UNIVERSITÀ DEGLI STUDI DI PADOVA

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Quantum fluctuations in atomic Josephson junctions:

the role of dimensionality

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Abstract

In this thesis we investigate the role of quantum fluctuations in atomic Josephson junctions in dimension $D = 1, 2, 3$. In particular, we study the impact of these fluctuations on two key quantities: the Josephson frequency and the critical strength of macroscopic quantum self-trapping.

Initially, we investigate the inter-atomic potential in the s-wave scattering approximation, exploring the relationship between contact and finite-range coupling constants with the s-wave scattering length and the effective range in dimensions $D = 1, 2, 3$. In the second chapter, we illustrate the mean-field behavior of Josephson junction systems and the calculation of key parameters, namely the Josephson frequency and the critical strength of macroscopic quantum self-trapping (MQST). After that, we present the derivation of quantum fluctuations correction to the grand potential of a bosonic system considering also a finite range term in the potential. This is done systematically for a system of dimension $D = 1, 2, 3$. Finally, in the last two chapters, we investigate the effects of quantum fluctuations on the Josephson frequency and the critical strength of macroscopic quantum self-trapping beyond the mean-field approximation, respectively with and without finite range correction to the potential.

Our main findings reveal that, when compared to the mean-field Josephson frequency, the Josephson frequency is higher in 2D and 3D, while it is lower in 1D. On the other hand, the MQST critical strength is higher in 1D and lower in 2D and 3D. These results highlight the crucial role of quantum fluctuations in determining the properties of atomic Josephson junctions and show that the behavior of these systems is different in different dimensions.

Contents

Introduction

The goal of this thesis is to find correction driven by gaussian fluctuation to the mean-field values of two key quantities of an atomic Josephson junction: the Josephson frequency and the macroscopic quantum self trapping (MQST) critical strength.

A Josephson junction is a device composed of a pair of superconductors or superfluids. Its behaviour was first theorized for superconductors by Brian Josephson in 1962 [1]. The main Josephson effects observed in this device are the DC Josephson effect and the AC Josephson effect. In the DC Josephson effect, a supercurrent passes through an insulator at zero voltage. In the AC Josephson effect, in addition to the usual DC current, there is an AC supercurrent whose frequency is proportional to the applied voltage. These effects were demonstrated the following year [2] and have been further studied since then. In recognition of his groundbreaking work, Brian Josephson was awarded the Nobel Prize in Physics in 1973. Nowadays the Josephson junction is employed in sophisticated technologies. For example, a superconducting quantum interference device (SQUID) is a magnetometer with a very high sensitivity and find primary applications in diverse fields including scientific research, medical diagnostics, and industrial settings [3]. Furthermore, Josephson junction are implented in the realization of qubits for quantum computers [4, 5] and to detect axions, a possible candidate for dark matter [6].

The achievement of Bose-Einstein condensation (BEC) with ultracold and dilute alkali-metal atoms in the 1990s [7, 8] marked a major milestone in the field of physics, resulting in Eric Cornell, Carl Wieman, and Wolfgang Ketterle, being awarded the 2001 Nobel Prize in Physics for their groundbreaking work. This breakthrough revitalized and significantly enhanced the interest in studying macroscopic quantum phenomena, leading to a renewed focus on various research areas. In particular, there has been renewed focus on studying the Josephson effect [9], where in this case the Josephson junction is made up of two superfluids. This case is interesting because it exhibits a new macroscopic phenomenon, which do not occur in the superconductive Josephson junction. The phenomenon known as macroscopic quantum self trapping (MQST) is characterized by a self-maintained population imbalance [10]. In 2004, MQST was experimentally detected in a single bosonic Josephson junction consisting of two Bose-Einstein condensate of ^{87}Rb atoms [11]. Later, in 2015, the same effect was empirically observed also for a Josephson junction made of fermionic superfluids in the BEC-BCS crossover with ^{6}Li atoms [12]. This observation extended the understanding of MQST to a different physical system, further supporting its generality and relevance in various contexts.

To describe the quantum behaviour of a Josephson junction usually one use the phase model [13], based upon the commutation rule between the number operator $\hat{N}$ and the phase operator $\hat{\phi}$ [14]. Since this commutation rule is approximately good for systems containing a large number of bosonic atoms or condensed electronic Cooper pairs, the phase model is used as a starting point for studying of beyond mean-field quantum effects. [15, 16, 17]. To give an example, finite size effects on an atomic bosonic Josephson junction have been studied using atomic coherent states [18]. Another possible approach involves path integral formalism, which is used to obtain an effective action depending only on the relative phase between the bosonic field of the two sites. The quantum correction to the Josephson frequency oscillation are then found through the quantum effective action [19].

Within this dissertation, the atomic bosonic Josephson junction is systematically investigated in dimensions $D = 1, 2, 3$. The main goal of this research is to analyse the influence of quantum fluctuations on two key phenomena observed in the system: Josephson oscillations and macroscopic quantum self-trapping. These phenomena are regulated by two important parameters: the Josephson frequency, which governs the oscillations between the two weakly coupled Bose-Einstein condensates, and the MQST critical strength, which determines the conditions for the occurrence of self-trapping at a macroscopic scale.

Since the bosonic atomic Josephson junction is a system involving interacting bosonic atoms, the first part of the study investigates the inter-atomic potential in dimensions $D = 1, 2, 3$. The inter-atomic potential can be approximated well as a sum of a contact-like potential and a finite range correction. The coupling costants of these terms are systematically calculated for each dimension $D = 1, 2, 3$ using two types of approximations, namely the s-wave approximation and the on-shell approximation [20]. Once the structure of the inter-atomic potential is understood, the mean-field model for a bosonic atomic Josephson junction is described. Specifically, the Josephson-Smerzi equations [10] are utilized to calculate the Josephson frequency and the MQST critical strength. These results are independent of the dimensionality of the system. The central chapter cover the beyond mean-field calculations to retrieve the correction to the Grand Potential driven by Gaussian fluctuation. This is done methodically in $D = 1, 2, 3$ by means regularization [21]. From the zero temperature grand potential involving Gaussian fluctuation correction are then found the number density and the energy density. Note that, neglecting finite-range effects, one retrieve the Lee-Huang-Yang correction for $D = 3$ [22], the Schick-Popov correction for $D = 2$ [23, 24], and the next-to-leading term of the Lieb-Liniger theory for $D = 1$ [25]. Finally, implementing the corrected energy density to the Lagrangian density of the bosonic atomic Josephson junction, it is possible to investigate how the quantum fluctuation acts on the Josephson junction and therefore to obtain the beyond mean-field contribution to the Josephson frequency and the MQST critical strength. In the chapter four only the contact term of the inter-atomic potential is considered, while in the last chapter also the finite range correction are taken in consideration.

Chapter 1

Inter-atomic Potential

In the following, many-body systems involving bosonic atoms are studied. It is therefore mandatory to spend few words about the interaction potential between atoms. An almost realistic model that describes this kind of interaction is the Lennard-Jones potential [26, 27, 28], that is given by

$$V(r) = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^6 \right] \tag{1.1}$$

where r is the inter-atomic distance, ϵ is the bond energy and σ the bond length. The first term is short-range and repulsive, its origin descends form the Pauli exclusion principle. The second term is long-range and attractive, and it is given by electric dipole fluctuations. The Lennard-Jones potential
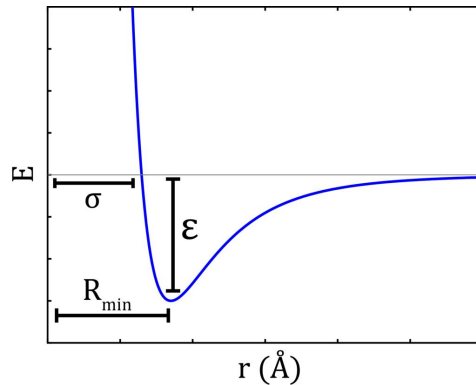

Figure 1.1: Lennard-Jones potential as a function of distance

is such that $V(r) \to +\infty$ for $r \to 0$ and $|\int V(r) d^3 \vec{r}| = +\infty$ (hard-core). In several applications are used soft-core versions of this potential, characterized by $|\int V(r) d^3 \vec{r}| < +\infty$.

The Lennard-Jones potential or its soft-core versions are not the potentials used in this work. In fact under certain conditions (low temperature, diluite gas) a simpler potential, the Fermi pseudo-potential [29], can be a good approximation for the inter-atomic interaction. This pseudo-potential is a contact potential given by

$$V(\vec{r}) = g \, \delta^{(3)}(\vec{r}) \tag{1.2}$$

From this formula it follows after integration with respect to $\vec{r}$ that

$$g = \int V(\vec{r}) \, d^3 \vec{r} \tag{1.3}$$

Thus, the strength g of the Fermi pseudo-potential is directly related to the "real" interaction potential $V(r)$. Moreover, this result can also be interpreted as the first term of a Taylor expansion of the Fourier transform of such potential:

$$g = \mathcal{F}[V(\vec{k} = 0)] \tag{1.4}$$

Generalizing to a D-dimensional space, the coupling constant is given by

$$g = \int V(\vec{r}) \, d^D \vec{r} \tag{1.5}$$

The dimensional units of the coupling constant are therefore

$$[g] = Energy \times Length^D \tag{1.6}$$

In particular, from scattering theory it follows that, within the Born approximation, calling a_s the scattering length in D dimensions and m the mass of an atom [30, 31, 32]:

$$g = \frac{4\pi\hbar^2 a_s}{m} \qquad (D = 3) \tag{1.7}$$

$$g = -\frac{4\pi\hbar^2}{m} \frac{1}{\ln\left(\Lambda^2 a_s^2 e^\gamma\right)} \qquad (D = 2) \tag{1.8}$$

$$g = -\frac{2\hbar^2}{m a_s} \qquad (D = 1) \tag{1.9}$$

Note that in $D = 1$ the coupling constant g and the scattering length a_s have opposite sign. The case $D = 2$ is very peculiar: the strength g is always positive and depend on an energy cut-off Λ and on the Euler-Mascheroni constant $\gamma \simeq -0.5772156649$. Another observation is on the dependence of the coupling constant to the scattering length. The dependence is linear for the $D = 3$ case, inverse proportional for the $D = 1$ case and inverse logarithmic for the $D = 2$ case. Nevertheless in all the three dimensional cases there is a proportional dependence to $\frac{\hbar^2}{m}$.

In the following we consider the Grand Potential accounting Gaussian fluctuation. This approximation, called Gaussian approximation or Bogoliubov approximation is valid only if $\tilde{n} \ll 1$ [21], where $\tilde{n} \equiv \frac{m}{\hbar^2} g n^{\frac{D-2}{D}}$ is an adimensional parameter of the expansion. In particular, in $D = 3$ the condition translates into $a^3 n \ll 1$, in $D = 2$ into $a_s^2 \Lambda^2 \gg 1$ or $a_s^2 \Lambda^2 \ll 1$, while in $D = 1$ into $|a| n \gg 1$.

As said before, the coupling constant g is the first term of a Taylor expansion of the Fourier transform of such potential. In fact one could express the Fourier transform of the potential as

$$\mathcal{F}[V] = \tilde{V}(k) = \sum_n g_{2n} k^{2n} \tag{1.10}$$

where, g_0 is the coupling constant g. Note that due to the symmetry of the system, only even powers of the momentum appear.

A correction of the Fermi pseudo-potential, that is contact-like, occurs with the introduction of a finite range term given by the second term of the expansion written above. The potential in the Fourier space is thus

$$\tilde{V}(k) = g_0 + g_2 k^2 \tag{1.11}$$

and the dimensional units of g_2 are therefore

$$[g_2] = Energy \times Length^{D+2} \tag{1.12}$$

In particular, calling r_s the D-dimensional s-wave effective range [30, 33]

$$g_2 = \frac{\pi \hbar^2}{m} a_s^2 r_s \qquad (D = 3) \tag{1.13}$$

$$g_2 = \frac{\pi^2 \hbar^2}{m} \frac{r_s^2}{\ln^2 \left(\Lambda^2 a_s^2 e^\gamma\right)} \qquad (D = 2) \tag{1.14}$$

$$g_2 = -\frac{\hbar^2}{2m} r_s \qquad (D = 1) \tag{1.15}$$

Note that while the $D = 3$ g_2 constant has a quadratic dependence on the scattering length a_s, the $D = 2$ g_2 constant has a logarithmic dependence while the $D = 1$ one is independent of a_s. Furthermore, while g_2 in $D = 3$ has the same sign of r_s, in $D = 2$ it is always positive due to the quadratic dependence on the effective range r_s and in $D = 1$ has opposite sign with respect to r_s. Last but not least, in $D = 2$ appear a UV cut-off Λ as in the g_0 case.

1.1 Coupling coefficients determination

To find the terms g_0 and g_2 let us consider the two-body problem. Given the transition operator $\hat{T}$, and defining its matrix element as $T_{\vec{k}\vec{k}'} = \langle \vec{k}|\hat{T}|\vec{k}'\rangle$ where $|k\rangle$ and $|k\rangle'$ are the initial and the final state respectively. In D dimension, one has the relation [32]

$$\hat{T}_{\vec{k}\vec{k}'} = V_{\vec{k}\vec{k}'} + \int d^D \vec{k}'' \frac{V_{\vec{k}\vec{k}''}}{\frac{\hbar^2 k^2}{2m_r} - \frac{\hbar^2 k''^2}{2m_r}} T_{\vec{k}''\vec{k}'} \tag{1.16}$$

where $|k\rangle''$ is an intermediate state, $m_r = \frac{m}{2}$ is the reduced mass, while $V_{\vec{k}\vec{k}'} = \langle \vec{k}|\hat{V}|\vec{k}'\rangle$ is the matrix element of the potential operator $\hat{V}$ which is given by the Fourier transform of the potential V, namely $V_{\vec{k}\vec{k}'} = \frac{\tilde{V}(k-k')}{(2\pi)^D}$. The next step is considering the s-wave approximation. First, one define the partial wave decomposition

$$V_{\vec{k}\vec{k}'} = \frac{1}{(2\pi)^D} \sum_l V_l(k,k') N(D,l) P_l^D(\hat{k} \cdot \hat{k}') \tag{1.17}$$

where $N(D,l$ is the number of spherical harmonics in D dimension and P_l^D is the Legendre polynomial in dimension D. Then, considering $k = k'$ and selecting $l = 0$ (s-wave approximation) one finds

$$T_0(k) = V_0(k) + S_D \int_0^\infty \frac{dk''}{(2\pi)^D} \frac{V_0(k,k'')}{\frac{\hbar^2 k^2}{m} - \frac{\hbar^2 k''^2}{m}} T_0(k'',k) \tag{1.18}$$

where $T_0(k) \equiv T_0(k,k)$, $V_0(k) \equiv V_0(k,k)$ and $S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(D/2)}$ is the solid angle in D dimension with $\Gamma(x)$ is the Euler gamma function.
A further approximation, called "on-shell approximation" is done [34]. One assumes, due to the presence of a singularity in $k = k''$ in (1.18) that

$$\begin{cases} V_0(k,k'') \simeq V_0(k,k) \equiv V_0(k) \\ T_0(k'',k) \simeq T_0(k,k) \equiv T_0(k) \end{cases} \tag{1.19}$$

where $V_0(k) = g_0 + 2g_2 k^2$. Therefore the expression (1.18) becomes

$$T_0(k) = V_0(k) + V_0(k)C(k)T_0(k) \tag{1.20}$$

where

$$C(k) \equiv \int \frac{d^D \vec{k}''}{(2\pi)^D} \frac{1}{\frac{\hbar^2 k^2}{m} - \frac{\hbar^2 k''^2}{m}} \tag{1.21}$$

Finally, one finds [20]

$$T_0(k) = \frac{1}{\frac{1}{V_0(k)} - C(k)} \tag{1.22}$$

From this result is possible to calculate the terms for g_0 and g_2 upon renormalization of $C(k)$ which has a ultraviolet divergence for any integer dimension D, however it is possible to eliminate the divergence through dimensional regularization [21, 35].

Using spherical coordinates in (1.21) one gets

$$C(k) = -\frac{S_D}{(2\pi)^D} \frac{m}{\hbar^2} \int_0^\infty dk'' k''^{D-1} \frac{1}{k''^2 + (-ik)^2} \tag{1.23}$$

Defining $t = -\frac{k''^2}{k^2}$, so that $k'' = (-ik)\sqrt{t}$ and $dk'' = \frac{(-ik)}{2\sqrt{t}}dt$ the expression becomes

$$C(k) = -\frac{S_D}{(2\pi)^D} \frac{m}{2\hbar^2} (-ik)^{D-2} \int_0^\infty dt \frac{t^{\frac{D}{2}-1}}{t+1} \tag{1.24}$$

Finally, using the Euler Beta function definition

$$B(x, y) = \int_0^\infty dt \frac{t^{x-1}}{(t+1)^{x+y}} \tag{1.25}$$

and the solid angle definition $S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ one obtains

$$C(k) = -\frac{m}{\hbar^2}(-ik)^{D-2} \frac{B(\frac{D}{2}, 1 - \frac{D}{2})}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \tag{1.26}$$

which, since $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ can be rewritten as

$$C(k) = -\frac{m}{\hbar^2}(-ik)^{D-2} \frac{\Gamma(1 - \frac{D}{2})}{(4\pi)^{\frac{D}{2}}} \tag{1.27}$$

1.1.1   D=3 case

The $D = 3$ case is simple. Remembering that $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$

$$C^{3D}(k) = (-ik)\frac{m}{4\pi\hbar^2} \tag{1.28}$$

Returning to the equation for the s-wave transition element $T_0(k)$, given by (1.22), one has in the $D = 3$ case

$$T_0(k) = \frac{1}{\frac{1}{V_0(k)} + ik\frac{m}{4\pi\hbar^2}} \tag{1.29}$$

and from the scattering theory is known that the s-wave transition element $T_0(k)$ is related to the s-wave phase shift $\delta_0(k)$ by the formula [32]

$$T_0(k) = -\frac{4\pi\hbar^2}{m}\frac{1}{k\cot\delta_0(k) - ik} \tag{1.30}$$

The s-wave phase shift $\delta_0(k)$ can be expanded at low momenta in terms of the $D = 3$ s-wave scattering length a_s and the $D = 3$ s-wave effective range r_s [32]

$$k\cot\delta_0(k) = -\frac{1}{a_s} + \frac{1}{2}r_s k^2 + \dots \tag{1.31}$$

Therefore one obtains

$$T_0(k) = \frac{1}{\frac{1}{V_0(k)} + ik\frac{m}{4\pi\hbar^2}}$$

$$\frac{1}{\frac{m}{4\pi\hbar^2 a_s} - \frac{1mr_s}{8\pi\hbar^2}k^2 + im\frac{m}{4\pi\hbar^2}} = \frac{1}{\frac{1}{g_0+2g_2 k^2} + ik\frac{m}{4\pi\hbar^2}} \tag{1.32}$$

$$\frac{m}{4\pi\hbar^2 a_s} - \frac{1mr_s}{8\pi\hbar^2}k^2 = \frac{1}{g_0 + 2g_2 k^2}$$

and expanding the RHS term one the expression reduces to

$$\frac{m}{4\pi\hbar^2 a_s} - \frac{1mr_s}{8\pi\hbar^2}k^2 = \frac{1}{g_0} - \frac{2g_2}{g_0^2}k^2 \tag{1.33}$$

Hence one retrieves the expressions for g_0 and g_2 written previously

$$g_0 = \frac{4\pi\hbar^2 a_s}{m} \qquad \text{and} \qquad g_2 = \frac{\pi\hbar^2}{m}a_s^2 r_s \tag{1.34}$$

1.1.2 D=2 case

The $D = 2$ is more complicate since $\Gamma(0)$ diverges. To solve this problem, admitting a non-integer dimension, the calculation is done for $D = 2 - \varepsilon$ and at the end of the calculation ε is sent to zero. Therefore, we need to calculate

$$
\begin{aligned}
C(k) &= -\frac{m}{\hbar^2} k_0^\varepsilon (-ik)^{-\varepsilon} \frac{\Gamma(\frac{\varepsilon}{2})}{(4\pi)^{1-\frac{\varepsilon}{2}}} \\
&= -\frac{m}{4\pi\hbar^2} \left(\frac{2\sqrt{\pi}k_0}{-ik} \right)^\varepsilon \Gamma\left(\frac{\varepsilon}{2}\right)
\end{aligned}
\tag{1.35}
$$

Expanding around $\varepsilon \to 0$ one obtains that $x^\varepsilon \simeq 1 + \varepsilon \ln x + O(\varepsilon^2)$ and

$$
\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)
\tag{1.36}
$$

Therefore the expression (1.35) can be rewritten as

$$
\begin{aligned}
C(k) &= -\frac{m}{4\pi\hbar^2} \left(\frac{2}{\varepsilon} - \gamma + 2\ln\left(\frac{2\sqrt{\pi}k_0}{k}\right) + 2\ln i \right) \\
&= -\frac{m}{4\pi\hbar^2} \left(2\ln\left(\frac{2\sqrt{\pi}k_0}{ke^{\frac{\gamma}{2}}}\right) + i\pi \right)
\end{aligned}
\tag{1.37}
$$

where the divergent term $\frac{2}{\varepsilon}$ is removed through minimal subtraction scheme. Finally, setting $\Lambda = \sqrt{\pi}k_0$, one obtains the formula for C(k) in the $D = 2$ case

$$
C^{2D}(k) = \frac{m}{2\pi\hbar^2} \ln\left(\frac{ke^{\frac{\gamma}{2}}}{2\Lambda}\right) - \frac{m}{4\hbar^2} i
\tag{1.38}
$$

The formula linking the s-wave transition element $T_0(k)$ to the s-wave phase shift $\delta_0(k)$ is given by [36]

$$
T_0(k) = -\frac{4\hbar^2}{m} \frac{1}{\cot\delta_0(k) - i}
\tag{1.39}
$$

where the cotangent of the s-wave phase shift $\delta_0(k)$ at low momenta can be expressed as a function of the $D = 2$ s-wave scattering length a_s and the $D = 2$ s-wave effective range r_s as follow [37]

$$
\cot\delta_0(k) = \frac{2}{\pi} \ln\left(\frac{k}{2}a_s e^\gamma\right) + \frac{1}{2}r_s^2 k^2
\tag{1.40}
$$

To retrieve the coupling constants g_0 and g_2 one inserts (1.38) and (1.39), with the aforementioned expansion, into the formula relating $T_0(k)$ with $V_0(k)$, namely (1.22), obtaining

$$
T_0(k) = \frac{1}{\frac{1}{V_0(k)} - \frac{m}{2\pi\hbar^2}\ln\left(\frac{ke^{\frac{\gamma}{2}}}{2\Lambda}\right) + \frac{m}{4\hbar^2}i}
$$

$$
\frac{1}{-\frac{m}{2\pi\hbar^2}\ln\left(\frac{k}{2}a_s e^\gamma\right) - \frac{mr_s^2}{8\hbar^2}k^2 + \frac{m}{4\hbar^2}i} = \frac{1}{\frac{1}{g_0 + 2g_2 k^2} - \frac{m}{2\pi\hbar^2}\ln\left(\frac{ke^{\frac{\gamma}{2}}}{2\Lambda}\right) + \frac{m}{4\hbar^2}i}
\tag{1.41}
$$

$$
-\frac{m}{2\pi\hbar^2}\ln\left(\Lambda a_s e^{\frac{\gamma}{2}}\right) - \frac{mr_s^2}{8\hbar^2}k^2 = \frac{1}{g_0} - \frac{2g_2}{g_0^2}k^2
$$

Finding the expressions for g_0 and g_2 mentioned above

$$
g_0 = -\frac{4\pi\hbar^2}{m} \frac{1}{\ln\left(\Lambda^2 a_s^2 e^\gamma\right)} \qquad \text{and} \qquad g_2 = \frac{\pi^2\hbar^2}{m} \frac{r_s^2}{\ln^2\left(\Lambda^2 a_s^2 e^\gamma\right)}
\tag{1.42}
$$

1.1.3   D=1 case

Finally, the $D = 1$ case is similar to the $D = 3$ one. Remembering that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ then

$$C^{1D}(k) = -i\frac{1}{k}\frac{m}{2\hbar^2} \tag{1.43}$$

From $1D$ scattering theory it is known that the s-wave transition element $T_0(k)$ is related to the s-wave phase shift $\delta_0(k)$ through the following relation [38, 39]

$$T_0(k) = -\frac{2\hbar^2}{m}\frac{k}{\cot\delta_0(k) - i} \tag{1.44}$$

By definition the ratio between the momentum and the cotangent of the s-wave phase shift $\delta_0(k)$ can be expressed at low momenta by means of the $D = 1$ s-wave scattering length a_s and the $D = 1$ s-wave effective range r_s [38, 39]

$$\frac{k}{\cot\delta_0(k)} = \frac{1}{a_s} + \frac{1}{2}r_s k^2 + \dots \tag{1.45}$$

There are now all the necessary quantities to calculate the coupling constants g_0 and g_2 starting from the relation (1.22)

$$T_0(k) = \frac{1}{\frac{1}{V_0(k)} + i\frac{1}{k}\frac{m}{2\hbar^2}}$$

$$\frac{1}{-\frac{m}{2\hbar^2}\frac{\cot\delta_0(k)}{k} + \frac{i}{k}\frac{m}{2\hbar^2}} = \frac{1}{\frac{1}{g_0 + 2g_2 k^2} + \frac{i}{k}\frac{m}{2\hbar^2}} \tag{1.46}$$

$$-\frac{2\hbar^2}{m a_s} - \frac{2\hbar^2}{m}\frac{r_s}{2}k^2 = g_0 + 2g_2 k^2$$

Therefore the coupling constants g_0 and g_2 are given respectively by

$$g_0 = -\frac{2\hbar^2}{m a_s} \qquad \text{and} \qquad g_2 = -\frac{\hbar^2}{2m}r_s \tag{1.47}$$

Chapter 2

Josephson Junction: Mean-field results

2.1 D-dimensional case

We want to construct an effective Lagrangian for a bosonic system with two sites of volume $V = L^D$ each [19]. The corresponding Lagrangian density is made of three terms:

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_J \tag{2.1}$$

The first and the second term are given by

$$\mathscr{L}_k = i\hbar\Phi_k^*(t)\partial_t\Phi_k(t) - \frac{1}{2}g|\Phi_k(t)|^4 \qquad k = 1, 2 \tag{2.2}$$

where $\Phi_k(t)$ is a complex time-dependent field describing the bosons in the k-th site and g is the coupling constant. The third term phenomenologically introduce the tunneling (hopping) and it is given by

$$\mathscr{L}_J = \frac{J}{2}\left(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t)\right) \tag{2.3}$$

the constant J is connected to the exchange of particles between the two sites. Integrating in space the Lagrangian density one obtains the Lagrangian [19]

$$\begin{aligned}
\mathcal{L} &= \int_V \mathscr{L} \, d^3\vec{r} \\
&= L^D \mathscr{L} \\
&= \sum_k \left(i\hbar\varphi_k^*(t)\partial_t\varphi_k(t) - \frac{U}{2}|\varphi_k(t)|^4\right) + \frac{J}{2}(\varphi_1^*(t)\varphi_2(t) + \varphi_2^*(t)\varphi_1(t))
\end{aligned} \tag{2.4}$$

where the new renormalized functions describing the system are

$$\varphi_k(t) \equiv \sqrt{L^D}\Phi_k(t) \quad k = 1, 2 \tag{2.5}$$

and the constant U, which has the dimension of an energy, is defined as

$$U \equiv \frac{g}{L^D} \tag{2.6}$$

Through the Madelung transformation [40], given by

$$\varphi_k(t) = \sqrt{N_k(t)}e^{i\phi_k(t)} \qquad k = 1, 2 \tag{2.7}$$

the complex function describing the bosons can be rewritten in terms of its phase $\phi_k(t)$ and its modulus $\sqrt{N_k(t)}$, where the square of the latter correspond to the number of boson in the $k-$th site.

Calculating the time derivative and the complex conjugate of the field

$$\partial_t \varphi_k(t) = \left(\frac{\dot{N}_k(t)}{2N_k(t)} + i\dot{\phi}_k(t) \right) \sqrt{N_k(t)} e^{i\phi_k(t)}$$

$$= \left(\frac{\dot{N}_k(t)}{2N_k(t)} + i\dot{\phi}_k(t) \right) \varphi_k(t) \tag{2.8}$$

$$\varphi_k^*(t) = \sqrt{N_k(t)} e^{-i\phi_k(t)} \tag{2.9}$$

Then

$$i\hbar \varphi_k^*(t) \partial_t \varphi_k(t) = i\hbar \varphi_k^*(t) \left(\frac{\dot{N}_k(t)}{2N_k(t)} + i\dot{\phi}_k(t) \right) \varphi_k(t)$$

$$= i\hbar N_k(t) \left(\frac{\dot{N}_k(t)}{2N_k(t)} + i\dot{\phi}_k(t) \right) \tag{2.10}$$

$$= i\hbar \frac{\dot{N}_k(t)}{2} - \hbar N_k(t)\dot{\phi}_k(t)$$

and

$$\varphi_1^*(t)\varphi_2(t) + \varphi_2^*(t)\varphi_1(t) = \sqrt{N_1(t)N_2(t)} e^{i(\phi_2(t)-\phi_1(t))} + \sqrt{N_2(t)N_1(t)} e^{i(\phi_1(t)-\phi_2(t))}$$

$$= 2\sqrt{N_1(t)N_2(t)} \cos\left(\phi_1(t) - \phi_2(t) \right) \tag{2.11}$$

Omitting the time dependence for simplicity of notation, the Lagrangian becomes

$$\mathcal{L} = \sum_k \left(i\hbar \frac{\dot{N}_k}{2} - \hbar \dot{\phi}_k N_k - \frac{U}{2} N_k^2 \right) + J \cos\left(\phi_1 - \phi_2 \right) \sqrt{N_1 N_2} \tag{2.12}$$

Introducing the total number of particles $N \equiv N_1(t) + N_2(t)$, the relative phase $\phi(t) \equiv \phi_2(t) - \phi_1(t)$, the total phase $\bar{\phi}(t) \equiv \phi_1(t) + \phi_2(t)$ and the population imbalance $z(t) \equiv \frac{N_1(t) - N_2(t)}{N}$ then the Lagrangian can be written as a function of N, ϕ, $\bar{\phi}$ and z. To express the number of particles on each site in terms of these parameters, we reverse the definitions of the population imbalance z and the total number of particles N, resulting in the following expressions:

$$N_1 = \frac{N}{2}(1 + z) \tag{2.13}$$

$$N_2 = \frac{N}{2}(1 - z) \tag{2.14}$$

Similarly, to find the expressions for the phases ϕ_1 and ϕ_2 of each site, we have:

$$\phi_1 = \frac{\bar{\phi} - \phi}{2} \tag{2.15}$$

$$\phi_2 = \frac{\bar{\phi} + \phi}{2} \tag{2.16}$$

Let us consider the terms of the Lagrangian separately:

$$\sum_k i\hbar \frac{\dot{N}_k}{2} = i\hbar \frac{\dot{N}}{2} = 0 \tag{2.17}$$

Since the total number of particles is a constant of motion, the first term is equal to zero.

The second term is instead given by

$$-\hbar \sum_k \dot{\phi}_k N_k = -\hbar \dot{\phi}_1 N_1 - \hbar \dot{\phi}_2 N_2$$

$$= -\hbar \frac{N}{2}(1+z)\frac{\dot{\bar{\phi}} - \dot{\phi}}{2} - \hbar \frac{N}{2}(1-z)\frac{\dot{\bar{\phi}} + \dot{\phi}}{2}$$

$$= -\hbar \frac{N}{4}\left[(1+z)(\dot{\bar{\phi}} - \dot{\phi}) + (1-z)(\dot{\bar{\phi}} + \dot{\phi})\right]$$

$$= -\hbar \frac{N}{4}\left[(\dot{\bar{\phi}} - \dot{\phi}) + z(\dot{\bar{\phi}} - \dot{\phi}) + (\dot{\bar{\phi}} + \dot{\phi}) - z(\dot{\bar{\phi}} + \dot{\phi})\right] \qquad (2.18)$$

$$= \frac{\hbar N}{2}\left(z\dot{\phi} - \dot{\bar{\phi}}\right)$$

$$= \frac{\hbar N}{2} z\dot{\phi}$$

where, the term $-\frac{\hbar N \dot{\bar{\phi}}}{2}$ is an exact differential and therefore it get cancelled by boundary conditions. For the third term we have

$$-\sum_k \frac{U}{2} N_k^2 = -\frac{U}{2}\left(N_1^2 + N_2^2\right)$$

$$= -\frac{U}{2}\frac{N^2}{4}\left[(1+z)^2 + (1-z)^2\right]$$

$$= -\frac{UN^2}{8}\left[1 + 2z + z^2 + 1 - 2z + z^2\right] \qquad (2.19)$$

$$= -\frac{UN^2}{4}\left(1 + z^2\right)$$

$$= -\frac{UN^2}{4} z^2$$

and we remove the term $-\frac{UN^2}{4}$ since it is a constant term. Finally the last term results in

$$J\cos(\phi_1 - \phi_2)\sqrt{N_1 N_2} = J\cos\phi\sqrt{\frac{N^2}{2}(1+z)(1-z)}$$

$$= \frac{JN}{2}\sqrt{1 - z^2}\cos\phi \qquad (2.20)$$

The Lagrangian then depends only on two dynamical variables, the population imbalance z and the phase difference ϕ and it is independent on the total phase $\bar{\phi}$. The complete form of the Lagrangian is thus

$$\mathcal{L} = \frac{N\hbar}{2}z\dot{\phi} - \frac{UN^2}{4}z^2 + \frac{JN}{2}\sqrt{1 - z^2}\cos\phi \qquad (2.21)$$

and the corresponding Euler-Lagrangian equations, called Josephson-Smerzi equations [10] are

$$\begin{cases} \dot{z} = -\frac{J}{\hbar}\sqrt{1 - z^2}\sin\phi \\ \dot{\phi} = \frac{J}{\hbar}\frac{z}{\sqrt{1-z^2}}\cos\phi + \frac{UNz}{\hbar} \end{cases} \qquad (2.22)$$

The Josephson-Smerzi equations are the starting point to calculate the Josephson frequency and the Macroscopic Quantum Self Trapping (MQST) critical value.

2.1.1 Josephson Frequency

To obtain the Josephson frequency we need a quadratic Lagrangian in the population imbalance z and the phase difference ϕ. Hence, one consider the limit in which the dynamical variables are small, namely $|\phi(t)| \ll 1$ and $|z(t)| \ll 1$. Under this limit, the following approximations are valid:

$$\sqrt{1 - z^2} = 1 - \frac{x^2}{2} + o(x^3) \qquad \cos\phi = 1 - \frac{\phi^2}{2} + o(\phi^2) \tag{2.23}$$

and quadratic Lagrangian has the form:

$$\mathcal{L} = \frac{N\hbar}{2} z\dot{\phi} - \frac{UN^2 + JN}{4} z^2 - \frac{JN}{4}\phi^2 \tag{2.24}$$

Therefore the linearized Josephson-Smerzi equations are

$$\begin{cases} \dot{z} = -\frac{J}{\hbar}\phi \\ \dot{\phi} = \frac{J+UN}{\hbar}z \end{cases} \tag{2.25}$$

and from these equations one can get the harmonic oscillator equations for the population imbalance $z(t)$ and the relative phase $\phi(t)$:

$$\begin{cases} \ddot{z} + \Omega_{mf}^2 z = 0 \\ \ddot{\phi} + \Omega_{mf}^2 \phi = 0 \end{cases} \tag{2.26}$$

where the Josephson frequency is introduced [10], its expression is given by

$$\Omega_{mf} = \frac{1}{\hbar}\sqrt{J^2 + UNJ} \tag{2.27}$$

which can be written also in faction of g and n as

$$\Omega_{mf} = \frac{1}{\hbar}\sqrt{J^2 + Jgn} \tag{2.28}$$

Note that there are two particular regimes. If $J \gg UN$ then the frequency can be approximated with the Rabi frequency Ω_R [10]

$$\Omega_{mf} \simeq \Omega_R = \frac{J}{\hbar} \tag{2.29}$$

Vice versa, if $J \ll UN$ then the frequency can be approximated to

$$\Omega_{mf} \simeq \Omega_J = \frac{\sqrt{UNJ}}{\hbar} \tag{2.30}$$

Hence the frequency (2.27) can be rewritten as a function of these two particular cases as

$$\Omega_{mf} = \sqrt{\Omega_R^2 + \Omega_J^2} \tag{2.31}$$

2.1.2 Macroscopic Quantum Self Trapping

Another interesting phenomenon that occurs in Josephson junction is the macroscopic quantum self trapping (MQST) [18].

Firstly, one need to find the conserved energy of the system, which is given by

$$E = \dot{z}\frac{\partial \mathcal{L}}{\partial \dot{z}} + \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \tag{2.32}$$

however the Lagrangian is independent from $\dot{z}$, so one gets from

$$E = \frac{N\hbar}{2}\dot{z\phi} - \frac{N\hbar}{2}\dot{z\phi} + \frac{UN^2}{4}z^2 - \frac{JN}{2}\sqrt{1-z^2}\cos\phi \tag{2.33}$$

Therefore the conserved energy is given by

$$E(z,\phi) = \frac{UN^2}{4}z^2 - \frac{JN}{2}\sqrt{1-z^2}\cos\phi \tag{2.34}$$

The macroscopic quantum self trapped happens when $\langle z \rangle \neq 0$ and the condition to have MQST is given, calling $z_0 = z(0)$ and $\phi_0 = \phi(0)$, by the following inequality

$$E(z_0, \phi_0) > E(0, \pi) \tag{2.35}$$

since $z(t)$ cannot become zero during an oscillation cycle. The MQST condition can be expressed also with a dimensionless parameter, known as strength, defined as

$$\Xi \equiv \frac{NU}{J} \tag{2.36}$$

In fact, inserting (2.34) and (2.36) into (2.35), one has [10]

$$\begin{aligned}
\frac{UN^2}{4}z_0^2 - \frac{JN}{2}\sqrt{1-z_0^2}\cos\phi_0 &> \frac{JN}{2} \\
\frac{UN}{2}z_0^2 - J\sqrt{1-z_0^2}\cos\phi_0 &> J \\
\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0 &> 1 \\
\Xi &> \frac{1+\sqrt{1-z_0^2}\cos\phi_0}{z_0^2/2}
\end{aligned} \tag{2.37}$$

and defining the critical value of the strength Ξ above which the macroscopic quantum self trapping occurs as

$$\Xi_{MQST} \equiv \frac{1+\sqrt{1-z_0^2}\cos\phi_0}{z_0^2/2} \tag{2.38}$$

the inequality condition translates to

$$\Xi > \Xi_{MQST} \tag{2.39}$$

which is the familiar mean-field condition to achieve MQST in Bose-Einstein condensates.

Chapter 3

Equation of state

The objective of this section is to employ dimensional regularization in order to renormalize the Grand Potential [21] to obtain the beyond mean-field Gaussian corrections in $D = 1, 2, 3$ spatial dimensions correction. These quantum fluctuations, disregarding finite-range effects, result in the Lee-Huang-Yang correction for $D = 3$ [22], the Schick-Popov correction for $D = 2$ [23, 24], and the next-to-leading term of the Lieb-Liniger theory for $D = 1$ [25].

The starting framework is an interacting system of identical bosons in the grand canonical ensemble. To study it the path integral formalism is adopted, where a complex field $\Phi(\vec{r}, \tau)$ describes the bosons [41].
The complex field depends on the position and of the Euclidean time (obtained Wick rotating the real time $\tau = -it$) . For a system of interacting atomic bosons in a $D-$dimensional box of volume $V = L^D$, chemical potential μ and $2-$body interaction potential $V(\vec{r}, \vec{r'})$ the Lagrangian density of the system is

$$\mathscr{L} = \Phi^*(\vec{r}, \tau) \left(\hbar \partial_\tau - \frac{\hbar^2}{2m} \vec{\nabla}^2 - \mu \right) \Phi(\vec{r}, \tau) + \frac{1}{2} \int d^3\vec{r} \, |\Phi(\vec{r}, \tau)|^2 V(\vec{r}, \vec{r'}) |\Phi(\vec{r'}, \tau)|^2 \tag{3.1}$$

Since the bosonic gas is diluite and at low temperature, it is possible to approximate the inter-atomic interaction with the Fermi pseudo-potential corrected with the finite range term, obtaining:

$$\mathscr{L} = \Phi^*(\vec{r}, \tau) \left(\hbar \partial_\tau - \frac{\hbar^2}{2m} \vec{\nabla}^2 - \mu \right) \Phi(\vec{r}, \tau) + \frac{g_0}{2} |\Phi(\vec{r}, \tau)|^4 - \frac{g_2}{2} |\Phi(\vec{r}, \tau)|^2 \vec{\nabla}^2 |\Phi(\vec{r}, \tau)|^2 \tag{3.2}$$

From the Lagrangian, one can construct the partition function of the system using the Euclidean action

$$S[\Phi, \Phi^*] = \int_0^{\hbar\beta} d\tau \int d^3\vec{r} \, \mathscr{L} \tag{3.3}$$

where $\beta \equiv \frac{1}{k_B T}$, k_B is the Boltzmann constant and T the temperature of the system. The partition function is thus [41]

$$\mathcal{Z} = \int \mathcal{D}[\Phi, \Phi^*] \, e^{-\frac{S[\Phi, \Phi^*]}{\hbar}} \tag{3.4}$$

and from the logarithm of it one retrieves the Grand Potential

$$\Omega = -\frac{1}{\beta} \ln \mathcal{Z} \tag{3.5}$$

Since the framework is in the superfluid phase, where there is a spontaneously symmetry breaking of the global $U(1)$ gauge symmetry [42], it is reasonable to set

$$\Phi(\vec{r}, \tau) = \psi_0 + \eta(\vec{r}, \tau) \tag{3.6}$$

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where ψ_0 is the order parameter of the system, it is supposed to be real, constant in time and uniform in space; and η is the complex field describing the bosonic fluctuation around ψ_0. Under this assumption the Grand Potential can be decomposed as follow [33, 30, 43]:

$$\Omega(\mu, \psi_0) = \Omega_0(\mu, \psi_0) + \Omega_G^{(0)}(\mu, \psi_0) + \Omega_G^{(T)}(\mu, \psi_0) \tag{3.7}$$

The first term is the grand potential of the order parameter Ω_0, which is given by

$$\Omega_0(\mu, \psi_0) \equiv -\mu \psi_0^2 L^D + \frac{1}{2} g_0 \psi_0^4 L^D \tag{3.8}$$

The second term is the zero-temperature contribution to quantum Gaussian fluctuations,

$$\Omega_G^{(0)}(\mu, \psi_0) \equiv \frac{1}{2} \sum_{\vec{q}} E_{\vec{q}}(\mu, \psi_0) \tag{3.9}$$

and the third term accounts thermal Gaussian fluctuations

$$\Omega_G^{(T)}(\mu, \psi_0) \equiv \frac{1}{\beta} \sum_{\vec{q}} \ln \left(1 - e^{-\beta E_{\vec{q}}(\mu, \psi_0)} \right) \tag{3.10}$$

The last two terms, which hold the Gaussian fluctuations contributions, depend on the Bogoliubov spectrum $E_{\vec{q}}$, which is given by

$$E_{\vec{q}}(\mu, \psi_0) = \sqrt{\left(\frac{\hbar^2 q^2}{2m} - \mu + 2g_0 \psi_0^2 + g_2 \psi_0^2 q^2 \right)^2 - \psi_0^4 (g_0 + g_2 q^2)^2} \tag{3.11}$$

However this is not the usual form of the Bogoliubov spectrum, to recover this form one needs to remove the dependence on ψ_0. To find an expression for the order parameter, one minimizes the Grand Potential with respect to such order parameter

$$\frac{\partial \Omega_0(\mu, \psi_0)}{\partial \psi_0} = 0 \tag{3.12}$$

and the relation obtained between the order parameter and the chemical potential is therefore

$$\psi_0 = \sqrt{\frac{\mu}{g_0}} \tag{3.13}$$

Hence the Bogoliubov spectrum has the more familiar form [33, 30, 43]

$$E_{\vec{q}}(\mu, g) = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\lambda(\mu) \frac{\hbar^2 q^2}{2m} + 2\mu \right)} \tag{3.14}$$

The finite range effects of the potential are taken into account through the term $\lambda(\mu)$ [33, 30, 43], defined as

$$\lambda(\mu) \equiv 1 + \chi \mu \tag{3.15}$$

where the coefficient of the chemical potential is given by $\chi \equiv \frac{4m}{\hbar^2} \frac{g_2}{g_0}$. If one neglects the finite range effects (setting $g_2 = 0$), $\lambda \equiv 1$ and the Bogoliubov sprectrum has the usual form [44]

$$E_{\vec{q}}(\mu, g) = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\frac{\hbar^2 q^2}{2m} + 2\mu \right)} \tag{3.16}$$

Using the relation (3.13), the mean-field grand potential can be rewritten as

$$\Omega_0(\mu, g) = -\frac{\mu^2}{2g_0} L^D \tag{3.17}$$

Now, taking the continuum limit for $\Omega_G^{(0)}$ a UV divergence appears [21]

$$\frac{\Omega_G^{(0)}}{L^D} = \frac{1}{2}\sum_{\vec{q}} E_{\vec{q}}(\mu, \psi_0) \longrightarrow \frac{1}{2}\frac{S_D}{(2\pi)^D}\int_0^{+\infty} dq q^{D-1}\sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda(\mu)\frac{\hbar^2 q^2}{2m} + 2\mu\right)} \tag{3.18}$$

where S_D is the surface area of a $(D-1)$-sphere with unitary radius.

Due to the divergence we need to perform a regularization, we achieve this through dimensional regularization, thinking at D as a continuous variable.

$$\int_0^{+\infty} dq q^{D-1}\sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} + 2\mu\right)} = \int_0^{+\infty} dq q^{D-1}\frac{\hbar^2 q^2}{2m}\sqrt{\lambda(\mu) + \frac{4\mu m}{\hbar^2 q^2}}$$

$$= \frac{\hbar^2\sqrt{\lambda(\mu)}}{2m}\int_0^{+\infty} dq q^{D+1}\sqrt{1 + \frac{4\mu m}{\hbar^2 q^2 \lambda(\mu)}} \tag{3.19}$$

Performing a change of variable, calling $t = \frac{4m\mu}{\hbar^2 q^2 \lambda(\mu)}$, then $dq = -\sqrt{\frac{m\mu}{\hbar^2\lambda(\mu)t^3}}dt$, so

$$\frac{\hbar^2\sqrt{\lambda}}{2m}\int_0^{+\infty} dq q^{D+1}\sqrt{1 + \frac{4\mu m}{\hbar^2 q^2 \lambda}} = \frac{\hbar^2\sqrt{\lambda}}{2m}\int_0^{+\infty} dt\sqrt{\frac{m\mu}{\hbar^2\lambda t^3}}\left(\frac{4m\mu}{\hbar^2\lambda t}\right)^{\frac{D+1}{2}}\sqrt{1 + t}$$

$$= \frac{1}{2\sqrt{\lambda}}(2\mu)^{(\frac{D}{2}+1)}\left(\frac{2m}{\hbar^2\lambda}\right)^{\frac{D}{2}}\int_0^{+\infty} dt\, t^{-\frac{D+4}{2}}\sqrt{1 + t}$$

$$= \frac{1}{2\sqrt{\lambda}}(2\mu)^{(\frac{D}{2}+1)}\left(\frac{2m}{\hbar^2\lambda}\right)^{\frac{D}{2}}\int_0^{+\infty} dt\, t^{-\frac{D+2}{2}-1}\sqrt{1 + t}$$

$$= \frac{1}{2\sqrt{\lambda}}(2\mu)^{(\frac{D}{2}+1)}\left(\frac{2m}{\hbar^2\lambda}\right)^{\frac{D}{2}} B\left(-\frac{D+2}{2}, \frac{D+1}{2}\right) \tag{3.20}$$

where $B(x, y)$ is the Euler Beta Function, related to the Gamma function by the following formula

$$B(x, y) = \int_0^{+\infty} dt\frac{t^{x-1}}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.21}$$

Thus, the zero-temperature contribution to quantum Gaussian fluctuation can be express in term of the Euler Beta function as [21]

$$\frac{\Omega_G^{(0)}}{L^D} = \frac{S_D(2\mu)^{\frac{D}{2}+1}}{4\sqrt{\lambda}(2\pi)^D}\left(\frac{2m}{\hbar^2\lambda}\right)^{\frac{D}{2}} B\left(-\frac{D+2}{2}, \frac{D+1}{2}\right) \tag{3.22}$$

which can be rewritten, remembering the definition of solid angle in D-dimension $S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ as

$$\frac{\Omega_G^{(0)}}{L^D} = \frac{\mu}{\sqrt{\lambda}\Gamma(D/2)}\left(\frac{\mu m}{\pi\hbar^2\lambda}\right)^{\frac{D}{2}} B\left(-\frac{D+2}{2}, \frac{D+1}{2}\right) \tag{3.23}$$

and writing the Euler Beta Function in terms of Gamma functions

$$\frac{\Omega_G^{(0)}}{L^D} = \frac{\mu}{\sqrt{\lambda}}\left(\frac{\mu m}{\pi\hbar^2\lambda}\right)^{\frac{D}{2}} \frac{\Gamma\left(-\frac{D+2}{2}\right)\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(-\frac{1}{2}\right)} \tag{3.24}$$

To compute it we need the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \qquad \text{Re}\{(z)\} > 0 \tag{3.25}$$

or rather, some of its properties (n is an integer):

$$\Gamma(n) = (n-1)!$$
$$\Gamma(1/2 - n) = \frac{(-2)^n\sqrt{\pi}}{(2n-1)!!}$$
$$\Gamma(1/2 + n) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}$$

(3.26)

where $\Gamma(1/2) = \sqrt{\pi}$ and the symbol "!!" denotes the semifactorial.

3.1   D=3 case

The $D = 3$ is rather simple, using the relations (3.26) the zero temperature Gaussian contribution to the Grand Potential is given by

$$\frac{\Omega_G^{(0)}}{L^3} = \frac{\mu}{\sqrt{\lambda}} \left(\frac{\mu m}{\pi \hbar^2 \lambda}\right)^{\frac{3}{2}} \frac{\Gamma(2)\,\Gamma\left(-\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\right)}$$

$$= \frac{\mu^{\frac{5}{2}}}{\lambda} \left(\frac{m}{\pi\hbar^2}\right)^{\frac{3}{2}} 1! \frac{(-2)^3 \sqrt{\pi}}{5!!} \frac{2}{1!!\sqrt{\pi}} \frac{1!!}{(-2)\sqrt{\pi}}$$

$$= \frac{\mu^{\frac{5}{2}}}{\lambda} \left(\frac{m}{\pi\hbar^2}\right)^{\frac{3}{2}} \frac{8}{15\sqrt{\pi}}$$

$$= \frac{8}{15\pi^2\lambda^2} \left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}} \mu^{\frac{5}{2}}$$

(3.27)

which is positive. The $D = 3$ zero-temperature Grand Potential is then [33]

$$\left.\frac{\Omega}{L^3}\right|_{T=0} = -\frac{\mu^2}{2g_0} + \frac{8}{15\pi^2\lambda^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\mu^{\frac{5}{2}}$$

(3.28)

The number density at $T = 0$ [33] can be obtained from the Grand Potential deriving respect to the chemical potential μ

$$n(\mu) = -\frac{1}{L^3}\frac{\partial\Omega}{\partial\mu}$$

$$= \frac{\mu}{g_0} - \frac{4}{3\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{\mu^{\frac{3}{2}}}{(1+\chi\mu)^2} + \frac{16}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{\chi\mu^{\frac{5}{2}}}{(1+\chi\mu)^3}$$

(3.29)

Considering $na_s^3 \ll 1$, an explicit expression for the chemical potential can be found inverting the relation using a perturbative approach (for more details see App. A)

$$\mu(n) = g_0 n + \frac{4g_0}{3\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{(g_0 n)^{\frac{3}{2}}}{(1+\chi g_0 n)^2} - \frac{16g_0}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{\chi(g_0 n)^{\frac{5}{2}}}{(1+\chi g_0 n)^3}$$

(3.30)

Finally, since $\mu = \frac{d\mathcal{E}(n)}{dn}$, the energy density is found by integration of the chemical potential in

$$\mathcal{E}(n) = \int \mu(n)dn$$

$$= \frac{1}{2}g_0 n^2 + \frac{8}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 \frac{(g_0 n)^{\frac{5}{2}}}{(1+\chi g_0 n)^2}$$

(3.31)

and writing the expression in terms of a_s instead of g_0 one has

$$\mathcal{E}(n) = \frac{2\pi\hbar^2 a_s n^2}{m}\left(1 + \frac{128}{15\sqrt{\pi}}\frac{\sqrt{a_s^3 n}}{(1+4\pi a_s^2 r_s n)^2}\right)$$

(3.32)

The expression in the parenthesis, upon setting $r_s = 0$, gives us the correction that Lee, Huang and Yang found in 1957 [22].

3.2   D=2 case

The $D = 2$ case is more complicated. The zero temperature Gaussian contribution is given by

$$
\begin{aligned}
\frac{\Omega_G^{(0)}}{L^2} &= \frac{\mu}{\sqrt{\lambda}} \left(\frac{\mu m}{\pi \hbar^2 \lambda} \right) \frac{\Gamma(-2)\, \Gamma\left(\frac{3}{2}\right)}{\Gamma(1)\, \Gamma\left(-\frac{1}{2}\right)} \\
&= \frac{\mu^2}{\lambda^{\frac{3}{2}}} \frac{m}{\pi \hbar^2} \Gamma(-2) \frac{1!! \sqrt{\pi}}{2} 0! \frac{1!!}{(-2)\sqrt{\pi}} \\
&= \frac{\mu^2}{\lambda^{\frac{3}{2}}} \frac{m}{\pi \hbar^2} \Gamma(-2) \frac{1}{-4} \\
&= -\frac{\mu^2}{4\pi \lambda^{\frac{3}{2}}} \frac{m}{\hbar^2} \Gamma(-2)
\end{aligned}
\tag{3.33}
$$

However, the term $\Gamma(-2)$ is divergent. To solve this problem, a non-integer dimension $D = 2 - \varepsilon$ is considered and at the end of the calculation the limit $\varepsilon \to 0$ is taken. To calculate the $D = 2 - \varepsilon$ case the zero temperature Gaussian contribution is necessary to step back to the general dimension formula

$$
\frac{\Omega_G^{(0)}}{L^D} = \frac{\mu}{\sqrt{\lambda}\, \Gamma\left(\frac{D}{2}\right)} \left(\frac{\mu m}{\pi \hbar^2 \lambda} \right)^{\frac{D}{2}} \frac{\Gamma\left(-\frac{D+2}{2}\right) \Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}
\tag{3.34}
$$

which in the $D = 2 - \varepsilon$ case is given by

$$
\begin{aligned}
\frac{\Omega_G^{(0)}}{L^D} &= \frac{\mu k_0^\varepsilon}{\sqrt{\lambda}\, \Gamma(1-\frac{\varepsilon}{2})} \left(\frac{\mu m}{\pi \hbar^2 \lambda} \right)^{1-\frac{\varepsilon}{2}} \frac{\Gamma\left(-2+\frac{\varepsilon}{2}\right) \Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \\
&= -\frac{m\mu^2}{2\pi \hbar^2 \lambda^{\frac{3}{2}} \sqrt{\pi}} \left(\frac{k_0 \pi \hbar^2 \lambda}{\mu m} \right)^{\frac{\varepsilon}{2}} \frac{\Gamma\left(-2+\frac{\varepsilon}{2}\right) \Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(1-\frac{\varepsilon}{2}\right)}
\end{aligned}
\tag{3.35}
$$

where k_0 is a scale wave-number introduced for dimensional reasons.
It is necessary to recall same expansions around $\varepsilon \to 0$, firstly

$$
\Gamma\left(-n+\frac{\varepsilon}{2}\right) = \frac{(-1)^n}{n!} \left[\frac{2}{\varepsilon} + \Psi(n+1) + \frac{\varepsilon}{4}\left(\frac{\pi^2}{3} + \Psi(n+1)^2 - \Psi'(n+1) \right) + O(\varepsilon^2) \right]
\tag{3.36}
$$

where $\Psi(z)$ is the digamma function, defined as the logarithmic derivative of the gamma function

$$
\Psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}
\tag{3.37}
$$

which has two useful properties, for a natural n different from zero

$$
\Psi(n) = \Psi(1) + \sum_{l=1}^{n-1} \frac{1}{l}
\tag{3.38}
$$

with $\Psi(1) = -\gamma$ where $\gamma = -0.5772156649$ is the Euler-Mascheroni constant and

$$
\Psi'(n) = \Psi'(1) - \sum_{l=1}^{n-1} \frac{1}{l^2}
\tag{3.39}
$$

where $\Psi'(1) = \frac{\pi^2}{6}$. Therefore the expansion of the gamma function $\Gamma(-2+\frac{\varepsilon}{2})$ around $\varepsilon \to 0$ is given by

$$
\Gamma\left(-2+\frac{\varepsilon}{2}\right) = \frac{1}{2} \left[\frac{2}{\varepsilon} - \gamma + \frac{3}{2} + \frac{\varepsilon}{4}\left(\gamma^2 - 3\gamma + \frac{7}{2} + \frac{\pi^2}{6} \right) + O(\varepsilon^2) \right]
\tag{3.40}
$$

Secondly, given a rational number q, an expansion of $\Gamma(q - \frac{\varepsilon}{2})$ around $\varepsilon \to 0$ is given by

$$\Gamma\left(q - \frac{\varepsilon}{2}\right) = \Gamma(q) - \frac{\varepsilon}{2}\Gamma'(n) + O(\varepsilon^2)$$
$$= \Gamma(q) - \frac{\varepsilon}{2}\Psi(q)\Gamma(q) + O(\varepsilon^2) \tag{3.41}$$
$$= \Gamma(q)\left(1 - \frac{\varepsilon}{2}\Psi(q)\right) + O(\varepsilon^2)$$

Therefore one gets for the expansion of $\Gamma^{-1}\left(1 - \frac{\varepsilon}{2}\right)$ around $\varepsilon \to 0$ is

$$\Gamma^{-1}\left(1 - \frac{\varepsilon}{2}\right) = \left(1 + \gamma\frac{\varepsilon}{2} + O(\varepsilon^2)\right)^{-1}$$
$$= 1 - \gamma\frac{\varepsilon}{2} + O(\varepsilon^2) \tag{3.42}$$

and the one of $\Gamma\left(\frac{3-\varepsilon}{2}\right)$ is given by

$$\Gamma\left(\frac{3-\varepsilon}{2}\right) = \frac{\sqrt{\pi}}{2}\left(1 - \frac{\varepsilon}{2}\Psi\left(\frac{3}{2}\right)\right) + O(\varepsilon^2)$$
$$= \frac{\sqrt{\pi}}{2}\left(1 + \frac{\varepsilon}{2}(\gamma + 2\ln 2 - 2)\right) + O(\varepsilon^2) \tag{3.43}$$

since for half-integer arguments the digamma function is given by

$$\Psi\left(n + \frac{1}{2}\right) = -\gamma - 2\ln 2 + \sum_{k=1}^{n}\frac{2}{2k-1} \tag{3.44}$$

Finally, a power of ε, x^ε, can be expressed as

$$x^\varepsilon = e^{\ln x^\varepsilon}$$
$$= e^{\varepsilon \ln x} \tag{3.45}$$
$$= 1 + \varepsilon \ln x + O(\varepsilon^2)$$

therefore the expansion of the term $\left(\frac{k_0\pi\hbar^2\lambda}{\mu m}\right)^{\frac{\varepsilon}{2}}$ around $\varepsilon \to 0$ is

$$\left(\frac{k_0\pi\hbar^2\lambda}{\mu m}\right)^{\frac{\varepsilon}{2}} = 1 + \frac{\varepsilon}{2}\ln\left(\frac{k_0\pi\hbar^2\lambda}{\mu m}\right) + O(\varepsilon^2) \tag{3.46}$$

which, setting the UV cut-off $\Lambda \equiv \sqrt{\pi}k_0$, can be rewritten as

$$\left(\frac{k_0\pi\hbar^2\lambda}{\mu m}\right)^{\frac{\varepsilon}{2}} = 1 + \frac{\varepsilon}{2}\ln\left(\frac{\hbar^2\Lambda^2\lambda}{\mu m}\right) + O(\varepsilon^2) \tag{3.47}$$

Hence, inserting the expressions (3.40), (3.42), (3.43) and (3.47) into (3.35) one gets

$$\frac{\Omega_G^{(0)}}{L^D} = -\frac{m\mu^2}{2\pi\hbar^2\lambda^{\frac{3}{2}}\sqrt{\pi}}\frac{1}{2}\frac{\sqrt{\pi}}{2}\left[1 + \frac{\varepsilon}{2}\ln\left(\frac{\hbar^2\Lambda^2\lambda}{\mu m}\right)\right]\times$$
$$\times\left[\frac{2}{\varepsilon} - \gamma + \frac{3}{2}\right]\left[1 + \frac{\varepsilon}{2}(\gamma + 2\ln 2 - 2)\right]\left[1 - \gamma\frac{\varepsilon}{2}\right] + O(\varepsilon)$$
$$= -\frac{m\mu^2}{8\pi\hbar^2\lambda^{\frac{3}{2}}}\left[\frac{2}{\varepsilon} - \gamma + \frac{3}{2} + \ln\left(\frac{\hbar^2\Lambda^2\lambda}{\mu m}\right)\right]\left[1 + \frac{\varepsilon}{2}(2\ln 2 - 2)\right] + O(\varepsilon) \tag{3.48}$$
$$= -\frac{m\mu^2}{8\pi\hbar^2\lambda^{\frac{3}{2}}}\left[\frac{2}{\varepsilon} - \gamma + \frac{3}{2} + \ln\left(\frac{\hbar^2\Lambda^2\lambda}{\mu m}\right) + 2\ln 2 - 2\right] + O(\varepsilon)$$
$$= -\frac{m\mu^2}{8\pi\hbar^2\lambda^{\frac{3}{2}}}\left[\frac{2}{\varepsilon} + \ln\left(\frac{4\hbar^2\Lambda^2\lambda}{\mu m e^{\gamma + \frac{1}{2}}}\right)\right] + O(\varepsilon)$$

After removing the divergent term proportional to $\frac{1}{\varepsilon}$ and sending $\varepsilon \to 0$ the expression above reduces to [45]

$$\frac{\Omega_G^{(0)}}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2\lambda^{\frac{3}{2}}} \ln\left(\frac{4\hbar^2\Lambda^2\lambda}{\mu m e^{\gamma+\frac{1}{2}}}\right) \tag{3.49}$$

Calculating explicitly the finite range factor λ

$$\begin{aligned}
\lambda &= 1 + \chi\mu \\
&= 1 + \frac{4m}{\hbar^2}\frac{g_2}{g_0}\mu \\
&= 1 - \frac{\pi m}{\hbar^2}\frac{r_s^2}{\ln\left(\Lambda^2 a_s^2 e^\gamma\right)}\mu
\end{aligned} \tag{3.50}$$

one may observe that the factor χ is small. This is also motivated from the fact that the finite range constant g_2 is a correction to the contact term constant g_0 and so $g_2 \ll g_0$, thus $\chi \ll 1$. Therefore one can do the following approximations

$$\frac{1}{(1+\chi\mu)^{\frac{3}{2}}} = 1 - \frac{3}{2}\chi\mu + \sum_{\ell=2}^{+\infty}(-1)^\ell\frac{(2\ell+1)!!}{2^\ell\ell!}\chi^\ell\mu^\ell \tag{3.51}$$

and

$$\ln(1+\chi\mu) = \sum_{k=1}^{+\infty}(-1)^{k+1}\frac{\chi^k\mu^k}{k} \tag{3.52}$$

Putting all together one has

$$\begin{aligned}
\frac{1}{\lambda^{\frac{3}{2}}}\ln\left(\frac{4\hbar^2\Lambda^2\lambda}{\mu m e^{\gamma+\frac{1}{2}}}\right) &= \frac{1}{\lambda^{\frac{3}{2}}}\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right) + \frac{\ln\lambda}{\lambda^{\frac{3}{2}}} \\
&= \left(1 - \frac{3}{2}\chi\mu + \sum_{\ell=2}^{+\infty}(-1)^\ell\frac{(2\ell+1)!!}{2^\ell\ell!}\chi^\ell\mu^\ell\right)\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right) + \\
&\quad + \left(1 - \frac{3}{2}\chi\mu + \sum_{\ell=2}^{+\infty}(-1)^\ell\frac{(2\ell+1)!!}{2^\ell\ell!}\chi^\ell\mu^\ell\right)\left(\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{\chi^k\mu^k}{k}\right) \\
&= \left(1 - \frac{3}{2}\chi\mu\right)\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right) + \sum_{\ell=2}^{+\infty}(-1)^\ell\frac{(2\ell+1)!!}{2^\ell\ell!}\chi^\ell\mu^\ell\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right) + \\
&\quad + \sum_{k=1}^{+\infty}(-1)^{k+1}\frac{\chi^k\mu^k}{k} - \frac{3}{2}\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{\chi^{k+1}\mu^{k+1}}{k} + \\
&\quad + \sum_{\ell=2}^{+\infty}\sum_{k=1}^{+\infty}(-1)^{k+\ell+1}\frac{(2\ell+1)!!}{2^\ell\ell!k}\chi^{k+\ell}\mu^{k+\ell}
\end{aligned} \tag{3.53}$$

However, since χ is inversely proportional to $\ln\left(\Lambda^2 a_s^2 e^\gamma\right)$, only the first two term are not discarded once the limit $\Lambda \to +\infty$ is taken. Hence the zero temperature Gaussian contribution to the Grand Potential si given by

$$\frac{\Omega_G^{(0)}}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right)\left(1 - \frac{3}{2}\chi\mu\right) \tag{3.54}$$

The beyond mean-field Grand Potential is obtained summing up the mean-field Grand Potential and the zero temperature Gaussian contribution

$$\begin{aligned}
\frac{\Omega}{L^2}\bigg|_{T=0} &= \frac{\Omega_0}{L^2} + \frac{\Omega_G^{(0)}}{L^2} \\
&= -\frac{\mu^2}{2g_0} - \frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right)\left(1 - \frac{3}{2}\chi\mu\right)
\end{aligned} \tag{3.55}$$

Recalling that the $D = 2$ coupling constant g_0 also depends on the ultraviolet cut-off as follow

$$g_0 = -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(\Lambda^2 a_s^2 e^\gamma)} \tag{3.56}$$

one obtains

$$
\begin{aligned}
\frac{\Omega}{L^2}\Big|_{T=0} &= \frac{m\mu^2}{8\pi\hbar^2}\ln(\Lambda^2 a_s^2 e^\gamma) - \frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right) - \frac{3}{2}\chi\frac{m}{8\pi\hbar^2}\mu^3\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right) \\
&= -\frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{4\hbar^2}{\mu m a_s^2 e^{2\gamma+\frac{1}{2}}}\right) - \frac{3}{16}\frac{m^2}{\hbar^4}r_s^2\mu^3\frac{\ln\left(\frac{4\hbar^2\Lambda^2}{\mu m e^{\gamma+\frac{1}{2}}}\right)}{\ln(\Lambda^2 a_s^2 e^\gamma)}
\end{aligned}
\tag{3.57}
$$

and sending the cut-off to infinity $\Lambda \to +\infty$, one obtains the $D = 2$ renormalized Grand Potential

$$\frac{\Omega}{L^2}\Big|_{T=0} = -\frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{4\hbar^2}{\mu m a_s^2 e^{2\gamma+\frac{1}{2}}}\right) - \frac{3}{16}\frac{m^2}{\hbar^4}r_s^2\mu^3 \tag{3.58}$$

This is the same $D = 2$ renormalized Grand Potential found by Mora and Castin [46] if one set $r_s = 0$. Note that the first term of the beyond mean-field Grand Potential is independent of the ultraviolet cut-off even without taking the limit.

Another observation is that the first term of the beyond mean-field Grand Potential is equal in form to the mean-field one if one defines a renormalized coupling g_r, in fact

$$\frac{\Omega}{L^2}\Big|_{T=0} = -\frac{\mu^2}{2g_r} - \frac{3}{16}\frac{m^2}{\hbar^4}r_s^2\mu^3 \tag{3.59}$$

where the renormalized coupling g_r is given by

$$g_r = -\frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{\mu}{\varepsilon_0}\right)} \tag{3.60}$$

and to retrieve the same results of [21] one sets

$$\varepsilon_0 \equiv \frac{4\hbar^2}{m a_s^2 e^{2\gamma+\frac{1}{2}}} \tag{3.61}$$

Also the second term of the beyond mean-field Grand Potential can be written in terms of the renormalized coupling g_r. To do so one defines also a renormalized finite term coupling g_{2r} given by

$$g_{2r} = \frac{\pi^2\hbar^2}{m}\frac{r_s^2}{\ln^2\left(\frac{\varepsilon_0}{\mu}\right)} \tag{3.62}$$

and a renormalized χ_r which uses the renormalized coupling constants g_r and g_{2r}

$$\chi_r \equiv \frac{4m}{\hbar^2}\frac{g_{2r}}{g_r} = -\frac{\pi m}{\hbar^2}\frac{r_s^2}{\ln\left(\frac{\varepsilon_0}{\mu}\right)} \tag{3.63}$$

Hence, the second term of the beyond mean-field Grand Potential can be rewritten in terms of g_r and χ_r as follow

$$
\begin{aligned}
-\frac{3}{16}\frac{m^2}{\hbar^4}r_s^2 &= \frac{3}{4}\left(-\frac{\pi m}{\hbar^2}\frac{r_s^2}{\ln\left(\frac{\varepsilon_0}{\mu}\right)}\right)\frac{m}{4\pi\hbar^2}\ln\left(\frac{\varepsilon_0}{\mu}\right) \\
&= \frac{3}{4}\frac{\chi_r}{g_r}
\end{aligned}
\tag{3.64}
$$

Finally, rewriting the beyond mean-field Grand potential in a more compact form

$$\frac{\Omega}{L^2}\bigg|_{T=0} = -\frac{\mu^2}{2g_r} + \frac{3}{4}\frac{\chi_r}{g_r}\mu^3 \tag{3.65}$$

To find the beyond mean-field chemical potential first the $T = 0$ number density is computed

$$
\begin{aligned}
n &= -\frac{1}{L^2}\frac{\partial\Omega}{\partial\mu} \\
&= \frac{m\mu}{8\pi\hbar^2}\left(2\ln\left(\frac{\varepsilon_0}{\mu}\right) - 1\right) + \frac{9}{16}\frac{m^2}{\hbar^4}r_s^2\mu^2
\end{aligned}
\tag{3.66}
$$

If the terms g_r and χ_r are redefined such that

$$\varepsilon_0\varepsilon_0 = \frac{4\hbar^2}{ma_s^2 e^{2\gamma+\frac{1}{2}}} \longrightarrow \varepsilon_0 = \frac{4\hbar^2}{ma_s^2 e^{2\gamma+1}} \tag{3.67}$$

then the number density can then be rewritten as

$$n = \frac{m\mu}{4\pi\hbar^2}\ln\left(\frac{\varepsilon_0}{\mu}\right) + \frac{9}{16}\frac{m^2}{\hbar^4}r_s^2\mu^2 \tag{3.68}$$

which can be rewritten in terms of χ_r and g_r as

$$n = \frac{\mu}{g_r} - \frac{9}{4}\frac{\chi_r}{g_r}\mu^2 \tag{3.69}$$

One can recognise in the expression for the number density n a quadratic equation in the chemical potential μ

$$\frac{9}{4}\chi_r\mu^2 - \mu + g_r n = 0 \tag{3.70}$$

Therefore the solutions are given by

$$\mu_\pm = \frac{2}{9\chi_r}(1 \pm \sqrt{1 - 9\chi_r g_r n}) \tag{3.71}$$

Having assumed that χ_r is small one could rewrite the square root as a binomial series

$$\sqrt{1 - 9\chi_r g_r n} = 1 + \sum_{k=1}^{+\infty}(-1)^k\binom{\frac{1}{2}}{k}(9\chi_r g_r n)^k \tag{3.72}$$

where the binomial coefficients are given by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} \tag{3.73}$$

while $\binom{\frac{1}{2}}{1} = \frac{1}{2}$, the binomial coefficients for $k \geq 2$ are given by

$$\binom{\frac{1}{2}}{k} = (-1)^k\frac{(2k-3)!!}{2^k k!} \tag{3.74}$$

Inserting the binomial series into the solutions (3.71) of the quadratic equation in μ one has for the

$$
\begin{aligned}
\mu_+ &= \frac{2}{9\chi_r}\left(2 - \frac{9\chi_r g_r n}{2} - \sum_{k=2}^{+\infty}(-1)^{2k}\frac{(2k-3)!!}{2^k k!}(9\chi_r g_r n)^k\right) \\
&= \frac{4}{9\chi_r} - g_r n - \sum_{k=2}^{+\infty}\frac{(2k-3)!!}{2^{k-1}k!}(9\chi_r)^{k-1}(g_r n)^k
\end{aligned}
\tag{3.75}
$$

and

$$\mu_- = \frac{2}{9\chi_r} \left(\frac{9\chi_r g_r n}{2} + \sum_{k=2}^{+\infty} \frac{(2k-3)!!}{2^k k!} (9\chi_r g_r n)^k \right)$$
$$= g_r n + \sum_{k=2}^{+\infty} \frac{(2k-3)!!}{2^{k-1} k!} (9\chi_r)^{k-1} (g_r n)^k \tag{3.76}$$

However the solution μ_+ is nonphysical since, when the finite range interaction is neglected ($\chi_r = 0$), μ_+ becomes divergent. Vice versa for the μ_- solution one obtains $\mu_- = g_r n$, which upon substituting g_r with g_0, corresponds to the mean-field result. The beyond mean-field chemical potential is therefore given by

$$\mu = g_r n \left(1 + \sum_{k=2}^{+\infty} C_k (\chi_r g_r n)^{k-1} \right) \tag{3.77}$$

where the numerical coefficient C_k is defined as

$$C_k \equiv \left(\frac{9}{2} \right)^{k-1} \frac{(2k-3)!!}{k!} \tag{3.78}$$

However μ appears still in the logarithms of the RHS. To find an explicit expression for the chemical potential μ as a function of n a further approximation is needed. Starting form

$$y = \frac{x}{\ln\left(\frac{A}{y}\right)} \left(1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left(\frac{A}{y}\right)} \right) \tag{3.79}$$

and inserting in the right-hand side of the equation the expression itself one obtains

$$y = \frac{x}{\ln\left[\frac{A}{x} \frac{\ln\left(\frac{A}{y}\right)}{1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left(\frac{A}{y}\right)}} \right]} \left(1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left[\frac{A}{x} \frac{\ln\left(\frac{A}{y}\right)}{1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left(\frac{A}{y}\right)}} \right]} \right) \tag{3.80}$$

Focusing now on the denominator of the first fraction

$$\ln\left[\frac{A}{x} \frac{\ln\left(\frac{A}{y}\right)}{1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left(\frac{A}{y}\right)}} \right] = \ln\left(\frac{A}{x}\right) + \ln\ln\left(\frac{A}{y}\right) - \ln\left(1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left(\frac{A}{y}\right)} \right) \tag{3.81}$$

and using the property that the logarithm of the product of two quantities is equal to the sum of their logarithms, it is possible to neglect the last two term making some assumption. Firstly, the second term at the denominator is much smaller than the first one due to the presence of the logarithm of a logarithm, and secondly the third term is negligible since it consists of the logarithm of the finite range interaction corrections. The equation (3.79) is therefore approximated to

$$y \approx \frac{x}{\ln\left(\frac{A}{x}\right)} \left(1 + \sum_{k=2}^{+\infty} c_k \frac{x^{k-1}}{\ln^{2k-2}\left(\frac{A}{x}\right)} \right) \tag{3.82}$$

Namely, the approximation does the following substitution inside g_r

$$\frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{\varepsilon_0}{\mu}\right)} \rightarrow \frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{m\varepsilon_0}{4\pi\hbar^2 n}\right)}$$

$$= \frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{m4\hbar^2}{4ma_s^2 e^{2\gamma+1}\pi\hbar^2 n}\right)}$$

$$= \frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{1}{a_s^2 e^{2\gamma+1}\pi n}\right)} \tag{3.83}$$

$$= -\frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(Cn\right)}$$

where the constant C is defined as

$$C \equiv \pi e^{2\gamma+1}a_s^2 \tag{3.84}$$

The same approximation is assumed to hold for $g_{2r}\ \chi_r$, too.

Now we need to find the corrected energy density. To do so in the $D = 2$ case a different approach is used instead of integrating the chemical potential. The grand potential (divided by the $D = 2$ volume) is defined by

$$\frac{\Omega}{L^2} \equiv \frac{E}{L^2} - T\frac{S}{L^2} - \mu\frac{N}{L^2} \tag{3.85}$$

which at $T = 0$ reduces to

$$\frac{\Omega}{L^2} = \mathcal{E} - \mu n \tag{3.86}$$

where $\mathcal{E}$ and n are the energy density and the number density, respectively.
Either the grand potential Ω and the number density depends on μ, so one has

$$\mathcal{E}(\mu) = \frac{\Omega(\mu)}{L^2} + \mu n(\mu)$$

$$= -\frac{\mu^2}{2g_r} + \frac{3}{4}\frac{\chi_r}{g_r}\mu^3 + \frac{\mu^2}{g_r} - \frac{9}{4}\frac{\chi_r}{g_r}\mu^2 \tag{3.87}$$

$$= \frac{\mu^2}{2g_r} - \frac{3}{2}\frac{\chi_r}{g_r}\mu^3$$

The energy density as a function of the chemical potential μ is therefore given by

$$\mathcal{E}(\mu) = \frac{\mu^2}{2g_r} - \frac{3}{2}\frac{\chi_r}{g_r}\mu^3 \tag{3.88}$$

However we want an expression of the energy density as a function of the number density n, to obtain it one inserts (3.77) into the formula above

$$\mathcal{E}(n) = \frac{1}{2g_r}\left[g_r n\left(1 + \sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)\right]^2 - \frac{3}{2}\frac{\chi_r}{g_r}\left[g_r n\left(1 + \sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)\right]^3$$

$$= \frac{g_r n^2}{2}\left[\left(1 + \sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)^2\right] - \frac{3}{2}\chi_r g_r^2 n^3\left[\left(1 + \sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)^3\right] \tag{3.89}$$

$$= \frac{g_r n^2}{2}\left[\left(1 + \sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)^2 - 3\chi_r g_r n\left(1 + \sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)^3\right]$$

To proceed with the calculation is necessary to perform the square and the cube of the series above. For the square one has

$$\left(1+\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)^2 = 1 + 2\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1} + \sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}C_k C_j(\chi_r g_r n)^{k+j-2} \tag{3.90}$$

while for the cube one has

$$\left(1+\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1}\right)^3 = 1 + 3\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1} + 3\sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}C_k C_j(\chi_r g_r n)^{k+j-2} +$$
$$+\sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}\sum_{\ell=2}^{+\infty}C_k C_j C_\ell(\chi_r g_r n)^{k+j+\ell-3} \tag{3.91}$$

Inserting the expansions into the energy density formula one obtains

$$\mathcal{E}(n) = \frac{g_r n^2}{2}\left[1 + 2\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1} + \sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}C_k C_j(\chi_r g_r n)^{k+j-2} +\right.$$
$$- 3\chi_r g_r n\left(1 + \sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}\sum_{\ell=2}^{+\infty}C_k C_j C_\ell(\chi_r g_r n)^{k+j+\ell-3} +\right.$$
$$\left.\left.+ 3\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1} + 3\sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}C_k C_j(\chi_r g_r n)^{k+j-2}\right)\right]$$
$$= \frac{g_r n^2}{2}\left[1 + 2\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k-1} + \sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}C_k C_j(\chi_r g_r n)^{k+j-2} +\right. \tag{3.92}$$
$$- 3\chi_r g_r n - 3\sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}\sum_{\ell=2}^{+\infty}C_k C_j C_\ell(\chi_r g_r n)^{k+j+\ell-2} +$$
$$\left.- 9\sum_{k=2}^{+\infty}C_k(\chi_r g_r n)^{k} - 9\sum_{k=2}^{+\infty}\sum_{j=2}^{+\infty}C_k C_j(\chi_r g_r n)^{k+j-1}\right]$$

The next step is to rewrite the series in such a way that they can be summed up, obtaining

$$\mathcal{E}(n) = \frac{g_r n^2}{2}\left[1 - 3\chi_r g_r n + 2\sum_{i=1}^{+\infty}C_{i+1}(\chi_r g_r n)^i - 9\sum_{i=2}^{+\infty}C_i(\chi_r g_r n)^i +\right.$$
$$+ \sum_{i=2}^{+\infty}\sum_{j=2}^{i}C_j C_{i+2-j}(\chi_r g_r n)^i - 9\sum_{i=3}^{+\infty}\sum_{j=2}^{i-1}C_j C_{i+1-j}(\chi_r g_r n)^i +$$
$$\left.- 3\sum_{i=4}^{+\infty}\sum_{j=2}^{i-2}\sum_{k=2}^{i-j}C_j C_k C_{i+2-j-k}(\chi_r g_r n)^i\right] \tag{3.93}$$
$$= \frac{g_r n^2}{2}\left(1 + \sum_{i=1}^{+\infty}\mathcal{C}_i(\chi_r g_r n)^i\right)$$

where the first three coefficients $\mathcal{C}_i$ are defined in the following way

$$\mathcal{C}_1 \equiv 2C_2 - 3 \qquad \mathcal{C}_2 \equiv 2C_3 - 9C_2 + C_2^2 \qquad \mathcal{C}_3 \equiv 2C_4 - 9C_3 + 2C_2 C_3 - 9C_2^2 \tag{3.94}$$

and for a generic $i \geq 4$ index the coefficient $\mathcal{C}_i$ is defined by

$$\mathcal{C}_i \equiv 2C_{i+1} - 9C_i + \sum_{j=2}^{i}C_j C_{i+2-j} - 9\sum_{j=2}^{i-1}C_j C_{i+1-j} - 3\sum_{j=2}^{i-2}\sum_{k=2}^{i-j}C_j C_k C_{i+2-j-k} \tag{3.95}$$

Remembering that $C_k \equiv \left(\frac{9}{2}\right)^{k-1} \frac{(2k-3)!!}{k!}$, in particular

$$C_2 = \frac{9}{2^2} \qquad C_3 = \frac{9^2}{2^3} \qquad C_4 = \frac{5 \cdot 9^3}{2^6} \tag{3.96}$$

then the coefficients $\mathcal{C}_i$ are given explicitly by

$$\begin{aligned}
\mathcal{C}_1 &= \frac{9}{2} - 3 = \frac{3}{2} \\
\mathcal{C}_2 &= \frac{9^2}{2^2} - \frac{9^2}{2^2} + \frac{9^2}{2^4} = \frac{81}{16} \\
\mathcal{C}_3 &= \frac{5 \cdot 9^3}{2^6} - \frac{9^3}{2^3} + \frac{9^3}{2^4} - \frac{9^3}{2^4} = -\frac{2187}{64}
\end{aligned} \tag{3.97}$$

and for a generic $i \geq 4$ index

$$\begin{aligned}
\mathcal{C}_i =& \frac{9^i}{2^{i-1}} \Bigg(\frac{(2i-1)!!}{(i+1)!} - \frac{(2i-3)!!}{i!} + \frac{1}{2} \sum_{j=2}^{i} \frac{(2j-3)!!(2i-2j+1)!!}{j!(i-j+2)!} + \\
&- \sum_{j=2}^{i-1} \frac{(2j-3)!!(2i-2j-1)!!}{j!(i-j+1)!} - \frac{1}{3} \sum_{j=2}^{i-2} \sum_{k=2}^{i-j} \frac{(2k-3)!!(2j-3)!!(2i-2j-2k+1)!!}{k!j!(i+2-j-k)!} \Bigg)
\end{aligned} \tag{3.98}$$

3.3 D=1 case

The procedure in this case is analogue to the $D = 3$ case. Using the relations (3.26) one finds

$$
\begin{aligned}
\frac{\Omega_G^{(0)}}{L} &= \frac{\mu}{\sqrt{\lambda}} \left(\frac{\mu m}{\pi \hbar^2 \lambda}\right)^{\frac{1}{2}} \frac{\Gamma(1) \Gamma\left(-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(-\frac{1}{2}\right)} \\
&= \frac{\mu^{\frac{3}{2}}}{\lambda} \left(\frac{m}{\pi \hbar^2}\right)^{\frac{1}{2}} 0! \frac{(-2)^2 \sqrt{\pi}}{3!!} \frac{1}{\sqrt{\pi}} \frac{1!!}{(-2)\sqrt{\pi}} \\
&= \frac{\mu^{\frac{3}{2}}}{\lambda} \left(\frac{m}{\pi \hbar^2}\right)^{\frac{1}{2}} \frac{(-2)}{3\sqrt{\pi}} \\
&= -\frac{2}{3\pi \lambda} \left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}} \mu^{\frac{3}{2}}
\end{aligned}
\tag{3.99}
$$

which, differently from the $D = 3$ case, is negative. The one dimensional $T = 0$ Grand Potential is thus [30]

$$
\frac{\Omega}{L}\bigg|_{T=0} = -\frac{\mu^2}{2g_0} - \frac{2}{3\pi \lambda} \left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}} \mu^{\frac{3}{2}}
\tag{3.100}
$$

As before, the $T = 0$ number density is given by [30]

$$
\begin{aligned}
n(\mu) &= -\frac{1}{L} \frac{\partial \Omega}{\partial \mu} \\
&= \frac{\mu}{g_0} + \frac{1}{\pi} \left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}} \frac{\mu^{\frac{1}{2}}}{1 + \chi \mu} - \frac{2}{3\pi} \left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}} \frac{\chi \mu^{\frac{3}{2}}}{(1 + \chi \mu)^2}
\end{aligned}
\tag{3.101}
$$

and the explicit expression of the chemical potential as a function of the number density is

$$
\mu(n) = g_0 n - \frac{g_0}{\pi} \left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}} \frac{(g_0 n)^{\frac{1}{2}}}{1 + \chi g_0 n} + \frac{2g_0}{3\pi} \left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}} \frac{\chi (g_0 n)^{\frac{3}{2}}}{(1 + \chi g_0 n)^2}
\tag{3.102}
$$

which is again found by perturbative expansion (for more details see App. C. Finally, the energy density is

$$
\begin{aligned}
\mathcal{E}(n) &= \int \mu(n) dn \\
&= \frac{1}{2} g_0 n^2 - \frac{2}{3\pi} \sqrt{\frac{m}{\hbar^2}} \frac{(g_0 n)^{\frac{3}{2}}}{1 + \chi g_0 n}
\end{aligned}
\tag{3.103}
$$

and writing it in terms of the scattering length a_s we obtain the next-to-leading term of the Lieb-Liniger theory for $D = 1$ [25] which accounts also for finite range corrections

$$
\begin{aligned}
\mathcal{E}(n) &= \int \mu(n) dn \\
&= -\frac{\hbar^2 n^2}{m a_s} \left(1 - \frac{2\sqrt{2}}{3\pi} \frac{1}{\sqrt{-a_s n(1 - 2r_s n)}}\right)
\end{aligned}
\tag{3.104}
$$

3.4 D=0 case

Similarly to the $D = 2$ case simply setting $D = 0$ in the equation (3.24) results in a indeterminate result, indeed

$$
\begin{aligned}
\Omega_G^{(0)} &= \mu \left(\frac{\mu m}{\pi \hbar^2} \right)^0 \frac{\Gamma(-1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma(0)} \\
&= -\frac{\mu}{2}\frac{\Gamma(-1)}{\Gamma(0)}
\end{aligned}
\tag{3.105}
$$

and both $\Gamma(-1)$ and $\Gamma(0)$ are divergent. The strategy to obtain a finite result is, as in the $D = 2$ case, regularization. We recall that regularization consist in setting $D = -\varepsilon$ and then performing the necessary calculations. After completing the calculations, the limit $\varepsilon \to 0$ is taken. Therefore the equation (3.24) takes the following form

$$
\Omega_G^{(0)} = \mu \left(\frac{\mu m}{k_0 \pi \hbar^2} \right)^{\frac{-\varepsilon}{2}} \frac{\Gamma\left(-1+\frac{\varepsilon}{2}\right)\Gamma\left(\frac{1-\varepsilon}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{-\varepsilon}{2}\right)}
\tag{3.106}
$$

where the parameter k_0 is a scale wave-number introduced for dimensional reasons.
The expansion of the exponential is given by the relation (3.45), which gives

$$
\left(\frac{k_0 \pi \hbar^2}{\mu m} \right)^{\frac{\varepsilon}{2}} = 1 + \frac{\varepsilon}{2} \ln \left(\frac{k_0 \pi \hbar^2}{\mu m} \right) + O(\varepsilon^2)
\tag{3.107}
$$

To expand the first factor in the numerator we exploit the expansion for the gamma distribution $\Gamma\left(-n+\frac{\varepsilon}{2}\right)$ in (3.36), for $n = 1$ one has

$$
\begin{aligned}
\Gamma\left(-1+\frac{\varepsilon}{2}\right) &= -\left[\frac{2}{\varepsilon} + \Psi(2) + \frac{\varepsilon}{4}\left(\frac{\pi^2}{3} + \Psi(2)^2 - \Psi'(2) \right) + O(\varepsilon^2) \right] \\
&= -\left[\frac{2}{\varepsilon} - \gamma + 1 + \frac{\varepsilon}{4}\left(\frac{\pi^2}{6} + \gamma^2 - 2\gamma + 2 \right) + O(\varepsilon^2) \right]
\end{aligned}
\tag{3.108}
$$

where, from (3.38) and (3.39) follow that

$$
\Psi(2) = \Psi(1) + 1 = -\gamma + 1 \qquad \Psi'(2) = \Psi'(1) - 1 = \frac{\pi^2}{6} - 1
\tag{3.109}
$$

The expansion of the second factor in the numerator is given by (3.41), which presents the general expansion of $\Gamma(q - \frac{\varepsilon}{2})$. Setting $q = \frac{1}{2}$, the expansion results in

$$
\begin{aligned}
\Gamma\left(\frac{1}{2} - \frac{\varepsilon}{2}\right) &= \Gamma\left(\frac{1}{2}\right)\left(1 - \frac{\varepsilon}{2}\Psi\left(\frac{1}{2}\right)\right) + O(\varepsilon^2) \\
&= \Gamma\left(\frac{1}{2}\right)\left(1 + \frac{\varepsilon}{2}(\gamma + 2\ln 2)\right) + O(\varepsilon^2)
\end{aligned}
\tag{3.110}
$$

where the digamma function in $\frac{1}{2}$ is given by

$$
\Psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2
\tag{3.111}
$$

Lastly, the expansion of the second factor in the denominator is given by the expansion around 0 of the reciprocal gamma function, namely

$$
\begin{aligned}
\Gamma^{-1}\left(1 - \frac{\varepsilon}{2}\right) &= \left(1 + \gamma\frac{\varepsilon}{2} + O(\varepsilon^2)\right)^{-1} \\
&= 1 - \gamma\frac{\varepsilon}{2} + O(\varepsilon^2)
\end{aligned}
\tag{3.112}
$$

Inserting all the expansions (3.107), (3.108), (3.110), (3.112) into (3.106)

$$
\begin{aligned}
\Omega_G^{(0)} =& \mu \left(\frac{\mu m}{k_0 \pi \hbar^2} \right)^{\frac{-\varepsilon}{2}} \frac{\Gamma\left(-1+\frac{\varepsilon}{2}\right) \Gamma\left(\frac{1-\varepsilon}{2}\right)}{\Gamma\left(-\frac{1}{2}\right) \Gamma\left(\frac{-\varepsilon}{2}\right)} \\
=& -\frac{\mu}{2} \left[1 + \frac{\varepsilon}{2} \ln\left(\frac{\hbar^2 k_0 \pi}{\mu m}\right)\right] \left(1 + \frac{\varepsilon}{2}(\gamma + 2\ln 2)\right) \times \\
& \times \frac{\varepsilon}{2} \left[\frac{2}{\varepsilon} - \gamma + 1 + \frac{\varepsilon}{4}\left(\frac{\pi^2}{6} + \gamma^2 - 2\gamma + 2\right)\right] + O(\varepsilon^2) \\
=& -\frac{\mu}{2} \left[1 + \frac{\varepsilon}{2} \ln\left(\frac{\hbar^2 k_0 \pi}{\mu m}\right) + \frac{\varepsilon}{2}(\gamma + 2\ln 2)\right] \left[1 - \frac{\varepsilon}{2}(\gamma - 1)\right] + O(\varepsilon^2) \\
=& -\frac{\mu}{2} \left[1 + \frac{\varepsilon}{2} \ln\left(\frac{\hbar^2 k_0 \pi}{\mu m}\right) + \frac{\varepsilon}{2}(\gamma + 2\ln 2) - \frac{\varepsilon}{2}(\gamma - 1)\right] + O(\varepsilon^2) \\
=& -\frac{\mu}{2} \left[1 + \frac{\varepsilon}{2} \ln\left(\frac{4\hbar^2 k_0 \pi e}{\mu m}\right)\right] + O(\varepsilon^2)
\end{aligned}
\tag{3.113}
$$

and taking the limit $\varepsilon \to 0$ one get the Gaussian fluctuations correction to the mean-field Grand Potential at $T = 0$

$$
\Omega_G^{(0)} = -\frac{\mu}{2} \tag{3.114}
$$

Note that differently from the $D = 2$ case, it is not needed no minimal subtraction.
The zero temperature Grand Potential in $D = 0$ is therefore

$$
\Omega\Big|_{T=0} = -\frac{\mu^2}{2g_0} - \frac{\mu}{2} \tag{3.115}
$$

Since we are at dimension $D = 0$, then $L^0 = 1$. As a consequence of that, the number density coincide with the number of particles and the energy density coincide with the energy of the system. Respectively, $n \equiv N$ and $\mathcal{E} \equiv E$. To determine the number of particles at $T = 0$, we derive the opposite corrected Grand Potential with respect to the chemical potential μ, obtaining

$$
N(\mu) = -\frac{\partial \Omega}{\partial \mu} = \frac{\mu}{g_0} + \frac{1}{2} \tag{3.116}
$$

This expression, contrary to all the previous dimensional cases, can be exactly reversed. The exact expression for the chemical potential as a function of the number of particle is thus

$$
\mu(N) = g_0 N - \frac{g_0}{2} \tag{3.117}
$$

Since $\mu = \frac{dE(N)}{dN}$, then integrating in N

$$
\begin{aligned}
E(N) &= \int \mu(N) dN \\
&= \frac{g_0 N^2 - g_0 N}{2} \\
&= \frac{g_0 N^2}{2}\left(1 - \frac{1}{N}\right)
\end{aligned}
\tag{3.118}
$$

where we collect as a common factor the term $g_0 N^2/2$, which represent the mean-field value of the energy.

Chapter 4

Josephson Junction: Beyond mean-field results

In this chapter, we incorporate the corrected energy density into the Lagrangian density of the bosonic atomic Josephson junction, it is possible to investigate the effects of quantum fluctuations on the Josephson junction in different dimensions.

While in $D = 0$ the quantum corrections do not affect the results, in $D = 1, 2, 3$ the corrections to the Josephson frequency, K_Ω, and the MQST critical strength, K_{Ξ_c}, can be expressed as follows:

$$\Omega = \frac{J}{\hbar}\sqrt{1 + \Xi K_\Omega} \qquad \Xi_c = \frac{1 + cos\theta_0 \sqrt{1 - z_0^2}}{z_0^2/2 + K_{\Xi_c}} \tag{4.1}$$

The values of the corrections for each dimension are shown in Table 4.1 and Table 4.2.

D	K_Ω
3	$1 + \frac{8\sqrt{2a_s^3 n}}{\sqrt{\pi}}$
2	$1 - \frac{3}{2\ln{(Cn)}} + \frac{1}{\ln^2{(Cn)}}$
1	$1 - \frac{1}{\pi\sqrt{-a_s n}}$

Table 4.1: Beyond mean-field correction to the Josephson frequency in dimension $D = 1, 2, 3$. The correction is determined by the s-wave scattering length a_s, and the number density n. In the case of $D = 2$, the result depends also on the parameters $C = \pi e^{2\gamma+1} a_s^2$, where $\gamma = -0.5772156649$ is the Euler-Mascheroni constant.

D	K_{Ξ_c}
3	$\frac{2\sqrt{2}}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 g_0^{\frac{3}{2}} n^{\frac{1}{2}}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}} - 2\right]$
2	$\frac{K(z_0,n) - \left(1 + z_0^2\right)}{2}$
1	$-\frac{\sqrt{2}}{3\pi}\sqrt{\frac{m}{\hbar^2}} g_0^{\frac{1}{2}} n^{-\frac{1}{2}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}} - 2\right]$

Table 4.2: Beyond mean-field correction to the MQST critical value in dimension $D = 1, 2, 3$. The correction is a function of the s-wave scattering length a_s, the number density n and the initial population imbalance z_0. In $D = 2$ the result depends also on a function of z_0 and n given by
$K(z_0, n) \equiv \frac{1}{2}\left(\frac{(1+z_0)^2}{1 + \frac{\ln{(1+z_0)}}{\ln{(Cn)}}} + \frac{(1-z_0)^2}{1 + \frac{\ln{(1-z_0)}}{\ln{(Cn)}}}\right)$ where $C = \pi e^{2\gamma+1} a_s^2$.

The chapter is structured into four sections, with each section providing a detailed demonstration of the calculations undertaken to obtain the results pertaining to each dimension.

4.1   D=3 case

Substituting the mean-field energy density with the Lee-Huang-Yang energy density [22] found in (3.31) upon setting $r_s = 0$, as we are neglecting finite range corrections for now, given by

$$\mathcal{E} = \frac{1}{2}g_0 n^2 + \frac{8}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 (g_0 n)^{\frac{5}{2}} \tag{4.2}$$

the Lagrangian density acquires a new term

$$\mathscr{L} = \sum_{k=1}^{2}\left(i\hbar\Phi_k^*(t)\partial_t\Phi_k(t) - \frac{1}{2}g_0|\Phi_k(t)|^4 - \frac{8g_0^{\frac{5}{2}}}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 |\Phi_k(t)|^5\right) +$$
$$+ \frac{J}{2}(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t))$$
$$\equiv \mathscr{L}_0 - \frac{8g_0^{\frac{5}{2}}\sqrt{m^3}}{15\pi^2\hbar^3}\left(|\Phi_1(t)|^5 + |\Phi_2(t)|^5\right) \tag{4.3}$$

where $\mathscr{L}_0$ is the mean-field Lagrangian density and the remaining part is the beyond mean-field contribution. Integrating the Lagrangian density in space one obtains the beyond mean-field Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - \frac{8g_0^{\frac{5}{2}}\sqrt{m^3}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\left(|\varphi_1(t)|^5 + |\varphi_2(t)|^5\right) \tag{4.4}$$

where $\mathcal{L}_0$ denotes the Lagrangian density within the mean-field approximation, while the additional term captures the correction arising from quantum fluctuations.

Performing the Madelung transformation (2.7) the new term is a function of the number of particles in the two sites

$$\mathcal{L} = \mathcal{L}_0 - \frac{8\sqrt{m^3 g_0^5}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\left(N_1^{\frac{5}{2}}(t) + N_2^{\frac{5}{2}}(t)\right) \tag{4.5}$$

and expressing the number of particles in each site as a function of the total number of particles N and the population imbalance z, specifically as $N_{1,2} = N(1 \pm z)/2$ the Lagrangian can be rewritten as

$$\mathcal{L} = \mathcal{L}_0 - \frac{\sqrt{2m^3 g_0^5}N^{\frac{5}{2}}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\left[(1+z)^{\frac{5}{2}} + (1-z)^{\frac{5}{2}}\right] \tag{4.6}$$

4.1.1   Josephson Frequency

To obtain the Josephson frequency, we make use of the assumption that we are in the low population imbalance limit, i.e. $|z| \ll 1$. Under this assumption, we can proceed with an expansion that disregards terms in z beyond quadratic order

$$(1 \pm z)^n = 1 \pm nz + \frac{n(n-1)z^2}{2} + O(z^3) \tag{4.7}$$

and summing the two contributions one obtains

$$(1+z)^n + (1-z)^n = 2 + n(n-1)z^2 + O(z^3) \tag{4.8}$$

Inserting it into (4.6) the Lagrangian reads

$$\mathcal{L} = \mathcal{L}_0 - \frac{\sqrt{2m^3 g_0^5}N^{\frac{5}{2}}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\left(2 + \frac{15}{4}z^2\right) \tag{4.9}$$

Since the term independent of z is constant, it is canceled out, leading to:

$$\mathcal{L} = \mathcal{L}_0 - \frac{\sqrt{m^3 g_0^5} N^{\frac{5}{2}}}{2\sqrt{2}\pi^2 \hbar^3 L^{\frac{9}{2}}} z^2 \tag{4.10}$$

Finally, in the case of $D = 3$, the beyond mean-field Lagrangian for the system has the form

$$\mathcal{L} = \frac{N\hbar}{2} z\dot{\phi} - \left(\frac{UN^2 + JN}{4}\right) z^2 - \frac{JN}{4}\phi^2 - \frac{\sqrt{m^3 g_0^5} N^{\frac{5}{2}}}{2\sqrt{2}\pi^2 \hbar^3 L^{\frac{9}{2}}} z^2 \tag{4.11}$$

and the corresponding Euler-Lagrangian equations are

$$\begin{cases} 0 = \frac{N\hbar}{2}\dot{\phi} - \left(\frac{UN^2 + JN}{2}\right) z - \frac{\sqrt{m^3 g_0^5} N^{\frac{5}{2}}}{\sqrt{2}\pi^2 \hbar^3 L^{\frac{9}{2}}} z \\ 0 = -\frac{N\hbar}{2}\dot{z} - \frac{JN}{2}\phi \end{cases} \tag{4.12}$$

In a similar manner to the mean-field case, we can rearrange the equation to recover the Euler-Lagrange equations of a harmonic oscillator, in fact

$$\begin{cases} \dot{\phi} = \left(\frac{UN + J}{\hbar} + \frac{\sqrt{2m^3 g_0^5} N^{\frac{3}{2}}}{\pi^2 \hbar^4 L^{\frac{9}{2}}}\right) z \\ \dot{z} = -\frac{J}{\hbar}\phi \end{cases} \implies \begin{cases} \ddot{\phi} + \Omega^2 \phi = 0 \\ \ddot{z} + \Omega^2 z = 0 \end{cases} \tag{4.13}$$

where the corrected Josephson frequency Ω is

$$\Omega \equiv \frac{1}{\hbar}\sqrt{J^2 + JUN + \frac{J\sqrt{2g_0^5 n^3 m^3}}{\pi^2 \hbar^3}} \tag{4.14}$$

The Josephson frequency Ω can also be written as a function of the coupling constant g_0 and number density n

$$\begin{aligned} \Omega &= \frac{1}{\hbar}\sqrt{J^2 + Jg_0 n + \frac{J\sqrt{2g_0^5 n^3 m^3}}{\pi^2 \hbar^3}} \\ &= \frac{J}{\hbar}\sqrt{1 + \frac{g_0 n}{J}\left(1 + \frac{\sqrt{2}}{\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 \sqrt{g_0^3 n}\right)} \end{aligned} \tag{4.15}$$

Remembering the definition of Rabi Frequency Ω_R, i.e. $\Omega_R \equiv= J/\hbar$, one obtains

$$\Omega = \Omega_R\sqrt{1 + \frac{g_0 n}{J}\left(1 + \frac{\sqrt{2}}{\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 \sqrt{g_0^3 n}\right)} \tag{4.16}$$

and writing explicitly the frequency as a function of the s-wave scattering length a_s

$$\Omega = \Omega_R\sqrt{1 + \frac{4\pi\hbar^2 a_s n}{mJ}\left(1 + \frac{8\sqrt{2a_s^3 n}}{\sqrt{\pi}}\right)} \tag{4.17}$$

Moreover, defining the reference energy ε_s and the gas parameter γ, which is adimensional, as

$$\varepsilon_s \equiv \frac{\hbar^2}{ma_s^2} \tag{4.18}$$

and

$$\gamma \equiv a_s^3 n \tag{4.19}$$

the Josephson Frequency can also be written as

$$\Omega = \Omega_R \sqrt{1 + 4\pi\gamma\frac{\varepsilon_s}{J}\left(1 + 8\sqrt{\frac{2\gamma}{\pi}}\right)} \tag{4.20}$$

To understand the magnitude of the beyond mean-field correction to the Josephson Frequency the ratio between the beyond mean-field Josephson Frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter, given by $\Xi = 4\pi\gamma\varepsilon_s/J$, is done. Namely it is considered

$$\frac{\Omega}{\Omega_{mf}} = \sqrt{\frac{1 + \Xi\left(1 + 8\sqrt{\frac{2\gamma}{\pi}}\right)}{1 + \Xi}} \tag{4.21}$$

Looking at the Fig. 4.1 one observes the following behavior: the correction increases the value of the Josephson frequency and it is more significant at higher strength parameters Ξ. For strength parameters $\Xi \to 0$ the beyond mean-field correction is irrelevant regardless of the gas parameter. Instead, for larger Ξ, the relative correction is given by

$$\left.\frac{\Omega}{\Omega_{mf}}\right|_{\Xi\gg 1} = \sqrt{1 + 8\sqrt{\frac{2\gamma}{\pi}}} \tag{4.22}$$

Focusing now on the bounds of the gas parameter γ, while the lower bound is $\gamma = 0$ and this is given by the fact that both the quantities defining γ, namely the s-wave scattering length a_s and the number density n are non-negative quantities. Instead, the upper bound limit is due to the fact that to obtain (3.29) from (3.30) we used a perturbative approach, assuming $\gamma \ll 1$, for this reason the upper bound limit is given by $\gamma = 3 \times 10^{-4}$ (for more details see App. A).

As pictured in Fig. 4.1 the relative correction is higher for larger values of the gas parameter γ, while for $\gamma = 0$ one retrieves the mean-field case.
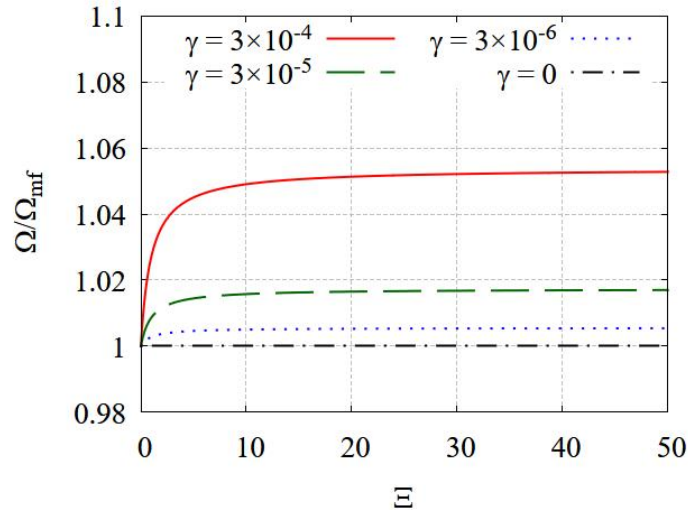

Figure 4.1: 3D beyond mean-field relative correction to the Josephson Frequency.
In the plot is pictured the ratio between the beyond mean-field Josephson frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter $\Xi = \frac{g_0 n}{J}$ for different values of the gas parameter $\gamma = a_s^3 n$: $\gamma = 3 \times 10^{-4}$ (red solid line), $\gamma = 3 \times 10^{-5}$ (green dashed line) $\gamma = 3 \times 10^{-6}$ (blue dotted line) and $\gamma = 0$ (black dash-dotted line). The last line corresponds to the mean-field case.

4.1.2 Macroscopic Quantum Self Trapping

Starting from the beyond mean-field Lagrangian written in terms of the total number of particles N, the population imbalance z and the phase difference ϕ

$$\mathcal{L} = \frac{N\hbar}{2} z\dot{\phi} - \frac{UN^2}{4} z^2 + \frac{JN}{2}\sqrt{1-z^2}\cos\phi - \frac{\sqrt{2m^3 g_0^5}N^{\frac{5}{2}}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\left[(1+z)^{\frac{5}{2}} + (1-z)^{\frac{5}{2}}\right] \tag{4.23}$$

one finds that the conserved energy is

$$E(z,\phi) = \frac{UN^2}{4} z^2 - \frac{JN}{2}\sqrt{1-z^2}\cos\phi + \frac{L^3\sqrt{2m^3}}{15\pi^2\hbar^3}(UN)^{\frac{5}{2}}\left[(1+z)^{\frac{5}{2}} + (1-z)^{\frac{5}{2}}\right] \tag{4.24}$$

Imposing the inequality condition, given by $E(z_0,\phi_0) > E(0,\pi)$, to have the Macroscopic Quantum Self Trapping phenomenon one gets

$$\frac{UN^2}{4} z_0^2 - \frac{JN}{2}\sqrt{1-z_0^2}\cos\phi_0+$$
$$+\frac{L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{5}{2}}N^{\frac{5}{2}}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}}\right] > \frac{JN}{2} + \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{5}{2}}N^{\frac{5}{2}}$$
$$\frac{\Xi}{2} z_0^2 - \sqrt{1-z_0^2}\cos\phi_0+$$
$$+\frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\frac{U^{\frac{5}{2}}N^{\frac{3}{2}}}{J}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}}\right] > 1 + \frac{4L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\frac{U^{\frac{5}{2}}N^{\frac{3}{2}}}{J} \tag{4.25}$$
$$\frac{\Xi}{2} z_0^2 - \sqrt{1-z_0^2}\cos\phi_0+$$
$$+\frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\Xi U^{\frac{3}{2}}N^{\frac{1}{2}}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}}\right] > 1 + \frac{4L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\Xi U^{\frac{3}{2}}N^{\frac{1}{2}}$$

and finally

$$\Xi > \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{3}{2}}N^{\frac{1}{2}}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}} - 2\right]} \tag{4.26}$$

The inequality condition still has the form

$$\Xi > \Xi_{c,3D} \tag{4.27}$$

where the critical value defined as

$$\Xi_{c,3D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{3}{2}}N^{\frac{1}{2}}U^{\frac{3}{2}}N^{\frac{1}{2}}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}} - 2\right]} \tag{4.28}$$

which can also be written as a function of the coupling constant g_0 and the number density n

$$\Xi_{c,3D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{2\sqrt{2}}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 g_0^{\frac{3}{2}}n^{\frac{1}{2}}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}} - 2\right]} \tag{4.29}$$

or as a function of the gas parameter γ

$$\Xi_{c,3D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{16\sqrt{2}}{15\sqrt{\pi}}\sqrt{\gamma}\left[(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}} - 2\right]} \tag{4.30}$$

To understand the significance of the beyond mean-field correction to the MQST critical value, one can compare it to the mean-field critical value $\Xi_{c,mf}$. This comparison can be expressed as the ratio:

$$\frac{\Xi_{c,3D}}{\Xi_{c,mf}} = \frac{1}{1 + \frac{32\sqrt{2}}{15\sqrt{\pi}}\sqrt{\gamma}\frac{(1+z_0)^{\frac{5}{2}} + (1-z_0)^{\frac{5}{2}} - 2}{z_0^2}} \tag{4.31}$$

Note that since the denominator of $\Xi_{c,3D}$ is larger than the $\Xi_{c,mf}$ one, then the beyond mean-field macroscopic quantum self trapping critical value is smaller than the mean-field one, i.e. $\Xi_{c,3D} < \Xi_{c,mf}$, as pictured in Fig. 4.2. Furthermore, the relative correction grows as the gas parameter decreases and it is slightly more significant for lower values of $|z_0|$. Additionally, the beyond mean-field correction is more important as the gas parameter γ increases.
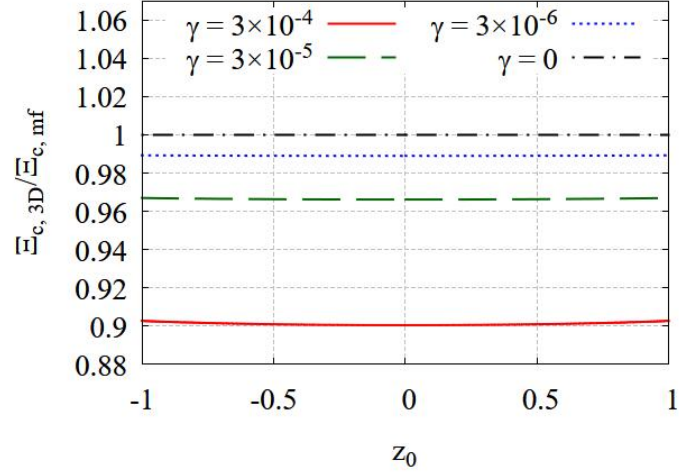

Figure 4.2: 3D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,3D}$ and the mean-field one $\Xi_{c,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t = 0) = (n_1(0) - n_2(0))/(n_1(0) + n_2(0))$ for different values of the gas parameter $\gamma = a_s^3 n$: $\gamma = 3 \times 10^{-4}$ (red solid line), $\gamma = 3 \times 10^{-5}$ (green dashed line) $\gamma = 3 \times 10^{-6}$ (blue dotted line) and $\gamma = 0$ (black dash-dotted line). The last line corresponds to the mean-field case.

4.2   D=2 case

The 2-dimensional case is very different from the 1-dimensional and the 3-dimensional cases for many reasons.

Firstly, the coupling depends on the number density of the site as follow

$$g_{r,k} = -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(Cn_k)} \qquad k = 1, 2 \tag{4.32}$$

where $C \equiv \pi e^{2\gamma+1}a_s^2$.

Secondly, the corrected Lagrangian is no more composed of a mean-field part and a correction, but rather it is equal in form to the mean-field Lagrangian although with the renormalized coupling g_r replacing g_0. Therefore, before computing the potential term, a discussion on the coupling is needed.

Since the coupling $g_{k,r}$ is different for each site, it is useful to define a coupling g_r for the entire system

$$g_r = -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(Cn)} \tag{4.33}$$

where n is the number density of the system and it is given by the mean of the number densities of the sites

$$n = \frac{n_1 + n_2}{2} \tag{4.34}$$

Then the number densities of the sites $n_{1,2}$ can be expressed in terms of the population imbalance variable $z(t) = \frac{n_1-n_2}{n_1+n_2}$ and the number density of the system n as

$$\begin{aligned} n_1 &= n(1+z) \\ n_2 &= n(1-z) \end{aligned} \tag{4.35}$$

The sites couplings $g_{r,k}$ can rewritten in terms of population imbalance $z(t)$, number density of the system n and coupling of the entire system g_r

$$\begin{aligned} g_{r,k} &= -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(Cn_k)} \qquad k = 1, 2 \\ &= -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(Cn(1\pm z))} \\ &= -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(Cn) + \ln(1\pm z)} \\ &= -\frac{4\pi\hbar^2}{m}\frac{1}{\ln(Cn)}\frac{1}{1 + \frac{\ln(1\pm z)}{\ln(Cn)}} \\ &= \frac{g_r}{1 + \frac{\ln(1\pm z)}{\ln(Cn)}} \end{aligned} \tag{4.36}$$

and therefore

$$g_{r,k} = \frac{g_r}{1 + \frac{\ln(1\pm z)}{\ln(Cn)}} \qquad k = 1, 2 \tag{4.37}$$

As seen in the previous chapter, the $D = 2$ beyond mean-field energy density is given by (3.93)

$$\mathcal{E}(n) = \frac{g_r n^2}{2} \tag{4.38}$$

Hence, the Lagrangian density is given by

$$\mathscr{L} = \sum_{k=1}^{2} \left(i\hbar \Phi_k^*(t) \partial_t \Phi_k(t) - \frac{1}{2} g_{r,k} |\Phi_k(t)|^4 \right) + \frac{J}{2} \left(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t) \right) \tag{4.39}$$

and it is obtained substituting g_0 with $g_{r,k}$. Integrating in space the corresponding Lagrangian is

$$\mathcal{L} = \sum_k \left(i\hbar \varphi_k^*(t) \partial_t \varphi_k(t) - \frac{U_k}{2} |\varphi_k(t)|^4 \right) + \frac{J}{2} (\varphi_1^*(t)\varphi_2(t) + \varphi_2^*(t)\varphi_1(t)) \tag{4.40}$$

where as a reminder

$$U_k \equiv \frac{g_{r,k}}{L^2} \qquad \varphi_k(t) \equiv L\Phi_k(t) \quad k = 1, 2 \tag{4.41}$$

In the mean-field $D = 2$ case the rewriting of the potential term in terms of N and z is simple and it gives

$$-\sum_k \frac{U}{2} N_k^2 \longrightarrow -\frac{U\cancel{N^2}}{4} - \frac{UN^2}{4} z^2 \tag{4.42}$$

In the beyond mean-field case it is not so simple since $g_{r,k}$ and so n depend on n_k, hence the potential term transforms differently. For $k = 1$ we have

$$\begin{aligned}
\frac{1}{2} U_1 |\varphi_1(t)|^4 &= \frac{g_{r,1}}{2L^2} |\varphi_1(t)|^4 \\
&= \frac{g_r}{2L^2} \frac{1}{1 + \frac{\ln(1+z)}{\ln(Cn)}} N_1^2 \\
&= \frac{U_r}{2} \frac{1}{1 + \frac{\ln(1+z)}{\ln(Cn)}} \frac{N^2(1+z)^2}{4}
\end{aligned} \tag{4.43}$$

where in the last step the constant U_r is defined as

$$U_r \equiv \frac{g_r}{L^2} \tag{4.44}$$

Similarly for $k = 2$

$$\begin{aligned}
\frac{1}{2} U_2 |\varphi_2(t)|^4 &= \frac{g_{r,2}}{2L^2} |\varphi_2(t)|^4 \\
&= \frac{g_r}{2L^2} \frac{1}{1 + \frac{\ln(1-z)}{\ln(Cn)}} N_2^2 \\
&= \frac{U_r}{2} \frac{1}{1 + \frac{\ln(1-z)}{\ln(Cn)}} \frac{N^2(1-z)^2}{4}
\end{aligned} \tag{4.45}$$

4.2.1 Josephson Frequency

Since we are interested in computing the Josephson frequency, we move to the low population imbalance limit, i.e. $|z(t)| \ll 1$, and the terms (4.43) and (4.45) are then Taylor expanded in z. For $k = 1$ the expansion is given by

$$\frac{U_r}{2} \frac{1}{1 + \frac{\ln(1+z)}{\ln(Cn)}} \frac{N^2(1+z)^2}{4} = \frac{U_r N^2}{8} \left(1 + \frac{z^2 - 2z}{2\ln(Cn)} + \frac{z^2}{\ln^2(Cn)}\right)(1 + 2z + z^2) + O(z^3) \qquad (4.46)$$

while for $k = 2$ one has

$$\frac{U_r}{2} \frac{1}{1 + \frac{\ln(1-z)}{\ln(Cn)}} \frac{N^2(1-z)^2}{4} = \frac{U_r N^2}{8} \left(1 + \frac{z^2 + 2z}{2\ln(Cn)} + \frac{z^2}{\ln^2(Cn)}\right)(1 - 2z + z^2) + O(z^3) \qquad (4.47)$$

Summing the two contributes one obtains, disregarding

$$\sum_k \frac{U_{r,k}}{2}|\varphi_k(t)|^4 = \frac{U_r N^2}{4}\left(1 + z^2 - \frac{3z^2}{2\ln(Cn)} + \frac{z^2}{\ln^2(Cn)}\right) + O(z^3) \qquad (4.48)$$

Writing this result in terms of the system coupling, inverting the relation (4.33)

$$\frac{1}{\ln(Cn)} = -\frac{mg_r}{4\pi\hbar^2} \qquad (4.49)$$

one gets an expression of the beyond-mean field correction written in terms of the system coupling constant g_r

$$\sum_k \frac{U_{r,k}}{2}|\varphi_k(t)|^4 = \frac{U_r N^2}{4}\left[1 + z^2\left(1 + \frac{3}{8}\frac{mg_r}{\pi\hbar^2} + \frac{1}{16}\frac{m^2 g_r^2}{\pi^2\hbar^4}\right)\right] \qquad (4.50)$$

The term is similar to the mean-field one, given by $\frac{UN^2 z^2}{4}$ with caution to substitute U with U_r and add the beyond the mean-field corrections to the contact interaction term. Therefore to obtain the Josephson frequency in the 2-dimensional beyond mean-field framework is sufficient to substituting inside the mean-field Josephson frequency (2.27) the constant U with

$$U \to U_r \left(1 + \frac{3}{8}\frac{mg_r}{\pi\hbar^2} + \frac{1}{16}\frac{m^2 g_r^2}{\pi^2\hbar^4}\right) \qquad (4.51)$$

Doing so, the 2-dimensional beyond mean-field Josephson frequency is given by

$$\Omega = \frac{1}{\hbar}\sqrt{J^2 + JU_r N\left(1 + \frac{3}{8}\frac{mg_r}{\pi\hbar^2} + \frac{1}{16}\frac{m^2 g_r^2}{\pi^2\hbar^4}\right)} \qquad (4.52)$$

or, alternatively, expressing it as a function of system coupling constant g_r and the number density n

$$\Omega = \frac{1}{\hbar}\sqrt{J^2 + Jg_r n\left(1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)}\right)} \qquad (4.53)$$

Note that, in the limit of low density $n \ll 1$, powers of the term

$$\frac{1}{\ln(Cn)} \qquad (4.54)$$

become smaller and smaller, the higher is the power, therefore keeping only terms of the order $\frac{1}{\ln(Cn)}$ then (4.51) can be approximated to U_r and so the Josephson frequency in $D = 2$ reduces to mean-field one (2.27)

$$\Omega = \frac{1}{\hbar}\sqrt{J^2 + U_r N J} \qquad (4.55)$$

with the care of substituting U with U_r.

Writing explicitly the Rabi frequency, the s-wave scattering length a_s and the number density n one obtains

$$\Omega = \Omega_R \sqrt{1 - \frac{4\pi\hbar^2 n}{mJ\ln(Cn)}\left(1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)}\right)} \tag{4.56}$$

Introducing the reference energy ε_s and the gas parameter in the 2-dimensional case

$$\varepsilon_s \equiv \frac{\hbar^2}{ma_s^2} \qquad \gamma \equiv a_s^2 n \tag{4.57}$$

and calling $C^* \equiv C/a_s^2 = \pi e^{2\gamma+1}$ where, as a reminder, in the exponential $\gamma = -0.5772156649$ is the Euler-Mascheroni constant and not the gas parameter. The Josephson frequency can be written as

$$\Omega = \Omega_R \sqrt{1 - \frac{4\pi\gamma}{\ln(C^*\gamma)}\frac{\varepsilon_s}{J}\left(1 - \frac{3}{2\ln(C^*\gamma)} + \frac{1}{\ln^2(C^*\gamma)}\right)} \tag{4.58}$$

The beyond mean-field relative correction to the Josephson Frequency Ω is given by

$$\frac{\Omega}{\Omega_{mf}} = \sqrt{\frac{1 + \Xi\left(1 - \frac{3}{2\ln(C^*\gamma)} + \frac{1}{\ln^2(C^*\gamma)}\right)}{1 + \Xi}} \tag{4.59}$$

where the strength parameter is given by $\Xi = -(4\pi\gamma/\ln(C^*\gamma))(\varepsilon_s/J)$.

As pictured in Fig. 4.3, the relative Ω/Ω_{mf} correction at fixed gas parameter γ is more significant for higher values of the strength parameter Ξ. As a matter of fact, for larger values of the strength parameter Ξ, the relative correction Ω/Ω_{mf} is independent on Ξ and it is given by

$$\left.\frac{\Omega}{\Omega_{mf}}\right|_{\Xi \gg 1} = \sqrt{1 - \frac{3}{2\ln(C^*\gamma)} + \frac{1}{\ln^2(C^*\gamma)}} \tag{4.60}$$

Instead, focusing on the gas parameter γ dependence one has an increment of the relative Ω/Ω_{mf} correction for higher value of γ. Conversely for $\gamma = 0$ one retrieves the mean-field result.

As one can see in App. B, the expression for the beyond-mean Josephson frequency stated above is not valid for all values of the gas parameter. There are indeed two ranges in which the approximations we have made hold true: $0 \le \gamma \lesssim 3 \times 10^{-18}$ and $0.01283 \lesssim \gamma \lesssim 0.01421$. In the first range, the beyond-mean field correction is almost negligible, meaning that the contribution from fluctuations beyond the mean-field is very small. However, in the second range, the contribution due to Gaussian fluctuations is significant. For instance, in the case of high Ξ, the beyond-mean field Josephson frequency is almost double that of the mean-field frequency.
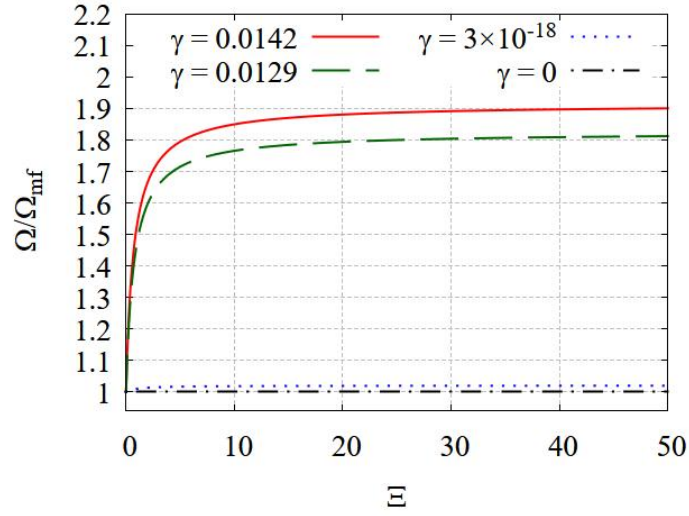
Figure 4.3: 2D beyond mean-field relative correction to the Josephson Frequency.
In the plot is pictured the ratio between the beyond mean-field Josephson frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter $\Xi = g_r n/J$ for different values of the gas parameter $\gamma = a_s^2 n$: $\gamma = 1.42 \times 10^{-2}$ (red solid line), $\gamma = 1.29 \times 10^{-2}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). The last line corresponds to the mean-field case.

4.2.2   Macroscopic Quantum Self Trapping

Unlike the Josephson frequency calculation, in the MQST one the low population imbalance limit is not taken. Therefore it is necessary to see how the interaction terms transform in the case the population imbalance is generic. Taking into account the contact interaction terms

$$\sum_k \frac{1}{2} U_k |\varphi_k(t)|^4 = \frac{N^2}{8}[U_1(1+z)^2 + U_2(1-z^2)] \tag{4.61}$$

Keeping in mind about the definitions of U_r and U_k, they can be linked by the followed relation

$$U_k = U_r \frac{1}{1 + \frac{\ln(1 \pm z)}{\ln(Cn)}} \qquad k = 1, 2 \tag{4.62}$$

Then the interaction term reduces to

$$\sum_k \frac{1}{2} U_k |\varphi_k(t)|^4 = \frac{U_r N^2}{8} \left[\frac{(1+z)^2}{1 + \frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1 + \frac{\ln(1-z)}{\ln(Cn)}} \right] \tag{4.63}$$

Hence, the 2-dimensional beyond mean-field Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{N\hbar}{2} z\dot{\phi} + \frac{JN}{2} \sqrt{1-z^2} \cos\phi - \frac{U_r N^2}{8} \left[\frac{(1+z)^2}{1 + \frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1 + \frac{\ln(1-z)}{\ln(Cn)}} \right] \\
&= \frac{N\hbar}{2} z\dot{\phi} + \frac{JN}{2} \sqrt{1-z^2} \cos\phi - K(z, n) \frac{U_r N^2}{4}
\end{aligned} \tag{4.64}$$

where, for simplicity of notation, we introduce the beyond-mean field energy correction to the 2-dimensional Lagrangian K

$$K(z, n) \equiv \frac{1}{2} \left[\frac{(1+z)^2}{1 + \frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1 + \frac{\ln(1-z)}{\ln(Cn)}} \right] \tag{4.65}$$

From the Lagrangian one can compute the conserved energy

$$E = K(z, n) \frac{U_r N^2}{4} - \frac{JN}{2} \sqrt{1-z^2} \cos\phi \tag{4.66}$$

and imposing the MQST inequality condition, given by $E(z_0, \phi_0) > E(0, \pi)$ one obtains

$$\begin{aligned}
K(z_0, n) \frac{U_r N^2}{4} - \frac{JN}{2} \sqrt{1-z_0^2} \cos\phi_0 &> \frac{JN}{2} + \frac{U_r N^2}{4} \\
(K(z_0, n) - 1) \frac{U_r N^2}{4} - \frac{JN}{2} \sqrt{1-z_0^2} \cos\phi_0 &> \frac{JN}{2} \\
(K(z_0, n) - 1) \frac{U_r N}{2J} - \sqrt{1-z_0^2} \cos\phi_0 &> 1 \\
(K(z_0, n) - 1) \frac{\Xi_r}{2} - \sqrt{1-z_0^2} \cos\phi_0 &> 1
\end{aligned} \tag{4.67}$$

where in the last step we define the the dimensionless constant Ξ_r as

$$\Xi_r \equiv \frac{U_r N}{J} \tag{4.68}$$

the inequality is equivalent to

$$\Xi_r > \frac{1 + \sqrt{1 - z_0^2} \cos \phi_0}{\left(K(z_0, n) - 1\right)/2}$$
$$\Xi_r > \frac{1 + \sqrt{1 - z_0^2} \cos \phi_0}{\frac{z_0^2}{2} + \frac{K(z_0, n) - \left(1 + z_0^2\right)}{2}} \tag{4.69}$$

where we make explicit the mean-field term $z_0^2/2$ at the denominator. Furthermore, we can simplify the inequality to the following form:

$$\Xi_r > \Xi_{c, 2D} \tag{4.70}$$

where the critical value $\Xi_{c, 2D}$ is given by

$$\Xi_{c, 2D} \equiv \frac{1 + \sqrt{1 - z_0^2} \cos \phi_0}{\frac{z_0^2}{2} + \frac{K(z_0, n) - \left(1 + z_0^2\right)}{2}} \tag{4.71}$$

By considering the definition of the gas parameter, we can further express $\Xi_{c, 2D}$ as

$$\Xi_{c, 2D} \equiv \frac{1 + \sqrt{1 - z_0^2} \cos \phi_0}{\frac{z_0^2}{2} + \frac{K(z_0, \gamma) - \left(1 + z_0^2\right)}{2}} \tag{4.72}$$

where

$$K(z_0, \gamma) \equiv \frac{1}{2} \left(\frac{(1 + z_0)^2}{1 + \frac{\ln (1 + z_0)}{\ln (C^* \gamma)}} + \frac{(1 - z_0)^2}{1 + \frac{\ln (1 - z_0)}{\ln (C^* \gamma)}} \right) \tag{4.73}$$

Dividing by the mean-field MQST critical value $\Xi_{c, mf}$ we obtain the expression

$$\frac{\Xi_{c, 2D}}{\Xi_{c, mf}} \equiv \frac{1}{1 + \frac{K(z_0, \gamma) - \left(1 + z_0^2\right)}{z_0^2}} \tag{4.74}$$

Upon examining Fig. 4.4, one can observe that the ratio is equal to 1 as the gas parameter approaches zero, $\gamma = 0$, thereby recovering the mean-field result. It is noteworthy that the ratio is consistently lower than 1, implying that the beyond-mean-field correction reduces the MQST critical value. Another significant observation is that as the gas parameter increases, the relative correction becomes more substantial, as evidenced by the decreasing ratio.

In particular, when considering the beyond-mean-field Josephson frequency, we observe different effects depending on the gas parameter range. For the range $0 \leq \gamma \lesssim 3 \times 10^{-18}$, the contribution of Gaussian fluctuations is relatively limited, resulting in only minor modifications to the MQST critical value. Viceversa, for the range $0.01283 \lesssim \gamma \lesssim 0.01421$, the presence of Gaussian fluctuations leads to a significant reduction in the beyond-mean-field MQST critical value. In this regime, the impact of these fluctuations becomes substantial, resulting in a notable deviation from the mean-field result.

Lastly, it is interesting to note that the reduction in the beyond-mean-field critical value is more prominent for higher values of $|z_0|$. This is in contrast to the case of $D = 3$, where the critical values were smaller for smaller $|z_0|$. These differences highlight the distinct behavior of the system in two dimensions compared to three dimensions.
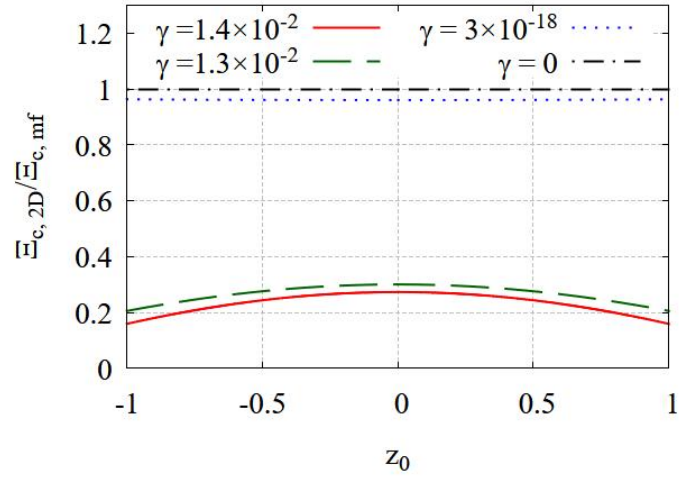
Figure 4.4: 2D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,\,2D}$ and the mean-field one $\Xi_{c,\,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t=0) = \frac{n_1(0)-n_2(0)}{n_1(0)+n_2(0)}$ for different values of the gas parameter $\gamma = a_s^2 n$: $\gamma = 1 \times 10^{-2}$ (red solid line), $\gamma = 1 \times 10^{-3}$ (green dashed line) $\gamma = 1 \times 10^{-4}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). The last line corresponds to the mean-field case.

4.3   D=1 case

As for the contribution of Gaussian fluctuations to the Grand Potential calculation, the procedure to find the beyond mean-field Josephson frequency is analogous to the one used for the $D = 3$ case. Starting from the next-to-leading term of the Lieb-Liniger theory [25], which is given by (3.103) when the effective range r_s is set to zero, the expression for the energy density is:

$$\mathcal{E} = \frac{1}{2}g_0 n^2 - \frac{2}{3\pi}\sqrt{\frac{m}{\hbar^2}}(g_0 n)^{\frac{3}{2}} \tag{4.75}$$

the Lagrangian density is obtained as

$$\mathcal{L} = \sum_{k=1}^{2} \left(i\hbar \Phi_k^*(t)\partial_t \Phi_k(t) - \frac{1}{2}g_0|\Phi_k(t)|^4 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar}|\Phi_k(t)|^3 \right) +$$
$$+ \frac{J}{2}(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t)) \tag{4.76}$$
$$\equiv \mathcal{L}_0 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar}\left(|\Phi_1(t)|^3 + |\Phi_2(t)|^3\right)$$

where $\mathcal{L}_0$ represents the mean-field Lagrangian density, and the second term accounts for the beyond mean-field calculation, taking into account quantum fluctuations.

Integrating in space, the Lagrangian takes the following form

$$\mathcal{L} = \mathcal{L}_0 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar L^{\frac{1}{2}}}\left(|\varphi_1(t)|^3 + |\varphi_2(t)|^3\right) \tag{4.77}$$

where $\mathcal{L}_0$ represents the mean-field Lagrangian, and the other term accounts for the correction due to quantum fluctuations.

Similarly to the previous dimensional cases, a Madelung transformation (2.7) is performed, resulting in

$$\mathcal{L} = \mathcal{L}_0 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar L^{\frac{1}{2}}}\left(N_1^{\frac{3}{2}}(t) + N_2^{\frac{3}{2}}(t)\right) \tag{4.78}$$

which can be rewritten in terms of the total number of particles N and the population imbalance z as

$$\mathcal{L} = \mathcal{L}_0 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{3\sqrt{2}\pi\hbar L^{\frac{1}{2}}}\left[(1+z)^{\frac{3}{2}} + (1-z)^{\frac{3}{2}}\right] \tag{4.79}$$

4.3.1   Josephson Frequency

To calculate the Josephson frequency, we make use of the assumption that the population imbalance is small, i.e., $|z| \ll 1$. This allows us to expand the terms $\sqrt{1 \pm z}$ in the expression (4.7) up to quadratic order in z. By summing the two expanded terms, we obtain the following approximation

$$\mathcal{L} = \mathcal{L}_0 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{3\sqrt{2}\pi\hbar L^{\frac{1}{2}}}\left(2 + \frac{3}{4}z^2\right) \tag{4.80}$$

where the first term in the parenthesis is disregarded since it is a constant, obtaining

$$\mathcal{L} = \mathcal{L}_0 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{4\sqrt{2}\pi\hbar L^{\frac{1}{2}}}z^2 \tag{4.81}$$

Finally, the beyond mean-field Lagrangian in $D = 1$ case is given by

$$\mathcal{L} = \frac{N\hbar}{2}z\dot{\phi} - \left(\frac{UN^2 + JN}{4}\right)z^2 - \frac{JN}{4}\phi^2 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{4\sqrt{2}\pi\hbar L^{\frac{1}{2}}}z^2 \tag{4.82}$$

Therefore the Euler-Lagrangian equations are

$$\begin{cases} 0 = \frac{N\hbar}{2}\dot{\phi} - \left(\frac{UN^2 + JN}{2}\right)z + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{2\sqrt{2}\pi\hbar L^{\frac{1}{2}}}z \\ 0 = -\frac{N\hbar}{2}\dot{z} - \frac{JN}{2}\phi \end{cases} \tag{4.83}$$

Similarly to the mean-field case, we can manipulate the equation to retrieve the Euler-Lagrange equations that resemble those of a harmonic oscillator

$$\begin{cases} \dot{\phi} = \left(\frac{UN + J}{\hbar} - \frac{\sqrt{mg_0^3}N^{\frac{1}{2}}}{\sqrt{2}\pi\hbar^2 L^{\frac{1}{2}}}\right)z \\ \dot{z} = -\frac{J}{\hbar}\phi \end{cases} \implies \begin{cases} \ddot{\phi} + \Omega^2\phi = 0 \\ \ddot{z} + \Omega^2 z = 0 \end{cases} \tag{4.84}$$

where, the beyond mean-field Josephson frequency is given by

$$\Omega \equiv \frac{1}{\hbar}\sqrt{J^2 + JUN - \frac{J\sqrt{g_0^3 nm}}{\sqrt{2}\pi\hbar}} \tag{4.85}$$

Expressing it as a function of the coupling constant g_0 and number density n one has

$$\Omega \equiv \frac{1}{\hbar}\sqrt{J^2 + Jg_0 n - \frac{J\sqrt{g_0^3 nm}}{\sqrt{2}\pi\hbar}}$$

$$= \frac{J}{\hbar}\sqrt{1 + \frac{g_0 n}{J}\left(1 - \frac{1}{\pi}\sqrt{\frac{m}{\hbar^2}\frac{g_0}{2n}}\right)} \tag{4.86}$$

and remembering the definition of Rabi Frequency Ω_R, given by $\Omega_R \equiv = J/\hbar$, one obtains

$$\Omega = \Omega_R\sqrt{1 + \frac{g_0 n}{J}\left(1 - \frac{1}{\pi}\sqrt{\frac{m}{\hbar^2}\frac{g_0}{2n}}\right)} \tag{4.87}$$

Writing now the Josephson frequency as a function of the s-wave scattering length a_s

$$\Omega = \Omega_R\sqrt{1 - \frac{2\hbar^2 n}{ma_s J}\left(1 - \frac{1}{\pi\sqrt{-a_s n}}\right)} \tag{4.88}$$

and defining the reference energy ε_s and the gas parameter γ in the 1-dimensional case as

$$\varepsilon_s \equiv \frac{\hbar^2}{ma_s^2} \tag{4.89}$$

and

$$\gamma \equiv a_s n \tag{4.90}$$

the Josephson Frequency can also be written as

$$\Omega = \Omega_R\sqrt{1 - 2\gamma\frac{\varepsilon_s}{J}\left(1 - \frac{1}{\pi\sqrt{-\gamma}}\right)} \tag{4.91}$$

Note that the gas parameter γ must be negative due to the presence of the inverse of the square root of it.

Analogously to the 3-dimensional case to acknowledge the degree of the beyond mean-field correction to the mean-field Josephson frequency is taking in account the ratio between the corrected frequency and the mean-field one as a function of the strength parameter $\Xi = -2\gamma\varepsilon_s/J$

$$\frac{\Omega}{\Omega_{mf}} = \sqrt{\frac{1 + \Xi\left(1 - \frac{1}{\pi\sqrt{-\gamma}}\right)}{1 + \Xi}} \tag{4.92}$$

Let us focus now on the bounds of the gas parameter γ. The upper bound limit is due to the fact that to obtain (3.102) from (3.101) we used a perturbative approach, assuming $-\gamma \gg 1$, for this reason the upper bound limit is given by $\gamma = -20.3$, while there is no lower bound limit (see more on App.C).

Contrary to the $D = 3$ case, the beyond mean-field Josephson frequency is lower than the mean-field one. Indeed, as pictured in Fig. 4.5 the relative correction $\Omega/\Omega_{mf} \leq 1$, where the equality is obtained for $\gamma \to -\infty$ or when the strength parameter is $\Xi = 0$. Furthermore, the beyond mean-field correction becomes more important as Ξ grows. In fact, for larger strength parameters, $\Xi \gg 1$, the asymptotic behaviour of the relative correction is given by

$$\left.\frac{\Omega}{\Omega_{mf}}\right|_{\Xi\gg1} = \sqrt{1 - \frac{1}{\pi\sqrt{-\gamma}}} \tag{4.93}$$

The relative correction Ω/Ω_{mf} behaviour also depends on the gas parameter γ, for higher values of the gas parameter the correction is more important and so the value of Ω/Ω_{mf} decreases.
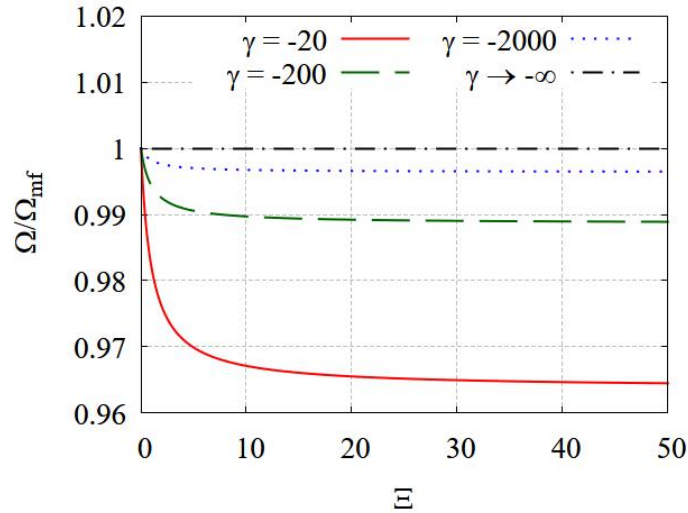

Figure 4.5: 1D beyond mean-field relative correction to the Josephson Frequency.
In the plot is pictured the ratio between the beyond mean-field Josephson frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter $\Xi = \frac{g_0 n}{J}$ for different values of the gas parameter $\gamma = a_s n$: $\gamma = -20$ (red solid line), $\gamma = -200$ (green dashed line) $\gamma = -2000$ (blue dotted line) and $\gamma \to -\infty$ (black dash-dotted line), which corresponds to the mean-field case.

4.3.2   Macroscopic Quantum Self Trapping

To compute the conserved energy one consider the beyond mean-field Lagrangian in $D = 1$ case given by

$$\mathcal{L} = \frac{N\hbar}{2}z\dot{\phi} - \frac{UN^2}{4}z^2 + \frac{JN}{2}\sqrt{1-z^2}\cos\phi + \frac{\sqrt{m g_0^3}N^{\frac{3}{2}}}{3\sqrt{2}\pi\hbar L^{\frac{1}{2}}}\left[(1+z)^{\frac{3}{2}} + (1-z)^{\frac{3}{2}}\right] \tag{4.94}$$

one finds that the conserved energy is

$$E(z,\phi) = \frac{UN^2}{4}z^2 - \frac{JN}{2}\sqrt{1-z^2}\cos\phi - \frac{L\sqrt{m}}{3\sqrt{2}\pi\hbar}(UN)^{\frac{3}{2}}\left[(1+z)^{\frac{3}{2}} + (1-z)^{\frac{3}{2}}\right] \tag{4.95}$$

Imposing the inequality condition, i.e. $E(z_0,\phi_0) > E(0,\pi)$, to observe the Macroscopic Quantum Self Trapping phenomenon (2.35), one gets

$$\begin{aligned}
&\frac{UN^2}{4}z_0^2 - \frac{JN}{2}\sqrt{1-z_0^2}\cos\phi_0+ \\
&- \frac{L\sqrt{m}}{3\sqrt{2}\pi\hbar}(UN)^{\frac{3}{2}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}}\right] > \frac{JN}{2} - \frac{L\sqrt{2m}}{3\pi\hbar}(UN)^{\frac{3}{2}} \\
&\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0+ \\
&- \frac{L\sqrt{2m}}{3\pi\hbar}\frac{U^{\frac{3}{2}}N^{\frac{1}{2}}}{J}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}}\right] > 1 - \frac{2L\sqrt{2m}}{3\pi\hbar}\frac{U^{\frac{3}{2}}N^{\frac{1}{2}}}{J} \\
&\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0+ \\
&- \frac{L\sqrt{2m}}{3\pi\hbar}\Xi U^{\frac{1}{2}}N^{-\frac{1}{2}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}}\right] > 1 - \frac{2L\sqrt{2m}}{3\pi\hbar}\Xi U^{\frac{1}{2}}N^{-\frac{1}{2}}
\end{aligned} \tag{4.96}$$

and finally

$$\Xi > \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - \frac{L\sqrt{2m}}{3\pi\hbar}U^{\frac{1}{2}}N^{-\frac{1}{2}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}} - 2\right]} \tag{4.97}$$

Defining the critical value $\Xi_{c,1D}$ as

$$\Xi_{c,1D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - \frac{2L\sqrt{2m}}{3\pi\hbar}U^{\frac{1}{2}}N^{-\frac{1}{2}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}} - 2\right]} \tag{4.98}$$

or, alternatively, as a function of g_0 and n

$$\Xi_{c,1D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - \frac{\sqrt{2}}{3\pi}\sqrt{\frac{m}{\hbar^2}}g_0^{\frac{1}{2}}n^{-\frac{1}{2}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}} - 2\right]} \tag{4.99}$$

or, also, as a function of the gas parameter γ

$$\Xi_{c,1D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - \frac{2}{3\pi}\frac{1}{\sqrt{-\gamma}}\left[(1+z_0)^{\frac{3}{2}} + (1-z_0)^{\frac{3}{2}} - 2\right]} \tag{4.100}$$

One obtains the same inequality condition of the mean-field case

$$\Xi > \Xi_{c,1D} \tag{4.101}$$

In this case, it is interesting to note that the critical value is reached for larger values of Ξ since $\Xi_{c,1D} > \Xi_{c,mf}$, namely the beyond mean-field critical value $\Xi_{c,1D}$ is larger than the mean-field one. Indeed, dividing $\Xi_{c,1D}$ by the mean-field critical value $\Xi_{c,mf}$ one gets

$$\frac{\Xi_{c,1D}}{\Xi_{c,mf}} = \frac{1}{1 - \frac{4}{3\pi}\frac{1}{\sqrt{-\gamma}}\frac{(1+z_0)^{\frac{3}{2}}+(1-z_0)^{\frac{3}{2}}-2}{z_0^2}} \tag{4.102}$$

and looking at Fig. 4.6 one finds $\Xi_{c,\,1D}/\Xi_{c,\,mf} \geq 1$, where the equality is verified for $\gamma \to -\infty$, which corresponds to the mean-field case. It is evident that the beyond-mean-field correction becomes less significant as the gas parameter decreases. Additionally, at a fixed γ value, the correction is more substantial for higher values of $|z_0|$.

Drawing a parallel to the beyond-mean-field correction in other dimensions, we find similarities between the 1-dimensional and 2-dimensional cases. In both scenarios, higher values of $|z_0|$ correspond to increased quantum fluctuation effects. However, unlike the 3-dimensional and 2-dimensional cases, the beyond-mean-field corrections in the 1-dimensional case lead to an increase in the strength parameter above which Macroscopic Quantum Self Trapping occurs.
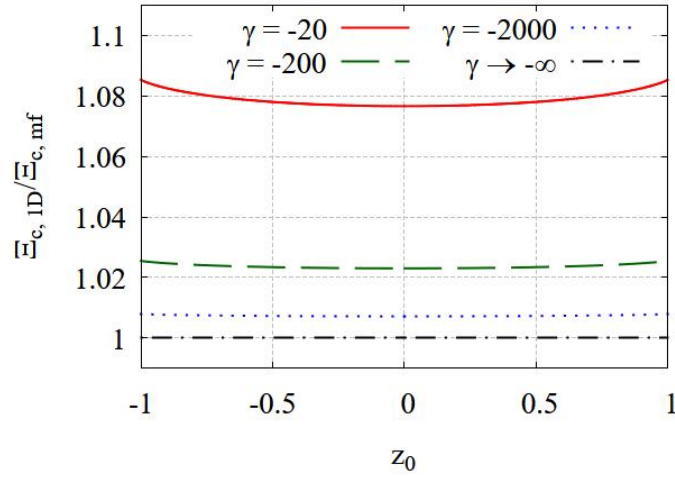

Figure 4.6: 1D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,\,1D}$ and the mean-field one $\Xi_{c,\,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t = 0) = (n_1(0) - n_2(0))/(n_1(0) + n_2(0))$ for different values of the gas parameter $\gamma = a_s n$: $\gamma = -20$ (red solid line), $\gamma = -200$ (green dashed line) $\gamma = -2000$ (blue dotted line) and $\gamma \to -\infty$ (dark dash-dotted line). The last line corresponds to the mean-field case.

4.4   D=0 case

The beyond mean-field energy in $D = 0$, given by (3.118), is

$$E(N) = \frac{g_0 N^2}{2}\left(1 - \frac{1}{N}\right)$$
(4.103)

The beyond mean-field lagrangian is thus

$$\mathcal{L} = \sum_{k=1}^{2}\left[i\hbar\varphi_k^*(t)\partial_t\varphi_k(t) - \frac{g_0}{2}|\varphi_k(t)|^4\left(1 - \frac{1}{|\varphi_k(t)|^2}\right)\right]$$
$$+ \frac{J}{2}\left(\varphi_1^*(t)\varphi_2(t) + \varphi_2^*(t)\varphi_1(t)\right)$$
(4.104)

Performing the Madelung transformation (2.7) on each of the potential terms, one obtains

$$\frac{g_0}{2}|\varphi_k(t)|^4\left(1 - \frac{1}{|\varphi_k(t)|^2}\right) = \frac{g_0 N_k^2}{2}\left(1 - \frac{1}{N_k}\right)$$
$$= \frac{g_0 N^2}{8}(1 \pm z)^2\left(1 - \frac{1}{N(1 \pm z)}\right)$$
(4.105)
$$= \frac{g_0 N^2}{8}(1 \pm z)^2 - \frac{g_0 N}{8}(1 \pm z)$$

where in the second step we rewrite the number of particles in each site as a function of the total number of particles and the population imbalance $N_{1,2} = N(1 \pm z)/2$.

Summing up the two contribution one has

$$\sum_{k=1}^{2}\frac{g_0}{2}|\varphi_k(t)|^4\left(1 - \frac{1}{|\varphi_k(t)|^2}\right) = \frac{g_0 N^2}{4}(1 + z^2) - \frac{g_0 N}{4}$$
$$= \frac{g_0 N^2 z^2}{4}) + \frac{g_0 N(N-1)}{4}$$
(4.106)

However, the second term, representing the beyond-mean field term is a constant term and therefore can be eliminated. The final result is

$$\sum_{k=1}^{2}\frac{g_0}{2}|\varphi_k(t)|^4\left(1 - \frac{1}{|\varphi_k(t)|^2}\right) = \frac{g_0 N^2 z^2}{4}$$
(4.107)

which is the equal to the mean-field one, hence quantum fluctuation does not provide any correction to either the Josephson frequency or the MQST critical value.

Chapter 5

Josephson Junction: Beyond mean-field with finite range results

In this chapter, we examine the corrected energy density, taking into account also the finite range corrections. We incorporate these corrections into the Lagrangian density of the bosonic atomic Josephson junction. By doing so, we can investigate the combined effects of quantum fluctuations and the finite range potential on the Josephson junction in various dimensions. We exclude the $D = 0$ case from our analysis since there are no finite range corrections to the potential. In this case, the contact potential is exact. Conversely, in dimensions $D = 1, 2, 3$, the finite range corrections to the Josephson frequency, denoted as K_{Ω, r_s}, and the critical strength of the MQST (Macroscopic Quantum Self-Trapping), denoted as K_{Ξ_c, r_s}, can be expressed as follows:

$$\Omega = \frac{J}{\hbar}\sqrt{1 + \Xi K_{\Omega, r_s}} \qquad \Xi_c = \frac{1 + \sqrt{1 - z_0^2}\cos\theta_0}{z_0^2/2 + K_{\Xi_c, r_s}} \tag{5.1}$$

The values of the corrections for each dimension are shown in Table 5.1 and Table 5.2.

D	K_{Ω, r_s}
3	$1 + \frac{8\sqrt{2a_s^3 n}}{\sqrt{\pi}(1+2\pi a_s^2 r_s n)^2}\left(1 - \frac{8}{3}\frac{2\pi a_s^2 r_s n}{1+2\pi a_s^2 r_s n} + \frac{8}{5}\frac{(2\pi a_s^2 r_s n)^2}{(1+2\pi a_s^2 r_s n)^2}\right)$
2	$1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)} + \sum_{\ell=1}^{+\infty}\mathcal{C}_\ell\left(\frac{2\pi^2 r_s^2 n}{\ln^2(Cn)}\right)^\ell\left(\frac{(\ell+2)(\ell+1)}{2} - \frac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)}{\ln^2(Cn)}\right)$
1	$1 - \frac{1}{\pi(1-r_s n)\sqrt{-a_s n}}\left(1 + 4\frac{r_s n}{1-r_s n} + \frac{8}{3}\frac{(r_s n)^2}{(1-r_s n)^2}\right)$

Table 5.1: Beyond mean-field correction to the Josephson frequency in dimension $D = 1, 2, 3$. The correction is determined by the s-wave scattering length a_s, the s-wave effective range r_s, and the number density n. In the case of $D = 2$, the result depends also on the parameters $C = \pi e^{2\gamma+1}a_s^2$ and $\mathcal{C}_\ell$, with the discussion of the latter's dependence on ℓ being presented in Chapter 3.

D	Ξ_{c, r_s}
3	$\frac{2\sqrt{2}}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 g_0^{\frac{3}{2}} n^{\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{5}{2}}}{[1+2\pi a_s^2 r_s n(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+2\pi a_s^2 r_s n(1-z_0)]^2} - \frac{2}{(1+2\pi a_s^2 r_s n)^2}\right)$
2	$\frac{K(z_0,n) - \left(1 + z_0^2 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell\left(2\pi^2 r_s^2 n \ln^{-2}(Cn)\right)^\ell\right)}{2}$
1	$-\frac{\sqrt{2}}{3\pi}\sqrt{\frac{m}{\hbar^2}} g_0^{\frac{1}{2}} n^{-\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{3}{2}}}{1-r_s n(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1-r_s n(1-z_0)} - \frac{2}{1-r_s n}\right)$

Table 5.2: Beyond mean-field correction to the MQST critical value in dimension $D = 1, 2, 3$. The correction is a function of the s-wave scattering length a_s, the s-wave effective range r_s, the number density n and the initial population imbalance z_0. In $D = 2$ the result depends also on a function of z_0 and n given by $K(z, n) \equiv \frac{1}{2}\left[\frac{(1+z)^2}{1+\frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1+\frac{\ln(1-z)}{\ln(Cn)}} + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell\left(\frac{2\pi^2 r_s^2 n}{\ln^2(Cn)}\right)^\ell\left(\frac{(1+z)^{\ell+2}}{\left[1+\frac{\ln(1+z)}{\ln(Cn)}\right]^{2\ell+1}} + \frac{(1-z)^{\ell+2}}{\left[1+\frac{\ln(1-z)}{\ln(Cn)}\right]^{2\ell+1}}\right)\right]$

5.1 D=3 case

Substituting the mean-field energy density with the beyond mean-field energy density found in (3.31), which account also finite range correction and its form is given by

$$\mathcal{E} = \frac{1}{2}g_0 n^2 + \frac{8}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 \frac{(g_0 n)^{\frac{5}{2}}}{(1 + 4\pi a_s^2 r_s n)^2} \tag{5.2}$$

The Lagrangian density obtains a new term

$$\mathcal{L} = \sum_{k=1}^{2} \left(i\hbar\Phi_k^*(t)\partial_t\Phi_k(t) - \frac{1}{2}g_0|\Phi_k(t)|^4 - \frac{8g_0^{\frac{5}{2}}}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 \frac{|\Phi_k(t)|^5}{(1 + 4\pi a_s^2 r_s|\Phi_k(t)|^2)^2} \right) + $$
$$+ \frac{J}{2}(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t)) \tag{5.3}$$
$$\equiv \mathcal{L}_0 - \frac{8g_0^{\frac{5}{2}}\sqrt{m^3}}{15\pi^2\hbar^3} \left(\frac{|\Phi_1(t)|^5}{(1 + 4\pi a_s^2 r_s|\Phi_1(t)|^2)^2} + \frac{|\Phi_2(t)|^5}{(1 + 4\pi a_s^2 r_s|\Phi_2(t)|^2)^2} \right)$$

where $\mathcal{L}_0$ is the mean-field Lagrangian density and the remaining part is the beyond mean-field contribution accounting also finite range corrections. Integrating the Lagrangian density in space one obtains the beyond mean-field Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - \frac{8g_0^{\frac{5}{2}}\sqrt{m^3}}{15\pi^2\hbar^3 L^{\frac{9}{2}}} \left(\frac{|\varphi_1(t)|^5}{(1 + \frac{4\pi a_s^2 r_s}{L^3}|\varphi_1(t)|^2)^2} + \frac{|\varphi_2(t)|^5}{(1 + \frac{4\pi a_s^2 r_s}{L^3}|\varphi_2(t)|^2)^2} \right) \tag{5.4}$$

Note that the Lagrangian is equal in form to the one which not account finite range terms (4.4). However, each beyond mean-field term is attenuated by a factor that depends on the s-wave parameters a_s and r_s, the corresponding wave function, and the size of the site.

By performing the Madelung transformation (2.7), the new term becomes dependent on the number of particles present in the two sites N_k

$$\mathcal{L} = \mathcal{L}_0 - \frac{8\sqrt{m^3 g_0^5}}{15\pi^2\hbar^3 L^{\frac{9}{2}}} \left(\frac{N_1^{\frac{5}{2}}(t)}{[1 + \frac{4\pi a_s^2 r_s}{L^3}N_1(t)]^2} + \frac{N_2^{\frac{5}{2}}(t)}{[1 + \frac{4\pi a_s^2 r_s}{L^3}N_2(t)]^2} \right) \tag{5.5}$$

By expressing the number of particles in each site as a function of the total number of particles N and the population imbalance z, namely $N_{1,2} = N(1 \pm z)/2$, the Lagrangian can be reformulated as follows

$$\mathcal{L} = \mathcal{L}_0 - \frac{\sqrt{2m^3 g_0^5}N^{\frac{5}{2}}}{15\pi^2\hbar^3 L^{\frac{9}{2}}} \left[\frac{(1+z)^{\frac{5}{2}}}{[1 + R(1+z)]^2} + \frac{(1-z)^{\frac{5}{2}}}{[1 + R(1-z)]^2} \right] \tag{5.6}$$

where the common factor R multiplying $(1 \pm z)$ at the denominators is dimensionless and it is defined in terms of the s-wave scattering length a_s, the s-wave finite range r_s and the system number density n

$$R \equiv 2\pi a_s^2 r_s n = \frac{1}{2}\chi g_0 n \tag{5.7}$$

5.1.1 Josephson Frequency

Now, since we are in the low population imbalance limit, i.e. $|z| \ll 1$, it is possible to expand the numerator of the beyond mean-field term of the Lagrangian (5.6) as

$$(1 \pm z)^{\frac{5}{2}} = 1 \pm \frac{5}{2}z + \frac{15}{8}z^2 + O(z^3) \tag{5.8}$$

and the denominator of the same fraction as

$$\frac{1}{[1+R(1\pm z)]^2} = \frac{1}{(1+R)^2} \mp \frac{2Rz}{(1+R)^3} + \frac{3R^2z^2}{(1+R)^4} + O(z^3)$$

$$= \frac{1}{(1+R)^2}\left(1 \mp \frac{2Rz}{1+R} + \frac{3R^2z^2}{(1+R)^2}\right) + O(z^3)$$

(5.9)

Putting the approximations (5.8) and (5.9) together one obtains

$$\frac{(1\pm z)^{\frac{5}{2}}}{[1+R(1\pm z)]^2} = \frac{1\pm\frac{5}{2}z+\frac{15z^2}{8}}{(1+R)^2}\left(1 \mp \frac{2Rz}{1+R} + \frac{3R^2z^2}{(1+R)^2}\right) + O(z^3)$$

$$= \frac{1}{(1+R)^2}\left(1 \pm \frac{5}{2}z \mp \frac{2Rz}{1+R} + \frac{15z^2}{8} - \frac{5Rz^2}{1+R} + \frac{3R^2z^2}{(1+R)^2}\right) + O(z^3)$$

(5.10)

and the sum of the two contributions is given by

$$\frac{(1+z)^{\frac{5}{2}}}{[1+R(1+z)]^2} + \frac{(1-z)^{\frac{5}{2}}}{[1+R(1-z)]^2} = \frac{1}{(1+R)^2}\left(2 + \frac{15z^2}{4} - \frac{10Rz^2}{1+R} + \frac{6R^2z^2}{(1+R)^2}\right) + O(z^3) \quad (5.11)$$

Inserting this approximation in (5.6) the beyond mean-field Lagrangian reads

$$\mathcal{L} = \mathcal{L}_0 - \frac{\sqrt{2m^3g_0^5}N^{\frac{5}{2}}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\frac{1}{(1+R)^2}\left(2 + \frac{15z^2}{4} - \frac{10Rz^2}{1+R} + \frac{6R^2z^2}{(1+R)^2}\right)$$

(5.12)

where the term independent of z is removed since it is constant. Factoring out $15z^2/4$ one obtains

$$\mathcal{L} = \mathcal{L}_0 - \frac{\sqrt{m^3g_0^5}N^{\frac{5}{2}}}{2\sqrt{2}\pi^2\hbar^3 L^{\frac{9}{2}}}\frac{z^2}{(1+R)^2}\left(1 - \frac{8}{3}\frac{R}{1+R} + \frac{8}{5}\frac{R^2}{(1+R)^2}\right)$$

(5.13)

Finally, the beyond-mean-field Lagrangian in the $D=3$ case is

$$\mathcal{L} = \frac{N\hbar}{2}z\dot{\phi} - \left(\frac{UN^2+JN}{4}\right)z^2 - \frac{JN}{4}\phi^2 - \frac{\sqrt{m^3g_0^5}N^{\frac{5}{2}}}{2\sqrt{2}\pi^2\hbar^3 L^{\frac{9}{2}}(1+R)^2}\left(1 - \frac{8}{3}\frac{R}{1+R} + \frac{8}{5}\frac{R^2}{(1+R)^2}\right)z^2 \quad (5.14)$$

and the Euler-Lagrangian equations are given by

$$\begin{cases} 0 = \frac{N\hbar}{2}\dot{\phi} - \left(\frac{UN^2+JN}{2}\right)z - \frac{\sqrt{m^3g_0^5}N^{\frac{5}{2}}}{\sqrt{2}\pi^2\hbar^3 L^{\frac{9}{2}}(1+R)^2}\left(1 - \frac{8}{3}\frac{R}{1+R} + \frac{8}{5}\frac{R^2}{(1+R)^2}\right)z \\ 0 = -\frac{N\hbar}{2}\dot{z} - \frac{JN}{2}\phi \end{cases}$$

(5.15)

Similarly to the to the mean-field case, we can rearrange the equation to recover the Euler-Lagrange equations of a harmonic oscillator, in fact

$$\begin{cases} \dot{\phi} = \left[\frac{UN+J}{\hbar} + \frac{\sqrt{2m^3g_0^5}N^{\frac{3}{2}}}{\pi^2\hbar^4 L^{\frac{9}{2}}(1+R)^2}\left(1 - \frac{8}{3}\frac{R}{1+R} + \frac{8}{5}\frac{R^2}{(1+R)^2}\right)\right]z \\ \dot{z} = -\frac{J}{\hbar}\phi \end{cases} \implies \begin{cases} \ddot{\phi} + \Omega^2\phi = 0 \\ \ddot{z} + \Omega^2 z = 0 \end{cases}$$

(5.16)

where the frequency of the harmonic oscillators Ω, which has the form

$$\Omega \equiv \sqrt{\frac{1}{\hbar^2}(J^2+JUN) + \frac{J\sqrt{2g_0^5 n^3 m^3}}{\pi^2\hbar^5(1+R)^2}\left(1 - \frac{8}{3}\frac{R}{1+R} + \frac{8}{5}\frac{R^2}{(1+R)^2}\right)}$$

(5.17)

is the corrected Josephson frequency. This quantity can also be written as a function of the coupling constant g_0, the ratio χ and the number density n

$$\Omega \equiv \sqrt{\frac{1}{\hbar^2}(J^2+Jg_0n) + \frac{4J\sqrt{2g_0^5 n^3 m^3}}{\pi^2\hbar^5(2+\chi g_0 n)^2}\left(1 - \frac{8}{3}\frac{\chi g_0 n}{2+\chi g_0 n} + \frac{8}{5}\frac{(\chi g_0 n)^2}{(2+\chi g_0 n)^2}\right)}$$

$$= \sqrt{\frac{J^2}{\hbar^2} + \frac{Jg_0n}{\hbar^2}\left[1 + \frac{4\sqrt{2}}{\pi^2}\sqrt{\frac{m^3}{\hbar^2}}\frac{\sqrt{g_0^3 n}}{(2+\chi g_0 n)^2}\left(1 - \frac{8}{3}\frac{\chi g_0 n}{2+\chi g_0 n} + \frac{8}{5}\frac{(\chi g_0 n)^2}{(2+\chi g_0 n)^2}\right)\right]}$$

(5.18)

Remembering the definition of Rabi Frequency Ω_R, one obtains

$$\Omega = \Omega_R \sqrt{1 + \frac{g_0 n}{J}\left[1 + \frac{4\sqrt{2}}{\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 \frac{\sqrt{g_0^3 n}}{(2+\chi g_0 n)^2}\left(1 - \frac{8}{3}\frac{\chi g_0 n}{2+\chi g_0 n} + \frac{8}{5}\frac{(\chi g_0 n)^2}{(2+\chi g_0 n)^2}\right)\right]} \qquad (5.19)$$

and writing explicitly the frequency as a function of the s-wave scattering length a_s and the s-wave effective range r_s

$$\Omega = \Omega_R \sqrt{1 + \frac{4\pi\hbar^2 a_s n}{mJ}\left[1 + \frac{8\sqrt{2a_s^3 n}}{\sqrt{\pi}(1+2\pi a_s^2 r_s n)^2}\left(1 - \frac{8}{3}\frac{2\pi a_s^2 r_s n}{1+2\pi a_s^2 r_s n} + \frac{8}{5}\frac{(2\pi a_s^2 r_s n)^2}{(1+2\pi a_s^2 r_s n)^2}\right)\right]} \qquad (5.20)$$

Moreover, remember the definitions of the reference energy ε_s and the gas parameter γ, and defining the adimensional ratio α as

$$\epsilon_s \equiv \frac{\hbar^2}{ma_s^2} \qquad (5.21)$$

$$\gamma \equiv a_s^3 n \qquad (5.22)$$

and

$$\alpha \equiv \frac{r_s}{a_s} \qquad (5.23)$$

the Josephson Frequency can also be written as

$$\Omega = \Omega_R \sqrt{1 + 4\pi\gamma\frac{\varepsilon_s}{J}\left[1 + \frac{8}{(1+2\pi\alpha\gamma)^2}\sqrt{\frac{2\gamma}{\pi}}\left(1 - \frac{8}{3}\frac{2\pi\alpha\gamma}{1+2\pi\alpha\gamma} + \frac{8}{5}\frac{(2\pi\alpha\gamma)^2}{(1+2\pi\alpha\gamma)^2}\right)\right]} \qquad (5.24)$$

To understand the magnitude of the beyond mean-field correction to the Josephson Frequency the ratio between the beyond mean-field Josephson Frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter, given by $\Xi = 4\pi\gamma\varepsilon_s/J$, is done. Namely it is considered

$$\frac{\Omega}{\Omega_{mf}} = \frac{\sqrt{1 + \Xi\left[1 + \frac{8}{(1+2\pi\alpha\gamma)^2}\sqrt{\frac{2\gamma}{\pi}}\left(1 - \frac{8}{3}\frac{2\pi\alpha\gamma}{1+2\pi\alpha\gamma} + \frac{8}{5}\frac{(2\pi\alpha\gamma)^2}{(1+2\pi\alpha\gamma)^2}\right)\right]}}{\sqrt{1+\Xi}} \qquad (5.25)$$

Looking at the Fig. 5.1 one observes the following behavior: similarly to the case in which finite range correction are not involved the correction is more significant at higher strength parameters Ξ. When the strength parameters approach zero ($\Xi \to 0$), the beyond mean-field correction becomes irrelevant regardless of the ratio α, which means that for small values of Ξ the correction has a very weak dependence on the finite range corrections. Conversely, for larger Ξ, the relative correction is given by

$$\left.\frac{\Omega}{\Omega_{mf}}\right|_{\Xi\gg1} = \sqrt{1 + \frac{8}{(1+2\pi\alpha\gamma)^2}\sqrt{\frac{2\gamma}{\pi}}\left(1 - \frac{8}{3}\frac{2\pi\alpha\gamma}{1+2\pi\alpha\gamma} + \frac{8}{5}\frac{(2\pi\alpha\gamma)^2}{(1+2\pi\alpha\gamma)^2}\right)} \qquad (5.26)$$

At a fixed value of γ, Fig. 5.1 demonstrates that finite range corrections are more significant for higher magnitudes of $|\alpha|$. Specifically, for $\alpha < 0$, the correction is greater than the one without considering finite range corrections (i.e., for $\alpha = 0$), while the opposite is observed for $\alpha > 0$, where the correction is lower than the $\alpha = 0$ case. This can be observed more precisely by considering the specific finite range correction illustrated in Fig. 5.2. It can be noted that this correction becomes more prominent as the absolute value of the ratio α increases. Furthermore, the sign of the correction is opposite to that of the ratio α.

Since the sign of the ratio α is the same as that of the finite range r_s (due to the positive scattering length a_s), this can be interpreted as follows: if the finite range correction to the repulsive contact-like potential is repulsive ($\alpha > 0$), then the finite range correction mitigates the effects of quantum fluctuations. Conversely, if the finite range correction to the repulsive contact-like potential is attractive ($\alpha < 0$), then the finite range correction amplifies the corrections due to quantum fluctuations. However, regardless of sign of the ratio α, the beyond mean-field Josephson frequency is greater then the mean-field one.
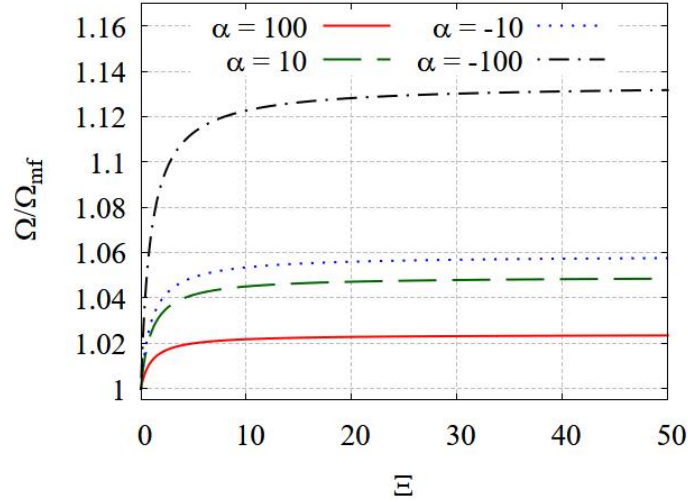

Figure 5.1: 3D beyond mean-field relative correction to the Josephson Frequency.
In the plot is pictured the ratio between the beyond mean-field Josephson frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter $\Xi = \frac{g_0 n}{J}$ for different values of the ratio $\alpha = \frac{r_s}{a_s}$: $\alpha = 100$ (red solid line), $\alpha = 10$ (green dashed line) $\alpha = -10$ (blue dotted line) and $\alpha = -100$ (dark dash-dotted line). In all the cases the gas parameter is $\gamma = a_s^3 n = 0.0003$.
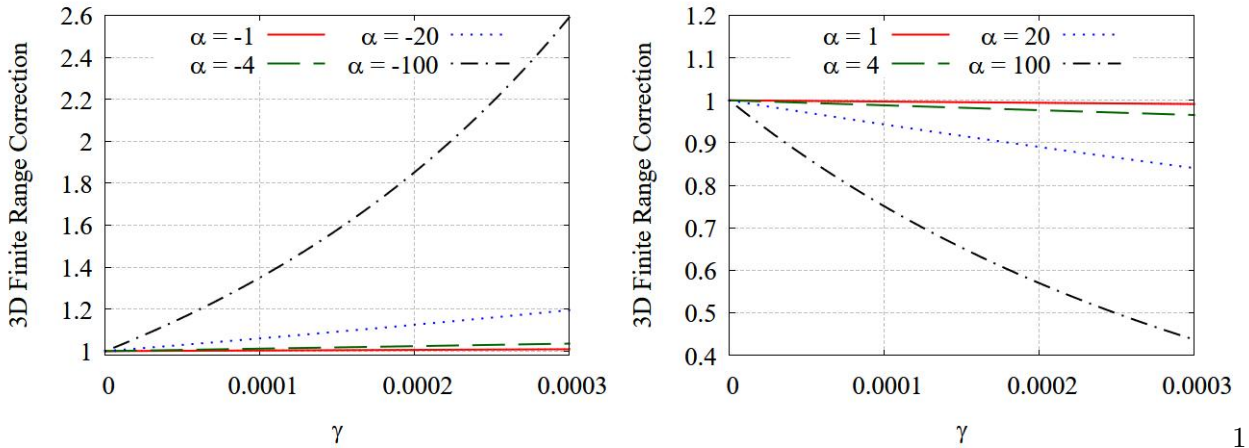

Figure 5.2: 3D Finite range correction $\frac{1}{(1+2\pi\alpha\gamma)^2}\left[1 - \frac{8}{3}\frac{2\pi\alpha\gamma}{1+2\pi\alpha\gamma} + \frac{8}{5}\left(\frac{2\pi\alpha\gamma}{1+2\pi\alpha\gamma}\right)^2\right]$ as a function of the gas parameter $\gamma = a_s^3 n$ at fixed $\alpha = r_s/a_s$.
Left: $\alpha = -1$ (red solid line), $\alpha = -4$ (green dashed line) $\alpha = -20$ (blue dotted line) and $\alpha = -100$ (dark dash-dotted line). **Right**: $\alpha = 1$ (red solid line), $\alpha = 4$ (green dashed line) $\alpha = 20$ (blue dotted line) and $\alpha = 100$ (dark dash-dotted line).

5.1.2 Macroscopic Quantum Self Trapping

Starting from the beyond mean-field Lagrangian written in terms of the total number of particles N, the population imbalance z and the phase difference ϕ

$$\mathcal{L} = \frac{N\hbar}{2}z\dot{\phi} - \frac{UN^2}{4}z^2 + \frac{JN}{2}\sqrt{1-z^2}\cos\phi - \frac{\sqrt{2m^3 g_0^5}N^{\frac{5}{2}}}{15\pi^2\hbar^3 L^{\frac{9}{2}}}\left[\frac{(1+z)^{\frac{5}{2}}}{[1+R(1+z)]^2} + \frac{(1-z)^{\frac{5}{2}}}{[1+R(1-z)]^2}\right] \quad (5.27)$$

one finds that the beyond mean-field conserved energy is

$$E = \frac{UN^2}{4}z^2 - \frac{JN}{2}\sqrt{1-z^2}\cos\phi + \frac{L^3\sqrt{2m^3}}{15\pi^2\hbar^3}(UN)^{\frac{5}{2}}\left[\frac{(1+z)^{\frac{5}{2}}}{[1+R(1+z)]^2} + \frac{(1-z)^{\frac{5}{2}}}{[1+R(1-z)]^2}\right] \quad (5.28)$$

Imposing the inequality condition (2.35), i.e. $E(z_0,\phi_0) > E(0,\pi)$, to have MQST one gets

$$\frac{UN^2}{4}z_0^2 - \frac{JN}{2}\sqrt{1-z_0^2}\cos\phi_0 +$$
$$+ \frac{L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{5}{2}}N^{\frac{5}{2}}\left[\frac{(1+z_0)^{\frac{5}{2}}}{[1+R(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+R(1-z_0)]^2}\right] > \frac{JN}{2} + \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\frac{U^{\frac{5}{2}}N^{\frac{5}{2}}}{(1+R)^2}$$

$$\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0 +$$
$$+ \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\frac{U^{\frac{5}{2}}N^{\frac{3}{2}}}{J}\left[\frac{(1+z_0)^{\frac{5}{2}}}{[1+R(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+R(1-z_0)]^2}\right] > 1 + \frac{4L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\frac{U^{\frac{5}{2}}N^{\frac{3}{2}}}{J(1+R)^2} \quad (5.29)$$

$$\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0 +$$
$$+ \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\Xi U^{\frac{3}{2}}N^{\frac{1}{2}}\left[\frac{(1+z_0)^{\frac{5}{2}}}{[1+R(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+R(1-z_0)]^2}\right] > 1 + \frac{4L^3\sqrt{2m^3}}{15\pi^2\hbar^3}\frac{U^{\frac{3}{2}}N^{\frac{1}{2}}}{(1+R)^2}\Xi$$

and finally

$$\Xi > \frac{1+\sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{3}{2}}N^{\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{5}{2}}}{[1+R(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+R(1-z_0)]^2} - \frac{2}{(1+R)^2}\right)} \quad (5.30)$$

The inequality condition still has the form

$$\Xi > \Xi_{c,\,3D} \quad (5.31)$$

where the critical value defined as

$$\Xi_{c,\,3D} \equiv \frac{1+\sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{2L^3\sqrt{2m^3}}{15\pi^2\hbar^3}U^{\frac{3}{2}}N^{\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{5}{2}}}{[1+R(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+R(1-z_0)]^2} - \frac{2}{(1+R)^2}\right)} \quad (5.32)$$

which can also be written as a function of g_0 and n

$$\Xi_{c,\,3D} \equiv \frac{1+\sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{2\sqrt{2}}{15\pi^2}\sqrt{\frac{m}{\hbar^2}}^3 g_0^{\frac{3}{2}}n^{\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{5}{2}}}{[1+2\pi a^2 r_s n(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+2\pi a^2 r_s n(1-z_0)]^2} - \frac{2}{(1+2\pi a^2 r_s n)^2}\right)} \quad (5.33)$$

or as a function of the gas parameter γ and the ratio α

$$\Xi_{c,\,3D} \equiv \frac{1+\sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{16\sqrt{2}}{15\sqrt{\pi}}\sqrt{\gamma}\left(\frac{(1+z_0)^{\frac{5}{2}}}{[1+2\pi\alpha\gamma(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+2\pi\alpha\gamma(1-z_0)]^2} - \frac{2}{(1+2\pi\alpha\gamma)^2}\right)} \quad (5.34)$$

Note that since the denominator of $\Xi_{c,\,3D}$ is larger than $\Xi_{c,\,mf}$, the beyond mean-field critical value for macroscopic quantum self-trapping is smaller than the mean-field critical value, i.e., $\Xi_{c,\,3D} < \Xi_{c,\,mf}$, just as in the case where finite range corrections are not considered, namely the ratio is given by

$$\frac{\Xi_{c,\,3D}}{\Xi_{c,\,mf}} = \frac{1}{1 + \frac{32\sqrt{2}}{15\sqrt{\pi}z_0^2}\sqrt{\gamma}\left(\frac{(1+z_0)^{\frac{5}{2}}}{[1+2\pi\alpha\gamma(1+z_0)]^2} + \frac{(1-z_0)^{\frac{5}{2}}}{[1+2\pi\alpha\gamma(1-z_0)]^2} - \frac{2}{(1+2\pi\alpha\gamma)^2}\right)} \tag{5.35}$$

By examining the graph in Fig. 5.3, a comparison can be made between the behavior of the beyond mean-field MQST critical value and of the beyond mean-field Josephson frequency in $D = 3$ as a function of the ratio α. Likewise, the correction due to the finite range term in the inter-atomic potential is more significant as the absolute value of the ratio $|\alpha|$ increases. However, there is an opposite behavior observed between the two quantities. While the finite range correction increases the value of the Josephson frequency compared to the mean-field value if the ratio is negative, and decreases it if the ratio is positive, for the beyond-mean-field MQST critical value, the effect is opposite.

Nevertheless, the interpretation given for the Josephson frequency remains valid: if the finite range correction to the repulsive contact-like potential is repulsive ($\alpha > 0$), then the finite range correction mitigates the effects of quantum fluctuations. Conversely, if the finite range correction to the repulsive contact-like potential is attractive ($\alpha < 0$), then the finite range correction amplifies the corrections due to quantum fluctuations.
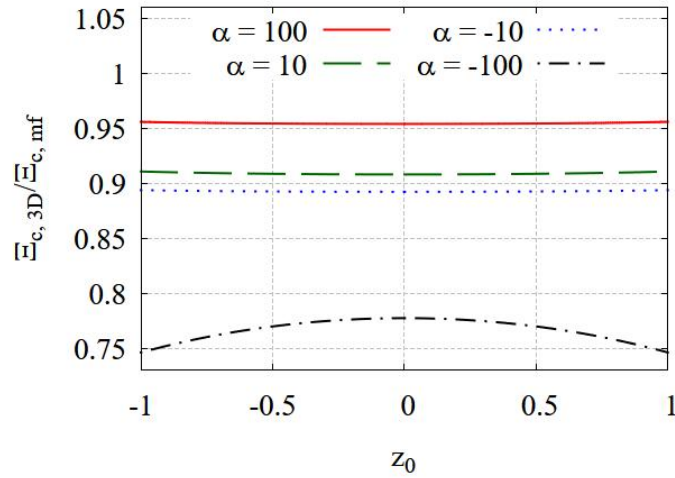

Figure 5.3: 3D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,\,3D}$ and the mean-field one $\Xi_{c,\,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t = 0) = (n_1(0) - n_2(0))/(n_1(0) + n_2(0))$ for different values of the ratio $\alpha = r_s/a_s$: $\alpha = 100$ (red solid line), $\alpha = 10$ (green dashed line) $\alpha = -10$ (blue dotted line) and $\alpha = -100$ (dark dash-dotted line). In all the cases the gas parameter is $\gamma = a_s^3 n = 0.0003$.

5.2   D=2 case

As in the previous chapters, the 2-dimensional case differs significantly from the other cases and therefore requires careful treatment and consideration.

The $D = 2$ beyond mean-field energy density is given by (3.93)

$$\mathcal{E}(n) = \frac{g_r n^2}{2}\left(1 + \sum_{\ell=1}^{+\infty} \mathcal{C}_i(\chi_r g_r n)^\ell\right) \tag{5.36}$$

Hence, the renormalized Lagrangian density is given by

$$\mathscr{L} = \sum_{k=1}^{2}\left[\hbar\Phi_k^*(t)\partial_t\Phi_k(t) - \frac{1}{2}g_{r,k}|\Phi_k(t)|^4\left(1 + \sum_{\ell=1}^{+\infty}\mathcal{C}_\ell(\chi_{r,k}g_{r,k}|\Phi_k(t)|^2)^\ell\right)\right] + \\ + \frac{J}{2}(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t)) \tag{5.37}$$

and it is obtained substituting g_0 with $g_{r,k}$ and adding the finite range term. The coupling constant was computed in the last chapter (4.37) and it is given by

$$g_{r,k} = \frac{g_r}{1 + \frac{\ln(1\pm z)}{\ln(Cn)}} \qquad k = 1,2 \tag{5.38}$$

where, as a reminder, the constant C is defined as $C \equiv \pi e^{2\gamma+1}a_s^2$.

Integrating in space, the corresponding beyond mean-field Lagrangian is

$$\mathcal{L} = \sum_k\left[i\hbar\varphi_k^*(t)\partial_t\varphi_k(t) - \frac{U_k}{2}|\varphi_k(t)|^4\left(1 + \sum_{\ell=1}^{+\infty}\mathcal{C}_\ell\left(\frac{\chi_{r,k}g_{r,k}}{L^2}\right)^\ell|\varphi_k(t)|^{2\ell}\right)\right] + \\ + \frac{J}{2}(\varphi_1^*(t)\varphi_2(t) + \varphi_2^*(t)\varphi_1(t)) \tag{5.39}$$

where as a reminder

$$U_k \equiv \frac{g_{r,k}}{L^2} \qquad \varphi_k(t) \equiv L\Phi_k(t) \quad k = 1,2 \tag{5.40}$$

How the terms $U_1|\varphi_1|/2$ and $U_1|\varphi_k|/2$ transforms through the Madelung transformation is calculated in (4.43) (for $k = 1$) and (4.45) (for k=2) and as a reminder its expression is

$$\frac{1}{2}U_k|\varphi_k(t)|^4 = \frac{U_r N^2}{8}\frac{(1\pm z)^2}{1 + \frac{\ln(1\pm z)}{\ln(Cn)}} \tag{5.41}$$

where for the ℓ-th finite range correction the procedure is similar and result in

$$\mathcal{C}_\ell\left(\frac{\chi_{r,k}g_{r,k}}{L^2}\right)^\ell|\varphi_k(t)|^{2\ell} = \mathcal{C}_\ell\left(\frac{4\pi^2 r_s^2}{L^2}\right)^\ell\frac{1}{\ln^{2\ell}(Cn_k)}N_k^\ell \\ = \mathcal{C}_\ell\left(\frac{4\pi^2 r_s^2}{L^2}\right)^\ell\frac{1}{\ln^{2\ell}(Cn(1\pm z))}\frac{N^\ell}{2^\ell}(1\pm z)^\ell \\ = \mathcal{C}_\ell\left(\frac{2\pi^2 r_s^2 n}{\ln^2(Cn)}\right)^\ell\frac{1}{\left[1 + \frac{\ln(1\pm z)}{\ln(Cn)}\right]^{2\ell}}(1\pm z)^\ell \tag{5.42}$$

and as did for the the 3-dimensional case we define a parameter R, given by

$$R \equiv \frac{2\pi^2 r_s^2 n}{\ln^2(Cn)} = \frac{1}{2}\chi_r g_r n \tag{5.43}$$

5.2.1   Josephson Frequency

Considering our interest in computing the Josephson frequency, we focus on the low population imbalance limit, where $|z(t)| \ll 1$. In this regime, the term (5.41) is Taylor expanded with respect to z as

$$\frac{U_r N^2}{8} \frac{(1 \pm z)^2}{1 + \frac{\ln(1 \pm z)}{\ln(Cn)}} = \frac{U_r N^2}{8} \left(1 + \frac{z^2 \pm 2z}{2\ln(Cn)} + \frac{z^2}{\ln^2(Cn)}\right)(1 \pm 2z + z^2) + O(z^3) \quad (5.44)$$

Summing the two contributes one obtains the beyond-mean field results that neglects the finite range correction

$$\sum_k \frac{U_{r,k}}{2}|\varphi_k(t)|^4 = \frac{U_r N^2}{4}\left(1 + z^2 - \frac{3z^2}{2\ln(Cn)} + \frac{z^2}{\ln^2(Cn)}\right) + O(z^3) \quad (5.45)$$

Writing this result in terms of the system coupling, inverting the relation (4.33)

$$\frac{1}{\ln(Cn)} = -\frac{mg_r}{4\pi\hbar^2} \quad (5.46)$$

one gets

$$\sum_k \frac{U_{r,k}}{2}|\varphi_k(t)|^4 = \frac{U_r N^2}{4}\left[1 + z^2\left(1 + \frac{3}{8}\frac{mg_r}{\pi\hbar^2} + \frac{1}{16}\frac{m^2 g_r^2}{\pi^2 \hbar^4}\right)\right] \quad (5.47)$$

For the ℓ-th finite range correction the procedure is similar and the expansion is given by

$$\mathcal{C}_\ell R^\ell \frac{(1 \pm z)^\ell}{\left[1 + \frac{\ln(1 \pm z)}{\ln(Cn)}\right]^{2\ell}} = \mathcal{C}_\ell R^\ell \left(1 + \frac{2\ell z^2 \mp 4\ell z}{2\ln(Cn)} + \frac{2\ell(2\ell + 1)z^2}{2\ln^2(Cn)}\right)\left(1 \pm \ell z + \frac{\ell(\ell-1)}{2}z^2\right) + O(z^3) \quad (5.48)$$

where z^3 term or higher are discarded since they are higher order terms.
Multiplying (5.48) by (5.44) one obtains the ℓ-th finite range contribution

$$\frac{U_r N^2 \mathcal{C}_\ell R^\ell}{8} \frac{(1 \pm z)^{\ell+2}}{\left[1 + \frac{\ln(1 \pm z)}{\ln(Cn)}\right]^{2\ell+1}} = \frac{U_r N^2 \mathcal{C}_\ell R^\ell}{8}\left(1 \pm (\ell + 2)z + \frac{(\ell+2)(\ell+1)}{2}z^2\right) \times$$

$$\times \left(1 + \frac{(2\ell+1)z^2 \mp 2(2\ell+1)z}{2\ln(Cn)} + \frac{(2\ell+1)(2\ell+2)z^2}{2\ln^2(Cn)}\right) + O(z^3)$$

$$= \frac{U_r N^2 \mathcal{C}_\ell R^\ell}{8}\left(1 \pm (\ell + 2)z + \frac{(\ell+2)(\ell+1)}{2}z^2 + \frac{(2\ell+1)z^2 \mp 2(2\ell+1)z}{2\ln(Cn)}\right.$$

$$\left.+ \frac{(2\ell+1)(2\ell+2)z^2}{2\ln^2(Cn)} - \frac{2(2\ell+1)(\ell+2)z^2}{2\ln(Cn)}\right) \quad (5.49)$$

Summing up the two contributions one gets

$$\frac{\mathcal{C}_\ell}{2}\sum_{k=1}^{2} U_k \left(\frac{\chi_{r,k}g_{r,k}}{L^2}\right)^\ell |\varphi_k(t)|^{2\ell+4} = \frac{U_r N^2 \mathcal{C}_\ell R^\ell}{4}\left(1 + \frac{(\ell+2)(\ell+1)}{2}z^2 + \right.$$

$$\left.- \frac{(2\ell+1)(2\ell+3)z^2}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)z^2}{\ln^2(Cn)}\right) + O(z^3) \quad (5.50)$$

As done previously, we remove the term $z-$independent, since it is a constant, obtaining the $\ell - th$ finite range correction term to the 2-dimensional beyond mean-field Lagrangian

$$\frac{\mathcal{C}_\ell}{2}\sum_{k=1}^{2} U_k \left(\frac{\chi_{r,k}g_{r,k}}{L^2}\right)^\ell |\varphi_k(t)|^{2\ell+4} = \frac{U_r N^2 \mathcal{C}_\ell R^\ell z^2}{4}\left(\frac{(\ell+2)(\ell+1)}{2} + \right.$$

$$\left.- \frac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)}{\ln^2(Cn)}\right) \quad (5.51)$$

The whole interaction term for the 2-dimensional beyond mean-field Lagrangian which account also for finite range correction to the inter-atomic potential is

$$\sum_{k=1}^{2} \frac{U_k}{2} |\varphi_k(t)|^4 \left(1 + \sum_{\ell=1}^{+\infty} \mathcal{C}_\ell \left(\frac{\chi_{r,k} g_{r,k}}{L^2} \right)^\ell |\varphi_k(t)|^{2\ell} \right) \tag{5.52}$$

is therefore given by

$$\frac{U_r N^2 z^2}{4} \left[1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)} + \sum_{\ell=1}^{+\infty} \mathcal{C}_\ell R^\ell \left(\frac{(\ell+2)(\ell+1)}{2} - \frac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)}{\ln^2(Cn)} \right) \right] \tag{5.53}$$

The term is similar to the mean-field one, given by $\frac{U N^2 z^2}{4}$ with caution to substitute U with U_r and add all the corrections, the second and the third terms inside the parenthesis are the beyond the mean-field corrections due to the contact interaction terms while the subsequent terms are the correction given by the inclusion of the finite range interaction. Therefore to obtain the Josephson frequency in the 2-dimensional beyond mean-field framework is sufficient to substituting inside (2.27) the energy constant U with

$$U_r \left[1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)} + \sum_{\ell=1}^{+\infty} \mathcal{C}_\ell R^\ell \left(\frac{(\ell+2)(\ell+1)}{2} - \frac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)}{\ln^2(Cn)} \right) \right] \tag{5.54}$$

Doing so, the Josephson frequency is given by

$$\begin{aligned} \Omega^2 = & \frac{J^2}{\hbar^2} + \frac{JNU_r}{\hbar^2} \left(1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)} \right) + \\ & + \frac{JNU_r}{\hbar^2} \left[\sum_{\ell=1}^{+\infty} \mathcal{C}_\ell R^\ell \left(\frac{(\ell+2)(\ell+1)}{2} - \frac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)}{\ln^2(Cn)} \right) \right] \end{aligned} \tag{5.55}$$

Note that, in the limit of low density $n \ll 1$, the term

$$\frac{1}{\ln^\ell Cn} \tag{5.56}$$

become more and more smaller higher is ℓ, therefore keeping only terms of the order $\frac{1}{\ln Cn}$, since

$$U_r \propto \frac{1}{\ln(Cn)} \qquad R \propto \frac{1}{\ln^2(Cn)} \tag{5.57}$$

then (5.54) can be approximated to U_r and so the Josephson frequency in $D = 2$ reduces to mean-field one (2.27)

$$\Omega = \frac{1}{\hbar} \sqrt{J^2 + U_r N J} \tag{5.58}$$

with the care of substituting U with U_r. Writing explicitly the Rabi frequency, the strength parameter Ξ and $\ln(Cn)$ inside the Josephson frequency

$$\Omega = \Omega_R \sqrt{1 + \Xi \left[1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)} + \sum_{\ell=1}^{+\infty} \mathcal{C}_\ell R^\ell \left(\frac{(\ell+2)(\ell+1)}{2} - \frac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \frac{(2\ell+1)(\ell+1)}{\ln^2(Cn)} \right) \right]} \tag{5.59}$$

For simplicity we analyze in details only the first contribution due to finite range correction, i.e. $\ell = 1$, obtaining

$$\Omega = \Omega_R \sqrt{1 + \Xi \left[1 - \frac{3}{2\ln(Cn)} + \frac{1}{\ln^2(Cn)} + \mathcal{C}_1 R \left(3 - \frac{15}{2\ln(Cn)} + \frac{6}{\ln^2(Cn)} \right) \right]} \tag{5.60}$$

Expressing then the finite range Josephson frequency as a function of the gas parameter γ and the ratio α, namely

$$\Omega = \Omega_R \sqrt{1 + \Xi \left[1 - \frac{3}{2\ln(C^*\gamma)} + \frac{1}{\ln^2(C^*\gamma)} + \frac{9\pi^2 \alpha^2 \gamma}{\ln^2(C^*\gamma)} \left(1 - \frac{5}{2\ln(C^*\gamma)} + \frac{2}{\ln^2(C^*\gamma)} \right) \right]} \tag{5.61}$$

where $C^* = C/a_s^2 = \pi e^{2\gamma+1}$. Finally, to evaluate the finite range correction contribute to the beyond mean-field Josephson frequency we divide it by the mean-field one obtaining

$$\frac{\Omega}{\Omega_{mf}} = \sqrt{\frac{1 + \Xi \left[1 - \frac{3}{2\ln(C^*\gamma)} + \frac{1}{\ln^2(C^*\gamma)} + \frac{9\pi^2 \alpha^2 \gamma}{\ln^2(C^*\gamma)} \left(1 - \frac{5}{2\ln(C^*\gamma)} + \frac{2}{\ln^2(C^*\gamma)} \right) \right]}{1 + \Xi}} \tag{5.62}$$

The values of the gas parameter γ in which the approximation that we have done are valid depends on the value of α (to see more on this dependence see App.B). The boundary of the range where the approximation holds is presented in Tab.B.1. The ratio Ω/Ω_{mf} is then plotted in the Fig. 5.4 for $\alpha = 1, 4, 20, 100$.

As for the other dimensional cases, including or not the finite range correction, the dependence on the strength parameter is the usual one. As Ξ approaches zero, we recover the mean-field result, while for higher values of Ξ, i.e., $\Xi \gg 1$, the relative correction beyond mean-field becomes independent of Ξ. This correction can be expressed as:

$$\left. \frac{\Omega}{\Omega_{mf}} \right|_{\Xi \gg 1} = \sqrt{1 - \frac{3}{2\ln(C^*\gamma)} + \frac{1}{\ln^2(C^*\gamma)} + \frac{9\pi^2 \alpha^2 \gamma}{\ln^2(C^*\gamma)} \left(1 - \frac{5}{2\ln(C^*\gamma)} + \frac{2}{\ln^2(C^*\gamma)} \right)} \tag{5.63}$$

Now focusing on the dependence on the ratio α, it can be observed from (5.63) that the dependence is quadratic. Therefore, the sign of α, and thus of the s-wave effective range r_s, do not affect the finite range correction. This is consistent with the fact that the constant coupling of the finite range correction to the inter-atomic potential, i.e. g_{2r}, is also unaffected from the sign of the s-wave effective range due to the quadratic dependence on r_s. For this reason we only consider positive values of the ratio α.

As in the case where the finite range corrections were absent ($\alpha = 0$) the beyond-mean field Josephson junction is approximately the mean-field one for the range $0 \leq \gamma \lesssim 3 \times 10^{-18}$, while for the second range, which varies when changing α, we have important finite range corrections. In particular the finite range contributions are higher and higher when the ratio increases.

It is important to note that the second range depends on the ratio α. Specifically, for higher values of α, the boundaries of the range are smaller. Thus, the dependence on the ratio α is stronger than that of the gas parameter γ since the beyond mean-field correction increases with the ratio α while the gas parameter decreases.

Furthermore, an interesting observation arises regarding the beyond-mean field correction for systems with a gas parameter at the boundaries of the second range of suitable parameters. For different values of α, when the ratio α is small, the correction due to Gaussian fluctuations and finite range effects at the boundaries of the second range are similar, resulting in a less pronounced dependence on the gas parameter. However, for higher values of α, the correction to the Josephson frequency becomes more reliant on the gas parameter value. Specifically, at the upper limit of the second range,

the beyond-mean field Josephson frequency is more than triple the mean-field value, while at the lower limit, the corrected Josephson frequency is just double the mean-field value.
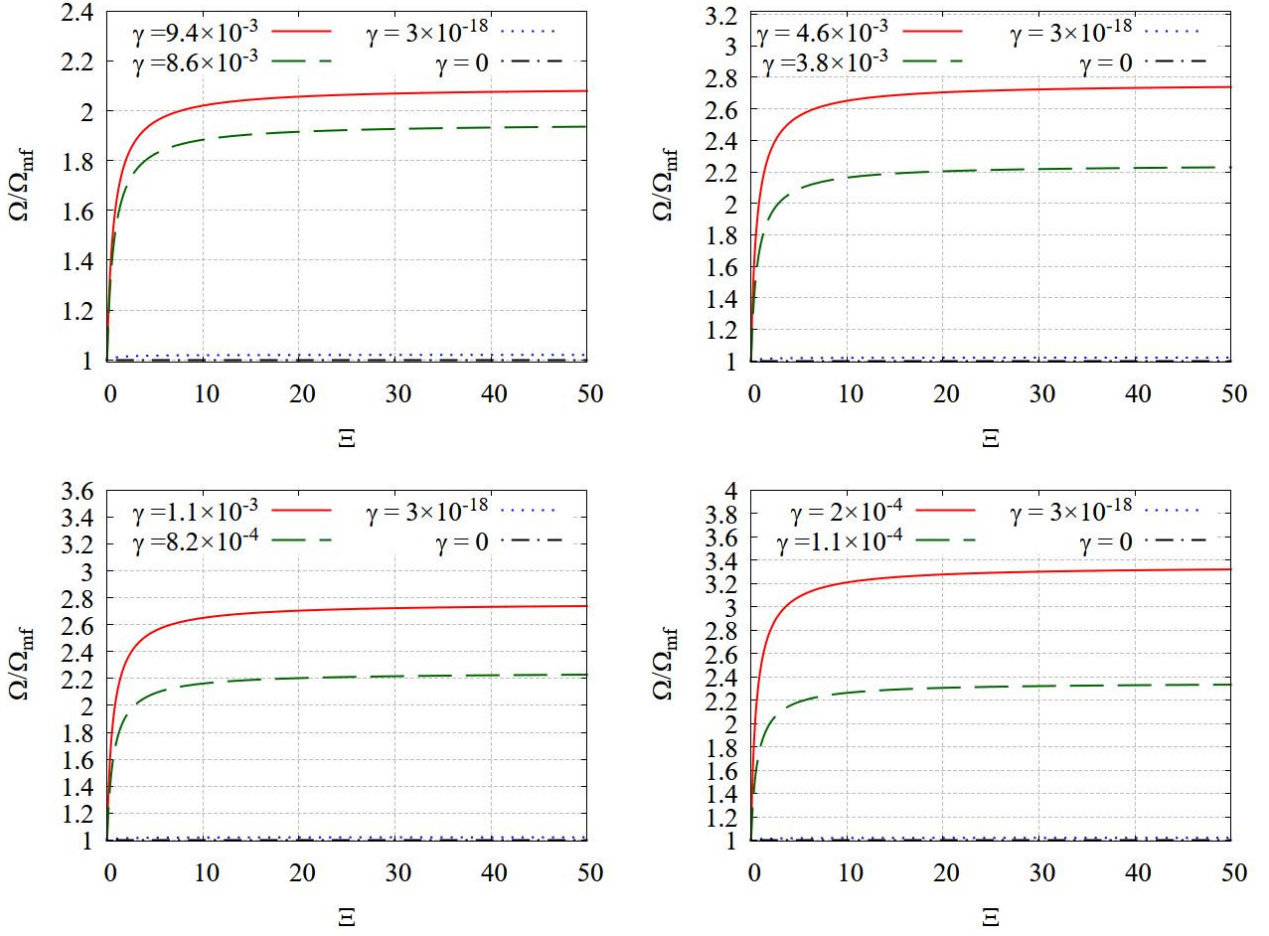

Figure 5.4: 2D beyond mean-field relative correction to the Josephson Frequency.
In the plots is pictured the ratio between the beyond mean-field Josephson frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter $\Xi = \frac{g_r n}{J}$ for different values of the ratio $\alpha = r_s/a_s$. **(Top Left)**: At fixed $\alpha = 1$, $\gamma = 9.4 \times 10^{-3}$ (red solid line), $\gamma = 8.6 \times 10^{-3}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). **(Top Rigth)**: At fixed $\alpha = 4$, $\gamma = 4.4 \times 10^{-3}$ (red solid line), $\gamma = 3.8 \times 10^{-3}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). **(Bottom Left)**: At fixed $\alpha = 20$, $\gamma = 1.1 \times 10^{-3}$ (red solid line), $\gamma = 8.2 \times 10^{-4}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). **(Bottom Right)**: At fixed $\alpha = 100$, $\gamma = 2 \times 10^{-4}$ (red solid line), $\gamma = 1.1 \times 10^{-4}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). In all the cases the last line represent the mean-field case.

5.2.2 Macroscopic Quantum Self Trapping

Unlike the Josephson frequency calculation, in the MQST one the low population imbalance limit is not taken. Therefore it is necessary to see how the interaction terms transform in the case the population imbalance is generic. Taking into account the contact interaction terms

$$\sum_k \frac{1}{2} U_k |\varphi_k(t)|^4 = \frac{N^2}{8} [U_1(1+z)^2 + U_2(1-z^2)] \tag{5.64}$$

Keeping in mind about the definitions of U_r and U_k, they can be linked by the followed relation

$$U_k = U_r \frac{1}{1 + \frac{\ln(1 \pm z)}{\ln(Cn)}} \qquad k = 1, 2 \tag{5.65}$$

Then the interaction term reduces to

$$\sum_k \frac{1}{2} U_k |\varphi_k(t)|^4 = \frac{U_r N^2}{8} \left(\frac{(1+z)^2}{1 + \frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1 + \frac{\ln(1-z)}{\ln(Cn)}} \right) \tag{5.66}$$

Instead for the ℓ-th finite range interaction correction one has

$$\sum_{k=1}^{2} \frac{1}{2} U_k \mathcal{C}_\ell \left(\frac{\chi_{r,k} g_{r,k}}{L^2} \right)^\ell |\varphi_k(t)|^{2\ell+4} = \frac{1}{2} \sum_{k=1}^{2} U_k \mathcal{C}_\ell \left(\frac{4\pi^2 r_s^2}{L^2} \right)^\ell \frac{N_k^{\ell+2}}{\ln^{2\ell}(Cn_k)}$$

$$= \frac{U_r N^2}{8} \mathcal{C}_\ell \left(\frac{2\pi^2 r_s^2 n}{\ln^2(Cn)} \right)^\ell \left(\frac{(1+z)^{\ell+2}}{\left[1 + \frac{\ln(1+z)}{\ln(Cn)} \right]^{2\ell+1}} + \frac{(1-z)^{\ell+2}}{\left[1 + \frac{\ln(1-z)}{\ln(Cn)} \right]^{2\ell+1}} \right)$$

$$= \frac{U_r N^2}{8} \mathcal{C}_\ell R_r^\ell \left(\frac{(1+z)^{\ell+2}}{\left[1 + \frac{\ln(1+z)}{\ln(Cn)} \right]^{2\ell+1}} + \frac{(1-z)^{\ell+2}}{\left[1 + \frac{\ln(1-z)}{\ln(Cn)} \right]^{2\ell+1}} \right) \tag{5.67}$$

Hence, the beyond mean-field Lagrangian in the $D = 2$ case is

$$L = \frac{N\hbar}{2} z\dot{\phi} + \frac{JN}{2} \sqrt{1-z^2} \cos\phi +$$
$$- \frac{U_r N^2}{8} \left[\frac{(1+z)^2}{1 + \frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1 + \frac{\ln(1-z)}{\ln(Cn)}} + \sum_{\ell=1}^{\infty} \mathcal{C}_\ell R_r^\ell \left(\frac{(1+z)^{\ell+2}}{\left[1 + \frac{\ln(1+z)}{\ln(Cn)} \right]^{2\ell+1}} + \frac{(1-z)^{\ell+2}}{\left[1 + \frac{\ln(1-z)}{\ln(Cn)} \right]^{2\ell+1}} \right) \right]$$
$$\frac{N\hbar}{2} z\dot{\phi} + \frac{JN}{2} \sqrt{1-z^2} \cos\phi - K(z,n) \frac{U_r N^2}{4} \tag{5.68}$$

where we introduce the beyond-mean field energy correction to the 2-dimensional Lagrangian K

$$K(z,n) \equiv \frac{1}{2} \left[\frac{(1+z)^2}{1 + \frac{\ln(1+z)}{\ln(Cn)}} + \frac{(1-z)^2}{1 + \frac{\ln(1-z)}{\ln(Cn)}} + \sum_{\ell=1}^{\infty} \mathcal{C}_\ell R_r^\ell \left(\frac{(1+z)^{\ell+2}}{\left[1 + \frac{\ln(1+z)}{\ln(Cn)} \right]^{2\ell+1}} + \frac{(1-z)^{\ell+2}}{\left[1 + \frac{\ln(1-z)}{\ln(Cn)} \right]^{2\ell+1}} \right) \right] \tag{5.69}$$

From the beyond mean-field Lagrangian one can compute the conserved energy

$$E = K(z,n) \frac{U_r N^2}{4} - \frac{JN}{2} \sqrt{1-z^2} \cos\phi \tag{5.70}$$

and imposing the MQST inequality condition, given by $E(z_0, \phi_0) > E(0, \pi)$ one obtains

$$K(z_0, n)\frac{U_r N^2}{4} - \frac{JN}{2}\sqrt{1 - z_0^2}\cos\phi_0 > \frac{JN}{2} + K(0, n)\frac{U_r N^2}{4}$$

$$K(z_0, n)\frac{U_r N^2}{4} - \frac{JN}{2}\sqrt{1 - z_0^2}\cos\phi_0 > \frac{JN}{2} + \left(1 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell\right)\frac{U_r N^2}{4}$$

$$\left[K(z_0, n) - \left(1 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell\right)\right]\frac{U_r N^2}{4} - \frac{JN}{2}\sqrt{1 - z_0^2}\cos\phi_0 > \frac{JN}{2}$$

$$\left[K(z_0, n) - \left(1 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell\right)\right]\frac{\Xi_r}{2} - \sqrt{1 - z_0^2}\cos\phi_0 > 1$$

(5.71)

where we define the adimensional constant Ξ_r as

$$\Xi_r \equiv \frac{U_r N}{J}$$

(5.72)

the inequality reduces to

$$\Xi_r > \frac{1 + \sqrt{1 - z_0^2}\cos\phi_0}{\left[K(z_0, n) - (1 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell)\right]/2}$$

$$\Xi_r > \frac{1 + \sqrt{1 - z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{K(z_0, n) - \left(1 + z_0^2 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell\right)}{2}}$$

(5.73)

where we make explicit the mean-field term $z_0^2/2$ at the denominator. The inequality condition can be expressed as

$$\Xi_r > \Xi_{c,\, 2D}$$

(5.74)

where the critical value $\Xi_{c,\, 2D}$ is given by

$$\Xi_{c,\, 2D} \equiv \frac{1 + \sqrt{1 - z_0^2}\cos\phi_0}{\frac{z_0^2}{2} + \frac{K(z_0, n) - \left(1 + z_0^2 + \sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell\right)}{2}}$$

(5.75)

The finite range correction is contained inside the function $K(z_0, n)$, which expression is given by (5.69) and inside the series $\sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell$. If we consider only the first term of the finite range correction, the function $K(z_0, n)$ assumes the following form

$$K_1(z_0, n) \equiv \frac{1}{2}\left\{\frac{(1 + z_0)^2}{1 + \frac{\ln(1 + z_0)}{\ln(Cn)}} + \frac{(1 - z_0)^2}{1 + \frac{\ln(1 - z_0)}{\ln(Cn)}} + \frac{3\pi^2 r_s^2 n}{\ln^2(Cn)}\left[\left(\frac{1 + z_0}{1 + \frac{\ln(1 + z_0)}{\ln(Cn)}}\right)^3 + \left(\frac{1 + z_0}{1 + \frac{\ln(1 + z_0)}{\ln(Cn)}}\right)^3\right]\right\}$$

(5.76)

will the first term of the series $\sum_{\ell=1}^{\infty}\mathcal{C}_\ell R^\ell$ is given by

$$C_1 R = \frac{3\pi^2 r_s^2 n}{\ln^2(Cn)}$$

(5.77)

To understand the significance of the beyond mean-field correction which accounts also for finite range contribution in the following we consider the ratio between the beyond-mean field MQST critical value and the mean-field one, namely

$$\frac{\Xi_{c,\, 2D}}{\Xi_{c,\, mf}} \equiv \frac{1}{1 + \frac{K_1(z_0, n) - \left(1 + z_0^2 + \frac{3\pi^2 r_s^2 n}{\ln^2(Cn)}\right)}{z_0^2}}$$

(5.78)

Writing it in terms of the adimensional quantities γ and α the ratio is given by

$$\frac{\Xi_{c,2D}}{\Xi_{c,mf}} \equiv \frac{1}{1 + \frac{K_1(z_0,\gamma) - \left(1 + z_0^2 + \frac{3\pi^2\alpha^2\gamma}{\ln^2(C^*\gamma)}\right)}{z_0^2}} \tag{5.79}$$

where $C^* = e^{2\gamma+1}\pi$ and

$$K_1(z_0,\gamma) \equiv \frac{1}{2}\left\{\frac{(1+z_0)^2}{1 + \frac{\ln(1+z_0)}{\ln(C^*\gamma)}} + \frac{(1-z_0)^2}{1 + \frac{\ln(1-z_0)}{\ln(C^*\gamma)}} + \frac{3\pi^2\alpha^2\gamma}{\ln^2(C^*\gamma)}\left[\left(\frac{1+z_0}{1 + \frac{\ln(1+z_0)}{\ln(C^*\gamma)}}\right)^3 + \left(\frac{1+z_0}{1 + \frac{\ln(1+z_0)}{\ln(C^*\gamma)}}\right)^3\right]\right\} \tag{5.80}$$

Looking at the pictures in Fig. 5.6 we can observe that as for the 2-dimensional beyond mean-field Josephson frequency Ω, there are two distinct behaviours for the two ranges of gas parameter for which the approximation used for calculating the expression for the chemical potential are valid. In the first range, i.e. $0 \leq \gamma \lesssim 3 \times 10^{-18}$, the correction are almost negligible. This is because the system has a very low number density in this range, making the beyond-mean-field correction marginal. In the second range instead, the contributes due to finite range contributes and Gaussian fluctuations are of great significance. Similar to the scenario with a contact-like inter-atomic potential, the critical values are higher for initial population imbalances closer to zero, at a fixed gas parameter γ and fixed ratio α. This gives the plots of the ratio $\Xi_{c,2D}/\Xi_{c,mf}$ a concave shape, which becomes more pronounced for smaller values of the ratio α. As shown in Fig. 5.5, when the ratio increases, the finite range correction flattens the ratio $\Xi_{c,2D}/\Xi_{c,mf}$, reducing the dependence of the ratio on the population imbalance.

Furthermore, with the increasing of the ratio α there is also an enhancing of the beyond mean-field effects as the ratio is lower and lower than 1.

Additionally, comparing the two plots in Fig. 4.4 the effects of the finite range correction combined with the Gaussian fluctuations ones are higher, i.e. the ratio is smaller, for higher values of the gas parameter γ. This observation also holds true for the concavity at the same ratio α, as the concavity decreases as the gas parameter increases. However, it's important to note that the stronger corrections for higher values of γ only apply at a fixed ratio α. In general, the ratio $\Xi_{c,2D}/\Xi_{c,mf}$ exhibits a stronger dependence on α due to its quadratic dependence, while the dependence on γ is linear and logarithmic.
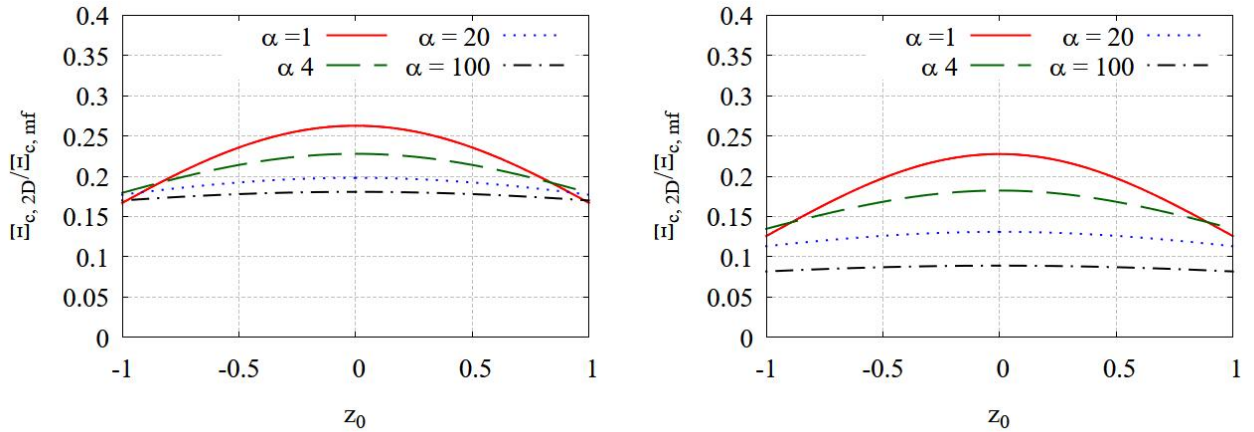

Figure 5.5: 2D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,2D}$ and the mean-field one $\Xi_{c,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t=0) = (n_1(0) - n_2(0))/(n_1(0) + n_2(0))$ for different values of the ratio $\alpha = r_s/a_s$ and the gas parameter $\gamma = a_s^2 n$. For both the plots $\alpha = 100$ (red solid line), $\alpha = 20$ (green dashed line) $\alpha = 4$ (blue dotted line) and $\alpha = 1$ (dark dash-dotted line).
(Left): Each curve as the gas parameter γ equal to the γ_{min} presented in Tab.B.1.
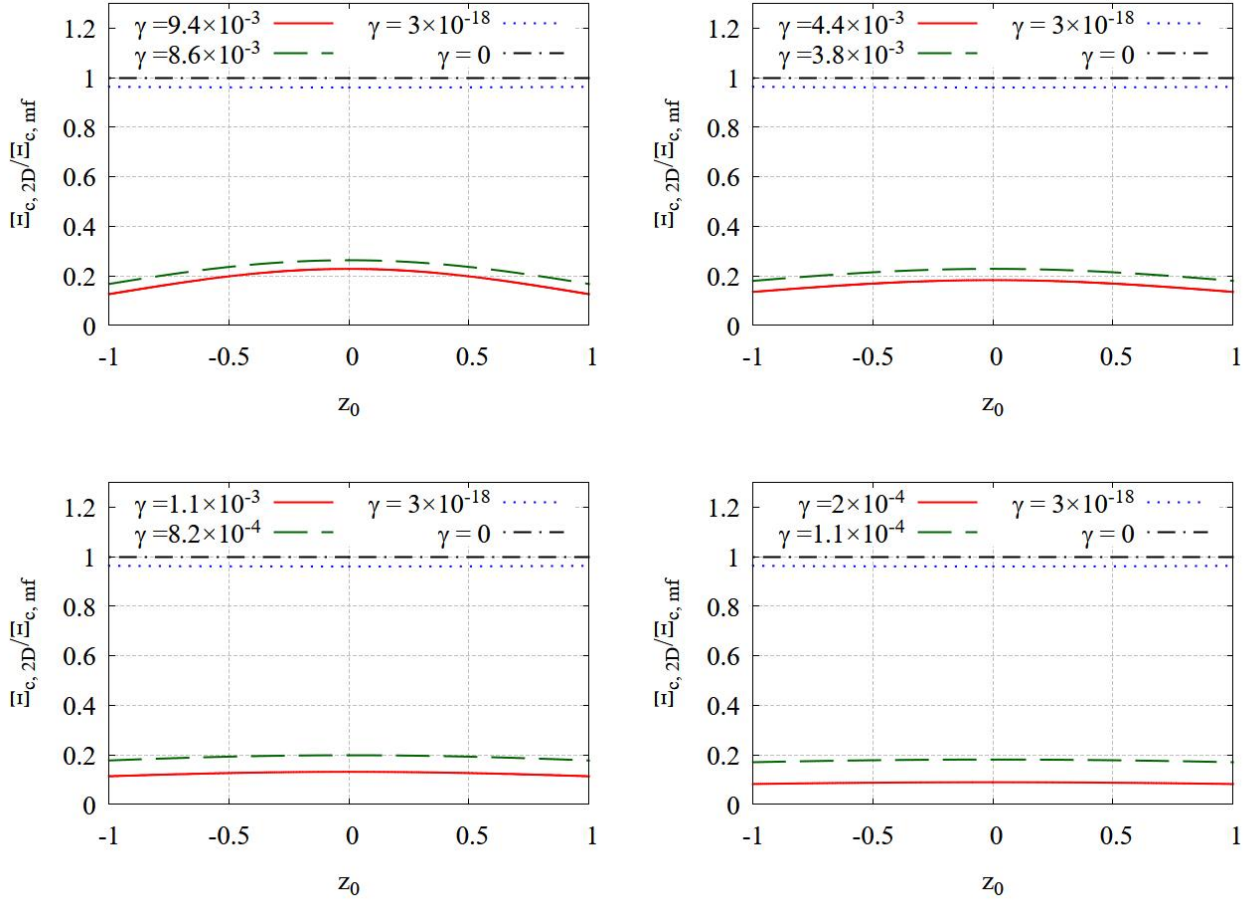**(Right)**: Each curve as the gas parameter γ equal to the γ_{max} presented in Tab.B.1.

Figure 5.6: 2D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,\,2D}$ and the mean-field one $\Xi_{c,\,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t=0) = (n_1(0) - n_2(0))/(n_1(0) + n_2(0))$ for different values of the ratio $\alpha = r_s/a_s$ and the gas parameter $\gamma = a_s^2 n$ **(Top Left)**: At fixed $\alpha = 1$, $\gamma = 9.4 \times 10^{-3}$ (red solid line), $\gamma = 8.6 \times 10^{-3}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). **(Top Right)**: At fixed $\alpha = 4$, $\gamma = 4.4 \times 10^{-3}$ (red solid line), $\gamma = 3.8 \times 10^{-3}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). **(Bottom Left)**: At fixed $\alpha = 20$, $\gamma = 1.1 \times 10^{-3}$ (red solid line), $\gamma = 8.2 \times 10^{-4}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). **(Bottom Right)**: At fixed $\alpha = 100$, $\gamma = 2 \times 10^{-4}$ (red solid line), $\gamma = 1.1 \times 10^{-4}$ (green dashed line) $\gamma = 3 \times 10^{-18}$ (blue dotted line) and $\gamma = 0$ (dark dash-dotted line). In all the cases the last line represent the mean-field case.

5.3 D=1 case

The procedure to find the corrected Josephson frequency is analogous to the one used before. From the corrected $D = 1$ energy density, accounting also for the finite range corrections, found in (3.103)

$$\mathcal{E} = \frac{1}{2}g_0 n^2 - \frac{2}{3\pi}\sqrt{\frac{m}{\hbar^2}}\frac{(g_0 n)^{\frac{3}{2}}}{1 - 2r_s n} \tag{5.81}$$

the Lagrangian density is given by

$$\mathscr{L} = \sum_{k=1}^{2}\left(\hbar\Phi_k^*(t)\partial_t\Phi_k(t) - \frac{1}{2}g_0|\Phi_k(t)|^4 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar}\frac{|\Phi_k(t)|^3}{1 - 2r_s|\Phi_k(t)|^2}\right) +$$

$$+ \frac{J}{2}(\Phi_1^*(t)\Phi_2(t) + \Phi_2^*(t)\Phi_1(t)) \tag{5.82}$$

$$\equiv\mathscr{L}_0 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar}\left(\frac{|\Phi_1(t)|^3}{1 - 2r_s|\Phi_1(t)|^2} + \frac{|\Phi_2(t)|^3}{1 - 2r_s|\Phi_k(t)|^2}\right)$$

where $\mathscr{L}_0$ is the mean-field Lagrangian density and the second term arises from the beyond mean-field calculation. The Lagrangian is thus

$$\mathcal{L} = \mathcal{L}_0 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar L^{\frac{1}{2}}}\left(\frac{|\varphi_1(t)|^3}{1 - \frac{2r_s}{L}|\varphi_1(t)|^2} + \frac{|\varphi_2(t)|^3}{1 - \frac{2r_s}{L}|\varphi_2(t)|^2}\right) \tag{5.83}$$

where $\mathscr{L}_0$ represents the mean-field Lagrangian density, and the second term accounts for the beyond mean-field calculation, taking into account quantum fluctuations.

Then, a Madelung transformation (2.7) is performed and the resulting Lagrangian is dependent on the number of particle in each sites N_k

$$\mathcal{L} = \mathcal{L}_0 + \frac{2\sqrt{mg_0^3}}{3\pi\hbar L^{\frac{1}{2}}}\left(\frac{N_1^{\frac{3}{2}}(t)}{1 - \frac{2r_s}{L}N_1(t)} + \frac{N_2^{\frac{3}{2}}(t)}{1 - \frac{2r_s}{L}N_2(t)}\right) \tag{5.84}$$

which can be rewritten in terms of the total number of particles N and the population imbalance z as

$$\mathcal{L} = \mathcal{L}_0 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{3\sqrt{2}\pi\hbar L^{\frac{1}{2}}}\left[\frac{(1+z)^{\frac{3}{2}}}{1 - r_s n(1+z)} + \frac{(1-z)^{\frac{3}{2}}}{1 - r_s n(1-z)}\right] \tag{5.85}$$

Defining the adimensional common factor multiplying $(1 \pm z)$ at the denominators as

$$R \equiv -r_s n = \frac{1}{2}\chi g_0 n \tag{5.86}$$

as the opposite product between the constant r_s and the number density.

5.3.1 Josephson Frequency

As before, one exploits the low population imbalance limit, namely $z \ll 1$, that allows the following expansion for the numerator

$$(1 \pm z)^{\frac{3}{2}} = 1 \pm \frac{3}{2}z + \frac{3}{8}z^2 + O(z^3) \tag{5.87}$$

and for the denominator

$$\frac{1}{1 + R(1 \pm z)} = \frac{1}{1 + R} \mp \frac{Rz}{(1 + R)^2} + \frac{R^2 z^2}{(1 + R)^3} + O(z^3)$$

$$= \frac{1}{1 + R}\left(1 \mp \frac{Rz}{1 + R} + \frac{R^2 z^2}{(1 + R)^2}\right) + O(z^3) \tag{5.88}$$

Putting (5.88) and (5.87) together one obtains

$$
\frac{(1 \pm z)^{\frac{3}{2}}}{1 + R(1 \pm z)} = \frac{1 \pm \frac{3}{2}z + \frac{3z^2}{8}}{1 + R}\left(1 \mp \frac{Rz}{1 + R} + \frac{R^2 z^2}{(1 + R)^2}\right) + O(z^3)
$$
$$
= \frac{1}{1 + R}\left(1 \pm \frac{3}{2}z \mp \frac{Rz}{1 + R} + \frac{3z^2}{8} - \frac{3Rz^2}{2(1 + R)} + \frac{R^2 z^2}{(1 + R)^2}\right) + O(z^3)
$$
(5.89)

and the sum of the two contribution results in

$$
\frac{(1 + z)^{\frac{3}{2}}}{1 + R(1 + z)} + \frac{(1 - z)^{\frac{3}{2}}}{1 + R(1 - z)} = \frac{1}{1 + R}\left(2 + \frac{3z^2}{4} - \frac{3Rz^2}{1 + R} + \frac{2R^2 z^2}{(1 + R)^2}\right) + O(z^3)
$$
(5.90)

Inserting it (5.85), one gets the Lagrangian in the low population imbalance limit, which is given by

$$
\mathcal{L} = \mathcal{L}_0 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{3\sqrt{2}\pi\hbar L^{\frac{1}{2}}}\frac{1}{1 + R}\left(\not{2} + \frac{3}{4}z^2 - \frac{3Rz^2}{1 + R} + \frac{2R^2 z^2}{(1 + R)^2}\right)
$$
(5.91)

where the first term in the parenthesis is disregarded since it is a constant. Factoring out $3z^2/4$ one obtains

$$
\mathcal{L} = \mathcal{L}_0 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{4\sqrt{2}\pi\hbar L^{\frac{1}{2}}}\frac{z^2}{1 + R}\left(1 - 4\frac{R}{1 + R} + \frac{8}{3}\frac{R^2}{(1 + R)^2}\right)
$$
(5.92)

Finally, the beyond mean-field Lagrangian in $D = 1$ case is given by

$$
\mathcal{L} = \frac{N\hbar}{2}z\dot{\phi} - \left(\frac{UN^2 + JN}{4}\right)z^2 - \frac{JN}{4}\phi^2 + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{4\sqrt{2}\pi\hbar L^{\frac{1}{2}}(1 + R)}\left(1 - 4\frac{R}{1 + R} + \frac{8}{3}\frac{R^2}{(1 + R)^2}\right)z^2
$$
(5.93)

Therefore the Euler-Lagrangian equations are given by

$$
\begin{cases}
0 = \frac{N\hbar}{2}\dot{\phi} - \left(\frac{UN^2 + JN}{2}\right)z + \frac{\sqrt{mg_0^3}N^{\frac{3}{2}}}{2\sqrt{2}\pi\hbar L^{\frac{1}{2}}(1 + R)}\left(1 - 4\frac{R}{1 + R} + \frac{8}{3}\frac{R^2}{(1 + R)^2}\right)z \\
0 = -\frac{N\hbar}{2}\dot{z} - \frac{JN}{2}\phi
\end{cases}
$$
(5.94)

Similarly to the to the mean-field case, we can rearrange the equation to recover the Euler-Lagrange equations of a harmonic oscillator, in fact

$$
\begin{cases}
\dot{\phi} = \left[\frac{UN + J}{\hbar} - \frac{\sqrt{mg_0^3}N^{\frac{1}{2}}}{\sqrt{2}\pi\hbar^2 L^{\frac{1}{2}}(1 + R)}\left(1 - 4\frac{R}{1 + R} + \frac{8}{3}\frac{R^2}{(1 + R)^2}\right)\right]z \\
\dot{z} = -\frac{J}{\hbar}\phi
\end{cases}
\implies
\begin{cases}
\dot{\phi} + \Omega^2\phi = 0 \\
\dot{z} + \Omega^2 z = 0
\end{cases}
$$
(5.95)

where

$$
\Omega \equiv \sqrt{\frac{1}{\hbar^2}(J^2 + JUN) - \frac{J\sqrt{g_0^3 nm}}{\sqrt{2}\pi\hbar^3(1 + R)}\left(1 - 4\frac{R}{1 + R} + \frac{8}{3}\frac{R^2}{(1 + R)^2}\right)}
$$
(5.96)

is the corrected Josephson frequency and expressing it as a function of g, χ and n one has

$$
\Omega \equiv \sqrt{\frac{1}{\hbar^2}(J^2 + Jg_0 n) - \frac{J\sqrt{2g_0^3 nm}}{\pi\hbar^3(2 + \chi g_0 n)}\left(1 - 4\frac{\chi g_0 n}{2 + \chi g_0 n} + \frac{8}{3}\frac{(\chi g_0 n)^2}{(2 + \chi g_0 n)^2}\right)}
$$
$$
= \frac{J}{\hbar}\sqrt{1 + \frac{g_0 n}{J}\left[1 - \frac{1}{\pi(2 + \chi g_0 n)}\sqrt{\frac{m}{\hbar^2}\frac{2g_0}{n}}\left(1 - 4\frac{\chi g_0 n}{2 + \chi g_0 n} + \frac{8}{3}\frac{(\chi g_0 n)^2}{(2 + \chi g_0 n)^2}\right)\right]}
$$
(5.97)

and remembering the definition of Rabi Frequency Ω_R

$$
\Omega = \Omega_R\sqrt{1 + \frac{g_0 n}{J}\left[1 - \frac{1}{\pi(2 + \chi g_0 n)}\sqrt{\frac{m}{\hbar^2}\frac{2g_0}{n}}\left(1 - 4\frac{\chi g_0 n}{2 + \chi g_0 n} + \frac{8}{3}\frac{(\chi g_0 n)^2}{(2 + \chi g_0 n)^2}\right)\right]}
$$
(5.98)

Writing now the Josephson frequency as a function of the s-wave scattering length a_s and the s-wave effective range r_s one get

$$\Omega = \Omega_R \sqrt{1 - \frac{2\hbar^2 n}{ma_s J}\left[1 - \frac{1}{\pi(1 - r_s n)\sqrt{-a_s n}}\left(1 + 4\frac{r_s n}{1 - r_s n} + \frac{8}{3}\frac{(r_s n)^2}{(1 - r_s n)^2}\right)\right]}$$ (5.99)

and remembering the definition of the reference energy ε_s, the gas parameter γ, and defining the ratio α in the 1-dimensional case as

$$\epsilon_s \equiv \frac{\hbar^2}{ma_s^2} \qquad \gamma \equiv a_s n \qquad \alpha \equiv \frac{r_s}{a_s}$$ (5.100)

the Josephson Frequency can also be written as

$$\Omega = \Omega_R \sqrt{1 - 2\gamma\frac{\varepsilon_s}{J}\left[1 - \frac{1}{\pi(1 - \alpha\gamma)\sqrt{-\gamma}}\left(1 + 4\frac{\alpha\gamma}{1 - \alpha\gamma} + \frac{8}{3}\frac{(\alpha\gamma)^2}{(1 - \alpha\gamma)^2}\right)\right]}$$ (5.101)

Note that the gas parameter γ must be negative due to the presence of the inverse of the square root of it. Analogously to the 3-dimensional case to acknowledge the degree of the beyond mean-field correction to the mean-field Josephson frequency we are taking in account the ratio between the corrected frequency and the mean-field one as a function of the strength parameter $\Xi = -2\gamma\varepsilon_s/J$

$$\frac{\Omega}{\Omega_{mf}} = \frac{\sqrt{1 + \Xi\left[1 - \frac{1}{\pi(1-\alpha\gamma)\sqrt{-\gamma}}\left(1 + 4\frac{\alpha\gamma}{1-\alpha\gamma} + \frac{8}{3}\frac{(\alpha\gamma)^2}{(1-\alpha\gamma)^2}\right)\right]}}{\sqrt{1 + \Xi}}$$ (5.102)

Looking at the Fig. 5.7 one observes the following behavior: similarly to the case in which finite range correction are not involved the correction is more significant at higher strength parameters Ξ. When the strength parameters approach zero ($\Xi \to 0$), the beyond mean-field correction becomes irrelevant regardless of the ratio α, which means that for small values of Ξ the correction has a very weak dependence on the finite range corrections. Conversely, for larger Ξ, the relative correction is given by

$$\frac{\Omega}{\Omega_{mf}}\bigg|_{\Xi\gg1} = \sqrt{1 - \frac{1}{\pi(1 - \alpha\gamma)\sqrt{-\gamma}}\left(1 + 4\frac{\alpha\gamma}{1 - \alpha\gamma} + \frac{8}{3}\frac{(\alpha\gamma)^2}{(1 - \alpha\gamma)^2}\right)}$$ (5.103)

In Fig. 5.7, it is plotted the ratio Ω/Ω_{mf} for different values of α at fixed gas parameter $\gamma = -50$. One can observe that finite range corrections are more significant for higher magnitudes of $|\alpha|$. Specifically, for $\alpha < 0$, the correction is greater than the one without considering finite range corrections (i.e., for $\alpha = 0$) as the ratio is lowered even more than the mean-field case, while the opposite is true for $\alpha > 0$, where the correction is lower than the $\alpha = 0$ case, indeed in this case the Josephson frequency is closer to the mean-field value when compared with those having $\alpha = 0$ or the ones with negative α.

However, this behaviour is only present when the product between α and the gas parameter is tiny, as one can see in Fig. 5.8, in which the pure finite range effects are pictured. The figure on the left illustrate the behaviour described above: when $\alpha\gamma$ is positive ($\alpha < 0$), then the finite range correction is greater than one and so amplify the effect of correction driven by quantum fluctuations, vice versa when $\alpha\gamma$ is negative ($\alpha > 0$) then the quantum fluctuations effects are quietened down by the finite range corrections. In either case the finite range correction is more important as the absolute value $|\alpha|$ increases.

From the figure on the right in Fig. 5.8 instead one can observe the general behaviour of the finite range correction. For high value of the product $|\alpha\gamma|$ the finite range correction is close to zero, in particular if $\alpha > 0$ the finite range correction approaches zero from below, while the opposite is true for $\alpha < 0$, since the finite range correction reaches the horizontal axis from above. Note that there are two points in which the finite range correction is exactly zero, namely $\alpha\gamma = 3 \pm 2\sqrt{3}$. Therefore

for high values of the product $|\alpha\gamma|$ the mean-field Josephson frequency is a good approximation even accounting Gaussian fluctuations and finite range corrections, while for the particular value aforementioned the mean-field Josephson frequency is even exact, since the finite range correction cancels out the Gaussian fluctuations one. Furthermore, in $\alpha\gamma = 1$ there is an asymptote, however for the reasoning did in App. C the region near the asympote, where the finite range correction should be really high, is not physical, since there aren't couples of (α, γ) for which the approximation (3.102) is valid. Finally, note that in the cases in which the finite range correction is negative the resulting beyond mean-field Josephson frequency is greater than the mean-field one, like in the 3-dimensional and 2-dimensional cases, however the total correction is small since the finite range one is close to zero.
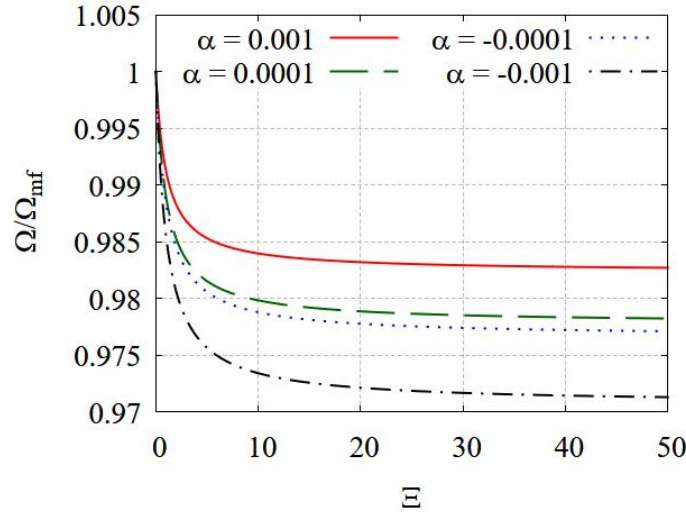

Figure 5.7: 1D beyond mean-field relative correction to the Josephson Frequency.
In the plot is pictured the ratio between the beyond mean-field Josephson frequency Ω and the mean-field one Ω_{mf} as a function of the strength parameter $\Xi = g_0 n/J$ for different values of the ratio $\alpha = \frac{r_s}{a_s}$: $\alpha = 0.001$ (red solid line), $\alpha = 0.0001$ (green dashed line) $\alpha = -0.0001$ (blue dotted line) and $\alpha = -0.001$ (dark dash-dotted line). In all the cases the gas parameter is $\gamma = a_s n = -50$.
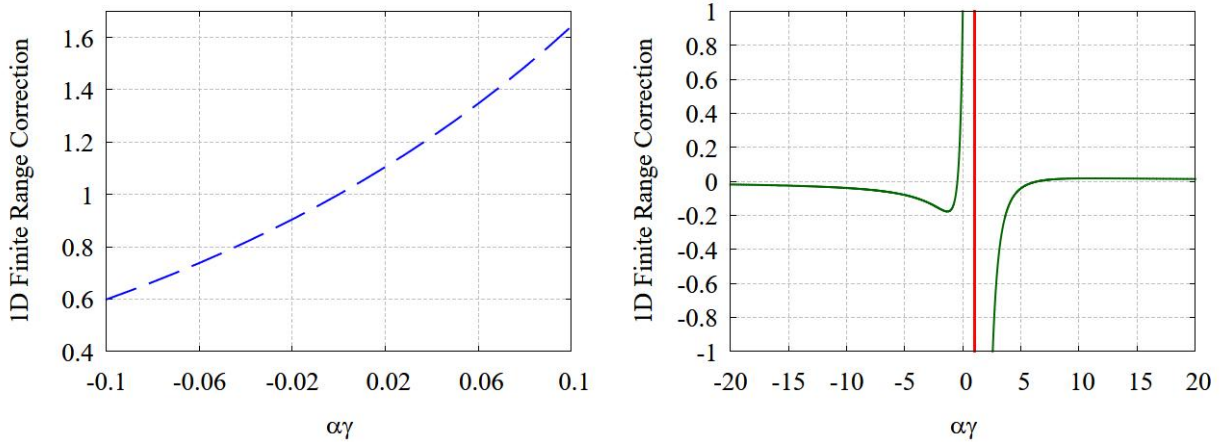

Figure 5.8: 1D Finite range correction $\frac{1}{(1-\alpha\gamma)}\left(1 + 4\frac{\alpha\gamma}{1-\alpha\gamma} + \frac{8}{3}\frac{(\alpha\gamma)^2}{(1-\alpha\gamma)^2}\right)$ as a function of the product between the gas parameter $\gamma = a_s n$ and the ratio $\alpha = r_s/a_s$ (blue line/green line), the red line $x = 1$ represents the zeroes of the denominators present in the correction.

5.3.2 Macroscopic Quantum Self Trapping

To determine the conserved energy, we consider the beyond mean-field Lagrangian in the $D = 1$ case, given by

$$L = \frac{N\hbar}{2}z\dot{\phi} - \frac{UN^2}{4}z^2 + \frac{JN}{2}\sqrt{1-z^2}\cos\phi + \frac{BN^{\frac{3}{2}}}{2\sqrt{2}}\left[\frac{(1+z)^{\frac{3}{2}}}{1+R(1+z)} + \frac{(1-z)^{\frac{3}{2}}}{1+R(1-z)}\right] \quad (5.104)$$

By analyzing the Lagrangian, we can determine the conserved energy, accounting for the corrections arising from quantum fluctuations and finite range effects. The conserved energy is given by

$$E = \frac{UN^2}{4}z^2 - \frac{JN}{2}\sqrt{1-z^2}\cos\phi - \frac{L\sqrt{m}}{3\sqrt{2\pi}\hbar}(UN)^{\frac{3}{2}}\left[\frac{(1+z)^{\frac{3}{2}}}{1+R(1+z)} + \frac{(1-z)^{\frac{3}{2}}}{1+R(1-z)}\right] \quad (5.105)$$

Imposing the inequality condition to have MQST (2.35), given by $E(z_0, \phi_0) > E(0, \pi)$, one gets

$$\frac{UN^2}{4}z_0^2 - \frac{JN}{2}\sqrt{1-z_0^2}\cos\phi_0+$$
$$-B'\frac{(UN)^{\frac{3}{2}}}{2}\left[\frac{(1+z_0)^{\frac{3}{2}}}{1+R(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1+R(1-z_0)}\right] > \frac{JN}{2} - B'\frac{(UN)^{\frac{3}{2}}}{1+R}$$

$$\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0+$$
$$-B'\frac{U^{\frac{3}{2}}N^{\frac{1}{2}}}{J}\left[\frac{(1+z_0)^{\frac{3}{2}}}{1+R(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1+R(1-z_0)}\right] > 1 - 2B'\frac{U^{\frac{3}{2}}N^{\frac{1}{2}}}{J(1+R)} \quad (5.106)$$

$$\frac{\Xi}{2}z_0^2 - \sqrt{1-z_0^2}\cos\phi_0+$$
$$-B'\Xi U^{\frac{1}{2}}N^{-\frac{1}{2}}\left[\frac{(1+z_0)^{\frac{3}{2}}}{1+R(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1+R(1-z_0)}\right] > 1 - 2B'\Xi\frac{U^{\frac{1}{2}}N^{-\frac{1}{2}}}{1+R}$$

and finally

$$\Xi > \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - B'U^{\frac{1}{2}}N^{-\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{3}{2}}}{1+R(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1+R(1-z_0)} - \frac{2}{1+R}\right)} \quad (5.107)$$

Defining the critical value $\Xi_{c,\,1D}$ as

$$\Xi_{c,\,1D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - B'U^{\frac{1}{2}}N^{-\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{3}{2}}}{1+R(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1+R(1-z_0)} - \frac{2}{1+R}\right)} \quad (5.108)$$

or, alternatively, as a function of g_0 and n

$$\Xi_{c,\,1D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - \frac{\sqrt{2}}{3\pi}\sqrt{\frac{m}{\hbar^2}}g_0^{\frac{1}{2}}n^{-\frac{1}{2}}\left(\frac{(1+z_0)^{\frac{3}{2}}}{1-r_sn(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1-r_sn(1-z_0)} - \frac{2}{1-r_sn}\right)} \quad (5.109)$$

or, also, as a function of the gas parameter γ and the ratio α

$$\Xi_{c,\,1D} \equiv \frac{1 + \sqrt{1-z_0^2}\cos\phi_0}{\frac{z_0^2}{2} - \frac{2}{3\pi}\frac{1}{\sqrt{-\gamma}}\left(\frac{(1+z_0)^{\frac{3}{2}}}{1-\gamma\alpha(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1-\gamma\alpha(1-z_0)} - \frac{2}{1-\gamma\alpha}\right)} \quad (5.110)$$

One obtains the same inequality condition of the mean-field case

$$\Xi > \Xi_{c,\,1D} \quad (5.111)$$

However in this case the critical value is reached for larger values of Ξ since $\Xi_{c,1D} > \Xi_{c,mf}$, namely the beyond mean-field critical value $\Xi_{c,1D}$ is larger than the mean-field one, as in the case in which the finite range corrections are not considered. To understand the beyond-mean field correction importance is useful to consider the ratio between the beyond-mean field critical value $\Xi_{c,1D}$ and the mean-field one $\Xi_{c,mf}$, which is given by

$$\frac{\Xi_{c,1D}}{\Xi_{c,mf}} = \frac{1}{1 - \frac{4}{3\pi}\frac{1}{z_0^2\sqrt{-\gamma}}\left(\frac{(1+z_0)^{\frac{3}{2}}}{1-\gamma\alpha(1+z_0)} + \frac{(1-z_0)^{\frac{3}{2}}}{1-\gamma\alpha(1-z_0)} - \frac{2}{1-\gamma\alpha}\right)} \tag{5.112}$$

Examining Fig. 5.9, we can observe the behavior of the relative beyond-mean field correction in relation to the mean-field MQST critical value for lower values of the ratio α. Similar to the 1-dimensional beyond-mean field Josephson frequency, the correction arising from the finite range term in the inter-atomic potential becomes more pronounced as the absolute value of the ratio $|\alpha|$ increases. Despite the apparent opposite behavior due to the sign of α compared to the Josephson frequency, where higher values of the ratio $\Xi_{c,1D}/\Xi_{c,mf}$ and lower values of Ω/Ω_{mf} are associated with negative values of α, the interpretation remains the same. Since in this case, Gaussian fluctuations increase the critical parameter while decreasing the Josephson frequency. Hence, we can conclude that the finite range contribution to the beyond-mean field correction works constructively with Gaussian fluctuations when the ratio is negative ($\alpha < 0$), indicating an attractive finite range correction. Conversely, when the finite range correction to the repulsive contact-like inter-atomic potential is also repulsive ($\alpha > 0$), the finite range contribution to the beyond-mean field quantities Ω and $\Xi_{c,1D}$ acts destructively, reducing the effects of quantum fluctuations and bringing the quantities closer to their mean-field value.

Moreover, the inclusion of finite range corrections does not change the concave shape of the ratio $\Xi_{c,1D}/\Xi_{c,mf}$. Specifically, $\Xi_{c,1D}/\Xi_{c,mf}$ remains lower for a low initial population imbalance z_0 at a fixed γ and α.

However, the aforementioned behavior holds true only for tiny values of the ratio α. For larger values of $|\alpha| \gtrsim 1$, the finite range correction dampens the beyond-mean field correction, and we retrieve the mean-field critical value $\Xi_{c,mf}$. This is analogous to what we observe for the 1-dimensional beyond-mean field Josephson frequency.
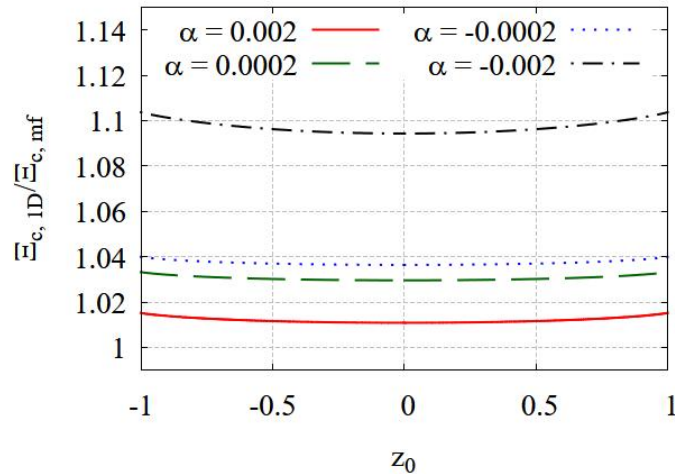

Figure 5.9: 1D beyond mean-field relative correction to the MQST critical value.
In the plot is pictured the ratio between the beyond mean-field MQST critical value $\Xi_{c,1D}$ and the mean-field one $\Xi_{c,mf}$ as a function of the initial population imbalance $z_0 \equiv z(t = 0) = z(t = 0) = (n_1(0) - n_2(0))/(n_1(0) + n_2(0))$ for different values of the ratio $\alpha = r_s/a_s$: $\alpha = 0.002$ (red solid line), $\alpha = 0.0002$ (green dashed line) $\alpha = -0.0002$ (blue dotted line) and $\alpha = -0.002$ (dark dash-dotted line). In all the cases the gas parameter is $\gamma = a_s n = -100$.

Conclusions

In this thesis, we have investigated the correction to the Josephson frequency in $D = 0, 1, 2, 3$ dimensions, considering quantum Gaussian fluctuations and finite range corrections to the inter-atomic potential. Initially assuming a contact-like potential, we calculated the contact coupling constant g_0 and the finite range correction coupling constant g_2 in relation to the s-wave scattering parameters a_s and r_s for $D = 1, 2, 3$ (whose values are in the table below). We excluded $D = 0$ as there is no scattering and, therefore, no finite range correction.

D	g_0	g_2
3	$\frac{4\pi\hbar^2 a_s}{m}$	$\frac{\pi\hbar^2}{m} a_s^2 r_s$
2	$-\frac{4\pi\hbar^2}{m} \frac{1}{\ln(\Lambda^2 a_s^2 e^\gamma)}$	$\frac{\pi^2 \hbar^2}{m} \frac{r_s^2}{\ln^2(\Lambda^2 a_s^2 e^\gamma)}$
1	$-\frac{2\hbar^2}{m a_s}$	$-\frac{\hbar^2}{2m} r_s$

Next, we examined the mean-field Josephson junction, which is influenced by two significant quantities. In the low population imbalance limit, the population imbalance $z(t)$ follows the equation of motion of a harmonic oscillator, and the oscillation frequency, known as the Josephson frequency Ω_{mf}, depends on the strength parameter $\Xi \equiv UN/J$. The critical value for Macroscopic Quantum Self Trapping (MQST) is the threshold above which MQST occurs. The mean-field expressions for these quantities are given by

$$\Omega_{mf} = \frac{J}{\hbar}\sqrt{1 + \Xi} \qquad \Xi_{c,\,mf} \equiv \frac{1 + \sqrt{1 - z_0^2}\cos\phi_0}{z_0^2/2}$$

To go beyond the mean-field values of these quantities, we replaced the mean-field energy density in the Lagrangian density with energy densities accounting for Gaussian fluctuations and finite range corrections.

D	$\mathcal{E}(n)$
3	$\frac{2\pi\hbar^2 a_s n^2}{m}\left(1 + \frac{128}{15\sqrt{\pi}}\frac{\sqrt{a_s^3 n}}{(1 + 4\pi a_s^2 r_s n)^2}\right)$
2	$-\frac{2\pi\hbar^2}{m}\frac{n^2}{\ln(c a_s^2 n)}\left(1 + \sum_k^\infty \mathcal{C}_\ell \left(\frac{4\pi^2 r_s^2 n}{\ln(c a_s^2 n)}\right)^\ell\right)$
1	$-\frac{\hbar^2 n^2}{m a_s}\left(1 - \frac{2\sqrt{2}}{3\pi}\frac{1}{\sqrt{-a_s n}(1 - 2r_s n)}\right)$
0	$\frac{g_0 n^2}{2}\left(1 - \frac{1}{n}\right)$

By doing so, we obtained new terms in the expressions for the Josephson frequency and the critical value of MQST in $D = 1, 2, 3$. However, for $D = 0$, the quantities remained unchanged. These corrections, denoted by $K_{\Omega,:r_s}$ and $K_{\Xi_c,:r_s}$, driven by Gaussian fluctuations and finite range effects, represent the original contributions of this work and can be expressed as follows:

$$\Omega = \frac{J}{\hbar}\sqrt{1 + \Xi K_{\Omega,\,r_s}} \qquad \Xi_c = \frac{1 + \sqrt{1 - z_0^2}cos\theta_0}{z_0^2/2 + K_{\Xi_c,\,r_s}} \tag{5.113}$$

D	$K_{\Omega,\,r_s}$
3	$1 + \dfrac{8\sqrt{2a_s^3 n}}{\sqrt{\pi}(1+2\pi a_s^2 r_s n)^2}\left(1 - \dfrac{8}{3}\dfrac{2\pi a_s^2 r_s n}{1+2\pi a_s^2 r_s n} + \dfrac{8}{5}\dfrac{(2\pi a_s^2 r_s n)^2}{(1+2\pi a_s^2 r_s n)^2}\right)$
2	$1 - \dfrac{3}{2\ln(Cn)} + \dfrac{1}{\ln^2(Cn)} + \sum_{\ell=1}^{+\infty} C_\ell \left(\dfrac{2\pi^2 r_s^2 n}{\ln^2(Cn)}\right)^\ell \left(\dfrac{(\ell+2)(\ell+1)}{2} - \dfrac{(2\ell+1)(2\ell+3)}{2\ln(Cn)} + \dfrac{(2\ell+1)(\ell+1)}{\ln^2(Cn)}\right)$
1	$1 - \dfrac{1}{\pi(1-r_s n)\sqrt{-a_s n}}\left(1 + 4\dfrac{r_s n}{1-r_s n} + \dfrac{8}{3}\dfrac{(r_s n)^2}{(1-r_s n)^2}\right)$

D	$\Xi_{c,\,r_s}$
3	$\dfrac{2\sqrt{2}}{15\pi^2}\sqrt{\dfrac{m}{\hbar^2}}^{\,3} g_0^{\frac{3}{2}} n^{\frac{1}{2}}\left(\dfrac{(1+z_0)^{\frac{5}{2}}}{[1+2\pi a_s^2 r_s n(1+z_0)]^2} + \dfrac{(1-z_0)^{\frac{5}{2}}}{[1+2\pi a_s^2 r_s n(1-z_0)]^2} - \dfrac{2}{(1+2\pi a_s^2 r_s n)^2}\right)$
2	$\dfrac{1}{4}\left[\dfrac{(1+z_0)^2}{1+\frac{\ln(1+z_0)}{\ln(Cn)}} + \dfrac{(1-z_0)^2}{1+\frac{\ln(1-z_0)}{\ln(Cn)}} - 2(1+z_0^2) + \sum_{\ell=1}^{\infty} C_\ell\left(\dfrac{2\pi^2 r_s^2 n}{\ln^2(Cn)}\right)^\ell\left(\dfrac{(1+z_0)^{\ell+2}}{\left[1+\frac{\ln(1+z_0)}{\ln(Cn)}\right]^{2\ell+1}} + \dfrac{(1-z_0)^{\ell+2}}{\left[1+\frac{\ln(1-z_0)}{\ln(Cn)}\right]^{2\ell+1}} - 2\right)\right]$
1	$-\dfrac{\sqrt{2}}{3\pi}\sqrt{\dfrac{m}{\hbar^2}} g_0^{\frac{1}{2}} n^{-\frac{1}{2}}\left(\dfrac{(1+z_0)^{\frac{3}{2}}}{1-r_s n(1+z_0)} + \dfrac{(1-z_0)^{\frac{3}{2}}}{1-r_s n(1-z_0)} - \dfrac{2}{1-r_s n}\right)$

Ignoring the finite range correction, while the 3-dimensional and 2-dimensional cases the Josephson junction is higher than the mean-field value, in the 1-dimensional case we observe an opposite behaviour. In particular the beyond mean-field correction grows as the gas parameter ($\gamma = a_s^D n$) increases in the first two cases while is smaller and smaller in the last case. Vice versa, a common behaviour of the three cases is the one depending on the strength parameter Ξ, in fact the Gaussian fluctuation corrections are increased for larger strength parameter.

Instead, the MQST critical value dependence on the gas parameter is the opposite of the one observed for the Josephson frequency in each dimension. Increasing the gas parameter the beyond mean-field critical value is smaller than the mean-field one in the 3-dimensional and 2-dimensional cases, while it is higher for the 1-dimensional case. Furthermore for higher population imbalance the correction is greater in $D = 3, 2$ and lower in $D = 1$.

Including also the finite range corrections there are plenty of interesting behaviours. First of all, one can observe that in all the dimensional cases ($D = 3, 2, 1$), the finite range contribution to the beyond mean-field correction is higher when the absolute value of the ratio $\alpha = r_s/a_s$ increases. In the 2-dimensional case there is an independence on the sign of the ratio α, which comes from the quadratic dependence on r_s of the 2-dimensional g_2. However, in the 3-dimensional and 1-dimensional cases, the finite range contributions have a different effect on the beyond mean-field quantities depending on the sign of the ratio α. These contributions can either amplify or reduce the overall contribution, introducing a sign-dependent behavior. In particular, if the finite range correction to the repulsive contact-like potential is positive ($\alpha > 0$), the effect of quantum fluctuations is mitigated by the finite range correction. Conversely, if the finite range correction to the repulsive contact-like potential is negative ($\alpha < 0$), the finite range correction amplifies the corrections arising from quantum fluctuations. In the 1-dimensional case, the statement holds true, but with a caveat. When the product of the ratio α and the gas parameter γ is small, the finite range contributions to the beyond mean-field terms exhibit the expected behavior, where they amplify or reduce the quantum fluctuation contributions. However, for higher values of this product, an interesting phenomenon occurs. The finite range contributions become such that they cancel out with the quantum fluctuation contributions resulting in an approximation that closely matches the mean-field result.

Concerning further research, it would be intriguing to study the system at finite temperature, taking into account the correction due to thermal Gaussian fluctuations. Additionally, a more in-depth analysis of the 2-dimensional beyond mean-field Lagrangian, considering all finite range contributions beyond the first one, would provide a deeper understanding of the system's behavior.

In summary, this thesis has contributed to the understanding of the correction to the Josephson frequency and the MQST critical value in different dimensions, incorporating quantum Gaussian fluctuations and finite range corrections. The results obtained pave the way for future investigations

and open up exciting possibilities for exploring the system's behavior under additional conditions and considering higher-order corrections.

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Appendices

Appendix A

3D perturbative expansion and gas parameter limits

A perturbative approach is necessary to find the expression of the chemical potential μ as a function of the 3-dimensional number density n (3.30), namely

$$\mu(n) = g_0 n + \frac{4g_0}{3\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{(g_0 n)^{\frac{3}{2}}}{(1+\chi g_0 n)^2} - \frac{16g_0}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{\chi(g_0 n)^{\frac{5}{2}}}{(1+\chi g_0 n)^3} \tag{A.1}$$

starting from the relation (3.29), given by

$$n(\mu) = \frac{\mu}{g_0} - \frac{4}{3\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{\mu^{\frac{3}{2}}}{(1+\chi\mu)^2} + \frac{16}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}}\frac{\chi\mu^{\frac{5}{2}}}{(1+\chi\mu)^3} \tag{A.2}$$

It is convenient to work with adimensional quantities, in particular the gas parameter γ, the ratio α and the chemical potential per energy reference ν, which are defined as

$$\gamma \equiv a_s^3 n \tag{A.3}$$

$$\alpha \equiv \frac{r_s}{a_s} \tag{A.4}$$

and

$$\nu \equiv \frac{\mu}{\varepsilon_s} = \frac{\mu m a_s^2}{\hbar^2} \tag{A.5}$$

Then, expressing the constant coupling g_0 and χ as a function of α and ν

$$g_0 \equiv \frac{4\pi\hbar^2 a_s}{m} = \frac{4\pi\mu a_s^3}{\nu} \tag{A.6}$$

and

$$\chi \equiv \frac{4m}{\hbar^2}\frac{g_2}{g_0} = \frac{m}{\hbar^2}a_s r_s = \alpha\frac{\nu}{\mu} \tag{A.7}$$

and multiplying each side of the equation (A.2) for the cube of s-wave scattering length a_s^3 one has

$$a_s^3 n = a_s^3\mu\frac{\nu}{4\pi\mu a_s^3} - a_s^3\frac{4}{3\pi^2}\left(\frac{m\mu}{\hbar^2}\right)^{\frac{3}{2}}\frac{1}{(1+\alpha\nu)^2} + a_s^3\frac{16}{15\pi^2}\left(\frac{m\mu}{\hbar^2}\right)^{\frac{3}{2}}\frac{\alpha\nu}{(1+\alpha\nu)^3}$$

$$\gamma = \frac{\nu}{4\pi} - \frac{4}{3\pi^2}\frac{\nu^{\frac{3}{2}}}{(1+\alpha\nu)^2} + \frac{16}{15\pi^2}\frac{\alpha\nu^{\frac{5}{2}}}{(1+\alpha\nu)^3} \tag{A.8}$$

$$\gamma = \frac{\nu}{4\pi}\left(1 - \frac{16}{3\pi}\frac{\nu^{\frac{1}{2}}}{(1+\alpha\nu)^2} + \frac{64}{15\pi}\frac{\alpha\nu^{\frac{3}{2}}}{(1+\alpha\nu)^3}\right)$$

To find an expression for ν then we do some approximations

$$
\begin{aligned}
\nu &= 4\pi\gamma \left(1 - \frac{16}{3\pi} \frac{\nu^{\frac{1}{2}}}{(1+\alpha\nu)^2} + \frac{64}{15\pi} \frac{\alpha\nu^{\frac{3}{2}}}{(1+\alpha\nu)^3} \right)^{-1} \\
&\simeq 4\pi\gamma \left(1 + \frac{16}{3\pi} \frac{\nu^{\frac{1}{2}}}{(1+\alpha\nu)^2} - \frac{64}{15\pi} \frac{\alpha\nu^{\frac{3}{2}}}{(1+\alpha\nu)^3} \right) \\
&\simeq 4\pi\gamma \left(1 + \frac{32}{3\sqrt{\pi}} \frac{\gamma^{\frac{1}{2}}}{(1+4\pi\alpha\gamma)^2} - \frac{512\sqrt{\pi}}{15} \frac{\alpha\gamma^{\frac{3}{2}}}{(1+4\pi\alpha\gamma)^3} \right)
\end{aligned} \tag{A.9}
$$

and this equation is the analogue of the (A.1), written in terms of adimensional variables. Note that these approximations are valid only if the following inequality is satisfied

$$
-0.1 < \frac{32}{3\sqrt{\pi}} \frac{\gamma^{\frac{1}{2}}}{(1+4\pi\alpha\gamma)^2} - \frac{512\sqrt{\pi}}{15} \frac{\alpha\gamma^{\frac{3}{2}}}{(1+4\pi\alpha\gamma)^3} < 0.1 \tag{A.10}
$$

Unfortunately, this inequality is analytical in the gas parameter γ and in α, only for $\alpha = 0$, in other words when the finite range correction is disregarded. In this case, setting $\alpha = 0$, one has the following inequality, with the left inequality always satisfied

$$
-0.1 < \frac{32}{3\sqrt{\pi}}\gamma^{\frac{1}{2}} < 0.1 \tag{A.11}
$$

Therefore in the case of a contact-like inter-atomic potential the gas parameter is bounded from above

$$
\gamma < \frac{9\pi}{102400} \simeq 2.8 \times 10^{-4} \tag{A.12}
$$

Since including the finite range correction give us a non analytical inequation, to find the domain of the gas parameter in which the approximation used above holds we consider the graphs in Fig. A.1 and Fig. A.2.
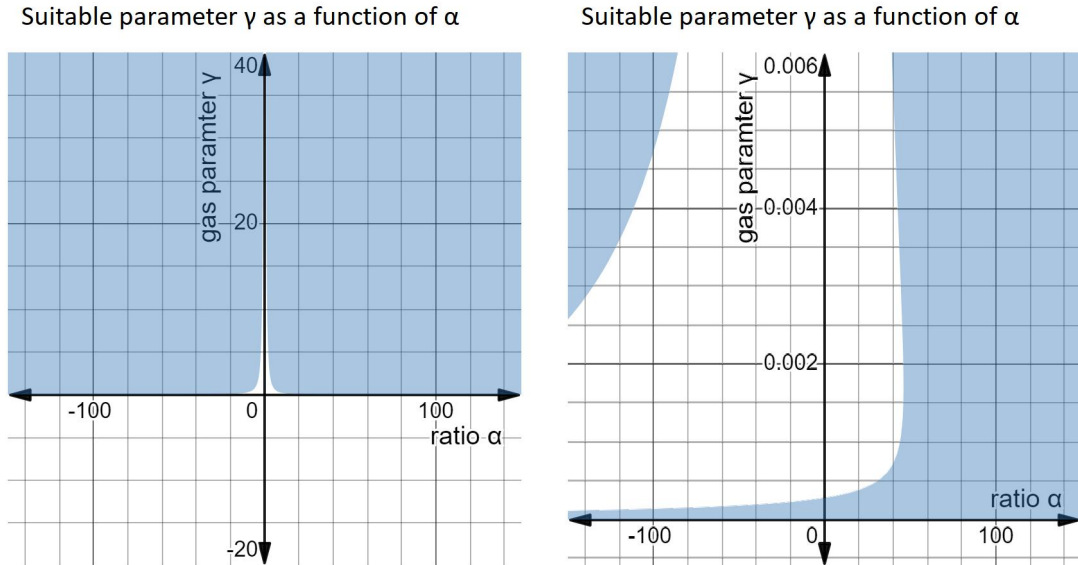

Figure A.1: Suitable $\gamma = a_s^3 n$ parameter as a function of $\alpha = r_s/a_s$.

The blue regions represent solutions of the inequality $-0.1 < \frac{32}{3\sqrt{\pi}} \frac{\gamma^{\frac{1}{2}}}{(1+4\pi\alpha\gamma)^2} - \frac{512\sqrt{\pi}}{15} \frac{\alpha\gamma^{\frac{3}{2}}}{(1+4\pi\alpha\gamma)^3} < 0.1$

The left figure focuses on high values of the gas parameter γ, while the right focuses on smaller ones.
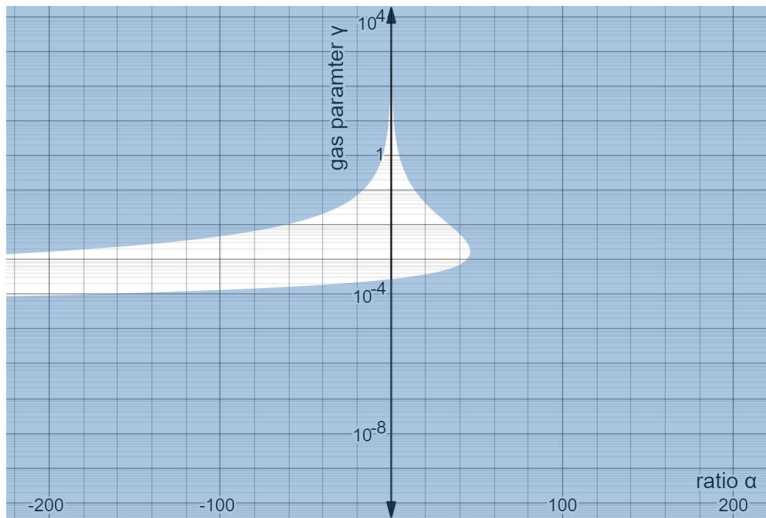
Figure A.2: Logarithmic plot of suitable $\gamma = a_s^3 n$ parameter as a function of $\alpha = r_s/a_s$.
The blue region represents solutions of the inequality $-0.1 < \frac{32}{3\sqrt{\pi}} \frac{\gamma^{\frac{1}{2}}}{(1+4\pi\alpha\gamma)^2} - \frac{512\sqrt{\pi}}{15} \frac{\alpha\gamma^{\frac{3}{2}}}{(1+4\pi\alpha\gamma)^3} < 0.1$,
the γ axis is in logarithmic scale.

From the graphs, particularly the logarithmic one, it can be observed that the range of γ in which the approximation does not hold is not only bounded from below, but also from above. When considering positive ratios, $\alpha > 0$, the lower limit is raised while the upper limit is lowered. As a result, for sufficiently high ratios, $(\alpha \gtrsim 46)$ all values of the gas parameter become suitable for the perturbative approach used above. On the other hand, for negative ratios $\alpha < 0$, both the upper and lower limits are lowered. However, it should be noted that the forbidden region shrinks as α decreases.

Appendix B

2D perturbative expansion and gas parameter limits

To find the expression of the chemical potential μ as a function of the number density n a perturbative calculations are involved as follow. Let's first concentrate in the case of a contact-like potential, setting for the moment the s-wave effective range r_s equal to zero ($r_s = 0$) in the equation of the 2-dimensional number density retrieved from the corresponding Grand Potential (3.66), i.e.

$$n = \frac{m\mu}{8\pi\hbar^2}\left(2\ln\left(\frac{\varepsilon_0}{\mu}\right) - 1\right) \tag{B.1}$$

Since we want to work with adimensional quantities, we define the gas parameter γ

$$\gamma \equiv a_s^2 n \tag{B.2}$$

and the ratio between the chemical potential μ and the reference energy ε_s

$$\nu \equiv \frac{\mu}{\varepsilon_s} = \frac{ma_s^2\mu}{\hbar^2} \tag{B.3}$$

which is related to μ/ε_0 in such a way that

$$\frac{\mu}{\varepsilon_0} = \frac{\nu e^{2\gamma + \frac{1}{2}}}{4} \tag{B.4}$$

Hence the equation (B.1) becomes

$$\begin{aligned}
\gamma &= \frac{\nu}{8\pi}\left(2\ln\left(\frac{4}{e^{2\gamma+\frac{1}{2}}\nu}\right) - 1\right) \\
&= -\frac{\nu}{4\pi}\ln\left(\frac{e^{2\gamma+1}\nu}{4}\right)
\end{aligned} \tag{B.5}$$

We want to write ν as a function of γ, therefore

$$\begin{aligned}
\nu &= -\frac{4\pi\gamma}{\ln\left(\frac{e^{2\gamma+1}\nu}{4}\right)} \\
&= -\frac{4\pi\gamma}{\ln\left(\frac{e^{2\gamma+1}\pi\gamma}{\ln\left(\frac{4}{e^{2\gamma+1}\nu}\right)}\right)} \\
&= -\frac{4\pi\gamma}{\ln\left(e^{2\gamma+1}\pi\gamma\right) - \ln\left(\ln\left(\frac{4}{e^{2\gamma+1}\nu}\right)\right)}
\end{aligned} \tag{B.6}$$

However there is still the dependence on ν on the second logarithm in the denominator, and thus an approximation is required

$$\nu = -\frac{4\pi\gamma}{\ln\left(e^{2\gamma+1}\pi\gamma\right)\left(1 - \frac{\ln\left(\ln\left(\frac{4}{e^{2\gamma+1}\nu}\right)\right)}{\ln\left(e^{2\gamma+1}\pi\gamma\right)}\right)}$$

$$\simeq -\frac{4\pi\gamma}{\ln\left(e^{2\gamma+1}\pi\gamma\right)}$$

(B.7)

The range in which the approximation is valid is given by the following inequalities

$$-0.1 < \frac{-\ln\left(\ln\left(\frac{4}{e^{(2\gamma+1)}\nu}\right)\right)}{\ln\left(e^{(2\gamma+1)}\pi\gamma\right)} < 0.1$$

(B.8)

which solutions are found graphically using the plot in Fig. B.1 where the approximation is valid if $0.01283 \lesssim \gamma \lesssim 0.01421$ or $0 \leq \gamma \lesssim 3 \times 10^{-18}$.
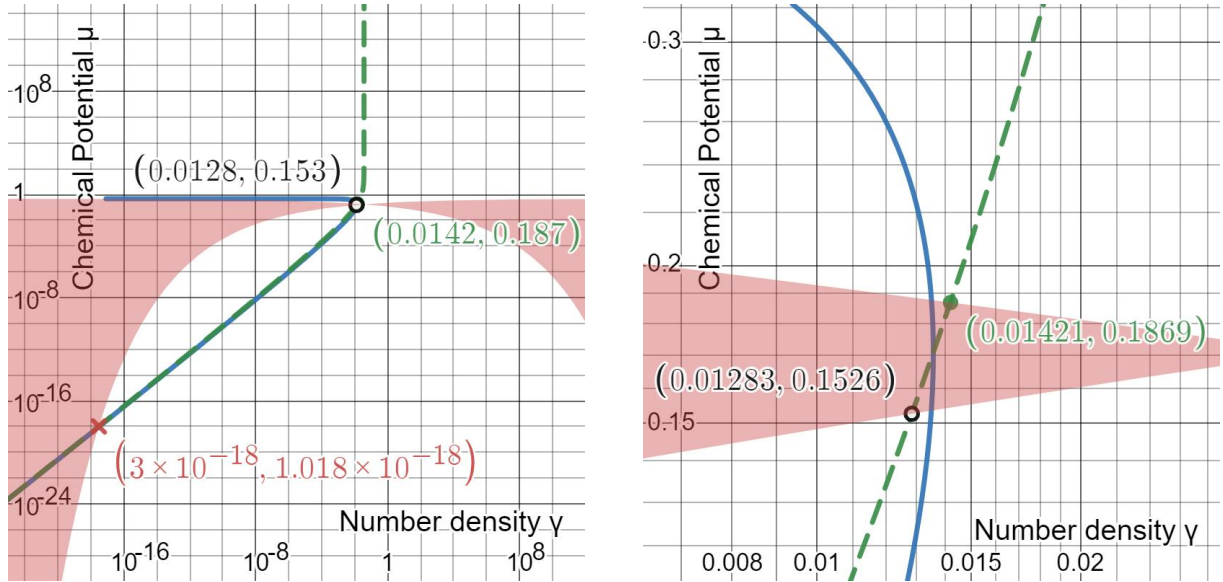

Figure B.1: Suitable $\gamma = a_s^2 n$ parameter for the approximation of the adimensional chemical potential ν in logarithmic scale.

The blue solid line represent the exact expression of the adimensional chemical potential ν as a function of the gas parameter γ, i.e. $\nu = -4\pi\gamma/\ln\left(e^{2\gamma+1}\nu/4\right)$; while the green dashed line represents the approximated chemical potential, namely $\nu = -4\pi\gamma/\ln\left(e^{2\gamma+1}\pi\gamma\right)$. The red regions represent portion of the solution of the inequality $-0.1 < -\ln\left(\ln\left(4/(e^{(2\gamma+1)}\nu)\right)\right)/\ln\left(e^{(2\gamma+1)}\pi\gamma\right) < 0.1$. The points in the graph correspond to the boundary of the domain in which the approximation is appropriate.

Including also the finite range corrections, the chemical potential μ is given by (3.77), namely

$$\mu = g_r n \left(1 + \sum_{k=2}^{+\infty} C_k (\chi_r g_r n)^{k-1} \right) \tag{B.9}$$

For simplicity in the following we limit to consider only the first finite range contribution, i.e. $k = 2$, which gives

$$\begin{aligned} \mu &= g_r n \left(1 + C_2(\chi_r g_r n) \right) \\ &= g_r n + \frac{9}{4} \chi_r g_r^2 n^2 \\ &= -\frac{4\pi\hbar^2}{m \ln\left(\frac{\mu}{\varepsilon_0}\right)} n - \frac{36\pi^3 \hbar^2 r_s^2}{m \ln^3\left(\frac{\mu}{\varepsilon_0}\right)} n^2 \end{aligned} \tag{B.10}$$

and rewriting it in adimensional quantities, upon defining the ratio α as

$$\alpha = \frac{r_s}{a_s} \tag{B.11}$$

one has, multiplying the LHS and the RHS for the s-wave scattering length a_s^2

$$\begin{aligned} \frac{m a_s^2 \mu}{\hbar^2} &= -\frac{4\pi a_s^2 n}{\ln\left(\frac{\mu}{\varepsilon_0}\right)} - \frac{36\pi^3 r_s^2 a_s^4 n^2}{a_s^2 \ln^3\left(\frac{\mu}{\varepsilon_0}\right)} \\ \nu &= -\frac{4\pi\gamma}{\ln\left(\frac{e^{2\gamma+1}\nu}{4}\right)} - \frac{36\pi^3 \alpha^2 \gamma^2}{\ln^3\left(\frac{e^{2\gamma+1}\nu}{4}\right)} \\ &= -\frac{4\pi\gamma}{\ln\left(\frac{e^{2\gamma+1}\nu}{4}\right)} \left(1 + \frac{9\pi^2 \alpha^2 \gamma}{\ln^2\left(\frac{e^{2\gamma+1}\nu}{4}\right)} \right) \end{aligned} \tag{B.12}$$

From the above equation we can rewrite the term below substituting the expression for ν

$$\begin{aligned} \ln\left(\frac{e^{2\gamma+1}\nu}{4}\right) &= \ln\left(e^{2\gamma+1}\pi\gamma\right) - \ln\ln\frac{4}{e^{2\gamma+1}\nu} + \ln\left(1 + \frac{9\pi^2\alpha^2\gamma}{\ln^2\left(\frac{e^{2\gamma+1}\nu}{4}\right)}\right) \\ &= \ln\left(e^{2\gamma+1}\pi\gamma\right)\left(1 + \frac{-\ln\ln\frac{4}{e^{2\gamma+1}\nu} + \ln\left(1 + \frac{9\pi^2\alpha^2\gamma}{\ln^2\left(\frac{e^{2\gamma+1}\nu}{4}\right)}\right)}{\ln\left(e^{2\gamma+1}\pi\gamma\right)}\right) \end{aligned} \tag{B.13}$$

which can be approximated to

$$\ln\left(\frac{e^{2\gamma+1}\nu}{4}\right) \simeq \ln\left(e^{2\gamma+1}\pi\gamma\right) \tag{B.14}$$

only if the following inequalities are satisfied

$$-0.1 < \frac{-\ln\left(\ln\left(\frac{4}{e^{(2\gamma+1)}\nu}\right)\right) + \ln\left(1 + \frac{9\pi^2\alpha^2\gamma}{\ln^2\left(\frac{e^{(2\gamma+1)}\nu}{4}\right)}\right)}{\ln\left(e^{(2\gamma+1)}\pi\gamma\right)} < 0.1 \tag{B.15}$$

If the inequalities above are satisfied then the approximated expression for the adimensional chemical potential ν is given by

$$\nu = -\frac{4\pi\gamma}{\ln\left(e^{2\gamma+1}\pi\gamma\right)} \left(1 + \frac{9\pi^2\alpha^2\gamma}{\ln^2\left(e^{2\gamma+1}\pi\gamma\right)} \right) \tag{B.16}$$

As for the contact-like case, the solutions of the inequalities were obtained graphically. While $0 \leq \gamma \lesssim 3 \times 10^{-18}$ remains a valid solution for the inequalities, the range $0.01283 \lesssim \gamma \lesssim 0.01421$ is no longer valid. However, there still exists a solution within the range $\gamma_{\min} \lesssim \gamma \lesssim \gamma_{\max}$. The boundaries of these ranges for selected values of α are presented in Table B.1, and the corresponding range is illustrated in Fig. B.2.

α	$\gamma_{\min}$	$\gamma_{\max}$
1	8.58×10^{-3}	9.43×10^{-3}
4	3.76×10^{-3}	4.37×10^{-3}
20	8.17×10^{-4}	1.12×10^{-3}
100	1.09×10^{-4}	1.97×10^{-4}

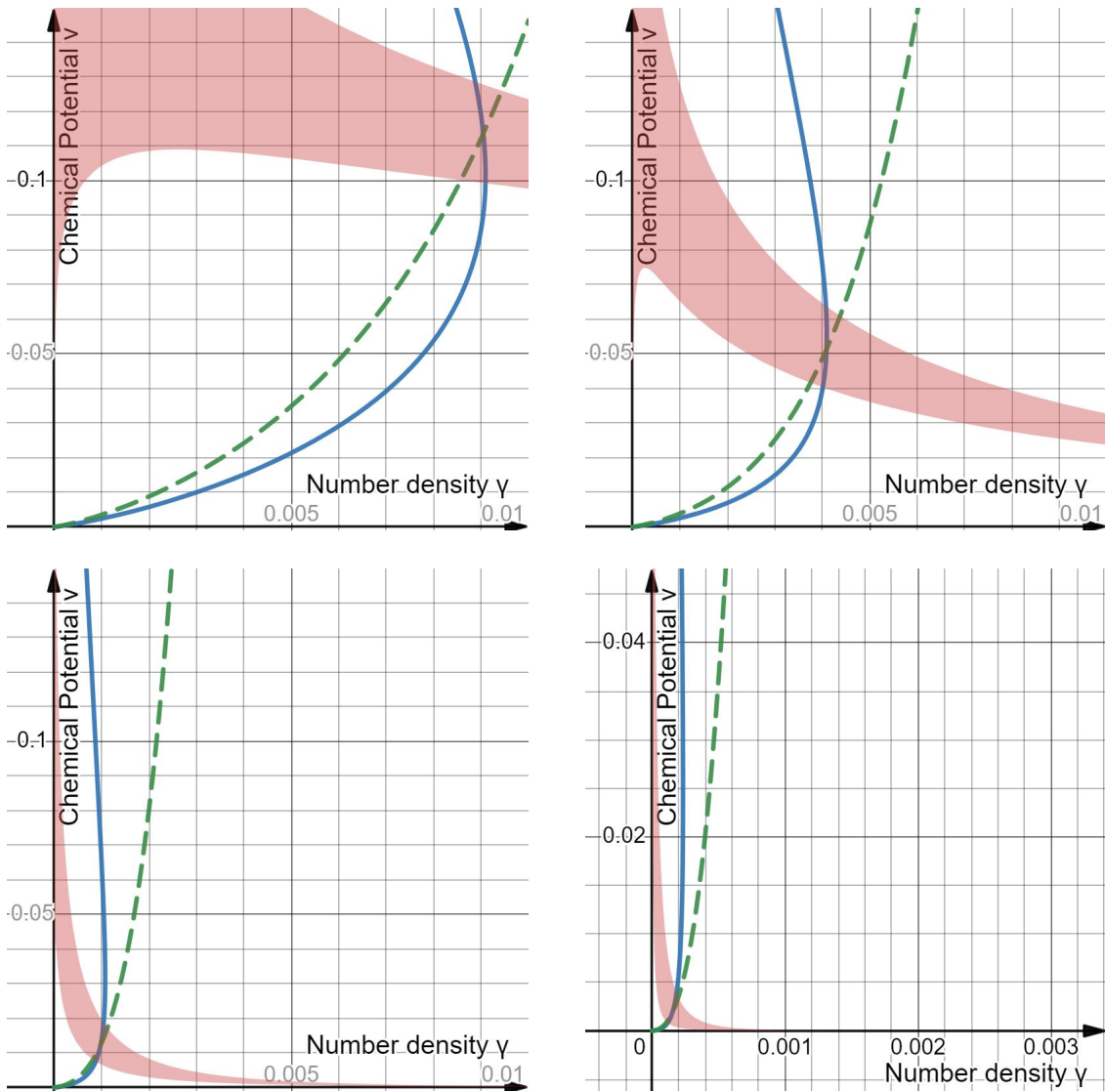Table B.1: Boundaries of suitable value of γ for certain α.

Figure B.2: Suitable $\gamma = a_s^2 n$ parameter for the approximation of the adimensional chemical potential ν for $\alpha = 1$ (top left), $\alpha = 4$ (top right), $\alpha = 20$ (bottom left) and $\alpha = 100$ (bottom right). The blue solid line represent the exact expression of the adimensional chemical potential ν as a function of the gas parameter γ, while the green dashed line represents the approximated chemical potential. The red regions represent portion of the solution of the inequality $-0.1 < \left[-\ln \left(\ln \left(4/(e^{(2\gamma+1)}\nu) \right) \right) + \ln \left(1 + 9\pi^2 \alpha^2 \gamma / \ln^2 \left(e^{(2\gamma+1)}\nu/4 \right) \right) \right] / \ln \left(e^{(2\gamma+1)} \pi \gamma \right) < 0.1$.

Appendix C

1D perturbative expansion and gas parameter limits

Also in the 1-dimensional case, a perturbative approach is necessary to find the expression of the chemical potential μ as a function of the 1-dimensional number density n (3.102), namely

$$\mu(n) = g_0 n - \frac{g_0}{\pi}\left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}}\frac{(g_0 n)^{\frac{1}{2}}}{1+\chi g_0 n} + \frac{2g_0}{3\pi}\left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}}\frac{\chi(g_0 n)^{\frac{3}{2}}}{(1+\chi g_0 n)^2} \tag{C.1}$$

starting from the relation (3.101), given by

$$n(\mu) = \frac{\mu}{g_0} + \frac{1}{\pi}\left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}}\frac{\mu^{\frac{1}{2}}}{1+\chi\mu} - \frac{2}{3\pi}\left(\frac{m}{\hbar^2}\right)^{\frac{1}{2}}\frac{\chi\mu^{\frac{3}{2}}}{(1+\chi\mu)^2} \tag{C.2}$$

As done in the $D = 3$ case, since it is more convenient, we work with adimensional quantities, in particular the gas parameter γ, the ratio α and the chemical potential per energy reference ν, which are defined as

$$\gamma \equiv a_s n \tag{C.3}$$

$$\alpha \equiv \frac{r_s}{a_s} \tag{C.4}$$

and

$$\nu \equiv \frac{\mu}{\varepsilon_s} = \frac{\mu m a_s^2}{\hbar^2} \tag{C.5}$$

Then, expressing the constant coupling g_0 and χ as a function of α and ν

$$g_0 \equiv -\frac{2\hbar^2}{m a_s} = -\frac{2\mu a_s}{\nu} \tag{C.6}$$

and the parameter χ is the same of the $D = 3$ case, indeed

$$\chi \equiv \frac{4m}{\hbar^2}\frac{g_2}{g_0} = \frac{m}{\hbar^2}a_s r_s = \alpha\frac{\nu}{\mu} \tag{C.7}$$

and multiplying each side of the equation (C.2) for the s-wave scattering length a_s one has

$$a_s n(\mu) = -a_s\mu\frac{\nu}{2\mu a_s} + a_s\frac{1}{\pi}\left(\frac{m\mu}{\hbar^2}\right)^{\frac{1}{2}}\frac{1}{1+\alpha\nu} - a_s\frac{2}{3\pi}\left(\frac{m\mu}{\hbar^2}\right)^{\frac{1}{2}}\frac{\alpha\nu}{(1+\alpha\nu)^2}$$

$$\gamma = -\frac{\nu}{2} + \frac{1}{\pi}\frac{\nu^{\frac{1}{2}}}{1+\alpha\nu} - \frac{2}{3\pi}\frac{\alpha\nu^{\frac{3}{2}}}{(1+\alpha\nu)^2} \tag{C.8}$$

$$\gamma = -\frac{\nu}{2}\left(1 - \frac{2}{\pi}\frac{1}{\sqrt{\nu}(1+\alpha\nu)} + \frac{4}{3\pi}\frac{\alpha\sqrt{\nu}}{(1+\alpha\nu)^2}\right)$$

To find an expression for ν then we do some approximations

$$
\begin{aligned}
\nu &= -2\gamma \left(1 - \frac{2}{\pi} \frac{1}{\sqrt{\nu}(1 + \alpha\nu)} + \frac{4}{3\pi} \frac{\alpha\sqrt{\nu}}{(1 + \alpha\nu)^2} \right)^{-1} \\
&\simeq -2\gamma \left(1 + \frac{2}{\pi} \frac{1}{\sqrt{\nu}(1 + \alpha\nu)} - \frac{4}{3\pi} \frac{\alpha\sqrt{\nu}}{(1 + \alpha\nu)^2} \right) \\
&\simeq -2\gamma \left(1 + \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{-\gamma}(1 - 2\alpha\gamma)} - \frac{4\sqrt{2}}{3\pi} \frac{\alpha\sqrt{-\gamma}}{(1 - 2\alpha\gamma)^2} \right)
\end{aligned}
\tag{C.9}
$$

and this equation is the analogue of the (C.1), written in terms of adimensional variables. Note that these approximations are valid only if the following inequality is satisfied

$$
-0.1 < \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{-\gamma}(1 - 2\alpha\gamma)} - \frac{4\sqrt{2}}{3\pi} \frac{\alpha\sqrt{-\gamma}}{(1 - 2\alpha\gamma)^2} < 0.1
\tag{C.10}
$$

Unfortunately, also this inequality is analytical in the gas parameter γ and in α, only for $\alpha = 0$, in other words when the finite range correction is disregarded. In this case, setting $\alpha = 0$, one has the following inequality, with the left inequality always satisfied

$$
-0.1 < \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{-\gamma}} < 0.1
\tag{C.11}
$$

Therefore in the case of a contact-like inter-atomic potential the gas parameter is bounded from below by

$$
\gamma < -\frac{200}{\pi^2} \simeq -20.3
\tag{C.12}
$$

Since including the finite range correction give us a non analytical inequation, to find the domain of the gas parameter in which the approximation used above holds we consider the graphs in Fig. C.1 and Fig. C.2.
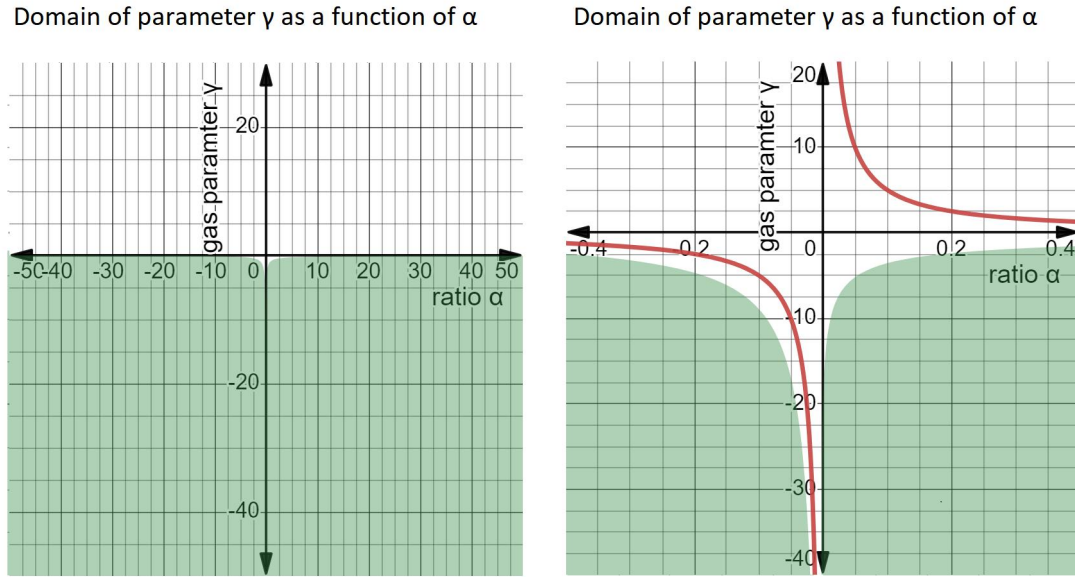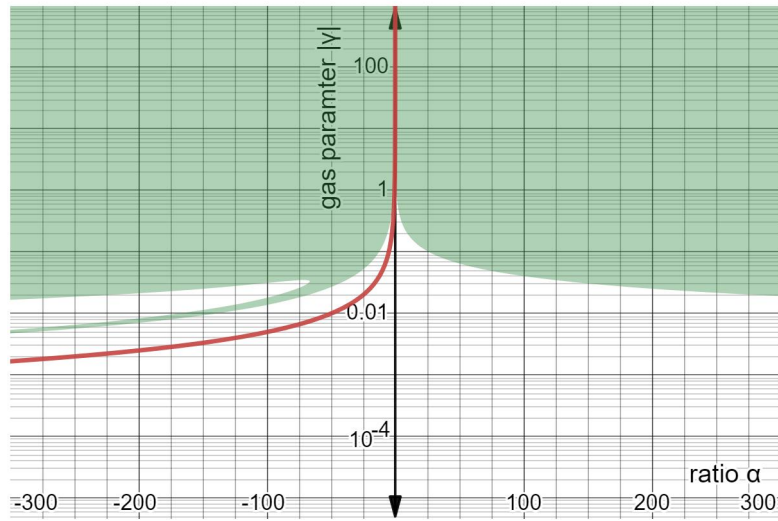

Figure C.1: Domain of $\gamma = a_s n$ parameter as a function of the ratio $\alpha = r_s / a_s$.
The green region represents the solution of the inequality $-0.1 < \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{-\gamma}(1-2\alpha\gamma)} - \frac{4\sqrt{2}}{3\pi} \frac{\alpha\sqrt{-\gamma}}{(1-2\alpha\gamma)^2} < 0.1$
The left figure focuses on high values of the ratio α, while the right focuses on smaller ones. The red lines represents the hyperbolas $2\alpha\gamma = 1$.

Figure C.2: Logarithmic plot of suitable $|\gamma| = |a_s n|$ parameter as a function of $\alpha = r_s/a_s$.
The green region represents solutions of the inequality $-0.1 < \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{|\gamma|(1-2\alpha|\gamma|)}} - \frac{4\sqrt{2}}{3\pi} \frac{\alpha\sqrt{|\gamma|}}{(1-2\alpha|\gamma|)^2} < 0.1$,
the γ axis is in logarithmic scale. The red lines represents the hyperbolas $2\alpha|\gamma| = 1$. The substitution
of γ with $|\gamma|$ is necessary to take the logarithmic scale of the gas parameter axis, since γ is negative.

From the logarithmic graph, it can be observed that when considering positive ratios, $\alpha > 0$, the
the domain of $|\gamma|$ is bounded from below and the limit drops as α increases. On the other hand,
for negative ratios $\alpha < 0$, we observe a similar behaviour but there is also a thin strip of permitted
value of $|\gamma|$ for smaller value of the parameter. The strip thins out and goes down as the value of α
decreases. Therefore for $\alpha < 0$ also some tiny value of the gas parameter $|\gamma|$ are admitted.

Note that, as shows the red line representing the hyperbolas $2\alpha\gamma = 1$ in Fig. C.1 and $2\alpha|\gamma| = -1$
in Fig. C.2 no values close to the zeroes of the denominator $1 - 2\alpha\gamma$ are admitted.