribution $\lambda = (\lambda_i : i \in I)$ and transition matrix P is denoted by $(X_n)_{n \in \mathbb{N}} \sim \text{Markov}(\lambda, P)$.

Let us also notice that if $(X_n)_{n \in \mathbb{N}} \sim \text{Markov}(\lambda, P)$ then, by Markov property and time-homogeneity, the probability that after n steps the Markov chain is in state j is $\mathbb{P}(X_n = j) = (\lambda P^n)_j$.

We now want to consider the continuous time case.

Let $(X_t)_{t \geq 0}$ be a random variable with values in a finite or countable set I . Let $\mathcal{F}_s := \sigma(X_u : u \leq s)$ be the σ -algebra generated by the random variables X_u for which $u \leq s$.

Definition 2.2. $(X_t)_{t \geq 0}$ is called a *Markov chain* if for every $i \in I$ and $0 \leq s \leq t$ it holds that

$$\mathbb{P}(X_t = i | \mathcal{F}_s) = \mathbb{P}(X_t = i | X_s)$$

Let us notice that this definition is the continuous time analogous of the discrete time case, since conditioning on \mathcal{F}_s is the same as conditioning on X_s , which means that the process is memoryless as in the discrete case.

We are interested in the case in which $\mathbb{P}(X_t = i | X_s = j)$ depends on s, t only through $t - s$: when this happens we say that the chain is *time-homogeneous*, and we define the operator $S(t) : \mathbb{R}^I \rightarrow \mathbb{R}^I$ for $t \geq 0$ by

$$S(t)f(j) := \mathbb{E}(f(X_t) | X_0 = j) = \sum_{i \in I} f(i) \mathbb{P}(X_t = i | X_0 = j)$$

where \mathbb{R}^I denotes the set of all the functions f from I to \mathbb{R} .

Proposition 2.1. For $t \geq 0$, $S(t)$ is a semigroup, i.e. $S(0) = I$ and $S(t + s) = S(t)S(s)$.

The thesis of the previous proposition follows directly from the definitions:

$$\begin{aligned} S(0)f(j) &= \mathbb{E}(f(X_0) | X_0 = j) = \sum_{i \in I} f(i) \mathbb{P}(X_0 = i | X_0 = j) = \\ &= \sum_{i \in I} f(i) \delta_{ij} = f(j) \end{aligned}$$

so $S(0) = I$, and also $S(t + s) = S(t)S(s)$ is a direct consequence of the definition and Markov property.

Now, let us notice that $S(t)$ is a linear operator from \mathbb{R}^I to itself, so it can be seen as a matrix

$$(S(t)_{ij})_{i,j \in I} = \mathbb{P}(X_t = j | X_0 = i) \tag{1}$$

Moreover, if $t \mapsto S(t)$ is continuous, it can be shown that

$$\lim_{t \downarrow 0} \frac{S(t) - I}{t}$$

exists and we call it L .

Using also Markov property, it holds that

$$\frac{d}{dt} S(t) |_{t=0} = L \iff S(t) = e^{tL}$$

since

$$\frac{d}{dt}S(t) = \lim_{h \rightarrow 0} \frac{S(t+h) - S(t)}{h} = \lim_{h \rightarrow 0} S(t) \frac{S(h) - I}{h} = S(t)L.$$

Now, from (1), for $0 < t_1 < t_2 < \dots < t_n$ and $i_0, i_1, \dots, i_n \in I$, it follows that:

$$\begin{aligned} & \mathbb{P}(X_{t_n} = i_n, X_{t_{n-1}} = i_{n-1}, \dots, X_0 = i_0) = \\ & = (S(t_n - t_{n-1}))_{i_n i_{n-1}} (S(t_{n-1} - t_{n-2}))_{i_{n-2} i_{n-1}} \dots (S(t_1))_{i_0 i_1} \mathbb{P}(X_0 = i_0) \end{aligned}$$

which means that the distribution of X_0 and the semigroup $S(t)_{t \geq 0}$ identify the law of the process.

Definition 2.3. We call L the *generator* of the Markov chain.

Notation 2.2. A continuous time Markov chain $(X_t)_{t \geq 0}$ with initial distribution $\lambda = (\lambda_i : i \in I)$ and semigroup S is denoted by $(X_t) \sim \text{Markov}(\lambda, S)$.

The generator L of a semigroup $S(t)$ has some properties that we want to underline:

- (i) $L_{ij} \geq 0$ for $i \neq j$,
- (ii) $\sum_{j \in I} L_{ij} = 0 \ \forall i$,
- (iii) $0 \leq -L_{ii} < \infty \ \forall i$.

where (i) holds because

$$0 \geq \mathbb{P}(X_t = j | X_0 = i) = (S(t))_{ij} = tL_{ij} + o(t) \Rightarrow L_{ij} \geq 0 \quad (2)$$

, (ii) is true since

$$\sum_{j \in I} L_{ij} = \lim_{t \downarrow 0} \sum_{j \in I} \frac{(S(t))_{ij} - \delta_{ij}}{t} = 0$$

, and (iii) follows from (i) and (ii) because by (ii) $\sum_{j \in I} L_{ij} = 0$ and by (i) $L_{ij} \geq 0$, so that

$$\sum_{j \in I} L_{ij} = L_{ii} + \sum_{j \neq i} L_{ij} = 0 \Rightarrow L_{ii} = - \sum_{j \neq i} L_{ij} \leq 0.$$

From these properties we can obtain a general expression for the generator of the semigroup:

$$\begin{aligned} Lf(i) &= \sum_{j \in I} L_{ij} f(j) = L_{ii} f(i) + \sum_{j \neq i} L_{ij} f(j) = \\ &= - \sum_{j \neq i} L_{ij} f(i) + \sum_{j \neq i} L_{ij} f(j) = \sum_{j \neq i} L_{ij} [f(j) - f(i)]. \end{aligned} \quad (3)$$

This expression gives us another way to define a continuous time Markov chain, just like (2), which is an infinitesimal characterisation of the transition probability (in other words L_{ij} can be seen as the rate of going from state i to state j).

2.2 Convergence and stationary distribution

We now investigate the limiting behaviour of the continuous time Markov chain law.

Let $\pi_t(j) = \mathbb{P}(X_t = j)$. It holds that

$$\begin{aligned}\pi_t(j) &= \sum_i \mathbb{P}(X_t = j, X_0 = i) = \\ &= \sum_i \mathbb{P}(X_t = j | X_0 = i) \mathbb{P}(X_0 = i) = \sum_i (S(t))_{ij} \pi_0(i)\end{aligned}$$

so we have

$$\pi_t = \pi_0 S(t) \iff \begin{cases} \dot{\pi}_t = \pi_0 \dot{S}(t) = \pi_0 S(t) L = \pi_t L, \\ \pi_0 \end{cases}$$

Definition 2.4. A probability π on I is called a *stationary* or *invariant distribution* if for every $t \geq 0$ $\pi S(t) = \pi$ or equivalently $\pi L = 0$.

The idea is that the stationary distribution π is the candidate limiting distribution of our chain.

It can be shown that if I is finite, then at least one stationary measure exists. This is not necessarily true if I is countable. We hence introduce the following concept, which will serve as necessary condition for the existence of a stationary measure even if I is countable.

Definition 2.5. A Markov chain is called *irreducible* if for every $i, j \in I$ there exists a path $i = i_0, i_1, \dots, i_n = j$ such that for every $k = 0, \dots, n-1$ $i_k \neq i_{k+1}$ and $L_{i_k i_{k+1}} > 0$.

Given this definition, the following result holds:

Theorem 2.1. (ergodic theorem) For irreducible chains there exists at most one stationary distribution π . When it exists $\pi(i) > 0$ for every $i \in I$ and for every probability π_0 we have that

$$\pi = \lim_{t \rightarrow \infty} \pi_0 S(t).$$

2.3 The Poisson process

Poisson processes are some of the simplest examples of continuous time Markov chains, indeed they may also be used as building blocks for other continuous time Markov chains that are more general (and this is in fact the case of the contact process).

Definition 2.6. A *Poisson process* $(N_t)_{t \geq 0}$ of rate λ , $0 < \lambda < \infty$, is a continuous time Markov chain which takes values in $I = \mathbb{N}$, with $N_0 = 0$, and $L_{ij} = \lambda \delta_{i+1, j}$.

Notation 2.3. $N_t \sim PP(\lambda)$.

Hence, representing L with a matrix, we have:

$$L = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & & & \\ 0 & -\lambda & \lambda & 0 & \dots & & \\ \dots & 0 & -\lambda & \lambda & 0 & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where we have used properties (i), (ii), and (iii) in the representation.

It is possible to define the Poisson process in other equivalent ways: we now give an equivalent definition in terms of its increments.

Definition 2.7. The *increment* of a real-valued process $(X_t)_{t \geq 0}$ on an interval $(s, t]$ is given by $X_t - X_s$. We say that $(X_t)_{t \geq 0}$ has *stationary* increments if the distribution of $X_{s+t} - X_s$ depends only on $t \geq 0$, i.e. $X_{t+s} - X_s \sim X_t - X_s$. We say that $(X_t)_{t \geq 0}$ has *independent* increments if its increments over any finite collection of disjoint intervals are independent, i.e. $X_{t+s} - X_s$ is independent of $X_u : u \leq s$.

Definition 2.8. A *Poisson process* $(N_t)_{t \geq 0}$ of rate λ , $0 < \lambda < \infty$, is a continuous time Markov chain that has stationary independent increments and such that, for each t , N_t has Poisson distribution of parameter λt .

A property of the Poisson process that we want to underline is the following:

Theorem 2.2. *If $(N_t)_{t \geq 0}$ and $(M_t)_{t \geq 0}$ are independent Poisson processes of rates λ and μ respectively, then their sum $(N_t + M_t)_{t \geq 0}$ is a Poisson process of rate $\lambda + \mu$.*

We end this paragraph giving an example and a graphical construction of what we obtain building a Markov chain starting from Poisson processes.

We hence consider a *continuous time simple random walk* $(\eta_t : t \geq 0)$ on $I = \mathbb{Z}$, which is defined by $\eta_t = N_t - M_t$, where $(N_t : t \geq 0) \sim PP(\lambda)$ and $(M_t : t \geq 0) \sim PP(\mu)$, so as the difference of two Poisson processes.

Therefore, the generator L of $(\eta_t : t \geq 0)$ is:

$$L = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \mu & -\lambda - \mu & \lambda & 0 & \dots & \dots & \dots \\ \dots & 0 & \mu & -\lambda - \mu & \lambda & 0 & \dots & \dots \\ \dots & \dots & 0 & \mu & -\lambda - \mu & \lambda & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (4)$$

As for the graphical construction of the process η_t , we show it in Figure 1, where in each column N_t is represented by \rightarrow , M_t by \leftarrow , and the trajectory of the process η_t is coloured in red (it is hence uniquely determined).

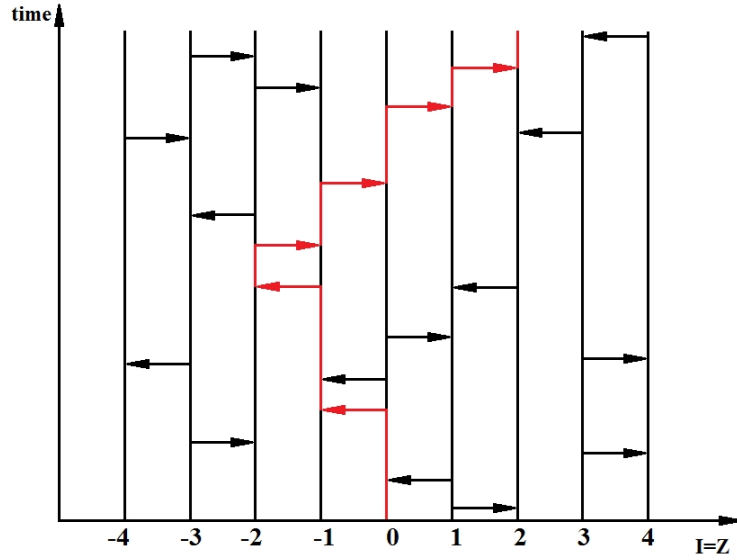


Figure 1: Continuous time simple random walk: graphical construction

As said at the end of paragraph 2.1, to define a continuous time Markov chain such as the simple random walk, we can also give the expression of the generator of its semigroup, i.e.:

$$Lf(\eta) = \lambda(f(\eta + 1) - f(\eta)) + \mu(f(\eta - 1) - f(\eta))$$

and indeed this is just another way to define matrix (4).

3 The contact process

[2]

We now want to describe the *contact process*. We start with a representation in terms of its jump rates L_{ij} and a graphical construction, then we give a definition declaring its generator.

The idea is that the contact process $(\eta_t)_{t \geq 0}$ represents the possible configurations of the spread of an illness in a population. We will hence have a graph $G = (V, E)$ whose vertices V will be seen as individuals of a population whose possible states are $\{0, 1\}$, where 0 means that the individual is healthy, 1 that it is infected.

Moreover, each infected vertex can recover with rate 1, while the healthy one can be infected with rate $\lambda > 0$ times the number of neighbours in the graph.

In other words, the contact process is made of Poisson processes: the recovering process (a Poisson process of rate 1) and the infection processes (Poisson processes of rate λ).

Notation 3.1. We denote with $y \sim x$ the fact that y is a neighbour of x in the graph.

The jump rates of the contact process are hence:

$$L_{x,\eta} = \begin{cases} 1, & \eta(x) = 1 \\ \lambda \sum_{y \sim x} \eta(y), & \eta(x) = 0 \end{cases}$$

To give a graphical construction of the contact process, we consider the case $V = \mathbb{Z}$: this means that each individual will have at most two infectious neighbours. We will hence introduce three independent Poisson processes: in Figure 2 $\times \sim PP(1)$ will mark on each line the recovery events, while the independent Poisson processes $\rightarrow \sim PP(\lambda)$, and $\leftarrow \sim PP(\lambda)$ will mark the infection events. The contact process will be coloured in red.

Notation 3.2. We will denote with $\xi_t^x := \{y \in V : (x, 0) \leftrightarrow (y, t)\}$ where \leftrightarrow indicates that x and y are connected by an infection path (which in figure 2 is coloured in red). This means that ξ_t^x is a set of vertices that are infectious. We also underline that hence $\xi_t^A = \cup_{x \in A} \xi_t^x$, and $\xi_{s+t}^A = \cup_{x \in \xi_t^A} \{y : (x, t) \leftrightarrow (y, t+s)\} \equiv \cup_{x \in \xi_t^A} (\xi_t^x \circ \theta_s)$.

From notation 3.2, we see that it holds:

$$\begin{aligned} \forall x \in \xi_t, \xi_t \rightarrow \xi_t \setminus \{x\} \text{ with rate } 1, \\ \forall x \notin \xi_t, \xi_t \rightarrow \xi_t \cup \{x\} \text{ with rate } \lambda |\{y \in \xi_t : y \sim x\}| \end{aligned}$$

Let us now define the contact process in terms of its generator.

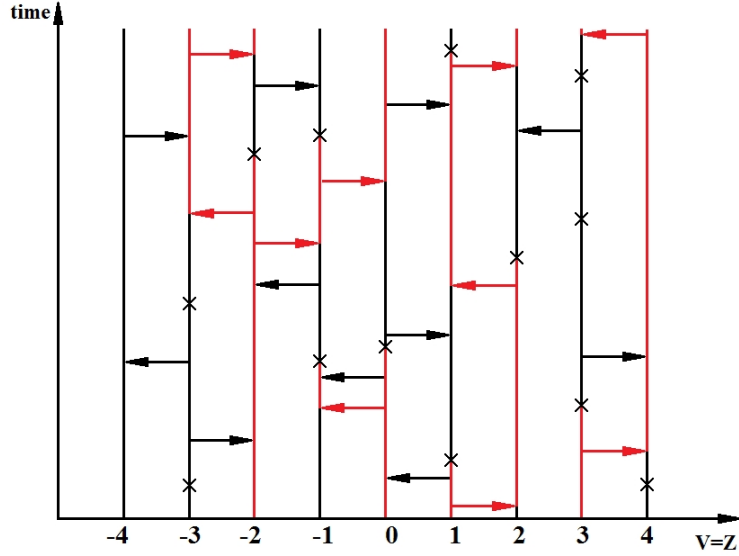


Figure 2: Contact process: graphical construction

Definition 3.1. Let $G = (V, E)$ be a strongly connected graph with vertex set V and edge set E . We call *contact process* the Markov chain with state space $I = \{0, 1\}^{|V|}$ and generator

$$Lf(\eta) = \sum_{z \in V} \left(\eta(z) + \lambda(1 - \eta(z)) \sum_{y \sim z} \eta(y) \right) (f(\eta^z) - f(\eta)) \quad (5)$$

where $\eta(z)$ denotes the value of the process in the component z and η^z is η with component z that has been updated.

3.1 Invariant measures and critical values

Let us start by giving some definitions and stating a property for the contact process. We usually say that a function $f : I \rightarrow \mathbb{R}$ is *increasing* if $x \leq y \Rightarrow f(x) \leq f(y)$. This concept can also be extended to Markov chains.

Definition 3.2. A continuous time Markov chain $(X_t)_{t \geq 0}$ with semigroup $(S(t) : t \geq 0)$ is said to be *attractive* or *monotone* if f increasing $\Rightarrow S(t)f$ increasing $\forall t \geq 0$.

Equivalently, using (1), i.e. the fact that $S(t)$ represents the time evolution of the distribution of a process, we can give the same definition in terms of distributions:

Definition 3.3. A continuous time Markov chain $(X_t)_{t \geq 0}$ with semigroup $(S(t) : t \geq 0)$ is said to be *attractive* or *monotone* if $\mu \leq \lambda \Rightarrow \mu S(t) \leq \lambda S(t)$, $\forall t \geq 0$, where μ and λ are distributions.

We now want to say something about the limiting behaviour of the contact process defined as in (5).

First, we give the following result, which can be used to prove proposition 3.1.

Lemma 3.1. *The contact process is attractive.*

Now, let δ_1 be the configuration that corresponds to $\eta(x) = 1 \forall x \in V$ (i.e. each individual of the population is infectious) and δ_0 the configuration such that $\eta(x) = 0 \forall x \in V$ (i.e. each individual of the population is healthy).

Let us notice that δ_0 is stationary, since if the process reaches the configuration in which each individual is healthy, no one will spread the illness anymore and hence everyone will remain healthy (we say indeed that δ_0 is an *absorbing* state).

Moreover, the following result holds:

Proposition 3.1. *For all $0 \leq s \leq t$ we have*

$$\delta_1 S(t) \leq \delta_1 S(s)$$

and that there exists the limit

$$\bar{\nu}_\lambda = \lim_{t \rightarrow \infty} \delta_1 S(t)$$

Moreover, $\bar{\nu}_\lambda$ is invariant and $\delta_0 \leq \pi \leq \bar{\nu}_\lambda$ for every π invariant.

Finally, $\lambda < \lambda'$ implies $\bar{\nu}_\lambda \leq \bar{\nu}_{\lambda'}$ and for each $x \in I$ $\rho_x(\lambda) := \bar{\nu}_\lambda(\eta(x))$ is monotone increasing in λ .

Definition 3.4. $\bar{\nu}_\lambda$ is called the *upper invariant measure*.

Since on a finite graph the configuration δ_0 can be reached in a finite time from any other configuration (a finite number of infectious individuals will no longer be infectious in a finite time), and since δ_0 is absorbing, we have that

$$\pi S(t) \rightarrow \delta_0 \text{ as } t \rightarrow \infty$$

for all π . Hence, taking $\pi = \delta_1$ we have, by the previous proposition, that $\bar{\nu}_\lambda = \delta_0$.

On the other hand, on an infinite graph it may be possible that $\bar{\nu}_\lambda \neq \delta_0$.

Definition 3.5. The *survival probability* α_η with initial configuration $\eta \in I$ is the probability defined by:

$$\alpha_\eta := \mathbb{P}(\eta_t \neq 0 \forall t \geq 0 | \eta_0 = \eta).$$

Notation 3.3. For each set $A \subseteq V$ we denote by α_A the survival probability α_{1_A} . We write α_x for $\alpha_{\{x\}}$.

Definition 3.6. The contact process $(\eta_t)_{t \geq 0}$ is said to *die out* if $\alpha_x = 0$ for some $x \in V$, otherwise it is said to *survive*.

Proposition 3.2. *If the contact process dies out for infection rate $\lambda' > 0$, then it dies out for all $\lambda \in [0, \lambda']$*

Therefore, we can introduce the following concept:

Definition 3.7. The *critical value* $\lambda_c \in [0, \infty]$ is defined by:

$$\lambda_c := \sup\{\lambda \geq 0 : \text{the contact process with infection rate } \lambda \text{ dies out}\}.$$

Our aim now is to show that there exists a phase transition varying λ ; in other words for $\lambda < \lambda_c$ the contact process will have a certain behaviour, while for $\lambda > \lambda_c$ the trend of the process will be different.

We hence state the next result:

Proposition 3.3. *For any set $A \subseteq V$ the survival probability is*

$$\alpha_A = \bar{\nu}_\lambda(\{1_B : B \cap A \neq \emptyset\}),$$

and for $\lambda < \lambda_c$ we have $\bar{\nu}_\lambda = \delta_0$, while for $\lambda > \lambda_c$, $\bar{\nu}_\lambda \neq \delta_0$.

To sum up, there exists a well defined critical value $\lambda_c \in [0, \infty]$ such that the contact process dies out and $\bar{\nu}_\lambda = \delta_0$ for $\lambda < \lambda_c$, while for $\lambda > \lambda_c$ it survives and $\bar{\nu}_\lambda \neq \delta_0$.

However, if for a finite graph we have seen that $\lambda_c = \infty$, on infinite graphs it is not clear if λ_c is non-trivial, i.e. if $\lambda_c \in (0, \infty)$. We hence state another result, which will give a lower bound for λ_c on infinite graphs.

Proposition 3.4. *Let us consider the contact process on a connected graph with maximal vertex degree m . Then $\lambda_c \geq \frac{1}{m}$.*

Giving an upper bound is more difficult, so in paragraph 3.2 and 3.3, we will focus on the case $V = \mathbb{Z}^d$ and $V = \mathbb{T}^d$ respectively, eventually stating other results.

3.1.1 Local and global extinction

[4] We have already defined (see definition 3.5) the survival probability of the contact process. However, in some situations it may happen that the infection survives weakly (for example, in paragraph 3.3 we will see that this is what happens when we consider the contact process on homogeneous connected trees), then a notion of weak survival probability or local extinction is necessary.

Definition 3.8. The *extinction time* τ_G^A of the contact process $(\xi_t^A)_{t \geq 0}$ on $G = (V, E)$, where $A \subseteq V$ is defined by

$$\tau_G^A := \inf\{t : \xi_t^A \neq \emptyset\}$$

Definition 3.9. The *local extinction time* τ_G^A of the contact process $(\xi_t^A)_{t \geq 0}$ on $G = (V, E)$, where $A \subseteq V$ is defined by

$$\tau_G^{A,loc} := \inf\{t : \xi_t^A \cap A = \emptyset\}$$

Notation 3.4. We write τ_G for τ_G^G , and τ^x for $\tau^{\{x\}}$.

Definition 3.10. Let $G = (V, E)$ be a graph, $A \subseteq V$, and $\lambda > 0$. Then, for the contact process ξ_t^A with parameter λ on G , we define:

- the *extinction probability* as

$$p_{G,A,\lambda}^{\text{ext}} := \mathbb{P}_{G,\lambda}[\tau_G^A < \infty]$$

- the *local extinction probability* as

$$p_{G,A,\lambda}^{\text{loc ext}} := \mathbb{P}_{G,\lambda}[\tau_G^{A,loc} < \infty].$$

Here we underline that the contact process has the same behaviour on all the subsets: it can either die out (locally) for all the subsets or survive (locally) for all the subsets, i.e. it holds:

$$\text{either } p_{G,A,\lambda}^{(\text{loc}) \text{ ext}} = 1 \ \forall A, \text{ or } p_{G,A,\lambda}^{(\text{loc}) \text{ ext}} < 1 \ \forall A$$

Definition 3.11. We say that the contact process

- *dies out* if $p_{G,A,\lambda}^{\text{ext}} = 1 \ \forall A$
- *survives weakly* or *locally* if $p_{G,A,\lambda}^{\text{ext}} < 1$ and $p_{G,A,\lambda}^{\text{loc ext}} = 1 \ \forall A$
- *survives strongly* or *globally* if $p_{G,A,\lambda}^{\text{loc ext}} < 1 \ \forall A$

Let now

$$\lambda_g(G) := \sup\{\lambda : p_{G,A,\lambda}^{\text{ext}} = 1\} \text{ and } \lambda_l(G) := \sup\{\lambda : p_{G,A,\lambda}^{\text{loc ext}} = 1\}$$

i.e. λ_g is the greatest λ for which the contact process dies out, while λ_l is the greatest λ for which the contact process survives locally.

Obviously, for the process to die out, we need an infection rate that is less or equal than the infection rate for the process to survive locally, hence $\lambda_g(G) \leq \lambda_l(G)$.

3.2 Basic results on \mathbb{Z}^d

[2] Let us consider the contact process on $V = \mathbb{Z}^d$.

Theorem 3.1. *For the critical value $\lambda_c(d)$ of a contact process on \mathbb{Z}^d , we have:*

$$\frac{1}{2d} \leq \lambda_c(d) \leq \frac{2}{d} \quad \forall d \geq 1.$$

Let us notice that the lower bound is the same as in the previous proposition, but $m = 2d$. It can also be shown that $d\lambda_c(d) \rightarrow \frac{1}{2}$ as $d \rightarrow \infty$.

The next result is about convergence of the contact process starting from a generic initial configuration $\eta \in I$

Theorem 3.2. (complete convergence theorem) *Let us consider the contact process on $V = \mathbb{Z}^d$. For every $\eta \in I$ we have*

$$\xi_t^\eta \xrightarrow{d} \alpha_\eta \bar{\nu}_\lambda + (1 - \alpha_\eta) \delta_0 \quad \text{as } t \rightarrow \infty, \quad (6)$$

where α_η is the survival probability, and for ξ_t^η we recall notation 3.2.

The extinction time represents hence the first time for which all the individuals recover starting from a given configuration. We are obviously interested in estimating the probability that the extinction time is finite when we are in the *supercritical regime* (i.e. the case $\lambda > \lambda_c$), for which we know that the process survives.

Theorem 3.3. *Let $\lambda > \lambda_c$ and $\tau_{\mathbb{Z}^d}$ be the extinction time of the contact process $(\eta_t)_{t \geq 0}$ on \mathbb{Z}^d . Then there exists $\epsilon > 0$ such that, for every $A \subseteq V$,*

$$\mathbb{P}(\tau_{\mathbb{Z}^d}^A < \infty) \leq e^{-\epsilon|A|}$$

However, if μ is translation invariant and $\mu(\eta(x)) > 0$, then $\mu(|\eta| = \infty) = 1$ by Borel Cantelli lemma, and hence $\mathbb{P}(\tau = \infty | \eta_0 = \eta) = \alpha_\eta = 1$, i.e. the process survives with probability 1. Now, by (6), setting $\alpha_\eta = 1$ and $\delta_\eta = \mu$ we obtain

$$\mu S(t) \rightarrow \bar{\nu}_\lambda \quad \text{as } t \rightarrow \infty.$$

The case left is the critical one, i.e. what happens if $\lambda = \lambda_c$.

In this situation, the following result holds:

Theorem 3.4. *The critical contact process dies out.*

3.3 Basic results on \mathbb{T}^d

[4] Here we want to describe the contact process on \mathbb{T}^d , therefore first of all we introduce the following concept:

Definition 3.12. The *homogeneous connected tree* \mathbb{T}^d is a tree in which each vertex has $d + 1$ neighbours, where we will assume that $d \geq 2$, since $\mathbb{T}^1 = \mathbb{Z}$.

It is often useful to think of \mathbb{T}^d as a branching tree in which each vertex has 1 parent and d children: to express this concept in a formal way, we define

Definition 3.13. Let $l : \mathbb{T}^d \rightarrow \mathbb{Z}$ be a function such that $\forall x \in \mathbb{T}^d$ it holds that $l(y) = l(x) - 1$ for exactly one neighbour y of x , and $l(y) = l(x) + 1$ for the other d neighbours y of x . The *generation number* of $x \in \mathbb{T}^d$ is $l(x)$.

Now, with the function $l(x)$ it is possible to generate all the elements of \mathbb{Z} , since for every generation we can either add or subtract 1. Conversely to $n \in \mathbb{Z}$, which represents generation n , can correspond more than one element x of \mathbb{T}^d , since x can have brothers. Hence, taking $e_n \in \mathbb{T}^d$ such that $l(e_n) = n$ and $|e_n - e_{n+1}| = 1$, this provides an embedding $k = l^{-1}|_{\mathbb{Z}}$ of \mathbb{Z} in \mathbb{T}^d

$$\begin{aligned} \mathbb{Z} &\xrightarrow{k} \mathbb{T}^d \\ n &\mapsto e_n. \end{aligned}$$

Taking this into account, we recall that on \mathbb{Z}^d , as we have seen in paragraph 3.2, we have that $\lambda_g = \lambda_l = \lambda_c$, hence there is no need to distinguish between local and global survival. Conversely, on \mathbb{T}^d we will see that an intermediate phase between λ_g and λ_l exists, so it is possible to have $\lambda_g < \lambda < \lambda_l$ such that the process survives locally.

This crucial result is shown in [4], here we will just describe the main results that are used in its proof. More precisely, we will follow the next steps:

- (i) Define the function Φ and show that it determines the asymptotics of $\mathbb{E}[\omega_\rho(\xi_t)]$ in a very strong sense, where $\omega_\rho(A) := \sum_{x \in A} \rho^{l(x)}$, $\rho \geq 0$ (proposition 3.5).
 - (ii) Give various monotonicity and continuity properties that are needed working with it (proposition 3.6).
 - (iii) State that if $\lambda = \lambda_1$ the process dies out and its expected size remains bounded (proposition 3.7).
 - (iv) State that properties of Φ determine whether or not the process survives strongly (proposition 3.8).
- (i) In order to define Φ , we will need the following theorem:

Theorem 3.5. *Suppose that $\alpha(t)$, $t \geq 0$ is locally bounded and satisfies the subadditivity property, i.e.*

$$\alpha(s+t) \leq \alpha(s) + \alpha(t), \quad s, t \geq 0.$$

Then

$$-\infty < \lim_{t \rightarrow \infty} \frac{\alpha(t)}{t} = \inf_{t > 0} \frac{\alpha(t)}{t} \leq \infty.$$

Let us recall notation 3.2 and consider

$$\xi_{t+s} = \cup_{x \in \xi_s} (\xi_t^x \circ \theta_s). \quad (7)$$

We have that, for $\rho > 0$,

$$\omega_\rho(\xi_{t+s}) = \sum_{y \in \xi_{t+s}} \rho^{l(y)} \leq \sum_{x \in \xi_s} \sum_{y \in \xi_t^x \circ \theta_s} \rho^{l(y)}$$

where in the last inequality we have used the fact that the union in (7) is not necessarily disjoint. We now take the conditional expectation with respect to \mathcal{F}_s on both members, obtaining:

$$\begin{aligned} \mathbb{E}[\omega_\rho(\xi_{t+s}) | \mathcal{F}_s] &\leq \mathbb{E}\left[\sum_{x \in \xi_s} \sum_{y \in \xi_t^x \circ \theta_s} \rho^{l(y)-l(x)+l(x)}\right] = \sum_{x \in \xi_s} \rho^{l(x)} \mathbb{E}\left[\sum_{y \in \xi_t^x} 1 \rho^{l(y)-l(x)}\right] = \\ &= \omega_\rho(\xi_s) \mathbb{E}[\omega_\rho(\xi_t)]. \end{aligned} \quad (8)$$

Taking expected values in inequality (8), we get:

$$\mathbb{E}[\omega_\rho(\xi_{t+s})] \leq \mathbb{E}[\omega_\rho(\xi_s)] \mathbb{E}[\omega_\rho(\xi_t)]$$

which implies

$$\log \mathbb{E}[\omega_\rho(\xi_{t+s})] \leq \log \mathbb{E}[\omega_\rho(\xi_s)] + \log \mathbb{E}[\omega_\rho(\xi_t)].$$

Hence, we can use theorem 3.5 to say that it exists

$$\lim_{t \rightarrow \infty} \frac{\log \mathbb{E}[\omega_\rho(\xi_t)]}{t}$$

so that it also exists

$$\lim_{t \rightarrow \infty} (E[\omega_\rho(\xi_t)])^{1/t}$$

therefore,

$$\Phi(\rho) = \lim_{t \rightarrow \infty} ((\mathbb{E}[\omega_\rho(\xi_t)])^{1/t}) \quad (9)$$

is well defined.

To end this first step, we also state

Proposition 3.5. (a) *The following symmetry properties hold:*

$$\mathbb{E}[\omega_{1/\rho d}(\xi_t)] = \mathbb{E}[\omega_\rho(\xi_t)] \text{ and } \Phi\left(\frac{1}{\rho d}\right) = \Phi(\rho).$$

(b) *There is a constant $C(\rho) \leq \infty$ depending only on d and ρ such that*

$$[\Phi(\rho)]^t \leq \mathbb{E}[\omega_\rho(\xi_t)] \leq C(\rho)[\Phi(\rho)]^t \quad t \geq 0$$

and $C(\rho) < \infty$ if $\rho \neq 1/\sqrt{d}$.

(ii) In the following proposition, we give monotonicity and continuity properties for Φ .

Proposition 3.6. (a) *Φ is nondecreasing in λ and is nondecreasing in ρ for $\rho \geq 1/\sqrt{d}$*

(b) *Φ is jointly continuous for $\lambda > 0$, $\rho > 0$.*

(iii) In the next result we see that for $\lambda = \lambda_g$ the process dies out and moreover its expected cardinality remains bounded.

Proposition 3.7. *If $\lambda = \lambda_g$, then $\Phi(1) = 1$, and hence*

$$\sup_{t>0} \mathbb{E}[|\xi_t|] < \infty$$

and ξ_t dies out.

(iv) The following proposition tells us that the properties of Φ determines if the process survives strongly or not.

Proposition 3.8. (a) *Suppose that ξ_t does not survive strongly. If $1/\sqrt{d} \leq \rho_1 < \rho_2$ and $\Phi(\rho_2) \geq 1$, then $\Phi(\rho_1) < \Phi(\rho_2)$.*

(b) *If $\Phi(\rho) < 1$ for some $\rho > 0$, then ξ_t does not survive strongly.*

Using propositions 3.5, 3.6, 3.7, and 3.8, we are now ready to prove the existence of an intermediate phase for the contact process on \mathbb{T}^d .

Theorem 3.6. *For all $d \geq 2$, $\lambda_g < \lambda_l$.*

Proof. If $\lambda = \lambda_g$, then by proposition 3.7, $\Phi(1) = 1$ and ξ_t dies out. Hence, by proposition 3.8 applied to $\rho_1 = \rho$ and $\rho_2 = 1$, it holds that $\Phi(\rho) < 1$. Let us now fix such a ρ . By proposition 3.6(b), there exists a $\lambda > \lambda_g$ such that $\Phi(\rho) < 1$ also for this λ . Now, proposition 3.8(b) implies that hence ξ_t does not survive strongly for this λ , so $\lambda \leq \lambda_l$. Therefore, $\lambda_l > \lambda_g$. \square

Let us now notice that these results does not only imply the existence of an intermediate phase, but also some statements about survival and strong survival. More precisely, we underline that:

- by proposition 3.5(b) and proposition 3.7 the contact process η_t survives if and only if $\Phi > 1$, and
- by proposition 3.8 $\Phi(1/\sqrt{d}) < 1$ implies that η_t does not survive strongly.

Our aim now is to see that the remaining implication holds, i.e. that it is true that if the process η_t does not survive strongly, then $\Phi(1/\sqrt{d}) \leq 1$.

We start by introducing the sequence $u(n)$

$$u(n) = \mathbb{P}(e_n \in \xi_t \text{ for some } t), n \geq 0$$

and letting

$$\tau = \inf\{t > 0 : e_n \in \xi_t\}.$$

We have that:

$$\begin{aligned} u(n+m) &= \mathbb{P}(e_{n+m} \in \xi_t) = \mathbb{E}[\mathbb{P}^{A_\tau}(e_{n+m} \in \xi_t \text{ for some } t), \tau < \infty] \geq \\ &\geq \mathbb{E}[\mathbb{P}^{e_n}(e_{n+m} \in \xi_t \text{ for some } t), \tau < \infty] = u(n)u(m) \end{aligned} \quad (10)$$

where the inequality is true because of monotonicity and the last equality comes from the strong Markov property.

Hence, by the discrete version of theorem 3.5 applied to the logarithm of u , reasoning as in the definition of Φ , we obtain that

$$\beta(\lambda) = \lim_{n \rightarrow \infty} [u(n)]^{\frac{1}{n}} \quad (11)$$

is well defined.

The next theorem collects some properties of β that we are going to need.

Theorem 3.7. (a) $\beta(\lambda_l) = \frac{1}{\sqrt{d}}$.

(b) If $\beta(\lambda) < \frac{1}{\sqrt{d}}$, then $\Phi(\frac{1}{\sqrt{d}}) < 1$.

To link β with Φ , let us also introduce

$$u(n, t) = \mathbb{P}(e_n \in \xi_t)$$

for $n \in \mathbb{N}$, and $t \geq 0$.

Repeating the reasoning of (10), we obtain

$$u(m+n, s+t) \geq u(m, s)u(n, t)$$

and hence, taking $s = mt$ and replacing the previous t with nt ,

$$u(m+n, (m+n)t) \geq u(m, mt)u(n, nt).$$

Taking again the logarithm, and using the discrete version of theorem 3.5, we obtain that

$$U(t) \lim_{n \rightarrow \infty} u(n, nt)^{\frac{1}{n}}$$

is well defined.

It holds that

Theorem 3.8. *The function $U(t)$ is such that $\log U$ is concave on $(0, \infty)$ and hence it is continuous there, and satisfies*

$$\sup_{t>0} U(t) = \beta(\lambda)$$

and

$$\lim_{t \rightarrow 0} U(t) = 0. \tag{12}$$

The introduction of U allows us to state the following results:

Theorem 3.9. *If $\beta(\lambda) < 1/\sqrt{d}$, then*

$$\Phi(\rho) = \sup_{0 < t < \infty} [\rho d U(t)]^{\frac{1}{t}} \text{ for } \rho > \frac{1}{\sqrt{d}}$$

and

$$\Phi(\beta(\lambda)) = \Phi\left(\frac{1}{\beta(\lambda)d}\right) = 1 \tag{13}$$

Corollary 3.1. (a) $\beta(\lambda)$ is continuous on $[0, \lambda_l]$.

(b) $\beta(\lambda_g) = \frac{1}{d}$.

(c) $\beta(\lambda) > \frac{1}{d}$ for $\lambda > \lambda_g$.

We are finally ready to state a theorem that will allow us to prove that if the process does not survive strongly, then $\Phi(1/\sqrt{d}) \leq 1$.

Theorem 3.10. *The function $\beta(\lambda)$ is strictly increasing on $[0, \lambda_l]$*

Using the previous result, theorem 3.7, theorem 3.10, and proposition 3.6, we now prove that if the process does not survive strongly, then $\Phi(1/\sqrt{d}) \leq 1$.

Let us take $\lambda < \lambda_l$. By theorems 3.7(a) and 3.10, $\beta < 1/\sqrt{d}$. By theorem 3.7(b), $\Phi(1/\sqrt{d}) < 1$. Finally, by proposition 3.6(b), we can let $\lambda \rightarrow \lambda_l$ to conclude that $\Phi(1/\sqrt{d}) \leq 1$. To see this, passing to the limit in equation (13) gives $\Phi(1/\sqrt{d}) = 1$ at $\lambda = \lambda_l$.

To sum up, up to now in this paragraph we have seen that:

- There exists an intermediate phase for the contact process on \mathbb{T}^d ,
- (i) The contact process on \mathbb{T}^d survives if and only if $\Phi > 1$.

- (ii) The contact process on \mathbb{T}^d does not survive strongly if and only if $\Phi(1/\sqrt{d}) < 1$.

As for the case $V = \mathbb{Z}^d$, where we have seen in theorem 3.2 that ξ_t^η converges in distribution to an expression depending on $\bar{\nu}$, α_η , and δ_0 , we have a similar result for $V = \mathbb{T}^d$, which is stated in the next theorem.

Theorem 3.11. (complete convergence theorem) *If $\lambda > \lambda_l$, then*

$$\xi_t^A \xrightarrow{d} \alpha_A(\lambda)\nu + [1 - \alpha_A(\lambda)]\delta_0 \text{ as } t \rightarrow \infty$$

for any initial configuration $A \subseteq \mathbb{T}^d$, where: $\bar{\nu}$ is the upper invariant measure of the process, and α_A the survival probability of the process started at A .

3.4 Extinction time of the contact process on \mathbb{T}_l^d

[6] Here we want to state the main result for the limiting behaviour of the extinction time of the contact process on the d -ary tree of length l , hence we premise the following definition.

Definition 3.14. *The d -ary tree of height l \mathbb{T}_l^d is a tree with a distinguished vertex o , called *root*, that has degree d , all vertices at distance between 1 and $l - 1$ from o with degree $d + 1$, and all vertices at distance l from o with degree 1*

In the next theorem we state the main result about the limiting behaviour of the extinction time $\tau_{\mathbb{T}_l^d}$: in particular we establish a phase transition with critical value $\lambda_l(\mathbb{T}^d)$, i.e. the greatest λ for which the contact process survives locally on the infinite $d + 1$ -tree.

Theorem 3.12. *(a) $\forall 0 < \lambda < \lambda_2(\mathbb{T}^d)$ there exists $C > 0$ such that, as $l \rightarrow \infty$,*

$$\frac{\tau_{\mathbb{T}_l^d}}{\log |\mathbb{T}_l^d|} \xrightarrow{\mathbb{P}} C$$

(b) $\forall \lambda > \lambda_2(\mathbb{T}^d)$ there exists $K > 0$ such that, as $l \rightarrow \infty$,

$$\frac{\log \tau_{\mathbb{T}_l^d}}{|\mathbb{T}_l^d|} \xrightarrow{\mathbb{P}} K.$$

Moreover,

$$\frac{\tau_{\mathbb{T}_l^d}}{\mathbb{E}[\tau_{\mathbb{T}_l^d}]} \xrightarrow{d} \exp(1).$$

Remark. It can be proved that this theorem still holds if we replace \mathbb{T}_l^d with $\hat{\mathbb{T}}_l^d$, which is the subgraph of \mathbb{T}^d that we obtain if we select a vertex o of \mathbb{T}^d that we consider as a root and all the vertices that are at distance at most l from o . There is hence a difference between \mathbb{T}_l^d and $\hat{\mathbb{T}}_l^d$, since we can obtain \mathbb{T}_l^d removing one of the subtrees that ramifies from the root o of $\hat{\mathbb{T}}_l^d$.

4 Random graphs

[3]

Since we are interested in studying the contact process on random regular graphs, the aim of this section is to describe what a random graph is.

With the term *random graphs* we refer to probability distributions over graphs.

We will start from analysing the simplest random graph imaginable, the *Erdős Renyi random graph* $ER_n(p)$, in which we have n vertices that connect with probability p , then we will introduce the *configuration model*, for which we have a fixed distribution of the degrees, and just in the end we will talk about *random regular graphs*, since they can be seen as a generalisation of the configuration model.

4.1 The Erdős Renyi random graph

4.1.1 Definition

Definition 4.1. The *Erdős Renyi random graph* $ER_n(p)$ is a graph $G = (V, E)$ with vertex set $V = \{1, \dots, n\}$, and edge set $E = \{st = \{s, t\} | s, t \in V\}$, where the edge st will be present or occupied in the graph with probability p and absent or vacant otherwise, independently of all other edges. p will be called the *edge probability*.

Notation 4.1. We write $s \longleftrightarrow t$ if there exists a path of occupied edges connecting s and t (by convention we also assume that $v \longleftrightarrow v$).

Since we want to talk about the properties of the Erdős Renyi random graph, we first have to introduce the connected component $\mathcal{C}(v)$ of a vertex v and give a procedure to find $\mathcal{C}(v)$ in a given graph.

Definition 4.2. For $v \in V$ we call *cluster of v* or *connecting component containing v* the set

$$\mathcal{C}(v) = \{x \in V | v \longleftrightarrow x\}. \quad (14)$$

This means that the size of $\mathcal{C}(v)$, denoted by $|\mathcal{C}(v)|$ will denote the number of vertices connected to v by an occupied path.

Definition 4.3. The *largest connected component* \mathcal{C}_{max} is defined by

$$\mathcal{C}_{max} = \max_{v \in V} |\mathcal{C}(v)| \quad (15)$$

so \mathcal{C}_{max} identifies any cluster $\mathcal{C}(v)$ for which $|\mathcal{C}(v)|$ is maximal. Since this definition does not identify \mathcal{C}_{max} uniquely, we specify the vertex with the smallest label contained in \mathcal{C}_{max} .

We now give a procedure to find the cluster of v $\mathcal{C}(v)$ in a given graph G , which is not necessarily $ER_n(p)$.

During our process, vertices can have three statuses: active, neutral, and inactive. The status of vertices can change in the exploration of the connected component of v , in the following way:

- At time $t = 0$ only v is active and all other vertices are neutral, and we set $S_0 = 1$.
- At each time $t \geq 1$ we pick the active vertex with the smallest label w and we explore all the edges ww' , where w' is a neutral vertex. If $ww' \in E(G)$, we set w' active, otherwise it remains neutral.
- After searching the whole set of neutral vertices w' , we set w inactive and we put S_t equal to the new number of active vertices at time t .
- When $S_t = 0$ for the first time, which means that there are no more active vertices, the procedure ends and $\mathcal{C}(v)$ is the set of all inactive vertices, which implies that $|\mathcal{C}(v)| = t$.

From this process we see that S_t is the total number of active vertices at time t .

We now define the variable X_t as the number of vertices that become active during the exploration of the t -th active vertex w_t . As described above, after its exploration w_t becomes inactive.

Therefore, it holds that:

$$S_0 = 1, S_t = S_{t-1} + X_t - 1 \quad (16)$$

The above procedure can be applied to any graph G .

We now focus on the Erdős Renyi graph $ER_n(p)$, whose main property gives us the possibility to determine the distribution of X_t , conditionally on S_{t-1} .

Indeed, in the Erdős Renyi graph $ER_n(p)$ each edge can be occupied or vacant independently of the others with probability p . A consequence of this property is the fact that the distribution of X_t depends only on S_{t-1} : more precisely, each neutral vertex w' at time $t - 1$ has probability p to become active at time t . Since the edges ww' are explored precisely once, we also know the conditional probability that ww' is an edge in $ER_n(p)$: it has to be always equal to p .

After $t - 1$ explorations of active vertices w , we have that the inactive vertices are $t - 1$, while the active vertices are S_{t-1} by definition. This means that the remaining $n - (t - 1) - S_{t-1}$ vertices are neutral.

Hence, conditionally on S_{t-1} , it holds that:

$$X_t \sim Bin(n - (t - 1) - S_{t-1}, p) \quad (17)$$

since to count the number of vertices that from neutral become active during the exploration, we have to do a number of trials that is equal to the number of neutral vertices.

Let now T be the first time t for which $S_t = 0$:

$$T = \inf\{t | S_t = 0\} \tag{18}$$

then we have that $|\mathcal{C}(v)| = T$, so the size of the connected component containing v actually describes the exploration of a single connected component.

The cluster exploration only makes sense when $S_{t-1} \geq 1$, since there is no point in continuing it formally for $t > T$.

Let us also notice that the statuses of the edges of the Erdős Renyi graph are i.i.d. random variables taking the value 1 (which corresponds to the status "occupied") with probability p and the value 0 (denoting the status "vacant") with probability $1 - p$.

A convention that we are going to use is that the edge probability p is set equal to $\frac{\lambda}{n}$: this choice will allow us to establish a phase transition on $|\mathcal{C}_{\max}|$ varying λ .

Always by convention, with \mathbb{P}_λ we are going to denote the distribution of $ER_n(\frac{\lambda}{n})$.

4.1.2 Properties

We are now going to talk about the properties of the Erdős Renyi random graph.

Monotonicity in the edge probabilities

We will show that the Erdős Renyi random graph is monotonically increasing in p .

Let us start by coupling the graph: to do this we assign independent uniform random variables U_{st} for each edge st on $[0, 1]$:

$$\mathbb{P}(U_{st} \leq x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

. We say that the edge is occupied if $U_{st} \leq p$ and since the edges are occupied independently of the others, the probability that an edge is occupied is hence $\mathbb{P}(U_{st} \leq p) = p$.

Therefore, the resulting graph of occupied edges has the same distribution as $ER_n(p)$, and moreover the coupling shows that the number of occupied edges increases when p increases. Hence, formally:

Definition 4.4. We say that *an event is increasing* when, if the event occurs for a set of occupied edges, it remains to hold when we make some more edges occupied.

We say that a random variable X is increasing if the events $\{X \geq x\}$ are increasing $\forall x \in \mathbb{R}$.

Example 4.1. An example of an increasing event is $\{s \leftrightarrow t\}$, and examples of increasing random variables are $|\mathcal{C}(v)|$, and $|\mathcal{C}_{max}|$.

Stochastic domination of connected components

In the following two theorems we will show that branching processes with binomial offspring distribution give a stochastic upper bound and a lower bound on the cluster tail for each connected component of $ER_n(\frac{\lambda}{n})$.

We hence premise the definition of branching process and we explain what an offspring distribution means.

Definition 4.5. A *Branching process* is the simplest model for a population evolving in time, in which each individual independently gives birth to a number of children with the same distribution.

Notation 4.2. • We write $(p_i)_{i \geq 0}$ for the offspring distribution, where

$$p_i = \mathbb{P}(\text{individual has } i \text{ children})$$

- We denote by Z_n the number of individuals in the n -th generation, where $Z_0 = 1$ by convention.

Let us notice that hence the following relation holds:

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}$$

where $(X_{n,i})_{n,i \geq 1}$ is a doubly infinite array of i.i.d. random variables.

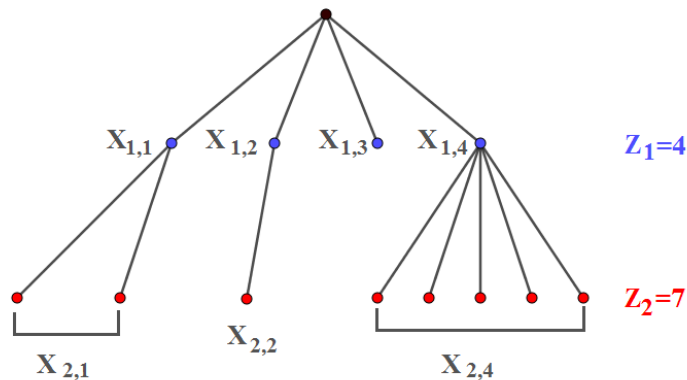


Figure 3: Branching processes: example of generation 2

Definition 4.6. The law $(p_i)_{i \geq 0}$ of $X_{n,i} \forall n, i \geq 0$ is called the *offspring distribution* of the branching process.

We are now ready to state the following result:

Theorem 4.1. *The following relation holds:*

$$|\mathcal{C}(1)| \preceq T^{\geq}$$

where \preceq means that, $\forall k \geq 1$,

$$\mathbb{P}_{np}(|\mathcal{C}(1)| \geq k) \leq \mathbb{P}_{n,p}(T^{\geq} \geq k)$$

and T^{\geq} is the total progeny of a binomial branching process with parameters n and p .

Here we recall that \mathbb{P}_λ denotes the law of $ER_n(\frac{\lambda}{n})$

Theorem 4.2. $\forall k \in V$, it holds that:

$$\mathbb{P}_{np}(|\mathcal{C}(1)| \geq k) \geq \mathbb{P}_{n-k,p}(T^{\geq} \geq k)$$

where T^{\geq} is the total progeny of a branching process with binomial distribution with $n - k$ trials and $p = \frac{\lambda}{n}$ as the success probability.

In the next sections we will investigate the behaviour of $|\mathcal{C}_{\max}|$ in subcritical and supercritical regime, i.e. when λ is less or greater than a critical value $\bar{\lambda}$ respectively.

4.1.3 Phase transition

Because of the stochastic domination of the connected components described in paragraph 4.1.2, it can be shown that the critical value $\bar{\lambda}$ for $|\mathcal{C}_{\max}|$ is equal to 1, as it is for the critical value of a branching process expected offspring. Hence we study the behaviour of $|\mathcal{C}_{\max}|$ for $\lambda < 1$ and $\lambda > 1$.

The subcritical regime

Let $I_\lambda = \lambda - 1 - \log(\lambda)$. Then it is possible to derive an upper and a lower bound on the largest subcritical component, as we state in the next results.

Theorem 4.3. *Let $\lambda < 1$ be fixed. Then, for every $a > \frac{1}{I_\lambda}$, there exists $\delta = \delta(a, \lambda) > 0$ such that*

$$\mathbb{P}_\lambda(|\mathcal{C}_{\max}| \geq a \log n) = O(n^{-\delta}).$$

Theorem 4.4. *Let $\lambda < 1$ be fixed. Then, for every $a < \frac{1}{I_\lambda}$, there exists $\delta = \delta(a, \lambda) > 0$ such that*

$$\mathbb{P}_\lambda(|\mathcal{C}_{\max}| \leq a \log n) = O(n^{-\delta}).$$

These two theorems together imply that $\frac{|\mathcal{C}_{\max}|}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{I_\lambda}$.

The supercritical regime

The main result for $\lambda > 1$ is a law of large numbers for the size of the maximal connected component. Before stating this theorem, we premise a definition:

Definition 4.7. The *survival probability* of a branching process, denoted by $\beta = 1 - \gamma$, is defined by

$$\beta := \mathbb{P}(Z_n > 0 \ \forall n \geq 0).$$

Notation 4.3. We denote by $\beta_\lambda = 1 - \gamma_\lambda$ the survival probability of a Poisson branching process with mean offspring λ .

We are now ready to give the next result:

Theorem 4.5. *Let $\lambda > 1$ be fixed. Then, for every $\nu \in (\frac{1}{2}, 1)$, there exists $\delta = \delta(\nu, \lambda) > 0$ such that*

$$\mathbb{P}(|\mathcal{C}_{\max}| - \beta_\lambda n| \geq n^\nu) = O(n^{-\delta}).$$

This theorem can be interpreted as a weak law of large numbers. Indeed, for $\lambda > 1$ a Poisson branching process has $\beta_\lambda > 0$, hence

$$\mathbb{P}(|\mathcal{C}_{\max}| - \beta_\lambda n| \geq n^\nu) = O(n^{-\delta}) = \mathbb{P}\left(\left|\frac{|\mathcal{C}_{\max}|}{n} - \beta_\lambda\right| \geq \frac{1}{n^{1-\nu}}\right) = O(n^{-\delta})$$

which is a weak law of large numbers.

Moreover, the previous theorem tells us that $|\mathcal{C}_{\max}| \sim n$ as $n \rightarrow \infty$, and hence in the limit all vertices are in the same connected component, which is called *giant component*.

Using this theorem, we can derive a discrete duality principle. Before giving this result, we premise a definition.

Definition 4.8. Let $\mu < 1 < \lambda$. Then μ and λ are said to be a *conjugate pair* if

$$\mu e^{-\mu} = \lambda e^{-\lambda}.$$

Theorem 4.6. *Let $\mu_\lambda < 1 < \lambda$ be a conjugate pair. Then the vector \mathcal{C} of connected components in the graph $ER_n(\lambda/n)$ with the giant component removed is close in law to the random graph $ER_m(\lambda/m)$, with $m = \lceil n\mu_\lambda \rceil$ that is the asymptotic number of vertices outside the giant component.*

In the previous theorem, by *close in law* we mean that

$$\lim_{n \rightarrow \infty} \mathbb{P}'_{n,\lambda}(E) = \lim_{m \rightarrow \infty} \mathbb{P}_{m,\mu_\lambda}(E)$$

where $P'_{n,\lambda}$ denotes the law the graph obtained by $ER_n(\lambda/n)$ removing its giant component, P_{m,μ_λ} denotes the law of $ER_m(\mu/m)$, and E an event which is determined by connected components.

The discrete duality principle can be understood by the law of large numbers, which implies that the giant component has size $n - m = \beta_\lambda n(1 + o(1))$.

We end this paragraph stating a Central Limit Theorem for giant components, which can be proved again using the law of large numbers.

Theorem 4.7. *Let $\lambda > 1$ be fixed. Then,*

$$\frac{|\mathcal{C}_{\max}| - \beta_\lambda n}{\sqrt{n}} \xrightarrow{d} Z$$

where Z is a normal random variable with mean 0 and variance

$$\sigma_\lambda^2 = \frac{\beta_\lambda(1 - \beta_\lambda)}{(1 - \lambda + \lambda\beta_\lambda)^2}.$$

4.2 Configuration model

Let us start by giving the following definition:

Definition 4.9. A graph is called *simple* when it has no self-loops and no multiple edges between any pair of vertices.

Let us hence consider a vertex set V such that $|V| = n$ and a sequence of degrees $\mathbf{d} = (d_i)_{i \in V}$.

The idea to obtain the so called configuration model, is to construct a simple graph with n vertices belonging to V , and prescribed degree sequence \mathbf{d} . Hence, we want that vertex j has degree d_j , where, without loss of generality, we can assume that $d_j \geq 1$ for all $j \in V$, since $d_j = 0$ means that vertex j is isolated.

Now, let us notice that it is not always possible to construct a simple graph with prescribed degree sequence since it may not exist. Therefore, we construct a *multigraph* instead, that is a graph possibly having self-loops and multiple edges between pairs of vertices.

One way of obtaining such a multigraph, is to see each vertex j with degree d_j as a labelled point from which d_j segments are attached. These segments have to be seen as half-edges, in the sense that if we pair a segment belonging to a vertex with another segment, the two segments create an edge of the multigraph. In other words our segments, that from now on we will call half-edges, can be seen as the right or left half of an edge.

Hence, we now describe how the half-edges are paired. Let $l_n = \sum_{j \in V} d_j$ and the half-edges be numbered arbitrarily from 1 to l_n . We start by randomly connecting the first half-edge with one of the remaining $l_n - 1$. In this

way we obtain one of the edges of the graph, so the two paired half-edges are removed from the list of half-edges to be paired.

We continue the procedure as above until all half-edges are connected, we call the resulting graph the *configuration model with degree sequence \mathbf{d}* , and we denote it by $\text{CM}_n(\mathbf{d})$.

Obviously this procedure does not prevent self-loops nor multiple edges, since the second half of the edge can be picked among half-edges of the same vertex or of an already chosen other vertex, as represented in figure 4.

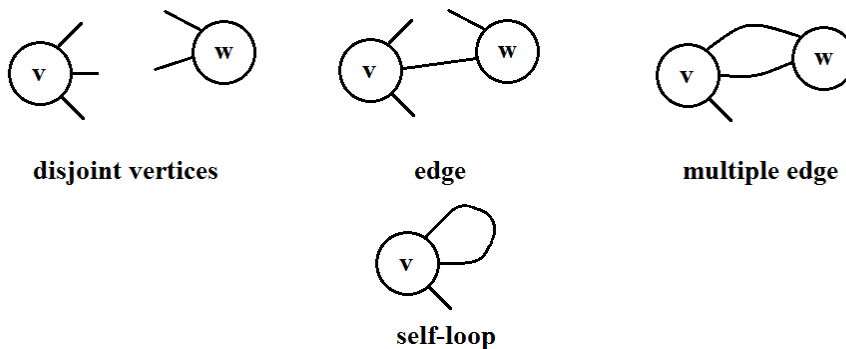


Figure 4: Configuration model: possible connections

The idea is hence that we can merge multiple edges and erase self-loops in a second moment (*Erased configuration model*) or we can repeat the procedure until there exists a simple graph (*Repeated configuration model*). However, we will see that under certain hypothesis self-loops and multiple edges are relatively scarce as $n \rightarrow \infty$.

Let us also notice that the total degree l_n has to be even, otherwise we cannot couple all the half-edges during the pairing procedure.

One may wonder about the influence of the arbitrary ordering of the half-edges in the pairing procedure: what we would like to have is that the process is *exchangeable*, i.e. such that the order does not matter for the distribution of the final outcome. Hence, to pair two half-edges we have to make sure that, conditionally on the previous paired half-edges, the new pairing occurs with equal probability. This allows us to pair half-edges in a random order.

The configuration model can also be constructed in another way, in terms of *uniform matchings*.

For this procedure, we draw a second graph, with vertex set $W = \{1, \dots, l_n\}$, which hence is such that the vertices in the new graph correspond to the half-edges of the random multigraph in the configuration model.

We pair the vertices of the new graph in a uniform way to produce a uniform matching, i.e. we pair vertex 1 with a uniform other vertex, then we pair the first not yet paired vertex to another uniform vertex which is

not yet paired. We stop the procedure when all the vertices are paired with another, unique vertex. The resulting graph is denoted by $\text{Conf}_n(\mathbf{d})$.

We can hence define $\text{Conf}_n(\mathbf{d}) := \{i\sigma(i) : i \in W\}$, where $\sigma(i)$ is the label of the vertex to which vertex $i \in W$ is paired.

We hence understand why we choose the name configuration model in the next definition:

Definition 4.10. The pairing of the vertices $1, \dots, l_n$ is called a *configuration*.

By construction, each configuration has the same probability.

We now focus on the vice versa: we want to draw the graph of the configuration model starting from a configuration.

We hence identify vertices $1, \dots, d_1$ in $\text{Conf}_n(\mathbf{d})$ to form vertex 1 in $CM_n(\mathbf{d})$, vertices $d_1 + 1, \dots, d_1 + d_2$ in $\text{Conf}_n(\mathbf{d})$ to form vertex 2 in $CM_n(\mathbf{d})$, and so on. In this way d_j vertices in $\text{Conf}_n(\mathbf{d})$ are identified precisely with vertex j in $CM_n(\mathbf{d})$, and the degree of vertex j in $CM_n(\mathbf{d})$ is precisely equal to d_j . Also in this construction self-loops and multiple edges are possible.

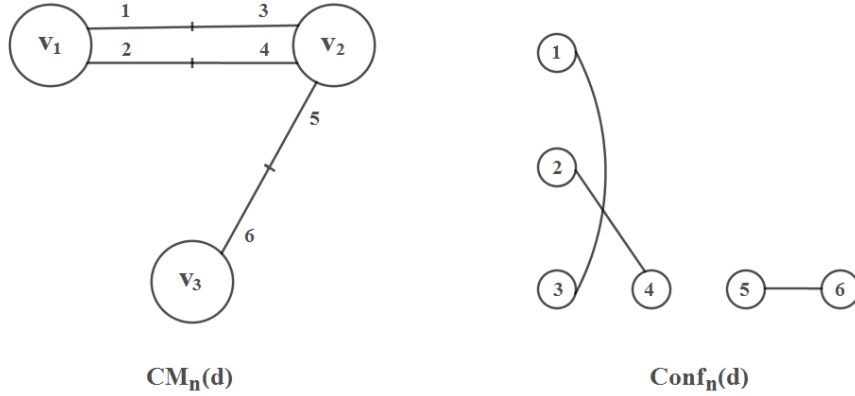


Figure 5: Example for $n = 3$ and $\mathbf{d} = (2, 3, 1)$

In this case it also holds that, as long as each configuration has probability $\frac{1}{(l_n-1)!!}$, the order in which the pairings take place is irrelevant. This is made formal in the next definition:

Definition 4.11. A *pairing scheme* is a sequence $(x_i)_{i \in A}$, where A is a set such that $|A| = \frac{l_n}{2}$, where x_i denotes the half-edge to be paired. We call a pairing scheme *adaptable* if the choice of x_m only depends on the previous pairings $(X_j, y_j)_{j=1}^{m-1}$, where y_j denotes the half-edge to which x_j is paired. An adaptable pairing scheme is said to be *uniform* when

$$\mathbb{P}(x_m \text{ is paired to } y_m | x_m, (x_j, y_j)_{j=1}^{m-1}) = \frac{1}{l_n - 2m + 1}, \quad (19)$$

for every $y_m \notin \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_{m-1}\}$.

When we use a uniform adaptable pairing scheme, the resulting graph has the same law as the configuration model, as stated in the next lemma:

Lemma 4.1. *For every uniform adaptable pairing, every configuration $\sigma \in \text{Conf}_n(\mathbf{d})$ has probability*

$$\frac{1}{(l_n - 1)!!}$$

. *Consequently, the resulting multigraph has the same distribution as the configuration model $CM_n(\mathbf{d})$.*

If we consider the multigraphs that can be obtained from a uniform adaptable pairing scheme, not all the multigraphs have the same probability (we say that they are not produced *uniformly at random*) since for each loop the number of corresponding pairings is divided by 2, while for each multiple edge of multiplicity j it is divided by $1/j!$.

For example, if we consider a multigraph with a triple edge between vertex v and w , the triple edge can be generated in $3!$ ways by pairing the three half-edges of v with the three half-edges of w .

This is formalized in the next result:

Proposition 4.1. the law of $CM_n(\mathbf{d})$ *Let $G = (x_{ij})_{i,j \in V}$ be a multigraph with vertex set V such that $|V| = n$, and such that*

$$d_i = x_{ii} + \sum_{j \in V} x_{ij}.$$

Then

$$\mathbb{P}(CM_n(\mathbf{d}) = G) = \frac{1}{(l_n - 1)!!} \frac{\prod_{i \in V} d_i!}{\prod_{i \in V} 2^{x_{ii}} \prod_{1 \leq i < j \leq n} x_{ij}!} \quad (20)$$

We end this paragraph introducing some notation and giving regularity conditions for the vertex degrees.

We denote the degree of a uniformly chosen vertex U in the vertex set V of cardinality n by $D_n = d_U$. The random variable D_n has hence distribution F_n given by

$$F_n(x) = \frac{1}{n} \sum_{j \in V} 1_{\{d_j \leq x\}}$$

which we call *empirical distribution of the degrees*, since it gives the number of vertices with degree less or equal than x over the total number of vertices.

From now on, we assume that the vertex degree satisfy the following regularity conditions.

Regularity conditions for vertex degrees

- (a) **Weak convergence of vertex weight** There exists a distribution function F such that

$$D_n \xrightarrow{d} D,$$

where D_n and D have distribution functions F_n and F respectively. Equivalently, for any x ,

$$\lim_{n \rightarrow \infty} F_n(x) = F(x).$$

Moreover, $F(0) = 0$, i.e. $\mathbb{P}(D \geq 1) = 1$.

- (b) **Convergence of average vertex degrees**

$$\lim_{n \rightarrow \infty} \mathbb{E}[D_n] = \mathbb{E}[D]$$

where D_n and D have the same distributions as in (a)

- (c) **Convergence of second moment vertex degrees**

$$\lim_{n \rightarrow \infty} \mathbb{E}[D_n^2] = \mathbb{E}[D^2]$$

where again D_n and D have the same distributions as in (a).

These regularity conditions can also be strengthened if our degrees d_i are random variables themselves, for example in the case in which they are the realizations of i.i.d. random variables.

If we are in this more general context, we can replace (a) with

$$\mathbb{P}_n(D_n = k) \xrightarrow{\mathbb{P}} \mathbb{P}(D = k)$$

where \mathbb{P}_n denotes the conditional probability given the random degrees, and (b) and (c) with

$$\mathbb{E}_n[D_n] \xrightarrow{\mathbb{P}} \mathbb{E}[D], \quad \mathbb{E}_n[D_n^2] \xrightarrow{\mathbb{P}} \mathbb{E}[D^2]$$

where \mathbb{E}_n denotes the expectation with respect to \mathbb{P}_n .

4.2.1 Erased configuration model

Now that we have constructed our multigraph, what is left is understanding how it can be made simple. We hence start by defining the erased configuration model.

Let the sequence \mathbf{d} of vertex degrees be fixed. We start from the multigraph $\text{CM}_n(\mathbf{d})$ and erase all self-loops. After this, we merge all multiple edges into single edges. We hence obtain a simple graph.

In order to state the main result concerning the degree sequence of this model, we premise some notations.

Notation 4.4. • We denote the degrees in the erased configuration model by $\mathbf{D}^{(er)} = (D_i^{(er)})_{i \in V}$, where $|V| = n$, so that

$$D_i^{(er)} = d_i - 2s_i - m_i,$$

where $(d_i)_{i \in V}$ are the degrees in $\text{CM}_n(\mathbf{d})$, $s_i = x_{ii}$ is the number of self-loops of vertex i in $\text{CM}_n(\mathbf{d})$, and

$$m_i = \sum_{j \neq i} (x_{ij} - 1) \mathbf{1}_{\{x_{ij} \geq 2\}}$$

is the number of multiple edges removed from i .

- We denote the empirical degree sequence $(p_k^{(n)})_{k \geq 1}$ in $\text{CM}_n(\mathbf{d})$ by

$$p_k^{(n)} = \mathbb{P}(D_n = k) = \frac{1}{n} \sum_{i \in V} \mathbf{1}_{\{d_i = k\}}$$

and the related degree sequence in the erased configuration model $(P_k^{(er)})_{k \geq 1}$ by

$$P_k^{(er)} = \mathbb{P}(D_n = k) = \frac{1}{n} \sum_{i \in V} \mathbf{1}_{\{D_i^{(er)} = k\}}.$$

Let us notice that $(p_k^{(n)})_{k \geq 1}$ is a deterministic sequence when \mathbf{d} is deterministic, while $(P_k^{(er)})_{k \geq 1}$ is a random sequence even if \mathbf{d} is deterministic, since $(D_i^{(er)})_{i \in V}$ is a random vector.

We are now ready to state the main result about this model:

Theorem 4.8. *For fixed degrees \mathbf{d} satisfying the regularity conditions, the degree sequence of the erased configuration model $(P_k^{(er)})_{k \geq 1}$ converges in probability to $(p_k)_{k \geq 1}$.*

4.2.2 Repeated configuration model

The idea of the repeated configuration model is that we repeat the procedure of generating $\text{CM}_n(\mathbf{d})$ until we obtain a simple graph..

Hence, we are interested in the probability that the graph produced in the configuration model is simple. This probability will be computed in a theorem which is a consequence of the next result, which we will state after some notation.

Notation 4.5. We denote the number of self-loops by S_n , and the number of multiple edges by M_n , where

$$S_n = \sum_{i \in V} s_i, M_n = \frac{1}{2} \sum_{i \in V} m_i.$$

Proposition 4.2. *Let $\mathbf{d} = (d_i)_{i \in V}$, $|V| = n$, satisfy the regularity conditions of paragraph 4.2. Then*

$$(S_n, M_n) \xrightarrow{d} (S, M)$$

where S and M are two independent Poisson random variables with means $\nu/2$ and $\nu^2/4$, where

$$\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}.$$

Now, if we are interested in the case in which the $CM_n(\mathbf{d})$ is simple, this corresponds to the fact that $S_n = M_n = 0$, since simple means that it has no self-loops nor multiple edges.

So, using the weak convergence stated in the proposition and the independence of S and M , we have

$$\mathbb{P}(S_n = M_n = 0) \longrightarrow \mathbb{P}(S = M = 0) = e^{-\mathbb{E}[S] - \mathbb{E}[M]} = e^{-\nu/2 - \nu^2/4}.$$

Hence, as a consequence of the proposition it holds that:

Theorem 4.9. *Let $\mathbf{d} = (d_i)_{i \in V}$, $|V| = n$, satisfy the regularity conditions of paragraph 4.2. Then the probability that $CM_n(\mathbf{d})$ is a simple graph is asymptotically equal to $e^{-\nu/2 - \nu^2/4}$, where $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D]$.*

We end this section giving the probability that $CM_n(\mathbf{d})$ is simple in the case in which $\mathbb{E}[D^2] = \infty$.

Proposition 4.3. *Let $\mathbf{d} = (d_i)_{i \in V}$, $|V| = n$, satisfy the regularity conditions of paragraph 4.2 and let $\mathbb{E}[D^2] = \infty$. Then $\mathbb{P}(CM_n(\mathbf{d}) \text{ is simple}) = o(1)$ when $\mathbb{E}[D^2] = \infty$.*

4.3 Random regular graphs

In this section, we will define a random d -regular graph and give its law. We will then focus on a way to generate it and on the so called *small cycles*.

4.3.1 Definition, law, and construction

Definition 4.12. A *d -regular graph* is a graph with the same degree d at each vertex.

Definition 4.13. A *random d -regular graph* is a random graph which is chosen uniformly at random among all d -regular graphs with vertex set $V = \{1, \dots, n\}$.

In order to give the law of a d -regular graph, we recall from paragraph 4.2 that it is the same as the law of the corresponding configuration model if we use a uniform adaptable pairing scheme.

Hence, starting from (20), taking $\mathbf{d} = (d_i)_{i \in V}$ such that $d_i \equiv d$, and $l_n = dn$, as it is in the regular case, since the regularity conditions of vertex degrees are obviously satisfied, we obtain:

$$\mathbb{P}(\text{CM}_n(\mathbf{d}) = G) = \frac{1}{(dn-1)!!} \frac{(d!)^n}{\prod_{i \in V} 2^{x_{ii}} \prod_{1 \leq i < j \leq n} x_{ij}!}. \quad (21)$$

where we recall that $(dn-1)!!$ is the number of all the possible pairings, and the necessary pairings to obtain a regular d -multigraph are

$$\frac{(d!)^n}{\prod_{i \in V} 2^{x_{ii}} \prod_{1 \leq i < j \leq n} x_{ij}!}$$

since for each loop the pairings have to be divided by 2 and for each multiple edge of multiplicity k by $k!$ not to count the same multigraph more than once.

From equation (21) we can see that, if G is simple, then

$$\mathbb{P}(\text{CM}_n(\mathbf{d}) = G) = \frac{(d!)^n}{(dn-1)!!} \quad (22)$$

hence in the simple case every graph has the same probability since it corresponds precisely to $(d!)^n$ pairings. This means that it would be possible to choose a simple d -regular graph uniformly at random by generating a uniform adaptable pairing scheme and rejecting it if the result has loops or multiple edges. In fact in this way we could make a uniform choice since we would pick a graph just among simple graphs, that have all the same probability.

In other words, a way to generate a random d -regular graph is to adopt the repeated configuration model.

4.3.2 Small cycles

An important part of the analysis of random regular graphs is the study of the number of their small cycles.

Notation 4.6. Given a (multi)graph G , we denote with $Z_k = Z_k(G)$ the number of cycles of length k in G .

Example 4.2. $Z_1 = S_n$ is the number of self-loops, $Z_2 = M_n$ is the number of multiple edges.

Let us notice that Z_k is a random variable that allows us to generalize proposition 4.2, in the d -regular case, for k also greater than 2.

Theorem 4.10. Let $\lambda_k = \frac{1}{2k}(d-1)^k$, let G be a random d -regular multigraph and let $l_k \sim \text{Poisson}(\lambda_k)$ be independent Poisson random variables, where $k = 1, 2, 3, \dots$. Then

$$Z_k(G) \xrightarrow{d} l_k \text{ as } n \rightarrow \infty$$

Directly from this result, conditioning on the event $Z_1 = Z_2 = 0$, we obtain:

Corollary 4.1. *Let λ_k and l_k be as in theorem 4.10 and G be a simple random d -regular graph. Then*

$$Z_k(G) \xrightarrow{d} l_k \text{ as } n \rightarrow \infty \forall k \geq 3.$$

Another corollary of theorem 4.10 is the following.

Corollary 4.2. *Let G be a d -regular random multigraph. Then, as $n \rightarrow \infty$,*

$$\mathbb{P}(G \text{ is simple}) \rightarrow e^{-(d^2-1)/4} > 0.$$

That can be shown using the fact that

$$\begin{aligned} \mathbb{P}(G \text{ is simple}) &= \mathbb{P}(Z_1 = Z_2 = 0) \rightarrow \mathbb{P}(l_1 = l_2 = 0) = \\ &= e^{-\lambda_1 - \lambda_2} = e^{-\frac{d-1}{2} - \frac{(d-1)^2}{4}} = e^{-(d^2-1)/4} \end{aligned}$$

since $l_k \sim \text{Poisson}(\lambda_k)$.

Here we underline that indeed this result is a particular case of the simplicity probability computed for the repeated configuration model: indeed $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D]$ in the d -regular case is $d(d-1)/d$, hence substituting this value in the simplicity probability computed in theorem 4.9 we obtain $e^{-(d^2-1)/4}$.

We are now also ready to compute the asymptotic number of d -regular graphs on n vertices.

Corollary 4.3. *The number of d -regular graphs on n vertices as $n \rightarrow \infty$ is:*

$$e^{-(d^2-1)/4} \frac{(dn-1)!!}{(d!)^n} (1 + o(1))$$

That can be shown starting from equation (22), since, taking G simple:

$$\begin{aligned} \mathbb{P}(\text{CM}_n(d) = G) &= \frac{(d!)^n}{(dn-1)!!} \Rightarrow \\ \Rightarrow \mathbb{P}(\text{CM}_n(d) \text{ is simple}) &= \frac{(d!)^n}{(dn-1)!!} |\{\text{simple } d\text{-regular graphs}\}| \end{aligned}$$

and hence

$$\begin{aligned} |\{\text{simple } d\text{-regular graphs}\}| &= \frac{\mathbb{P}(\text{CM}_n(d) \text{ is simple})(dn-1)!!}{(d!)^n} = \\ &= \frac{e^{-(d^2-1)/4}(dn-1)!!}{(d!)^n} (1 + o(1)). \end{aligned}$$

Now we see that an asymptotic property that holds for the multigraph holds also for the corresponding simple graph.

Theorem 4.11. *Any property that holds asymptotically almost surely for the multigraph $CM_n(\mathbf{d})$, holds asymptotically almost surely for the corresponding simple graph G too.*

Proof. Let \mathcal{P} be a property that holds asymptotically almost surely for $CM_n(\mathbf{d})$. Then

$$\begin{aligned} \mathbb{P}(G \text{ does not have } \mathcal{P}) &= \mathbb{P}(CM_n(\mathbf{d}) \text{ does not have } \mathcal{P} | CM_n(\mathbf{d}) \text{ is simple}) = \\ &= \frac{\mathbb{P}(CM_n(\mathbf{d}) \text{ does not have } \mathcal{P} \text{ and is simple})}{\mathbb{P}(CM_n(\mathbf{d}) \text{ is simple})} \leq \\ &\leq \frac{\mathbb{P}(CM_n(\mathbf{d}) \text{ does not have } \mathcal{P})}{\mathbb{P}(CM_n(\mathbf{d}) \text{ is simple})} = \frac{\mathbb{P}(CM_n(\mathbf{d}) \text{ does not have } \mathcal{P})}{e^{-(d^2-1)/4}} \rightarrow 0 \end{aligned}$$

where in the last equality we have used corollary 4.2. □

The converse does not hold as the example of containing a loop shows.

5 Contact process on random graphs

[5] Here we want to study and model the spread of an illness in a population. In particular, we are interested in the limiting behaviour of the infection. To study it, we consider the contact process with rate λ , which represents the rate of spreading the infection to a neighbour, on a random $d + 1$ -regular graph with n vertices G_n , and we analyze what happens to the extinction time τ_{G_n} of the process as $n \rightarrow \infty$.

Let us notice that this will imply a double randomness, due to the fact that we are on a random graph G_n , and to the contact process.

We will see that there is a phase transition, i.e. a change in the behaviour of the extinction time, depending on λ . More precisely, if $\lambda < \lambda_g(\mathbb{T}^d)$, where $\lambda_g(\mathbb{T}^d)$ is the lower critical value for the contact process on the infinite $(d+1)$ -regular tree, τ_{G_n} will grow logarithmically with n , while if $\lambda > \lambda_g(\mathbb{T}^d)$ it will grow exponentially with n .

We will then generalize to the contact process on a graph G_n with degrees bounded by $d + 1$, as it is more likely to have this situation in reality (the individuals of a population can be infected by a number of neighbours that is finite, but not necessarily constant as it is in the regular case). In this situation we will see that there is again a phase transition depending on λ and that for $\lambda < \lambda_g(\mathbb{T}^d)$ the corresponding contact process dies out.

5.1 Main result

To investigate the limiting behaviour of the extinction time on random regular graphs, we are going to fix λ and the parameters n and $d + 1$ of the graph, that are the number of vertices and the degree of each vertex respectively.

Then, we let $n \rightarrow \infty$ and we obtain a sequence of graphs G_n to which the random variable τ_{G_n} will be associated. We denote with \mathbb{P}_n the law of G_n , and with $\mathbb{P}_{n,\lambda}$ the probability measure under which both G_n and the contact process on G_n are defined.

The following theorem is our main result about the phase transition in the limiting behaviour of the extinction time of the contact process on random regular graphs, and it will be proved in the next section.

Theorem 5.1. (a) $\forall 0 < \lambda < \lambda_g(\mathbb{T}^d)$ there exists a constant $C < \infty$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{n,\lambda}(\tau_{G_n} < C \log n) = 1$$

(b) $\forall \lambda > \lambda_g(\mathbb{T}^d)$ there exists a constant $K > 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{n,\lambda}(\tau_{G_n} > e^{Kn}) = 1$$

Now, since we will prove that $\mathbb{P}(G_n \text{ is isomorphic to } \hat{\mathbb{T}}_R^d) \rightarrow 1$ as $n \rightarrow \infty$ (see the remark after proposition 5.1), the above result may seem to contrast

theorem 3.12, that holds also for $\hat{\mathbb{T}}_l^d$ as we have seen in the remark, for which the critical λ is $\lambda_l(\mathbb{T}^d)$ instead of $\lambda_g(\mathbb{T}^d)$.

This apparent contradiction is solved by the fact that in both cases the critical λ is actually $\lambda_g(\lim_{l \rightarrow \infty} X(l))$ where $X(l) = \mathbb{T}_l^d$ in theorem 3.12, while $X(l) = \hat{\mathbb{T}}_l^d$ in theorem 5.1. Indeed, $\lim_{l \rightarrow \infty} \hat{\mathbb{T}}_l^d = \mathbb{T}^d$ while $\lim_{l \rightarrow \infty} \mathbb{T}_l^d = \mathbb{C}\mathbb{T}^d$, and $\lambda_g(\mathbb{C}\mathbb{T}^d) = \lambda_l(\mathbb{T}^d)$, where $\mathbb{C}\mathbb{T}^d$ is the canopy tree (for a proof and a definition of the canopy tree, see [4] and [5]).

However, this observation underlines the difference between \mathbb{T}_l^d and $\hat{\mathbb{T}}_l^d$, i.e. the fact that \mathbb{T}_l^d and $\hat{\mathbb{T}}_l^d$ do not look alike locally and hence converge to different limits.

Notation 5.1. From now on we fix $d \geq 2$ and omit d from \mathbb{T}^d and \mathbb{T}_l^d , hence writing \mathbb{T} and \mathbb{T}_l respectively. The root of \mathbb{T}_l will be denoted by o .

5.2 Proof

In this section we will prove part (a) and (b) of theorem 5.1 on the subcritical and critical regime respectively.

5.2.1 Proof of (b): supercritical regime

To prove theorem 5.1 part (b) we will use theorem 5.2 and the two lemmas 5.2 and 5.3.

We will hence proceed as follows:

- we prove lemma 5.1
- we prove proposition 5.1 and corollary 5.1 using 5.1
- we prove theorem 5.2 using corollary 5.1
- we prove lemma 5.2
- we prove lemma 5.3 using lemma 5.1
- we prove theorem 5.1 part (b) using theorem 5.2.

Before giving these statements and their proofs, we premise some definitions.

Definition 5.1. A *rooted graph* is a pair ρ, G where G is a graph and $\rho \in G$. Given two rooted graphs (ρ, G) and (ρ', G') , with $G = (V, E)$ and $G' = (V', E')$, we say that $f : V \rightarrow V'$ is an *embedding* of (ρ, G) if

1. $f(\rho) = \rho'$,
2. f is injective,
3. for every $u, v \in V$ the number of edges in G containing u and v is less than or equal to the number of edges of G' containing $f(u)$ and $f(v)$.

where 1. means that (ρ, G) and (ρ', G') have the same root, 2. that distinguished vertices of (ρ, G) remain distinguished also in (ρ', G') , and 3. means that the neighbours of vertices in (ρ, G) are equal or less than the neighbours of (ρ', G') .

Definition 5.2. We say that (ρ', G') *embeds* (ρ, G) if there exists an embedding of (ρ, G) into ρ', G' , and that (ρ, G) and ρ', G' are isomorphic if each embeds the other.

Definition 5.3. We say that a set of vertices $W \subseteq V_n$ is *l-regenerative* if there exists a family $(G'_w)_{w \in W}$ of subgraphs of G_n that are pairwise disjoint and such that for every $w \in W$ the following two conditions hold:

- G'_w contains w
- there exists $x \in G'_w$ such that the distance in G'_w between x and w is 4 and (x, G'_w) embeds (o, \mathbb{T}_l)

Definition 5.4. A set of vertices $W \subseteq V$ is said to be *well separated* if:

1. for every $w \in W$ the 3-neighbourhood of w is cycle-free, where with r -neighbourhood of a vertex $v \in V$ we mean a set of vertices x whose distance to v is at most r ,
2. for every two distinct vertices $v, w \in W$ the 3-neighbourhoods of v and w are disjoint.

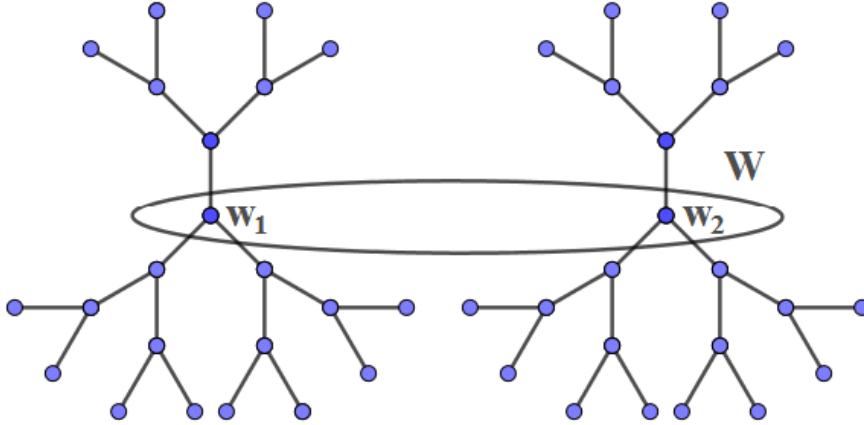


Figure 6: Example of W well-separated

We are now ready to start the proof of theorem 5.1 part (b).

Lemma 5.1. *For every $\lambda > \lambda_g(\mathbb{T})$ there exist $R, p_0 > 0$, and $\alpha_0 > 1$ such that, for l large enough,*

$$\mathbb{P}_{\mathbb{T}_l, \lambda}[|\eta_{Rl}^o| \geq (\alpha_0)^l] \geq p_0.$$

Proof. Let \mathbb{T}_∞ be the infinite, rooted, d -ary tree, i.e. the infinite rooted tree in which the root o has degree d and all the other vertices have degree $d+1$. In [5] it is shown that, for every $\lambda > \lambda_g(\mathbb{T})$, there exists $U > 0$, $\gamma > 1$, and $p > 0$ such that, for $l \in \mathbb{N}$ large enough,

$$\mathbb{P}_{\mathbb{T}_\infty, \lambda}[|\xi_{Ul}^o| \geq \gamma^l] \geq p. \quad (23)$$

Using equation (12), we can also show that there exists $S > 0$ such that, for all $t > 0$,

$$\begin{aligned} & \mathbb{P}_{\mathbb{T}_\infty, \lambda}[\xi_s^o(y) = 0 \text{ for all } s \leq t \text{ and } y \text{ with } \text{dist}(o, y) \geq St] \geq \\ & \geq \mathbb{P}_{\mathbb{T}, \lambda}[\xi_s^o(y) = 0 \text{ for all } s \leq t \text{ and } y \text{ with } \text{dist}(o, y) \geq St] > 1 - \frac{p}{2}. \end{aligned} \quad (24)$$

Combining equations (23) and (24), and setting $R = S^{-1}$, and $\alpha_0 = \gamma^{R/U} > 1$, we get

$$\begin{aligned} \frac{p}{2} & \leq \mathbb{P}_{\mathbb{T}_{infly}, \lambda}[|\xi_{Rl}^o| \geq \alpha_0^l, \xi_s^o(y) = 0 \text{ for all } s \leq Rl \text{ and } y \text{ with } \text{dist}(o, y) \geq l] \leq \\ & \leq \mathbb{P}_{\mathbb{T}_l, \lambda}[|\xi_{Rl}^o| \geq \alpha_0^l]. \end{aligned}$$

□

Proposition 5.1. *For every $l \in \mathbb{N}$ there exists $c > 0$ such that for every $\epsilon > 0$ sufficiently small (depending on l) and every n large enough, if E is the event $E =$ "the set $\{1, \dots, \epsilon n\}$ is well separated but has no l -regenerative subsets of size $\epsilon n/5$ ", then:*

$$\mathbb{P}[E] \leq (c\epsilon^{6/5})^{\epsilon n}.$$

Proof. In what follows, we will pretend that certain quantities, such as ϵn , are integers. To be formal we should take the proper integer part of such quantities adding or subtracting 1 if needed, but this would make the notation too heavy without changing the proof in any relevant way.

In this proof we are going to give a procedure to generate a regular random graph starting from a set of vertices with degree $d+1$. Before doing this, we premise some notation. By *semi-graph* $g := (V, \mathcal{E}, \mathcal{H})$ we mean a triple consisting of a set of vertices V , a set of edges between points of V and a set \mathcal{H} of half-edges, each half-edge being attached to some vertex in V . Given two half-edges h and h' , we write $h + h'$ to denote the graph obtained by gluing together h and h' (we will assume that $(d+1)n$ is even), and by

distance in the semigraph g we mean simply the distance in the graph (V_n, \mathcal{E}) . The initial configuration will be $\mathcal{E} = \emptyset$, since at the beginning we imagine to have just $|V|$ vertices with $d + 1$ half-edges attached, without formed edges. To construct a random $(d + 1)$ -regular graph with distribution \mathbb{P}_n we will use the following recursive procedure:

- we take an arbitrary half-edge h in \mathcal{H} (we will call it the *elected* half-edge),
- we choose a half-edge $h' \in \mathcal{H} \setminus \{h\}$ uniformly at random,
- we add the edge $h + h'$ to the set \mathcal{E} and remove h, h' from \mathcal{H}
- we repeat the previous steps until the set \mathcal{H} of half-edges is empty.

Notation 5.2. In a more symbolic form, since each step we replace \mathcal{E} with $\mathcal{E} \cup \{h + h'\}$ and \mathcal{H} with $\mathcal{H} \setminus \{h, h'\}$, we write:

$$\mathcal{E} \leftarrow \mathcal{E} \cup \{h, h'\}, \mathcal{H} \leftarrow \mathcal{H} \setminus \{h, h'\}.$$

Now that notation 5.2 is clear, we want to apply the procedure specified above to prove the proposition.

We hence fix $l \in \mathbb{N}$, $\epsilon \in (0, 1)$, and $n \in \mathbb{N}$ arbitrarily. We take the semigraph $g = (V_n, \mathcal{E}, \mathcal{H})$ such that $\mathcal{E} = \emptyset$, and such that every site has exactly $d + 1$ half-edges. We let \mathcal{W} be the set of vertices $\{1, \dots, \epsilon n\}$. As long as there is a half-edge h attached to a vertex at distance 2 or less from \mathcal{W} , we choose this as the elected half-edge, then we pick another half edge $h' \in \mathcal{H} \setminus \{h\}$ uniformly at random and we do the operations written in notation 5.2. At the end of this procedure, we have obtained the 3-neighbourhood of every vertex of \mathcal{W} : these 3-neighbourhoods may be well-separated or not. If not, then we stop. If on the contrary they are well-separated, we continue the construction of the graph, with the aim of showing that \mathcal{W} will contain an l -regenerative subset with high probability for n large enough.

Let \mathcal{F} be the set of vertices still having $d + 1$ half-edges at this point. We call *fresh vertices* the elements of \mathcal{F} , while elements of \mathcal{W} will be called seeds.

Below we will give a procedure to continue electing half-edges, and thus continue the construction of the graph. At any point of the construction, we will say that a seed $\rho \in \mathcal{W}$ is *active* if there are at least 3 half edges attached to vertices at distance 3 from ρ . As of now, there are $(d + 1)d^3$ half-edges. On the contrary, a point is said to be *quiet* if it is not active. As of now, every seed is active. We let ρ be any active seed, that as of now means that $\rho \in \mathcal{W}$, and run the step described below.

The step Take the active seed ρ .

Let \bar{V} be the set of vertices at distance 3 or less from ρ and let $\bar{\mathcal{E}}$ be the set of vertices with both extremities in \bar{V} . Let $\bar{\mathcal{H}}$ be the set of half-edges attached to vertices of \bar{V} . By definition of active seed, $|\bar{\mathcal{H}}| \geq 3$.

✕) Let $v \in \bar{V}$ be at distance strictly less than $l + 4$ from ρ as measured in the graph $(\bar{V}, \bar{\mathcal{E}})$ and such that there is a half-edge $h \in \bar{\mathcal{H}}$ attached to it.

If such v does not exist, declare that the pass is a **success** and stop.

Otherwise, pick h' uniformly at random in $\bar{\mathcal{H}} \setminus \{h\}$ and let v' be the vertex to which it is attached. If $v' \notin \mathcal{F}$, then declare that a *collision* occurs. More precisely, if $v' \in \bar{V}$ say that a *short collision* occurs, otherwise that a *long collision* occurs.

- If it is the second time during the step that a collision occurs, then declare that the step is a **failure**, do the updates

$$\mathcal{E} \leftarrow \mathcal{E} \cup \{h + h'\}, \mathcal{H} \leftarrow \mathcal{H} \setminus \{h, h'\},$$

and then stop the pass.

- If it is the first time during the step that a collision occurs, do the updates

$$\mathcal{E} \leftarrow \mathcal{E} \cup \{h + h'\}, \mathcal{H} \leftarrow \mathcal{H} \setminus \{h, h'\}, \bar{\mathcal{H}} \leftarrow \bar{\mathcal{H}} \setminus \{h, h'\}$$

where more precisely we subtract h' from $\bar{\mathcal{H}}$ just in case of a short collision, since for a long collision h' is not an element of $\bar{\mathcal{H}}$ at this point, and then go back to ✕).

- If a collision does not occur, i.e. $v' \in \mathcal{F}$, then do the updates

$$\mathcal{E} \leftarrow \mathcal{E} \cup \{h + h'\}, \mathcal{H} \leftarrow \mathcal{H} \setminus \{h, h'\}, \bar{\mathcal{H}} \leftarrow \bar{\mathcal{H}} \setminus h,$$

$$\bar{V} \leftarrow \bar{V} \cup \{v'\}, \mathcal{F} \leftarrow \mathcal{F} \setminus \{v'\}, \bar{\mathcal{E}} \leftarrow \bar{\mathcal{E}} \cup \{h + h'\},$$

and finally add the remaining half-edges of v' to $\bar{\mathcal{H}}$. Then go back to ✕).

When we iterate this step **we always start with an active seed that has not been the starting point of any previous step**, until every seed is quiet or has already been used as the starting point of a step.

Observation 5.1. Let us notice that in the case of a failure, we do not update $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$, and in the case of a first collision we do not update $\bar{\mathbb{E}}$. In this way, the graphs $G'_\rho(\bar{V}, \bar{\mathcal{E}})$ obtained $\forall \rho \in \mathcal{W}$ will continue to be disjoint and cycle-free if a step is a success. In other words, at the end of our procedure a success will mean that our seeds will have become either $l+4$ cycle-free neighbourhoods or $l+4$ cycle-free neighbourhoods with some missing edges, due to collisions occurred during the step.

More formally, if the pass is a success, we define $G'_\rho = (\bar{V}, \bar{\mathcal{E}})$. This subgraph of G_n is a tree satisfying:

1. ρ is the vertex set,
2. there exists $x \in \bar{V}$ such that $\text{dist}_{G'_\rho}(x, \rho) = 4$ and x, G'_ρ embeds (o, \mathbb{T}_l) .

To verify (b), given a vertex $x \in \bar{V}$, let \bar{V}_x be the set of vertices $y \in \bar{V}$ whose path to ρ contains x , let $\bar{\mathcal{E}}_x$ be the set of edges with extremities in \bar{V}_x , and let $G'_{\rho,x} \equiv (\bar{V}_x, \bar{\mathcal{E}}_x)$.

If during the step there were no collisions, then there are at least three distinct vertices x_1, x_2, x_3 (this is due to the fact that an active seed has at least three half-edges attached by definition) such that $\text{dist}_{G'_\rho}(x_i, \rho) = 4$, and (x_i, G'_{ρ,x_i}) is isomorphic to (o, \mathbb{T}_l) for each i , so (b) holds.

If one collision occurred in the step, then at most two of these subtrees is compromised by missing edges (one in case of a long collision, two in case of a short collision), so there still exists some x with $\text{dist}_{G'_\rho}(x, \rho) = 4$ and $(x, G'_{\rho,x})$ is isomorphic to (o, \mathbb{T}_l) . This is also the reason why we do not declare that the step is a failure if it is just the first time that a collision occurs.

Apart from satisfying the above properties, G'_ρ is disjoint from any other subgraph G'_ρ obtained from the previous passes.

Observation 5.2. During one step, the instruction \blackboxtimes is iterated at most

$$c_l := (d+1)d^3 + (d+1)d^4 + \dots + (d+1)d^{l+4}$$

times, since this is the maximum number of half-edges that are free to be connected to other half-edges. This also means that at most c_l vertices are removed from the set \mathcal{F} of fresh vertices. The number of fresh vertices at the beginning of the step, since the 3-neighbourhoods are already part of the seeds, and since we cannot run the step more than $|\mathcal{W}|$ times having just $|\mathcal{W}|$ seeds, is at least

$$n - |\mathcal{W}|(1 + (d+1) + (d+1)d + (d+1)d^2).$$

Therefore, the number of fresh vertices at any given time will be at least

$$\begin{aligned}
& n - |\mathcal{W}|(1 + (d+1) + (d+1)d + (d+1)d^2) - |\mathcal{W}|c_l = \\
& = n - |\mathcal{W}|(1 + (d+1) + (d+1)d + \dots + (d+1)d^{l+4}) = \\
& = n - |\mathcal{W}|c'_l \tag{25}
\end{aligned}$$

where $c'_l = 1 + (d+1) + (d+1)d + \dots + (d+1)d^{l+4}$.

Observation 5.3. During the step it may happen that a seed becomes inactive without having been used in a step, but just because of half-edges removed for collisions. We will say that such a seed is then *ruined*.

It is possible to count the minimum number of steps started from other seeds to ruin a seed ρ . Indeed, at the beginning ρ has $(d+1)d^3$ half-edges attached to vertices at distance 3 from ρ , and we know that it remains active as long as at least 3 of these half-edges are still present. Hence $(d+1)d^3 - 2$ is the number of half-edges that should be removed to the graph for the occurrence of collisions in order to ruin the seed. Moreover, at each step that starts from another seed ρ' , we can remove at most two of these half-edges (two in case of a short collisions, one in case of a long collision).

In conclusion, it takes at least

$$\frac{(d+1)d^3 - 2}{2}$$

steps started from other seeds to ruin the seed ρ .

Taking this into account, if we run the step t times, then at most

$$t + t \frac{2}{(d+1)d^3 - 3}$$

seeds become inactive: those that have been explored in each step, and those ruined by collisions.

Hence, starting from $|\mathcal{W}| = \epsilon n$ seeds, we can run the pass at least

$$\left(1 + \frac{2}{(d+1)d^3 - 2}\right)^{-1} |\mathcal{W}| \geq \frac{4}{5} \tag{26}$$

where the inequality holds since $\left(1 + \frac{2}{(d+1)d^3 - 2}\right)^{-1}$ is increasing and $d \geq 2$.

We can now estimate the probability that a pass is a failure.

We start with estimating the probability of having a collision, that is

$$\mathbb{P}(\text{a collision occurs}) \leq \frac{dc'_l|\mathcal{W}|}{(d+1)(n-|\mathcal{W}|c'_l)}$$

as a result of equation (25) and the definition of c'_l .

Now, since $|\mathcal{W}| = \epsilon n$

$$\begin{aligned} \frac{dc'_l|\mathcal{W}|}{(d+1)(n-|\mathcal{W}|c'_l)} &= \frac{dc'_l\epsilon n}{(d+1)(n-\epsilon nc'_l)} = \frac{c'_l\epsilon}{1-\epsilon c'_l-1/d-\epsilon c'_l/c} = \\ &= \frac{c'_l\epsilon}{(1+\frac{1}{d})(1-\epsilon c'_l)} \leq \frac{c'_l\epsilon}{1-\epsilon c'_l} \end{aligned}$$

hence it holds that:

$$\mathbb{P}(\text{a failure occurs}) = \mathbb{P}(\text{two collisions occur}) \leq \mathbb{P}[X \geq 2] \quad (27)$$

where $X \sim \text{Bin}(c_l, \frac{c'_l\epsilon}{1-c'_l\epsilon})$.

Since X has a binomial distribution of known parameters, we can easily estimate the probability that is greater or equal than 2, indeed

$$\begin{aligned} \mathbb{P}[X \geq 2] &= 1 - \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = \\ &= 1 - (1 - \frac{1-c'_l\epsilon}{1-c'_l\epsilon})^{c_l} - c_l(\frac{c'_l\epsilon}{1-c'_l\epsilon})(1 - \frac{c'_l\epsilon}{1-c'_l\epsilon})^{c_l-1} \leq \\ &\leq c(c_l, c'_l)\epsilon^2 = C\epsilon^2 \end{aligned} \quad (28)$$

where we have obtained the final inequality using a Taylor expansion.

Therefore, combining (27) and (28), we have

$$\mathbb{P}(\text{a failure occurs}) \leq C\epsilon^2.$$

From this result it follows that, if $Y \sim \text{Bin}(|\mathcal{W}|, C\epsilon^2) = (\epsilon n, C\epsilon^2)$ and $\delta = \frac{3}{5} - C\epsilon^2$, then

$$\mathbb{P}(\text{at least } \frac{3}{5} \text{ steps fail}) = \mathbb{P}(Y \geq \frac{3}{5}\epsilon n) \leq e^{-\epsilon n \psi_{C\epsilon^2}(\delta)} \quad (29)$$

where in the last inequality we have used lemma 5.2. In this case, after some computations, we get:

$$\psi_{C\epsilon^2}(\delta) = \frac{3}{5} \log\left(\frac{3}{5C\epsilon^2}\right) + \frac{2}{5} \log\left(\frac{2}{5(1-C\epsilon^2)}\right)$$

and hence

$$e^{\epsilon n \psi_{c\epsilon^2(\delta)}} \leq c\epsilon^{6/5|\mathcal{W}|} \quad (30)$$

, where $c = \frac{5^{3/5|\mathcal{W}|} 5^{2/5|\mathcal{W}|}}{2} C^{3/5|\mathcal{W}|}$.

Therefore, combining (29) and (30), we obtain:

$$\mathbb{P}(\text{at least } \frac{3}{5} \text{ steps fail}) \leq c\epsilon^{6/5|\mathcal{W}|}$$

which, taking the complementary event, becomes

$$\mathbb{P}(\text{at most } \frac{3}{5}\epsilon n \text{ steps fail}) \geq 1 - c\epsilon^{6/5|\mathcal{W}|}. \quad (31)$$

Now, from observation 5.3, and equation (31), we have respectively that:

- we run our procedure for at least $\frac{4}{5}$ steps,
- we have at most $\frac{3}{5}$ failed passes with probability larger than $1 - c\epsilon^{6/5|\mathcal{W}|}$.

From this it follows that

$$\mathbb{P}(\text{at most } \frac{\epsilon n}{5} \text{ steps have success}) \geq 1 - c\epsilon^{6/5|\mathcal{W}|}$$

which, taking the complementary is our thesis. \square

Remark. This proposition tells us that a random regular graph, for n large enough and ϵ small enough is locally $\hat{\mathbb{T}}_l^d$, so it is locally cycle-free. Indeed, since in the limit the probability that a small enough well-separated subset of V_n does not have l -regenerative subsets goes to 0, this means that with probability tending to 1 a small enough well-separated subset of V_n has l -regenerative subsets, hence it embeds (o, \mathbb{T}_l) .

Corollary 5.1. *For every $l \in \mathbb{N}$ and $\epsilon > 0$ sufficiently small, if E is the event $E =$ "from every well-separated set $W \subseteq V_n$ of size ϵn , one can extract an l -regenerative subset of size $\epsilon n/5$ ", then:*

$$\mathbb{P}[E] \rightarrow 1 \text{ as } n \rightarrow \infty$$

Proof. To show that $\mathbb{P}[E] \rightarrow 1$ as $n \rightarrow \infty$, we will compute the probability that taking all the well-separated sets of size ϵn in V_n one cannot extract any l -regenerative set, and we will show that this probability tends to 0 as $n \rightarrow \infty$.

We will need the following estimates:

$$\binom{m}{k} \leq \frac{m^k}{k!} \quad (32)$$

$$\log(m!) \geq m \log m - m \tag{33}$$

where (33) can be proved by induction.

Now, the total number of sets $W \subseteq V_n$ of size ϵn is obviously $\binom{n}{\epsilon n}$, which can also be seen as $e^{\log \binom{n}{\epsilon n}}$. By estimate (32) we have:

$$\binom{n}{\epsilon n} \leq \frac{n^{\epsilon n}}{(\epsilon n)!}$$

hence

$$\begin{aligned} \log \binom{n}{\epsilon n} &\leq \log \frac{n^{\epsilon n}}{(\epsilon n)!} = \epsilon n \log n - \log (\epsilon n)! \leq \\ &\leq \epsilon n \log n - \epsilon n \log \epsilon n + \epsilon n = \epsilon n \left(1 + \log \left(\frac{1}{\epsilon}\right)\right) \end{aligned}$$

where in the second inequality we have used estimate (33).

Taking $\epsilon > 0$ sufficiently small, and using also proposition 5.1, our probability is equal to

$$(c\epsilon^{\frac{6}{5}})^{\epsilon n} e^{\epsilon n \left(1 + \log \left(\frac{1}{\epsilon}\right)\right)}$$

which tends to 0 as $n \rightarrow \infty$. □

Theorem 5.2. *For any $l \in \mathbb{N}$ and $\epsilon > 0$ sufficiently small (depending on l), the following holds with \mathbb{P}_n -probability tending to 1 as $n \rightarrow \infty$. From every $W \subseteq V_n$ of cardinality at least ϵn , one can extract an l -regenerative subset of cardinality at least $\frac{\epsilon}{40d^6}n$.*

Proof. In this proof we want to use corollary 5.1, hence we will start from any subset $W \subseteq V_n$ such that $|W| \geq \epsilon n$ as in our hypothesis, but we will show that it is possible to find a subset \bar{W} of W which is well-separated, so that we can apply the corollary and obtain the thesis.

Let $W \subseteq V_n$ such that $|W| \geq \epsilon n$. We want to show that from W it is possible to extract \bar{W} well-separated, i.e. \bar{W} such that

1. for every $\bar{w} \in \bar{W}$ its 3-neighbourhood is cycle-free
2. for every $\bar{v}, \bar{w} \in \bar{W}$ the 3-neighbourhoods of \bar{v} and \bar{w} are disjoint.

In order to show that we can extract \bar{W} from W such that 1. holds, let us define the random variable $X_3 :=$ the number of cycles in G_n bounded above by 3. If $\mathbb{E}[X_3] < \infty$, then

$$\mathbb{P}(X_3 \leq \sqrt{n}) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

since by Markov inequality

$$\mathbb{P}(X_3 \geq \sqrt{n}) \leq \frac{\mathbb{E}[X_3]}{\sqrt{n}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, if we show that $\mathbb{E}[X_3] < \infty$, we have that for n large enough the event $E = "X_3 \geq \sqrt{n}"$ does not occur almost surely.

Now, let Y_k be the random variable that represents the number of cycles of length k , with $k \in \mathbb{N}$. Since for $x, y \in V_n$

$$\mathbb{P}(\text{there is a cycle between } x \text{ and } y) = \frac{r^2(r-1)^2}{(nr-1)!!}$$

and there are $\binom{n}{2}$ possible couples of vertices, we have that

$$\begin{aligned} \mathbb{E}[Y_2] &= \mathbb{E}\left[\sum_{x,y \in V_n} 1_{\text{there is a cycle between } x \text{ and } y}\right] = \\ &= \sum_{x,y \in V_n} \mathbb{P}(\text{there is a cycle between } x \text{ and } y) = \binom{n}{2} \frac{r^2(r-1)^2}{(nr-1)!!} \leq \\ &\quad \frac{n^2 r^2 (r-1)^2}{2 (nr-1)^2} \rightarrow c_2(r) \text{ for } n \text{ large enough.} \end{aligned}$$

Analogously, for n large enough, it holds that $\mathbb{E}[Y_3] \leq c_3(r)$ since

$$\mathbb{E}[Y_3] = \binom{n}{3} \frac{C(r)}{(nr-1)!!} \leq \frac{n^3}{3} \frac{C(r)}{(nr-1)^3} \rightarrow c_3(r).$$

Therefore,

$$\mathbb{E}[X_3] = \mathbb{E}[Y_2 + Y_3] \leq K_3(r) = c_2(r) + c_3(r) < \infty.$$

We hence take \bar{W} such that $|\bar{W}| > \sqrt{n}$: in this way we are sure that \bar{W} has cycle-free 3-neighbourhoods.

It remains to make sure that also 2. holds for \bar{W} .

Let us notice that $\forall v \in V_n$ the number of vertices that compose the 6-neighbourhood of v is

$$1 + (d+1) + (d+1)d + \dots + (d+1)d^5$$

since to enumerate the cardinality of the k -neighbourhood of v we start from v , then we can choose a neighbour among $(d+1)$ vertices and from that step on we can only choose d vertices since one is already connected to the neighbourhood.

It also holds that

$$1 + (d+1) + (d+1)d + \dots + (d+1)d^5 \leq 4d^6$$

since

$$1 + (d+1) + (d+1)d + \dots + (d+1)d^5 = 1 + d + 1 + 2^2 + d + \dots + d^6 + d^5 =$$

$$= 2 + 2d + 2d^2 + \dots + 2d^5 + d^6 = 2\left(\frac{d^6 - 1}{d - 1}\right) + d^6 < 2d^6 + d^6 = 3d^6 \leq 4d^6$$

Now, taking $U \subseteq V_n$ such that $|U| = m$, we can extract from U a subset \bar{U} of $\frac{m}{(4d^6)}$ vertices whose 3-neighbourhoods are pairwise disjoint. To see this, let $x_1 \in U$ be arbitrary and remove its 6-neighbourhood from U , obtaining U_1 . Obviously U_1 will have cardinality at least $m - 4d^6$. Then, take $x_2 \in U_1$ arbitrarily and remove its 6-neighbourhood from U_1 , obtaining U_2 , where similarly to before $|U_2| \geq m - 8d^6$. Continue this procedure until $x_{m/4d^6}$ is defined and let $\bar{U} := \{x_1, x_2, \dots, x_{m/4d^6}\}$. With this construction, the 3-neighbourhood of any point of U_{i+1} will be disjoint from the 3-neighbourhood of any point of U_i , hence the elements of \bar{U} will have pairwise disjoint 3-neighbourhoods.

We hence take $m = \epsilon n$, so that $\bar{U} \subseteq W$ has at least $\frac{\epsilon n}{4d^6}$ vertices whose 3-neighbourhoods are pairwise disjoint, i.e. 2. holds for \bar{U} .

Now, since we want \bar{W} to be well-separated, and above $\frac{\epsilon n}{4d^6}$ this may not be true, we take \bar{W} such that $|\bar{W}| < \epsilon n / (4d^6)$ and also $\bar{W} \subseteq \bar{U}$ so that also property 2. is satisfied.

Combining 1. and 2. we have that

$$\sqrt{n} < \bar{W} < \frac{\epsilon n}{4d^6},$$

so if we pick \bar{W} such that $|\bar{W}| = \frac{\epsilon n}{8d^6}$, and n large enough for which $\sqrt{n} < \frac{\epsilon n}{8d^6}$, we have that \bar{W} is well-separated.

By corollary 5.1 we have that from a well-separated set \bar{W} of cardinality $\frac{\epsilon n}{8d^6}$ we can extract an l -regenerative subset of cardinality at least $|\bar{W}|/5 = \frac{\epsilon n}{40d^6}$.

Therefore, starting from $W \subseteq V_n$ of cardinality at least ϵn we are able to extract an l -regenerative subset L of cardinality $\frac{\epsilon n}{40d^6}$, since we can first extract \bar{W} from W such that corollary 5.1 holds, and then extract L from \bar{W} using the corollary. \square

Lemma 5.2. *Let $X \sim \text{Bin}(m, p)$ be a binomial random variable with parameters $m \in \mathbb{N}$ and $p \in [0, 1]$. Then, for every $\delta \geq 0$, it holds that*

$$\mathbb{P}(X \geq (p + \delta)m) \leq e^{-m\psi_p(\delta)}$$

where

$$\begin{aligned} \psi_p(\delta) &= \sup_{\lambda} [\lambda(p + \delta) - \log(1 - p + pe^{\lambda})] = \\ &= (p + \delta) \log\left(\frac{p + \delta}{p}\right) + (1 - p - \delta) \log\left(\frac{1 - p - \delta}{1 - p}\right). \end{aligned}$$

Proof.

$$\mathbb{P}(X \geq m(p + \delta)) = \mathbb{P}(e^{\lambda X} \geq e^{\lambda m(p + \delta)}) \quad \forall \lambda,$$

and by Markov inequality

$$\mathbb{P}(e^{\lambda X} \geq e^{\lambda m(p+\delta)}) = \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda m(p+\delta)}} = (1-p+pe^\lambda)^m e^{-\lambda m(p+\delta)}$$

hence

$$\begin{aligned} \mathbb{P}(X \geq m(p+\delta)) &= e^{m \log(1-p+pe^\lambda)} e^{-\lambda m(p+\delta)} = e^{-m(\lambda(p+\delta) - \log(1-p+pe^\lambda))} \leq \\ &\leq e^{-m \sup_{\lambda>0} \lambda(p+\delta) - \log(1-p+pe^\lambda)} = e^{-m\psi_p(\delta)} \end{aligned}$$

□

Here we recall notation 3.2, which will be used in the next lemma.

Lemma 5.3. *For every $\lambda > \lambda_g(\mathbb{T})$ and $r > 0$, there exist $r, \sigma > 0$ and $\alpha > 1$ such that for every l large enough, the following holds. For any graph G with vertices x, y such that $\text{dist}(x, y) \leq r$ and (y, G) embeds (o, \mathbb{T}_l) , we have*

$$\mathbb{P}_{G,\lambda}[|\xi_{Rl}^x| \geq (\alpha)^l] > \sigma.$$

Proof. Let R, p_0 and α_0 be as in the previous lemma. Then:

$$\begin{aligned} &\mathbb{P}_{G,\lambda}[|\xi_{Rl+R}^x| \geq \alpha_o^l] \geq \\ &\geq \mathbb{P}_{G,\lambda}[\xi_R^x(y) = 1] \mathbb{P}_{G,\lambda}[|\xi_{Rl}^y| \geq \alpha_o^l] \geq \mathbb{P}_{G,\lambda}[\xi_r^x(y) = 1] p_0. \end{aligned}$$

In the first inequality we have used the fact that all the possible vertices connected by an infectious path to x after a time $Rl + R$ are more than the vertices connected by an infection path to y after a time Rl , since y is a *fixed* vertex connected by an infection path to x after a time R . In the second inequality we have used the embedding of (o, \mathbb{T}_l) in (y, G) and lemma 5.1.

Since $\mathbb{P}[\xi_R^x(y) = 1] > 0$, we have that

$$\mathbb{P}_{G,\lambda}[|\xi_{Rl+R}^x| \geq \alpha_o^l] > \sigma > 0. \quad (34)$$

We now rewrite equation (34) for $\bar{l} = l - 1$.

$$\mathbb{P}_{G,\lambda}[|\xi_{R\bar{l}+R}^x| \geq \alpha_o^{\bar{l}}] > \sigma > 0$$

$$\mathbb{P}_{G,\lambda}[|\xi_{R(l-1)+R}^x| \geq \alpha_o^{l-1}] > \sigma > 0$$

$$\mathbb{P}_{G,\lambda}[|\xi_{Rl}^x| \geq \alpha_o^{l-1}] > \sigma > 0.$$

Now, since

$$\alpha_o^{\frac{l-1}{l}} = \alpha_o^{1-1/l} \rightarrow \alpha_o,$$

choosing $\alpha \in (1, \alpha_0)$ and l large enough, it holds that

$$\alpha < \alpha_0^{\frac{l-1}{l}}, \text{ i.e. } \alpha_0^{l-1} > \alpha^l$$

We hence have that

$$\mathbb{P}_{G,\lambda}[|\xi_{Rl}^W| \geq \alpha^l] \geq \mathbb{P}_{G,\lambda}[|\xi_{Rl}^x| \geq \alpha_0^{l-1}] > \sigma > 0.$$

□

Proof of theorem 5.1 part (b): supercritical regime

Let us fix $\lambda > \lambda_g(\mathbb{T})$ and choose:

1. α, R, σ that correspond to λ and $r = 4$ as in lemma 5.3;
2. l large enough as required by lemma 5.3, and also such that $\alpha^l > \frac{80d^6}{\sigma}$;
3. ϵ corresponding to l as in theorem 5.2.

Let us assume that G_n satisfies the property of theorem 5.2, i.e.

every $W \subseteq V_n$ with $|W| \geq \epsilon n$ has an l -regenerative subset of cardinality $\frac{\epsilon}{40d^6}n$.

We will now prove that, if it holds that for some constant $c > 0$ which does not depend on n ,

$$\forall W \subseteq V_n \text{ with } |W| \geq \epsilon n, \mathbb{P}_{G_n}[|\xi_{Rl}^W| \geq \epsilon n] \geq 1 - e^{-cn}, \quad (35)$$

then this implies the statement of theorem 5.1. Eventually, we will prove equation (35).

Equation (35) means that there is a high probability to have at least ϵn vertices that are connected by an infection path to the elements of W after the time Rl . Hence, we can iterate our process, so that after n repetitions, it will hold that $\mathbb{P}_{G_n}[|\xi_{(Rl)^n}^W| \geq \epsilon n] \geq (1 - e^{-cn})^n$ and, since $(Rl)^n = e^{n \log Rl}$, setting $Rl \equiv c$, we obtain that

$$\mathbb{P}_{G_n}[|\xi_{e^{nc}}^W| \geq \epsilon n] \geq (1 - e^{-cn})^n.$$

Then, we underline the following inclusions between events:

$$\{\tau_{G_n} > e^{nc}\} \supseteq \{\tau_{G_n}^W > e^{nc}\} = \{|\xi_{e^{nc}}^W| > 0\} \supseteq \{|\xi_{e^{nc}}^W| > \epsilon n\}.$$

where for the first inclusion we recall that $\tau_{G_n} \equiv \tau_{G_n}^{G_n}$ and the only equality comes from the fact that an extinction time greater than e^{nc} means that after e^{nc} the contact process has not died out yet, therefore after e^{nc} the cardinality of the infected vertices is greater than 0.

We hence obtain

$$\mathbb{P}_{G_n}(\tau_{G_n} > e^{nc}) \geq \mathbb{P}_{G_n}(|\xi_{e^{nc}}^W| > \epsilon n) \geq (1 - e^{-cn})^n \rightarrow 1 \text{ as } n \rightarrow \infty$$

where we have computed the limit $\log(1 - e^{-cn})^n = n \log(1 - e^{-cn}) = n(-e^{-cn} + o(e^{-cn})) \rightarrow 0$ as $n \rightarrow \infty$ and used continuity of the function logarithm.

It remains to show (35).

Let us fix W , with $|W| \geq \epsilon n$, and extract from it an l -regenerative subset W' of cardinality $n' = \frac{\epsilon}{40d^6}n$, so that $W' = \{v_1, \dots, v_{n'}\}$ (this is possible thanks to theorem 5.2). By definition 5.3, there exist pairwise disjoint subgraphs $G'_{v_1}, \dots, G'_{v_{n'}}$ such that for $i = 1, \dots, n'$

- $v_i \in G'_{v_i}$
- there exists $x_i \in G'_{v_i}$ such that $\text{dist}(v_i, x_i) = 4$ and (x_i, G'_{v_i}) embeds (o, \mathbb{T}_l) .

Let hence define a new contact process $(\zeta_t^{v_i})_{t \geq 0}$ on G'_{v_i} for each i , where $(\zeta_t^{v_i})_{t \geq 0}$ and $(\zeta_t^{v_j})_{t \geq 0}$ are independent since G_{v_i} and G_{v_j} are disjoint.

Let E_i be the event $E_i := \{|\zeta_{Rl}^{v_i}| \geq \alpha^l\}$, $1 \leq i \leq n'$. Then, $\mathbb{P}(E_i) \geq \sigma > 0$ by lemma 5.3, and

$$\mathbb{P}_{G_n} \left[\sum_{i=1}^{n'} 1_{E_i} \geq \frac{\sigma}{2} n' \right] \geq 1 - e^{-cn} \quad (36)$$

by lemma 5.2 with $X = \sum_{i=1}^{n'} 1_{E_i}$.

Equation (36) obviously still holds if we multiply by α^l both members inside the probability, hence we have

$$\mathbb{P}_{G_n} \left[\alpha^l \sum_{i=1}^{n'} 1_{E_i} \geq \alpha^l \frac{\sigma}{2} n' \right] \geq 1 - e^{-cn} \quad (37)$$

Moreover, let us notice that $\xi_t^W \supseteq \xi_t^{W'} \supseteq \zeta_t^{W'} = \cup_{v \in W'} \zeta_t^v$ by definition, so it holds that

$$|\xi_{Rl}^W| \geq |\cup_{v \in W'} \zeta_t^v| \geq \alpha^l \sum_{i=1}^{n'} 1_{E_i} \quad (38)$$

where in the first inequality we have used that $\zeta_t^{v_i}$ and $\zeta_t^{v_j}$ are disjoint.

Therefore, we have that

$$\mathbb{P}_{G_n} [|\xi_{Rl}^W| \geq \epsilon n] \geq \mathbb{P}_{G_n} \left[\alpha^l \sum_{i=1}^{n'} 1_{E_i} \geq \epsilon n \right]. \quad (39)$$

To obtain our thesis from equation (37), we want $\alpha^l \frac{\sigma}{2} n' > \epsilon n$, so remembering that $n' = \frac{\epsilon n}{40d^6}$, we want

$$\alpha^l \frac{\sigma}{2} \frac{\epsilon n}{40d^6} > \epsilon n. \quad (40)$$

Since in 2. we have said that l was chosen such that $\alpha^l > \frac{80d^6}{\sigma}$, inequality (40) is satisfied, hence equation (37) implies that

$$\mathbb{P}_{G_n} \left[\alpha^l \sum_{i=1}^{n'} 1_{E_i} \geq \epsilon n \right] \geq 1 - e^{-cn} \quad (41)$$

is true a fortiori.

Therefore, combining (39) and (41), we obtain

$$\mathbb{P}_{G_n} [|\xi_{Rl}^W| \geq \epsilon n] \geq \mathbb{P}_{G_n} \left[\alpha^l \sum_{i=1}^{n'} 1_{E_i} \geq \epsilon n \right] \geq 1 - e^{-cn}$$

which is our thesis.

5.2.2 Proof of (a): subcritical regime

The proof of theorem 5.1(a) is a consequence proposition 5.2 and lemma 5.4.

Proposition 5.2. *For any $\lambda < \lambda_g(\mathbb{T})$, there exists $C > 0$ such that*

$$\lim_{n \rightarrow \infty} \sup_{A \subseteq \mathbb{T}: |A|=n} \mathbb{P}_{\mathbb{T}, \lambda} [\tau_{\mathbb{T}}^A > C \log n] = 0.$$

Proof. Let $\lambda < \lambda_g(\mathbb{T})$: by theorem 3.10 and corollary 3.1, this implies that $\beta(\lambda) < \frac{1}{d}$. Now, by theorem 3.9, this shows that $\Phi(\rho) = 1$ for some ρ . Hence, using proposition 3.8(a), we obtain $\Phi(1) < 1$ that, together with proposition 3.5(b), implies

$$\mathbb{E}[|\xi_t^o|] \leq C_0 e^{c_0 t}, \quad t \geq 0. \quad (42)$$

Now, since

$$|\xi_t^A| = |\cup_{x \in A} \xi_t^x| \leq \sum_{x \in A} |\xi_t^x|$$

where the last inequality is due to the fact that ξ_t^x are not necessarily disjoint, we have:

$$\begin{aligned} \mathbb{P}_{\mathbb{T}, \lambda} (|\xi_t^A| \neq 0) &\leq \mathbb{E}_{\mathbb{T}, \lambda} [|\xi_t^A|] = \mathbb{E}_{\mathbb{T}, \lambda} [|\cup_{x \in A} \xi_t^x|] \leq \mathbb{E}_{\mathbb{T}, \lambda} \left[\sum_{x \in A} |\xi_t^x| \right] \leq \\ &\leq \sum_{x \in A} C_0 e^{-c_0 t} = |A| C_0 e^{-c_0 t} \end{aligned} \quad (43)$$

where in the penultimate inequality we have used (42), and in the last (5.2.2).

We now set $C = 2/c_0$, so that equation 43 becomes:

$$\mathbb{P}_{\mathbb{T}, \lambda} (|\xi_t^A| \neq 0) \leq |A| C_0 e^{-\frac{2}{C} t}. \quad (44)$$

Now, noticing that

$$\{|\xi_t^A| \neq 0\} = \{\tau_{\mathbb{T}}^A > t\}$$

, (44) becomes

$$\mathbb{P}_{\mathbb{T},\lambda}(\tau_{\mathbb{T}}^A > t) \leq |A|C_0 e^{-\frac{2}{c}t}$$

hence taking $t = C \log n$ we get

$$\mathbb{P}_{\mathbb{T},\lambda}(\tau_{\mathbb{T}}^A > C \log n) \leq |A|C_0 e^{-\frac{2}{c}C \log n} = \frac{|A|C_0}{n^2}.$$

This implies that

$$\sup_{A \subseteq \mathbb{T}: |A|=n} \mathbb{P}_{\mathbb{T},\lambda}(\tau_{\mathbb{T}}^A > C \log n) \leq \frac{|A|C_0}{n^2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

holds a fortiori. \square

Lemma 5.4. *For any finite graph $G = (V, E)$ with degree bounded by $d + 1$, $A \subseteq V$ and $t > 0$,*

$$\mathbb{P}_{G,\lambda}[\tau_G^A > t] \leq \sup_{B \subseteq \mathbb{T}: |B|=|A|} \mathbb{P}_{\mathbb{T},\lambda}[\tau_{\mathbb{T}}^B > t]$$

For the proof of lemma 5.4 we refer to [5].

Proof of theorem 5.1(a): subcritical regime By lemma 5.4 with $G = G_n$, $A = G_n$, and $t = C \log n$, we have:

$$\mathbb{P}_{G_n,\lambda}[\tau_{G_n} > C \log n] \leq \sup_{B \subseteq \mathbb{T}: |B|=|G_n|} \mathbb{P}_{\mathbb{T},\lambda}[\tau_{\mathbb{T}}^B > C \log n]$$

that goes to 0 as $n \rightarrow \infty$ by proposition 5.2.

5.3 Generalisations

In these section we state some generalisations to graphs with bounded degree. For the proofs we refer to [5].

We first give a result that generalizes theorem 5.1 to connected graphs with vertex degree bounded by $d + 1$, in the sense that it establishes a phase transition varying λ also in this case, that is more general than the regular one.

Theorem 5.3. *Let $\mathcal{G}_{n,d}$ be the set of of the connected graphs with n vertices with degree bounded by $d + 1$. Then:*

(a) $\forall 0 < \lambda < \lambda_g(\mathbb{T}^d)$ there exists a constant $C < \infty$ such that

$$\lim_{n \rightarrow \infty} \inf_{G \in \mathcal{G}_{n,d}} \mathbb{P}_{G,\lambda}(\tau_G < C \log n) = 1$$

(b) $\forall \lambda > \lambda_c(\mathbb{Z})$ there exists a constant $K > 0$ such that

$$\lim_{n \rightarrow \infty} \inf_{G \in \mathcal{G}_{n,d}} \mathbb{P}_{G,\lambda}(\tau_G > e^{Kn}) = 1$$

The last result we give is the fact that the contact process on general graphs with vertex degrees bounded by $d + 1$ dies out in the subcritical regime.

Theorem 5.4. *If $\lambda \leq \lambda_g(\mathbb{T}^d)$, then the contact process of parameter λ on any graph with vertex degree limited by $d + 1$ dies out.*

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